The critical 2d Stochastic Heat Flow and related models

Francesco Caravenna (Milano-Bicocca)

with **Rongfeng Sun** (NUS) and **Nikos Zygouras** (Warwick)

Seoul National University $\sim$ *Intensive Lecture Series*

Lecture 2 • 22 October 2024

Summary

From the previous episode...

Stochastic Heat Equation (SHE) $\leftrightsquigarrow$ Directed Polymers

discretized solution $\leftrightsquigarrow$ partition function

- Role of dimension d — sub-critical $d = 1$ — critical $d = 2$
- Role of disorder strength β — critical scaling $\beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$ in $d = 2$

Today's lecture: $d = 1$

"disorder relevance"

Main Result. Scaling limit of partition function toward SHE solution

Tools. Polynomial chaos, Lindeberg principle

Outline

White noise

Formally: $\big(\xi(t,x)\big)_{t>0,x\in\mathbb{R}^d}$ centered Gaussian "i.i.d. in space-time"

$$\mathbb{E}\big[\xi(t,x)\,\xi(s,y)\big] = \delta(t-s)\,\delta(x-y)$$

Rigorously: $\big(\langle\xi,\varphi\rangle\big)_{\varphi\in L^2}$ centered Gaussian "$\int \varphi(t,x)\,\xi(t,x)\,\mathrm{d}t\,\mathrm{d}x$"

$$\mathbb{E}\big[\langle\xi,\varphi\rangle\,\langle\xi,\psi\rangle\big] = \langle\varphi,\psi\rangle_{L^2} = \int \varphi(t,x)\,\psi(t,x)\,\mathrm{d}t\,\mathrm{d}x$$

$\varphi \longmapsto \langle\xi,\varphi\rangle$ is a random distribution $\varphi \in C_c^\infty$

$(d=0)$ $\xi(t) = \frac{\mathrm{d}}{\mathrm{d}t}B(t)$ derivative of BM

Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x) + \beta_{\mathrm{SHE}}\, \xi(t,x)\, u(t,x) \\ u(0,x) = 1 \ \ \text{(say)} \end{cases} \qquad t \geq 0,\ x \in \mathbb{R}^d$$

$$g_t(x) = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi\, t)^{d/2}} \qquad \text{heat kernel on } \mathbb{R}^d$$

"Mild" integral formulation (Duhamel's principle)

$$u(t,x) = 1 + \beta_{\mathrm{SHE}} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, g_{t-s}(x-y)\, \xi(s,y)\, u(s,y)$$

Finding the solution

Expoiting linearity of SHE "Picard iteration"

$$
\begin{aligned}
u(t,x) &= 1 + \beta_{\mathrm{SHE}} \int\limits_{0<s<t} \mathrm{d}s \int\limits_{\mathbb{R}^d} \mathrm{d}y \; g_{t-s}(x-y)\, \xi(s,y)\, \{1 + \ldots\} \\
&= 1 + \sum_{k=1}^{\infty} \underbrace{u^{(k)}(t,x)}_{\displaystyle (\beta_{\mathrm{SHE}})^k \int \cdots \int_{\substack{0<s_k<\ldots<s_1<t \\ y_k,\ldots,y_1 \in \mathbb{R}^d}} \mathrm{d}\vec{s}\, \mathrm{d}\vec{y} \prod_{i=1}^{k} g_{s_{i-1}-s_i}(y_{i-1}-y_i)\, \xi(s_i,y_i)}
\end{aligned}
$$

Wiener chaos expansion: meaningful for $d < 2$ $\qquad g_t(x) \in L^2$

Wiener chaos expansion

Multiple Wiener Integral for $\varphi(\vec{s},\vec{y}) \in L^2$ [Ito '51]

$$\int \cdots \int_{\substack{0<s_k<\ldots<s_1<t \\ y_k,\ldots,y_1 \in \mathbb{R}^d}} \mathrm{d}\vec{s}\, \mathrm{d}\vec{y}\; \varphi(\vec{s},\vec{y}) \prod_{i=1}^{k} \xi(s_i,y_i) \;=\; \langle \varphi \,,\, \xi^{\otimes k} \rangle$$

$(d=0)$ Iterated Ito integral $\displaystyle \int_0^t \left(\int_0^{s_1} \left(\int_0^{s_2} \varphi(\vec{s},\vec{y})\, \mathrm{d}B_{s_3} \right) \mathrm{d}B_{s_2} \right) \mathrm{d}B_{s_1}$

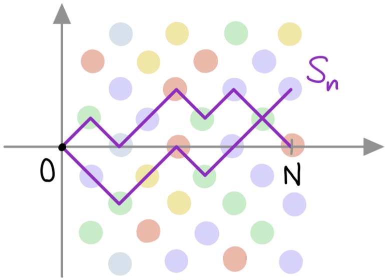
SHE solution in $d=1$

$$u(t,x) \;=\; 1 + \sum_{k=1}^{\infty} u^{(k)}(t,x)$$

series of stochastic integrals

Directed Polymer partition function

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}$ starting from $S_0 = 0$
- Independent Gaussians $\omega(n, z) \sim \mathcal{N}(0, 1)$
- $H(S, \omega) := \sum_{n=1}^{N} \omega(n, S_n)$



Partition Function

starting from $(0,0)$

$$Z_{N,\beta}^{\omega} = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N} \,\middle|\, S_0 = 0 \right] \quad \rightsquigarrow \quad \mathbb{E}\left[Z_{N,\beta}^{\omega}\right] = 1$$

Main result of this lecture

Stochastic Heat Equation in $d = 1$

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x) + \beta_{\text{SHE}}\, u(t,x)\, \xi(t,x) \\ u(0,x) \equiv 1 \end{cases} \tag{SHE}$$

Theorem ($d = 1$) [Alberts–Khanin–Quastel *AoP* 2014]

Rescaling $\beta_N = \dfrac{\beta_{\text{SHE}}}{N^{1/4}}$ $\qquad Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{d} u(1,0) = \text{Wiener chaos exp.}$

Proof applies to other disorder relevant models [C.S.Z. *JEMS* 2017]

Strategy of the proof

1. Express partition function Z^{ω}_{N,β_N} as a "discrete chaos expansion"

   Polynomial Chaos

   $\rightsquigarrow$ Qualitatively similar to Wiener chaos expansion of $u(t,x)$

2. Prove convergence of polynomial chaos to Wiener chaos

   Lindeberg Principle

   $\rightsquigarrow$ Far reaching generalization of the classical CLT

Polynomial chaos

- New disorder variables $\quad X_{n,z} := e^{\beta\omega(n,z) - \frac{1}{2}\beta^2} - 1$

  i.i.d. $\quad \mathbb{E}[X] = 0 \qquad \mathbb{V}\mathrm{ar}[X] = e^{\beta^2} - 1 \sim \beta^2$ as $\beta \downarrow 0$

- Random walk kernel $\quad q_n(z) := \mathrm{P}(S_n = z) \sim \frac{1}{N^{d/2}} g_{\frac{n}{N}}\left(\frac{x}{\sqrt{N}}\right)$ local CLT

Proposition (polynomial chaos)

$$Z_{N,\beta}^{\omega} = 1 + \sum_{k=1}^{N} \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ z_1,\ldots,z_k \in \mathbb{Z}^d}} \underbrace{q_{\vec{n}}(\vec{z})}_{\prod_{i=1}^{k} q_{n_i - n_{i-1}}(z_i - z_{i-1})} \prod_{i=1}^{k} X_{n_i, z_i}$$

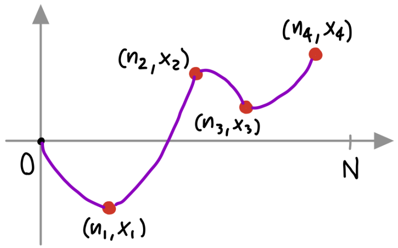
Proof: Mayer expansion

$$
\begin{aligned}
e^{\beta H(S,\omega)-\frac{1}{2}\beta^2 N} &= e^{\sum_{n=1}^{N}\{\beta\,\omega(n,S_n)-\frac{1}{2}\beta^2\}} = e^{\sum_{n=1}^{N}\sum_{z\in\mathbb{Z}^d}\{\beta\,\omega(n,z)-\frac{1}{2}\beta^2\}\,\mathbb{1}_{S_n=z}} \\
&= \prod_{n=1}^{N}\prod_{z\in\mathbb{Z}^d} e^{\{\beta\,\omega(n,z)-\frac{1}{2}\beta^2\}\,\mathbb{1}_{S_n=z}} \\
&= \prod_{n=1}^{N}\prod_{z\in\mathbb{Z}^d}\left\{1+\left(e^{\beta\,\omega(n,z)-\frac{1}{2}\beta^2}-1\right)\mathbb{1}_{S_n=z}\right\} \\
&= \prod_{n=1}^{N}\prod_{z\in\mathbb{Z}^d}\left\{1+X_{n,z}\,\mathbb{1}_{S_n=z}\right\} \\
&= 1+\sum_{k=1}^{N}\sum_{\substack{0<n_1<\ldots<n_k\le N\\ z_1,\ldots,z_k\in\mathbb{Z}^d}}\prod_{i=1}^{k}X_{n_i,z_i}\cdot\underbrace{\prod_{i=1}^{k}\mathbb{1}_{S_{n_i}=z_i}}_{\mathrm{E}[\ldots]\,=\,q_{\vec{n}}(\vec{z})}
\end{aligned}
$$

Summarizing

Partition function $Z^{\omega}_{N,\beta}$ is a **multi-linear polynomial** in the variables $X_{n,z}$

$$Z^{\omega}_{N,\beta} = 1 + \sum_{k=1}^{N} Z^{(k)}_{N,\beta} = 1 + \sum_{k=1}^{N} \sum_{|\boldsymbol{I}|=k} q(\boldsymbol{I})\, X^{\boldsymbol{I}}$$



with $\boldsymbol{I} = \{(n_1, z_1), \ldots, (n_k, z_k)\}$

- $q(\boldsymbol{I}) = q_{\vec{\boldsymbol{n}}}(\vec{\boldsymbol{z}}) = \prod_{i=1}^{k} q_{n_i - n_{i-1}}(z_i - z_{i-1})$
- $X^{\boldsymbol{I}} = \prod_{i=1}^{k} X_{n_i, z_i}$

Heuristics for $d = 1$

Rescale diffusively $(n, z) \rightsquigarrow (s, y) = (n/N, z/\sqrt{N})$ "Riemann sums"

Replace $X_{n,z}$ by Gaussians with matching variance $\mathbb{V}\mathrm{ar}[X] \sim \beta^2$

$$\tilde{X}_{n,z} = \beta_{\mathrm{SHE}} \cdot \sqrt{N} \int_{(\frac{n-1}{N},\frac{n}{N}]\times(\frac{z-1}{\sqrt{N}},\frac{z}{\sqrt{N}}]} \xi(s,y)\, \mathrm{d}s\, \mathrm{d}y \qquad \beta = \beta_N = \frac{\beta_{\mathrm{SHE}}}{N^{1/4}}$$

$$\tilde{Z}^{(k)}_{N,\beta_N} = (\beta_{\mathrm{SHE}})^k \int \cdots \int_{\substack{0<s_1<\ldots<s_k<1 \\ y_1,\ldots,y_k\in\mathbb{R}}} \mathrm{d}\vec{s}\, \mathrm{d}\vec{y} \prod_{i=1}^{k} \underbrace{\sqrt{N}\, q_{\lfloor N(s_i-s_{i-1})\rfloor}\big(\lfloor\sqrt{N}(y_i-y_{i-1})\rfloor\big)}_{\xrightarrow[N\to\infty]{} g_{s_i-s_{i-1}}(y_i-y_{i-1}) \text{ by local CLT}} \xi(s_i,y_i)$$

$$\xrightarrow[N\to\infty]{L^2} u^{(k)}(1,0)$$

Turning heuristics into a proof

Replacing $X_{n,z}$ by $\tilde{X}_{n,z}$ needs to be justified (Lindeberg Principle)

- Term-by-term convergence $\tilde{Z}^{(k)}_{N,\beta_N} \xrightarrow[N\to\infty]{L^2} u^{(k)}(1,0)$ is OK ($\forall k \in \mathbb{N}$)
- Tail contribution $\sum_{k>K} Z^{(k)}_{N,\beta_N}$ is small uniformly in N (L^2 bounds)

Theorem ($d=1$) [Alberts–Khanin–Quastel *AoP* 2014]

$$Z^{\omega}_{N,\beta_N} = 1 + \sum_{k=1}^{\infty} Z^{(k)}_{N,\beta_N} \xrightarrow[N\to\infty]{d} u(1,0) = 1 + \sum_{k=1}^{\infty} u^{(k)}(1,0) \qquad \left(\beta_N = \frac{\beta_{\mathrm{SHE}}}{N^{1/4}} \right)$$

Rough formulation

Families $X = (X_i)_{i \in \mathbb{T}}$ and $\tilde{X} = (\tilde{X}_i)_{i \in \mathbb{T}}$ of independent RVs ($\mathbb{T}$ countable)

- Matching moments $\quad \mathbb{E}[X_i] = \mathbb{E}[\tilde{X}_i] = 0 \qquad \mathbb{E}[X_i^2] = \mathbb{E}[\tilde{X}_i^2] = 1$
- 3rd moment bound $\quad m_3 := \max_{i \in \mathbb{T}} \left\{ \mathbb{E}\left[|X_i|^3\right] \vee \mathbb{E}\left[|\tilde{X}_i|^3\right] \right\} < \infty$

Functional $\Psi(x)$ depending on many variables x_i, but not too much on any x_i

Lindeberg Principle

$\Psi(X)$ and $\Psi(\tilde{X})$ are close in distribution

Polynomial chaos

$$\Psi(X) = \sum_{I \subseteq \mathbb{T}} q(I)\, X^I \qquad \Big(\text{with} \quad X^I := \prod_{i \in I} X_i\Big)$$

$$= \mathbb{E}\big[\Psi(X)\big] + \sum_{i \in \mathbb{T}} q(i)\, X_i + \sum_{i \neq j \in \mathbb{T}} q(i,j)\, X_i\, X_j + \dots$$

Variance and Influences

$$\mathrm{V}[\Psi] := \mathbb{V}\mathrm{ar}\big[\Psi(X)\big] = \sum_{I \neq \varnothing} q(I)^2$$

$$\mathrm{Inf}_i[\Psi] := \mathbb{E}\Big[\mathbb{V}\mathrm{ar}\big[\Psi(X)\,\big|\,X_{\mathbb{T}\setminus\{i\}}\big]\Big] = \sum_{I \ni i} q(I)^2$$

Noise sensitivity: [Benjaimini–Kalai–Schramm 2001] [Garban–Steif 2012]

Lindeberg Principle

$$d_{\text{weak}}\big(\Psi(X), \Psi(\tilde{X})\big) := \sup_{\substack{\varphi \in C^3(\mathbb{R} \to \mathbb{R}): \\ \|\varphi'\|_\infty, \|\varphi''\|_\infty, \|\varphi'''\|_\infty \le 1}} \Big| \mathbb{E}\big[\varphi(\Psi(X))\big] - \mathbb{E}\big[\varphi(\Psi(\tilde{X}))\big] \Big|$$

Theorem [Mossel–O'Donnel–Oleszkiewicz *Ann. Math.* 2010]

Ψ polynomial chaos of degree $\le K$

$$d_{\text{weak}}\big(\Psi(X), \Psi(\tilde{X})\big) \le \big(30\, m_3\big)^K \, \mathrm{V}[\Psi] \sqrt{\max_{i \in \mathbb{T}} \mathrm{Inf}_i[\Psi]}$$

U.I. of $(X_i^2)_{i \in \mathbb{T}}$ and $(\tilde{X}_i^2)_{i \in \mathbb{T}}$ is enough [C.S.Z. *AAP* 2017]

Back to SHE

For $\Psi(X) = \sum_{k=1}^{K} Z_{N,\beta_N}^{(k)}$ we compute

$$\mathrm{Inf}_{(m,x)}[\Psi] = \ldots = O(\beta_N^2) \xrightarrow[N\to\infty]{} 0$$

Replacing $X_{n,z}$ by $\tilde{X}_{n,z}$ is justified

$$d_{\mathrm{weak}}\left(\sum_{k=1}^{K} Z_{N,\beta_N}^{(k)}, \sum_{k=1}^{K} \tilde{Z}_{N,\beta_N}^{(k)} \right) \xrightarrow[N\to\infty]{} 0$$

The proof of $Z_{N,\beta_N}^{\omega} \xrightarrow[N\to\infty]{d} u(1,0)$ is completed $\square$

Summarizing

SHE and DP partition function in the sub-critical dimension $d = 1$

- SHE solution $u(t, x)$ well-defined and "explicit" (Wiener chaos)
- Partition function $Z^{\omega}_{N,\beta}$ enjoys similar representation (Polynomial chaos)

Replacing X by Gaussian $\tilde{X}$ $\rightsquigarrow$ coupling with white noise $\xi(t, x)$

justified by Lindeberg principle for polynomial chaos

Weak convergence of Z^{ω}_{N,β_N} toward $u(1, 0)$ (for $\beta_N = \frac{\beta_{\mathrm{SHE}}}{N^{1/4}}$)

Next lectures

Rescaled noise strength in $d = 2$: $\qquad \beta_N \sim \hat{\beta}\, \frac{\sqrt{\pi}}{\sqrt{\log N}} \qquad (\,\hat{\beta}_c = 1\,)$

Lectures 3–4. (R.S.) Critical dimension and phase transition $\qquad (d = 2,\ \hat{\beta} < \hat{\beta}_c)$

- Variance computation, log-normality, E-W fluctuations (log-corr. Gaussian)
- *Tools: hypercontractivity, concentration of measure*

Lectures 5–6. (N.Z.) The critical 2d Stochastic Heat Flow $\qquad (d = 2,\ \hat{\beta} = \hat{\beta}_c)$

- Coarse-graining, sharp variance asymptotics, high moment bounds
- *Tools: refined Lindeberg, renewal theory, functional inequalities*

감사합니다