

Scaling and Multiscaling in Financial Indexes: a Simple Model

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Joint work with Alessandro Andreoli (Padova),
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Additional results by Paolo Pigato e Mario Bonino.

Roma Tor Vergata ~ September 23, 2011

Outline

1. Black & Scholes and beyond
2. The Model
3. Main Results
4. Estimation and Simulations
5. Bivariate Model
6. Conclusions

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Black & Scholes model

Black & Scholes model for the price S_t of a stock price or index:

$$dS_t = S_t (r dt + \sigma dW_t)$$

- ▶ σ (the **volatility**) and r (the **interest rate**) are constant
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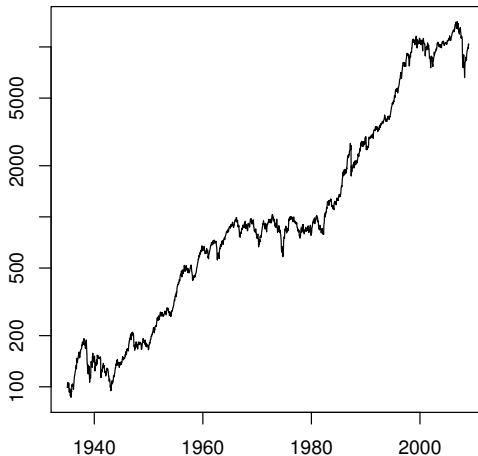
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Basic example: **Dow Jones Industrial Average (DJIA)**.

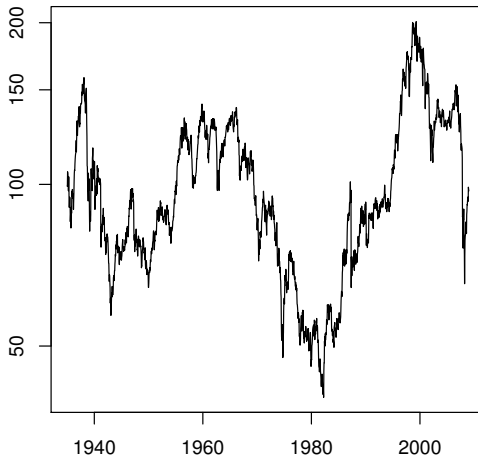
DJIA time series (1935-2009)

Exponential growth of the DJIA [log plot]:



DJIA time series (1935-2009)

DJIA after linear detrend [log plot]:



Beyond the Black & Scholes model

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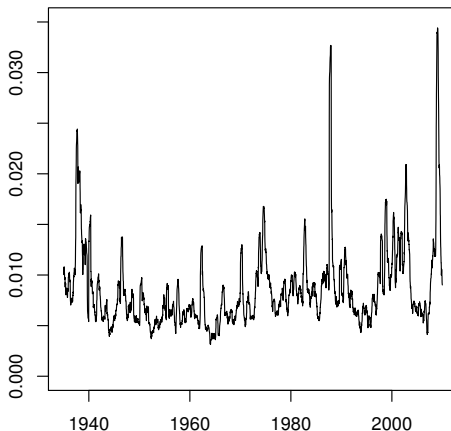
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- ▶ The **volatility σ is not constant**: it may have high peaks (“**shocks**” in the market).

DJIA time series (1935-2009)

Empirical volatility



Local standard deviation of log-returns in a window of 100 days

Beyond the Black & Scholes model

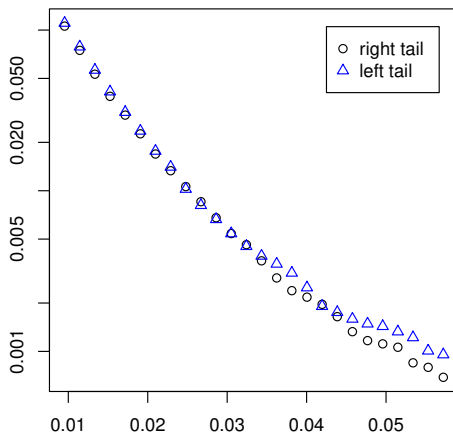
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DJIA time series (1935-2009)

Tails of daily log-return distribution [log plot]



Daily log-return **standard deviation ≈ 0.01** \rightarrow Range: 1 to 6 st. dev.

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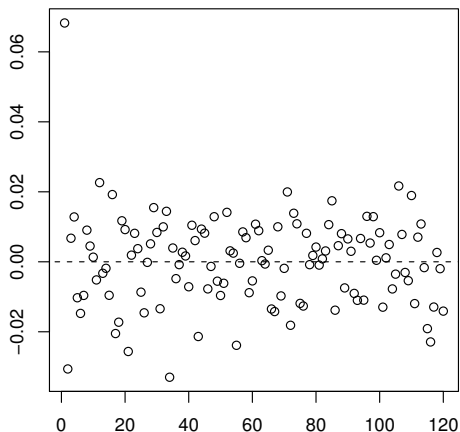
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- ▶ Log-returns corresponding to disjoint time intervals are **uncorrelated**...

DJIA time series (1935-2009)

Decorrelation of daily log-returns over 1–120 days



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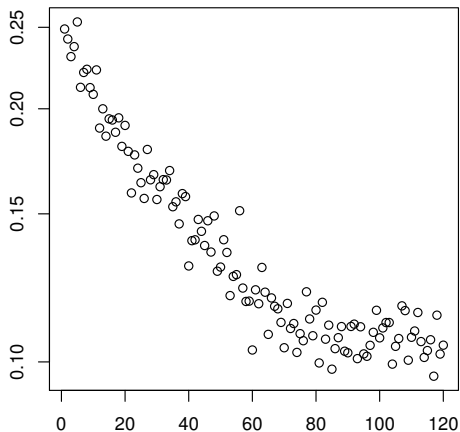
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The correlation between $|X_{t+h} - X_t|$ and $|X_{s+h} - X_s|$, called **volatility autocorrelation**, has a **slow decay** in $|t - s|$, up to moderate values of $|t - s|$ (**clustering of volatility**).

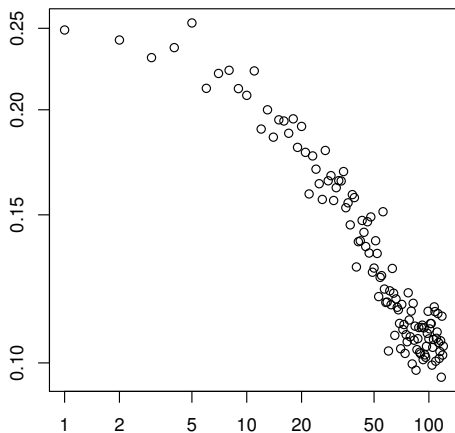
DJIA time series (1935-2009)

Volatility autocorrelation over 1–120 days [log plot]



DJIA time series (1935-2009)

Volatility autocorrelation over 1–120 days [log-log plot]



Alternative models: GARCH

Autoregressive models in such as the **GARCH** are widely used:

$$\varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \varepsilon_{t-1}^2$$

where $\varepsilon_t := X_{t+1} - X_t$ and $(z_t)_{t \in \mathbb{N}}$ are i.i.d. $N(0, 1)$.

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This model can also be built in [continuous time](#) ([COGARCH](#)).

Alternative models: stochastic volatility

Stochastic volatility processes: the constant σ is replaced by a stochastic process $(\sigma_t)_{t \geq 0}$ independent of W :

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$$d\sigma_t^2 = -\alpha \sigma_t^2 dt + dL_t,$$

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Anticipation: our model is a stochastic volatility process, that solves a SDE similar to the O-U but with a non-linear “drift” term.

Scaling properties

More recently, some striking **scaling properties** of stock indexes of developed markets have been emphasized.

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- ▶ Diffusive scaling of log-regurns
- ▶ Multiscaling (or anomalous scaling) of moments

Diffusive scaling of log-returns

Denote by $\hat{p}_h(\cdot)$ the **empirical distribution of the log-return** over h days, for an observed time series $(x_t)_{1 \leq t \leq T}$ of the detrended log-index X :

$$\hat{p}_h(\cdot) := \frac{1}{T-h} \sum_{t=1}^{T-h} \delta_{x_{t+h}-x_t}(\cdot),$$

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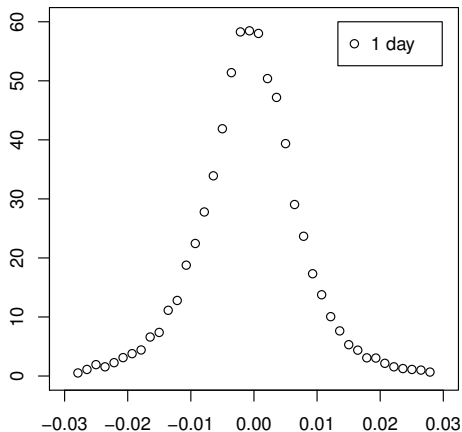
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$$X_{t+h} - X_t \stackrel{d}{\approx} \sqrt{h} (X_{t+1} - X_t) \quad \rightarrow \quad \hat{p}_h(dr) \simeq \frac{1}{\sqrt{h}} g\left(\frac{r}{\sqrt{h}}\right) dr$$

where g is a **non-Gaussian** density.

DJIA time series (1935-2009)

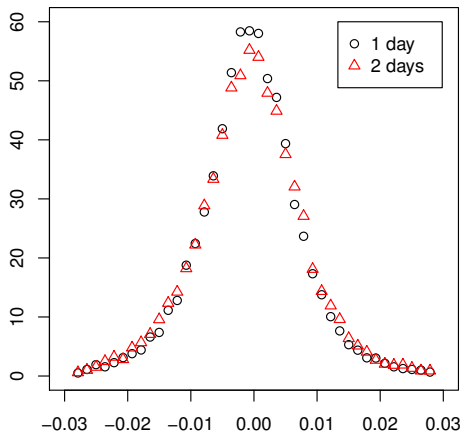
Rescaled empirical density of log-returns (1 day)



Daily log-return **standard deviation ≈ 0.01** \rightarrow Range: -3 to +3 st. dev.

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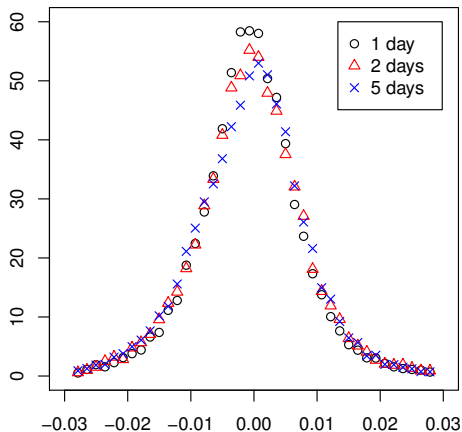
Rescaled empirical density of log-returns (1-2 days)



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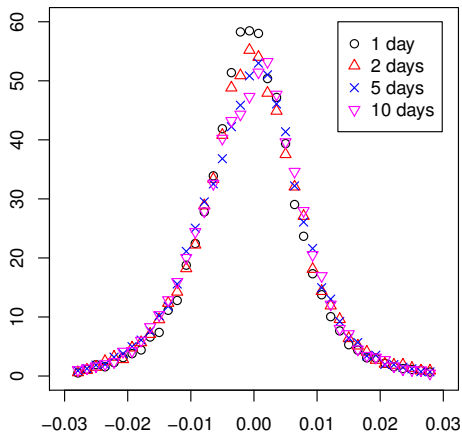
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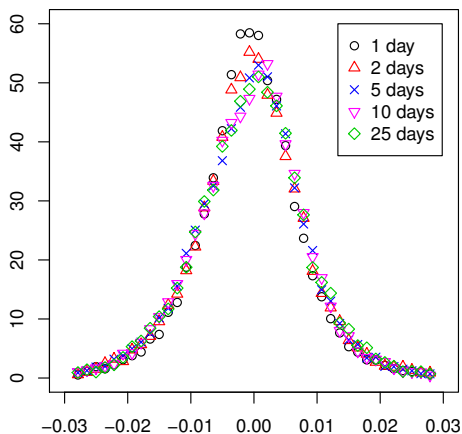
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Rescaled empirical density of log-returns (1-2-5-10-25 days)



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Multiscaling of moments

Consider the empirical q -th moment of the log-return over h days:

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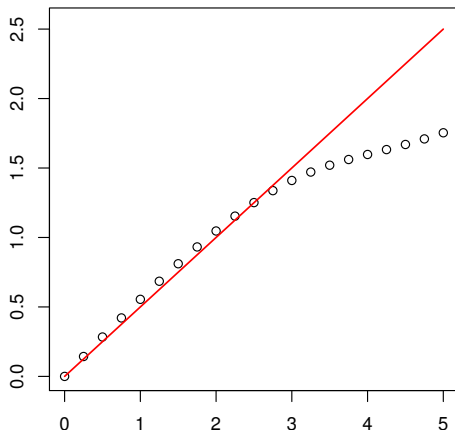
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For $q > q^*$ we have the **anomalous scaling** (or **multiscaling**)

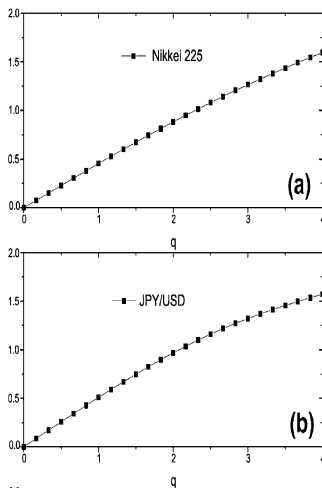
$$\hat{m}_q(h) \approx h^{A(q)}, \quad \text{with } A(q) < \frac{q}{2}.$$

DJIA time series (1935-2009)

Scaling exponent $A(q)$ (linear regression of $\log \hat{m}_q(h)$ vs. $\log h$)



Other data series (from [Di Matteo, Aste & Dacorogna, 2005])



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(We don't consider finer "stylized facts", such as leverage.)

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For $t \geq 0$ we set

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} \quad (\sim \text{Po}(\lambda t)),$$

so that $\tau_{i(t)}$ is the last point in \mathcal{T} before t .

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In order for the SDE $dX_t = v_t dB_t$ to make sense, the trajectories $t \mapsto v_t^2$ must be locally integrable \longrightarrow we must impose $\gamma > 2$.

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$$v_t^2 = \frac{1}{(\alpha(\gamma-1))^{1/(\gamma-1)}} (t - \tau_n)^{-1/(\gamma-1)}.$$

In order for the SDE $dX_t = v_t dB_t$ to make sense, the trajectories $t \mapsto v_t^2$ must be locally integrable \longrightarrow we must impose $\gamma > 2$.

We can now complete the definition of our process, expressing α and γ in terms of our parameters $D \in (0, \frac{1}{2}]$ and $\sigma \in (0, \infty)$.

More generally, $\sigma \longrightarrow (\sigma_n)_{n \in \mathbb{N}}$ i.i.d. random variables in $(0, \infty)$.

Definition of our model

We define $\gamma = \gamma(D) \in (2, \infty)$ and $\alpha = \alpha(\sigma, D) \in (0, \infty)$ by

$$\gamma = 2 + \frac{2D}{1-2D}, \quad \alpha = \frac{1-2D}{(2D)^{1/(1-2D)}} \frac{1}{\sigma^{1/(1-2D)}}.$$

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More generally:

$$dv_t^2 = -\alpha(\sigma_{i(t)}) (v_t^2)^\gamma dt + \infty di(t).$$

The value of the constant α is renewed at each jump of $i(t)$.

An alternative description

Fact: every stochastic volatility process is an **independent random time change** of a (different) Brownian motion $W = (W_t)_{t \geq 0}$.

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Remark: explicit formula for $v_t^2 \implies$ explicit formula for I_t

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Recall $D \in (0, 1/2]$, $\lambda \in (0, \infty)$, $\sigma \in \mathcal{M}_1((0, \infty))$ and (W, \mathcal{T}, Σ)

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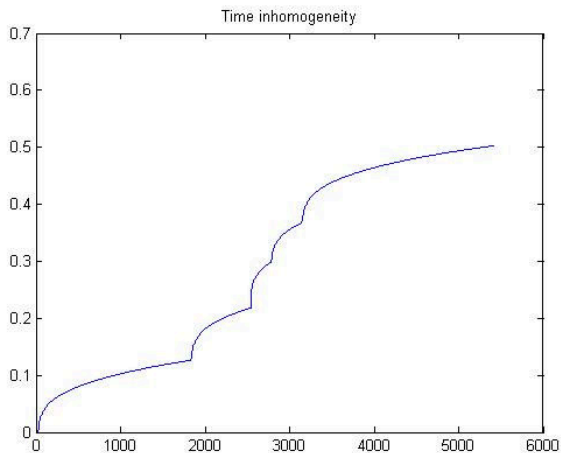
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$I_t = I_t(\mathcal{T}, \Sigma)$ explicit function of \mathcal{T}, Σ (hence independent of W).

The process $(I_t)_{t \geq 0}$



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- ▶ $E[|X_t|^q] < +\infty$ iff $E(\sigma^q) < +\infty$.

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Approximate Diffusive Scaling

Theorem

As $h \downarrow 0$ we have the convergence in distribution

$$\frac{(X_{t+h} - X_t)}{\sqrt{h}} \xrightarrow[h \downarrow 0]{d} f(x) dx,$$

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$$f(x) = \int_0^\infty \nu(d\sigma) \int_0^\infty ds \lambda e^{-\lambda s} \frac{s^{1/2-D}}{\sigma \sqrt{4D\pi}} \exp\left(-\frac{s^{1-2D} x^2}{4D\sigma^2}\right) .$$

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Heavy tails of f related to **multiscaling** of $E[|X_{t+h} - X_t|^q]$.

Multiscaling of Moments

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Assume $E(\sigma^q) := \int \sigma^q \nu(d\sigma) < +\infty$.

The moment $m_q(h) := E(|X_{t+h} - X_t|^q) = E(|X_h|^q)$ is finite and has the following asymptotic behavior as $h \downarrow 0$:

$$m_q(h) \sim \begin{cases} C_q h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q h^{Dq+1} & \text{if } q > q^* \end{cases}, \quad \text{where } q^* := \frac{1}{(\frac{1}{2} - D)}.$$

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- ▶ C_q explicit function of D , λ and $E(\sigma^q)$ (used in estimation)
- ▶ We can write $m_q(h) \approx h^{A(q)}$ with scaling exponent $A(q)$

$$A(q) := \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} q/2 & \text{if } q \leq q^* \\ Dq + 1 & \text{if } q \geq q^* \end{cases}.$$

Decay of Correlations

Theorem

The correlation of the absolute values of the increments of the process X has the following asymptotic behavior as $h \downarrow 0$:

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- The function $\phi(\cdot)$ has a slower than exponential decay.

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Estimation of the Parameters

Loss function: $(T = 40)$

$$\begin{aligned} L(D, \lambda, E(\sigma), E(\sigma^2)) &= \frac{1}{2} \left\{ \left(\frac{\hat{C}_1}{C_1} - 1 \right)^2 + \left(\frac{\hat{C}_2}{C_2} - 1 \right)^2 \right\} \\ &+ \int_0^5 \left(\frac{\hat{A}(q)}{A(q)} - 1 \right)^2 \frac{dq}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\hat{\rho}(t)}{\rho(t)} - 1 \right)^2 \end{aligned}$$

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Estimator: minimization constrained on $E(\sigma^2) \geq E(\sigma)^2$.

$$(\hat{D}, \hat{\lambda}, \widehat{E(\sigma)}, \widehat{E(\sigma^2)}) = \arg \min L(D, \lambda, E(\sigma), E(\sigma^2))$$

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$$L(D, \lambda, E(\sigma), E(\sigma^2)) = \frac{1}{2} \left\{ \left(\frac{\hat{C}_1}{C_1} - 1 \right)^2 + \left(\frac{\hat{C}_2}{C_2} - 1 \right)^2 \right\} \\ + \int_0^5 \left(\frac{\hat{A}(q)}{A(q)} - 1 \right)^2 \frac{dq}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\hat{\rho}(t)}{\rho(t)} - 1 \right)^2$$

Estimator: minimization constrained on $E(\sigma^2) \geq E(\sigma)^2$.

$$(\hat{D}, \hat{\lambda}, \widehat{E(\sigma)}, \widehat{E(\sigma^2)}) = \arg \min L(D, \lambda, E(\sigma), E(\sigma^2))$$

$$\hat{D} \simeq 0.16 \quad \hat{\lambda} \simeq 0.00097 \quad \widehat{E(\sigma)} \simeq 0.108 \quad \widehat{E(\sigma^2)} \simeq (\widehat{E(\sigma)})^2$$

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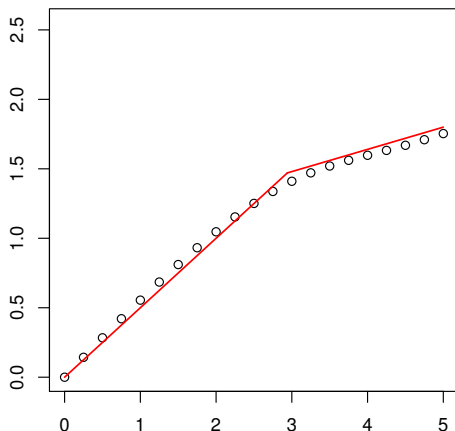
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The fit turns out to be very satisfactory, as we now show.

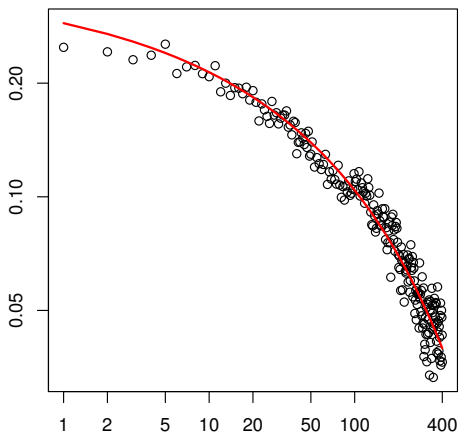
DJIA Time Series (1935-2009)

Empirical (circles) and theoretical (line) scaling exponent $A(q)$



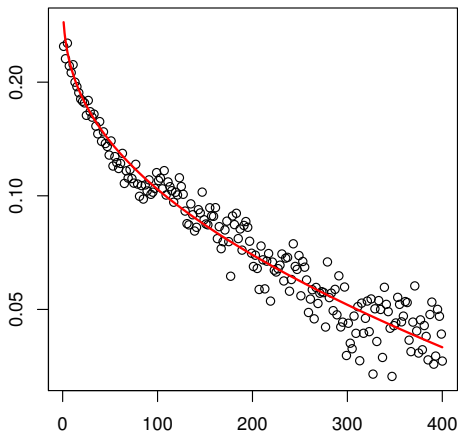
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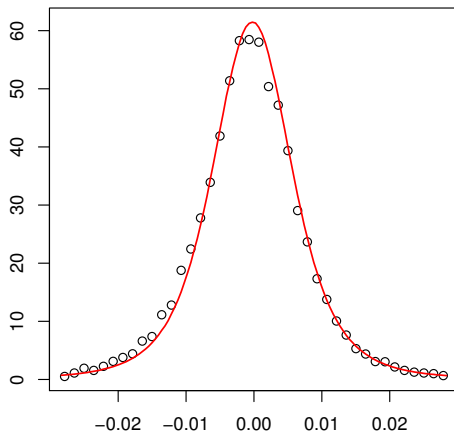
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The agreement is **remarkably good** (both **bulk** and **tails**).

DJIA Time Series (1935-2009)

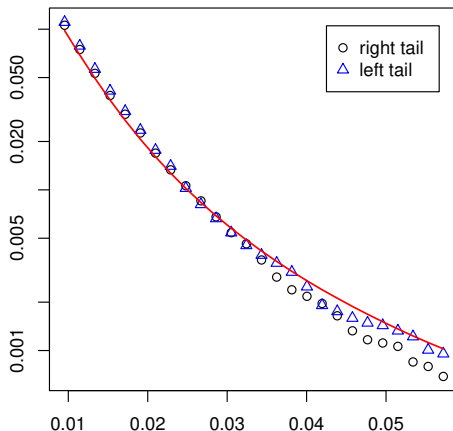
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Daily log-return **standard deviation ≈ 0.01** \rightarrow Range: -3 to 3 st. dev.

DJIA Time Series (1935-2009)

Empirical and theoretical tails of daily log return [log plot]



Daily log-return **standard deviation ≈ 0.01** \rightarrow Range: 1 to 6 st. dev.

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Different laws for σ with the same $E(\sigma)$ and $E(\sigma^2)$ give very similar results.

The law of the log-returns (in the range of interest) is effectively determined by the t^{2D} time scaling at the points of \mathcal{T} .

Outline

1. Black & Scholes and beyond
2. The Model
3. Main Results
4. Estimation and Simulations
5. Bivariate Model
6. Conclusions

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Which [joint distribution](#) for $(W^X, \mathcal{T}^X, \Sigma^X), (W^Y, \mathcal{T}^Y, \Sigma^Y)$?

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How do **cross correlations** behave for such a model?

$$\rho^{X,Y}(t-s) := \lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |Y_{t+h} - Y_t|)$$

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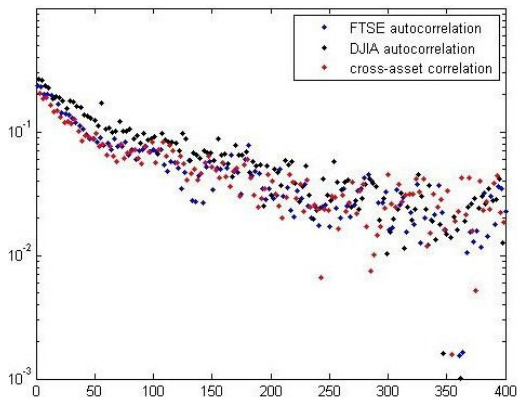
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- ▶ This is indeed what one observes! (Not expected a priori.)

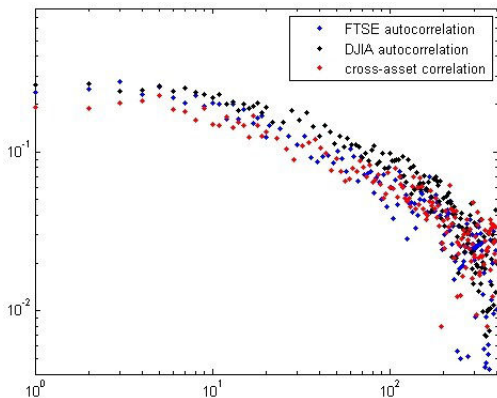
DJIA and FTSE Time Series (1984-2011)

Empirical autocorrelations ρ^X , ρ^Y and cross correlations $\rho^{X,Y}$: log plot



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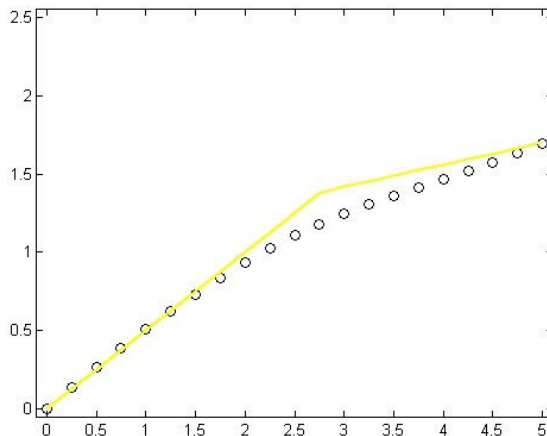
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For both indexes, the agreement is very satisfactory.

Again, the fit of the law of the log-returns is very good, even with no explicit calibration on it.

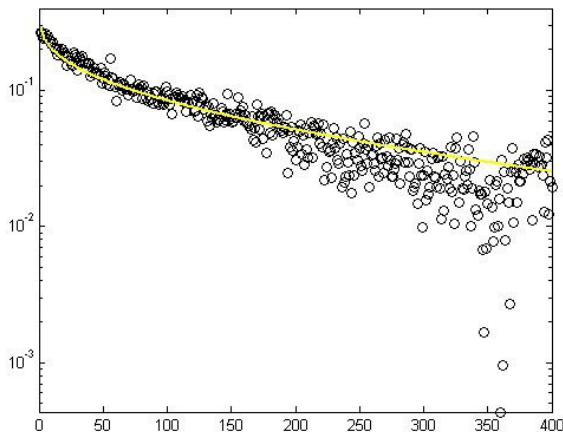
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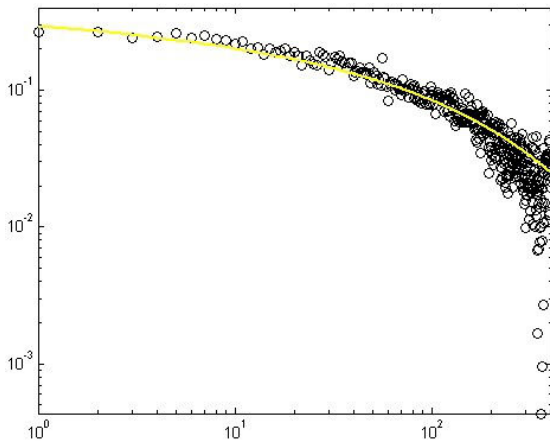
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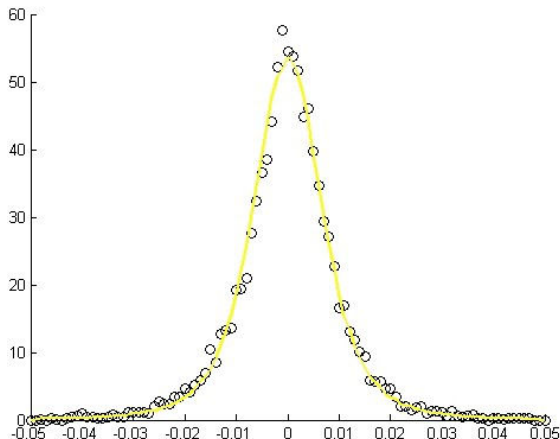
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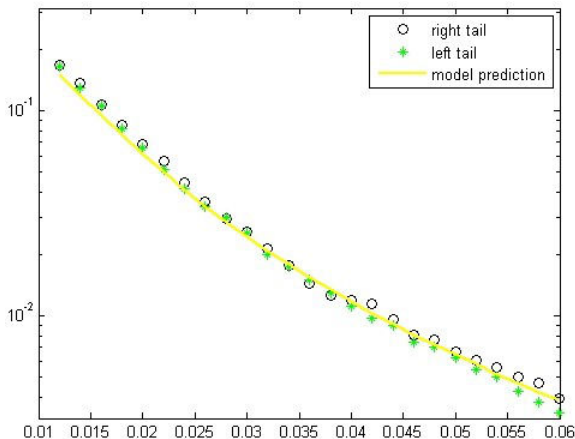
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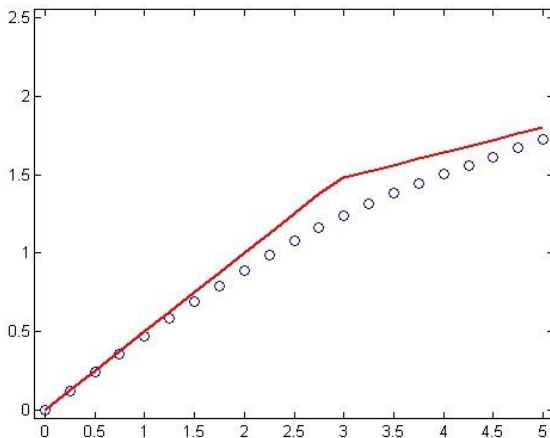
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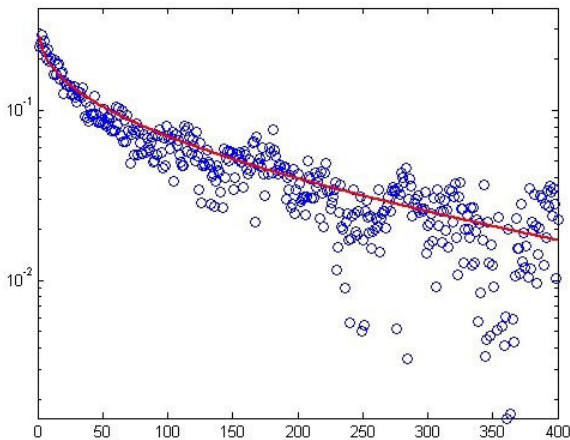
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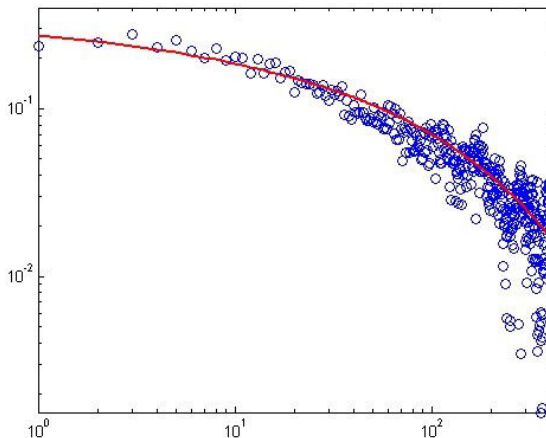
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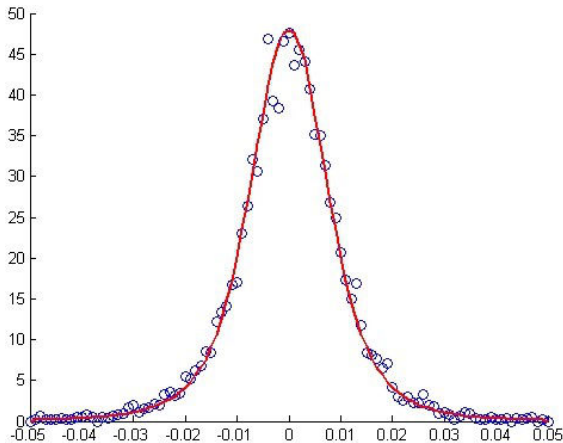
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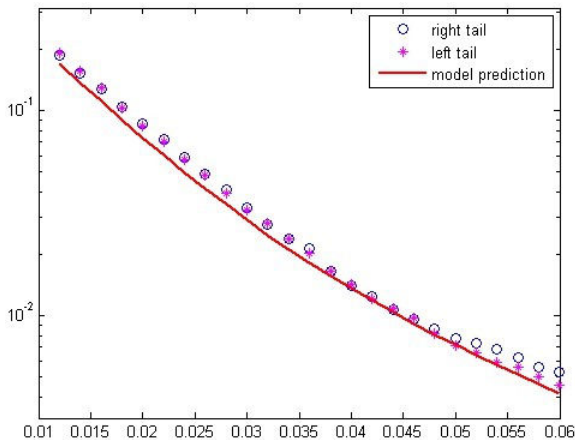
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Recall that $dX_t = v_t dB_t$ with $v_t^2 = I'(t) \propto (t - \tau_{i(t)})^{2D-1}$.

Basic observation: the volatility v_t^2 diverges precisely on the set $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of shock times.

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Basic observation: the volatility v_t^2 diverges precisely on the set $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of shock times.

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Unfortunately, due to fluctuations, there may be several local maxima... How to locate the right one?

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Idea: compare locations of the maxima for **different values of T** .

If \bar{t} is a “true” shock time, it should be detected as a maximum of $V_T(t)$ for (almost) every fixed value of $T > \bar{t}$.

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If our predictions are right, the set $\{g_T, g_{T-1}, g_{T-2}, \dots\}$ should consist of only **few distinct values** (each attained by several g_i 's) corresponding to the shock points, i.e. the points of \mathcal{T} .

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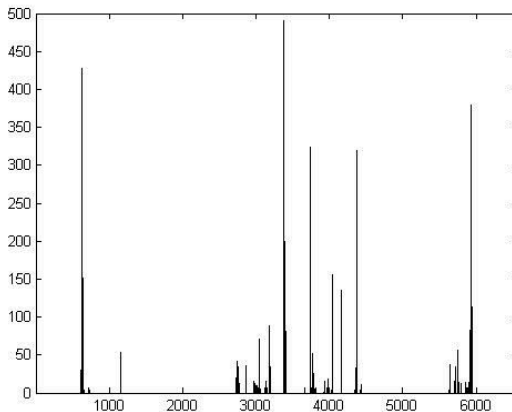
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This is indeed (almost) the case! We just need to identify couples of **very close** (< 20 days) shock points.

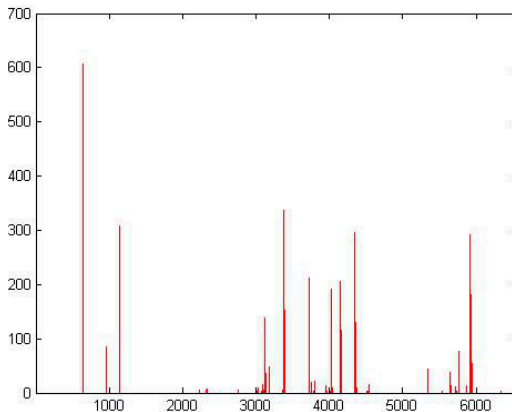
DJIA Time Series (1984-2011)

Shock times \mathcal{T}^X for the DJIA



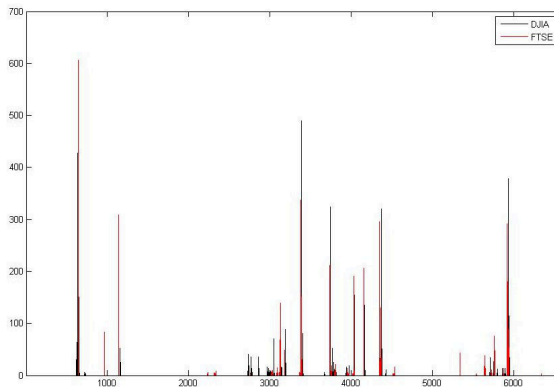
FTSE Time Series (1984-2011)

Shock times \mathcal{T}^Y for the FTSE



DJIA and FTSE Time Series (1984-2011)

Shock times \mathcal{T}^X and \mathcal{T}^Y for the DJIA and FTSE



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Recall the estimated values

$$\lambda^X \simeq 0.0013, \quad \lambda^Y \simeq 0.0018,$$

and we want to find λ_3 such that $\lambda^X = \lambda_1 + \lambda_3$, $\lambda^Y = \lambda_2 + \lambda_3$.

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Guess: **large value of λ_3** . More quantitatively, the cross correlation

$$\rho^{X,Y}(t-s) := \lim_{h \downarrow 0} \rho(|X_{s+h} - X_s|, |Y_{t+h} - Y_t|)$$

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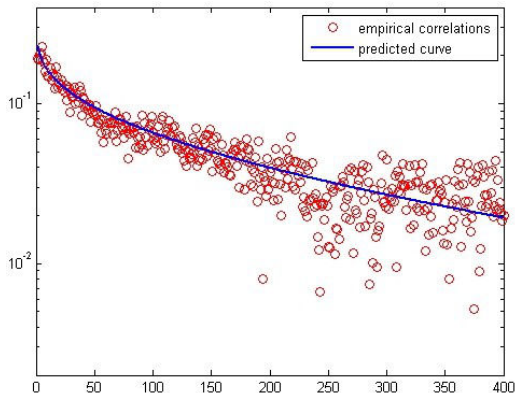
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Result: $\lambda_1 \simeq 0.0001, \quad \lambda_2 \simeq 0.0006, \quad \lambda_3 \simeq 0.0012$

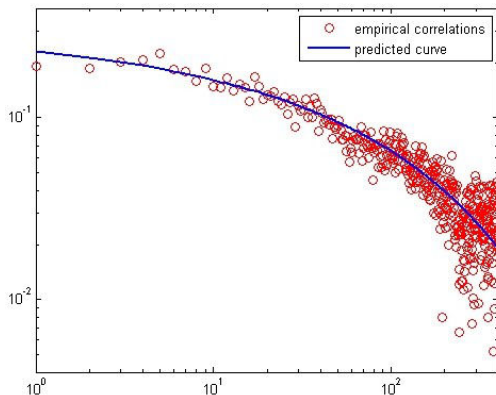
DJIA and FTSE Time Series (1984-2011)

Empirical (circles) and theoretical (lines) cross correlations: log plot



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Locating the shock times

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Actually, it would be good even with $\lambda_3 = 0$, i.e. if every shock time of DJIA were a shock time of FTSE.

Outline

1. Black & Scholes and beyond
2. The Model
3. Main Results
4. Estimation and Simulations
5. Bivariate Model
6. Conclusions

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- ▶ Several generalizations can be considered: correlations between Σ , \mathcal{T} and W can be introduced, or the nonlinear time change $t \mapsto t^{2D}$ can be modified in many ways.

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Next steps:

- ▶ Solve specific problems by using this model: pricing of options, portfolio management, . . .

Thanks.

Variability in subperiods

A natural question on the [DJIA time series](#) is the amount of variability of the data set in subperiods. Is the period 1935-2009 long enough to be close to the ergodic limit?

More concretely: are the statistics of the DJIA time series in (large) subperiods close to those of the whole period 1935-2009?

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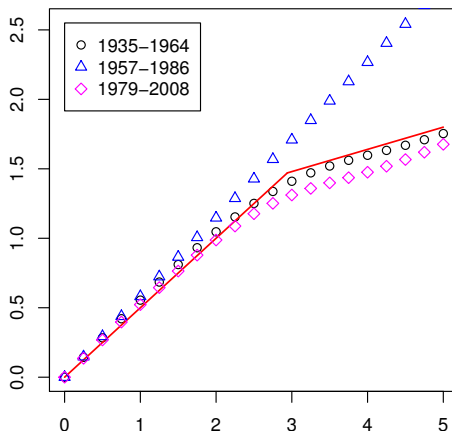
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It turns out that a **considerable variability** is present for all the quantities we observe ([multiscaling of moments](#), [decay of correlations](#) and [empirical distribution](#)) if one takes different (suitably chosen) large time windows of 30 years.

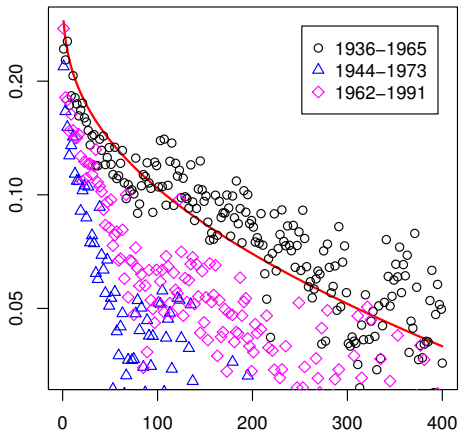
DJIA Time Series (1935-2009)

Empirical scaling exponent $A(q)$ over sub-periods of 30 years.



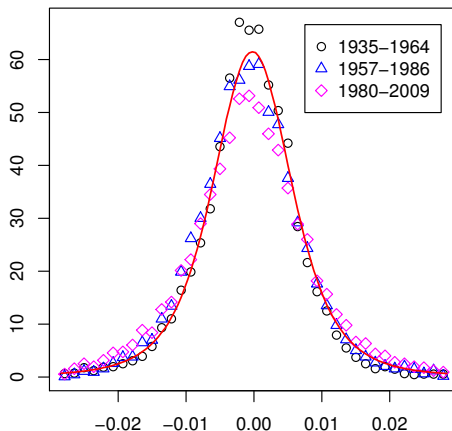
DJIA Time Series (1935-2009)

Volatility autocorrelation over sub-periods of 30 years [log plot]



DJIA Time Series (1935-2009)

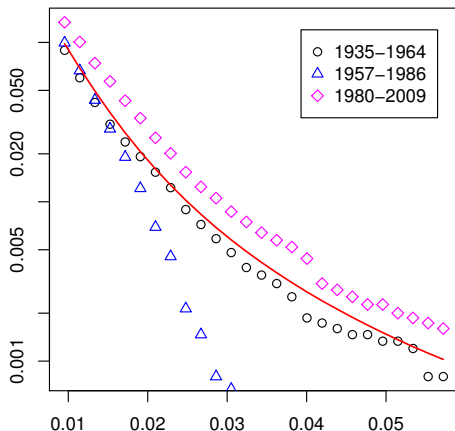
Variability of the distribution in sub-periods of 30 years



Daily log-return $\text{standard deviation} \approx 0.01 \rightarrow \text{Range: } -3 \text{ to } 3 \text{ st. dev.}$

DJIA Time Series (1935-2009)

Variability of the left tail in sub-periods of 30 years



Daily log-return **standard deviation ≈ 0.01** \rightarrow Range: 1 to 6 st. dev.

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These plots show that the DJIA time series in the period 1935-2009 is **not so close** to the ergodic limit: empirical averages over subperiods of 30 years exhibit non negligible fluctuations.

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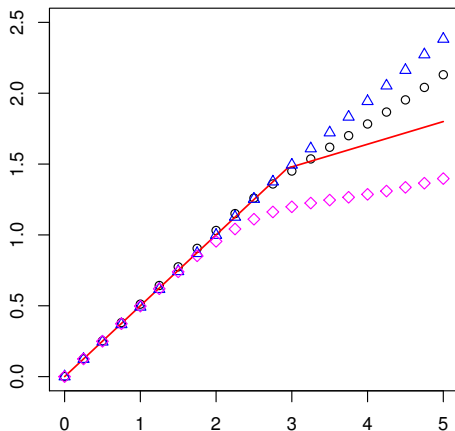
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A **significant, comparable variability** is present also in our model.

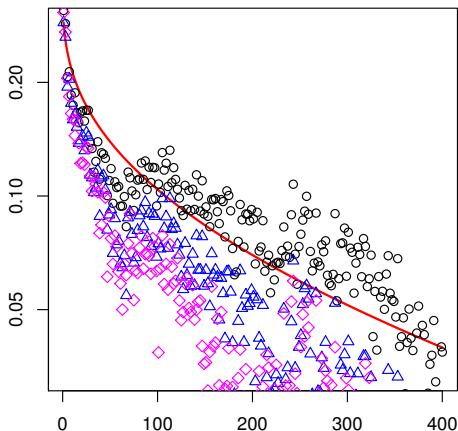
Simulated Data (75 years)

Simulated scaling exponent of our model over sub-periods of 30 years



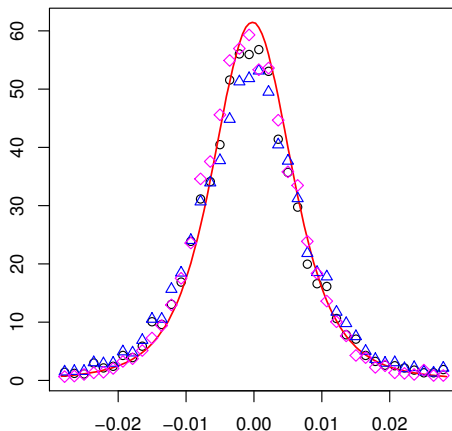
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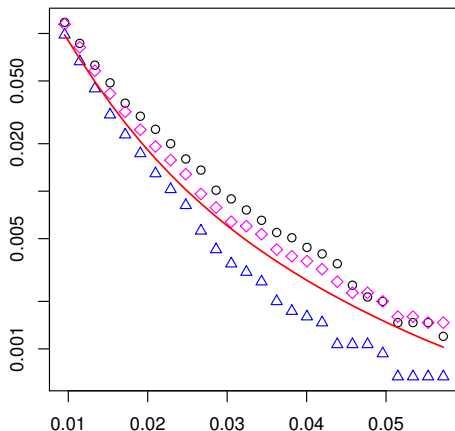
Simulated Data (75 years)

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- ▶ If g is standard Gaussian $\rightarrow (Y_t)_{t \geq 0}$ is Brownian motion.
- ▶ Is the definition well-posed? Conditions on g .

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Apart from this issue, there is still **no multiscaling** of moments.
This is solved introducing a **time inhomogeneity** in the model.

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Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \dots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \quad \text{for } t \in [0, \tau_1),$$

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- ▶ Interpretation: $(\tau_n)_{n \geq 1}$ linked to “shocks” in the market.

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Despite the time-change, the process $(X_t)_{t \geq 0}$ remains **not ergodic**. However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits **all mentioned stylized facts**. They actually simulate **a different model**: an **autoregressive version** of $(X_t)_{t \geq 0}$.

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However, Baldovin & Stella show by simulations that this model (with $(\tau_n)_n$ a periodic sequence) fits **all mentioned stylized facts**.

They actually simulate **a different model**: an **autoregressive version** of $(X_t)_{t \geq 0}$.

Other issue: the density of Y_1 is $g(\cdot)$ by construction. However, the density of X_1 **is not $g(\cdot)$** and depends on the choice of $(\tau_n)_n$.

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Our aims

- Define a simple model capturing the essential features of Baldovin & Stella's construction.

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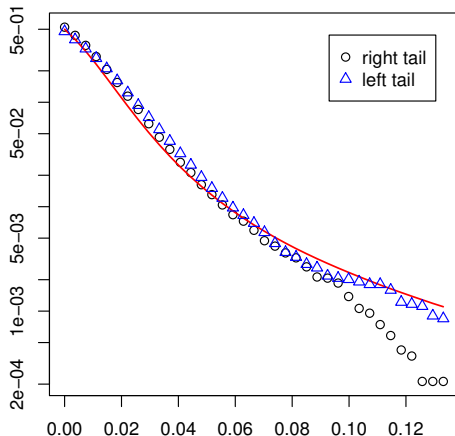
- ▶ Define a simple model capturing the essential features of Baldovin & Stella's construction.
- ▶ Easy to describe and to **simulate**.
- ▶ **Rigorous proofs** of the mentioned stylized facts.

Other observables

Is everything going as expected?

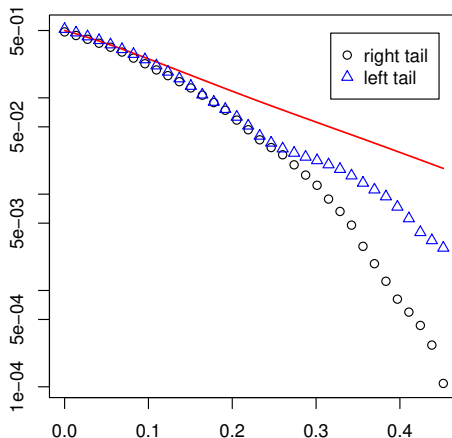
DJIA Time Series (1935-2009)

Empirical and theoretical tails of 5-day log return [log plot]



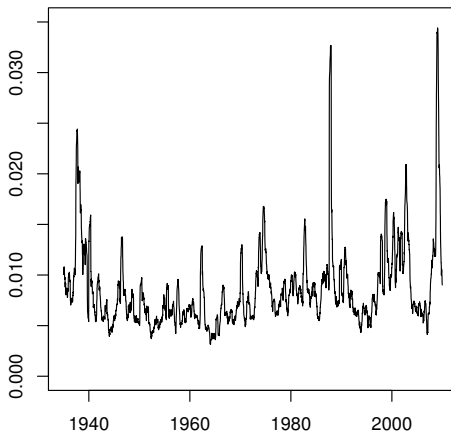
DJIA Time Series (1935-2009)

Empirical and theoretical tails of 400-day log return [log plot]



DJIA Time Series (1935-2009)

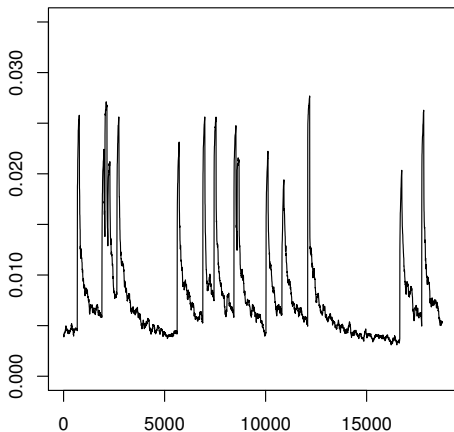
Empirical volatility



Local standard deviation of log-returns in a window of 100 days

Simulated Data (75 years)

Empirical volatility



Local standard deviation of log-returns in a window of 100 days