

# ON THE MAXIMUM OF RANDOM WALKS CONDITIONED TO STAY POSITIVE AND TIGHTNESS FOR PINNING MODELS

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**ABSTRACT.** We consider real random walks with finite variance. We prove an optimal integrability result for the diffusively rescaled maximum, when the walk or its bridge is conditioned to stay positive, or to avoid zero. As an application, we prove tightness under diffusive rescaling for general pinning and wetting models based on random walks.

## 1. INTRODUCTION

In this paper we deal with random walks on  $\mathbb{R}$ , with zero mean and finite variance.

In Section 2 we consider the random walks, or their bridges, conditioned to stay positive on a finite time interval. We prove that the maximum of the walk, diffusively rescaled, has a *uniformly integrable square*. The same result is proved under the conditioning that the walk avoids zero.

In Section 3 we present an application to *pinning and wetting models* built over random walks. More generally, we consider probabilities which admit suitable regeneration epochs, which cut the path into independent “excursions”. We prove that these models, under diffusive rescaling, are tight in the space of continuous functions. This fills a gap in the proof of [DGZ05, Lemma 4].

Sections 4, 5, 6 contain the proofs.

This paper generalizes and supersedes the unpublished manuscript [CGZ07b].

## 2. RANDOM WALKS CONDITIONED TO STAY POSITIVE, OR TO AVOID ZERO

We use the conventions  $\mathbb{N} := \{1, 2, 3, \dots\}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Let  $(X_i)_{i \in \mathbb{N}}$  be i.i.d. real random variables. Let  $(S_n)_{n \in \mathbb{N}_0}$  be the associated random walk:

$$S_0 := 0, \quad S_n := X_1 + \dots + X_n \quad \text{for } n \in \mathbb{N}.$$

**Assumption 2.1.**  $E[X_1] = 0$ ,  $E[X_1^2] = \sigma^2 < \infty$ , and one of the following cases hold.

- Discrete case. *The law of  $X_1$  is integer valued and, for simplicity, aperiodic.*
- Continuous case. *The law of  $X_1$  has a density with respect to the Lebesgue measure, and the density of  $S_n$  is essentially bounded for some  $n \in \mathbb{N}$ :*

$$f_n(x) := \frac{P(S_n \in dx)}{dx} \in L^\infty.$$

Then  $P(S_n = 0) > 0$  (discrete case),  $f_n(0) > 0$  (continuous case) for large  $n$ , say  $n \geq n_0$ .

Let us denote by  $P_n$  the law of the first  $n$  steps of the walk:

$$P_n := P((S_0, S_1, \dots, S_n) \in \cdot). \quad (2.1)$$

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Next we define the laws of the *meander* and (for  $n \geq n_0$ ) of the *bridge* and *excursion*:

$$\begin{aligned} P_n^{\text{mea}}(\cdot) &:= P((S_0, S_1, \dots, S_n) \in \cdot \mid S_1 > 0, S_2 > 0, \dots, S_n > 0), \\ P_n^{\text{bri}}(\cdot) &:= P((S_0, S_1, \dots, S_n) \in \cdot \mid S_n = 0), \\ P_n^{\text{exc}}(\cdot) &:= P((S_0, S_1, \dots, S_n) \in \cdot \mid S_1 > 0, S_2 > 0, \dots, S_{n-1} > 0, S_n = 0). \end{aligned} \quad (2.2)$$

Our main result concerns the integrability of the absolute maximum:

$$M_n := \max_{0 \leq i \leq n} |S_i|. \quad (2.3)$$

**Theorem 2.2.** *Let Assumption 2.1 hold. Then  $M_n^2/n$  is uniformly integrable under any of the laws  $Q \in \{P_n, P_n^{\text{bri}}, P_n^{\text{mea}}, P_n^{\text{exc}}\}$ .*

$$\lim_{K \rightarrow \infty} \sup_{n \geq n_0} E_Q \left[ \frac{M_n^2}{n} \mathbf{1}_{\left\{ \frac{M_n^2}{n} > K \right\}} \right] = 0. \quad (2.4)$$

The proof of Theorem 2.2, given in Section 4, comes in three steps. First we exploit *local limit theorems*, to remove the conditioning on  $\{S_n = 0\}$  and just deal with  $P_n, P_n^{\text{mea}}$ . Then we use *martingale arguments*, to get rid of the maximum  $M_n$  and focus on  $S_n$ . Finally we use *fluctuation theory*, to perform sharp computations on the law of  $S_n$ .

**Remark 2.3.** *For a symmetric random walk, the bound  $M_n^2 \geq X_n^2 \mathbf{1}_{\{S_{n-1} \geq 0, X_n \geq 0\}}$  gives*

$$E \left[ \frac{M_n^2}{n} \mathbf{1}_{\left\{ \frac{M_n^2}{n} > K \right\}} \right] \geq \frac{1}{4} E \left[ \frac{X_1^2}{n} \mathbf{1}_{\left\{ \frac{X_1^2}{n} > K \right\}} \right]. \quad (2.5)$$

Given  $n \in \mathbb{N}$ , we can choose the law of  $X_1$  so that the right hand side vanishes as slow as we wish, as  $K \rightarrow \infty$ . This shows that (2.4) cannot be improved, without further assumptions on the walk.

We next introduce the laws of the random walk and bridge *conditioned to avoid zero*:

$$\begin{aligned} P_n^{\text{mea}2}(\cdot) &:= P((S_0, S_1, \dots, S_n) \in \cdot \mid S_1 \neq 0, S_2 \neq 0, \dots, S_n \neq 0), \\ P_n^{\text{exc}2}(\cdot) &:= P((S_0, S_1, \dots, S_n) \in \cdot \mid S_1 \neq 0, S_2 \neq 0, \dots, S_{n-1} \neq 0, S_n = 0). \end{aligned} \quad (2.6)$$

In the continuous case  $P(S_n \neq 0) = 1$ , so we have trivially  $P_n^{\text{mea}2} = P_n$  and  $P_n^{\text{exc}2} = P_n^{\text{bri}}$ . In the discrete case, however, the conditioning on  $\{S_n \neq 0\}$  has a substantial effect:  $P_n^{\text{mea}2}$  and  $P_n^{\text{exc}2}$  are close to “two-sided versions” of  $P_n^{\text{mea}}$  and  $P_n^{\text{exc}}$  (see [Bel72, Kai76]).

We prove the following analogue of Theorem 2.2.

**Theorem 2.4.** *Let Assumption 2.1 hold. Then  $M_n^2/n$  under  $P_n^{\text{exc}2}$  or  $P_n^{\text{mea}2}$  is uniformly integrable.*

Theorem 2.4 is proved in Section 5. We first use local limit theorems to reduce the analysis to  $P_n^{\text{mea}2}$ , as for Theorem 2.2, but we can no longer apply martingale techniques. We then exploit direct path arguments to *deduce* Theorem 2.4 from Theorem 2.2.

### 3. TIGHTNESS FOR PINNING AND WETTING MODELS

We prove tightness under diffusive rescaling for *pinning and wetting models*, see Subsection 3.2, exploiting the property that these models have independent excursions<sup>†</sup>, conditionally on their zero level set. It is simpler and more transparent to work with general probabilities which enjoy (a generalization of) this property, that we now define.

<sup>†</sup>In this section the word “excursion” has a more general meaning than in Section 2.

**3.1. A sharp criterion for tightness based on excursions.** Given  $t \in \mathbb{N}$ , we use the shorthands

$$[t] := \{0, 1, \dots, t\}, \quad \mathbb{R}^{[t]} = \{x = (x_0, x_1, \dots, x_t) : x_i \in \mathbb{R}\} \simeq \mathbb{R}^{t+1}.$$

We consider probabilities  $\mathbf{P}_N$  on paths  $x = (x_0, \dots, x_N) \in \mathbb{R}^{[N]}$  which admit *regeneration epochs in their zero level set*. To define  $\mathbf{P}_N$ , we need three ingredients:

- the *regeneration law*  $p_N$  is a probability on the space of subsets of  $[N]$  which contain 0;
- the *bulk excursion laws*  $P_t^{\text{bulk}}$ ,  $t \in \mathbb{N}$ , are probabilities on  $\mathbb{R}^{[t]}$  with  $P_t^{\text{bulk}}(x_0 = x_t = 0) = 1$ ;
- the *final excursion laws*  $P_t^{\text{fin}}$ ,  $t \in \mathbb{N}$ , are probabilities on  $\mathbb{R}^{[t]}$  with  $P_t^{\text{fin}}(x_0 = 0) = 1$ .

**Definition 3.1.**  $\mathbf{P}_N$  is the probability on  $\mathbb{R}^{[N]}$  under which the path  $x = (x_0, x_1, \dots, x_N)$  is built as follows.

- (1) First sample the number  $n$  and the locations  $0 =: t_1 < \dots < t_n \leq N$  of the regeneration epochs, with probabilities  $p_N(\{t_1, \dots, t_n\})$ .
- (2) Then write the path  $x$  as a concatenation of  $n$  excursions  $x^{(i)}$ , with  $i = 1, \dots, n$ :

$$x^{(i)} := (x_{t_i}, \dots, x_{t_{i+1}}), \quad \text{with } t_{n+1} := N.$$

- (3) Finally, given the regeneration epochs, sample the excursions  $x^{(i)}$  independently, with marginal laws  $P_{t_{i+1}-t_i}^{\text{bulk}}$  for  $i = 1, \dots, n-1$  and (in case  $t_n < N$ )  $P_{N-t_n}^{\text{fin}}$  for  $i = n$ .

Let  $C([0, 1])$  be the space of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ , with the topology of uniform convergence. We define the *diffusive rescaling* operator  $\mathcal{R}_N : \mathbb{R}^{[N]} \rightarrow C([0, 1])$

$$\mathcal{R}_N(x) := \left\{ \text{linear interpolation of } \frac{1}{\sqrt{N}} x_{Nt} \text{ for } t \in \left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\right\} \right\}$$

We give conditions under which the laws  $\mathbf{P}_N \circ \mathcal{R}_N^{-1}$ , that we call *diffusive rescalings* of  $\mathbf{P}_N$ , are tight. Remarkably, we make *no assumption on the regeneration laws*  $p_N$ .

**Theorem 3.2.** Let  $\mathbf{P}_N$  be as in Definition 3.1. The diffusive rescalings  $(\mathbf{P}_N \circ \mathcal{R}_N^{-1})_{N \in \mathbb{N}}$  are tight in  $C([0, 1])$ , for arbitrary regeneration laws, if the following conditions hold:

- (1) the diffusive rescalings  $(P_t^{\text{bulk}} \circ \mathcal{R}_t^{-1})_{t \in \mathbb{N}}$  and  $(P_t^{\text{fin}} \circ \mathcal{R}_t^{-1})_{t \in \mathbb{N}}$  are tight in  $C([0, 1])$ ;
- (2) the bulk excursion law satisfies the following integrability bound:

$$\sup_{t \in \mathbb{N}} P_t^{\text{bulk}} \left( \frac{\max_{0 \leq i \leq t} |x_i|}{\sqrt{t}} > a \right) = o\left(\frac{1}{a^2}\right) \quad \text{as } a \uparrow \infty. \quad (3.1)$$

This result is *optimal*: if condition (1) or (2) fails, then one can build regeneration laws  $(p_N)_{N \in \mathbb{N}}$  such that  $\mathbf{P}_N \circ \mathcal{R}_N^{-1}$  are not tight. We omit the proof for brevity.

To make a link with the previous section, we set  $M_t := \max_{0 \leq i \leq t} |x_i|$  and observe that

$$P_t^{\text{bulk}} \left( \frac{\max_{0 \leq i \leq t} |x_i|}{\sqrt{t}} > a \right) \leq \frac{1}{a^2} E_t^{\text{bulk}} \left[ \frac{M_t^2}{t} \mathbf{1}_{\left\{ \frac{M_t^2}{t} > a \right\}} \right].$$

Thus condition (2) in Theorem 3.2 is satisfied if  $M_t^2/t$  is uniformly integrable under  $P_t^{\text{bulk}}$ . We then obtain the following corollary of Theorems 2.2 and 2.4.

**Proposition 3.3.** Condition (2) in Theorem 3.2 is satisfied if  $P_t^{\text{bulk}}$  is chosen among  $\{P_t^{\text{bri}}, P_t^{\text{exc}}, P_t^{\text{exc2}}\}$ , see (2.2) and (2.6), for a random walk satisfying Assumption 2.1.

**Remark 3.4.** Condition (1) in Theorem 3.2 is satisfied too, if  $P_t^{\text{bulk}}$  is chosen among  $\{P_t^{\text{bri}}, P_t^{\text{exc}}, P_t^{\text{exc2}}\}$  and  $P_t^{\text{fin}}$  is chosen among  $\{P_n, P_n^{\text{mea}}, P_n^{\text{mea2}}\}$ , under Assumption 2.1. Indeed, the diffusive rescalings of  $P_n, P_n^{\text{bri}}, P_n^{\text{mea}}$  and  $P_n^{\text{exc}}$  converge weakly to Brownian motion [Don51], bridge [Lig68, DGZ05], meander [Bol76] and excursion [CC13]; in the discrete case, the diffusive rescalings of  $P_n^{\text{mea2}}$  and  $P_n^{\text{exc2}}$  converge weakly to two-sided Brownian meander [Bel72] and excursion [Kai76].

**3.2. Pinning and wetting models.** An important class of laws  $\mathbf{P}_N$  to which Theorem 3.2 applies is given by pinning and wetting models (see [Gia07, Gia11, Hol09] for background).

Fix a random walk  $(S_n)_{n \in \mathbb{N}_0}$  as in Assumption 2.1 and a real sequence  $\xi = (\xi_n)_{n \in \mathbb{N}}$  (*environment*). For  $N \in \mathbb{N}$ , the *pinning model*  $\mathbf{P}_N^\xi$  is the law on  $\mathbb{R}^{[N]}$  defined as follows.

- *Discrete case.* We define

$$\frac{\mathbf{P}_N^\xi((S_0, \dots, S_N) = (s_0, \dots, s_N))}{\mathbf{P}((S_1, \dots, S_N) = (s_1, \dots, s_N))} := \frac{e^{\sum_{n=1}^N \xi_n \mathbb{1}_{\{s_n=0\}}}}{Z_N^\xi},$$

where  $Z_N^\xi$  is a suitable normalizing constant, called *partition function*.

- *Continuous case.* We assume that  $\xi_n \geq 0$  for all  $n \in \mathbb{N}$  and we define  $\mathbf{P}_N^\xi$  by

$$\mathbf{P}_N^\xi((S_0, \dots, S_N) \in (ds_0, \dots, ds_N)) := \delta_0(ds_0) \frac{\prod_{n=1}^N (f(s_n - s_{n-1}) ds_n + \xi_n \delta_0(ds_n))}{Z_N^\xi},$$

where  $f(\cdot)$  is the density of  $S_1$  and  $\delta_0(\cdot)$  is the Dirac mass at 0.

Note that  $\mathbf{P}_N^\xi$  fits Definition 3.1 with regeneration epochs  $\{k \in [N] : s_k = 0\}$  (the whole zero level set) and  $P_t^{\text{bulk}} = P_t^{\text{exc}2}$ ,  $P_t^{\text{fin}} = P_t^{\text{mea}2}$  (which means  $P_t^{\text{bulk}} = P_t^{\text{bri}}$ ,  $P_t^{\text{fin}} = P_t$  in the continuous case).

Another example of law  $\mathbf{P}_N$  as in Definition 3.1 is the *wetting model*  $\mathbf{P}_N^{\xi,+}$ , defined by

$$\mathbf{P}_N^{\xi,+}(\cdot) := \mathbf{P}_N^\xi(\cdot \mid s_1 \geq 0, s_2 \geq 0, \dots, s_N \geq 0).$$

The bulk excursion law is now  $P_t^{\text{bulk}} = P_t^{\text{exc}}$ , while the final excursion law is  $P_t^{\text{fin}} = P_t^{\text{mea}}$ .

Finally, *constrained* versions of the pinning and wetting models also fit Definition 3.1:

$$\mathbf{P}_N^{\xi,c}(\cdot) := \mathbf{P}_N^\xi(\cdot \mid s_N = 0), \quad \mathbf{P}_N^{\xi,+,c}(\cdot) := \mathbf{P}_N^{\xi,+}(\cdot \mid s_N = 0).$$

The final and bulk excursion laws coincide ( $P_t^{\text{fin}} = P_t^{\text{exc}2}$  for  $\mathbf{P}_N^{\xi,c}$ ,  $P_t^{\text{fin}} = P_t^{\text{exc}}$  for  $\mathbf{P}_N^{\xi,+,c}$ ).

Proposition 3.3 and Remark 3.4 yield immediately the following result.

**Theorem 3.5** (Tightness for pinning and wetting models). *Fix a real sequence  $\xi = (\xi_n)_{n \in \mathbb{N}}$ . Under Assumption 2.1, the diffusive rescalings  $(\mathbf{P}_N \circ \mathcal{R}_N^{-1})_{N \in \mathbb{N}}$  of pinning or wetting models  $\mathbf{P}_N \in \{\mathbf{P}_N^\xi, \mathbf{P}_N^{\xi,+}, \mathbf{P}_N^{\xi,c}, \mathbf{P}_N^{\xi,+,c}\}$  are tight in  $C([0, 1])$ .*

This result fills a gap in the proof of [DGZ05, Lemma 4], which was also used in the works [CGZ06], [CGZ07a]. A recent application of Theorem 3.5 can be found in [DO18].

Pinning and wetting models are challenging models, which display a rich behavior. This complexity is hidden in the regeneration law  $p_N = p_N^\xi$ . This explains the importance of having criteria for tightness, such as Theorem 3.2, which only looks at excursions.

**Remark 3.6.** *There are models where regeneration epochs are a strict subset of the zero level set. For instance, in presence of a Laplacian interaction [BC10, CD08, CD09], couples of adjacent zeros are regeneration epochs. Theorem 3.2 can cover these cases.*

#### 4. PROOF OF THEOREM 2.2

We fix a random walk  $(S_n)_{n \in \mathbb{N}_0}$  which satisfies Assumption 2.1. For simplicity, we assume that the variance  $\sigma^2$  equals one. We split the proof of Theorem 2.2 in three steps.

**Step 1.** We use the shorthand UI for “uniformly integrable”. In this step assume that

$$\frac{M_n^2}{n} \text{ under } \mathbf{P}_n \text{ (resp. under } \mathbf{P}^{\text{mea}}) \text{ is UI,} \quad (4.1)$$

and we show that

$$\frac{M_n^2}{n} \text{ under } \mathbf{P}_n^{\text{bri}} \text{ (resp. under } \mathbf{P}_n^{\text{exc}}) \text{ is UI.} \quad (4.2)$$

Let us set  $M_{[a,b]} := \max_{a \leq i \leq b} |S_i|$ . Since  $M_n \leq \max\{M_{[0,n/2]}, M_{[n/2,n]}\}$ , it suffices to prove that  $M_{[0,n/2]}^2/n$  and  $M_{[n/2,n]}^2/n$  are UI. By symmetry, (4.2) is equivalent to

$$\frac{M_{n/2}^2}{n} \text{ under } \mathbf{P}_n^{\text{bri}} \text{ (resp. under } \mathbf{P}_n^{\text{exc}}) \text{ is UI.} \quad (4.3)$$

We take  $n$  even (for simplicity). We show that the laws of  $\mathbf{V}_{n/2} := (S_1, \dots, S_{n/2})$  under  $\mathbf{P}_n^{\text{bri}}$  (resp.  $\mathbf{P}_n^{\text{exc}}$ ) and under  $\mathbf{P}_n$  (resp.  $\mathbf{P}_n^{\text{mea}}$ ) have a bounded Radon-Nikodym density:

$$\sup_{n \geq n_0} \sup_{\mathbf{z} \in \mathbb{R}^{n/2}} \frac{\mathbf{P}_n^{\text{bri}}(\mathbf{V}_{n/2} \in d\mathbf{z})}{\mathbf{P}_n(\mathbf{V}_{n/2} \in d\mathbf{z})} < \infty \quad \left( \text{resp. } \sup_{n \geq n_0} \sup_{\mathbf{z} \in \mathbb{R}^{n/2}} \frac{\mathbf{P}_n^{\text{exc}}(\mathbf{V}_{n/2} \in d\mathbf{z})}{\mathbf{P}_n^{\text{mea}}(\mathbf{V}_{n/2} \in d\mathbf{z})} < \infty \right). \quad (4.4)$$

Since  $M_{n/2}$  is a function of  $\mathbf{V}_{n/2}$ , it follows that (4.1) implies (4.3) (note that  $M_{n/2} \leq M_n$ ).

It remains to prove (4.4). By Gnedenko's local limit theorem, in the discrete case

$$\forall n \geq n_0 : \quad \mathbf{P}(S_n = 0) \geq \frac{c}{\sqrt{n}}, \quad \sup_{x \in \mathbb{Z}} \mathbf{P}(S_n = x) \leq \frac{C}{\sqrt{n}}, \quad (4.5)$$

hence

$$\frac{\mathbf{P}_n^{\text{bri}}(\mathbf{V}_{n/2} = \mathbf{z})}{\mathbf{P}_n(\mathbf{V}_{n/2} = \mathbf{z})} = \frac{\mathbf{P}(S_{n/2} = -z_{n/2})}{\mathbf{P}(S_n = 0)} \leq \frac{C}{c} < \infty,$$

which proves the first relation in (4.4) in the discrete case. The continuous case is similar, since  $f_n(0) \geq \frac{c}{\sqrt{n}}$  and  $\sup_{x \in \mathbb{R}} f_n(x) \leq \frac{C}{\sqrt{n}}$  for  $n \geq n_0$ , under Assumption 2.1.

To prove the second relation in (4.4), in the discrete case we compute

$$\frac{\mathbf{P}_n^{\text{exc}}(\mathbf{V}_{n/2} = \mathbf{z})}{\mathbf{P}_n^{\text{mea}}(\mathbf{V}_{n/2} = \mathbf{z})} = \frac{\mathbf{P}_0(S_1 > 0, \dots, S_n > 0) \mathbf{P}_{z_{n/2}}(S_1 > 0, \dots, S_{n/2-1} > 0, S_{n/2} = 0)}{\mathbf{P}_0(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) \mathbf{P}_{z_{n/2}}(S_1 > 0, \dots, S_{n/2} > 0)},$$

where  $\mathbf{P}_x$  is the law of the random walk started at  $S_0 = x$ . For some  $c_1 < \infty$  we have

$$\mathbf{P}_0(S_1 > 0, \dots, S_n > 0) \leq \frac{c_1}{\sqrt{n}},$$

by [Fel71, Th.1 in §XII.7, Th.1 in §XVIII.5]. Next we apply [CC13, eq. (4.5) in Prop. 4.1] (with  $a_n = \sqrt{n}(1 + o(1))$ ), which summarizes [AD99, VV09]: for some  $c_2 \in (0, \infty)$

$$\mathbf{P}_0(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) = \hat{q}_n(0, 0) \geq \frac{c_2}{n^{3/2}}.$$

As a consequence, if we rename  $n/2$  as  $n$  and  $z_{n/2}$  as  $x$ , it remains to show that

$$\sup_{n \geq n_0} \sup_{x \geq 0} \frac{n \mathbf{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0)}{\mathbf{P}_x(S_1 > 0, \dots, S_n > 0)} < \infty. \quad (4.6)$$

By contradiction, if (4.6) does not hold, there are subsequences  $n = n_k \in \mathbb{N}$ ,  $x = x_k \geq 0$ , for  $k \in \mathbb{N}$ , such that the ratio in (4.6) diverges as  $k \rightarrow \infty$ . By extracting subsequences, assume that either  $x_k \geq \eta \sqrt{n_k}$  with  $\eta > 0$  (case 1), or  $x_k = o(\sqrt{n_k})$  as  $k \rightarrow \infty$  (case 2).

In case 1, i.e. for  $x \geq \eta \sqrt{n}$ , the denominator in (4.6) is bounded away from zero:

$$\mathbf{P}_x(S_1 > 0, \dots, S_n > 0) \geq \mathbf{P}_{\lfloor \eta \sqrt{n} \rfloor}(S_1 > 0, \dots, S_n > 0) \xrightarrow{n \rightarrow \infty} \mathbf{P}_\eta(B_t > 0 \forall t \in [0, 1]) > 0,$$

by Donsker's invariance principle [Don51] ( $B = (B_t)_{t \geq 0}$  is Brownian motion started at  $\eta$ ). For the numerator, by [CC13, eq. (4.4) in Prop. 4.1] which summarizes [Car05, VV09],

$$\mathbf{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) = \hat{q}_n^+(x, 0) \leq \frac{c_3}{\sqrt{n}} \mathbf{P}(S_1 < 0, \dots, S_n < 0) \leq \frac{c'_3}{n},$$

for suitable  $c_3, c'_3 \in (0, \infty)$ . Then the ratio in (4.6) is bounded, which is a contradiction.

In case 2, i.e. for  $x = o(\sqrt{n})$ , by [CC13, eq. (4.5) in Prop. 4.1] we have

$$\mathbf{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) = \hat{q}_n(x, 0) \underset{n \rightarrow \infty}{\sim} \underline{V}^-(x) \frac{c_4}{n^{3/2}}, \quad (4.7)$$

for a suitable  $\underline{V}^-(x)$ . Summing (4.7) for  $n > \bar{n}$ , we obtain (see also [Don12, Cor. 3])

$$\mathbf{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n > 0) \underset{n \rightarrow \infty}{\sim} \underline{V}^-(x) \frac{2c_4}{\sqrt{n}}.$$

Thus the ratio in (4.6) is bounded, which is the desired contradiction.

This completes the proof of the second relation in (4.4) in the discrete case. The continuous case is dealt with with identical arguments, exploiting [CC13, Th. 5.1].  $\square$

**Step 2.** In this step we assume that

$$\frac{S_n^2}{n} \text{ under } P_n \text{ (resp. under } P_n^{\text{mea}}) \text{ is UI,} \quad (4.8)$$

and we deduce that

$$\frac{M_n^2}{n} \text{ under } P_n \text{ (resp. under } P_n^{\text{mea}}) \text{ is UI.} \quad (4.9)$$

Observe that  $(|S_i|)_{0 \leq i \leq n}$  is a submartingale under  $P_n$ . Let us show that  $(|S_i|)_{0 \leq i \leq n}$  is a submartingale also under  $P_n^{\text{mea}}$  (for every fixed  $n \in \mathbb{N}$ ). We set for  $m \in \mathbb{N}$  and  $x \in \mathbb{R}$

$$q_m(x) := P(x + S_1 > 0, x + S_2 > 0, \dots, x + S_m > 0),$$

with  $q_0(x) := 1$ . Then we can write, for any  $n \in \mathbb{N}$ ,  $i \in \{0, 1, \dots, n-1\}$  and  $x \geq 0$ ,

$$P_n^{\text{mea}}[S_{i+1} \in dy \mid S_i = x] = \frac{\mathbf{1}_{(0, \infty)}(y) q_{n-(i+1)}(y)}{q_{n-i}(x)} P(X_1 \in dy - x).$$

Since  $y \mapsto (y - x)$  and  $y \mapsto \mathbf{1}_{(0, \infty)}(y) q_{n-(i+1)}(y)$  are non-decreasing functions, it follows by the Harris inequality (a special case of the FKG inequality) and  $E[X_1] = 0$  that

$$E_n^{\text{mea}}[S_{i+1} - S_i \mid S_i = x] \geq \int_{\mathbb{R}} (y - x) P(X_1 \in dy - x) \cdot \int_0^\infty \frac{q_{n-(i+1)}(y)}{q_{n-i}(x)} P(X_1 \in dy - x) = 0.$$

Since  $(|S_i|)_{0 \leq i \leq n}$  is a submartingale, also  $(Z_i := (|S_i| - K)^+)_{0 \leq i \leq n}$  is a submartingale, for any  $K \in (0, \infty)$ . Doob's  $L^2$  inequality yields, for  $P_n = P_n$  or  $P_n = P_n^{\text{mea}}$  (recall (2.3)),

$$E_n[(M_n - K)^2 \mathbf{1}_{\{M_n > K\}}] = E_n\left[\left(\max_{0 \leq i \leq n} Z_i\right)^2\right] \leq 4 E_n[Z_n^2] = 4 E_n[(S_n - K)^2 \mathbf{1}_{\{S_n > K\}}].$$

For  $M_n > 2K$  we can bound  $M_n^2 \leq 4(M_n - K)^2$ . Since  $(S_n - K)^2 \leq S_n^2$  for  $S_n > K$ , we get

$$E_n[M_n^2 \mathbf{1}_{\{M_n > 2K\}}] \leq 16 E_n[S_n^2 \mathbf{1}_{\{S_n > K\}}].$$

We finally choose  $K = \frac{t}{2}\sqrt{n}$ , for  $t \in (0, \infty)$ , to obtain

$$E_n\left[\frac{M_n^2}{n} \mathbf{1}_{\{\frac{M_n^2}{n} > t\}}\right] \leq 16 E_n\left[\frac{S_n^2}{n} \mathbf{1}_{\{\frac{S_n^2}{n} > \frac{t}{2}\}}\right], \quad \forall t > 0.$$

This relation for  $P_n = P_n$  (resp.  $P_n = P_n^{\text{mea}}$ ) shows that (4.8) implies (4.9).  $\square$

**Step 3.** In this step we prove that (4.8) holds, completing the proof of Theorem 2.2. We are going to apply the following standard result, proved below.

**Proposition 4.1.** *Let  $(Y_n)_{n \in \mathbb{N}}$ ,  $Y$  be real random variables such that  $Y_n \rightarrow Y$  in law. Then  $(Y_n)_{n \in \mathbb{N}}$  is UI if and only if  $\lim_{n \rightarrow \infty} E[|Y_n|] = E[|Y|]$ .*

Let us define

$$Y_n := \frac{S_n^2}{n}.$$

Since  $S_n/\sqrt{n}$  under  $P_n$  converges in law to  $Z \sim N(0, 1)$ , we have  $Y_n \rightarrow Z^2$  in law. Since  $E_n[|Y_n|] = 1 = E[Z^2]$  for all  $n \in \mathbb{N}$ , relation (4.8) under  $P_n$  follows by Proposition 4.1.

Next we focus on  $P_n^{\text{mea}}$ . It is known [Bol76] that  $S_n/\sqrt{n}$  under  $P_n^{\text{mea}}$  converges in law toward the Brownian meander at time 1, that is a random variable  $V$  with law  $P(V \in dx) := x e^{-x^2/2} \mathbf{1}_{(0, \infty)}(x) dx$ . Therefore  $Y_n \rightarrow V^2$  in law, under  $P_n^{\text{mea}}$ . Since  $E[V^2] = 2$ , relation (4.8) under  $P_n^{\text{mea}}$  is proved once we show that

$$\lim_{n \rightarrow \infty} E_n^{\text{mea}}\left[\frac{S_n^2}{n}\right] = 2. \quad (4.10)$$

To evaluate this limit, we express the law of  $S_n/\sqrt{n}$  under  $P_n^{\text{mea}}$  using fluctuation theory for random walks. By [Car05, equations (3.1) and (2.6)], as  $n \rightarrow \infty$

$$P_n^{\text{mea}}\left(\frac{S_n}{\sqrt{n}} \in dx\right) = (\sqrt{2\pi} + o(1)) \int_0^1 \int_0^\infty P\left(\frac{S_{\lfloor n(1-\alpha) \rfloor}}{\sqrt{n}} \in dx - \beta\right) \mathbb{1}_{[0,x)}(\beta) d\mu_n(\alpha, \beta),$$

where  $\mu_n$  is a finite measure on  $[0, 1) \times [0, \infty)$ , defined in [Car05, eq. (3.2)]. Then

$$E_n^{\text{mea}}\left[\frac{S_n^2}{n}\right] = (\sqrt{2\pi} + o(1)) \int_0^1 \int_0^\infty \left\{ E\left[\frac{(S_{\lfloor n(1-\alpha) \rfloor}^+)^2}{n}\right] + 2\beta E\left[\frac{S_{\lfloor n(1-\alpha) \rfloor}^+}{\sqrt{n}}\right] + \beta^2 P\left(\frac{S_{\lfloor n(1-\alpha) \rfloor}}{\sqrt{n}} > 0\right) \right\} d\mu_n.$$

By the convergence in law (under  $P$ )  $S_n/\sqrt{n} \rightarrow Z \sim N(0, 1)$ , together with the uniform integrability of  $(S_n/\sqrt{n})^2$  that we already proved, we have as  $n \rightarrow \infty$

$$\begin{aligned} E\left[\frac{(S_{\lfloor n(1-\alpha) \rfloor}^+)^2}{n}\right] &\longrightarrow (1-\alpha) E[(Z^+)^2] = \frac{1-\alpha}{2}, \\ E\left[\frac{S_{\lfloor n(1-\alpha) \rfloor}^+}{\sqrt{n}}\right] &\longrightarrow \sqrt{1-\alpha} E[Z^+] = \frac{\sqrt{1-\alpha}}{\sqrt{2\pi}}, \quad P\left(\frac{S_{\lfloor n(1-\alpha) \rfloor}}{\sqrt{n}} > 0\right) \longrightarrow P(Z > 0) = \frac{1}{2}, \end{aligned}$$

uniformly for  $\alpha \in [0, 1-\delta]$ , for  $\delta > 0$ . By [Car05, Prop. 5] we have the weak convergence

$$\mu_n(d\alpha, d\beta) \implies \mu(d\alpha, d\beta) := \frac{\beta}{\sqrt{2\pi} \alpha^{3/2}} e^{-\frac{\beta^2}{2\alpha}} d\alpha d\beta,$$

and note that  $\mu$  is a finite measure on  $[0, 1) \times [0, \infty)$ . Then  $\lim_{n \rightarrow \infty} E_n^{\text{mea}}\left[\frac{S_n^2}{n}\right]$  equals

$$\int_0^1 \left( \int_0^\infty \left\{ \frac{1-\alpha}{2} + 2\beta \frac{\sqrt{1-\alpha}}{\sqrt{2\pi}} + \frac{1}{2}\beta^2 \right\} \frac{\beta}{\alpha^{3/2}} e^{-\frac{\beta^2}{2\alpha}} d\beta \right) d\alpha = \int_0^1 \left\{ \frac{1-\alpha}{2\sqrt{\alpha}} + \sqrt{1-\alpha} + \sqrt{\alpha} \right\} d\alpha = 2,$$

which completes the proof of (4.10).  $\square$

*Proof of Proposition 4.1.* We assume that  $Y_n \rightarrow Y$  a.s., by Skorokhod's representation theorem. If  $(Y_n)_{n \in \mathbb{N}}$  is UI, then  $Y_n \rightarrow Y$  in  $L^1$ , hence  $E[|Y_n|] \rightarrow E[|Y|]$ .

Assume now that  $\lim_{n \rightarrow \infty} E[|Y_n|] = E[|Y|]$ . Since  $Y_n \rightarrow Y$  a.s., dominated convergence yields  $\lim_{n \rightarrow \infty} E[|Y_n| \mathbb{1}_{\{|Y_n| \leq T\}}] = E[|Y| \mathbb{1}_{\{|Y| \leq T\}}]$  for  $T \in (0, \infty)$  with  $P(|Y| = T) = 0$ . Then

$$\limsup_{n \rightarrow \infty} E[|Y_n| \mathbb{1}_{\{|Y_n| > T\}}] = \limsup_{n \rightarrow \infty} (E[|Y_n|] - E[|Y_n| \mathbb{1}_{\{|Y_n| \leq T\}}]) = E[|Y| \mathbb{1}_{\{|Y| > T\}}].$$

Since  $\lim_{T \rightarrow \infty} E[|Y| \mathbb{1}_{\{|Y| > T\}}] = 0$ , this shows that  $(Y_n)_{n \in \mathbb{N}}$  is UI.  $\square$

## 5. PROOF OF THEOREM 2.4

We fix a random walk  $(S_n)_{n \in \mathbb{N}_0}$  which satisfies Assumption 2.1 in the discrete case (the continuous case is covered by Theorem 2.2), with  $\sigma^2 = 1$ . We proceed in two steps.

**Step 1.** We assume that  $M_n^2/n$  under  $P_n^{\text{mea}2}$  is UI and we prove that  $M_n^2/n$  under  $P_n^{\text{exc}2}$  is UI. As in Section 4, it suffices to show that, with  $\mathbf{V}_{n/2} := (S_1, \dots, S_{n/2})$ ,

$$\sup_{n \geq n_0} \sup_{\mathbf{z} \in \mathbb{Z}^{n/2}} \frac{P_n^{\text{exc}2}(\mathbf{V}_{n/2} = \mathbf{z})}{P_n^{\text{mea}2}(\mathbf{V}_{n/2} = \mathbf{z})} < \infty. \quad (5.1)$$

If we define  $T := \min\{n \in \mathbb{N} : S_n = 0\}$ , we can compute (recall (2.6))

$$\frac{P_n^{\text{exc}2}(\mathbf{V}_{n/2} = \mathbf{z})}{P_n^{\text{mea}2}(\mathbf{V}_{n/2} = \mathbf{z})} = \frac{P(T > n) P_{z_{n/2}}(T = n/2)}{P(T = n) P_{z_{n/2}}(T = n/2)},$$

where  $P_x$  is the law of the random walk started at  $S_0 = x$ . By [Kes63], as  $n \rightarrow \infty$

$$P(T = n) = \frac{\sigma}{\sqrt{2\pi} n^{3/2}} (1 + o(1)), \quad (5.2)$$

hence, summing over  $n$ , we get  $P(T > n) = 2n P(T = n) (1 + o(1))$ . Then (5.1) reduces to

$$\sup_{n \geq n_0} \sup_{x \geq 0} \frac{n P_x(T = n)}{P_x(T > n)} < \infty. \quad (5.3)$$

Arguing as in the lines after (4.6), we need to show that the ratio in (5.3) is bounded in two cases: when  $x \geq \eta\sqrt{n}$  for fixed  $\eta > 0$  (*case 1*) and when  $x = x_n = o(\sqrt{n})$  (*case 2*).

In case 1, i.e. for  $x \geq \eta\sqrt{n}$ , the denominator in (5.3) is bounded away from zero:

$$P_x(T > n) \geq P_{\lfloor \eta\sqrt{n} \rfloor}(S_1 > 0, \dots, S_n > 0) \xrightarrow{N \rightarrow \infty} P_\eta(B_t > 0 \forall t \in [0, 1]) > 0,$$

where  $(B_t)_{t \geq 0}$  is a Brownian motion [Don51]. Then the ratio in (5.3) is bounded because  $\sup_{x \in \mathbb{Z}} P_x(T = n) \leq \frac{c'}{n}$  for some  $c' \in (0, \infty)$ , by [Kai75, Cor. 1].

In case 2, i.e. for  $x = o(\sqrt{n})$ , we apply [Uch11, Thm. 1.1], which generalizes (5.2):

$$P_x(T = n) = a^*(x) \frac{\sigma}{\sqrt{2\pi} n^{3/2}} (1 + o(1)),$$

for a suitable  $a^*(x)$  (the potential kernel of the walk). Then  $P_x(T > n) = 2n P_x(T = n) (1 + o(1))$ , hence the ratio in (5.3) is bounded. This completes the proof of (5.1).  $\square$

**Step 2.** We prove that  $M_n^2/n$  under  $P_n^{\text{mea2}}$  is UI. We argue by contradiction: if this does not hold, then there are  $\eta > 0$  and  $(n_i)_{i \in \mathbb{N}}, (K_i)_{i \in \mathbb{N}}$ , with  $\lim_{i \rightarrow \infty} K_i = \infty$ , such that

$$E_{n_i}^{\text{mea2}} \left[ \frac{M_{n_i}^2}{n_i} \mathbf{1}_{\left\{ \frac{M_{n_i}^2}{n_i} > K_i \right\}} \right] \geq \eta, \quad \forall i \in \mathbb{N}. \quad (5.4)$$

We are going to deduce that  $M_n^2/n$  under  $P_n$  is not UI, which contradicts Theorem 2.2.

We show below that we can strengthen (5.4), replacing  $E_{n_i}^{\text{mea2}}$  by  $E_m^{\text{mea2}}$  for any  $m \in \{n_i, \dots, 2n_i\}$ : more precisely, there exists  $\eta' > 0$  such that

$$E_m^{\text{mea2}} \left[ \frac{M_{n_i}^2}{n_i} \mathbf{1}_{\left\{ \frac{M_{n_i}^2}{n_i} > K_i \right\}} \right] \geq \eta', \quad \forall i \in \mathbb{N}, \forall m \in \{n_i, \dots, 2n_i\}. \quad (5.5)$$

To exploit (5.5), we work on the time horizon  $2n$ , for some fixed  $n \in \mathbb{N}$ . We define  $\sigma := \sigma_{2n} := \max\{i \leq 2n : S_i = 0\}$  and we split the path  $S = (S_1, \dots, S_{2n})$  in two parts  $\tilde{S} = (S_0, S_1, \dots, S_\sigma)$  and  $\hat{S} = (S_\sigma, S_{\sigma+1}, \dots, S_{2n})$ . The key observation is that, conditionally on  $\sigma$ , the path  $\hat{S}$  has law  $P_{2n-\sigma}^{\text{mea2}}$ . Then, if we define  $\tilde{M}_{2n} := \max |\hat{S}| = \max_{\sigma \leq i \leq 2n} |S_i|$ , the bound  $M_{2n} \geq \tilde{M}_{2n}$  gives

$$E \left[ \frac{(M_{2n})^2}{2n} \mathbf{1}_{\left\{ \frac{(M_{2n})^2}{2n} > \frac{K}{2} \right\}} \right] \geq E \left[ \frac{(\tilde{M}_{2n})^2}{2n} \mathbf{1}_{\left\{ \frac{(\tilde{M}_{2n})^2}{2n} > \frac{K}{2} \right\}} \right] = \sum_{r=0}^{2n} E \left[ E_{2n-r}^{\text{mea2}} \left[ \frac{M_{2n-r}^2}{2n} \mathbf{1}_{\left\{ \frac{M_{2n-r}^2}{2n} > K \right\}} \right] \mathbf{1}_{\{\sigma=r\}} \right].$$

We now restrict the sum to  $r \leq n$ , so that  $M_{2n-r}^2 \geq M_n^2$ , to get

$$E \left[ \frac{(M_{2n})^2}{2n} \mathbf{1}_{\left\{ \frac{(M_{2n})^2}{2n} > \frac{K}{2} \right\}} \right] \geq \frac{1}{2} P(\sigma_{2n} \leq n) \inf_{n \leq m \leq 2n} E_m^{\text{mea2}} \left[ \frac{M_n^2}{n} \mathbf{1}_{\left\{ \frac{M_n^2}{n} > K \right\}} \right].$$

Note that  $\lim_{n \rightarrow \infty} P(\sigma_{2n} \leq n) = P(B_t \neq 0 \forall t \in (\frac{1}{2}, 1]) =: p > 0$  (actually  $p = \frac{1}{2}$ , by the arcsine law), hence  $\gamma := \inf_{n \in \mathbb{N}} P(\sigma_{2n} \leq n) > 0$ . If we take  $n = n_i$  and  $K = K_i$ , by (5.5)

$$\liminf_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} E \left[ \frac{(M_n)^2}{n} \mathbf{1}_{\left\{ \frac{(M_n)^2}{n} > \frac{K}{2} \right\}} \right] \geq \inf_{i \in \mathbb{N}} E \left[ \frac{(M_{2n_i})^2}{2n_i} \mathbf{1}_{\left\{ \frac{(M_{2n_i})^2}{2n_i} > \frac{K_i}{2} \right\}} \right] \geq \frac{\gamma \eta'}{2} > 0.$$

This means that  $M_n^2/n$  under  $P_n$  is not UI, which contradicts Theorem 2.2.

It remains to prove (5.5). We fix  $C \in (0, \infty)$ , to be determined later. We may assume that  $K_i \geq C$  for all  $i \in \mathbb{N}$ . To deduce (5.5) from (5.4), we show that for some  $c > 0$

$$\inf_{n \in \mathbb{N}, m \in \{n, \dots, 2n\}, z \in \mathbb{Z}: z \geq C\sqrt{n}} \frac{P_m^{\text{mea2}}(M_n = z)}{P_n^{\text{mea2}}(M_n = z)} \geq c. \quad (5.6)$$

Fix  $m \geq n$  and  $z > 0$ . If we sum over the last  $\ell \leq n$  for which  $M_n = |S_\ell|$ , we can write

$$P_m^{\text{mea2}}(M_n = z) = \sum_{\ell=1}^n P_m^{\text{mea2}}(M_{\ell-1} \leq z, |S_\ell| = z, |S_i| < z \forall i = \ell+1, \dots, n).$$

We write  $P_m^{\text{mea2}}(\cdot) = P(\cdot | E_m)$ , with  $E_m := \{S_1 \neq 0, \dots, S_m \neq 0\}$ , and we apply the Markov property at time  $\ell$ . The cases  $S_\ell = z$  and  $S_\ell = -z$  give a similar contribution and, for brevity, we do not distinguish them (e.g. assume that the walk is symmetric). Then

$$P_m^{\text{mea2}}(M_n = z) = \frac{1}{P(T > m)} \sum_{\ell=1}^n P(M_{\ell-1} \leq z, |S_\ell| = z, E_\ell) \underbrace{P_z(|S_i| < z \forall 1 \leq i \leq n - \ell, E_{m-\ell})}_A.$$

A very similar expression holds if we replace  $P_m^{\text{mea2}}$  by  $P_n^{\text{mea2}}$ , namely

$$P_n^{\text{mea2}}(M_n = z) = \frac{1}{P(T > n)} \sum_{\ell=1}^n P(M_{\ell-1} \leq z, |S_\ell| = z, E_\ell) \underbrace{P_z(|S_i| < z \forall 1 \leq i \leq n - \ell, E_{n-\ell})}_B.$$

Since  $P(T > m) \leq P(T > n)$ , to prove (5.6) we show that  $A \geq cB$ , with  $c > 0$ . We bound

$$B \leq P_z(S_i < z \forall i = 1, \dots, n - \ell) = P_0(E_{n-\ell}^-),$$

where we set  $E_k^- := \{S_1 < 0, \dots, S_k < 0\}$ . Similarly, for  $z \geq C\sqrt{n}$  we bound

$$\begin{aligned} A &\geq P_z(S_i < z \forall i = 1, \dots, n - \ell, S_i > 0 \forall i = 1, \dots, m - \ell) \\ &= P_0(E_{n-\ell}^-, S_i > -z \forall i = 1, \dots, m - \ell) \geq P_0(E_{n-\ell}^-) \underbrace{P_0((-S_i) < C\sqrt{n} \forall i = 1, \dots, m - \ell | E_{n-\ell}^-)}_D. \end{aligned}$$

It remains to show that  $D \geq c$ . Let us set  $\tilde{S}_i := -S_i$  and  $\tilde{E}_k^+ := E_k^- = \{\tilde{S}_1 > 0, \dots, \tilde{S}_k > 0\}$ . If we write  $r := n - \ell$ , for  $m \in \{n, \dots, 2n\}$ , we have  $m - \ell = r + (m - n) \leq r + n$ , hence

$$D \geq P(\tilde{S}_i < \frac{1}{2}C\sqrt{r} \forall i = 1, \dots, r | \tilde{E}_r^+) \cdot P(\tilde{S}_i < \frac{1}{2}C\sqrt{n} \forall i = 1, \dots, n), \quad (5.7)$$

by the Markov property, since  $(\tilde{S}_j)_{j \geq r}$  under  $P(\cdot | \tilde{E}_r^+)$  is the random walk  $\tilde{S}$  started at  $\tilde{S}_r$ .

By [Bol76, Don51], as  $r \rightarrow \infty$  the two probabilities in the right hand side of (5.7) converge respectively to  $P(\sup_{t \in [0,1]} m_t < \frac{1}{2}C)$  and  $P(\sup_{t \in [0,1]} B_t < \frac{1}{2}C)$ , where  $B = (B_t)_{t \geq 0}$  is Brownian motion and  $m = (m_t)_{t \in [0,1]}$  is Brownian meander. Then, if we fix  $C > 0$  large enough, the right hand side of (5.7) is  $\geq c > 0$  for all  $r, n \in \mathbb{N}_0$ .  $\square$

## 6. PROOF OF THEOREM 3.2

We set for short  $\mathbf{Q}_N := \mathbf{P}_N \circ \mathcal{R}_N^{-1}$ . We assume that conditions (1) and (2) in Theorem 3.2 hold and we prove that  $(\mathbf{Q}_N)_{N \in \mathbb{N}}$  is tight in  $C([0, 1])$ , that is

$$\forall \eta > 0 : \quad \lim_{\delta \downarrow 0} \sup_{N \in \mathbb{N}} \mathbf{Q}_N(\Gamma(\delta) > \eta) = 0, \quad (6.1)$$

where  $\Gamma(\delta)(f) := \sup_{|t-s| \leq \delta} |f_t - f_s|$  denotes the continuity modulus of  $f \in C([0, 1])$ .

Given a finite subset  $U = \{u_1 < \dots < u_n\} \subseteq [0, 1]$  and points  $s, t \in [0, 1]$ , we write  $s \sim_U t$  iff no point  $u_i \in U$  lies between  $s$  and  $t$ . Then we define

$$\tilde{\Gamma}_U(\delta)(f) := \sup_{s, t \in [0, 1]: s \sim_U t, |t-s| \leq \delta} |f_t - f_s|.$$

Plainly, if  $f(u_i) = 0$  for all  $u_i \in U$ , then  $\Gamma(\delta)(f) \leq 2\tilde{\Gamma}_U(\delta)(f)$ . This means that in (6.1) we can replace  $\Gamma(\delta)(f)$  by  $\tilde{\Gamma}_U(\delta)(f)$ , where  $U$  is any subset of  $[0, 1]$  on which  $f$  vanishes. We fix  $U = \{\frac{t_1}{N}, \dots, \frac{t_n}{N}\}$ , where  $t_i$  are the regeneration epochs of  $\mathbf{P}_N$ . It remains to show that

$$\forall \eta > 0 : \quad \lim_{\delta \downarrow 0} \sup_{N \in \mathbb{N}} \mathbf{Q}_N(\tilde{\Gamma}_U(\delta) > \eta) = 0. \quad (6.2)$$

We set for short  $Q_t^{\text{fin}} := P_t^{\text{fin}} \circ \mathcal{R}_t^{-1}$  and  $Q_t^{\text{bulk}} := P_t^{\text{bulk}} \circ \mathcal{R}_t^{-1}$ . By Definition 3.1

$$\mathbf{Q}_N(\tilde{\Gamma}_U(\delta) \leq \eta) = \sum_{n=1}^{N+1} \sum_{0=t_1 < \dots < t_n \leq N} p_N(\{t_1, \dots, t_n\}) \prod_{i=1}^{n-1} Q_{t_{i+1}-t_i}^{\text{bulk}} \left( \Gamma\left(\frac{N}{t_{i+1}-t_i}\delta\right) \leq \eta \sqrt{\frac{N}{t_{i+1}-t_i}} \right) \times \\ \times Q_{N-t_n}^{\text{fin}} \left( \Gamma\left(\frac{N}{N-t_n}\delta\right) \leq \eta \sqrt{\frac{N}{N-t_n}} \right).$$

Note that we have the original continuity modulus  $\Gamma$ . Let us set

$$g_\eta^{\text{bulk}}(\delta) := \inf_{\substack{N \in \mathbb{N}, 2 \leq n \leq N+1, \\ 0=t_1 < \dots < t_n \leq N}} \prod_{i=1}^{n-1} Q_{t_{i+1}-t_i}^{\text{bulk}} \left( \Gamma\left(\frac{N}{t_{i+1}-t_i}\delta\right) \leq \eta \sqrt{\frac{N}{t_{i+1}-t_i}} \right) \quad (6.3) \\ g_\eta^{\text{fin}}(\delta) := \inf_{N \in \mathbb{N}, 1 \leq t < N} Q_{N-t}^{\text{fin}} \left( \Gamma\left(\frac{N}{N-t}\delta\right) \leq \eta \sqrt{\frac{N}{N-t}} \right),$$

so that we can bound  $\mathbf{Q}_N(\tilde{\Gamma}_U(\delta) \leq \eta) \geq g_\eta^{\text{bulk}}(\delta) g_\eta^{\text{fin}}(\delta)$  (we recall that  $p_N(\cdot)$  is a probability). We complete the proof of (6.2) by showing that

$$\forall \eta > 0 : \quad \lim_{\delta \downarrow 0} g_\eta^{\text{bulk}}(\delta) g_\eta^{\text{fin}}(\delta) \geq 1.$$

We first show that  $\lim_{\delta \downarrow 0} g_\eta^{\text{fin}}(\delta) \geq 1$ , for every  $\eta > 0$ . We fix  $\theta \in (0, 1)$  and consider two regimes. For  $t < (1 - \theta)N$  we can bound (recall that  $(Q_\ell^{\text{fin}})_{\ell \in \mathbb{N}}$  is tight by assumption)

$$\inf_{N \in \mathbb{N}, 1 \leq t < (1-\theta)N} Q_{N-t}^{\text{fin}} \left( \Gamma\left(\frac{N}{N-t}\delta\right) \leq \eta \sqrt{\frac{N}{N-t}} \right) \geq \inf_{\ell \in \mathbb{N}} Q_\ell^{\text{fin}} \left( \Gamma\left(\frac{\delta}{\theta}\right) \leq \eta \right) \xrightarrow{\delta \downarrow 0} 1.$$

On the other hand, for  $t \geq (1 - \theta)N$  we can bound

$$\inf_{N \in \mathbb{N}, (1-\theta)N \leq t < N} Q_{N-t}^{\text{fin}} \left( \Gamma\left(\frac{N}{N-t}\delta\right) \leq \eta \sqrt{\frac{N}{N-t}} \right) \geq \inf_{\ell \in \mathbb{N}} Q_\ell^{\text{fin}} \left( \max_{s \in [0,1]} |f_s| \leq \frac{1}{2} \frac{\eta}{\sqrt{\theta}} \right) =: h_\eta(\theta).$$

For any  $\eta > 0$ , we have  $\lim_{\delta \downarrow 0} g_\eta^{\text{fin}}(\delta) \geq \lim_{\delta \downarrow 0} h_\eta(\theta) = 1$ , by the tightness of  $(Q_\ell^{\text{fin}})_{\ell \in \mathbb{N}}$ .

To complete the proof, we show that  $\lim_{\delta \downarrow 0} g_\eta^{\text{bulk}}(\delta) \geq 1$ , for every  $\eta > 0$ . Note that

$$\inf_{t \in \mathbb{N}} Q_t^{\text{bulk}} \left( \max_{s \in [0,1]} |f_s| \leq a \right) = \inf_{t \in \mathbb{N}} P_t^{\text{bulk}} \left( \max_{i=0, \dots, t} |x_i| \leq a \sqrt{t} \right) \geq 1 - \frac{\epsilon(a)}{a^2},$$

where  $\lim_{a \uparrow \infty} \epsilon(a) = 0$ , by assumption (2). We may assume that  $a \mapsto \epsilon(a)$  is non increasing. Fix  $\theta \in (0, 1)$ . Given a family of epochs  $0 \leq t_1 < \dots < t_n \leq N$ , we distinguish two cases.

- For  $\theta N < t_{i+1} - t_i \leq N$  we can bound

$$Q_{t_{i+1}-t_i}^{\text{bulk}} \left( \Gamma\left(\frac{N}{t_{i+1}-t_i}\delta\right) \leq \eta \sqrt{\frac{N}{t_{i+1}-t_i}} \right) \geq \inf_{t \in \mathbb{N}} Q_t^{\text{bulk}} \left( \Gamma\left(\frac{\delta}{\theta}\right) \leq \eta \right) =: F_{\eta, \theta}(\delta),$$

and note that for fixed  $\eta, \theta$  we have  $\lim_{\delta \downarrow 0} F_{\eta, \theta}(\delta) = 1$ , because  $(Q_t^{\text{bulk}})_{t \in \mathbb{N}}$  is tight.

- For  $t_{i+1} - t_i \leq \theta N$  we can bound

$$Q_{t_{i+1}-t_i}^{\text{bulk}} \left( \Gamma\left(\frac{N}{t_{i+1}-t_i}\delta\right) \leq \eta \sqrt{\frac{N}{t_{i+1}-t_i}} \right) \geq Q_{t_{i+1}-t_i}^{\text{bulk}} \left( \max_{s \in [0,1]} |f_s| \leq \frac{\eta}{2} \sqrt{\frac{N}{t_{i+1}-t_i}} \right) \\ \geq 1 - \frac{4(t_{i+1}-t_i)}{\eta^2 N} \epsilon\left(\frac{\eta}{2\sqrt{\theta}}\right) \geq \exp\left(-\frac{4(t_{i+1}-t_i)}{\eta^2 N} \epsilon\left(\frac{\eta}{2\sqrt{\theta}}\right)\right),$$

where the last inequality holds for  $\theta > 0$  small, by  $1 - z \geq e^{-2z}$  for  $z \in [0, \frac{1}{2}]$ .

We can have  $t_{i+1} - t_i > \theta N$  for at most  $\lfloor 1/\theta \rfloor$  values of  $i$ , hence

$$g_\eta^{\text{bulk}}(\delta) \geq F_{\eta, \theta}(\delta)^{\frac{1}{\theta}} \prod_{i=1}^{n-1} \exp\left(-\frac{4(t_{i+1}-t_i)}{\eta^2 N} \epsilon\left(\frac{\eta}{2\sqrt{\theta}}\right)\right) \geq F_{\eta, \theta}(\delta)^{\frac{1}{\theta}} \exp\left(-\frac{4}{\eta^2} \epsilon\left(\frac{\eta}{2\sqrt{\theta}}\right)\right).$$

Given  $\eta > 0$ , we first fix  $\theta > 0$  small enough, so that the exponential is close to one; then we let  $\delta \rightarrow 0$ , so that  $F_{\eta, \theta}(\delta)^{\frac{1}{\theta}} \rightarrow 1$ . This yields  $\lim_{\delta \downarrow 0} g_\eta^{\text{bulk}}(\delta) \geq 1$ .  $\square$

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