# GENERAL SMILE ASYMPTOTICS WITH BOUNDED MATURITY

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Abstract. We provide explicit conditions on the distribution of risk-neutral log-returns which yield sharp asymptotic estimates on the implied volatility smile. Our results extend previous work of Benaim and Friz [BF09] and are valid in great generality, both for extreme strike (with arbitrary bounded maturity, possibly varying with the strike) and for small maturity (with arbitrary strike, possibly varying with the maturity). Applications to popular models are discussed, including Carr-Wu finite moment logstable model, Heston's model and Merton's jump diffusion model.

## 1. Introduction

The price of a European option is typically expressed in terms of the Black&Scholes *implied volatility* $\sigma_{\text{imp}}(\kappa, t)$ (where $\kappa$ denotes the log-strike and $t$ the maturity), cf. [Gat06]. Benaim and Friz [BF09] provide explicit conditions on the log-return distribution to obtain the asymptotic behavior of $\sigma_{\text{imp}}(\kappa, t)$, in the special regime $\kappa \to \pm\infty$ for fixed $t > 0$. In this paper we strengthen and extend their results, *allowing $\kappa$ and $t$ to vary simultaneously along an arbitrary curve* such that either $|\kappa| \to \infty$ with bounded $t$, or $t \to 0$ with arbitrary $\kappa$.

This flexibility allows to determine *the asymptotics of $\sigma_{\text{imp}}(\kappa, t)$ as a surface*, when $(\kappa, t)$ vary in open regions of the plane. We illustrate this fact in Section 2, where we apply our results to some concrete models (see Remarks 2.2, 2.4 and 2.5 below).

Our results are organized as follows.

- First we provide universal formulas that link the asymptotic behavior of the implied volatility $\sigma_{\text{imp}}(\kappa, t)$ to that of the call $c(\kappa, t)$ and put $p(\kappa, t)$ option prices, cf. §1.2.
- Then we show that the asymptotic behavior of the option prices $c(\kappa, t)$ and $p(\kappa, t)$ can be linked explicitly to the *tail probabilities* $\overline{F}_t(\kappa) := \mathrm{P}(X_t > \kappa)$ and $F_t(\kappa) := \mathrm{P}(X_t \le \kappa)$, where $X_t$ denotes the risk-neutral log-return, cf. §1.3.

Combining these results, whenever enough information on the tail probabilities is available, it is possible to write down explicitly the asymptotic behavior of the implied volatility.

### 1.1. The setting.

**The setting.** We consider a generic stochastic process $(X_t)_{t\ge 0}$ representing the log-price of an asset, normalized by $X_0 := 0$. We work under the risk-neutral measure, that is (assuming zero interest rate) the price process $(S_t := e^{X_t})_{t\ge 0}$ is a martingale. European call and put options, with maturity $t > 0$ and a log-strike $\kappa \in \mathbb{R}$, are priced respectively

$$c(\kappa, t) = \mathrm{E}[(e^{X_t} - e^{\kappa})^+], \qquad p(\kappa, t) = \mathrm{E}[(e^{\kappa} - e^{X_t})^+], \tag{1.1}$$

and are linked by the *call-put parity* relation:

$$c(\kappa, t) - p(\kappa, t) = 1 - e^{\kappa}. \tag{1.2}$$

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In all of our results, *we take limits along an arbitrary family (or "path") of values of* $(\kappa, t)$. It is immaterial whether this is a sequence $((\kappa_n, t_n))_{n\in\mathbb{N}}$ or a curve $((\kappa_s, t_s))_{s\in[0,\infty)}$, therefore we omit subscripts. Without loss of generality, we assume that all the $\kappa$'s have the same sign (just consider separately the subfamilies with positive and negative $\kappa$'s). To simplify notation, we only consider positive families $\kappa \geq 0$ and give results for both $\kappa$ and $-\kappa$.

Our main interest is for families of values of $(\kappa, t)$ such that

$$\text{either } \kappa \to \infty \text{ with bounded } t\,, \quad \text{ or } t \to 0 \text{ with arbitrary } \kappa \geq 0\,. \tag{1.3}$$

Note that (1.3) gathers many interesting regimes, namely:

(1) $\kappa \to \infty$ and $t \to \bar{t} \in (0, \infty)$;

(2) $\kappa \to \infty$ and $t \to 0$;

(3) $\kappa \to \bar{\kappa} \in (0, \infty)$ and $t \to 0$;

(4) $\kappa \to 0$ and $t \to 0$.

Remarkably, while regime (4) needs to be handled separately, regimes (1)-(2)-(3) will be analyzed at once, as special instances of the case "$\kappa$ is bounded away from zero".

Whenever (1.3) holds, one has (see §A.1)

$$c(\kappa, t) \to 0\,, \qquad p(-\kappa, t) \to 0\,, \tag{1.4}$$

but relation (1.4) is more general, as it can be satisfied also when $t \to \infty$. Except for the results in §1.2, which are valid in complete generality under (1.4), we stick to the case of bounded $t$ (we refer to [T09, JKM13] for results in the regime $t \to \infty$).

A key quantity of interest is the *implied volatility* $\sigma_{\mathrm{imp}}(\kappa, t)$ of the model, defined as the value of the volatility parameter $\sigma \in [0, \infty)$ that plugged into the Black&Scholes formula yields the given call and put prices $c(\kappa, t)$ and $p(\kappa, t)$ (see §3.2-§3.3 below). Note that $\sigma_{\mathrm{imp}}(\kappa, t) = 0$ if $c(\kappa, t) = 0$ and, likewise, $\sigma_{\mathrm{imp}}(-\kappa, t) = 0$ if $p(-\kappa, t) = 0$. Consequently, to avoid trivialities, we focus on families of $(\kappa, t)$ such that $c(\kappa, t) > 0$ and $p(-\kappa, t) > 0$.

Throughout the paper, we write $f(\kappa, t) \sim g(\kappa, t)$ to mean $f(\kappa, t)/g(\kappa, t) \to 1$. Let us recall a useful standard device, referred to as *subsequence argument*: to prove an asymptotic relation, such as e.g. $f(\kappa, t) \sim g(\kappa, t)$, along a given family of values of $(\kappa, t)$, it suffices to show that from every subsequence one can extract a further sub-subsequence along which the relation holds. As a consequence, *in the proofs we may always assume that all quantities of interest have a (possibly infinite) limit*, e.g. $\kappa \to \bar{\kappa} \in [0, \infty]$ and $t \to \bar{t} \in [0, \infty)$, because this is always true extracting a suitable subsequence.

1.2. **From option price to implied volatility.** We first show that, whenever the option prices $c(\kappa, t)$ or $p(-\kappa, t)$ vanish, they determine the asymptotic behavior of the implied volatility through *explicit universal formulas.*

We need to introduce some notation. Denote by $\phi(\cdot)$ and $\Phi(\cdot)$ respectively the density and distribution function of a standard Gaussian (see (3.1) below), and define the function

$$D(z) := \frac{1}{z}\phi(z) - \Phi(-z), \qquad \forall z > 0\,. \tag{1.5}$$

As we shown in §3.1 below, $D$ is a smooth and strictly decreasing bijection from $(0, \infty)$ to $(0, \infty)$. Its inverse $D^{-1} : (0, \infty) \to (0, \infty)$ is also smooth, strictly decreasing and satisfies

$$D^{-1}(y) \sim \sqrt{2\,(-\log y)} \quad \text{ as } y \downarrow 0\,, \qquad D^{-1}(y) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{y} \quad \text{ as } y \uparrow \infty\,. \tag{1.6}$$

The following theorem, proved in Section 3, describes the link between option price and implied volatility asymptotics, extending Benaim and Friz [BF09, Lemma 3.3]. As we discuss in Remark 1.3 below, it overlaps with recent results by Gao and Lee [GL14].

**Theorem 1.1** (From option price to implied volatility). *Consider an arbitrary family of values of $(\kappa, t)$ with $\kappa \geq 0$ and $t > 0$, such that $c(\kappa, t) \to 0$, resp. $p(-\kappa, t) \to 0$.*

- Case of $\kappa$ bounded away from zero (i.e. $\liminf \kappa > 0$).

$$\begin{aligned} \sigma_{\text{imp}}(\kappa, t) &\sim \left( \sqrt{\frac{-\log c(\kappa, t)}{\kappa} + 1} - \sqrt{\frac{-\log c(\kappa, t)}{\kappa}} \right) \sqrt{\frac{2\kappa}{t}}, \qquad \textit{resp.} \\ \sigma_{\text{imp}}(-\kappa, t) &\sim \left( \sqrt{\frac{-\log p(-\kappa, t)}{\kappa}} - \sqrt{\frac{-\log p(-\kappa, t)}{\kappa} - 1} \right) \sqrt{\frac{2\kappa}{t}}. \end{aligned} \tag{1.7}$$

- Case of $\kappa \to 0$, with $\kappa > 0$.

$$\begin{aligned} \sigma_{\text{imp}}(\kappa, t) &\sim \frac{1}{D^{-1}\left(\frac{c(\kappa,t)}{\kappa}\right)} \frac{\kappa}{\sqrt{t}}, \qquad \textit{resp.} \\ \sigma_{\text{imp}}(-\kappa, t) &\sim \frac{1}{D^{-1}\left(\frac{p(-\kappa,t)}{\kappa}\right)} \frac{\kappa}{\sqrt{t}}. \end{aligned} \tag{1.8}$$

- Case of $\kappa = 0$.

$$\sigma_{\text{imp}}(0, t) \sim \sqrt{2\pi}\, \frac{c(0, t)}{\sqrt{t}} = \sqrt{2\pi}\, \frac{p(0, t)}{\sqrt{t}}. \tag{1.9}$$

Note that Theorem 1.1 requires *no assumption on the model*: the link between $\sigma_{\text{imp}}(\kappa, t)$ and the option prices $c(\kappa, t)$ and $p(\kappa, t)$ is *universal*, being essentially a statement about the inversion of Black&Scholes formula (see Theorem 3.3 for an explicit reformulation).

**Remark 1.2.** The formulas in Theorem 1.1 become more explicit when additional information on the asymptotic behavior of $c(\kappa, t)$ and $p(-\kappa, t)$ is available. For instance, whenever $\frac{-\log c(\kappa,t)}{\kappa} \to \infty$, resp. $\frac{-\log p(-\kappa,t)}{\kappa} \to \infty$, formula (1.7) reduces to

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t\,(-\log c(\kappa, t))}}, \qquad \text{resp.} \qquad \sigma_{\text{imp}}(-\kappa, t) \sim \frac{\kappa}{\sqrt{2t\,(-\log p(-\kappa, t))}}. \tag{1.10}$$

Likewise, using the estimates in (1.6), formula (1.8) can be rewritten as follows:

$$\sigma_{\text{imp}}(\kappa, t) \sim \begin{cases} \dfrac{\kappa}{\sqrt{2t\,(-\log(c(\kappa, t)/\kappa))}} & \text{if } \dfrac{c(\kappa, t)}{\kappa} \to 0\,; \\ \dfrac{\kappa}{D^{-1}(a)\,\sqrt{t}} & \text{if } \dfrac{c(\kappa, t)}{\kappa} \to a \in (0, \infty)\,; \\ \sqrt{2\pi}\,\dfrac{c(\kappa, t)}{\sqrt{t}} & \text{if } \dfrac{c(\kappa, t)}{\kappa} \to \infty, \text{ or if } \kappa = 0\,, \end{cases} \tag{1.11}$$

and analogously for $\sigma_{\text{imp}}(-\kappa, t)$, just replacing $c(\kappa, t)$ by $p(-\kappa, t)$.

It is interesting to observe that the first relation in (1.10) coincides with the first line of (1.11) when $\kappa \to 0$ slowly enough, more precisely when

$$-\log \kappa = o\big(-\log c(\kappa, t)\big)\,, \tag{1.12}$$

so that $-\log(c(\kappa,t)/\kappa) \sim -\log c(\kappa,t)$. This means that *relations* (1.7) *and* (1.8) *match at the boundary* of their respective domain of validity. On the other hand, when $\kappa \to 0$ fast enough so that (1.12) fails, relation (1.7) must be replaced by (1.8)-(1.9).

**Remark 1.3.** Taking the square of both sides of the first line of (1.7), one can rewrite it as

$$\frac{\sigma_{\mathrm{imp}}(\kappa,t)^2\, t}{\kappa} \sim \psi\bigg(\frac{-\log c(\kappa,t)}{\kappa}\bigg), \qquad \text{with} \qquad \psi(x) := 2 - 4\big[\sqrt{x^2+x} - x\big],$$

which is the key formula in Lemma 3.3 of Benaim and Friz [BF09] (who considered the regime $\kappa \to \infty$ for fixed $t$ and made some additional assumptions). Theorem 1.1 provides a substantial extension, allowing for *any regime of* $(\kappa,t)$ and making no extra assumptions.

We point out that equation (1.7) in full generality has been recently proved by Gao and Lee [GL14] (extending previous results of Lee [L04], Roper and Rutkowski [RR09], Gulisashvili [G10]). Actually, Gao and Lee *prove much more than* (1.7), since their approach provides explicit estimates for the error and allows to obtain higher order asymptotics. On the other hand, in [GL14] condition (1.12) is assumed (cf. equation (4.2) therein), which means that all regimes in which $\kappa \to 0$ "fast enough" are excluded from their analysis.

Summarizing, our Theorem 1.1 provides a simple and comprehensive account of first order asymptotics for the implied volatility as a function of the option price, which can be applied to *any* family of $(\kappa,t)$ such that $c(\kappa,t) \to 0$, resp. $p(-\kappa,t) \to 0$ (with no restriction such as (1.12)). This is especially useful for the results in the next subsection, which cover all possible regimes of $(\kappa,t)$ with bounded $t$. For these reasons, despite the overlap with [GL14], we give a complete and self-contained proof of Theorem 1.1 in Section 3.

1.3. **From tail probability to option price.** For Theorem 1.1 to be concretely useful, one needs to control the asymptotic behavior of $c(\kappa,t)$ and $p(-\kappa,t)$. We are going to show that this can be extracted from the asymptotic behavior of the *tail probabilities* of the risk-neutral log-price $(X_t)_{t\ge 0}$, defined by:

$$\overline{F}_t(\kappa) := \mathrm{P}(X_t > \kappa), \qquad F_t(-\kappa) := \mathrm{P}(X_t \le -\kappa). \tag{1.13}$$

We need to distinguish two regimes for $(\kappa,t)$, namely when the tail probabilities converge to a strictly positive limit (*typical deviations*) or when they vanish (*atypical deviations*).

*Atypical deviations.* We first focus on families of values of $(\kappa,t)$ such that

$$\overline{F}_t(\kappa) \to 0, \qquad \text{resp.} \qquad F_t(-\kappa) \to 0. \tag{1.14}$$

We stress that this includes regimes (1), (2) and (3) on page 2, and also regime (4) provided $\kappa \to 0$ sufficiently slow. We need to formulate a regularity assumption on the decay of $\overline{F}_t(\kappa)$, resp. $F_t(-\kappa)$, which is a natural generalization of the *regular variation* condition of Benaim and Friz [BF09] (see Remark 1.7 below for more details).

**Hypothesis 1.4** (Regular decay of tail probability)**.** *The family of values of* $(\kappa,t)$ *with* $\kappa > 0$*,* $t > 0$ *satisfies* (1.14)*, and for every* $\varrho \in [1,\infty)$ *the following limit exists in* $[0,+\infty]$*:*

$$I_+(\varrho) := \lim \frac{\log \overline{F}_t(\varrho\kappa)}{\log \overline{F}_t(\kappa)}, \qquad \textit{resp.} \qquad I_-(\varrho) := \lim \frac{\log F_t(-\varrho\kappa)}{\log F_t(-\kappa)}, \tag{1.15}$$

*and moreover*

$$\lim_{\varrho\downarrow 1} I_+(\varrho) = 1, \qquad \textit{resp.} \qquad \lim_{\varrho\downarrow 1} I_-(\varrho) = 1. \tag{1.16}$$

*(The limits in* (1.15) *are taken along the given family of values of* $(\kappa,t)$*.)*

Further assumptions on $I_\pm(\cdot)$, depending on the regime of $\kappa$, will be required below, coupled to suitable *moment conditions*, that we state here for convenience.

- Given $\eta \in (0,\infty)$, the first moment condition is
$$\limsup \mathrm{E}[e^{(1+\eta)X_t}] < \infty\,, \tag{1.17}$$
where the lim sup is taken along the given family of values of $(\kappa,t)$ (however, only $t$ enters in this relation). Note that if $t \le T$ it suffices to require that
$$\mathrm{E}[e^{(1+\eta)X_T}] < \infty\,, \tag{1.18}$$
because $(e^{(1+\eta)X_t})_{t\ge 0}$ is a submartingale and hence $\mathrm{E}[e^{(1+\eta)X_t}] \le \mathrm{E}[e^{(1+\eta)X_T}]$.
- Always for $\eta \in (0,\infty)$, the second moment condition is
$$\limsup \mathrm{E}\left[\left|\frac{e^{X_t}-1}{\kappa}\right|^{1+\eta}\right] < \infty\,, \tag{1.19}$$
along the given family of values of $(\kappa,t)$. Note that for $\eta = 1$ this simplifies to
$$\exists C \in (0,\infty): \qquad \mathrm{E}[e^{2X_t}] \le 1 + C\kappa^2\,. \tag{1.20}$$

The following theorems, proved in §4.1, give the link between tail probabilities and option prices. Due to different assumptions, we present separately the results on $c(\kappa,t)$ and $p(-\kappa,t)$.

**Theorem 1.5** (Right-tail atypical deviations)**.** *Consider a family of values of* $(\kappa,t)$ *with* $\kappa > 0$, $t > 0$ *such that Hypothesis 1.4 is satisfied by the right tail probability* $\overline{F}_t(\kappa)$.

- Case of $\kappa$ bounded away from zero and $t$ bounded away from infinity ($\liminf \kappa > 0$, $\limsup t < \infty$). *Let the moment condition* (1.17) *hold for* every $\eta > 0$, *or alternatively let it hold only for* some $\eta > 0$ *but in addition assume that*
$$I_+(\varrho) \ge \varrho\,, \qquad \forall \varrho \ge 1\,. \tag{1.21}$$
*Then*
$$\log c(\kappa,t) \sim \log \overline{F}_t(\kappa) + \kappa\,, \tag{1.22}$$
*which yields, by Theorem 1.1,*
$$\sigma_{\mathrm{imp}}(\kappa,t) \sim \left(\sqrt{\frac{-\log \overline{F}_t(\kappa)}{\kappa}} - \sqrt{\frac{-\log \overline{F}_t(\kappa)}{\kappa} - 1}\right)\sqrt{\frac{2\kappa}{t}}\,. \tag{1.23}$$
*In the special case when* $-\log \overline{F}_t(\kappa)/\kappa \to \infty$, *assumption* (1.21) *can be relaxed to*
$$\lim_{\varrho\to\infty} I_+(\varrho) = \infty\,, \tag{1.24}$$
*relation* (1.22) *reduces to*
$$\log c(\kappa,t) \sim \log \overline{F}_t(\kappa)\,, \tag{1.25}$$
*and* (1.23) *simplifies to*
$$\sigma_{\mathrm{imp}}(\kappa,t) \sim \frac{\kappa}{\sqrt{2t\,(-\log \overline{F}_t(\kappa))}}\,. \tag{1.26}$$

- Case of $\kappa \to 0$ and $t \to 0$. *Let the moment condition* (1.19) *hold for* every $\eta > 0$, *or alternatively let it hold only for* some $\eta > 0$ *but in addition assume* (1.24). *Then*
$$\log\big(c(\kappa,t)/\kappa\big) \sim \log \overline{F}_t(\kappa)\,, \tag{1.27}$$
*which yields, by Theorem 1.1, precisely the same asymptotics* (1.26) *for* $\sigma_{\text{imp}}(\kappa,t)$.

**Theorem 1.6** (Left-tail atypical deviations)**.** *Consider a family of values of* $(\kappa,t)$ *with* $\kappa > 0$, $t > 0$ *such that Hypothesis 1.4 is satisfied by the left tail probability* $F_t(-\kappa)$.

- Case of $\kappa$ bounded away from zero and $t$ bounded away from infinity ($\liminf \kappa > 0$, $\limsup t < \infty$). *With no moment condition and no extra assumption on* $I_-(\cdot)$, *one has*
$$\log p(-\kappa,t) \sim \log F_t(-\kappa) - \kappa\,, \tag{1.28}$$
*which yields, by Theorem 1.1,*
$$\sigma_{\text{imp}}(-\kappa,t) \sim \left(\sqrt{\frac{-\log F_t(-\kappa)}{\kappa} + 1} - \sqrt{\frac{-\log F_t(-\kappa)}{\kappa}}\right)\sqrt{\frac{2\kappa}{t}}\,. \tag{1.29}$$
*In the special case when* $-\log F_t(-\kappa)/\kappa \to \infty$, *relation* (1.28) *reduces to*
$$\log p(-\kappa,t) \sim \log F_t(-\kappa)\,, \tag{1.30}$$
*and* (1.29) *simplifies to*
$$\sigma_{\text{imp}}(-\kappa,t) \sim \frac{\kappa}{\sqrt{2t\,(-\log F_t(-\kappa))}}\,. \tag{1.31}$$
- Case of $\kappa \to 0$ and $t \to 0$. *Let the moment condition* (1.19) *hold for* every $\eta > 0$, *or alternatively let it hold only for* some $\eta > 0$ *but in addition assume that*
$$\lim_{\varrho\uparrow\infty} I_-(\varrho) = \infty\,. \tag{1.32}$$
*Then*
$$\log\big(p(-\kappa,t)/\kappa\big) \sim \log F_t(-\kappa)\,, \tag{1.33}$$
*which yields, by Theorem 1.1, precisely the same asymptotics* (1.31) *for* $\sigma_{\text{imp}}(-\kappa,t)$.

**Remark 1.7.** Let us compare our Hypothesis 1.4 with the key assumption of Benaim and Friz [BF09], the *regular variation of the tail probabilities*, i.e. there exist $\alpha > 0$ and a slowly varying function† $L(\cdot) = L_t(\cdot)$ such that, as $\kappa \to \infty$ for *fixed* $t > 0$,
$$\log \overline{F}_t(\kappa) \sim -L(\kappa)\,\kappa^\alpha\,, \qquad \text{resp.} \qquad \log F_t(-\kappa) \sim -L(\kappa)\,\kappa^\alpha\,. \tag{1.34}$$
If (1.34) holds, conditions (1.15) and (1.16) are satisfied, with $I_\pm(\varrho) = \varrho^\alpha$. Remarkably, in the special regime $\kappa \to \infty$ with fixed $t$, conditions (1.15) and (1.16) are actually *equivalent* to (1.34), by [BGT89, Theorem 1.4.1]. Thus Hypothesis 1.4 is a natural extension of the regular variation assumption of Benaim and Friz, when the maturity $t$ is allowed to vary.

**Remark 1.8.** The assumptions for left-tail asymptotics in Theorem 1.6 are weaker than those for right-tail asymptotics in Theorem 1.5. For instance, the left-tail condition $\mathrm{E}[e^{-\eta X_T}] < \infty$ required in [BF09, Theorem 1.2] is not needed, allowing to treat the case of a *polynomially decaying* left tail, like in the Carr-Wu model described in Section 2.

†A positive function $L(\cdot)$ is slowly varying if $\lim_{x\to\infty} L(\varrho x)/L(x) = 1$ for all $\varrho > 0$.

**Remark 1.9.** We stress that the "special case" conditions

$$-\frac{\log \overline{F}_t(\kappa)}{\kappa} \to \infty\,, \qquad \text{resp.} \qquad -\frac{\log F_t(-\kappa)}{\kappa} \to \infty\,, \tag{1.35}$$

are always fulfilled if $t \to 0$ and $\kappa$ is bounded away from infinity (say $\kappa \to \bar{\kappa} \in (0,\infty)$).

For families of $(\kappa, t)$ with $\kappa \to \infty$, conditions (1.35) are satisfied if $\limsup \mathrm{E}[e^{(1+\eta)X_t}] < \infty$, resp. $\limsup \mathrm{E}[e^{-\eta X_t}] < \infty$, *for every* $\eta \in (0,\infty)$, by Markov's inequality (see (4.5) below).

*Typical deviations.* We next focus on the case when $\kappa \to 0$ and $t \to 0$ in such a way that the tail probability $\overline{F}_t(\kappa)$, resp. $F_t(-\kappa)$ converges to a strictly positive limit. To deal with this regime, we make the following natural assumption.

**Hypothesis 1.10** (Small time scaling)**.** *There is a positive function* $(\gamma_t)_{t>0}$ *with* $\lim_{t\downarrow 0} \gamma_t = 0$ *such that* $X_t/\gamma_t$ *converges in law as* $t \downarrow 0$ *to some random variable* $Y$*:*

$$\frac{X_t}{\gamma_t} \xrightarrow[t\downarrow 0]{d} Y\,. \tag{1.36}$$

Note that (1.36) is a condition on the tail probabilities: for all $a \geq 0$ with $\mathrm{P}(Y = a) = 0$,

$$\overline{F}_t(a\gamma_t) \to \mathrm{P}(Y > a)\,, \qquad F_t(a\gamma_t) \to \mathrm{P}(Y \leq a)\,. \tag{1.37}$$

In particular, the limits in (1.37) are strictly positive for every $a \geq 0$ if the support of the law of $Y$ is unbounded from above and below.

We can finally state the following result, proved in §4.2 below.

**Theorem 1.11** (Right- and left-tail typical deviations)**.** *Assume that Hypothesis 1.10 is satisfied, and moreover the moment condition* (1.19) *holds for some* $\eta > 0$ *with* $\kappa = \gamma_t$*:*

$$\exists \eta > 0: \qquad \limsup_{t\to 0} \mathrm{E}\left[\left|\frac{e^{X_t} - 1}{\gamma_t}\right|^{1+\eta}\right] < \infty\,. \tag{1.38}$$

*Then the random variable* $Y$ *in* (1.36) *is in* $L^1$ *and satisfies* $\mathrm{E}[Y] = 0$.

*For any family of values of* $(\kappa, t)$ *such that*

$$t \to 0 \quad \textit{and} \quad \frac{\kappa}{\gamma_t} \to a \in [0,\infty)\,,$$

*assuming that* $\mathrm{P}(Y > a) > 0$*, resp.* $\mathrm{P}(Y < -a) > 0$*, one has*

$$c(\kappa, t) \sim \gamma_t\, \mathrm{E}[(Y-a)^+]\,, \qquad \textit{resp.} \qquad p(-\kappa, t) \sim \gamma_t\, \mathrm{E}[(Y+a)^-]\,. \tag{1.39}$$

*This yields, by Theorem 1.1,*

$$\sigma_{\mathrm{imp}}(\pm\kappa, t) \sim C_\pm(a)\, \frac{\gamma_t}{\sqrt{t}}\,, \qquad \textit{with} \qquad C_\pm(a) = \begin{cases} \dfrac{a}{D^{-1}\big(\frac{\mathrm{E}[(Y\mp a)^\pm]}{a}\big)} & \textit{if} \;\; a > 0\,, \\ \sqrt{2\pi}\, \mathrm{E}[Y^\pm] & \textit{if} \;\; a = 0\,. \end{cases} \tag{1.40}$$

1.4. **Discussion and structure of the paper.** Theorems 1.5, 1.6 and 1.11 are useful because their assumptions, involving asymptotics for the tail probabilities $\overline{F}_t(\kappa)$ and $F_t(-\kappa)$, can be verified for concrete models (see Section 2 for some examples). The difference between the regimes of typical and atypical deviations can be described as follows:

- for typical deviations, the key assumption is Hypothesis 1.10, which concerns the *weak convergence* of $X_t$, cf. (1.36)-(1.37);
- for atypical deviations, the key assumption is Hypotehsis 1.4, which concerns the *large deviations* properties of $X_t$, cf. (1.15)-(1.16).

In particular, it is worth stressing that Hypotehsis 1.4 requires sharp asymptotics only for *the logarithm of the tail probabilities* $\log \overline{F}_t(\kappa)$ *and* $\log F_t(-\kappa)$, and not for the tail probabilities themselves, which would be a considerably harder task (out of reach for many models). As a consequence, Hypotehsis 1.4 can often be checked through the celebrated *Gärtner-Ellis Theorem* [DZ98, Theorem 2.3.6], which yields sharp asymptotics on $\log \overline{F}_t(\kappa)$ and $\log F_t(-\kappa)$ under suitable conditions on the moment generating function of $X_t$.

The rest of the paper is structured as follows.

- In Section 2 we apply our results to the finite moment logstable model of Carr-Wu, determining the complete asymptotic behavior of the implied volatility smile for bounded maturity. We then discuss the models of Heston and Merton, as well as a stochastic volatility model recently introduced in [ACDP12].
- In Section 3 we prove Theorem 1.1.
- In Section 4 we prove Theorems 1.5, 1.6 and 1.11.
- Finally, a few technical points have been deferred to the Appendix A.

## 2. Examples

We apply our main results to some models: the the Carr-Wu model in §2.1, the Heston model in §2.2, the Merton model in §2.3 and, finally, a stochastic volatility model which exhibits multiscaling of moments, recently introduced in [ACDP12], in §2.4.

**2.1. Carr-Wu's Finite Moment Logstable Model.** Carr and Wu [CW04] consider a model where the log-strike $X_t$ has characteristic function

$$\mathrm{E}\big[e^{iuX_t}\big] = e^{t\left[iu\mu - |u|^\alpha \sigma^\alpha \left(1+i\,\mathrm{sign}(u)\tan(\frac{\pi\alpha}{2})\right)\right]}, \tag{2.1}$$

where $\sigma \in (0,\infty)$, $\alpha \in (1,2]$, while $\mu := \sigma^\alpha / \cos(\frac{\pi\alpha}{2})$ in the risk-neutral measure, cf. [CW04, Proposition 1]. The moment generating function of $X_t$ is

$$\mathrm{E}\big[e^{\lambda X_t}\big] = \begin{cases} e^{[\lambda\mu - \frac{(\lambda\sigma)^\alpha}{\cos(\frac{\pi\alpha}{2})}]\,t} & \text{if } \lambda \ge 0\,, \\ +\infty & \text{if } \lambda < 0\,. \end{cases} \tag{2.2}$$

Note that as $\alpha \to 2$ one recovers Black&Scholes model with volatility $\sqrt{2}\sigma$, cf. §3.2 below.

Let $Y$ denote a random variable with characteristic function

$$\mathrm{E}[e^{iuY}] = e^{-|u|^\alpha\left(1+i\,\mathrm{sign}(u)\tan(\frac{\pi\alpha}{2})\right)}, \tag{2.3}$$

i.e. $Y$ has a strictly stable law with index $\alpha$ and skewness parameter $\beta = -1$, and $\mathrm{E}[Y] = 0$. Applying Theorems 1.5, 1.6 and 1.11, we obtain the following complete characterization of the volatility smile asymptotics with bounded maturity for this model.

**Theorem 2.1** (Smile asymptotics of Carr-Wu model). *The following asymptotics hold.*

- Atypical deviations. *Consider any family of $(\kappa, t)$ such that*

$$0 < t \le T \quad \textit{for some fixed } T < \infty, \textit{ and } \quad \frac{\kappa}{t^{1/\alpha}} \to \infty\,. \tag{2.4}$$

*(This includes, in particular, the regimes* (1), (2), (3) *on page 2, as well as part of regime* (4)*.) Then one has the right-tail asymptotics*

$$\sigma_{\mathrm{imp}}(\kappa,t) \sim B_\alpha \left(\frac{\kappa}{t}\right)^{-\frac{2-\alpha}{2(\alpha-1)}}, \qquad \textit{where } B_\alpha := \frac{(\alpha\sigma)^{\frac{\alpha/2}{\alpha-1}}}{\sqrt{2}\,|\cos(\frac{\pi\alpha}{2})|^{\frac{1/2}{\alpha-1}}}\,, \tag{2.5}$$

*and the left-tail asymptotics*

$$\sigma_{\text{imp}}(-\kappa,t) \sim \left( \sqrt{\frac{\log \frac{\kappa^\alpha}{t}}{\kappa} + 1} - \sqrt{\frac{\log \frac{\kappa^\alpha}{t}}{\kappa}} \right) \sqrt{\frac{2\kappa}{t}}\,, \tag{2.6}$$

*which can be made more explicit as follows:*

$$\sigma_{\text{imp}}(-\kappa,t) \sim \begin{cases} \sqrt{\dfrac{2\kappa}{t}} & \textit{if } \dfrac{\kappa}{\log \frac{1}{t}} \to \infty\,, \\ \dfrac{\sqrt{1+a}-1}{\sqrt{a}} \sqrt{\dfrac{2\kappa}{t}} & \textit{if } \dfrac{\kappa}{\log \frac{1}{t}} \to a \in (0,\infty)\,, \\ \dfrac{\kappa}{\sqrt{2t \log \frac{\kappa^\alpha}{t}}} & \textit{if } \dfrac{\kappa}{\log \frac{1}{t}} \to 0\,. \end{cases} \tag{2.7}$$

- Typical deviations. *For any family of* $(\kappa,t)$ *with*

$$t \to 0\,, \qquad \frac{\kappa}{t^{1/\alpha}} \to a \in [0,\infty)\,, \tag{2.8}$$

*one has*

$$\sigma_{\text{imp}}(\pm\kappa,t) \sim C_\pm(a)\, t^{\frac{2-\alpha}{2\alpha}}\,, \quad \textit{with} \quad C_\pm(a) := \begin{cases} \dfrac{a}{D^{-1}\big(\frac{\mathrm{E}[(\sigma Y \mp a)^\pm]}{a}\big)} & \textit{if } a > 0\,, \\ \sqrt{2\pi}\,\sigma\,\mathrm{E}[Y^\pm] & \textit{if } a = 0\,. \end{cases} \tag{2.9}$$

**Remark 2.2** (Surface asymptotics for the Carr-Wu model)**.** The fact that relations (2.5) and (2.6) hold for *any* family of $(\kappa,t)$ satisfying (2.4) yields interesting consequences. In fact, for any $T \in (0,\infty)$ and $\varepsilon > 0$, we claim that there exists $M = M(\varepsilon,T) \in (0,\infty)$ such that the following inequalities hold *for all* $(\kappa,t)$ *in the region* $\mathcal{A}_{T,M} := \{0 < t \le T,\ \kappa > Mt^{1/\alpha}\}$:

$$\big(1-\varepsilon\big) B_\alpha \left(\frac{\kappa}{t}\right)^{-\frac{2-\alpha}{2(\alpha-1)}} \le \sigma_{\text{imp}}(\kappa,t) \le \big(1+\varepsilon\big) B_\alpha \left(\frac{\kappa}{t}\right)^{-\frac{2-\alpha}{2(\alpha-1)}}\,; \tag{2.10}$$

analogous inequalities can be written for $\sigma_{\text{imp}}(-\kappa,t)$, using relations (2.6)-(2.7). Likewise, in the typical deviations regime, by (2.9), we claim that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that *for all* $(\kappa,t)$ *in the region* $\mathcal{B}_\delta := \{0 < t < \delta,\ 0 \le \kappa < \delta\}$ one has

$$\big(1-\varepsilon\big) C_\pm\left(\frac{\kappa}{t^{1/\alpha}}\right) t^{\frac{2-\alpha}{2\alpha}} \le \sigma_{\text{imp}}(\pm\kappa,t) \le \big(1+\varepsilon\big) C_\pm\left(\frac{\kappa}{t^{1/\alpha}}\right) t^{\frac{2-\alpha}{2\alpha}}\,, \tag{2.11}$$

Relations like (2.10) and (2.11) provide *uniform approximations* of the volatility surface $\sigma_{\text{imp}}(\kappa,t)$ that hold *for* $(\kappa,t)$ *in open regions of the plane*, and not only along "thin lines".

The proof of the above relations is simple. Let us focus on (2.10), for definiteness, and assume by contradiction that there exist $T,\varepsilon \in (0,\infty)$ such that for every $M \in (0,\infty)$ relation (2.10) *fails* for some $(\kappa_M,t_M) \in \mathcal{A}_{T,M}$; then the family $((\kappa_M,t_M))_{M\in(0,\infty)}$ satisfies (2.4) but (2.5) does not hold, contradicting Theorem 2.1.

*Proof of Theorem 2.1.* If we set

$$Y_t := \frac{X_t - \mu t}{\sigma t^{1/\alpha}}\,, \tag{2.12}$$

it follows by (2.1) that $Y_t$ has the same distribution as $Y$ in (2.3), because

$$\mathrm{E}[e^{iuY_t}] = \mathrm{E}[e^{iuY}] = e^{-|u|^\alpha\left(1+i\,\mathrm{sign}(u)\tan(\frac{\pi\alpha}{2})\right)}\,. \tag{2.13}$$

It follows by (2.12) that

$$\frac{X_t}{t^{1/\alpha}} \xrightarrow[t\downarrow 0]{d} \sigma Y\,, \tag{2.14}$$

hence Hypothesis 1.10 is satisfied with $\gamma_t := t^{1/\alpha}$.

It is well-known that $Y$ has a density which is strictly positive everywhere, hence $\mathrm{P}(Y > a) > 0$ and $\mathrm{P}(Y < -a) > 0$ for all $a \in \mathbb{R}$. We also note that the right tail of $Y$ has a super-exponential decay: as $\kappa \to \infty$

$$\log \mathrm{P}(Y > k) \sim -\widetilde{B}_\alpha \, \kappa^{\alpha/(\alpha-1)} \qquad \text{where} \qquad \widetilde{B}_\alpha := \frac{\alpha-1}{\alpha}\bigg(\frac{|\cos(\frac{\pi\alpha}{2})|}{\alpha}\bigg)^{1/(\alpha-1)}\,, \tag{2.15}$$

cf. [CW04, Property 1 and references therein]. On the other hand the left tail is polynomial: there exists $c = c_\alpha \in (0,\infty)$ such that

$$\mathrm{P}(Y \le -\kappa) \sim \frac{c}{\kappa^\alpha}\,, \qquad \text{hence} \qquad \log \mathrm{P}(Y \le -\kappa) \sim -\alpha \log \kappa\,. \tag{2.16}$$

Recalling that $\overline{F}_t(\kappa) := \mathrm{P}(X_t > \kappa)$ and $F_t(-\kappa) := \mathrm{P}(X_t \le \kappa)$, by (2.12) we can write

$$\overline{F}_t(\kappa) = \mathrm{P}\bigg(Y > \frac{k-\mu t}{\sigma t^{1/\alpha}}\bigg)\,, \qquad F_t(-\kappa) = \mathrm{P}\bigg(Y \le \frac{-k-\mu t}{\sigma t^{1/\alpha}}\bigg)\,, \tag{2.17}$$

hence we can transfer the estimates (2.15) and (2.16) to $X_t$.

Henceforth we consider separately the regimes of atypical deviations (2.4), and that of typical deviations (2.8). Note that it is easy to check that (2.6) is equivalent to (2.7).

*Atypical deviations.* Let us fix an arbitrary family of values of $(\kappa, t)$ satisfying (2.4). Then also $\kappa/t \to \infty$ (because $\alpha > 1$ and $0 < t \le T$), hence

$$\frac{\kappa-\mu t}{\sigma t^{1/\alpha}} \sim \frac{\kappa}{\sigma t^{1/\alpha}} \to \infty\,, \qquad \frac{-\kappa-\mu t}{\sigma t^{1/\alpha}} \sim \frac{-\kappa}{\sigma t^{1/\alpha}} \to -\infty\,.$$

By (2.15), (2.16) and (2.17) we then obtain

$$\log \overline{F}_t(\kappa) \sim -\widetilde{B}_\alpha \left(\frac{\kappa}{\sigma t^{1/\alpha}}\right)^{\alpha/(\alpha-1)}\,, \qquad \log F_t(-\kappa) \sim -\log\frac{\kappa^\alpha}{t}\,. \tag{2.18}$$

Let us now check the assumptions of Theorem 1.5.

- The first relation in (2.18) shows that Hypothesis 1.4 is satisfied by the right tail $\overline{F}_t(\kappa)$, with $I_+(\varrho) = \varrho^{\alpha/(\alpha-1)}$. Note that $I_+(\varrho) \ge \varrho$ for all $\varrho \ge 1$, since $\alpha > 1$, hence also condition (1.21) is satisfied.
- Condition (1.17) is satisfied because (1.18) holds for all $T > 0$ and $\eta > 0$, by (2.2).
- It remains to check condition (1.19). As we prove below, for all $\eta \in (0, \alpha-1)$ and $T > 0$ there are constants $A, B, C \in (0,\infty)$, depending on $\eta, T$ and on the parameters $\alpha, \sigma$, such that for all $0 < t \le T$ and $\kappa \ge 0$ the following inequality holds:

$$\mathrm{E}\bigg[\bigg|\frac{e^{X_t}-1}{\kappa}\bigg|^{1+\eta}\bigg] \le A\bigg(\bigg(\frac{t^{1/\alpha}}{\kappa}\bigg)^B + C\bigg)\,. \tag{2.19}$$

  In particular, since $\kappa/t^{1/\alpha} \to \infty$ by assumption (2.4), condition (1.19) is satisfied.

Applying Theorem 1.5, since $-\log \overline{F}_t(\kappa)/\kappa \to \infty$ by the first relation in (2.18), the asymptotic behavior of $\sigma_{\text{imp}}(\kappa, t)$ is given by (1.26), which by (2.18) coincides with (2.5).

Next we want to apply Theorem 1.6. By the second relation in (2.18), Hypothesis 1.4 is satisfied by the left tail $F_t(-\kappa)$, with $I_-(\varrho) \equiv 1$. If $\kappa$ is bounded away from zero, the asymptotic behavior of $\sigma_{\mathrm{imp}}(\kappa, t)$ is given by (1.29), which by (2.18) yields precisely (2.6).

If $\kappa \to 0$ we cannot apply directly Theorem 1.6, because the moment condition (1.19) is satisfied only for some $\eta > 0$, and condition (1.32) is not satisfied, since $I_-(\varrho) \equiv 1$. However, we can show that (1.33) still holds by direct estimates. By (1.1)

$$p(-\kappa, t) = \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t < -\kappa\}}] \geq \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t < -2\kappa\}}] \geq (e^{-\kappa} - e^{-2\kappa}) F_t(-2\kappa) \,,$$

and since $(e^{-\kappa} - e^{-2\kappa}) = e^{-2\kappa}(e^{\kappa} - 1) \geq e^{-2\kappa}\kappa$, we can write by (2.18) (recall that $\kappa \to 0$)

$$\log \big( p(-\kappa, t)/\kappa \big) \geq -2\kappa - \log \frac{(2\kappa)^\alpha}{t} \sim -\log \frac{\kappa^\alpha}{t} \,. \tag{2.20}$$

Next we give a matching upper bond on $p(-\kappa, t)$. Since $\mu t \leq \kappa$ eventually (recall that $\kappa / t^{1/\alpha} \to \infty$, hence $\kappa / t \to \infty$), by (2.17) and (2.16) we obtain, for all $y \geq 1$

$$F_t(-\kappa y) \leq \mathrm{P}\bigg( Y \leq -\frac{2\kappa y}{\sigma t^{1/\alpha}} \bigg) \leq c' \frac{t}{\kappa^\alpha y^\alpha} \,,$$

for some $c' = c'_{\alpha,\sigma,\mu} \in (0, \infty)$. Then by Fubini's theorem

$$\begin{aligned} p(-\kappa, t) &= \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t < -\kappa\}}] = \mathrm{E}\bigg[ \int_\kappa^\infty e^{-x} 1_{\{x < -X_t\}} \mathrm{d}x \bigg] = \int_\kappa^\infty e^{-x} \, F_t(-x) \, \mathrm{d}x \\ &= \kappa \int_1^\infty e^{-\kappa y} \, F_t(-\kappa y) \, \mathrm{d}y \leq c' \kappa \, \frac{t}{\kappa^\alpha} \int_1^\infty \frac{1}{y^\alpha} \mathrm{d}y =: c'' \kappa \, \frac{t}{\kappa^\alpha} \,, \end{aligned}$$

hence

$$\log \big( p(-\kappa, t)/\kappa \big) \leq \log c'' - \log \frac{\kappa^\alpha}{t} \sim -\log \frac{\kappa^\alpha}{t} \,.$$

This relation, together with (2.20), yields

$$\log \big( p(-\kappa, t)/\kappa \big) \sim -\log \frac{\kappa^\alpha}{t} \,.$$

Since $\kappa / t^{1/\alpha} \to \infty$, this shows that we are in the regime when $\kappa \to 0$ and $p(-\kappa, t)/\kappa \to 0$. We can thus apply equation (1.8) in Theorem 1.1, which recalling Remark 1.2 simplifies as the first line in (1.11) (with $p(-\kappa, t)$ instead of $c(\kappa, t)$), yielding

$$\sigma_{\mathrm{imp}}(-\kappa, t) \sim \frac{\kappa}{\sqrt{2t \, (-\log(p(-\kappa, t)/\kappa))}} \sim \frac{\kappa}{\sqrt{2t \, \log \frac{\kappa^\alpha}{t}}} \,,$$

hence (2.6) is proved also when $\kappa \to 0$ (cf. the last line of (2.7)).

*Typical deviations.* Let us fix an arbitrary family of values of $(\kappa, t)$ satisfying (2.8). Relation (2.19) for $\kappa = \gamma_t = t^{1/\alpha}$ shows that condition (1.38) is satisfied, and Hypothesis 1.10 holds by (2.14). We can then apply Theorem 1.11, and relation (1.40) gives precisely (2.9). □

*Proof of* (2.19). Since $|\frac{e^x - 1}{x}| \leq 1$ if $x < 0$ and $|\frac{e^x - 1}{x}| \leq e^x$ if $x \geq 0$, we have $|\frac{e^x - 1}{x}| \leq 1 + e^x$ for all $x \in \mathbb{R}$. If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, Young's inequality $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$ yields

$$\bigg| \frac{e^{X_t} - 1}{\kappa} \bigg| = \bigg| \frac{X_t}{\kappa} \bigg| \bigg| \frac{e^{X_t} - 1}{X_t} \bigg| \leq \frac{1}{p} \bigg( \frac{|X_t|}{\kappa} \bigg)^p + \frac{1}{q} (1 + e^{X_t})^q \,.$$

Noting that $(a+b)^r \le 2^{r-1}(a^r+b^r)$ for $r \ge 1$, by Hölder's inequality, and denoting by $c = c_{p,\eta}$ a suitable constant depending only on $p, \eta$, we can write

$$\left|\frac{e^{X_t}-1}{\kappa}\right|^{1+\eta} \le c\left(\frac{|X_t|^{p(1+\eta)}}{\kappa^{p(1+\eta)}} + 1 + e^{q(1+\eta)X_t}\right).$$

Given $0 < \eta < \alpha - 1$, we fix $p = p_{\eta,\alpha} > 1$ such that $B := p(1+\eta) < \alpha$. (Note that $B$ depends only on $\eta, \alpha$.) Moreover, it follows by (2.12) that

$$\mathrm{E}[|X_t|^B] = (\sigma t^{1/\alpha})^B\, \mathrm{E}[|Y|^B]\left(1 + O(t^{B(1-1/\alpha)})\right),$$

and note that $\mathrm{E}[|Y|^B] < \infty$, because $Y$ has finite moments of all orders strictly less than $\alpha$, cf. [CW04, Property 1]. Since for $t \le T$ one has $\mathrm{E}[e^{q(1+\eta)X_t}] \le \mathrm{E}[e^{q(1+\eta)X_T}] < \infty$, by (2.2), relation (2.19) is proved. $\square$

2.2. **The Heston Model.** Given the parameters $\lambda, \vartheta, \eta, \sigma_0 \in (0,\infty)$ and $\varrho \in [-1,1]$, the Heston model [H93] is a stochastic volatility model $(S_t)_{t\ge0}$ defined by the following SDEs

$$\begin{cases} \mathrm{d}S_t \,=\, S_t\sqrt{V_t}\,\mathrm{d}W_t^1\,, \\ \mathrm{d}V_t \,=\, -\lambda(V_t-\vartheta)\,\mathrm{d}t \,+\, \eta\sqrt{V_t}\,\mathrm{d}W_t^2\,, \\ X_0 = 0\,, \quad V_0 = \sigma_0\,, \end{cases}$$

where $(W_t^1)_{t\ge0}$ and $(W_t^2)_{t\ge0}$ are standard Brownian motions with $\langle \mathrm{d}W_t^1\,, \mathrm{d}W_t^2\rangle = \varrho\,\mathrm{d}t$.

Unlike the Carr-Wu model, here $S_t$ displays explosion of moments, i.e. $\mathrm{E}[S_T^p] = \infty$ for $p > 1$ large enough (note that $\mathrm{E}[S_t] = 1$, since $(S_t)_{t\ge0}$ is a martingale). In particular for any fixed $t \ge 0$ we define the explosion moment $p^*(t)$ as

$$p^*(t) := \sup\{p > 0 : \ \mathrm{E}[S_t^p] < \infty\},$$

so that $\mathrm{E}[S_t^p] < \infty$ for $p < p^*(t)$ while $\mathrm{E}[S_t^p] = \infty$ for $p > p^*(t)$. The behavior of the explosion moment $p^*(t)$ is described in the following Lemma, proved below.

**Lemma 2.3.** *If $\varrho = -1$, then $p^*(t) = +\infty$ for every $t \ge 0$.*
*If $\varrho > -1$, then $p^*(t) \in (1,+\infty)$ for every $t > 0$. Moreover, as $t \downarrow 0$*

$$p^*(t) \sim \frac{C}{t},$$

*where*

$$C = C(\varrho,\eta) := \begin{cases} \dfrac{2}{\eta\sqrt{1-\varrho^2}}\left(\arctan\dfrac{\sqrt{1-\varrho^2}}{\varrho} + \pi 1_{\varrho<0}\right) & \text{if } \varrho < 1 \\ \dfrac{2}{\eta} & \text{if } \varrho = 1 \end{cases}. \tag{2.21}$$

The asymptotic behavior of the implied volatility $\sigma_{\mathrm{imp}}(\kappa,t)$ for the Heston model is known in the regimes of large strike (with fixed maturity) and small maturity (with fixed strike).

- In [BF08], Benaim and Friz show that for fixed $t > 0$, when $\kappa \to +\infty$

$$\sigma_{\mathrm{imp}}(\kappa,t) \underset{k\uparrow\infty}{\sim} \frac{\sqrt{2\kappa}}{\sqrt{t}}\left(\sqrt{p^*(t)} - \sqrt{p^*(t)-1}\right), \tag{2.22}$$

based on the estimate (cf. also [AP07])

$$-\log \mathrm{P}(X_t > \kappa) \underset{k\uparrow\infty}{\sim} p^*(t)\,\kappa\,. \tag{2.23}$$

- In [FJ09], Forde and Jacquier have proved that for any fixed $\kappa > 0$, as $t \downarrow 0$

$$\sigma_{\mathrm{imp}}(\kappa, t) \underset{t\downarrow 0}{\sim} \frac{\kappa}{\sqrt{2\,\Lambda^*(\kappa)}}\,, \tag{2.24}$$

where $\Lambda^*(\cdot)$ is the Legendre transform of the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ given by

$$\Lambda(p) := \begin{cases} \dfrac{\sigma_0 p}{\eta\left(\sqrt{1-\varrho^2}\cot\left(\frac{1}{2}\eta p\sqrt{1-\varrho^2}\right) - \varrho\right)} & \text{if } p < C\,, \\ \infty & \text{if } p \geq C\,, \end{cases} \tag{2.25}$$

where $C$ is the constant in (2.21). Their analysis is based on the estimate

$$-\log \mathrm{P}(X_t \geq \kappa) \underset{t\downarrow 0}{\sim} \frac{1}{t}\Lambda^*(\kappa)\,, \tag{2.26}$$

obtained by showing that the log-price $(X_t)_{t\geq 0}$ in the Heston model satisfies a large deviations principle as $t \downarrow 0$, with rate $1/t$ and good rate function $\Lambda^*(\kappa)$.

We first note that the asymptotics (2.22) and (2.24) follow easily from our Theorem 1.5, plugging the estimates (2.23) and (2.26) into relations (1.23) and (1.26), respectively.

We also observe that the estimates (2.22) and (2.24) match, in the following sense: if we take the limit $t \to 0$ of the right hand side of (2.22) (i.e. we first let $\kappa \uparrow +\infty$ and then $t \downarrow 0$ in $\sigma_{\mathrm{imp}}(\kappa, t)$), we obtain

$$(2.22) \underset{t\downarrow 0}{\sim} \frac{\sqrt{2\kappa}}{\sqrt{t}}\frac{1}{2\sqrt{p^*(t)}} \sim \frac{\sqrt{2\kappa}}{\sqrt{t}}\frac{1}{2\sqrt{\frac{C}{t}}} = \frac{\sqrt{\kappa}}{\sqrt{2\,C}}\,. \tag{2.27}$$

If, on the other hand, we take the limit $\kappa \uparrow 0$ of the right hand side of (2.24) (i.e. we first let $t \downarrow 0$ and then $\kappa \uparrow +\infty$ in $\sigma_{\mathrm{imp}}(\kappa, t)$), since $\Lambda^*(\kappa) \sim C\kappa$,† we obtain

$$(2.24) \underset{\kappa\uparrow+\infty}{\sim} \frac{\kappa}{\sqrt{2C\kappa}} = \frac{\sqrt{\kappa}}{\sqrt{2\,C}}\,, \tag{2.28}$$

which coincides with (2.27). Analogously, also the estimates (2.23) and (2.26) match.

It is then natural to conjecture that, for *any* family of values of $(\kappa, t)$ such that $\kappa \uparrow +\infty$ and $t \downarrow 0$ jointly, one should have

$$\log \mathrm{P}(X_t \geq \kappa) \sim -C\,\frac{\kappa}{t}\,, \tag{2.29}$$

where $C$ is the constant in (2.21). If this holds, applying Theorem 1.5, relation (1.26) yields

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \frac{\sqrt{\kappa}}{\sqrt{2\,C}}\,, \tag{2.30}$$

providing a smooth interpolation between (2.22) and (2.24).

**Remark 2.4** (Surface asymptotics for the Heston model). If (2.30) holds for any family of values of $(\kappa, t)$ with $\kappa \to \infty$ and $t \to 0$, it follows that for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in (0, \infty)$ such that the following inequalities hold:

$$\big(1-\varepsilon\big)\frac{\sqrt{\kappa}}{\sqrt{2\,C}} \leq \sigma_{\mathrm{imp}}(\kappa, t) \leq \big(1+\varepsilon\big)\frac{\sqrt{\kappa}}{\sqrt{2\,C}}\,,$$

*for all* $(\kappa, t)$ *in the region* $\mathcal{A}_{T,M} := \{0 < t \leq \frac{1}{M},\ \kappa > M\}$, as it follows easily by contradiction (cf. Remark 2.2 for a similar argument).

†This is because $\Lambda(p) \uparrow +\infty$ as $p \uparrow C$, hence the slope of $\Lambda^*(\kappa)$ converges to $C$ as $\kappa \to \infty$.

*Proof of Lemma 2.3.* Given any number $p > 1$ we define the explosion time $T^*(p)$ as

$$T^*(p) := \sup\{t > 0 : \ \mathrm{E}[S_t^p] < \infty\}\,.$$

Note that if $T^*(p) = t \in (0, +\infty)$ then $p^*(t) = p$. By [AP07] (see also [FK09])

$$T^*(p) = \begin{cases} +\infty & \text{if } \ \Delta(p) \geq 0, \ \chi(p) < 0\,, \\ \frac{1}{\sqrt{\Delta(p)}} \log\left(\frac{\chi(p)+\sqrt{\Delta(p)}}{\chi(p)-\sqrt{\Delta(p)}}\right) & \text{if } \ \Delta(p) \geq 0, \ \chi(p) > 0\,, \\ \frac{2}{\sqrt{-\Delta(p)}} \left(\arctan \frac{\sqrt{-\Delta(p)}}{\chi(p)} + \pi 1_{\chi(p)<0}\right) & \text{if } \ \Delta(p) < 0\,, \end{cases} \tag{2.31}$$

where

$$\chi(p) := \varrho\eta p \, - \, \lambda\,, \qquad \Delta(p) := \chi^2(p) - \eta^2(p^2 - p)\,,$$

Observe that if $\varrho = -1$, then $\chi(p) = -\eta p - \lambda < 0$ and $\Delta(p) = \lambda^2 + p\left(2\eta\lambda + \eta^2\right) \geq 0$, which implies $T^*(p) = +\infty$ for every $p > 1$, or equivalently $p^*(t) = +\infty$ for every $t > 0$.

On the other hand, since

$$\Delta(p) \, = \, \varrho^2\eta^2 p^2 + \lambda^2 - 2\eta\varrho\lambda p - \eta^2 p^2 + \eta^2 p \, = \, \eta^2 p^2(\varrho^2 - 1) + p(\eta^2 - 2\eta\varrho\lambda) + \lambda^2\,,$$

we observe that if $\varrho \neq 1$, then $\Delta p < 0$ as $p \to +\infty$, which implies

$$\begin{aligned} T^*(p) &\underset{p\uparrow\infty}{\sim} \frac{2}{p(\eta\sqrt{1-\varrho^2})} \left(\arctan \frac{\eta p\sqrt{1-\varrho^2}}{\varrho\eta p} + \pi 1_{\varrho<0}\right) \\ &= \frac{1}{p} \frac{2}{\eta\sqrt{1-\varrho^2}} \left(\arctan \frac{\sqrt{1-\varrho^2}}{\varrho} + \pi 1_{\varrho<0}\right)\,. \end{aligned} \tag{2.32}$$

In particular this leads to the conclusion that, if $|\varrho| \neq 1$, then

$$p^*(t) \underset{t\downarrow 0}{\sim} \frac{C}{t}$$

where $C$ was defined in (2.21).

It remains to study the case $\varrho = 1$, in which $\chi(p) > 0$ for every $p$. We have two possibilities: if $\eta > 2\lambda$ then $\Delta(p) > 0$ when $p \to +\infty$, and so by (2.31)

$$T^*(p) \underset{p\uparrow\infty}{\sim} \frac{1}{\sqrt{p(\eta^2 + 2\eta\lambda)}} \log\left(1 + 2\frac{\sqrt{p(\eta^2 + 2\eta\lambda)}}{\eta p - \sqrt{p(\eta^2 + 2\eta\lambda)}}\right) \sim \frac{2}{\eta}\frac{1}{p}\,.$$

On the other hand, if $\eta < 2\lambda$, then $\Delta(p) < 0$ when $p \to \infty$ and so

$$T^*(p) \underset{p\uparrow\infty}{\sim} \frac{2}{\sqrt{p(2\eta\lambda - \eta^2)}} \left(\arctan \frac{\sqrt{p(2\eta\lambda - \eta^2)}}{p\eta}\right) \sim \frac{2}{\eta}\frac{1}{p}\,.$$

Finally if $\eta = 2\lambda$, $\Delta(p) = \lambda^2$, and so

$$T^*(p) = \frac{1}{\lambda} \log\left(1 + \frac{2\lambda}{\eta p - 2\lambda}\right) \underset{p\uparrow\infty}{\sim} \frac{2}{\eta}\frac{1}{p}\,.$$

In all the cases we obtain $p^*(t) \underset{t\downarrow 0}{\sim} \frac{2}{\eta}\frac{1}{t}$, in agreement with (2.21). □

2.3. **Merton's Jump Diffusion Model.** Consider a model [M76] where the log-return $X_t$ has an infinitely divisible distribution, whose moment generating function is given by

$$\mathrm{E}\left[\exp(zX_t)\right] = \exp\left( t \left\{ z\mu + \frac{1}{2} z^2 \sigma^2 + \lambda \left( e^{z\alpha + z^2 \frac{\delta^2}{2}} - 1 \right) \right\} \right), \tag{2.33}$$

where $\mu, \alpha \in \mathbb{R}$ and $\sigma, \lambda, \delta \in (0, \infty)$ are fixed parameters.

Benaim and Fritz [BF09] observed that for fixed $t > 0$, as $\kappa \to \infty$,

$$\log \mathrm{P}(X_t \geq \kappa) \sim -\frac{\kappa}{\delta} \sqrt{2 \log \frac{\kappa}{t}}\,, \tag{2.34}$$

deducing that

$$\sigma^2_{\mathrm{imp}}(\kappa, t) \sim \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}}\,. \tag{2.35}$$

Remarkably, formula (2.35) holds *for any family of* $(\kappa, t)$ *such that* $t$ *is bounded, say* $0 < t \leq T$, *and* $\frac{\kappa}{t} \uparrow \infty$, by our Theorem 1.5, because the asymptotic relation (2.34) also holds for any such family (we thank Stefan Gerhold for this observation). In fact, for any $c \in (1, \infty)$ such that $\mathrm{E}[e^{cX_t}] < \infty$, we can write

$$\mathrm{P}(X_t \geq \kappa) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{\mathrm{E}[e^{sX_t}]\, e^{-\kappa s}}{s} \, \mathrm{d}s\,.$$

The asymptotic evaluation of this integral can be done by saddle point methods: the relevant estimate for the saddle point $\hat{s}$, taken from [FGY14],† reads as follows:

$$\hat{s} = \frac{\sqrt{2 \log \kappa}}{\delta} - \frac{\mu}{\delta^2} + \mathcal{O}\left( \frac{\log \log \kappa}{\sqrt{\log \kappa}} \right),$$

which gives precisely (2.34):

$$\log \mathrm{P}(X_t \geq \kappa) \sim -\kappa \hat{s} + \log M_{X_t}(\hat{s}) \sim -\frac{\kappa}{\delta} \sqrt{2 \log \frac{\kappa}{t}} + \frac{\kappa}{\delta \sqrt{2 \log \frac{\kappa}{t}}} \sim -\frac{\kappa}{\delta} \sqrt{2 \log \frac{\kappa}{t}}\,.$$

**Remark 2.5** (Surface asymptotics for the Merton model)**.** In analogy with Remark 2.4, since formula (2.35) holds for any family of $(\kappa, t)$ with $t$ bounded and $\kappa / t \to \infty$, for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in (0, \infty)$ such that the following inequalities hold *for all* $(\kappa, t)$ *in the region* $\mathcal{A}_{T,M} := \{0 < t \leq T,\ \kappa > Mt\}$:

$$\big(1 - \varepsilon\big) \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}} \leq \sigma^2_{\mathrm{imp}}(\kappa, t) \leq \big(1 + \varepsilon\big) \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}}\,.$$

2.4. **A multiscaling stochastic volatility model.** Our results can be also applied to a stochastic volatility model exhibiting multiscaling of moments, cf. [ACDP12]. Under the risk-neutral measure, the price $(S_t)_{t \geq 0}$ in this model evolves according to the SDE

$$\frac{\mathrm{d}S_t}{S_t} = \sigma_t \, \mathrm{d}B_t\,, \tag{2.36}$$

where $(B_t)_{t \geq 0}$ is a Brownian motion and $(\sigma_t)_{t \geq 0}$ is an *independent* process, defined as follows. Given the parameters $D \in (0, \frac{1}{2})$ and $V, \lambda \in (0, \infty)$, denote by $(N_t)_{t \geq 0}$ a Poisson process of

†The formula for $\hat{s}^2$ at the end of the section "The Merton Model" in [FGY14] contains a misprint, since the term $-\log(\lambda T \delta^2)$ should be replaced by $-\log(\lambda \delta^2)$. We also refer to [GMZ14] for the special case $\kappa \to \infty$ with fixed $t$, with a more detailed computation.

rate $\lambda$ independent of $(B_t)_{t\geq 0}$, with jump times $0 < \tau_1 < \tau_2 < \ldots$, and define $\sigma_t$ by

$$\sigma_t = V \, \frac{\lambda^{D-\frac{1}{2}}}{\Gamma(2D)} \, (t - \tau_{N_t})^{D-\frac{1}{2}} \,. \tag{2.37}$$

Note that $\tau_{N_t}$ is the epoch of the last jump of the Poisson process before time $t$. In other terms, the volatility $\sigma_t$ explodes at each jump time of the Poisson process, after which it decays as an inverse power, with exponent tuned by $D < \frac{1}{2}$.

The properties of this model (under the historical measure) have been investigated in [ACDP12], and it was shown that interesting features emerge, namely:

- *Heavy tails*: the distribution of the log-price $X_t := \log(S_t/S_0)$ is asymptotically Gaussian for large time $t$, but asymptotically *heavy tailed for short time.*
- *Clustering of volatility*: the covariance between $|X_{t+h} - X_t|$ and $|X_{t+\Delta t+h} - X_{t+\Delta t}|$ decays exponentially fast for large $\Delta t$, but slower (polynomially) for $\Delta t = O(1)$.
- *Multiscaling of moments*: one has $\mathrm{E}(|X_{t+\Delta t} - X_t|^q) = (\Delta t)^{A(q)+o(1)}$ as $\Delta t \downarrow 0$, where the diffusive exponent $A(q) = \frac{q}{2}$ holds until a critical moment $q < q^* := \left(\frac{1}{2} - D\right)^{-1} \in (2, \infty)$, while for $q > q^*$ one has the anomalous exponent $A(q) < \frac{q}{2}$.

Unlike the models described previously, no closed expression is available for the moment generating function of the log-price. Nevertheless, as we show in a separate paper [CC], it is possible to determine *explicitly* the asymptotic behavior of the tail probabilities for this model. This allows to apply Theorems 1.5, 1.6 and 1.11, extracting the asymptotic behavior of the implied volatility along any family of values of $(\kappa, t)$ satisfying (1.3).

Denoting by $A \in (0, \infty)$ an explicit constant (depending only on the parameters $D, V, \lambda$ of the model), we show in [CC] that the following asymptotic relation:

$$\sigma_{\mathrm{imp}}(\kappa, t) \;\sim\; A \left( \frac{|\kappa|/t}{\sqrt{\log(|\kappa|/t)}} \right)^{\frac{1-2D}{2-2D}} \tag{2.38}$$

holds for any family of values of $(\kappa, t)$ satisfying either of the regimes (1), (2), (3) listed on page 2, as well as regime (4) provided $\kappa \to 0$ slowly enough, i.e.

$$t \to 0 \qquad \text{and} \qquad \frac{\kappa}{t^D \sqrt{\log \frac{1}{t}}} \to \infty \,.$$

In particular, the implied volatility diverges in the *short maturity* regime (as $t \downarrow 0$ for fixed $\kappa \neq 0$), despite the price having continuous paths. It also diverges *deep out-of-the-money* (as $|\kappa| \to \infty$ for fixed $t > 0$) with an explicit limiting shape, displaying a very pronounced smile.

We refer to [CC] for the complete results, which include all the cases when $\kappa, t \to 0$.

## 3. From option price to implied volatility

In this section we prove Theorem 1.1. We start with some background on Black&Scholes model and on related quantities.

### 3.1. **Mills ratio.**

We let $Z$ be a standard Gaussian random variable and denote by $\phi$ and $\Phi$ its density and distribution functions:

$$\phi(z) := \frac{\mathrm{P}(Z \in \mathrm{d}z)}{\mathrm{d}z} = \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \,, \qquad \Phi(z) := \mathrm{P}(Z \leq z) = \int_{-\infty}^{z} \phi(t) \, \mathrm{d}t \,. \tag{3.1}$$

The Mills ratio $U : \mathbb{R} \to (0, \infty)$ is defined by

$$U(z) := \frac{1 - \Phi(z)}{\phi(z)} = \frac{\Phi(-z)}{\phi(z)}, \qquad \forall z \in \mathbb{R}. \tag{3.2}$$

The next lemma summarizes the main properties of $U$ that will be used in the sequel.

**Lemma 3.1.** *The function $U$ is smooth, strictly decreasing, strictly convex and satisfies*

$$U'(z) \sim -\frac{1}{z^2} \qquad \textit{as } z \uparrow \infty. \tag{3.3}$$

*Proof.* Since $\Phi'(z) = \phi(z)$ and $\phi$ is an analytic function, $U$ is also analytic. Since $\phi'(z) = -z\phi(z)$, one obtains

$$U'(z) = zU(z) - 1, \qquad U''(z) = U(z) + zU'(z) = (1 + z^2)U(z) - z. \tag{3.4}$$

Recalling that $U(z) > 0$, these relations already show that $U'(z) < 0$ and $U''(z) > 0$ for all $z \leq 0$. For $z > 0$, the following bounds hold [S54, eq. (19)], [P01, Th. 1.5]:

$$\frac{z}{z^2 + 1} = \frac{1}{z + \frac{1}{z}} < U(z) < \frac{1}{z + \frac{1}{z + \frac{2}{z}}} = \frac{z^2 + 2}{z^3 + 3z}, \qquad \forall z > 0. \tag{3.5}$$

Applying (3.4) yields $U''(z) > 0$ and $-\frac{1}{1+z^2} < U'(z) < -\frac{1}{3+z^2}$ for all $z > 0$, hence (3.3). □

We recall that the smooth function $D : (0, \infty) \to (0, \infty)$ was introduced in (1.5). Since

$$D'(z) = -\frac{1}{z^2}\phi(z) < 0, \tag{3.6}$$

$D(\cdot)$ is a strictly decreasing bijection (note that $\lim_{z \downarrow 0} D(z) = \infty$ and $\lim_{z \to \infty} D(z) = 0$). Its inverse $D^{-1} : (0, \infty) \to (0, \infty)$ is then smooth and strictly decreasing as well. Writing $D(z) = \phi(z)(\frac{1}{z} - U(z))$, it follows by (3.5) that $\frac{1}{z} - U(z) \sim \frac{1}{z^3}$ as $z \uparrow \infty$, hence

$$D(z) \sim \frac{1}{z^3}\phi(z) \sim \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}\, z^3} \quad \text{as } z \uparrow \infty, \qquad D(z) \sim \frac{1}{z}\phi(0) = \frac{1}{\sqrt{2\pi} z} \quad \text{as } z \downarrow 0.$$

It follows easily that $D^{-1}(\cdot)$ satisfies (1.6).

3.2. **Black&Scholes formula.** The Black&Scholes model is defined by a risk-neutral log-price $(X_t := \sigma B_t - \frac{1}{2}\sigma^2 t)_{t \geq 0}$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion and the parameter $\sigma \in (0, \infty)$ represents the volatility. The Black&Scholes formula for the price of a normalized European call is $\mathsf{C}_{\mathsf{BS}}(\kappa, \sigma\sqrt{t})$, where $\kappa$ is the log-strike, $t$ is the maturity and we define

$$\mathsf{C}_{\mathsf{BS}}(\kappa, v) := \mathrm{E}[(e^{vZ - \frac{1}{2}v^2} - e^\kappa)^+] = \begin{cases} (1 - e^\kappa)^+ & \text{if } v = 0, \\ \Phi(d_1) - e^\kappa \Phi(d_2) & \text{if } v > 0, \end{cases} \tag{3.7}$$

where $\Phi$ is defined in (3.1), and we set

$$\begin{cases} d_1 = d_1(\kappa, v) := -\frac{\kappa}{v} + \frac{v}{2}, \\ d_2 = d_2(\kappa, v) := -\frac{\kappa}{v} - \frac{v}{2}, \end{cases} \qquad \text{so that} \qquad \begin{cases} d_2 = d_1 - v, \\ d_2^2 = d_1^2 + 2\kappa. \end{cases} \tag{3.8}$$

Note that $\mathsf{C}_{\mathsf{BS}}(\kappa, v)$ is a continuous function of $(\kappa, v)$. Since $e^\kappa \phi(d_2) = \phi(d_1)$, for all $v > 0$ one easily computes

$$\frac{\partial \mathsf{C}_{\mathsf{BS}}(\kappa, v)}{\partial v} = \phi(d_1) > 0, \qquad \frac{\partial \mathsf{C}_{\mathsf{BS}}(\kappa, v)}{\partial \kappa} = -e^\kappa \Phi(d_2) < 0,$$

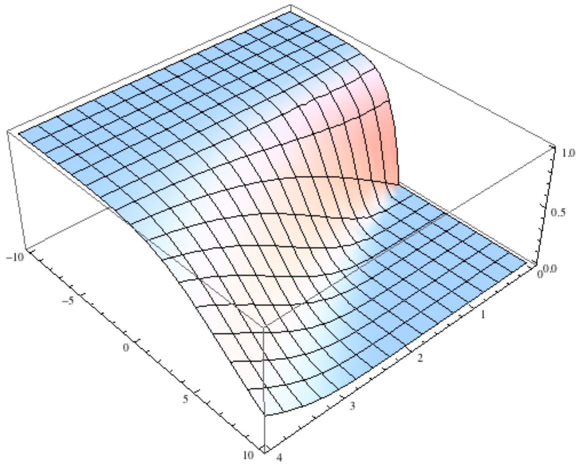


FIGURE 1. A plot of $(\kappa, v) \mapsto \mathsf{C_{BS}}(\kappa, v)$, for $\kappa \in [-10, 10]$ and $v \in [0, 4]$.

hence $\mathsf{C_{BS}}(\kappa, v)$ is strictly increasing in $v$ and strictly decreasing in $\kappa$ (see Figure 1). It is also directly checked that for all $\kappa \in \mathbb{R}$ and $v \geq 0$ one has

$$\mathsf{C_{BS}}(\kappa, v) = 1 - e^{\kappa} + e^{\kappa}\mathsf{C_{BS}}(-\kappa, v)\,. \tag{3.9}$$

In the following key proposition, proved in Appendix A.2, we show that when $\kappa \geq 0$ the Black&Scholes call price $\mathsf{C_{BS}}(\kappa, v)$ vanishes precisely when $v \to 0$ or $d_1 \to -\infty$ (or, more generally, in a combination of these two regimes, when $\min\{d_1, \log v\} \to -\infty$). We also provide sharp estimates for each regime, that will play a crucial role in the sequel.

**Proposition 3.2.** *For any family of values of $(\kappa, v)$ with $\kappa \geq 0$, $v > 0$, one has*

$$\mathsf{C_{BS}}(\kappa, v) \to 0 \qquad \text{if and only if} \qquad \min\{d_1, \log v\} \to -\infty\,, \tag{3.10}$$

*that is, $\mathsf{C_{BS}}(\kappa, v) \to 0$ if and only if from any subsequence of $(\kappa, v)$ one can extract a sub-subsequence along which eiter $d_1 \to -\infty$ or $v \to 0$. Moreover:*

- *if $d_1 := -\frac{\kappa}{v} + \frac{v}{2} \to -\infty$, then*

$$\mathsf{C_{BS}}(\kappa, v) \sim \phi(d_1)\,\frac{v}{-d_1(-d_1 + v)}\,; \tag{3.11}$$

- *if $v \to 0$, then*

$$\mathsf{C_{BS}}(\kappa, v) \sim -U'(-d_1)\,\phi(d_1)\,v\,; \tag{3.12}$$

*where $\phi(\cdot)$ and $U(\cdot)$ are defined in (3.1) and (3.2).*

3.3. **Implied volatility.** Since the function $v \mapsto \mathsf{C_{BS}}(\kappa, v)$ is a strictly increasing bijection from $[0, \infty)$ to $[(1 - e^{\kappa})^+, 1)$, it admits an inverse function $c \mapsto \mathsf{V_{BS}}(\kappa, c)$, defined by

$$\mathsf{C_{BS}}(\kappa, \mathsf{V_{BS}}(\kappa, c)) = c\,. \tag{3.13}$$

By construction, $\mathsf{V_{BS}}(\kappa, \cdot)$ is a strictly increasing bijection from $[(1 - e^{\kappa})^+, 1)$ to $[0, \infty)$. We will mainly focus on the case $\kappa \geq 0$, for which $\mathsf{V_{BS}}(\kappa, \cdot) : [0, 1) \to [0, \infty)$.

Consider an arbitrary model, with a risk-neutral log-price $(X_t)_{t \geq 0}$, and let $c(\kappa, t)$ be the corresponding price of a normalized European call option, cf. (1.1). Since $z \mapsto (z - e^{\kappa})^+$ is a convex function, one has $c(\kappa, t) \geq (\mathrm{E}[e^{X_t}] - e^{\kappa})^+ = (1 - e^{\kappa})^+$ by Jensen's inequality; since $(z - e^{\kappa})^+ < z^+$, one has $c(\kappa, t) < \mathrm{E}[e^{X_t}] = 1$. Having shown that $c(\kappa, t) \in [(1 - e^{\kappa})^+, 1)$, one defines the *implied volatility* $\sigma_{\mathrm{imp}}(\kappa, t)$ of the model as the unique value of $\sigma \in [0, \infty)$ for which the Black&Scholes call price $\mathsf{C}_{\mathsf{BS}}(\kappa, \sigma\sqrt{t})$ equals $c(\kappa, t)$. Equivalently, by (3.13),

$$\sigma_{\mathrm{imp}}(\kappa, t) := \frac{\mathsf{V}_{\mathsf{BS}}(\kappa, c(\kappa, t))}{\sqrt{t}} \,. \tag{3.14}$$

It is now convenient to rewrite Theorem 1.1 more transparently in terms of the function $\mathsf{V}_{\mathsf{BS}}$. To this purpose, inspired by (1.2), let us define a new variable $p = p(\kappa, c)$ by

$$p := c - (1 - e^{\kappa}) \,. \tag{3.15}$$

**Theorem 3.3.** *Consider a family of values of $(\kappa, c)$, such that either $\kappa \geq 0$, $c \in (0,1)$ and $c \to 0$, or alternatively $\kappa \leq 0$, $p \in (0,1)$ and $p \to 0$, where $p$ is defined in* (3.15)*.*

- *If $\kappa$ bounded away from zero* ($\liminf |\kappa| > 0$)*, one has*

$$\mathsf{V}_{\mathsf{BS}}(\kappa, c) \sim \begin{cases} \sqrt{2\,(-\log c + \kappa)} - \sqrt{2\,(-\log c)} & \text{if } \kappa > 0 \,, \\ \sqrt{2\,(-\log p)} - \sqrt{2\,(-\log p + \kappa)} & \text{if } \kappa < 0 \,. \end{cases} \tag{3.16}$$

- *If $\kappa$ is bounded away from infinity* ($\limsup |\kappa| < \infty$)*, one has*

$$\mathsf{V}_{\mathsf{BS}}(\kappa, c) \sim \begin{cases} \dfrac{\kappa}{D^{-1}(\frac{c}{\kappa})} & \text{if } \kappa > 0 \,, \\ \sqrt{2\pi}\, c = \sqrt{2\pi}\, p & \text{if } \kappa = 0 \,, \\ \dfrac{-\kappa}{D^{-1}(\frac{p}{-\kappa})} & \text{if } \kappa < 0 \,, \end{cases} \tag{3.17}$$

*where $D^{-1}(\cdot)$ is the inverse of the function $D(\cdot)$ defined in* (1.5)*, and satisfies* (1.6)*.*

We give the proof in a moment (see §3.4 below), *restricting to the case* $\kappa \geq 0$, because the complementary case $\kappa \leq 0$ follows by symmetry, as we now briefly discuss. It follows by (3.9) and (3.13) that for all $k \in \mathbb{R}$ and $c \in [(1 - e^{\kappa})^+, 1)$ one has

$$\mathsf{V}_{\mathsf{BS}}(\kappa, c) = \mathsf{V}_{\mathsf{BS}}(-\kappa, 1 - e^{-\kappa} + e^{-\kappa} c) = \mathsf{V}_{\mathsf{BS}}(-\kappa, e^{-\kappa} p) \,,$$

where we recall that $p$ is defined in (3.15). As a consequence, in the case $\kappa \leq 0$, replacing $\kappa$ by $-\kappa$ and $c$ by $e^{-\kappa} p$ in the first line of (3.16), one obtains the second line of (3.16). Performing the same replacements in the first line of (3.17) yields

$$\mathsf{V}_{\mathsf{BS}}(\kappa, c) \sim \frac{-\kappa}{D^{-1}(e^{-\kappa} \frac{p}{-\kappa})} \,,$$

which is slightly different with respect to the third line of (3.17). However, the discrepacy is only apparent, because we claim that $D^{-1}(e^{-\kappa} \frac{p}{-\kappa}) \sim D^{-1}(\frac{p}{-\kappa})$. This is checked as follows: if $\kappa \to 0$, then $e^{-\kappa} \frac{p}{-\kappa} \sim \frac{p}{-\kappa}$; if, on the other hand, $\kappa \to \bar{\kappa} \in (-\infty, 0)$, since $p \to 0$ by assumption, the first relation in (1.6) yields $D^{-1}(e^{-\kappa} \frac{p}{-\kappa}) \sim \sqrt{2(-\log(\frac{p}{-\bar{\kappa}}) + \bar{\kappa})} \sim \sqrt{2(-\log(\frac{p}{-\bar{\kappa}}))} \sim D^{-1}(\frac{p}{-\kappa})$, as requested. (For more details, see the end of the proof of Theorem 3.3, cf. (3.26) and the following lines.)

In conclusion, it suffices to prove Theorem 3.3 in the case $\kappa \geq 0$, and Theorem 1.1 follows.

3.4. **Proof of Theorem 3.3 for $\kappa \ge 0$.** We prove separately relations (3.16) and (3.17).

*Proof of* (3.16). We fix a family of values of $(\kappa, c)$ with $c \to 0$ and $\kappa$ bounded away from zero, say $\kappa \ge \delta$ for some fixed $\delta > 0$. Our goal is to prove that relation (3.16) holds. If we set $v := \mathsf{V_{BS}}(\kappa, c)$, by definition (3.13) we have $\mathsf{C_{BS}}(\kappa, v) = c \to 0$.

Let us first show that $d_1 := -\frac{\kappa}{v} + \frac{v}{2} \to -\infty$. By Proposition 3.2, $\mathsf{C_{BS}}(\kappa, v) \to 0$ implies $\min\{d_1, \log v\} \to -\infty$, which means that every subsequence of values of $(\kappa, c)$ admits a further sub-subsequence along which either $d_1 \to \infty$ or $v \to 0$. The key point is that $v \to 0$ implies $d_1 \to -\infty$, because $d_1 \le -\frac{\delta}{v} + \frac{v}{2}$ (recall that $\kappa \ge \delta$). Thus $d_1 \to -\infty$ along every sub-subsequence, which means that $d_1 \to -\infty$ along the whole family of values of $(\kappa, c)$.

Since $d_1 \to -\infty$, we can apply relation (3.11). Taking log of both sides of that relation, recalling the definition (3.1) of $\phi$ and the fact that $\mathsf{C_{BS}}(\kappa, v) = c$, we can write

$$\log c \sim -\frac{1}{2} d_1^2 - \log \sqrt{2\pi} + \log \frac{v}{-d_1(-d_1 + v)} \,. \tag{3.18}$$

We now show that the last term in the right hand side is $o(d_1^2)$ and can therefore be neglected. Note that $-d_1 \ge 1$ eventually, because $d_1 \to -\infty$, hence

$$\log \frac{v}{-d_1(-d_1+v)} \le \log \frac{v}{1+v} \le 0 \,.$$

Since $v \mapsto \frac{-d_1+v}{v}$ is decreasing for $-d_1 > 0$, in case $v \ge -d_1$ one has

$$\left| \log \frac{v}{-d_1(-d_1+v)} \right| = \log \frac{-d_1(-d_1+v)}{v} \le \log(-2d_1) = o(d_1^2) \,.$$

On the other hand, recalling that $d_1 \le -\frac{\delta}{v} + \frac{v}{2}$, in case $v < -d_1$ one has $d_1 \le -\frac{\delta}{v} - \frac{d_1}{2}$, which can be rewritten as $v \ge \frac{2\delta}{-3d_1}$ and together with $v < -d_1$ yields

$$\left| \log \frac{v}{-d_1(-d_1+v)} \right| = \log \frac{-d_1(-d_1+v)}{v} \le \log \frac{-d_1(-d_1-d_1)}{\frac{2\delta}{-3d_1}} = \log \left( \frac{3(-d_1)^3}{2\delta} \right) = o(d_1^2) \,.$$

In conclusion, (3.18) yields $\log c \sim -\frac{1}{2} d_1^2$, that is there exists $\gamma = \gamma(\kappa, c) \to 0$ such that $(1+\gamma) \log c = -\frac{1}{2} d_1^2$, and since $\log c \le 0$ we can write

$$(1+\gamma) |\log c| = \frac{1}{2} d_1^2 = \frac{1}{2} \left( \frac{\kappa^2}{v^2} + \frac{v^2}{4} - \kappa \right) .$$

This is a second degree equation in $v^2$, whose solutions (both positive) are

$$v^2 = 2\kappa \left[ 1 + 2\frac{(1+\gamma)|\log c|}{\kappa} \pm 2 \sqrt{ \left( \frac{(1+\gamma)|\log c|}{\kappa} \right)^2 + \frac{(1+\gamma)|\log c|}{\kappa} } \right] . \tag{3.19}$$

Since $d_1 \to -\infty$, eventually one has $d_1 < 0$: since $d_1 = -\frac{\kappa}{v} + \frac{v}{2} = -\frac{1}{2v}(\sqrt{2\kappa} - v)(\sqrt{2\kappa} + v)$, it follows that $v^2 < 2\kappa$, which selects the "$-$" solution in (3.19). Taking square roots of both sides of (3.19) and recalling that $v = \mathsf{V_{BS}}(\kappa, c)$ yields the equality

$$\mathsf{V_{BS}}(\kappa, c) = \sqrt{2(1+\gamma)|\log c| + 2\kappa} - \sqrt{2(1+\gamma)|\log c|} \,, \tag{3.20}$$

as one checks squaring both sides of (3.20).

Finally, since $\gamma \to 0$, it is quite intuitive that relation (3.20) yields (3.16). To prove this fact, we observe that by (3.20) we can write

$$\frac{\mathsf{V_{BS}}(\kappa, c)}{\sqrt{2|\log c| + 2\kappa} - \sqrt{2|\log c|}} = f_\gamma \left( \frac{\kappa}{|\log c|} \right) , \tag{3.21}$$

where for fixed $\gamma > -1$ we define the function $f_\gamma : [0,\infty) \to (0,\infty)$ by

$$f_\gamma(x) := \frac{\sqrt{1+\gamma+x}-\sqrt{1+\gamma}}{\sqrt{1+x}-1} \quad \text{for } x > 0\,, \qquad f_\gamma(0) := \lim_{x\downarrow 0} f_\gamma(x) = \frac{1}{\sqrt{1+\gamma}}\,.$$

By direct computation, when $\gamma > 0$ (resp. $\gamma < 0$) one has $\frac{\mathrm{d}}{\mathrm{d}x} f_\gamma(x) > 0$ (resp. $< 0$) for all $x > 0$. Since $\lim_{x\to+\infty} f_\gamma(x) = 1$, it follows that for every $x \ge 0$ one has $f_\gamma(0) \le f_\gamma(x) \le 1$ if $\gamma > 0$, while $1 \le f_\gamma(x) \le f_\gamma(0)$ if $\gamma < 0$; consequently, for any $\gamma$,

$$\frac{1}{\sqrt{1+|\gamma|}} \le f_\gamma(x) \le \frac{1}{\sqrt{1-|\gamma|}}\,, \qquad \forall x \ge 0\,,$$

which yields $\lim_{\gamma\to 0} f_\gamma(x) = 1$ *uniformly over* $x \ge 0$. By (3.21), relation (3.16) is proved. □

*Proof of* (3.17). We now fix a family of values of $(\kappa, c)$ with $c \to 0$ and $\kappa$ bounded away from infinity, say $0 \le \kappa \le M$ for some fixed $M \in (0,\infty)$, and we prove relation (3.17).

We set $v := \mathsf{V_{BS}}(\kappa, c)$ so that $\mathsf{C_{BS}}(\kappa, v) = c \to 0$, cf. (3.13). (Note that $v > 0$, because $c > 0$ by assumption.) Applying Proposition 3.2 we have $\min\{d_1, \log v\} \to -\infty$, i.e. either $d_1 \to -\infty$ or $v \to 0$ along sub-subsequences. However, this time $d_1 \to -\infty$ implies $v \to 0$, because $d_1 \ge -\frac{M}{v} + \frac{v}{2}$ (recall that $\kappa \le M$), which means that $v \to 0$ along the whole given family of values of $(\kappa, c)$.

Since $v \to 0$, relation (3.12) yields

$$c \sim -U'(-d_1)\,\phi(d_1)\,v\,. \tag{3.22}$$

Let us focus on $U'(-d_1)$: recalling that $d_1 = -\frac{\kappa}{v} + \frac{v}{2}$ and $v \to 0$, we first show that

$$U'(-d_1) \sim U'\left(\frac{\kappa}{v}\right). \tag{3.23}$$

By a subsequence argument, we may assume that $\frac{\kappa}{v} \to \varrho \in [0,\infty]$, and we recall that $v \to 0$:

- if $\varrho < \infty$, $U'(-d_1)$ and $U'(\frac{\kappa}{v})$ converge to $U'(\varrho) \neq 0$, hence $U'(-d_1)/U'(\frac{\kappa}{v}) \to 1$;
- if $\varrho = \infty$, $-d_1$ and $\frac{\kappa}{v}$ diverge to $\infty$ and (3.3) yields $U'(-d_1)/U'(\frac{\kappa}{v}) \sim (\frac{\kappa}{v})/(-d_1) \to 1$.

The proof of (3.23) is completed. Next we observe that, again by $v \to 0$,

$$\phi(-d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\kappa^2}{v^2}+\frac{v^2}{2}-\kappa)} \sim e^{\frac{1}{2}\kappa}\,\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\kappa^2}{v^2}} = e^{\frac{1}{2}\kappa}\phi\left(\frac{\kappa}{v}\right).$$

We can thus rewrite (3.22) as

$$c \sim -U'\left(\frac{\kappa}{v}\right)\phi\left(\frac{\kappa}{v}\right) e^{\frac{1}{2}\kappa}\, v\,. \tag{3.24}$$

If $\kappa = 0$, recalling (3.4) we obtain $c \sim \phi(0) v = \frac{1}{\sqrt{2\pi}} v$, which is the second line of (3.17).

Next we assume $\kappa > 0$. By (3.4), (3.2) and (1.5), for all $z > 0$ we can write

$$-U'(z)\,\phi(z) = -\phi(z)\big(zU(z)-1\big) = \phi(z) - z\Phi(-z) = zD(z)\,,$$

hence (3.24) can be rewritten as

$$c \sim \kappa\, e^{\frac{1}{2}\kappa}\, D\left(\frac{\kappa}{v}\right), \qquad \text{i.e.} \qquad (1+\gamma)c = \kappa\, e^{\frac{1}{2}\kappa}\, D\left(\frac{\kappa}{v}\right),$$

for some $\gamma = \gamma(\kappa, c) \to 0$. Recalling that $v = \mathsf{V_{BS}}(\kappa, c)$, we have shown that

$$\mathsf{V_{BS}}(\kappa, c) = \frac{\kappa}{D^{-1}\left(\frac{(1+\gamma)c}{\kappa e^{\frac{1}{2}\kappa}}\right)}\,. \tag{3.25}$$

We now claim that
$$D^{-1}\bigg(\frac{(1+\gamma)c}{\kappa e^{\frac{1}{2}\kappa}}\bigg) \sim D^{-1}\Big(\frac{c}{\kappa}\Big) . \tag{3.26}$$
By a subsequence argument, we may assume that $\frac{c}{\kappa} \to \eta \in [0,\infty]$ and $\kappa \to \bar\kappa \in [0,M]$.

- If $\eta \in (0,\infty)$, then $\bar\kappa = 0$ (recall that $c \to 0$) hence $(1+\gamma)c/(\kappa e^{\frac{1}{2}\kappa}) \to \eta$; then both sides of (3.26) converge to $D^{-1}(\eta) \in (0,\infty)$, hence their ratio converges to 1.
- If $\eta = \infty$, then again $\bar\kappa = 0$, hence $(1+\gamma)c/(\kappa e^{\frac{1}{2}\kappa}) \to \infty$: since $D^{-1}(y) \sim \frac{1}{\sqrt{2\pi}} y^{-1}$ as $y \to \infty$, cf. (1.6), it follows immediately that (3.26) holds.
- If $\eta = 0$, then $(1+\gamma)c/(\kappa e^{\frac{1}{2}\kappa}) \to 0$: since $D^{-1}(y) \sim \sqrt{2|\log y|}$ as $y \to 0$, cf. (1.6),
$$D^{-1}\bigg(\frac{(1+\gamma)c}{\kappa e^{\frac{1}{2}\kappa}}\bigg) \sim \sqrt{2\bigg|\Big(\log\frac{c}{\kappa}\Big) + \Big(\log\frac{1+\gamma}{e^{\frac{1}{2}\kappa}}\Big)\bigg|} \sim \sqrt{2\Big|\log\frac{c}{\kappa}\Big|} ,$$
because $|\log\frac{c}{\kappa}| \to \infty$ while $|\log[(1+\gamma)/e^{\frac{1}{2}\kappa}]| \to \frac{1}{2}\bar\kappa \in [0,\frac{M}{2}]$, hence (3.26) holds.

Having proved (3.26), we can plug it into (3.25), obtaining precisely the first line of (3.17). This completes the proof of Theorem 3.3. □

## 4. From tail probability to option price

In this section we prove Theorems 1.5, 1.6 and 1.11.

4.1. **Proof of Theorem 1.5 and 1.6.** We prove Theorem 1.5 and 1.6 at the same time. We recall that the tail probabilities $\overline{F}_t(\kappa)$, $F_t(-\kappa)$ are defined in (1.13). Throughout the proof, we fix a family of values of $(\kappa,t)$ with $\kappa > 0$ and $0 < t < T$, for some fixed $T \in (0,\infty)$, such that Hypothesis 1.4 is satisfied.

Extracting subsequences, we may distinguish three regimes for $\kappa$:

- if $\kappa \to \infty$ our goal is to prove (1.22), resp. (1.28);
- if $\kappa \to \bar\kappa \in (0,\infty)$ our goal is to prove (1.25), resp. (1.30), because in this case, plainly, one has $-\log\overline{F}_t(\kappa)/\kappa \to \infty$, resp. $-\log F_t(-\kappa)/\kappa \to \infty$, by (1.14);
- if $\kappa \to 0$, our goal is to prove (1.27), resp. (1.33).

Of course, each regime has different assumptions, as in Theorem 1.5 and 1.6.

*Step 0. Preparation.* It follows by conditions (1.15) and (1.16) that
$$\forall\varepsilon > 0 \quad \exists\varrho_\varepsilon \in (1,\infty): \qquad I_\pm(\varrho_\varepsilon) < 1+\varepsilon , \tag{4.1}$$
therefore for every $\varepsilon > 0$ one has eventually
$$\begin{aligned} \log\overline{F}_t(\varrho_\varepsilon\kappa) &\ge (1+\varepsilon)\log\overline{F}_t(\kappa) , \qquad \text{resp.} \\ \log F_t(-\varrho_\varepsilon\kappa) &\ge (1+\varepsilon)\log F_t(-\kappa) , \end{aligned} \tag{4.2}$$
where the inequality is "$\ge$" instead of "$\le$", because both sides are negative quantities.

We stress that $\overline{F}_t(\kappa) \to 0$, resp. $F_t(-\kappa) \to 0$, by (1.14), hence
$$\log\overline{F}_t(\kappa) \to -\infty , \qquad \text{resp.} \qquad \log F_t(-\kappa) \to -\infty . \tag{4.3}$$
Moreover, we claim that in any of the regimes $\kappa \to \infty$, $\kappa \to \bar\kappa \in (0,\infty)$ and $\kappa \to 0$ one has
$$\log\overline{F}_t(\kappa) + \kappa \to -\infty . \tag{4.4}$$

This follows readily by (4.3) if $\kappa \to 0$ or $\kappa \to \bar{\kappa} \in (0, \infty)$. If $\kappa \to \infty$ we argue as follows: by Markov's inequality, for $\eta > 0$

$$\overline{F}_t(\kappa) \le \mathrm{E}[e^{(1+\eta)X_t}]e^{-(1+\eta)\kappa}, \tag{4.5}$$

hence

$$\log \overline{F}_t(\kappa) + \kappa \le -\eta\kappa + \log \mathrm{E}[e^{(1+\eta)X_t}].$$

Since in the regime $\kappa \to \infty$ we assume that the moment condition (1.17) holds for some or every $\eta > 0$, the term $\log \mathrm{E}[e^{(1+\eta)X_t}]$ is bounded from above, hence eventually

$$\log \overline{F}_t(\kappa) + \kappa \le -\frac{\eta}{2}\,\kappa, \tag{4.6}$$

which proves relation (4.4).

The rest of the proof is divided in four steps, in each of which we prove lower and upper bounds on $c(\kappa, t)$ and $p(-\kappa, t)$, respectively.

*Step 1. Lower bounds on $c(\kappa, t)$.* We are going to prove sharp lower bounds on $c(\kappa, t)$, that will lead to relations (1.22), (1.25) and (1.27).

By (1.1) and (4.1), for every $\varepsilon > 0$ we can write

$$c(\kappa, t) \ge \mathrm{E}[(e^{X_t} - e^{\kappa})1_{\{X_t > \varrho_\varepsilon \kappa\}}] \ge (e^{\varrho_\varepsilon \kappa} - e^{\kappa})\overline{F}_t(\varrho_\varepsilon \kappa), \tag{4.7}$$

and applying (4.2) we get

$$\log c(\kappa, t) \ge \log\left(e^{\varrho_\varepsilon \kappa} - e^{\kappa}\right) + (1+\varepsilon)\log \overline{F}_t(\kappa). \tag{4.8}$$

If $\kappa \to \infty$, since $\log(e^{\varrho_\varepsilon \kappa} - e^{\kappa}) = \kappa + \log(e^{(\varrho_\varepsilon - 1)\kappa} - 1) \ge \kappa$ eventually, we obtain

$$\begin{aligned} \log c(\kappa, t) &\ge \kappa + (1+\varepsilon)\log \overline{F}_t(\kappa) = (1+\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big) - \varepsilon\kappa \\ &\ge (1 + 2\varepsilon + \tfrac{2}{\eta}\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big), \end{aligned} \tag{4.9}$$

where in the last inequality we have applied (4.6). It follows that

$$\limsup \frac{\log c(\kappa, t)}{\log \overline{F}_t(\kappa) + \kappa} \le 1 + 2\varepsilon + \tfrac{2}{\eta}\varepsilon, \tag{4.10}$$

where the lim sup is taken along the given family of values of $(\kappa, t)$ (note that $\log c(\kappa, t)$ and $\log \overline{F}_t(\kappa) + \kappa$ are negative quantities, cf. (4.4), hence the reverse inequality with respect to (4.9)). Since $\varepsilon > 0$ is arbitrary and $\eta > 0$ is fixed, we have shown that

$$\limsup \frac{\log c(\kappa, t)}{\log \overline{F}_t(\kappa) + \kappa} \le 1, \tag{4.11}$$

that is we have obtained a sharp bound for (1.22).

If $\kappa \to \bar{\kappa} \in (0, \infty)$, since $\log(e^{\varrho_\varepsilon \kappa} - e^{\kappa}) \to \log(e^{\varrho_\varepsilon \bar{\kappa}} - e^{\bar{\kappa}})$ is bounded while $\log \overline{F}_t(\kappa) \to -\infty$, relation (4.8) gives

$$\limsup \frac{\log c(\kappa, t)}{\log \overline{F}_t(\kappa)} \le 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that when $\kappa \to \bar{\kappa} \in (0, \infty)$

$$\limsup \frac{\log c(\kappa, t)}{\log \overline{F}_t(\kappa)} \le 1, \tag{4.12}$$

obtaining a sharp bound for (1.25).

Finally, if $\kappa \to 0$, since for $\kappa \geq 0$ by convexity $\log(e^{\varrho_\varepsilon \kappa} - e^\kappa) = \kappa + \log(e^{(\varrho_\varepsilon - 1)\kappa} - 1) \geq \kappa + \log((\varrho_\varepsilon - 1)\kappa) = \kappa + \log(\varrho_\varepsilon - 1) + \log \kappa$, relation (4.8) yields

$$\log \frac{c(\kappa,t)}{\kappa} = \log c(\kappa,t) - \log \kappa \geq \log(\varrho_\varepsilon - 1) + (1+\varepsilon) \log \overline{F}_t(\kappa) \,.$$

Again, since $\log(\varrho_\varepsilon - 1)$ is constant and $\log \overline{F}_t(\kappa) \to -\infty$, and $\varepsilon > 0$ is arbitrary, we get

$$\limsup \frac{\log \big(c(\kappa,t)/\kappa\big)}{\log \overline{F}_t(\kappa)} \leq 1 \,, \tag{4.13}$$

proving a sharp bound for (1.27).

*Step 2. Lower bounds on $p(-\kappa,t)$.* We are going to prove sharp lower bounds on $p(-\kappa,t)$, that will lead to relations (1.28), (1.30) and (1.33).

Recalling (1.1) and (4.1), for every $\varepsilon > 0$ we can write

$$p(-\kappa,t) \geq \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t \leq -\varrho_\varepsilon \kappa\}}] \geq (e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) F_t(-\varrho_\varepsilon \kappa) \,, \tag{4.14}$$

and applying (4.2) we obtain

$$\log p(-\kappa,t) \geq \log \big(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}\big) + (1+\varepsilon) \log F_t(-\kappa) \,. \tag{4.15}$$

If $\kappa \to \infty$, since $\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) = -\kappa + \log(1 - e^{-(\varrho_\varepsilon - 1)\kappa}) \sim -\kappa$, eventually one has $\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) \geq -(1+\varepsilon)\kappa$ and we obtain

$$\log p(-\kappa,t) \geq (1+\varepsilon)\big(\log F_t(-\kappa) - \kappa\big) \,.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup \frac{\log p(-\kappa,t)}{\log F_t(-\kappa) - \kappa} \leq 1 \,, \tag{4.16}$$

which is a sharp bound for (1.28).

If $\kappa \to \bar\kappa \in (0,\infty)$, since $\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) \to \log(e^{-\bar\kappa} - e^{-\varrho_\varepsilon \bar\kappa})$ is bounded while $\log F_t(-\kappa) \to -\infty$, and $\varepsilon > 0$ is arbitrary, relation (4.15) gives

$$\limsup \frac{\log p(-\kappa,t)}{\log F_t(-\kappa)} \leq 1 \,, \tag{4.17}$$

which is a sharp bound for (1.30).

Finally, if $\kappa \to 0$, since $e^{-\kappa} - e^{-\varrho_\varepsilon \kappa} = e^{-\varrho_\varepsilon \kappa}(e^{(\varrho_\varepsilon - 1)\kappa} - 1) \geq e^{-\varrho_\varepsilon \kappa}(\varrho_\varepsilon - 1)\kappa$ by convexity, since $\kappa \geq 0$, one has eventually

$$\log \big(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}\big) \geq \log \kappa + \log \big(e^{-\varrho_\varepsilon \kappa}(\varrho_\varepsilon - 1)\big) \geq \log \kappa + \varepsilon \log F_t(-\kappa) \,,$$

because $\log \big(e^{-\varrho_\varepsilon \kappa}(\varrho_\varepsilon - 1)\big) \to \log(\varrho_\varepsilon - 1) > -\infty$ while $\log F_t(-\kappa) \to -\infty$. Relation (4.15) then yields, eventually,

$$\log \frac{p(-\kappa,t)}{\kappa} = \log p(-\kappa,t) - \log \kappa \geq (1+2\varepsilon) \log F_t(-\kappa) \,.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\limsup \frac{\log \big(p(-\kappa,t)/\kappa\big)}{\log F_t(-\kappa)} \leq 1 \,, \tag{4.18}$$

obtaining a sharp bound for (1.33).

*Step 3. Upper bounds on $c(\kappa,t)$.* We are going to prove sharp upper bounds on $c(\kappa,t)$, that will complete the proof of relations (1.22), (1.25) and (1.27). We first consider the case when *the moment assumptions* (1.17) *and* (1.19) *hold for every* $\eta > 0$.

Let us look at the regimes $\kappa \to \infty$ and $\kappa \to \bar\kappa \in (0,\infty)$ (i.e. $\kappa$ is bounded away from zero), assuming that condition (1.17) holds *for every* $\eta > 0$. By Hölder's inequality,

$$c(\kappa,t) = \mathrm{E}[(e^{X_t} - e^{\kappa})1_{\{X_t>\kappa\}}] \le \mathrm{E}[e^{X_t}1_{\{X_t>\kappa\}}] \le \mathrm{E}[e^{(1+\eta)X_t}]^{\frac{1}{1+\eta}}\,\overline{F}_t(\kappa)^{\frac{\eta}{1+\eta}}\,. \tag{4.19}$$

Let us fix $\varepsilon > 0$ and choose $\eta = \eta_\varepsilon$ large enough, so that $\frac{\eta}{1+\eta} > 1-\varepsilon$. By assumption (1.17), for some $C \in (0,\infty)$ one has

$$\mathrm{E}[e^{(1+\eta)X_t}]^{\frac{1}{1+\eta}} \le C\,,$$

hence eventually, recalling that $\log \overline{F}_t(\kappa) \to -\infty$, by (4.3),

$$\log c(\kappa,t) \le \log C + (1-\varepsilon)\log \overline{F}_t(\kappa) \le (1-2\varepsilon)\log \overline{F}_t(\kappa)\,. \tag{4.20}$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\liminf \frac{\log c(\kappa,t)}{\log \overline{F}_t(\kappa)} \ge 1\,, \tag{4.21}$$

which together with (4.11) completes the proof of (1.22), if $\kappa \to \infty$, because $\log \overline{F}_t(\kappa) \sim \log \overline{F}_t(\kappa) - \kappa$ when condition (1.17) holds for every $\eta > 0$, by (4.5) (cf. also Remark 1.9). If $\kappa \to \bar\kappa \in (0,\infty)$, relation (4.21) together with (4.12) completes the proof of (1.25).

We then consider the regime $\kappa \to 0$, assuming that condition (1.19) holds *for every* $\eta > 0$. We modify (4.19) as follows: since $(e^{X_t} - e^{\kappa}) \le (e^{X_t} - 1) \le |e^{X_t} - 1|$,

$$c(\kappa,t) \le \mathrm{E}[|e^{X_t} - 1|1_{\{X_t>\kappa\}}] \le \kappa\,\mathrm{E}\left[\left|\frac{e^{X_t}-1}{\kappa}\right|^{1+\eta}\right]^{\frac{1}{1+\eta}} \overline{F}_t(\kappa)^{\frac{\eta}{1+\eta}}\,. \tag{4.22}$$

Let us fix $\varepsilon > 0$ and choose $\eta = \eta_\varepsilon$ large enough, so that $\frac{\eta}{1+\eta} > 1-\varepsilon$. By assumption (1.19), for some $C \in (0,\infty)$ one has

$$\mathrm{E}\left[\left|\frac{e^{X_t}-1}{\kappa}\right|^{1+\eta}\right]^{\frac{1}{1+\eta}} \le C\,, \tag{4.23}$$

hence relation (4.22) yields eventually

$$\log \frac{c(\kappa,t)}{\kappa} \le \log C + (1-\varepsilon)\log \overline{F}_t(\kappa) \le (1-2\varepsilon)\log \overline{F}_t(\kappa)\,. \tag{4.24}$$

Since $\varepsilon > 0$ is arbitrary, we have proved that

$$\liminf \frac{\log\big(c(\kappa,t)/\kappa\big)}{\log \overline{F}_t(\kappa)} \ge 1\,, \tag{4.25}$$

which together with (4.13) completes the proof of (1.27).

It remains to consider the case when the moment assumptions (1.17) and (1.19) holds *for some* $\eta > 0$, but in addition conditions (1.21) (if $\kappa \to \infty$ or $\kappa \to \bar\kappa \in (0,\infty)$) or (1.24) (if $\kappa \to 0$) holds. We start with considerations that are valid in any regime of $\kappa$.

Defining the constant

$$A := \limsup \left\{ \frac{-\kappa}{\log \overline{F}_t(\kappa) + \kappa} \right\} + 1\,, \tag{4.26}$$

where the lim sup is taken along the given family of values of $(\kappa,t)$, we claim that $A < \infty$. This follows by (4.4) if $\kappa \to 0$ or if $\kappa \to \bar\kappa \in (0,\infty)$ (in which case, plainly, $A = 1$), while if $\kappa \to +\infty$ it suffices to apply (4.6) to get $A \le 2/\eta + 1$. It follows by (4.26) that eventually

$$\kappa \le -A(\log \overline{F}_t(\kappa) + \kappa)\,. \tag{4.27}$$

Next we show that, for all fixed $\varepsilon > 0$ and $1 < M < \infty$, eventually one has

$$\log\Bigg(\sup_{y\in[1,M]} e^{\kappa y}\,\overline{F}_t(\kappa y)\Bigg) \le (1-\varepsilon)\big(\log\overline{F}_t(\kappa)+\kappa\big)\,, \tag{4.28}$$

which means that the sup is approximately attained for $y = 1$. This is easy if $\kappa \to 0$ or if $\kappa \to \bar\kappa \in (0,\infty)$: in fact, since $\kappa \to \overline{F}_t(\kappa)$ is non-increasing, we can write

$$\begin{aligned}\log\Bigg(\sup_{y\in[1,M]} e^{\kappa y}\,\overline{F}_t(\kappa y)\Bigg) &\le \log\big(e^{\kappa M}\overline{F}_t(\kappa)\big) = \kappa M + \log\overline{F}_t(\kappa)\\ &= \big(\log\overline{F}_t(\kappa)+\kappa\big) \,+\, (M-1)\kappa\,,\end{aligned}$$

and since $\log\overline{F}_t(\kappa)+\kappa \to -\infty$ by (4.4), while $(M-1)\kappa$ is bounded, (4.28) follows.

To prove (4.28) in the regime $\kappa \to \infty$, we are going to exploit the assumption (1.21). First we fix $\delta > 0$, to be defined later, and set $\bar n := \lceil \frac{M-1}{\delta}\rceil$ and $a_n := 1 + n\delta$ for $n = 0,\ldots,\bar n$, so that $[1,M] \subseteq \bigcup_{n=1}^{\bar n}[a_{n-1},a_n]$. For all $y \in [a_{n-1},a_n]$ one has, by (1.15),

$$\log\overline{F}_t(\kappa y) \le \log\overline{F}_t(\kappa a_{n-1}) \sim I_+(a_{n-1})\log\overline{F}_t(\kappa) \le a_{n-1}\log\overline{F}_t(\kappa)\,,$$

having used that $I_+(\varrho) \ge \varrho$, by (1.21), hence eventually

$$\log\overline{F}_t(\kappa y) \le (1-\delta)a_{n-1}\log\overline{F}_t(\kappa)\,, \qquad \forall y \in [a_{n-1},a_n]\,.$$

Recalling that $a_n = a_{n-1}+\delta$, we can write $a_n \le (1-\delta)a_{n-1}+\delta(1+M)$, because $a_{n-1} \le M$ by construction, and since $e^{\kappa y} \le e^{\kappa a_n}$ for $y \in [a_{n-1},a_n]$, it follows that

$$\begin{aligned}\log\Bigg(\sup_{y\in[1,M]} e^{\kappa y}\,\overline{F}_t(\kappa y)\Bigg) &\le \max_{n=1,\ldots,\bar n}\big(a_n\kappa + (1-\delta)a_{n-1}\log\overline{F}_t(\kappa)\big)\\ &= \max_{n=1,\ldots,\bar n}\big((1-\delta)a_{n-1}\big(\log\overline{F}_t(\kappa)+\kappa\big)+\delta(1+M)\kappa\big)\,.\end{aligned}$$

Plainly, the max is attained for $n = 1$, for which $a_{n-1} = a_0 = 1$. Recalling (4.27), we get

$$\log\Bigg(\sup_{y\in[1,M]} e^{\kappa y}\,\overline{F}_t(\kappa y)\Bigg) \le (1-\delta(1+A+AM))\big(\log\overline{F}_t(\kappa)+\kappa\big)\,.$$

Choosing $\delta := \varepsilon/(1+A+AM)$, the claim (4.28) is proved.

We are ready to give sharp upper bounds on $c(\kappa,t)$, refining (4.19). For fixed $M \in (0,\infty)$, we write

$$c(\kappa,t) = \mathrm{E}[(e^{X_t}-e^{\kappa})\mathbb{1}_{\{\kappa<X_t\le\kappa M\}}] + \mathrm{E}[(e^{X_t}-e^{\kappa})\mathbb{1}_{\{X_t>\kappa M\}}]\,, \tag{4.29}$$

and we estimate the first term as follows: by Fubini-Tonelli's theorem and (4.28),

$$\begin{aligned}\mathrm{E}[(e^{X_t}-e^{\kappa})\mathbb{1}_{\{\kappa<X_t\le\kappa M\}}] &= \mathrm{E}\Bigg[\Bigg(\int_\kappa^\infty e^x\,\mathbb{1}_{\{x<X_t\}}\,\mathrm{d}x\Bigg)\mathbb{1}_{\{\kappa<X_t\le\kappa M\}}\Bigg]\\ &= \int_\kappa^{\kappa M} e^x\,\mathrm{P}(x<X_t\le\kappa M)\,\mathrm{d}x \le \int_\kappa^{\kappa M} e^x\,\overline{F}_t(x)\,\mathrm{d}x\\ &= \kappa\int_1^M e^{\kappa y}\,\overline{F}_t(\kappa y)\,\mathrm{d}y \le \kappa\,(M-1)\,e^{(1-\varepsilon)(\log\overline{F}_t(\kappa)+\kappa)}\,.\end{aligned} \tag{4.30}$$

To estimate the second term in (4.29), we start with the cases $\kappa \to \infty$ and $\kappa \to \bar\kappa \in (0,\infty)$, where we assume that (1.17) holds for some $\eta > 0$, as well as (1.24), hence we can fix $M > 1$ such that $I_+(M) > \frac{1+\eta}{\eta}$. Bounding $(e^{X_t}-e^{\kappa}) \le e^{X_t}$, Hölder's inequality yields

$$\mathrm{E}[(e^{X_t}-e^{\kappa})\mathbb{1}_{\{X_t>\kappa M\}}] \le \mathrm{E}[e^{(1+\eta)X_t}]^{\frac{1}{1+\eta}}\,\overline{F}_t(\kappa M)^{\frac{\eta}{1+\eta}} = C\,\overline{F}_t(\kappa M)^{\frac{\eta}{1+\eta}}\,,$$

where $C \in (0, \infty)$ is an absolute constant, by (1.17). Applying relation (1.15) together with $I_+(M) > \frac{1+\eta}{\eta}$ we obtain

$$\frac{\eta}{1+\eta} \log \overline{F}_t(\kappa M) \sim \frac{\eta}{1+\eta} I_+(M) \log \overline{F}_t(\kappa) \leq \log \overline{F}_t(\kappa)\,, \tag{4.31}$$

hence eventually

$$\log \mathrm{E}[(e^{X_t} - e^{\kappa}) 1_{\{X_t > \kappa M\}}] \leq (1-\varepsilon) \log \overline{F}_t(\kappa) \leq (1-\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big)\,. \tag{4.32}$$

Recalling (4.6) and (4.4), eventually $\kappa(M-1) \leq e^{-\varepsilon(\log \overline{F}_t(\kappa)+\kappa)}$, hence by (4.30)

$$\log \mathrm{E}[(e^{X_t} - e^{\kappa}) 1_{\{\kappa < X_t \leq \kappa M\}}] \leq (1-2\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big)\,. \tag{4.33}$$

Looking back at (4.29), since

$$\log(a+b) \leq \log 2 + \max\{\log a, \log b\}\,, \qquad \forall a, b > 0\,, \tag{4.34}$$

by (4.32), (4.33) and again (4.4) one has eventually

$$\log c(\kappa, t) \leq \log 2 + (1-2\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big) \leq (1-3\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big)\,.$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\liminf \frac{\log c(\kappa, t)}{\log \overline{F}_t(\kappa) + \kappa} \geq 1\,, \tag{4.35}$$

which together with (4.11) completes the proof of (1.22), if $\kappa \to \infty$. Since $\log \overline{F}_t(\kappa) + \kappa \sim \log \overline{F}_t(\kappa)$ if $\kappa \to \bar{\kappa} \in (0, \infty)$, by (4.3), we can rewrite (4.35) in this case as

$$\liminf \frac{\log c(\kappa, t)}{\log \overline{F}_t(\kappa)} \geq 1\,, \tag{4.36}$$

which together with (4.12) completes the proof of (1.25).

It remains to consider the case when $\kappa \to 0$, where we assume that relation (1.19) holds for some $\eta \in (0, \infty)$, together with (1.24). As before, we fix $M > 1$ such that $I_+(M) > \frac{1+\eta}{\eta}$. Since

$$\mathrm{E}\left[\left(\frac{e^{X_t} - e^{\kappa}}{\kappa}\right)^{1+\eta} 1_{\{X_t > \kappa\}}\right] \leq \mathrm{E}\left[\left|\frac{e^{X_t} - 1}{\kappa}\right|^{1+\eta}\right] \leq C\,, \tag{4.37}$$

for some absolute constant $C \in (0, \infty)$, by (1.19), the second term in (4.29) is bounded by

$$\mathrm{E}[(e^{X_t} - e^{\kappa}) 1_{\{X_t > \kappa M\}}] \leq \kappa \mathrm{E}\left[\left|\frac{e^{X_t} - e^{\kappa}}{\kappa}\right|^{1+\eta}\right]^{\frac{1}{1+\eta}} \overline{F}_t(\kappa M)^{\frac{\eta}{1+\eta}} \leq \kappa\, C\, \overline{F}_t(\kappa M)^{\frac{\eta}{1+\eta}}\,. \tag{4.38}$$

In complete analogy with (4.31)-(4.32), we obtain that eventually

$$\log \frac{\mathrm{E}[(e^{X_t} - e^{\kappa}) 1_{\{X_t > \kappa M\}}]}{\kappa} \leq (1-\varepsilon) \log \overline{F}_t(\kappa)\,. \tag{4.39}$$

By (4.4), eventually $(M-1) \leq e^{-\varepsilon(\log \overline{F}_t(\kappa)+\kappa)}$, hence by (4.30)

$$\log \frac{\mathrm{E}[(e^{X_t} - e^{\kappa}) 1_{\{\kappa < X_t \leq \kappa M\}}]}{\kappa} \leq (1-2\varepsilon)(\log \overline{F}_t(\kappa) + \kappa)\,. \tag{4.40}$$

Recalling (4.29) and (4.34), we can finally write

$$\log \frac{c(\kappa, t)}{\kappa} \leq \log 2 + (1-2\varepsilon)\big(\log \overline{F}_t(\kappa) + \kappa\big) \leq (1-3\varepsilon) \log \overline{F}_t(\kappa)\,,$$

because $\kappa \to 0$ and $\log \overline{F}_t(\kappa) \to -\infty$. Since $\varepsilon > 0$ is arbitrary, we have proved that

$$\liminf \frac{\log \big(c(\kappa,t)/\kappa\big)}{\log \overline{F}_t(\kappa)} \geq 1\,, \tag{4.41}$$

which together with (4.13) completes the proof of (1.27).

*Step 4. Upper bounds on $p(-\kappa,t)$.* We are going to prove sharp upper bounds on $p(-\kappa,t)$, that will complete the proof of relations (1.28), (1.30) and (1.33).

By (1.1) we can write

$$p(-\kappa,t) = \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t \leq -\kappa\}}] \leq e^{-\kappa}\, F_t(-\kappa)\,,$$

therefore

$$\frac{\log p(-\kappa,t)}{\log F_t(-\kappa) - \kappa} \geq 1\,, \tag{4.42}$$

which together with (4.16) completes the proof of (1.33), if $\kappa \to \infty$. On the other hand, if $\kappa \to \bar\kappa \in (0,\infty)$, since relation (4.42) implies (recall that $\kappa \geq 0$)

$$\frac{\log p(-\kappa,t)}{\log F_t(-\kappa)} \geq 1\,, \tag{4.43}$$

in view of (4.17), the proof of (1.30) is completed.

It remains to consider the case $\kappa \to 0$. If relation (1.19) holds *for every* $\eta \in (0,\infty)$, we argue in complete analogy with (4.22)-(4.23)-(4.24), getting

$$\liminf \frac{\log \big(p(-\kappa,t)/\kappa\big)}{\log F_t(-\kappa)} \geq 1\,, \tag{4.44}$$

which together with (4.18) completes the proof of (1.33). If, on the other hand, relation (1.19) holds only *for some* $\eta \in (0,\infty)$, we also assume that condition (1.32) holds, hence we can fix $M > 1$ such that $I_-(M) > \frac{1+\eta}{\eta}$. Let us write

$$p(-\kappa,t) = \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{-\kappa M < X_t \leq -\kappa\}}] + \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t \leq -\kappa M\}}]\,. \tag{4.45}$$

In analogy with (4.30), for every fixed $\varepsilon > 0$, the first term in the right hand side can be estimated as follows (note that $y \mapsto F_t(-\kappa y)$ is decreasing):

$$\begin{aligned} \mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{-\kappa M < X_t \leq -\kappa\}}] &\leq \int_{-\kappa M}^{-\kappa} e^x\, F_t(x)\, \mathrm{d}x = \kappa \int_1^M e^{-\kappa y}\, F_t(-\kappa y)\, \mathrm{d}y \\ &\leq \kappa (M-1)\, F_t(-\kappa) \leq \kappa\, e^{(1-\varepsilon)\log F_t(-\kappa)}\,. \end{aligned}$$

The second term in (4.45) is estimated in complete analogy with (4.37)-(4.38)-(4.39), yielding

$$\log \frac{\mathrm{E}[(e^{-\kappa} - e^{X_t}) 1_{\{X_t \leq -\kappa M\}}]}{\kappa} \leq (1-\varepsilon) \log F_t(-\kappa)\,.$$

Recalling (4.34), we obtain from (4.45)

$$\log \frac{p(-\kappa,t)}{\kappa} \leq \log 2 + (1-\varepsilon)\log F_t(-\kappa) \leq (1-2\varepsilon)\log F_t(-\kappa)\,,$$

and since $\varepsilon > 0$ is arbitrary we have proved that relation (4.44) still holds, which together with (4.18) completes the proof of (1.33), and of the whole Theorem 1.5. $\square$

4.2. **Proof of Theorem 1.11.** By Skorokhod's representation theorem, we can build a coupling of the random variables $(X_t)_{t\geq 0}$ and $Y$ such that relation (1.36) holds a.s.. Since the function $z \mapsto z^+$ is continuous, recalling that $\gamma_t \to 0$, for $\kappa \sim a\gamma_t$ we have a.s.

$$\frac{(e^{X_t} - e^{\kappa})^+}{\gamma_t} = \left( \frac{e^{Y\gamma_t(1+o(1))} - 1}{\gamma_t} - \frac{e^{a\gamma_t(1+o(1))} - 1}{\gamma_t} \right)^+ \xrightarrow[t\downarrow 0]{a.s.} (Y - a)^+ \,, \tag{4.46}$$

and analogously for $\kappa \sim -a\gamma_t$

$$\frac{(e^{\kappa} - e^{X_t})^+}{\gamma_t} \xrightarrow[t\downarrow 0]{a.s.} (-a - Y)^+ = (Y + a)^- \,. \tag{4.47}$$

Taking the expectation of both sides of these relations, one would obtain precisely (1.39). To justify the interchanging of limit and expectation, we observe that the left hand sides of (4.46) and (4.47) are uniformly integrable, being bounded in $L^{1+\eta}$. In fact

$$\frac{|e^{X_t} - e^{\kappa}|}{\gamma_t} \leq \frac{|e^{X_t} - 1|}{\gamma_t} + \frac{|e^{\kappa} - 1|}{\gamma_t} \,,$$

and the second term in the right hand side is uniformly bounded (recall that $\kappa \sim a\gamma_t$ by assumption), while the first term is bounded in $L^{1+\eta}$, by (1.38). ☐

## Appendix A. Miscellanea

A.1. **About conditions** (1.3) **and** (1.4)**.** Recall from §1.1 that $(X_t)_{t\geq 0}$ denotes the risk-neutral log-price, and assume that $X_t \to X_0 := 0$ in distribution as $t \to 0$ (which is automatically satisfied if $X$ has right-continuous paths). For an arbitrary family of values of $(\kappa, t)$, with $t > 0$ and $\kappa \geq 0$, we show that condition (1.3) implies (1.4).

Assume first that $t \to 0$ (with no assumption on $\kappa$). Since $\kappa \geq 0$, one has $(e^{X_t} - e^{\kappa})^+ \to (1 - e^{\kappa})^+ = 0$ in distribution, hence $c(\kappa, t) \to 0$ by (1.1) and Fatou's lemma. With analogous arguments, one has $p(-\kappa, t) \to 0$, hence (1.4) is satisfied.

Next we assume that $\kappa \to \infty$ and $t$ is bounded, say $t \in (0, T]$ for some fixed $T > 0$. Since $z \mapsto (z - c)^+$ is a convex function and $(e^{X_t})_{t\geq 0}$ is a martingale, the process $((e^{X_t} - e^{\kappa})^+)_{t\geq 0}$ is a submartingale and by (1.1) we can write

$$0 \leq c(\kappa, t) \leq \mathrm{E}[(e^{X_T} - e^{\kappa})^+] = \mathrm{E}[(e^{X_T} - e^{\kappa})1_{\{X_T > \kappa\}}] \leq \mathrm{E}[e^{X_T} 1_{\{X_T > \kappa\}}] \,.$$

It follows that, if $\kappa \to +\infty$, then $c(\kappa, t) \to 0$. With analogous arguments, one shows that $p(-\kappa, t) \to 0$, hence condition (1.4) holds.

A.2. **Proof of Proposition 3.2.** Let us first prove (3.11) and (3.12). Since $\phi(d_2)e^k = \phi(d_1)$, cf. (3.1) and (3.8), recalling (3.2) we can rewrite the Black&Scholes formula (3.7) as follows:

$$\mathsf{C}_{\mathsf{BS}}(\kappa, v) = \phi(d_1)\big(U(-d_1) - U(-d_2)\big) = \phi(d_1)\big(U(-d_1) - U(-d_1 + v)\big) \,. \tag{A.1}$$

If $d_1 \to -\infty$, applying (3.3) we get

$$U(-d_1) - U(-d_1 + v) = -\int_{-d_1}^{-d_1+v} U'(z)\, \mathrm{d}z \sim \int_{-d_1}^{-d_1+v} \frac{1}{z^2}\, \mathrm{d}z = \frac{v}{-d_1(-d_1 + v)} \,,$$

and (3.11) is proved. Next we assume that $v \to 0$. By convexity of $U(\cdot)$ (cf. Lemma 3.1),

$$-U'(-d_1 + v) \leq \frac{U(-d_1) - U(-d_1 + v)}{v} \leq -U'(-d_1) \,,$$

hence to prove (3.12) it suffices to show that $U'(-d_1 + v) \sim U'(-d_1)$. To this purpose, by a subsequence argument, we may assume that $d_1 \to \overline{d_1} \in \mathbb{R} \cup \{\pm\infty\}$. Since $d_1 \leq \frac{v}{2}$ for $\kappa \geq 0$, when $v \to 0$ necessarily $\overline{d_1} \in [-\infty, 0]$. If $\overline{d_1} = -\infty$, i.e. $-d_1 \to +\infty$, then

$-d_1 + v \sim -d_1 \to +\infty$ and $U'(-d_1 + v) \sim U'(-d_1)$ follows by (3.3). On the other hand, if $\overline{d_1} \in (-\infty, 0]$ then both $U'(-d_1)$ and $U'(-d_1 + v)$ converge to $U'(-\overline{d_1}) \neq 0$, by continuity of $U'$, hence $U'(-d_1)/U'(-d_1 + v) \to 1$, i.e. $U'(-d_1 + v) \sim U'(-d_1)$ as requested.

Let us now prove (3.10). Assume that $\min\{d_1, \log v\} \to -\infty$, and note that for every subsequence we can extract a sub-subsequence along which either $d_1 \to -\infty$ or $v \to 0$. We can then apply (3.11) and (3.12) to show that $\mathsf{C}_{\mathsf{BS}}(\kappa, v) \to 0$:

- if $d_1 \to -\infty$, the right hand side of (3.11) is bounded from above by $\phi(d_1)/(-d_1) \to 0$;
- If $\kappa \geq 0$ and $v \to 0$, then $d_1 \leq \frac{v}{2} \to 0$ and consequently $\phi(d_1)U'(-d_1)$ is uniformly bounded from above, hence the right hand side of (3.12) vanishes (since $v \to 0$).

Finally, we assume that $\min\{d_1, \log v\} \not\to -\infty$ and show that $\mathsf{C}_{\mathsf{BS}}(\kappa, v) \not\to 0$. Extracting a subsequence, we have $\min\{d_1, \log v\} \geq -M$ for some fixed $M \in (0, \infty)$, i.e. both $v \geq \varepsilon := e^{-M} > 0$ and $d_1 \geq -M$, and we may assume that $v \to \overline{v} \in [\varepsilon, +\infty]$ and $d_1 \to \overline{d_1} \in [-M, +\infty]$. Consider first the case $\overline{v} = +\infty$, i.e. $v \to +\infty$: by (3.8) one has $-d_1 + v = -d_2 \geq \frac{v}{2} \to +\infty$, hence $\phi(d_1)U(-d_1 + v) \to 0$ (because $\phi$ is bounded), and recalling (3.2) relation (A.1) yields

$$\mathsf{C}_{\mathsf{BS}}(\kappa, v) = \Phi(d_1) - \phi(d_1)U(-d_1 + v) \to \Phi(\overline{d_1}) > 0\,.$$

Next consider the case $\overline{v} < +\infty$: since $d_1 \leq \frac{v}{2}$, we have $\overline{d_1} \leq \frac{\overline{v}}{2}$ and again by (A.1) we obtain $\mathsf{C}_{\mathsf{BS}}(\kappa, v) \to \phi(\overline{d_1})(U(-\overline{d_1}) - U(-\overline{d_1} + \overline{v})) > 0$. In both cases, $\mathsf{C}_{\mathsf{BS}}(\kappa, v) \not\to 0$. □

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