An Additive Noise Approximation to Keller–Segel–Dean–Kawasaki Dynamics

Adrian Martini

University of Oxford

In collaboration with Avi Mayorcas (Bath)

Statistical mechanics, interacting particle systems and SPDE

Statistical mechanics, interacting particle systems and SPDE

- Inverse temperature $\beta > 0$;

Statistical mechanics, interacting particle systems and SPDE

- Inverse temperature $\beta > 0$;
- Population size $N \in \mathbb{N}$ (conserved);

Statistical mechanics, interacting particle systems and SPDE

- Inverse temperature $\beta > 0$;
- Population size $N \in \mathbb{N}$ (conserved);
- Family of i.i.d. standard Brownian motions $(B^i)_{i=1,\ldots,N}$;

Statistical mechanics, interacting particle systems and SPDE

- Inverse temperature $\beta > 0$;
- Population size $N \in \mathbb{N}$ (conserved);
- Family of i.i.d. standard Brownian motions $(B^i)_{i=1,\ldots,N}$;
- Confinement potential V (under appropriate assumptions);

Statistical mechanics, interacting particle systems and SPDE

- Inverse temperature $\beta > 0$;
- Population size $N \in \mathbb{N}$ (conserved);
- Family of i.i.d. standard Brownian motions $(B^i)_{i=1,\dots,N}$;
- Confinement potential V (under appropriate assumptions);
- Coulomb potential

$$K(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ \frac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2}}\frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}$$

Consider the system of weakly-interacting, overdamped Langevin diffusions

Consider the system of weakly-interacting, overdamped Langevin diffusions

$$dX_t^i = -\frac{2\beta}{N} \sum_{j=1,\ j\neq i}^{N} \nabla K(X_t^i - X_t^j)\, dt - 2\beta \nabla V(X_t^i)\, dt + \sqrt{2}\, dB_t^i, \qquad i = 1, \dots, N,$$

Consider the system of weakly-interacting, overdamped Langevin diffusions

$$\mathrm{d}X_t^i = -\frac{2\beta}{N}\sum_{j=1,\ j\neq i}^{N} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t - 2\beta\nabla V(X_t^i)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1,\dots,N,$$

whose infinitesimally invariant measure is the Coulomb gas

$$\mu(\mathrm{d}x) \propto \mathrm{e}^{-\beta\left(\frac{1}{N}\sum_{i\neq j} K(x^i - x^j) + 2\sum_{i=1}^N V(x^i)\right)}\,\mathrm{d}x, \qquad x = (x^1,\dots,x^N) \in \mathbb{R}^N.$$

Consider the system of weakly-interacting, overdamped Langevin diffusions

$$\mathrm{d}X_t^i = -\frac{2\beta}{N}\sum_{j=1,\ j\neq i}^{N} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t - 2\beta\nabla V(X_t^i)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1,\dots,N,$$

whose infinitesimally invariant measure is the Coulomb gas

$$\mu(\mathrm{d}x) \propto \mathrm{e}^{-\beta\left(\frac{1}{N}\sum_{i\neq j} K(x^i - x^j) + 2\sum_{i=1}^{N} V(x^i)\right)}\,\mathrm{d}x, \qquad x = (x^1,\dots,x^N)\in\mathbb{R}^N.$$

Understand fluctuations of those models:

Consider the system of weakly-interacting, overdamped Langevin diffusions

$$\mathrm{d}X_t^i = -\frac{2\beta}{N}\sum_{j=1,\ j\neq i}^{N} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t - 2\beta\nabla V(X_t^i)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1,\ldots,N,$$

whose infinitesimally invariant measure is the Coulomb gas

$$\mu(\mathrm{d}x) \propto \mathrm{e}^{-\beta\left(\frac{1}{N}\sum_{i\neq j} K(x^i - x^j) + 2\sum_{i=1}^{N} V(x^i)\right)}\,\mathrm{d}x, \qquad x = (x^1,\ldots,x^N) \in \mathbb{R}^N.$$

Understand fluctuations of those models:

- Propagation of chaos for particle system [Liu, Yang '16, $d = 2$]

Consider the system of weakly-interacting, overdamped Langevin diffusions

$$\mathrm{d}X_t^i = -\frac{2\beta}{N}\sum_{j=1,\ j\neq i}^{N} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t - 2\beta\nabla V(X_t^i)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1,\ldots,N,$$

whose infinitesimally invariant measure is the Coulomb gas

$$\mu(\mathrm{d}x) \propto \mathrm{e}^{-\beta\left(\frac{1}{N}\sum_{i\neq j} K(x^i - x^j) + 2\sum_{i=1}^{N} V(x^i)\right)}\,\mathrm{d}x, \qquad x = (x^1,\ldots,x^N) \in \mathbb{R}^N.$$

Understand fluctuations of those models:

- Propagation of chaos for particle system [Liu, Yang '16, $d = 2$]
- LDP for Coulomb gas [Serfaty survey '15]

Q.: How about attractive models / negative inverse temperatures?

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

$$\mathrm{d}X_t^i = \frac{\chi}{N} \sum_{j \neq i} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \ldots, N.$$

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

$$\mathrm{d}X_t^i = \frac{\chi}{N} \sum_{j \neq i} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \dots, N.$$

Mean-field approximation of Keller–Segel model of chemotaxis [Keller, Segel ’70; Fournier, Jourdain ’17; Tardy ’22]

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

$$\mathrm{d}X_t^i = \frac{\chi}{N} \sum_{j \neq i} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \ldots, N.$$

Mean-field approximation of Keller–Segel model of chemotaxis [Keller, Segel ’70; Fournier, Jourdain ’17; Tardy ’22]

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi \nabla \cdot (\rho \nabla c), \\ (\tau \partial_t - \Delta)c = \rho, \end{cases}$$

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

$$\mathrm{d}X_t^i = \frac{\chi}{N} \sum_{j \neq i} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \ldots, N.$$

Mean-field approximation of Keller–Segel model of chemotaxis [Keller, Segel '70; Fournier, Jourdain '17; Tardy '22]

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho\nabla c), \\ (\tau\partial_t - \Delta)c = \rho, \end{cases} \quad \underset{\tau = 0}{\rightsquigarrow} \quad (\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho\nabla K * \rho).$$

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

$$\mathrm{d}X_t^i = \frac{\chi}{N}\sum_{j\neq i} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \dots, N.$$

Mean-field approximation of Keller–Segel model of chemotaxis [Keller, Segel '70; Fournier, Jourdain '17; Tardy '22]

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla c), \\ (\tau\partial_t - \Delta)c = \rho, \end{cases} \quad \underset{\tau=0}{\rightsquigarrow} \quad (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla K * \rho).$$

(McKean–Valsov equation / nonlinear Fokker–Planck equation)

Q.: How about attractive models / negative inverse temperatures?

Let $\beta = -\chi/2$, where we call $\chi > 0$ the *chemotactic sensitivity*, and $V \equiv 0$.

$$\mathrm{d}X_t^i = \frac{\chi}{N} \sum_{j \neq i} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \qquad i = 1, \dots, N.$$

Mean-field approximation of Keller–Segel model of chemotaxis [Keller, Segel '70; Fournier, Jourdain '17; Tardy '22]

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho \nabla c), \\ (\tau\partial_t - \Delta)c = \rho, \end{cases} \quad \underset{\tau = 0}{\rightsquigarrow} \quad (\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho\nabla K * \rho).$$

(McKean–Valsov equation / nonlinear Fokker–Planck equation)

In biology: Emergence of spatial structure

Competition between Diffusion and Aggregation

Competition between Diffusion and Aggregation

If $d = 2$

Competition between Diffusion and Aggregation

If $d = 2$ and

- $\chi < 8\pi$, then particle system and PDE exist globally;

Competition between Diffusion and Aggregation

If $d = 2$ and

- $\chi < 8\pi$, then particle system and PDE exist globally;
- $\chi > 8\pi$, then particles collide. The system and PDE exist only locally [Fournier, Tardy '23; Blanchet, Dolbeaut, Perthame '06].

Competition between Diffusion and Aggregation

If $d = 2$ and

- $\chi < 8\pi$, then particle system and PDE exist globally;
- $\chi > 8\pi$, then particles collide. The system and PDE exist only locally [Fournier, Tardy '23; Blanchet, Dolbeaut, Perthame '06].

CLT and LDP are still open for the particle system.

Competition between Diffusion and Aggregation

If $d = 2$ and

- $\chi < 8\pi$, then particle system and PDE exist globally;
- $\chi > 8\pi$, then particles collide. The system and PDE exist only locally [Fournier, Tardy '23; Blanchet, Dolbeaut, Perthame '06].

CLT and LDP are still open for the particle system.

Our Goal: Understand the influence of fluctuations on models of aggregation.

Fluctuating hydrodynamics

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - X_t^i)$.

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - X_t^i)$. By a formal application of Itô's formula,

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^N \delta(x - X_t^i)$. By a formal application of Itô's formula,

$$\mathrm{d}\rho_t(x) = \Delta\rho_t(x)\,\mathrm{d}t - \chi\nabla\cdot(\rho_t\nabla K * \rho_t)(x)\,\mathrm{d}t - \frac{\sqrt{2}}{N}\nabla\cdot\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i.$$

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^N \delta(x - X_t^i)$. By a formal application of Itô's formula,

$$\mathrm{d}\rho_t(x) = \Delta\rho_t(x)\,\mathrm{d}t - \chi\nabla\cdot(\rho_t\nabla K * \rho_t)(x)\,\mathrm{d}t - \frac{\sqrt{2}}{N}\nabla\cdot\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i.$$

Problem: Expression not closed.

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - X_t^i)$. By a formal application of Itô's formula,

$$\mathrm{d}\rho_t(x) = \Delta\rho_t(x)\,\mathrm{d}t - \chi\nabla\cdot(\rho_t\nabla K * \rho_t)(x)\,\mathrm{d}t - \frac{\sqrt{2}}{N}\nabla\cdot\sum_{i=1}^{N}\delta(x - X_t^i)\,\mathrm{d}B_t^i.$$

Problem: Expression not closed.

Dean's Trick: The noise $1/\sqrt{N}\sum_{i=1}^{N}\delta(x - X_t^i)\,\mathrm{d}B_t^i$ has the same covariance as $\sqrt{\rho_t(x)}\boldsymbol{\xi}(t,x)$, where $\boldsymbol{\xi}$ is a vector-valued space-time white noise.

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^N \delta(x - X_t^i)$. By a formal application of Itô's formula,

$$\mathrm{d}\rho_t(x) = \Delta\rho_t(x)\,\mathrm{d}t - \chi\nabla\cdot(\rho_t\nabla K * \rho_t)(x)\,\mathrm{d}t - \frac{\sqrt{2}}{N}\nabla\cdot\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i.$$

Problem: Expression not closed.

Dean's Trick: The noise $1/\sqrt{N}\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i$ has the same covariance as $\sqrt{\rho_t(x)}\boldsymbol{\xi}(t,x)$, where $\boldsymbol{\xi}$ is a vector-valued space-time white noise.

Keller–Segel–Dean–Kawasaki equation

$$(\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla K * \rho) - \sqrt{2/N}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}).$$

Fluctuating hydrodynamics

Define empirical density $\rho_t(x) := \frac{1}{N}\sum_{i=1}^N \delta(x - X_t^i)$. By a formal application of Itô's formula,

$$\mathrm{d}\rho_t(x) = \Delta\rho_t(x)\,\mathrm{d}t - \chi\nabla\cdot(\rho_t\nabla K * \rho_t)(x)\,\mathrm{d}t - \frac{\sqrt{2}}{N}\nabla\cdot\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i.$$

Problem: Expression not closed.

Dean's Trick: The noise $1/\sqrt{N}\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i$ has the same covariance as $\sqrt{\rho_t(x)}\boldsymbol{\xi}(t,x)$, where $\boldsymbol{\xi}$ is a vector-valued space-time white noise.

Keller–Segel–Dean–Kawasaki equation

$$(\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla K * \rho) - \sqrt{2/N}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}).$$

Conjecture: Dean–Kawasaki SPDE has same LDP as particle system.

Parametrize KSDK by *noise intensity* $\varepsilon > 0$,

$$(\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho\nabla K * \rho) - \varepsilon^{1/2}\nabla \cdot (\sqrt{\rho}\boldsymbol{\xi}).$$

Parametrize KSDK by *noise intensity* $\varepsilon > 0$,

$$(\partial_t - \Delta)\rho = -\chi \nabla \cdot (\rho \nabla K * \rho) - \varepsilon^{1/2} \nabla \cdot (\sqrt{\rho}\boldsymbol{\xi}).$$

Problem: (KS)DK eqn is unstable:

Parametrize KSDK by *noise intensity* $\varepsilon > 0$,

$$(\partial_t - \Delta)\rho = -\chi \nabla \cdot (\rho \nabla K * \rho) - \varepsilon^{1/2} \nabla \cdot (\sqrt{\rho}\boldsymbol{\xi}).$$

Problem: (KS)DK eqn is unstable:

- if $\varepsilon = 2/N$, then only solution is particle system;

Parametrize KSDK by *noise intensity* $\varepsilon > 0$,

$$(\partial_t - \Delta)\rho = -\chi \nabla \cdot (\rho \nabla K * \rho) - \varepsilon^{1/2} \nabla \cdot (\sqrt{\rho} \boldsymbol{\xi}).$$

Problem: (KS)DK eqn is unstable:

- if $\varepsilon = 2/N$, then only solution is particle system;
- if $\varepsilon \neq 2/N$, then no solution exists [Konarovskyi, Lehmann, von Renesse '19 & '20].

Parametrize KSDK by *noise intensity* $\varepsilon > 0$,

$$(\partial_t - \Delta)\rho = -\chi \nabla \cdot (\rho \nabla K * \rho) - \varepsilon^{1/2} \nabla \cdot (\sqrt{\rho} \boldsymbol{\xi}).$$

Problem: (KS)DK eqn is unstable:

- if $\varepsilon = 2/N$, then only solution is particle system;
- if $\varepsilon \neq 2/N$, then no solution exists [Konarovskyi, Lehmann, von Renesse '19 & '20].

Instead: Use (KS)DK as recipe or template [Dirr, Fehrman, Gess '20; Fehrman, Gess '23], [Cornalba et al '20–'23], [Djurdjevac, Kremp, Perkowski '22].

Propose additive-noise approximation $\rho_{\delta}^{(\varepsilon)}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

Propose additive-noise approximation $\rho_\delta^{(\varepsilon)}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla\cdot(\sigma\boldsymbol{\xi}^\delta) \\ \quad -\Delta\Phi_\rho = \rho - \langle\rho, 1\rangle_{L^2(\mathbb{T}^2)} \\ \quad \rho|_{t=0} = \rho_0 \end{cases}$$

Propose additive-noise approximation $\rho_\delta^{(\varepsilon)}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla\cdot(\sigma\boldsymbol{\xi}^\delta) \\ \qquad -\Delta\Phi_\rho = \rho - \langle\rho, 1\rangle_{L^2(\mathbb{T}^2)} \\ \qquad \rho|_{t=0} = \rho_0 \end{cases}$$

- $\delta > 0$ *correlation length*, mollify noise in space: $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$;

Propose additive-noise approximation $\rho_\delta^{(\varepsilon)}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla\cdot(\sigma\boldsymbol{\xi}^\delta) \\ -\Delta\Phi_\rho = \rho - \langle\rho, 1\rangle_{L^2(\mathbb{T}^2)} \\ \rho|_{t=0} = \rho_0 \end{cases}$$

- $\delta > 0$ *correlation length*, mollify noise in space: $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$;
- $\sigma \in C_T\mathcal{H}^2$ space-time *inhomogeneity*.

Propose additive-noise approximation $\rho_\delta^{(\varepsilon)}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla\cdot(\sigma\boldsymbol{\xi}^\delta) \\ \quad -\Delta\Phi_\rho = \rho - \langle\rho, 1\rangle_{L^2(\mathbb{T}^2)} \\ \quad \rho|_{t=0} = \rho_0 \end{cases}$$

- $\delta > 0$ *correlation length*, mollify noise in space: $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$;
- $\sigma \in C_T\mathcal{H}^2$ space-time *inhomogeneity*.

Can show: Same LLN and CLT as (conjectured for) the particle system, if we choose $\sigma = \sqrt{\rho_{\mathrm{det}}}$ with ρ_{det} the hydrodynamic limit.

Mild formulation

Mild formulation

Denote $P_t := \mathrm{e}^{t\Delta}$ and $\mathcal{I}[f]_t := \int_0^t P_{t-s} f_s \, \mathrm{d}s$.

Mild formulation

Denote $P_t \coloneqq \mathrm{e}^{t\Delta}$ and $\mathcal{I}[f]_t \coloneqq \int_0^t P_{t-s} f_s \,\mathrm{d}s$.

Mild solution

$$\rho_\delta^{(\varepsilon)} = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho_\delta^{(\varepsilon)}\nabla\Phi_{\rho_\delta^{(\varepsilon)}}] - \varepsilon^{1/2}\nabla\cdot\mathcal{I}[\sigma\boldsymbol{\xi}^\delta].$$

Mild formulation

Denote $P_t := \mathrm{e}^{t\Delta}$ and $\mathcal{I}[f]_t := \int_0^t P_{t-s} f_s \, \mathrm{d}s$.

Mild solution

$$\rho_\delta^{(\varepsilon)} = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho_\delta^{(\varepsilon)} \nabla \Phi_{\rho_\delta^{(\varepsilon)}}] - \varepsilon^{1/2} \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta].$$

Driven by additive SHE

$$\Psi^\delta := \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta].$$

Mild formulation

Denote $P_t := \mathrm{e}^{t\Delta}$ and $\mathcal{I}[f]_t := \int_0^t P_{t-s} f_s \,\mathrm{d}s$.

Mild solution

$$\rho_\delta^{(\varepsilon)} = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho_\delta^{(\varepsilon)} \nabla \Phi_{\rho_\delta^{(\varepsilon)}}] - \varepsilon^{1/2} \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta].$$

Driven by additive SHE

$$\text{¡}^\delta := \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta].$$

Behaviour of ¡^δ as $\delta \to 0$ determines behaviour of $\rho_\delta^{(\varepsilon)}$ as $\varepsilon, \delta \to 0$.

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\dot{\mathbf{I}}^\delta\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1 + \delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^\gamma}.$$

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^{\gamma}$. Then

$$\mathbb{E}[\|{}^{\delta}\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1 + \delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^{\gamma}}.$$

Example. Choose $\gamma' = 0$, obtain ${}^{\delta} \in C_T L^2$ with blow-up $1 + \delta^{-1-}$.

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\Psi^\delta\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1 + \delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^\gamma}.$$

Example. Choose $\gamma' = 0$, obtain $\Psi^\delta \in C_T L^2$ with blow-up $1 + \delta^{-1-}$.

Intuition. Recall $\Psi^\delta = \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta]$.

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T \mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\text{\textbf{¡}}^{\delta}\|^2_{C_T \mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1 + \delta^{-1-\gamma}) \|\sigma\|_{C_T \mathcal{H}^\gamma}.$$

Example. Choose $\gamma' = 0$, obtain $\text{\textbf{¡}}^{\delta} \in C_T L^2$ with blow-up $1 + \delta^{-1-}$.

Intuition. Recall $\text{\textbf{¡}}^{\delta} = \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta]$. Space-time white noise has parabolic regularity $\boldsymbol{\xi} \in \mathcal{C}_{\mathfrak{s}}^{-2-}([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ with scaling $\mathfrak{s} = (2, 1, 1)$.

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\text{⫯}^\delta\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1 + \delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^\gamma}.$$

Example. Choose $\gamma' = 0$, obtain $\text{⫯}^\delta \in C_T L^2$ with blow-up $1 + \delta^{-1-}$.

Intuition. Recall $\text{⫯}^\delta = \nabla \cdot \mathcal{I}[\sigma\boldsymbol{\xi}^\delta]$. Space-time white noise has parabolic regularity $\boldsymbol{\xi} \in \mathcal{C}_\mathfrak{s}^{-2-}([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ with scaling $\mathfrak{s} = (2,1,1)$. This means it has

$$-1/2 - \text{ regularity in time} \qquad -1 \text{ regularity in space.}$$

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^{\gamma}$. Then

$$\mathbb{E}[\|⫯^{\delta}\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1+\delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^{\gamma}}.$$

Example. Choose $\gamma' = 0$, obtain $⫯^{\delta} \in C_T L^2$ with blow-up $1+\delta^{-1-}$.

Intuition. Recall $⫯^{\delta} = \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^{\delta}]$. Space-time white noise has parabolic regularity $\boldsymbol{\xi} \in \mathcal{C}_{\mathfrak{s}}^{-2-}([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ with scaling $\mathfrak{s} = (2,1,1)$. This means it has

$-1/2-$ regularity in time $\qquad -1$ regularity in space.

Use smoothing in space $\boldsymbol{\xi}^{\delta} = \psi_{\delta} * \boldsymbol{\xi}$ to add $1+\gamma$ regularity in space, pay with a blow-up of $1+\delta^{-1-\gamma}$.

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\mathbf{I}^\delta\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1+\delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^\gamma}.$$

Example. Choose $\gamma' = 0$, obtain $\mathbf{I}^\delta \in C_T L^2$ with blow-up $1+\delta^{-1-}$.

Intuition. Recall $\mathbf{I}^\delta = \nabla \cdot \mathcal{I}[\sigma\boldsymbol{\xi}^\delta]$. Space-time white noise has parabolic regularity $\boldsymbol{\xi} \in \mathcal{C}_{\mathfrak{s}}^{-2-}([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ with scaling $\mathfrak{s} = (2,1,1)$. This means it has

$$-1/2- \text{ regularity in time} \qquad -1 \text{ regularity in space.}$$

Use smoothing in space $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$ to add $1+\gamma$ regularity in space, pay with a blow-up of $1+\delta^{-1-\gamma}$. Obtain

$$-1/2- \text{ regularity in time} \qquad \gamma \text{ regularity in space.}$$

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\text{¡}^{\delta}\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1+\delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^\gamma}.$$

Example. Choose $\gamma' = 0$, obtain $\text{¡}^{\delta} \in C_T L^2$ with blow-up $1+\delta^{-1-}$.

Intuition. Recall $\text{¡}^{\delta} = \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}^\delta]$. Space-time white noise has parabolic regularity $\boldsymbol{\xi} \in \mathcal{C}_{\mathfrak{s}}^{-2-}([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ with scaling $\mathfrak{s} = (2,1,1)$. This means it has

$-1/2-$ regularity in time $\qquad -1$ regularity in space.

Use smoothing in space $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$ to add $1+\gamma$ regularity in space, pay with a blow-up of $1+\delta^{-1-\gamma}$. Obtain

$-1/2-$ regularity in time $\qquad \gamma$ regularity in space.

Spend 1+ smoothing of $\mathcal{I}$ to establish continuity in time

Lemma. Let $T > 0$, $\gamma > 0$, $\gamma' < \gamma$, $\delta > 0$ and $\sigma \in C_T\mathcal{H}^\gamma$. Then

$$\mathbb{E}[\|\overset{\bullet}{|}^{\delta}\|^2_{C_T\mathcal{H}^{\gamma'}}]^{1/2} \lesssim (1+\delta^{-1-\gamma})\|\sigma\|_{C_T\mathcal{H}^\gamma}.$$

Example. Choose $\gamma' = 0$, obtain $\overset{\bullet}{|}^{\delta} \in C_T L^2$ with blow-up $1 + \delta^{-1-}$.

Intuition. Recall $\overset{\bullet}{|}^{\delta} = \nabla \cdot \mathcal{I}[\sigma\boldsymbol{\xi}^\delta]$. Space-time white noise has parabolic regularity $\boldsymbol{\xi} \in \mathcal{C}_{\mathfrak{s}}^{-2-}([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ with scaling $\mathfrak{s} = (2,1,1)$. This means it has

$$-1/2- \text{ regularity in time} \qquad -1 \text{ regularity in space.}$$

Use smoothing in space $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$ to add $1+\gamma$ regularity in space, pay with a blow-up of $1+\delta^{-1-\gamma}$. Obtain

$$-1/2- \text{ regularity in time} \qquad \gamma \text{ regularity in space.}$$

Spend 1+ smoothing of $\mathcal{I}$ to establish continuity in time and remaining 1− smoothing of $\mathcal{I}$ to cancel (most of) $\nabla\cdot$.

‘Regular’ LDP

‘Regular’ LDP

The variational approach based on the Boué–Dupuis formula and the contraction principle yield:

‘Regular’ LDP

The variational approach based on the Boué–Dupuis formula and the contraction principle yield:

Theorem. Let $T > 0$, $\gamma > 0$ and $\sigma \in C_T L^\infty \cap C_T \mathcal{H}^\gamma$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-1} \to 0$ as $\varepsilon \to 0$.

‘Regular’ LDP

The variational approach based on the Boué–Dupuis formula and the contraction principle yield:

Theorem. Let $T > 0$, $\gamma > 0$ and $\sigma \in C_T L^\infty \cap C_T \mathcal{H}^\gamma$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-1} \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T L^2$ with rate ε and good rate function

$$\mathscr{I}(\rho) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = -\chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] + \nabla\cdot\mathcal{I}[\sigma h]\Big\}.$$

Observe: Regularity $⫯^{\delta} \in C_T\mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Observe: Regularity $^{\delta} \in C_T\mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish ‘natural regularity’, resp. uniform bound

Observe: Regularity $\text{¡}^{\delta} \in C_T\mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish ‘natural regularity’, resp. uniform bound

$$\mathbb{E}[\|\text{¡}^{\delta}\|^2_{C_T\mathcal{C}^{-1-}}]^{1/2} \lesssim \|\sigma\|_{C_T L^\infty},$$

Observe: Regularity $\text{¡}^{\delta} \in C_T \mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish ‘natural regularity’, resp. uniform bound

$$\mathbb{E}[\|\text{¡}^{\delta}\|^2_{C_T \mathcal{C}^{-1-}}]^{1/2} \lesssim \|\sigma\|_{C_T L^\infty},$$

which corresponds to the choice $\gamma = -1$ in the previous lemma (with a different proof).

Observe: Regularity $\mathbf{I}^{\delta} \in C_T\mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish ‘natural regularity’, resp. uniform bound

$$\mathbb{E}[\|\mathbf{I}^{\delta}\|^2_{C_T\mathcal{C}^{-1-}}]^{1/2} \lesssim \|\sigma\|_{C_T L^\infty},$$

which corresponds to the choice $\gamma = -1$ in the previous lemma (with a different proof).

Idea: Solve the mild equation

Observe: Regularity $\Psi^{\delta} \in C_T \mathcal{H}^{\gamma-}$ is 'unnatural', as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish 'natural regularity', resp. uniform bound

$$\mathbb{E}[\|\Psi^{\delta}\|^2_{C_T \mathcal{C}^{-1-}}]^{1/2} \lesssim \|\sigma\|_{C_T L^\infty},$$

which corresponds to the choice $\gamma = -1$ in the previous lemma (with a different proof).

Idea: Solve the mild equation

$$\rho^{(\varepsilon)}_{\delta} = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho^{(\varepsilon)}_{\delta} \nabla \Phi_{\rho^{(\varepsilon)}_{\delta}}] - \varepsilon^{1/2} \Psi^{\delta}$$

Observe: Regularity $\mathbf{I}^{\delta} \in C_T\mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish ‘natural regularity’, resp. uniform bound

$$\mathbb{E}[\|\mathbf{I}^{\delta}\|^2_{C_T\mathcal{C}^{-1-}}]^{1/2} \lesssim \|\sigma\|_{C_T L^\infty},$$

which corresponds to the choice $\gamma = -1$ in the previous lemma (with a different proof).

Idea: Solve the mild equation

$$\rho^{(\varepsilon)}_\delta = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho^{(\varepsilon)}_\delta \nabla\Phi_{\rho^{(\varepsilon)}_\delta}] - \varepsilon^{1/2}\mathbf{I}^{\delta}$$

using only $\mathbf{I}^{\delta} \in C_T\mathcal{C}^{-1-}$.

Observe: Regularity $\Psi^{\delta} \in C_T\mathcal{H}^{\gamma-}$ is ‘unnatural’, as seen by the blow-up $1 + \delta^{-1-\gamma}$.

Can establish ‘natural regularity’, resp. uniform bound

$$\mathbb{E}[\|\Psi^{\delta}\|^2_{C_T\mathcal{C}^{-1-}}]^{1/2} \lesssim \|\sigma\|_{C_T L^\infty},$$

which corresponds to the choice $\gamma = -1$ in the previous lemma (with a different proof).

Idea: Solve the mild equation

$$\rho_\delta^{(\varepsilon)} = P\rho_0 - \chi\nabla \cdot \mathcal{I}[\rho_\delta^{(\varepsilon)}\nabla\Phi_{\rho_\delta^{(\varepsilon)}}] - \varepsilon^{1/2}\Psi^{\delta}$$

using only $\Psi^{\delta} \in C_T\mathcal{C}^{-1-}$.

$\rightsquigarrow$ ‘Rough’ LDP under more general scaling assumption on $(\varepsilon, \delta(\varepsilon))$.

‘Rough’ LDP

‘Rough’ LDP

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

‘Rough’ LDP

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Idea of Proof: Decompose equation with the Da Prato–Debussche trick and paracontrolled Ansatz [Gubinelli, Imkeller, Perkowski ’15].

Contributions

Contributions

- Established the local well-posedness of an additive noise approximation to Keller–Segel–Dean–Kawasaki dynamics.

Contributions

- Established the local well-posedness of an additive noise approximation to Keller–Segel–Dean–Kawasaki dynamics.
- Established an LDP in regular spaces under restrictive scaling.

Contributions

- Established the local well-posedness of an additive noise approximation to Keller–Segel–Dean–Kawasaki dynamics.
- Established an LDP in regular spaces under restrictive scaling.
- Established an LDP in rough spaces under general scaling.

Contributions

- Established the local well-posedness of an additive noise approximation to Keller–Segel–Dean–Kawasaki dynamics.
- Established an LDP in regular spaces under restrictive scaling.
- Established an LDP in rough spaces under general scaling.

A.M. & Avi Mayorcas. *An Additive Noise Approximation to Keller-Segel-Dean-Kawasaki Dynamics Part I: Local Well-Posedness of Paracontrolled Solutions*. arXiv:2207.10711.

— *Part II: Small-Noise Results*. On arXiv in ’23.

Contributions

- Established the local well-posedness of an additive noise approximation to Keller–Segel–Dean–Kawasaki dynamics.
- Established an LDP in regular spaces under restrictive scaling.
- Established an LDP in rough spaces under general scaling.

A.M. & Avi Mayorcas. *An Additive Noise Approximation to Keller-Segel-Dean-Kawasaki Dynamics Part I: Local Well-Posedness of Paracontrolled Solutions*. arXiv:2207.10711.

— *Part II: Small-Noise Results*. On arXiv in '23.

I am in the academic job market for autumn '24 ;)

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Sketch of Proof:

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Sketch of Proof:

1. Apply Da Prato–Debussche trick, define remainder $u := \rho + \varepsilon^{1/2}\text{¡}^{\delta}$;

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Sketch of Proof:

1. Apply Da Prato–Debussche trick, define remainder $u := \rho + \varepsilon^{1/2}\Psi^{\delta}$;
2. Remainder solves equation (for $\chi = 1$)

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Sketch of Proof:

1. Apply Da Prato–Debussche trick, define remainder $u := \rho + \varepsilon^{1/2}\Psi^\delta$;
2. Remainder solves equation (for $\chi = 1$)

$$u = P\rho_0 - \nabla\cdot\mathcal{I}[u\nabla\Phi_u] + \varepsilon^{1/2}\nabla\cdot\mathcal{I}[u\nabla\Phi_{\Psi^\delta}] + \varepsilon^{1/2}\nabla\cdot\mathcal{I}[\Psi^\delta\nabla\Phi_u] - \varepsilon\nabla\cdot\mathcal{I}[\Psi^\delta\nabla\Phi_{\Psi^\delta}];$$

Theorem. Let $T > 0$ and $\sigma \in C_T\mathcal{H}^2$. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then, the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{C}^{-1-}$ with rate ε and good rate function $\mathscr{I}$.

Sketch of Proof:

1. Apply Da Prato–Debussche trick, define remainder $u := \rho + \varepsilon^{1/2}\text{¡}^{\delta}$;
2. Remainder solves equation (for $\chi = 1$)

$$u = P\rho_0 - \nabla\cdot\mathcal{I}[u\nabla\Phi_u] + \varepsilon^{1/2}\nabla\cdot\mathcal{I}[u\nabla\Phi_{\text{¡}^{\delta}}] + \varepsilon^{1/2}\nabla\cdot\mathcal{I}[\text{¡}^{\delta}\nabla\Phi_u] - \varepsilon\nabla\cdot\mathcal{I}[\text{¡}^{\delta}\nabla\Phi_{\text{¡}^{\delta}}];$$

3. To solve equation, need to define the stochastic term $\mathsf{Y}^{\delta}_{\mathrm{can}} := \nabla\cdot\mathcal{I}[\text{¡}^{\delta}\nabla\Phi_{\text{¡}^{\delta}}]$ and the singular product $u\nabla\Phi_{\text{¡}^{\delta}} + \text{¡}^{\delta}\nabla\Phi_u$;

4. Construct $\mathbf{Y}^{\delta}_{\mathrm{can}}$ with Itô’s formula;

4. Construct $\mathbf{Y}^{\delta}_{\mathrm{can}}$ with Itô’s formula;
5. Construct $u\nabla\Phi_{\mathbf{I}^{\delta}} + \mathbf{I}^{\delta}\nabla\Phi_{u}$ with another application of Da Prato–Debussche, followed by paracontrolled calculus [Gubinelli–Imkeller–Perkowski ’15].

Q: Why the log blow-up?

Q: Why the log blow-up?

Recall $^{\delta} \in C_T \mathcal{C}^{-1-}$ with uniform bound as $\delta \to 0$.

Q: Why the log blow-up?

Recall $\overset{\bullet}{|}^{\delta} \in C_T \mathcal{C}^{-1-}$ with uniform bound as $\delta \to 0$.

Power counting:

Q: Why the log blow-up?

Recall $\overset{\bullet}{|}^{\delta} \in C_T \mathcal{C}^{-1-}$ with uniform bound as $\delta \to 0$.

Power counting:

$$\overset{\bullet\,\bullet}{\curlyvee}{}^{\delta}_{\mathrm{can}} := \nabla \cdot \mathcal{I}\,[\underbrace{\overset{\bullet}{|}^{\delta}}_{-1-}\ \underbrace{\nabla\Phi_{\overset{\bullet}{|}^{\delta}}}_{0-}\,].$$

Q: Why the log blow-up?

Recall $\Psi^{\delta} \in C_T\mathcal{C}^{-1-}$ with uniform bound as $\delta \to 0$.

Power counting:

$$\Upsilon^{\delta}_{\mathrm{can}} \coloneqq \nabla \cdot \mathcal{I}\,[\underbrace{\Psi^{\delta}}_{-1-}\,\underbrace{\nabla\Phi_{\Psi^{\delta}}}_{0-}\,].$$

Gap in regularity is $-1-$; suggests power-law divergence.

Idea: Use product rule.

Idea: Use product rule.

Analogue in one dimension:

Idea: Use product rule.

Analogue in one dimension:

$$\underbrace{f^\delta}_{-1/2-} \underbrace{\partial_x \partial_x^{-2} f^\delta}_{1/2-} = \frac{1}{2}\Big(\partial_x^2 (\underbrace{\partial_x^{-1} f^\delta}_{1/2-} \underbrace{\partial_x^{-2} f^\delta}_{3/2-}) - \partial_x (\underbrace{f^\delta}_{-1/2-} \underbrace{\partial_x^{-2} f^\delta}_{3/2-})\Big).$$

Idea: Use product rule.

Analogue in one dimension:

$$\underbrace{f^\delta}_{-1/2-}\,\underbrace{\partial_x \partial_x^{-2} f^\delta}_{1/2-} = \frac{1}{2}\Big(\partial_x^2 \big(\underbrace{\partial_x^{-1} f^\delta}_{1/2-}\,\underbrace{\partial_x^{-2} f^\delta}_{3/2-}\big) - \partial_x \big(\underbrace{f^\delta}_{-1/2-}\,\underbrace{\partial_x^{-2} f^\delta}_{3/2-}\big)\Big).$$

gap 0–

no gaps

Idea: Use product rule.

Analogue in one dimension:

$$\underbrace{f^\delta}_{-1/2-} \underbrace{\partial_x \partial_x^{-2} f^\delta}_{1/2-} = \frac{1}{2}\Big(\partial_x^2(\underbrace{\partial_x^{-1} f^\delta}_{1/2-} \underbrace{\partial_x^{-2} f^\delta}_{3/2-}) - \partial_x(\underbrace{f^\delta}_{-1/2-} \underbrace{\partial_x^{-2} f^\delta}_{3/2-})\Big).$$

gap 0– no gaps

In higher dimensions: replace with symmetry of $\mathcal{F}(\nabla\Phi)(\omega) = \frac{2\pi i\omega}{|2\pi\omega|^2}$.

$$
\begin{aligned}
H^j_t(\omega) &:= 2\pi \mathrm{i}\omega^j \exp(-t|2\pi\omega|^2)\mathbb{1}_{t\geq 0}, \quad \text{Fourier multiplier of } \nabla\mathcal{I} \\
G^j(\omega) &:= 2\pi \mathrm{i}\omega^j |2\pi \mathrm{i}\omega|^{-2}\mathbb{1}_{\omega\neq 0}, \quad \text{Fourier multiplier of } \nabla\Phi,\ \omega\in\mathbb{Z}^2,\ j=1,2, t\in\mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
&\widehat{\bullet^{\delta}}(t,\omega) \\
&:= \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \sum_{j_1,j_3=1}^{2} \sum_{m_1\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_1 \widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_1,\omega_2+m_1)|\varphi(\delta m_1)|^2 \\
&\qquad\qquad \times H^{j_3}_{t-u_3}(\omega)H^{j_1}_{u_3-u_1}(\omega_1)H^{j_1}_{u_3-u_1}(\omega_2)G^{j_3}(\omega_2) \\
&= \frac{1}{2} \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \sum_{j_1,j_3=1}^{2} \sum_{m_1\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_1 \widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_1,\omega_2+m_1)|\varphi(\delta m_1)|^2 \\
&\qquad\qquad \times H^{j_3}_{t-u_3}(\omega)H^{j_1}_{u_3-u_1}(\omega_1)H^{j_1}_{u_3-u_1}(\omega_2)(G^{j_3}(\omega_1)+G^{j_3}(\omega_2)).
\end{aligned}
$$

$$
G^{j_3}(\omega_1)+G^{j_3}(\omega_2) = G^{j_3}(\omega-\omega_2) - G^{j_3}(-\omega_2).
$$

$$u^\delta \partial_x \partial_x^{-2} u^\delta = \frac{1}{2}(\partial_x^2(\partial_x^{-1} u^\delta \partial_x^{-2} u^\delta) - \partial_x(u^\delta \partial_x^{-2} u^\delta)).$$

$$H_t^j(\omega) := 2\pi \mathrm{i} \omega^j \exp(-t|2\pi\omega|^2) \mathbb{1}_{t \geq 0}, \quad \text{Fourier multiplier of } \nabla \mathcal{I}$$
$$G^j(\omega) := 2\pi \mathrm{i} \omega^j |2\pi \mathrm{i} \omega|^{-2} \mathbb{1}_{\omega \neq 0}, \quad \text{Fourier multiplier of } \nabla \Phi, \ \omega \in \mathbb{Z}^2, \ j = 1, 2, t \in \mathbb{R}$$

We bound

$$\begin{aligned} \left| \frac{\omega - \omega_2}{|\omega - \omega_2|^2} - \frac{-\omega_2}{|-\omega_2|^2} \right| &= \frac{|\omega |\omega_2|^2 + |\omega|^2 \omega_2 - 2\langle \omega, \omega_2 \rangle \omega_2|}{|\omega - \omega_2|^2 |\omega_2|^2} \\ &\lesssim |\omega|^2 |\omega_2|^{-1} |\omega - \omega_2|^{-2} + |\omega| |\omega - \omega_2|^{-2}. \end{aligned}$$

Assume $\sigma \equiv 1$ so that $\widehat{\sigma}(u_1, \omega) = \mathbb{1}_{\omega=0}$, then

$$\begin{aligned}
&\widehat{\mathbf{o}^{\delta}}(t, \omega, j) \\
&= \frac{1}{2} \sum_{\substack{\omega_1, \omega_2 \in \mathbb{Z}^2 \\ \omega = \omega_1 + \omega_2}} \sum_{j_1=1}^{2} \sum_{m_1 \in \mathbb{Z}^2} \int_0^t \mathrm{d}u_1 \widehat{\sigma}(u_1, \omega_1 - m_1) \widehat{\sigma}(u_1, \omega_2 + m_1) |\varphi(\delta m_1)|^2 \\
&\qquad\qquad \times H_{t-u_1}^{j_1}(\omega_1) H_{t-u_1}^{j_1}(\omega_2) (G^j(\omega_1) + G^j(\omega_2)) \\
&= \mathbb{1}_{\omega=0} \frac{1}{2} \sum_{\substack{\omega_1, \omega_2 \in \mathbb{Z}^2 \\ 0 = \omega_1 + \omega_2}} \sum_{j_1=1}^{2} \int_0^t \mathrm{d}u_1 |\varphi(\delta \omega_1)|^2 H_{t-u_1}^{j_1}(\omega_1) H_{t-u_1}^{j_1}(\omega_2) (G^j(\omega_1) + G^j(\omega_2)).
\end{aligned}$$

We use the symmetry of G (resp. $\nabla\Phi$) to deduce

$$G^j(\omega_1) + G^j(\omega_2) = G^j(\omega_1) + G^j(-\omega_1) = G^j(\omega_1) - G^j(\omega_1) = 0$$

hence $\widehat{\mathbf{o}^{\delta}}(t, \omega, j) = 0$.