

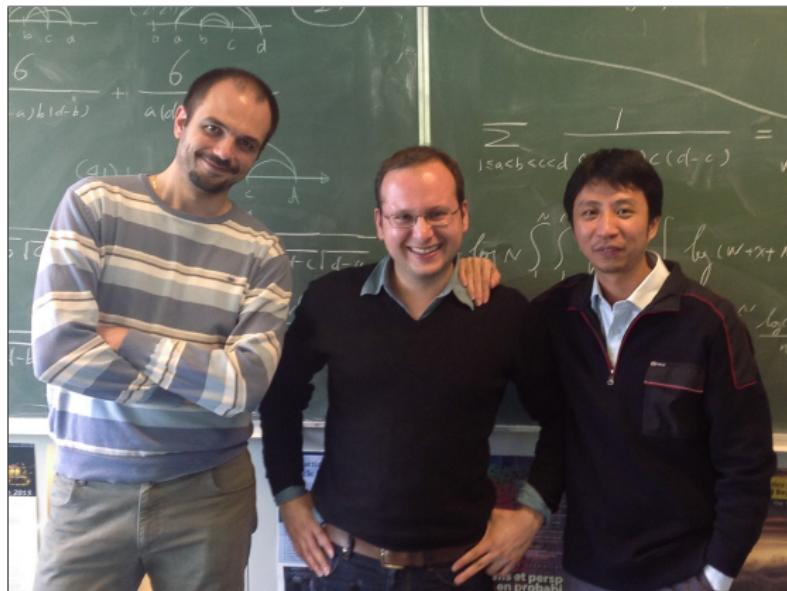
# Polynomial Chaos and Scaling Limits of Disordered Systems

Francesco Caravenna

Università degli Studi di Milano-Bicocca

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# Coworkers



Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

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Inspired by recent work of Alberts, Quastel and Khanin on DPRE

# Outline

1. Disordered Systems and their Scaling Limits
2. Main Results (I): Partition Function
3. Main Results (II): Continuum Disordered Pinning Model
4. Further Developments

# General Framework

Lattice  $\Omega \subseteq \mathbb{R}^d$     “spins”  $\sigma = (\sigma_x)_{x \in \Omega} \in \{0, 1\}^\Omega$  or  $\{-1, +1\}^\Omega$

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## Disordered law

Random Gibbs measure on spin configurations  $\sigma$ , indexed by disorder  $\omega$

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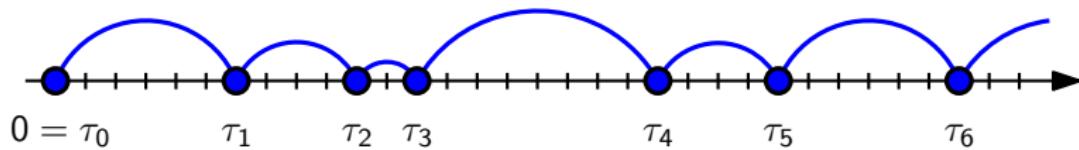
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$$\mathbf{P}_{\Omega, \lambda, h}^\omega(\sigma) := \frac{1}{Z_{\Omega, \lambda, h}^\omega} \exp \left( \sum_{x \in \Omega} (\lambda\omega_x + h)\sigma_x \right) \mathbf{P}_\Omega^{\text{ref}}(\sigma)$$

Partition function  $Z_{\Omega, \lambda, h}^\omega = \mathbf{E}_\Omega^{\text{ref}}[e^{\sum_{x \in \Omega} (\lambda\omega_x + h)\sigma_x}]$

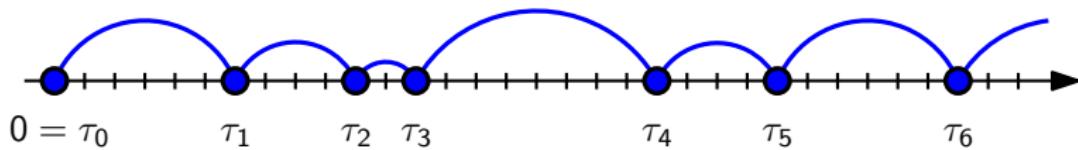
# 1. Disordered pinning model



Reference law: **renewal process**  $\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \dots\} \subseteq \mathbb{N}_0$

$$\mathbf{P}^{\text{ref}}((\tau_{i+1} - \tau_i) = n) \sim \frac{C}{n^{1+\alpha}}, \quad \text{tail exponent } \alpha \in (0, 1)$$

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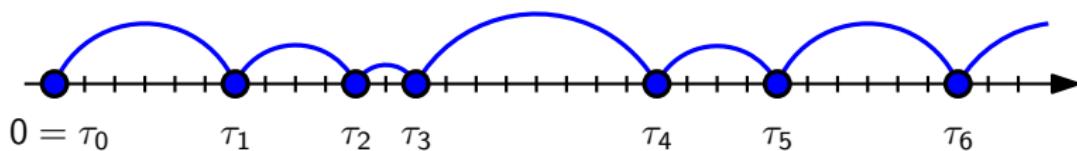


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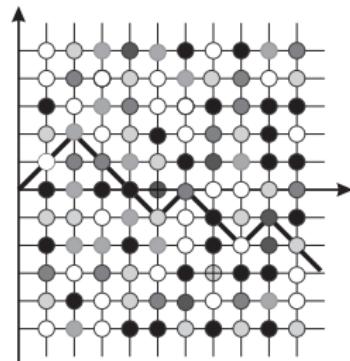
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## 2. Directed polymer in random environment



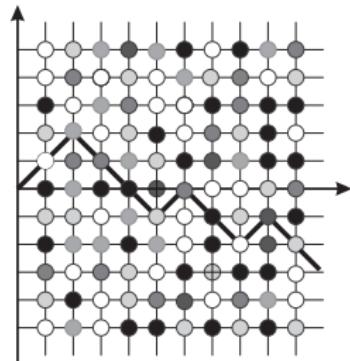
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$X = (X_n)_{n \geq 0}$  on  $\mathbb{Z}$ , in the domain of attraction  
of a **stable Lévy process** with index  $\alpha \in (0, 2]$

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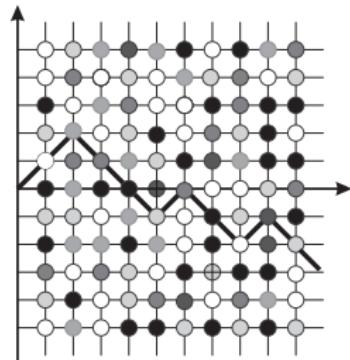
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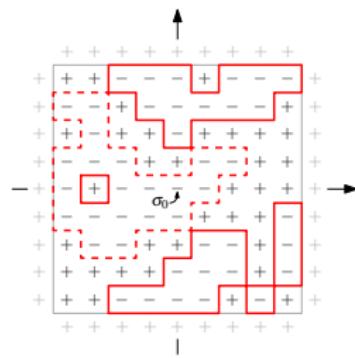
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Reference law: **critical** 2d Ising model with “+” boundary conditions



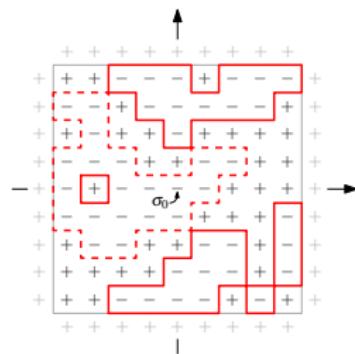
Lattice  $\Omega := \{-N, \dots, N\} \times \{-N, \dots, N\}$

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where  $\sigma_x = \pm 1$ ,  $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$

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Fix  $\Omega \subset \mathbb{R}^d$  bounded open with smooth boundary, and consider the lattice

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i.e. rescale space by a factor  $\delta > 0$  (in the examples  $\delta = \frac{1}{N}$ )

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Does the disordered model  $\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega$  admit a non-trivial continuum limit?

# A direct approach?

Recall the definition of the (discrete) disordered law:

$$P_{\Omega_\delta, \lambda, h}^\omega(d\sigma) \propto \exp\left(\sum_{x \in \Omega_\delta} (\lambda \omega_x + h) \sigma_x\right) P_{\Omega_\delta}^{\text{ref}}(d\sigma)$$

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Difficulty is substantial:  $\mathcal{P}_{\Omega, \lambda, h}^{\omega}$  can be singular w.r.t.  $\mathcal{P}_{\Omega}^{\text{ref}}$  !

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# The partition function

The disordered system  $P_{\Omega_\delta, \lambda, h}^{\omega}$  is a difficult object (a random probability)

Let us be less ambitious and focus on the partition function

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YES, for Pinning and DPRE (and hopefully for Ising too)

# Assumptions

*k*-point function  $\mathbf{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]$  defined on  $(\Omega_\delta)^k \rightsquigarrow$  extended on  $\Omega^k$

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$$\exists \gamma > 0 : \quad \frac{\mathbf{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]}{(\delta^\gamma)^k} \xrightarrow[\delta \downarrow 0]{\text{in } L^2(\Omega^k)} \psi_\Omega^{(k)}(x_1, \dots, x_k) \quad (\star)$$

$\forall k \in \mathbb{N}$ .

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The *k*-point functions of  $\mathbf{P}_{\Omega_\delta}^{\text{ref}}$  converge in  $L^2$  under polynomial rescaling

$$\exists \gamma > 0 : \quad \frac{\mathbf{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]}{(\delta^\gamma)^k} \xrightarrow[\delta \downarrow 0]{\text{in } L^2(\Omega^k)} \psi_\Omega^{(k)}(x_1, \dots, x_k) \quad (*)$$

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$L^2$  convergence then requires that

$$\boxed{\gamma < \frac{d}{2}}$$

# Main result (I): partition function

Theorem [C., Sun, Zygouras '13]

Assume that  $\mathbf{P}_{\Omega_\delta}^{\text{ref}}$  satisfies  $(\star)$  with exponent  $\gamma$  (and dimension  $d$ )

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- Case  $\sigma_x \in \{-1, 1\}$ . The same, up to minor modifications (cf. below)

# Motivating models: Pinning and DPRE

- ▶ **Pinning.** Dimension  $d = 1$ , exponent  $\gamma = 1 - \alpha$ ,

$$\psi_{\Omega}^{(k)}(x_1, \dots, x_k) = \frac{c^k}{x_1^{1-\alpha}(x_2 - x_1)^{1-\alpha} \cdots (x_k - x_{k-1})^{1-\alpha}}$$

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These restrictions are not technical, but substantial (physical)!

# Motivating models: Ising

Pointwise convergence of  $k$ -point function, with exponent  $\gamma = \frac{1}{8}$ , toward

$\psi_{\Omega}^{(k)}(x_1, \dots, x_k)$  conformally covariant,

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We fix continuous functions  $\hat{\lambda} : \overline{\Omega} \rightarrow (0, \infty)$  and  $\hat{h} : \overline{\Omega} \rightarrow \mathbb{R}$  and set

$$\lambda = \hat{\lambda}(x) \delta^{7/8} \quad h = \hat{h}(x) \delta^{15/8}$$

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Theorem [C., Sun, Zygouras '13]

As  $\delta \downarrow 0$  one has the convergence in law

$$e^{-\frac{1}{2}\|\hat{\lambda}\|_2^2\delta^{-1/4}}Z_{\Omega_\delta,\lambda,h}^{\omega} \implies Z_{\Omega;\hat{\lambda},\hat{h}}^W$$

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Conformal covariance: if  $\phi : \tilde{\Omega} \rightarrow \Omega$  is a conformal map,

$$Z_{\Omega;\hat{\lambda},\hat{h}}^W \stackrel{dist.}{=} Z_{\tilde{\Omega};\tilde{\lambda},\tilde{h}}^W$$

where  $\tilde{\lambda}(x) := |\phi'(x)|^{7/8} \hat{\lambda}(\phi(x))$  and  $\tilde{h}(x) := |\phi'(x)|^{15/8} \hat{h}(\phi(x))$

# Sketch of the proof

1. Linearization. Since  $\sigma_x \in \{0, 1\}$ , every function of  $\sigma_x$  is linear

$$Z_{\Omega_\delta, \lambda, h}^{\omega} = \mathbf{E}_{\Omega_\delta}^{\text{ref}} \left[ \prod_{x \in \Omega_\delta} e^{(\lambda \omega_x + h) \sigma_x} \right] = \mathbf{E}_{\Omega_\delta}^{\text{ref}} \left[ \prod_{x \in \Omega_\delta} (1 + \epsilon_x \sigma_x) \right]$$

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Partition function is a multilinear polynomial of the random variables  $\epsilon_x$ , with coefficient given by the  $k$ -point functions of  $\mathbf{P}^{\text{ref}}$

# Sketch of the proof

3. Lindeberg principle, extending [Mossel, O'Donnell, Oleszkiewicz '10]

The law of a multilinear polynomial is insensitive toward the distribution of the  $\epsilon_x$  (keeping same mean and variance)  $\rightsquigarrow$  independent Gaussians

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## 4. Wiener chaos expansion. Plugging the assumption

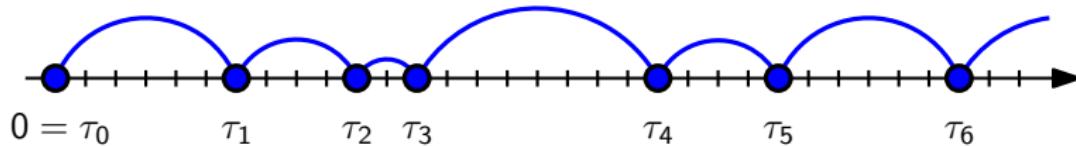
$$\mathbf{E}_{\Omega_\delta}^{\text{ref}} [\sigma_{x_1} \cdots \sigma_{x_k}] \simeq (\delta^\gamma)^k \psi_{\Omega}^{(k)}(x_1, \dots, x_k)$$

yields a Wiener chaos expansion with  $\hat{\lambda} = \lambda \delta^{\gamma - \frac{d}{2}}$  and  $\hat{h} = h' \delta^{\gamma - d}$

# Outline

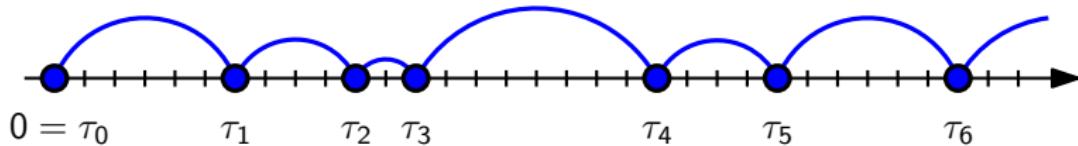
1. Disordered Systems and their Scaling Limits
2. Main Results (I): Partition Function
3. Main Results (II): Continuum Disordered Pinning Model
4. Further Developments

# Back to pinning models



$\tau = \{\tau_0 < \tau_1 < \tau_2 < \dots\}$  random element of  $E := \{\text{closed subsets of } \mathbb{R}\}$

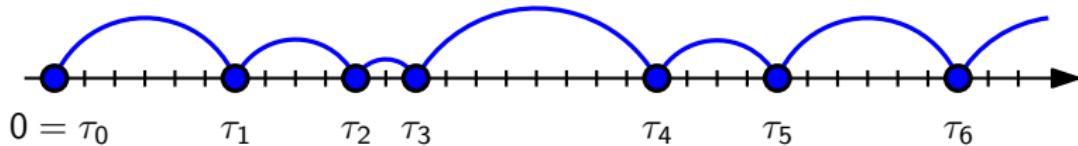
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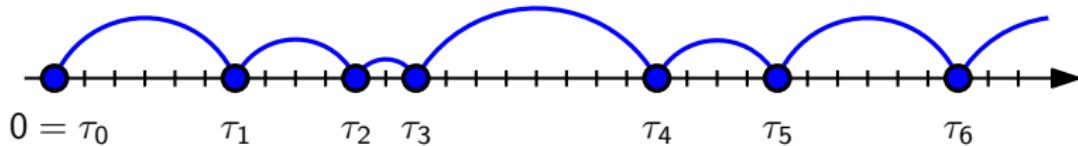


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What happens for the disordered model  $\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega$ ?  $(\Omega = (0, 1))$

Restrict  $\alpha \in (\frac{1}{2}, 1)$ . Fix  $\hat{\lambda} > 0$ ,  $\hat{h} \in \mathbb{R}$  and set

$$\lambda := \hat{\lambda} \delta^{\alpha - \frac{1}{2}} \quad h := \hat{h} \delta^\alpha - \frac{1}{2} \lambda^2$$

# Continuum Disordered Pinning Model [C., Sun, Zygouras '14]

$E := \{\text{closed subsets of } \mathbb{R}\}$  equipped with the Hausdorff distance

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As  $\delta \downarrow 0$ , the rescaled discrete set  $(\delta\tau, \mathbf{P}_{\Omega_\delta, \lambda, h}^\omega)$  converges in distribution on  $E$  to a universal random closed set  $(\tau, \mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W)$ , called CDPM

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The CDPM has any a.s. property of the  $\alpha$ -stable regenerative set  $\mathcal{P}^{\text{ref}}$

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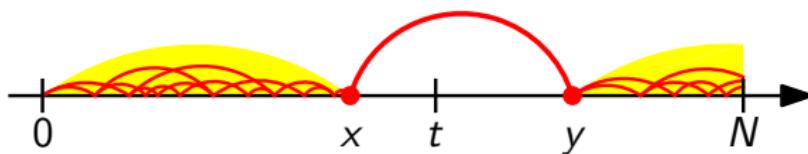
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## Theorem (singularity)

The CDPM  $\mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W$  law is singular w.r.t.  $\mathcal{P}^{\text{ref}}$  for  $\mathbb{P}$ -a.e.  $W$

# Construction strategy

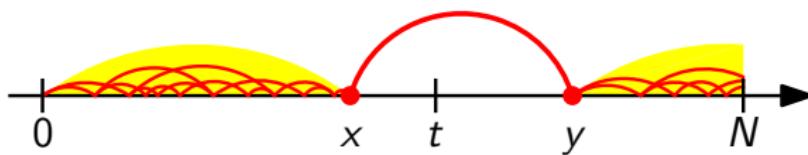
Macroscopic observables ([finite-dimensional distributions](#)) expressed using partition functions with suitable boundary conditions



$$\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega(\dots) = \frac{Z_{0,x}^{\text{cond}} \frac{C}{(y-x)^{1+\alpha}} Z_{y,N}}{Z_{0,N}}$$

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Scaling limit (at the process level) of  $(Z_{x,y}^{\text{cond}}, Z_{x,y})_{0 \leq x < y \leq N}$   $\rightsquigarrow$   
Definition of CDPM via “finite-dimensional distributions”

The same can be done for DPRE, cf. [Alberts, Khanin, Quastel '12]

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Possible alternative approach: define continuum disordered law  $\mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W$  assigning its  $k$ -point function  $\mathcal{E}_{\Omega, \hat{\lambda}, \hat{h}}^W[\sigma_{x_1} \cdots \sigma_{x_k}]$ ?

A generalization of our theorem about the scaling limit of partition functions yields the corresponding **scaling limit of correlations**:

$$\mathbb{E}_{\Omega_\delta, \lambda, h}^{\omega} [\sigma_{x_1} \cdots \sigma_{x_k}] \xrightarrow[\delta \downarrow 0]{d} \mathcal{E}_{\Omega, \hat{\lambda}, \hat{h}}^W [\sigma_{x_1} \cdots \sigma_{x_k}] := \text{Wiener chaos expansion}$$

# Outline

1. Disordered Systems and their Scaling Limits
2. Main Results (I): Partition Function
3. Main Results (II): Continuum Disordered Pinning Model
4. Further Developments

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Our restriction involving  $L^2$  convergence of  $k$ -point function ( $\gamma < \frac{d}{2}$ ) matches with Harris criterion  $\nu < \frac{2}{d}$  for disorder relevance

$$(\nu \text{ correlation length exponent } \rightsquigarrow \nu = \frac{1}{d-\gamma})$$

# Continuum free energy and critical exponents

Continuum partition function  $Z_{\Omega, \hat{\lambda}, \hat{h}}^W \rightsquigarrow$  continuum free energy

$$F(\hat{\lambda}, \hat{h}) := \lim_{\Omega \uparrow \mathbb{R}^d} \frac{1}{Leb(\Omega)} \log Z_{\Omega, \hat{\lambda}, \hat{h}}^W$$

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## Conjecture

$$\lim_{h \downarrow 0} \frac{\langle \sigma_0 \rangle_{\hat{\lambda} h^{\frac{7}{15}}, h}}{h^{\frac{1}{15}}} = \frac{\partial \mathbf{F}}{\partial h}(\hat{\lambda}, 1) \quad \text{refining [Camia, Garban, Newman '12]}$$

# Thanks