THE QUENCHED CRITICAL POINT
OF A DILUTED DISORDERED POLYMER MODEL

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ABSTRACT. We consider a model for a polymer interacting with an attractive wall through a random sequence of charges. We focus on the so-called diluted limit, when the charges are very rare but have strong intensity. In this regime, we determine the quenched critical point of the model, showing that it is different from the annealed one. The proof is based on a rigorous renormalization procedure. Applications of our results to the problem of a copolymer near a selective interface are discussed.

1. INTRODUCTION

The issue addressed in this work is the determination of the quenched critical point for the localization/delocalization phase transition of a polymer interacting with an attractive wall through a diluted disordered potential. The model we consider was first introduced by Bodineau and Giacomin in [4], as a *reduced model* for the so-called *copolymer near a selective interface model* [5], with the hope that it would have the same behavior as the full copolymer model, in the limit of weak coupling constants. As we will see, our main result shows that this is not the case.

The cornerstone of our approach is a rigorous renormalization procedure. We point out that the same result has recently been obtained by Fabio Toninelli [13], with a rather different approach, see the discussion following Theorem 5 below for more details.

1.1. **The model and the free energy.** Let $S = \{S_n\}_{n \geq 0}$ be the simple symmetric random walk on $\mathbb{Z}$, and denote by P its law. More explicitly, $S_0 = 0$ and $\{S_n - S_{n-1}\}_{n \geq 1}$ are i.i.d. random variables with $P(S_1 = +1) = P(S_1 = -1) = \frac{1}{2}$. For $N \in \mathbb{N} := \{1, 2, \ldots\}$ we denote by $P_N^+(\,\cdot\,) = P(\,\cdot\,|S_n \geq 0$ for $n = 1, \ldots, N)$ the law of the random walk conditioned to stay non-negative up to time N. The trajectories $\{(n, S_n)\}_{0 \leq n \leq N}$ under P_N^+ model the configurations of a polymer chain of length N above an *impenetrable wall*.

The interaction of the polymer with the wall is tuned by two parameters $\beta \geq 0$, and $p \in [0, 1]$. For fixed β and p, we introduce a sequence $\omega = (\omega_n)_{n \geq 1}$ of i.i.d. random variables, taking values in $\{0, \beta\}$ and with law $\mathbb{P}$ given by:

$$\mathbb{P}(\omega_1 = \beta) = p, \quad \mathbb{P}(\omega_1 = 0) = 1 - p. \tag{1.1}$$

We are ready to define our model: for a fixed (typical) realization ω and $N \in \mathbb{N}$, we introduce the probability measure $\mathbf{P}_{N,\omega}^{\beta,p}$ defined by:

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}^{\beta,p}}{\mathrm{d}P_N^+}(S) = \frac{1}{Z_{N,\omega}^{\beta,p}} \exp\left(\sum_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}}\right), \tag{1.2}$$
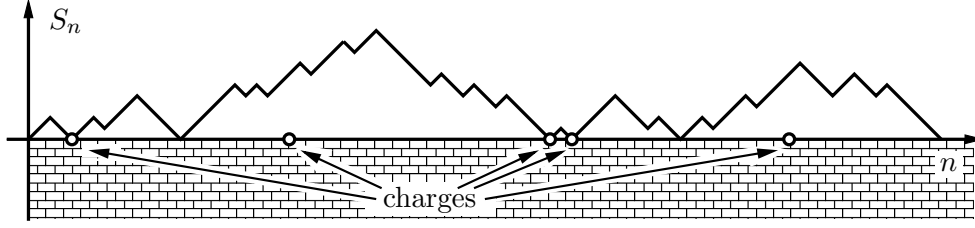
FIGURE 1. A typical path of the polymer measure $\mathbf{P}_{N,\omega}^{\beta,c}$.

where the normalization constant $Z_{N,\omega}^{\beta,p} := E_N^+\big(\exp\big(\sum_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}}\big)\big)$ is usually called the *partition function*.

Notice that the polymer measure $\mathbf{P}_{N,\omega}^{\beta,p}$ and the partition function $Z_{N,\omega}^{\beta,p}$ are functions of N and ω only; the superscripts β, p are there to indicate that we are interested in the case when the sequence ω follows the law $\mathbb{P}$, which depends on β, p.

In this paper we focus on the regime of large β and small p: then ω represents a random sequence of charges sitting on the wall (i.e. on the x-axis), which are rare, but of strong intensity, and which attract the polymer, see Figure 1. We are interested in the behavior of the polymer measure $\mathbf{P}_{N,\omega}^{\beta,p}$ in the limit of large N: in particular, we want to understand whether the attractive effect of the *environment* ω is strong enough to pin the polymer at the wall (*localization*), or whether it is still more convenient for the polymer to wander away from it (*delocalization*), as it happens when there are no charges. It should be clear that we are facing a competition between energy and entropy.

The classical way of detecting the transition between the two regimes mentioned above, is to study the free energy of the model, which is defined by:

$$f(\beta, p) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}^{\beta,p}. \tag{1.3}$$

The existence of this limit, both $\mathbb{P}(d\omega)$–a.s. and in $L^1(\mathbb{P})$, and the fact that $f(\beta, p)$ is non-random are proved in [8] via super-additivity arguments. Notice that trivially $Z_{N,\omega}^{\beta,p} \geq 1$ and hence $f(\beta, p) \geq 0$ for all β, p. Zero is in fact the contribution to the free energy of the paths that never touch the wall: indeed, by restricting to the set of random walk trajectories that stay strictly positive until time N, one has

$$Z_{N,\omega}^{\beta,p} \geq \frac{P\big(S_i > 0 \text{ for } i=1,\ldots,N\big)}{P\big(S_i \geq 0 \text{ for } i=1,\ldots,N\big)} = \frac{\frac{1}{2}P\big(S_i \geq 0 \text{ for } i=1,\ldots,N-1\big)}{P\big(S_i \geq 0 \text{ for } i=1,\ldots,N\big)} \xrightarrow{N\to\infty} \frac{1}{2},$$

where we use the well-known fact that $P\big(S_i \geq 0 \text{ for } i=1,\ldots,N\big) \sim (const.)/\sqrt{N}$ as $N \to \infty$, cf. [7, Ch. 3]. Based on this observation, we partition the phase space into:

- the Localized region $\mathcal{L} := \{(\beta, p) : f(\beta, p) > 0\}$
- the Delocalized region $\mathcal{D} := \{(\beta, p) : f(\beta, p) = 0\}$.

By a standard coupling on the environment, it is clear that the function $p \mapsto f(\beta, p)$ is non-decreasing. Therefore for every $\beta \geq 0$, there exists a critical value $p_c(\beta) \in [0,1]$ such that the model is localized for $p > p_c(\beta)$ and delocalized for $p < p_c(\beta)$ (in fact for $p \leq p_c(\beta)$, since the function $f(\beta, p)$ is continuous). The main goal of this work is to study the asymptotic behavior of $p_c(\beta)$, as $\beta \to \infty$.

Remark 1. One may reasonably ask to what extent the definition of (de)localization given above in terms of the free energy corresponds to a real (de)localized behavior of the typical paths of $\mathbf{P}_{N,\omega}^{\beta,p}$. Let us just mention that, by convexity arguments, one can prove that when $(\beta, p) \in \mathcal{L}$ the typical paths of $\mathbf{P}_{N,\omega}^{\beta,p}$, for large N, touch the wall a positive fraction of time, while this does not happen when (β, p) are in the interior of $\mathcal{D}$. We do not focus on path properties in this paper: for deeper results, we refer to [8]. $\square$

Remark 2. Models like $\mathbf{P}_{N,\omega}^{\beta,p}$ are known in the literature as (disordered) *wetting models*, the terminology referring to the interpretation of $\{S_n\}_{n \geq 0}$ as the interface of separation between a liquid and a gazeous phase, when the liquid is above an impenetrable wall.

More generally, $\mathbf{P}_{N,\omega}^{\beta,p}$ belongs to the class of the so-called *disordered pinning models*, which have received a lot of attention in the recent probabilistic literature, cf. [2, 10, 1, 12, 13] (see also [8] for an overview). In our case we prefer to refer to $\mathbf{P}_{N,\omega}^{\beta,p}$ as a polymer model, because of its original interpretation as a simplified model for a copolymer near a selective interface [4] (the link with the copolymer model is discussed below). $\square$

1.2. **The main result.** Some bounds on $p_c(\beta)$ can be obtained quite easily, as shown in [4, §4.1]. These results are stated in the following two lemmas, whose (easy) proofs are given in detail here, since they provide some insight into the problem. Our main result is then stated in Theorem 5 below.

Lemma 3. *The following relation holds:*

$$-\liminf_{\beta \to \infty} \frac{1}{\beta} \log p_c(\beta) \ \leq \ 1 \,. \tag{1.4}$$

Proof. Since the limit in (1.3) holds also in $L^1(\mathbb{P})$, by Jensen's inequality we get

$$f(\beta, p) \ = \ \lim_{N \to \infty} \frac{1}{N} \, \mathbb{E}\big(\log Z_{N,\omega}^{\beta,p} \big) \ \leq \ \lim_{N \to \infty} \frac{1}{N} \, \log \mathbb{E}\big(Z_{N,\omega}^{\beta,p} \big) \,. \tag{1.5}$$

This is usually called the *annealed bound*, and the limit in the r.h.s. above (whose existence follows by a standard super-additivity argument) is the *annealed free energy*. It can be evaluated using the definition (1.2) of the model, and Fubini's Theorem:

$$\mathbb{E}\big(Z_{N,\omega}^{\beta,p} \big) \ = \ E_N^+ \, \mathbb{E} \left(\exp \left(\sum_{n=1}^{N} \omega_n \mathbf{1}_{\{S_n=0\}} \right) \right) \ = \ E_N^+ \left(\exp \left(\sum_{n=1}^{N} \log \mathrm{M}(\beta, p) \mathbf{1}_{\{S_n=0\}} \right) \right),$$

where $\mathrm{M}(\beta, p) = \mathbb{E}(e^{\omega_1}) = p \, e^{\beta} + (1-p)$. Therefore $\mathbb{E}\big(Z_{N,\omega}^{\beta,p} \big)$ is the partition function of the simple random walk conditioned to stay non-negative, and given a constant reward $\log \mathrm{M}(\beta, p)$ each time it touches zero. This model is exactly solvable, see [8, §1.3], and in particular we have:

$$\lim_{N \to \infty} \frac{1}{N} \, \log \mathbb{E}\big(Z_{N,\omega}^{\beta,p} \big) \ = \ 0 \qquad \text{if and only if} \qquad \mathrm{M}(\beta, p) \ \leq \ 2 \,.$$

Looking back to (1.5) and recalling the definition of $\mathrm{M}(\beta, p)$, we have shown that

$$p \ \leq \ p^a(\beta) \ := \ \frac{1}{e^{\beta} - 1} \qquad \Longrightarrow \qquad (\beta, p) \in \mathcal{D} \,,$$

where $p^a(\beta)$ is the *annealed critical point*. Therefore equation (1.4) is proved. $\square$

Lemma 4. *The following relation holds:*

$$- \limsup_{\beta \to \infty} \frac{1}{\beta} \log p_c(\beta) \ \geq \ \frac{2}{3} \,. \tag{1.6}$$

Proof. We have to bound the partition function from below. To this aim, we compute the contribution of the set of trajectories that touch the wall wherever there is a non-zero charge (on even sites, because of the periodicity of the random walk). We need some notations: we introduce the subset of paths

$$\Omega_N^\omega \ := \ \big\{ S : \ S_n = 0 \iff \omega_n > 0, \ \forall n \leq N, \ n \in 2\mathbb{N} \big\} \,,$$

and the locations $\{\xi_n\}_{n \geq 0}$ of the positive charges sitting on even sites:

$$\xi_0 \ := \ 0 \qquad \xi_{n+1} \ := \ \inf\{ k > \xi_n, \ k \in 2\mathbb{N} : \ \omega_k > 0 \} \,, \quad n \in \mathbb{N} \,.$$

We denote by $\iota_N := \max\big\{ k \geq 0 : \ \xi_k \leq N \big\}$ the number of positive charges (sitting on even sites) up to time N. Finally, we introduce the distribution of the first return time to zero of the simple random walk restricted to the non-negative half-line:

$$K^+(n) \ := \ P\big(S_i > 0 \text{ for } i = 1, \dots, n-1, \ S_n = 0 \big) \tag{1.7}$$

(observe that $K^+(n) = 0$ for n odd) and we recall that [7, Ch. 3]

$$K^+(2n) \ \overset{n \to \infty}{\sim} \ \frac{C_K}{n^{3/2}} \quad \text{where} \quad C_K \ = \ \frac{1}{2\sqrt{\pi}} \,, \qquad \sum_{n \in \mathbb{N}} K^+(2n) \ = \ \frac{1}{2} \,. \tag{1.8}$$

Then we have

$$Z_{N,\omega}^{\beta,p} \ \geq \ E_N^+\Bigg(\exp\Bigg(\sum_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}} \Bigg) \mathbf{1}_{\{S \in \Omega_N^\omega\}} \Bigg)$$

$$= \ \frac{1}{P(S_1 \geq 0, \dots, S_N \geq 0)} \cdot e^{\beta \, \iota_N} \cdot \Bigg(\prod_{\ell=1}^{\iota_N} K^+\big(\xi_\ell - \xi_{\ell-1}\big) \Bigg) \cdot \Bigg(\sum_{n=N-\xi_{\iota_N}+1}^{\infty} K^+(n) \Bigg) \,.$$

Note that $\{(\xi_\ell - \xi_{\ell-1})/2\}_{\ell \geq 1}$ is an i.i.d. sequence of geometric random variables with parameter p. Therefore, by the strong law of large numbers, we have:

$$\lim_{N \to \infty} \frac{\iota_N}{N} \ = \ \frac{p}{2} \quad \text{and} \quad \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{\iota_N} \log K^+(\xi_\ell - \xi_{\ell-1}) \ = \ \frac{p}{2} \, \mathbb{E}\big(\log K^+(\xi_1) \big), \quad \mathbb{P}(\mathrm{d}\omega)\text{–a.s.}$$

Hence, from the last equation we get, $\mathbb{P}(\mathrm{d}\omega)$–a.s.,

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\omega}^{\beta,p} \ \geq \ \frac{p}{2} \Big(\beta \ + \ \mathbb{E}\big(\log K^+(\xi_1) \big) \Big) \,.$$

By (1.8) there exists a positive constant c_1 such that $K^+(n) \geq c_1/n^{3/2}$, for all $n \in 2\mathbb{N}$. Using this bound and Jensen's inequality yields:

$$\lim_{N \to \infty} \frac{1}{N} \log Z_{N,\omega}^{\beta,p} \ \geq \ \frac{p}{2} \Big(\beta \ + \ \log c_1 \ - \ \frac{3}{2}\mathbb{E}\big(\log \xi_1 \big) \Big) \ \geq \ \frac{p}{2} \Big(\beta \ + \ \log c_1 \ - \ \frac{3}{2} \log \mathbb{E}(\xi_1) \Big) \,.$$

Since $\mathbb{E}\big(\xi_1\big) = 2p^{-1}$, setting $c_2 := \log c_1 - \frac{3}{2}\log 2$, we get

$$\lim_{N \to \infty} \frac{1}{N} Z_{N,\omega}^{\beta,p} \ \geq \ \frac{p}{2} \Big(\beta \ + \ \frac{3}{2} \log p \ + \ c_2 \Big) \,,$$

so that

$$p \ \geq \ e^{-\frac{2}{3}(\beta + c_2)} \qquad \Longrightarrow \qquad (\beta, p) \in \mathcal{L} \,,$$

and equation (1.6) is proved. □

We can summarize Lemmas 3 and 4 in the following way: if we knew that

$$p_c(\beta) \ \asymp \ e^{-c_{red}\beta} \qquad\qquad (\beta \to +\infty)\,,$$

then $\frac{2}{3} \le c_{red} \le 1$ (the subscript red stands for $reduced\ model$, see the discussion below). The main result of this paper is that in fact $c_{red} = \frac{2}{3}$. More precisely:

Theorem 5. *For every $c > \frac{2}{3}$ there exists $\beta_0 = \beta_0(c)$ such that*

$$f(\beta, e^{-c\beta}) \ = \ 0 \qquad for\ all\ \beta \ge \beta_0\,,$$

i.e. $(\beta, e^{-c\beta}) \in \mathcal{D}$ for $\beta \ge \beta_0$. Therefore

$$-\lim_{\beta\to\infty} \frac{1}{\beta} \log p_c(\beta) \ = \ \frac{2}{3}\,.$$

Let us discuss some consequences of this Theorem. We recall that the model $\mathbf{P}_{N,\omega}^{\beta,p}$ was first introduced in [4], as a simplified version ('reduced model') of the so-called *copolymer near a selective interface model*, cf. [5] (see also [8] for a recent overview). It is known that the copolymer model undergoes a localization/delocalization phase transition. An interesting object is the *critical line* separating the two phases, in particular in the limit of weak coupling constants, where it becomes a straight line with positive slope C_{cop}.

A lot of effort has been put in finding the exact value of C_{cop}. This is motivated by the fact that C_{cop} appears to be a *universal* quantity: it is independent of the law of the environment [9, Section 3] and it determines the phase transition of a continuous copolymer model, arising as the scaling limit of the discrete one [5, §0.3]. What is known up to now is that $\frac{2}{3} \le C_{cop} \le 1$. Notice that $\frac{2}{3}$ and 1 are exactly the bounds that were previously known for c_{red}, and this is not by chance: indeed the definition of the model $\mathbf{P}_{N,\omega}^{\beta,p}$ is inspired by the strategy behind the proof of $C_{cop} \ge \frac{2}{3}$, cf. [4].

The reason for introducing a reduced model is to have a more tractable model, which would possibly have the same behavior as the full copolymer model in the limit of weak coupling constants, i.e. for which possibly $c_{red} = C_{cop}$. However, the numerical results obtained in [6] provide strong indications for the fact that $C_{cop} > \frac{2}{3}$. If this is indeed the case, our result shows that the reduced model does not catch the full complexity of the copolymer model, i.e. the 'missing free energy' should come from a different strategy than the one which is at the basis of the lower bound $C_{cop} \ge \frac{2}{3}$.

By Theorem 5, our model provides also a non-trivial example of a linear chain pinning model where, for large β, the quenched critical point $p_c(\beta)$ is different from the annealed one $p^a(\beta) = 1/(e^\beta - 1)$ (see the proof of Lemma 3).

What we actually prove in this paper is a stronger version of Theorem 5, i.e. Theorem 6, stated in the next section. The proof relies on *quenched* arguments, based on a rigorous *renormalization procedure* (somewhat in the spirit of [11]). The idea is to remove from the environment sequence ω the positive charges that are well-spaced (that therefore give no sensible contribution to the partition function) and to cluster together the positive charges that are very close. This procedure produces a new environment sequence ω', which has fewer charges but of stronger intensity. The key point is that replacing ω by ω' in the partition function yields an upper bound on the free energy. Then, by iterating this transformation, we obtain environment sequences for which the free energy can be

estimated and shown to be arbitrarily small. A detailed description of this approach, together with the organization of the paper, is given in Section 2.

We point out that Theorem 5 has recently been obtained by Fabio Toninelli [13] with a simpler (though more indirect) argument, avoiding the renormalization procedure we apply. We however believe that our direct procedure, eliminating 'bad' regions in a recursive way, is also of value for other problems, e.g. for proving that $C_{cop} < 1$.

1.3. **Beyond the simple random walk.** Theorem 5 can actually be extended to a broader class of models. Namely, let $\big(\tau = \{\tau_n\}_{n\geq 0}, \mathbf{P}\big)$ be a renewal process, i.e. $\tau_0 = 0$ and $\{\tau_n - \tau_{n-1}\}_{n\geq 1}$ under $\mathbf{P}$ are i.i.d. random variables taking positive values (including $+\infty$). It is convenient to look at τ also as the (random) subset $\cup_{n\geq 0}\{\tau_n\}$ of $\mathbb{N} \cup \{0\}$, so that expressions like $\{k \in \tau\}$ make sense. We assume that τ is *terminating*, i.e.

$$\delta := \mathbf{P}(\tau_1 < \infty) < 1, \tag{1.9}$$

that it is *aperiodic*, i.e. $\gcd\{n \in \mathbb{N} : \mathbf{P}(n \in \tau) > 0\} = 1$, and that for some positive constant C_K we have:

$$K(n) := \mathbf{P}(\tau_1 = n) \sim \frac{C_K}{n^{3/2}} \qquad (n \to \infty). \tag{1.10}$$

We introduce $\ell_N := \max\{k \geq 0 : \tau_k \leq N\}$, that gives the number of renewal epochs up to N, and the renewal function $U(\cdot)$ associated to τ, defined for $n \in \mathbb{N}$ by

$$U(n) := \mathbf{P}(n \in \tau) = \sum_{k=0}^{\infty} \mathbf{P}(\tau_k = n). \tag{1.11}$$

By (1.9) and (1.10), the asymptotic behavior of $U(n)$ is [8, Th. A.4]

$$U(n) \sim \frac{C_K}{(1-\delta)^2} \frac{1}{n^{3/2}} \qquad (n \to \infty),$$

so that in particular there exists a positive constant $\mathcal{C}$ such that

$$U(n) \leq \frac{\mathcal{C}}{n^{3/2}} \qquad \text{for every } n \in \mathbb{N}. \tag{1.12}$$

We stress that $U(n)$ has the same polynomial behavior as $K(n)$: this is a consequence of equation (1.9) and is a crucial fact for us.

Keeping the same environment $\omega = \{\omega_n\}_n$ as in (1.1), we define the new partition function

$$\mathbf{Z}_{N,\omega}^{\beta,p} := \mathbf{E}\left(\exp\left(\sum_{n=1}^{N} \omega_n \mathbf{1}_{\{n\in\tau\}}\right)\mathbf{1}_{\{N\in\tau\}}\right) = \mathbf{E}\left(\exp\left(\sum_{k=1}^{\ell_N} \omega_{\tau_k}\right)\mathbf{1}_{\{N\in\tau\}}\right), \tag{1.13}$$

and we call $\mathtt{f}(\beta, p)$ the corresponding free energy:

$$\mathtt{f}(\beta, p) := \lim_{N\to\infty} \frac{1}{N} \log \mathbf{Z}_{N,\omega}^{\beta,p} \qquad \mathbb{P}(\mathrm{d}\omega)\text{–a.s. and in } L^1(\mathbb{P}). \tag{1.14}$$

Then we have the following extension of Theorem 5.

Theorem 6. *For every $c > \frac{2}{3}$ there exists $\beta_0 = \beta_0(c)$ such that*

$$\mathtt{f}(\beta, e^{-c\beta}) = 0 \qquad \text{for all } \beta \geq \beta_0.$$

Let us show that Theorem 5 can easily be deduced from Theorem 6. To this purpose, we choose $(\tau, \mathbf{P})$ to be the renewal process with inter-arrival law $K^+(\cdot)$ defined in (1.7). Notice that τ is not aperiodic, but this is a minor point (it suffices to focus on the even sites to recover aperiodicity), and that $K^+(\cdot)$ satisfies (1.9) and (1.10) (restricted to even sites). Then we can write the original partition function as

$$Z_{N,\omega}^{\beta,p} \ = \ E_N^+\left(\exp\left(\sum_{n=1}^{N} \mathbf{1}_{\{S_n=0\}} \right) \right) \ = \ \frac{1}{P(S_1 \geq 0, \ldots, S_N \geq 0)} \, \mathbf{E}\left(\exp\left(\sum_{k=1}^{\ell_N} \omega_{\tau_k} \right) \right).$$

This formula looks slightly different from (1.13). First, there is a pre-factor, due to the fact that $K^+(\cdot)$ is defined under the *restricted law* $P\left(\cdot \ \mathbf{1}_{\{S_1 \geq 0, \ldots, S_N \geq 0\}} \right)$ while $Z_{N,\omega}^{\beta,p}$ is defined as an average with respect to the *conditioned law* P_N^+. We have already noted that $P\left(S_i \geq 0 \text{ for } i = 1, \ldots, N \right) \sim (const.)/\sqrt{N}$, therefore this pre-factor is irrelevant for the purpose of computing the free energy. The second difference is the presence in (1.13) of the indicator function $\mathbf{1}_{\{N \in \tau\}}$, but again this *boundary condition* does not change the Laplace asymptotic behavior, as is shown in [8, Rem. 1.2]. Therefore $\mathtt{f}(\beta, p)$ defined by (1.14) coincides with $f(\beta, p)$ defined by (1.3), and Theorem 5 follows from Theorem 6.

1.4. **On more general return exponents.** Another natural extension is to let the renewal process $(\tau, \mathbf{P})$ have inter-arrival distribution $K(\cdot)$ such that

$$K(n) \ \sim \ \frac{L(n)}{n^{1+\alpha}} \qquad\qquad (n \to \infty), \qquad\qquad (1.15)$$

with $\alpha \geq 0$ and $L : (0, \infty) \to (0, \infty)$ a *slowly varying function* (cf. [3]), i.e. such that $L(tx)/L(x) \to 1$ as $x \to \infty$, for every $t > 0$. In this paper we stick to the case $\alpha = \frac{1}{2}$ and $L(n) \to C_K$, for ease of notations, but we stress that our proof of Theorem 6 goes through for the general case with easy modifications, provided one replaces $\frac{2}{3}$ by $\frac{1}{1+\alpha}$ in the statement of Theorem 6. Lemma 4 also generalizes immediately, again with $\frac{2}{3}$ replaced by $\frac{1}{1+\alpha}$, hence we have $p_c(\beta) \asymp e^{-c_{red}^\alpha \beta}$ with $c_{red}^\alpha = \frac{1}{1+\alpha}$. In particular, for large β the quenched critical point is different from the annealed one, for any value of $\alpha > 0$.

2. STRATEGY OF THE PROOF: A RENORMALIZATION PROCEDURE

In this section, we explain the strategy behind the proof of Theorem 6 in details, and describe the organization of the paper. Before doing that, we introduce some notations, and make some preliminary transformations of the partition function $\mathtt{Z}_{N,\omega}^{\beta,p}$.

2.1. **Some basic notations.** Let us first introduce some notations to be used throughout the paper. Given an arbitrary sequence $\omega = \{\omega_n\}_{n \geq 1}$, the sequence $\{t_n\}_{n \geq 0} = \{t_n(\omega)\}_{n \geq 0}$, is defined as follows:

$$t_0(\omega) \ := \ 0 \qquad\qquad t_n(\omega) \ := \ \min\left\{k > t_{n-1}(\omega) : \omega_k > 0\right\}. \qquad\qquad (2.1)$$

In our case, ω is the sequence of charges, and $\{t_n\}_{n \geq 1}$ are the locations of the positive charges. However we are not assuming that ω is a typical realization of the law (1.1) (in particular, the size of the positive elements of ω is not necessarily β).

Notice that the $\{t_n\}_{n \geq 0}$ are finite iff $\#\{i \in \mathbb{N} : \omega_i > 0\} = +\infty$, which is always the case for us. The increments of $\{t_n\}_{n \geq 0}$ are denoted by $\{\Delta_n\}_{n \geq 1} = \{\Delta_n(\omega)\}_{n \geq 1}$:

$$\Delta_n(\omega) \ := \ t_n(\omega) - t_{n-1}(\omega) \qquad (n \in \mathbb{N}). \qquad\qquad (2.2)$$

Finally, we introduce the sequence $\{\eta_n\}_{n\geq 1} = \{\eta_n(\omega)\}_{n\geq 1}$ giving the intensities of the positive charges, i.e.

$$\eta_n(\omega) := \omega_{t_n(\omega)} \qquad (n \in \mathbb{N}). \tag{2.3}$$

We stress that one can easily reconstruct the sequence ω in terms of $\{t, \eta\} = \{t_n, \eta_n\}_n$ (or equivalently of $\{\Delta_n, \eta_n\}_n$). Therefore we look at ω, $\{t, \eta\}$ and $\{\Delta, \eta\}$ as equivalent ways of describing the sequence of charges.

We also mention an elementary fact that will be used in the next section: the sequence $\{\omega_n\}_{n\geq 1}$ is i.i.d. if and only if the variables $\{\Delta_n, \eta_n\}_{n\geq 1}$ are all independent, the $\{\eta_n\}_{n\geq 1}$ are i.i.d. and the $\{\Delta_n\}_{n\geq 1}$ are i.i.d. geometrically distributed.

2.2. **A preliminary transformation.** In the partition function $\mathsf{Z}_{N,\omega}^{\beta,p}$, the parameter N represents the size of the system. However it turns out to be more convenient to consider a partition function with a *fixed* number of positive charges.

Let us be more precise. Since the limit in (1.14) exists $\mathbb{P}(\mathrm{d}\omega)$–a.s., we can take it along the (random) subsequence $\{t_n\}_{n\geq 0} = \{t_n(\omega)\}_{n\geq 0}$, i.e. we can write, $\mathbb{P}(\mathrm{d}\omega)$–a.s.,

$$\mathtt{f}(\beta, p) = \lim_{n\to\infty} \frac{1}{t_n} \log \mathsf{Z}_{t_n,\omega}^{\beta,p}, \tag{2.4}$$

Let us focus on $\mathsf{Z}_{t_n,\omega}^{\beta,p}$: by summing over the locations of the positive charges that are visited and recalling the definition (1.11) of the renewal function $U(\cdot)$, we obtain the explicit expression

$$\mathsf{Z}_{t_n,\omega}^{\beta,p} \leq \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_{k-1}\in\mathbb{N} \\ 0=:j_0<j_1<\ldots<j_{k-1}<j_k=n}} \prod_{\ell=1}^{k} e^{\eta_{j_\ell}} \cdot U(t_{j_\ell} - t_{j_{\ell-1}}). \tag{2.5}$$

We stress that in the r.h.s. we have $U(t_{j_\ell} - t_{j_{\ell-1}})$ and not $K(t_{j_\ell} - t_{j_{\ell-1}})$: in fact we are fixing the positive charges that are visited, but the path is still free to touch the wall in between the positive charges. Also notice that, when ω is distributed according to (1.1), we have $\eta_{j_\ell} = \beta$, but we keep the notation implicit for later convenience.

Formula (2.5) leads us to the following definition: for $n \in \mathbb{N}$ and $C \in \mathbb{R}^+$ we set:

$$\mathcal{Z}_n(\omega, C) := \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_{k-1}\in\mathbb{N} \\ 0=:j_0<j_1<\ldots<j_{k-1}<j_k:=n}} \prod_{\ell=1}^{k} e^{\eta_{j_\ell}} \cdot \frac{C}{(t_{j_\ell} - t_{j_{\ell-1}})^{3/2}}, \tag{2.6}$$

so that applying the upper bound (1.12) on $U(\cdot)$ we have

$$\mathsf{Z}_{t_n,\omega}^{\beta,p} \leq \mathcal{Z}_n(\omega, \mathcal{C}). \tag{2.7}$$

Notice that $\mathcal{Z}_n(\omega, \mathcal{C})$ carries no explicit dependence of the parameters β, p. In fact, we look at $\mathcal{Z}_n(\omega, \mathcal{C})$ as a deterministic function of the constant $\mathcal{C}$ and of the sequence ω. In the sequel, $\mathcal{C}$ will always denote the constant appearing in the r.h.s. of (1.12).

Now assume that $(\omega = \{\omega_n\}_n, \mathbb{P}_\mu)$ is an i.i.d. sequence of random variables with marginal law μ: $\mathbb{P}_\mu(\omega_1 \in \mathrm{d}x) = \mu(\mathrm{d}x)$. Then for $C > 0$ we define the free energy $\mathrm{F}(\mu, C)$ as the limit

$$\mathrm{F}(\mu, C) := \lim_{n\to\infty} \frac{1}{t_n(\omega)} \log \mathcal{Z}_n(\omega, C) \qquad \mathbb{P}_\mu(\mathrm{d}\omega)\text{–a.s.}. \tag{2.8}$$

If we denote by $\mu_{\beta,p} := (1-p)\,\delta_{\{0\}} + p\,\delta_{\{\beta\}}$, then by (2.4) and (2.7) we can write

$$\mathtt{f}(\beta, p) \leq \mathrm{F}\big(\mu_{\beta,p}, \mathcal{C}\big).$$

Then to prove Theorem 6 it suffices to prove the following: *for every $\mathcal{C} > 0$ and $c > \frac{2}{3}$ there exists $\beta_0 = \beta_0(\mathcal{C}, c)$ such that $\mathrm{F}\big(\mu_{\beta, \exp(-c\beta)}, \mathcal{C}\big) = 0$ for every $\beta \geq \beta_0$.*

Therefore, from now on, we focus on $\mathscr{Z}_n(\omega, \mathcal{C})$ and $\mathrm{F}(\mu, \mathcal{C})$. The constants $\mathcal{C} > 0$ and $c > \frac{2}{3}$ are fixed throughout the sequel. We also use the shorthand μ_β for $\mu_{\beta, \exp(-c\beta)}$, i.e.

$$\mu_\beta := \big(1 - e^{-c\beta}\big)\,\delta_{\{0\}} + e^{-c\beta}\,\delta_{\{\beta\}}\,. \tag{2.9}$$

Our goal is to show that $\mathrm{F}(\mu_\beta, \mathcal{C}) = 0$ for β large. For ease of notation, *we only consider the case $\beta \in \mathbb{N}$* (the general case can be recovered with minor modifications).

2.3. **The renormalization procedure.** The proof of Theorem 6 is achieved through an induction argument. The steps of the induction are labeled by $\{\beta, \beta + 1, \beta + 2, \ldots\}$, and we call them *level β, level $\beta + 1$, $\ldots$*

Each induction step consists of a renormalization procedure that acts both on the environment ω, and on the partition function $\mathscr{Z}_n(\omega, \mathcal{C})$, and produces an upper bound on the free energy $\mathrm{F}(\mu, \mathcal{C})$. Let us be more precise, by describing in detail how this procedure works.

Renormalizing the environment (Section 3). At the starting point (level β) the environment ω is i.i.d. with marginal law μ_β defined in (2.9), supported on $\{0\} \cup \{\beta\}$. More generally, at level $b \geq \beta$ the environment ω will be i.i.d. with marginal law μ_b supported on $\{0\} \cup \{b, b+1, \ldots\}$. If we are at level b, we define a renormalization map T_b acting on ω that produces a new sequence $T_b(\omega)$ as follows.

We first need to define *isolated charges*, *good charges* and *bad blocks* at level b. To this purpose, we fix the threshold $L_b := \big\lfloor e^{\frac{2}{3}(b+K_b)} \big\rfloor$, where the constant K_b is defined explicitly in (3.2) below. A positive charge is said to be an *isolated charge* if both its neighboring positive charges are at distance greater than L_b. Among the isolated charges, we call *good charges* those that have intensity exactly equal to b, i.e. the least possible intensity. Finally, a group of adjacent positive charges is said to be a *bad block* if all the distances between neighboring charges inside the group are smaller than L_b. Note that a charge is either isolated, or it belongs to a bad block (see Figure 2 for a graphical illustration).
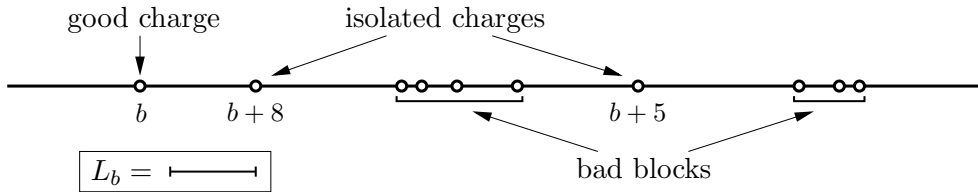

FIGURE 2. *Good charges, isolated charges and bad blocks at level b.*

Then the renormalized environment $\omega' = T_b(\omega)$ is obtained from ω in the following way: each *bad block* is clustered into one single larger charge, each *good charge* is erased, the *isolated charges* that are not good are left unchanged and finally the distances between charges are suitably shortened. In Section 3 we show that the new environment ω' constructed in this way is still i.i.d. and we obtain an explicit expression for the marginal law of ω'_1, denoted by $\mu_{b+1} := T_b(\mu_b)$. Observe that by construction μ_{b+1} is supported by $\{0\} \cup \{b+1, b+2, \ldots\}$.

Renormalizing the partition function (Section 4). The idea behind the definition of good charges and bad blocks is the following:

- if a charge is good, it is not worth for the polymer to visit it, because this would entail a substantial entropy loss;
- on the other hand, if a charge belongs to a bad block and the polymer visits it, it is extremely convenient for the polymer to visit all the charges in the block.

These rough considerations are made precise in Section 4 (as a side remark, notice that the choice of what is 'good' and what is 'bad' is biased by the fact that we aim at proving delocalization). As a consequence, if we replace the environment ω with the renormalized one $T_b(\omega)$, we get an upper bound on the partition function. More precisely, if we also denote by T_b the transformation acting on $C > 0$ by $T_b(C) := C \cdot (1 + B \, e^{-K_b} \, C)$ (where B is an absolute constant defined in Lemma 10 below), then we show that the partition function satisfies, for every $N \in \mathbb{N}$,

$$\mathscr{Z}_n(\omega, C) \;\leq\; (const.) \, \mathscr{Z}_N\big(T_b(\omega), T_b(C)\big)\,, \tag{2.10}$$

for a suitable $n = n(\omega, N)$ such that $n \geq N$ and $t_n(\omega) \geq t_N(T_b(\omega))$. Taking '$\frac{1}{t_n} \log$' on both sides of (2.10), and letting $n \to \infty$, we then obtain for every $b \geq \beta$,

$$\textsc{f}\big(\mu_b, C\big) \;\leq\; \textsc{f}\big(\mu_{b+1}, T_b(C)\big)$$

(recall that $\mu_{b+1} = T_b(\mu_b)$) and by iteration we have for $b \geq \beta$

$$\textsc{f}\big(\mu_\beta, \mathcal{C}\big) \;\leq\; \textsc{f}\big(\mu_b, \mathcal{C}_b\big) \qquad \text{where} \qquad \mathcal{C}_b := \big(T_{b-1} \cdot T_{b-2} \cdots T_\beta\big)(\mathcal{C})\,. \tag{2.11}$$

Completing the proof (Section 5). The last step is to get a control on the law μ_b and on the constant $\mathcal{C}_b$, in order to extract explicit bounds from (2.11). By easy estimates, we show that $\mathcal{C}_b \leq 2\mathcal{C}$ for every b, so that this yields no problem. The crucial point is rather in estimating the law μ_b: in Proposition 12 we prove (when β is large but fixed) an explicit stochastic domination of μ_b, which allows to show that

$$\lim_{b \to \infty} \textsc{f}\big(\mu_b, \mathcal{C}_b\big) \;=\; 0\,.$$

By (2.11) this implies that $\textsc{f}\big(\mu_\beta, \mathcal{C}\big) = 0$, and Theorem 6 is proved.

3. Renormalization of the environment

In this section we describe the renormalization transformation performed on the environment, outlined in the previous section. At level $b \in \{\beta, \beta + 1, \ldots\}$ the sequence $\omega = \{\omega_1, \omega_2, \ldots\}$ is i.i.d. where the ω_i have law μ_b supported on $\{0\} \cup \{b, b+1, \ldots\}$. We set

$$c_b := \mu_b\left([b, \infty)\right)\,, \qquad \tilde{c}_b := \mu_b\left([b+1, \infty)\right)\,. \tag{3.1}$$

Then we act on ω by clustering and removing charges, in order to obtain a new environment sequence $\omega' = T_b(\omega)$ which is still i.i.d. but with a new charge distribution $\mu_{b+1} = T_b(\mu_b)$ supported by $\{0\} \cup \{b+1, b+2, \ldots\}$. The definition of T_b depends on

$$K_b := \big\lfloor K_0 + \log^+(2\mathcal{C}) + 2 \log b \big\rfloor\,, \qquad L_b := \big\lfloor e^{\frac{2}{3}(b + K_b)} \big\rfloor\,, \tag{3.2}$$

where $\log^+(x) := \log(x) \vee 0$, K_0 is an absolute positive constant defined in Lemma 10 below, and $\mathcal{C}$ is the constant appearing in (1.12).

We remind the reader that the sequence $\{\omega_k\}_{k \geq 1}$ is in one-to-one correspondence with the pair of sequences $\big(\{t_k\}_{k \geq 1}, \{\eta_k\}_{k \geq 1}\big)$, where $0 < t_1 < t_2 < \cdots$ is the sequence of
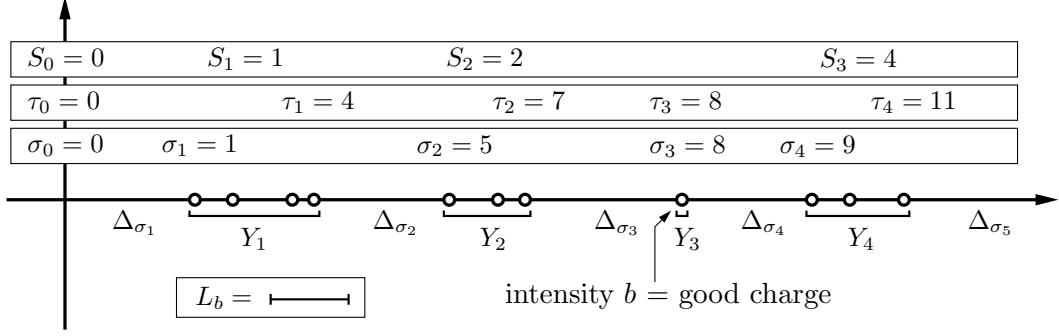
FIGURE 3. A sample configuration of charges, with the corresponding values of the variables σ_k, τ_k and S_k. Also indicated are the blocks Y_k and the spacing Δ_{σ_k} between blocks.

successive times where $\omega_{t_k} > 0$ and $\eta_k := \omega_{t_k}$. We also set for convenience $\eta_0 := 0$. Alternatively, we can replace $\{t_k\}_{k \geq 1}$ by the sequence $\{\Delta_k\}_{k \geq 1}$, $\Delta_k := t_k - t_{k-1}$, where $t_0 := 0$. We will freely switch from one representation to the other without special mentioning (see §2.1 for more details).

We define two sequences $\{\sigma_n\}_{n \geq 0}$, $\{\tau_n\}_{n \geq 0}$ of random times by $\sigma_0 := 0$,

$$\tau_n := \inf\{k \geq \sigma_n : \Delta_{k+1} > L_b\}, \qquad \sigma_{n+1} = \tau_n + 1.$$

Note that $\tau_0 = 0$ if and only if $\Delta_1 > L_b = \lfloor e^{\frac{2}{3}(b+K_b)} \rfloor$. In words, the sequence $\{\tau_n\}_{n \geq 0}$ represents the indices of those positive charges that have a 'distant' $(\Delta > L_b)$ neighboring charge on the right. Of course, one could define the sequence $\{\tau_n\}_{n \geq 0}$ alone, without the need of introducing $\{\sigma_n\}_{n \geq 0}$, but in the sequel it will be convenient to deal with both $\{\tau_n\}_{n \geq 0}$ and $\{\sigma_n\}_{n \geq 0}$. Next we define a sequence of random variables Y_k, $k \geq 0$, taking values in the space

$$\Gamma := \left(\{1\} \times \mathbb{R}_+\right) \cup \bigcup_{n \geq 2} \left(\{n\} \times \mathbb{N}^{n-1} \times \mathbb{R}_+^n\right),$$

by

$$Y_k := \left(\tau_k - \sigma_k + 1, (\Delta_{\sigma_k+1}, \ldots, \Delta_{\tau_k}), (\eta_{\sigma_k}, \eta_{\sigma_k+1}, \ldots, \eta_{\tau_k})\right)$$

(the meaning of these definitions will be explained in a moment). We occasionally write $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$. Here we understand that in case $\tau_k = \sigma_k$, the Δ-part is absent, and the variable takes values in the Γ_1-part of Γ. It should be remarked that in case $\sigma_0 = \tau_0 = 0$ we have $Y_0 = (1, 0)$ (recall that $\eta_0 := 0$). See Figure 3 for a graphical illustration of the variables just introduced.

Let us give some insight into these definitions. Each Y_k represents a block of adjacent positive charges, possibly reducing to one single charge. More precisely:

- If Y_k contains more than one charge, i.e. $Y_k \in \Gamma_n$ with $n \geq 2$, then Y_k corresponds to a *bad block* as defined in §2.3, because by construction the distances between the positive charges contained in it, $\{\Delta_i\}_{\sigma_k+1 \leq i \leq \tau_k}$, are such that $\Delta_i \leq L_b$.
- If on the other hand $Y_k \in \Gamma_1$, then Y_k is an *isolated charge*, because the distances $\Delta_{\sigma_k}, \Delta_{\sigma_{k+1}}$ with its neighboring blocks are by construction larger than L_b.
- If $Y_k \in \Gamma_1$ and in addition $\eta_{\sigma_k} = b$, i.e. $Y_k \in \{1\} \times \{b\}$, then Y_k is a *good charge*.

The variables σ_k and τ_k give the indexes of the first and last positive charge appearing in the k-th block, and therefore $\tau_n - \sigma_n + 1$ is the number of positive charges in a bad block.

Being larger than L_b, the variables Δ_{σ_k} are not geometrically distributed. We therefore subtract L_b and put

$$\widehat{\Delta}_k := \Delta_{\sigma_k} - L_b, \qquad k \geq 1.$$

Note that the two sequences $\{Y_k\}_{k \geq 0}$ and $\{\widehat{\Delta}_k\}_{k \geq 1}$ contain all the information of the original sequence ω. The basic properties of the variables Y_k, $\widehat{\Delta}_k$ are given in the following Lemma, whose (elementary) proof is omitted for conciseness.

Lemma 7.

a) *The random variables $Y_0, Y_1, Y_2, \ldots, \widehat{\Delta}_1, \widehat{\Delta}_2, \ldots,$ are independent.*

b) *The random variables $Y_k, k \geq 1$, are identically distributed with the following distribution:*

- *$\tau_k - \sigma_k + 1$ is geometrically distributed with parameter:*

$$q_b := \mathbb{P}_{\mu_b}(\Delta_1 > L_b) = (1 - c_b)^{L_b}, \qquad (3.3)$$

i.e. for $n \in \mathbb{N}$

$$P\left(\tau_k - \sigma_k + 1 = n\right) = q_b \left(1 - q_b\right)^{n-1}.$$

- *Conditionally on $\{\tau_k - \sigma_k + 1 = n\}$, we have that $\eta_{\sigma_k}, \eta_{\sigma_k+1}, \ldots, \eta_{\tau_k}$ are n i.i.d. random variables with distribution $\overline{\mu}_b$ given by*

$$\overline{\mu}_b(x) := \frac{\mu_b(x)}{c_b} 1_{\{x \geq b\}}.$$

- *When $n \geq 2$, $\Delta_{\sigma_k+1}, \ldots, \Delta_{\tau_k}$ are $n - 1$ i.i.d. random variables distributed like Δ_1 conditionally on $\{\Delta_1 \leq L_b\}$.*

c) *The random variables $\widehat{\Delta}_1, \widehat{\Delta}_2, \ldots$ are i.i.d. geometrically distributed with parameter c_b.*

d) *The distribution of Y_0 is given by an obvious modification: $\tau_0 + 1$ is geometrically distributed as in b), and conditionally on $\{\tau_0 + 1 = n\}$ the distribution of the Δ_i, η_i is the same as described above, except that there is one η_i less, since $\eta_0 := 0$.*

Next we define a mapping $\Phi : \Gamma \to \mathbb{N} \cup \{0\}$. On Γ_1, we simply put $\Phi((1, \eta)) := \eta$, while on Γ_n, $n \geq 2$, we put

$$\Phi\left(\left(n, (\Delta_1, \ldots, \Delta_{n-1}), (\eta_1, \ldots, \eta_n)\right)\right) := \sum_{i=1}^{n} \eta_i - \sum_{i=1}^{n-1} \left\lfloor \frac{3}{2} \log \Delta_i \right\rfloor + 2(n-1) K_b. \quad (3.4)$$

The interpretation is as follows: if Y_k is a *bad block*, i.e. if $Y_k \in \Gamma_n$ with $n \geq 2$, then Y_k will be replaced by a single charge in the new environment sequence ω', and $\Phi(Y_k)$ is exactly the value of this *clustered charge*. The reason why the size of the clustered charge should be given by (3.4) will be clear in the next section. Since $\eta_i \geq b$ and $\Delta_i \leq e^{\frac{2}{3}(b+K_b)}$, it follows from (3.4) that the value of the clustered charge is always greater than $b + (n-1)K_b$, hence it is strictly greater than b, if $n \geq 2$.

We are ready to define the new sequence $\omega' = T_b(\omega)$. First we set

$$\widehat{\eta}_0 := \Phi(Y_0). \qquad (3.5)$$

Then we introduce a sequence of stopping times by setting $S_0 := 0$ and for $k \geq 1$

$$S_k := \inf\left\{n > S_{k-1} : \{\tau_n > \sigma_n\} \text{ or } \{\tau_n = \sigma_n \text{ and } \eta_{\sigma_n} > b\}\right\}$$
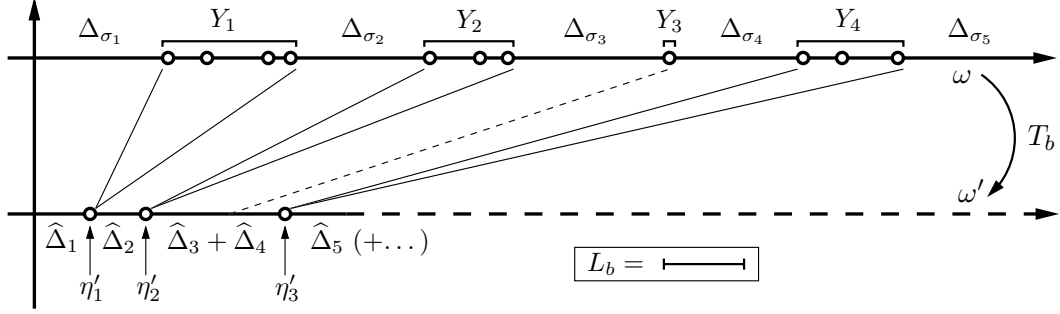
FIGURE 4. An illustrative example of the map T_b sending ω to ω' (the starting ω is the same as in Figure 3). The distances Δ_{σ_k} between blocks are shortened by L_b and become $\widehat{\Delta}_k$, the bad blocks (Y_1, Y_2, Y_4 in this example) are clustered into one single charge, the good charges (Y_3 in this example) are erased, while the charges that are isolated but not good (none in this example) are left unchanged.

(see also Figure 3). The variables $\{S_k\}_{k\geq 0}$ indicate which blocks will survive after the renormalization: the block Y_{S_k} will become the k-th positive charge of ω'. More precisely, we set

$$\eta'_k := \Phi\left(Y_{S_k}\right), \qquad \Delta'_k := \sum_{j=S_{k-1}+1}^{S_k} \widehat{\Delta}_j, \qquad (3.6)$$

and the sequence $(\Delta'_k, \eta'_k)_{k\geq 1}$ defines our new sequence $\omega' =: T_b(\omega)$ (see also Figure 4). Note that the effect of the sequence $\{S_k\}_{k\geq 0}$ is to erase all the *good charges* in ω.

In the next lemma we show that $\{\omega'_n\}_{n\geq 1}$ is indeed i.i.d., and we denote its one-marginal law by $\mu_{b+1} := T_b(\mu_b)$. Observe that $\widehat{\eta}_0$ is not included in the new sequence ω', but it will enter the estimate given below in Proposition 9 (remark that $\widehat{\eta}_0 = 0$ if $\tau_0 = 0$).

Lemma 8. *The random variables Δ'_k, η'_k are all independent. The η'_k are identically distributed and the Δ'_k are identically geometrically distributed, therefore the sequence $\{\omega'_n\}_{n\geq 1}$ is i.i.d. The law $\mu_{b+1}(x)$ of ω'_1 satisfies, for $x > 0$, the following relation:*

$$\mu_{b+1}(x) = q_b\,\mu_b(x)\,\mathbf{1}_{\{x\geq b+1\}} + q_b\,(Q_b\mu_b)(x), \qquad (3.7)$$

where

$$(Q_b\nu)(x) := \sum_{n=1}^{\infty} \sum_{\ell_1,\ldots,\ell_n=1}^{L_b} \left(\prod_{i=1}^{n}(1-c_b)^{\ell_i-1}\right) \sum_{\substack{x_1,\ldots,x_{n+1}\geq b \\ \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n}\lfloor\frac{3}{2}\log\ell_i\rfloor + 2nK_b = x}} \nu(x_1)\cdots\nu(x_{n+1}).$$
$$(3.8)$$

Proof. Introducing the subset A of Γ defined by $A := \{1\} \times [b+1, \infty) \cup \bigcup_{n\geq 2}\Gamma_n$, we see that S_k is nothing but the sequence of times when Y_n visits the set A. In particular, by Lemma 7, the increments $\{S_k - S_{k-1}\}_{k\geq 1}$ are i.i.d. geometrically distributed, independent of the $\widehat{\Delta}_n$, with parameter $\gamma = P(Y_1 \in A)$. (As a matter of fact, one can easily compute $\gamma = (1 - q_b) + q_b\,\tilde{c}_b/c_b$, cf. (3.1), but the precise value of γ is immaterial for the proof.) Therefore the variables Δ'_k are i.i.d. geometrically distributed, with parameter $c_b\gamma$.

Notice that the variables Y_{S_k} are i.i.d. and independent of the S_n. It follows easily that the variables η'_k are i.i.d. and independent from the Δ'_n. The sequence ω'_n is therefore i.i.d. and moreover for $x > 0$ we have $P(\omega'_1 = x) = (c_b \gamma) \cdot P(\eta'_1 = x)$.

It remains to determine the law of $\eta'_1 = \Phi(Y_{S_1})$. Plainly, Y_{S_1} is distributed like Y_1 conditionally on $\{Y_1 \in A\}$, hence

$$P(\omega'_1 = x) = c_b \gamma P(\Phi(Y_1) = x \mid Y_1 \in A) = c_b P(\Phi(Y_1) = x, Y_1 \in A). \qquad (3.9)$$

Since $\{Y_1 \in A\}$ is the disjoint union $\{\tau_1 = \sigma_1, \eta_{\sigma_1} > b\} \cup \bigcup_{n \geq 1}\{\tau_1 - \sigma_1 = n\}$ and since $\Phi(Y_1) = \eta_{\sigma_1}$ if $\tau_1 = \sigma_1$, we can write

$$P(\omega'_1 = x) = c_b P(\tau_1 = \sigma_1) P(\eta_{\sigma_1} = x, \eta_{\sigma_1} > b \mid \tau_1 = \sigma_1)$$

$$+ \sum_{n=1}^{\infty} c_b P(\tau_1 - \sigma_1 = n) P(\Phi(Y_1) = x \mid \tau_1 - \sigma_1 = n).$$

By Lemma 7, the first term in the r.h.s. equals

$$c_b q_b \mathbf{1}_{\{x \geq b+1\}} \bar{\mu}_b(x) = q_b \mathbf{1}_{\{x \geq b+1\}} \mu_b(x),$$

matching with (3.7). Using (3.4) and again Lemma 7, we rewrite the second term as

$$\sum_{n=1}^{\infty} c_b q_b (1-q_b)^n \sum_{\ell_1, \ldots, \ell_n = 1}^{L_b} \sum_{\substack{x_1, \ldots, x_{n+1} \geq b \\ \Phi(n+1, (\ell_i), (x_i)) = x}} P(\hat{\Delta}_{\sigma_1+i} = \ell_i, \eta_{\sigma_1+j-1} = x_j, \mid \tau_1 - \sigma_1 = n)$$

$$= \sum_{n=1}^{\infty} c_b q_b (1-q_b)^n \sum_{\ell_1, \ldots, \ell_n = 1}^{L_b} \sum_{\substack{x_1, \ldots, x_{n+1} \geq b \\ \Phi(n+1, (\ell_i), (x_i)) = x}} \prod_{i=1}^{n} \frac{c_b(1-c_b)^{\ell_i - 1}}{1 - q_b} \prod_{j=1}^{n+1} \frac{\mu_b(x_j)}{c_b}$$

$$= q_b \sum_{n=1}^{\infty} \sum_{\ell_1, \ldots, \ell_n = 1}^{L_b} \sum_{\substack{x_1, \ldots, x_{n+1} \geq b \\ \Phi(n+1, (\ell_i), (x_i)) = x}} \prod_{i=1}^{n} (1-c_b)^{\ell_i - 1} \prod_{j=1}^{n+1} \mu_b(x_j) = q_b (Q_b \mu_b)(x),$$

so that equation (3.7) is proved. $\qquad \square$

As a side remark, we observe that by summing equation (3.7) (or, more easily, equation (3.9)) over $x \geq b+1$ we obtain the following explicit formula for $\mu_{b+1}(0)$:

$$\mu_{b+1}(0) = 1 - (q_b \tilde{c}_b + (1-q_b)c_b),$$

cf. (3.1). Since $\tilde{c}_b \leq c_b = 1 - \mu_b(0)$, it follows that $\mu_{b+1}(0) \geq \mu_b(0)$, i.e. at each renormalization step the density of positive charges decreases.

4. Renormalization of the partition function

In the preceding section we have defined, at each level b of the induction, a renormalizing map T_b acting on the environment sequence ω and producing a renormalized sequence $\omega' = T_b(\omega)$. In this section we show that, by replacing ω by $T_b(\omega)$, one gets an upper bound on the free energy. This will be the key to the proof of Theorem 6 in Section 5. With some abuse of notation, we define the map T_b acting on the positive number C by

$$T_b(C) := (1 + B e^{-K_b} C) \cdot C, \qquad (4.1)$$

where K_b is defined in (3.2) and B is an absolute constant defined in Lemma 10 below. Then we have the following

Proposition 9. *There exists b_0 such that for every $b \geq b_0$, for every $C \in (0, 2\mathcal{C}]$ and for all $N \in \mathbb{N}$, there exists for $\mathbb{P}_{\mu_b}$–a.e. ω, a natural number $n(\omega, N) < \infty$ satisfying*

$$N \leq n(\omega, N), \qquad t_N(T_b(\omega)) \leq t_{n(\omega, N)}(\omega),$$

and such that

$$\mathcal{Z}_{n(\omega, N)}(\omega, C) \leq e^{\hat{\eta}_0} \cdot \mathcal{Z}_N\big(T_b(\omega), T_b(C)\big). \tag{4.2}$$

Proof. We set

$$n(\omega, N) := \tau_{S_N}.$$

The interpretation is as follows: by construction (see Section 3) Y_{S_N} is the N-th block of charges of ω that will survive after the renormalization, and $n(\omega, N)$ is the index of the last positive charge in that block. Therefore it is evident that $n(\omega, N) \geq N$. Also $t_{n(\omega, N)}(\omega) \geq t_N(T_b(\omega))$ is easy to check, because in the renormalization procedure leading from ω to $\omega' = T_b(\omega)$ the distances between charges are shortened (see also Figure 4).

For the rest of the proof, we fix $N \in \mathbb{N}$, and $n := n(\omega, N)$, and we typically drop them from notations. The estimate is purely deterministic and holds for any ω which has the property that $n < \infty$.

We are going to work with the subsets of $\{0, t_1, t_2, \ldots, t_n\}$ and we need some notation. We write $\mathcal{J}$ for the collection of intervals $I_j := \{t_{\sigma_j}, \ldots, t_{\tau_j}\}$, $1 \leq j \leq S_N$. Note that $\bigcup_{j=1}^{S_N} I_j = \{t_{\sigma_1}, \ldots, t_n\}$. In $\mathcal{J}$ there are the N 'bad' intervals $\hat{I}_j := I_{S_j}$, $1 \leq j \leq N$, and the other ones, we call 'good'. Note that the 'good' intervals correspond to what we have called *good charges* in Sections 2 and 3, while the 'bad' intervals correspond to the *bad blocks* and also to the *isolated charges* that are not good. In particular, the good intervals are just single points: for this reason, we also call them 'good points'. The bad intervals may be single points, too. We write $\mathcal{J}^{\mathrm{bad}}$ for the set $\left\{\hat{I}_1, \ldots, \hat{I}_N\right\}$ of bad intervals, and $\mathcal{G} \subset \{t_1, \ldots, t_n\}$ for the subset of good points. The first interval $I_0 = \{0, \ldots, t_{\tau_0}\}$ is somewhat special. In case $\tau_0 = 0$, there is no charge (because $\omega_0 = 0$). In case $\tau_0 > 0$, this interval is of course 'bad', but we keep it separate from the others (remark that we don't take it into $\mathcal{J}$).

We are in the situation where between the bad intervals, there may or may not be good points in $\mathcal{G}$. Also before the first bad interval, i.e. between I_0 and $\hat{I}_1$, there may or may not be good points. If $X \subset \mathcal{J}$, we write $\mathcal{P}(X)$ for the set of subsets $F \subset \{0, t_1, \ldots, t_n\}$ which contain 0 and t_n, and which have the property that it has non void intersection with any interval in X, and empty intersection with any interval in $\mathcal{J} \backslash X$. Then

$$\mathcal{Z}_n(\omega, C) = \sum_{X \subset \mathcal{J}, \, X \ni \hat{I}_N} \sum_{F \in \mathcal{P}(X)} C^{|F|-1} \zeta(F) R(F)$$

$$= \sum_{X \subset \mathcal{J}, \, X \ni \hat{I}_N} \sum_{F \in \mathcal{P}(X)} \zeta(F) \left(C^{|F \cap I_0|-1} R(F \cap I_0)\right) \prod_{I \in X} \left(C^{|F \cap I|} R(F \cap I)\right),$$

where for a finite set $A = \{s_0, \ldots, s_m\}$, $m \geq 1$, we set

$$\zeta(A) := \prod_{i=1}^{m} (s_i - s_{i-1})^{-3/2},$$

and we put $\zeta(A) := 1$ in case A reduces to a single point. We also set

$$R(F) := \exp\left[\sum_{x \in F} \omega_x\right],$$

where we recall that $\omega_0 := 0$. Note that the sum over X is only over those which contain the last interval in $\mathcal{J}$ (which is a bad one, namely $\hat{I}_N = I_{S_N}$), in agreement with (2.8).

The $\zeta(F)$ for $F \in \mathcal{P}(X)$ contains the parts inside the $F \cap I$, and the 'interaction part'. We want to split off this interaction part, and estimate it by a bound which depends only on X. If the intervals in X are (in increasing order) $I_{r_1}, \ldots, I_{r_k} = I_{S_N}$, we write M_j for the largest point of I_{r_j}, and m_j for the smallest. Put

$$\hat{\zeta}(X) := (m_1 - t_{\tau_0})^{-3/2} \prod_{j=2}^{k} (m_j - M_{j-1})^{-3/2}.$$

Then, if $F \in \mathcal{P}(X)$

$$\zeta(F) \leq \hat{\zeta}(X) \zeta(F \cap I_0) \prod_{I \in X} \zeta(F \cap I).$$

The inequality comes from the fact that if x is the largest element of $I \cap F$, and y is the smallest element of $I' \cap F$, I an interval below I', then $(y - x)^{-3/2} \leq (m' - M)^{-3/2}$, m' being the smallest point in I', and M the largest of I. We set for $I \in \mathcal{J}$

$$\alpha(I) := \sum_{F \subset I,\, F \neq \emptyset} \left(C^{|F|} R(F) \zeta(F) \right),$$

and $\alpha(I_0)$ with one C-factor less. With this notation, we have the estimate

$$\mathscr{Z}_n(\omega, C) \leq \sum_{X \subset \mathcal{J},\, X \ni \hat{I}_N} \hat{\zeta}(X) \alpha(I_0) \left[\prod_{I \in X} \alpha(I) \right].$$

Next we claim that

$$\alpha(I) \leq C \mathrm{e}^{2(|I|-1)K_b} R(I) \zeta(I) =: C \exp[\Phi_I], \tag{4.3}$$

where K_b is defined in (3.2). This is evident if I contains just one point, say t_j, in which case $\alpha(I) = C \mathrm{e}^{\eta_j}$. If I contains more than one point, then first observe that there are $2^{|I|} - 1 \leq \mathrm{e}^{(|I|-1)K_b}$ possibilities to choose a non-empty subset $F \subset I$, because $K_b \geq 2\log 2$ for $b \geq b_0$ and b_0 large. Assume $I = \{t_\sigma, t_{\sigma+1}, \ldots, t_{\sigma+R} = t_\tau\}$, and $F = \{t_{j_1}, \ldots, t_{j_m}\}$, so that

$$C^{|F|} R(F) \zeta(F) = C^m \exp\left[\sum_{r=1}^{m} \eta_{j_r} \right] \prod_{r=2}^{m} (t_{j_r} - t_{j_{r-1}})^{-3/2}. \tag{4.4}$$

We can bound this from above by replacing $(t_{j_r} - t_{j_{r-1}})^{-3/2}$ by $(t_{j_{r-1}+1} - t_{j_{r-1}})^{-3/2}$, and for the remaining gaps $t_{i+1} - t_i$, we simply use $1 \leq \mathrm{e}^{K_b} \mathrm{e}^{\eta_i} (t_{i+1} - t_i)^{-3/2}$ (we recall that $t_{i+1} - t_i \leq \mathrm{e}^{\frac{2}{3}(b+K_b)}$ and $\eta_i \geq b$ by construction). Therefore, the right hand side of (4.4) is bounded by

$$C^m \mathrm{e}^{K_b(R-m+1)} R(I) \zeta(I) \leq C \mathrm{e}^{(|I|-1)K_b} R(I) \zeta(I),$$

having used that $K_b \geq \log 2C \geq \log C$, and (4.3) follows. Note that the definition of Φ_I matches with (3.4).

For the first interval I_0, we have a factor C less on the right hand side of (4.3), and therefore from (4.3) it follows that $\alpha(I_0) \leq \mathrm{e}^{\hat{\eta}_0}$ (we recall that $\hat{\eta}_0$ is defined in (3.5)). Combining, we get

$$\mathscr{Z}_n(\omega, C) \leq \mathrm{e}^{\hat{\eta}_0} \sum_{X \subset \mathcal{J},\, X \ni \hat{I}_N} \hat{\zeta}(X) C^{|X|} \left[\prod_{I \in X} \exp[\Phi_I] \right].$$

We have thus succeeded in clustering all the bad blocks.

Let us now fix $X' \subset \mathcal{J}^{\mathrm{bad}}$, $X' \ni \hat{I}_N$. Summing over X with $X \cap \mathcal{J}^{\mathrm{bad}} = X'$ amounts to summing over all subsets of $\mathcal{G}$. Assume the intervals in X' are described by the sequence

$$0 \leq k_0 < j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_m \leq k_m = n$$

with $I_0 = \{t_0, \ldots, t_{k_0}\}$, $X' = \{\{t_{j_1}, \ldots, t_{k_1}\}, \ldots, \{t_{j_m}, \ldots, t_{k_m}\}\}$, $m = |X'|$. Write also $\mathcal{G}_r$, $1 \leq r \leq m$, for the set of good points between $t_{k_{r-1}}$ and t_{j_r}, and $\overline{\mathcal{G}}_r := \mathcal{G}_r \cup \{t_{k_{r-1}}, t_{j_r}\}$. Then we can write the summation over X with $X \cap \mathcal{J}^{\mathrm{bad}} = X'$ as the summation over $A_1 \subset \mathcal{G}_1, \ldots, A_m \subset \mathcal{G}_m$. For a single $t_j \in \mathcal{G}$ we have $\Phi_{\{t_j\}} = b$. We therefore get

$$\sum_{X \in \mathcal{J}: X \cap \mathcal{J}^{\mathrm{bad}} = X'} \hat{\zeta}(X) C^{|X|} \left[\prod_{I \in X} \exp[\Phi_I] \right] = \left[\prod_{I \in X'} \exp[\Phi_I] \right] \prod_{r=1}^{m} \Xi\left(b, C, \overline{\mathcal{G}}_r\right),$$

where for a finite subset $\mathcal{A} = \{s_0, s_1, \ldots, s_m\}$, the s_i ordered increasingly, we set

$$\Xi(b, C, \mathcal{A}) := \sum_{\substack{A \subset \mathcal{A} \\ A \supset \{s_0, s_m\}}} C^{|A|-1} \, e^{(|A|-2)b} \zeta(A). \tag{4.5}$$

Remark that the points in $\overline{\mathcal{G}}_r$ have inter-distances all $> e^{\frac{2}{3}(b+K_b)}$. We can therefore apply Lemma 11 below, and obtain

$$\Xi\left(b, C, \overline{\mathcal{G}}_r\right) \leq \frac{C\left(1 + BC e^{-K_b}\right)}{\left(t_{j_r} - t_{k_{r-1}}\right)^{3/2}}.$$

We thus get

$$\mathcal{Z}_n(\omega, C) \leq e^{\hat{\eta}_0} \sum_{X' \subset \mathcal{J}^{\mathrm{bad}}, \, X' \ni \hat{I}_N} \left[\prod_{I \in X'} \exp[\Phi_I] \right] \prod_{r=1}^{|X'|} \frac{C\left(1 + BC e^{-K_b}\right)}{\left(t_{j_r} - t_{k_{r-1}}\right)^{3/2}}.$$

Note that $C\left(1 + BC e^{-K_b}\right)$ is by definition $T_b(C)$, cf. (4.1).

We are almost done. For the renormalized environment $\omega' = T_b(\omega)$, defined in Section 3, we call $\{t'_k, \eta'_k\}_{k \in \mathbb{N}}$ the locations and intensities of the positive charges of ω' (see §2.1). Consider the following correspondence: to each bad interval $\hat{I}_l \in \mathcal{J}^{\mathrm{bad}}$ we associate the positive charge $\eta'_l \in \omega'$. Notice in fact that $\eta'_l = \Phi_{\hat{I}_l}$, see (4.3) and (3.6). Moreover, given two bad intervals $\hat{I}_l = \{t_{j_l}, \ldots, t_{k_l}\}$, $\hat{I}_m = \{t_{j_m}, \ldots, t_{k_m}\} \in \mathcal{J}^{\mathrm{bad}}$, with $\hat{I}_l$ below $\hat{I}_m$, we can bound $t_{j_m} - t_{k_l} > t'_m - t'_l$, because in passing from ω to ω' the distances Δ_{σ_i} between intervals have been shortened to $\widehat{\Delta}_i$ (it may be useful to look at Figure 4). Therefore we can bound $\mathcal{Z}_n(\omega, C)$ by

$$\mathcal{Z}_n(\omega, C) \leq e^{\hat{\eta}_0} \sum_{\substack{A = \{a_0, \ldots, a_k\} \subset \{0, \ldots, N\} \\ 0 = a_0 < \ldots < a_k = N}} \prod_{i=1}^{|A|} e^{\eta'_{a_i}} \frac{T_b(C)}{\left(t'_{a_i} - t'_{a_{i-1}}\right)^{3/2}} = e^{\hat{\eta}_0} \, \mathcal{Z}_N\left(\omega', T_b(C)\right),$$

where the last equality is just the definition (2.6) of the partition function, and the proof is completed. $\qquad \square$

We conclude this section with an auxiliary result (Lemma 11 below) that is used in the preceding proof. We first need a basic renewal theory lemma.

Lemma 10. *There exist positive constants B and K_0 such that for every $C > 0$ and for all $K \geq K_0 + \log C$ the following relation holds for every $N \in \mathbb{N}$:*

$$\Theta_N^+ := \sum_{k=1}^{N} \sum_{\substack{j_1, \ldots, j_{k-1} \in \mathbb{N} \\ 0 =: j_0 < j_1 < \ldots < j_{k-1} < j_k := N}} \prod_{\ell=1}^{k} \frac{Ce^{-K}}{(j_\ell - j_{\ell-1})^{3/2}} \leq \left(1 + BCe^{-K}\right) \frac{Ce^{-K}}{N^{3/2}}.$$

Proof. Defining the constant $A := \left(\sum_{n=1}^{\infty} n^{-3/2}\right)^{-1}$, we set

$$q(n) := \frac{A}{n^{3/2}}, \qquad \gamma := \frac{C}{A}e^{-K}.$$

Note that we can write

$$\Theta_n^+ = \sum_{k=1}^{\infty} \gamma^k q^{*k}(n), \tag{4.6}$$

where $q^{*k}(\cdot)$ denotes the k-fold convolution of the probability distribution $q(\cdot)$ with itself.

Let us prove by induction that $q^{*k}(n) \leq A k^{5/2}/n^{3/2}$ for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$. The case $k = 1$ holds by definition of $q(\cdot)$. For the inductive step, if k is even, $k = 2m$, we can write

$$q^{*(2m)}(n) \leq 2 \sum_{\ell=1}^{\lfloor n/2 \rfloor} q^{*m}(\ell)\, q^{*m}(n-\ell) \leq 2\, A\, m^{5/2} \sum_{\ell=1}^{\lfloor n/2 \rfloor} q^{*m}(\ell)\, \frac{1}{(n-\ell)^{3/2}}$$

$$\leq \frac{2\, A\, m^{5/2}}{(n/2)^{3/2}} \sum_{\ell=1}^{\infty} q^{*m}(\ell) = \frac{A\, (2m)^{5/2}}{n^{3/2}},$$

and the odd case follows analogously. Then by (4.6) we can bound Θ_n^+ by $Ag\,(\gamma)\, n^{-3/2}$, where

$$g\,(\gamma) := \sum_{k=1}^{\infty} k^{5/2} \gamma^k \leq \gamma + 8\gamma^2,$$

provided $0 < \gamma \leq \gamma_0$, γ_0 sufficiently small. Let us set $K_0 := -\log(A\,\gamma_0)$, then if $K \geq K_0 + \log C$, we have $\gamma \leq \gamma_0$, and therefore

$$\Theta_n^+ \leq \left(\gamma + 8\,\gamma^2\right) \frac{A}{n^{3/2}} \leq \left(1 + \frac{8C}{A}e^{-K}\right) \frac{Ce^{-K}}{n^{3/2}}.$$

The proof is completed by setting $B := 8/A$. $\qquad\square$

Lemma 11. *Let K_0 and B be the constants of Lemma 10. Then $\forall N \geq 2$, $\forall b > 0$, $\forall C > 0$, $\forall K \geq K_0 + \log C$ and for all $\mathcal{T} = (t_0, \ldots, t_N) \in \mathbb{N}^{N+1}$ with*

$$t_0 < t_1 < \ldots < t_N \qquad and \qquad t_n - t_{n-1} > e^{\frac{2}{3}(b+K)}, \quad \forall n = 1, \ldots, N,$$

the following relation holds for $\Xi(b, C, \mathcal{T})$ (defined in (4.5)):

$$\Xi(b, C, \mathcal{T}) \leq \left(1 + BCe^{-K}\right) \frac{C}{(t_n - t_1)^{3/2}}.$$

Proof. Expanding the definition of $\Xi(b, C, \mathcal{T})$ we can write:

$$\Xi(b, C, \mathcal{T}) \;=\; e^{-b} \sum_{k=1}^{N} \sum_{\substack{j_1,\ldots,j_{k-1}\in\mathbb{N} \\ 0=:j_0<j_1<\ldots<j_{k-1}<j_k:=N}} \prod_{\ell=1}^{k} \frac{C\,e^b}{(t_{j_\ell} - t_{j_{\ell-1}})^{3/2}}\,.$$

Fix a configuration $j_1 < \cdots < j_{k-1} \subset \{1, \cdots, N-1\}$. Then we have

$$\prod_{\ell=1}^{k} \frac{1}{t_{j_\ell} - t_{j_{\ell-1}}} = \frac{1}{t_N - t_0} \frac{\sum_{\ell=1}^{k}(t_{j_\ell} - t_{j_{\ell-1}})}{\prod_{\ell=1}^{k}(t_{j_\ell} - t_{j_{\ell-1}})}$$

$$= \frac{1}{t_N - t_0} \sum_{\ell=1}^{k} \prod_{\ell'\in\{1,\cdots,k\}\setminus\ell} \frac{1}{t_{j_{\ell'}} - t_{j_{\ell'-1}}}$$

$$\leq \frac{e^{-\frac{2}{3}(b+K)(k-1)}}{t_N - t_0} \sum_{\ell=1}^{k} \prod_{\ell'\in\{1,\cdots,k\}\setminus\ell} \frac{1}{j_{\ell'} - j_{\ell'-1}} \quad \left[\text{since } t_{j_\ell} - t_{j_{\ell-1}} \geq e^{2(b+K)/3}\,(j_\ell - j_{\ell-1})\right]$$

$$= \frac{e^{-\frac{2}{3}(b+K)(k-1)}}{t_N - t_0} \frac{\sum_{\ell=1}^{k}(j_\ell - j_{\ell-1})}{\prod_{\ell=1}^{k}(j_\ell - j_{\ell-1})} = \frac{e^{-\frac{2}{3}(b+K)(k-1)}}{t_N - t_0} \frac{N}{\prod_{\ell=1}^{k}(j_\ell - j_{\ell-1})}\,.$$

Therefore we get:

$$\Xi(b, C, \mathcal{T}) \;\leq\; e^{-b}\, \frac{e^{b+K}\,N^{3/2}}{(t_N - t_0)^{3/2}} \sum_{k=1}^{N} \sum_{\substack{j_1,\ldots,j_{k-1}\in\mathbb{N} \\ 0=:j_0<j_1<\ldots<j_{k-1}<j_k:=N}} \prod_{\ell=1}^{k} \frac{C\,e^{-K}}{(j_\ell - j_{\ell-1})^{3/2}}$$

$$\leq\; \left(1 + BCe^{-K}\right) \frac{C}{(t_N - t_0)^{3/2}}\,,$$

having used Lemma 10, and we are done. $\qquad\qquad\square$

5. Proof of Theorem 6

The starting point in the proof of Theorem 6 is Proposition 9, which immediately gives an upper bound on the free energy $\mathrm{F}(\mu_b, C)$, for every $b \geq b_0$ and $C \in (0, 2\mathcal{C}]$. In fact, since the limit in (2.8) holds $\mathbb{P}_{\mu_b}(\mathrm{d}\omega)$–a.s., we can take it along the (random) subsequence $\{n(\omega, N)\}_{N\in\mathbb{N}}$ and relation (4.2) yields

$$\mathrm{F}\big(\mu_b, C\big) \;\leq\; \liminf_{N\to\infty} \frac{1}{t_{n(\omega,N)}(\omega)} \log \mathcal{Z}_N\big(T_b(\omega), T_b(C)\big)\,, \qquad \mathbb{P}_{\mu_b}(\mathrm{d}\omega)\text{–a.s.,}$$

and since $t_{n(\omega,N)}(\omega) \geq t_N(T_b(\omega))$ we obtain, again by (2.8),

$$\mathrm{F}\big(\mu_b, C\big) \;\leq\; \mathrm{F}\big(\mu_{b+1}, T_b(C)\big)\,.$$

Note in fact that μ_{b+1} is by definition the one-marginal law of $T_b(\omega)$, when ω has law μ_b, see Section 3. We now iterate this relation starting from $b = \beta$: if we set

$$\mathcal{C}_\beta := \mathcal{C} \qquad \text{and} \qquad \mathcal{C}_b := \big(T_{b-1} \cdot T_{b-2} \cdots T_{\beta+1} \cdot T_\beta\big)(\mathcal{C}) \quad \text{for } b > \beta\,, \qquad (5.1)$$

since in §5.2 below we show that $\mathcal{C}_b \leq 2\mathcal{C}$ for every $b \geq \beta$, provided β is sufficiently large, we can write

$$\mathrm{F}\big(\mu_\beta, \mathcal{C}\big) \;\leq\; \mathrm{F}\big(\mu_b, \mathcal{C}_b\big)\,, \qquad \forall \beta \geq b_0,\ \forall b \geq \beta\,. \qquad (5.2)$$

We stress that, though not explicitly indicated, both the law μ_b and the constant $\mathcal{C}_b$ depend also on β, which is the starting level of our procedure: however β is kept fixed in all our

arguments. We recall that to prove Theorem 6 it suffices to show that $\mathrm{F}\big(\mu_\beta, \mathcal{C}\big) = 0$ when β is large (see §2.2). Hence by (5.2) we are left with showing that, if we fix β sufficiently large, $\mathrm{F}\big(\mu_b, \mathcal{C}_b\big)$ vanishes as $b \to \infty$.

To estimate $\mathrm{F}\big(\mu_b, \mathcal{C}_b\big)$, we start from a very rough upper bound on the partition function: from the definition (2.6) we can write for every $n \in \mathbb{N}$

$$
\mathcal{Z}_n(\omega, \mathcal{C}) \ \leq \ e^{\sum_{i=1}^n (\eta_i + \log(\mathcal{C}/A))} \left(\sum_{k=1}^n \ \sum_{\substack{j_1, \ldots, j_{k-1} \in \mathbb{N} \\ 0=:j_0 < j_1 < \ldots < j_{k-1} < j_k = n}} \ \prod_{\ell=1}^k \frac{A}{(t_{j_k} - t_{j_{k-1}})^{3/2}} \right),
$$

where $A := \big(\sum_{m=1}^\infty m^{-3/2} \big)^{-1} < 1$ is the constant that makes $m \mapsto A/m^{3/2}$ a probability law. With this choice, the term in parenthesis in the r.h.s. above is bounded from above by the probability that a renewal process with step law $A/m^{3/2}$ visits the point t_n, hence it is less than 1 and we have

$$
\mathcal{Z}_n(\omega, \mathcal{C}) \ \leq \ \exp\left(\sum_{i=1}^n \left(\eta_i + \log \frac{\mathcal{C}}{A} \right) \right) \ = \ \exp\left(\sum_{j=1}^{t_n(\omega)} \left(\omega_j + \log \frac{\mathcal{C}}{A} \, \mathbf{1}_{\{\omega_j > 0\}} \right) \right)
$$

Now, if the sequence ω is i.i.d. with marginal law μ, with $0 < \mu(0) < 1$, by (2.8) we have

$$
\mathrm{F}(\mu, \mathcal{C}) \ \leq \ \lim_{n \to \infty} \frac{1}{t_n(\omega)} \sum_{j=1}^{t_n(\omega)} \left(\omega_j + \log \frac{\mathcal{C}}{A} \, \mathbf{1}_{\{\omega_j > 0\}} \right) \ = \ \mathbb{E}_\mu(\omega_1) + \mathbb{P}_\mu\big(\omega_1 > 0\big) \cdot \log \frac{\mathcal{C}}{A},
$$

having used that $t_n(\omega) \to \infty$, $\mathbb{P}_\mu(\mathrm{d}\omega)$–a.s., and the strong law of the large numbers. Combining this bound with (5.2), we get

$$
\mathrm{F}\big(\mu_\beta, \mathcal{C}\big) \ \leq \ \mathrm{F}\big(\mu_b, \mathcal{C}_b\big) \ \leq \ \mathbb{E}_{\mu_b}(\omega_1) \ + \ \big(1 - \mu_b(0)\big) \cdot \log \frac{\mathcal{C}_b}{A}, \tag{5.3}
$$

for every $\beta \geq b_0$ and for every $b \geq \beta$.

We are left with estimating the r.h.s. of (5.3). To this purpose, we exploit the stochastic domination on μ_b given by the following

Proposition 12. *There exists a finite b_1 (depending on $\mathcal{C} > 0$ and $c > \frac{2}{3}$) such that for all $b \geq \beta \geq b_1$ we have:*

$$
\mu_b(x) \leq e^{-\frac{2}{3}x - \sqrt{x}} \qquad \forall x \geq b. \tag{5.4}
$$

Applying (5.4) to (5.3), for any fixed $\beta \geq \beta_0 := b_0 \vee b_1$, we obtain for every $b \geq \beta$

$$
\mathrm{F}\big(\mu_\beta, \mathcal{C}\big) \ \leq \ \sum_{x=b}^\infty x \, e^{-\frac{2}{3}x - \sqrt{x}} \ + \ \left(\log \frac{\mathcal{C}_b}{A} \right) \sum_{x=b}^\infty e^{-\frac{2}{3}x - \sqrt{x}}.
$$

It is clear that both the sums in the r.h.s. can be made arbitrarily small by taking b large. Moreover, in §5.2 below we show that $\mathcal{C}_b \leq 2\mathcal{C}$ for all b. Therefore, by letting $b \to \infty$, we have shown that $\mathrm{F}\big(\mu_\beta, \mathcal{C}\big) = 0$ for all $\beta \geq \beta_0$, and this completes the proof of Theorem 6. $\square$

The proof of Proposition 12 is given in §5.1 below. Before that, we need to establish two technical lemmas.

Lemma 13. *Let $m \in \mathbb{N}$, $m \geq 2$, let $b \in \mathbb{N}$, $b \geq \beta \vee 100$, and $z \geq mb$. Then*

$$A_{m,b}(z) := \sum_{\substack{x_1,\ldots,x_m \geq b \\ x_1+\cdots+x_m=z}} \exp\left[-\sum_{i=1}^{m} \sqrt{x_i}\right] \leq e^{-\sqrt{z} - \frac{m-1}{4}\sqrt{b}}.$$

Proof. We first treat the case $m = 2$. There we have for the left hand side

$$A_{2,b}(z) \leq 2 \sum_{x=b}^{\lceil z/2 \rceil} \exp\left[-\sqrt{x} - \sqrt{z-x}\right].$$

For $x \leq z/2$

$$\sqrt{z-x} = \sqrt{z}\sqrt{1 - \frac{x}{z}}.$$

For $t \leq 1/2$, we have from concavity of the square root function

$$\sqrt{1-t} \geq 1 - \left(2 - \sqrt{2}\right) t,$$

and therefore for $x \leq z/2$

$$\sqrt{z-x} \geq \sqrt{z} - \left(2 - \sqrt{2}\right) \frac{x}{\sqrt{z}}$$
$$\geq \sqrt{z} - \left(\sqrt{2} - 1\right) \sqrt{x}.$$

Hence,

$$\sum_{x=b}^{z-b} \exp\left[-\sqrt{x} - \sqrt{z-x}\right] \leq 2e^{-\sqrt{z}} \sum_{x=b}^{\infty} \exp\left[-\frac{1}{2}\sqrt{x}\right].$$

$$\sum_{x=b}^{\infty} \exp\left[-\frac{1}{2}\sqrt{x}\right] \leq \int_{b-1}^{\infty} e^{-\sqrt{t}/2} dt = 8 \int_{\sqrt{b-1}/2}^{\infty} t e^{-t} dt$$
$$= 8 \left(\frac{\sqrt{b-1}}{2} + 1\right) e^{-\sqrt{b-1}/2} \leq \frac{1}{2} \exp\left[-\frac{1}{4}\sqrt{b}\right],$$

if $b \geq 100$. This proves the claim for $m = 2$.

The general case follows by induction on m. Assume $m \geq 3$. Then

$$A_{m,b}(z) = \sum_{x_1=b}^{z-(m-1)b} e^{-\sqrt{x_1}} \sum_{\substack{x_2,\ldots,x_m \geq b \\ x_2+\cdots+x_m=z-x_1}} \exp\left[-\sum_{k=2}^{m} \sqrt{x_k}\right]$$
$$\leq \sum_{x_1=b}^{z-(m-1)b} e^{-\sqrt{x_1}} \exp\left[-\sqrt{z-x_1} - \frac{m-2}{4}\sqrt{b}\right]$$
$$\leq e^{-\frac{m-2}{4}\sqrt{b}} \sum_{x=b}^{z-b} \exp\left[-\sqrt{x} - \sqrt{z-x}\right] \leq e^{-\sqrt{z} - \frac{m-1}{4}\sqrt{b}},$$

by induction and the $m = 2$ case. $\qquad\square$

Lemma 14. *There exists $b_2 = b_2(\mathcal{C})$ such that for every $b \geq b_2$ and $n \in \mathbb{N}$ we have:*

$$B_{n,b}(x) = \sum_{\ell_1,\cdots,\ell_n=1}^{\lfloor e^{\frac{2}{3}(b+K_b)} \rfloor} e^{-\frac{2}{3}(x + \sum_{i=1}^{n}(\lfloor \frac{3}{2} \log \ell_i \rfloor - 2K_b))} A_{n+1,b}\left(x + \sum_{i=1}^{n}\left(\left\lfloor \frac{3}{2} \log \ell_i \right\rfloor - 2K_b\right)\right)$$
$$\leq e^{-\frac{2}{3}x - \sqrt{x}} e^{-\frac{n}{8}\sqrt{b}},$$

where $A_{m,b}(x)$ is defined in Lemma 13 and K_b is defined by (3.2).

Proof. By Lemma 13, we have:

$$A_{n+1,b}\left(x + \sum_{i=1}^{n}\left(\left\lfloor \frac{3}{2} \log \ell_i \right\rfloor - 2K_b\right)\right) \leq e^{-\frac{n}{4}\sqrt{b}} e^{-\sqrt{x + \sum_{i=1}^{n}(\lfloor \frac{3}{2} \log \ell_i \rfloor - 2K_b)}}.$$

Since for $0 \leq a \leq b$ we have $\sqrt{b-a} \geq \sqrt{b} - \sqrt{a}$, it follows that

$$\sqrt{x + \sum_{i=1}^{n}\left(\left\lfloor \frac{3}{2} \log \ell_i \right\rfloor - 2K_b\right)} \geq \sqrt{x} - \sqrt{\sum_{i=1}^{n}\left(2K_b - \left\lfloor \frac{3}{2} \log \ell_i \right\rfloor\right)} \geq \sqrt{x} - \sqrt{2nK_b},$$

because $\ell_i \geq 1$. Therefore

$$A_{n+1,b}\left(x + \sum_{i=1}^{n}\left(\left\lfloor \frac{3}{2} \log \ell_i \right\rfloor - 2K_b\right)\right) \leq e^{-\frac{n}{4}\sqrt{b}} e^{-\sqrt{x}} e^{\sqrt{2nK_b}}.$$

Plugging this in the definition of $B_{n,b}(x)$ and using the fact that $\lfloor \frac{3}{2} \log \ell_i \rfloor \geq \frac{3}{2} \log \ell_i - 1$ yields:

$$B_{n,b}(x) \leq e^{-\frac{2}{3}x - \sqrt{x}} e^{-\frac{n}{4}\sqrt{b}} e^{\frac{4}{3}nK_b + \sqrt{2nK_b}} \left(\sum_{\ell_1=1}^{\lfloor e^{\frac{2}{3}(b+K_b)} \rfloor} \frac{e}{\ell_1}\right)^n$$
$$\leq e^{-\frac{2}{3}x - \sqrt{x}} e^{-\frac{n}{4}\sqrt{b}} e^{\frac{4}{3}nK_b + \sqrt{2nK_b}} \left(e \cdot \frac{2}{3}(b+K_b)\right)^n$$
$$= e^{-\frac{2}{3}x - \sqrt{x}} e^{-\frac{n}{8}\sqrt{b}} e^{(-\frac{1}{8}\sqrt{b} + \frac{4}{3}K_b + \frac{\sqrt{2K_b}}{\sqrt{n}} + 1 + \log \frac{2}{3}(b+K_b))n}.$$

Observe that, for every fixed $\mathcal{C} > 0$, we have $-\frac{1}{8}\sqrt{b} + \frac{4}{3}K_b + \frac{\sqrt{K_b}}{\sqrt{n}} + 1 + \log \frac{2}{3}(b+K_b) \to -\infty$ as $b \to \infty$ (recall the definition (3.2) of K_b). Therefore there exists $b_2 = b_2(\mathcal{C})$, such that for all $b \geq b_2$, $B_{n,b}(x) \leq e^{-\frac{2}{3}x - \sqrt{x}} e^{-\frac{n}{8}\sqrt{b}}$. $\qquad \square$

5.1. **Proof of Proposition 12.** The law μ_b is defined recursively by equations (3.7) and (3.8), starting from $\mu_\beta = (1 - e^{-c\beta})\delta_{\{0\}} + e^{-c\beta}\delta_{\{\beta\}}$. In particular we have

$$\mu_\beta(\beta) = e^{-\beta c}, \qquad \mu_\beta(x) = 0, \quad \forall x \geq \beta + 1,$$

and for $x \geq \beta$

$$\mu_{b+1}(x) \leq \mu_b(x) + \sum_{n=1}^{\infty} \sum_{\ell_1,\cdots,\ell_n=1}^{\lfloor e^{\frac{2}{3}(b+K_b)} \rfloor} \sum_{\substack{x_1,\cdots,x_{n+1} \geq b \\ \sum_{i=1}^{n+1} x_i - \sum_{j=1}^{n}(\lfloor \frac{3}{2} \log \ell_j \rfloor - 2K_b) = x}} \mu_b(x_1) \cdots \mu_b(x_{n+1}). \quad (5.5)$$

We recall that μ_b is supported on $\{0\} \cup \{b, b+1, \ldots\}$.

Let us prove equation (5.4) by induction on $b = \beta, \beta + 1, \cdots$. For $b = \beta$ we have

$$\mu_\beta(\beta) = e^{-\beta c} = e^{-\frac{2}{3}\beta - (c - \frac{2}{3})\beta} \leq e^{-\frac{2}{3}\beta - \sqrt{\beta}},$$

whenever $\beta \geq b_3(c) := \frac{1}{(c - \frac{2}{3})^2}$. In particular, (5.4) holds for $b = \beta$, provided $\beta \geq b_3(c)$.

Assume now that $\mu_\alpha(x) \leq e^{-\frac{2}{3}x - \sqrt{x}}$, for all $\beta \leq \alpha \leq b$. Then, using (5.5) and the fact that $\mu_\beta(x) = 0$, $\forall x \geq \beta + 1$, we have:

$$\mu_{b+1}(x) \leq \sum_{\alpha=\beta}^{b} \sum_{n=1}^{\infty} \sum_{\ell_1, \cdots, \ell_n = 1}^{\lfloor e^{\frac{2}{3}(\alpha + K_b)} \rfloor} \sum_{\substack{x_1, \cdots, x_{n+1} \geq \alpha \\ \sum_{i=1}^{n+1} x_i - \sum_{j=1}^{n}(\lfloor \frac{3}{2} \log \ell_j \rfloor - 2K_b) = x}} \mu_\alpha(x_1) \cdots \mu_\alpha(x_{n+1}),$$

Plugging in the induction assumption yields:

$$\mu_{b+1}(x) \leq \sum_{\alpha=\beta}^{b} \sum_{n=1}^{\infty} B_{n,\alpha}(x),$$

where $B_{n,\alpha}(x)$ is defined in Lemma 14. Assuming $\beta \geq b_2$ and using Lemma 14 gives:

$$\mu_{b+1}(x) \leq e^{-\frac{2}{3}x - \sqrt{x}} \sum_{\alpha=\beta}^{b} \sum_{n=1}^{\infty} e^{-\frac{n}{8}\sqrt{\alpha}}$$

$$\leq e^{-\frac{2}{3}x - \sqrt{x}} \sum_{\alpha=\beta}^{\infty} e^{-\frac{\sqrt{\alpha}}{10}}$$

$$\leq e^{-\frac{2}{3}x - \sqrt{x}} \int_{\beta-1}^{\infty} e^{-\frac{\sqrt{\alpha}}{10}} = e^{-\frac{2}{3}x - \sqrt{x}} \left(20(10 + \sqrt{\beta-1}) e^{-\frac{\sqrt{\beta-1}}{10}} \right)$$

$$\leq e^{-\frac{2}{3}x - \sqrt{x}}, \text{ when } \beta \geq 6000.$$

Hence Proposition 12 holds for all $\beta \geq b_1(\mathcal{C}, c) := \max\{b_2(\mathcal{C}), b_3(c), 6000\}$. $\qquad \square$

5.2. Bounding the constant $\mathcal{C}_b$.

We are going to show that $\mathcal{C}_b \leq 2\mathcal{C}$ for all $b \geq \beta$. We recall the definition (2.11) of $\mathcal{C}_b$ (see also (5.1)):

$$\mathcal{C}_\beta := \mathcal{C}, \qquad \mathcal{C}_b = T^{b-1}(\mathcal{C}_{b-1}) = \left(1 + B\mathcal{C}_{b-1} e^{-K_{b-1}}\right) \cdot \mathcal{C}_{b-1}, \qquad b > \beta,$$

where $\mathcal{C}$ is the constant appearing in the definition (2.6) of the partition function. Passing to logarithms we get

$$\log \mathcal{C}_b = \log \mathcal{C} + \sum_{a=\beta}^{b-1} \log \left(1 + B\mathcal{C}_a e^{-K_a}\right).$$

We prove that $\mathcal{C}_b \leq 2\mathcal{C}$ by induction: the case $b = \beta$ is trivial since $\mathcal{C}_\beta = \mathcal{C}$. Assuming that $\mathcal{C}_a \leq 2\mathcal{C}$ for all $a \in \{\beta, \ldots, b-1\}$, we have

$$\log \mathcal{C}_b \leq \log \mathcal{C} + \sum_{a=\beta}^{b-1} \log \left(1 + 2B\mathcal{C} e^{-K_a}\right).$$

By the definition (3.2) of K_a we have $K_a \geq \big(K_0 + \log(2\mathcal{C}) + 2\log a\big) - 1$, therefore

$$\log \mathcal{C}_b \ \leq \ \log \mathcal{C} \ + \ \sum_{a=\beta}^{b-1} \log \big(1 + e^{1-K_0} \frac{B}{a^2}\big) \ \leq \ \log \mathcal{C} \ + \ B \, e^{1-K_0} \sum_{a=\beta}^{\infty} \frac{1}{a^2} \,.$$

Therefore, if we choose β sufficiently large, we get $\log \mathcal{C}_b \ \leq \ \log(2\mathcal{C})$ and we are done.

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