Level lines of the massive Gaussian free field

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The Gaussian free field

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Definition

The Gaussian free field in D with Dirichlet boundary conditions is a centered Gaussian process h indexed by smooth functions with compact support in D and whose covariance is given by, for f and g two such functions,

$$\mathbb{E}[(h,f)(h,g)] = \int_{D\times D} f(x) G_D(x,y) g(y) dxdy$$

where G_D is the Green function of (minus) the Laplacian in D with Dirichlet boundary conditions, that is G_D is the inverse (in the sense of distributions) of $-\Delta$ in D with Dirichlet boundary conditions.

The Gaussian free field with boundary conditions

Let $D \subset \mathbb{C}$ be a bounded, open and simply connected domain.

Definition

Let $f : \partial D \to \mathbb{R}$ be a bounded function with finitely many discontinuity points. A GFF in D is said to have boundary conditions f if it has the same law as $h + \phi_f$ where h has the law of a GFF in D with Dirichlet boundary conditions and ϕ_f is the unique harmonic extension of f in D.

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ϕ_f is the unique solution to the boundary value problem

$$\begin{cases} -\Delta u(z) = 0, & z \in D \\ u(z) = f(z), & z \in \partial D. \end{cases}$$

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$$\mathbb{E}[(h,f)(h,g)] = \int_{D\times D} f(x) G_D^m(x,y) g(y) dxdy$$

where G_D^m is the inverse (in the sense of distributions) of the operator $-\Delta + m^2(\cdot)$ in D with Dirichlet boundary conditions.

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Let $f : \partial D \to \mathbb{R}$ be a bounded function with finitely many discontinuity points. A massive GFF in D with mass m is said to have boundary conditions f if it has the same law as $h + \phi_f^m$ where h has the law of a massive GFF in D with mass m and Dirichlet boundary conditions and ϕ_f^m is the unique massive harmonic extension of f in D.

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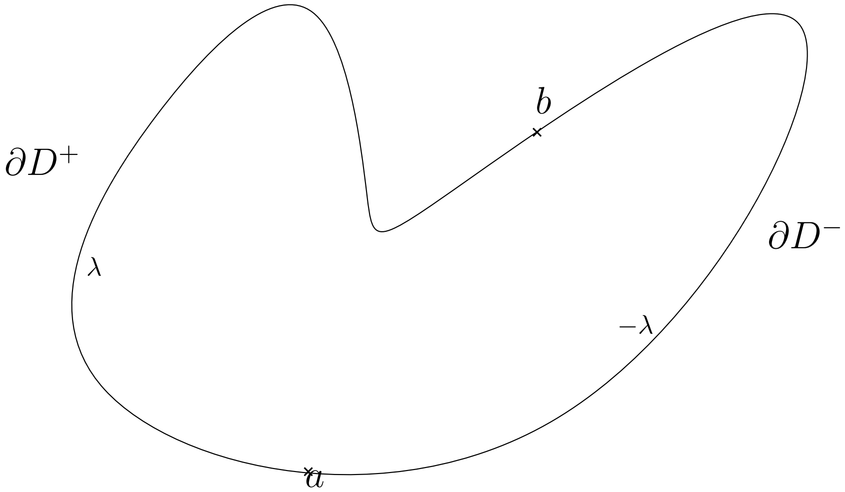
Coupling GFF-SLE_4

Set $\lambda := \sqrt{\pi/8}$. Let $a, b \in \partial D$ and let $\phi : D \to \mathbb{R}$ be the unique harmonic function with boundary conditions λ on ∂D^+ and $-\lambda$ on ∂D^-.



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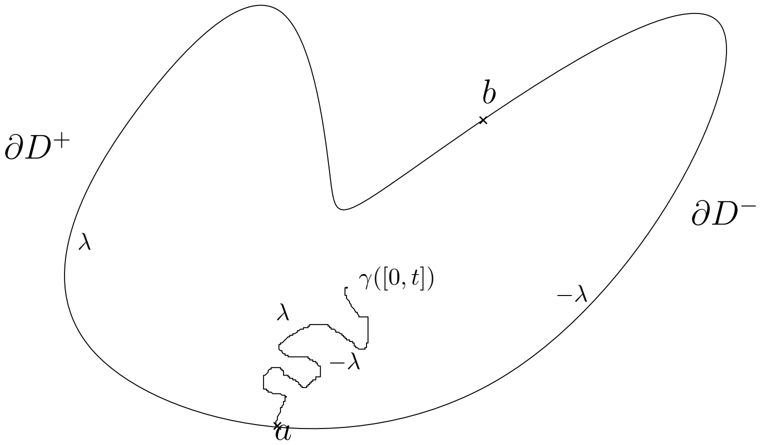
$$h + \phi = h_t + \phi_t$$

where h_t is a GFF with Dirichlet boundary conditions in $D \setminus \gamma([0, t])$ and ϕ_t is the unique harmonic function with boundary conditions λ on ∂D^+ and the left side of $\gamma([0, t])$ and $-\lambda$ on ∂D^- and the right side of $\gamma([0, t])$. See Dubédat 2015, Miller-Sheffield 2016.

Coupling GFF-SLE_4



Conditionally on $\gamma([0,t])$, $h + \phi = h_t + \phi_t$.

Absolute continuity

The massive GFF with mass m and boundary conditions λ on ∂D^+ and $-\lambda$ on ∂D^- is absolutely continuous with respect to the GFF with the same boundary conditions. The Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}\mathbb{P}_{\mathrm{mGFF}}}{\mathrm{d}\mathbb{P}_{\mathrm{GFF}}}(h+\phi) = \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \int m^2(z) : (h+\phi)^2(z) : dz \right)$$

where: $(h+\phi)^2(z)$: is the Wick square of $h+\phi$, i.e. a renormalized version of $(h+\phi)^2$ and the above random variable is defined as the L^2-limit of an appropriate sequence of approximations.

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Idea: use this fact to re-weight the law of the coupling GFF-SLE$_4$. (Bauer-Bernard-Cantini, 2009)

A level line of the massive GFF

Theorem (P., 23+)

Denote by $\mathbb{P}$ the law of the coupling between a GFF $h + \phi$ in D and an SLE_4 curve γ in D from a to b. Let $m : D \to \mathbb{R}_+$ be a continuous and bounded function. Define a new probability measure $\tilde{\mathbb{P}}$ by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}((h+\phi,\gamma)) := \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2}\int m^2(z) : (h+\phi)^2(z) : dz \right).$$

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Then, under $\tilde{\mathbb{P}}$, the marginal law of $h+\phi$ is that of a massive GFF in D with mass m and boundary conditions λ on ∂D^+ and $-\lambda$ on ∂D^-. Moreover, under $\tilde{\mathbb{P}}$, for $t \geq 0$, conditionally on $\gamma([0,t])$, $h+\phi = h_t+\phi_t$ where $h_t+\phi_t$ has the law of a massive GFF with mass m in $D_t = D \setminus \gamma([0,t])$ and boundary conditions λ on ∂D^+ and the left side of $\gamma([0,t])$ and $-\lambda$ on ∂D^- and the right side of $\gamma([0,t])$.

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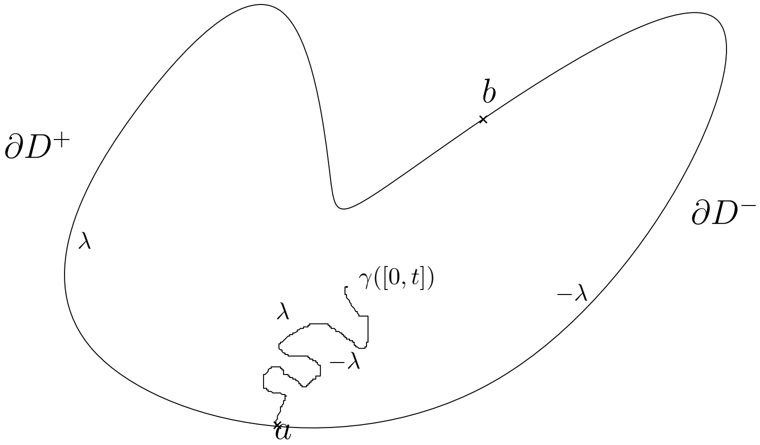

Under $\tilde{\mathbb{P}}$, conditionally on $\gamma([0,t])$, $h + \phi = h_t + \phi_t$, where $h_t + \phi_t$ has the law of a massive GFF in D_t with boundary conditions described in the figure above.

Idea of the proof

When we pretend that $(h+\phi)^2$ is well-defined and forget about the Wick renormalization $:(h+\phi)^2:$, we have, for $f : D \to \mathbb{R}$ a smooth function with compact support,

$$\begin{aligned}
&\tilde{\mathbb{E}}[\exp(i(h+\phi,f))|\gamma([0,t])] \\
&= \frac{1}{\mathcal{Z}_t}\mathbb{E}\left[\exp(i(h+\phi,f))\exp\left(-\frac{1}{2}\int_D m^2(z)(h+\phi)^2(z)dz\right)|\gamma([0,t])\right] \\
&= \frac{1}{\mathcal{Z}_t}\mathbb{E}\left[\exp(i(h_t+\phi_t,f))\exp\left(-\frac{1}{2}\int_{D_t} m^2(z)(h_t+\phi_t)^2(z)dz\right)|\gamma([0,t])\right].
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Similarly, we obtain that

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\mathcal{Z}_t &= \mathbb{E}\Big[\exp\Big(-\frac{1}{2}\int_D m^2(z)(h+\phi)^2(z)dz\Big)|\gamma([0,t])\Big] \\
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$$\tilde{\mathbb{E}}[\exp(i(h+\phi,f))|\gamma([0,t])]$$
$$= \frac{1}{\mathcal{Z}_t}\mathbb{E}\Bigg[\exp(i(h_t+\phi_t,f))\exp\Bigg(-\frac{1}{2}\int_{D_t} m^2(z)(h_t+\phi_t)^2(z)dz\Bigg)|\gamma([0,t])\Bigg].$$

Similarly, we obtain that

$$\mathcal{Z}_t = \mathbb{E}\Bigg[\exp\Bigg(-\frac{1}{2}\int_{D_t} m^2(z)(h_t+\phi_t)^2(z)dz\Bigg)|\gamma([0,t])\Bigg].$$

We can thus recognize the Radon-Nikodym derivative of the massive GFF in D_t with mass m and boundary conditions ϕ_t with respect to the GFF in D_t with the same boundary conditions. These computations can be made rigorous: these equalities actually hold when considering $:(h+\phi)^2:$.

massive SLE_4

In view of the previous theorem, we call the marginal law of γ under $\tilde{\mathbb{P}}$ massive SLE_4 with mass m in D from a to b. By absolute continuity of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, massive SLE_4 is absolutely continuous with respect to SLE_4.

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Question: can we get a less implicit definition of massive SLE_4?

Discrete approximation: the massive harmonic explorer

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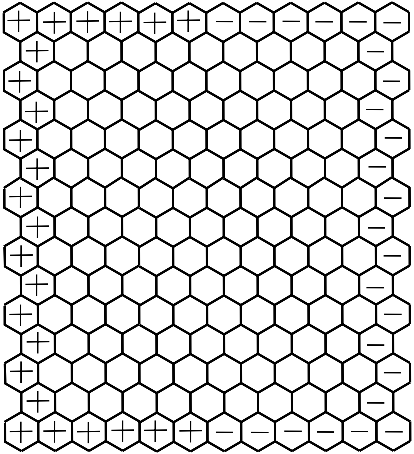

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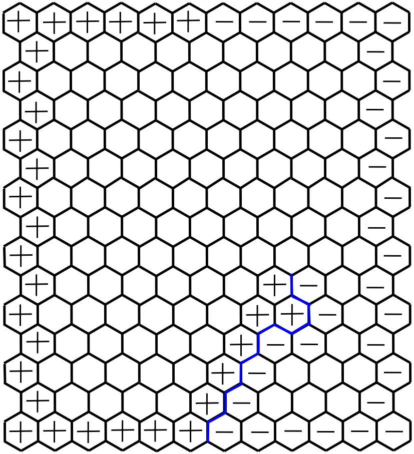

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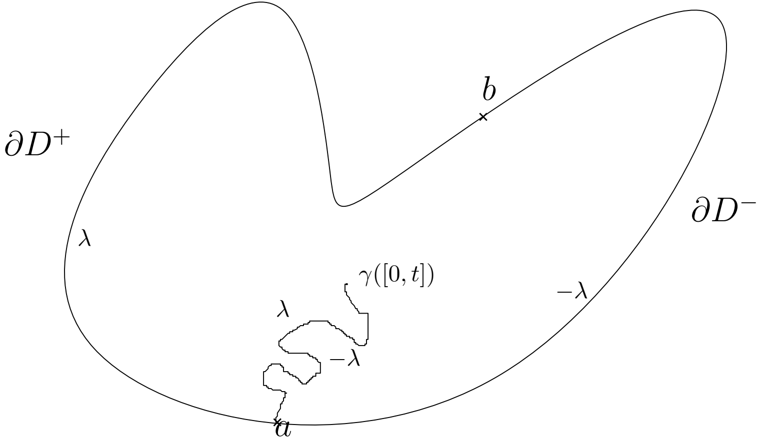
This model was proposed by Makarov and Smirnov (2009) as a massive perturbation of the harmonic explorer.

Theorem (P., 23)

Let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. Assume that $(D_\delta; a_\delta, b_\delta)_\delta$ converges to $(D; a, b)$ in the Carathéodory topology. Then the massive harmonic explorer with mass m converges weakly to massive SLE_4 with mass m in D from a to b.

Martingale characterization

For a simple curve $\gamma : [0, \infty) \to D$, denote by H_t^m the unique massive harmonic function with boundary conditions λ on ∂D^+ and the left side of $\gamma([0,t])$ and $-\lambda$ on ∂D^- and the right side of $\gamma([0,t])$.



Martingale characterization

Let $m : D \to \mathbb{R}_+$ be a bounded and continuous function. For a simple curve $\gamma : [0, \infty) \to D$, denote by H_t^m the unique massive harmonic function with mass m and boundary conditions $1/2$ on ∂D^+ and the left side of $\gamma([0, t])$ and $-\lambda$ on ∂D^- and the right side of $\gamma([0, t])$.

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Theorem (P., 23)

Let γ be a random simple curve in D from a to b and denote by $(\mathcal{F}_t)_t$ the filtration generated by γ. Assume that for any $z \in D$, $(H_t^m(z), t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t)_t$.

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Remark: the proof uses the martingale property of $(H_t^m(z), t \geq 0)$ to identify the driving function of $\varphi(\gamma)$, where $\varphi : D \to \mathbb{H}$ is a conformal map from D to the complex upper half-plane $\mathbb{H}$ such that $\varphi(a) = 0$ and $\varphi(b) = \infty$.

Another result and a question

- (P., 23+): using the same ideas as those behind the theorem about the coupling massive GFF-massive SLE_4, one can define a massive version of CLE_4 that can be coupled to a massive GFF with Dirichlet boundary conditions. This yields a "loop decomposition" of the massive GFF.

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- Question: would the same ideas work to define the level lines of other interacting Euclidean QFTs that are absolutely continuous with respect to the GFF?