

# Pinning Model, Universality and Rough Paths

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INHOMOGENEOUS RANDOM SYSTEMS ~ I.H.P.

PARIS, 23 JAN 2018

- Very preliminary, few "new" results
- Massimiliano Gubinelli, Giulia Comi
- (Rongfeng Sun, Nikas Zygouras)

DISORDER RELEVANCE & SINGULAR PDEs

## SUMMARY

1. Introduction and Results
2. Pinning Model
3. Rough Paths
4. Robust Analysis of our Equation
5. Conclusion

# 1. INTRODUCTION AND RESULTS

## OUR EQUATION

- Fix  $H \in (0, \frac{1}{2})$
- Singular Kernel  $q_t = \frac{1}{t^{\frac{1}{2}-H}} \mathbb{1}_{\{t>0\}}$  ( $L^2([0,1])$ )
- Our equation for  $Z: [0,1] \rightarrow \mathbb{R}$

$$\textcircled{\star} \quad Z_t = 1 + \int_0^\infty Z_n q_{t-n} dw_n \quad t \in [0,1]$$

$w = (w_n)_{n \in [0,1]}$  Brownian motion

- Explicit solution:  $Z_t = 1 + \underbrace{\int_0^\infty q_{t-n} dw_n}_{X_t^{(1)}} + \underbrace{\int_0^\infty \int_0^\infty q_{s-n} q_{t-s} dw_n dw_s}_{X_t^{(2)}} + \dots$   
(Picard iteration)

## OUR PLAN

- Give a robust ("pathwise") meaning to  $\star$

$$\star \quad Z_t = 1 + \underbrace{\int_0^\infty z_r q_{t-r} dw_r}_{= 1 + J_t(z)}$$

for a given path  $w \in C^{\frac{1}{2}-}$  (deterministic!)

- $x_t^{(1)} = J_t(1) = \int_0^\infty q_{t-r} dw_r$  is canonically defined
- Higher order terms  $x_t^{(n)}$  with  $n \geq 2$  are NOT CANONICAL
- Need to "enrich" the noise  $w$ : more information needed

## MAIN RESULT

- Recall that  $X_t^{(1)} = \int_0^\infty \frac{1}{(t-\tau)^{\frac{1}{2}-H}} dW_\tau$  is canonically defined

THEOREM Fix  $H > \frac{1}{4}$ . "Define"  $I_t = \int_0^t X_\tau^{(1)} dW_\tau$  (non canonical!)

Then  $J_t(z)$  is CANONICALLY defined for a wide space of  $z$ .

In this space one can solve  $z_t = 1 + J_t(z)$   $\star$

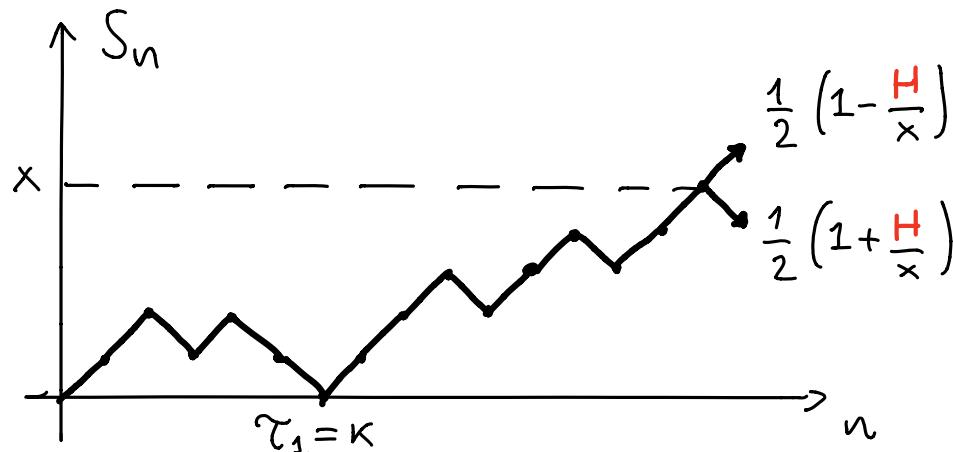
The solution  $z$  is a CONTINUOUS function of  $\underbrace{(w, I)}$   
"ENRICHED NOISE"

- Toy model for REGULARITY STRUCTURES

## 2. PINNING MODEL

# BESSEL RANDOM WALK

(K. Alexander)

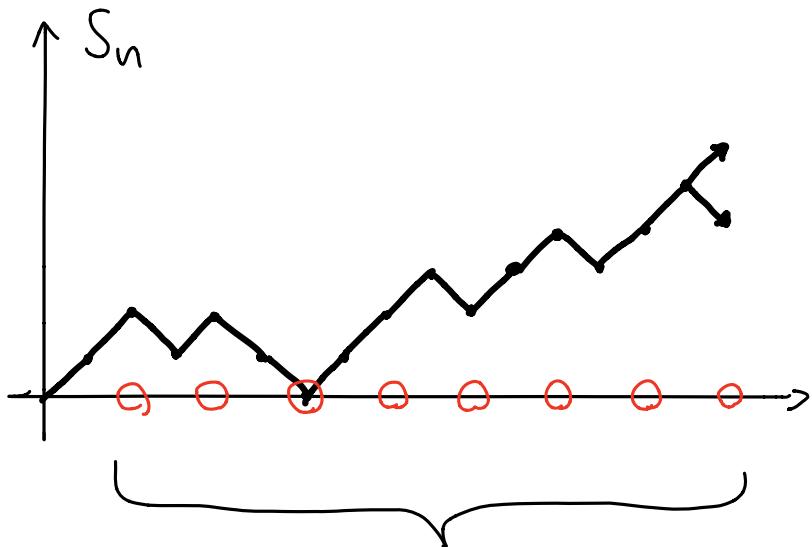


$$E[S_{n+1} | S_n = x] \sim \frac{H}{x}$$

$$\begin{aligned} P(\tau_1 = k) &\sim \frac{C}{k^{\frac{3}{2} + H}} \\ &= \frac{C}{k^{1+\alpha}} \end{aligned}$$

- $\alpha = \frac{1}{2} + H$  so  $H \in (0, \frac{1}{2}) \Leftrightarrow \alpha \in (\frac{1}{2}, 1)$
- RENEWAL FUNCTION  $U_k = P(S_k = 0) \sim \frac{\tilde{C}}{k^{\frac{1}{2}-H}} = \frac{\tilde{C}}{k^{1-\alpha}}$
- WE ONLY NEED  $\bar{\tau} = \{\tau_1, \tau_2, \dots\} = \{n \geq 0 : S_n = 0\}$   
(RENEWAL PROCESS)

## DISORDER



- $\omega = (\omega_n)_{n \in \mathbb{N}}$  I.I.D., law  $\mathbb{P}$
- $\mathbb{E}[\omega_n] = 0 \quad \text{VAR}[\omega_n] = 1$
- $\lambda(\beta) = \log \mathbb{E}[e^{\beta \omega_1}] \text{ FINITE}$

- REWARDS / PENALTIES  $e^{\beta \omega_n - \lambda(\beta) + h}$   $\beta > 0, h \in \mathbb{R}$

- $\mathbb{E} [ e^{\beta \omega_n - \lambda(\beta) + h} ] = e^h$

## PINNING MODEL

- Fix  $N \in \mathbb{N}$ ,  $\beta \geq 0$ ,  $h \in \mathbb{R}$ , a typical realization of  $\omega$

$$P_N^\omega(dS) = \frac{e^{\sum_{n=1}^N (\beta \omega_n - \lambda(\beta) + h) \mathbb{I}_{\{S_n=0\}}}}{Z_N^\omega} P(dS)$$

- Behavior of  $P_N^\omega$  as  $N \rightarrow \infty$   
$$\begin{cases} \text{LOCALIZED at zero} & (h < h_c(\beta)) \\ \text{DELOCALIZED away from zero} & (h > h_c(\beta)) \end{cases}$$
- Phase transition: for  $H > 0$  ( $\alpha > \frac{1}{2}$ ) DISORDER IS RELEVANT

Any fixed  $\beta > 0$  radically different from  $\beta = 0$  [...]

## WEAK DISORDER AND CONTINUUM LIMIT

- DISORDER RELEVANCE is linked to the existence of CONTINUUM LIMITS  
as  $N \rightarrow \infty$ , in a suitable WEAK DISORDER regime  $\beta = \beta_N \rightarrow 0$ ,  $h = h_N \rightarrow 0$

- PARTITION FUNCTION  $Z_N^\omega = E \left[ e^{\sum_{n=1}^N (\beta \omega_n - \lambda(\beta) + h) \mathbb{I}_{\{S_n=0\}}} \right]$

THEOREM (C., Sun, Zygouras)

Fix  $h=0$  (for simpl.),  $\beta = \beta_N = \frac{1}{N^H}$ . Reverse  $\tilde{\omega}_i := \omega_{N-i}$ . Then

$$(Z_{Nt}^{\tilde{\omega}})_{t \in [0,1]} \xrightarrow[N \rightarrow \infty]{d} (z_t)_{t \in [0,1]} \quad \text{in } C([0,1])$$

## POLYNOMIAL CHAOS

$$Z_N^\omega = E \left[ e^{\sum_{n=1}^N (\beta \omega_n - \lambda(\beta)) \mathbb{1}_{\{S_n=0\}}} \right]$$

$$= E \left[ \prod_{n=1}^N \left\{ 1 + \beta \cdot \eta_n \cdot \mathbb{1}_{\{S_n=0\}} \right\} \right]$$

$$\eta_n = \frac{e^{\beta \omega_n - \lambda(\beta)} - 1}{\beta}$$

$$E[\eta_n] = 0, \quad \text{VAR}[\eta_n] \approx 1$$

$$= 1 + \sum_{n=1}^N \underbrace{\beta \eta_n P(S_n=0)}_{\frac{1}{n^{\frac{1}{2}-H}}} + \sum_{1 \leq m < n \leq N} \underbrace{\beta^2 \eta_m \eta_n P(S_m=0, S_n=0)}_{\frac{1}{m^{\frac{1}{2}-H} (n-m)^{\frac{1}{2}-H}}} + \dots$$

$$\frac{1}{n^{\frac{1}{2}-H}}$$

$$\simeq 1 + \int_0^1 \frac{1}{r^{\frac{1}{2}-H}} dW_r + \iint_{0 < r < s < 1} \frac{1}{r^{\frac{1}{2}-H} (s-r)^{\frac{1}{2}-H}} dW_r dW_s + \dots$$

$$= Z_1$$

## RESULTS FOR PINNING MODEL

- The last step approximates MULTI-LINEAR POLYNOMIALS of ANY order with MULTIPLE WIENER INTEGRALS  $\rightsquigarrow \underline{\text{INDEPENDENCE}}$  of the  $\eta_n$  is used
- $Z_{Nt}^\omega$  solves approximately our DIFFERENTIAL EQUATION  $\star$

$$Z_{Nt}^\omega \approx 1 + \int_0^\infty Z_{Nr}^\omega q_{t-n}^{(N)} dw_n^{(N)}$$

where

$$w_t^{(N)} = \frac{1}{\sqrt{N}} \{ \eta_1 + \eta_2 + \dots + \eta_{Nt} \}$$

- We expect that  $Z_{Nt}^\omega$  is approx. a CONTINUOUS FUNCTION of  $w^{(N)}$  and

$$I_t^{(N)} = \frac{1}{N^{\frac{1}{2}+H}} \sum_{1 \leq m < n \leq Nt} \frac{\eta_m \eta_n}{(n-m)^{\frac{1}{2}-H}}$$

## RESULTS FOR PINNING MODEL

CONJECTURE. Assume that  $H \in (\frac{1}{4}, \frac{1}{2})$ , i.e.  $\alpha \in (\frac{3}{4}, 1)$

If  $(w^{(n)}, I^{(n)}) \xrightarrow[N \rightarrow \infty]{d(\dots)} (w, I)$ , then  $(Z_{Nt}^{\omega}) \xrightarrow[N \rightarrow \infty]{d} (z_t)$

- Convergence of  $(w^{(n)}, I^{(n)})$  holds much beyond independent  $\eta_n$   
 $\rightsquigarrow$  a form of UNIVERSALITY
- For  $H \in (0, \frac{1}{4}] \Leftrightarrow \alpha \in [\frac{1}{2}, \frac{3}{4}]$  a FINITE NUMBER of statistics beyond  $w^{(n)}$  and  $I^{(n)}$  should be relevant ( $\rightsquigarrow$  SINGULAR PDEs)
- The case  $H=0$ , i.e.  $\alpha = \frac{1}{2}$  is MARGINAL / CRITICAL ...

### 3. ROUGH PATHS

## OUR EQUATION

- All functions will be defined on  $[0,1]$ .
- Hölder space  $C^\alpha = \{ f: [0,1] \rightarrow \mathbb{R} : |f_t - f_s| \leq C|t-s|^\alpha\}$
- Fix a path  $w \in C^{\frac{1}{2}-}$

★ 
$$\begin{aligned} z_t &= 1 + \int_0^\infty z_r q_{t-r} dw_r \\ &= 1 + \int_0^\infty z_r \frac{1}{(t-r)^{\frac{1}{2}-H}} dw_r \end{aligned}$$

ROUGH:  $z \in C^{H-}$       SMOOTH, BUT DIVERGES AS  $r \rightarrow t$

## CONTROLLED PATHS (M. Gubinelli)

- Fix  $\alpha, \beta \in (0, 1)$ . Take  $f \in C^\alpha$  and  $g, h \in C^\beta$ .

- DEFINITION  $h$  is controlled by  $g$  (through  $f$ ) iff

$$h_t - h_s = f_s(g_t - g_s) + O((t-s)^{\alpha+\beta})$$

"Increments of  $h$  are not worse than those of  $g$ "

- EXAMPLE  $h_t = \Phi(g_t)$  is controlled by  $g_t$ , for  $\Phi$  smooth

## YOUNG INTEGRAL

- Fix  $x \in C^\alpha$ ,  $w \in C^\beta$  with  $\boxed{\alpha + \beta > 1}$

- Canonical definition of the integral

$$I_t = \int_0^t x_s dw_s \in C^\beta$$

$$= \lim_{|\Pi| \rightarrow 0} \sum_{t_i \in \Pi} x_{t_i} (w_{t_{i+1}} - w_{t_i})$$

- CHARACTERIZING PROPERTY:  $I$  is controlled by  $w$  through  $x$

$$I_t - I_s = x_s (w_t - w_s) + O(|t-s|^{\alpha+\beta})$$

# Rough Paths (T. Lyons)

- For  $\boxed{\alpha + \beta < 1}$ , NO CANONICAL DEFINITION of  $\int x_n dw_n$   
 Ex: Ito vs. Stratonovich for  $x_n = w_n$
- We "choose"  $I_t = \int_0^t x_n dw_n \in C^\beta$  as some path satisfying  

$$I_t - I_s = x_s(w_t - w_s) + \underbrace{O(|t-s|^{\alpha+\beta})}_{\mathbb{X}W_{s,t}}$$
- $I \iff$  Remainder  $\mathbb{X}W_{s,t} = I_t - I_s - x_s(w_t - w_s) = " \int_s^t (x_n - x_s) dw_n "$   

$$\begin{cases} \mathbb{X}W_{s,t} = O(|t-s|^{\alpha+\beta}) \\ \mathbb{X}W_{s,t} - (\mathbb{X}W_{s,u} + \mathbb{X}W_{u,t}) = (x_u - x_s)(w_t - w_u) \end{cases} \quad (\text{CHEN})$$

## ROUGH INTEGRAL

- Assume now  $\alpha + \beta < 1$  but  $2\alpha + \beta > 1$

For us  $\alpha = H-$ ,  $\beta = \frac{1}{2} -$   $\Rightarrow H > \frac{1}{4}$

- ROUGH INTEGRAL. There is a CANONICAL DEFINITION of  $\int z_n dw_n$  for a wide class  $z \in D_x$  (depending on the choice  $I_t = \int_0^t x_n dw_n$ )

$$D_x = \left\{ (z, z') \in C^\alpha \times C^\alpha : z_t - z_s = z'_s (x_t - x_s) + O(|t-s|^{2\alpha}) \right\}$$
$$= \text{paths } z \in C^\alpha \text{ controlled by } x \in C^\alpha \text{ through } z' \in C^\alpha$$

(This is the space where we will solve our equation  $\star$ )

## SINGULAR KERNEL (Hairev)

- It remains to introduce a SINGULAR KERNEL

$$q_r \approx \frac{1}{|r|^\gamma} = \sum_{n \geq 0} (2^n)^\gamma \cdot \underbrace{\varphi_n(r)}_{\text{SMOOTH}} \cdot \prod_{\{|w| \leq \frac{1}{2^n}\}}$$

THEOREM. Assume that  $\gamma < \beta$ , where  $w \in C^\beta$ .

Then there is a canonical definition of

$$J_t(z) = \int_0^\infty z_n q_{t-n} dw_n \quad \text{for all } z \in D_x$$

such that  $J_t(z) \in C^{\beta - \gamma}$

LOSS OF REGULARITY!

## 4. ROBUST ANALYSIS OF OUR EQUATION

## BACK TO OUR EQUATION

★

$$z_t = 1 + \int_0^\infty z_n q_{t-n} dw_n$$

$$\begin{cases} q_s = \frac{1}{s^{\frac{1}{2}-H}} \mathbf{1}_{\{s>0\}} \\ w \in C^{\frac{1}{2}-} = C^\beta \end{cases}$$

1. Define  $x \in C^{H-} = C^\alpha$  canonically by

$$x_t = J_t(1) = \int_0^\infty q_{t-n} dw_n$$

2. Choose  $I_t = " \int_0^t x_n dw_n "$  (non canonical), equivalently

$$W_{s,t} = " \int_s^t (x_n - x_s) dw_n " = O(|t-s|^{H+\frac{1}{2}-}) + (\text{CHEN})$$

# TOPOLOGIES

- ENRICHED NOISE  $(w, \mathbb{X}w)$

$$\|w\|_{\frac{1}{2}-} := \sup_{0 \leq s < t \leq 1} \frac{|w_t - w_s|}{|t-s|^{\frac{1}{2}-}}$$

$$\|\mathbb{X}w\|_{H^{+\frac{1}{2}-}} := \sup_{0 \leq s < t \leq 1} \frac{|\mathbb{X}w_{s,t}|}{|t-s|^{H^{+\frac{1}{2}-}}}$$

- CONTROLLED PATHS

$$D_x = \left\{ (z, z'): z_t - z_s = z'_s (x_t - x_s) + \underbrace{o(|t-s|^{2H})}_{z^R_{s,t}} \right\}$$

$$\|z\|_{D_x} := \|z'\|_{H_-} + \|z^R\|_{2H_-}$$

## MAIN RESULT

$$\textcircled{\star} \quad z_t = 1 + \int_0^\infty z_r q_{t-r} dw_r \\ = 1 + J_t(z)$$

• THEOREM Assume  $H \in (\frac{1}{4}, 1)$ . Fix  $(w, \mathbb{X}\mathbb{W})$  and  $x$  as above.

- ▶  $J_t(z)$  is canonically defined for all  $z \in D_x$ ; moreover it defines a controlled path with  $J_t'(z) = z_t$ .
- ▶ The map  $z \mapsto 1 + J(z)$  is a contraction on  $D_x$  for short time
- ▶ There exists a unique solution  $z = 1 + J(z)$  of  $\textcircled{\star}$ , which depends continuously on  $(w, \mathbb{X}\mathbb{W})$ .

## CONCLUSIONS

- 1-dimensional stochastic equation, motivated by PINNING
- Robust analysis for  $H \in (\frac{1}{4}, \frac{1}{2})$  (and extend to  $H \in (0, \frac{1}{2})$ )
- Rough Paths techniques + ideas from Regularity Structures
- Ultimate goal (dream?): push the analysis to  $H=0$

Merci