The critical 2d Stochastic Heat Flow and related models

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Seoul National University $\sim$ *Intensive Lecture Series*

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Lecture notes (work in progress)

THE CRITICAL 2D STOCHASTIC HEAT FLOW AND RELATED MODELS

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Abstract. This is a **preliminary draft** of lecture notes, in which we review recent progress in the study of the stochastic heat equation and its discrete analogue, the directed polymer model, in spatial dimension 2. It was discovered that a phase transition exists on an intermediate disorder scale, with Edwards-Wilkinson (Gaussian) fluctuations in the sub-critical regime. In the critical window, a unique scaling limit has been identified and was named the *critical 2d stohcastic heat flow*. This gives a meaning to the solution of the SHE in the critical dimension 2, which lies beyond existing solution theories for singular SPDEs. We will outline the proof ideas for these results, introduce the key ingredients, and discuss related literature on disordered systems and singular SPDEs. A list of open questions is also provided.

Plan of the course

- Lectures 1–2 (Caravenna)

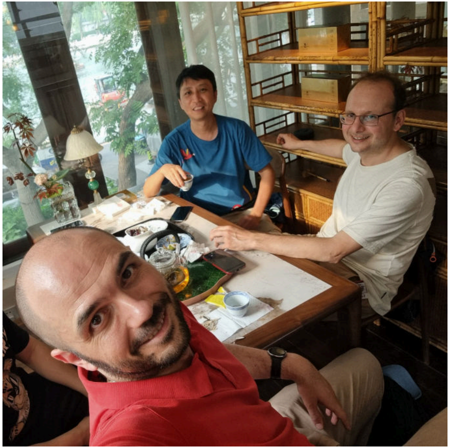
  Overview, sub-critical dimension

- Lectures 3–4 (Sun)

  Critical dimension, sub-critical disorder

- Lectures 5–6 (Zygouras)

  Critical dimension and critical disorder



In a nuthshell

Stochastic Heat Equation (SHE)

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} u(t,x)} + \beta\, u(t,x)\, \xi(t,x) \qquad t \geq 0,\ x \in \mathbb{R}^d$$

Singular random potential $\xi(t,x)$ "space-time white noise"

Main result

We construct a natural candidate solution of SHE in space dimension $d = 2$

called the critical 2d Stochastic Heat Flow

Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

- Fundamental PDE + universal random potential $\xi(t,x)$

  white noise = "continuum" i.i.d. random variables

- KPZ equation [Kardar–Parisi–Zhang *PRL* 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x) \qquad \text{(KPZ)}$$

Cole–Hopf transformation $h(t,x) = \log u(t,x)$

Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

$\xi(t,x)$ is a distribution $\quad\rightsquigarrow\quad$ u(t,x) expected $\begin{cases} \text{non-smooth function} & d = 1 \\ \text{genuine distribution} & d \geq 2 \end{cases}$

Product $u(t,x)\,\xi(t,x)$ unclear: no classical space to solve SHE (as a PDE)

Stochastic integral for $d = 1$ [Ito–Walsh, Da Prato–Zabczyk]

SHE solution $u(t,x) > 0$

starting from $u(0,\cdot) \geq 0$

$\rightsquigarrow$ "KPZ solution" $h(t,x) = \log u(t,x)$

The role of dimension

Revolution in 2010s: **robust solution theories** for **sub-critical SPDEs**

[Hairer *Invent. Math.* 14] [Gubinelli–Imkeller–Perkowski *Forum Math Pi* 15] [...]

SHE and KPZ: robust theories apply only for $d = 1$

Space-time blow-up $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$

$$\partial_t \tilde{u}(t,x) = \Delta_x \tilde{u}(t,x) + \varepsilon^{\frac{2-d}{2}} \, \beta \, \tilde{u}(t,x) \, \tilde{\xi}(t,x)$$

as $\varepsilon \downarrow 0$ the noise strength $\begin{cases} \text{vanishes} & d < 2 & \text{sub-critical / super-renorm.} \\ \text{unchanged} & d = 2 & \textbf{critical} \text{ / renormalizable} \\ \text{diverges} & d > 2 & \text{super-critical / sub-renorm.} \end{cases}$

What can we do?

Main results: critical dimension $d = 2$ (other dimensions: later)

Regularized noise $\xi_N(t,x)$ $\rightsquigarrow$ well-defined solution $u_N(t,x)$

(discretization, mollification, Fourier cutoff, ...)

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, u_N(t,x)\, \xi_N(t,x) \qquad \text{(reg-SHE)}$$

Fix $\xi_N(t,x) \xrightarrow[N\to\infty]{} \xi(t,x)$

Does $u_N(t,x)$ converge to some interesting limit?

No! Unless some kind of renormalization is performed

Which notion of convergence?

Do not expect pointwise convergence to a limiting random field

Space-average $\varphi : \mathbb{R}^2 \to \mathbb{R}$

$$u_N(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\varphi) \qquad (?)$$

i.e. convergence as random measure on $\mathbb{R}^2$ $u_N(t,x) \geq 0$

$$u_N(t,x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\mathrm{d}x) \qquad (?)$$

Still no interesting limit without renormalization

What do we mean by renormalization?

We take $\beta = \beta_N \sim \dfrac{\hat{\beta}}{\sqrt{\log N}} \xrightarrow[N\to\infty]{} 0$ for specific $\hat{\beta} \in (0,\infty)$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta_N \, u_N(t,x) \, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \quad \text{(say)} \end{cases} \qquad \text{(reg-SHE)}$$

Formally $\beta_N \, u_N(t,x) \, \xi_N(t,x) \longrightarrow 0$ *but actually not!* (singular product)

Proposition [Bertini–Cancrini *J. Phys. A* 98] [C.S.Z. *EJP* 19]

for $\hat{\beta} = \sqrt{\pi}$ $\qquad \mathbb{V}\mathrm{ar}\big[\, u_N(t,\varphi) \,\big] \xrightarrow[N\to\infty]{} K_t(\varphi,\varphi) > 0$

Main result

Theorem [C.S.Z. *Invent. Math.* 2023]

Take $\beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$ more precisely for some $\vartheta \in \mathbb{R}$

$$\beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N} \right)$$

Then u_N converges in law to a unique and non-trivial limit $\mathscr{U}^\vartheta$

$$\big(u_N(t,x)\,\mathrm{d}x \big)_{t \ge 0} \xrightarrow[N\to\infty]{d} \big(\mathscr{U}^\vartheta(t,\mathrm{d}x) \big)_{t \ge 0}$$

$\mathscr{U}^\vartheta$ = critical 2d **Stochastic Heat Flow (SHF)** = stochastic process of random measures on $\mathbb{R}^2$

Link with the Stochastic Heat Equation

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad \text{"initial condition 1 at time 0"}$$

We actually build the SHF as a two-parameter space-time process

$$\Big(\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x)\Big)_{0 \le s \le t < \infty} \qquad \text{"starting at time } s \text{ from } \mathrm{d}y\text{"}$$

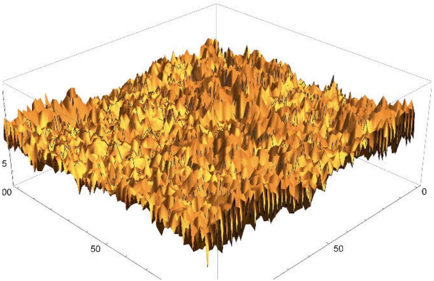
Why "flow"? Chapman-Kolmogorov property for $s < u < t$ [Clark–Mian 2024+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x) = \int_{z \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y; \underbrace{u, \mathrm{d}z)\, \mathscr{U}^{\vartheta}(u, \mathrm{d}z}_{\substack{\text{non-trivial "product"} \\ \text{of measures}}}; t, \mathrm{d}x)$$

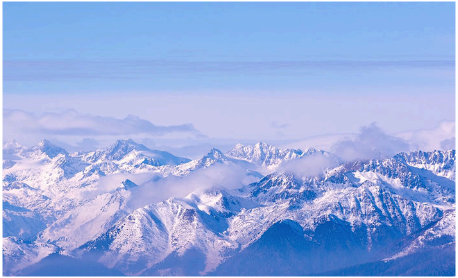
How does the SHF look like?

We can efficiently simulate the SHF via $u_N(t,x)$ time $O(N^2)$

Some not-so-randomly picked realizations [M. Mucciconi, N. Zygouras]



KPZ $\approx \log u_N(t,x)$



Alps, Italy (vecteezy.com)

Key properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. *in preparation*]

  "not a function"

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)

  "*barely* not a function"

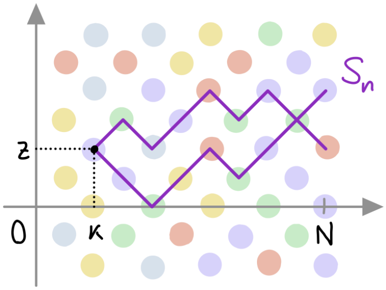
- $\mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)] = \mathrm{d}x$ $\qquad$ $\mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)\,\mathscr{U}^{\vartheta}(t, \mathrm{d}y)] = \underbrace{K_t^{\vartheta}(x-y)}_{\approx \log|x-y|^{-1}}\,\mathrm{d}x\,\mathrm{d}y$

- Formulas for higher moments [C.S.Z. *CMP* 19] [Gu–Quastel–Tsai *PMP* 21]

- Diffusive rescaling $\qquad a^{-1}\,\mathscr{U}^{\vartheta}(a\,t, \mathrm{d}(\sqrt{a}\,x)) \overset{d}{=} \mathscr{U}^{\vartheta + \log a}(t, \mathrm{d}x)$

Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \ \sim \mathcal{N}(0, N-k)$



Partition Functions

$(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Polymer model

Polymer measure $\quad \mathrm{P}_{N,\beta}^{\omega}(S) = \dfrac{\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N}}{Z_{N,\beta}^{\omega}} \mathrm{P}(S) \qquad \left[Z_{N,\beta}^{\omega} = Z_{N,\beta}^{\omega}(0,0) \right]$

Phase transition: critical point $\beta_c = \beta_c^{(d)} \geq 0$

- For $\beta < \beta_c^{(d)}$ weak disorder / delocalized phase (as $N \to \infty$)

$$Z_{N,\beta}^{\omega} \to Z_{\infty,\beta}^{\omega} > 0 \qquad |S_N| = O(\sqrt{N}) \text{ diffusive under } \mathrm{P}_{N,\beta}^{\omega}$$

- For $\beta > \beta_c^{(d)}$ strong disorder / localized phase

$$Z_{N,\beta}^{\omega} \to Z_{\infty,\beta}^{\omega} = 0 \qquad |S_N| \gg \sqrt{N} \text{ super-diffusive under } \mathrm{P}_{N,\beta}^{\omega} \quad \text{(conj.)}$$

Disorder (ir)relevance

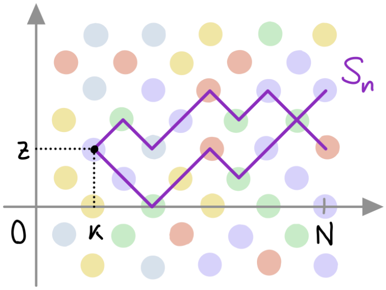
Critical point $\beta_c = \beta_c^{(d)}$ [Comets 2017 book]

- $\beta_c^{(d)} = 0$ for $d < 2$: disorder relevance
- $\beta_c^{(d)} > 0$ for $d > 2$: disorder irrelevance
- $\beta_c^{(d)} = 0$ for $d = 2$: marginal relevance (critical dimension)

Disordered Systems	Singular SPDEs	QFT
disorder relevance	sub-criticality	super-renormalizable
disorder irrelevance	super-criticality	sub-renormalizable
marginal (ir)relevance	criticality	renormalizable

Back to partition functions

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathscr{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n)$



Partition Functions $(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z_{N,\beta}^{\omega}(k,z) \;=\; \mathrm{E}\left[\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Partition functions and SHE

Diff. rescaled partition functions $=$ discretized SHE solutions

$$Z^{\omega}_{N,\beta}\big(N(1-t), \sqrt{N}x\big) \;=\; u_N(t,x) \qquad \text{(time rev.)}$$

Partition functions solve a difference equation: with $\xi_N \approx \omega$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}_{\beta_{\mathrm{SHE}}}\, u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(reg-SHE)}$$

Discrete analogue of Feynman-Kac

$$u_N(t,x) \;\approx\; \mathrm{E}\Big[\mathrm{e}^{\beta_{\mathrm{SHE}} \int_{1-t}^{1} \xi(s,B_s) - \frac{1}{2}\beta_{\mathrm{SHE}}^2\, t} \,\Big|\, B_{1-t} = x \Big]$$

Scaling limit: sub-critical dimension

Recall the Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x) + \beta_{\mathrm{SHE}}\, u(t,x)\, \xi(t,x) \\ u(0,x) \equiv 1 \end{cases} \tag{SHE}$$

Well defined in $d=1$ for any $\beta_{\mathrm{SHE}} > 0$ (e.g. by stochastic integration)

Theorem ($d=1$) [Alberts–Khanin–Quastel *AoP* 2014]

Rescaling $\beta_N = \dfrac{\beta_{\mathrm{SHE}}}{N^{1/4}}$ $\qquad Z^{\omega}_{N,\beta}\big(N(1-t), \sqrt{N}x\big) \xrightarrow[N\to\infty]{d} u(t,x)$

Extended to a class of disorder relevant models [C.S.Z. *JEMS* 2017]

Scaling limit: critical dimension

SHE is ill-posed in $d = 2$ $\rightsquigarrow$ no available solution u(t,x)

Candidate solution defined by scaling limit of $u_N(t,x)$ (Directed Polymers)

Renormalization

$$\beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right)$$

Critical window
of a phase transition

Theorem ($d = 2$)

[C.S.Z. *Invent. Math.* 2023]

$$\big(\, u_N(t,x)\, \mathrm{d}x \,\big)_{t\geq 0} \xrightarrow[N\to\infty]{d} \big(\, \mathscr{U}^{\vartheta}(t, \mathrm{d}x) \big)_{t\geq 0} \quad \text{(SHF)}$$

Gaussian Multiplicative Chaos?

Much studied class of random measures: Gaussian Multiplicative Chaos (GMC)

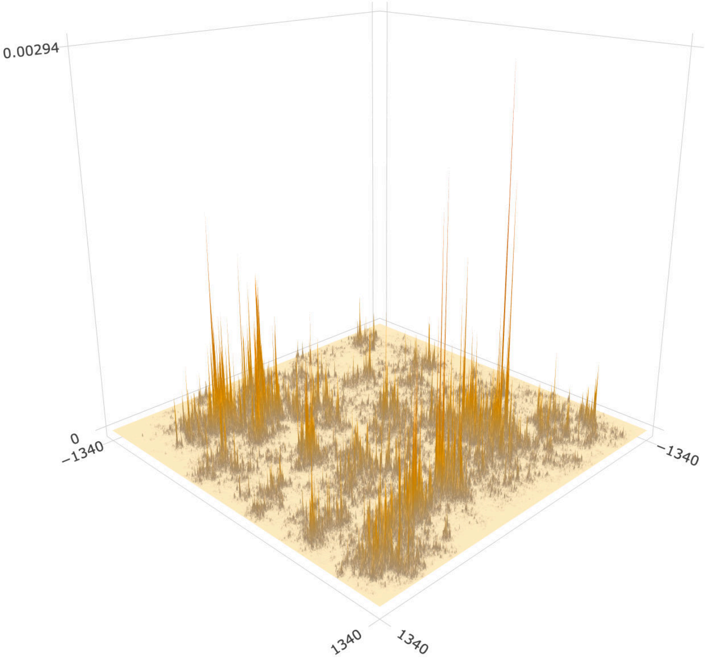
$$\mathscr{M}(\mathrm{d}x) = \text{“}\, \mathrm{e}^{X(x) - \frac{1}{2}\mathbb{V}\mathrm{ar}[X(x)]}\, \mathrm{d}x \,\text{”} \qquad X(\cdot) \text{ generalized Gaussian field}$$

Comparison with (non-usual) GMC: $\mathscr{U}^{\vartheta}$ log-correlated $\rightsquigarrow$ X log-log-correlated

Theorem [C.S.Z. *AoP* 23]

The critical 2d Stochastic Heat Flow $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is not a GMC

Simulations: zooming in $\mathscr{U}^{\vartheta}(1, \mathrm{d}x) \approx u_N(1, x)$ for $N = 50\,000$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Summarizing

Singular Stochastic PDEs such as SHE and KPZ

closely linked to Directed Polymers

(discretized solutions $\leftrightsquigarrow$ partition functions)

- Role of dimension d (sub-, super-) criticality
- Role of disorder strength β phase transition

Goal of this course

Scaling limit of partition functions (diffusively rescaled)

Plan of the course

Lecture 2. Polynomial chaos, sub-criticality and disorder relevance $(d = 1)$

- Convergence of partition functions to SHE solution in dimension 1
- *Tools: polynomial chaos, Lindeberg principles*

Lectures 3–4. Critical dimension and phase transition $(d = 2,\ \beta < \beta_c)$

- Variance computation, log-normality, E-W fluctuations (log-corr. Gaussian)
- *Tools: hypercontractivity, concentration of measure*

Lectures 5–6. The critical 2d Stochastic Heat Flow $(d = 2,\ \beta = \beta_c)$

- Coarse-graining, sharp variance asymptotics, high moment bounds
- *Tools: refined Lindeberg, renewal theory, functional inequalities*

감사합니다