LOCAL LARGE DEVIATIONS AND THE STRONG RENEWAL THEOREM

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Abstract. We establish two different, but related results for random walks in the domain of attraction of a stable law of index α. The first result is a local large deviation upper bound, valid for $\alpha \in (0,1) \cup (1,2)$, which improves on the classical Gnedenko and Stone local limit theorems. The second result, valid for $\alpha \in (0,1)$, is the derivation of necessary and sufficient conditions for the random walk to satisfy the *strong renewal theorem* (SRT). This solves a long standing problem, which dates back to the 1963 paper of Garsia and Lamperti [GL63] for renewal processes (i.e. random walks with non-negative increments), and to the 1968 paper of Williamson [Wil68] for general random walks.

1. Introduction and results

This paper contains new results about asymptotically stable random walks. We first present a local large deviation estimate which improves the error term in the classical local limit theorems, without making any further assumptions (cf. Theorem 1.1). Then we exploit this bound to solve a long-standing problem, namely we establish necessary and sufficient conditions for the validity of the *strong renewal theorem* (SRT), both for renewal processes (cf. Theorem 1.4) and for general random walks (cf. Theorem 1.12). The corresponding result for Lévy processes is also presented (cf. Theorem 1.18).

This paper supersedes the individual preprints [Car15] and [Don15].

Notation. We set $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $RV(\gamma)$ the class of regularly varying functions with index γ, namely $f \in RV(\gamma)$ if and only if $f(x) = x^\gamma \ell(x)$ for some slowly varying function $\ell \in RV(0)$, cf. [BGT89]. Given $f, g : [0,\infty) \to (0,\infty)$ we write $f \sim g$ as a shorthand for $\lim_{s\to\infty} f(s)/g(s) = 1$.

1.1. **Local large deviations.** Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. real-valued random variables, with law F. Let $S_0 := 0$, $S_n := X_1 + \ldots + X_n$ be the associated random walk and

$$M_n := \max\{X_1, X_2, \ldots, X_n\}\,. \tag{1.1}$$

We assume that the law F is in the domain of attraction of a strictly stable law with index $\alpha \in (0,1) \cup (1,2)$. More explicitly, setting $\overline{F}(x) := F((x,\infty))$ and $F(x) := F((-\infty, x])$,

$$\overline{F}(x) \underset{x\to\infty}{\sim} \frac{p}{A(x)} \qquad \text{and} \qquad F(-x) \underset{x\to\infty}{\sim} \frac{q}{A(x)} \qquad \text{for some } \ A \in RV(\alpha)\,, \tag{1.2}$$

with $p > 0$ and $q \geq 0$. For $\alpha > 1$ we further assume that $\mathrm{E}[X] = 0$.

We may assume that $A \in RV(\alpha)$ is strictly increasing. Introducing the norming sequence

$$a_n := A^{-1}(n)\,, \qquad n \in \mathbb{N}\,, \tag{1.3}$$

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S_n/a_n converges in law to a random variable Y with a stable law of index α and positivity parameter $\rho = \mathrm{P}(Y > 0) = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan(\frac{p-q}{p+q}\tan\frac{\pi\alpha}{2}) > 0$ (since we assume that $p > 0$).

Our first result is a local large deviation estimate for S_n, constrained on M_n. This extends a result for $\alpha = 2$ due to Negaev [Nag79].

Theorem 1.1 (Local Large Deviations). *Let F satisfy* (1.2) *with $\alpha \in (0,1) \cup (1,2)$ and $p > 0$, and $\mathrm{E}[X] = 0$ if $\alpha > 1$. Fix a bounded measurable $J \subseteq \mathbb{R}$. Given $\gamma \in (0,\infty)$, there is $C_0 = C_0(\gamma, J) < \infty$ such that, for all $n \in \mathbb{N}$ and $x \geq 0$, the following relations hold:*

$$\mathrm{P}(S_n \in x + J,\ M_n \leq \gamma x) \leq C_0 \, \frac{1}{a_n} \left(\frac{n}{A(x)}\right)^{\lceil 1/\gamma \rceil}, \tag{1.4}$$

where $\lceil x \rceil := \min\{n \in \mathbb{N} : \ n \geq x\}$ is the upper integer part of x. More explicitly:

$$\forall k \in \mathbb{N}, \quad \forall \gamma \in [\tfrac{1}{k}, \tfrac{1}{k-1}) : \qquad \mathrm{P}(S_n \in x + J,\ M_n \leq \gamma x) \leq C_0 \, \frac{1}{a_n} \left(\frac{n}{A(x)}\right)^{k}. \tag{1.5}$$

Moreover, for some $C_0' = C_0'(J) < \infty$,

$$\mathrm{P}(S_n \in x + J) \leq C_0' \, \frac{1}{a_n} \, \frac{n}{A(x)}. \tag{1.6}$$

Remark 1.2. *The classical Gnedenko and Stone local limit theorems (that we recall in Subsection 2.3) yield the bound $\mathrm{P}(S_n \in x + J) = O(\frac{1}{a_n})$. Relation* (1.6) *improves on this bound as soon as $A(x)/n \to \infty$, that is when $x/a_n \to \infty$.*

A heuristic explanation of (1.6) goes as follows: for large x, if $S_n \in x + J$, it is likely that *a single step X_i takes values comparable to x*. Since $\mathrm{P}(X_i > cx) \approx 1/A(x)$ by (1.2), and since there are n available steps, we get the factor $n/A(x)$ in (1.6).

A similar argument sheds light on (1.5). Under the constraint $M_n \leq \gamma x$, with $\gamma \in [\frac{1}{k}, \frac{1}{k-1})$, the most likely way to have $S_n \in x + J$ is that *exactly k steps $X_{i_1}, \ldots, X_{i_k}$ take values comparable to x/k*, and this yields the factor $(n/A(x))^k$ in (1.5).

1.2. **The strong renewal theorem.** Henceforth we assume that $\alpha \in (0,1)$. We say that F is *arithmetic* if it is supported by $h\mathbb{Z}$ for some $h > 0$, in which case the maximal value of $h > 0$ with this property is called the *arithmetic span* of F. It is convenient to set

$$I = (-h, 0] \qquad \text{where} \qquad h := \begin{cases} \text{arithmetic span of } F & \text{(if } F \text{ is arithmetic)} \\ \text{any fixed number} > 0 & \text{(if } F \text{ is non-arithmetic)}. \end{cases} \tag{1.7}$$

The renewal measure $U(\cdot)$ associated to F is defined by

$$U(\mathrm{d}x) := \sum_{n \geq 0} F^{*n}(\mathrm{d}x) = \sum_{n \geq 0} \mathrm{P}(S_n \in \mathrm{d}x). \tag{1.8}$$

It is well known [BGT89, Theorem 8.6.3] that (1.2) implies the following renewal theorem:

$$U([0,x]) \underset{x\to\infty}{\sim} \frac{\mathsf{C}}{\alpha} A(x), \qquad \text{with} \qquad \mathsf{C} = \mathsf{C}(\alpha, \rho) = \alpha \, \mathrm{E}[Y^{-\alpha} \, \mathbb{1}_{\{Y>0\}}] \tag{1.9}$$

(recall that Y denotes a random variable with the limiting stable law). In the special case when $p = 1$ and $q = 0$ in (1.2) (so that $\rho = 1$) one has $\mathsf{C} = \frac{1}{\pi}\alpha\sin(\pi\alpha)$.

It is natural to wonder whether the local version of (1.9) holds, namely

$$U(x + I) = U((x-h, x]) \underset{x\to\infty}{\sim} \mathsf{C}\, h \, \frac{A(x)}{x}. \tag{SRT}$$

This relation, called *strong renewal theorem* (SRT), follows from (1.2) for $\alpha > \frac{1}{2}$, cf. [GL63, Wil68, Eri70, Eri71], while for $\alpha \le \frac{1}{2}$ there are F satisfying (1.2) but not (SRT).

Our next results are *necessary and sufficient conditions for the SRT*. Let us set for $k \ge 0$ and $x \in \mathbb{R}$

$$b_k(x) := \frac{A(|x|)^k}{|x| \vee 1}\,. \tag{1.10}$$

Definition 1.3 (Asymptotic Negligibility). *A function $J(\delta; x)$ is* asymptotically negligible *(a.n.) if and only if*

$$\lim_{\delta\to 0} \limsup_{x\to+\infty} \frac{J(\delta;x)}{b_1(x)} = \lim_{\delta\to 0} \limsup_{x\to+\infty} \frac{x}{A(x)}\, J(\delta;x) = 0\,. \tag{1.11}$$

The SRT turns out to be equivalent to the a.n. of suitable integral quantities defined in terms of F and b_k. We start with the case of renewal processes, which is simpler.

1.3. **The renewal process case.** Assume that F is a law on $[0,\infty)$ such that

$$\overline{F}(x) \underset{x\to\infty}{\sim} \frac{1}{A(x)} \qquad \text{for some} \quad A \in RV(\alpha)\,. \tag{1.12}$$

For $\delta > 0$ and $x \ge 0$ we set

$$I_1^+(\delta;x) := \int_{0\le z\le \delta x} F(x - \mathrm{d}z)\, b_2(z) = \int_{0\le z\le \delta x} F(x-\mathrm{d}z)\, \frac{A(z)^2}{z \vee 1}\,. \tag{1.13}$$

The following is our main result for renewal processes.

Theorem 1.4 (SRT for Renewal Processes). *Let F be a probability on $[0,\infty)$ satisfying* (1.12) *with $\alpha \in (0,1)$. Define $I = (-h, 0]$ with $h > 0$ as in* (1.7).

- *If $\alpha > \frac{1}{2}$, the SRT holds with no extra assumption on F.*
- *If $\alpha \le \frac{1}{2}$, the SRT holds if and only if $I_1^+(\delta;x)$ is a.n. (cf. Definition 1.3).*

In the "boundary" case $\alpha = \frac{1}{2}$, we can say precisely under which conditions on $A(x)$ the SRT holds with no further assumption on F besides (1.12) (like it happens for $\alpha > \frac{1}{2}$).

Theorem 1.5 (SRT for Renewal Processes with $\alpha = \frac{1}{2}$). *Let F be a probability on $[0,\infty)$ satisfying* (1.12) *with $\alpha = \frac{1}{2}$ (so that $A(x)/\sqrt{x}$ is a slowly varying function). If*

$$\sup_{1\le s\le x} \frac{A(s)}{\sqrt{s}} \underset{x\to\infty}{=} O\!\left(\frac{A(x)}{\sqrt{x}}\right), \tag{1.14}$$

then the SRT holds with no extra assumption on F. (This includes the case $A(x) \sim c\sqrt{x}$.) If condition (1.14) *fails, there are examples of F for which the SRT fails.*

1.4. **Sufficient conditions for renewal processes.** Given a probability F on $[0,\infty)$ which satisfies (1.12), a classical sufficient condition for the SRT is

$$F(x+I) \underset{x\to\infty}{=} O\!\left(\frac{1}{xA(x)}\right), \tag{1.15}$$

as proved by Doney [Don97] in the arithmetic case (extending previous results of Williamson [Wil68]), and by Vatutin and Topchii [VT13] in the non-arithmetic case.

Interestingly, if one only looks at the growth of the "local" probabilities $F(x+I)$, *no sharper condition than* (1.15) *can ensure that the SRT holds*, as the following result shows.

Proposition 1.6. *Fix $A \in RV(\alpha)$ with $\alpha \in (0, \frac{1}{2})$, and let $\zeta : (0, \infty) \to (0, \infty)$ be an arbitrary non-decreasing function with $\lim_{x\to\infty} \zeta(x) = \infty$. Then there exists a probability F on $[0, \infty)$ which satisfies* (1.12)*, such that $F(x+I) = O(\frac{\zeta(x)}{xA(x)})$, for which the SRT fails.*

Intuitively, when condition (1.15) is *not* satisfied, in order for the SRT to hold, the points x for which $F(x + I) \gg \frac{1}{xA(x)}$ must not be "too cluttered". We can give a precise formulation of this loose statement by looking at the probability of intervals $F((x - y, x])$.

Proposition 1.7. *Let F be a probability on $[0, \infty)$ satisfying* (1.12) *with $\alpha \in (0, \frac{1}{2}]$.*

- *A sufficient condition for the SRT is that there exists $\gamma > 1 - 2\alpha$ such that*

$$F((x - y, x]) \underset{x\to\infty}{=} o\left(\frac{1}{A(x)} \left(\frac{y}{x}\right)^{\gamma}\right), \qquad \textit{for any } y = y_x \geq 1 \textit{ with } y_x = o(x) . \tag{1.16}$$

- *A necessary condition for the SRT is that* (1.16) *holds for any $\gamma < 1 - 2\alpha$.*

Remark 1.8. *In the sufficient condition* (1.16)*, one can equivalently replace $o(\cdot)$ by $O(\cdot)$. In fact, if* (1.16) *holds with $O(\cdot)$ for some $\gamma > 1 - 2\alpha$, it holds with $o(\cdot)$ for any $\gamma' < \gamma$.*

Remark 1.9. *The sufficient condition* (1.16) *is a generalization of* (1.15)*. In fact, if* (1.15) *holds, then* (1.16) *holds with $\gamma = 1$, since $F((x-y, x]) \leq \sum_{j=0}^{\lceil y/h \rceil} F(x-j+I) \leq \frac{2y}{h} O(\frac{1}{xA(x)})$.*

Remark 1.10. *Other sufficient conditions for the SRT, which generalize* (1.15)*, were given by Chi in* [Chi15, Chi13]*. These can be deduced from Theorem 1.4.*

Conditions similar to (1.16)*, in a different context, appear in* [CSZ16]*.*

We point out that if F satisfies (1.12), we can easily deduce that $F((x-y, x]) = o(\frac{1}{A(x)})$ for any $y = o(x)$, that is (1.16) holds with $\gamma = 0$. However, with no extra assumption, one cannot hope to improve this estimate, as Lemma 9.2 below shows.

To see how condition (1.16) appears, let us introduce the following variant of (1.13):

$$\tilde{I}_1^+(\delta; x) := \int_1^{\delta x} \frac{F((x - z, x])}{z} \, b_2(z) \, \mathrm{d}z = \int_1^{\delta x} \frac{F((x - z, x])}{z} \frac{A(z)^2}{z} \, \mathrm{d}z . \tag{1.17}$$

Our next result shows that one can look at $\tilde{I}_1^+(\delta; x)$ instead of $I_1^+(\delta; x)$.

Proposition 1.11. *Let F be a probability on $[0, \infty)$ satisfying* (1.12) *for some $\alpha \in (0, \frac{1}{2}]$.*

- *If $\tilde{I}_1^+(\delta; x)$ is a.n., then also $I_1^+(\delta; x)$ is a.n., hence the SRT holds.*
- *When $\alpha < \frac{1}{2}$, the converse is also true: $\tilde{I}_1^+(\delta; x)$ is a.n. if and only if $I_1^+(\delta; x)$ is a.n..*

1.5. **The general random walk case.** We now turn to the general random walk case, that is we assume that F is a probability on $\mathbb{R}$ which satisfies (1.2) for some $\alpha \in (0, 1)$, $p > 0$ and $q \geq 0$. Let us generalize (1.13) as follows: for $\delta > 0$ and $x \geq 0$ we set:

$$I_1(\delta; x) := \int_{|y| \leq \delta x} F(x + \mathrm{d}y) \, b_2(y) . \tag{1.18}$$

For $k \in \mathbb{N}$ with $k \geq 2$, we introduce a further parameter $\eta \in (0, 1)$ and we set

$$I_k(\delta, \eta; x) := \int_{|y_1| \leq \delta x} F(x + \mathrm{d}y_1) \int \cdots \int_{|y_j| \leq \eta |y_{j-1}| \text{ for } 2 \leq j \leq k} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k) \, b_{k+1}(y_k) , \tag{1.19}$$

$$\text{where} \quad P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k) := F(-y_1 + \mathrm{d}y_2) F(-y_2 + \mathrm{d}y_3) \cdots F(-y_{k-1} + \mathrm{d}y_k) .$$

Note that $P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)$ is the law of $(S_2, \ldots, S_k)$ conditionally on $S_1 = y_1$, hence

$$I_k(\delta, \eta; x) = \mathrm{E}\left[b_{k+1}(S_k)\, \mathbb{1}_{\{|S_j| \le \eta |S_{j-1}| \text{ for } 2 \le j \le k\}}\, \mathbb{1}_{\{|S_1| \le \delta x\}} \,\middle|\, S_0 = -x\right]. \tag{1.20}$$

The same formula holds also for $k = 1$ (where the first indicator function equals 1).

Let us define

$$\kappa_\alpha := \left\lfloor \frac{1}{\alpha} \right\rfloor - 1 = \begin{cases} 0 & \text{if } \alpha \in (\frac{1}{2}, 1) \\ 1 & \text{if } \alpha \in (\frac{1}{3}, \frac{1}{2}] \\ 2 & \text{if } \alpha \in (\frac{1}{4}, \frac{1}{3}] \\ \vdots & \\ m & \text{if } \alpha \in (\frac{1}{m+2}, \frac{1}{m+1}] \end{cases}, \tag{1.21}$$

We are going to see that, when $1/\alpha \notin \mathbb{N}$, necessary and sufficient conditions for the SRT involve the a.n. of $I_k(\delta; x)$ for $k = \kappa_\alpha$. The case $1/\alpha \in \mathbb{N}$ is slightly more involved. We need to introduce a suitable modification of (1.10), namely

$$\tilde{b}_k(z, x) := \tilde{b}_k(|z|, |x|) := \int_{|x|}^{|z|} \frac{b_k(t)}{t \vee 1}\, \mathrm{d}t, \tag{1.22}$$

where the integral vanishes if $|x| > |z|$. We then define $\tilde{I}_1(\delta; x)$ and $\tilde{I}_k(\delta, \eta; x)$ in analogy with (1.18) and (1.19), replacing $b_2(y)$ by $\tilde{b}_2(\delta x, y)$ and $b_{k+1}(y_k)$ by $\tilde{b}_{k+1}(y_{k-1}, y_k)$:

$$\tilde{I}_1(\delta; x) := \int_{|y| \le \delta x} F(x + \mathrm{d}y)\, \tilde{b}_2(\delta x, y), \tag{1.23}$$

and for $k \ge 2$:

$$\tilde{I}_k(\delta, \eta; x) := \int_{|y_1| \le \delta x} F(x + \mathrm{d}y_1) \underset{|y_j| \le \eta |y_{j-1}| \text{ for } 2 \le j \le k}{\int \cdots \int} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, \tilde{b}_{k+1}(y_{k-1}, y_k). \tag{1.24}$$

Note that, by Fubini's theorem, we can equivalently rewrite (1.23) as

$$\tilde{I}_1(\delta; x) = \int_0^{\delta x} \frac{F((x - t, x + t])}{t \vee 1}\, b_2(t)\, \mathrm{d}t, \tag{1.25}$$

which is a natural random walk generalization of (1.17).

We can now state our main result for random walks.

Theorem 1.12 (SRT for Random Walks)**.** *Let F be a probability on $\mathbb{R}$ satisfying* (1.12) *with $\alpha \in (0, 1)$ and with $p, q > 0$. Define $I = (-h, 0]$ with $h > 0$ as in* (1.7)*.*

- *If $\alpha > \frac{1}{2}$, the SRT holds with no extra assumption on F.*
- *If $\alpha \le \frac{1}{2}$ and $\frac{1}{\alpha} \notin \mathbb{N}$, we distinguish two cases:*
  - *if $\alpha \in (\frac{1}{3}, \frac{1}{2})$, i.e. $\kappa_\alpha = 1$, the SRT holds if and only if $I_1(\delta; x)$ is a.n..*
  - *if $\alpha \in (\frac{1}{k+2}, \frac{1}{k+1})$ for some $k = \kappa_\alpha \ge 2$, the SRT holds if and only if $I_{\kappa_\alpha}(\delta, \eta; x)$ is a.n., for every fixed $\eta \in (0, 1)$.*
- *If $\alpha \le \frac{1}{2}$ and $\frac{1}{\alpha} \in \mathbb{N}$, the same statement holds if we replace I_k by $\tilde{I}_k$, namely:*
  - *if $\alpha = \frac{1}{2}$, i.e. $\kappa_\alpha = 1$, the SRT holds if and only if $\tilde{I}_1(\delta; x)$ is a.n..*
  - *if $\alpha = \frac{1}{k+1}$, for some $k = \kappa_\alpha \ge 2$, the SRT holds if and only if $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n., for every fixed $\eta \in (0, 1)$.*

In Appendix A we show some relations between the quantities I_k and $\tilde{I}_k$. These lead to the following clarifying remarks.

Remark 1.13. *When $\frac{1}{\alpha} \notin \mathbb{N}$, $\tilde{I}_{\kappa_\alpha}$ is a.n. if and only if I_{κ_α} is a.n. (cf. Corollary A.5 below). As a consequence, we could rephrase Theorem 1.12 in a more compact way as follows:*

$$\text{The SRT holds: } \begin{cases} \text{with no extra assumption} & \text{if } \alpha > \frac{1}{2} \\ \text{iff } \tilde{I}_1(\delta; x) \text{ is a.n.} & \text{if } \frac{1}{3} < \alpha \leq \frac{1}{2} \\ \text{iff } \tilde{I}_{\kappa_\alpha}(\delta, \eta; x) \text{ is a.n. for every } \eta \in (0,1) & \text{if } \alpha \leq \frac{1}{3} \end{cases} \tag{1.26}$$

When $\alpha \leq \frac{1}{3}$, our proof actually shows that if $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n. for some *$\eta > 1 - \frac{\alpha}{1-\alpha}$, then (the SRT holds and consequently) it is a.n. for* every *$\eta \in (0,1)$. It is not clear whether the a.n. of $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ for some $\eta \leq 1 - \frac{\alpha}{1-\alpha}$ also implies the a.n. for any $\eta \in (0,1)$.*

Remark 1.14. *If $\frac{1}{\alpha} \notin \mathbb{N}$, the condition that $I_{\kappa_\alpha}(\delta, \eta; x)$ is a.n. is equivalent to the seemingly stronger condition that $I_k(\delta, \eta; x)$ is a.n. for all $k \in \mathbb{N}$ (cf. Corollary A.2 below). Similarly, $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n. if and only if $\tilde{I}_k(\delta, \eta; x)$ is a.n. for all $k \in \mathbb{N}$ (cf. Corollary A.4 below).*

Remark 1.15. *In Theorem 1.12 we require $q > 0$ (that is the positivity index ρ is strictly less than one), but a large part of it actually extends to $q = 0$. More precisely, when $q = 0$, our proof shows that if $\alpha > \frac{1}{2}$ the SRT holds with no extra assumption on F, while if $\alpha \leq \frac{1}{2}$ the a.n. of I_{κ_α} (if $\frac{1}{\alpha} \notin \mathbb{N}$) or $\tilde{I}_{\kappa_\alpha}$ (if $\frac{1}{\alpha} \in \mathbb{N}$) are sufficient conditions for the SRT. However, when $q = 0$, we do not expect the a.n. of I_{κ_α} or $\tilde{I}_{\kappa_\alpha}$ to be necessary, in general.*

1.6. **Sufficient conditions for random walks.** Necessary and sufficient conditions for the SRT in the random walk case involve the a.n. of $\tilde{I}_k$ for a suitable $k = \kappa_\alpha \in \mathbb{N}$. Unlike the renewal process case, this cannot be reduced to the a.n. of just $\tilde{I}_1$.

Proposition 1.16. *For any $\alpha \in (0, \frac{1}{3})$, there is a probability F on $\mathbb{R}$ which satisfies* (1.12)*, such that $\tilde{I}_1(\delta; x)$ is a.n. but $\tilde{I}_2(\delta, \eta; x)$ is not a.n., for any $\eta \in (0,1)$ (hence the SRT fails).*

Let us now give simpler sufficient conditions which ensure the a.n. of $\tilde{I}_k$. Note that the condition that $\tilde{I}_1(\delta; x)$ is a.n. only involves the *right tail* of F (cf. Definition 1.3). To express conditions on the *left tail* of F, we define

$$\tilde{I}_1^*(\delta; x) := \int_0^{\delta x} \frac{F((-x-t, -x+t])}{t \vee 1} \, b_2(t) \, \mathrm{d}t \,, \tag{1.27}$$

which is nothing but $\tilde{I}_1(\delta; x)$ in (1.25) applied to the *reflected probability* $F^*(A) := F(-A)$.

Proposition 1.17. *Let F be a probability on $\mathbb{R}$ satisfying* (1.12) *with $\alpha \in (0, \frac{1}{2}]$ and $p > 0$, $q \geq 0$. If both $\tilde{I}_1(\delta; x)$ and $\tilde{I}_1^*(\delta; x)$ are a.n., then the SRT holds.*

In particular, a sufficient condition for the SRT is that there exists $\gamma > 1 - 2\alpha$ such that relation (1.16) *holds both for F and for F^* (i.e., both as $x \to +\infty$ and as $x \to -\infty$). As a special case, the SRT holds when the classical condition* (1.15) *holds both for F and F^**

1.7. **Lévy processes.** Let $X = (X_t)_{t \geq 0}$ be a Lévy process with Lévy measure Π, Brownian coefficient σ^2 and linear term μ in its Lévy-Khintchine representation, that is

$$\log \mathrm{E}[e^{i\theta X_1}] = i\mu\theta - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R} \setminus \{0\}} (e^{i\theta x} - 1 - i\theta x \mathbb{1}_{\{|x| \leq 1\}}) \, \Pi(\mathrm{d}x) \,. \tag{1.28}$$

Whenever X is transient, we can define its potential or renewal measure by

$$G(\mathrm{d}x) := \int_0^\infty \mathrm{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t\,.$$

We assume that X is asymptotically stable: more precisely, there is a norming function $a(t)$ such that $X_t/a(t)$ converges in law as $t \to \infty$ to a random variable Y with a stable law of index $\alpha \in (0,1)$ and positivity parameter $\rho > 0$. In this case

$$A(x) := \frac{1}{\overline{\Pi}(x)} = \frac{1}{\Pi((x,\infty))} \in RV(\alpha) \qquad \text{as } x \to +\infty\,,$$

and we can take $a(\cdot) = A^{-1}(\cdot)$. Under these assumptions, the renewal theorem (1.9) holds, just replacing $U([0,x])$ by $G([0,x])$. It is natural to wonder whether the corresponding local version (SRT) holds as well, in which case we say that X satisfies the SRT.

Our next result shows that this question can be reduced to the validity of the SRT for a random walk whose step distribution F only depends on the Lévy measure Π, namely:

$$F(\mathrm{d}x) := \begin{cases} \dfrac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R}\setminus(-1,1))} & \text{for } |x| \geq 1 \\ 0 & \text{for } |x| < 1 \end{cases}. \tag{1.29}$$

Theorem 1.18 (SRT for Lévy Processes)**.** *Let X be any Lévy process that is in the domain of attraction of a stable law of index $\alpha \in (0,1)$ and positivity parameter $\rho > 0$ as $t \to \infty$. Suppose also that its Lévy measure is non-arithmetic. Then X satisfies the SRT, i.e.*

$$\lim_{x\to\infty} x\,\overline{\Pi}(x)\,G((x-h,x]) = h\,\alpha\,\mathrm{E}[Y^{-\alpha}\,\mathbb{1}_{\{Y>0\}}]\,, \qquad \forall h > 0\,, \tag{1.30}$$

if and only if the random walk with step distribution F defined in (1.29) *satisfies the SRT.*

As a consequence, Theorems 1.4 and 1.12 can be applied to X.

The proof of Theorem 1.18, given in Section 9, is obtained comparing the Lévy process X with a compound Poisson process with step distribution F.

Remark 1.19. *It is known, cf.* [Ber96, Proof of Theorem 21 on page 38]*, that the potential measure $G(\mathrm{d}x)$ of any Lévy process X coincides for $x \neq 0$ with the renewal measure of a random walk $(S_n)_{n\geq 0}$ with step distribution $\mathrm{P}(S_1 \in \mathrm{d}x) := \int_0^\infty e^{-t}\,\mathrm{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t$. It is also easy to see that X is in the domain of attraction of a stable law of index $\alpha \in (0,1)$ and positivity parameter $\rho > 0$, with norming function $a(t)$, if and only if the random walk S is in the domain of attraction of a the same stable law with norming function $a(n)$.*

So, if we write down necessary and sufficient conditions for S to verify the SRT, these will be necessary and sufficient conditions for X to verify the SRT. However this approach is unsatisfactory, because one would like conditions expressed in terms of the characteristics of X, i.e. the quantities Π, σ^2, μ appearing in the Lévy-Khintchine representation (1.28)*, and the technical problem of expressing our necessary and sufficient conditions for S to satisfy the SRT in terms of these characteristics seems quite challenging.*

1.8. **Structure of the paper.** The paper is organized as follows.

- In Section 2 we recall some standard results.
- In Section 3 we prove Theorem 1.1.
- Sections 4–8 are devoted to the proofs of Theorems 1.4 and 1.12.
    - In Section 4 we reformulate the SRT and we give two key bounds.

  - In Section 5 we prove the necessity part for both Theorems 1.4 and 1.12.
  - In Section 6 we prove the sufficiency part of Theorem 1.4.
  - The sufficiency part of Theorem 1.12 is proved in Section 7 for the case $\alpha > \frac{1}{3}$. The case $\alpha \le \frac{1}{3}$ is treated in Section 8 and is much more technical.
- In Section 9 we prove some "soft" results (such as Theorem 1.5, Propositions 1.6, 1.7, 1.11, 1.16, 1.17, and Theorem 1.18) which are corollaries of our main results.
- In Appendix A we prove some technical results.

2. Setup

2.1. **Notation.** We recall that $f(s) \lesssim g(s)$ or $f \lesssim g$ means $f(s) = O(g(s))$, i.e. for a suitable constant $C < \infty$ one has $f(s) \le C\, g(s)$ for all s in the range under consideration. The constant C may depend on the probability F (in particular, on α) and on h. When some extra parameter ϵ enters the constant $C = C_\epsilon$, we write $f(s) \lesssim_\epsilon g(s)$. If both $f \lesssim g$ and $g \lesssim f$, we write $f \approx g$. We recall that $f(s) \sim g(s)$ means $\lim_{s\to\infty} f(s)/g(s) = 1$.

2.2. **Regular variation.** Without loss of generality [BGT89, §1.3.2], we can assume that $A : [0,\infty) \to (0,\infty)$ is differentiable, strictly increasing and such that

$$A'(s) \sim \alpha \frac{A(s)}{s}\,, \qquad \text{as } s \to \infty\,. \tag{2.1}$$

We fix $A(0) := \frac{1}{2}$ and $A(1) := 1$, so that both A and A^{-1} map $[1,\infty)$ onto itself. We also write $a_u = A^{-1}(u)$ for all $u \in [\frac{1}{2},\infty)$, in agreement with (1.3).

We observe that, by Potter's bounds, for every $\epsilon > 0$ one has

$$\rho^{\alpha+\epsilon} \lesssim_\epsilon \frac{A(\rho x)}{A(x)} \lesssim_\epsilon \rho^{\alpha-\epsilon}\,, \qquad \forall \rho \in (0,1],\ x \in (0,\infty) \ \text{ such that } \ \rho x \ge 1\,. \tag{2.2}$$

More precisely, part (i) of [BGT89, Theorem 1.5.6] shows that relation (2.2) holds for $\rho x \ge \bar{x}_\epsilon$, for a suitable $\bar{x}_\epsilon < \infty$; the extension to $1 \le \rho x \le \bar{x}_\epsilon$ follows as in part (ii) of the same theorem, because $A(y)$ is bounded away from zero and infinity for $y \in [1, \bar{x}_\epsilon]$.

We also recall Karamata's Theorem [BGT89, Propositions 1.5.8 and 1.5.10]:

$$\text{if } f(n) \in RV(\zeta) \text{ with } \zeta > -1: \qquad \sum_{n \le t} f(n) \underset{t\to\infty}{\sim} \frac{1}{\zeta+1}\, t\, f(t)\,, \tag{2.3}$$

$$\text{if } f(n) \in RV(\zeta) \text{ with } \zeta < -1: \qquad \sum_{n > t} f(n) \underset{t\to\infty}{\sim} \frac{-1}{\zeta+1}\, t\, f(t)\,. \tag{2.4}$$

2.3. **Local limit theorems.** We call a probability F on $\mathbb{R}$ *lattice* if it is supported by $v\mathbb{Z} + a$ for some $v > 0$ and $0 \le a < v$, and the maximal value of $v > 0$ with this property is called the *lattice span* of F. If F is arithmetic (i.e. supported by $h\mathbb{Z}$), then it is also lattice, but the spans might differ (for instance, $F(\{-1\}) = F(\{+1\}) = \frac{1}{2}$ has arithmetic span $h = 1$ and lattice span $v = 2$). A lattice distribution is not necessarily arithmetic.†

† If F is lattice, say supported by $v\mathbb{Z} + a$ where v is the lattice span and $a \in [0, v)$, then F is arithmetic if and only if $a/v \in \mathbb{Q}$, in which case its arithmetic span equals $h = v/m$ for some $m \in \mathbb{N}$.

Recall that, under (1.2), S_n/a_n converges in distribution as $n \to \infty$ toward a stable law, whose density we denote by ϕ (the norming sequence a_n is defined in (1.3)). If we set

$$J = (-v, 0] \qquad \text{with} \qquad v = \begin{cases} \text{lattice span of } F & \text{(if } F \text{ is lattice)} \\ \text{any fixed number} > 0 & \text{(if } F \text{ is non-lattice)} \end{cases}, \tag{2.5}$$

Gnedenko's and Stone's local limit theorems [BGT89, Theorems 8.4.1 and 8.4.2] yield

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| a_n \, \mathrm{P}(S_n \in x + J) - v \, \phi\left(\frac{x}{a_n}\right) \right| = 0 \,. \tag{2.6}$$

Since $\sup_{z\in\mathbb{R}} \phi(z) < \infty$, we obtain the useful estimate

$$\sup_{x\in\mathbb{R}} \mathrm{P}(S_n \in (x - w, x]) \lesssim_w \frac{1}{a_n} \,, \tag{2.7}$$

which, plainly, holds for *any* fixed $w > 0$ (not necessarily the lattice span of F).

3. Proof of Theorem 1.1

We prove (1.4), equivalently (1.5), by steps. Without loss of generality, we assume that $J \subseteq [0, \infty)$ (it suffices to redefine $x \mapsto x' := x + \min J$ and $J \mapsto J' := J - \min J$).

Step 1. We first prove an integrated version of (1.4), namely:

$$\forall \gamma \in (0, \infty) : \qquad \mathrm{P}(S_n \geq x, \, M_n \leq \gamma x) \lesssim_\gamma \left(\frac{n}{A(x)}\right)^{1/\gamma} . \tag{3.1}$$

By Markov's inequality, for any $\mu \geq 0$

$$\mathrm{P}(S_n \geq x, \, M_n \leq \gamma x) \leq e^{-\mu x} \, \mathrm{E}\left[e^{\mu S_n} \mathbb{1}_{\{M_n \leq \gamma x\}}\right] = e^{-\mu x} \, (m_0)^n \,,$$

where $m_0 := \mathrm{E}[e^{\mu X_1} \, \mathbb{1}_{\{X_1 \leq \gamma x\}}]$. We are going to fix

$$\mu := \frac{1}{\gamma x} \log \Lambda \,, \qquad \text{where} \qquad \Lambda := \frac{A(x)}{n} \,.$$

Since $e^{-\mu x} = \Lambda^{-1/\gamma}$, it remains to show that $m_0 \leq 1 + \frac{c}{n}$ for some $c < \infty$.

For $x = O(a_n)$ we have $n/A(x) \geq (const.) > 0$ and (3.1) holds trivially. Then we take $x \geq C \, a_n$, with C large so that $\Lambda = A(x)/n \geq e^2$ and consequently $\frac{2}{\mu} \leq \gamma x$. If we split

$$m_0 = \int_{(-\infty, -\frac{2}{\mu}]} e^{\mu y} \, F(\mathrm{d}y) + \int_{(-\frac{2}{\mu}, \frac{2}{\mu}]} e^{\mu y} \, F(\mathrm{d}y) + \int_{(\frac{2}{\mu}, \gamma x]} e^{\mu y} \, F(\mathrm{d}y) \,,$$

by $1 \geq F(\frac{2}{\mu})$ we obtain

$$m_0 - 1 \leq \int_{(-\frac{2}{\mu}, \frac{2}{\mu}]} (e^{\mu y} - 1) \, F(\mathrm{d}y) + \int_{(\frac{2}{\mu}, \gamma x]} e^{\mu y} \, F(\mathrm{d}y) \,. \tag{3.2}$$

Assume first that $\alpha \in (0, 1)$. Writing $e^{\mu y} = y \, e^{\mu y - \log y}$ and observing that $\mu y - \log y$ is increasing for $y \geq \frac{1}{\mu}$, we get

$$m_0 - 1 \lesssim \mu \int_{(-\frac{2}{\mu}, \frac{2}{\mu}]} |y| \, F(\mathrm{d}y) + \frac{e^{\mu \gamma x}}{\gamma x} \int_{(0, \gamma x]} y \, F(\mathrm{d}y) \,.$$

Note that $\int_{(0,w]} y\, F(\mathrm{d}y) = \int_0^w \overline{F}(z)\,\mathrm{d}z - w\overline{F}(w) \lesssim w\,\overline{F}(w)$ for $w \ge 0$ by (2.3), because $\alpha < 1$, hence

$$m_0 - 1 \lesssim F(-\tfrac{2}{\mu}) + \overline{F}(\tfrac{2}{\mu}) + e^{\mu\gamma x}\,\overline{F}(\gamma x) \lesssim_\gamma \frac{1}{A(\frac{1}{\mu})} + \frac{\Lambda}{A(x)} = \frac{1}{n}\left(\frac{A(a_n)}{A(\frac{1}{\mu})} + 1\right). \tag{3.3}$$

It remains to show that $A(a_n) \lesssim A(\frac{1}{\mu})$. Since $A \in RV(\alpha)$, it is enough to show that $a_n \lesssim \frac{1}{\mu}$. First we note that, by Potter's bounds (2.2), for $x \ge Ca_n$

$$\left(\frac{a_n}{x}\right)^{3\alpha/2} \lesssim \Lambda^{-1} = \frac{A(a_n)}{A(x)} \lesssim \left(\frac{a_n}{x}\right)^{\alpha/2}, \tag{3.4}$$

and we conclude that, always for $x \ge Ca_n$,

$$\frac{a_n}{\frac{1}{\mu}} = \frac{1}{\gamma}\frac{\log\Lambda}{x/a_n} \lesssim_\gamma \frac{\log(x/a_n)^{3\alpha/2}}{x/a_n} \lesssim 1\,.$$

Next we assume that $\alpha \in (1,2)$. Subtracting $\mu\,\mathrm{E}[X_1] = 0$ from (3.2), we can write

$$\begin{aligned} m_0 - 1 &\le -\mu \int_{(-\infty,-\frac{2}{\mu}]} y\,F(\mathrm{d}y) + \int_{(-\frac{2}{\mu},\frac{2}{\mu}]} (e^{\mu y} - 1 - \mu y)\,F(\mathrm{d}y) + \int_{(\frac{2}{\mu},\gamma x]} e^{\mu y}\,F(\mathrm{d}y) \\ &\lesssim \mu \int_{(-\infty,-\frac{2}{\mu}]} |y|\,F(\mathrm{d}y) + \mu^2 \int_{(-\frac{2}{\mu},\frac{2}{\mu}]} y^2\,F(\mathrm{d}y) + \frac{e^{\mu\gamma x}}{(\gamma x)^2}\int_{(0,\gamma x]} y^2\,F(\mathrm{d}y)\,, \end{aligned} \tag{3.5}$$

where the last inequality holds by $e^{\mu y} = y^2\,e^{\mu y - 2\log y}$, because $\mu y - 2\log y$ is increasing for $y \ge \frac{2}{\mu}$. Since $\alpha > 1$, it follows by (2.4) that for $w \ge 0$

$$\int_w^\infty y\,F(\mathrm{d}y) = \int_w^\infty \left(\int_0^\infty \mathbb{1}_{\{z<y\}}\,\mathrm{d}z\right) F(\mathrm{d}y) = w\overline{F}(w) + \int_w^\infty \overline{F}(z)\,\mathrm{d}z \lesssim w\,\overline{F}(w)\,,$$

while using also $\alpha < 2$ and (2.3), always for $w \ge 0$,

$$\int_0^w y^2\,F(\mathrm{d}y) = 2\int_0^w \left(\int_0^\infty z\,\mathbb{1}_{\{z<y\}}\,\mathrm{d}z\right) F(\mathrm{d}y) \lesssim \int_0^w z\,\overline{F}(z)\,\mathrm{d}z \lesssim w^2\,\overline{F}(w)\,.$$

Similar bounds hold for the integrals over $y \le 0$. Then from (3.5) we get

$$m_0 - 1 \lesssim F(-\tfrac{2}{\mu}) + \overline{F}(\tfrac{2}{\mu}) + e^{\mu\gamma x}\,\overline{F}(\gamma x) \lesssim_\gamma \frac{1}{A(\frac{1}{\mu})} + \frac{\Lambda}{A(x)} \lesssim \frac{1}{n}\,,$$

where the last inequality holds as we showed in (3.3). This completes the proof of (3.1).

Step 2. Next we deduce from (3.1) a rougher version of (1.4), namely

$$\mathrm{P}(S_n \in x + J,\, M_n \le \tfrac{1}{2}\gamma x) \lesssim \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{1/\gamma}. \tag{3.6}$$

Define $\hat{X}_i := X_{n+1-i}$, for $1 \le i \le n$, and let $(\hat{S}_k := \hat{X}_1 + \ldots + \hat{X}_k = S_n - S_{n-k})_{1\le k\le n}$ be the corresponding random walk, which has the same law as $(S_k)_{1\le k\le n}$. Then

$$\begin{aligned} \mathrm{P}(S_n \in x+J,\, S_{\lfloor n/2\rfloor} < \tfrac{x}{2},\, M_n \le \tfrac{1}{2}\gamma x) &= \mathrm{P}(\hat{S}_n \in x+J,\, \hat{S}_{\lfloor n/2\rfloor} < \tfrac{x}{2},\, \hat{M}_n \le \tfrac{1}{2}\gamma x) \\ &= \mathrm{P}(S_n \in x+J,\, S_n - S_{\lceil n/2\rceil} < \tfrac{x}{2},\, M_n \le \tfrac{1}{2}\gamma x) \\ &\le \mathrm{P}(S_n \in x+J,\, S_{\lceil n/2\rceil} > \tfrac{x}{2},\, M_n \le \tfrac{1}{2}\gamma x)\,, \end{aligned}$$

where the second equality holds because $\hat{S}_n = S_n$ and $\hat{M}_n = M_n$, while for the inequality note that $S_n \geq x$ (by $J \subseteq [0,\infty)$). To lighten notation, henceforth we assume that n is even (the odd case is analogous). It follows from the previous inequality that

$$\begin{aligned}\mathrm{P}(S_n \in x+J,\, M_n \leq \tfrac{1}{2}\gamma x) &\leq 2\,\mathrm{P}(S_n \in x+J,\, S_{n/2} \geq \tfrac{x}{2},\, M_n \leq \tfrac{1}{2}\gamma x) \\ &\leq 2\int_{z\geq\frac{x}{2}} \mathrm{P}(S_{n/2}\in \mathrm{d}z,\, M_{n/2} \leq \tfrac{1}{2}\gamma x)\,\mathrm{P}(S_{n/2}\in x-z+J) \\ &\lesssim \frac{1}{a_{n/2}}\,\mathrm{P}(S_{n/2}\geq \tfrac{1}{2}x,\, M_{n/2}\leq \tfrac{1}{2}\gamma x) \lesssim \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{1/\gamma},\end{aligned}$$

where we have used (2.7) and (3.1).

Step 3. Next we show that

$$\mathrm{P}(S_n \in x+J) \lesssim \frac{1}{a_n}\,\frac{n}{A(x)}\,. \tag{3.7}$$

This is easy: if we fix $\epsilon = \frac{1}{2}$, by (2.7) we can write

$$\begin{aligned}\mathrm{P}(S_n \in x+J, M_n > \epsilon\, x) &\leq n\,\mathrm{P}(S_n\in x+J,\, X_1>\epsilon\, x) \\ &= n\int_{y>\epsilon\, x} F(\mathrm{d}y)\,\mathrm{P}(S_{n-1}\in x-y+J) \lesssim \frac{n}{a_{n-1}}\,\overline{F}(\epsilon\, x) \lesssim_\epsilon \frac{1}{a_n}\,\frac{n}{A(x)}\,.\end{aligned} \tag{3.8}$$

Applying (3.6) with $\gamma = 1$, we see that (3.7) holds.

Step 4. Finally we prove (1.5). The case $k=1$, that is $\gamma\in[1,\infty)$, follows by (3.7). Inductively, we fix $k\in\mathbb{N}$ and we prove that (1.5) holds for $\gamma\in[\frac{1}{k+1},\frac{1}{k})$, assuming that it holds for $\gamma\in[\frac{1}{k},\frac{1}{k-1})$. Let us fix $\epsilon := \frac{1}{2(k+1)}$. By (3.6) (where we choose $\gamma = 2\epsilon$) we get

$$\mathrm{P}(S_n\in x+J,\ M_n\leq \epsilon x) \lesssim \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{1/(2\epsilon)} = \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{k+1}.$$

It remains to consider

$$\begin{aligned}\mathrm{P}(S_n\in x+J,\ \epsilon x < M_n \leq \gamma x) &\leq n\,\mathrm{P}(S_n\in x+J,\, X_1>\epsilon x\,, M_n\leq\gamma x) \\ &\leq n\int_{y\in(\epsilon\, x,\gamma x]} F(\mathrm{d}y)\,\mathrm{P}(S_{n-1}\in x-y+J,\, M_{n-1}\leq\gamma x) \\ &\leq n\,\overline{F}(\epsilon x)\sup_{z\geq(1-\gamma)x}\mathrm{P}(S_{n-1}\in z+J,\, M_{n-1}\leq \gamma x)\,.\end{aligned}$$

Observe that, for $z\geq(1-\gamma)x$, we can bound

$$\mathrm{P}(S_{n-1}\in z+J,\, M_{n-1}\leq\gamma x)\leq \mathrm{P}(S_{n-1}\in z+J,\, M_{n-1}\leq\gamma' z)\,,\qquad \text{with}\quad \gamma' := \frac{\gamma}{1-\gamma}\,.$$

The key observation is that $\gamma'\in[\frac{1}{k},\frac{1}{k-1})$, since $\gamma\in[\frac{1}{k+1},\frac{1}{k})$. By our inductive assumption, relation (1.5) holds for γ', so $\mathrm{P}(S_{n-1}\in z+J,\, M_{n-1}\leq\gamma' z)\lesssim\frac{1}{a_n}(\frac{n}{A(z)})^k$ and we get

$$\mathrm{P}(S_n\in x+J,\ \epsilon x<M_n\leq\gamma x)\lesssim_\gamma n\,\overline{F}(\epsilon x)\,\frac{1}{a_n}\left(\frac{n}{A(x)}\right)^k \lesssim_\epsilon \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{k+1},$$

which completes the proof. ∎

4. Strategy and key bounds for Theorems 1.4 and 1.12

4.1. **Reformulation of the SRT.** It turns out that proving the SRT amounts to showing that *small values of* n *give a negligible contribution to the renewal measure*. More precisely, if F is a probability on $\mathbb{R}$ satisfying (1.2), it is known that (SRT) holds if and only if

$$T(\delta;x) := \sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I) \qquad \text{is a.n.}\,, \tag{4.1}$$

see [Chi15, Appendix] or Remark 4.1 below.

Applying Theorem 1.1, it is easy to show that (4.1) always holds for $\alpha > \frac{1}{2}$. Since n/a_n is regularly varying with index $1-1/\alpha > -1$, by (1.6) and (2.3)

$$\sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n\in x+I) \lesssim \frac{1}{A(x)} \sum_{1\le n\le A(\delta x)} \frac{n}{a_n} \lesssim \frac{1}{A(x)} \frac{A(\delta x)^2}{\delta x} \underset{x\to\infty}{\sim} \delta^{2\alpha-1}\frac{A(x)}{x}\,,$$

from which (4.1) follows, since $2\alpha-1>0$. *We have just proved Theorems 1.4 and 1.12 for* $\alpha > \frac{1}{2}$. In the next sections, we will focus on the case $\alpha \le \frac{1}{2}$.

Remark 4.1. *It is easy to see how* (4.1) *arises. For fixed* $\delta>0$*, by* (1.8) *we can write*

$$U(x+I) \ge \sum_{A(\delta x)<n\le A(\frac{1}{\delta}x)} \mathrm{P}(S_n\in x+I)\,. \tag{4.2}$$

Since $\mathrm{P}(S_n\in x+I) \sim \frac{h}{a_n}\,\phi(\frac{x}{a_n})$ *by* (2.6) *(where we take* $h=v$ *for simplicity), a Riemann sum approximation yields (see* [Chi15, Lemma 3.4]*)*

$$\sum_{A(\delta x)<n\le A(\frac{1}{\delta}x)} \mathrm{P}(S_n\in x+I) \sim h\,\frac{A(x)}{x}\,\mathsf{C}(\delta)\,, \qquad \textit{with} \qquad \mathsf{C}(\delta) = \alpha\int_\delta^{\frac{1}{\delta}} z^{\alpha-2}\phi(\tfrac{1}{z})\,\mathrm{d}z\,.$$

Since $\lim_{\delta\to 0}\mathsf{C}(\delta) = \mathsf{C}$*, proving* (SRT) *amounts to controlling the ranges excluded from* (4.2)*, i.e.* $\{n\le A(\delta x)\}$ *and* $\{n > A(\frac{1}{\delta}x)\}$*. The latter gives a negligible contribution by* $\mathrm{P}(S_n\in x+I)\le C/a_n$ *(recall* (2.7)*), while the former is controlled precisely by* (4.1)*.*

4.2. **Some key bounds.** First we note that, for any bounded $J\subseteq\mathbb{R}$,

$$I_1(\delta;x)\ \text{ is a.n.} \qquad \Longrightarrow \qquad F(x+J) \underset{x\to\infty}{=} o\big(b_1(x)\big)\,, \tag{4.3}$$

as it follows by (1.18), because $I_1(\delta;x) \ge F(x+J)(\inf_{z\in J} b_2(z)) \gtrsim_J F(x+J)$ for large x. Relation (4.3) implies that

$$I_1(\delta;x)\ \text{ is a.n.} \qquad \Longrightarrow \qquad \text{for every fixed } \ell\in\mathbb{N}: \quad \mathrm{P}(S_\ell\in x+J) \underset{x\to\infty}{=} o\big(b_1(x)\big)\,, \tag{4.4}$$

just because (assuming for simplicity that $J\subseteq(-\infty,0]$)

$$\mathrm{P}(S_\ell\in x+J) \le \ell\,\mathrm{P}(S_\ell\in x+J,\, X_1 = \max\{X_1,\ldots,X_\ell\}) \le \ell \sup_{z\in[\frac{1}{\ell}x,\infty)} F(z+J)\,. \tag{4.5}$$

Next we give two useful lemmas. We recall that κ_α was defined in (1.21).

Lemma 4.2 (Big jumps)**.** *Let* F *satisfy* (1.2) *for some* $A\in RV(\alpha)$*, with* $\alpha\in(0,1)$*. There is* $\eta=\eta_\alpha>0$ *such that for all* $\delta\in(0,1]$*,* $\gamma\in(0,1)$ *and* $x\in[0,\infty)$ *the following holds:*

$$\forall \ell\ge\kappa_\alpha: \qquad \sum_{1\le n\le A(\delta x)} n^\ell \left\{\sup_{z\in\mathbb{R}} \mathrm{P}\left(S_n\in z+I,\, M_n > \gamma x\right)\right\} \lesssim_{\gamma,\ell} \delta^\eta\, b_{\ell+1}(x)\,. \tag{4.6}$$

Proof. For $x < 1$ one has $\delta x < 1$, hence the left hand side of (4.6) vanishes (since $A(1) = 1$). Henceforth we assume that $x \geq 1$. Recalling (2.7), we can write

$$\begin{aligned} \mathrm{P}\big(S_n \in z + I,\ M_n > \gamma x\big) &\leq n\, \mathrm{P}\left(S_n \in z + I,\ X_1 > \gamma x\right) \\ &= n \int_{w > \gamma x} \mathrm{P}\left(X \in \mathrm{d}w\right)\, \mathrm{P}(S_{n-1} \in z - w + I) \\ &\leq n\, \mathrm{P}\left(X > \gamma x\right) \left\{ \sup_{y \in \mathbb{R}} \mathrm{P}(S_{n-1} \in y + I) \right\} \\ &\lesssim \frac{n}{A(\gamma x)} \frac{1}{a_n} \lesssim_\gamma \frac{n}{A(x)} \frac{1}{a_n}\,, \end{aligned} \tag{4.7}$$

therefore

$$\begin{aligned} \sum_{1 \leq n \leq A(\delta x)} n^\ell \left\{ \sup_{z \in \mathbb{R}} \mathrm{P}\left(S_n \in z + I,\ M_n > \gamma x\right) \right\} &\lesssim_\gamma \frac{1}{A(x)} \sum_{1 \leq n \leq A(\delta x)} \frac{n^{\ell+1}}{a_n} \\ &\lesssim_\ell \frac{1}{A(x)} \frac{A(\delta x)^{\ell+2}}{\delta x}\,, \end{aligned} \tag{4.8}$$

by (2.3), because $n^{\ell+1}/a_n$ is regularly varying with index $(\ell+1) - \frac{1}{\alpha} \geq (\kappa_\alpha + 1) - \frac{1}{\alpha} = \lfloor \frac{1}{\alpha} \rfloor - \frac{1}{\alpha} > -1$. Let us introduce a parameter $b = b_\alpha \in (0,1)$, depending only on α, that will be fixed in a moment. Since (4.6) holds trivially for $\delta x < 1$ (the left hand side vanishes, by $A(0) < 1$), we may assume that $\delta x \geq 1$. We can then apply the upper bound in (2.2) with $\epsilon = (1-b)\alpha$ and $\rho = \delta$, that is $A(\delta x) \lesssim \delta^{b\alpha} A(x)$, which shows that (4.8) is

$$\lesssim \delta^{b\alpha(\ell+2)-1} \frac{A(x)^{\ell+1}}{x} \lesssim \delta^{b\alpha(\kappa_\alpha+2)-1} \frac{A(x)^{\ell+1}}{x}\,,$$

because $\delta \leq 1$ and $\ell \geq \kappa_\alpha$ by assumption. Since $\alpha(\kappa_\alpha + 2) > 1$ (because $\kappa_\alpha + 2 = \lfloor \frac{1}{\alpha} \rfloor + 1 > \frac{1}{\alpha}$), we can choose $b = b_\alpha < 1$ so that the exponent of δ is strictly positive (e.g. $b_\alpha = \{\alpha(\kappa_\alpha+2)\}^{-1/2}$). This completes the proof. ∎

Our second lemma analyzes the case of "no big jump". The proof exploits in an essential way the large deviation estimate provided by Theorem 1.1.

Lemma 4.3 (No big jump)**.** *Let F satisfy* (1.2) *with $\alpha \in (0,1)$. For any $\gamma \in (0, \frac{\alpha}{1-\alpha})$ there is $\theta = \theta_{\alpha,\gamma} > 0$ such that for all $\delta \in (0,1]$ and $x \in [0,\infty)$ the following holds:*

$$\forall \ell \geq 0: \qquad \sum_{1 \leq n \leq A(\delta x)} n^\ell\, \mathrm{P}\left(S_n \in x + I,\ M_n \leq \gamma x\right) \lesssim_{\gamma,\ell} \delta^\theta\, b_{\ell+1}(x)\,. \tag{4.9}$$

Proof. We may assume that $x \geq 1$, since for $x < 1$ there is nothing to prove (the left hand side of (4.9) vanishes). By (1.4)

$$\sum_{1 \leq n \leq A(\delta x)} n^\ell\, \mathrm{P}\left(S_n \in x + I,\ M_n \leq \gamma x\right) \lesssim \frac{1}{A(x)^{\frac{1}{\gamma}}} \sum_{1 \leq n \leq A(\delta x)} \frac{n^{\ell+\frac{1}{\gamma}}}{a_n} \lesssim \frac{1}{A(x)^{\frac{1}{\gamma}}} \frac{A(\delta x)^{\ell+\frac{1}{\gamma}+1}}{\delta x}\,,$$

where we applied (2.3), because the sequence $n^{\ell+\frac{1}{\gamma}}/a_n$ is regularly varying with index $\ell + \frac{1}{\gamma} - \frac{1}{\gamma} \geq 0 > -1$. Since the left hand side of (4.6) vanishes for $\delta x < 1$, by $A(0) < 1$, we may assume that $\delta x \geq 1$. By the upper bound in (2.2), since $\ell \geq 0$ and $\delta \leq 1$ we can write

$$A(\delta x)^{\ell+\frac{1}{\gamma}+1} \lesssim_\epsilon \delta^{(\alpha-\epsilon)(\ell+\frac{1}{\gamma}+1)} A(x)^{\ell+\frac{1}{\gamma}+1} \lesssim \delta^{(\alpha-\epsilon)(\frac{1}{\gamma}+1)} A(x)^{\ell+\frac{1}{\gamma}+1}\,.$$

Since $\gamma < \frac{\alpha}{1-\alpha}$, we can choose $\epsilon = \epsilon_{\alpha,\gamma} > 0$ small enough so that $(\alpha - \epsilon)(\frac{1}{\gamma} + 1) > 1$. ∎

5. Proof of Theorems 1.4 and 1.12: necessity

5.1. **Necessity for Theorem 1.4.** Let F be a probability on $[0,\infty)$ satisfying (1.12) with $\alpha \in (0,1)$. We assume (4.1), which is equivalent to the strong renewal theorem (SRT), and we deduce that $I_1^+(\delta; x)$ is a.n. (recall (1.13)).

If F is lattice (cf. section 2.3), we actually assume (4.1) with $I = (-h, 0]$ replaced by $J = (-v, 0]$, where v is the lattice span of F. This is no problem, since

$$\mathrm{P}(S_n \in x + J) \le \sum_{\ell=0}^{\lfloor v/h \rfloor} \mathrm{P}(S_n \in x_\ell + I)\,, \qquad \text{where} \qquad x_\ell := x - \ell h\,. \tag{5.1}$$

We also observe that, by (4.1), it follows that

$$F(x + J) = \mathrm{P}(S_1 \in x + J) \underset{x\to\infty}{=} o(b_1(x))\,, \tag{5.2}$$

and, by subadditivity, this relation actually holds for any fixed $J \subseteq \mathbb{R}$.

We need some preparation. Let us define the compact interval

$$K := [\tfrac{1}{2}, 1]\,. \tag{5.3}$$

By (2.6), since $\inf_{z\in K} \phi(z) > 0$, there are $n_1 \in \mathbb{N}$ and $c_1, c_2 \in (0,\infty)$ such that

$$\forall n \ge n_1: \qquad \inf_{z\in\mathbb{R}:\ z/a_n \in K} \mathrm{P}(S_n \in z + J) \ge \frac{c_1}{a_n}\,, \tag{5.4}$$

$$\forall n \in \mathbb{N}: \qquad \sup_{z\in\mathbb{R}} \mathrm{P}(S_n \in z + J) \le \frac{c_2}{a_n}\,. \tag{5.5}$$

Then, since $F((-\infty, -x] \cup [x, \infty)) \lesssim 1/A(x)$, we can fix $C \in (0,\infty)$ such that

$$\forall n \in \mathbb{N}: \qquad F((-\infty, -Ca_n] \cup [Ca_n, \infty)) \le \frac{c_1}{2c_2}\,\frac{1}{n}\,. \tag{5.6}$$

(Of course, we could just take $F([Ca_n, \infty))$, since $F((-\infty, 0)) = 0$, but this estimate will be useful later for random walks.) We also claim that

$$\forall n \ge n_1: \qquad \inf_{z\in\mathbb{R}:\ z/a_n\in K} \mathrm{P}\left(S_n \in z + J,\, \max\{|X_1|, \ldots, |X_n|\} < Ca_n\right) \ge \frac{c_1}{2}\,\frac{1}{a_n}\,. \tag{5.7}$$

This follows because $\mathrm{P}(S_n \in z + J) \ge c_1/a_n$, by (5.4), and applying (5.5), (5.6) we get

$$\begin{aligned}
&\mathrm{P}(S_n \in z + J\,,\ \exists 1 \le j \le n \text{ with } |X_j| \ge Ca_n) \\
&\le n \int_{|y|\ge Ca_n} F(\mathrm{d}y)\, \mathrm{P}(S_{n-1} \in z - y + J) \le \frac{n\, F((-\infty, -Ca_n] \cup [Ca_n, \infty))\, c_2}{a_n} \le \frac{c_1}{2\, a_n}.
\end{aligned}$$

We can now start the proof. The events $B_i := \{X_i \ge Ca_n,\ \max_{j\in\{1,\ldots,n+1\}\setminus\{i\}} X_j < Ca_n\}$ are disjoint for $i = 1, \ldots, n$, hence for $n \ge n_1$ we can write

$$\begin{aligned}
&\mathrm{P}(S_{n+1} \in x + J) \\
&\quad \ge (n+1)\, \mathrm{P}\left(S_{n+1} \in x + J,\ \max\{X_1, \ldots, X_n\} < Ca_n,\ X_{n+1} \ge Ca_n\right) \\
&\quad \ge n \int_{\{z \le x - Ca_n\}} \mathrm{P}\left(X_{n+1} \in x - \mathrm{d}z\right) \mathrm{P}\left(S_n \in z + J,\ \max\{X_1, \ldots, X_n\} < Ca_n\right) \qquad (5.8) \\
&\quad \ge \int_{\{z \le x - Ca_n\}} F\left(x - \mathrm{d}z\right)\, \frac{c_1}{2}\, \frac{n}{a_n}\, \mathbb{1}_{\{z/a_n \in K\}}\,,
\end{aligned}$$

where the last inequality holds by (5.7). We are going to choose $n \le A(\delta x)$, in particular $x - Ca_n \ge x - C\delta x \ge \frac{\delta}{2}x$ for $\delta > 0$ small enough. Restricting the integral, we get

$$\sum_{n=n_1}^{A(\delta x)} \mathrm{P}(S_{n+1} \in x + J) \gtrsim \int_{\{a_{n_1} \le z \le \frac{\delta}{2}x\}} F(x - \mathrm{d}z) \left(\sum_{n=n_1}^{A(\delta x)-1} \frac{n}{a_n} \mathbb{1}_{\{z/a_n \in K\}} \right).$$

Note that $z/a_n \in K$ means $\frac{1}{2}a_n \le z \le a_n$, that is $A(z) \le n \le A(2z)$. In the range of integration, we have $A(z) \ge n_1$ and $A(2z) \le A(\delta)$, hence

$$\sum_{n=n_1}^{A(\delta x)} \frac{n}{a_n} \mathbb{1}_{\{z/a_n \in K\}} \ge \sum_{n=A(z)}^{A(2z)} \frac{n}{a_n} \gtrsim \frac{A(z)}{2z} \big(A(2z) - A(z)\big) \gtrsim \frac{A(z)^2}{z \vee 1} = b_2(z),$$

where the last inequality holds since $z \ge a_{n_1} \ge 1$ (if we take n_1 large enough). Then

$$\begin{aligned} \sum_{n_1 \le n \le A(\delta x)} \mathrm{P}(S_{n+1} \in x + J) &\gtrsim \int_{z \in [a_{n_1}, \frac{\delta}{2}x]} F(x - \mathrm{d}z)\, b_2(z) \\ &\ge I_1^+(\tfrac{\delta}{2}; x) - \hat{C}\, F([x - a_{n_1}, x]), \end{aligned}$$

where $\hat{C} := \sup_{|z| \le a_{n_1}} b_2(z) < \infty$. The left hand side is a.n. by (4.1), hence the right hand side is a.n. too. Since $F([x - a_{n_1}, x])$ is a.n. by (5.2), it follows that $I_1^+(\delta; x)$ is a.n.. ∎

5.2. **Necessity for Theorem 1.12.** Let F be a probability on $\mathbb{R}$ satisfying (1.2) with $\alpha \in (0,1)$ and $p, q > 0$. We assume (4.1) and we deduce that $\tilde{I}_1(\delta; x)$ is a.n. and, moreover, $\tilde{I}_k(\delta, \eta; x)$ is a.n., for any $k \ge 2$ and for all fixed $\eta \in (0,1)$. This completes the proof of the necessity part in Theorem 1.12, see the discussion in Remark 1.14.

Remark 5.1. *For* $|x| \ge 1$ *we can rewrite* (1.22) *as*

$$\tilde{b}_k(z,x) = \int_{|x|}^{|z|} \frac{b_k(t)}{t}\,\mathrm{d}t = \int_{A(|x|)}^{A(|z|)} \frac{b_k(A^{-1}(s))}{A^{-1}(s)} \frac{1}{A'(A^{-1}(s))}\,\mathrm{d}s \simeq \int_{A(|x|)}^{A(|z|)} \frac{s^{k-1}}{A^{-1}(s)}\,\mathrm{d}s,$$

where we have used (2.1) *and* (1.10)*. Recalling also* (1.3)*, we can write*

$$\tilde{b}_k(z,x) \simeq \sum_{n=A(|x|)}^{A(|z|)} \frac{n^{k-1}}{a_n}. \tag{5.9}$$

Since we assume that $p, q > 0$ in (1.2), the density $\phi(\cdot)$ of the limiting Lévy process is strictly positive on the whole real line. In particular, instead of (5.3), we can define

$$K := [-1, 1], \tag{5.10}$$

and relations (5.4), (5.5), (5.6), (5.7) still hold.

Let us show that $\tilde{I}_1(\delta; x)$ is a.n.. This is similar to the case of renewal processes in subsection 5.1. In fact, relation (5.8) with X_i replaced by $|X_i|$ and z replaced by $-y$ gives

$$\mathrm{P}(S_{n+1} \in x + J) \gtrsim \int_{|x+y| \ge Ca_n} F(x + \mathrm{d}y)\, \frac{n}{a_n}\, \mathbb{1}_{\{|y| \le a_n\}}, \tag{5.11}$$

because $K = [-1, 1]$. Note that for $n \le A(\delta x)$ we have $a_n \le \delta x \le x - Ca_n$ for $\delta > 0$ small, hence we can ignore the restriction $|x + y| \ge Ca_n$. Then we can write

$$\sum_{n=n_1}^{A(\delta x)} \frac{n}{a_n} \mathbb{1}_{\{|y| \le a_n\}} = \mathbb{1}_{\{|y| \le \delta x\}} \sum_{n=A(y) \vee n_1}^{A(\delta x)} \frac{n}{a_n} \gtrsim \mathbb{1}_{\{|y| \le \delta x\}}\, \tilde{b}_2(\delta x, |y| \vee a_{n_1}).$$

For $|y| < a_{n_1}$, $\tilde{b}_2(\delta x, |y| \vee a_{n_1}) = \tilde{b}_2(\delta x, a_{n_1})$ differs from $\tilde{b}_2(\delta x, |y|)$ by at most the constant $\sum_{n \le n_1} \frac{n}{a_n}$. It follows that $\tilde{b}_2(\delta x, |y| \vee a_{n_1}) \gtrsim \tilde{b}_2(\delta x, |y|)$ for large x, hence, by (1.23),

$$\sum_{n=n_1}^{A(\delta x)} \mathrm{P}(S_{n+1} \in x + J) \gtrsim \tilde{I}_1(\delta; x)\,.$$

Since we assume that (4.1) holds, it follows that $\tilde{I}_1(\delta; x)$ is a.n..

Next we fix $k \ge 2$ and $\eta \in (0,1)$ and we generalize the previous arguments in order to show that $\tilde{I}_k(\delta, \eta; x)$ is a.n., cf. (1.24). (Inductively, we assume that we already know that $\tilde{I}_1(\delta; x)$, $\tilde{I}_2(\delta, \eta; x)$, $\ldots$, $\tilde{I}_{k-1}(\delta, \eta; x)$ are a.n..) Suppose that $z_1, \cdots z_k \in \mathbb{R}$ satisfy

$$\min_{1 \le j \le k} |z_j| \ge C a_n\,, \qquad |(z_1 + \ldots + z_k) - x| \le a_n\,,$$

and set $y_k := x - (z_1 + \ldots + z_k)$. Then, for $n \ge n_1$, we can write

$$\begin{aligned}
&P(\exists 1 \le j_1 < j_2 < \cdots < j_k \le n \text{ with } X_{j_1} \in \mathrm{d}z_1\,, \ldots, X_{j_k} \in \mathrm{d}z_k\,, \text{ and } S_{n+k} \in x + I) \\
&\quad \ge \binom{n+k}{k} \mathrm{P}(X_r \in \mathrm{d}z_r\,,\ \forall 1 \le r \le k\,,\ X_j \notin \{\mathrm{d}z_1, \cdots \mathrm{d}z_k\},\ \forall k < j \le n+k\,,\ S_{n+k} \in x + I) \\
&\quad \gtrsim n^k\, \mathrm{P}(X_r \in \mathrm{d}z_r\,,\ \forall 1 \le r \le k)\, \mathrm{P}(|X_j| \le C a_n\,,\ \forall 1 \le j \le n\,,\ S_n \in y_k + I) \\
&\quad \gtrsim \frac{n^k}{a_n} \mathrm{P}(X_r \in \mathrm{d}z_r\,,\ \forall 1 \le r \le k)\,,
\end{aligned}$$

It follows that for $n \ge n_1$ we have the the bound

$$\begin{aligned}
\mathrm{P}(S_{n+k} \in x + I) &\gtrsim \frac{n^k}{a_n} \mathrm{P}\left(\min_{1 \le r \le k} |X_r| \ge C a_n\,,\ |(X_1 + \ldots + X_k) - x| \le\ a_n \right) \\
&= \frac{n^k}{a_n} \mathrm{P}_{-x}\left(\min_{1 \le r \le k} |S_r - S_{r-1}| \ge C a_n\,,\ |S_k| \le\ a_n \right)\,,
\end{aligned}$$

where P_{-x} denotes the law of the random walk $S_r := -x + (X_1 + \ldots + X_r)$, $r \ge 1$, which starts from $S_0 := -x$.

If we fix $\eta \in (0,1)$, and define $\overline{\eta} := 1 - \eta$, we can write

$$\left\{ \min_{1 \le r \le k} |S_r - S_{r-1}| \ge C a_n \right\} \supseteq \left\{ |S_r - S_{r-1}| \ge \overline{\eta} |S_{r-1}| \text{ and } |S_{r-1}| \ge \tfrac{C}{\overline{\eta}}\, a_n\,, \forall 1 \le r \le k \right\}\,.$$

For $r = 1$, $|S_{r-1}| \ge \frac{C}{\overline{\eta}}\, a_n$ reduces to $x \ge \frac{C}{\overline{\eta}}\, a_n$, which holds automatically, since we take $n \le A(\delta x)$ with $\delta > 0$ small, while $|S_r - S_{r-1}| \ge \overline{\eta}|S_{r-1}|$ becomes $|S_1 + x| \ge \overline{\eta} x$, which is implied by $|S_1| \le \frac{C}{\overline{\eta}} \delta x$, for $\delta > 0$ small. For $r \ge 2$, $|S_r - S_{r-1}| \ge \overline{\eta}|S_{r-1}|$ is implied by $|S_r| \le \eta |S_{r-1}|$, since $\overline{\eta} = 1 - \eta$. Thus

$$\left\{ \min_{1 \le r \le k} |S_r - S_{r-1}| \ge C a_n \right\} \supseteq \left\{ |S_1| \le \tfrac{C}{\overline{\eta}} \delta x\,,\ \ |S_r| \le \eta |S_{r-1}|\,, \forall 2 \le r \le k\,,\ \ |S_{k-1}| \ge \tfrac{C}{\overline{\eta}} a_n \right\}\,,$$

where the last term is justified because $|S_{k-1}| = \min_{2 \le r \le k-1} |S_{r-1}|$ on the event. Thus

$$\mathrm{P}(S_{n+k} \in x + I) \gtrsim \mathrm{E}_{-x}\left[\mathbb{1}_{\{|S_1| \le \frac{C}{\overline{\eta}} \delta x,\ |S_r| \le \eta |S_{r-1}|\,, \forall 2 \le r \le k\}} \left(\frac{n^k}{a_n} \mathbb{1}_{\{A(|S_k|) \le n \le A(\frac{\overline{\eta}}{C} |S_{k-1}|)\}} \right) \right]\,.$$

Let us now sum over $n_1 \le n \le A(\delta x)$. Note that $A(\frac{\eta}{C}|S_{k-1}|) \le A(\frac{\eta}{C}|S_1|) \le A(\delta x)$, hence

$$\begin{aligned}\sum_{n_1 \le n \le A(\delta x)} & \mathrm{P}(S_{n+k} \in x + I) \\ & \gtrsim \mathrm{E}_{-x}\Big[\mathbb{1}_{\{|S_1| \le \frac{C}{\eta}\delta x,\ |S_r| \le \eta |S_{r-1}|,\, \forall 2 \le r \le k\}}\, \tilde{b}_{k+1}\big(\tfrac{\eta}{C}|S_{k-1}|,\, |S_k| \vee a_{n_1}\big)\Big]\,,\end{aligned} \tag{5.12}$$

where we recall that $\tilde{b}_{k+1}$ is given by (5.9). The right hand side can be rewritten as

$$\begin{aligned}\int\limits_{\substack{|y_1| \le \delta' x \\ |y_r| \le \eta |y_{r-1}| \text{ for all } 2 \le r \le k}} & F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, \tilde{b}_{k+1}\big(\epsilon |y_{k-1}|,\, |y_k| \vee c\big)\,, \\ & \text{where} \quad \delta' = \tfrac{C}{\eta}\delta\,, \quad \epsilon = \tfrac{\eta}{C}\,, \quad c = a_{n_1}\,.\end{aligned} \tag{5.13}$$

This looks very similar to $\tilde{I}_k(\delta', \eta; x)$, cf. (1.24), the only difference being that the term $b_{k+1}(y_{k-1}, y_k) = b_{k+1}(|y_{k-1}|, |y_k|)$ in (1.24) is replaced by $b_{k+1}(\epsilon|y_{k-1}|, |y_k| \vee c)$. It remains to show that this difference is immaterial. More precisely, since we assume (4.1), it follows by (5.12) that (5.13) is a.n.. From this, we now deduce that $\tilde{I}_k(\delta, \eta; x)$ is a.n..

For any $B \in (0, \infty)$, the contribution of $|y_{k-1}| \le B$ to $\tilde{I}_k(\delta, \eta; x)$ in (1.24) is at most

$$\tilde{b}_{k+1}(B, 0)\, \mathrm{P}(|-x + S_{k-1}| \le B) \lesssim_B \mathrm{P}(S_{k-1} \in [x - B, x + B])$$

and this is a.n. as $x \to \infty$, in analogy with (5.2), because we assume that (4.1) holds.

We are left with the contribution of $\{|y_{k-1}| > B\}$ to $\tilde{I}_k(\delta, \eta; x)$ in (1.24). We claim that we can bound $\tilde{b}_{k+1}\big(|y_{k-1}|, |y_k|\big) \lesssim \tilde{b}_{k+1}\big(|y_{k-1}|, |y_k| \vee c\big)$, in this regime. In fact, the two terms differ only when $|y_k| < c$, in which case their ratio is bounded, for $|y_{k-1}| > B$, by

$$\frac{\tilde{b}_{k+1}\big(|y_{k-1}|,\, |y_k|\big)}{\tilde{b}_{k+1}\big(|y_{k-1}|,\, |y_k| \vee c\big)} \le \frac{\tilde{b}_{k+1}\big(|y_{k-1}|,\, c\big) + \tilde{b}_{k+1}\big(c,\, |y_k|\big)}{\tilde{b}_{k+1}\big(|y_{k-1}|,\, c\big)} \le 1 + \frac{\tilde{b}_{k+1}\big(c,\, 0\big)}{\tilde{b}_{k+1}\big(B,\, c\big)}\,.$$

If we fix $B > c$ so that $\tilde{b}_{k+1}\big(B,\, c\big) > 0$, we have $\tilde{b}_{k+1}\big(|y_{k-1}|,\, |y_k|\big) \lesssim \tilde{b}_{k+1}\big(|y_{k-1}|,\, |y_k| \vee c\big)$.

Finally, we write $b_{k+1}\big(|y_{k-1}|,\, |y_k| \vee c\big) \le b_{k+1}\big(\epsilon|y_{k-1}|,\, |y_k| \vee c\big) + b_{k+1}\big(|y_{k-1}|,\, \epsilon|y_{k-1}|\big)$ and we note that the contribution of the first term to $\tilde{I}_k(\delta, \eta; x)$ in (1.24) is a.n., because we know that (5.13) is a.n.. For the second term, just note that

$$b_{k+1}\big(|y_{k-1}|,\, \epsilon|y_{k-1}|\big) = \sum_{n = A(\epsilon|y_{k-1}|)}^{A(|y_{k-1}|)} \frac{n^k}{a_n} \le \frac{A(|y_{k-1}|)^{k+1}}{\epsilon|y_{k-1}|} \lesssim_\epsilon b_{k+1}(y_{k-1})\,,$$

hence

$$\begin{aligned}\int_{|y_k| \le \eta |y_{k-1}|} F(-y_{k-1} + \mathrm{d}y_k)\, \tilde{b}_{k+1}(|y_{k-1}|, \epsilon|y_{k-1}|) & \lesssim_\epsilon b_{k+1}(y_{k-1})\, F(-(1-\eta)|y_{k-1}|) \\ & \lesssim_\eta b_k(y_{k-1})\,,\end{aligned}$$

and the corresponding contribution to $\tilde{I}_k(\delta, \eta; x)$ is $\lesssim_{\epsilon,\eta} I_{k-1}(\delta, \eta; x)$. Since $\tilde{I}_{k-1}$ is a.n. by our inductive assumption, also I_{k-1} is a.n., cf. (A.7), hence we are done. ∎

6. Proof of Theorem 1.4: sufficiency

In this section we prove the sufficiency part of Theorem 1.4: we assume that $I_1^+(\delta;x)$ is a.n. and we deduce (4.1), which is equivalent to the SRT. Let us set

$$T_\ell(\delta;x) := \sum_{1\le n\le A(\delta x)} n^\ell \, \mathrm{P}(S_n \in x+I)\,. \tag{6.1}$$

Theorem 6.1. *Let F be a probability on $[0,\infty)$ satisfying (1.12) with $\alpha \in (0,1)$. Assume that $I_1^+(\delta;x)$ is a.n.. Then for every $\ell \in \mathbb{N}_0$:*

$$\lim_{\delta\to 0} \limsup_{x\to\infty} \frac{T_\ell(\delta;x)}{b_{\ell+1}(x)} = 0\,. \tag{6.2}$$

In particular, setting $\ell = 0$, relation (4.1) holds.

The proof exploits the general bounds provided by Lemmas 4.2 and 4.3, together with the next Lemma, which is specialized to renewal processes.

Lemma 6.2. *If F is a probability on $[0,\infty)$ which satisfies (1.12) with $\alpha \in (0,1)$, there are $C, c \in (0,\infty)$ such that for all $n \in \mathbb{N}_0$ and $z \in [0,\infty)$*

$$\mathrm{P}(S_n \in z+I) \le \frac{C}{a_n}\, e^{-c\frac{n}{A(z)}}\,. \tag{6.3}$$

Remark 6.3. *We observe that T_ℓ is uniformly bounded, by Lemma 6.2: for some $C<\infty$*

$$T_\ell(\delta;x) \le C\,. \tag{6.4}$$

Proof of Lemma 6.2. Assume that n is even (the odd case is analogous). By (2.7), we get

$$\begin{aligned}\mathrm{P}\left(S_n \in z+I\right) &= \int_{y\in[0,z]} \mathrm{P}(S_{\frac{n}{2}} \in \mathrm{d}y)\ \mathrm{P}(S_{\frac{n}{2}} \in z-y+I) \lesssim \frac{1}{a_{\frac{n}{2}}}\,\mathrm{P}(S_{\frac{n}{2}} \le z) \\ &\lesssim \frac{1}{a_n}\,\mathrm{P}\left(\max_{1\le i\le \frac{n}{2}} X_i \le z\right) = \frac{(1-\mathrm{P}(X>z))^{\frac{n}{2}}}{a_n} \le \frac{e^{-\frac{n}{2}\mathrm{P}(X>z)}}{a_n} \le \frac{e^{-c\frac{n}{A(z)}}}{a_n}\,,\end{aligned}$$

provided $c>0$ is chosen such that $\mathrm{P}(X>z) \ge 2c/A(z)$ for all $z \ge 0$. This is possible by (1.12) and because $z \mapsto A(z)$ is increasing and continuous, with $A(0)>0$ (see §2.2). ∎

Proof of Theorem 6.1. We fix, once and for all, $\gamma \in (0, \frac{\alpha}{1-\alpha})$, and we decompose

$$\begin{aligned}T_\ell(\delta;x) = &\sum_{1\le n\le A(\delta x)} n^\ell \, \mathrm{P}(S_n \in x+I,\, M_n > \gamma x) \\ &+ \sum_{1\le n\le A(\delta x)} n^\ell \, \mathrm{P}(S_n \in x+I,\, M_n \le \gamma x)\,.\end{aligned}$$

Then it follows by Lemma 4.2 and Lemma 4.3 that (6.2) holds for every $\ell \ge \kappa_\alpha$.

It remains to prove that (6.2) holds for $\ell < \kappa_\alpha$. We proceed by backward induction: we fix $\ell \in \{0,1,\ldots,\kappa_\alpha - 1\}$ and, *assuming* that

$$\lim_{\delta\to 0} \limsup_{x\to\infty} \frac{T_{\ell+1}(\delta;x)}{b_{\ell+2}(x)} = 0\,, \tag{6.5}$$

we deduce (6.2). We need to estimate $T_\ell(\delta;x)$ and we split it in some pieces.

We start by writing

$$\mathrm{P}\left(S_n \in x+I\right) = \mathrm{P}\left(S_n \in x+I,\, M_n \le \gamma x\right) + \mathrm{P}\left(S_n \in x+I,\, M_n > \gamma x\right)\,,$$

and note that the contribution of the first term in the right hand side is negligible for (6.2), by Lemma 4.3. Next we bound

$$\begin{aligned}\mathrm{P}\left(S_n \in x+I,\, M_n > \gamma x\right) &\leq n\,\mathrm{P}\left(S_n \in x+I,\, X_1 > \gamma x\right) \\ &= n \int_{0 \leq z < (1-\gamma)x} F(x-\mathrm{d}z)\,\mathrm{P}\left(S_{n-1} \in z+I\right).\end{aligned} \tag{6.6}$$

Looking back at (6.1), we may restrict the sum to $n \geq 2$, because the contribution of the term $n=1$ is negligible for (6.2), since $F(x+I) = o(b_1(x)) = o(b_{\ell+1}(x))$ by (4.3). As a consequence, it remains to prove that (6.2) holds with $T_\ell(\delta;x)$ replaced by

$$\widetilde{T}_\ell(\delta;x) := \sum_{2 \leq n \leq A(\delta x)} n^{\ell+1} \int_{0 \leq z < (1-\gamma)x} F(x-\mathrm{d}z)\,\mathrm{P}\left(S_{n-1} \in z+I\right).$$

We can bound $n^{\ell+1} \lesssim (n-1)^{\ell+1}$, since $n \geq 2$, and rename $n-1$ as n, to get

$$\widetilde{T}_\ell(\delta;x) \leq \int_{1 \leq z < (1-\gamma)x} F(x-\mathrm{d}z) \left\{ \sum_{1 \leq n \leq A(\delta x)-1} n^{\ell+1}\,\mathrm{P}\left(S_n \in z+I\right) \right\} + o(b_{\ell+1}(x)), \tag{6.7}$$

where we have restricted the integral to $z \geq 1$ and estimated the contribution of $z \in [0,1)$ as $o(b_1(x)) = o(b_{\ell+1}(x))$, thanks to (4.3) and (6.4).

Let us fix $\epsilon \in (0,1)$ and consider the contribution to the sum in (6.7) given by $n > A(\epsilon z)$. Applying Lemma 6.2, since $a_n \geq \epsilon z \gtrsim_\epsilon z$, we get

$$\begin{aligned}\sum_{A(\epsilon z) < n \leq A(\delta x)} n^{\ell+1}\,\mathrm{P}\left(S_n \in z+I\right) &\lesssim \sum_{A(\epsilon z) < n \leq A(\delta x)} \frac{n^{\ell+1}}{a_n}\, e^{-c\frac{n}{A(z)}} \\ &\lesssim_\epsilon \frac{A(z)^{\ell+2}}{z}\, \mathbb{1}_{\{z < \frac{\delta}{\epsilon}x\}} \left\{ \sum_{n \in \mathbb{N}} \frac{1}{A(z)} \left(\frac{n}{A(z)}\right)^{\ell+1} e^{-c\frac{n}{A(z)}} \right\}.\end{aligned} \tag{6.8}$$

The bracket is a Riemann sum which converges to $\int_0^\infty t^{\ell+1}\, e^{-ct}\,\mathrm{d}t < \infty$ as $z \to \infty$, hence it is uniformly bounded for $z \in [0,\infty)$. The contribution to the integral in (6.7) is then

$$\lesssim_\epsilon \int_{1 \leq z < \frac{\delta}{\epsilon}x} F(x-\mathrm{d}z)\,\frac{A(z)^{\ell+2}}{z} \leq A(\tfrac{\delta}{\epsilon}x)^\ell\, I_1^+(\tfrac{\delta}{\epsilon};x) \lesssim_\epsilon A(x)^\ell\, I_1^+(\tfrac{\delta}{\epsilon};x).$$

This is negligible for (6.2), for any fixed $\epsilon > 0$, by the assumption that I_1^+ is a.n..

Finally, the contribution to the integral in (6.7) given by $n \leq A(\epsilon z)$ is, by (6.1),

$$\int_{1 \leq z < (1-\gamma)x} F(x-\mathrm{d}z)\, T_{\ell+1}(\epsilon;z). \tag{6.9}$$

By the inductive assumption (6.5), for every $\eta > 0$ we can choose $\epsilon > 0$ small enough and $\bar{x}_\epsilon < \infty$ large enough so that $T_{\ell+1}(\epsilon;x) \leq \eta b_{\ell+2}(x)$ for every $x \geq \bar{x}_\epsilon$. Recalling (6.4), we see that (6.9) is bounded by

$$C\,F((x-\bar{x}_\epsilon, x-1]) + \eta\, b_{\ell+2}(x)\, F([\gamma x,\infty)) \lesssim_{\epsilon,\gamma} o(b_{\ell+1}(x)) + \eta\, b_{\ell+1}(x).$$

If we let $x \to \infty$ and then $\eta \to 0$, we have proved (6.2). ∎

7. Proof of Theorem 1.12: sufficiency in case $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

Let F be a probability on $\mathbb{R}$ that satisfies (1.2) with $p, q \geq 0$ and $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. In this section, we assume that $\tilde{I}_1(\delta, x)$ is a.n., cf. (1.23), and we deduce (4.1), which is equivalent to the SRT. By Remark 1.14, this proves the sufficiency part in Theorem 1.12 when $\kappa_\alpha = 1$.

Let us set

$$Z_1 := M_n = \max\{X_1, \ldots, X_n\} . \tag{7.1}$$

We fix $\gamma \in (0, \frac{\alpha}{1-\alpha})$ and define the events

$$\begin{aligned} &E_1^{(1)} := \{Z_1 \leq \gamma x\} , \qquad E_1^{(2)} := \{|Z_1 - x| \leq a_n\} , \\ &E_1^{(3)} := \{Z_1 > \gamma x, \, |Z_1 - x| > a_n\} \end{aligned} \tag{7.2}$$

By Lemma 4.3 with $\ell = 0$, with no extra assumptions on F, we already know that

$$\sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I, E_1^{(1)}) \quad \text{is always a.n.} . \tag{7.3}$$

Next we look at $E_1^{(2)}$. Note that by (2.7)

$$\begin{aligned} \mathrm{P}(S_n \in x + I, \, E_1^{(2)}) &\leq \sum_{i=1}^{n} \mathrm{P}(S_n \in x + I, \, |X_i - x| \leq a_n) \\ &= n \int_{|y| \leq a_n} F(x + \mathrm{d}y) \, \mathrm{P}(S_{n-1} \in I - y) \lesssim \frac{n}{a_n} \int_{|y| \leq a_n} F(x + \mathrm{d}y) . \end{aligned} \tag{7.4}$$

Using the fact that n/a_n is regularly varying and recalling (5.9), we obtain

$$\begin{aligned} \sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I, \, E_1^{(2)}) &\lesssim \int_{|y| \leq \delta x} F(x + \mathrm{d}y) \sum_{A(|y|) \leq n \leq A(\delta x)} \frac{n}{a_n} \\ &\lesssim \int_{|y| \leq \delta x} F(x + \mathrm{d}y) \, \tilde{b}_2(\delta x, y) = \tilde{I}_1(\delta; x) . \end{aligned} \tag{7.5}$$

Recalling (1.23), we have shown that

$$\tilde{I}_1(\delta; x) \text{ is a.n.} \qquad \Longrightarrow \qquad \sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I, E_1^{(2)}) \quad \text{is a.n.} . \tag{7.6}$$

(Actually, the reverse implication also holds, as shown in Section 5.)

We finally turn to $E_1^{(3)}$. Arguing as in (7.4) and setting $\overline{\gamma} := 1 - \gamma$, we have by (1.6)

$$\begin{aligned} \mathrm{P}(S_n \in x + I, E_1^{(3)}) &\lesssim n \int_{|y| > a_n, \, y > -\overline{\gamma} x} F(x + \mathrm{d}y) \, \mathrm{P}(S_{n-1} \in I - y) \\ &\lesssim \frac{n^2}{a_n} \int_{|y| > a_n, \, y > -\overline{\gamma} x} F(x + \mathrm{d}y) \, \frac{1}{A(y)} , \end{aligned}$$

hence, recalling (1.10), we get

$$\begin{aligned} \sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I, \, E_1^{(3)}) &\lesssim \int_{y > -\overline{\gamma} x} F(x + \mathrm{d}y) \, \frac{1}{A(y)} \sum_{1 \leq n \leq A(\delta x \wedge |y|)} \frac{n^2}{a_n} \\ &\lesssim \int_{y > -\overline{\gamma} x} F(x + \mathrm{d}y) \, \frac{1}{A(y)} \, b_3(\delta x \wedge |y|) , \end{aligned}$$

where the last inequality holds for $\alpha > \frac{1}{3}$, thanks to (2.3), because n^2/a_n is regularly varying with index $2 - 1/\alpha > -1$. The right hand side can be estimated by

$$\int_{|y|\le\delta x} F(x+\mathrm{d}y)\, b_2(y) \;+\; b_3(\delta x) \int_{y>-\overline{\gamma}x,\, |y|>\delta x} F(x+\mathrm{d}y)\, \frac{1}{A(y)}$$
$$\lesssim I_1(\delta;x) \;+\; \frac{b_3(\delta x)}{A(\delta x)}\, F((\gamma x,\infty)) \lesssim I_1(\delta;x) \;+\; b_1(\delta x)\,.$$

Therefore

$$\sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I, E_1^{(3)}) \quad \text{is a.n.} \quad \text{if } \alpha > \frac{1}{3} \text{ and } I_1(\delta;x) \text{ is a.n.}. \tag{7.7}$$

Relations (7.3), (7.6), (7.7) prove Theorem 6.1 in case $\kappa_\alpha = 1$ (recall Lemma A.3).

8. Proof of Theorem 1.12: sufficiency in case $\alpha \le \frac{1}{3}$

Let F be a probability on $\mathbb{R}$ that satisfies (1.2) with $p, q \ge 0$ and $\alpha \in (0, \frac{1}{3}]$. In this section, we assume that $\tilde{I}_{\kappa_\alpha}(\delta,\eta;x)$ is a.n. and we deduce (4.1), which is equivalent to the SRT. By Remark 1.14, this proves the sufficiency part in Theorem 1.12, in case $\kappa_\alpha \ge 2$.

By Remark 1.14, $\tilde{I}_1(\delta;x)$ is a.n., hence also $I_1(\delta;x)$ is a.n. (cf. Lemma A.3). Throughout this section we fix $\gamma \in (0, \frac{\alpha}{1-\alpha})$, we set $\eta = \overline{\gamma} := 1-\gamma$ and we drop η from notations.

8.1. **Preparation.**

Let us rewrite $I_k(\delta;x)$ as follows (recall (1.18), (1.19)):

$$I_k(\delta;x) := \int_{|y_1|\le\delta x} F(x+\mathrm{d}y_1)\, g_k(y_1)\,, \tag{8.1}$$

$$\text{where we set} \qquad g_k(y_1) := \begin{cases} b_2(y_1) & \text{if } k=1 \\ \displaystyle\int_{\Omega_k(y_1)} P_{y_1}(\mathrm{d}y_2,\ldots,\mathrm{d}y_k)\, b_{k+1}(y_k) & \text{if } k\ge 2 \end{cases}, \tag{8.2}$$

$$\Omega_k(y_1) := \big\{|y_j| \le \overline{\gamma}|y_{j-1}| \text{ for } 2\le j\le k\big\}\,, \tag{8.3}$$

and we recall that $P_{y_1}(\mathrm{d}y_2,\ldots,\mathrm{d}y_k) := F(-y_1+\mathrm{d}y_2)F(-y_2+\mathrm{d}y_3)\cdots F(-y_{k-1}+\mathrm{d}y_k)$.

If $I_r(\delta,x)$ is a.n., then for $\delta > 0$ small we have the bound $I_r(\delta,x) \lesssim b_1(x)$ for all $x \ge 0$. The following crucial Proposition shows that the same bound holds also when the integral in (8.1) is enlarged to $\{y_1 > -\kappa x\}$, for any fixed $\kappa < 1$.

Proposition 8.1. *If $I_r(\delta,x)$ is a.n. for some $r \ge 1$, then for any $0<\kappa<1$*

$$\int_{y>-\kappa x} F(x+\mathrm{d}y)\, g_r(y) \lesssim_{\kappa,\gamma} b_1(x) \qquad (\forall x \ge 0)\,. \tag{8.4}$$

Proof. The case $r=1$ is easy: since $b_2 \in RV(2\alpha-1)$ and $2\alpha - 1 < 0$, for any fixed $\delta_0 > 0$

$$\int_{|y|>\delta_0 x,\, y>-\kappa x} F(x+\mathrm{d}y)\, b_2(y) \le \Big(\sup_{|y|>\delta_0 x} b_2(y)\Big)\, \overline{F}((1-\kappa)x) \lesssim_{\delta_0,\kappa} \frac{b_2(x)}{A(x)} = b_1(x)\,. \tag{8.5}$$

On the other hand, the contribution to the integral of $|y| \le \delta_0 x$ gives $I_1(\delta_0;x)$ which is $\lesssim b_1(x)$ for $\delta_0 > 0$ small enough, as we already observed, because $I_1(\delta;x)$ is a.n..

Next we fix $k \ge 1$, we assume that $I_{k+1}(\delta;x)$ is a.n.. By induction, we can assume that (8.4) holds for $1 \le r \le k$ and our goal is to prove it for $r = k+1$.

We start estimating $g_{k+1}(-z)$ for $z \geq 0$. In case $k = 1$, we can write

$$g_2(-z) = \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2)\, b_3(y_2) \leq A(z) \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2)\, b_2(y_2) \lesssim b_2(z)\,,$$

where in the last inequality we applied the inductive hypothesis (8.4) for $r = 1$ (using the fact that $z \geq 0$). Similarly, for $k \geq 2$,

$$\begin{aligned} g_{k+1}(-z) &= \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2) \int_{\Omega_k(y_2)} P_{y_2}(\mathrm{d}y_3, \cdots, \mathrm{d}y_{k+1})\, b_{k+2}(y_{k+1}) \\ &\leq A(z) \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2) \int_{\Omega_k(y_2)} P_{y_2}(\mathrm{d}y_3, \cdots, \mathrm{d}y_{k+1})\, b_{k+1}(y_{k+1}) \\ &= A(z) \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2)\, g_k(y_2) \lesssim b_2(z)\,, \end{aligned}$$

by (8.4) for $r = k$. As a consequence, the contribution of $y \leq 0$ to (8.4) for $r = k + 1$ is

$$\int_{0 \leq z < \kappa x} F(x - \mathrm{d}z)\, g_{k+1}(-z) \lesssim \int_{0 \leq z < \kappa x} F(x - \mathrm{d}z)\, b_2(z) \lesssim b_1(x)\,,$$

where the last inequality holds again by (8.4) for $r = 1$.

It remains to look at the contribution of $y > 0$ in (8.4) for $r = k + 1$. By (8.1), the contribution of $\{0 < y \leq \delta_1 x\}$ is bounded by $I_{k+1}(\delta_1, x)$ which is a.n. by assumption, hence it is $\lesssim b_1(x)$ provided $\delta_1 > 0$ is small enough. It remains to focus on $\{y > \delta_1 x\}$.

We need a simple observation; let $I = (a_1, a_2)$ where $0 \leq a_1 < a_2 \leq \infty$ and put $I' = (\gamma a_1, (2 - \gamma) a_2)$. Then for all non-negative f, g

$$\begin{aligned} &\int_{y \in xI} F(x + \mathrm{d}y)\, f(y) \int_{|z| \leq \overline{\gamma} y} F(-y + \mathrm{d}z)\, g(z) \\ &= \int_{y \in xI} F(x + \mathrm{d}y)\, f(y) \int_{\gamma y \leq w \leq (2-\gamma) y} F(-\mathrm{d}w)\, g(y - w) \\ &\leq \int_{w \in xI'} F(-\mathrm{d}w) \int_{(2-\gamma)^{-1} w \leq y \leq \gamma^{-1} w} F(x + \mathrm{d}y)\, f(y)\, g(y - w) \\ &= \int_{w \in xI'} F(-\mathrm{d}w) \int_{-\gamma_1 w \leq v \leq \gamma_2 w} F(x + w + \mathrm{d}v)\, f(w + v)\, g(v), \end{aligned} \tag{8.6}$$

where $\gamma_1 = 1 - (2 - \gamma)^{-1}$ and $\gamma_2 = \gamma^{-1} - 1$. Using this, we can write the contribution of $\{y > \delta_1 x\}$ to (8.4) for $r = k + 1$ as follows:

$$\begin{aligned} \int_{\delta_1 x}^{\infty} F(x + \mathrm{d}y)\, g_{k+1}(y) &\leq \int_{\delta_1 x}^{\infty} F(x + \mathrm{d}y) \int_{-\overline{\gamma} y}^{\overline{\gamma} y} F(-y + \mathrm{d}z)\, A(z)\, g_k(z) \\ &= \int_{\gamma \delta_1 x}^{\infty} F(-\mathrm{d}w) \int_{-\gamma_1 w}^{\gamma_2 w} F(x + w + \mathrm{d}v)\, A(v)\, g_k(v) \\ &\lesssim_{\gamma} \int_{\gamma \delta_1 x}^{\infty} F(-\mathrm{d}w)\, A(x + w) \int_{-\gamma_1 (x+w)}^{\gamma_2 (x+w)} F(x + w + \mathrm{d}v)\, g_k(v)\,. \end{aligned}$$

Applying (8.4) for $r = k$, and the fact that $b_2(\cdot)$ is asymptotically decreasing, we obtain

$$\int_{\delta_1 x}^{\infty} F(x + \mathrm{d}y)\, g_{k+1}(y) \lesssim \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w)\, A(x+w)\, b_1(x+w) = \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w)\, b_2(x+w)$$
$$\lesssim b_2(x) \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w) \lesssim_{\delta_1,\gamma} b_1(x)\,. \ \blacksquare$$

We need to generalize 8.2: for any non-negative, even function $f : \mathbb{R} \to \mathbb{R}$ we denote by $g_k(y_1, f)$ what we get by replacing $b_{k+1}(y_k)$ by $b_k(y_k) f(y_k)$ in 8.2, that is

$$g_k(y_1, f) := \begin{cases} b_1(y_1)\, f(y_1) & \text{if } k = 1 \\ \displaystyle\int_{\Omega_k(y_1)} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, b_k(y_k)\, f(y_k) & \text{if } k \geq 2 \end{cases}. \tag{8.7}$$

In particular, $g_k(y)$ is $g_k(y, A)$. The following result is in the same spirit as Proposition 8.1.

Proposition 8.2. *Assume that $I_{\ell-1}(\delta, x)$ is a.n. for some $\ell \geq 1$ (if $\ell = 1$, no assumption is made). For any $f \in RV(\beta)$ with $0 < \beta < 1 - \alpha$, and for all $0 < \delta_0 < \kappa < 1$,*

$$\int_{|y|>\delta_0 x,\, y>-\kappa x} F(x + \mathrm{d}y)\, g_\ell(y, f) \lesssim_{\delta_0,\kappa,\gamma} \frac{f(x)}{x \vee 1} \qquad (\forall x \geq 0)\,. \tag{8.8}$$

Proof. Since $g_1(\cdot) = b_1(\cdot) f(\cdot) \in RV(\alpha + \beta - 1)$ is asymptotically decreasing,

$$\int_{|y|>\delta_0 x,\, y>-\kappa x} F(x + \mathrm{d}y)\, g_1(y, f) \lesssim_{\delta_0} b_1(x) f(x) \int_{y>-\kappa x} F(x + \mathrm{d}y)$$
$$\lesssim_\kappa \frac{b_1(x) f(x)}{A(x)} = \frac{f(x)}{x \vee 1}\,,$$

which proves (8.8) if $\ell = 1$. Henceforth we assume that $\ell \geq 2$ and proceed by induction.

For $\ell = 2$ and $z \geq 0$, since $f(\cdot)$ is asymptotically increasing, we can bound

$$g_2(-z, f) = \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2)\, b_2(y_2)\, f(y_2) \lesssim f(z) \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2)\, b_2(y_2)$$
$$\lesssim f(z) \int_{y_2 \geq -\overline{\gamma} z} F(z + \mathrm{d}y_2)\, b_2(y_2) \lesssim f(z)\, b_1(z)\,,$$

where for the last inequality we apply Proposition 8.1 for $r = \ell - 1 = 1$ (since $z \geq 0$). Similarly, for $\ell \geq 3$ and $z \geq 0$,

$$g_\ell(-z, f) = \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2) \int_{\Omega_{\ell-1}(y_2)} P_{y_2}(\mathrm{d}y_3, \cdots, \mathrm{d}y_\ell)\, b_\ell(y_\ell)\, f(y_\ell)$$
$$\lesssim f(z) \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2) \int_{\Omega_{\ell-1}(y_2)} P_{y_2}(\mathrm{d}y_3, \cdots, \mathrm{d}y_\ell)\, b_\ell(y_\ell)$$
$$= f(z) \int_{|y_2| \leq \overline{\gamma} z} F(z + \mathrm{d}y_2)\, g_{\ell-1}(y_2) \lesssim_\gamma f(z)\, b_1(z)\,,$$

again by Proposition 8.1 for $r = \ell - 1$, because $z \geq 0$. Then the contribution of $y \leq 0$ to (8.8) is easily estimated (recall that $f(\cdot) b_1(\cdot)$ is asymptotically increasing):

$$\int_{\delta_0 x \leq z \leq \kappa x} F(x - \mathrm{d}z)\, g_\ell(-z, f) \lesssim_{\delta_0} f(x)\, b_1(x) \int_{\delta_0 x \leq z \leq \kappa x} F(x - \mathrm{d}z) \lesssim_\kappa \frac{f(x)\, b_1(x)}{A(x)} = \frac{f(x)}{x \vee 1}\,.$$

It remains to control the contribution to (8.8) of $y > 0$. Let us consider the case $\ell \geq 3$: by (8.7), since $f(\cdot)$ is asymptotically increasing,

$$\begin{aligned}
g_\ell(y, f) &= \int_{\Omega_\ell(y)} P_y(\mathrm{d}y_2, \ldots, \mathrm{d}y_\ell)\, b_\ell(y_\ell)\, f(y_\ell) \\
&= \int_{|z| \leq \overline{\gamma} y} F(-y + \mathrm{d}z) \int_{\Omega_{\ell-1}(z)} P_z(\mathrm{d}y_3, \ldots, \mathrm{d}y_\ell)\, b_\ell(y_\ell)\, f(y_\ell) \\
&\lesssim \int_{|z| \leq \overline{\gamma} y} F(-y + \mathrm{d}z)\, f(z) \int_{\Omega_{\ell-1}(z)} P_z(\mathrm{d}y_3, \ldots, \mathrm{d}y_\ell)\, b_\ell(y_\ell) \\
&= \int_{|z| \leq \overline{\gamma} y} F(-y + \mathrm{d}z)\, f(z)\, g_{\ell-1}(z)\,.
\end{aligned}$$

and the same bound holds also for $\ell = 2$. Then, applying (8.6),

$$\begin{aligned}
\int_{\delta_0 x}^{\infty} F(x + \mathrm{d}y)\, g_\ell(y, f) &\leq \int_{\delta_0 x}^{\infty} F(x + \mathrm{d}y) \int_{-\overline{\gamma} y}^{\overline{\gamma} y} F(-y + \mathrm{d}z)\, f(z)\, g_{\ell-1}(z) \\
&\lesssim \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w) \int_{-\gamma_1 w}^{\gamma_2 w} F(x + w + \mathrm{d}z)\, f(z)\, g_{\ell-1}(z) \\
&\lesssim_\gamma \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w)\, f(w) \int_{-\gamma_1(x+w)}^{\infty} F(x + w + \mathrm{d}z)\, g_{\ell-1}(z)\,.
\end{aligned} \tag{8.9}$$

Applying again Proposition 8.1 for $r = \ell - 1$ we get

$$\begin{aligned}
\int_{\delta_0 x}^{\infty} F(x + \mathrm{d}y)\, g_\ell(y, f) &\lesssim \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w)\, f(x + w)\, b_1(x + w) \\
&\lesssim f(x)\, b_1(x) \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w) \lesssim_{\gamma, \delta_0} \frac{f(x)}{x \vee 1}\,. \qquad \blacksquare
\end{aligned} \tag{8.10}$$

We need to look at a generalization of the quantities that we have studied so far. Let us enlarge the set $\Omega_k(y_1)$ in (8.3), defining for $k \geq 2$ a new set $\Theta_k(y_1) \subseteq \mathbb{R}^{k-1}$ by

$$\begin{aligned}
\Theta_k(y_1) &:= \big\{ |y_j - y_{j-1}| \geq \gamma |y_{j-1}| \text{ for } 2 \leq j \leq k \big\}\,, \\
&:= \big\{ y_j \in \theta(y_{j-1}) \text{ for } 2 \leq j \leq k \big\}\,, \\
\text{where} \qquad \theta(y) &:= \begin{cases} (-\infty, \overline{\gamma} y] & \text{if } y \geq 0 \\ [\overline{\gamma} y, +\infty) & \text{if } y < 0 \end{cases}\,.
\end{aligned} \tag{8.11}$$

Then we define $h_k(y_1)$ in analogy with $g_k(y_1)$, cf. (8.2), just replacing $\Omega_k(y_1)$ by $\Theta_k(y_1)$:

$$h_k(y_1) := \begin{cases} b_2(y_1) & \text{if } k = 1 \\ \displaystyle\int_{\Theta_k(y_1)} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, b_{k+1}(y_k) & \text{if } k \geq 2 \end{cases}\,. \tag{8.12}$$

Similarly to (8.7), for any non-negative, even function $f : \mathbb{R} \to \mathbb{R}$ we also define

$$h_k(y_1, f) := \begin{cases} b_1(y_1)\, f(y_1) & \text{if } k = 1 \\ \displaystyle\int_{\Omega_k(y_1)} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, b_k(y_k)\, f(y_k) & \text{if } k \geq 2 \end{cases}\,, \tag{8.13}$$

and note that $h_k(y_1, A) = h_k(y_1)$. Finally, recalling (8.1), we set

$$J_k(\delta; x) := \int_{|y|\le\delta x} F(x + \mathrm{d}y)\, h_k(y)\,. \tag{8.14}$$

Note that $h_1 = g_1$, hence $J_1(\delta; x) = I_1(\delta; x)$. For $k \ge 2$, since $\Theta_k \supseteq \Omega_k$, we have $h_k \ge g_k$, hence $J_k(\delta; x) \ge I_k(\delta; x)$.

Out next result shows that we can often work with J_k rather than I_k.

Proposition 8.3. *Fix $r \in \mathbb{N}$ and assume $\alpha < \frac{1}{r+1}$. If $I_r(\delta, x)$ is a.n., then*

$$J_r(\delta, x) \text{ is a.n..} \tag{8.15}$$

Also if $f \in RV(\beta)$ with $0 < \beta < 1 - r\alpha$ then for any $0 < \delta_0 < \kappa < 1$

$$\int_{|y|>\delta_0 x,\, y>-\kappa x} F(x + \mathrm{d}y)\, h_r(y, f) \lesssim_{\delta_0,\kappa,\gamma} \frac{f(x)}{x \vee 1} \qquad (\forall x \ge 0)\,. \tag{8.16}$$

Proof. We claim that for all f satisfying the assumptions the following bound holds:

$$h_r(y, f) \lesssim_\gamma \sum_{j=1}^{r} g_j(y, f) \qquad (\forall y \in \mathbb{R})\,. \tag{8.17}$$

Applying this bound, relation (8.16) follows immediately by Proposition 8.2 (compare (8.7) and (8.13)). If we choose $f = A$ in (8.17) (this is possible because $\beta = \alpha$ satisfies $\beta < 1-r\alpha$, since we assume $\alpha < \frac{1}{r+1}$) we obtain $h_r(y) \lesssim_\gamma \sum_{j=1}^r g_j(y)$, which plugged into (8.14) shows that $J_r(\delta; x) \lesssim_\gamma \sum_{j=1}^r I_r(\delta; x)$, from which (8.15) follows.

Note that (8.17) holds trivially for $r = 1$, since $h_1(y, f) = g_1(y, f)$. Henceforth we fix $r \ge 2$ and, inductively, we assume that (8.15), (8.16) and (8.17) have been proved when r is replaced by $r - 1$, $r - 2$, etc.. Our goal is to prove (8.17).

Let us first consider the case $y < 0$. By (8.13) we can write

$$h_r(y, f) = \int_{y_2\ge-\overline{\gamma}|y|} F(-y + \mathrm{d}y_2)\, h_{r-1}(y_2, Af) \lesssim_\gamma \sum_{j=1}^{r-1} \int_{y_2\ge-\overline{\gamma}|y|} F(-y + \mathrm{d}y_2)\, g_j(y_2, Af)\,,$$

by the induction hypothesis, applying (8.17) with $r-1$ instead of r (this is possible because $Af \in RV(\beta')$ with $\beta' = \alpha + \beta$ which satisfies $0 < \beta' < 1 - (r - 1)\alpha$). We now split the domain of integration in the two subsets $[-\overline{\gamma}|y|, \overline{\gamma}|y|] \cup (\overline{\gamma}|y|, \infty)$. The first subset gives

$$\int_{|y_2|\le\overline{\gamma}|y|} F(-y + \mathrm{d}y_2)\, g_j(y_2, Af) = g_{j+1}(y, f)\,,$$

by (8.7). For the second subset we can apply Proposition 8.2 (since $-y \ge 0$), getting

$$\int_{y_2>\overline{\gamma}|y|} F(-y + \mathrm{d}y_2)\, g_j(y_2, Af) \lesssim_\gamma \frac{A(y)f(y)}{y \vee 1} = b_1(y)\, f(y) = g_1(y, f)\,,$$

hence (8.17) is proved.

Next we consider the case $y \ge 0$. If we restrict the domain of integration $\Theta_r(y)$ in (8.13) to $y_2 \ge 0, y_3 \ge 0, \ldots, y_r \ge 0$, then the domain becomes $\{0 \le y_j \le \overline{\gamma} y_{j-1} \text{ for } 2 \le j \le r\}$ which is included in $\Omega_r(y)$, cf. (8.3). This contribution to $h_r(y, f)$ is then bounded from above by $g_r(y, f)$, cf. (8.7), in accordance with (8.17).

It remains to estimate $h_r(y, f)$ for $y \ge 0$, when some of the coordinates $y_2, y_3, \ldots, y_r$ in the integral in (8.13) are negative. Let us define $H := \min\{j \in \{2, \ldots, r\} : y_j \le 0\}$.

In the extreme case $H = r$, the corresponding contribution to $h_r(y, f)$ is

$$\begin{aligned}
&\int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{r-1}=0}^{\overline{\gamma} y_{r-2}} \int_{y_r=-\overline{\gamma} y_{r-1}}^{0} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_r)\, b_r(y_r) f(y_r) \\
&+ \int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{r-1}=0}^{\overline{\gamma} y_{r-2}} \int_{y_r=-\infty}^{-\overline{\gamma} y_{r-1}} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_r)\, b_r(y_r) f(y_r)\,.
\end{aligned} \tag{8.18}$$

If $r = 2$, one should ignore the first integrals, that is we have

$$\int_{y_2=-\overline{\gamma} y}^{0} F(-y + \mathrm{d}y_2)\, b_2(y_2)\, f(y) \;+\; \int_{y_2=-\infty}^{\overline{\gamma} y} F(-y + \mathrm{d}y_2)\, b_2(y_2)\, f(y)\,. \tag{8.19}$$

The first integral in (8.18)-(8.19) is bounded by $g_r(y, f)$ (recall (8.7)). For the second integral, we use the fact that $b_r(\cdot) f(\cdot) \in \mathcal{R}_{r\alpha - 1 + \beta}$ is asymptotically decreasing (since $r\alpha - 1 + \beta < 0$ by assumption), hence we can bound $b_r(y_r) f(y_r) \lesssim b_r(y_{r-1}) f(y_{r-1})$. Since $P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_r) = P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1}) F(-y_{r-1} + \mathrm{d}y_r)$, when we integrate over $y_r \in (-\infty, -\overline{\gamma} y_{r-1}]$ we get a factor $\lesssim_\gamma 1/A(y_{r-1})$. Overall, we can bound (8.18) by

$$\begin{aligned}
&\le g_r(y, f) + \int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{r-1}=0}^{\overline{\gamma} y_{r-2}} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1}) \frac{1}{A(y_{r-1})}\, b_r(y_{r-1}) f(y_{r-1}) \\
&\le g_r(y, f) + g_{r-1}(y, f)\,,
\end{aligned}$$

and the same bound holds also for $r = 2$. This proves (8.17) when $H = r$.

Finally, if $H = j \in \{2, \dots, r-1\}$, we can estimate

$$\begin{aligned}
&\int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{j-1}=0}^{\overline{\gamma} y_{j-2}} \int_{y_j=-\infty}^{0} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_j) \int_{y_{j+1} > -\overline{\gamma}|y_j|} F(|y_j| + \mathrm{d}y_{j+1})\, h_{r-j}(y_{j+1}, A^j f) \\
&\lesssim_\gamma \int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{j-1}=0}^{\overline{\gamma} y_{j-2}} \int_{y_j=-\infty}^{0} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_j)\, b_j(y_j)\, f(y_j)\,,
\end{aligned}$$

thanks to the induction hypothesis (8.16). Splitting the integral over y_j in the domains $(-\infty, -\overline{\gamma} y_{j-1}) \cup [-\overline{\gamma} y_{j-1}, 0]$, we obtain precisely (8.18) with j in place of r. With the same arguments, we see that this contribution is $\lesssim g_j(y, f) + g_{j-1}(y, f)$. ∎

We finally need to consider, in the case $\alpha = \frac{1}{k+1}$ for some $k \ge 2$, the following quantity:

$$\tilde{J}_k(\delta, x) = \int_{|y_1| \le \delta x} F(x + \mathrm{d}y_1)\, \tilde{h}_k(y_1)\,, \tag{8.20}$$

with

$$\tilde{h}_k(y_1) = \int_{\Theta_{k-1}(y_1) \cap \{|y_k| \le \overline{\gamma} |y_{k-1}|\}} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_k)\, \tilde{b}_{k+1}(y_{k-1}, y_k), \tag{8.21}$$

where $\Theta_k(y_1)$ and $\tilde{b}(x, z)$ are defined in (8.11) and (1.22) (or equivalently (5.9)). In case $k = 2$, the constraint $\Theta_{k-1}(y_1)$ in (8.21) should be ignored (since $\Theta_r(y_1)$ is only defined for $r \ge 2$) and $\tilde{J}_2(\delta, x)$ coincides with $\tilde{I}_2(\delta, x)$, defined in (1.24) (recall that $\overline{\gamma} = 1 - \gamma = \eta$).

From (5.9), (2.4) and (1.10), for any $\lambda \in (0, k)$, with $\alpha = \frac{1}{k+1}$, we have the bound

$$\tilde{b}_{k+1}(y, z) \le A(y)^\lambda\, \tilde{b}_{k+1-\lambda}(y, z) \le A(y)^\lambda \sum_{m=A(z)}^{\infty} \frac{m^{(k-\lambda)}}{a_m} \lesssim A(y)^\lambda\, b_{k+1-\lambda}(z). \tag{8.22}$$

Proposition 8.4. *Fix $k \in \mathbb{N}$ with $k \ge 2$ and assume that $\alpha = \frac{1}{k+1}$. If $\tilde{I}_k(\delta, x)$ is a.n., then $\tilde{J}_k(\delta, x)$ is a.n..*

Proof. In case $k=2$ there is nothing to prove, since $\tilde{J}_2(\delta,x)=\tilde{I}_2(\delta,x)$.

We now fix $k\geq 3$. If we consider the contribution to the integrals in (8.20)-(8.21) of $y_1\geq 0, y_2\geq 0,\ldots,y_{k-1}\geq 0$, the domain of integration, cf. (8.11), reduces to

$$\{0\leq y_1\leq \delta x\}\cap\{0\leq y_i\leq \overline{\gamma}y_{i-1} \text{ for } 2\leq i\leq k-1\}\cap\{|y_k|\leq \overline{\gamma}y_{k-1}\}\,.$$

This contribution is bounded from above by $\tilde{I}_k(\delta,x)$, cf. (1.24), which is a.n. by assumption.

Next we consider the contribution to (8.20)-(8.21) coming from $y_1,\ldots,y_{k-1}$ such that $y_i<0$ for some $1\leq i\leq k-1$. Let us define $\tilde{H}=\max\{j\in\{1,\ldots,k-1\}:\ y_j<0\}$. If $\tilde{H}=k-1$, the bound (8.22) with $\lambda=k-1$ and the fact that $y_{k-1}<0$ show that

$$\begin{aligned}&\int_{|y_k|\leq\overline{\gamma}|y_{k-1}|} F(-y_{k-1}+\mathrm{d}y_k)\,\tilde{b}_{k+1}(y_{k-1},y_k)\\ &\lesssim A(y_{k-1})^{k-1}\int_{|y_k|\leq\overline{\gamma}|y_{k-1}|} F(-y_{k-1}+\mathrm{d}y_k)\,b_2(y_k)\lesssim A(y_{k-1})^{k-1}b_1(y_{k-1})=b_k(y_{k-1}),\end{aligned}$$

where the second inequality comes from Proposition 8.1 with $r=1$ (we recall that $I_k(\delta,x)$ is a.n. follows from $\tilde{I}_k(\delta,x)$ is a.n., thanks to Lemma A.3). Plugging this bound into (8.21), we see that the contribution to $\tilde{h}_k(y)$ is at most $h_{k-1}(y)$ (recall (8.12)), hence the contribution to $\tilde{H}_k(\delta,x)$ is at most $J_{k-1}(\delta,x)$ (recall (8.14)), which is a.n. by Proposition 8.3.

We finally consider the contribution of $\tilde{H}=k-j$ with $j\geq 2$. This means that $y_{k-j}<0$, while $y_{k-j+1}\geq 0,\ldots,y_{k-1}\geq 0$, and the range of integration in (8.21) is a subset of

$$\Theta_{k-j-1}(y_1)\cap\{y_{k-j}<0\}\cap\{y_{k-j+1}\geq 0\}\cap\{|y_{k-j+1+r}|\leq\overline{\gamma}|y_{k-j+r}|, r=1,\cdots j-1\}.$$

We split this into the two subsets $\{0\leq y_{k-j+1}\leq\overline{\gamma}|y_{k-j}|\}$ and $\{y_{k-j+1}>\overline{\gamma}|y_{k-j}|\}$.

On the first subset $\{0\leq y_{k-j+1}\leq\overline{\gamma}|y_{k-j}|\}$, we bound $\tilde{b}_{k+1}(y_{k-1},y_k)\lesssim A(y_{k-1})^{k-j}b_{j+1}(y_k)$, by (8.22) with $\lambda=k-j$, and then $A(y_{k-1})\lesssim A(y_{k-j})$. Recalling the definition (8.2) of $g_j(\cdot)$, we see that this part of the integral with respect to $y_{k-j+1},\cdots,y_k$ is

$$\begin{aligned}&\lesssim A(y_{k-j})^{k-j}\int\limits_{0\leq y_{k-j+1}\leq\overline{\gamma}|y_{k-j}|} F(|y_{k-j}|+\mathrm{d}y_{k-j+1})\,g_j(y_{k-j+1})\\ &\leq A(y_{k-j})^{k-j}\int\limits_{z\geq-\overline{\gamma}|y_{k-j}|} F(|y_{k-j}|+\mathrm{d}z)\,g_j(z)\\ &\lesssim A(y_{k-j})^{k-j}\,b_1(y_{k-j})=b_{k-j+1}(y_{k-j})\,,\end{aligned}$$

where the last inequality follows by Proposition 8.1. The contribution to (8.21) is

$$\lesssim\int_{\Theta_{k-j}(y_1)\cap\{y_{k-j}<0\}} P_y(\mathrm{d}y_2,\ldots,\mathrm{d}y_{k-j})\,b_{k-j+1}(y_{k-j})\leq h_{k-j}(y_1)\,, \tag{8.23}$$

hence the contribution to $\tilde{J}_k(\delta,x)$ is $\lesssim J_{k-j}(\delta,x)$, which is a.n. by Proposition 8.3.

On the second subset $\{y_{k-j+1}>\overline{\gamma}|y_{k-j}|\}$, we we use the bound $\tilde{b}_{k+1}(y_{k-1},y_k)\lesssim A(y_{k-1})^{k-j+1}b_j(y_k)$, by (8.22) with $\lambda=k-j+1$, and then $A(y_{k-1})\lesssim A(y_{k-j+1})$, getting

$$\begin{aligned}&\lesssim\int\limits_{y_{k-j+1}>\overline{\gamma}|y_{k-j}|} F(|y_{k-j}|+\mathrm{d}y_{k-j+1})\,A(y_{k-j+1})^{k-j+1}\int\limits_{|y_{k-j+2}|\leq\overline{\gamma}y_{k-j+1}} F(-y_{k-j+1}+\mathrm{d}y_{k-j+2})\,g_{j-1}(y_{k-j+2})\\ &=\int\limits_{y>\overline{\gamma}|y_{k-j}|} F(|y_{k-j}|+\mathrm{d}y)\,A(y)^{k-j+1}\int\limits_{|z|\leq\overline{\gamma}y} F(-y+\mathrm{d}z)\,g_{j-1}(z)\,,\end{aligned}$$

where we have set $y = y_{k-j+1}$ and $z = y_{k-j+2}$ for short. Applying (8.6), where $\gamma_1 = 1-(2-\gamma)^{-1}$ and $\gamma_2 = \gamma^{-1}-1$, we get

$$\begin{aligned}
&\lesssim \int\limits_{w \ge \gamma\overline{\gamma}|y_{k-j}|} F(-\mathrm{d}w) \int\limits_{-\gamma_1 w \le v \le \gamma_2 w} F(|y_{k-j}| + w + \mathrm{d}v)\, A(w+v)^{k-j+1}\, g_{j-1}(v) \\
&\lesssim_\gamma \int\limits_{w \ge \gamma\overline{\gamma}|y_{k-j}|} F(-\mathrm{d}w)\, A(w)^{k-j+1} \int\limits_{v \ge -\gamma_1(|y_{k-j}|+w)} F(|y_{k-j}| + w + \mathrm{d}v)\, g_{j-1}(v) \\
&\lesssim_\gamma \int\limits_{w \ge \gamma\overline{\gamma}|y_{k-j}|} F(-\mathrm{d}w)\, A(w)^{k-j+1}\, b_1(|y_{k-j}| + w)\,,
\end{aligned}$$

by Proposition 8.1 with $r = j-1$. Finally, this is easily bounded by

$$\int\limits_{w \ge \gamma\overline{\gamma}|y_{k-j}|} F(-\mathrm{d}w)\, b_{k-j+2}(|y_{k-j}| + w) \lesssim F(-\gamma\overline{\gamma}|y_{k-j}|)\, b_{k-j+2}(|y_{k-j}|) \lesssim b_{k-j+1}(|y_{k-j}|)\,,$$

because $b_{k-j+2}(\cdot) \in RV(\alpha(k-j+2)-1)$ is asymptotically decreasing, since $j \ge 2$ and $\alpha = \frac{1}{k+1}$. Arguing as in (8.23), the contribution to $\tilde{J}_k(\delta, x)$ is $\lesssim J_{k-j+1}(\delta, x)$, which is a.n. by Proposition 8.3. This completes the proof. ∎

8.2. **Proof of Sufficiency.** Throughout the proof we fix $k = \kappa_\alpha = \lfloor 1/\alpha \rfloor - 1$ (cf. (1.21)).

We generalize (7.1), defining two sequences $Z_1, Z_2, \ldots Z_k$ and $Y_1, Y_2, \ldots, Y_k$ as follows:

$$Z_1 := \max\{X_1, \ldots, X_n\}\,, \qquad Y_1 := Z_1 - x\,,$$

and for $r \in \{2, \ldots, k\}$

$$Z_r := \begin{cases} \max\big\{\{X_j, 1 \le j \le n\} \setminus \{Z_j, 1 \le j \le r-1\}\big\} & \text{if } Y_{r-1} \le 0, \\ \min\big\{\{X_j, 1 \le j \le n\} \setminus \{Z_j, 1 \le j \le r-1\}\big\} & \text{if } Y_{r-1} > 0. \end{cases} \tag{8.24}$$

$$Y_r := \sum_{i=1}^r Z_i - x \tag{8.25}$$

Intuitively, Z_r *is the largest available step towards* x *from* $Z_1 + \ldots + Z_{r-1}$. In fact, we may assume that the following holds:

$$Z_r > 0 \quad \text{if } Y_{r-1} \le 0\,, \qquad \text{while} \qquad Z_r < 0 \quad \text{if } Y_{r-1} > 0\,, \tag{8.26}$$

because the event that (8.26) fails to be true can be ignored. In fact, this event occurs if, for some $r \le k$, either $Y_{r-1} \le 0$ and $\{X_j, 1 \le j \le n\} \setminus \{Z_j, 1 \le j \le r-1\}$ contains no positive terms or $Y_{r-1} > 0$ and this set contains no negative terms. Call $\mathcal{E}_{n,r}$ such event. Clearly $\mathrm{P}(\mathcal{E}_{n,r},\, S_n \in x + I,\, Y_{r-1} \notin I) = 0$ for all $n \ge r$ (we recall that $I = (-h, 0]$), while

$$\begin{aligned}
\mathrm{P}(\mathcal{E}_{n,r},\, Y_{r-1} \in I) &\le \binom{n}{r-1} \mathrm{P}(S_{r-1} \in x + I,\ X_r, X_{r+1}, \ldots, X_n \le 0) \\
&\le n^{r-1}\, c^{n-(r-1)}\, \mathrm{P}(S_{r-1} \in x + I)\,,
\end{aligned}$$

with $c = \mathrm{P}(X_1 \le 0) < 1$, hence by (4.4)

$$\sum_{n=r}^\infty \mathrm{P}(\mathcal{E}_{n,r},\, S_n \in x + I) \le \sum_{n=r}^\infty \mathrm{P}(\mathcal{E}_{n,r},\, Y_{r-1} \in I) \lesssim_r \mathrm{P}(S_{r-1} \in x + I) \underset{x\to\infty}{=} o\big(b_1(x)\big)\,.$$

We recall from (7.2) that

$$E_1^{(3)} = \{Z_1 > \gamma x,\ |Y_1| > a_n\}\,.$$

Next we define events $E_r^{(1)}, E_r^{(2)}, E_r^{(3)}$ for $r \ge 2$ inductively by

$$E_r^{(1)} = E_{r-1}^{(3)} \cap \{|Z_r| \le \gamma |Y_{r-1}|\}\,, \qquad E_r^{(2)} = E_{r-1}^{(3)} \cap \{|Y_r| \le a_n\}\,, \tag{8.27}$$
$$E_r^{(3)} = E_{r-1}^{(3)} \cap \{|Z_r| > \gamma |Y_{r-1}|,\ |Y_r| > a_n\}.$$

We can partition the probability space Ω as follows:

$$\Omega = \bigcup_{r=1}^{k} E_r^{(1)} \ \cup\ \bigcup_{r=1}^{k} E_r^{(2)} \ \cup\ E_k^{(3)}\,.$$

The argument to show that $\mathrm{P}(S_n \in x + I, E_1^{(1)} \cup E_1^{(2)})$ is a.n. presented in Section 7 is still valid, so it suffices to show that $\mathrm{P}(S_n \in x + I, E_r^{(1)})$ and $\mathrm{P}(S_n \in x + I, E_r^{(2)})$ are a.n. for $2 \le r \le k$, and furthermore $\mathrm{P}(S_n \in x + I, E_k^{(3)})$ is a.n..

Remark 8.5. *We can rewrite $E_\ell^{(3)}$ more explicitly as follows:*

$$E_\ell^{(3)} = \{Z_1 > \gamma x\,,\ |Z_i| > \gamma |Y_{i-1}| \text{ for } 2 \le i \le \ell\} \cap \{|Y_i| > a_n\,,\ \text{for } 1 \le i \le \ell\}\,.$$

Recalling the definition (8.11) *of $\Theta_k(\cdot)$, we claim that $E_\ell^{(3)}$ can also be rewritten as*

$$E_\ell^{(3)} = \{Y_1 > -\overline{\gamma} x,\ (Y_2, \ldots, Y_\ell) \in \Theta_\ell(Y_1)\} \cap \{|Y_i| > a_n\,,\ \text{for } 1 \le i \le \ell\}\,. \tag{8.28}$$

To prove the claim, we show that $|Z_i| > \gamma |Y_{i-1}|$ is equivalent to $Y_i \in \theta(Y_{i-1})$, for $2 \le i \le k$, with $\theta(\cdot)$ defined in (8.11)*. We recall that $Z_i = Y_i - Y_{i-1}$, cf.* (8.25)*. If $Y_{i-1} > 0$, then $Z_i \le 0$ by* (8.26)*, hence $|Z_i| > \gamma |Y_{i-1}|$ becomes $Y_{i-1} - Y_i > \gamma Y_{i-1}$, which is precisely $Y_i \in (-\infty, -\overline{\gamma} Y_{i-1}) = \theta(Y_{i-1})$. Similar arguments apply if $Y_{i-1} \le 0$, in which case $Z_i \ge 0$.*

Estimate of $E_r^{(1)}$. We fix $r \in \{2, \ldots, k\}$. By exchangeability,

$$\mathrm{P}(S_n \in x + I,\ E_r^{(1)}) \le n^{r-1}\, \mathrm{P}\big((Z_1, \ldots Z_{r-1}) = (X_1, \ldots, X_{r-1}),\ S_n \in x + I,\ E_r^{(1)}\big)\,. \tag{8.29}$$

Conditionally on $(X_1, \ldots, X_{r-1}) = (z_1, \ldots, z_{r-1})$, we have $S_n = (z_1 + \ldots + z_{r-1}) + S'_{n-(r-1)}$, where we set $S'_k := X'_1 + \ldots + X'_k$ with $X'_i := X_{(r-1)+i}$. Motivated by (8.25), if we set

$$y_{r-1} := (z_1 + \ldots + z_{r-1}) - x\,,$$

then $\{S_n \in x + I\} = \{S'_{n-(r-1)} \in -y_{r-1} + I\}$. Assume first that $y_{r-1} \le 0$. Then $Z_r = M'_{n-(r-1)} := \max\{X'_i, 1 \le i \le n - (r-1)\}$. Recalling (8.27) and (8.24), we need to evaluate

$$\mathrm{P}\Big(S'_{n-(r-1)} \in -y_{r-1} + I,\ |M'_{n-(r-1)}| \le \gamma\, |y_{r-1}|\Big)\,. \tag{8.30}$$

Since this probability is increasing in γ, applying (1.4), we get the bound

$$\lesssim \frac{1}{a_n} \left(\frac{n}{A(|y_{r-1}|)} \right)^d, \qquad \text{for all} \quad d \le \frac{1}{\gamma}\,. \tag{8.31}$$

In case $y_{r-1} > 0$, relation (8.30) holds with M'_k replaced by $(M'_k)^* := \min_{1\le i\le k} X'_i$. Applying (1.4) to the reflected walk $(S')^* = -S'$, we see that the bound (8.31) still holds. Looking back at (8.29), and recalling (8.27) and (8.28), we have the key bound

$$\mathrm{P}(S_n \in x{+}I,\, E_r^{(1)}) \lesssim \int_{\substack{y_1 > -\overline{\gamma} x, \\ (y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1), \\ \min\{|y_1|,\ldots,|y_{r-1}|\} \ge a_n}} F(x{+}\mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_{r-1})\, \frac{n^{r+d-1}}{a_n\, A(|y_{r-1}|)^d}\,. \tag{8.32}$$

Henceforth we fix $d \in (\frac{1}{\alpha} - r, \frac{1}{\alpha} - r + 1)$. Since $\gamma < \frac{\alpha}{1-\alpha}$ by assumption, and $r \ge 2$, we have $\frac{1}{\gamma} > \frac{1}{\alpha} - 1 \ge \frac{1}{\alpha} - (r-1) > d$, hence the constraint $d \le \frac{1}{\gamma}$ is satisfied. The sequence n^{d+r-1}/a_n is regularly varying with exponent $(d+r-1) - \frac{1}{\alpha} > -1$, hence by (2.3)

$$\sum_{1\le n\le A(w)} \frac{n^{r+d-1}}{a_n} \lesssim \frac{A(w)^{r+d}}{w \wedge 1} = b_{r+d}(w)\,, \qquad \forall w \ge 0\,,$$

where we recall that $b_k(\cdot)$ was defined in (1.10). It is convenient to set

$$\tilde{y}_{r-1} := \min\{|y_1|, \ldots, |y_{r-1}|\}\,, \tag{8.33}$$

so that the integral in (8.32) is restricted to $n \le A(\tilde{y}_{r-1})$. It follows that

$$\begin{aligned} &\sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x + I,\, E_r^{(1)}) \\ &\quad\lesssim \int_{\substack{y_1 > -\overline{\gamma} x \\ (y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1)}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_{r-1})\, \frac{b_{r+d}(\tilde{y}_{r-1} \wedge \delta x)}{A(|y_{r-1}|)^d}\,. \end{aligned} \tag{8.34}$$

To show that this integral is a.n., we separate the contributions of $|y_1| \le \delta x$ and $|y_1| > \delta x$.

- Since $\tilde{y}_{r-1} \le |y_{r-1}|$, and $b_{r+d}(\cdot)$ is increasing (because $r + d > \frac{1}{\alpha}$), we have

$$\frac{b_{r+d}(\tilde{y}_{r-1} \wedge \delta x)}{A(|y_{r-1}|)^d} \le \frac{b_{r+d}(|y_{r-1}|)}{A(|y_{r-1}|)^d} = b_r(|y_{r-1}|)\,,$$

  hence the contribution of $|y_1| \le \delta x$ to the integral in (8.34) is bounded by

$$\int_{\substack{|y_1| \le \delta x \\ (y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1)}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_{r-1})\, b_r(|y_{r-1}|) =: J_{r-1}(\delta, x)\,,$$

  where $J_k(\delta, x)$ was defined in (8.14). By Proposition 8.3, this is a.n..
- If we define $f_1(y) := 1/b_{r+d-1}(y)$, then recalling (8.13) we can write, for fixed y_1,

$$\int_{(y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1)} P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_{r-1})\, \frac{1}{A(|y_{r-1}|)^d} = h_{r-1}(y_1, f_1)\,.$$

  As a consequence, the contribution of $|y_1| > \delta x$ to the integral in (8.34) is at most

$$b_{r+d}(\delta x) \int_{|y_1| > \delta x,\, y_1 > -\overline{\gamma} x} F(x + \mathrm{d}y_1)\, h_{r-1}(y_1, f_1)\,.$$

Note that $f_1(\cdot) \in RV(\beta)$ with $\beta = 1-(r+d-1)\alpha$. Our choice of d implies that $\frac{1}{\alpha} < r+d < \frac{1}{\alpha}+1$, hence $0<\beta<\alpha$. Since we are in the regime $\alpha<\frac{1}{2}$, we also have $0<\beta<1-\alpha$, hence Proposition 8.2 with $f=f_1$, shows that this term is a.n.

Estimate of $E_r^{(2)}$. Always for $r \in \{2,\ldots,k\}$, in analogy with (8.29), we have

$$\begin{aligned} \mathrm{P}(S_n \in x+I,\, E_r^{(2)}) &\le n^r\, \mathrm{P}\big((Z_1,\ldots Z_r)=(X_1,\ldots,X_r),\, S_n\in x+I,\, E_r^{(2)}\big) \\ &\le n^r\, \mathrm{P}\big((Z_1,\ldots Z_r)=(X_1,\ldots,X_r),\, E_r^{(2)}\big)\left(\sup_{z\in\mathbb{R}} \mathrm{P}(S_{n-r}\in z+I)\right) \\ &\lesssim \frac{n^r}{a_n}\, \mathrm{P}\big((Z_1,\ldots Z_r)=(X_1,\ldots,X_r),\, E_r^{(2)}\big)\,, \end{aligned} \tag{8.35}$$

where the last inequality follows by (2.7). By (8.27) and (8.28), we obtain

$$\mathrm{P}(S_n\in x+I,\, E_r^{(2)}) \lesssim \int\limits_{\substack{y_1>-\overline{\gamma}x,\ (y_2,\ldots,y_{r-1})\in\Theta_{r-1}(y_1),\\ \min\{|y_1|,\ldots,|y_{r-1}|\}\ge a_n,\ |y_r|<a_n}} F(x+\mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_r)\,\frac{n^r}{a_n}\,. \tag{8.36}$$

Recalling (8.33) and (5.9), we can write

$$\begin{aligned} \sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n\in x+I,\, E_r^{(2)}) &\lesssim \int\limits_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\ldots,y_{r-1})\in\Theta_{r-1}(y_1)}} F(x+\mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_r) \sum_{n=A(|y_r|)}^{A(\delta x\wedge\tilde{y}_{r-1})} \frac{n^r}{a_n} \\ &= \int\limits_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\ldots,y_{r-1})\in\Theta_{r-1}(y_1)}} F(x+\mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_r)\, \tilde{b}_{r+1}(\delta x\wedge\tilde{y}_{r-1},|y_r|)\,. \end{aligned}$$

- Note that $\tilde{b}_{r+1}(\delta x\wedge\tilde{y}_{r-1},|y_r|) \le \tilde{b}_{r+1}(|y_{r-1}|,|y_r|)\,\mathbb{1}_{\{|y_r|\le\delta x\}}$, hence the contribution of $|y_1|\le\delta x$ is bounded by $\tilde{J}_r(\delta,x)$, cf. (8.20), which is a.n. by Proposition 8.4.
- Next we deal with $|y_1|>\delta x$. Note that $r\in\{2,\ldots,k\}$, hence $\alpha(r+1)\le\alpha(k+1)\le 1$ (since $k=\kappa_\alpha$ means $\alpha\in(\frac{1}{k+2},\frac{1}{k+1}]$). If $\alpha(r+1)<1$ we set $\psi=0$, while if $\alpha(r+1)=1$ we fix arbitrarily $\psi\in(0,1)$, so that $\alpha(r+1-\psi)<1$ and $f_2(y):=A(y)^{1-\psi}\in RV(\beta)$, with $\beta=\alpha(1-\psi)$, satisfies the conditions of Proposition 8.3. Then in both cases
$$\tilde{b}_{r+1}(\delta x\wedge\tilde{y}_{r-1},|y_r|) \lesssim A(\delta x)^{\psi}\,\tilde{b}_{r+1-\psi}(\tilde{y}_{r-1},|y_r|) \lesssim A(\delta x)^{\psi}\, b_{r+1-\psi}(y_r),$$
and we conclude that the contribution of $|y_1|>\delta x$ is
$$\lesssim A(\delta x)^{\psi}\int_{y_1>-\overline{\gamma}x,|y_1|>\delta x} F(x+\mathrm{d}y_1)\, h_r(y_1,f_2),$$
which is a.n. by Proposition 8.3.

Estimate of $E_k^{(3)}$. Finally we write

$$P(S_n\in x+I,\, E_k^{(3)}) \lesssim \int\limits_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\ldots,y_k)\in\Theta_k(y_1),\\ \min\{|y_1|,\ldots,|y_k|\}\ge a_n}} F(x+\mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_k)\,\frac{n^{k+1}}{a_n A(y_k)}.$$

Note that $n^{k+1}/a_n \in RV(\zeta)$ with $\zeta = k+1-\frac{1}{\alpha} > -1$, by $k = \kappa_\alpha$. Therefore, by (2.3),

$$\sum_{1\le n\le A(\delta x)} P(S_n \in x+I, E_k^{(3)}) \lesssim \int_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\dots,y_k)\in\Theta_k(y_1)}} F(x+\mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_k) \frac{b_{k+2}(\delta x \wedge \tilde{y}_k)}{A(y_k)}\,.$$

- Note that $b_{k+2}(\delta x \wedge \tilde{y}_k)/A(y_k) \le b_{k+1}(y_k)$, hence the contribution of $\{|y_1| \le \delta x\}$ is bounded by $J_k(\delta; x)$, cf. (8.14), which is a.n. by Proposition 8.3.
- To deal with $\{|y_1| > \delta x\}$ we put $\nu = 1$ if $\alpha(k+1) < 1$, while if $\alpha(k+1) = 1$ we choose $\nu \in (0,1)$ with $\alpha(k+\nu) < 1$. We also put $f_3(y) = 1/b_{k+\nu}(y) \in RV(\beta)$ with $\beta = 1-\alpha(k+\nu)$ and note that f_3 satisfies the assumptions of Proposition 8.3 with $r = k$. Then in both cases we have the bound
$$\frac{b_{k+2}(\delta x \wedge \tilde{y}_k)}{A(y_k)} \lesssim \frac{b_{k+1+\nu}(\delta x) A(y_k)^{1-\nu}}{A(y_k)} = \frac{b_{k+1+\nu}(\delta x)}{A(y_k)^\nu} = b_{k+1+\nu}(\delta x)\, b_k(y_k)\, f_3(y_k)\,,$$
and we conclude that the contribution of $\{|y_1| > \delta x\}$ is
$$\lesssim b_{r+1+\nu}(\delta x) \int_{y_1>-\overline{\gamma}x, |y_1|>\delta_0 x} F(x+\mathrm{d}y_1)\, h_k(y_1, f_3)\,,$$
which is also a.n. by Proposition 8.3 with $r = k$.

9. Soft results

In this section we prove several corollaries of our main results. Let us first describe a practical way to build counter-examples.

Remark 9.1. *Let us fix $A \in RV(\alpha)$. Let F_1 be a probability on $(0,\infty)$ which satisfies*

$$\overline{F_1}(x) \underset{x\to\infty}{\sim} \frac{2}{A(x)}\,, \qquad F_1((x-h,x]) \underset{x\to\infty}{=} O\bigg(\frac{1}{xA(x)}\bigg)\,, \quad \forall h > 0\,. \tag{9.1}$$

(For instance, fix $n_0 \in \mathbb{N}$ such that $c_1 := \sum_{n>n_0} \frac{2\,\alpha}{n\,A(n)} < 1$ and define $F_1(\{n_0\}) := 1-c_1$, $F_1(\{n\}) := \frac{2\,\alpha}{n\,A(n)}$ for $n \in \mathbb{N}$ with $n > n_0$.) Let F_2 be a probability on $(0,\infty)$ such that

$$\overline{F_2}(x) \underset{x\to\infty}{=} o\bigg(\frac{1}{A(x)}\bigg)\,. \tag{9.2}$$

If we define $F := \frac{1}{2}(F_1+F_2)$, we obtain a new probability on $(0,\infty)$ which satisfies

$$\overline{F}(x) \underset{x\to\infty}{\sim} \frac{1}{A(x)}\,, \qquad F(x+I) \ge \frac{1}{2}F_2(x+I)\,. \tag{9.3}$$

Next we state a useful result. If F satisfies (1.12), then necessarily $F(x+I) = o(\frac{1}{A(x)})$ as $x\to\infty$ (because $\overline{F}(x-h) \sim \overline{F}(x) \sim \frac{1}{A(x)}$). Interestingly, this bound can be approached as close as one wishes, in the following sense.

Lemma 9.2. *Fix two arbitrary positive positive sequences $x_n \to \infty$ and $\epsilon_n \to 0$. For any $A \in RV(\alpha)$, with $\alpha \in (0,1)$, there is a probability F on $(0,\infty)$ satisfying (1.12) such that*

$$F(\{x_n\}) \ge \frac{\epsilon_n}{A(x_n)} \qquad \text{for infinitely many } n \in \mathbb{N}\,. \tag{9.4}$$

Proof. Let us fix $A \in RV(\alpha)$. By Remark 9.1, it is enough to build a probability F_2 on $(0,\infty)$, supported on the sequence $\{x_n\}_{n\in\mathbb{N}}$, which satisfies (9.2) and

$$F_2(\{x_n\}) \geq 2\frac{\epsilon_n}{A(x_n)} \quad \text{for infinitely many } n \in \mathbb{N}\,. \tag{9.5}$$

Then, if we define $F := \frac{1}{2}(F_1 + F_2)$, the proof is completed (recall (9.3)).

By assumption $x_n \to \infty$ and $\epsilon_n \to 0$, hence we can fix a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\frac{\epsilon_{n_{k+1}}}{A(x_{n_{k+1}})} \leq \frac{1}{2}\,\frac{\epsilon_{n_k}}{A(x_{n_k})}\,, \qquad \forall k \in \mathbb{N}\,. \tag{9.6}$$

This ensures that $\sum_{k\in\mathbb{N}} \frac{\epsilon_{n_k}}{A(x_{n_k})} < \infty$ (the series converges geometrically) and we fix $k_0 \in \mathbb{N}$ so that $\sum_{k\geq k_0} \frac{\epsilon_{n_k}}{A(x_{n_k})} \leq \frac{1}{2}$. We now define F_2, supported on the set $\{x_{n_k} :\ k \geq k_0\}$, by

$$F_2(\{x_{n_k}\}) := c_2\,\frac{\epsilon_{n_k}}{A(x_{n_k})} \quad \text{for } k \geq k_0\,, \qquad \text{where} \quad c_2 := \left(\sum_{k\geq k_0} \frac{\epsilon_{n_k}}{A(x_{n_k})}\right)^{-1} \geq 2\,.$$

In this way, (9.5) is satisfied. It remains to check that (9.2) holds. Given $x \in (0,\infty)$, if we set $\bar{k} := \min\{k \geq k_0 :\ x_{n_k} > x\}$, we can write

$$F_2((x,\infty)) = \sum_{k\geq\bar{k}} c_2\,\frac{\epsilon_{n_k}}{A(x_{n_k})} \leq c_2\,\frac{\epsilon_{n_{\bar{k}}}}{A(x_{n_{\bar{k}}})} \sum_{k\geq\bar{k}} \frac{1}{2^{k-\bar{k}(x)}} \leq 2\,c_2\,\frac{\epsilon_{n_{\bar{k}}}}{A(x)}\,,$$

where we used (9.6), and the last inequality holds because $x_{n_{\bar{k}}} > x$, by definition of $\bar{k}$. Since $\epsilon_n \to 0$ by assumption, and $\bar{k} \to \infty$ as $x \to \infty$, the proof is completed. ∎

9.1. **Proof of Theorem 1.5.** Assume that condition (1.14) holds. By (1.10) we can write

$$\sup_{1\leq z\leq x} b_2(z) = \sup_{1\leq z\leq x} \frac{A(z)^2}{z} \lesssim \frac{A(x)^2}{x}\,.$$

For $0 \leq z \leq 1$ we can also write $b_2(z) \leq A(1)^2 = b_2(1) \lesssim \frac{A(x)^2}{x}$ hence by (1.13)

$$\begin{aligned} I_1^+(\delta;x) &\lesssim \frac{A(x)^2}{x}\,F([x-\delta x,x]) \underset{x\to\infty}{\sim} \frac{A(x)^2}{x}\left(\frac{1}{A((1-\delta)x)} - \frac{1}{A(x)}\right) \\ &\underset{x\to\infty}{\sim} \frac{A(x)}{x}\left(\frac{1}{(1-\delta)^\alpha} - 1\right) \underset{\delta\to 0}{=} \frac{A(x)}{x}\,O(\delta)\,. \end{aligned}$$

This shows that $I_1^+(\delta;x)$ is a.n., hence the SRT holds by Theorem 1.4.

Next we prove the second part of Theorem 1.5: we assume that condition (1.14) is not satisfied, and we build a probability F for which the SRT fails. Since $A \in RV(\frac{1}{2})$, we can write $A(x) = L(x)\sqrt{x}$ where L is slowly varying. By assumption, cf. (1.14), there is a subsequence $x_n \to \infty$ such that $\sup_{1\leq s\leq x_n} L(s) \gg L(x_n)$, hence we can find $1 \leq s_n \leq x_n$ for which $L(s_n) \gg L(x_n)$. We have necessarily $s_n = o(x_n)$, because $L(s)/L(x_n) \to 1$ uniformly for $s \in [\epsilon x_n, x_n]$, for any fixed $\epsilon > 0$, by the uniform convergence theorem of slowly varying functions [BGT89, Theorem 1.2.1]. Summarizing:

$$x_n \to \infty\,, \qquad s_n = o(x_n)\,, \qquad \epsilon_n := \frac{L(x_n)}{L(s_n)} \to 0\,. \tag{9.7}$$

By Lemma 9.2, there is a probability F on $(0,\infty)$, which satisfies (1.12), such that (9.4) holds. Since $A(x) = L(x)\sqrt{x}$, recalling (1.13), for infinitely many $n \in \mathbb{N}$ we can write

$$I_1^+(\delta; x_n + s_n) \geq \frac{A(s_n)^2}{s_n} F(\{x_n\}) \geq L(s_n)^2 \frac{\epsilon_n}{A(x_n)} = \frac{L(s_n)}{\sqrt{x_n}} = \frac{1}{\epsilon_n} \frac{A(x_n)}{x_n} \gg \frac{A(x_n + s_n)}{x_n + s_n},$$

where the last inequality holds because $\epsilon_n \to 0$ and $x_n + s_n \sim x_n$, cf. (9.7). This shows that $I_1^+(\delta; x)$ is not a.n., hence the SRT fails, by Theorem 1.4. ∎

9.2. **Proof of Proposition 1.6.** Let us fix $A \in RV(\alpha)$ with $\alpha \in (0, \frac{1}{2})$. By Remark 9.1, it is enough to build a probability F_2 on $(0,\infty)$ which satisfies (9.2) and moreover

$$F_2(x+I) = O\bigg(\frac{\zeta(x)}{xA(x)}\bigg), \qquad I_1^+(\delta; x; F_2) \text{ is not a.n.}, \tag{9.8}$$

where $I_1^+(\delta; x; F_2)$ denotes the quantity $I_1^+(\delta; x)$ in (1.13) with F replaced by F_2. Once this is done, we can set $F := \frac{1}{2}(F_1 + F_2)$ and the proof is completed (recall (9.3)).

We recall that $\zeta(\cdot)$ is non-decreasing with $\lim_{x\to\infty} \zeta(x) = \infty$. Let us define $x_n := 2^n$, and fix $n_0 \in \mathbb{N}$ large enough so that $\zeta(x_{n_0-1}) \geq 1$. Let us define

$$z_n := \frac{1}{2} \frac{x_n}{\zeta(x_{n-1})^{1+\theta}}, \qquad \text{where } \theta > 0 \text{ will be fixed later}. \tag{9.9}$$

Note that $z_n \leq \frac{1}{2}x_n$ for $n \geq n_0$ (because $\zeta(x_{n-1}) \geq 1$), hence the intervals $(x_n - z_n, x_n]$ are disjoint. We may also assume that $z_n \geq 1$ (if we decrease $\zeta(\cdot)$ we get a stronger statement, so we can replace $\zeta(x)$ by $\min\{\zeta(x), \log x\}$, which ensures that $z_n \to \infty$).

We define a probability F_2 supported on the set $\bigcup_{n \geq n_0} (x_n - z_n, x_n]$, with a constant density on each interval, as follows:

$$F_2(x_n - \mathrm{d}s) := c\, \frac{\zeta(x_{n-1})}{x_n\, A(x_n)}\, \mathbb{1}_{[0,z_n]}(s)\, \mathrm{d}s, \qquad \forall n \geq n_0, \tag{9.10}$$

where $c \in (0,\infty)$ is the normalizing constant that makes F_2 a probability. Note that

$$F_2((x_n - z_n, x_n]) = c\, \frac{\zeta(x_{n-1})}{x_n\, A(x_n)}\, z_n = \frac{c}{2\, A(x_n)\, \zeta(x_{n-1})^\theta}.$$

Since $A(x_n) = A(2x_{n-1}) \sim 2^\alpha A(x_{n-1})$ as $n \to \infty$, we may assume that $A(x_n) \geq 2^{\alpha/2} A(x_{n-1})$ for all $n \geq n_0 + 1$ (possibly enlarging n_0). Since $\zeta(\cdot)$ is non-decreasing, we obtain

$$F_2((x_n - z_n, x_n]) \leq 2^{-\alpha/2} F_2((x_{n-1} - z_{n-1}, x_{n-1}]), \qquad \forall n \geq n_0 + 1.$$

It follows that, for every $n \geq n_0$,

$$\sum_{m \geq n} F_2((x_m - z_m, x_m]) \leq F_2((x_n - z_n, x_n]) \sum_{m \geq n} (2^{-\alpha/2})^{m-n} = C\, F_2((x_n - z_n, x_n]),$$

where $C := (1 - 2^{-\alpha/2})^{-1} < \infty$. This relation, for $n = n_0$, shows that F_2 is indeed a finite measure, so we can make it a probability by suitably choosing c in (9.10).

For every $x \in (0,\infty)$ large enough, we have $x_{n-1} < x \leq x_n$ for a unique $n \geq n_0$ and

$$\overline{F_2}(x) \leq \sum_{m \geq n} F_2((x_m - z_m, x_m]) \leq C\, F_2((x_n - z_n, x_n]) \leq \frac{c\, C}{2\, A(x)\, \zeta(x_{n-1})^\theta} \underset{x\to\infty}{=} o\bigg(\frac{1}{A(x)}\bigg),$$

so that (9.2) holds. Similarly, for $x_{n-1} < x \leq x_n$ we can write

$$F_2(x+I) = F_2((x-h, x]) \leq c\, h\, \frac{\zeta(x_{n-1})}{x_n\, A(x_n)} \leq c\, h\, \frac{\zeta(x)}{x\, A(x)},$$

because $\zeta(\cdot)$ and $A(\cdot)$ are non-decreasing, hence the first relation in (9.8) holds. Finally, for $x = x_n$ and $\delta < \frac{1}{2}$, since $z_n \le \delta x_n$ for n large enough, we have by (2.3)

$$I_1^+(\delta; x_n; F_2) = c\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)} \int_{0\le z\le z_n} \frac{A(z)^2}{z \vee 1} \underset{n\to\infty}{\sim} c\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)}\,A(z_n)^2\,.$$

Recalling (9.9), we can apply Potter's bounds (2.2), since $z_n \ge 1$, to get, for any $\epsilon > 0$,

$$I_1^+(\delta; x_n; F_2) \gtrsim_\epsilon \frac{\zeta(x_{n-1})}{x_n\,A(x_n)}\,\frac{A(x_n)^2}{\zeta(x_{n-1})^{2(1+\theta)(\alpha+\epsilon)}} = \zeta(x_{n-1})^{1-2(1+\theta)(\alpha+\epsilon)}\,\frac{A(x_n)}{x_n} \gg \frac{A(x_n)}{x_n}\,,$$

where the last inequality holds provided we choose $\theta > 0$ and $\epsilon > 0$ small enough, depending only on α, so that $1-2(1+\theta)(\alpha+\epsilon) > 0$ (we recall that $\alpha < \frac{1}{2}$). This shows that $I_1^+(\delta; x_n; F_2)$ is not a.n. and completes the proof. ∎

9.3. **Proof of Proposition 1.7.** We first prove that relation (1.16) for $\gamma < 1-2\alpha$ is a necessary condition for the SRT. We can assume that $\alpha < \frac{1}{2}$, because for $\alpha = \frac{1}{2}$ relation (1.16) follows directly from (1.12). If we restrict the integral (1.13) to $z \in [0, y)$, where $y = y_x = o(x)$, for arbitrary $\delta > 0$ and large x we obtain $I_1^+(\delta; x) \gtrsim b_2(y)\,F((x-y, x])$, because $b_2(z) \in RV(2\alpha - 1)$ is asymptotically decreasing. If the SRT holds, $I_1^+(\delta; x)$ is a.n. by Theorem 1.4. It follows that $b_2(y)\,F((x-y,x]) = o(b_1(x))$, or equivalently

$$F((x-y,x]) \underset{x\to\infty}{=} o\bigg(\frac{1}{A(x)}\,\frac{b_2(x)}{b_2(y)}\bigg)\,, \qquad \text{for any } y = y_x \ge 1 \text{ with } y_x = o(x)\,.$$

Since $b_2 \in RV(2\alpha-1)$, it follows by Potter's bounds (2.2) that, for any given $\gamma < 1-2\alpha$, we have $\frac{b_2(x)}{b_2(y)} \lesssim_\gamma (\frac{y}{x})^\gamma$, hence (1.16) holds as claimed.

We now turn to the sufficiency part. Let F be a probability on $[0,\infty)$ which satisfies (1.12), for some $\alpha \in (0, \frac{1}{2}]$, such that relation (1.16) holds for some $\gamma > 1-2\alpha$. We prove that the SRT holds by showing that $\tilde{I}_1^+(\delta; x)$ defined in (1.17) is a.n., by Proposition 1.11.

By Remark 1.8, we can assume that (1.16) holds with $o(\cdot)$ replaced by $O(\cdot)$. We claim that this is equivalent to the following relation:

$$\exists x_0, C < \infty: \qquad F((x-y,x]) \le C\frac{1}{A(x)}\Big(\frac{y}{x}\Big)^\gamma\,, \qquad \forall x \ge x_0,\ \forall y \in [1, \tfrac{1}{2}x]\,. \tag{9.11}$$

It is clear that (9.11) implies (1.16) with $o(\cdot)$ replaced by $O(\cdot)$. On the other hand, if (9.11) fails, there are sequences $x_n \to \infty$, $C_n \to \infty$ and $y_n \in [1, \frac{1}{2}x_n]$ such that

$$F((x_n - y_n, x_n]) > C_n\frac{1}{A(x_n)}\Big(\frac{y_n}{x_n}\Big)^\gamma\,. \tag{9.12}$$

By extracting subsequences, we may assume that $\frac{y_n}{x_n} \to \rho \in [0, \frac{1}{2}]$. If $\rho = 0$, then $y_n = o(x_n)$ and (9.12) contradicts (1.16). If $\rho > 0$, then (9.12) contradicts (1.12), because it yields

$$\overline{F}(\tfrac{1}{2}x_n) \ge F((x_n-y_n,x_n]) > C_n\frac{1}{A(x_n)}\big(\rho^\gamma + o(1)\big) \gg \frac{1}{A(\frac{1}{2}x_n)}\,.$$

Applying (9.11) and recalling (1.17), for $\delta < \frac{1}{2}$ we get

$$\tilde{I}_1^+(\delta; x) \le \frac{C}{A(x)\,x^\gamma}\int_1^{\delta x} \frac{A(z)^2}{z^{2-\gamma}}\,\mathrm{d}z \underset{x\to\infty}{\sim} \frac{C'}{A(x)\,x^\gamma}\,\frac{A(\delta x)^2}{(\delta x)^{1-\gamma}} \underset{x\to\infty}{\sim} C'\,\delta^{2\alpha-1+\gamma}\,\frac{A(x)}{x}\,,$$

where $C' := C/(2\alpha-1-\gamma)$ and the first asymptotic equivalence holds by (2.3), because $z \mapsto A(z)^2/z^{2-\gamma}$ is regularly varying with index $2\alpha-(2-\gamma) > -1$ (since $\gamma > 1-2\alpha$). This shows that $\tilde{I}_1^+(\delta; x)$ is a.n. and completes the proof of Proposition 1.7. ∎

9.4. **Proof of Proposition 1.11.** By (1.17) we can write

$$\begin{aligned}\tilde{I}_1^+(\delta;x) &= \int_1^{\delta x} \bigg(\int_{\mathbb{R}} \mathbb{1}_{\{y\in[0,z)\}}\, F(x-\mathrm{d}y) \bigg) \frac{b_2(z)}{z}\,\mathrm{d}z \\ &= \int_{y\in[0,\delta x)} F(x-\mathrm{d}y) \bigg(\int_{1\vee y}^{\delta x} \frac{b_2(z)}{z}\,\mathrm{d}z \bigg) .\end{aligned} \tag{9.13}$$

We recall that b_k is defined in (1.10). Assume that $\alpha < \frac{1}{2}$. Then the function $z \mapsto b_2(z)/z$ is regularly varying with index $2\alpha - 2 < -1$, hence by (2.4), for $y \ge 0$ we can write

$$\int_{1\vee y}^{\delta x} \frac{b_2(z)}{z}\,\mathrm{d}z \le \int_{1\vee y}^{\infty} \frac{b_2(z)}{z}\,\mathrm{d}z \lesssim b_2(1\vee y) \lesssim b_2(y)\,,$$

because for $0 \le y < 1$ we have $b_2(y) \ge A(0)^2 > 0$, cf. §2.2. Recalling (1.13), we have shown that $\tilde{I}_1^+(\delta;x) \lesssim I_1^+(\delta;x)$ when $\alpha < \frac{1}{2}$. Then, if $I_1^+(\delta;x)$ is a.n., also $\tilde{I}_1^+(\delta;x)$ is a.n..

We now work for $\alpha \le \frac{1}{2}$. Let us restrict the outer integral in (9.13) to $y \in [0, \frac{\delta}{2}x)$, and the inner integral to $z \in [1\vee y, 2\vee 2y)$. For $y \ge 1$ we have

$$\int_{1\vee y}^{2\vee 2y} \frac{b_2(z)}{z}\,\mathrm{d}z = \int_y^{2y} \frac{b_2(z)}{z}\,\mathrm{d}z \ge \frac{b_2(y)}{2y}(2y-y) = \frac{1}{2} b_2(y)\,,$$

while for $0 \le y < 1$ we can write $\int_{1\vee y}^{2\vee 2y} \frac{b_2(z)}{z}\,\mathrm{d}z = \int_1^2 \frac{b_2(z)}{z}\,\mathrm{d}z = C \gtrsim b_2(y)$. Overall, it follows from (9.13) that $\tilde{I}_1^+(\delta;x) \gtrsim I_1^+(\frac{\delta}{2};x)$. This completes the proof. ∎

9.5. **Proof of Proposition 1.16.** We fix $\alpha \in (0, \frac{1}{3})$ and choose for simplicity $A(x) := x^\alpha$. We are going to build a probability F on $\mathbb{R}$ which satisfies (1.2) with $p = q = 1$, such that $\tilde{I}_1(\delta;x)$ is a.n. but $\tilde{I}_2(\delta,\eta;x)$ is not a.n., for any $\eta \in (0,1)$. It suffices to show that $I_1(\delta;x)$ is a.n. but $I_2(\delta,\eta;x)$ is not a.n., thanks to (A.5) and (A.6).

In analogy with Remark 9.1, we fix a probability F_1, *this time on the whole real line* $\mathbb{R}$, which satisfies (1.2) with $p = q = 3$ and such that $F_1((x-h,x]) = O(\frac{1}{|x|A(x)})$ as $x \to \pm\infty$. Then we define two probabilities F_2, F_3 on $(0,\infty)$ which both satisfy (9.2), and we set

$$F := \frac{1}{3}(F_1 + F_2 + F_3^*)\,, \tag{9.14}$$

where $F_3^*(A) := F_3(-A)$ is the reflection of F_3 (so that it is a probability on $(-\infty,0)$). Clearly, (1.2) holds for F with $p = q = 1$. It remains to build F_2 and F_3.

We are going to define F_2 so that

$$I_1(\delta;x;F_2) \text{ is a.n.}\,, \tag{9.15}$$

(where $I_1(\delta;x;F_2)$ denotes the quantity in (1.18) with F replaced by F_2). This implies that $I_1(\delta;x) = I_1(\delta;x;F)$ is a.n., because $I_1(\delta;x;F_1)$ is clearly a.n., while F_3^* is supported on $(-\infty,0)$ and gives no contribution.

We fix a parameter $p \in (1, \frac{1}{3\alpha})$. We set $E_{n,k} := [2^n + 2^k, 2^n + 2^k + \frac{2^k}{2k^p})$ for $n \in \mathbb{N}$ with $n \ge 2$ and for $1 \le k \le n-1$. Note that $E_{n,k} \subseteq [2^n + 2^k, 2^n + 2^{k+1})$ are disjoint intervals, and moreover $\bigcup_{k=1}^{n-1} E_{n,k} \subseteq [2^n, 2^{n+1})$. We define F_2 by stipulating that it has a constant density in each interval $E_{n,k}$ (for $n \ge 2$ and $1 \le k \le n-1$) and zero otherwise, given by

$$F_2(2^n + 2^k + \mathrm{d}w) := \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \, \frac{1}{(2^k)^{2\alpha}} \, \mathbb{1}_{[0,\frac{2^k}{2k^p})}(w)\,\mathrm{d}w\,, \tag{9.16}$$

where $c \in (0,\infty)$ is a suitable normalizing constant and we set for short

$$\ell(n) := \log(1+n)\,. \tag{9.17}$$

Note that

$$F_2(E_{n,k}) = \frac{c}{\ell(n)\,(2^n)^{1-\alpha}}\,\frac{1}{(2^k)^{2\alpha}}\,\frac{2^k}{2k^p} = \frac{c}{\ell(n)\,(2^n)^{1-\alpha}}\,\frac{(2^k)^{1-2\alpha}}{2k^p}\,, \tag{9.18}$$

hence

$$\begin{aligned} F_2([2^n,2^{n+1})) &= \sum_{k=1}^{n-1} F_2(E_{n,k}) \le \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \sum_{k=1}^{n-1} (2^k)^{1-2\alpha} \le \frac{c\,(2^n)^{1-2\alpha}}{\ell(n)\,(2^n)^{1-\alpha}} \\ &= \frac{c}{\ell(n)\,(2^n)^{\alpha}} \underset{n\to\infty}{=} o\bigg(\frac{1}{(2^n)^\alpha}\bigg)\,. \end{aligned}$$

Note that $F_2([2^n,2^{n+1}))$ decreases exponentially fast in n, hence for $x \in [2^n,2^{n+1})$ we have $\overline{F_2}(x) \le \overline{F_2}(2^n) \lesssim F_2([2^n,2^{n+1})) = o(1/A(x))$, which shows that (9.2) is fulfilled. It remains to check (9.15). We do this by showing that, for any $\delta < \frac14$,

$$I_1(\delta;x;F_2) \underset{x\to\infty}{=} o\big(b_1(x)\big) = o\bigg(\frac{1}{x^{1-\alpha}}\bigg)\,. \tag{9.19}$$

This is elementary but slightly technical, and it is shown below.

Finally, we define F_3, We introduce the disjoint intervals $G_k := [2^k, 2^k + \frac{2^k}{k^p})$ for $k \ge 2$. We let F_3 have a constant density on every G_k (for $k \ge 2$) and zero otherwise, given by

$$F_3(2^k + \mathrm{d}z) := \frac{c'}{\ell(k)}\,\frac{k^p}{(2^k)^{1+\alpha}}\,\mathbb{1}_{[0,\frac{2^k}{k^p})}(z)\,\mathrm{d}z\,, \tag{9.20}$$

where $c' \in (0,\infty)$ is a normalizing constant. Then

$$F_3([2^k,2^{k+1})) = F_3(G_k) = \frac{c'}{\ell(k)}\,\frac{k^p}{(2^k)^{1+\alpha}}\,\frac{2^k}{k^p} = \frac{c'}{\ell(k)}\,\frac{1}{(2^k)^\alpha} = o\bigg(\frac{1}{(2^k)^\alpha}\bigg)\,.$$

Then for $x \in [2^k,2^{k+1})$ we have $\overline{F_3}(x) \le \overline{F_3}(2^k) \lesssim \overline{F_3}([2^k,2^{k+1})) = o(1/A(x))$ as $x \to \infty$, hence (9.2) holds. Given $\eta \in (0,1)$, fix $k_0 = k_0(\eta)$ large enough so that $\frac{1}{2k^p} < \eta$ for $k \ge k_0$. Then, recalling (1.19) and (9.14), we can write

$$\begin{aligned} I_2(\delta,\eta;2^n;F) &\ge \int_{0\le y\le \delta 2^n} F(2^n+\mathrm{d}y) \int_{|z|\le \eta y} F(-y+\mathrm{d}z)\, b_3(z) \\ &\ge \frac19 \sum_{k=k_0}^{\lfloor \log_2(\delta 2^n)\rfloor} \int_{0\le w\le \frac{2^k}{2k^p}} F_2(2^n+2^k+\mathrm{d}w) \int_{0\le z\le \frac{2^k}{2k^p}} F_3\big(2^k+w+\mathrm{d}z\big)\,(1\vee z)^{3\alpha-1}\,. \end{aligned}$$

Note that $(1\vee z)^{3\alpha-1} \ge (\frac{2^k}{2k^p})^{3\alpha-1}$ (we recall that $\alpha < \frac13$) and by (9.20)

$$\forall 0 \le w \le \tfrac{2^k}{2k^p}: \qquad F_3\big(2^k+w+[0,\tfrac{2^k}{2k^p})\big) = \frac{c'}{\ell(k)}\,\frac{k^p}{(2^k)^{1+\alpha}}\,\frac{2^k}{2k^p} \gtrsim \frac{1}{\ell(k)}\,\frac{1}{(2^k)^\alpha}\,.$$

Since $\log_2(\delta 2^n) = n + \log_2\delta$, recalling (9.18), we can write for large n

$$I_2(\delta,\eta;2^n;F) \gtrsim \sum_{k=k_0}^{n/2} F_2(E_{n,k})\,\frac{1}{\ell(k)}\,\frac{1}{(2^k)^\alpha}\left(\frac{2^k}{2k^p}\right)^{3\alpha-1} \gtrsim \frac{1}{\ell(n)\,(2^n)^{1-\alpha}} \sum_{k=k_0}^{n/2} \frac{1}{\ell(k)\,k^{3\alpha p}}\,.$$

Since we have fixed $p < \frac{1}{3\alpha}$, applying (2.3) and recalling (9.17) we finally obtain

$$I_2(\delta, \eta; 2^n; F) \gtrsim \frac{n^{1-3\alpha p}}{\ell(n)^2} \frac{1}{(2^n)^{1-\alpha}} \gg \frac{1}{(2^n)^{1-\alpha}} = b_1(2^n) .$$

This shows that $I_2(\delta, \eta; x; F)$ is not a.n.. ∎

Proof of (9.19). We recall that F_2 is supported on the intervals $E_{n,k} := [2^n + 2^k, 2^n + 2^k + \frac{2^k}{2k^p})$ with $n \geq 2$ and $1 \leq k \leq n-1$. Let us set $E_n := \bigcup_{k=1}^{n-1} E_{n,k} \subseteq [2^n, 2^{n+1})$.

For large $x \geq 0$, we define $n \geq 2$ such that $2^n \leq x < 2^{n+1}$. For small $\delta \in (0, \frac{1}{4})$ and large x, the interval $(x - \delta x, x + \delta x)$ can intersect at most E_n and E_{n+1} (because the rightmost point in E_{n-1} is $2^{n-1} + 2^{n-2} + \frac{2^{n-2}}{2(n-2)^p} \sim \frac{3}{4}\, 2^n$ as $n \to \infty$). Consequently we can write

$$I_1(\delta; x; F_2) = \int_{|y| \leq \delta x} F_2(x + \mathrm{d}y)\, b_2(y) \leq \int_{z \in E_n \cup E_{n+1}} F_2(\mathrm{d}z)\, b_2(z - x) . \tag{9.21}$$

For $z \in E_{n+1}$ we have $z \in E_{n+1,k}$ for some $1 \leq k \leq n$, in which case $z \geq 2^{n+1} + 2^k$. Since $x < 2^{n+1}$, we have $|z - x| = z - x > 2^k$ which yields $b_2(z - x) \leq b_2(2^k) = (2^k)^{2\alpha - 1}$. Recalling (9.18), we see that the contribution of E_{n+1} to (9.21) is at most

$$\sum_{k=1}^{n} \frac{c}{\ell(n)\, (2^{n+1})^{1-\alpha}} \frac{(2^k)^{1-2\alpha}}{2k^p} (2^k)^{2\alpha-1} \lesssim \frac{1}{\ell(n)\, (2^{n+1})^{1-\alpha}} \sum_{k=1}^{\infty} \frac{1}{k^p} = o\bigg(\frac{1}{(2^{n+1})^{1-\alpha}}\bigg) ,$$

which is $o\big(\frac{1}{x^{1-\alpha}}\big)$, so it is negligible for (9.19).

Then we look at the contribution of E_n to (9.21). Assume first that $2^n + 2 \leq x < 2^{n+1}$. Then we can write $2^n + 2^{\tilde{k}} \leq x < 2^n + 2^{\tilde{k}+1}$ for a unique $\tilde{k} \in \{1, \ldots, n-1\}$. For $z \in E_n$ we have $z \in E_{n,k}$ for some $1 \leq k \leq n-1$. We distinguish three cases.

- If $k \leq \tilde{k} - 1$, then

$$|z - x| = x - z \gtrsim (2^n + 2^{\tilde{k}}) - (2^n + 2^{\tilde{k}-1} + \tfrac{2^{\tilde{k}-1}}{2(\tilde{k}-1)^p}) \gtrsim 2^{\tilde{k}} ,$$

  hence $b_2(z - x) \lesssim b_2(2^{\tilde{k}}) = (2^{\tilde{k}})^{2\alpha-1}$. By (9.18), the contribution to (9.21) is at most

$$\sum_{k=1}^{\tilde{k}-1} F_2(E_{n,k})\, (2^{\tilde{k}})^{2\alpha-1} \leq \frac{c}{\ell(n)\, (2^n)^{1-\alpha}} (2^{\tilde{k}})^{2\alpha-1} \sum_{k=1}^{\tilde{k}-1} (2^k)^{1-2\alpha} \lesssim \frac{c}{\ell(n)\, (2^n)^{1-\alpha}} ,$$

  which is negligible for (9.19).

- If $k \geq \tilde{k} + 2$, then $|z - x| = z - x \gtrsim (2^n + 2^k) - (2^n + 2^{\tilde{k}+1}) \geq 2^k - 2^{k-1} \gtrsim 2^k$, hence $b_2(z - x) \lesssim b_2(2^k) = (2^k)^{2\alpha-1}$ and we get

$$\sum_{k=\tilde{k}+1}^{n-1} F_2(E_{n,k})\, (2^k)^{2\alpha-1} \leq \frac{c}{\ell(n)\, (2^n)^{1-\alpha}} \sum_{k=\tilde{k}+1}^{\infty} \frac{1}{2k^p} \lesssim \frac{c}{\ell(n)\, (2^n)^{1-\alpha}} ,$$

  because $p > 1$, hence this contribution is also negligible for (9.19).

- If $k \in \{\tilde{k}, \tilde{k}+1\}$, then $|z - x| \leq 2^{\tilde{k}+2} - 2^{\tilde{k}} = 3 \cdot 2^{\tilde{k}}$. By (9.16), since the density of F_2 is larger in $E_{n,\tilde{k}}$ then in $E_{n,\tilde{k}+1}$, we see the contribution to (9.21) is at most

$$\frac{c}{\ell(n)\, (2^n)^{1-\alpha}} \frac{1}{(2^{\tilde{k}})^{2\alpha}} \int_{|w| \leq 3 \cdot 2^{\tilde{k}}} (|w| \vee 1)^{2\alpha-1}\, \mathrm{d}w \lesssim \frac{1}{\ell(n)\, (2^n)^{1-\alpha}} ,$$

  which is negligible for (9.19).

Finally, the regime $2^n \le x < 2^n + 2$ is treated similarly. For $z \in E_{n,k}$, we distinguish the cases $k \ge 2$ and with $k = 1$. If we set $\tilde{k} := 0$, the estimates in the two cases $k \ge \tilde{k} + 2$ and $k \in \{\tilde{k}, \tilde{k}+1\}$ treated above apply with no change. ∎

9.6. **Proof of Proposition 1.17.** Assume that both $\tilde{I}_1(\delta; x)$ and $\tilde{I}_1^*(\delta; x)$ are a.n., cf. (1.25) and (1.27). We claim that, for any $\eta \in (0,1)$,

$$\forall z \in \mathbb{R},\ \forall \ell \in \mathbb{N}: \qquad \int_{|y| \le \eta |z|} F(-z + \mathrm{d}y)\, \tilde{b}_{\ell+1}(z,y) \lesssim_\eta b_\ell(z)\,. \tag{9.22}$$

Since $\tilde{b}_{\ell+1}(z,y) \le A(z)^{\ell-1} \tilde{b}_2(z,y)$ and $b_\ell(z) = A(z)^{\ell-1} b_1(z)$, cf. (1.22) and (1.10), it is enough to prove (9.22) for $\ell = 1$. Let us fix $0 < \delta_0 < \eta$. For $|y| > \delta_0 |z|$ we can bound $\tilde{b}_2(z,y) \lesssim \tilde{b}_2(z, \delta_0 z) \lesssim_{\delta_0} b_2(z)$ and $\int_{\delta_0|z|<|y|\le\eta|z|} F(-z+\mathrm{d}y) \lesssim_{\delta_0,\eta} 1/A(z)$. It remains to prove (9.22) for $\ell = 1$ and with η replaced by an arbitrary $\delta_0 > 0$. The left hand side of (9.22) equals $\tilde{I}_1(\eta; z)$ for $z \ge 0$ and $\tilde{I}_1^*(\eta; -z)$ for $z \le 0$ (recall (1.23)), which are a.n. by assumption, hence we can fix $\eta = \delta_0 > 0$ small enough so that the inequality (9.22) holds for $|z| > x_0$, for a suitable $x_0 \in (0, \infty)$. Finally, for $|z| \le x_0$ both sides of (9.22) are uniformly bounded away from 0 and ∞, hence the inequality (9.22) still holds.

Observe that, for $|z| \le \eta |w|$, we can bound $b_\ell(z) \lesssim_\eta \tilde{b}_\ell(\frac{1}{\eta} z, z) \le \tilde{b}_\ell(w, z)$, so (9.22) yields

$$\forall z, w \in \mathbb{R} \text{ with } |z| \le \eta|w|,\ \forall \ell \in \mathbb{N}: \qquad \int_{|y|\le\eta|z|} F(-z+\mathrm{d}y)\, \tilde{b}_{\ell+1}(z,y) \lesssim_\eta \tilde{b}_\ell(w,z)\,. \tag{9.23}$$

If we plug this inequality into (1.24), we see that $\tilde{I}_2(\delta, \eta; x) \lesssim_\eta \tilde{I}_1(\delta; x)$ and, similarly, $\tilde{I}_k(\delta, \eta; x) \lesssim_\eta \tilde{I}_{k-1}(\delta, \eta; x)$ for any $k \ge 3$. Since $\tilde{I}_1(\delta; x)$ is a.n. by assumption, it follows that $\tilde{I}_k(\delta, \eta; x)$ is a.n. for any $k \ge 2$, hence the SRT holds by Theorem 1.12.

Finally, if relation (1.16) holds both for F and for F^*, the same arguments as in the proof of Proposition 1.7, cf. §9.3, show that both $\tilde{I}_1(\delta; x)$ and $\tilde{I}_1^*(\delta; x)$ are a.n.. ∎

9.7. **Proof of Theorem 1.18.** Since Stone's local limit theorem applies equally to Lévy processes, cf. [BD97, Proposition 2], an argument similar to the random walk case (see Subsection 4.1) shows that the SRT (1.30) holds if and only if $\widehat{T}(\delta; x)$ is a.n., where

$$\widehat{T}(\delta; x) := \int_0^{\delta A(x)} \mathrm{P}(X_t \in (x-h, x])\, \mathrm{d}t\,. \tag{9.24}$$

Let $J_s := X_s - X_{s-}$ be the jump of X at time $s > 0$. If we write

$$X_t = X_t^{(1)} + X_t^{(2)}\,, \qquad \text{where} \qquad X_t^{(1)} := \sum_{s \le t} J_s \, \mathbb{1}_{\{|J_s| \ge 1\}}\ ,$$

then $X^{(1)}$ and $X^{(2)}$ are independent Lévy processes.

- The process $X^{(1)}$ is compound Poisson: we can write $X_t^{(1)} = S_{N_{\lambda t}}$, where $N = (N_t)_{t \ge 0}$ is a standard Poisson process, $S = (S_n)_{n \in \mathbb{N}_0}$ is a random walk with step distribution $\mathrm{P}(S_1 \in \mathrm{d}x) = F(\mathrm{d}x)$ given in (1.29), and $\lambda = \Pi(\mathbb{R} \setminus (-1,1)) \in (0, \infty)$.
- The process $X_t^{(2)}$ can be written as $X_t^{(2)} = \sigma B_t + \mu t + M_t$, where M is the martingale formed from the compensated sum of jumps with modulus less than 1.

To complete the proof, we show that the SRT holds for the random walk S if and only if it holds for $X^{(1)}$ (step 1) if and only if it holds for X (step 2).

Step 1. Since $X^{(1)}_t = S_{N_{\lambda t}}$, we have

$$\mathrm{P}(X^{(1)}_t \in (x-h, x]) = \sum_{n\in\mathbb{N}_0} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\mathrm{P}(S_n \in (x-h, x])\,.$$

Note that $\int_0^z e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,\mathrm{d}t = \frac{1}{\lambda}\mathrm{P}(Z_{n,\lambda} \le z)$, where $Z_{n,\lambda}$ denotes a random variable with a Gamma(n, λ) distribution. Then the quantity $\widehat{T}(\delta; x) = \widehat{T}_{X^{(1)}}(\delta; x)$ for $X^{(1)}$ equals

$$\widehat{T}_{X^{(1)}}(\delta; x) = \frac{1}{\lambda}\sum_{n\in\mathbb{N}_0} \mathrm{P}(Z_{n,\lambda} \le \delta A(x))\,\mathrm{P}(S_n \in (x-h, x])\,. \tag{9.25}$$

For $n \le \lambda\delta A(x)$ we have $\mathrm{P}(Z_{n,\lambda} \le \delta A(x)) \ge \mathrm{P}(Z_{n,\lambda} \le \frac{n}{\lambda}) \to \frac{1}{2}$ as $n \to \infty$, by the central limit theorem (recall that $Z_{n,\lambda} \sim \frac{1}{\lambda}(Y_1 + \ldots + Y_n)$, where Y_i are i.i.d. Exp(1) random variables). Denoting by $T_S(\delta; x)$ the quantity in (4.1) for the random walk S, and restricting the sum in (9.25) to $n \le \lambda\delta A(x)$, we get

$$\widehat{T}_{X^{(1)}}(\delta; x) \gtrsim T_S(\lambda\delta; x)\,.$$

To prove a reverse inequality, we observe that for all $z \le \frac{1}{2}\frac{n}{\lambda}$ we can write, for $\epsilon > 0$,

$$\mathrm{P}(Z_{n,\lambda} \le z) \le e^{\epsilon\lambda z}\,\mathrm{E}[e^{-\epsilon\lambda Z_{n,\lambda}}] = \frac{e^{\epsilon\lambda z}}{(1+\epsilon)^n} \le \left(\frac{e^{\frac{1}{2}\epsilon}}{1+\epsilon}\right)^n \le e^{-cn}\,,$$

where the last inequality holds with $c = c_\epsilon > 0$, provided we fix $\epsilon > 0$ small. Then, splitting the sum in (9.25) according to $n \le 2\lambda\delta A(x)$ and $n > 2\lambda\delta A(x)$, we get

$$\widehat{T}_{X^{(1)}}(\delta; x) \le \frac{1}{\lambda}\sum_{n\le 2\lambda\delta A(x)} \mathrm{P}(S_n \in (x-h, x]) + \sum_{n > 2\lambda\delta A(x)} e^{-cn} \lesssim T_S(2\lambda\delta; x) + e^{-2c\lambda\delta A(x)}\,.$$

These inequalities show that $\widehat{T}_{X^{(1)}}(\delta; x)$ is a.n. if and only if $T_S(\delta; x)$ is a.n., that is, the SRT holds for $X^{(1)}$ if and only if it holds for S.

Step 2. Assume that $X = X^{(1)} + X^{(2)}$ and the SRT holds for $X^{(1)}$, that is $\widehat{T}_{X^{(1)}}(\delta; x)$ is a.n.. Then, given $\varepsilon > 0$, there are δ_0, x_0 such that, for all $0 < \delta < \delta_0$,

$$\forall y > x_0 : \qquad \widehat{T}_{X^{(1)}}(\delta; y) = \int_0^{\delta A(y)} \mathrm{P}(X^{(1)}_t \in (y-h, y])\,\mathrm{d}t \le \varepsilon\,\frac{A(y)}{y}. \tag{9.26}$$

Let us now write

$$\begin{aligned}\int_0^{\delta A(x)} &\mathrm{P}(X_t \in (x-h, x],\, X^{(2)}_t \le x/2)\,\mathrm{d}t \\ &= \int_{-\infty}^{x/2} \mathrm{P}(X^{(2)}_t \in \mathrm{d}z) \int_0^{\delta A(x)} \mathrm{P}(X^{(1)}_t \in (x-z-h, x-z])\,\mathrm{d}t\,.\end{aligned}$$

For $z \le x/2$ we can write $A(x) \le cA(x/2)$, for any $c > 2^\alpha$ and for large x. Then the inner integral is bounded by $\widehat{T}_{X^{(1)}}(c\delta; x-z) \le \varepsilon\frac{A(x-z)}{x-z} \lesssim \varepsilon\frac{A(x)}{x}$, by (9.26). This shows that

$$\int_0^{\delta A(x)} \mathrm{P}(X_t \in (x-h, x],\, X^{(2)}_t \le x/2)\,\mathrm{d}t \lesssim \varepsilon\,\frac{A(x)}{x}\,. \tag{9.27}$$

Note that $X^{(2)}$ has finite exponential moments, because its Lévy measure $\Pi(\,\cdot \cap (-1,1))$ is compactly supported, hence $\mathrm{E}[e^{|X^{(2)}_t|}] \le \mathrm{E}[e^{X^{(2)}_t}] + \mathrm{E}[e^{-X^{(2)}_t}] \le e^{ct}$ for a suitable $c \in (0, \infty)$.

This yields the exponential bound $\mathrm{P}(|X_t^{(2)}| > a) \le e^{-a}\, e^{ct}$, for all $a \ge 0$, hence

$$\begin{aligned}\int_0^{\delta A(x)} & \mathrm{P}(X_t \in (x-h, x],\, X_t^{(2)} > x/2)\, \mathrm{d}t \\ & \le \int_0^{\delta A(x)} \mathrm{P}(X_t^{(2)} > x/2)\, \mathrm{d}t \lesssim e^{-x/2}\, e^{c\delta A(x)} \underset{x\to\infty}{=} o\left(\frac{A(x)}{x}\right).\end{aligned}$$

Together with (9.27), this shows that $\widehat{T}_X(\delta; x)$ is a.n., that is the SRT holds for X.

If the SRT holds for X, to show that it holds for $X^{(1)}$ we can repeat the previous arguments switching X and $X^{(1)}$ (no special feature of $X^{(1)}$ was used in this step). ∎

Appendix A. Some technical results

The next Lemmas show some relations between the quantities I_k and $\tilde{I}_k$ defined in (1.18), (1.19) and in (1.23), (1.24), respectively.

Lemma A.1. *Let F be a probability on $\mathbb{R}$ satisfying* (1.12) *with $\alpha \in (0,1)$ and with $p, q > 0$. Fix $k \in \mathbb{N}$ with $k \ge 2$ and $\eta \in (0,1)$. For $x \ge 0$, the following relations hold:*

$$\text{if } k < \tfrac{1}{\alpha} - 1: \qquad I_{k-1}(\delta,\eta;x) \lesssim_\eta I_k(\delta,\eta;x)\,, \tag{A.1}$$

$$\text{if } k > \tfrac{1}{\alpha} - 1: \qquad I_k(\delta,\eta;x) \lesssim_\eta I_{k-1}(\delta,\eta;x)\,. \tag{A.2}$$

Corollary A.2. *If $\frac{1}{\alpha} \notin \mathbb{N}$, $I_{\kappa_\alpha}(\delta,\eta;x)$ is a.n. implies $I_k(\delta,\eta;x)$ is a.n. for all $k \in \mathbb{N}$.*

It follows by (A.1) and (A.2) that

$$\max\big\{I_{k-1}(\delta,\eta;x),\, I_k(\delta,\eta;x)\big\} \approx_\eta \begin{cases} I_k(\delta,\eta;x) & \text{if } k < \frac{1}{\alpha} - 1 \\ I_{k-1}(\delta,\eta;x) & \text{if } k > \frac{1}{\alpha} - 1 \end{cases}. \tag{A.3}$$

Lemma A.3. *Let F be a probability on $\mathbb{R}$ satisfying* (1.12) *with $\alpha \in (0,1)$ and with $p, q > 0$. For $x \ge 0$, the following relations hold:*

$$I_1(\tfrac{\delta}{2}; x) \lesssim \tilde{I}_1(\delta; x)\,, \tag{A.4}$$

$$\text{if } \alpha < \tfrac{1}{2}: \qquad \tilde{I}_1(\delta;x) \lesssim I_1(\delta;x)\,. \tag{A.5}$$

Fix now $k \in \mathbb{N}$ with $k \ge 2$ and $\eta \in (0,1)$. For $x \ge 0$, the following relations hold:

$$\max\big\{I_{k-1}(\delta,\eta;x),\, I_k(\delta,\eta;x)\big\} \lesssim_\eta \tilde{I}_k(\delta,\eta;x)\,, \tag{A.6}$$

$$\text{if } k \neq \tfrac{1}{\alpha} - 1: \qquad \tilde{I}_k(\delta,\eta;x) \lesssim_\eta \max\big\{I_{k-1}(\delta,\eta;x),\, I_k(\delta,\eta;x)\big\}\,. \tag{A.7}$$

It follows that

$$\text{if } k \le \tfrac{1}{\alpha} - 1: \qquad \tilde{I}_{k-1}(\delta,\eta;x) \lesssim_\eta \tilde{I}_k(\delta,\eta;x)\,, \tag{A.8}$$

$$\text{if } k > \tfrac{1}{\alpha} - 1: \qquad \tilde{I}_k(\tfrac{\delta}{2},\eta;x) \lesssim_\eta \tilde{I}_{k-1}(\delta,\eta;x)\,, \tag{A.9}$$

where we stress that $k = \frac{1}{\alpha} - 1$ is included in (A.8).

Corollary A.4. *$\tilde{I}_{\kappa_\alpha}(\delta,\eta;x)$ is a.n. implies $\tilde{I}_k(\delta,\eta;x)$ is a.n. for all $k \in \mathbb{N}$.*

Corollary A.5. *If $\frac{1}{\alpha} \notin \mathbb{N}$, $\tilde{I}_{\kappa_\alpha}(\delta,\eta;x)$ is a.n. if and only if $I_{\kappa_\alpha}(\delta,\eta;x)$ is a.n.. If $\frac{1}{\alpha} \in \mathbb{N}$, $\tilde{I}_{\kappa_\alpha}(\delta,\eta;x)$ is a.n. implies $I_{\kappa_\alpha}(\delta,\eta;x)$ is a.n. (but not the other way round).*

A.1. **Proof of Lemma A.1.** For $k < \frac{1}{\alpha} - 1$, the function $b_{k+1}(y)$, cf. (1.10), is regularly varying with index $(k+1)\alpha - 1 < 0$, hence it is asymptotically decreasing: bounding $b_{k+1}(y_k) \gtrsim b_{k+1}(\eta y_{k-1})$ for $|y_k| \le \eta |y_{k-1}|$ gives

$$\int_{|y_k| \le \eta |y_{k-1}|} F(-y_{k-1} + \mathrm{d}y_k)\, b_{k+1}(y_k) \gtrsim F(-(1-\eta)|y_{k-1}|)\, b_{k+1}(\eta y_{k-1}) \gtrsim_\eta b_k(y_{k-1})\,,$$

which plugged into (1.19) shows that $I_k(\delta, \eta; x) \gtrsim_\eta I_{k-1}(\delta, \eta; x)$, proving (A.1). When $\alpha > \frac{1}{k+1}$, the function $b_{k+1}(y)$ is asymptotically increasing, and we use a similar argument to get $I_k(\delta, \eta; x) \lesssim_\eta I_{k-1}(\delta, \eta; x)$, that is (A.2). ∎

A.2. **Proof of Corollary A.2.** If $\frac{1}{\alpha} \notin \mathbb{N}$, then $\kappa_\alpha < \frac{1}{\alpha} - 1$, hence if I_{κ_α} is a.n. also $I_{\kappa_\alpha - 1}, I_{\kappa_\alpha - 2}, \ldots$ are a.n., by (A.1), while $I_{\kappa_\alpha + 1}, I_{\kappa_\alpha + 2}, \ldots$ are a.n., by (A.2). ∎

A.3. **Proof of Lemma A.3.** By (5.9), for $|y_k| \le \eta |y_{k-1}|$ with $\eta \in (0,1)$ we can write

$$\tilde{b}_{k+1}(y_{k-1}, y_k) \ge \sum_{m=A(|y_k|)}^{A(|y_k|/\eta)} \frac{m^k}{a_m} \gtrsim \frac{A(|y_k|)^k}{(|y_k|/\eta) \vee 1} \big(1 + A(|y_k|/\eta) - A(|y_k|)\big) \gtrsim_\eta b_{k+1}(y_k)\,,$$

The same arguments show that, for $|y| \le \frac{\delta}{2} x$, we have $\tilde{b}_2(\delta x, y) \ge \tilde{b}_2(2y, y) \gtrsim b_2(y)$. Plugging these bounds into (1.23) and (1.24) proves (A.4) and also $\tilde{I}_k(\delta, \eta; x) \gtrsim_\eta I_k(\delta, \eta; x)$, which is half of (A.6). For the other half, note that for $|y_k| \le \eta |y_{k-1}|$, always by (5.9),

$$\tilde{b}_{k+1}(y_{k-1}, y_k) \gtrsim \sum_{m=A(\eta|y_{k-1}|)}^{A(|y_{k-1}|)} \frac{m^k}{a_m} \gtrsim \frac{A(\eta |y_{k-1}|)^k}{|y_{k-1}| \vee 1} \big(1 + A(|y_{k-1}|) - A(\eta|y_{k-1}|)\big) \gtrsim_\eta b_{k+1}(y_{k-1})\,,$$

hence

$$\int_{|y_k| \le \eta |y_{k-1}|} F(-y_{k-1} + \mathrm{d}y_k)\, \tilde{b}_{k+1}(y_{k-1}, y_k) \gtrsim_\eta b_{k+1}(y_{k-1}) F(-(1-\eta)|y_{k-1}|) \gtrsim_\eta b_k(y_{k-1})\,. \tag{A.10}$$

From (1.23) we get $\tilde{I}_k(\delta, \eta; x) \gtrsim_\eta I_{k-1}(\delta, \eta; x)$, which completes the proof of (A.6).

Next we prove (A.5) and (A.7). Recalling (A.3), we distinguish two cases.

- If $k < \frac{1}{\alpha} - 1$, the sequence m^k / a_m is regularly varying with index $k - \frac{1}{\alpha} < -1$. By (2.4), we can write

$$\tilde{b}_{k+1}(y_{k-1}, y_k) \le \sum_{m=A(|y_k|)}^{\infty} \frac{m^k}{a_m} \lesssim \frac{A(|y_k|)^{k+1}}{|y_k| \vee 1} = b_{k+1}(y_k)\,,$$

  which yields $\tilde{I}_k(\delta, \eta; x) \lesssim_\eta I_k(\delta, \eta; x)$. For $k = 1$, we have proved (A.5), since $k < \frac{1}{\alpha} - 1$ means precisely $\alpha < \frac{1}{2}$, while for $k \ge 2$ we have proved half of (A.7).

- If $k > \frac{1}{\alpha} - 1$, with $k \ge 2$, the sequence m^k / a_m is regularly varying with index $k - \frac{1}{\alpha} > -1$ and by (2.3) we get

$$\tilde{b}_{k+1}(y_{k-1}, y_k) \le \sum_{m=1}^{A(|y_{k-1}|)} \frac{m^k}{a_m} \lesssim \frac{A(|y_{k-1}|)^{k+1}}{|y_{k-1}| \vee 1} = b_{k+1}(y_{k-1})\,,$$

  and in analogy with (A.10) we get $\tilde{I}_k(\delta, \eta; x) \lesssim_\eta I_{k-1}(\delta, \eta; x)$. Relation (A.7) is proved.

We finally prove (A.8) and (A.9). Fix $k \le \frac{1}{\alpha} - 1$ and assume first that $k \ge 3$. By (A.7) and (A.3) (with $k-1$ in place of k; note that $k - 1 < \frac{1}{\alpha} - 1$), and then by (A.6), we have

$$\tilde{I}_{k-1} \lesssim_\eta \max\{I_{k-2}, I_{k-1}\} \approx_\eta I_{k-1} \le \max\{I_{k-1}, I_k\} \lesssim_\eta \tilde{I}_k\,.$$

If $k = 2$, the assumption $k \le \frac{1}{\alpha} - 1$ means $\alpha \le \frac{1}{3}$, hence we can apply (A.5) followed by (A.6) to get $\tilde{I}_1 \lesssim I_1 \le \max\{I_1, I_2\} \lesssim_\eta \tilde{I}_2$. This completes the proof of (A.8).

Fix now $k > \frac{1}{\alpha} - 1$ with $k \ge 2$. By (A.7) and (A.3), we can write

$$\tilde{I}_k(\tfrac{\delta}{2}) \lesssim_\eta \max\{I_{k-1}(\tfrac{\delta}{2}), I_k(\tfrac{\delta}{2})\} \lesssim_\eta I_{k-1}(\tfrac{\delta}{2})\,.$$

If $k \ge 3$, we apply (A.6) with $k-1$ in place of k, to get

$$I_{k-1}(\tfrac{\delta}{2}) \le \max\{I_{k-2}(\tfrac{\delta}{2}), I_{k-1}(\tfrac{\delta}{2})\} \lesssim_\eta \tilde{I}_{k-1}(\tfrac{\delta}{2}) \le \tilde{I}_{k-1}(\delta)\,.$$

This yields $\tilde{I}_k(\frac{\delta}{2}) \lesssim_\eta \tilde{I}_{k-1}(\delta)$, which is precisely (A.9). If $k = 2$, we apply (A.4) to see that $I_{k-1}(\frac{\delta}{2}) = I_1(\frac{\delta}{2}) \le \tilde{I}_1(\delta)$. This completes the proof of (A.9). ∎

A.4. **Proof of Corollary A.4.** Assume that $\tilde{I}_{\kappa_\alpha}$ is a.n.. Since $\kappa_\alpha \le \frac{1}{\alpha} - 1$, we can apply (A.8) iteratively to see that $\tilde{I}_{\kappa_\alpha - 1}$, $\tilde{I}_{\kappa_\alpha - 2}$, ... are a.n.. Similarly, since $\kappa_\alpha + 1 > \frac{1}{\alpha} - 1$, relation (A.9) shows that $\tilde{I}_{\kappa_\alpha + 1}$, $\tilde{I}_{\kappa_\alpha + 2}$, ... are a.n.. ∎

A.5. **Proof of Corollary A.5.** With no restriction on α, if $\tilde{I}_{\kappa_\alpha}$ is a.n. then I_{κ_α} is also a.n., by (A.4) and (A.6). The reverse implication holds when $\frac{1}{\alpha} \notin \mathbb{N}$, because we can apply (A.5) if $\kappa_\alpha = 1$ (in which case $\alpha < \frac{1}{2}$, since $\frac{1}{\alpha} \notin \mathbb{N}$) or (A.7) if $\kappa_\alpha > 1$. ∎

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