ON CONSTRAINED ANNEALED BOUNDS FOR LINEAR CHAIN PINNING MODELS

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ABSTRACT. The free energy of quenched disordered systems is bounded above by the free energy of the corresponding annealed system. This bound may be improved by applying the annealing procedure, which is just Jensen inequality, after having modified the Hamiltonian in a way that the quenched expressions are left unchanged. This procedure is often viewed as a partial annealing or as a constrained annealing, in the sense that the term that is added may be interpreted as a Lagrange multiplier on the disorder.

We point out in this note that for a family of models, some of which have attracted much attention, the multipliers of the form of empirical averages of local functions cannot improve on the basic annealed bound from the viewpoint of characterizing the phase diagram. This class of multipliers is the one that is suitable for computations and it is often believed that in this class one can approximate arbitrarily well the quenched free energy.

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1. The framework and the main result

1.1. **The general set–up.** A number of disordered models of linear chains undergoing localization or pinning effects can be put into the following general framework. Let $S := \{S_n\}_{n=0,1,\dots}$ be a process with S_n taking values in $\mathbb{Z}^d$, $d \in \mathbb{N} := \{1, 2, \dots\}$ and law $\mathbf{P}$.

The disorder in the system is given by a sequence $\omega := \{\omega_n\}_n$ of IID random variables taking values in a finite set Γ with law $\mathbb{P}$, acting on the path of S via an Hamiltonian that, for a system of size N, is a function $H_{N,\omega}$ of the trajectory S, but depending only on $S_0, S_1, \dots, S_N$. One is interested in the properties of the probability measures $\mathbf{P}_{N,\omega}$ defined by giving the density with respect to $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_{N,\omega}} \exp\left(H_{N,\omega}(S)\right), \tag{1.1}$$

where $Z_{N,\omega} := \mathbf{E}\left[\exp\left(H_{N,\omega}(S)\right)\right]$ is the normalization constant. Our attention focuses on the asymptotic behavior of $\log Z_{N,\omega}$.

In the sequel we will assume:

Basic Hypothesis. There exists a sequence $\{D_n\}_n$ of subsets of $\mathbb{Z}^d$ such that $\mathbf{P}(S_n \in D_n \text{ for } n = 1, 2, \dots, N) \overset{N\to\infty}{\asymp} 1$, namely

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \dots, N\right) = 0, \tag{1.2}$$

and $H_{N,\omega}(S) = 0$ if $S_n \in D_n$ for $n = 1, 2, \dots, N$.

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One sees directly that this hypothesis implies

$$\liminf_{N\to\infty} \frac{1}{N} \log Z_{N,\omega} \geq \lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \dots, N\right) = 0, \tag{1.3}$$

$\mathbb{P}(\mathrm{d}\omega)$–a.s.. We will assume that $\{(1/N) \log Z_{N,\omega}\}_N$ is a sequence of integrable random variables that converges in the $L^1(\mathbb{P}(\mathrm{d}\omega))$ sense and $P(\mathrm{d}\omega)$–almost surely to a constant, the free energy, that we will call f. These assumptions are verified in the large majority of the interesting situations, for example whenever super/sub–additivity tools are applicable.

Of course (1.3) says that $f \geq 0$ and one is lead to the natural question of whether $f = 0$ or $f > 0$. In the instances that we are going to consider the free energy may be zero or positive according to some parameters from which the $H_{N,\omega}$ depends: $f = 0$ and $f > 0$ are associated to sharply different behaviors of the system.

In order to establish upper bounds on f one may apply directly Jensen inequality (*annealed bound*) obtaining

$$f \leq \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[Z_{N,\omega}\right] =: \widetilde{f}, \tag{1.4}$$

and, in our context, if $\widetilde{f} = 0$ then $f = 0$. The annealed bound may be improved by adding to $H_{N,\omega}(S)$ an integrable function $A_N : \Gamma^{\mathbb{N}} \to \mathbb{R}$ such that $\mathbb{E}\left[A_N(\omega)\right] = 0$: while the left–hand side is unchanged, $\widetilde{f}$ may depend on the choice of $\{A_N\}_N$. We stress that not only f is left unchanged by $H_{N,\omega}(S) \to H_{N,\omega}(S) + A_N(\omega)$, but $\mathbf{P}_{N,\omega}$ itself is left unchanged (for every N). Notice that the choice $A_N(\omega) = -\log Z_{N,\omega} + \mathbb{E}\left[\log Z_{N,\omega}\right]$ yields the equality in (1.4).

In the sequel when we refer to $\widetilde{f}$ we mean that $Z_{N,\omega}$ is defined with respect to $H_{N,\omega}$ satisfying the Basic Hypothesis (no A_N term added).

1.2. **The result.** What we prove in this note is that

Proposition 1.1. *If $\widetilde{f} > 0$ then for every local function $F : \Gamma^{\mathbb{N}} \longrightarrow \mathbb{R}$ such that $\mathbb{E}\left[F(\omega)\right] = 0$ one has*

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S) + \sum_{n=0}^{N} F(\theta_n\omega)\right)\right] > 0, \tag{1.5}$$

where $(\theta_n\omega)_m = \omega_{n+m}$.

We can sum up this result by saying that when $f = 0$ but $\widetilde{f} > 0$ it is of no use modifying the Hamiltonian by adding the empirical average of a (centered) local function.

On a mathematical level it is clear that we are playing with an exchange of limits and that it is not obvious that the free energy, recall the optimal choice of A_N above, may be approximated via empirical averages of a local function of the disorder. But we remark that in the physical literature the approach of approximating the free energy via what can be viewed as a constrained annealed computation, the term $\sum_{n=0}^{N} F(\theta_n\omega)$ being interpreted as a Lagrange multiplier, is often considered as an effective way of approximating the quenched free energy. Here we mention in particular [16] and [13] in which this approach is taken up in a systematic way: the aim is to approach the quenched free energy by constrained annealing via local functions F that are more and more complex, the most natural example being linear combinations of correlations of higher and higher order.

The proof of Proposition 1.1 is based on the simple observation that whenever A_N is centered

$$\frac{1}{N}\log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S)+A_N(\omega)\right)\right] \geq$$
$$\frac{1}{N}\log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right]+\frac{1}{N}\log \mathbf{P}\left(S_n\in D_n \text{ for } n=1,2,\dots,N\right) =: Q_N+P_N. \quad (1.6)$$

By hypothesis $P_N = o(1)$ so one has to consider the asymptotic behavior of Q_N. If $\liminf_N Q_N > 0$ there is nothing to prove. So let us assume that $\liminf_N Q_N = 0$: in this case the inferior limit of the left–hand side of (1.6) may be zero and we want to exclude this possibility when $\widetilde{f} > 0$ and $A_N(\omega) = \sum_{n=0}^{N} F(\theta_n\omega)$, F local and centered (of course in this case $\lim_N Q_N$ does exist). And in Theorem 2.1 below in fact we show that if $\log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right] = o(N)$, then there exists a local function G such that $F(\omega) = G(\theta_1\omega) - G(\omega)$ so that $\{\sum_{n=0}^{N} F(\theta_n\omega)\}_N$ is just a boundary term and the corresponding constrained annealing is just the standard annealing.

Notice that having chosen Γ finite frees us from integrability conditions.

Remark 1.2. We stress that our Basic Hypothesis is more general than it may look at first. As already observed, one has the freedom of adding to the Hamiltonian $H_{N,\omega}(S)$ any term that does not depend on S (but possibly does depend on ω and N) without changing the model $\mathbf{P}_{N,\omega}$. It may therefore happen that the *natural* formulation of the Hamiltonian does not satisfy our Basic Hypothesis, but it does after a suitable additive correction. This happens for example in §1.5 below: the additive correction in that case is linear in ω and it corresponds to what in [17] is called *first order* Morita approximation. In these terms, Proposition 1.1 is saying that *higher order* Morita approximations cannot improve the bound on the critical curve found with the first order computation.

Let us now look at applications of Proposition 1.1.

1.3. **Random rewards or penalties at the origin.** Let S, $S_0 = 0 \in \mathbb{Z}^d$, be a random walk with centered IID non degenerate increments $\{X_n\}_n$, $(X_n)_j \in \{-1, 0, 1\}$ for $j = 1, 2, \dots, d$, and

$$H_{N,\omega} = \beta \sum_{n=1}^{N} (1+\varepsilon\omega_n)\, \mathbf{1}_{\{S_n=0\}}. \quad (1.7)$$

for $\beta \geq 0$ and $\varepsilon \geq 0$. The random variable ω_1 is chosen such that $\mathbb{E}[\exp(\lambda\omega_1)] < \infty$ for every $\lambda \in \mathbb{R}$, and centered. We write $f(\beta, \varepsilon)$ for f: by super–additive arguments f exists and it is self–averaging (this observation is valid for all the models we consider and will not be repeated). We note that for $\varepsilon = 0$ the model can be solved, see e.g. [10], and in particular $f(\beta, 0) = 0$ if and only if $\beta \leq \beta_c(d) := -\log(1 - \mathbf{P}(S \text{ never comes back to } 0))$. Adding the disorder makes this model much more complex: the annealed bound yields $f(\beta, \varepsilon) = 0$ if $\beta \leq \beta_c(d) - \log \mathbb{E}\left[\exp(\varepsilon\omega_1)\right] =: \widetilde{\beta}_c$. It is an open question whether $\widetilde{\beta}_c$ coincides with the quenched critical value or not, that is whether $f(\beta, \varepsilon) = 0$ implies $\beta \leq \widetilde{\beta}_c$ or not. For references about this issue we refer to [2], see however also the next paragraph: the model we are considering can in fact be exactly mapped to the wetting problem ([2], [10]). Proposition 1.1 applies to this context with $D_n = \{0\}^{\complement}$ for every n [8, Ch. 3] and says that one cannot answer this question via constrained annealed bounds.

1.4. **Wetting models in $1+d$ dimensions.** Let S and ω as in the previous example and

$$H_{N,\omega} = \begin{cases} \beta \sum_{n=1}^{N} (1+\varepsilon\omega_n)\, \mathbf{1}_{\{(S_n)_d=0\}} & \text{it } (S_n)_d \geq 0 \text{ for } n=1,2,\dots,N \\ -\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

with $\beta \geq 0$ and $\varepsilon \geq 0$. If one takes the directed walk viewpoint, that is if one considers the walk $\{(n, S_n)\}_n$, then this is a model of a walk constrained above the (hyper–)plane $x_d = 0$ and rewarded β, on the average, when touching this plane. If $d = 1$ then this is an effective model for a (1+1)–dimensional interface above a wall which mostly attracts it. As a matter of fact in this case there is no loss of generality in considering $d = 1$, since in the directions parallel to the wall the model is just the original walk. Once again if $\varepsilon = 0$ the model can be solved in detail, see e.g. [10]. Computing the critical β and deciding whether the annealed bound is sharp, at least for small ε, is an unresolved and disputed question in the physical literature, see e.g. [9], [7] and [21]. Proposition 1.1 applies with the choice $D_n = \mathbb{Z}^{d-1} \times \mathbb{N}$.

1.5. **Copolymer and adsorption models.** Choose S as above and take the directed walk viewpoint. Imagine that above the axis ($x_d > 0$) is filled of a solvent A, while below ($x_d < 0$) there is a solvent B. At $x_d = 0$ there is the interface. We choose $\omega = \{A, B\}$ and for example

$$H^{AB}_{N,\omega}(S) \,=\, \sum_{n=1}^{N} \big(a\mathbf{1}_{\{\mathrm{sign}(S_n)=+1,\, \omega_n=A\}} + b\mathbf{1}_{\{\mathrm{sign}(S_n)=-1,\, \omega_n=B\}} + c\mathbf{1}_{\{S_n=0\}} \big) \tag{1.9}$$

with a, b and c real parameters and $\mathrm{sign}(S_n) = \mathrm{sign}(S_{n-1})$ if $S_n = 0$ (this is just a trick to reward the bonds rather than the sites). In order to apply Proposition 1.1 one has to subtract a disorder dependent term, cf. Remark 1.2: if $a \geq b$ we change the Hamiltonian

$$H_{N,\omega}(S) \,:=\, H^{AB}_{N,\omega}(S) - \sum_{n=1}^{N} a\mathbf{1}_{\{\omega_n=A\}}. \tag{1.10}$$

without changing the measure $\mathbf{P}_{N,\omega}$ while the free energy has the trivial shift from f to $f - a\mathbb{P}(\omega_1 = A)$. One can therefore choose $D_n = \mathbb{Z}^{d-1} \times \mathbb{N}$ and Proposition 1.1 applies. This model has been considered for example in [17].

Note that if $c = 0$ the model can be cast in a form that has been considered by a variety of authors (see e.g. [12], [19], [1], [4], [20], [22], [15], [6]):

$$H_{N,\omega}(S) \,=\, \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}(S_n), \tag{1.11}$$

with ω taking values in $\mathbb{R}$. Once again the Hamiltonian has to be corrected by subtracting the term $\lambda \sum_n (\omega_n + h)$ in order to apply Proposition 1.1. One readily sees that (1.10) and (1.11) are the same model when in the second case ω takes only the values ± 1, $A = +1$ and $B = -1$, and $h = (a-b)/(a+b)$, $\lambda = (a+b)/4$.

Proposition 1.1 acquires some interest in this context given the fact that the physical literature is rather split on the precise value of the critical curve and on whether the annealed bound is sharp or not, see [6] for details on this issue. In [5] we present numerical evidence on the fact that the annealed curve does not coincide with the quenched one, and in view of Proposition 1.1 this would mean that constrained annealing via local functions cannot capture the phase diagram of the quenched system.

1.6. **Further models and observations.** In spite of substantial numerical evidence that in several instances $f = 0$ but $\widetilde{f} > 0$, we are unaware of an *interesting* model for which this situation is rigorously known to happen. Consider however the case $\mathbb{P}(\omega_n = \pm 1) = 1/2$ and

$$H_{N,\omega}(S) = \beta \sum_{n=1}^{N} (1 + \varepsilon\omega_n)\, \mathbf{1}_{\{S_n = n\}}, \tag{1.12}$$

with β and ε real numbers and S the simple random walk on $\mathbb{Z}$. We observe that Proposition 1.1 applies to this case with $D_n = \{n\}^{\complement}$ and that the model is solvable in detail. In particular $f(\beta, \varepsilon) = (\beta - \log 2) \vee 0$, regardless of the value of ε. The annealed computation instead yields $\widetilde{f}(\beta, \varepsilon) = (\beta + \log\cosh(\varepsilon) - \log 2) \vee 0$. Notice in particular that the critical values of β, respectively $\log 2$ and $\log 2 - \log\cosh(\varepsilon)$, differ as long as there is disorder in the system ($\varepsilon \neq 0$). It is interesting to see in this toy model how A_N has to be chosen *very non local* in order to improve on the annealed bound.

Remark 1.3. We point out that we restricted our examples only to cases in which S is a simple random walk, but in principle our approach goes through for much more general models, like walks with correlated increments or self–interacting walks, see [18] for an example. And of course S_n takes values in $\mathbb{Z}^d$ only for ease of exposition and can be easily generalized. It is however unclear whether our argument applies to the disordered wetting problem in $d+1$ dimensions, $d > 1$. In this case S is a random interface, the Hamiltonian is like in (1.8), but $n \in \{0, 1, 2, \dots\}^d$, $S_n \in \mathbb{Z}$ or $\mathbb{R}$. We set for example $S_n = 0$ when one of the coordinates of n is zero. The missing ingredient is an analog of Theorem 2.1 in higher dimensions.

2. On cocycles with null free energy

Let $\{\boldsymbol{\omega}_n\}_{n\in\mathbb{N}}$ be an IID sequence of random variables under the probability measure $\mathbb{P}$, taking values in a finite space Γ (we have switched the notation $\omega \to \boldsymbol{\omega}$ for clarity). The law of $\boldsymbol{\omega}_1$ on Γ is denoted by ν: we will assume that $\nu(\alpha) > 0$ for all $\alpha \in \Gamma$.

We are interested in families $A = \{A_N\}_{N\in\mathbb{N}}$ of random variables of the form of empirical averages of a centered local function F, that is

$$A_N = \sum_{n=1}^{N} F(\boldsymbol{\omega}_n, \dots, \boldsymbol{\omega}_{n+k}), \tag{2.1}$$

where $k \in \{0\} \cup \mathbb{N}$ and F is a real function defined on Γ^{k+1} such that $\int F \mathrm{d}\nu^{*(k+1)} = 0$. We will call $A = \{A_N\}_{N\in\mathbb{N}}$ a centered *cocycle*, and with some abuse of notation we will speak of *the cocycle* F to mean the cocycle $\{A_N\}_{N\in\mathbb{N}}$ defined by (2.1).

A cocycle $F : \Gamma^{k+1} \to \mathbb{R}$ is said to be a *coboundary* if (when $k \geq 1$) there exists a function $G : \Gamma^k \to \mathbb{R}$ such that

$$F(\alpha_1, \dots, \alpha_{k+1}) = G(\alpha_2, \dots \alpha_{k+1}) - G(\alpha_1, \dots, \alpha_k) \tag{2.2}$$

for all $\alpha_1, \dots, \alpha_{k+1} \in \Gamma$. When $k = 0$, we say that F is a *coboundary* if it is identically zero: $F(\alpha) = 0$ for every $\alpha \in \Gamma$.

For $\beta \in \mathbb{R}$ we define the free energy $L^F(\beta)$ of a cocycle F as

$$L^F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[e^{\beta A_N}\right]. \tag{2.3}$$

The limit above is easily seen to exist by a standard superadditive argument, and Jensen's inequality yields immediately $L^F(\beta) \geq 0$. Of course, if F is a coboundary then the corresponding free energy vanishes for all $\beta \in \mathbb{R}$. That also the converse is true is the object of the following theorem.

Theorem 2.1. *Let F be a centered cocycle, and let $L^F(\beta)$ be the corresponding free energy, defined by (2.3). The following conditions are equivalent:*

(1) *F is a coboundary;*
(2) *$L^F(\beta) = 0$ for all $\beta \in \mathbb{R}$;*
(3) *$L^F(\beta_0) = 0$ for some $\beta_0 \in \mathbb{R} \setminus \{0\}$.*

The proof is obtained combining convexity ideas with the following combinatorial reformulation of the condition that a function be a coboundary.

Lemma 2.2. *A function $F : \Gamma^{k+1} \to \mathbb{R}$ is a coboundary if and only if for every $N \in \mathbb{N}$ and for every $(\eta_1, \dots, \eta_N) \in \Gamma^N$ the following relation holds:*

$$\sum_{i=1}^{N} F(\eta_i, \eta_{i \oplus_N 1}, \dots, \eta_{i \oplus_N k}) = 0\,, \tag{2.4}$$

where for $a, b \in \mathbb{N}$ we have set $a \oplus_N b := (a + b) \mod N$.

Proof. The *if* part trivially follows from the definition of a coboundary (see (2.2)), so we can focus on the *only if* part. As a matter of fact, we will use the hypothesis of the Lemma only for two values of N, namely $N = 2k$ and $N = 2k + 1$.

Let us take k elements $\gamma_1, \dots, \gamma_k \in \Gamma$, arbitrarily chosen, that will be kept *fixed* throughout the proof; moreover, let $\alpha_1, \dots, \alpha_{k+1}$ denote generic elements of Γ. We start rewriting equation (2.4) for $N = 2k + 1$, with $(\eta_1, \dots, \eta_N) = (\alpha_1, \dots, \alpha_{k+1}, \gamma_1, \dots, \gamma_k)$, as

$$F(\alpha_1, \dots, \alpha_{k+1}) = -\sum_{i=1}^{k} F(\alpha_{i+1}, \dots, \alpha_{k+1}, \gamma_1, \dots, \gamma_i) - \sum_{i=1}^{k} F(\gamma_i, \dots, \gamma_k, \alpha_1, \dots, \alpha_i)\,. \tag{2.5}$$

In order to determine an alternative expression for the second sum in the r.h.s., we use again equation (2.4), this time with $N = 2k$ and $(\eta_1, \dots, \eta_N) = (\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_k)$, getting

$$\sum_{i=1}^{k} F(\gamma_i, \dots, \gamma_k, \alpha_1, \dots, \alpha_i) = -\sum_{i=1}^{k} F(\alpha_i, \dots, \alpha_k, \gamma_1, \dots, \gamma_i)\,. \tag{2.6}$$

If now we introduce a function $G : \Gamma^k \to \mathbb{R}$, defined by

$$G(\zeta_1, \dots, \zeta_k) := -\sum_{i=1}^{k} F(\zeta_i, \dots, \zeta_k, \gamma_1, \dots, \gamma_i)\,,$$

we can combine equations (2.5) and (2.6) to get

$$F(\alpha_1, \dots, \alpha_{k+1}) = G(\alpha_2, \dots \alpha_{k+1}) - G(\alpha_1, \dots \alpha_k)\,,$$

so that the proof is completed. □

Proof of Theorem 2.1. It has already been remarked that (1) ⇒ (2), and of course (2) ⇒ (3) holds trivially. In the following we are going to prove that (3) ⇒ (2) ⇒ (1).

We start determining an explicit expression for the free energy. For this, we define a slight modification of the cocycle A defined by (2.1), by setting

$$\widetilde{A}_N := \sum_{n=1}^{N} F(\boldsymbol{\omega}_n, \boldsymbol{\omega}_{n\oplus_N 1}, \dots, \boldsymbol{\omega}_{n\oplus_N k}), \tag{2.7}$$

where by $\oplus_N$ we mean addition modulo N. Of course, only the last k addends in the sum are really changed: as F is a bounded function (the space Γ is finite), it easily follows that the free energies of A and $\widetilde{A}$ are the same, so that we can write

$$L^F(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) \qquad \text{where} \qquad Z_N(\beta) = Z_N^F(\beta) = \mathbb{E}\Big[e^{\beta \widetilde{A}_N}\Big]. \tag{2.8}$$

Now we introduce the $\Gamma^{k+1} \times \Gamma^{k+1}$ matrix A_β, defined for $\alpha_i, \gamma_i \in \Gamma,\ i = 1, \dots, k+1$ by

$$A_\beta\big[(\alpha_1, \dots, \alpha_{k+1}), (\gamma_1, \dots, \gamma_{k+1})\big] := \delta_{\gamma_1,\alpha_2} \cdots \delta_{\gamma_k,\alpha_{k+1}} \cdot e^{\beta F(\gamma_1,\dots,\gamma_{k+1})} \cdot \nu(\gamma_{k+1}). \tag{2.9}$$

Developing the expectation defining $Z_N(\beta)$ we get

$$\begin{aligned} Z_N(\beta) &= \sum_{\zeta_1,\dots,\zeta_N \in \Gamma} e^{\beta \sum_{i=1}^N F(\zeta_i, \zeta_{i\oplus_N 1}, \dots, \zeta_{i\oplus_N k})} \cdot \nu(\zeta_1) \cdots \nu(\zeta_N) \\ &= \operatorname{Tr}\big[A_\beta^N\big] = \sum_{i=1}^{|\Gamma|^{2(k+1)}} e_i(\beta)^N, \end{aligned} \tag{2.10}$$

where $\{e_i(\beta),\ i = 1, \dots, |\Gamma|^{2(k+1)}\}$ are the (possibly complexes) eigenvalues of the matrix A_β (counted repeatedly according to their algebraic multiplicity). It's immediate to check that A_β is an irreducible, aperiodic matrix, and since its entries are nonnegative we can apply Perron–Frobenius theory [3]: there exists a real positive simple eigenvalue, say $e_1(\beta)$, such that $|e_i(\beta)| < e_1(\beta)$ for every $i \geq 2$. To lighten the notation, from now on we will let $e(\beta) := e_1(\beta)$. Combining (2.8) with (2.10) we get

$$Z_N(\beta) = e(\beta)^N \cdot \left(1 + \sum_{i=2}^{|\Gamma|^{2(k+1)}} \left(\frac{e_i(\beta)}{e(\beta)}\right)^N\right), \tag{2.11}$$

so that

$$Z_N(\beta) \cdot e(\beta)^{-N} \to 1 \quad \text{as } N \to \infty.$$

From this *sharp asymptotics* for $Z_N(\beta)$ we obtain in particular the explicit expression of $L^F(\beta)$ we were looking for:

$$L^F(\beta) = \log e(\beta). \tag{2.12}$$

This equation shows that $L^F(\beta)$ is a *real analytic* function of $\beta \in \mathbb{R}$, since $e(\beta)$ is so: this is because the Perron–Frobenius eigenvalue is a simple root of the characteristic polynomial and the entries of A_β are real–analytic functions of $\beta \in \mathbb{R}$.

From (2.8) it is clear that $\log Z_N(\beta)$ is a convex function of $\beta \in \mathbb{R}$, for every $N \in \mathbb{N}$. Moreover, we have $Z_N(\beta) \geq 1$ for every $\beta \in \mathbb{R}$ by Jensen's inequality, and trivially $Z_N(0) = 1$. It follows immediately that $L^F(\beta)$ is a convex function too, being the pointwise limit of $\log Z_N(\beta)/N$, that $L^F(\beta) \geq 0$ for every $\beta \in \mathbb{R}$, and $L^F(0) = 0$.

Let's assume that condition (3) in the statement of the theorem holds, that is $L^F(\beta_0) = 0$ for some $\beta_0 > 0$ (the case $\beta_0 < 0$ is completely analogous): the preceding observations yield $L^F(\beta) = 0$ for every $\beta \in [0, \beta_0]$, and by analyticity we conclude that indeed $L^F(\beta) = 0$ for every $\beta \in \mathbb{R}$. We have thus shown that (3) $\Rightarrow$ (2).

Now we assume that condition (2) holds: by (2.12) this means $e(\beta) = 1$ for every $\beta \in \mathbb{R}$, and (2.11) we have that

$$|Z_N(\beta)| \le e(\beta)^N \cdot |\Gamma|^{2(k+1)} = |\Gamma|^{2(k+1)} \qquad \forall N \in \mathbb{N}\,,\ \forall \beta \in \mathbb{R}\,. \tag{2.13}$$

Since $\log Z_N(\beta)$ is a convex function, $Z_N(\beta)$ is convex too; furthermore, we have already remarked that $Z_N(\beta) \ge 1$ for every $\beta \in \mathbb{R}$ and that $Z_N(0) = 1$. Since (2.13) shows that $|Z_N(\beta)|$ is bounded, by elementary convex analysis it follows that Z_N must be constant, therefore $Z_N(\beta) = 1$ for all $\beta \in \mathbb{R}$ and $N \in \mathbb{N}$. This means that for every $\beta \in \mathbb{R}$ Jensen's inequality for $Z_N(\beta)$ it's not strict: since for any $\beta > 0$ the function $\{x \mapsto e^{\beta x}\}$ is a strictly convex function, this can happen if and only if $\widetilde{A}_N$ is $\mathbb{P}$–a.s. constant, for every $N \in \mathbb{N}$. Recalling (2.7) and the fact that by hypothesis $\nu(\alpha) > 0$ for every $\alpha \in \Gamma$, this amounts to saying that

$$\sum_{i=1}^{N} F(\eta_i, \eta_{i \oplus_N 1}, \dots, \eta_{i \oplus_N k}) = 0\,,$$

for every $N \in \mathbb{N}$ and for every $\eta_1, \dots, \eta_N \in \Gamma$: applying Lemma 2.2 we conclude that F is a coboundary, and the proof is complete. □

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