Weyl's law for Liouville quantum gravity

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Joint work with Nathanaël Berestycki (arXiv:2307.05407).

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

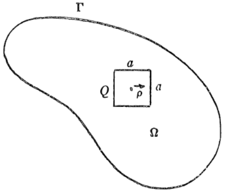

FIG. 1

Kac (1966): can one hear the shape of a drum?

Let $D \subset \mathbb{R}^d$ be an open bounded domain, and consider the eigenvalue problem with Dirichlet boundary condition:

$$\begin{aligned} -\frac{1}{2}\Delta\psi_k(x) &= \lambda_k\psi_k(x) && \text{for } x \in D, \\ \psi_k(x) &= 0 && \text{for } x \in \partial D. \end{aligned}$$

- $-\frac{1}{2}\Delta$: non-negative self-adjoint operator $\Rightarrow$ non-negative eigenvalues $\lambda_k \geq 0$.
- Compact resolvent $\Rightarrow$ discrete spectrum: $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$.
- Q: if we know all the eigenvalues λ_k's, what can we learn about D?

Weyl's law: asymptotic behaviour of eigenvalues

If we consider the eigenvalue counting function

$$N(\lambda) := \sum_{k \geq 1} 1\{\lambda_k \leq \lambda\} = \#\text{eigenvalues not greater than } \lambda,$$

then it is generally expected that

$$N(\lambda) \overset{\lambda \to \infty}{\sim} a_d \lambda^{\frac{d}{2}} |D|.$$

- One of the first results was due to Weyl (1911) and (1912) (for $d = 2, 3$).

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Weyl's law: compact manifold

If $-\frac{1}{2}\Delta_g$ is the Laplace-Beltrami operator on some compact manifold (D, g) and

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Q: what if we consider a compact manifold with a **rough** metric?

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Liouville quantum gravity (LQG) surface

$D \subset \mathbb{R}^2$: bounded, simply connected domain with "Riemmanian metric tensor"

$$e^{\gamma h(x)} \left(dx_1^2 + dx_2^2\right) \qquad \forall x = (x_1, x_2) \in D$$

where h is the (Dirichlet) Gaussian free field. (Note: $\mathrm{vol}_g(dx) = \underbrace{e^{\gamma h(x)} dx}_{=:\mu_\gamma(dx)}$.)

- Polyakov (1981): model of 2d quantum gravity.

Q: Weyl's law for $-\frac{1}{2}\Delta_\gamma$?

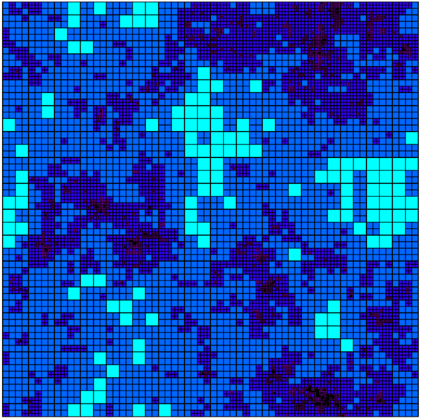

Figure: Realisation of an LQG surface: each mini-square has comparable mass w.r.t. $\mu_\gamma(dx)$. Simulation by Sheffield.

Making sense of $\mathrm{vol}_g(dx) = e^{\gamma h(x)} dx$

Recall h is a Dirichlet GFF on D if it is a centred Gaussian field on D with

$$\mathbb{E}[h(x)h(y)] = G_0^D(x, y) \overset{|x-y|\to 0}{\sim} -\log|x-y| + \mathcal{O}(1).$$

Theorem (Kah1985, RV2010, Ber2017)

Let $h_\epsilon := h * \theta_\epsilon$ be a collection of mollified fields. Then for $\gamma \in (0, 2)$

$$\mu_{\gamma,\epsilon}(dx) := \epsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\epsilon(x)} dx$$

converges in probability to some non-trivial random measure $\mu_\gamma(dx)$ as $\epsilon \to 0^+$.

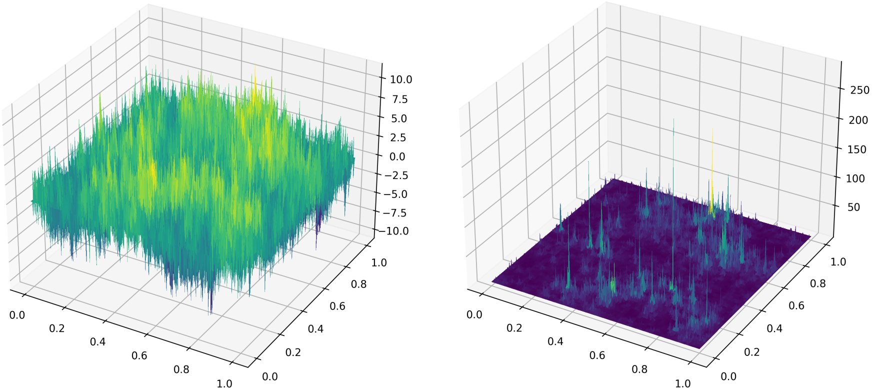

Figure: A realisation of approximate GFF (left) / GMC (right) on $[0,1]^2$ with $\gamma = 0.5$.

How to interpret $-\frac{1}{2}\Delta_\gamma$?

Let us formally express Δ_γ in terms of local coordinate:

$$\Delta_\gamma := \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right) = e^{-\gamma h(x)} \underbrace{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)}_{=\Delta}.$$

If $-\frac{1}{2}\Delta_\gamma \psi = \lambda \psi$ then for any suitable g with compact support:

$$\begin{aligned} \lambda \int_D \psi(x) g(x) dx &= \int_D \left[-\frac{1}{2} \Delta_\gamma \psi(x) \right] g(x) dx \\ &= \int_D \left[-\frac{1}{2} \Delta \psi(x) \right] \left[e^{-\gamma h(x)} g(x) \right] dx \\ &= \frac{1}{2} \int_D \nabla \psi(x) \cdot \nabla \left[e^{-\gamma h(x)} g(x) \right] dx. \end{aligned}$$

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\end{aligned}$$

The probabilistic approach

Weak formulation: (λ, ψ) is an eigenpair for $-\frac{1}{2}\Delta_\gamma$ if

$$\frac{1}{2}\int_D \nabla\psi \cdot \nabla g dx = \lambda \int_D \psi g \mu_\gamma(dx) \qquad \forall g \in H_0^1(D) \cap L^2(\mu_\gamma).$$

- Dirichlet form $\mathcal{E}(u,v) := \frac{1}{2}\int_D \nabla u \cdot \nabla v$ with Revuz measure μ_γ.
- Formal Laplacian $-\frac{1}{2}\Delta_\gamma \leftrightarrow$ generator of **Liouville Brownian motion** (AK2016, Ber2015, GRV2016), i.e.

$$\mathcal{B}_t = W_{A_\gamma^{-1}(t)} \qquad \text{with} \qquad A_\gamma(t) \text{``}=\text{''} \int_0^{t \wedge \tau_{\partial D}} e^{\gamma h(W_s)} ds$$

  where $(W_t)_t$ is a standard planar BM and $\tau_{\partial D}$ is its hitting time of ∂D.
- Existence of jointly continuous Liouville heat kernel:

$$\mathcal{P}_x^\gamma(\mathcal{B}_t \in dy) = \mathbf{p}_t^\gamma(x,y)\mu_\gamma(dy).$$

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Weyl's law for LQG surfaces

Known (AK2016): the generator $-\frac{1}{2}\Delta_\gamma$ has compact resolvent and hence a discrete spectrum

$$0 \le \lambda_{\gamma,1} \le \lambda_{\gamma,2} \le \lambda_{\gamma,3} \le \dots$$

Question: any geometric information encoded in the spectrum?

Theorem (Berestycki, W' 2023+)

Let $\gamma \in (0,2)$, and $\mathbf{N}_\gamma(\lambda) := \sum_{j\ge 1} 1\{\lambda_{\gamma,j} \le \lambda\}$ be the counting function of the eigenvalues associated to $-\frac{1}{2}\Delta_\gamma$. Then there exists some constant $c_\gamma \in (0,\infty)$ such that

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in the sense of L^1 convergence (and hence convergence in probability).

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Generalisation: heat trace asymptotics

Theorem (Berestycki, W' 2023+)

For each $t > 0$ and fixed open set $A \subset D$, consider

$$\mathbf{S}_\gamma(t; A) := \int_A \mathbf{p}_t^\gamma(x, x)\mu_\gamma(dx).$$

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Note: $S_\gamma(t; D) = \int_0^\infty e^{-t\lambda} d\mathbf{N}_\gamma(\lambda)$.

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What else do we know?

- Q: is it the case that for μ_γ-a.e. $x \in D$ we have the existence of

  $$\lim_{t \to 0^+} t\mathbf{p}_t^\gamma(x,x) \qquad \text{in the sense of convergence in probability?}$$

  A: No, the appearance of c_γ is a consequence of stochastic homogenisation.

- Q: explicit formula for c_γ?

  A: Yes: $c_\gamma = \left[\pi(2-\gamma^2/2)\right]^{-1} \xrightarrow{\gamma \to 0^+} (2\pi)^{-1} = a_2$ (Riemannian constant).

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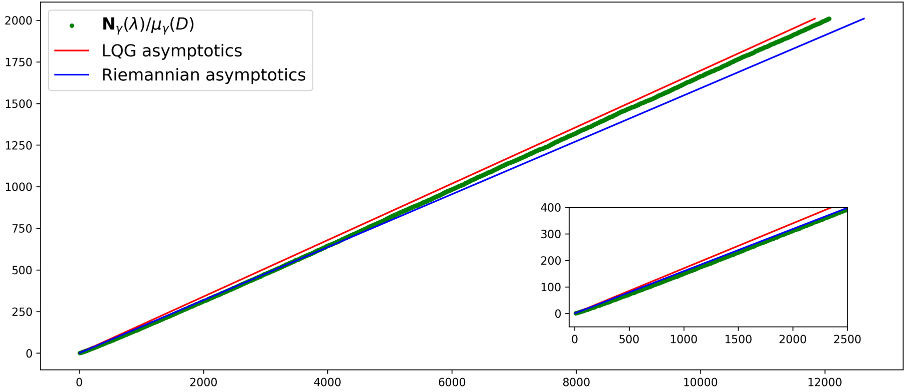
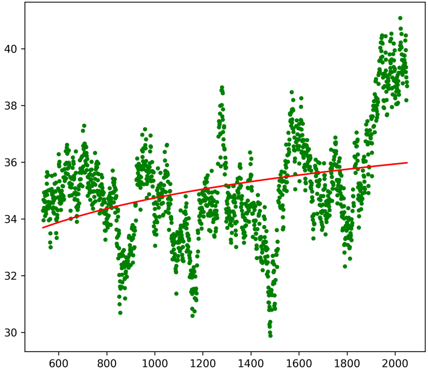


Figure: A realisation of LQG surface with $\gamma = 0.5$ and the first 2000 eigenvalues.

Open problems: subleading order?



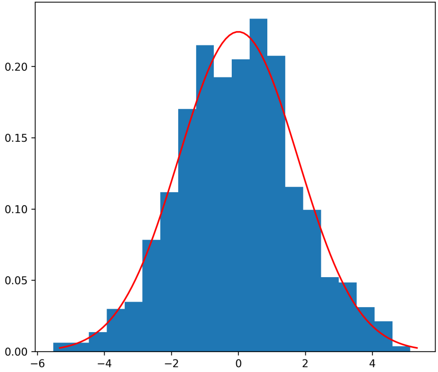


Figure: Left: scatter plot of $c_\gamma \mu_\gamma(D)\lambda - \mathbf{N}_\gamma(\lambda)$ (green) versus fitted power-law curve (red). Right: histogram of deviations from the power-law curve (blue) versus fitted Gaussian density.

Open problems: spacing distribution?

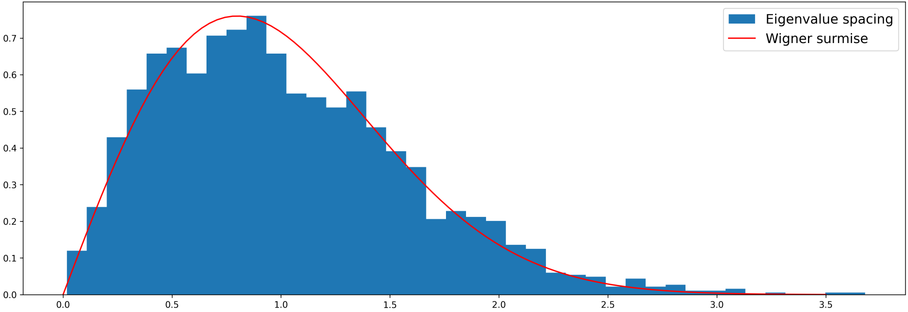


Figure: Empirical spacing distribution based on the first 2000 LQG eigenvalues (blue) versus GOE statistics approximated by Wigner surmise (red).

Hearing the shape of LQG?

Conjecture (Berestycki, W' 2023+)

One can almost surely hear the shape of Liouville quantum gravity. More precisely, the GFF h is a measurable function of the eigenvalues given D, i.e. there exists a measurable function ϕ such that

$$h = \phi\left((\lambda_{\gamma,k})_k\right) \qquad \textit{almost surely.}$$

Thank you