The weak coupling limit of disordered copolymer models

Francesco Caravenna
Università degli Studi di Padova

Joint work with Giambattista Giacomin (Université Paris Diderot)

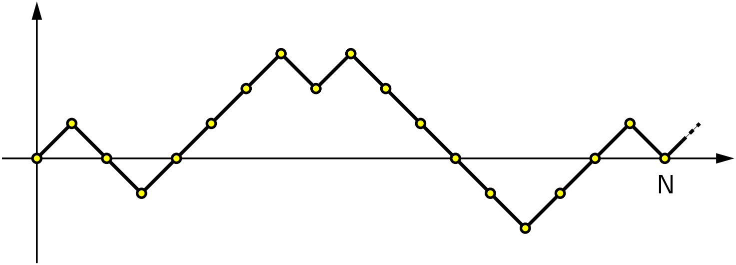
Erwin Bolthausen's birthday conference

TU Berlin $\sim$ October 16, 2010

Outline

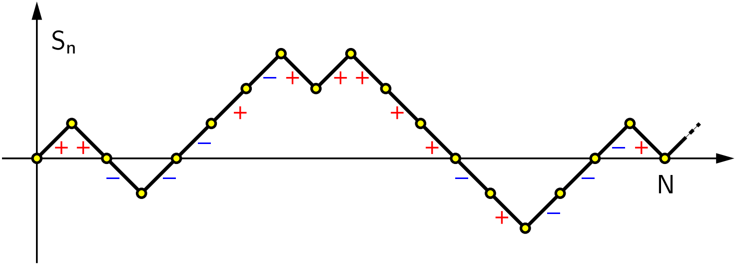
Outline

A random walk with a random potential



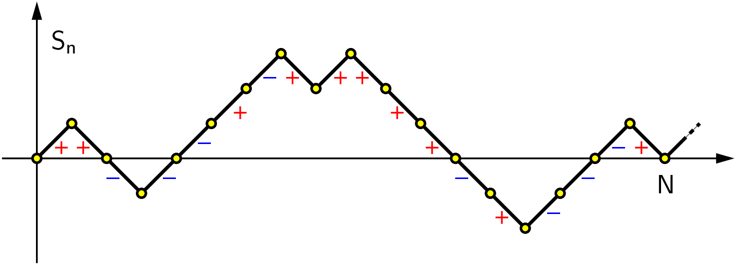
- $S = \{S_n\}_{n \in \mathbb{N}_0}$ simple symmetric random walk on $\mathbb{Z}$ $\rightarrow$ law $\mathbf{P}$

A random walk with a random potential



- $S = \{S_n\}_{n \in \mathbb{N}_0}$ simple symmetric random walk on $\mathbb{Z}$ $\rightarrow$ law $\mathbf{P}$
- $\omega = \{\omega_n\}_{n \in \mathbb{N}} \in \{+, -\}^{\mathbb{N}}$ fair coin tossing $\rightarrow$ quenched law $\mathbb{P}$

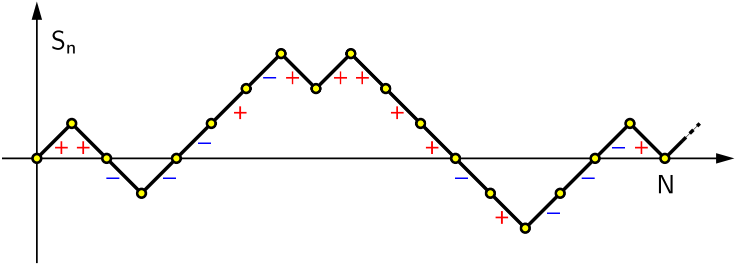
A random walk with a random potential



- $S = \{S_n\}_{n \in \mathbb{N}_0}$ simple symmetric random walk on $\mathbb{Z}$ $\rightarrow$ law $\mathbf{P}$
- $\omega = \{\omega_n\}_{n \in \mathbb{N}} \in \{+, -\}^{\mathbb{N}}$ fair coin tossing $\rightarrow$ quenched law $\mathbb{P}$

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z_{N,\omega}} \exp\left(\sum_{n=1}^{N} \omega_n \cdot \mathrm{sign}\left((S_{n-1}, S_n)\right)\right)$$

A random walk with a random potential

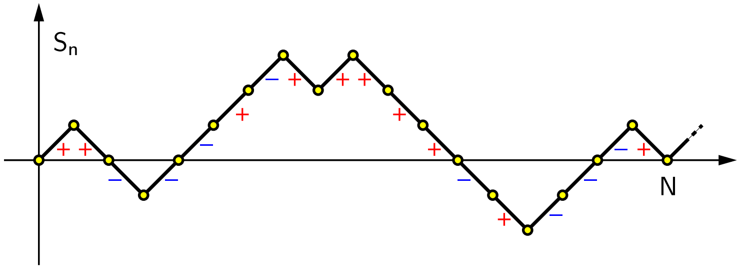


- $S = \{S_n\}_{n \in \mathbb{N}_0}$ simple symmetric random walk on $\mathbb{Z}$ $\to$ law $\mathbf{P}$
- $\omega = \{\omega_n\}_{n \in \mathbb{N}} \in \{+, -\}^{\mathbb{N}}$ fair coin tossing $\to$ quenched law $\mathbb{P}$

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z_{N,\omega}} \exp\left(\sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}((S_{n-1}, S_n)) \right)$$

- $h \geq 0$ asymmetry

A random walk with a random potential

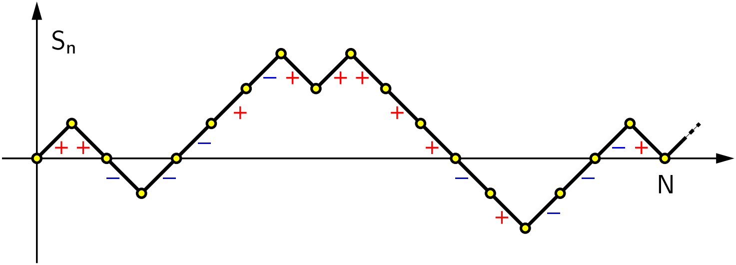


- $S = \{S_n\}_{n \in \mathbb{N}_0}$ simple symmetric random walk on $\mathbb{Z}$ $\rightarrow$ law $\mathbf{P}$
- $\omega = \{\omega_n\}_{n \in \mathbb{N}} \in \{+, -\}^{\mathbb{N}}$ fair coin tossing $\rightarrow$ quenched law $\mathbb{P}$

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z_{N,\omega}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}\left((S_{n-1}, S_n)\right) \right)$$

- $\lambda \geq 0$ interaction strength $\sim$ (temperature)$^{-1}$
- $h \geq 0$ asymmetry

A random walk with a random potential



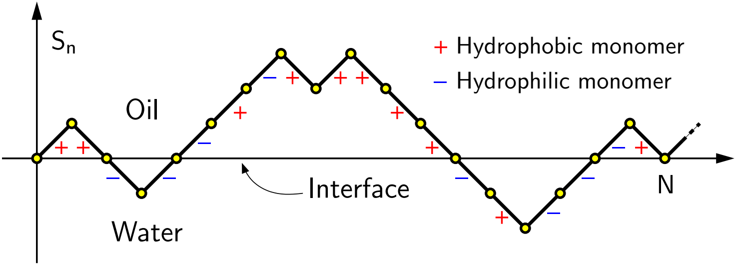
- $S = \{S_n\}_{n \in \mathbb{N}_0}$ simple symmetric random walk on $\mathbb{Z}$ $\rightarrow$ law $\mathbf{P}$
- $\omega = \{\omega_n\}_{n \in \mathbb{N}} \in \{+, -\}^{\mathbb{N}}$ fair coin tossing $\rightarrow$ quenched law $\mathbb{P}$

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{Z_{N,\omega}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h) \operatorname{sign}\left((S_{n-1}, S_n)\right) \right)$$

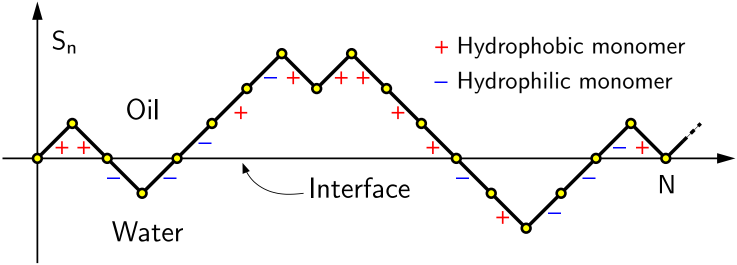
- $\lambda \geq 0$ interaction strength $\sim (\text{temperature})^{-1}$
- $h \geq 0$ asymmetry

Localization or **Delocalization** ?

A polymer model interpretation

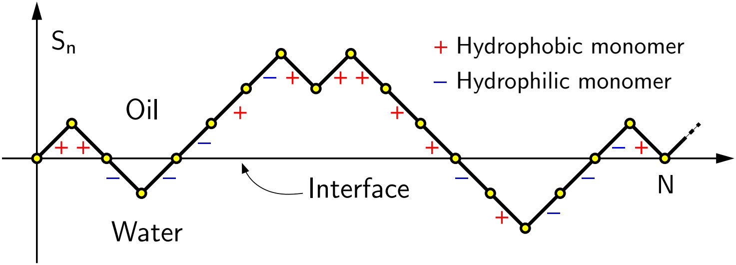


A polymer model interpretation



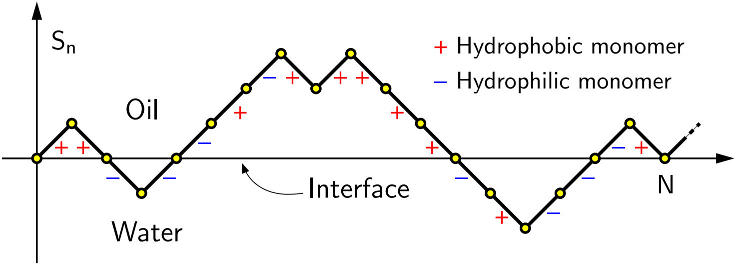
- T. Garel, D. A. Huse, S. Leibler and H. Orland, *Europhys. Lett.* (1989)

A polymer model interpretation



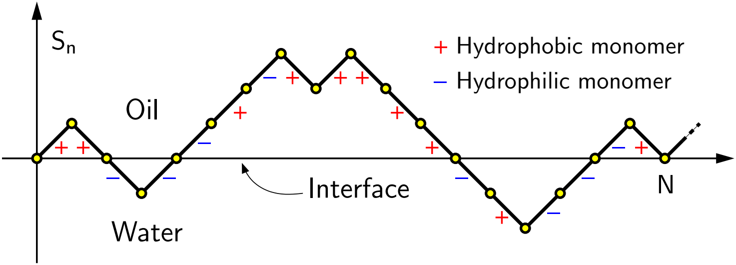
- T. Garel, D. A. Huse, S. Leibler and H. Orland, *Europhys. Lett.* (1989)
- Ya.G. Sinai, *Theory Probab. Appl.* (1993)

A polymer model interpretation



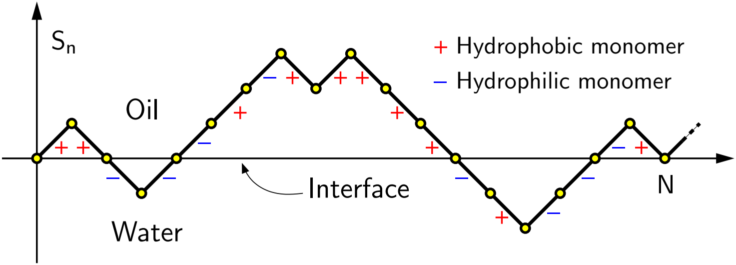
- T. Garel, D. A. Huse, S. Leibler and H. Orland, *Europhys. Lett.* (1989)
- Ya.G. Sinai, *Theory Probab. Appl.* (1993)
- S. Albeverio and X. Y. Zhou, *J. Statist. Phys.* (1996)

A polymer model interpretation



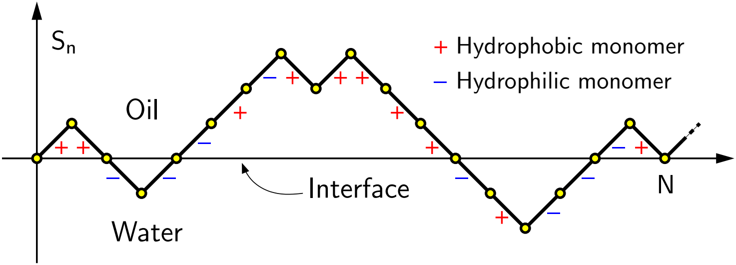
- T. Garel, D. A. Huse, S. Leibler and H. Orland, *Europhys. Lett.* (1989)
- Ya.G. Sinai, *Theory Probab. Appl.* (1993)
- S. Albeverio and X. Y. Zhou, *J. Statist. Phys.* (1996)
- E. Bolthausen and F. den Hollander, *Ann. Probab.* (1997)

A polymer model interpretation



- T. Garel, D. A. Huse, S. Leibler and H. Orland, *Europhys. Lett.* (1989)
- Ya.G. Sinai, *Theory Probab. Appl.* (1993)
- S. Albeverio and X. Y. Zhou, *J. Statist. Phys.* (1996)
- E. Bolthausen and F. den Hollander, *Ann. Probab.* (1997)
- M. Biskup, G. Giacomin, T. Bodineau, F.L. Toninelli, F.C., M. Gubinelli, N. Pétrélis, L. Zambotti, B. de Tilière, H. Lacoin, D. Cheliotis, . . .

A polymer model interpretation



- T. Garel, D. A. Huse, S. Leibler and H. Orland, *Europhys. Lett.* (1989)
- Ya.G. Sinai, *Theory Probab. Appl.* (1993)
- S. Albeverio and X. Y. Zhou, *J. Statist. Phys.* (1996)
- E. Bolthausen and F. den Hollander, *Ann. Probab.* (1997)
- M. Biskup, G. Giacomin, T. Bodineau, F.L. Toninelli, F.C., M. Gubinelli, N. Pétrélis, L. Zambotti, B. de Tilière, H. Lacoin, D. Cheliotis, . . .
- Physics literature: C. Monthus, S. Stepanow, J.-U. Sommer, I. Ya. Erukhimovich, A. Trovato, A. Maritan, S. G. Whittington, C. E. Soteros, A. Rechnitzer, G. K. Iliev, . . .

The basic copolymer model

Definition of the model: $\dfrac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \dfrac{1}{Z_{N,\omega}} \exp\big(- H_{N,\omega}(S)\big)$

$$-H_{N,\omega}(S) := \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}\,\big((S_{n-1}, S_n)\big)$$

The basic copolymer model

Definition of the model: $\dfrac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \dfrac{1}{Z_{N,\omega}} \exp\left(- H_{N,\omega}(S)\right)$

$$-H_{N,\omega}(S) := \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}\left((S_{n-1}, S_n)\right)$$

Slight modification: $H_{N,\omega}(S) \rightsquigarrow H_{N,\omega}(S) + \lambda \sum_{n=1}^{N} (\omega_n + h)$.

The law $\mathbf{P}_{N,\omega}$ remains the same.

The basic copolymer model

Definition of the model: $\dfrac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \dfrac{1}{Z_{N,\omega}} \exp\left(-H_{N,\omega}(S)\right)$

$$-H_{N,\omega}(S) := \lambda \sum_{n=1}^{N} (\omega_n + h)\, \mathrm{sign}\left((S_{n-1}, S_n)\right)$$

Slight modification: $H_{N,\omega}(S) \rightsquigarrow H_{N,\omega}(S) + \lambda \sum_{n=1}^{N} (\omega_n + h)$.

The law $\mathbf{P}_{N,\omega}$ remains the same. New Hamiltonian (and $Z_{N,\omega}$):

$$-H_{N,\omega}(S) := -2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}$$

The basic copolymer model

Definition of the model: $\dfrac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \dfrac{1}{Z_{N,\omega}} \exp\left(- H_{N,\omega}(S)\right)$

$$-H_{N,\omega}(S) := \lambda \sum_{n=1}^{N} (\omega_n + h) \,\mathrm{sign}\left((S_{n-1}, S_n)\right)$$

Slight modification: $H_{N,\omega}(S) \rightsquigarrow H_{N,\omega}(S) + \lambda \sum_{n=1}^{N} (\omega_n + h)$.

The law $\mathbf{P}_{N,\omega}$ remains the same. New Hamiltonian (and $Z_{N,\omega}$):

$$-H_{N,\omega}(S) := -2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}$$

Monomer (S_{n-1}, S_n) is rewarded: $\begin{cases} 0 & \text{if } (S_{n-1}, S_n) > 0 \\ & \end{cases}$

The basic copolymer model

Definition of the model: $\dfrac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) := \dfrac{1}{Z_{N,\omega}} \exp\left(- H_{N,\omega}(S)\right)$

$$-H_{N,\omega}(S) := \lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}\left((S_{n-1}, S_n)\right)$$

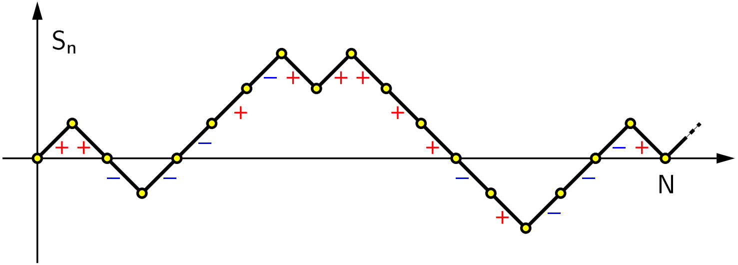
Slight modification: $H_{N,\omega}(S) \ \rightsquigarrow\ H_{N,\omega}(S) + \lambda \sum_{n=1}^{N} (\omega_n + h)$.

The law $\mathbf{P}_{N,\omega}$ remains the same. New Hamiltonian (and $Z_{N,\omega}$):

$$-H_{N,\omega}(S) := -2\lambda \sum_{n=1}^{N} (\omega_n + h)\mathbf{1}_{\{(S_{n-1},S_n)<0\}}$$

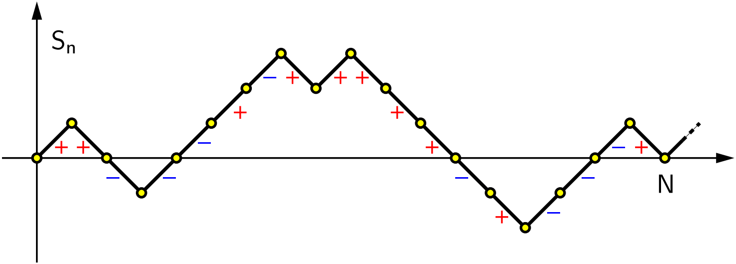
Monomer (S_{n-1}, S_n) is rewarded: $\begin{cases} 0 & \text{if } (S_{n-1}, S_n) > 0 \\ -2\lambda(\omega_n + h) & \text{if } (S_{n-1}, S_n) < 0 \end{cases}$

The continuum (Brownian) copolymer model



We now define a continuum model.

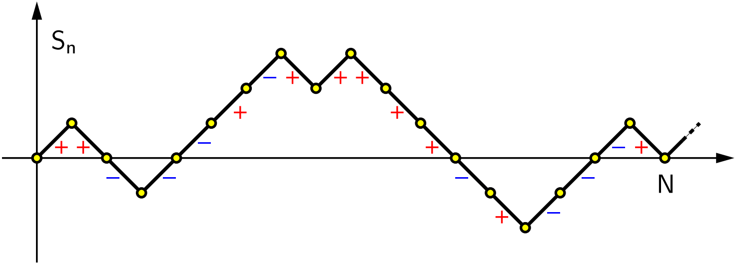
The continuum (Brownian) copolymer model



We now define a continuum model. Replace:

- the SRW $\{S_n\}_{n\in\mathbb{N}_0}$ by Brownian motion $B = \{B_t\}_{t\geq 0}$ (law $\tilde{\mathbf{P}}$)

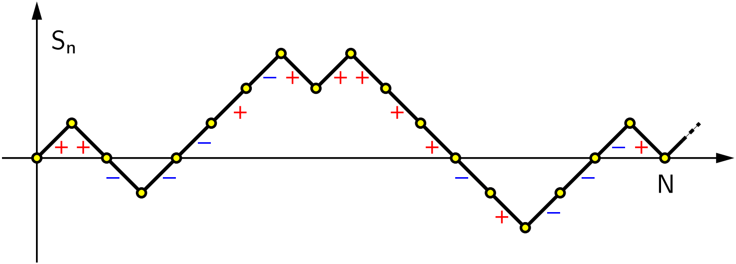
The continuum (Brownian) copolymer model



We now define a continuum model. Replace:

- the SRW $\{S_n\}_{n\in\mathbb{N}_0}$ by Brownian motion $B = \{B_t\}_{t\geq 0}$ (law $\tilde{\mathbf{P}}$)
- the charges $\{\omega_n\}_{n\in\mathbb{N}}$ by white noise $\mathrm{d}\beta = \{\mathrm{d}\beta_t\}_{t\geq 0}$ (law $\tilde{\mathbb{P}}$)

The continuum (Brownian) copolymer model

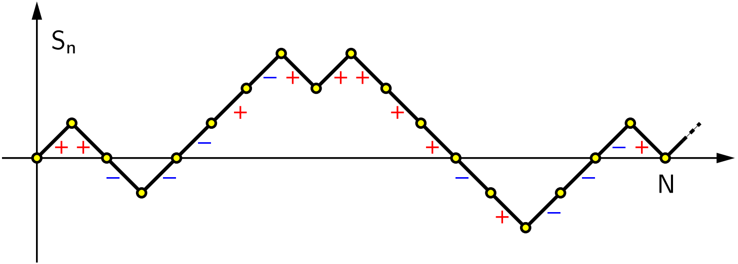


We now define a continuum model. Replace:

- the SRW $\{S_n\}_{n\in\mathbb{N}_0}$ by Brownian motion $B = \{B_t\}_{t\geq 0}$ (law $\tilde{\mathbf{P}}$)
- the charges $\{\omega_n\}_{n\in\mathbb{N}}$ by white noise $\mathrm{d}\beta = \{\mathrm{d}\beta_t\}_{t\geq 0}$ (law $\tilde{\mathbb{P}}$)

$$\frac{\mathrm{d}\tilde{\mathbf{P}}_{t,\beta}}{\mathrm{d}\tilde{\mathbf{P}}}(B) := \frac{1}{\tilde{Z}_{t,\omega}} \exp\left(-2\lambda \int_0^t (\mathrm{d}\beta_s + h\,\mathrm{d}s)\,\mathbf{1}_{\{B_s<0\}}\right)$$

The continuum (Brownian) copolymer model



We now define a continuum model. Replace:

- the SRW $\{S_n\}_{n\in\mathbb{N}_0}$ by Brownian motion $B = \{B_t\}_{t\geq 0}$ (law $\tilde{\mathbf{P}}$)
- the charges $\{\omega_n\}_{n\in\mathbb{N}}$ by white noise $\mathrm{d}\beta = \{\mathrm{d}\beta_t\}_{t\geq 0}$ (law $\tilde{\mathbb{P}}$)

$$\frac{\mathrm{d}\tilde{\mathbf{P}}_{t,\beta}}{\mathrm{d}\tilde{\mathbf{P}}}(B) := \frac{1}{\tilde{Z}_{t,\omega}} \exp\left(-2\lambda \int_0^t (\mathrm{d}\beta_s + h\,\mathrm{d}s)\,\mathbf{1}_{\{B_s<0\}}\right)$$

Discrete model: $\quad -2\lambda \sum_{n=1}^N (\omega_n + h)\,\mathbf{1}_{\{(S_{n-1},S_n)<0\}}$

Outline

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ localized or delocalized for large N (and typical ω)?

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log Z_{N,\omega}\right)$$

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$Z_{N,\omega} = \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}\right)\right]$$

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}^{\lambda,h}_{N,\omega}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$Z_{N,\omega} = \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\mathbf{1}_{\{(S_{n-1},S_n)<0\}}\right)\right]$$

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$Z_{N,\omega} = \mathbf{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n + h)\mathbf{1}_{\{(S_{n-1},S_n)<0\}}\right)\right]$$

Restricting on paths that stay always in the upper half-plane gives

$$Z_{N,\omega} \;\geq\; \mathbf{P}\left[S_1 > 0, \ldots, S_N > 0\right] \;\sim\; C\,N^{-1/2}$$

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$Z_{N,\omega} = \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}\right)\right]$$

Restricting on paths that stay always in the upper half-plane gives

$$Z_{N,\omega} \geq \mathbf{P}\left[S_1 > 0, \ldots, S_N > 0\right] \sim C\, N^{-1/2} \quad \Longrightarrow \quad \mathtt{F}(\lambda, h) \geq 0$$

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$Z_{N,\omega} = \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}\right)\right]$$

Restricting on paths that stay always in the upper half-plane gives

$$Z_{N,\omega} \geq \mathbf{P}\left[S_1 > 0, \ldots, S_N > 0\right] \sim C\, N^{-1/2} \quad \Longrightarrow \quad \mathtt{F}(\lambda, h) \geq 0$$

The model is: $\mathcal{L}$ocalized if $\mathtt{F}(\lambda, h) > 0$, $\mathcal{D}$elocalized if $\mathtt{F}(\lambda, h) = 0$.

Localization and Delocalization

Is $\mathbf{P}_{N,\omega} = \mathbf{P}^{\lambda,h}_{N,\omega}$ localized or delocalized for large N (and typical ω)?

It depends on λ, h. Look at the free energy:

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$Z_{N,\omega} = \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h) \mathbf{1}_{\{(S_{n-1},S_n)<0\}}\right)\right]$$

Restricting on paths that stay always in the upper half-plane gives

$$Z_{N,\omega} \geq \mathbf{P}\left[S_1 > 0, \ldots, S_N > 0\right] \sim C\, N^{-1/2} \quad \Longrightarrow \quad \mathtt{F}(\lambda, h) \geq 0$$

The model is: $\mathcal{L}$ocalized if $\mathtt{F}(\lambda, h) > 0$, $\mathcal{D}$elocalized if $\mathtt{F}(\lambda, h) = 0$.

This definition does correspond to sharply different path behaviors!

The phase diagram: discrete model

Theorem

The regions $\mathcal{L}$ and $\mathcal{D}$ are separated by a strictly increasing, continuous critical line $h_c(\cdot)$:

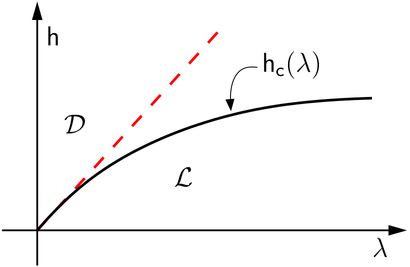
$$\mathcal{L} = \{(\lambda, h): \; h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda, h): \; h \geq h_c(\lambda)\}.$$

The phase diagram: discrete model

Theorem

The regions $\mathcal{L}$ and $\mathcal{D}$ are separated by a strictly increasing, continuous critical line $h_c(\cdot)$:

$$\mathcal{L} = \{(\lambda, h) : \ h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda, h) : \ h \geq h_c(\lambda)\}\,.$$

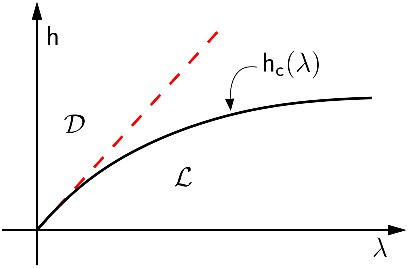


The phase diagram: discrete model

Theorem

The regions $\mathcal{L}$ and $\mathcal{D}$ are separated by a strictly increasing, continuous critical line $h_c(\cdot)$:

$$\mathcal{L} = \{(\lambda, h) : \; h < h_c(\lambda)\} \qquad \mathcal{D} = \{(\lambda, h) : \; h \geq h_c(\lambda)\}\,.$$



Theorem

$h_c(0) = 0$ and as $\lambda \downarrow 0$

$$h_c(\lambda) = m\,\lambda + o(\lambda)\,,$$

with

$$\frac{2}{3} \leq m < 1\,.$$

The phase diagram: continuum model

The continuum free energy $\tilde{\mathtt{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathtt{F}}(\lambda, h) := \lim_{t\to\infty} \frac{1}{t}\, \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta} \right)$$

The phase diagram: continuum model

The continuum free energy $\tilde{\mathtt{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathtt{F}}(\lambda, h) := \lim_{t\to\infty} \frac{1}{t} \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta} \right)$$

The phase diagram: continuum model

The continuum free energy $\tilde{\mathrm{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathrm{F}}(\lambda, h) := \lim_{t\to\infty} \frac{1}{t} \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta}\right)$$

This time existence is highly non-trivial!

The phase diagram: continuum model

The continuum free energy $\tilde{\mathrm{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathrm{F}}(\lambda, h) := \lim_{t \to \infty} \frac{1}{t} \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta} \right)$$

This time existence is highly non-trivial!

Again $\tilde{\mathrm{F}}(\lambda, h) \geq 0$. We then define $\widetilde{\mathcal{L}}$ocalization and $\widetilde{\mathcal{D}}$elocalization:

$$\widetilde{\mathcal{L}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} \qquad \widetilde{\mathcal{D}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\}$$

The phase diagram: continuum model

The continuum free energy $\tilde{\mathrm{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathrm{F}}(\lambda, h) := \lim_{t\to\infty} \frac{1}{t} \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta} \right)$$

This time existence is highly non-trivial!

Again $\tilde{\mathrm{F}}(\lambda, h) \geq 0$. We then define $\widetilde{\mathcal{L}}$ocalization and $\widetilde{\mathcal{D}}$elocalization:

$$\begin{aligned} \widetilde{\mathcal{L}} &= \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} \\ &= \{(\lambda, h) : \ h < \tilde{h}_c(\lambda)\} \end{aligned} \qquad \begin{aligned} \widetilde{\mathcal{D}} &= \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\} \\ &= \{(\lambda, h) : \ h \geq \tilde{h}_c(\lambda)\} \end{aligned}$$

The phase diagram: continuum model

The continuum free energy $\tilde{\mathrm{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathrm{F}}(\lambda, h) := \lim_{t \to \infty} \frac{1}{t} \, \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta} \right)$$

This time existence is highly non-trivial!

Again $\tilde{\mathrm{F}}(\lambda, h) \geq 0$. We then define $\widetilde{\mathcal{L}}$ocalization and $\widetilde{\mathcal{D}}$elocalization:

$$\begin{aligned} \widetilde{\mathcal{L}} &= \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} \\ &= \{(\lambda, h) : \; h < \tilde{h}_c(\lambda)\} \end{aligned} \qquad \begin{aligned} \widetilde{\mathcal{D}} &= \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\} \\ &= \{(\lambda, h) : \; h \geq \tilde{h}_c(\lambda)\} \end{aligned}$$

By Brownian scaling $\tilde{\mathrm{F}}(a\lambda, ah) = a^2 \, \tilde{\mathrm{F}}(\lambda, h)$ for all $a, \lambda, h \geq 0$.

The phase diagram: continuum model

The continuum free energy $\tilde{\mathrm{F}}(\lambda, h)$ is defined analogously:

$$\tilde{\mathrm{F}}(\lambda, h) := \lim_{t \to \infty} \frac{1}{t} \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta} \right)$$

This time existence is highly non-trivial!

Again $\tilde{\mathrm{F}}(\lambda, h) \geq 0$. We then define $\widetilde{\mathcal{L}}$ocalization and $\widetilde{\mathcal{D}}$elocalization:

$$\widetilde{\mathcal{L}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} = \{(\lambda, h) : \ h < \tilde{h}_c(\lambda)\}$$

$$\widetilde{\mathcal{D}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\} = \{(\lambda, h) : \ h \geq \tilde{h}_c(\lambda)\}$$

By Brownian scaling $\tilde{\mathrm{F}}(a\lambda, ah) = a^2\, \tilde{\mathrm{F}}(\lambda, h)$ for all $a, \lambda, h \geq 0$.

Therefore $\tilde{h}_c(\cdot)$ is a straight line: $\tilde{h}_c(\lambda) = \tilde{m}\, \lambda$.

The weak coupling limit

Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathtt{F}(a\lambda, ah) = \tilde{\mathtt{F}}(\lambda, h) .$$

The weak coupling limit

Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathtt{F}(a\lambda, ah) = \tilde{\mathtt{F}}(\lambda, h) .$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m} .$$

The weak coupling limit

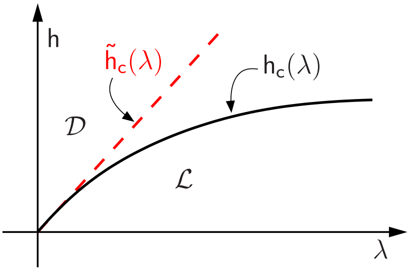
Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathtt{F}(a\lambda, ah) = \tilde{\mathtt{F}}(\lambda, h) \,.$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m} \,.$$



The weak coupling limit

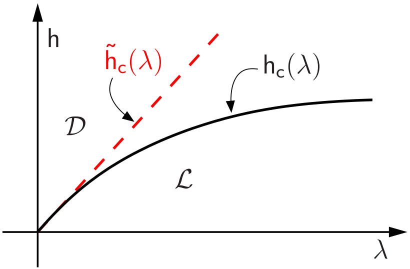
Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathtt{F}(a\lambda, ah) = \tilde{\mathtt{F}}(\lambda, h)\,.$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m}\,.$$



- Long, difficult, technical (though beautiful) proof.

The weak coupling limit

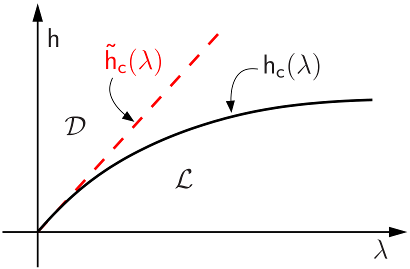
Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathrm{F}(a\lambda, ah) = \tilde{\mathrm{F}}(\lambda, h) \,.$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m} \,.$$



- Long, difficult, technical (though beautiful) proof.
  Not an easy consequece of invariance principles!

The weak coupling limit

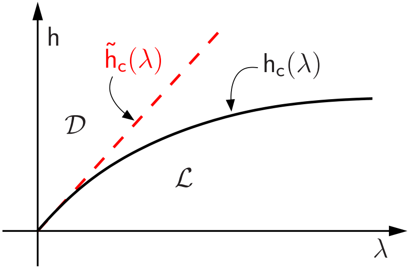
Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathrm{F}(a\lambda, ah) = \tilde{\mathrm{F}}(\lambda, h) \, .$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m} \, .$$



- Long, difficult, technical (though beautiful) proof.
  Not an easy consequece of invariance principles!
- Convergence of the slope does not follow from free energy.

The weak coupling limit

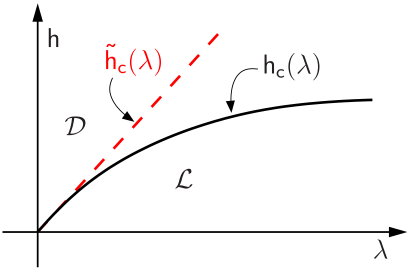
Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathtt{F}(a\lambda, ah) = \tilde{\mathtt{F}}(\lambda, h)\,.$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m}\,.$$



- Long, difficult, technical (though beautiful) proof.
  Not an easy consequece of invariance principles!
- Convergence of the slope does not follow from free energy.
- Universality phenomenon for small coupling constants

The weak coupling limit

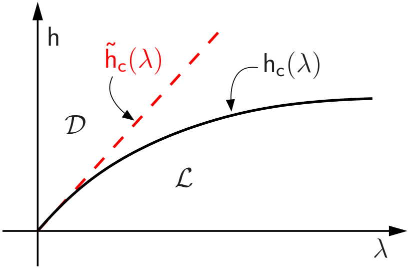
Theorem ([BdH 97])

For all $\lambda, h \geq 0$

$$\lim_{a \downarrow 0} \frac{1}{a^2} \mathtt{F}(a\lambda, ah) = \tilde{\mathtt{F}}(\lambda, h) .$$

Moreover $m = \tilde{m}$, that is

$$\lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m} .$$



- Long, difficult, technical (though beautiful) proof.
  Not an easy consequece of invariance principles!
- Convergence of the slope does not follow from free energy.
- Universality phenomenon for small coupling constants
  ...unfortunately for just one discrete model. Generalization?

Outline

1. The basic copolymer model

2. The free energy

3. Generalized copolymer models

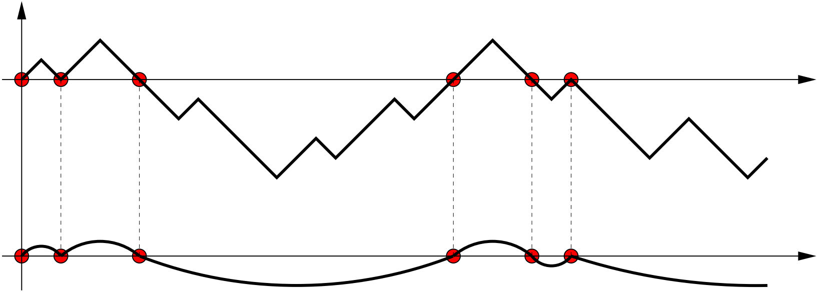
4. Strategy of the proof

Beyond the simple random walk

Which features of the SRW should we keep?
(Only the process $\Delta_n := \mathbf{1}_{\{(S_{n-1},S_n)<0\}}$ matters)

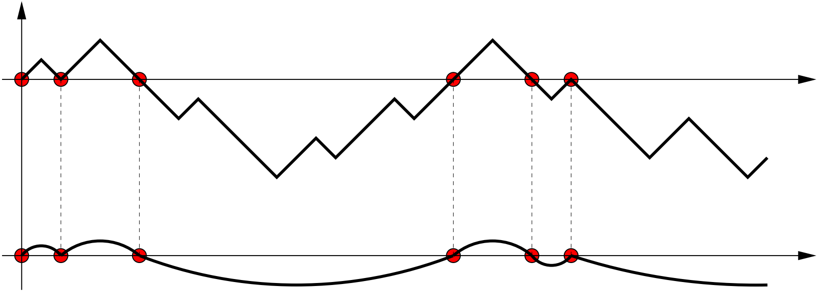
Beyond the simple random walk

Which features of the SRW should we keep?
(Only the process $\Delta_n := \mathbf{1}_{\{(S_{n-1},S_n)<0\}}$ matters)



Beyond the simple random walk

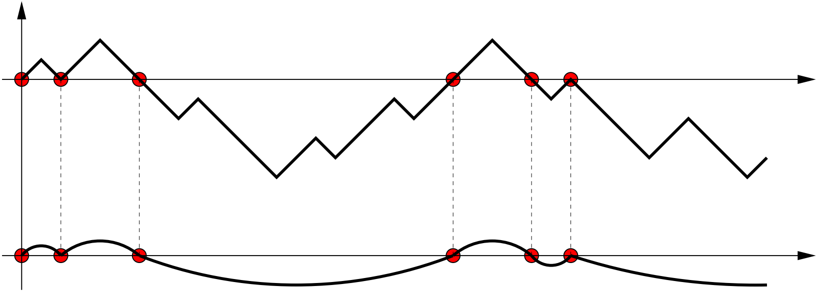
Which features of the SRW should we keep?
(Only the process $\Delta_n := \mathbf{1}_{\{(S_{n-1},S_n)<0\}}$ matters)



- Return times to zero $\{\tau_k\}_{k\geq 0}$: renewal process on $2\mathbb{N}_0$
  IID inter-arrivals with polynomial tails: $\mathbf{P}(\tau_1 = 2n) \sim cn^{-3/2}$

Beyond the simple random walk

Which features of the SRW should we keep?
(Only the process $\Delta_n := \mathbf{1}_{\{(S_{n-1},S_n)<0\}}$ matters)



- Return times to zero $\{\tau_k\}_{k\geq 0}$: renewal process on $2\mathbb{N}_0$
  IID inter-arrivals with polynomial tails: $\mathbf{P}(\tau_1 = 2n) \sim cn^{-3/2}$
- Excursions signs: fair coin tossing (independent of $\{\tau_k\}_{k\geq 0}$)

Generalized discrete α-copolymer models

- Renewal process $\tau = \{\tau_k\}_{k \geq 0}$ on $2\mathbb{N}_0$ with

$$\mathbf{P}(\tau_1 = 2n) = \frac{L(n)}{n^{1+\alpha}}, \qquad \alpha \in (0,1), \quad L(\cdot) \text{ slowly varying} \qquad (\star)$$

Generalized discrete α-copolymer models

- Renewal process $\tau = \{\tau_k\}_{k \geq 0}$ on $2\mathbb{N}_0$ with

$$\mathbf{P}(\tau_1 = 2n) = \frac{L(n)}{n^{1+\alpha}}, \qquad \alpha \in (0,1), \quad L(\cdot) \text{ slowly varying} \qquad (\star)$$

- IID sequence $\{\xi_n\}_{n \geq 1}$ of Be($\frac{1}{2}$) random variables ($\xi_n \in \{0,1\}$)

Generalized discrete α-copolymer models

- Renewal process $\tau = \{\tau_k\}_{k \geq 0}$ on $2\mathbb{N}_0$ with

$$\mathbf{P}(\tau_1 = 2n) = \frac{L(n)}{n^{1+\alpha}}, \qquad \alpha \in (0,1), \; L(\cdot) \text{ slowly varying} \quad (\star)$$

- IID sequence $\{\xi_n\}_{n \geq 1}$ of $\mathrm{Be}(\frac{1}{2})$ random variables ($\xi_n \in \{0,1\}$)

The generalized process $(\Delta = \{\Delta_n\}_{n \geq 0}, \mathbf{P})$ (repl. $\mathbf{1}_{\{(S_{n-1}, S_n) < 0\}}$) is

$$\Delta_n := \xi_k \qquad \text{for } \tau_{k-1} < n \leq \tau_k$$

Generalized discrete α-copolymer models

- Renewal process $\tau = \{\tau_k\}_{k\geq 0}$ on $2\mathbb{N}_0$ with

$$\mathbf{P}(\tau_1 = 2n) = \frac{L(n)}{n^{1+\alpha}}\,, \qquad \alpha \in (0,1)\,, \quad L(\cdot) \text{ slowly varying} \qquad (\star)$$

- IID sequence $\{\xi_n\}_{n\geq 1}$ of Be($\frac{1}{2}$) random variables ($\xi_n \in \{0,1\}$)

The generalized process $(\Delta = \{\Delta_n\}_{n\geq 0}, \mathbf{P})$ (repl. $\mathbf{1}_{\{(S_{n-1},S_n)<0\}}$) is

$$\Delta_n := \xi_k \qquad \text{for } \tau_{k-1} < n \leq \tau_k$$

One always has $\Delta_n = \mathbf{1}_{\{(S_{n-1},S_n)<0\}}$ with $\{S_n\}_n$ Markov chain on $\mathbb{Z}$

Generalized discrete α-copolymer models

- Renewal process $\tau = \{\tau_k\}_{k \ge 0}$ on $2\mathbb{N}_0$ with

$$\mathbf{P}(\tau_1 = 2n) = \frac{L(n)}{n^{1+\alpha}}, \qquad \alpha \in (0,1), \quad L(\cdot) \text{ slowly varying} \quad (\star)$$

- IID sequence $\{\xi_n\}_{n \ge 1}$ of $\mathrm{Be}(\frac{1}{2})$ random variables ($\xi_n \in \{0,1\}$)

The generalized process $(\Delta = \{\Delta_n\}_{n \ge 0}, \mathbf{P})$ (repl. $\mathbf{1}_{\{(S_{n-1},S_n)<0\}}$) is

$$\Delta_n := \xi_k \qquad \text{for } \tau_{k-1} < n \le \tau_k$$

One always has $\Delta_n = \mathbf{1}_{\{(S_{n-1},S_n)<0\}}$ with $\{S_n\}_n$ Markov chain on $\mathbb{Z}$

Discrete Bessel-like process $(c_\alpha = 1/2 - \alpha)$

$$\mathbf{P}(S_{n+1} = x \pm 1 | S_n = x) = \frac{1}{2}\left(1 \pm \frac{c_\alpha}{x} + o\left(\frac{1}{x}\right)\right) \text{ yields } (\star) \text{ asymp.}$$

Generalized discrete α-copolymer models

The charges are generalized to any real IID sequence $\{\omega_n\}_{n\in\mathbb{N}}$ with

$$\mathbb{E}\left(e^{t\omega_1}\right) < \infty \quad \forall t \in (-\epsilon, +\epsilon)\,, \qquad \mathbb{E}(\omega_1) = 0\,, \quad \mathbb{E}(\omega_1^2) = 1\,.$$

Generalized discrete α-copolymer models

The charges are generalized to any real IID sequence $\{\omega_n\}_{n\in\mathbb{N}}$ with

$$\mathbb{E}\left(e^{t\omega_1}\right) < \infty \quad \forall t \in (-\epsilon, +\epsilon)\,, \qquad \mathbb{E}(\omega_1) = 0\,, \quad \mathbb{E}(\omega_1^2) = 1\,.$$

Fix $(\Delta = \{\Delta_n\}_{n\geq 0} = \{\mathbf{1}_{\{(S_{n-1},S_n)<0\}}\}_{n\geq 0}, \mathbf{P})$ & $(\omega = \{\omega_n\}_{n\geq 0}, \mathbb{P})$.
The generalized discrete copolymer model is defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\Delta) := \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n\right)$$

Generalized discrete α-copolymer models

The charges are generalized to any real IID sequence $\{\omega_n\}_{n\in\mathbb{N}}$ with

$$\mathbb{E}\left(e^{t\omega_1}\right) < \infty \quad \forall t \in (-\epsilon, +\epsilon)\,, \qquad \mathbb{E}(\omega_1) = 0\,, \quad \mathbb{E}(\omega_1^2) = 1\,.$$

Fix $(\Delta = \{\Delta_n\}_{n\geq 0} = \{\mathbf{1}_{\{(S_{n-1},S_n)<0\}}\}_{n\geq 0}, \mathbf{P})$ & $(\omega = \{\omega_n\}_{n\geq 0}, \mathbb{P})$.
The generalized discrete copolymer model is defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\Delta) := \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n\right)$$

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}\left(\log Z_{N,\omega}\right)$$

Generalized discrete α-copolymer models

The charges are generalized to any real IID sequence $\{\omega_n\}_{n\in\mathbb{N}}$ with

$$\mathbb{E}\left(e^{t\omega_1}\right) < \infty \quad \forall t \in (-\epsilon, +\epsilon)\,, \qquad \mathbb{E}(\omega_1) = 0\,, \quad \mathbb{E}(\omega_1^2) = 1\,.$$

Fix $(\Delta = \{\Delta_n\}_{n\geq 0} = \{\mathbf{1}_{\{(S_{n-1},S_n)<0\}}\}_{n\geq 0}, \mathbf{P})$ & $(\omega = \{\omega_n\}_{n\geq 0}, \mathbb{P})$.
The generalized discrete copolymer model is defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\Delta) := \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n\right)$$

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$\mathcal{L} = \{(\lambda, h) : \mathtt{F}(\lambda, h) > 0\} = \{(\lambda, h) :\ h < h_c(\lambda)\}$$

$$\mathcal{D} = \{(\lambda, h) : \mathtt{F}(\lambda, h) = 0\} = \{(\lambda, h) :\ h \geq h_c(\lambda)\}$$

Generalized discrete α-copolymer models

The charges are generalized to any real IID sequence $\{\omega_n\}_{n\in\mathbb{N}}$ with

$$\mathbb{E}\left(e^{t\omega_1}\right) < \infty \quad \forall t \in (-\epsilon, +\epsilon)\,, \qquad \mathbb{E}(\omega_1) = 0\,, \quad \mathbb{E}(\omega_1^2) = 1\,.$$

Fix $(\Delta = \{\Delta_n\}_{n\geq 0} = \{\mathbf{1}_{\{(S_{n-1},S_n)<0\}}\}_{n\geq 0}, \mathbf{P})$ & $(\omega = \{\omega_n\}_{n\geq 0}, \mathbb{P})$.
The generalized discrete copolymer model is defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\Delta) := \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n\right)$$

$$\mathtt{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}\left(\log Z_{N,\omega}\right)$$

$$\mathcal{L} = \{(\lambda, h) : \mathtt{F}(\lambda, h) > 0\} = \{(\lambda, h) : \ h < h_c(\lambda)\}$$

$$\mathcal{D} = \{(\lambda, h) : \mathtt{F}(\lambda, h) = 0\} = \{(\lambda, h) : \ h \geq h_c(\lambda)\}$$

Note that $\mathtt{F}(\cdot,\cdot)$ and $h_c(\cdot)$ depend on the choice of $\mathbf{P}$ and $\mathbb{P}$

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \geq 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \geq 0}$

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \geq 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \geq 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \; B_t = 0\}$

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \ge 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \ge 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \; B_t = 0\}$

Attaching independent fair coins to the excursions $I \subseteq \tilde{\tau}^c$ yields $\tilde{\Delta}$

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \geq 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \geq 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \ B_t = 0\}$

Attaching independent fair coins to the excursions $I \subseteq \tilde{\tau}^c$ yields $\tilde{\Delta}$

For $\alpha \in (0, 1)$ let $\tilde{\tau}^\alpha$ be the stable regenerative set of index α

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \ge 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \ge 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \; B_t = 0\}$

Attaching independent fair coins to the excursions $I \subseteq \tilde{\tau}^c$ yields $\tilde{\Delta}$

For $\alpha \in (0, 1)$ let $\tilde{\tau}^\alpha$ be the stable regenerative set of index α, i.e.

- The (closure of the) image of the α-stable subordinator (increasing Lévy process with Lévy measure $x^{-1-\alpha} \mathrm{d}x$ on $\mathbb{R}^+$)

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \ge 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \ge 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \; B_t = 0\}$

Attaching independent fair coins to the excursions $I \subseteq \tilde{\tau}^c$ yields $\tilde{\Delta}$

For $\alpha \in (0,1)$ let $\tilde{\tau}^\alpha$ be the stable regenerative set of index α, i.e.

- The (closure of the) image of the α-stable subordinator (increasing Lévy process with Lévy measure $x^{-1-\alpha} \mathrm{d}x$ on $\mathbb{R}^+$)
- The limit in law as $\epsilon \downarrow 0$ of the random sets $\{\epsilon \, \tau_n\}_{n \ge 0}$ in $(\star)$

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \ge 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \ge 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \; B_t = 0\}$

Attaching independent fair coins to the excursions $I \subseteq \tilde{\tau}^c$ yields $\tilde{\Delta}$

For $\alpha \in (0,1)$ let $\tilde{\tau}^\alpha$ be the stable regenerative set of index α, i.e.

- The (closure of the) image of the α-stable subordinator (increasing Lévy process with Lévy measure $x^{-1-\alpha}\mathrm{d}x$ on $\mathbb{R}^+$)
- The limit in law as $\epsilon \downarrow 0$ of the random sets $\{\epsilon \, \tau_n\}_{n \ge 0}$ in $(\star)$
- The zero level set of the Bessel(δ) process with $\delta = 2(1-\alpha)$

$$B_0^\delta = 0 \,, \qquad \mathrm{d}B_t^\delta = \mathrm{d}W_t + \frac{(\delta - 1)}{2B_t^\delta}\,\mathrm{d}t \quad \text{on } B_t^\delta > 0$$

The continuum α-copolymer model

For $\alpha = \frac{1}{2}$ we have BM $\{B_t\}_{t \ge 0}$, or better $\tilde{\Delta} = \{\tilde{\Delta}_t = \mathbf{1}_{\{B_t < 0\}}\}_{t \ge 0}$

It suffice to know the zero level set $\tilde{\tau} = \{t \in [0, \infty) : \; B_t = 0\}$

Attaching independent fair coins to the excursions $I \subseteq \tilde{\tau}^c$ yields $\tilde{\Delta}$

For $\alpha \in (0,1)$ let $\tilde{\tau}^\alpha$ be the stable regenerative set of index α, i.e.

- The (closure of the) image of the α-stable subordinator (increasing Lévy process with Lévy measure $x^{-1-\alpha} \mathrm{d}x$ on $\mathbb{R}^+$)
- The limit in law as $\epsilon \downarrow 0$ of the random sets $\{\epsilon \, \tau_n\}_{n \ge 0}$ in $(\star)$
- The zero level set of the Bessel(δ) process with $\delta = 2(1-\alpha)$

$$B_0^\delta = 0 \, , \qquad \mathrm{d}B_t^\delta = \mathrm{d}W_t + \frac{(\delta - 1)}{2 B_t^\delta} \, \mathrm{d}t \quad \text{on } B_t^\delta > 0$$

From $\tilde{\tau}^\alpha$ we obtain $(\tilde{\Delta}^\alpha = \{\tilde{\Delta}^\alpha_t\}_{t \ge 0}, \tilde{\mathbf{P}})$ (For $\alpha = \frac{1}{2}$ we recover BM)

The continuum α-copolymer model: free energy

The continuum charges $(\{\mathrm{d}\beta_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ are always white noise

The continuum α-copolymer model: free energy

The continuum charges $(\{\mathrm{d}\beta_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ are always white noise

For the α-continuum free energy, replace $\mathbf{1}_{\{B_s<0\}}$ by $\tilde{\Delta}_s^\alpha$:

$$\tilde{\mathrm{F}}^\alpha(\lambda, h) := \lim_{t\to\infty} \frac{1}{t}\, \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta}^\alpha \right)$$

$$\tilde{Z}_{t,\omega}^\alpha = \tilde{\mathbf{E}} \left[\exp \left(-2\lambda \int_0^t (\mathrm{d}\beta_s + h\, \mathrm{d}s)\, \tilde{\Delta}_s^\alpha \right) \right]$$

The continuum α-copolymer model: free energy

The continuum charges $(\{\mathrm{d}\beta_s\}_{s \ge 0}, \tilde{\mathbb{P}})$ are always white noise

For the α-continuum free energy, replace $\mathbf{1}_{\{B_s<0\}}$ by $\tilde{\Delta}_s^\alpha$:

$$\tilde{\mathrm{F}}^\alpha(\lambda, h) := \lim_{t\to\infty} \frac{1}{t}\,\tilde{\mathbb{E}}\left(\log \tilde{Z}^\alpha_{t,\beta}\right)$$

$$\tilde{Z}^\alpha_{t,\omega} = \tilde{\mathbf{E}}\left[\exp\left(-2\lambda\int_0^t (\mathrm{d}\beta_s + h\,\mathrm{d}s)\,\tilde{\Delta}_s^\alpha\right)\right]$$

Theorem [CG]

$\tilde{\mathrm{F}}^\alpha(\lambda, h)$ exists and is self-averaging ($\tilde{\mathbb{E}}$ can be dropped)

The continuum α-copolymer model: free energy

The continuum charges $(\{\mathrm{d}\beta_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ are always white noise

For the α-continuum free energy, replace $\mathbf{1}_{\{B_s<0\}}$ by $\tilde{\Delta}_s^\alpha$:

$$\tilde{\mathrm{F}}^\alpha(\lambda, h) := \lim_{t\to\infty} \frac{1}{t}\, \tilde{\mathbb{E}} \left(\log \tilde{Z}_{t,\beta}^\alpha \right)$$

$$\tilde{Z}_{t,\omega}^\alpha = \tilde{\mathbf{E}} \left[\exp \left(-2\lambda \int_0^t (\mathrm{d}\beta_s + h\, \mathrm{d}s)\, \tilde{\Delta}_s^\alpha \right) \right]$$

Theorem [CG]

$\tilde{\mathrm{F}}^\alpha(\lambda, h)$ exists and is self-averaging ($\tilde{\mathbb{E}}$ can be dropped)

- Non-trivial, highly technical proof (also for $\alpha = \frac{1}{2}$)

The continuum α-copolymer model: free energy

The continuum charges $(\{\mathrm{d}\beta_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ are always white noise

For the α-continuum free energy, replace $\mathbf{1}_{\{B_s<0\}}$ by $\tilde{\Delta}_s^\alpha$:

$$\tilde{\mathrm{F}}^\alpha(\lambda, h) := \lim_{t\to\infty} \frac{1}{t} \tilde{\mathbb{E}} \left(\log \tilde{Z}^\alpha_{t,\beta} \right)$$

$$\tilde{Z}^\alpha_{t,\omega} = \tilde{\mathbf{E}} \left[\exp \left(-2\lambda \int_0^t (\mathrm{d}\beta_s + h\, \mathrm{d}s)\, \tilde{\Delta}_s^\alpha \right) \right]$$

Theorem [CG]

$\tilde{\mathrm{F}}^\alpha(\lambda, h)$ exists and is self-averaging ($\tilde{\mathbb{E}}$ can be dropped)

- Non-trivial, highly technical proof (also for $\alpha = \frac{1}{2}$)
- Kingman's super-additive ergodic theorem for a modified $\tilde{Z}^\alpha_{t,\omega}$

The continuum α-copolymer model: free energy

The continuum charges $(\{\mathrm{d}\beta_s\}_{s\geq 0}, \tilde{\mathbb{P}})$ are always white noise

For the α-continuum free energy, replace $\mathbf{1}_{\{B_s<0\}}$ by $\tilde{\Delta}_s^\alpha$:

$$\tilde{\mathrm{F}}^\alpha(\lambda, h) := \lim_{t\to\infty} \frac{1}{t}\,\tilde{\mathbb{E}}\left(\log \tilde{Z}_{t,\beta}^\alpha\right)$$

$$\tilde{Z}_{t,\omega}^\alpha = \tilde{\mathbf{E}}\left[\exp\left(-2\lambda\int_0^t (\mathrm{d}\beta_s + h\,\mathrm{d}s)\,\tilde{\Delta}_s^\alpha\right)\right]$$

Theorem [CG]

$\tilde{\mathrm{F}}^\alpha(\lambda, h)$ exists and is self-averaging ($\tilde{\mathbb{E}}$ can be dropped)

- Non-trivial, highly technical proof (also for $\alpha = \frac{1}{2}$)
- Kingman's super-additive ergodic theorem for a modified $\tilde{Z}_{t,\omega}^\alpha$
- Continuity theory of Gaussian processes plays essential role

The continuum α-copolymer model: scaling limit

Scaling properties of β_s and $\tilde{\Delta}_s^\alpha \longrightarrow \tilde{\mathrm{F}}^\alpha(a\lambda, ah) = a^2 \tilde{\mathrm{F}}^\alpha(\lambda, h)$.

The continuum α-copolymer model: scaling limit

Scaling properties of β_s and $\tilde{\Delta}_s^\alpha$ $\longrightarrow$ $\tilde{\mathtt{F}}^\alpha(a\lambda, ah) = a^2 \tilde{\mathtt{F}}^\alpha(\lambda, h)$.

Therefore

$$\begin{aligned} \widetilde{\mathcal{L}} &= \{(\lambda, h) : \tilde{\mathtt{F}}(\lambda, h) > 0\} \\ &= \{(\lambda, h) : \ h < \tilde{m}^\alpha \lambda\} \end{aligned} \qquad \begin{aligned} \widetilde{\mathcal{D}} &= \{(\lambda, h) : \tilde{\mathtt{F}}(\lambda, h) = 0\} \\ &= \{(\lambda, h) : \ h \geq \tilde{m}^\alpha \lambda\} \end{aligned}$$

The continuum α-copolymer model: scaling limit

Scaling properties of β_s and $\tilde{\Delta}_s^\alpha$ $\longrightarrow$ $\tilde{\mathrm{F}}^\alpha(a\lambda, ah) = a^2 \tilde{\mathrm{F}}^\alpha(\lambda, h)$.

Therefore

$$\widetilde{\mathcal{L}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} = \{(\lambda, h) : \; h < \tilde{m}^\alpha \lambda\} \qquad \widetilde{\mathcal{D}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\} = \{(\lambda, h) : \; h \geq \tilde{m}^\alpha \lambda\}$$

Theorem [CG] (For any discrete α-copolymer model)

For all $\lambda, h > 0$ and $\epsilon \in (0,1)$ there exists $a_0 > 0$ s.t. for all $a < a_0$

$$\tilde{\mathrm{F}}^\alpha\left(\frac{\lambda}{1+\epsilon}, \frac{h}{1-\epsilon}\right) \leq \frac{\mathrm{F}\,(a\lambda, ah)}{a^2} \leq \tilde{\mathrm{F}}^\alpha\,((1+\epsilon)\lambda, (1-\epsilon)h)$$

The continuum α-copolymer model: scaling limit

Scaling properties of β_s and $\tilde{\Delta}_s^\alpha \longrightarrow \tilde{\mathrm{F}}^\alpha(a\lambda, ah) = a^2 \tilde{\mathrm{F}}^\alpha(\lambda, h)$.

Therefore

$$\widetilde{\mathcal{L}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} \qquad \widetilde{\mathcal{D}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\}$$

$$= \{(\lambda, h) : \ h < \tilde{m}^\alpha \lambda\} \qquad = \{(\lambda, h) : \ h \geq \tilde{m}^\alpha \lambda\}$$

Theorem [CG] (For any discrete α-copolymer model)

For all $\lambda, h > 0$ and $\epsilon \in (0,1)$ there exists $a_0 > 0$ s.t. for all $a < a_0$

$$\tilde{\mathrm{F}}^\alpha\left(\frac{\lambda}{1+\epsilon}, \frac{h}{1-\epsilon}\right) \leq \frac{\mathrm{F}\,(a\lambda, ah)}{a^2} \leq \tilde{\mathrm{F}}^\alpha\,((1+\epsilon)\lambda, (1-\epsilon)h)$$

Corollary

$$\lim_{a \downarrow 0} \frac{\mathrm{F}\,(a\lambda, ah)}{a^2} = \tilde{\mathrm{F}}^\alpha(\lambda, h) \qquad \lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m}^\alpha$$

The continuum α-copolymer model: scaling limit

Scaling properties of β_s and $\tilde{\Delta}_s^\alpha \longrightarrow \tilde{\mathrm{F}}^\alpha(a\lambda, ah) = a^2 \tilde{\mathrm{F}}^\alpha(\lambda, h)$.

Therefore

$$\widetilde{\mathcal{L}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) > 0\} = \{(\lambda, h) : \; h < \tilde{m}^\alpha \lambda\}$$

$$\widetilde{\mathcal{D}} = \{(\lambda, h) : \tilde{\mathrm{F}}(\lambda, h) = 0\} = \{(\lambda, h) : \; h \geq \tilde{m}^\alpha \lambda\}$$

Theorem [CG] (For any discrete α-copolymer model)

For all $\lambda, h > 0$ and $\epsilon \in (0,1)$ there exists $a_0 > 0$ s.t. for all $a < a_0$

$$\tilde{\mathrm{F}}^\alpha\left(\frac{\lambda}{1+\epsilon}, \frac{h}{1-\epsilon}\right) \leq \frac{\mathrm{F}\,(a\lambda, ah)}{a^2} \leq \tilde{\mathrm{F}}^\alpha\left((1+\epsilon)\lambda, (1-\epsilon)h\right)$$

Corollary

$$\lim_{a \downarrow 0} \frac{\mathrm{F}\,(a\lambda, ah)}{a^2} = \tilde{\mathrm{F}}^\alpha(\lambda, h) \qquad \lim_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} = \tilde{m}^\alpha \in \left[\tfrac{1}{1+\alpha}, 1\right)$$

Outline

1. The basic copolymer model

2. The free energy

3. Generalized copolymer models

4. Strategy of the proof

Strategy of the proof

Let $f^i(\lambda, h)$ be the free energy of an Hamiltonian $H_N^i = H_N^i(\lambda, h)$

$$f^i(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log \mathbf{E}\left[\exp\left(-H_N^i\right)\right]\right)$$

Strategy of the proof

Let $f^i(\lambda, h)$ be the free energy of an Hamiltonian $H_N^i = H_N^i(\lambda, h)$

$$f^i(\lambda, h) := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left(\log \mathbf{E}\left[\exp\left(-H_N^i\right)\right]\right)$$

We write $H_N^i \simeq H_N^j$ if for all $\lambda, h > 0$, $\epsilon \in (0,1)$ one has for $a \ll 1$

$$\frac{1}{a^2} f^j\left(\frac{a\lambda}{1+\epsilon}, \frac{ah}{1-\epsilon}\right) \le \frac{f^i\left(a\lambda, ah\right)}{a^2} \le \frac{f^j\left((1+\epsilon)a\lambda, (1-\epsilon)ah\right)}{a^2}$$

Strategy of the proof

Let $f^i(\lambda, h)$ be the free energy of an Hamiltonian $H_N^i = H_N^i(\lambda, h)$

$$f^i(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \left(\log \mathbf{E} \left[\exp\left(-H_N^i \right) \right] \right)$$

We write $H_N^i \simeq H_N^j$ if for all $\lambda, h > 0$, $\epsilon \in (0,1)$ one has for $a \ll 1$

$$\frac{1}{a^2} f^j \left(\frac{a\lambda}{1+\epsilon}, \frac{ah}{1-\epsilon} \right) \le \frac{f^i\,(a\lambda, ah)}{a^2} \le \frac{f^j\,((1+\epsilon)a\lambda, (1-\epsilon)ah)}{a^2}$$

Proof in four steps: $H_N^0 \simeq H_N^1 \simeq H_N^2 \simeq H_N^3 \simeq H_N^4$ where

$$-H_N^0 := -2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n \qquad -H_N^4 := -2\lambda \int_0^t (\mathrm{d}\beta_s + h\,\mathrm{d}s)\,\tilde{\Delta}_s^\alpha$$

Strategy of the proof

Let $f^i(\lambda, h)$ be the free energy of an Hamiltonian $H_N^i = H_N^i(\lambda, h)$

$$f^i(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\log \mathbf{E}\left[\exp\left(-H_N^i\right)\right]\right)$$

We write $H_N^i \simeq H_N^j$ if for all $\lambda, h > 0$, $\epsilon \in (0,1)$ one has for $a \ll 1$

$$\frac{1}{a^2} f^j\left(\frac{a\lambda}{1+\epsilon}, \frac{ah}{1-\epsilon}\right) \le \frac{f^i(a\lambda, ah)}{a^2} \le \frac{f^j((1+\epsilon)a\lambda, (1-\epsilon)ah)}{a^2}$$

Proof in four steps: $\quad H_N^0 \simeq H_N^1 \simeq H_N^2 \simeq H_N^3 \simeq H_N^4 \quad$ where

$$-H_N^0 := -2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n \qquad -H_N^4 := -2\lambda \int_0^t (\mathrm{d}\beta_s + h\,\mathrm{d}s)\,\tilde{\Delta}_s^\alpha$$

$H_N^i \simeq H_N^j$ if for all $C > 0$ and $a \ll 1$ [also with $i \leftrightarrow j$]

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathbf{E}\left[\exp\left(-C\left(H_N^i(a\lambda, ah) - H_N^j(a\lambda, (1-\epsilon)ah)\right)\right)\right] \le 0$$

Strategy of the proof

1. Coarse-graining of the renewal process

Strategy of the proof

1. Coarse-graining of the renewal process
2. From discrete charges to the white noise

Strategy of the proof

1. Coarse-graining of the renewal process
2. From discrete charges to the white noise
3. From the renewal process to the regenerative set

Strategy of the proof

1. Coarse-graining of the renewal process
2. From discrete charges to the white noise
3. From the renewal process to the regenerative set
4. Coarse-graining of the regenerative set

Happy birthday, Erwin!