A Polymer in a Multi-Interface Medium

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Università di Roma 3 $\sim$ July 21, 2009

References

- [CP1] F. Caravenna and N. Pétrélis
  A polymer in a multi-interface medium
  AAP (2009), to appear

- [CP2] F. Caravenna and N. Pétrélis
  Depinning of a polymer in a multi-interface medium
  preprint (2009) [arXiv.org: 0901.2902]

Outline

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What is a polymer?

A polymer is a large molecule composed of repeating smaller units, called monomers, linked together to form a chain.

Polymer configurations $\longleftrightarrow$ Trajectories of a random process

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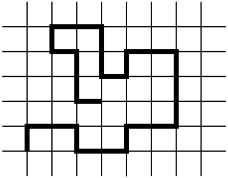
Self-avoiding walks

Simple symmetric random walk on $\mathbb{Z}^d$ conditioned to visit each site at most once $\longrightarrow$ very difficult!



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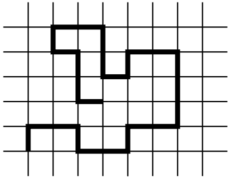
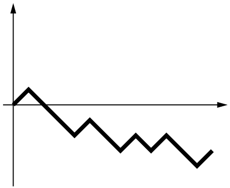


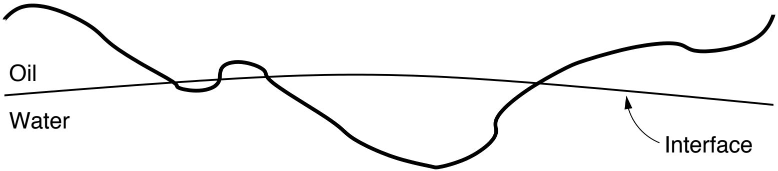
Directed walks

Deterministic component: (n, S_n) where S_n is the simple symmetric random walk on $\mathbb{Z}^{d-1}$ $\longrightarrow$ tractable models (interaction with the environment)



Copolymer and pinning at an interface

A polymer interacting with two solvents and with the interface that separates them:



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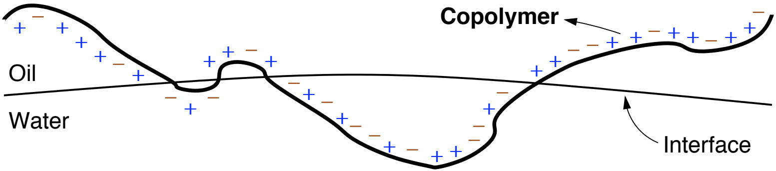

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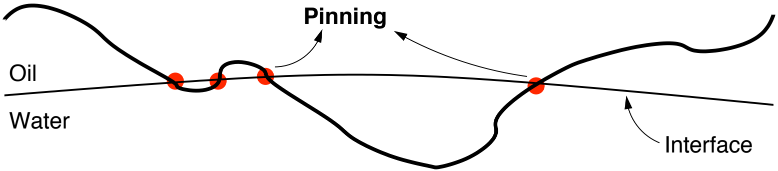

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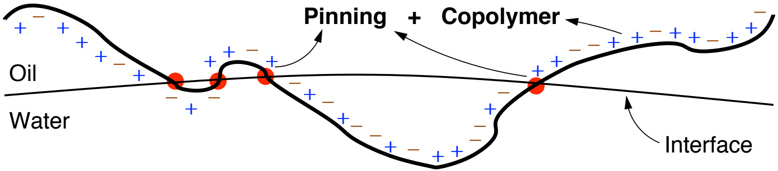

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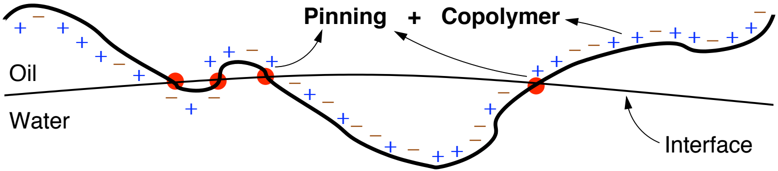

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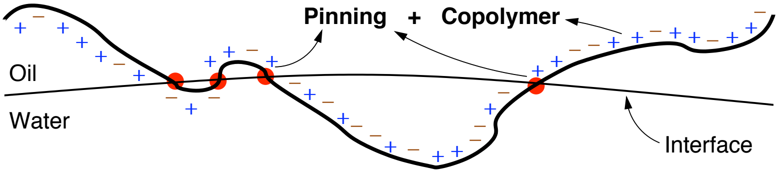

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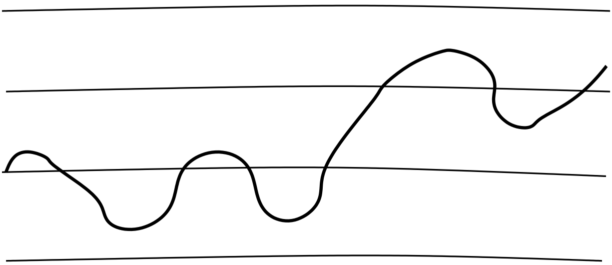
Recent results: very good comprehension (survey: [Giacomin '07])

Multi-interface media

More general environments: a multi-interface medium



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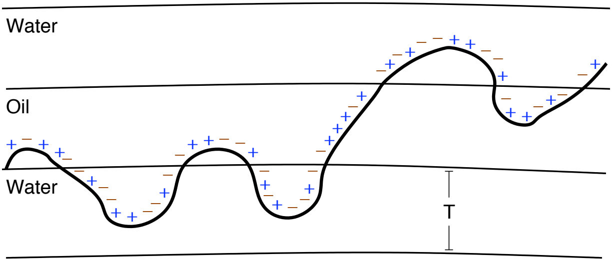

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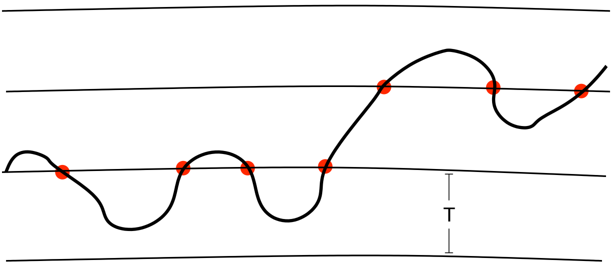

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- We focus on the pinning case: homogeneous interaction (attractive or repulsive), general T_N. Path behavior ?

Polymer in a slit

Recent physical literature:

Polymer confined between two walls and interacting with them

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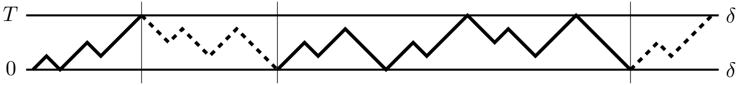

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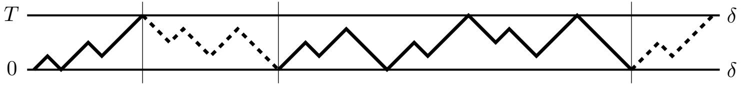

Attraction/repulsion of interfaces by polymers

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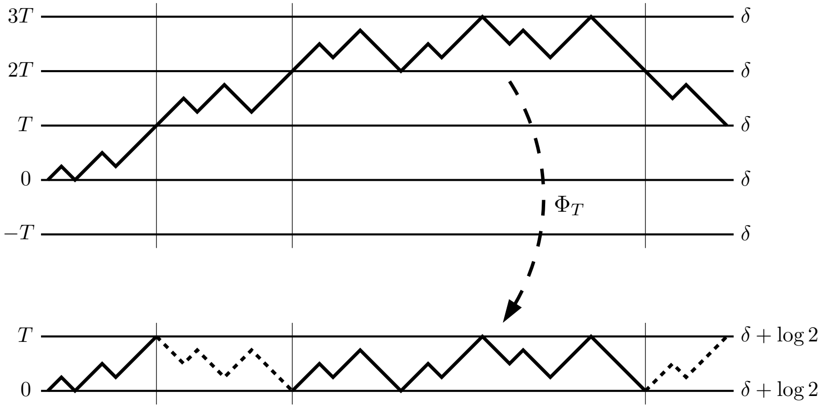

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Definition of the model

Ingredients:

- Simple symmetric random walk $S = \{S_n\}_{n\ge 0}$ on $\mathbb{Z}$:

$$S_0 := 0\,, \qquad S_n := X_1 + \ldots + X_n\,,$$

with $\{X_i\}_i$ i.i.d. and $P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$.

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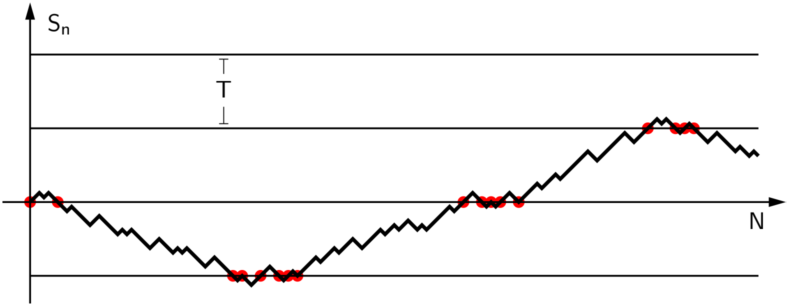

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Penalisation of a random walk

The free energy

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If $\phi(\delta, \{T_n\}_n)$ is non-analytic in $\delta \in \mathbb{R}$ there is a phase transition

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Theorem ([CP1])

$$\phi(\delta, \{T_n\}_n) \;=\; \phi(\delta, T_\infty) \;=\; \begin{cases} \left(Q_{T_\infty}\right)^{-1}(e^{-\delta}) & \textit{if } T_\infty < +\infty \\ \left(Q_\infty\right)^{-1}(e^{-\delta} \wedge 1) & \textit{if } T_\infty = +\infty \end{cases}$$

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- $\phi'(\delta, T_\infty) > 0$ for every $\delta \in \mathbb{R}$: positive density of contacts

$L_N \sim \phi'(\delta, T_\infty) \cdot N$ (conj. diffusive behavior of S_N)

The free energy: further results

If $T_N \to \infty$

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- Same path behavior? NO!
- If $\delta < 0$ then $Z_{N,\delta}^{T_N} = \exp(o(N))$. In fact

$$Z_{N,\delta}^{T_N} \approx \frac{(const.)}{N^{3/2}} f\left(\frac{\sqrt{N}}{T_N}\right) g\left(\frac{N^{1/3}}{T_N}\right),$$

improving known results for the polymer in a slit.

Path results: the attractive case $\delta > 0$

Assume $\delta > 0$ and $T_N \to \infty$. Since $\phi'(\delta, \infty) > 0$, the polymer visits the interfaces a positive fraction of times: $L_N \sim \phi'(\delta, \infty) \cdot N$.

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$$\frac{S_N}{T_N} \Longrightarrow S_\Gamma \qquad \Gamma \sim \text{Poisson}(f(\delta, \zeta))$$

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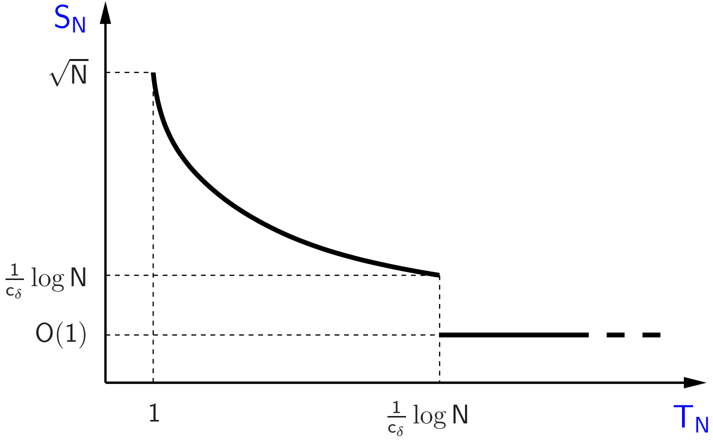

- Sub-diffusive scaling ($T_N \to \infty$)
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with $Z \sim N(0,1)$

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- If $T_N \sim (const.)N^{1/3} \longrightarrow O(1)$ visited interfaces.
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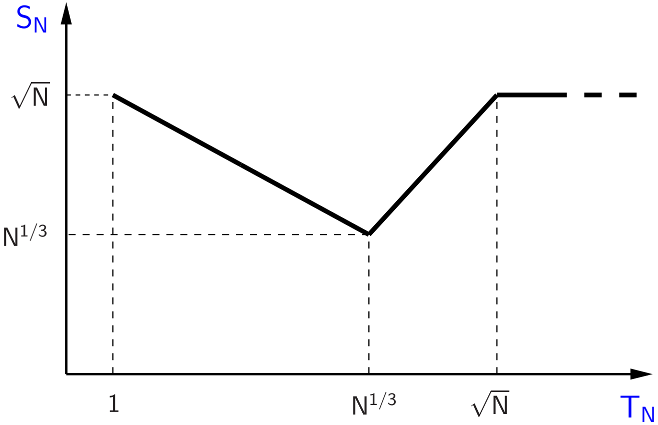

- Sub-diffusive if $1 \ll T_N \ll \sqrt{N}$
- Transitions $T_N \approx N^{1/3}, \sqrt{N}$

Outline

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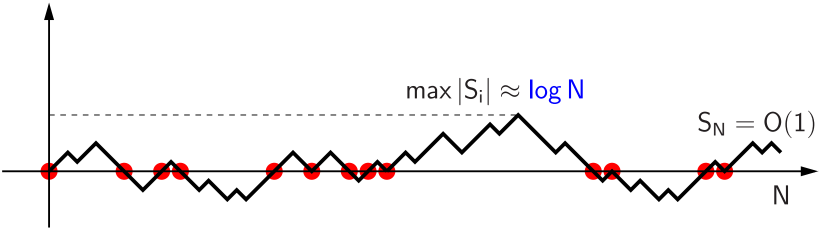

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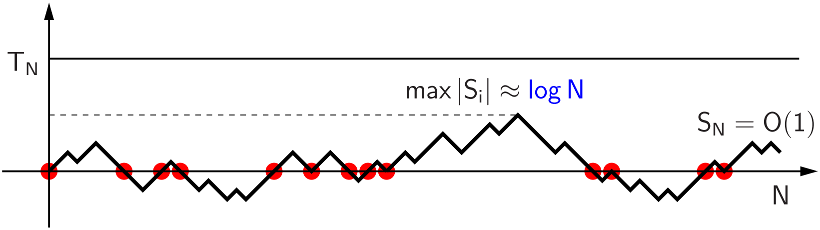

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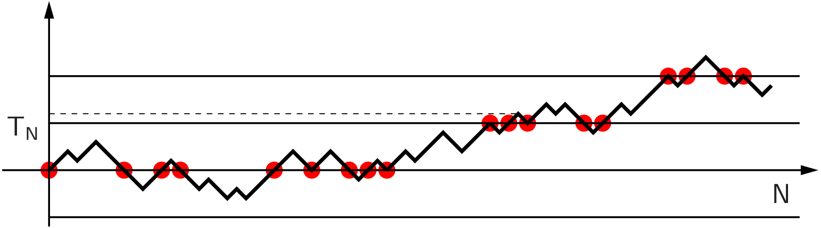

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- If $T_N \ll \log N$ it is worth to visit different interfaces: sub-diffusive behavior

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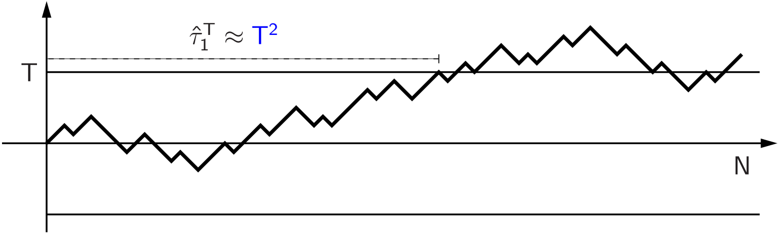

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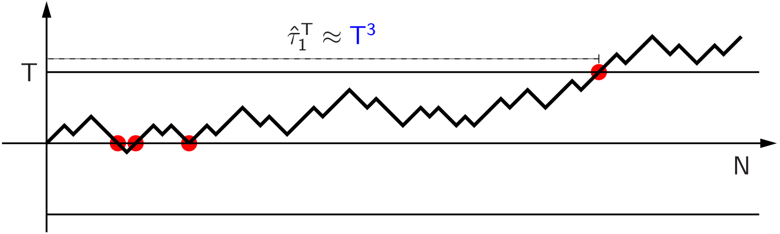

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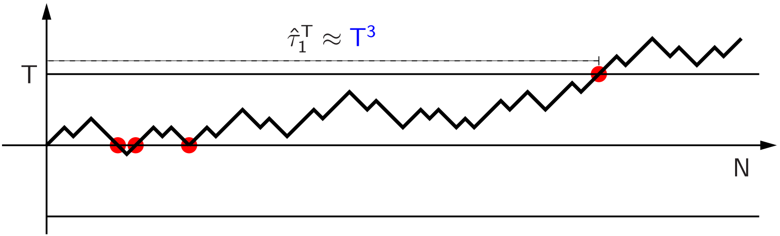

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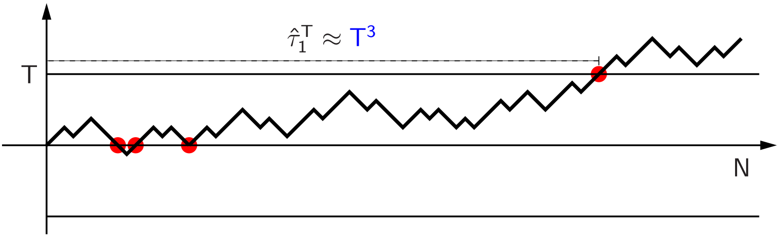

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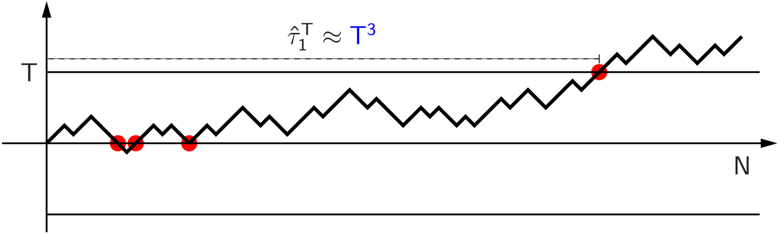

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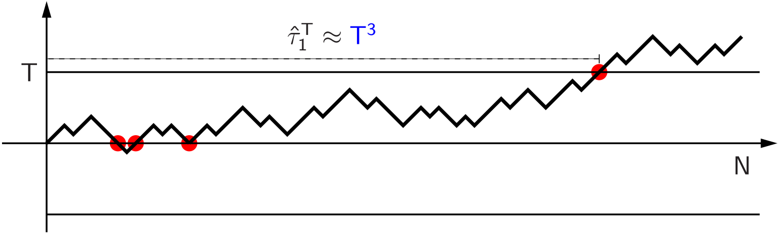

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A renewal theory approach

Let $\tau_1^T, \tau_2^T, \tau_3^T \ldots$ be the points at which S_n visits an interface

$$\tau_{k+1}^T \;:=\; \inf\left\{ n > \tau_k^T \,:\; S_n - S_{\tau_k^T} \in \{-T, 0, T\} \right\} \qquad (T \text{ is fixed})$$

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However we want to study $\mathbf{P}_{N,\delta}^{T_N}$ with T_N varying with N

1. Study $\tau^{T_N} \cap [0, N]$ under $\mathcal{P}_{\delta, T_N}$

Strategy of the proof

The law of $\tau^T \cap [0, N] = \{\tau_1^T, \ldots, \tau_{L_N}^T\}$ is the same

$$\text{under} \quad \mathbf{P}_{N,\delta}^T\big(\,\cdot\,\big|\,N \in \tau^T\big) \quad \text{and} \quad \mathcal{P}_{\delta,T}\big(\,\cdot\,\big|\,N \in \tau^T\big)$$

$\mathcal{P}_{\delta,T}$ does not depend explicitly on N $\quad (\mathbf{P}_{N,\delta}^{T_N} \Longrightarrow \mathcal{P}_{\delta,T})$

However we want to study $\mathbf{P}_{N,\delta}^{T_N}$ with T_N varying with N

1. Study $\tau^{T_N} \cap [0, N]$ under $\mathcal{P}_{\delta,T_N}$
2. Transfer the results to $\mathcal{P}_{\delta,T_N}(\,\cdot\,|\,N \in \tau^{T_N})$ that is to $\mathbf{P}_{N,\delta}^{T_N}(\,\cdot\,|\,N \in \tau^{T_N})$ (hard part)

Strategy of the proof

The law of $\tau^T \cap [0, N] = \{\tau_1^T, \dots, \tau_{L_N}^T\}$ is the same

$$\text{under} \quad \mathbf{P}_{N,\delta}^T\big(\,\cdot\, \big| N \in \tau^T \big) \quad \text{and} \quad \mathcal{P}_{\delta,T}\big(\,\cdot\, \big| N \in \tau^T \big)$$

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1. Study $\tau^{T_N} \cap [0, N]$ under $\mathcal{P}_{\delta,T_N}$
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3. Remove the conditioning on $\{N \in \tau^{T_N}\}$

Strategy of the proof

The law of $\tau^T \cap [0,N] = \{\tau_1^T, \ldots, \tau_{L_N}^T\}$ is the same

$$\text{under} \quad \mathbf{P}_{N,\delta}^T\big(\,\cdot\,\big|\,N \in \tau^T\big) \quad \text{and} \quad \mathcal{P}_{\delta,T}\big(\,\cdot\,\big|\,N \in \tau^T\big)$$

$\mathcal{P}_{\delta,T}$ does not depend explicitly on N $\quad (\mathbf{P}_{N,\delta}^{T_N} \Longrightarrow \mathcal{P}_{\delta,T})$

However we want to study $\mathbf{P}_{N,\delta}^{T_N}$ with T_N varying with N

1. Study $\tau^{T_N} \cap [0,N]$ under $\mathcal{P}_{\delta,T_N}$
2. Transfer the results to $\mathcal{P}_{\delta,T_N}(\,\cdot\,|\,N \in \tau^{T_N})$ that is to $\mathbf{P}_{N,\delta}^{T_N}(\,\cdot\,|\,N \in \tau^{T_N})$ (hard part)
3. Remove the conditioning on $\{N \in \tau^{T_N}\}$

- Good estimates on $q_T(n)$ and on the free energy $\phi(\delta, T)$

Strategy of the proof

The law of $\tau^T \cap [0, N] = \{\tau_1^T, \dots, \tau_{L_N}^T\}$ is the same

under $\mathbf{P}_{N,\delta}^T(\,\cdot\,|\, N \in \tau^T)$ and $\mathcal{P}_{\delta,T}(\,\cdot\,|\, N \in \tau^T)$

$\mathcal{P}_{\delta,T}$ does not depend explicitly on N $(\mathbf{P}_{N,\delta}^{T_N} \Longrightarrow \mathcal{P}_{\delta,T})$

However we want to study $\mathbf{P}_{N,\delta}^{T_N}$ with T_N varying with N

1. Study $\tau^{T_N} \cap [0, N]$ under $\mathcal{P}_{\delta,T_N}$
2. Transfer the results to $\mathcal{P}_{\delta,T_N}(\,\cdot\,|\, N \in \tau^{T_N})$ that is to $\mathbf{P}_{N,\delta}^{T_N}(\,\cdot\,|\, N \in \tau^{T_N})$ (hard part)
3. Remove the conditioning on $\{N \in \tau^{T_N}\}$

- Good estimates on $q_T(n)$ and on the free energy $\phi(\delta, T)$
- Uniform renewal theorems

Thanks.