THE STRONG RENEWAL THEOREM

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Abstract. We consider real random walks with positive increments (renewal processes) in the domain of attraction of a stable law with index $\alpha \in (0, 1)$. The famous local renewal theorem of Garsia and Lamperti [GL63], also called *strong renewal theorem*, is known to hold in complete generality only for $\alpha > \frac{1}{2}$. Understanding when the strong renewal theorem holds for $\alpha \leq \frac{1}{2}$ is a long-standing problem, with sufficient conditions given by Williamson [W68], Doney [D97] and Chi [C15, C13]. In this paper we give a complete solution, providing explicit necessary and sufficient conditions. We also show that these conditions fail to be sufficient if the random walk is allowed to take negative values.

An analogous result has been independently and simultaneously proved by Doney [D15].

1. Introduction

We use the notations $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given two functions $f, g : [0, \infty) \to (0, \infty)$ we write $f \sim g$ to mean $\lim_{s\to\infty} f(s)/g(s) = 1$.

We denote by $\mathcal{R}_\gamma$ the space of regularly varying functions with index $\gamma \in \mathbb{R}$, that is $f \in \mathcal{R}_\gamma$ if and only if $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\gamma$ for all $\lambda \in (0, \infty)$. Functions in $\mathcal{R}_0$ are called slowly varying. Note that $f \in \mathcal{R}_\gamma$ if and only if $f(x) = x^\gamma \ell(x)$ for some slowly varying function $\ell \in \mathcal{R}_0$. We refer to [BGT89] for more details.

1.1. **Main result.**

We fix a probability F on $[0, \infty)$ and we let $X, (X_i)_{i\in\mathbb{N}}$ be independent and identically distributed (i.i.d.) random variables with law F. The associated random walk (renewal process) will be denoted by $S_n := X_1 + \ldots + X_n$, with $S_0 := 0$. We say that F is *arithmetic* if it is supported by $h\mathbb{Z}$ for some $h > 0$, and the maximal value of $h > 0$ with this property is called the *arithmetic span* of F.

Our key assumption is that there exist $\alpha \in (0, 1)$ and $A \in \mathcal{R}_\alpha$ such that

$$\overline{F}(x) := F((x, \infty)) = \mathrm{P}(X > x) \sim \frac{1}{A(x)} \qquad \text{as} \quad x \to \infty\,. \tag{1.1}$$

We can write $A(x) = L(x)\, x^\alpha$, for a suitable slowly varying function $L \in \mathcal{R}_0$. By [BGT89, §1.3.2], we may take $A : [0, \infty) \to (0, \infty)$ to be differentiable, strictly increasing and

$$A'(s) \sim \alpha \frac{A(s)}{s}\,, \qquad \text{as } s \to \infty\,. \tag{1.2}$$

Let us introduce the renewal measure

$$U(\mathrm{d}x) := \sum_{n\geq 0} \mathrm{P}(S_n \in \mathrm{d}x)\,, \tag{1.3}$$

so that $U(I)$ is the expected number of variables S_n that fall inside $I \subseteq \mathbb{R}$. It is well known [BGT89, Theorem 8.6.3] that (1.1) implies the following infinite-mean version of the renewal

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theorem, with $\mathsf{C} = \mathsf{C}(\alpha) = \frac{\alpha \sin(\pi\alpha)}{\pi}$:

$$U([0,x]) \sim \frac{\mathsf{C}}{\alpha} A(x) \quad \text{as} \quad x \to \infty . \tag{1.4}$$

Let us introduce the shorthand

$$I = (-h, 0] \qquad \text{where} \qquad h := \begin{cases} \text{arithmetic span of } F & (\text{if } F \text{ is arithmetic}) \\ \text{any fixed number} > 0 & (\text{if } F \text{ is non-arithmetic}) . \end{cases} \tag{1.5}$$

Recalling (1.2), it is natural to look for a local version of (1.4), namely

$$U(x+I) = U((x-h, x]) \sim \mathsf{C}\, h \, \frac{A(x)}{x} \quad \text{as} \quad x \to \infty . \tag{1.6}$$

This relation, called *strong renewal theorem (SRT)*, is known to hold in complete generality under (1.1) when $\alpha > \frac{1}{2}$, cf. [GL63, W68, E70]. On the other hand, when $\alpha \le \frac{1}{2}$ there are examples of F satisfying (1.1) but not (1.6). It is therefore of great theoretical and practical interest to find conditions on F, in addition to (1.1), ensuring the validity of (1.6) for $\alpha \le \frac{1}{2}$.

Let us introduce the function

$$\mathsf{r}(x) := \frac{F(x+I)}{\overline{F}(x)/x} \sim x\, A(x)\, F(x+I) . \tag{1.7}$$

By (1.1), one expects $\mathsf{r}(x)$ to be bounded for "typical" values of x, although there might be exceptional values for which it is much larger. It is by now a classical result that a sufficient condition for the SRT (1.6) is the global boundedness of r:

$$\sup_{x \ge 0} \mathsf{r}(x) < \infty , \tag{1.8}$$

as proved by Doney [D97] in the arithmetic case (extending Williamson [W68], who assumed $\alpha > \frac{1}{4}$) and by Vatutin and Topchii [VT13] in the non-arithmetic case.

More recently [C15, C13], Chi showed that (1.8) can be substantially relaxed, through suitable integral criteria. To mention the simplest [C13, Theorem 1.1], if one defines

$$\mathsf{R}_T(a,b) := \int_a^b \big(\mathsf{r}(y) - T\big)^+ \, \mathrm{d}y , \qquad \text{where} \quad z^+ := \max\{z, 0\} , \tag{1.9}$$

a sufficient condition for the SRT (1.6), for $\alpha \le \frac{1}{2}$, is that for some $\eta \in (0,1)$, $T \in [0,\infty)$

$$\mathsf{R}_T((1-\eta)x, x) \underset{x\to\infty}{=} \begin{cases} o(A(x)^2) & \text{if } \alpha < \frac{1}{2} \\ o\Big(\frac{A(x)^2}{u(x)}\Big) & \text{if } \alpha = \frac{1}{2} \end{cases} \qquad \text{where} \quad u(x) := \int_1^x \frac{A(s)^2}{s^2} \, \mathrm{d}s . \tag{1.10}$$

This clearly improves (1.8) (just note that $\mathsf{R}_T(a,b) \equiv 0$ for $T = \sup_{x\ge 0} \mathsf{r}(x)$). We refer to [C15, C13] for a variety of more general (and more technical) sufficient conditions.

Integral criteria like (1.10) are appealing, because they are very explicit and can be easily checked in concrete examples. It is natural to ask whether more refined integral criteria can provide *necessary and sufficient* conditions for the SRT (1.6). Our main result shows that this is indeed the case, giving a complete solution to the SRT problem.

Theorem 1.1 (Strong Renewal Theorem). *Let F be a probability on $[0,\infty)$ satisfying (1.1) with $A \in \mathcal{R}_\alpha$, for $\alpha \in (0,1)$. Define $I = (-h, 0]$ with $h > 0$ as in (1.5).*

- *If $\alpha > \frac{1}{2}$, the SRT (1.6) holds with no extra assumption on F.*

- *If $\alpha \le \frac{1}{2}$, the SRT* (1.6) *holds if and only if*

$$\lim_{\eta\to 0}\left\{\limsup_{x\to\infty}\frac{1}{A(x)^2}\left(\int_1^{\eta x}\frac{A(s)^2}{s}\,\mathsf{r}(x-s)\,\mathrm{d}s\right)\right\}=0\,. \tag{1.11}$$

For $\alpha < \frac{1}{2}$, setting $\mathsf{R}_0(a,b) := \int_a^b \mathsf{r}(y)\,\mathrm{d}y$ (see (1.9)*), relation* (1.11) *is equivalent to*

$$\lim_{\eta\to 0}\left\{\limsup_{x\to\infty}\frac{1}{A(x)^2}\left(\int_1^{\eta x}\frac{A(s)^2}{s^2}\,\mathsf{R}_0(x-s,x)\,\mathrm{d}s\right)\right\}=0\,, \tag{1.12}$$

while for $\alpha = \frac{1}{2}$ relation (1.12) *is stronger than (i.e. it implies)* (1.11).

In Section 3 we reformulate conditions (1.11)-(1.12) more explicitly in terms of the probability F (see Lemma 3.1). We also present an overview on the strategy of the proof of Theorem 1.1, which is a refinement of the probabilistic approach of Chi [C15, C13] and allows to treat the arithmetic and non-arithmetic cases in a unified way, avoiding characteristic functions (except for their implicit use in local limit theorems).

In the rest of the introduction, after some remarks, we derive some consequences of conditions (1.11)-(1.12), see §1.2. Then we discuss the case of two-sided random walks, showing that *condition* (1.11) *is not sufficient for the SRT*, see §1.3.

Remark 1.2. A result analogous to Theorem 1.1 has been independently and simultaneously proved by Doney [D15].

Remark 1.3. When $\alpha > \frac{1}{2}$ condition (1.11) follows from (1.1) (see the Appendix §A.4). As a consequence, we can reformulate Theorem 1.1 as follows: *assuming* (1.1)*, condition* (1.11) *is necessary and sufficient for the SRT* (1.6) *for any $\alpha \in (0,1)$.* ☐

Remark 1.4. The double limit $x \to \infty$ followed by $\eta \to 0$ can be reformulated as follows: relations (1.11)-(1.12) are equivalent to asking that, for any fixed function $g(x) = o(x)$,

$$\int_1^{g(x)}\frac{A(s)^2}{s}\,\mathsf{r}(x-s)\,\mathrm{d}s \underset{x\to\infty}{=} o\big(A(x)^2\big)\,, \tag{1.11'}$$

$$\int_1^{g(x)}\frac{A(s)^2}{s^2}\,\mathsf{R}_0(x-s,x)\,\mathrm{d}s \underset{x\to\infty}{=} o\big(A(x)^2\big)\,, \tag{1.12'}$$

as an easy contradiction argument shows. ☐

Remark 1.5. Relations (1.11)-(1.12) contain no cutoff parameter T, unlike (1.10). This can be introduced replacing $\mathsf{r}(x-s)$ by $(\mathsf{r}(x-s)-T)^+$ and $\mathsf{R}_0(x-s,x)$ by $\mathsf{R}_T(x-s,x)$, respectively, because (1.11)-(1.12) are equivalent to the following:

$$\exists T\in[0,\infty): \qquad \lim_{\eta\to 0}\left\{\limsup_{x\to\infty}\frac{1}{A(x)^2}\int_1^{\eta x}\left(\frac{A(s)^2}{s}\big(\mathsf{r}(x-s)-T\big)^+\right)\mathrm{d}s\right\}=0\,, \tag{1.11''}$$

$$\exists T\in[0,\infty): \qquad \lim_{\eta\to 0}\left\{\limsup_{x\to\infty}\frac{1}{A(x)^2}\int_1^{\eta x}\left(\frac{A(s)^2}{s^2}\,\mathsf{R}_T(x-s,x)\right)\mathrm{d}s\right\}=0\,. \tag{1.12''}$$

This is easily checked, by writing

$$\mathsf{r}(x-s)\le T+\big(\mathsf{r}(x-s)-T\big)^+\,, \qquad \mathsf{R}_0(x-s,x)\le Ts+\mathsf{R}_T(x-s,x)\,,$$

and noting that the terms T and Ts give a negligible contribution to (1.11) and (1.12), respectively, because by Karamata's Theorem [BGT89, Proposition 1.5.8]

$$\int_1^{\eta x}\frac{A(s)^2}{s}\,\mathrm{d}s\sim\frac{1}{2\alpha}A(\eta x)^2\sim\frac{1}{2\alpha}\eta^{2\alpha}A(x)^2 \qquad \text{as } x\to\infty\,. \tag{1.13}$$

Nothing is really gained with the cutoff T, since relations (1.11")-(1.12") are *equivalent* to the $T = 0$ versions (1.11)-(1.12). However, in concrete examples it is often convenient to use (1.11")-(1.12"), because they allow to focus one's attention on the "large" values of r. $\square$

1.2. **Some consequences.** An immediate corollary of Theorem 1.1 is the sufficiency of conditions (1.8) and (1.10) for the SRT (1.6).

- For condition (1.8), note that it implies (1.11), thanks to (1.13).
- For condition (1.10), note that it implies (1.12"), since $\mathsf{R}_T(x-s,x) \le \mathsf{R}_T((1-\eta)x,x)$ and, moreover, $\lim_{x\to\infty} u(x) = \int_1^\infty (A(s)^2/s^2)\,\mathrm{d}s < \infty$ for $\alpha < \frac{1}{2}$, because $A(s)^2/s^2$ is regularly varying with index $2\alpha - 2 < -1$ (see [BGT89, Proposition 1.5.10]).

More generally, all the sufficient conditions presented in [C15, C13] can be easily derived from Theorem 1.1. We present alternative sufficient conditions, in terms of "smoothness" properties of F. Observe that, if (1.1) holds, for any $s_x = o(x)$ one has

$$\frac{F((x, x+s_x])}{F((x,\infty))} = \frac{\mathrm{P}(X \in (x, x+s_x])}{\mathrm{P}(X \in (x,\infty))} \xrightarrow[x\to\infty]{} 0\,. \tag{1.14}$$

Our next result shows that a suitable polynomial rate of decay in (1.14) ensures the validity of the SRT (1.6). (Analogous conditions, in a different context, appear in [CSZ15+]).

Proposition 1.6. *Let F be a probability on $[0,\infty)$ satisfying* (1.1) *for some $\alpha \in (0, \frac{1}{2}]$. A sufficient condition for the SRT* (1.6) *is that there is $\varepsilon > 0$ such that, for any $1 \le s_x = o(x)$,*

$$\frac{F((x, x+s_x])}{F((x,\infty))} = O\bigg(\Big(\frac{s_x}{x}\Big)^{1-2\alpha+\varepsilon}\bigg) \qquad \textit{as} \quad x \to \infty\,. \tag{1.15}$$

We finally focus on the case $\alpha = \frac{1}{2}$. Our next result unravels this case, by stating under which conditions on $A(x)$ the SRT (1.6) holds with no further assumption on F than (1.1) (like it happens for $\alpha > \frac{1}{2}$). Given a function L, let us define

$$L^*(x) := \sup_{1\le s\le x} L(s)\,. \tag{1.16}$$

Theorem 1.7 (Case $\alpha = \frac{1}{2}$)**.** *Let F be a probability on $[0,\infty)$ satisfying* (1.1) *with $\alpha = \frac{1}{2}$, that is $A \in \mathcal{R}_{1/2}$. Write $A(x) = L(x)\sqrt{x}$, where $L \in \mathcal{R}_0$ is slowly varying.*

- *If $A(x)$ satisfies the following condition:*

$$L^*(x) \underset{x\to\infty}{=} O(L(x))\,, \tag{1.17}$$

  the SRT (1.6) *holds with no extra assumption on F.*
- *If condition* (1.17) *fails, there are examples of F for which the SRT* (1.6) *fails.*

Remark 1.8. Condition (1.17) is satisfied, in particular, when $A(x) \sim c\sqrt{x}$ for some $c \in (0,\infty)$, hence the SRT (1.6) holds with no extra assumption on F, in this case.

In order to understand how (1.17) arises, we bound the integral in (1.12") from above by $L^*(x)^2 R((1-\eta)x,x)$, hence a sufficient condition for the SRT (1.6) is

$$\exists T \in [0,\infty): \qquad \lim_{\eta\to 0}\bigg(\limsup_{x\to\infty} \frac{L^*(x)^2}{A(x)^2}\,\mathsf{R}_T\big((1-\eta)x,x\big)\bigg) = 0\,, \tag{1.18}$$

and a slightly weaker (but more explicit) sufficient condition is

$$\exists \eta \in (0,1),\ T \in [0,\infty): \qquad \mathsf{R}_T\big((1-\eta)x,x\big) \underset{x\to\infty}{=} o\left(\frac{A(x)^2}{L^*(x)^2}\right)\,. \tag{1.19}$$

It is worth observing that (1.18)-(1.19) refine Chi's condition (1.10) for $\alpha = \frac{1}{2}$, because it is easy to show that $u(x) \geq c\, L^*(x)^2$ for some $c \in (0, \infty)$.

1.3. **Beyond renewal processes.** It is natural to consider the two-sided version of (1.1), i.e. to take a probability F on the real line $\mathbb{R}$ which is in the domain of attraction of a stable law with index $\alpha \in (0,1)$ and positivity parameter $\varrho \in (0,1]$. More explicitly, setting $F(x) := F((-\infty, x])$ and $\overline{F}(x) := F((x,\infty))$, assume that

$$\overline{F}(x) \sim \frac{p}{A(x)} \qquad \text{and} \qquad F(-x) \sim \frac{q}{A(x)} \qquad \text{as} \quad x \to \infty\,, \tag{1.20}$$

where $A \in \mathcal{R}_\alpha$ and $p > 0$, $q \geq 0$ are finite constants. As usual, let $S_n = X_1 + \ldots + X_n$ be the random walk associated to F and define the renewal measure $U(\cdot)$ as in (1.3).

The "integrated" renewal theorem (1.4) still holds (with a different value of $\mathsf{C} = \mathsf{C}(\alpha, \varrho)$) and, for $\alpha > \frac{1}{2}$, the SRT (3.4) follows again by (1.20) with no additional assumptions cf. [GL63, W68, E70, E71] (we give an independent proof in Section 4).

For $\alpha \leq \frac{1}{2}$, our next result gives a necessary condition for the SRT, which is shown to be strictly stronger than (1.11), when $q > 0$. This means that *condition* (1.11) *is not sufficient for the SRT* in the two-sided case (1.20).

Theorem 1.9 (Two-sided case). *Let F be a probability on $\mathbb{R}$ satisfying* (1.20) *for some $A \in \mathcal{R}_\alpha$, with $\alpha \in (0,1)$ and $p > 0$, $q \geq 0$. Define $I = (-h, 0]$ with $h > 0$ as in* (1.5).

- *If $\alpha > \frac{1}{2}$, the SRT* (1.6) *holds with no extra assumption on F.*
- *If $\alpha \leq \frac{1}{2}$, a necessary condition for the SRT* (1.6) *is the following:*

$$\lim_{\eta \to 0} \left\{ \limsup_{x \to \infty} \frac{1}{A(x)^2} \left(\int_1^{\eta x} \frac{A(s)^2}{s} \big(\mathsf{r}(x-s) + \mathbb{1}_{\{q>0\}} \mathsf{r}(x+s) \big)\, \mathrm{d}s \right) \right\} = 0\,. \tag{1.21}$$

There are examples of F satisfying (1.11) *but not* (1.21)*, for which the SRT fails.*

It is not clear whether (1.21) is also sufficient for the SRT, or whether additional conditions (possibly on the left tail of F) need to be imposed.

1.4. **Structure of the paper.** The paper is organized as follows.

- In Section 2 we recall some standard background results.
- In Section 3 we reformulate conditions (1.11)-(1.12) and (1.21) more explicitly in terms of F (see Lemma 3.1) and we describe the general strategy underlying the proof of Theorem 1.1, which is carried out in the following Sections 4, 5 and 6.
- In Section 7 we prove Proposition 1.6 and Theorems 1.7 and 1.9, while the Appendix A contains the proofs of some auxiliary results.

2. Setup

2.1. **Notation.** We write $f(s) \lesssim g(s)$ or $f \lesssim g$ to mean $f(s) = O(g(s))$, i.e. for a suitable constant $C < \infty$ one has $f(s) \leq C\, g(s)$ for all s in the range under consideration. The constant C may depend on the probability F (in particular, on α) and on h. When some extra parameter ε enters the constant $C = C_\varepsilon$, we write $f(s) \lesssim_\varepsilon g(s)$. If both $f \lesssim g$ and $g \lesssim f$, we write $f \simeq g$. We recall that $f(s) \sim g(s)$ means $\lim_{s\to\infty} f(s)/g(s) = 1$.

2.2. **Regular variation.** We recall that $A : [0, \infty) \to (0, \infty)$ in (1.1) is assumed to be differentiable, strictly increasing and such that (1.2) holds. For definiteness, let us fix $A(0) := \frac{1}{2}$ and $A(1) := 1$, so that both A and A^{-1} map $[1, \infty)$ onto itself.

We observe that, by Potter's bounds, for every $\varepsilon > 0$ one has

$$\varrho^{\alpha+\varepsilon} \lesssim_\varepsilon \frac{A(\varrho x)}{A(x)} \lesssim_\varepsilon \varrho^{\alpha-\varepsilon}, \qquad \forall \varrho \in (0,1],\ x \in (0,\infty) \ \text{ such that } \ \varrho x \geq 1\,. \tag{2.1}$$

More precisely, part (i) of [BGT89, Theorem 1.5.6] shows that relation (2.1) holds for $\varrho x \geq \bar{x}_\varepsilon$, for a suitable $\bar{x}_\varepsilon < \infty$; the extension to $1 \leq \varrho x \leq \bar{x}_\varepsilon$ follows as in part (ii) of the same theorem, because $A(y)$ is bounded away from zero and infinity for $y \in [1, \bar{x}_\varepsilon]$.

We also recall Karamata's Theorem [BGT89, Proposition 1.5.8]:

$$\text{if } f(n) \in \mathcal{R}_\zeta \text{ with } \zeta > -1: \qquad \sum_{n \leq t} f(n) \underset{t\to\infty}{\sim} \frac{1}{\zeta+1}\, t\, f(t)\,. \tag{2.2}$$

As a matter of fact, this relation holds also in the limiting case $\zeta = -1$, in the sense that $tf(t) = o(\sum_{n=1}^t f(n))$, by [BGT89, Proposition 1.5.9a].

2.3. **Local limit theorems.** We call a probability F on $\mathbb{R}$ *lattice* if it is supported by $v\mathbb{Z} + a$ for some $v > 0$ and $0 \leq a < v$, and the maximal value of $v > 0$ with this property is called the *lattice span* of F. If F is arithmetic (i.e. supported by $h\mathbb{Z}$, cf. §1.1), then it is also lattice, but the spans might differ (for instance, $F(\{-1\}) = F(\{+1\}) = \frac{1}{2}$ has arithmetic span $h = 1$ and lattice span $v = 2$). A lattice distribution is not necessarily arithmetic.[†]

Let us define

$$a_n := A^{-1}(n)\,, \qquad n \in \mathbb{N}_0\,,$$

so that $a_n \in \mathcal{R}_{1/\alpha}$. Under (1.1) or, more generally, (1.20), S_n/a_n converges in distribution as $n \to \infty$ toward a stable law, whose density we denote by φ. If we set

$$J = (-v, 0] \qquad \text{with} \qquad v = \begin{cases} \text{lattice span of } F & \text{(if } F \text{ is lattice)} \\ \text{any fixed number} > 0 & \text{(if } F \text{ is non-lattice)} \end{cases}, \tag{2.3}$$

by Gnedenko's and Stone's local limit theorems [BGT89, Theorems 8.4.1 and 8.4.2] we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| a_n \, \mathrm{P}(S_n \in x + J) - v\, \varphi\left(\frac{x}{a_n}\right) \right| = 0\,. \tag{2.4}$$

Since $\sup_{z\in\mathbb{R}} \varphi(z) < \infty$, we obtain the useful estimate

$$\sup_{z\in\mathbb{R}} \mathrm{P}(S_n \in (x-w, x]) \lesssim_w \frac{1}{a_n}\,, \tag{2.5}$$

which, plainly, holds for *any* fixed $w > 0$ (not necessarily the lattice span of F).

Besides the local limit theorem (2.4), a key tool in the proof will be a local large deviations estimate by Denisov, Dieker and Shneer [DDS08, Proposition 7.1] (see (4.14) below).

[†] If F is lattice, say supported by $v\mathbb{Z} + a$ where v is the lattice span and $a \in [0, v)$, then F is arithmetic if and only if $a/v \in \mathbb{Q}$, in which case its arithmetic span equals $h = v/m$ for some $m \in \mathbb{N}$.

3. Proof of Theorem 1.1: strategy

We start reformulating the key conditions (1.11)-(1.12) and (1.21) more explicitly in terms of F. We recall that X denotes a random variable with law F. The next Lemma is proved in Appendix A.1.

Lemma 3.1. *Assuming* (1.1)*, condition* (1.11) *is equivalent to*

$$\lim_{\eta\to 0}\left(\limsup_{x\to\infty}\frac{x}{A(x)}\int_{s\in[1,\eta x)}\frac{A(s)^2}{s}\,\mathrm{P}(X\in x-\mathrm{d}s)\right)=0\,, \tag{3.1}$$

where we set $\mathrm{P}(X\in x-\mathrm{d}s):=\mathrm{P}(x-X\in\mathrm{d}s)$*, and condition* (1.12) *is equivalent to*

$$\lim_{\eta\to 0}\left(\limsup_{x\to\infty}\frac{x}{A(x)}\int_{1}^{\eta x}\frac{A(s)^2}{s^2}\,\mathrm{P}(X\in(x-s,x])\,\mathrm{d}s\right)=0\,. \tag{3.2}$$

Analogously, assuming (1.20)*, condition* (1.21) *is equivalent to*

$$\lim_{\eta\to 0}\left(\limsup_{x\to\infty}\frac{x}{A(x)}\int_{s\in[1,\eta x)}\frac{A(s)^2}{s}\big(\mathrm{P}(X\in x-\mathrm{d}s)+\mathbb{1}_{\{q>0\}}\mathrm{P}(X\in x+\mathrm{d}s)\big)\right)=0\,, \tag{3.3}$$

It is now easy to prove the second part of Theorem 1.1: through a standard integration by parts, one shows that if $\alpha<\frac{1}{2}$ relation (3.2) is equivalent to (3.1), while if $\alpha=\frac{1}{2}$ it is stronger than (3.1). We refer to the Appendix §A.2 for the details.

Next we turn to the first part of Theorem 1.1, i.e. the fact that (1.11) is a necessary and sufficient condition for the SRT. The following general statement is known [C15, Appendix]: for F satisfying (1.1), or more generally (1.20), the SRT (1.6) is equivalent to

$$\lim_{\delta\to 0}\left(\limsup_{x\to\infty}\frac{x}{A(x)}\sum_{1\le n\le A(\delta x)}\mathrm{P}(S_n\in x+I)\right)=0\,, \tag{3.4}$$

which means that small values of n give a negligible contribution to the renewal measure (we refer to Remark 3.3 below for an intuitive explanation of (3.4)). By Lemma 3.1, it remains to show that *condition* (3.1) *is necessary and sufficient for* (3.4).

The necessity of (3.1) (or, if we assume (1.20), of (3.3)) is quite easy to check and is carried out in the Appendix A.3. Showing the sufficiency of (3.1) for (3.4) is much harder and is the core of the paper.

- In Section 4 we prove that (3.4) follows by (1.1) alone, if $\alpha>\frac{1}{2}$. The proof is based on the notion of "big jump" and on two key bounds, cf. Lemmas 4.1 and 4.2, that will be exploited in an essential way also for the case $\alpha\le\frac{1}{2}$.
- In Section 5 we prove that (3.1) implies (3.4) in the special regime $\alpha\in(\frac{1}{3},\frac{1}{2}]$. This case is technically simpler, because there is only one big jump to deal with, but it already contains all the ingredients of the general case.
- In Section 6 we complete the proof, showing that (3.1) implies (3.4) for any $\alpha\in(0,\frac{1}{2}]$. The strategy is conceptually analogous to the one of Section 5 but it is technically much more involved, because we have to deal with more than one big jump.

Remark 3.2. Condition (3.1), equivalently (1.11), implies that for any fixed $w>0$

$$\mathrm{P}(X\in(x-w,x])=o\left(\frac{A(x)}{x}\right)\qquad\text{as } x\to\infty\,, \tag{3.5}$$

as we prove in the Appendix §A.5. This is not surprising, since (3.5) is a necessary condition for the SRT (1.6), because $U(x+I) \geq \mathrm{P}(S_1 \in x+I) = \mathrm{P}(X \in x+I)$. In Appendix §A.5 we also prove the following easy consequence of (3.5): for all fixed $m \in \mathbb{N}$ and $w > 0$

$$\mathrm{P}(S_m \in (x-w, x]) = o\left(\frac{A(x)}{x}\right) \qquad \text{as } x \to \infty\,. \tag{3.6}$$

Relations (3.5)-(3.6) will be useful in the next sections. □

Remark 3.3. It is worth explaining how (3.4) arises. For fixed $\delta > 0$, by (1.3) we can write

$$U(x+I) \geq \sum_{A(\delta x) < n \leq A(\frac{1}{\delta} x)} \mathrm{P}(S_n \in x+I)\,. \tag{3.7}$$

Since $\mathrm{P}(S_n \in x+I) \sim \frac{h}{a_n}\,\varphi(\frac{x}{a_n})$ by (2.4) (where we take $h = v$ for simplicity), a Riemann sum approximation yields (see [C15, Lemma 3.4] for the details)

$$\sum_{A(\delta x) < n \leq A(\frac{1}{\delta} x)} \mathrm{P}(S_n \in x+I) \sim h\,\frac{A(x)}{x}\,\mathsf{C}(\delta)\,, \qquad \text{with} \qquad \mathsf{C}(\delta) = \alpha \int_{\delta}^{\frac{1}{\delta}} z^{\alpha-2} \varphi(\tfrac{1}{z})\,\mathrm{d}z\,.$$

One can show that $\lim_{\delta \to 0} \mathsf{C}(\delta) = \mathsf{C}$, therefore proving the SRT (1.6) amounts to controlling the ranges excluded from (3.7), i.e. $\{n \leq A(\delta x)\}$ and $\{n > A(\frac{1}{\delta} x)\}$. The latter always gives a negligible contribution, by the bound $\mathrm{P}(S_n \in x+I) \leq C/a_n$ (recall (2.5)), and the former is controlled precisely by (3.4). □

4. Proof of Theorems 1.1 and 1.9: the case $\alpha > \frac{1}{2}$

In this section we prove that, if $\alpha > \frac{1}{2}$, relation (3.4)*, which is equivalent to the SRT* (1.6)*, follows with no additional assumptions by* (1.1)*, or more generally by* (1.20) (we never use the positivity of the increments of the random walk in this section).

We have to estimate the probability of the event $\{S_n \in x+I\}$ with $n \leq A(\delta x)$, where $S_n = X_1 + X_2 + \ldots + X_n$. Let us call "big jump" any increment X_i strictly larger than a suitable threshold $\xi_{n,x}$, defined as a multiplicative average of a_n and x:

$$\xi_{n,x} := a_n^{\gamma_\alpha}\, x^{1-\gamma_\alpha} = a_n \left(\frac{x}{a_n}\right)^{1-\gamma_\alpha}\,, \qquad \text{with} \quad \gamma_\alpha := \frac{\alpha}{4}\left(1 - \left\{\frac{1}{\alpha}\right\}\right) > 0\,, \tag{4.1}$$

where $\{z\} := z - \lfloor z \rfloor \in [0,1)$ denotes the fractional part of z. The reason for the specific choice of $\gamma_\alpha > 0$ will be clear later (it is important that γ_α is small enough).

4.1. Bounding the number of big jumps.

As a first step, for every $\alpha \in (0,1)$, we show that, on the event $\{S_n \in x+I\}$ with $n \leq A(\delta x)$, the number of "big jumps" can be bounded by a deterministic number $\kappa_\alpha \in \mathbb{N}_0$, defined as follows:

$$\kappa_\alpha := \left\lfloor \frac{1}{\alpha} \right\rfloor - 1\,. \qquad \text{i.e.} \qquad \kappa_\alpha = m \quad \text{if} \quad \alpha \in (\tfrac{1}{m+2}, \tfrac{1}{m+1}] \quad \text{with} \;\; m \in \mathbb{N}_0\,. \tag{4.2}$$

Note that $\kappa_\alpha = 0$ if $\alpha > \frac{1}{2}$ and this is why the SRT holds with no additional assumption in this case. If $\alpha \leq \frac{1}{2}$, on the other hand, $\kappa_\alpha \geq 1$ and a more refined analysis is required.

Let us call $B^k_{n,x}$ the event "there are exactly k big jumps", i.e.

$$\begin{aligned} B^0_{n,x} &:= \left\{ \max_{1\le i\le n} X_i \le \xi_{n,x} \right\} , \\ B^k_{n,x} &:= \left\{ \exists I \subseteq \{1,\dots,n\},\ |I| = k :\ \min_{i\in I} X_i > \xi_{n,x} ,\ \max_{j\in\{1,\dots,n\}\setminus I} X_i \le \xi_{n,x} \right\} , \quad k \ge 1 , \end{aligned} \tag{4.3}$$

and correspondingly let $B^{\ge k}_{n,x}$ be the event "there are at least k big jumps":

$$B^{\ge k}_{n,x} := \bigcup_{\ell=k}^{n} B^{\ell}_{n,x} , \tag{4.4}$$

The following lemma shows that the event $B^{\ge\kappa_\alpha+1}_{n,x}$ gives a negligible contribution to (3.4) (just plug $\ell = 0$ and $m = \kappa_\alpha + 1$ into (4.5)). This sharpens [C13, Lemma 4.1], where κ was defined as $\lfloor \frac{1}{\alpha} \rfloor$, i.e. one unit *larger* than our choice (4.2) of κ_α. Furthermore, we allow for an extra parameter ℓ, that will be useful later.

Lemma 4.1. *Let F satisfy (1.20) for some $A \in \mathcal{R}_\alpha$, with $\alpha \in (0,1)$ and $p > 0$, $q \ge 0$. There is $\eta = \eta_\alpha > 0$ such that for all $\delta \in (0,1]$, $x \in [1,\infty)$, $\ell, m \in \mathbb{N}_0$ the following holds:*

$$\textit{if}\ \ \ell + m \ge \kappa_\alpha + 1 : \qquad \sum_{1\le n\le A(\delta x)} n^\ell\, \mathrm{P}\left(S_n \in x + I,\ B^{\ge m}_{n,x}\right) \lesssim_{\ell,m} \delta^\eta\, \frac{A(x)^{\ell+1}}{x} . \tag{4.5}$$

Proof. Throughout the proof we work for $n \le A(\delta x)$, hence $a_n \le \delta x \le x$ (since $\delta \le 1$). Consequently, recalling (4.1), we have $a_n \le \xi_{n,x} \le x$.

For $m \in \mathbb{N}$, recalling (2.5), we can write

$$\begin{aligned} \mathrm{P}\big(S_n \in x + I,\ B^{\ge m}_{n,x}\big) &= \mathrm{P}\left(S_n \in x + I,\ \exists A \subseteq \{1,\dots,n\}, |A| = m :\ \min_{i\in A} X_i > \xi_{n,x}\right) \\ &\le n^m\, \mathrm{P}\left(S_n \in x + I,\ \min_{1\le i\le m} X_i > \xi_{n,x}\right) \\ &= n^m \int_{w\in\mathbb{R}} \mathrm{P}\left(S_m \in \mathrm{d}w,\ \min_{1\le i\le m} X_i > \xi_{n,x}\right) \mathrm{P}(S_{n-m} \in x - w + I) \\ &\le n^m\, \mathrm{P}\left(\min_{1\le i\le m} X_i > \xi_{n,x}\right) \left\{\sup_{z\in\mathbb{R}} \mathrm{P}(S_{n-m} \in z + I)\right\} \\ &\lesssim n^m\, \mathrm{P}(X > \xi_{n,x})^m \frac{1}{a_{n-m}} \lesssim_m \frac{n^m}{A(\xi_{n,x})^m}\, \frac{1}{a_n} , \end{aligned} \tag{4.6}$$

and this estimate holds also for $m = 0$ (in which case $\mathrm{P}(S_n \in x + I,\ B^{\ge 0}_{n,x}) = \mathrm{P}(S_n \in x + I)$). Next we apply the lower bound in (2.1) with $\varepsilon = \alpha$ and $\varrho = \xi_{n,x}/x$ (note that the condition $\varrho x = \xi_{n,x} \ge 1$ is fulfilled because $\xi_{n,x} \ge a_n \ge 1$):

$$\frac{A(\xi_{n,x})}{A(x)} \gtrsim \left(\frac{\xi_{n,x}}{x}\right)^{2\alpha} = \left(\frac{a_n}{x}\right)^{2\alpha\gamma_\alpha} .$$

Looking back at (4.6), we get

$$\sum_{1\le n\le A(\delta x)} n^\ell\, \mathrm{P}\left(S_n \in x + I,\ B^{\ge m}_{n,x}\right) \lesssim \frac{x^{2\alpha\gamma_\alpha m}}{A(x)^m} \sum_{1\le n\le A(\delta x)} \frac{n^{m+\ell}}{(a_n)^{2\alpha\gamma_\alpha m+1}} . \tag{4.7}$$

Since $a_n \in \mathcal{R}_{1/\alpha}$, the sequence in the sum is regularly varying with index

$$J'_{m,\ell,\alpha} = (m+\ell) - \frac{1}{\alpha}(2\alpha\gamma_\alpha m + 1) = (m+\ell)(1-2\gamma_\alpha) - \frac{1}{\alpha} + 2\gamma_\alpha \ell \,.$$

By assumption $\ell \geq 0$ and $m+\ell \geq \kappa_\alpha + 1$, hence

$$J'_{m,\ell,\alpha} \geq J_\alpha \qquad \text{with} \qquad J_\alpha := (\kappa_\alpha + 1)(1-2\gamma_\alpha) - \frac{1}{\alpha}\,. \tag{4.8}$$

We claim that

$$J_\alpha > -1\,, \qquad \text{that is} \qquad \kappa_\alpha + 1 > \frac{1-\alpha}{\alpha}\frac{1}{1-2\gamma_\alpha}\,. \tag{4.9}$$

To verify it, write $\kappa_\alpha + 1 = \lfloor \frac{1}{\alpha} \rfloor = \frac{1}{\alpha} - \{\frac{1}{\alpha}\} = \frac{1-\alpha}{\alpha} + (1 - \{\frac{1}{\alpha}\})$, so that relation (4.9) becomes $1 - \{\frac{1}{\alpha}\} > \frac{1-\alpha}{\alpha}\frac{2\gamma_\alpha}{1-2\gamma_\alpha}$. Since $\gamma_\alpha < \frac{1}{4}$ by construction, cf. (4.1), we have $\frac{1-\alpha}{\alpha}\frac{2\gamma_\alpha}{1-2\gamma_\alpha} < \frac{1-\alpha}{\alpha} 4\gamma_\alpha < \frac{4\gamma_\alpha}{\alpha}$ and it remains to note that $\frac{4\gamma_\alpha}{\alpha} = 1 - \{\frac{1}{\alpha}\}$, by definition (4.1) of γ_α.

Coming back to (4.7), since the sequence in the sum is regularly varying with index $J'_{m,\ell,\alpha} \geq J_\alpha > -1$, we can apply relation (2.2), getting

$$\sum_{1 \leq n \leq A(\delta x)} n^\ell \, \mathrm{P}\left(S_n \in x + I,\ B^m_{n,x}\right) \lesssim_{\ell,m} \frac{x^{2\alpha\gamma_\alpha m}}{A(x)^m} \frac{A(\delta x)^{m+\ell+1}}{(\delta x)^{2\alpha\gamma_\alpha m + 1}} = \frac{A(\delta x)^{\ell+m+1}}{\delta^{2\alpha\gamma_\alpha m+1} A(x)^m \, x}\,. \tag{4.10}$$

It is convenient to introduce a parameter $b = b_\alpha \in [\frac{1}{2}, 1)$, depending only on α, that will be fixed later. Note that (4.5) holds trivially for $\delta x < 1$ (the left hand side vanishes, due to $A(0) < 1$), hence we may assume that $\delta x \geq 1$. We can then apply the upper bound in (2.1) with $\varepsilon = (1-b)\alpha$ and $\varrho = \delta$, that is $A(\delta x) \lesssim \delta^{b\alpha} A(x)$, which plugged into (4.10) gives

$$\begin{aligned} \sum_{1 \leq n \leq A(\delta x)} n^\ell \, \mathrm{P}\left(S_n \in x + I,\ B^m_{n,x}\right) &\lesssim_{\ell,m} \delta^{b\alpha(\ell+m+1) - (2\alpha\gamma_\alpha m + 1)} \, \frac{A(x)^{\ell+1}}{x} \\ &= \delta^{\alpha\{(m+\ell)(b-2\gamma_\alpha) - \frac{1}{\alpha} + 2\gamma_\alpha \ell + 1\}} \, \frac{A(x)^{\ell+1}}{x} \\ &\leq \delta^{\alpha\{(\kappa_\alpha+1)(b-2\gamma_\alpha) - \frac{1}{\alpha} + 1\}} \, \frac{A(x)^{\ell+1}}{x}\,, \end{aligned} \tag{4.11}$$

where the last inequality holds because $m+\ell \geq \kappa_\alpha + 1$ and $\ell \geq 0$ by assumption (recall that $\delta \leq 1$ and note that $b - 2\gamma_\alpha > 0$, because $b \geq \frac{1}{2}$ and $\gamma_\alpha < \frac{1}{4}$). Recalling (4.8), we get

$$\sum_{1 \leq n \leq A(\delta x)} n^\ell \, \mathrm{P}\left(S_n \in x + I,\ B^m_{n,x}\right) \lesssim_{\ell,m} \delta^{\alpha\{(J_\alpha+1) - (1-b)(\kappa_\alpha+1)\}} \, \frac{A(x)^{\ell+1}}{x}\,. \tag{4.12}$$

Since $J_\alpha + 1 > 0$, by (4.9), we can choose $b < 1$ so that the term in bracket is strictly positive. More explicitly, defining $b = b_\alpha := \max\{\frac{1}{2}, 1 - \frac{1}{2}\frac{J_\alpha+1}{\kappa_\alpha+1}\}$, the right hand side of (4.12) becomes $\lesssim \delta^{\alpha\{\frac{1}{2}(J_\alpha+1)\}} \frac{A(x)^{\ell+1}}{x}$. This shows that relation (4.5) holds with $\eta = \eta_\alpha := \frac{1}{2}\alpha(J_\alpha+1)$. $\square$

4.2. **The case of no big jumps.** Next we analyze the event $B^0_{n,x}$ of "no big jumps", showing that it gives a negligible contribution to (3.4), irrespective of $\alpha \in (0,1)$. (The extra parameter ℓ and the sup over z in (4.13) will be useful later.)

Lemma 4.2. *Let F satisfy* (1.20) *for some $A \in \mathcal{R}_\alpha$, with $\alpha \in (0,1)$ and $p > 0$, $q \geq 0$. For all $\delta \in (0,1]$, $x \in [1,\infty)$, $\ell \in \mathbb{N}_0$ the following holds, with $\gamma = \gamma_\alpha > 0$ defined in* (4.1)*:*

$$\sum_{1 \leq n \leq A(\delta x)} n^\ell \sup_{z \geq \delta^{\gamma/2} x} \mathrm{P}\left(S_n \in z + I,\ B^0_{n,x}\right) \lesssim e^{-\frac{1}{\delta^{\gamma/3}}} \, \frac{A(x)^{\ell+1}}{x}\,. \tag{4.13}$$

Proof. Throughout the proof we may assume that $\delta x \geq 1$, because for $\delta x < 1$ the left hand side of (4.13) vanishes (recall that $A(0) < 1$ by construction).

We need a refinement of (2.4), given by [DDS08, Proposition 7.1] (see also [C15, Lemma 3.2]): if F satisfies (1.1), or more generally (1.20), there are $C_1, C_2 < \infty$ such that for any sequence $s_n \to \infty$ and $z \geq 0$

$$\mathrm{P}\left(S_n \in z + I,\ \max_{1 \leq i \leq n} X_i \leq s_n \right) \leq C_1\, e^{C_2 \frac{n}{A(s_n)}} \left(\frac{1}{s_n} + \frac{1}{a_n} \right) e^{-\frac{z}{s_n}}\,. \tag{4.14}$$

We choose $s_n = \xi_{n,x}$, cf. (4.1). For $n \leq A(\delta x)$, with $\delta \leq 1$, we have $a_n \leq x$, hence by (4.1) we get $\xi_{n,x} \geq a_n$ and consequently $A(\xi_{n,x}) \geq n$. Applying (4.14), we obtain

$$\mathrm{P}\left(S_n \in z + I,\, B^0_{n,x} \right) \leq C_1\, e^{C_2 \frac{n}{A(\xi_{n,x})}} \left(\frac{1}{\xi_{n,x}} + \frac{1}{a_n} \right) e^{-\frac{z}{\xi_{n,x}}} \lesssim \frac{1}{a_n}\, e^{-\frac{z}{x} (\frac{x}{a_n})^\gamma}\,. \tag{4.15}$$

The function $\varphi(y) := \frac{1}{y} e^{-1/y^\gamma}$ is increasing for $y \in (0, c]$, with $c \in (0,1)$ a fixed constant (by direct computation $c = \gamma^{1/\gamma}$). Then, if $\delta \leq c^2$, for $z \geq \delta^{\gamma/2} x$ and $a_n \leq \delta x$ one has

$$\mathrm{P}\left(S_n \in z + I,\, B^0_{n,x} \right) \leq \frac{1}{a_n}\, e^{-(\frac{\sqrt{\delta} x}{a_n})^\gamma} = \frac{1}{\sqrt{\delta} x} \varphi\left(\frac{a_n}{\sqrt{\delta} x} \right) \leq \frac{1}{\sqrt{\delta} x} \varphi(\sqrt{\delta})\,,$$

hence, always for $\delta \leq c^2$, applying (2.1) with $\varepsilon = \alpha/2$ and $\varrho = \delta$,

$$\sum_{1 \leq n \leq A(\delta x)} n^\ell \sup_{z \geq \delta^{\gamma/2} x} \mathrm{P}\left(S_n \in z + I,\, B^0_{n,x} \right) \lesssim \frac{\varphi(\sqrt{\delta})}{\sqrt{\delta} x} A(\delta x)^{\ell+1} \lesssim \delta^{\ell \frac{\alpha}{2}} \frac{e^{-\frac{1}{\delta^{\gamma/2}}}}{\delta^{1-\frac{\alpha}{2}}} \frac{A(x)^{\ell+1}}{x}\,. \tag{4.16}$$

Bounding $\delta^{\ell\alpha} \leq 1$ since $\delta \leq 1$, relation (4.13) is proved for $\delta \leq c^2$.

In case $\delta \in (c^2, 1]$, the right hand side of (4.13) is $\simeq A(x)^{\ell+1}/x$, hence we have to show that the left hand side is $\lesssim A(x)^{\ell+1}/x$. The contribution of the terms with $n \leq A(c^2 x)$ is under control, by (4.16) with $\delta = c^2$. For the remaining terms, by (2.5),

$$\sum_{A(c^2 x) < n \leq A(\delta x)} n^\ell \sup_{z \geq \delta^{\gamma/2} x} \mathrm{P}\left(S_n \in z + I,\, B^0_{n,x} \right) \lesssim \sum_{A(c^2 x) < n \leq A(\delta x)} \frac{n^\ell}{a_n} \leq \frac{A(x)^{\ell+1}}{c^2 x}\,,$$

where we have bounded $a_n \geq a_{A(c^2 x)} = c^2 x$ and $n \leq A(\delta x) \leq A(x)$ (recall that $\delta \leq 1$). □

4.3. **Proof of Theorems 1.1 and 1.9 for $\alpha > \frac{1}{2}$.** Assume (1.1), or more generally (1.20), for some $\alpha > \frac{1}{2}$. We have already observed that $\kappa_\alpha = 0$ for $\alpha > \frac{1}{2}$, cf. (4.2). We can then apply Lemma 4.1 with $\ell = 0$ and $m = 1$, since $\ell + m \geq \kappa_\alpha + 1$ in this case. Together with Lemma 4.2 with $\ell = 0$ and $z = x$, this yields

$$\begin{aligned} \sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I) &= \sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I,\, B^{\geq 1}_{n,x}) + \sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I,\, B^0_{n,x}) \\ &\lesssim \left(\delta^\eta + e^{-\frac{1}{\delta^{\gamma/3}}} \right) \frac{A(x)}{x}\,, \end{aligned} \tag{4.17}$$

which shows that relation (3.4), and hence the SRT (1.6), holds true for $\alpha > \frac{1}{2}$. □

5. Proof of Theorem 1.1: sufficiency for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

For $\alpha \le \frac{1}{2}$ big jumps have to be taken into account, because $\kappa_\alpha \ge 1$, cf. (4.2), and we need to show that their contributions can be controlled using (3.1), which is equivalent to (1.11) by Lemma 3.1. In order to illustrate the main ideas, in this section we focus on the special case $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, which is technically simpler, because $\kappa_\alpha = 1$. The general case $\alpha \in (0, \frac{1}{2}]$ is treated in Section 6.

Throughout this section we assume condition (3.1) *and we show that, for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, it implies* (3.4)*, which is equivalent to the SRT* (1.6)*.*

We start with a basic estimate.

Lemma 5.1. *If F satisfies* (1.1) *with $\alpha \in (0, 1)$, there are $C, c \in (0, \infty)$ such that for all $n \in \mathbb{N}_0$ and $z \in [0, \infty)$*

$$\mathrm{P}(S_n \in z + I) \le \frac{C}{a_n} e^{-c\frac{n}{A(z)}} . \tag{5.1}$$

Proof. Assuming that n is even (the odd case is analogous) and applying (2.5), we get

$$\begin{aligned}\mathrm{P}\left(S_n \in z + I\right) &= \int_{y \in [0,z]} \mathrm{P}(S_{\frac{n}{2}} \in \mathrm{d}y)\, \mathrm{P}(S_{\frac{n}{2}} \in z - y + I) \le \frac{1}{a_{\frac{n}{2}}} \mathrm{P}(S_{\frac{n}{2}} \le z) \\ &\lesssim \frac{1}{a_n} \mathrm{P}\left(\max_{1 \le i \le \frac{n}{2}} X_i \le z \right) = \frac{(1 - \mathrm{P}(X > z))^{\frac{n}{2}}}{a_n} \le \frac{e^{-\frac{n}{2}\mathrm{P}(X>z)}}{a_n} \le \frac{e^{-c\frac{n}{A(z)}}}{a_n} ,\end{aligned}$$

provided $c > 0$ is chosen such that $\mathrm{P}(X > z) \ge 2c/A(z)$ for all $z \ge 0$. This is possible by (1.1) and because $z \mapsto A(z)$ is (increasing and) continuous, with $A(0) > 0$ (see §2.2). ☐

We are ready to prove that (3.4) follows by (3.1) for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. In analogy with (4.17), we apply Lemma 4.1 with $\ell = 0$ and, this time, with $m = 2$, so that $\ell + m \ge \kappa_\alpha + 1$ (because $\kappa_\alpha = 1$). Applying also Lemma 4.2 with $\ell = 0$ and $z = x$, we obtain

$$\sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I) \lesssim \left(\delta^\eta + e^{-\frac{1}{\delta^{\gamma/3}}} \right) \frac{A(x)}{x} + \sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I,\, B^1_{n,x}) . \tag{5.2}$$

The first term gives no problem for (3.4), hence we focus on $\mathrm{P}(S_n \in x + I,\, B^1_{n,x})$. Plainly,

$$\begin{aligned}\mathrm{P}(S_n \in x + I,\ B^1_{n,x}) &\le n\, \mathrm{P}\left(S_n \in x + I,\ X_n > \xi_{n,x},\ \max_{1 \le j \le n-1} X_i \le \xi_{n,x} \right) \\ &= n \int_{y \in (\xi_{n,x}, x]} \mathrm{P}(X \in \mathrm{d}y)\, \mathrm{P}\left(S_{n-1} \in x - y + I,\ B^0_{n-1,x} \right) ,\end{aligned} \tag{5.3}$$

where we recall that $B^0_{n-1,x} = \{\max_{1 \le j \le n-1} X_i \le \xi_{n,x}\}$.

We first consider the contribution to the integral given by $y \in (\xi_{n,x}, x(1 - \delta^{\gamma/2})]$ (where $\gamma = \gamma_\alpha > 0$ was defined in (4.1)): since $x - y \ge \delta^{\gamma/2} x$, this contribution is bounded by

$$n\, \mathrm{P}(X > \xi_{n,x}) \sup_{z \ge \delta^{\gamma/2} x} \mathrm{P}\left(S_{n-1} \in z + I,\ B^0_{n-1,x} \right) \lesssim \sup_{z \ge \delta^{\gamma/2} x} \mathrm{P}\left(S_{n-1} \in z + I,\ B^0_{n-1,x} \right) , \tag{5.4}$$

because $\mathrm{P}(X > \xi_{n,x}) \le \mathrm{P}(X > a_n) \sim 1/A(a_n) = 1/n$, since $\xi_{n,x} \ge a_n(x/a_n)^{1-\gamma} \ge a_n$ for $n \le A(\delta x)$ with $\delta \le 1$. Applying Lemma 4.2 with $\ell = 0$, by (5.3) we get

$$\begin{aligned} &\sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I,\ B^1_{n,x}) \lesssim e^{-\frac{1}{\delta^{\gamma/3}}}\,\frac{A(x)}{x} + \mathcal{I}_{\delta,x} \\ &\text{where}\qquad \mathcal{I}_{\delta,x} := \int_{y\in(x(1-\delta^{\gamma/2}),x]} \mathrm{P}(X\in \mathrm{d}y)\left(\sum_{n\in\mathbb{N}} n\,\mathrm{P}\left(S_{n-1}\in x-y+I\right)\right). \end{aligned} \tag{5.5}$$

Next we look at the contribution to $\mathcal{I}_{\delta,x}$ given by $y \in (x-1, x]$. Applying Lemma 5.1, recalling that $z \mapsto A(z)$ is increasing, for $x - y \le 1$ we have

$$\sum_{n\in\mathbb{N}} n\,\mathrm{P}\left(S_{n-1}\in x-y+I\right) \le \sum_{n\in\mathbb{N}} \frac{n}{a_{n-1}}\, e^{-c\frac{n-1}{A(1)}} =: C < \infty\,, \tag{5.6}$$

hence the contribution to $\mathcal{I}_{\delta,x}$ in (5.5) of $y \in (x-1, x]$ is bounded by

$$C\int_{y\in(x-1,x]} \mathrm{P}(X\in\mathrm{d}y) = C\,\mathrm{P}(X\in(x-1,x]) = o\left(\frac{A(x)}{x}\right),$$

where the last equality is a consequence of (3.1), see (3.5). We can thus rewrite (5.5) as

$$\mathcal{I}_{\delta,x} = o\left(\frac{A(x)}{x}\right) + \int_{y\in(x(1-\delta^{\gamma/2}),x-1]} \mathrm{P}(X\in\mathrm{d}y)\left(\sum_{n\in\mathbb{N}} n\,\mathrm{P}\left(S_{n-1}\in x-y+I\right)\right). \tag{5.7}$$

Finally, we show in a moment that the following estimate holds:

$$\sum_{n\in\mathbb{N}} n\,\mathrm{P}\left(S_{n-1}\in w+I\right) \lesssim \frac{A(w)^2}{w}\,, \qquad \forall w\ge 1\,. \tag{5.8}$$

Plugging this into (5.7), since $x - y \ge 1$, we get

$$\begin{aligned} \mathcal{I}_{\delta,x} &= o\left(\frac{A(x)}{x}\right) + \int_{y\in(x(1-\delta^{\gamma/2}),x-1]} \mathrm{P}(X\in\mathrm{d}y)\,\frac{A(x-y)^2}{(x-y)} \\ &= o\left(\frac{A(x)}{x}\right) + \int_{s\in[1,\delta^{\gamma/2}x)} \frac{A(s)^2}{s}\,\mathrm{P}(X\in x-\mathrm{d}s)\,, \end{aligned} \tag{5.9}$$

by the change of variable $s = x - y$. Gathering (5.2), (5.5) and (5.9), we have shown that relation (3.4), and hence the SRT (1.6), holds true for $\alpha \in (\frac{1}{2}, \frac{1}{3}]$.

It only remains to prove (5.8). The term $n = 1$ contributes only if $0 \in w + I = (w - h, w]$ (recall that $S_0 = 0$), i.e. if $w \le h$. Since $\inf_{w\in[0,h]} A(w)^2/w > 0$, this gives no problem for (5.8). For $n \ge 2$ we bound $n \le 2(n-1)$, and renaming $n - 1$ as m we rewrite (5.8) as

$$\sum_{m\in\mathbb{N}} m\,\mathrm{P}\left(S_m\in w+I\right) \lesssim \frac{A(w)^2}{w}\,, \qquad \forall w\ge 1\,. \tag{5.10}$$

- Let us first look at the contribution of the terms $m > A(w)$. By Lemma 5.1,

$$\sum_{m>A(w)} m\,\mathrm{P}\left(S_m\in w+I\right) \le \sum_{m>A(w)} \frac{m}{a_m}\,e^{-c\frac{m}{A(w)}} \le \frac{A(w)^2}{w}\left\{\sum_{m>A(w)} \frac{1}{A(w)}\,\frac{m}{A(w)}\,e^{-c\frac{m}{A(w)}}\right\},$$

  because $a_m > w$ for $m > A(w)$. The bracket is a Riemann sum which converges to $\int_1^\infty t\,e^{-ct}\,\mathrm{d}t < \infty$ as $w \to \infty$. It is also a continuous function of w (by dominated convergence), hence it is uniformly bounded for $w \in [1, \infty)$.

- For the terms with $m \le A(w)$, we distinguish the events $B^{\ge 1}_{\overline{m},w}$ and $B^0_{m,w}$, i.e. whether there are "big jumps" or not (recall (4.3)). Applying Lemma 4.1 with $\delta = 1$, $x = w$ and with $\ell = m = 1$ (note that $\kappa_\alpha = 1$ and hence $\ell + m \ge \kappa_\alpha + 1$), we get

$$\sum_{m \le A(w)} m \, \mathrm{P}\left(S_m \in w + I, \, B^{\ge 1}_{\overline{m},w}\right) \lesssim \frac{A(w)^2}{z} .$$

Likewise, by Lemma 4.2 with $\delta = 1$, $x = w$ and $\ell = 1$, we obtain

$$\sum_{m \le A(w)} m \, \mathrm{P}\left(S_m \in w + I, \, B^{0}_{m,w}\right) \lesssim e^{-1} \frac{A(w)^2}{z} .$$

Altogether, we have completed the proof of (5.10), hence of (5.8). □

6. Proof of Theorem 1.1: sufficiency for $\alpha \in (0, \frac{1}{2}]$

In this section we assume condition (3.1)*, which by Lemma 3.1 is equivalent to* (1.11)*, and we show that for any $\alpha \in (0, \frac{1}{2}]$ it implies* (3.4)*, which is equivalent to the SRT* (1.6)*.*

We stress that the strategy is analogous to the one adopted in Section 5 for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, but having to deal with more than one big jumps makes things more involved. In order to keep the exposition as streamlined as possible, we will use a "backward" induction, proving the following result, which is stronger than (3.4).

Theorem 6.1. *Let F be a probability on $[0, \infty)$ satisfying* (1.1) *with $\alpha \in (0, 1)$. Assume that condition* (3.1) *is satisfied. Then, for every $\ell \in \mathbb{N}_0$,*

$$\lim_{\delta \to 0} \left(\limsup_{x \to \infty} \frac{x}{A(x)^{\ell+1}} \sum_{1 \le n \le A(\delta x)} n^\ell \, \mathrm{P}(S_n \in x + I) \right) = 0 . \tag{6.1}$$

In particular, setting $\ell = 0$, relation (3.4) *holds.*

Proof. Writing $\mathrm{P}(S_n \in x + I) = \mathrm{P}(S_n \in x + I, \, B^{\ge 0}_{n,x})$, Lemma 4.1 with $m = 0$ shows that relation (6.1) holds for all $\ell \ge \kappa_\alpha + 1$.

We can now proceed by "backward induction": we fix $\bar{\ell} \in \{0, 1, \ldots, \kappa_\alpha\}$ and assume that (6.1) holds for all $\ell \ge \bar{\ell} + 1$. If we show that (6.1) holds for $\ell = \bar{\ell}$, Theorem 6.1 is proved.

Let us define $\bar{m} := \kappa_\alpha - \bar{\ell}$. Again by Lemma 4.1, for $\delta \le 1$ and $x \ge 1$

$$\sum_{1 \le n \le A(\delta x)} n^{\bar{\ell}} \, \mathrm{P}(S_n \in x + I, \, B^{\ge \bar{m}+1}_{n,x}) \lesssim \delta^\eta \, \frac{A(x)^{\bar{\ell}+1}}{x} .$$

Likewise, by Lemma 4.2,

$$\sum_{1 \le n \le A(\delta x)} n^{\bar{\ell}} \, \mathrm{P}(S_n \in x + I, \, B^{0}_{n,x}) \lesssim e^{-\frac{1}{\delta^{\gamma/3}}} \, \frac{A(x)^{\bar{\ell}+1}}{x} .$$

Therefore, the proof is completed if we show that for every fixed $m \in \{1, 2, \ldots, \bar{m}\}$

$$\lim_{\delta \to 0} \left(\limsup_{x \to \infty} \frac{x}{A(x)^{\bar{\ell}+1}} \sum_{1 \le n \le A(\delta x)} n^{\bar{\ell}} \, \mathrm{P}(S_n \in x + I, \, B^{m}_{n,x}) \right) = 0 . \tag{6.2}$$

Proof of (6.2). Note that $\mathrm{P}(S_n \in x+I,\, B^m_{n,x}) = 0$ if $n < m$. For $n \ge m$, plainly,

$$\begin{aligned}
\mathrm{P}\left(S_n \in x+I,\, B^m_{n,x}\right) &\le n^m\, \mathrm{P}\left(S_n \in x+I,\, \min_{1\le i\le m} X_i > \xi_{n,x},\, \max_{m+1\le j\le n} X_j \le \xi_{n,x}\right) \\
&= n^m \int_{(y,w)\in(0,x]^2} \mathrm{P}\left(S_m \in \mathrm{d}y,\, \min_{1\le i\le m} X_i \in \mathrm{d}w\right) \mathbb{1}_{\{w>\xi_{n,x}\}} \\
&\qquad\qquad \mathrm{P}\left(S_{n-m} \in x-y+I,\, B^0_{n-m,x}\right) .
\end{aligned}$$

Since $w > \xi_{n,x} := a_n^\gamma x^{1-\gamma}$ if and only if $a_n < (\frac{w}{x})^{1/\gamma} x$, i.e. $n < A((\frac{w}{x})^{1/\gamma} x)$, we obtain

$$\begin{aligned}
&\sum_{1\le n\le A(\delta x)} n^{\bar\ell}\, \mathrm{P}(S_n \in x+I,\, B^m_{n,x}) \\
&\quad \le \int_{(y,w)\in(0,x]^2} \left\{ \mathrm{P}\left(S_m \in \mathrm{d}y,\, \min_{1\le i\le m} X_i \in \mathrm{d}w\right) \right. \\
&\qquad\qquad \left. \sum_{m\le n\le A\left(\{(\frac{w}{x})^{1/\gamma}\wedge\delta\}x\right)} n^{\bar\ell+m}\, \mathrm{P}\left(S_{n-m} \in x-y+I,\, B^0_{n-m,x}\right) \right\},
\end{aligned} \tag{6.3}$$

where we set $a \wedge b := \min\{a,b\}$. The contribution to the sum of the single term $n = m$ can be bounded as follows: since $S_{n-m} = S_0 = 0$, by (3.6)

$$\int_{(y,w)\in(0,x]^2} \mathrm{P}\left(S_m \in \mathrm{d}y,\, \min_{1\le i\le m} X_i \in \mathrm{d}w\right) \mathbb{1}_{\{0\in x-y+I\}} \le \mathrm{P}(S_m \in x+I) = o\left(\frac{A(x)}{x}\right) ,$$

which is negligible for (6.2). Consequently, we can restrict the sum in (6.3) to $n \ge m+1$. In this case $n \le (m+1)(n-m) \lesssim_m (n-m)$, and renaming $n-m$ as n we simplify (6.3) as

$$\begin{aligned}
&\sum_{n\le A(\delta x)} n^{\bar\ell}\, \mathrm{P}(S_n \in x+I,\, B^m_{n,x}) \\
&\quad \lesssim_m \int_{(y,w)\in(0,x]^2} \left\{ \mathrm{P}\left(S_m \in \mathrm{d}y,\, \min_{1\le i\le m} X_i \in \mathrm{d}w\right) \right. \\
&\qquad\qquad \left. \sum_{1\le n\le A\left(\{(\frac{w}{x})^{1/\gamma}\wedge\delta\}x\right)} n^{\bar\ell+m}\, \mathrm{P}\left(S_n \in x-y+I,\, B^0_{n,x}\right) \right\}.
\end{aligned} \tag{6.4}$$

We split the domain of integration in (6.4) as $(0,x]^2 = J_1 \cup J_2 \cup J_3 \cup J_4$, where

$$\begin{gathered}
J_1 := \{y \le x - (\delta^\gamma x \wedge w)\} , \qquad J_2 := \{y > x-1\} , \\
J_3 := \{w > \delta^\gamma x,\, y \in (x-\delta^\gamma x, x-1]\} , \qquad J_4 := \{w \le \delta^\gamma x,\, y \in (x-w, x-1]\} .
\end{gathered}$$

and consider each sub-domain separately.

Contribution of J_1. Let us set

$$\hat\delta = \hat\delta(w,x,\delta) := \left(\frac{w}{x}\right)^{1/\gamma} \wedge \delta , \tag{6.5}$$

so that $J_1 = \{y \le x - \hat\delta^\gamma x\}$. Since $x - y \ge \hat\delta^\gamma x$ on J_1, the sum in (6.4) is bounded by

$$\sum_{1\le n\le A\left(\hat\delta x\right)} n^{\bar\ell+m} \sup_{z\ge\hat\delta^\gamma x} \mathrm{P}\left(S_n \in z+I,\, B^0_{n,x}\right) \lesssim e^{-\frac{1}{\hat\delta^{\gamma/3}}}\, \frac{A(x)^{\bar\ell+m+1}}{x} ,$$

where the inequality follows by Lemma 4.2, with δ replaced by $\hat{\delta}$ and ℓ replaced by $\bar{\ell}+m$. The contribution of J_1 to the integral in (6.4) is thus bounded by

$$\lesssim \frac{A(x)^{\bar{\ell}+m+1}}{x} \int_{w\in(0,x],\ y\in(0,x-(\delta^\gamma x\wedge w)]} \mathrm{P}\left(S_m \in \mathrm{d}y,\ \min_{1\le i\le m} X_i \in \mathrm{d}w\right) e^{-\frac{1}{(\frac{w}{x})^{1/3}\wedge \delta^{\gamma/3}}} . \tag{6.6}$$

We split this integral in the sub-domains $J_1^{\le} := \{w \le \delta^\gamma x\}$ and $J_1^{>} := \{w > \delta^\gamma x\}$. Bounding $\mathrm{P}(X > \delta^\gamma x) \lesssim 1/A(\delta^\gamma x) \lesssim \delta^{-2\gamma\alpha}/A(x)$, by the lower bound in (2.1) with $\varepsilon = \alpha$ and $\varrho = \delta$, the contribution of $J_1^{>}$ is controlled by

$$\begin{aligned}\frac{A(x)^{\bar{\ell}+m+1}}{x}\, e^{-\frac{1}{\delta^{\gamma/3}}}\, \mathrm{P}\left(\min_{1\le i\le m} X_i > \delta^\gamma x\right) &= e^{-\frac{1}{\delta^{\gamma/3}}}\, \frac{A(x)^{\bar{\ell}+m+1}}{x}\, \mathrm{P}\left(X > \delta^\gamma x\right)^m \\ &\lesssim \frac{e^{-\frac{1}{\delta^{\gamma/3}}}}{\delta^{2\gamma\alpha m}}\, \frac{A(x)^{\bar{\ell}+1}}{x} ,\end{aligned}$$

which gives no problem for (6.2). Next we bound the contribution of $J_1^{\le}$ to (6.6) by

$$\frac{A(x)^{\bar{\ell}+m+1}}{x} \int_{w\in(0,\delta^\gamma x]} \mathrm{P}\left(\min_{1\le i\le m} X_i \in \mathrm{d}w\right) \varphi\left(\frac{w}{x}\right) , \qquad \text{with} \qquad \varphi(t) := e^{-\frac{1}{t^{1/3}}} .$$

We set $G(w) := \mathrm{P}\left(\min_{1\le i\le m} X_i > w\right)$, so that $\mathrm{P}\left(\min_{1\le i\le m} X_i \in \mathrm{d}w\right) = -\mathrm{d}G(w)$. Integrating by parts, since the contribution of the boundary terms is negative, we get

$$\frac{A(x)^{\bar{\ell}+m+1}}{x} \int_0^{\delta^\gamma x} G(w)\, \varphi'\left(\frac{w}{x}\right) \frac{1}{x}\, \mathrm{d}w \lesssim \frac{A(x)^{\bar{\ell}+m+1}}{x} \int_0^{\delta^\gamma x} \frac{1}{A(w)^m}\, \varphi'\left(\frac{w}{x}\right) \frac{1}{x}\, \mathrm{d}w .$$

Performing the change of variable $v = w/x$, since $A(vx) \gtrsim A(x) v^{2\alpha}$ by (2.1), we obtain

$$\lesssim \frac{A(x)^{\bar{\ell}+1}}{x} \int_0^{\delta^\gamma} \frac{\varphi'(v)}{v^{2\alpha m}}\, \mathrm{d}v = \frac{A(x)^{\bar{\ell}+1}}{x} \int_0^{\delta^\gamma} \frac{e^{-\frac{1}{v^{1/3}}}}{3\, v^{2\alpha m + 4/3}}\, \mathrm{d}v \lesssim \frac{A(x)^{\bar{\ell}+1}}{x} \int_0^{\delta^\gamma} e^{-\frac{1}{2v^{1/3}}}\, \mathrm{d}v ,$$

which again gives no problem for (6.2). Overall, the contribution of J_1 is under control.

Contribution of J_2. By Lemma 5.1, for $x - y \le 1$ we have

$$\sum_{n\in\mathbb{N}} n^{\bar{\ell}+m}\, \mathrm{P}\left(S_{n-1} \in x - y + I\right) \le \sum_{n\in\mathbb{N}} \frac{n^{\bar{\ell}+m}}{a_{n-1}}\, e^{-c\frac{n-1}{A(1)}} =: C_{\bar{\ell}+m} < \infty , \tag{6.7}$$

because $z \mapsto A(z)$ is increasing, hence the contribution of J_2 to (6.4) is bounded by

$$\int_{(y,w)\in J_2} \mathrm{P}\left(S_m \in \mathrm{d}y,\ \min_{1\le i\le m} X_i \in \mathrm{d}w\right) C_{\bar{\ell}+m} \lesssim_{\bar{\ell},m} \mathrm{P}(S_m \in (x-1,x]) = o\left(\frac{A(x)}{x}\right) ,$$

where the last equality is a consequence of (3.1), see (3.6). This shows that J_2 gives a negligible contribution to (6.2).

Technical interlude. Before analyzing J_3 and J_4, let us elaborate on (6.1) (where we rename ℓ as k and x as z for later convenience). Our induction hypothesis that (6.1) holds for all $k \ge \bar{\ell}+1$ can be rewritten as follows: for every $\delta > 0$ there is $\bar{z}_k(\delta) < \infty$ such that

$$\sum_{1\le n\le A(\delta z)} n^k\, \mathrm{P}(S_n \in z + I) \le f_k(\delta)\, \frac{A(z)^{k+1}}{z} , \qquad \forall k \ge \bar{\ell}+1,\ \forall z \ge \bar{z}_k(\delta) , \tag{6.8}$$

where we set $f_k(\delta) := 2 \limsup_{x\to\infty}(\dots)$ in (6.1) (with ℓ replaced by k), so that

$$\lim_{\delta\to 0} f_k(\delta) = 0 . \tag{6.9}$$

We also claim that

$$\sum_{n\in\mathbb{N}} n^k \, \mathrm{P}(S_n \in z+I) \lesssim_k \frac{A(z)^{k+1}}{z}\,, \qquad \forall k \geq \bar{\ell}+1,\ \forall z \geq 1\,. \tag{6.10}$$

To show this, fix $\bar{\delta}_k \in (0,1]$ such that $f_k(\bar{\delta}_k) \leq 1$, by (6.9). If we restrict the sum to $n \leq A(\bar{\delta}_k z)$, relation (6.8) shows that (6.10) holds for $z \geq \bar{z}_k(\bar{\delta})$, while for $z \leq \bar{z}_k(\bar{\delta})$

$$\sum_{n\leq A(\bar{\delta}_k z)} n^k \, \mathrm{P}(S_n \in z+I) \leq \sum_{n \leq A(z)} n^k \leq A(z)^{k+1} \leq \bar{z}_k(\bar{\delta}_k)\, \frac{A(z)^{k+1}}{z} \lesssim_k \frac{A(z)^{k+1}}{z}\,.$$

It remains to prove that (6.10) holds for the sum restricted to the terms with $n > A(\bar{\delta}_k z)$: applying (5.1) followed by $a_n \geq a_{A(\bar{\delta}_k z)} = \bar{\delta}_k z \gtrsim_k z$, we can write

$$\sum_{n>A(\bar{\delta}_k z)} n^k \, \mathrm{P}(S_n \in z+I) \lesssim \sum_{n>A(\bar{\delta}_k z)} \frac{n^k}{a_n}\, e^{-c\frac{n}{A(z)}} \lesssim_k \frac{A(z)^{k+1}}{z} \left\{ \sum_{n\in\mathbb{N}} \frac{1}{A(z)} \left(\frac{n}{A(z)}\right)^k e^{-c\frac{n}{A(z)}} \right\}.$$

The bracket is a Riemann sum which converges to the integral $\int_0^\infty t^k\, e^{-ct}\, \mathrm{d}t < \infty$ as $z \to \infty$. Being a continuous function of z (by dominated convergence), the sum is uniformly bounded for $z \in [1,\infty)$. The proof of (6.10) is completed.

Let us finally rewrite (3.1), which is equivalent to our assumption (A.2), as follows: defining $g(\eta) := 2 \limsup_{x\to\infty}(\ldots)$ in (3.1), for every $\eta \in (0,1]$ there is $\widetilde{z}(\eta) < \infty$ such that

$$\int_{s\in[1,\eta z)} \frac{A(s)^2}{s}\, \mathrm{P}(X \in z - \mathrm{d}s) \leq g(\eta)\, \frac{A(z)}{z}\,, \qquad \forall z \geq \widetilde{z}(\eta)\,, \tag{6.11}$$

with

$$\lim_{\eta\to 0} g(\eta) = 0\,. \tag{6.12}$$

Moreover, in analogy with (6.10), we claim that

$$\int_{s\in[1,z)} \frac{A(s)^2}{s}\, \mathrm{P}(X \in z - \mathrm{d}s) \lesssim \frac{A(z)}{z}\,, \qquad \forall z \geq 1\,. \tag{6.13}$$

To show this, let us fix $\bar{\eta} \in (0,1)$ such that $g(\bar{\eta}) \leq 1$, and split $\int_{s\in[1,z)} = \int_{s\in[1,\bar{\eta}z)} + \int_{s\in[\bar{\eta}z,z)}$. The contribution of $[1,\bar{\eta}z)$ is controlled by relation (6.11) for $z \geq \widetilde{z}(\bar{\eta})$, while for $z < \widetilde{z}(\bar{\eta})$ it is enough to note that $c := \inf_{z\in[1,\widetilde{z}(\bar{\eta})]} \frac{A(z)}{z} > 0$ while

$$\sup_{z\in[1,\widetilde{z}(\bar{\eta})]} \int_{s\in[1,\bar{\eta}z)} \frac{A(s)^2}{s}\, \mathrm{P}(X \in z - \mathrm{d}s) \leq A(\widetilde{z}(\bar{\eta}))^2 =: C < \infty\,,$$

hence (6.13) holds restricted to $[1,\bar{\eta}z)$. Finally, for the integral over $[\bar{\eta}z, z)$ we estimate

$$\int_{s\in[\bar{\eta}z,z)} \frac{A(s)^2}{s}\, \mathrm{P}(X \in z - \mathrm{d}s) \leq \frac{A(z)^2}{\bar{\eta}\, z} \lesssim \frac{A(z)^2}{z}\,,$$

completing the proof of (6.13).

Contribution of J_3. We recall that

$$J_3 = \{w > \delta^\gamma x,\ y \in (x - \delta^\gamma x, x-1]\}\,.$$

For $m = 1$, since $S_m = \min_{1 \le i \le m} X_i = X_1$, we have $J_3 = \{y \in (x - \delta^\gamma x, x-1],\, w = y\}$. Applying (6.10) with $k = \bar\ell + 1$ and $z = x - y$, the contribution of J_3 to (6.4) is bounded by

$$\lesssim_{\bar\ell} \int_{y \in (x-\delta^\gamma x, x-1]} \mathrm{P}(X \in \mathrm{d}y)\, \frac{A(x-y)^{\bar\ell+2}}{x-y} \le A(x)^{\bar\ell} \int_{s \in [1, \delta^\gamma x)} \mathrm{P}(X \in x - \mathrm{d}s)\, \frac{A(s)^2}{s}\,,$$

where we have performed the change of variable $s = x - y$. Applying (3.1), or equivalently (6.11)-(6.12), it follows immediately that J_3 gives no problem for (6.2), when $m = 1$.

Next we assume that $m \ge 2$. It is convenient to set

$$\Lambda_m := \min_{1 \le i \le m} X_i\,, \qquad M_m := \max_{1 \le i \le m} X_i\,. \tag{6.14}$$

Applying (6.10) for $k = \bar\ell + m$, the contribution of J_3 to (6.4) is bounded by

$$\lesssim_{\bar\ell, m} \int_{y \in (x-\delta^\gamma x, x-1]} \left\{ \mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m > \delta^\gamma x\right) \frac{A(x-y)^{\bar\ell+m+1}}{x-y} \right\}. \tag{6.15}$$

We need to estimate $\mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m > \delta^\gamma x\right)$. The events $\{X_j \ge \max_{i \in \{1,\ldots,m\} \setminus \{j\}} X_i\}$ for $j = 1, \ldots, m$ cover the whole probability space and have the same probability, hence

$$\begin{aligned} &\mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m \in \mathrm{d}w\right) \le m\, \mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m \in \mathrm{d}w,\, M_{m-1} \le X_m\right) \\ &\quad \le m \int_{u,v \in (0,y]} \mathrm{P}\left(S_{m-1} \in \mathrm{d}u,\, \Lambda_{m-1} \in \mathrm{d}w,\, M_{m-1} \in \mathrm{d}v\right) \mathbb{1}_{\{v \le y-u\}}\, \mathrm{P}(X \in \mathrm{d}y - u)\,, \end{aligned} \tag{6.16}$$

where $\mathbb{1}_{\{v \le y-u\}}$ comes from $\{M_{m-1} \le X_m\}$. Note that

$$u = S_{m-1} \le (m-1) M_{m-1} = (m-1) v \le (m-1)(y-u)\,,$$

which yields the restriction $u \le \frac{m-1}{m} y$. In particular, for $y \le x$ we have $y \le \frac{m-1}{m} x$, which by (6.16) yields the bound

$$\begin{aligned} &\mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m \in \mathrm{d}w\right) \\ &\qquad \le m \int_{u \in (0, \frac{m-1}{m} x]} \mathrm{P}\left(S_{m-1} \in \mathrm{d}u,\, \Lambda_{m-1} \in \mathrm{d}w\right) \mathrm{P}(X \in \mathrm{d}y - u)\,. \end{aligned} \tag{6.17}$$

Plugging this into (6.15), the contribution of J_3 to (6.4) is bounded by

$$\begin{aligned} \lesssim_m \int_{u \in (0, \frac{m-1}{m} x]} &\mathrm{P}\left(S_{m-1} \in \mathrm{d}u,\, \Lambda_{m-1} > \delta^\gamma x\right) \\ &\left\{ \int_{y \in (x-\delta^\gamma x, x-1]} \mathrm{P}(X \in \mathrm{d}y - u) \frac{A(x-y)^{\bar\ell+m+1}}{x-y} \right\}. \end{aligned} \tag{6.18}$$

With the change of variables $s = x - y$, the term in bracket in (6.18) becomes

$$\begin{aligned} \int_{s \in [1, \delta^\gamma x)} \mathrm{P}(X \in x - u - \mathrm{d}s) \frac{A(s)^{\bar\ell+m+1}}{s} &\le A(\delta^\gamma x)^{\bar\ell+m-1} \int_{s \in [1, \delta^\gamma x)} \mathrm{P}(X \in x - u - \mathrm{d}s) \frac{A(s)^2}{s} \\ &\le A(\delta^\gamma x)^{\bar\ell+m-1} \int_{s \in [1, m\delta^\gamma (x-u))} \mathrm{P}(X \in x - u - \mathrm{d}s) \frac{A(s)^2}{s}\,, \end{aligned}$$

where in the last inequality we have enlarged the domain of integration, for $u \le \frac{m-1}{m} x$ (as in (6.18)). Since $x - u \ge \frac{1}{m} x$, we can apply (6.11) with $z = x - u$ and $\eta = m\delta^\gamma$, provided x

is large enough (so that $\frac{1}{m}x \geq \widetilde{z}(m\delta^\gamma)$). This allows to bound (6.18) by

$$\begin{aligned}
&\leq A(\delta^\gamma x)^{\bar{\ell}+m-1} \int_{u\in(0,\frac{m-1}{m}x]} \mathrm{P}\left(S_{m-1}\in \mathrm{d}u,\, \Lambda_{m-1} > \delta^\gamma x\right) \left\{ g(m\delta^\gamma)\, \frac{A(x-u)}{x-u} \right\} \\
&\leq A(\delta^\gamma x)^{\bar{\ell}+m-1} \mathrm{P}\left(\Lambda_{m-1} > \delta^\gamma x\right)\; g(m\delta^\gamma)\, \frac{A(x)}{\frac{1}{m}x}\,,
\end{aligned}$$

and since $\mathrm{P}\left(\Lambda_{m-1} > t\right) = \mathrm{P}(X>t)^{m-1} \sim 1/A(t)^{m-1}$ the last line is

$$\sim A(\delta^\gamma x)^{\bar{\ell}}\, g(m\delta^\gamma)\, \frac{m\, A(x)}{x} \lesssim_m g(m\delta^\gamma)\, \frac{A(x)^{\bar{\ell}+1}}{x}\,.$$

Plugging this bound into (6.2) and applying (6.12), we have shown that the contribution of J_3 is under control.

Contribution of J_4. Note that $J_4 := \{w \leq \delta^\gamma x,\, y \in (x-w, x-1]\}$ is empty for $m=1$, provided $\delta > 0$ is small enough: in fact, relations $y > x-w$ and $w \leq \delta^\gamma x$ cannot be fulfilled simultaneously, since $y = w$ for $m = 1$. Henceforth we assume that $m \geq 2$.

Recalling (6.14) and plugging (6.10) with $k = \bar{\ell}+m$ into (6.4), the contribution of J_4 is bounded as follows:

$$\lesssim_{\bar{\ell},m} \int_{w\in(1,\delta^\gamma x],\, y\in(x-w,x-1]} \mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m \in \mathrm{d}w\right) \frac{A(x-y)^{\bar{\ell}+m+1}}{x-y}\,. \tag{6.19}$$

Our goal is to show that this satisfies (6.2). It is convenient to set for $C, D \in (0,\infty)$

$$\Theta^{C,D}_{\bar{\ell},m}(x,\delta) := \int_{w\in[C,\delta^\gamma x],\, y\in[x-Dw,x-1]} \mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m \in \mathrm{d}w\right) \frac{A(x-y)^{\bar{\ell}+m+1}}{x-y}\,, \tag{6.20}$$

so that (6.19) is bounded from above by $\Theta^{C,D}_{\bar{\ell},m}(x,\delta)$ with $C = D = 1$. Consequently, to prove our goal (6.2) it is enough to show the following: recalling that $\bar{\ell} \in \{0,\dots,\kappa_\alpha\}$ is fixed,

$$\lim_{\delta\to 0}\left(\limsup_{x\to\infty} \frac{x}{A(x)^{\bar{\ell}+1}} \Theta^{C,D}_{\bar{\ell},m}(x,\delta) \right) = 0\,, \qquad \forall C,D \in (0,\infty)\,,\ \ \forall m \in \{1,2,\dots,\bar{m}\}\,. \tag{6.21}$$

Note that

$$\begin{aligned}
\mathrm{P}\left(S_m \in \mathrm{d}y,\, \Lambda_m \in \mathrm{d}w\right) &\leq m\, \mathrm{P}\left(S_m \in \mathrm{d}y,\, X_m \in \mathrm{d}w,\, \Lambda_{m-1} \geq w\right) \\
&= m\, \mathrm{P}(X \in \mathrm{d}w)\, \mathrm{P}\left(S_{m-1} \in \mathrm{d}y - w,\, \Lambda_{m-1} \geq w\right)\,,
\end{aligned}$$

therefore

$$\begin{aligned}
\Theta^{C,D}_{\bar{\ell},m}(x,\delta) \lesssim_m &\int_{w\in[C,\delta^\gamma x]} \mathrm{P}(X\in \mathrm{d}w) \\
&\left\{ \int_{y\in[x-Dw,x-1]} \mathrm{P}\left(S_{m-1} \in \mathrm{d}y - w,\, \Lambda_{m-1} \geq w\right)\, \frac{A(x-y)^{\bar{\ell}+m+1}}{x-y} \right\}.
\end{aligned}$$

Next we change variable from y to $s = (w+x) - y$ in the inner integral (for fixed w). Since $\mathrm{d}y - w = x - \mathrm{d}s$ and $x - y = s - w$, we get

$$\int_{w\in[C,\delta^\gamma x]} \mathrm{P}(X\in \mathrm{d}w) \left\{ \int_{s\in[1+w,(D+1)w]} \mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \geq w\right) \frac{A(s-w)^{\bar{\ell}+m+1}}{s-w} \right\}.$$

Writing $\mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \geq w\right) = \int_{u \in [0,\infty)} \mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \in \mathrm{d}u\right) \mathbb{1}_{\{u \geq w\}}$ and observing that

$$\{w \in [C, \delta^\gamma x],\, s \in [1+w, (1+D)w]\} \subseteq \{s \in [1+C, (1+D)\delta^\gamma x],\, w \in [\tfrac{s}{1+D}, s-1]\}\,,$$

we obtain by Fubini's theorem

$$\begin{aligned}\Theta^{C,D}_{\bar\ell,m}(x,\delta) \lesssim_m \int_{s \in [1+C,(1+D)\delta^\gamma x],\, u \in [0,\infty)} & \mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \in \mathrm{d}u\right) \\ & \left\{ \int_{w \in [\frac{s}{1+D}, s-1]} \mathrm{P}(X \in \mathrm{d}w)\, \frac{A(s-w)^{\bar\ell+m+1}}{s-w}\, \mathbb{1}_{\{w \leq u\}} \right\}.\end{aligned}$$

We can restrict the domain of integration for u to $[\frac{s}{1+D}, \infty)$, because for $u < \frac{s}{1+D}$ the inner integral vanishes, due to $\mathbb{1}_{\{w \leq u\}}$. After this restriction, we drop $\mathbb{1}_{\{w \leq u\}}$ and change variable from w to $t = s - w$ in the inner integral, getting

$$\begin{aligned}\Theta^{C,D}_{\bar\ell,m}(x,\delta) \lesssim_m \int_{s \in [1+C,(1+D)\delta^\gamma x],\, u \in [\frac{s}{1+D},\infty)} & \mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \in \mathrm{d}u\right) \\ & \left\{ \int_{t \in [1, \frac{Ds}{1+D}]} \mathrm{P}(X \in s - \mathrm{d}t)\, \frac{A(t)^{\bar\ell+m+1}}{t} \right\}.\end{aligned} \tag{6.22}$$

Since $[1, \frac{Ds}{1+D}] \subseteq [1, s)$, applying (6.13) with $z = s$ allows to bound the term in bracket by

$$A(s)^{\bar\ell+m-1} \int_{t \in (1,s]} \mathrm{P}(X \in s - \mathrm{d}t)\, \frac{A(t)^2}{t} \lesssim \frac{A(s)^{\bar\ell+m}}{s}\,, \tag{6.23}$$

hence from (6.22) we get the crucial estimate

$$\Theta^{C,D}_{\bar\ell,m}(x,\delta) \lesssim_m \int_{s \in [1+C,(1+D)\delta^\gamma x],\, u \in [\frac{s}{1+D},\infty)} \mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \in \mathrm{d}u\right) \frac{A(s)^{\bar\ell+m}}{s}\,. \tag{6.24}$$

Let us first consider the case $m = 2$. Then $S_{m-1} = X_1$, hence by (6.11) with $z = x$ and $\eta = (1+D)\delta^\gamma$ we get

$$\Theta^{C,D}_{\bar\ell,2}(x,\delta) \lesssim_m A(x)^{\bar\ell} \int_{s \in [1+C,(1+D)\delta^\gamma x]} \mathrm{P}\left(X \in x - \mathrm{d}s\right) \frac{A(s)^2}{s} \lesssim g\left((1+D)\delta^\gamma\right)\, \frac{A(x)^{\bar\ell+1}}{x}\,,$$

and recalling (6.12) it follows that (6.21) is proved.

Henceforth we assume that $m \geq 3$. We start focusing on the contribution to (6.24) given by $u \geq \delta^\gamma x$, which is bounded by

$$\int_{s \in [1+C,(1+D)\delta^\gamma x]} \mathrm{P}\left(S_{m-1} \in x - \mathrm{d}s,\, \Lambda_{m-1} \geq \delta^\gamma x\right) \frac{A(s)^{\bar\ell+m}}{s}\,,$$

and applying (6.17) with m replaced by $m-1$ we get, by Fubini's theorem,

$$
\begin{aligned}
&\lesssim_m \int_{s\in[1+C,(1+D)\delta^\gamma x]} \frac{A(s)^{\bar\ell+m}}{s} \\
&\qquad\qquad \left\{\int_{u\in(0,\frac{m-2}{m-1}x]} \mathrm{P}\left(S_{m-2}\in \mathrm{d}u,\, \Lambda_{m-2}\geq \delta^\gamma x\right) \mathrm{P}(X\in x-u-\mathrm{d}s)\right\} \\
&\leq A((1+D)\delta^\gamma x)^{\bar\ell+m-2} \int_{u\in(0,\frac{m-2}{m-1}x]} \mathrm{P}\left(S_{m-2}\in \mathrm{d}u,\, \Lambda_{m-2}\geq \delta^\gamma x\right) \\
&\qquad\qquad\qquad \left\{\int_{s\in[1+C,(1+D)\delta^\gamma x]} \frac{A(s)^2}{s} \mathrm{P}(X\in x-u-\mathrm{d}s)\right\}.
\end{aligned}
\tag{6.25}
$$

Concerning the inner integral, we enlarge the domain of integration to $[1,\hat\eta(x-u))$ with

$$
\hat\eta = \hat\eta_{m,D,\delta} := 2(1+D)\delta^\gamma \sup_{u\leq\frac{m-2}{m-1}x} \frac{x}{x-u} = 2(m-1)(1+D)\delta^\gamma\,, \tag{6.26}
$$

after which we can apply (6.11) with $z=x-u$ and $\eta=\hat\eta$ (which satisfies $z\geq\widetilde{z}(\hat\eta)$ provided x is large enough, since $x-u\geq\frac{x}{m-1}$). In this way,

$$
\left\{\int_{s\in[1+C,(1+D)\delta^\gamma x]} \frac{A(s)^2}{s} \mathrm{P}(X\in x-u-\mathrm{d}s)\right\} \leq g(\hat\eta)\,\frac{A(x-u)}{x-u} \lesssim_m g(\hat\eta)\,\frac{A(x)}{x}\,,
$$

where the last inequality holds again because $x-u\geq\frac{x}{m-1}$ (recall that $x\mapsto A(x)/x$ is regularly varying with index $\alpha-1<0$). Then (6.25) is bounded by

$$
\lesssim_D A(\delta^\gamma x)^{\bar\ell+m-2}\, g(\hat\eta)\,\frac{A(x)}{x}\, \mathrm{P}\left(\Lambda_{m-2}\geq\delta^\gamma x\right) \lesssim g(\hat\eta)\,\frac{A(x)^{\bar\ell+1}}{x}\,,
$$

because $\mathrm{P}(\Lambda_{m-2}\geq t)=\mathrm{P}(X\geq t)^{m-2}\sim 1/A(t)^{m-2}$. Looking back at our goal (6.21), and recalling (6.26) and (6.12), the contribution of $u\geq\delta^\gamma x$ to (6.24) is under control.

It finally remains to consider the contribution of $u<\delta^\gamma x$ to (6.24): since

$$
\{s\in[1+C,(1+D)\delta^\gamma x],\, u\in[\tfrac{s}{1+D},\delta^\gamma x]\} = \{u\in[\tfrac{1+C}{1+D},\delta^\gamma x],\, s\in[1+C,(1+D)u]\}\,,
$$

applying Fubini's theorem we can write such a contribution as follows:

$$
\begin{aligned}
&\int_{u\in[\frac{1+C}{1+D},\delta^\gamma x],\, s\in[1+C,(1+D)u]} \mathrm{P}\left(S_{m-1}\in x-\mathrm{d}s,\, \Lambda_{m-1}\in\mathrm{d}u\right) \frac{A(s)^{\bar\ell+m}}{s} \\
&= \int_{u\in[\frac{1+C}{1+D},\delta^\gamma x],\, y\in[x-(1+D)u,x-(1+C)]} \mathrm{P}\left(S_{m-1}\in\mathrm{d}y,\, \Lambda_{m-1}\in\mathrm{d}u\right) \frac{A(x-y)^{\bar\ell+m}}{x-y} \\
&\leq \Theta^{C',D'}_{\bar\ell,m-1}(x,\delta)\,, \qquad \text{with} \qquad C':=\tfrac{1+C}{1+D}\,,\ D':=1+D\,,
\end{aligned}
$$

where for the last inequality we recall (6.20). Therefore

$$
\lim_{\delta\to0}\left(\limsup_{x\to\infty} \frac{x}{A(x)^{\bar\ell+1}}\Theta^{C,D}_{\bar\ell,m}(x,\delta)\right) \leq \lim_{\delta\to0}\left(\limsup_{x\to\infty} \frac{x}{A(x)^{\bar\ell+1}}\Theta^{C',D'}_{\bar\ell,m-1}(x,\delta)\right). \tag{6.27}
$$

We can then conclude by induction on m. In fact, we have already proved that (6.21) holds for $m=2$, and relation (6.27) shows that if it holds for $m-1$ then it holds for m. $\square$

7. Proof of Proposition 1.6 and and of Theorems 1.7 and 1.9

7.1. **Proof of Proposition 1.6.** We can reformulate condition (1.15) equivalently as follows: there exist $x_0, C \in (0,\infty)$ such that (for the same $\varepsilon > 0$ as in (1.15))

$$\frac{F((x,x+s])}{F((x,\infty))} \le C \left(\frac{s}{x}\right)^{1-2\alpha+\varepsilon}, \qquad \forall x \ge x_0\,,\ \forall s \in [1,x]\,. \tag{7.1}$$

It is clear that (7.1) implies (1.15), and the converse also holds, by a contradiction argument.

Then it suffices to show that condition (7.1) implies (3.2) (which is equivalent to (1.12), by Lemma 3.1). For $x \ge 2x_0$ and $0 \le s \le \frac{1}{2}x$, by (7.1),

$$\mathrm{P}(X \in (x-s,x]) \le C\,\mathrm{P}(X > x-s) \left(\frac{s}{x}\right)^{1-2\alpha+\varepsilon} \lesssim \frac{1}{A(x)} \left(\frac{s}{x}\right)^{1-2\alpha+\varepsilon}. \tag{7.2}$$

Since $\frac{A(s)^2}{s^2}\, s^{1-2\alpha+\varepsilon}$ is regularly varying with index $(2\alpha-2)+1-2\alpha+\varepsilon = -1+\varepsilon > -1$, one has $\int_1^z \frac{A(s)^2}{s^2}\, s^{1-2\alpha+\varepsilon}\,\mathrm{d}s \lesssim A(z)^2\, z^{-2\alpha+\varepsilon}$ by [BGT89, Proposition 1.5.8], hence

$$\frac{x}{A(x)} \int_1^{\eta x} \frac{A(s)^2}{s^2}\,\mathrm{P}(X \in (x-s,x])\,\mathrm{d}s \lesssim \frac{A(\eta x)^2\,(\eta x)^{-2\alpha+\varepsilon}}{A(x)^2\,x^{-2\alpha+\varepsilon}} \xrightarrow[x\to\infty]{} \eta^{\varepsilon}\,.$$

Then (3.2) follows. □

7.2. **Proof of Theorem 1.7.** We recall that $A(x) = L(x)\sqrt{x}$ with $L \in \mathcal{R}_0$, and a sufficient condition for the SRT (1.6) when $\alpha = \frac{1}{2}$ is given by (1.18).

If (1.17) holds, we can write $L^*(x) \lesssim L(x) = A(x)/\sqrt{x}$, hence (1.18) is implied by

$$\exists T \in [0,\infty): \qquad \lim_{\eta\to 0}\left(\limsup_{x\to\infty} \frac{\mathsf{R}_T\big((1-\eta)x, x\big)}{x}\right) = 0\,. \tag{7.3}$$

It is easy to show that this holds for $T=0$, with no extra assumption on F. By (1.7)-(1.9)

$$\mathsf{R}_0\big((1-\eta)x,x\big) = \int_{(1-\eta)x}^{x} y\,A(y)\,F(y+I)\,\mathrm{d}y \le x\,A(x) \int_{(1-\eta)x}^{x} F(y+I)\,\mathrm{d}y\,, \tag{7.4}$$

and the last integral can be estimated as follows: by Fubini's theorem

$$\begin{aligned}
\int_{(1-\eta)x}^{x} F(y+I)\,\mathrm{d}y &= \int_{(1-\eta)x}^{x} \left(\int_{\mathbb{R}} \mathbb{1}_{\{t\in(y-h,y]\}}\,F(\mathrm{d}t)\right)\mathrm{d}y \\
&\le \int_{t\in((1-\eta)x-h,x]} \left(\int_{\mathbb{R}} \mathbb{1}_{\{y\in[t,t+h)\}}\,\mathrm{d}y\right) F(\mathrm{d}t) \\
&= h\,F\big((1-\eta)x-h,x]\big) \underset{x\to\infty}{\sim} h\left(\frac{1}{A((1-\eta)x)} - \frac{1}{A(x)}\right) \\
&\underset{x\to\infty}{\sim} h\,\frac{1}{A(x)}\left(\frac{1}{(1-\eta)^{\alpha}} - 1\right) \underset{\eta\to 0}{\sim} h\,\frac{1}{A(x)}\,\alpha\,\eta\,.
\end{aligned}$$

Recalling (7.4), it follows that (7.3) holds. This proves the first part of Theorem 1.7.

Next we observe that if F satisfies (1.1), then necessarily $F(x+I) = o(1/A(x))$ as $x\to\infty$. Interestingly, this bound can be approached as close as one wishes, in the following sense.

Lemma 7.1. *Fix two arbitrary positive sequences $(z_n)_{n\in\mathbb{N}}$, $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $z_n\to\infty$ and $\varepsilon_n \to 0$. For any $A(x) \in \mathcal{R}_\alpha$, with $\alpha\in(0,1)$, there are a constant $c\in(0,\infty)$, a subsequence*

$(n_k)_{k\in\mathbb{N}}$ *of* n *and a probability* F *on* $(0,\infty)$ *satisfying* (1.1) *such that*

$$F(\{z_{n_k}\}) \geq c\,\frac{\varepsilon_{n_k}}{A(z_{n_k})}\,, \qquad \forall k \in \mathbb{N}\,. \tag{7.5}$$

With Lemma 7.1 at hand, we prove the second part of Theorem 1.7. Assume that $A(x) \in \mathcal{R}_{1/2}$ is such that condition (1.17) fails, that is there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \infty$ such that

$$\zeta_n := \frac{L^*(x_n)}{L(x_n)} \to \infty\,. \tag{7.6}$$

By (1.16), since $L(\cdot)$ is continuous, we can write $L^*(x_n) = L(s_n)$ for some $1 \leq s_n \leq x_n$. We recall that, for any $\varepsilon > 0$, one has $L(s)/L(x_n) \to 1$ uniformly for $s \in [\varepsilon x_n, x_n]$, by the uniform convergence theorem of slowly varying functions [BGT89, Theorem 1.2.1]. Then it follows by (7.6) that necessarily $s_n = o(x_n)$. Summarizing:

$$x_n \to \infty\,, \qquad s_n = o(x_n)\,, \qquad \zeta_n \to \infty \qquad \text{with} \qquad \zeta_n = \frac{L(s_n)}{L(x_n)}\,.$$

Let us define

$$z_n := x_n - s_n\,, \qquad \varepsilon_n := \frac{1}{\zeta_n}\,,$$

so that $z_n \sim x_n \to \infty$ and $\varepsilon_n \to 0$. By Lemma 7.1, there are a subsequence $(n_k)_{k\in\mathbb{N}}$ of n and a probability F on $(0,\infty)$ such that (7.5) holds. Then, by $A(x) = L(x)\sqrt{x}$,

$$\begin{aligned}\int_1^{\eta x_{n_k}} \frac{A(s)^2}{s}\, F(x_{n_k} - \mathrm{d}s) &\geq \frac{A(s_{n_k})^2}{s_{n_k}}\, F(\{x_{n_k} - s_{n_k}\}) = L(s_{n_k})^2\, F(\{z_{n_k}\}) \\ &= \zeta_{n_k}^2\, L(x_{n_k})^2\, F(\{z_{n_k}\}) \geq \zeta_{n_k}^2\, \frac{A(x_{n_k})^2}{x_{n_k}}\, c\,\frac{\varepsilon_{n_k}}{A(z_{n_k})} \gtrsim \zeta_{n_k}\, c\,\frac{A(x_{n_k})}{x_{n_k}}\,,\end{aligned}$$

where in the last inequality we used the definition of ε_n and the fact that $A(x_n) \sim A(z_n)$, since $x_n \sim z_n$. Consequently, condition (3.1) is *not* satisfied, because for every $\eta > 0$

$$\limsup_{x\to\infty} \frac{x}{A(x)} \int_1^{\eta x} \frac{A(s)^2}{s}\, F(x_{n_k} - \mathrm{d}s) \geq \limsup_{k\to\infty} c\,\zeta_{n_k} = \infty\,.$$

Since (3.1) —which is equivalent to (1.11)— is necessary for the SRT (1.6), we have built an example of F satisfying (1.1) but not (1.6), completing the proof of Theorem 1.7. ☐

7.3. **Proof of Lemma 7.1.** Fix $n_0 \in \mathbb{N}$ such that $c_1 := \sum_{n\geq n_0+1} \frac{2\,\alpha}{n\,A(n)} < 1$. Then define a probability F_1 on $\mathbb{N}$ by

$$F_1(\{n\}) := (1 - c_1)\mathbb{1}_{\{n=n_0\}} + \frac{2\,\alpha}{n\,A(n)}\mathbb{1}_{\{n\geq n_0+1\}}\,, \tag{7.7}$$

so that

$$F_1((x,\infty)) \sim \frac{2}{A(x)} \qquad \text{as} \quad x \to \infty\,.$$

We may assume that $(x_n)_{n\in\mathbb{N}}$ is increasing. Fix a subsequence $(n_k)_{k\in\mathbb{N}}$ of n such that

$$\frac{\varepsilon_{n_{k+1}}}{A(x_{n_{k+1}})} \leq \frac{1}{2}\,\frac{\varepsilon_{n_k}}{A(x_{n_k})}\,, \qquad \forall k \in \mathbb{N}\,, \tag{7.8}$$

which is clearly possible since $A(x_{n_{k+1}}) \ge A(x_{n_k})$ and $\varepsilon_n \to 0$. Then define a probability F_2 supported by $E := \{x_{n_k} :\ k \in \mathbb{N}\}$ by

$$F_2(\{x_{n_k}\}) := c_2 \, \frac{\varepsilon_{n_k}}{A(x_{n_k})} \,, \qquad \text{where} \qquad c_2 := \left(\sum_{k\in\mathbb{N}} \frac{\varepsilon_{n_k}}{A(x_{n_k})} \right)^{-1} ,$$

and note that $c_2 > 0$ because the series converges, by (7.8). Given $x \in (0,\infty)$, if we define $\bar{k}(x) := \min\{k \in \mathbb{N} :\ x_{n_k} > x\}$, using (7.8) we can write

$$F_2((x,\infty)) = \sum_{k \ge \bar{k}(x)} c_2 \, \frac{\varepsilon_{n_k}}{A(x_{n_k})} \le c_2 \, \frac{\varepsilon_{n_{\bar{k}(x)}}}{A(x_{n_{\bar{k}(x)}})} \sum_{k\ge \bar{k}(x)} \frac{1}{2^{k-\bar{k}(x)}} \le c_2 \, \frac{\varepsilon_{n_{\bar{k}(x)}}}{A(x)} \, 2 \,,$$

where the last inequality holds because $x_{n_{\bar{k}(x)}} \ge x$ by construction. Since $\varepsilon_n \to 0$, we have shown that $F_2((x,\infty)) = o(1/A(x))$ as $x \to \infty$.

We can finally define the probability $F := \frac{1}{2}(F_1 + F_2)$, which satisfies (1.1) since

$$F((x,\infty)) \sim \frac{1}{2}\big(F_1((x,\infty)) + F_2((x,\infty))\big) \sim \frac{1}{2}\left(\frac{2}{A(x)} + o\left(\frac{1}{A(x)}\right)\right) \sim \frac{1}{A(x)} \,,$$

and by construction $F(\{x_{n_k}\}) \ge \frac{1}{2} F_2(\{x_{n_k}\})$, hence (7.5) holds with $c := c_2/2$. ☐

7.4. **Proof of Theorem 1.9.** The case $\alpha > \frac{1}{2}$ was already considered in Section 4, hence we focus on $\alpha \le \frac{1}{2}$. Since the necessity of (1.21) is proved in Appendix A.3, it remains to give examples of F satisfying (1.20) and (1.11) but not (1.21).

We first consider the case $\alpha < \frac{1}{2}$. We fix $A(x) := x^\alpha$ and, in analogy with (7.7), we define a *symmetric* probability F_1 on $\mathbb{Z}$ by

$$F_1(\{n\}) := c_1 \, \mathbb{1}_{\{|n| = n_0\}} \; + \; \frac{2\alpha}{n\,A(n)} \, \mathbb{1}_{\{|n| \ge n_0 + 1\}} \,, \tag{7.9}$$

where $c_1 \in (0,1)$ and $n_0 \in \mathbb{N}$ are chosen so that $\sum_{n\in\mathbb{Z}} F_1(\{n\}) = 1$. Note that

$$F_1((-\infty,-x]) \sim F_1((x,\infty)) \sim \frac{2}{x^\alpha} = \frac{2}{A(x)} \qquad \text{as} \quad x \to \infty \,. \tag{7.10}$$

For $n, k \in \mathbb{N}$ we define (recall that $\alpha < \frac{1}{2}$)

$$x_n := 2^n \,, \qquad z_k := k^{\frac{1}{1-2\alpha}} \,, \qquad E_n := \{x_{n,k} := x_n + z_k :\ 0 \le k < \hat{k}_n := \lfloor x_n^{1-2\alpha} \rfloor \} \tag{7.11}$$

so that E_n is a finite set of points in $[x_n, x_{n+1})$. Since $|E_n| \le 2 x_n^{1-2\alpha}$, we have

$$\sum_{y \in E_n} \frac{A(y)}{y\sqrt{\log y}} \le \frac{A(x_n)}{x_n \sqrt{\log x_n}} \, |E_n| \le \frac{2\, x_n^{1-2\alpha}}{x_n^{1-\alpha} \sqrt{\log x_n}} = \frac{2}{x_n^\alpha \sqrt{\log x_n}} =: d_n \,, \tag{7.12}$$

and note that $\sum_{n\in\mathbb{N}} d_n < \infty$, since $x_n = 2^n$. We can then define a probability F_2 by

$$F_2(\{y\}) := c_2 \, \frac{A(y)\, \mathbb{1}_{\{y \in E\}}}{y\sqrt{\log y}} = c_2 \, \frac{\mathbb{1}_{\{y\in E\}}}{y^{1-\alpha}\sqrt{\log y}} \,, \qquad \text{where} \qquad E := \bigcup_{n\in\mathbb{N}} E_n \,, \tag{7.13}$$

and c_2 is a normalizing constant. Note that for $x \in [x_\ell, x_{\ell+1})$ we have the upper bound

$$\begin{aligned} F_2((x,\infty)) &\le \sum_{n=\ell}^\infty F_2(E_n) \le \sum_{n=\ell}^\infty d_n \le \frac{c_2}{\sqrt{\log x_\ell}} \sum_{n=\ell}^\infty \frac{1}{2^{\alpha n}} \lesssim \frac{1}{\sqrt{\log x_\ell}} \, \frac{1}{2^{\alpha \ell}} = \frac{1}{x_\ell^\alpha \sqrt{\log x_\ell}} \\ &\lesssim \frac{1}{x^\alpha \sqrt{\log x}} = o\left(\frac{1}{x^\alpha}\right) = o\left(\frac{1}{A(x)}\right) \qquad \text{as } x \to \infty \,. \end{aligned} \tag{7.14}$$

Consequently, the probability $F := \frac{1}{2}(F_1 + F_2)$ satisfies (1.20) with $A(x) = x^\alpha$ and $p = q = 1$.

Let us show that F, does not satisfy (3.3), which is equivalent to (1.21). We focus on the second part of the integral. For $\eta < \frac{1}{2}$ and $x = x_n$, so that $[x_n + 1, x_n + \eta x_n) \subseteq E_n$, we have

$$\begin{aligned}\int_{[1,\eta x_n)} \frac{A(s)^2}{s} F_2(x_n + \mathrm{d}s) &= \sum_{y \in E_n} \frac{\mathbb{1}_{\{y \in [x_n+1, x_n+\eta x_n)\}}}{(y - x_n)^{1-2\alpha}} F_2(\{y\}) \\ &\ge F_2(\{x_{n+1}\}) \sum_{1 \le k < (\eta x_n)^{1-2\alpha}} \frac{1}{z_k^{1-2\alpha}} ,\end{aligned}$$

because $F_2(\{\cdot\})$ is decreasing on E. Recalling (7.11)-(7.13), since $\sum_{k=1}^z \frac{1}{k} \sim \log z$, we obtain

$$\int_{[1,\eta x_n)} \frac{A(s)^2}{s} F_2(x_n + \mathrm{d}s) \gtrsim_\eta F_2(\{x_{n+1}\}) \log x_n \gtrsim \frac{\sqrt{\log x_n}}{x_n^{1-\alpha}} = \sqrt{\log x_n}\, \frac{A(x_n)}{x_n} .$$

The $\limsup_{x\to\infty}$ in (3.3) then equals ∞ for every fixed $\eta > 0$, hence (3.3) does not hold.

Let us finally show that F does satisfy (3.1), which is equivalent to (1.11) by Lemma 3.1. Since F_1 clearly satisfies (3.1), it suffices to focus on F_2. Note that

$$\begin{aligned}\int_{[1,\eta x)} \frac{A(s)^2}{s} F_2(x - \mathrm{d}s) &\le \left\{ \sup_{z \in (x-\eta x, x-1]} F_2(\{z\}) \right\} \sum_{y \in E} \frac{1}{(x-y)^{1-2\alpha}} \mathbb{1}_{\{y \in (x-\eta x, x-1]\}} \\ &\lesssim \frac{1}{x^{1-\alpha}\sqrt{\log x}} \sum_{y\in E} \frac{1}{(x-y)^{1-2\alpha}} \mathbb{1}_{\{y \in (x-\eta x, x-1]\}} .\end{aligned} \tag{7.15}$$

For $x \ge 5$ we have $x - 1 \in [x_\ell, x_{\ell+1})$ for some $\ell \ge 2$. For $\eta < \frac{1}{2}$, certainly $x - \eta x > \frac{x}{2} \ge x_{\ell-1}$, hence we can replace $\mathbb{1}_{\{y\in(x-\eta x,x-1]\}}$ by $\mathbb{1}_{\{y\in[x_{\ell-1},x-1]\}}$ in (7.15), getting

$$\int_{[1,\eta x)} \frac{A(s)^2}{s} F_2(x - \mathrm{d}s) \lesssim \frac{A(x)}{x\sqrt{\log x}} \left\{ \sum_{y\in E_{\ell-1}} \frac{1}{(x-y)^{1-2\alpha}} + \sum_{y \in E_\ell} \frac{\mathbb{1}_{\{y \le x-1\}}}{(x-y)^{1-2\alpha}} \right\} . \tag{7.16}$$

It suffices to show that both sums are uniformly bounded, and relation (3.1) holds.

We start looking at the second sum. Writing $y = x_{\ell,k}$, by (7.11), the constraint $y \le x - 1$ becomes $k \le \bar{k}$ for a suitable $\bar{k} = \bar{k}_x$ (the precise value is immaterial), hence

$$\sum_{y\in E_\ell} \frac{\mathbb{1}_{\{y\le x-1\}}}{(x-y)^{1-2\alpha}} = \sum_{0 \le k \le \bar{k}} \frac{1}{(x - x_{\ell,k})^{1-2\alpha}} \le 1 + \sum_{0\le k\le \bar{k}-1} \frac{1}{(x_{\ell,\bar{k}} - x_{\ell,k})^{1-2\alpha}} , \tag{7.17}$$

where we have bounded the term $k = \bar{k}$ by $x - x_{\ell,\bar{k}} \ge x - (x-1) = 1$, while for the terms $k < \bar{k}$ we have replaced x by $x_{\ell,\bar{k}} < x$. Next observe that for $k = \bar{k} - i$

$$x_{\ell,\bar{k}} - x_{\ell,\bar{k}-i} = z_{\bar{k}} - z_{\bar{k}-i} = \bar{k}^{\frac{1}{1-2\alpha}} - (\bar{k}-i)^{\frac{1}{1-2\alpha}} = \bar{k}^{\frac{1}{1-2\alpha}} \left[1 - (1 - \tfrac{i}{\bar{k}})^{\frac{1}{1-2\alpha}} \right] .$$

Since $1 - (1-x)^\gamma \ge x$ for $0 \le x \le 1$ and $\gamma \ge 1$, we obtain $x_{\ell,\bar{k}} - x_{\ell,\bar{k}-i} \ge \bar{k}^{\frac{2\alpha}{1-2\alpha}} i$, hence

$$\sum_{y\in E_\ell} \frac{\mathbb{1}_{\{y\le x-1\}}}{(x-y)^{1-2\alpha}} \le 1 + \sum_{1\le i\le\bar{k}} \frac{1}{\left(\bar{k}^{\frac{2\alpha}{1-2\alpha}} i\right)^{1-2\alpha}} = 1 + \frac{1}{\bar{k}^{2\alpha}} \sum_{1\le i\le \bar{k}} \frac{1}{i^{1-2\alpha}} \lesssim 1 ,$$

uniformly over $\bar{k}$, by (2.2). Analogously, for the first sum in (7.16), we can write $y = x_{\ell-1,k}$ and sum over $0 \le k \le \hat{k}$ with $\hat{k} := \hat{k}_{\ell-1}$ (recall (7.11)). Arguing as before, we can bound

$$\sum_{y \in E_{\ell-1}} \frac{1}{(x-y)^{1-2\alpha}} = \sum_{0 \le k \le \hat{k}} \frac{1}{(x - x_{\ell-1,k})^{1-2\alpha}} \le 1 + \sum_{0 \le k \le \hat{k}-1} \frac{1}{(x_{\ell-1,\hat{k}} - x_{\ell-1,k})^{1-2\alpha}} ,$$

and also this sum is $\lesssim 1$, by the previous steps with $\hat{k}$ in place of $\bar{k}$.

We finally consider the case $\alpha = \frac{1}{2}$. We fix $A(x) := \sqrt{x}/\log(1+x)$ and we define F_1 as in (7.9) (with our current $A(x)$), so that (7.10) holds. Next we change (7.11) to

$$x_n := 2^n , \qquad z_k := e^{\sqrt{k}} - 1, \qquad E_n := \{x_{n,k} := x_n + z_k : \ 0 \le k < \hat{k}_n := \lfloor \log(1+x_n) \rfloor^2\} ,$$

and note that $E_n \subseteq [x_n, x_{n+1})$. We then define a probability F_2 supported by $E := \bigcup_{n\in\mathbb{N}} E_n$:

$$F_2(\{z\}) := c_2 \, \frac{A(y) \, \mathbb{1}_{\{y \in E\}}}{y \sqrt{\log\log(1+y)}} = c_2 \, \frac{\mathbb{1}_{\{y\in E\}}}{\sqrt{y} \, \log(1+y) \, \sqrt{\log\log(1+y)}} .$$

Since $|E_n| \le 2(\log(1+x_n))^2$, we can write

$$\sum_{y\in E_n} \frac{A(y)}{y\sqrt{\log\log(1+y)}} \le \frac{|E_n|}{\sqrt{x_n} \, \log(1+x_n) \, \sqrt{\log\log(1+x_n)}} \le \frac{2 \, \log(1+x_n)}{\sqrt{x_n} \, \sqrt{\log\log(1+x_n)}} =: d_n ,$$

hence for $x \in [x_\ell, x_{\ell+1})$ we have the upper bound

$$F_2((x,\infty)) \le \sum_{n=\ell}^{\infty} F_2(E_n) \le c_2 \sum_{n=\ell}^{\infty} d_n \lesssim d_\ell \lesssim \frac{\log(1+x)}{\sqrt{x} \, \sqrt{\log\log(1+x)}} = o\left(\frac{1}{A(x)}\right) .$$

It follows that $F := \frac{1}{2}(F_1 + F_2)$ satisfies (1.20) with $A(x) = \sqrt{x}/\log(1+x)$ and $p = q = 1$.

To show that F does not satisfy (3.3), note that for $\eta < \frac{1}{2}$ and $x = x_n$ we have

$$\begin{aligned}
\int_{[1,\eta x_n)} \frac{A(s)^2}{s} \, F(x_n + \mathrm{d}s) &\ge F_2(\{x_{n+1}\}) \sum_{1 \le k \le \lfloor \log(1+\eta x_n) \rfloor^2} \frac{1}{(\log(1+z_k))^2} \\
&\gtrsim \frac{A(x_n)}{x_n \sqrt{\log\log(1+x_n)}} \, \log\{\lfloor \log(1+\eta x_n)\rfloor^2\} \gtrsim_\eta \frac{A(x_n)}{x_n} \sqrt{\log\log x_n} .
\end{aligned}$$

Finally, to show that F satisfies (3.1), arguing as in (7.15) we get the analogue of (7.16):

$$\begin{aligned}
\int_{[1,\eta x)} \frac{A(s)^2}{s} \, F_2(x - \mathrm{d}s) \lesssim \frac{A(x)}{x\sqrt{\log\log(1+x)}} \Bigg\{ & \sum_{y\in E_{\ell-1}} \frac{1}{[\log(1+x-y)]^2} \\
& + \sum_{y \in E_\ell} \frac{\mathbb{1}_{\{y \le x-1\}}}{[\log(1+x-y)]^2} \Bigg\} ,
\end{aligned} \tag{7.18}$$

and it remains to show that both sums are bounded. For a suitable $\bar{k} = \bar{k}_x$ the second sum is

$$\sum_{0\le k\le \bar{k}} \frac{1}{[\log(1+x-x_{\ell,k})]^2} \le \frac{1}{(\log 2)^2} + \sum_{0 \le k \le \bar{k}-1} \frac{1}{[\log(1+x_{\ell,\bar{k}} - x_{\ell,k})]^2} , \tag{7.19}$$

where we have bounded the term $k = \bar{k}$ by $x - x_{\ell,\bar{k}} \ge x - (x-1) = 1$ and we have replaced x by $x_{\ell,\bar{k}}$ in the remaining terms. Next we note that for all $k \le \bar{k} - 1$

$$\log(1 + x_{\ell,\bar{k}} - x_{\ell,k}) \ge \log(1 + x_{\ell,\bar{k}} - x_{\ell,\bar{k}-1}) = \log\left(1 + e^{\sqrt{\bar{k}}} - e^{\sqrt{\bar{k}-1}}\right) \gtrsim \log \frac{e^{\sqrt{\bar{k}}}}{\sqrt{\bar{k}}} \gtrsim \sqrt{\bar{k}} ,$$

which plugged into (7.19) shows that the sum is uniformly bounded. The first sum in (7.18) is estimated similarly, replacing ℓ by $\ell-1$ and $\bar{k}$ by $\hat{k}_{\ell-1}$. This completes the proof. □

Appendix A. Miscellanea

A.1. **Proof of Lemma 3.1.** By (1.7), uniformly for $0 \le s \le \eta x$ and $\eta < \frac{1}{2}$, we can write

$$F(x-s+I) \sim \frac{\mathsf{r}(x-s)}{(x-s)\,A(x-s)} \simeq \frac{\mathsf{r}(x-s)}{x\,A(x)}\,, \tag{A.1}$$

and analogously with s replaced by $-s$. Then (1.21) is equivalent to the following relation:

$$\lim_{\eta\to 0}\left(\limsup_{x\to\infty} \frac{x}{A(x)} \int_1^{\eta x} \frac{A(s)^2}{s} \big(F(x-s+I) + \mathbb{1}_{\{q>0\}} F(x+s+I)\big)\,\mathrm{d}s\right) = 0\,. \tag{A.2}$$

We show below that (A.2) is equivalent to (3.3). Then (1.21) is equivalent to (3.3), i.e. the last statement in Lemma 3.1 holds. For $q=0$, we have the equivalence of (1.11) and (3.1).

Let us now prove the equivalence of relations (3.2) and (1.12). Since $h>0$ is fixed, uniformly for $0 \le s \le \eta x$ and $\eta < \frac{1}{2}$ we can write

$$\mathrm{P}(X \in (x-s,x]) \sim \mathrm{P}(X \in (x-s,x-h]) = \int_{\mathbb{R}} \mathbb{1}_{\{t\in[h,s)\}}\, F(x-\mathrm{d}t)$$

Writing $1 = \frac{1}{h}\int_{\mathbb{R}} \mathbb{1}_{\{u\in(t-h,t]\}}\,\mathrm{d}u$, for any fixed t, by Fubini's theorem we get

$$\mathrm{P}(X \in (x-s,x]) \sim \frac{1}{h}\int_0^s \left(\int_{\mathbb{R}} \mathbb{1}_{\{t\in[u,u+h)\}}\, F(x-\mathrm{d}t)\right) \mathrm{d}u = \frac{1}{h}\int_0^s F(x-u+I)\,\mathrm{d}u\,.$$

Applying (A.1) then gives

$$\mathrm{P}(X \in (x-s,x]) \sim \frac{1}{h}\,\frac{1}{x\,A(x)} \int_0^s \mathsf{r}(x-u+I)\,\mathrm{d}u = \frac{1}{h}\,\frac{1}{x\,A(x)}\,\mathsf{R}_0(x-s,x)\,,$$

which shows that (3.2) is equivalent to (1.12).

It remains to prove the equivalence of (A.2) and (3.3). We recall that $I=(-h,0]$ and, for this purpose, we can take $h>0$ arbitrarily also in the lattice case. We first claim that in (3.3) one can equivalently replace the domain of integration $[1,\eta x)$ by $[1+h,\eta x)$. For this it is enough to show that the interval $[1,1+h)$ gives a contribution to (3.3) which is dominated by that of $[1+h,1+2h)$. The function $A(s)^2/s$ is continuous and strictly positive, hence it is bounded away from zero and infinity in any compact interval. Then for x,x' large enough

$$\frac{x}{A(x)} \int_{s\in[1,1+h)} \frac{A(s)^2}{s}\, \mathrm{P}(X \in x-\mathrm{d}s) \lesssim \frac{x}{A(x)}\, \mathrm{P}(X \in (x-h-1,x-1])\,,$$

$$\frac{x'}{A(x')} \int_{s\in[1+h,1+2h)} \frac{A(s)^2}{s}\, \mathrm{P}(X \in x'-\mathrm{d}s) \gtrsim \frac{x'}{A(x')}\, \mathrm{P}(X \in (x'-2h-1,x'-h-1])\,.$$

Choosing $x'=x+h$ and letting $x\to\infty$, since $\frac{x'}{A(x')} \sim \frac{x}{A(x)}$, we have proved the claim. With analogous estimates one deals with $\mathrm{P}(X \in x+\mathrm{d}s)$ in (3.3).

Next we note that there are constants $0<c<C<\infty$ (depending on w) such that

$$c\left(\frac{1}{h}\int_{s-h}^s \frac{A(t)^2}{t}\,\mathrm{d}t\right) \le \frac{A(s)^2}{s} \le C\left(\frac{1}{h}\int_{s-h}^s \frac{A(t)^2}{t}\,\mathrm{d}t\right)\,, \qquad \forall s \ge 1+h\,. \tag{A.3}$$

Plugging this into (3.3), where the domain of integration has been changed to $[1+h,\eta x)$, shows precisely that (3.3) is equivalent to (A.2). □

A.2. **Proof of Theorem 1.1: second part.** We show that condition (3.2) is equivalent to (3.1) for $\alpha < \frac{1}{2}$, while it is stronger for $\alpha = \frac{1}{2}$. By Lemma 3.1, an analogous statement holds for (1.12) and (1.11), proving the second part of Theorem 1.1.

For fixed x, we define $G(s) := \mathrm{P}(x - X \in (1, s]) = \mathrm{P}(x - X \leq s) - \mathrm{P}(x - X \leq 1)$ and note that $\mathrm{P}(x - X \in \mathrm{d}s) = \mathrm{d}G(s)$. Integrating by parts, since $G(1) = 0$ we get

$$\int_{s\in[1,\eta x)} \frac{A(s)^2}{s} \, \mathrm{P}(x - X \in \mathrm{d}s) = G(\eta x-) \, \frac{A(\eta x)^2}{\eta x} - \int_1^{\eta x} G(s) \, \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{A(s)^2}{s} \right) \mathrm{d}s \,. \tag{A.4}$$

The first term in the right hand side equals

$$\begin{aligned} \mathrm{P}(X \in (x - \eta x, x - 1)) \, \frac{A(\eta x)^2}{\eta x} &\underset{x\to\infty}{\sim} \left(\frac{1}{A((1-\eta)x)} - \frac{1}{A(x-1)} \right) \frac{A(\eta x)^2}{\eta x} \\ &\underset{x\to\infty}{\sim} \left(\frac{1}{(1-\eta)^\alpha} - 1 \right) \eta^{2\alpha - 1} \, \frac{A(x)}{x} = O(\eta^{2\alpha}) \, \frac{A(x)}{x} \,, \end{aligned} \tag{A.5}$$

hence this terms always gives a negligible contribution to the limit in (3.1).

Next observe that by (1.2)

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{A(s)^2}{s} = \frac{2A(s)A'(s)}{s} - \frac{A(s)^2}{s^2} \begin{cases} \underset{s\to\infty}{\sim} (2\alpha - 1) \frac{A(s)^2}{s^2} & \text{if } \alpha < \frac{1}{2} \\ \underset{s\to\infty}{=} o\left(\frac{A(s)^2}{s^2} \right) & \text{if } \alpha = \frac{1}{2} \end{cases} .$$

If $\alpha < \frac{1}{2}$, for the second term in (A.4) we can write

$$\begin{aligned} - \int_1^{\eta x} G(s) \, \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{A(s)^2}{s} \right) &\simeq \int_1^{\eta x} \frac{A(s)^2}{s^2} \, \mathrm{P}(X \in [x - s, x - 1)) \, \mathrm{d}s \\ &= \int_1^{\eta x} \frac{A(s)^2}{s^2} \, \mathrm{P}(X \in (x - s, x - 1)) \, \mathrm{d}s \,. \end{aligned} \tag{A.6}$$

If relation (3.2) holds, it follows by (A.4)-(A.5)-(A.6) that relation (3.1) also holds. Viceversa, if (3.1) holds, applying again (A.4)-(A.5)-(A.6) together with (3.5) (which is a consequence of (3.1)), we see that (3.2) holds. Thus (3.2) and (3.1) are equivalent for $\alpha < \frac{1}{2}$.

For $\alpha = \frac{1}{2}$ we can replace $\simeq$ by $\lesssim$ in (A.6), hence (3.2) still implies (3.1). ☐

A.3. **Necessity of** (1.11) **and** (1.21) **for the SRT.** We assume that F is a probability on $\mathbb{R}$ satisfying (1.20). We show that relation (3.4), which is equivalent to the SRT (1.6), implies (3.3), hence it implies (1.21), by Lemma 3.1. In particular, the case $q = 0$ shows that, assuming (1.1), relation (3.4) implies (1.11).

Recall that $J = (-v, 0]$, cf. (2.3). Assume that F satisfies (1.20) and define $K \subseteq \mathbb{R}$ by

$$K := \begin{cases} [1, 2] & \text{if } q = 0 \\ [-2, -1] \cup [1, 2] & \text{if } q > 0 \end{cases} , \tag{A.7}$$

Here is a mild refinement of the local limit theorem (2.4), there are $c, C \in (0, \infty)$ such that

$$\inf_{z\in\mathbb{R}:\, z/a_n \in K} \mathrm{P}\left(S_n \in z + J, \, \max_{1\leq i\leq n} X_i \leq C a_n \right) \geq \frac{c}{a_n} \,, \qquad \forall n \in \mathbb{N}. \tag{A.8}$$

This follows by [C13, Lemma 4.5], but it is worth giving a direct proof. By (2.4), there is $c_1 > 0$ such that

$$\inf_{z\in\mathbb{R}:\, z/a_n \in K} \mathrm{P}(S_n \in z + J) \geq \frac{c_1}{a_n} \,, \qquad \forall n \in \mathbb{N}, \tag{A.9}$$

because $\min_{z\in K}\varphi(z)>0$. Next, for the maximum restricted to $i\le n/2$ (assuming that n is even for simplicity, the odd case is analogous), we can write

$$\begin{aligned}\mathrm{P}\left(S_n\in z+J,\ \max_{1\le i\le\frac{n}{2}}X_i>Ca_n\right)&=\int_{\mathbb{R}}\mathrm{P}\Big(S_{\frac{n}{2}}\in \mathrm{d}y,\ \max_{1\le i\le\frac{n}{2}}X_i>Ca_n\Big)\mathrm{P}\left(S_{\frac{n}{2}}\in z-y+J\right)\\&\le \mathrm{P}\left(\max_{1\le i\le\frac{n}{2}}X_i>Ca_n\right)\left\{\sup_{x\in\mathbb{R}}\mathrm{P}\left(S_{\frac{n}{2}}\in x+J\right)\right\}.\end{aligned}$$

The term in bracket is $\le c_2/a_n$, by (2.5). By Potter's bounds (2.1) and by $A(a_n)=n$ we have $\mathrm{P}(X>Ca_n)\le c_3/A(Ca_n)\le c_4/(C^{\alpha/2}n)$, hence

$$\sup_{z\in\mathbb{R}}\mathrm{P}\left(S_n\in z+J,\ \max_{1\le i\le\frac{n}{2}}X_i>Ca_n\right)\le\frac{n}{2}\,\frac{c_4}{C^{\alpha/2}\,n}\,\frac{c_2}{a_n}=\frac{c_2\,c_4}{2\,C^{\alpha/2}\,a_n}.$$

The contribution of $\{\max_{\frac{n}{2}\le i\le n}X_i>Ca_n\}$ is the same, by exchangeability, hence by (A.9)

$$\inf_{z\in\mathbb{R}:\,z/a_n\in K}\mathrm{P}\left(S_n\in z+J,\ \max_{1\le i\le n}X_i\le Ca_n\right)\ge\frac{c_1}{a_n}-2\frac{c_2\,c_4}{2\,C^{\alpha/2}a_n}=\left(c_1-\frac{c_2\,c_4}{C^{\alpha/2}}\right)\frac{1}{a_n},$$

which proves (A.8), provided C is chosen large enough.

Next we argue as in [C13, Proposition 2.2]. Since $\{X_i>t,\ \max_{j\in\{1,\ldots,n\}\setminus\{i\}}X_j\le t\}$ are disjoint events for $i=1,\ldots,n$, we can write

$$\mathrm{P}(S_n\in x+J)\ge n\,\mathrm{P}\left(S_n\in x+J,\ X_n>\frac{x}{2},\ \max_{1\le j\le n-1}X_j\le\frac{x}{2}\right).\tag{A.10}$$

If $n\le A(\delta x)$ then $a_n\le\delta x$, hence $\frac{x}{2}>Ca_n$ for $\delta<\frac{1}{2C}$. Therefore, by (A.7)-(A.8),

$$\begin{aligned}\mathrm{P}(S_n\in x+J)&\ge\int_{\frac{x}{2}}^{\infty}\mathrm{P}(X\in\mathrm{d}y)\,n\,\mathrm{P}\left(S_{n-1}\in x-y+J,\ \max_{1\le j\le n-1}X_j\le Ca_n\right)\\&\ge\int_{\frac{x}{2}}^{\infty}\mathrm{P}(X\in\mathrm{d}y)\,c\,\frac{n-1}{a_{n-1}}\,\mathbb{1}_{\{(x-y)/a_{n-1}\in K\}}\\&\ge\int_{\frac{x}{2}}^{\infty}\mathrm{P}(X\in\mathrm{d}y)\,c\,\frac{A\big(\frac{|x-y|}{2}\big)}{|x-y|}\,\mathbb{1}_{\{(x-y)/a_{n-1}\in K\}},\end{aligned}\tag{A.11}$$

where the last inequality follows because $a_{n-1}\le|x-y|\le 2a_{n-1}$, by the definition (A.7) of K, and we recall that $a_m=A^{-1}(m)$. Let us assume that $q>0$. If we restrict the integral to $y\in(x-\delta x,x-1]\cup[x+1,x+\delta x)$, i.e. $1\le|y-x|<\delta x$, summing over $n\le A(\delta x)$ we get

$$\sum_{1\le n\le A(\delta x)}\mathbb{1}_{\{(x-y)/a_{n-1}\in K\}}=\sum_{1\le n\le A(\delta x)}\mathbb{1}_{\{A(\frac{|x-y|}{2})\le n-1\le A(|x-y|)\}}\ge A(|x-y|)-A(\tfrac{|x-y|}{2}).$$

Since $A(z)-A(\frac{z}{2})\gtrsim A(z)$, and also $A(\frac{z}{2})\gtrsim A(z)$, we obtain from (A.11)

$$\begin{aligned}\sum_{1\le n\le A(\delta x)}\mathrm{P}(S_n\in x+J)&\gtrsim\int_{1\le|y-x|<\delta x}\mathrm{P}(X\in\mathrm{d}y)\,\frac{A(|x-y|)^2}{|x-y|}\\&=\int_{s\in[1,\delta x)}\frac{A(s)^2}{s}\left(\mathrm{P}(X\in x-\mathrm{d}s)+\mathbb{1}_{\{q>0\}}\mathrm{P}(X\in x+\mathrm{d}s)\right),\end{aligned}\tag{A.12}$$

where we performed the change of variables $s=x-y$ and we inserted $\mathbb{1}_{\{q>0\}}$ so that the formula holds also for $q=0$ (just restrict (A.11) to $y\in(x-\delta x,x-1]$).

Assume now that (3.4) holds. If we can replace $I = (-h, 0]$ by $J = (-v, 0]$ therein, (A.12) shows that (3.3) holds, completing the proof. To replace I by J, it suffices to write

$$\mathrm{P}(S_n \in x + J) \le \sum_{\ell=0}^{\lfloor v/h \rfloor} \mathrm{P}(S_n \in x_\ell + I) \,, \qquad \text{where} \qquad x_\ell := x - \ell h \,, \tag{A.13}$$

and note that relation (3.4) holds replacing $\mathrm{P}(S_n \in x + I)$ by $\mathrm{P}(S_n \in x_\ell + I)$, for fixed ℓ, because $x/A(x) \sim x_\ell / A(x_\ell)$. (Since $v > 0$ and $h > 0$ are fixed, $\lfloor v/h \rfloor$ is also fixed.) ☐

A.4. **On condition** (1.11) **for** $\alpha > \frac{1}{2}$**.** Let us show that condition (1.11) is always satisfied for $\alpha > \frac{1}{2}$. By Lemma 3.1, it is equivalent to prove (3.1). Plainly,

$$\begin{aligned} \int_{s \in [1, \eta x)} \frac{A(s)^2}{s} \, \mathrm{P}(X \in x - \mathrm{d}s) &\le \left(\sup_{s \in [1, \eta x)} \frac{A(s)^2}{s} \right) \mathrm{P}(X \in (x - \eta x, x]) \\ &\lesssim \frac{A(\eta x)^2}{\eta x} \left(\frac{1}{A((1-\eta)x)} - \frac{1}{A(x)} \right) \\ &\underset{x \to \infty}{\sim} \frac{A(x)}{x} \eta^{2\alpha - 1} \left(\frac{1}{(1-\eta)^\alpha} - 1 \right) = \frac{A(x)}{x} \, O(\eta^{2\alpha}) \,, \end{aligned} \tag{A.14}$$

where the second inequality holds because $A(s)^2/s$ is regularly varying with index $2\alpha - 1 > 0$ and we can apply [BGT89, Theorem 1.5.3]. Consequently relation (3.1) holds. ☐

A.5. **Necessity of conditions** (3.5) **and** (3.6)**.** We prove that (3.1) implies (3.5) and (3.6). Let us consider relation (3.1) with x replaced by $x + 1$: restricting the integral to $y \in [1, 1 + w)$, since $A(s)^2/s$ is bounded away from zero, we get

$$\begin{aligned} 0 &= \lim_{\eta \to 0} \left(\limsup_{x \to \infty} \frac{x+1}{A(x+1)} \int_{s \in [1, 1+w)} \frac{A(s)^2}{s} \, \mathrm{P}(X \in x + 1 - \mathrm{d}s) \right) \\ &\gtrsim \limsup_{x \to \infty} \frac{x+1}{A(x+1)} \, \mathrm{P}(X \in (x - w, x]) = \limsup_{x \to \infty} \frac{x}{A(x)} \, \mathrm{P}(X \in (x - w, x]) \,, \end{aligned}$$

which is precisely (3.5). In order to obtain (3.6), let us write

$$\begin{aligned} \mathrm{P}(S_m (x - w, x]) &\le m \, \mathrm{P}\left(S_m \in (x - w, x], \max_{1 \le i \le m-1} X_i \le X_m \right) \\ &\le m \int_{y \in [0, \frac{m-1}{m} x]} \mathrm{P}\left(S_{m-1} \in \mathrm{d}y, \max_{1 \le i \le m-1} X_i \le x - y \right) \mathrm{P}(X \in (x - y - w, x - y]) \\ &\le m \sup_{z \in [\frac{1}{m} x, x]} \mathrm{P}(X \in (z - w, z]) = \sup_{z \in [\frac{1}{m} x, x]} o\left(\frac{A(z)}{z} \right) \lesssim o\left(\frac{A(x)}{x} \right) \,, \end{aligned}$$

completing the proof. ☐

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