LOCAL LARGE DEVIATIONS AND THE STRONG RENEWAL THEOREM

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Abstract. We establish two different, but related results for random walks in the domain of attraction of a stable law of index α. The first result is a local large deviation upper bound, valid for $\alpha \in (0,1) \cup (1,2)$, which improves on the classical Gnedenko and Stone local limit theorems. The second result, valid for $\alpha \in (0,1)$, is the derivation of necessary and sufficient conditions for the random walk to satisfy the *strong renewal theorem* (SRT). This solves a long-standing problem, which dates back to the 1962 paper of Garsia and Lamperti [GL62] for renewal processes (i.e. random walks with non-negative increments), and to the 1968 paper of Williamson [Wil68] for general random walks.

1. Introduction and results

This paper contains new results about asymptotically stable random walks. We first present a local large deviation estimate which improves the error term in the classical local limit theorems, without making any further assumptions (see Theorem 1.1). Then we exploit this bound to solve a long-standing problem, namely we establish necessary and sufficient conditions for the validity of the *strong renewal theorem* (SRT), both for renewal processes (Theorem 1.4) and for general random walks (Theorem 1.12). The corresponding result for Lévy processes is also presented (see Theorem 1.18).

This paper supersedes the individual preprints [Car15] and [Don15].

Notation. We set $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $RV(\gamma)$ the class of regularly varying functions with index γ, namely $f \in RV(\gamma)$ if and only if $f(x) = x^\gamma \ell(x)$ for some slowly varying function $\ell \in RV(0)$, see [BGT89]. Given $f, g : [0,\infty) \to (0,\infty)$ we write $f \sim g$ to mean $\lim_{s\to\infty} f(s)/g(s) = 1$, and $f \ll g$ to mean $\lim_{s\to\infty} f(s)/g(s) = 0$.

1.1. **Local large deviations.**

Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. real-valued random variables, with law F. Let $S_0 := 0$, $S_n := X_1 + \ldots + X_n$ be the associated random walk and

$$M_n := \max\{X_1, X_2, \ldots, X_n\} . \tag{1.1}$$

We assume that the law F is in the domain of attraction of a strictly stable law with index $\alpha \in (0,1) \cup (1,2)$, that is, with $\overline{F}(x) := F((x,\infty))$ and $F(x) := F((-\infty, x])$,

$$\overline{F}(x) \underset{x\to\infty}{\sim} \frac{p}{A(x)} \qquad \text{and} \qquad F(-x) \underset{x\to\infty}{\sim} \frac{q}{A(x)} \qquad \text{for some } \ A \in RV(\alpha) . \tag{1.2}$$

More explicitly, if we write $A(x) = x^\alpha / L(x)$, with $L(\cdot)$ slowly varying,

$$\mathrm{P}(X > x) \underset{x\to\infty}{\sim} p\, \frac{L(x)}{x^\alpha} \qquad \text{and} \qquad \mathrm{P}(X \le -x) \underset{x\to\infty}{\sim} q\, \frac{L(x)}{x^\alpha} .$$

Date: October 17, 2018.

2010 *Mathematics Subject Classification.* 60K05; 60G50.

Key words and phrases. Renewal Theorem, Local Limit Theorem, Regular Variation.

We assume that $p > 0$ and $q \geq 0$ (when $q = 0$, the second relation in (1.2) should be understood as $F(-x) = o(1/A(x))$). For $\alpha > 1$, we further assume that $\mathrm{E}[X] = 0$.

Without loss of generality, we may assume that $A \in RV(\alpha)$ is continuous and strictly increasing. If we introduce the norming sequence $a_n \in RV(1/\alpha)$ defined by

$$a_n := A^{-1}(n) , \qquad n \in \mathbb{N} , \tag{1.3}$$

then S_n/a_n converges in law to a random variable Y with a stable law of index α and positivity parameter $\rho = \mathrm{P}(Y > 0) = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan(\frac{p-q}{p+q} \tan \frac{\pi\alpha}{2}) > 0$ (because $p > 0$).

Our first main result is a local large deviation estimate for S_n, constrained on M_n.

Theorem 1.1 (Local Large Deviations). *Let F satisfy (1.2) with $\alpha \in (0,1) \cup (1,2)$ and $p > 0$, and $\mathrm{E}[X] = 0$ if $\alpha > 1$. Fix a bounded measurable $J \subseteq \mathbb{R}$. Given $\gamma \in (0,\infty)$, there is $C_0 = C_0(\gamma, J) < \infty$ such that, for all $n \in \mathbb{N}$ and $x \geq 0$, the following relation holds:*

$$\mathrm{P}(S_n \in x + J,\ M_n \leq \gamma x) \leq C_0 \, \frac{1}{a_n} \left(\frac{n}{A(x)} \right)^{\lceil 1/\gamma \rceil} , \tag{1.4}$$

where $\lceil x \rceil := \min\{n \in \mathbb{N} :\ n \geq x\}$ is the upper integer part of x. More explicitly:

$$\forall k \in \mathbb{N}, \quad \forall \gamma \in [\tfrac{1}{k}, \tfrac{1}{k-1}) : \qquad \mathrm{P}(S_n \in x + J,\ M_n \leq \gamma x) \leq C_0 \, \frac{1}{a_n} \left(\frac{n}{A(x)} \right)^{k} . \tag{1.5}$$

Moreover, for some $C_0' = C_0'(J) < \infty$,

$$\mathrm{P}(S_n \in x + J) \leq C_0' \, \frac{1}{a_n} \, \frac{n}{A(x)} . \tag{1.6}$$

The non-local version of (1.4), where $S_n \in x + J$ is replaced by $S_n \geq x$, is known as a *Fuk-Nagaev inequality* [Nag79]. This is the starting point of our proof of Theorem 1.1, see Section 3. We prove (1.4) through direct path estimates, combined with local limit theorems. Relation (1.6) is obtained as a simple corollary of (1.4) with $\gamma = 1$.

A heuristic explanation of (1.6) goes as follows: for large x, if $S_n \in x + J$, it is likely that *a single step X_i takes a value y comparable to x*. Since $\mathrm{P}(X_i > cx) \approx 1/A(x)$ by (1.2), and since there are n available steps, we get the factor $n/A(x)$ in (1.6). The extra factor $1/a_n$ comes from Gnedenko and Stone local limit theorems.

A similar argument sheds light on (1.4)-(1.5). Under the constraint $M_n \leq \gamma x$, with $\gamma \in [\frac{1}{k}, \frac{1}{k-1})$, the most likely way to have $S_n \in x + J$ is that *exactly k steps $X_{i_1}, \ldots, X_{i_k}$ take values comparable to x/k*, and this yields the factor $(n/A(x))^k$ in (1.5).

Remark 1.2. *The classical Gnedenko and Stone local limit theorems only give the weak bound $\mathrm{P}(S_n \in x + J) = o(\frac{1}{a_n})$ as $x/a_n \to \infty$. The inequality (1.6) improves quantitatively on this bound, with no further assumptions besides (1.2).*

The Cauchy case $\alpha = 1$ is left out from our analysis, because of the extra care needed to handle the centering issues. However, an analogue of Theorem 1.1 holds also in this case, as shown by Q. Berger in the recent paper [Ber17].

Finally, it is worth stressing that the estimate (1.6) is essentially optimal, under the mere assumption (1.2). However, if one makes extra local requirements on the step distribution, such as e.g. (1.15) below, one can correspondingly sharpen (1.6) along the same line of proof, see [Ber17, Theorem 2.4] (which is valid for any $\alpha \in (0,2)$).

1.2. **The strong renewal theorem.** Henceforth we assume that $\alpha \in (0,1)$. We say that F is *arithmetic* if it is supported by $h\mathbb{Z}$ for some $h > 0$, in which case the maximal value of $h > 0$ with this property is called the *arithmetic span* of F. It is convenient to set

$$I = (-h, 0] \qquad \text{where} \qquad h := \begin{cases} \text{arithmetic span of } F & \text{(if } F \text{ is arithmetic)} \\ \text{any fixed number} > 0 & \text{(if } F \text{ is non-arithmetic)}\,. \end{cases} \tag{1.7}$$

The renewal measure $U(\cdot)$ associated to F is the measure on $\mathbb{R}$ defined by

$$U(\mathrm{d}x) := \sum_{n \ge 0} F^{*n}(\mathrm{d}x) = \sum_{n \ge 0} \mathrm{P}(S_n \in \mathrm{d}x)\,. \tag{1.8}$$

It is well known (see [BGT89, Theorem 8.6.3] and [Chi15, Appendix]) that (1.2) implies

$$U([0,x]) \underset{x\to\infty}{\sim} \frac{\mathsf{C}}{\alpha} A(x)\,, \qquad \text{with} \qquad \mathsf{C} = \mathsf{C}(\alpha,\rho) = \alpha\, \mathrm{E}[Y^{-\alpha}\, \mathbb{1}_{\{Y>0\}}] \tag{1.9}$$

(recall that Y denotes a random variable with the limiting stable law). In the special case when $p = 1$ and $q = 0$ in (1.2) (so that $\rho = 1$) one has $\mathsf{C} = \frac{1}{\pi}\sin(\pi\alpha)$.

It is natural to wonder whether the local version of (1.9) holds, namely

$$U(x+I) = U((x-h,x]) \underset{x\to\infty}{\sim} \mathsf{C}\, h\, \frac{A(x)}{x}\,. \tag{SRT}$$

For a more usual formulation, we can write $A(x) = x^\alpha / L(x)$ with $L(\cdot)$ slowly varying:

$$U(x+I) = U((x-h,x]) \underset{x\to\infty}{\sim} \mathsf{C}\, h\, \frac{1}{L(x)\, x^{1-\alpha}}\,. \tag{SRT}$$

This relation, called *strong renewal theorem* (SRT), is known to follow from (1.2) when $\alpha > \frac{1}{2}$, see [GL62, Wil68, Eri70, Eri71]. However, when $\alpha \le \frac{1}{2}$ there are examples of F satisfying (1.2) but not (SRT). The reason is that small values of n in (1.8) can give an anomalous contribution to the renewal measure (see Subsection 4.1 for more details).

In order for the SRT to hold, when $\alpha \le \frac{1}{2}$, extra assumptions are needed. Sufficient conditions have been derived along the years [Wil68, Don97, VT13, Chi15, Chi13], but none of these is necessary. In this paper we settle this problem, determining *necessary and sufficient conditions for the SRT*: see Theorem 1.4 for renewal processes and Theorem 1.12 for random walks. We also obtain very explicit and sharp sufficient conditions, which refine those in the literature, see Propositions 1.7 and 1.17. Our results are referred to in the recent papers [Ber17, Ber18, Chi18, DN17, DW18, FMMV18, Kev17, Kol17, MT17, Uch18].

Renewal theory is a very active area of current research. Besides its theoretical interest, the SRT for heavy tailed renewal processes has played a key role in applications, e.g. in pinning and related models of statistical mechanics, see [Gia07, Hol09, Gia11].

Let us proceed with our results. For $k \ge 0$ and $x \in \mathbb{R}$ we set

$$b_k(x) := \frac{A(|x|)^k}{|x| \vee 1}\,. \tag{1.10}$$

Note that $b_1(x) = A(x)/x = (L(x)x^{1-\alpha})^{-1}$ for $x \ge 1$ is precisely the rate in the right hand side of (SRT). In the sequel, we will often need to require that some quantity $J(\delta;x)$ is *much smaller than* $b_1(x)$*, when* $x \to \infty$ *followed by* $\delta \to 0$. This leads to the following

Definition 1.3. *A function* $J(\delta;x)$ *is* asymptotically negligible *(a.n.) if and only if*

$$\lim_{\delta\to 0} \limsup_{x\to+\infty} \frac{J(\delta;x)}{b_1(x)} = \lim_{\delta\to 0} \limsup_{x\to+\infty} \frac{J(\delta;x)}{A(x)/x} = 0\,. \tag{1.11}$$

We are ready to state our necessary and sufficient conditions for the SRT. We start with the case of renewal processes, which is simpler.

1.3. **The renewal process case.** Assume that F is a law on $[0,\infty)$ such that

$$\overline{F}(x) \underset{x\to\infty}{\sim} \frac{1}{A(x)} \qquad \text{for some} \quad A \in RV(\alpha)\,, \tag{1.12}$$

which is a special case of (1.2) with $p=1$, $q=0$. For $\delta > 0$ and $x \geq 0$ we set

$$I_1^+(\delta;x) := \int_{1\leq z\leq \delta x} F(x-\mathrm{d}z)\, b_2(z) = \int_{1\leq z\leq \delta x} F(x-\mathrm{d}z)\, \frac{A(z)^2}{z}\,. \tag{1.13}$$

The following is our main result for renewal processes.

Theorem 1.4 (SRT for Renewal Processes)**.** *Let F be a probability on $[0,\infty)$ satisfying* (1.12) *with $\alpha \in (0,1)$. Define $I = (-h, 0]$ with $h > 0$ as in* (1.7)*.*

- *If $\alpha > \frac{1}{2}$, the SRT holds with no extra assumption on F.*
- *If $\alpha \leq \frac{1}{2}$, the SRT holds if and only if $I_1^+(\delta;x)$ is a.n. (see Definition 1.3).*

Let us spell out the condition "$I_1^+(\delta;x)$ is a.n." for $A(x) = x^\alpha/L(x)$:

$$\text{“}I_1^+(\delta;x) \text{ is a.n.”} \qquad \Longleftrightarrow \qquad \lim_{\delta\to 0}\, \limsup_{x\to\infty} \frac{L(x)}{x^{\alpha-1}} \int_{1\leq z\leq\delta x} F(x-\mathrm{d}z)\, \frac{z^{2\alpha-1}}{L(z)^2} \;=\; 0\,.$$

This can be checked in concrete examples, if one has enough control on $F(\cdot)$. We will soon deduce more explicit sufficient conditions, see Proposition 1.7, which are almost optimal.

Interestingly, in the "boundary" case $\alpha = \frac{1}{2}$, we can characterize the class of $A(\cdot)$'s for which the SRT holds with no extra assumption on F besides (1.12) (like for $\alpha > \frac{1}{2}$).

Theorem 1.5 (SRT for Renewal Processes with $\alpha = \frac{1}{2}$)**.** *Let F be a probability on $[0,\infty)$ satisfying* (1.12) *with $\alpha = \frac{1}{2}$ (so that $A(x)/\sqrt{x}$ is a slowly varying function). If*

$$\sup_{1\leq s\leq x} \frac{A(s)}{\sqrt{s}} \underset{x\to\infty}{=} O\biggl(\frac{A(x)}{\sqrt{x}}\biggr)\,, \tag{1.14}$$

then the SRT holds with no extra assumption on F. (This includes the case $A(x) \sim c\sqrt{x}$.) If condition (1.14) *fails, there are examples of F for which the SRT fails.*

The proof of Theorem 1.4 is based on direct probabilistic arguments and is remarkably compact ($\simeq$ 6 pages). We start in Section 4 recalling a reformulation of the SRT, which can be paraphrased as follows: *the contribution of "small n" to the renewal measure* (1.8) *is asymptotically negligible* (see Subsection 4.1). In Section 4 we also derive two key bounds on the contribution of "big jumps", see Lemmas 4.2 and 4.3. We complete the proof of Theorem 1.4 in Subsection 5.1 (necessity) and in Section 6 (sufficiency).

1.4. **Sufficient conditions for renewal processes.** For a probability F on $[0,\infty)$ which satisfies (1.12), a sufficient condition for the SRT is that for some $x_0, C < \infty$ one has

$$F(x+I) \;\leq\; \frac{C}{x\,A(x)} \qquad \forall x \geq x_0\,, \tag{1.15}$$

as proved by Doney [Don97] in the arithmetic case (extending previous results of Williamson [Wil68]), and by Vatutin and Topchii [VT13] in the non-arithmetic case.

Interestingly, if one only looks at the growth of the "local" probabilities $F(x+I)$, *no sharper condition than* (1.15) *can ensure that the SRT holds*, as the following result shows.

Proposition 1.6. *Fix $A \in RV(\alpha)$ with $\alpha \in (0, \frac{1}{2})$, and let $\zeta : (0,\infty) \to (0,\infty)$ be an arbitrary non-decreasing function with $\lim_{x\to\infty} \zeta(x) = \infty$. Then there exists a probability F on $[0,\infty)$ which satisfies (1.12), such that $F(x+I) = O(\frac{\zeta(x)}{xA(x)})$, for which the SRT fails.*

Intuitively, when condition (1.15) is *not* satisfied, in order for the SRT to hold, the points x for which $F(x+I) \gg \frac{1}{xA(x)}$ must not be "too cluttered". We can make this loose statement precise by looking at the probability of intervals $F((x-y,x])$. The following result provides very explicit conditions on $F(\cdot)$ for the SRT.

Proposition 1.7. *Let F be a probability on $[0,\infty)$ satisfying (1.12) with $\alpha \in (0, \frac{1}{2}]$.*

- *A sufficient condition for the SRT is that for some $\gamma > 1 - 2\alpha$ and $x_0, C < \infty$ one has*
$$F((x-y,x]) \le \frac{C}{A(x)} \left(\frac{y}{x}\right)^{\gamma} \qquad \forall x \ge x_0\,, \quad \forall y \in \left[1, \tfrac{1}{2}x\right]. \tag{1.16}$$
- *A necessary condition for the SRT is that for every $\gamma < 1 - 2\alpha$ there are $x_0, C < \infty$ such that (1.16) holds.*

Remark 1.8. *The sufficient condition (1.16) is a generalization of (1.15). In fact, if (1.15) holds, then (1.16) holds with $\gamma = 1$, since $F((x-y,x]) \le \sum_{j=0}^{\lceil y/h \rceil} F(x-j+I) \le \frac{2y}{h} O(\frac{1}{xA(x)})$.*

Remark 1.9. *Other sufficient conditions for the SRT, which generalize and sharpen (1.15), were given by Chi in [Chi15, Chi13]. These can be deduced from Theorem 1.4.*

Remark 1.10. *Conditions similar to (1.16), in a different context, appear in [CSZ16].*

We point out that if F satisfies (1.12), then (1.16) holds with $\gamma = 0$. However, with no extra assumption, one cannot hope to improve this estimate, as Lemma 10.2 below shows.

To see how condition (1.16) appears, let us introduce the following variant of (1.13):
$$\tilde{I}_1^+(\delta;x) := \int_1^{\delta x} \frac{F((x-z,x])}{z}\, b_2(z)\, \mathrm{d}z = \int_1^{\delta x} \frac{F((x-z,x])}{z} \, \frac{A(z)^2}{z}\, \mathrm{d}z\,. \tag{1.17}$$
Our next result shows that one can look at $\tilde{I}_1^+(\delta;x)$ instead of $I_1^+(\delta;x)$.

Proposition 1.11. *Let F be a probability on $[0,\infty)$ satisfying (1.12) with $\alpha \in (0, \frac{1}{2}]$.*

- *If $\tilde{I}_1^+(\delta;x)$ is a.n., then also $I_1^+(\delta;x)$ is a.n., hence the SRT holds.*
- *When $\alpha < \frac{1}{2}$, the converse is also true: $\tilde{I}_1^+(\delta;x)$ is a.n. if and only if $I_1^+(\delta;x)$ is a.n..*

1.5. **The general random walk case.** We now turn to the general random walk case, which is more challenging. We assume that F is a probability on $\mathbb{R}$ which satisfies (1.2) with $\alpha \in (0,1)$, $p > 0$ and $q \ge 0$.

Let us generalize (1.13) as follows: for $\delta > 0$ and $x \ge 0$ we set:
$$I_1(\delta;x) := \int_{|y|\le \delta x} F(x+\mathrm{d}y)\, b_2(y)\,. \tag{1.18}$$
For $k \in \mathbb{N}$ with $k \ge 2$, we introduce a further parameter $\eta \in (0,1)$ and we set
$$I_k(\delta,\eta;x) := \int\limits_{|y_1|\le\delta x} F(x+\mathrm{d}y_1) \int\!\!\cdots\!\!\int\limits_{|y_j|\le\eta|y_{j-1}| \text{ for } 2\le j\le k} P_{y_1}(\mathrm{d}y_2,\ldots,\mathrm{d}y_k)\, b_{k+1}(y_k)\,, \tag{1.19}$$
$$\text{where} \quad P_{y_1}(\mathrm{d}y_2,\ldots,\mathrm{d}y_k) := F(-y_1+\mathrm{d}y_2)F(-y_2+\mathrm{d}y_3)\cdots F(-y_{k-1}+\mathrm{d}y_k)\,.$$

Note that $P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)$ is the law of $(S_2, \ldots, S_k)$ conditionally on $S_1 = y_1$, hence

$$I_k(\delta, \eta; x) = \mathrm{E}\left[b_{k+1}(S_k)\, \mathbb{1}_{|S_j| \le \eta |S_{j-1}| \text{ for } 2 \le j \le k\}}\, \mathbb{1}_{|S_1| \le \delta x\}} \,\middle|\, S_0 = -x\right]. \tag{1.20}$$

The same formula holds also for $k = 1$ (where the first indicator function equals 1).

Let us define

$$\kappa_\alpha := \left\lfloor \frac{1}{\alpha} \right\rfloor - 1 = \begin{cases} 0 & \text{if } \alpha \in (\frac{1}{2}, 1) \\ 1 & \text{if } \alpha \in (\frac{1}{3}, \frac{1}{2}] \\ 2 & \text{if } \alpha \in (\frac{1}{4}, \frac{1}{3}] \\ \vdots & \\ m & \text{if } \alpha \in (\frac{1}{m+2}, \frac{1}{m+1}] \end{cases}, \tag{1.21}$$

We are going to see that, when $1/\alpha \notin \mathbb{N}$, necessary and sufficient conditions for the SRT involve the a.n. of $I_k(\delta, \eta; x)$ for $k = \kappa_\alpha$. The case $1/\alpha \in \mathbb{N}$ is slightly more involved. We need to introduce a suitable modification of (1.10), namely

$$\tilde{b}_k(z, x) := \tilde{b}_k(|z|, |x|) := \int_{|x|}^{|z|} \frac{b_k(t)}{t \vee 1}\, \mathrm{d}t, \tag{1.22}$$

where the integral vanishes if $|x| > |z|$. We then define $\tilde{I}_1(\delta; x)$ and $\tilde{I}_k(\delta, \eta; x)$ in analogy with (1.18) and (1.19), replacing $b_2(y)$ by $\tilde{b}_2(\delta x, y)$ and $b_{k+1}(y_k)$ by $\tilde{b}_{k+1}(y_{k-1}, y_k)$:

$$\tilde{I}_1(\delta; x) := \int_{|y| \le \delta x} F(x + \mathrm{d}y)\, \tilde{b}_2(\delta x, y), \tag{1.23}$$

and for $k \ge 2$:

$$\tilde{I}_k(\delta, \eta; x) := \int\limits_{|y_1| \le \delta x} F(x + \mathrm{d}y_1) \int \cdots \int\limits_{|y_j| \le \eta |y_{j-1}| \text{ for } 2 \le j \le k} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, \tilde{b}_{k+1}(y_{k-1}, y_k). \tag{1.24}$$

Note that, by Fubini's theorem, we can equivalently rewrite (1.23) as follows:

$$\tilde{I}_1(\delta; x) = \int_0^{\delta x} \frac{F((x - t, x + t])}{t \vee 1}\, b_2(t)\, \mathrm{d}t, \tag{1.25}$$

which is a natural random walk generalization of (1.17).

We can now state our main result for random walks.

Theorem 1.12 (SRT for Random Walks)**.** *Let F be a probability on $\mathbb{R}$ satisfying* (1.12) *with $\alpha \in (0,1)$ and with $p, q > 0$. Define $I = (-h, 0]$ with $h > 0$ as in* (1.7)*.*

- *If $\alpha > \frac{1}{2}$, the SRT holds with no extra assumption on F.*
- *If $\alpha \le \frac{1}{2}$ and $\frac{1}{\alpha} \notin \mathbb{N}$, we distinguish two cases:*
    - *if $\alpha \in (\frac{1}{3}, \frac{1}{2})$, i.e. $\kappa_\alpha = 1$, the SRT holds if and only if $I_1(\delta; x)$ is a.n..*
    - *if $\alpha \in (\frac{1}{k+2}, \frac{1}{k+1})$ for some $k = \kappa_\alpha \ge 2$, the SRT holds if and only if $I_{\kappa_\alpha}(\delta, \eta; x)$ is a.n., for every fixed $\eta \in (0,1)$.*
- *If $\alpha \le \frac{1}{2}$ and $\frac{1}{\alpha} \in \mathbb{N}$, the same statement holds if we replace I_k by $\tilde{I}_k$, namely:*
    - *if $\alpha = \frac{1}{2}$, i.e. $\kappa_\alpha = 1$, the SRT holds if and only if $\tilde{I}_1(\delta; x)$ is a.n..*
    - *if $\alpha = \frac{1}{k+1}$, for some $k = \kappa_\alpha \ge 2$, the SRT holds if and only if $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n., for every fixed $\eta \in (0,1)$.*

In Appendix A we show some relations between the quantities I_k and $\tilde{I}_k$. These lead to the following clarifying remarks.

Remark 1.13. *The condition "$\tilde{I}_{\kappa_\alpha}$ is a.n." is stronger than "I_{κ_α} is a.n.", but for $\frac{1}{\alpha} \notin \mathbb{N}$ they are equivalent (see Lemma A.3). As a consequence, we can rephrase Theorem 1.12 in a more compact way as follows:*

$$\text{The SRT holds:} \begin{cases} \text{with no extra assumption} & \text{for } \alpha > \frac{1}{2} \\ \text{iff } \tilde{I}_1(\delta; x) \text{ is a.n.} & \text{for } \frac{1}{3} < \alpha \le \frac{1}{2} \\ \text{iff } \tilde{I}_{\kappa_\alpha}(\delta, \eta; x) \text{ is a.n. for every } \eta \in (0,1) & \text{for } \alpha \le \frac{1}{3} \end{cases} \tag{1.26}$$

When $\alpha \le \frac{1}{3}$, our proof actually shows that if $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n. for some $\eta > 1 - \frac{\alpha}{1-\alpha}$, *then (the SRT holds and consequently) it is a.n. for* every $\eta \in (0, 1)$. *It is not clear whether the a.n. of $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ for some $\eta \le 1 - \frac{\alpha}{1-\alpha}$ also implies its a.n. for any $\eta \in (0, 1)$.*

Remark 1.14. *If $\frac{1}{\alpha} \notin \mathbb{N}$, the condition "$I_{\kappa_\alpha}(\delta, \eta; x)$ is a.n." is equivalent to the seemingly stronger one "$I_k(\delta, \eta; x)$ is a.n. for all $k \in \mathbb{N}$" (see Lemma A.2). Similarly, the condition "$\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n." is equivalent to "$\tilde{I}_k(\delta, \eta; x)$ is a.n. for all $k \in \mathbb{N}$" (see Lemma A.1).*

Remark 1.15. *In Theorem 1.12 we require $q > 0$ (that is the positivity index ρ is strictly less than one), but a large part of it actually extends to $q = 0$. More precisely, when $q = 0$, our proof shows that if $\alpha > \frac{1}{2}$ the SRT holds with no extra assumption on F, while if $\alpha \le \frac{1}{2}$ the a.n. of I_{κ_α} (if $\frac{1}{\alpha} \notin \mathbb{N}$) or $\tilde{I}_{\kappa_\alpha}$ (if $\frac{1}{\alpha} \in \mathbb{N}$) are sufficient conditions for the SRT. However, when $q = 0$, we do not expect the a.n. of I_{κ_α} or $\tilde{I}_{\kappa_\alpha}$ to be necessary, in general.*

1.6. **Sufficient conditions for random walks.** Necessary and sufficient conditions for the SRT in the random walk case involve the a.n. of $\tilde{I}_k$ for a suitable $k = \kappa_\alpha \in \mathbb{N}$. Unlike the renewal process case, this cannot be reduced to the a.n. of just $\tilde{I}_1$.

Proposition 1.16. *For any $\alpha \in (0, \frac{1}{3})$, there is a probability F on $\mathbb{R}$ which satisfies* (1.12), *such that $\tilde{I}_1(\delta; x)$ is a.n. but $\tilde{I}_2(\delta, \eta; x)$ is not a.n., for any $\eta \in (0, 1)$ (hence the SRT fails).*

Let us now give simpler sufficient conditions which ensure the a.n. of $\tilde{I}_k$. Note that the condition that $\tilde{I}_1(\delta; x)$ is a.n. only involves the *right tail* of F (see Definition 1.3). To express conditions on the *left tail* of F, we define

$$\tilde{I}_1^*(\delta; x) := \int_0^{\delta x} \frac{F((-x-t, -x+t])}{t \vee 1}\, b_2(t)\, \mathrm{d}t\,, \tag{1.27}$$

which is nothing but $\tilde{I}_1(\delta; x)$ in (1.25) applied to the *reflected probability* $F^*(A) := F(-A)$.

Proposition 1.17. *Let F be a probability on $\mathbb{R}$ satisfying* (1.12) *with $\alpha \in (0, \frac{1}{2}]$ and $p > 0$, $q \ge 0$. If both $\tilde{I}_1(\delta; x)$ and $\tilde{I}_1^*(\delta; x)$ are a.n., then the SRT holds.*

In particular, a sufficient condition for the SRT is that there exists $\gamma > 1 - 2\alpha$ such that relation (1.16) *holds both for F and for F^* (i.e., both as $x \to +\infty$ and as $x \to -\infty$).*

In particular, the SRT holds when the classical condition (1.15) *holds both for F and F^**

1.7. **Lévy processes.** Let $X = (X_t)_{t \ge 0}$ be a Lévy process with Lévy measure Π, Brownian coefficient σ^2 and linear term μ in its Lévy-Khintchine representation, that is

$$\log \mathrm{E}[e^{i\theta X_1}] = i\mu\theta - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta x \mathbb{1}_{\{|x| \le 1\}})\, \Pi(\mathrm{d}x)\,. \tag{1.28}$$

Whenever X is transient, we can define its potential or renewal measure by

$$G(\mathrm{d}x) := \int_0^\infty \mathrm{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t\,.$$

We assume that X is asymptotically stable: more precisely, there is a norming function $a(t)$ such that $X_t/a(t)$ converges in law as $t \to \infty$ to a random variable Y with a stable law of index $\alpha \in (0,1)$ and positivity parameter $\rho > 0$. In this case

$$A(x) := \frac{1}{\overline{\Pi}(x)} = \frac{1}{\Pi((x,\infty))} \in RV(\alpha) \qquad \text{as } x \to +\infty\,,$$

and we can take $a(\cdot) = A^{-1}(\cdot)$. Under these assumptions, the renewal theorem (1.9) holds, just replacing $U([0,x])$ by $G([0,x])$. It is natural to wonder whether the corresponding local version (SRT) holds as well, in which case we say that X satisfies the SRT.

Our next result shows that this question can be reduced to the validity of the SRT for a random walk whose step distribution F only depends on the Lévy measure Π, namely:

$$F(\mathrm{d}x) := \begin{cases} \dfrac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R}\setminus(-1,1))} & \text{for } |x| \geq 1 \\ 0 & \text{for } |x| < 1 \end{cases}. \tag{1.29}$$

Theorem 1.18 (SRT for Lévy Processes)**.** *Let X be any Lévy process that is in the domain of attraction of a stable law of index $\alpha \in (0,1)$ and positivity parameter $\rho > 0$ as $t \to \infty$. Suppose also that its Lévy measure is non-arithmetic. Then X satisfies the SRT, i.e.*

$$\lim_{x\to\infty} x\,\overline{\Pi}(x)\,G((x-h,x]) = h\,\alpha\,\mathrm{E}[Y^{-\alpha}\,\mathbb{1}_{\{Y>0\}}]\,, \qquad \forall h > 0\,, \tag{1.30}$$

if and only if the random walk with step distribution F defined in (1.29) *satisfies the SRT.*

As a consequence, Theorems 1.4 and 1.12 can be applied to X.

The proof of Theorem 1.18, given in Section 9, is obtained comparing the Lévy process X with a compound Poisson process with step distribution F.

Remark 1.19. *It is known, see* [Ber96, Proof of Theorem 21 on page 38]*, that the potential measure $G(\mathrm{d}x)$ of any Lévy process X coincides for $x \neq 0$ with the renewal measure of a random walk $(S_n)_{n\geq 0}$ with step distribution* $\mathrm{P}(S_1 \in \mathrm{d}x) := \int_0^\infty e^{-t}\,\mathrm{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t$. *It is also easy to see that X is in the domain of attraction of a stable law of index $\alpha \in (0,1)$ and positivity parameter $\rho > 0$, with norming function $a(t)$, if and only if the random walk S is in the domain of attraction of a the same stable law with norming function $a(n)$.*

So, if we write down necessary and sufficient conditions for S to verify the SRT, these will be necessary and sufficient conditions for X to verify the SRT. However this approach is unsatisfactory, because one would like conditions expressed in terms of the characteristics of X, i.e. the quantities Π, σ^2, μ appearing in the Lévy-Khintchine representation (1.28)*, and the technical problem of expressing our necessary and sufficient conditions for S to satisfy the SRT in terms of these characteristics seems quite challenging.*

1.8. **Structure of the paper.** The paper is organized as follows.

- In Section 2 we recall some standard results.
- In Section 3 we prove Theorem 1.1.
- Sections 4–8 are devoted to the proofs of Theorems 1.4 and 1.12.
    - In Section 4 we reformulate the SRT and we give two key bounds.

  - In Section 5 we prove the necessity part for both Theorems 1.4 and 1.12.
  - In Section 6 we prove the sufficiency part of Theorem 1.4.
  - The sufficiency part of Theorem 1.12 is proved in Section 7 for the case $\alpha > \frac{1}{3}$. The case $\alpha \le \frac{1}{3}$ is treated in Section 8 and is much more technical.
- In Section 9 we prove "soft" results, such as Theorem 1.5, Propositions 1.7, 1.11, 1.17, and Theorem 1.18, which are corollaries of our main results.
- In Section 10 we prove Propositions 1.6 and 1.16, which provide counter-examples.
- In Appendix A we prove some technical results.

2. Setup

2.1. **Notation.** We recall that $f(s) \lesssim g(s)$ or $f \lesssim g$ means $f(s) = O(g(s))$, i.e. for a suitable constant $C < \infty$ one has $f(s) \le C\, g(s)$ for all s in the range under consideration. The constant C may depend on the probability F (in particular, on α) and on h. When some extra parameter ϵ enters the constant $C = C_\epsilon$, we write $f(s) \lesssim_\epsilon g(s)$. If both $f \lesssim g$ and $g \lesssim f$, we write $f \approx g$. We recall that $f(s) \sim g(s)$ means $\lim_{s\to\infty} f(s)/g(s) = 1$.

2.2. **Regular variation.** Without loss of generality [BGT89, §1.3.2], we can assume that $A : [0,\infty) \to (0,\infty)$ is differentiable, strictly increasing and such that

$$A'(s) \sim \alpha \frac{A(s)}{s}, \qquad \text{as } s \to \infty. \tag{2.1}$$

We fix $A(0) := \frac{1}{2}$ and $A(1) := 1$, so that both A and A^{-1} map $[1,\infty)$ onto itself. We also write $a_u = A^{-1}(u)$ for all $u \in [\frac{1}{2}, \infty)$, in agreement with (1.3).

We observe that, by Potter's bounds, for every $\epsilon > 0$ one has

$$\rho^{\alpha+\epsilon} \lesssim_\epsilon \frac{A(\rho s)}{A(s)} \lesssim_\epsilon \rho^{\alpha-\epsilon}, \qquad \forall \rho \in (0,1],\ s \in [1,\infty) \ \text{ such that } \ \rho s \ge 1. \tag{2.2}$$

More precisely, part (i) of [BGT89, Theorem 1.5.6] shows that relation (2.2) holds for $\rho s \ge \bar{x}_\epsilon$, for a suitable $\bar{x}_\epsilon < \infty$; the extension to $1 \le \rho s \le \bar{x}_\epsilon$ follows as in part (ii) of the same theorem, because $A(y)$ is bounded away from zero and infinity for $y \in [1, \bar{x}_\epsilon]$.

We also recall Karamata's Theorem [BGT89, Propositions 1.5.8 and 1.5.10]:

$$\text{if } f \in RV(\zeta) \text{ with } \zeta > -1: \qquad \int_{s \le t} f(s)\,\mathrm{d}s \underset{t\to\infty}{\sim} \sum_{n \le t} f(n) \underset{t\to\infty}{\sim} \frac{1}{\zeta+1}\, t\, f(t), \tag{2.3}$$

$$\text{if } f \in RV(\zeta) \text{ with } \zeta < -1: \qquad \int_{s > t} f(s)\,\mathrm{d}s \underset{t\to\infty}{\sim} \sum_{n > t} f(n) \underset{t\to\infty}{\sim} \frac{-1}{\zeta+1}\, t\, f(t). \tag{2.4}$$

2.3. **Local limit theorems.** We call a probability F on $\mathbb{R}$ *lattice* if it is supported by $v\mathbb{Z}+a$ for some $v > 0$ and $0 \le a < v$, and the maximal value of $v > 0$ with this property is called the *lattice span* of F. If F is arithmetic (i.e. supported by $h\mathbb{Z}$), then it is also lattice, but the spans might differ (for instance, $F(\{-1\}) = F(\{+1\}) = \frac{1}{2}$ has arithmetic span $h = 1$ and lattice span $v = 2$). A lattice distribution is not necessarily arithmetic.[†]

[†] If F is lattice, say supported by $v\mathbb{Z} + a$ where v is the lattice span and $a \in [0, v)$, then F is arithmetic if and only if $a/v \in \mathbb{Q}$, in which case its arithmetic span equals $h = v/m$ for some $m \in \mathbb{N}$.

Recall that, under (1.2), S_n/a_n converges in distribution as $n \to \infty$ toward a stable law, whose density we denote by ϕ (the norming sequence a_n is defined in (1.3)). If we set

$$J = (-v, 0] \qquad \text{with} \qquad v = \begin{cases} \text{lattice span of } F & \text{(if } F \text{ is lattice)} \\ \text{any fixed number} > 0 & \text{(if } F \text{ is non-lattice)} \end{cases}, \tag{2.5}$$

Gnedenko's and Stone's local limit theorems [BGT89, Theorems 8.4.1 and 8.4.2] yield

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}} \left| a_n \, \mathrm{P}(S_n \in x + J) - v \, \phi\left(\frac{x}{a_n}\right) \right| = 0 \,. \tag{2.6}$$

Since $\sup_{z \in \mathbb{R}} \phi(z) < \infty$, we obtain the useful estimate

$$\sup_{x \in \mathbb{R}} \mathrm{P}(S_n \in (x - w, x]) \lesssim_w \frac{1}{a_n} \,, \tag{2.7}$$

which, plainly, holds for *any* fixed $w > 0$ (not necessarily the lattice span of F).

3. Proof of Theorem 1.1

We prove (1.4), equivalently (1.5), by steps. Without loss of generality, we assume that $J \subseteq [0, \infty)$ (it suffices to redefine $x \mapsto x' := x + \min J$ and $J \mapsto J' := J - \min J$).

Step 1. Our starting point is an integrated version of (1.4):

$$\forall \gamma \in (0, \infty) : \qquad \mathrm{P}(S_n \geq x, \, M_n \leq \gamma x) \lesssim_\gamma \left(\frac{n}{A(x)}\right)^{1/\gamma} . \tag{3.1}$$

This is a Fuk-Nagaev inequality, which follows from [Nag79, Theorems 1.1 and 1.2] (see [Ber17, Theorem 5.1] for a more transparent statement). Let us be more precise.

- *Case* $\alpha \in (0, 1)$. We apply equation (1.1) from [Nag79, Theorem 1.1] (neglecting the first term in the right hand side, which is the contribution of $M_n > y$): for every $y \in (0, x]$ and $t \in (0, 1]$, if we define $A(t; 0, y) := n \int_0^y u^t \, F(\mathrm{d}u)$, we have

$$\mathrm{P}(S_n \geq x, \, M_n \leq y) \leq P_1 = \left(\frac{e}{1 + \frac{x y^{t-1}}{A(t;0,y)}}\right)^{\frac{x}{y}} \leq \left(e \, \frac{A(t; 0, y)}{x y^{t-1}}\right)^{\frac{x}{y}} .$$

  We fix $t \in (\alpha, 1]$, so that $A(t; 0, y) \leq n \int_0^y t z^{t-1} \, \overline{F}(z) \, \mathrm{d}z \lesssim n y^t / A(y)$, thanks to (1.2) and (2.3). Taking $y = \gamma x$, since $A(y) \gtrsim_\gamma A(x)$, we obtain (3.1).

- *Case* $\alpha \in (1, 2)$. We apply equation (1.3) from [Nag79, Theorem 1.2]: for $y \in (0, x]$ and $t \in [1, 2]$, setting $A(t; -y, y) := n \int_{-y}^y |u|^t \, F(\mathrm{d}u)$ and $\mu(-y, y) := n \int_{-y}^y u \, F(\mathrm{d}u)$,

$$\mathrm{P}(S_n \geq x, \, M_n \leq y) \leq P_3 = \frac{e^{\frac{x}{y}}}{\left(1 + \frac{x y^{t-1}}{A(t;-y,y)}\right)^{\frac{x - \mu(-y,y)}{y} + \frac{A(t,-y-y)}{y^t}}} .$$

  We drop the term $A(t, -y-y)/y^t \geq 0$ from the exponent and get an upper bound. Next we fix $t \in (\alpha, 2]$, so that $A(t; -y, y) \lesssim n y^t / A(y)$ as before, hence

$$\mathrm{P}(S_n \geq x, \, M_n \leq y) \leq \frac{e^{\frac{x}{y}}}{\left(1 + \frac{x}{y} \frac{A(y)}{n}\right)^{\frac{x - \mu(-y,y)}{y}}} \leq \left(\frac{en}{A(y)}\right)^{\frac{x}{y}} \left(1 + \frac{x}{y} \frac{A(y)}{n}\right)^{\frac{\mu(-y,y)}{y}} .$$

  If we fix $y = \gamma x$, the first term in the right hand side matches with (3.1). It remains to show that the second term is bounded. Since we assume that F has zero mean, we can write $|\mu(-y,y)| = |-n\int_{|u|\geq y} u\, F(\mathrm{d}u)| \lesssim ny/A(y)$, by (1.2) and (2.4), therefore for $y = \gamma x$ the second term is $\lesssim (1+\frac{A(y)}{\gamma n})^{n/A(y)} \leq \exp(\frac{1}{\gamma})$. This proves (3.1).

Step 2. Next we deduce from (3.1) the following relation

$$\mathrm{P}(S_n \in x+J,\, M_n \leq \tfrac{1}{2}\gamma x) \lesssim \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{1/\gamma}, \tag{3.2}$$

which is rougher than (1.4), due to the factor $\frac{1}{2}$ and to the exponent $1/\gamma$ instead of $\lfloor 1/\gamma \rfloor$.

Define $\hat{X}_i := X_{n+1-i}$, for $1 \leq i \leq n$, and let $(\hat{S}_k := \hat{X}_1 + \ldots + \hat{X}_k = S_n - S_{n-k})_{1\leq k\leq n}$ be the corresponding random walk, which has the same law as $(S_k)_{1\leq k\leq n}$. Then

$$\begin{aligned}
\mathrm{P}(S_n \in x+J,\, S_{\lfloor n/2\rfloor} < \tfrac{x}{2},\, M_n \leq \tfrac{1}{2}\gamma x) &= \mathrm{P}(\hat{S}_n \in x+J,\, \hat{S}_{\lfloor n/2\rfloor} < \tfrac{x}{2},\, \hat{M}_n \leq \tfrac{1}{2}\gamma x) \\
&= \mathrm{P}(S_n \in x+J,\, S_n - S_{\lceil n/2\rceil} < \tfrac{x}{2},\, M_n \leq \tfrac{1}{2}\gamma x) \\
&\leq \mathrm{P}(S_n \in x+J,\, S_{\lceil n/2\rceil} > \tfrac{x}{2},\, M_n \leq \tfrac{1}{2}\gamma x)\,,
\end{aligned}$$

where the second equality holds because $\hat{S}_n = S_n$ and $\hat{M}_n = M_n$, while for the inequality note that $S_n \geq x$ (by $J \subseteq [0,\infty)$). To lighten notation, henceforth we assume that n is even (the odd case is analogous). It follows from the previous inequality that

$$\begin{aligned}
\mathrm{P}(S_n \in x+J,\, M_n \leq \tfrac{1}{2}\gamma x) &\leq 2\,\mathrm{P}(S_n \in x+J,\, S_{n/2} \geq \tfrac{x}{2},\, M_n \leq \tfrac{1}{2}\gamma x) \\
&\leq 2\int_{z\geq \frac{x}{2}} \mathrm{P}(S_{n/2} \in \mathrm{d}z,\, M_{n/2} \leq \tfrac{1}{2}\gamma x)\,\mathrm{P}(S_{n/2} \in x-z+J) \\
&\lesssim \frac{1}{a_{n/2}}\,\mathrm{P}(S_{n/2} \geq \tfrac{1}{2}x,\, M_{n/2} \leq \tfrac{1}{2}\gamma x) \lesssim \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{1/\gamma},
\end{aligned}$$

where we have used (2.7) and (3.1).

Step 3. Next we prove relation (1.6), i.e. we show that

$$\mathrm{P}(S_n \in x+J) \lesssim \frac{1}{a_n}\,\frac{n}{A(x)}\,. \tag{3.3}$$

This is easy: if we fix $\epsilon = \frac{1}{2}$, by (2.7) we can write

$$\begin{aligned}
\mathrm{P}(S_n \in x+J, M_n > \epsilon\, x) &\leq n\,\mathrm{P}(S_n \in x+J,\, X_1 > \epsilon\, x) \\
&= n\int_{y>\epsilon\, x} F(\mathrm{d}y)\,\mathrm{P}(S_{n-1} \in x-y+J) \lesssim \frac{n}{a_{n-1}}\,\overline{F}(\epsilon\, x) \lesssim_\epsilon \frac{1}{a_n}\,\frac{n}{A(x)}\,.
\end{aligned} \tag{3.4}$$

Applying (3.2) with $\gamma = 1$, we see that (3.3) holds.

Step 4. Finally we prove (1.5). The case $k = 1$, that is $\gamma \in [1,\infty)$, follows by (3.3). Inductively, we fix $k \in \mathbb{N}$ and we prove that (1.5) holds for $\gamma \in [\frac{1}{k+1}, \frac{1}{k})$, assuming that it holds for $\gamma \in [\frac{1}{k}, \frac{1}{k-1})$. Let us fix $\epsilon := \frac{1}{2(k+1)}$. By (3.2) (where we choose $\gamma = 2\epsilon$) we get

$$\mathrm{P}(S_n \in x+J,\;M_n \leq \epsilon x) \lesssim \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{1/(2\epsilon)} = \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{k+1}.$$

It remains to consider

$$\begin{aligned}
\mathrm{P}(S_n \in x+J,\ \epsilon x < M_n \le \gamma x) &\le n\,\mathrm{P}(S_n \in x+J,\ X_1 > \epsilon x\,, M_n \le \gamma x) \\
&\le n \int_{y \in (\epsilon x, \gamma x]} F(\mathrm{d}y)\,\mathrm{P}(S_{n-1} \in x-y+J,\ M_{n-1} \le \gamma x) \\
&\le n\,\overline{F}(\epsilon x) \sup_{z \ge (1-\gamma)x} \mathrm{P}(S_{n-1} \in z+J,\ M_{n-1} \le \gamma x)\,.
\end{aligned}$$

Observe that, for $z \ge (1-\gamma)x$, we can bound

$$\mathrm{P}(S_{n-1} \in z+J,\ M_{n-1} \le \gamma x) \le \mathrm{P}(S_{n-1} \in z+J,\ M_{n-1} \le \gamma' z)\,, \qquad \text{with} \quad \gamma' := \frac{\gamma}{1-\gamma}\,.$$

The key observation is that $\gamma' \in [\frac{1}{k}, \frac{1}{k-1})$, since $\gamma \in [\frac{1}{k+1}, \frac{1}{k})$. By our inductive assumption, relation (1.5) holds for γ', so $\mathrm{P}(S_{n-1} \in z+J,\ M_{n-1} \le \gamma' z) \lesssim_\gamma \frac{1}{a_n}(\frac{n}{A(z)})^k$ and we get

$$\mathrm{P}(S_n \in x+J,\ \epsilon x < M_n \le \gamma x) \lesssim_\gamma n\,\overline{F}(\epsilon x)\,\frac{1}{a_n}\left(\frac{n}{A(x)}\right)^k \lesssim_\epsilon \frac{1}{a_n}\left(\frac{n}{A(x)}\right)^{k+1},$$

which completes the proof. ∎

4. Strategy and key bounds for Theorems 1.4 and 1.12

4.1. **Reformulation of the SRT.** It turns out that proving the SRT amounts to showing that *small values of n give a negligible contribution to the renewal measure*. More precisely, if F is a probability on $\mathbb{R}$ satisfying (1.2), it is known that (SRT) holds if and only if

$$T(\delta; x) := \sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x+I) \qquad \text{is a.n.}\,, \tag{4.1}$$

see [Chi15, Appendix] or Remark 4.1 below.

Applying Theorem 1.1, it is easy to show that (4.1) always holds for $\alpha > \frac{1}{2}$. Since n/a_n is regularly varying with index $1 - 1/\alpha > -1$, by (1.6) and (2.3)

$$\sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x+I) \lesssim \frac{1}{A(x)} \sum_{1 \le n \le A(\delta x)} \frac{n}{a_n} \lesssim \frac{1}{A(x)} \frac{A(\delta x)^2}{\delta x} \underset{x\to\infty}{\sim} \delta^{2\alpha-1} \frac{A(x)}{x}\,,$$

from which (4.1) follows, since $2\alpha - 1 > 0$. *We have just proved Theorems 1.4 and 1.12 for $\alpha > \frac{1}{2}$.* In the next sections, we will focus on the case $\alpha \le \frac{1}{2}$.

Remark 4.1. *It is easy to see how* (4.1) *arises. For fixed $\delta > 0$, by* (1.8) *we can write*

$$U(x+I) \ge \sum_{A(\delta x) < n \le A(\frac{1}{\delta}x)} \mathrm{P}(S_n \in x+I)\,. \tag{4.2}$$

Since $\mathrm{P}(S_n \in x+I) \sim \frac{h}{a_n}\,\phi(\frac{x}{a_n})$ *by* (2.6) *(where we take $h = v$ for simplicity), a Riemann sum approximation yields (see* [Chi15, Lemma 3.4]*)*

$$\sum_{A(\delta x) < n \le A(\frac{1}{\delta}x)} \mathrm{P}(S_n \in x+I) \sim h\,\frac{A(x)}{x}\,\mathsf{C}(\delta)\,, \qquad \textit{with} \qquad \mathsf{C}(\delta) = \alpha \int_\delta^{\frac{1}{\delta}} z^{\alpha-2} \phi(\tfrac{1}{z})\,\mathrm{d}z\,.$$

Since $\lim_{\delta\to 0} \mathsf{C}(\delta) = \mathsf{C}$, *proving* (SRT) *amounts to controlling the ranges excluded from* (4.2)*, i.e.* $\{n \le A(\delta x)\}$ *and* $\{n > A(\frac{1}{\delta}x)\}$*. The latter gives a negligible contribution by* $\mathrm{P}(S_n \in x+I) \le C/a_n$ *(recall* (2.7)*), while the former is controlled precisely by* (4.1)*.*

4.2. **Key bounds.** The next two lemmas estimate the contribution of the maximum M_n, see (1.1), to the probability $\mathrm{P}(S_n \in x+I)$. Recall that κ_α is defined in (1.21).

We first consider the case when there is a "big jump", i.e. $M_n > \gamma x$ for some $\gamma > 0$.

Lemma 4.2 (Big jumps)**.** *Let F satisfy* (1.2) *for some $A \in RV(\alpha)$, with $\alpha \in (0,1)$. There is $\eta = \eta_\alpha > 0$ such that for all $\delta \in (0,1]$, $\gamma \in (0,1)$ and $x \in [0,\infty)$ the following holds:*

$$\forall \ell \geq \kappa_\alpha : \qquad \sum_{1 \leq n \leq A(\delta x)} n^\ell \left\{ \sup_{z \in \mathbb{R}} \mathrm{P}\left(S_n \in z+I,\, M_n > \gamma x\right) \right\} \lesssim_{\gamma,\ell} \delta^\eta \, b_{\ell+1}(x)\,. \tag{4.3}$$

Proof. For $\delta x < 1$ the left hand side of (4.3) vanishes, because $A(\delta x) < A(1) = 1$. Then we can assume that $\delta x \geq 1$, hence $x \geq 1$. Recalling (2.7), we can write

$$\begin{aligned}
\mathrm{P}\big(S_n \in z+I,\ M_n > \gamma x\big) &\leq n\,\mathrm{P}\left(S_n \in z+I,\ X_1 > \gamma x\right) \\
&= n \int_{w > \gamma x} \mathrm{P}\left(X \in \mathrm{d}w\right)\, \mathrm{P}(S_{n-1} \in z-w+I) \\
&\leq n\,\mathrm{P}\left(X > \gamma x\right) \left\{ \sup_{y \in \mathbb{R}} \mathrm{P}(S_{n-1} \in y+I) \right\} \\
&\lesssim \frac{n}{A(\gamma x)} \frac{1}{a_n} \lesssim_\gamma \frac{n}{A(x)} \frac{1}{a_n}\,,
\end{aligned} \tag{4.4}$$

therefore

$$\begin{aligned}
\sum_{1 \leq n \leq A(\delta x)} n^\ell \left\{ \sup_{z \in \mathbb{R}} \mathrm{P}\left(S_n \in z+I,\, M_n > \gamma x\right) \right\} &\lesssim_\gamma \frac{1}{A(x)} \sum_{1 \leq n \leq A(\delta x)} \frac{n^{\ell+1}}{a_n} \\
&\lesssim_\ell \frac{1}{A(x)} \frac{A(\delta x)^{\ell+2}}{\delta x}\,,
\end{aligned} \tag{4.5}$$

by (2.3), because $n^{\ell+1}/a_n$ is regularly varying with index $(\ell+1) - \frac{1}{\alpha} \geq (\kappa_\alpha + 1) - \frac{1}{\alpha} = \lfloor \frac{1}{\alpha} \rfloor - \frac{1}{\alpha} > -1$. Let us introduce a parameter $b = b_\alpha \in (0,1)$, depending only on α, that will be fixed in a moment. Since we assume that $\delta x \geq 1$, we can apply the upper bound in (2.2) with $\epsilon = (1-b)\alpha$ and $\rho = \delta$, that is $A(\delta x) \lesssim \delta^{b\alpha} A(x)$, which shows that (4.5) is

$$\lesssim \delta^{b\alpha(\ell+2)-1} \frac{A(x)^{\ell+1}}{x} \lesssim \delta^{b\alpha(\kappa_\alpha+2)-1} \frac{A(x)^{\ell+1}}{x}\,,$$

because $\delta \leq 1$ and $\ell \geq \kappa_\alpha$ by assumption. Since $\alpha(\kappa_\alpha+2) > 1$ (because $\kappa_\alpha + 2 = \lfloor \frac{1}{\alpha} \rfloor + 1 > \frac{1}{\alpha}$), we can choose $b = b_\alpha < 1$ so that the exponent of δ is strictly positive (e.g. $b_\alpha = \{\alpha(\kappa_\alpha + 2)\}^{-1/2}$). This completes the proof. ∎

We next consider the case of "no big jump", i.e. $M_n < \gamma x$. The proof exploits in an essential way the large deviation estimate provided by Theorem 1.1.

Lemma 4.3 (No big jump)**.** *Let F satisfy* (1.2) *with $\alpha \in (0,1)$. For any $\gamma \in (0, \frac{\alpha}{1-\alpha})$ there is $\theta = \theta_{\alpha,\gamma} > 0$ such that for all $\delta \in (0,1]$ and $x \in [0,\infty)$ the following holds:*

$$\forall \ell \geq 0 : \qquad \sum_{1 \leq n \leq A(\delta x)} n^\ell \, \mathrm{P}\left(S_n \in x+I,\, M_n \leq \gamma x\right) \lesssim_{\gamma,\ell} \delta^\theta \, b_{\ell+1}(x)\,. \tag{4.6}$$

Proof. As in the proof of Lemma 4.2, we can assume that $x \ge 1$ and $\delta x \ge 1$ (since otherwise the left hand side of (4.6) vanishes). By (1.4)

$$\sum_{1\le n\le A(\delta x)} n^\ell \, \mathrm{P}\left(S_n \in x+I,\, M_n \le \gamma x\right) \lesssim \frac{1}{A(x)^{\frac{1}{\gamma}}} \sum_{1\le n\le A(\delta x)} \frac{n^{\ell+\frac{1}{\gamma}}}{a_n} \lesssim \frac{1}{A(x)^{\frac{1}{\gamma}}} \frac{A(\delta x)^{\ell+\frac{1}{\gamma}+1}}{\delta x}\,,$$

where we applied (2.3), because the sequence $n^{\ell+\frac{1}{\gamma}}/a_n$ is regularly varying with index $\ell + \frac{1}{\gamma} - \frac{1}{\alpha} > \frac{1-\alpha}{\alpha} - \frac{1}{\alpha} = -1$. By the upper bound in (2.2), since $\ell \ge 0$ and $\delta \le 1$ we get

$$A(\delta x)^{\ell+\frac{1}{\gamma}+1} \lesssim_\epsilon \delta^{(\alpha-\epsilon)(\ell+\frac{1}{\gamma}+1)} A(x)^{\ell+\frac{1}{\gamma}+1} \lesssim \delta^{(\alpha-\epsilon)(\frac{1}{\gamma}+1)} A(x)^{\ell+\frac{1}{\gamma}+1}\,.$$

Since $\gamma < \frac{\alpha}{1-\alpha}$, we can choose $\epsilon = \epsilon_{\alpha,\gamma} > 0$ small enough so that $(\alpha-\epsilon)(\frac{1}{\gamma}+1) > 1$. ∎

5. Proof of Theorems 1.4 and 1.12: necessity

In this section we assume (4.1), which is equivalent to the strong renewal theorem (SRT), and we deduce the necessary conditions in Theorems 1.4 and 1.12. We can actually assume (4.1) with $I = (-h, 0]$ replaced by *any fixed bounded interval* J. Indeed, if $J = (-v, 0]$ (for simplicity), we can bound $\mathrm{P}(S_n \in x+J) \le \sum_{\ell=0}^{\lfloor v/h \rfloor} \mathrm{P}(S_n \in x_\ell + I)$, with $x_\ell := x - \ell h$.

Note that, since we assume (4.1), the following holds:

$$\text{for any fixed } k \in \mathbb{N}: \qquad \mathrm{P}(S_k \in x+J) \underset{x\to\infty}{=} o(b_1(x))\,. \tag{5.1}$$

5.1. **Necessity for Theorem 1.4.** Let us fix a probability F on $[0,\infty)$ satisfying (1.12) with $\alpha \in (0,1)$. We assume (4.1) and we deduce that $I_1^+(\delta; x)$ is a.n. (recall (1.13)).

We need some preparation. Let us define the compact interval

$$K := [\tfrac{1}{2}, 1]\,. \tag{5.2}$$

By (2.6), since $\inf_{z\in K} \phi(z) > 0$, there are $n_1 \in \mathbb{N}$ and $c_1, c_2 \in (0,\infty)$ such that

$$\forall n \ge n_1: \qquad \inf_{z\in\mathbb{R}:\ z/a_n \in K} \mathrm{P}(S_n \in z+J) \ge \frac{c_1}{a_n}\,, \tag{5.3}$$

$$\forall n \in \mathbb{N}: \qquad \sup_{z\in\mathbb{R}} \mathrm{P}(S_n \in z+J) \le \frac{c_2}{a_n}\,. \tag{5.4}$$

Then, since $F((-\infty,-x]\cup[x,\infty)) \lesssim 1/A(x)$, we can fix $C \in (0,\infty)$ such that

$$\forall n \in \mathbb{N}: \qquad F((-\infty,-Ca_n]\cup[Ca_n,\infty)) \le \frac{c_1}{2c_2}\,\frac{1}{n}\,. \tag{5.5}$$

(Of course, we could just take $F([Ca_n,\infty))$, since $F((-\infty,0)) = 0$, but this estimate will be useful later for random walks.) We also claim that

$$\forall n \ge n_1: \qquad \inf_{z\in\mathbb{R}:\ z/a_n\in K} \mathrm{P}\left(S_n \in z+J,\, \max\{|X_1|,\ldots,|X_n|\} < Ca_n\right) \ge \frac{c_1}{2}\,\frac{1}{a_n}\,. \tag{5.6}$$

This follows because $\mathrm{P}(S_n \in z+J) \ge c_1/a_n$, by (5.3), and applying (5.4), (5.5) we get

$$\begin{aligned} &\mathrm{P}(S_n \in z+J\,,\ \exists 1\le j\le n \text{ with } |X_j| \ge Ca_n) \\ &\le n \int_{|y|\ge Ca_n} F(\mathrm{d}y)\, \mathrm{P}(S_{n-1} \in z-y+J) \le \frac{n\,F((-\infty,-Ca_n]\cup[Ca_n,\infty))\,c_2}{a_n} \le \frac{c_1}{2\,a_n}. \end{aligned}$$

We can now start the proof. The events $B_i := \{X_i \ge Ca_n,\ \max_{j\in\{1,\dots,n+1\}\setminus\{i\}} X_j < Ca_n\}$ are disjoint for $i = 1,\dots,n$, hence for $n \ge n_1$ we can write

$$\begin{aligned}
&\mathrm{P}(S_{n+1} \in x + J) \\
&\quad \ge (n+1)\,\mathrm{P}\left(S_{n+1} \in x+J,\ \max\{X_1,\dots,X_n\} < Ca_n,\ X_{n+1} \ge Ca_n\right) \\
&\quad \ge n \int_{\{z \le x - Ca_n\}} \mathrm{P}\left(X_{n+1} \in x - \mathrm{d}z\right) \mathrm{P}\left(S_n \in z+J,\ \max\{X_1,\dots,X_n\} < Ca_n\right) \qquad (5.7) \\
&\quad \ge \int_{\{z \le x - Ca_n\}} F\left(x - \mathrm{d}z\right) \frac{c_1}{2}\,\frac{n}{a_n}\,\mathbb{1}_{\{z/a_n \in K\}}\,,
\end{aligned}$$

where the last inequality holds by (5.6). We are going to choose $n \le A(\delta x)$, in particular $x - Ca_n \ge x - C\delta x \ge \frac{\delta}{2}x$ for $\delta > 0$ small enough. Restricting the integral, we get

$$\sum_{n_1 \le n \le A(\delta x)} \mathrm{P}(S_{n+1} \in x+J) \gtrsim \int_{\{z \le \frac{\delta}{2}x\}} F\left(x - \mathrm{d}z\right) \left(\sum_{n_1 \le n \le A(\delta x)} \frac{n}{a_n} \mathbb{1}_{\{z/a_n \in K\}} \right).$$

Note that $z/a_n \in K$ means $\frac{1}{2}a_n \le z \le a_n$, that is $A(z) \le n \le A(2z)$, so in the range of integration we have $A(2z) \le A(\delta x)$. If we further restrict the integration on $z \ge a_{n_1}$, we also have $A(z) \ge n_1$. This leads to the following lower bound:

$$\sum_{n_1 \le n \le A(\delta x)} \frac{n}{a_n} \mathbb{1}_{\{z/a_n \in K\}} \ge \sum_{A(z) \le n \le A(2z)} \frac{n}{a_n} \ge \frac{A(z)}{2z}\big(A(2z) - A(z) - 1\big) \gtrsim \frac{A(z)^2}{z} = b_2(z)\,,$$

where the last inequality holds for $z \ge a_{n_1}$ large (just take n_1 large enough). Then

$$\begin{aligned}
\sum_{n_1 \le n \le A(\delta x)} \mathrm{P}(S_{n+1} \in x+J) &\gtrsim \int_{\{a_{n_1} \le z \le \frac{\delta}{2}x\}} F\left(x - \mathrm{d}z\right)\, b_2(z) \\
&\ge I_1^+(\tfrac{\delta}{2}; x) - \hat{C}\, F([x - a_{n_1}, x-1])\,,
\end{aligned}$$

where $\hat{C} := \sup_{|z| \le a_{n_1}} b_2(z) < \infty$. The left hand side is a.n. by (4.1), hence the right hand side is a.n. too. Since $F([x - a_{n_1}, x])$ is a.n. by (5.1), it follows that $I_1^+(\delta; x)$ is a.n.. ∎

5.2. **Necessity for Theorem 1.12.** Let F be a probability on $\mathbb{R}$ satisfying (1.2) with $\alpha \in (0,1)$ and $p, q > 0$. We assume (4.1), which is equivalent to the (SRT), and we deduce that $\tilde{I}_1(\delta; x)$ is a.n. and, for any $k \ge 2$, that $\tilde{I}_k(\delta, \eta; x)$ is also a.n., for every fixed $\eta \in (0,1)$. This completes the proof of the necessity part in Theorem 1.12 (see Remarks 1.13-1.14).

Remark 5.1. *For $|x| \ge 1$ and $|z| \ge |x|$ we can rewrite* (1.22) *as*

$$\tilde{b}_k(z,x) = \int_{|x|}^{|z|} \frac{b_k(t)}{t}\,\mathrm{d}t = \int_{A(|x|)}^{A(|z|)} \frac{b_k(A^{-1}(s))}{A^{-1}(s)} \frac{1}{A'(A^{-1}(s))}\,\mathrm{d}s \approx \int_{A(|x|)}^{A(|z|)} \frac{s^{k-1}}{A^{-1}(s)}\,\mathrm{d}s\,,$$

by (2.1) *and* (1.10) *(we recall that $\approx$ means both $\lesssim$ and $\gtrsim$). Recalling also* (1.3)*, we obtain*

$$\tilde{b}_k(z,x) \approx \sum_{A(|x|) \le n \le A(|z|)} \frac{n^{k-1}}{a_n}\,. \qquad (5.8)$$

Since we assume that $p, q > 0$ in (1.2), the density $\phi(\cdot)$ of the limiting Lévy process is strictly positive on the whole real line. In particular, instead of (5.2), we can define

$$K := [-1, 1]\,, \qquad (5.9)$$

and relations (5.3), (5.4), (5.5), (5.6) still hold, where $n_1 \in \mathbb{N}$ is fixed (it depends on F).

Let us show that $\tilde{I}_1(\delta;x)$ is a.n.. This is similar to the case of renewal processes in Subsection 5.1. In fact, relation (5.7) with X_i replaced by $|X_i|$ and z replaced by $-y$ gives

$$\mathrm{P}(S_{n+1} \in x+J) \gtrsim \int_{|x+y|\geq Ca_n} F(x+\mathrm{d}y)\, \frac{n}{a_n}\, \mathbb{1}_{\{|y|\leq a_n\}}\,, \tag{5.10}$$

because $K=[-1,1]$. Note that for $n \leq A(\delta x)$ we have $a_n \leq \delta x \leq x - Ca_n$ for $\delta > 0$ small, hence *we can ignore the restriction* $|x+y| \geq Ca_n$. Next we write, by (5.8),

$$\sum_{n=n_1}^{A(\delta x)} \frac{n}{a_n}\, \mathbb{1}_{\{|y|\leq a_n\}} = \mathbb{1}_{\{|y|\leq \delta x\}} \sum_{n=A(y)\vee n_1}^{A(\delta x)} \frac{n}{a_n} \gtrsim \mathbb{1}_{\{|y|\leq \delta x\}}\, \tilde{b}_2(\delta x, |y| \vee a_{n_1})\,.$$

For $|y| < a_{n_1}$, $\tilde{b}_2(\delta x, |y|\vee a_{n_1}) = \tilde{b}_2(\delta x, a_{n_1})$ differs from $\tilde{b}_2(\delta x, |y|)$ at most by the constant $C := \sum_{n\leq n_1} \frac{n}{a_n}$, so $\tilde{b}_2(\delta x, |y|\vee a_{n_1}) \geq \tilde{b}_2(\delta x, |y|) - C\, \mathbb{1}_{\{|y|\leq K\}}$, with $K := a_{n_1}$. This yields

$$\sum_{1\leq n \leq A(\delta x)} \mathrm{P}(S_{n+1} \in x+J) \gtrsim \tilde{I}_1(\delta;x) - C\, F([x-K, x+K])\,.$$

Since we assume that (4.1) holds, and we have $F([x-K,x+K]) = o(b_1(x))$ as $x\to\infty$, as we already observed in (5.1), it follows that $\tilde{I}_1(\delta;x)$ is a.n..

Next we fix $k \geq 2$ and $\eta \in (0,1)$ and we generalize the previous arguments in order to show that $\tilde{I}_k(\delta,\eta;x)$ is a.n., see (1.24). *Inductively, we assume that we already know that* $\tilde{I}_1(\delta;x)$, $\tilde{I}_2(\delta,\eta;x)$, $\ldots$, $\tilde{I}_{k-1}(\delta,\eta;x)$ *are a.n.*. Suppose that $z_1,\cdots z_k \in \mathbb{R}$ satisfy

$$\min_{1\leq j\leq k} |z_j| \geq Ca_n\,, \qquad |(z_1+\ldots+z_k) - x| \leq a_n\,,$$

and set $y_k := x - (z_1+\ldots+z_k)$. Then, for $n \geq n_1$, we can write

$$\begin{aligned}
&P(\exists 1 \leq j_1 < j_2 < \cdots < j_k \leq n \text{ with } X_{j_1} \in \mathrm{d}z_1\,, \ldots,\, X_{j_k} \in \mathrm{d}z_k\,, \text{ and } S_{n+k} \in x+I)\\
&\geq \binom{n+k}{k} \mathrm{P}(X_r \in \mathrm{d}z_r\ \forall 1\leq r\leq k\,,\ X_j \notin \{\mathrm{d}z_1,\cdots \mathrm{d}z_k\}\ \forall k<j\leq n+k\,,\ S_{n+k}\in x+I)\\
&\gtrsim n^k\, \mathrm{P}(X_r \in \mathrm{d}z_r\,,\ \forall 1\leq r\leq k)\, \mathrm{P}(|X_j| \leq Ca_n\,,\ \forall 1\leq j\leq n\,,\ S_n \in y_k + I)\\
&\gtrsim \frac{n^k}{a_n}\, \mathrm{P}(X_r \in \mathrm{d}z_r\,,\ \forall 1\leq r\leq k)\,,
\end{aligned}$$

having used (5.6) in the last inequality. It follows that for $n \geq n_1$ we have the the bound

$$\begin{aligned}
\mathrm{P}(S_{n+k} \in x+I) &\gtrsim \frac{n^k}{a_n}\, \mathrm{P}\left(\min_{1\leq r\leq k} |X_r| \geq Ca_n\,,\ |(X_1+\ldots+X_k)-x| \leq\ a_n \right)\\
&= \frac{n^k}{a_n}\, \mathrm{P}_{-x}\left(\min_{1\leq r\leq k} |S_r - S_{r-1}| \geq Ca_n\,,\ |S_k| \leq\ a_n \right)\,,
\end{aligned}$$

where P_{-x} denotes the law of the random walk $S_r := -x + (X_1+\ldots+X_r)$, $r\geq 1$, which starts from $S_0 := -x$.

If we fix $\eta \in (0,1)$, and define $\overline{\eta} := 1-\eta$, we can write

$$\left\{ \min_{1\leq r\leq k} |S_r - S_{r-1}| \geq Ca_n \right\} \supseteq \left\{ |S_r - S_{r-1}| \geq \overline{\eta}|S_{r-1}| \text{ and } |S_{r-1}| \geq \tfrac{C}{\eta}\, a_n\,, \forall 1 \leq r \leq k \right\}\,.$$

For $r=1$, $|S_{r-1}| \geq \frac{C}{\eta}\, a_n$ reduces to $x \geq \frac{C}{\eta}\, a_n$, which holds automatically, since we take $n \leq A(\delta x)$ with $\delta > 0$ small, while $|S_r - S_{r-1}| \geq \overline{\eta}|S_{r-1}|$ becomes $|S_1 + x| \geq \overline{\eta} x$, which is implied by $|S_1| \le \frac{C}{\eta}\delta x$, for $\delta > 0$ small. For $r \ge 2$, $|S_r - S_{r-1}| \ge \overline{\eta}|S_{r-1}|$ is implied by $|S_r| \le \eta|S_{r-1}|$, since $\overline{\eta} = 1 - \eta$. Thus

$$\left\{ \min_{1\le r\le k} |S_r - S_{r-1}| \ge C a_n \right\} \supseteq \left\{ |S_1| \le \tfrac{C}{\eta}\delta x\,, \quad |S_r| \le \eta|S_{r-1}|\,, \forall 2 \le r \le k\,, \quad |S_{k-1}| \ge \tfrac{C}{\eta} a_n \right\},$$

where the last term is justified because $|S_{k-1}| = \min_{2\le r\le k-1} |S_{r-1}|$ on the event. Thus

$$\mathrm{P}(S_{n+k} \in x + I) \gtrsim \mathrm{E}_{-x}\left[\mathbb{1}_{\{|S_1|\le \frac{C}{\eta}\delta x,\ |S_r|\le\eta|S_{r-1}|\,,\forall 2\le r\le k\}} \left(\frac{n^k}{a_n}\, \mathbb{1}_{\{A(|S_k|)\le n\le A(\frac{\overline{\eta}}{C}|S_{k-1}|)\}} \right) \right].$$

Let us now sum over $n_1 \le n \le A(\delta x)$. Note that $A(\frac{\overline{\eta}}{C}|S_{k-1}|) \le A(\frac{\overline{\eta}}{C}|S_1|) \le A(\delta x)$, hence

$$\begin{aligned} &\sum_{n_1\le n\le A(\delta x)} \mathrm{P}(S_{n+k} \in x + I) \\ &\qquad \gtrsim \mathrm{E}_{-x}\left[\mathbb{1}_{\{|S_1|\le \frac{C}{\eta}\delta x,\ |S_r|\le\eta|S_{r-1}|\,,\forall 2\le r\le k\}}\, \tilde{b}_{k+1}\big(\tfrac{\overline{\eta}}{C}|S_{k-1}|,\, |S_k| \vee a_{n_1}\big) \right], \end{aligned} \tag{5.11}$$

where we recall that $\tilde{b}_{k+1}$ is given by (5.8). The right hand side can be rewritten as

$$\begin{gathered} \int\limits_{\substack{|y_1|\le\delta' x \\ |y_r|\le\eta|y_{r-1}| \text{ for all } 2\le r\le k}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, \tilde{b}_{k+1}\big(\epsilon|y_{k-1}|,\, |y_k| \vee c\big)\,, \\ \text{where} \quad \delta' = \tfrac{C}{\eta}\delta\,, \quad \epsilon = \tfrac{\overline{\eta}}{C}\,, \quad c = a_{n_1}\,. \end{gathered} \tag{5.12}$$

This is like $\tilde{I}_k(\delta', \eta; x)$, see (1.24), except that $\tilde{b}_{k+1}(y_{k-1}, y_k) = \tilde{b}_{k+1}(|y_{k-1}|, |y_k|)$ in (1.24) is replaced by $\tilde{b}_{k+1}(\epsilon|y_{k-1}|, |y_k| \vee c)$. We now show that this is immaterial. More precisely, by (4.1) and (5.11), we know that (5.12) is a.n.. We now deduce that $\tilde{I}_k(\delta, \eta; x)$ is a.n..

Since $\tilde{b}_{k+1}(|y_{k-1}|, |y_k|)$ differs from $\tilde{b}_{k+1}(|y_{k-1}|, |y_k| \vee c)$ at most by $C := \sum_{n=1}^{A(c)} \frac{n^{k+1}}{a_n}$, see (5.8), we can bound $\tilde{b}_{k+1}(|y_{k-1}|, |y_k|) \le \tilde{b}_{k+1}(|y_{k-1}|, |y_k| \vee c) + C\, \mathbb{1}_{\{|y_k|\le c\}}$. Plugging this into (1.24), we see that the contribution of $\mathbb{1}_{\{|y_k|\le c\}}$ is a.n., because it is at most

$$\int_{\mathbb{R}} F(x + \mathrm{d}y_1) \int_{\mathbb{R}^{k-1}} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, \mathbb{1}_{\{|y_k|\le c\}} = \mathrm{P}(S_k \in [x - c, x + c]) \underset{x\to\infty}{=} o(b_1(x))\,,$$

by (5.1). Then in (1.24) we can safely replace $\tilde{b}_{k+1}(y_{k-1}, y_k)$ by $\tilde{b}_{k+1}(|y_{k-1}|, |y_k| \vee c)$.

Finally, we write $\tilde{b}_{k+1}\big(|y_{k-1}|, |y_k| \vee c\big) = \tilde{b}_{k+1}\big(|y_{k-1}|, \epsilon|y_{k-1}|\big) + \tilde{b}_{k+1}\big(\epsilon|y_{k-1}|, |y_k| \vee c\big)$. Note that the contribution of the second term to $\tilde{I}_k(\delta, \eta; x)$ in (1.24) is a.n., because we already know that (5.12) is a.n.. For the first term, observe that by (5.8)

$$\tilde{b}_{k+1}\big(|y_{k-1}|,\, \epsilon|y_{k-1}|\big) \lesssim \sum_{n=A(\epsilon|y_{k-1}|)}^{A(|y_{k-1}|)} \frac{n^k}{a_n} \le \frac{A(|y_{k-1}|)^{k+1}}{\epsilon|y_{k-1}|} \lesssim_\epsilon b_{k+1}(y_{k-1})\,,$$

hence

$$\int\limits_{|y_k|\le\eta|y_{k-1}|} F(-y_{k-1} + \mathrm{d}y_k)\, \tilde{b}_{k+1}(|y_{k-1}|, \epsilon|y_{k-1}|) \lesssim_\epsilon b_{k+1}(y_{k-1})\, F(-(1-\eta)|y_{k-1}|) \lesssim_\eta b_k(y_{k-1})\,,$$

so the contribution to $\tilde{I}_k(\delta, \eta; x)$ in (1.24) is $\lesssim_{\epsilon,\eta} I_{k-1}(\delta, \eta; x)$. We know that $\tilde{I}_{k-1}$ is a.n., by our inductive assumption, and this implies that I_{k-1} is a.n. too, by the inequalities (A.3) and (A.5) in the Appendix (see (A.7)-(A.8) for their proof). We are done. ∎

6. Proof of Theorem 1.4: sufficiency

In this section we prove the sufficiency part of Theorem 1.4: we assume that $I_1^+(\delta;x)$ is a.n. and we deduce (4.1), which is equivalent to the SRT. Let us set

$$T_\ell(\delta;x) := \sum_{1\le n\le A(\delta x)} n^\ell \, \mathrm{P}(S_n \in x+I) \,. \tag{6.1}$$

We actually prove the following result.

Theorem 6.1. *Let F be a probability on $[0,\infty)$ satisfying (1.12) with $\alpha\in(0,1)$. Assume that $I_1^+(\delta;x)$ is a.n.. Then for every $\ell\in\mathbb{N}_0$:*

$$\lim_{\delta\to 0} \limsup_{x\to\infty} \frac{T_\ell(\delta;x)}{b_{\ell+1}(x)} = 0 \,. \tag{6.2}$$

In particular, setting $\ell=0$, relation (4.1) holds.

The proof exploits the general bounds provided by Lemmas 4.2 and 4.3, together with the next Lemma, which is specialized to renewal processes.

Lemma 6.2. *If F is a probability on $[0,\infty)$ which satisfies (1.12) with $\alpha\in(0,1)$, there are $C,c\in(0,\infty)$ such that for all $n\in\mathbb{N}_0$ and $z\in[0,\infty)$*

$$\mathrm{P}(S_n\in z+I) \le \frac{C}{a_n}\, e^{-c\frac{n}{A(z)}} \,. \tag{6.3}$$

Proof. Assume that n is even (the odd case is analogous). By (2.7), we get

$$\begin{aligned} \mathrm{P}\left(S_n\in z+I\right) &= \int_{y\in[0,z]} \mathrm{P}(S_{\frac{n}{2}}\in \mathrm{d}y)\ \mathrm{P}(S_{\frac{n}{2}}\in z-y+I) \lesssim \frac{1}{a_{\frac{n}{2}}}\,\mathrm{P}(S_{\frac{n}{2}}\le z) \\ &\lesssim \frac{1}{a_n}\,\mathrm{P}\left(\max_{1\le i\le\frac{n}{2}} X_i\le z\right) = \frac{(1-\mathrm{P}(X>z))^{\frac{n}{2}}}{a_n} \le \frac{e^{-\frac{n}{2}\mathrm{P}(X>z)}}{a_n} \le \frac{e^{-c\frac{n}{A(z)}}}{a_n}\,, \end{aligned}$$

provided $c>0$ is chosen such that $\mathrm{P}(X>z)\ge 2c/A(z)$ for all $z\ge 0$. This is possible by (1.12) and because $z\mapsto A(z)$ is increasing and continuous, with $A(0)>0$ (see §2.2). ∎

Remark 6.3. *Since $A(\cdot)$ is increasing, it follows by (6.3) that for any $\bar{x}>0$ and $\ell\in\mathbb{N}$*

$$\sup_{z\in[0,\bar{x}]}\left\{\sum_{n\in\mathbb{N}} n^\ell\,\mathrm{P}(S_n\in z+I)\right\} \lesssim C_{\bar{x},\ell}\,, \qquad \textit{with} \qquad C_{\bar{x},\ell} := \sum_{n\in\mathbb{N}} \frac{n^\ell}{a_n}\, e^{-c\frac{n}{A(\bar{x})}} \ < \ \infty\,. \tag{6.4}$$

Before proving Theorem 6.1, we state some easy consequences of “$I_1^+(\delta;x)$ is a.n.”.

- First we show that, for any bounded interval $J\subseteq\mathbb{R}$,

$$I_1^+(\delta;x) \ \text{ is a.n.} \qquad \Longrightarrow \qquad F(x+J) \underset{x\to\infty}{=} o\big(b_1(x)\big)\,. \tag{6.5}$$

  It is convenient to write $J=[-1-b,-1-a]$, for $a,b\in\mathbb{R}$ with $a<b$. Restricting (1.13) to $z\in -a-J=[1,1+(b-a)]$ we get $I_1^+(\delta;x)\ge F(x+a+J)\cdot\inf_{z\in -a-J} b_2(z) \gtrsim_J F(x+a+J)$, so if $I_1^+(\delta;x)$ is a.n. then $F(x+a+J)=o(b_1(x))$, hence (6.5) follows.

- Next we improve (6.5) as follows:

$$I_1^+(\delta;x) \ \text{ is a.n.} \qquad \Longrightarrow \qquad \text{for every fixed } \ell\in\mathbb{N}: \quad \mathrm{P}(S_\ell\in x+J) \underset{x\to\infty}{=} o\big(b_1(x)\big)\,. \tag{6.6}$$

To see this, we write $J = [a, b]$ and we note that on the event $\{S_\ell \in x + J\}$ we must have $M_\ell \ge (x-a)/\ell$, hence $S_\ell - M_\ell \le x + b - \frac{x-a}{\ell} \le x - (\frac{x}{\ell} - 2b)$. Then

$$\mathrm{P}(S_\ell \in x+J) \le \ell \int_{z > \frac{x}{\ell} - 2b} \mathrm{P}(S_{\ell-1} \in x - \mathrm{d}z)\, \mathrm{P}(X_1 \in z + J) \le \ell \sup_{z > \frac{x}{\ell} - 2b} F(z+J)\,, \tag{6.7}$$

and, by (6.5), as $x \to \infty$ the right hand side is $o(b_1(\frac{x}{\ell} - 2b)) = o(b_1(x))$, for fixed ℓ.

- Finally, we observe that for any fixed $\gamma > 0$

$$I_1^+(\delta; x) \;\text{ is a.n.} \qquad \Longrightarrow \qquad \text{for every fixed } \gamma \in (0,1): \quad I_1^+(1-\gamma; x) \underset{x\to\infty}{=} O\big(b_1(x)\big)\,. \tag{6.8}$$

First we fix $\bar\delta > 0$ small enough so that $I_1^+\big(\bar\delta; x\big) = O(b_1(x))$ (recall Definition 1.3). Then we consider the contribution to $I_1^+(1-\gamma, x)$ from $z \ge \bar\delta$, see (1.13), which is

$$\int_{\bar\delta x \le z \le (1-\gamma)x} F(x - \mathrm{d}z)\, \frac{A(z)^2}{z} \le \frac{A(x)^2}{\bar\delta\, x}\, \overline{F}(\gamma x) \lesssim_{\gamma, \bar\delta} \frac{A(x)}{x} = b_1(x)\,.$$

Proof of Theorem 6.1. We fix, once and for all, $\gamma \in (0, \frac{\alpha}{1-\alpha})$, and we decompose

$$\begin{aligned} T_\ell(\delta; x) = & \sum_{1 \le n \le A(\delta x)} n^\ell\, \mathrm{P}(S_n \in x + I,\, M_n > \gamma x) \\ & + \sum_{1 \le n \le A(\delta x)} n^\ell\, \mathrm{P}(S_n \in x + I,\, M_n \le \gamma x)\,. \end{aligned}$$

Then it follows by Lemma 4.2 and Lemma 4.3 that (6.2) holds for every $\ell \ge \kappa_\alpha$.

It remains to prove that (6.2) holds for $\ell < \kappa_\alpha$. We proceed by backward induction: we fix $\ell \in \{0, 1, \ldots, \kappa_\alpha - 1\}$ and, *assuming* that

$$\lim_{\delta \to 0} \limsup_{x \to \infty} \frac{T_{\ell+1}(\delta; x)}{b_{\ell+2}(x)} = 0\,, \tag{6.9}$$

we deduce (6.2). We need to estimate $T_\ell(\delta; x)$ and we split it in some pieces.

We start by writing

$$\mathrm{P}\left(S_n \in x + I\right) = \mathrm{P}\left(S_n \in x + I,\, M_n \le \gamma x\right) + \mathrm{P}\left(S_n \in x + I,\, M_n > \gamma x\right)\,,$$

and note that the contribution of the first term in the right hand side is negligible for (6.2), by Lemma 4.3. Next we bound

$$\begin{aligned} \mathrm{P}\left(S_n \in x + I,\, M_n > \gamma x\right) & \le n\, \mathrm{P}\left(S_n \in x + I,\, X_1 > \gamma x\right) \\ & = n \int_{0 \le z < (1-\gamma)x} F(x - \mathrm{d}z)\, \mathrm{P}\left(S_{n-1} \in z + I\right)\,. \end{aligned} \tag{6.10}$$

Looking back at (6.1), we may restrict the sum to $n \ge 2$, because the contribution of the term $n = 1$ is negligible for (6.2), since $F(x + I) = o(b_1(x)) = o(b_{\ell+1}(x))$ by (6.5). As a consequence, it remains to prove that (6.2) holds with $T_\ell(\delta; x)$ replaced by

$$\widetilde{T}_\ell(\delta; x) := \sum_{2 \le n \le A(\delta x)} n^{\ell+1} \int_{0 \le z < (1-\gamma)x} F(x - \mathrm{d}z)\, \mathrm{P}\left(S_{n-1} \in z + I\right)\,.$$

We can bound $n^{\ell+1} \lesssim (n-1)^{\ell+1}$, since $n \ge 2$, and rename $n - 1$ as n, to get

$$\widetilde{T}_\ell(\delta; x) \le \int_{1 \le z < (1-\gamma)x} F(x - \mathrm{d}z) \left\{ \sum_{1 \le n \le A(\delta x) - 1} n^{\ell+1}\, \mathrm{P}\left(S_n \in z + I\right) \right\} + o(b_{\ell+1}(x))\,, \tag{6.11}$$

where we have restricted the integral to $z \geq 1$, because the contribution of $z \in [0,1)$ can be estimated as $o(b_1(x)) = o(b_{\ell+1}(x))$, thanks to (6.4) and (6.5).

Let us fix $\epsilon \in (0,1)$ and consider the contribution to the sum in (6.11) given by $n > A(\epsilon z)$. Applying Lemma 6.2, since $a_n \geq \epsilon z \gtrsim_\epsilon z$, we get

$$\begin{aligned} \sum_{A(\epsilon z) < n \leq A(\delta x)} n^{\ell+1}\, \mathrm{P}\left(S_n \in z + I\right) &\lesssim \sum_{A(\epsilon z) < n \leq A(\delta x)} \frac{n^{\ell+1}}{a_n}\, e^{-c \frac{n}{A(z)}} \\ &\lesssim_\epsilon \frac{A(z)^{\ell+2}}{z}\, \mathbb{1}_{\{z < \frac{\delta}{\epsilon} x\}} \left\{ \frac{1}{A(z)} \sum_{n \in \mathbb{N}} \left(\frac{n}{A(z)}\right)^{\ell+1} e^{-c\frac{n}{A(z)}} \right\}. \end{aligned} \tag{6.12}$$

The bracket is a Riemann sum which converges to $\int_0^\infty t^{\ell+1}\, e^{-ct}\, \mathrm{d}t < \infty$ as $z \to \infty$, hence it is uniformly bounded for $z \in [0,\infty)$. The contribution of $n > A(\epsilon z)$ to (6.11) is then

$$\lesssim_\epsilon \int_{1 \leq z < \frac{\delta}{\epsilon} x} F(x - \mathrm{d}z)\, \frac{A(z)^{\ell+2}}{z} \leq A(\tfrac{\delta}{\epsilon} x)^\ell\, I_1^+(\tfrac{\delta}{\epsilon}; x) \lesssim_\epsilon A(x)^\ell\, I_1^+(\tfrac{\delta}{\epsilon}; x)\,.$$

This is negligible for (6.2), for any fixed $\epsilon > 0$, by the assumption that I_1^+ is a.n..

Finally, the contribution of $n \leq A(\epsilon z)$ to the integral in (6.11) is, by (6.1),

$$\int_{1 \leq z < (1-\gamma)x} F(x - \mathrm{d}z)\, T_{\ell+1}(\epsilon; z)\,. \tag{6.13}$$

By the inductive assumption (6.9), for every $\eta > 0$ we can choose $\epsilon > 0$ and $\bar{x}_\epsilon < \infty$ so that $T_{\ell+1}(\epsilon; z) \leq \eta\, b_{\ell+2}(z)$ for $z \geq \bar{x}_\epsilon$. Then the integral in (6.13) restricted to $z \geq \bar{x}_\epsilon$ is

$$\leq \eta \int_{\bar{x}_\epsilon \leq z < (1-\gamma)x} F(x - \mathrm{d}z)\, b_{\ell+2}(z) \leq \eta\, A(x)^\ell\, I_1^+(1-\gamma, x) \lesssim \eta\, A(x)^\ell\, b_1(x) = \eta\, b_{\ell+1}(x)\,,$$

where we have applied (6.8). If we let $x \to \infty$ and then $\eta \to 0$, this is negligible for (6.2). Finally, by (6.4) and (6.5), the integral in (6.13) restricted to $z \leq \bar{x}_\epsilon$ is, as $x \to \infty$

$$\leq C_{\bar{x}_\epsilon, \ell+1}\, F((x - \bar{x}_\epsilon, x-1]) = o(b_1(x)) = o(b_{\ell+1}(x))\,.$$

This completes the proof. ∎

7. Proof of Theorem 1.12: sufficiency in case $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

Let F be a probability on $\mathbb{R}$ that satisfies (1.2) with $p, q \geq 0$ and $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, i.e. $\kappa_\alpha = 1$. *We assume that $\tilde{I}_1(\delta; x)$ is a.n.* (hence also $I_1(\delta; x)$ is a.n., recall Remark 1.13), and we deduce (4.1), which is equivalent to the SRT. This proves the sufficiency in Theorem 1.12.

Let us set

$$Z_1 := M_n = \max\{X_1, \ldots, X_n\}\,. \tag{7.1}$$

We fix $\gamma \in (0, \frac{\alpha}{1-\alpha})$ and define the events

$$\begin{aligned} &E_1^{(1)} := \{Z_1 \leq \gamma x\}\,, \qquad E_1^{(2)} := \{|Z_1 - x| \leq a_n\}\,, \\ &E_1^{(3)} := \{Z_1 > \gamma x,\, |Z_1 - x| > a_n\} \end{aligned} \tag{7.2}$$

By Lemma 4.3 with $\ell = 0$, we already know that (with no extra assumptions on F)

$$\sum_{1 \leq n \leq A(\delta x)} \mathrm{P}(S_n \in x + I, E_1^{(1)}) \quad \text{is always a.n.}\,. \tag{7.3}$$

Next we look at $E_1^{(2)}$. Note that by (2.7)

$$\begin{aligned} \mathrm{P}(S_n \in x+I,\, E_1^{(2)}) &\le \sum_{i=1}^{n} \mathrm{P}(S_n \in x+I,\, |X_i - x| \le a_n) \\ &= n \int_{|y|\le a_n} F(x+\mathrm{d}y)\, \mathrm{P}(S_{n-1} \in I - y) \lesssim \frac{n}{a_n} \int_{|y|\le a_n} F(x+\mathrm{d}y)\,. \end{aligned} \tag{7.4}$$

Using the fact that n/a_n is regularly varying and recalling (5.8), we obtain

$$\begin{aligned} \sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I,\, E_1^{(2)}) &\lesssim \int_{|y|\le \delta x} F(x+\mathrm{d}y) \sum_{A(|y|)\le n\le A(\delta x)} \frac{n}{a_n} \\ &\lesssim \int_{|y|\le \delta x} F(x+\mathrm{d}y)\, \tilde{b}_2(\delta x, y) = \tilde{I}_1(\delta; x)\,. \end{aligned} \tag{7.5}$$

Recalling (1.23), we have shown that

$$\tilde{I}_1(\delta; x) \text{ is a.n.} \qquad \Longrightarrow \qquad \sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I, E_1^{(2)}) \quad \text{is a.n.}\,. \tag{7.6}$$

(The reverse implication also holds, as shown in Section 5.)

We finally turn to $E_1^{(3)}$. Arguing as in (7.4) and setting $\overline{\gamma} := 1-\gamma$, we have by (1.6)

$$\begin{aligned} \mathrm{P}(S_n \in x+I, E_1^{(3)}) &\lesssim n \int_{|y|>a_n,\, y>-\overline{\gamma}x} F(x+\mathrm{d}y)\, \mathrm{P}(S_{n-1} \in I - y) \\ &\lesssim \frac{n^2}{a_n} \int_{|y|>a_n,\, y>-\overline{\gamma}x} F(x+\mathrm{d}y)\, \frac{1}{A(y)}\,, \end{aligned}$$

hence, recalling (1.10), we get

$$\begin{aligned} \sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I,\, E_1^{(3)}) &\lesssim \int_{y>-\overline{\gamma}x} F(x+\mathrm{d}y)\, \frac{1}{A(y)} \sum_{1\le n\le A(\delta x\wedge |y|)} \frac{n^2}{a_n} \\ &\lesssim \int_{y>-\overline{\gamma}x} F(x+\mathrm{d}y)\, \frac{1}{A(y)}\, b_3(\delta x \wedge |y|)\,, \end{aligned}$$

where the last inequality holds for $\alpha > \frac{1}{3}$, thanks to (2.3), because n^2/a_n is regularly varying with index $2 - 1/\alpha > -1$. For fixed $\delta_0 > 0$, the right hand side can be estimated by

$$\begin{aligned} &\int_{|y|\le \delta_0 x} F(x+\mathrm{d}y)\, b_2(y) \;+\; b_3(\delta x) \int_{y>-\overline{\gamma}x,\, |y|>\delta_0 x} \frac{F(x+\mathrm{d}y)}{A(y)} \\ &\qquad \lesssim I_1(\delta_0; x) \;+\; \frac{b_3(\delta x)}{A(\delta_0 x)}\, F((\gamma x, \infty)) \lesssim I_1(\delta_0; x) \;+\; \frac{b_3(\delta x)}{A(\delta_0 x)\, A(x)}\,. \end{aligned} \tag{7.7}$$

By the a.n. of I_1, given $\epsilon > 0$, we can fix $\delta_0 > 0$ small so that $I_1(\delta_0; x) \le \epsilon\, b_1(x)$ for large x. Then we can fix $\delta > 0$ small (depending on δ_0) so that the second term in the right hand side of (7.7) is also $\le \epsilon\, b_1(x)$ for large x, because $b_3(\delta x) \sim \delta^{3\alpha-1}\, b_3(x)$ and $\alpha > \frac{1}{3}$. Thus

$$\sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n \in x+I, E_1^{(3)}) \quad \text{is a.n.} \quad \text{if } \alpha > \frac{1}{3} \text{ and } I_1(\delta; x) \text{ is a.n.}\,. \tag{7.8}$$

Relations (7.3), (7.6), (7.8) prove the sufficiency part in Theorem 6.1, when $\kappa_\alpha = 1$.

8. Proof of Theorem 1.12: sufficiency in case $\alpha \le \frac{1}{3}$

Let F be a probability on $\mathbb{R}$ that satisfies (1.2) with $p, q \ge 0$ and $\alpha \in (0, \frac{1}{3}]$. In this section, *we assume that* $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ *is a.n.* and we deduce (4.1), which is equivalent to the SRT. By Remark 1.13, this proves the sufficiency part in Theorem 1.12, in case $\kappa_\alpha \ge 2$.

We stress that our assumption that $\tilde{I}_{\kappa_\alpha}(\delta, \eta; x)$ is a.n. ensures that $\tilde{I}_r(\delta, \eta; x)$ *and* $I_r(\delta, \eta; x)$ *are a.n. for every* $r \in \mathbb{N}$, by Remark 1.14 (see Lemmas A.1-A.2-A.3).

Throughout this section we fix $\gamma \in (0, \frac{\alpha}{1-\alpha})$, we choose $\eta = \overline{\gamma} := 1 - \gamma$ and we drop it from notations. In particular, we write $I_k(\delta; x)$ instead of $I_k(\delta, \eta; x)$.

8.1. **Preparation.** We will prove that $T(\delta; x) := \sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I)$ is a.n. by direct path arguments, see Subsection 8.2. This will lead us to consider explicit quantities $J_k(\delta; x)$, $\tilde{J}_k(\delta; x)$ that generalize $I_k(\delta; x)$, $\tilde{I}_k(\delta; x)$. For clarity, in this subsection we define such quantities and show that they are a.n..

We recall that $I_k(\delta; x)$ is defined in (1.18), (1.19). Let us rewrite it as follows:

$$I_k(\delta; x) := \int_{|y_1| \le \delta x} F(x + \mathrm{d}y_1)\, g_k(y_1)\,, \tag{8.1}$$

$$\text{where we set} \qquad g_k(y_1) := \begin{cases} b_2(y_1) & \text{if } k = 1 \\ \displaystyle\int_{\Omega_k(y_1)} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, b_{k+1}(y_k) & \text{if } k \ge 2 \end{cases}, \tag{8.2}$$

$$\Omega_k(y_1) := \left\{ (y_2, \ldots, y_k) \in \mathbb{R}^{k-1} : \ |y_j| \le \overline{\gamma} |y_{j-1}| \text{ for } 2 \le j \le k \right\}, \tag{8.3}$$

and we recall that $P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k) := F(-y_1 + \mathrm{d}y_2) F(-y_2 + \mathrm{d}y_3) \cdots F(-y_{k-1} + \mathrm{d}y_k)$.

We define $J_k(\delta; x)$ by extending the integral in (8.2) to a larger subset $\Theta_k(y_1) \supseteq \Omega_k(y_1)$. We introduce the shortcut

$$\theta(y) := \begin{cases} (-\infty, \overline{\gamma} y] & \text{if } y \ge 0 \\ [\overline{\gamma} y, +\infty) & \text{if } y < 0 \end{cases}, \tag{8.4}$$

and note the important fact that (since $0 < \overline{\gamma} < 1$)

$$\int_{\theta(y)} F(-y + \mathrm{d}z) = O\left(\frac{1}{A(|y|)} \right) \qquad \text{as } |y| \to \infty\,. \tag{8.5}$$

Then, recalling that $\overline{\gamma} = 1 - \gamma$, we set for $k \ge 2$ and $y_1 \in \mathbb{R}$

$$\Theta_k(y_1) := \left\{ (y_2, \ldots, y_k) \in \mathbb{R}^{k-1} : \ y_j \in \theta(y_{j-1}) \text{ for } 2 \le j \le k \right\}. \tag{8.6}$$

We then define

$$J_k(\delta; x) := \int_{|y| \le \delta x} F(x + \mathrm{d}y)\, h_k(y)\,. \tag{8.7}$$

where $h_k(y_1)$ is nothing but $g_k(y_1)$, see (8.2), with $\Omega_k(y_1)$ replaced by $\Theta_k(y_1)$:

$$h_k(y_1) := \begin{cases} b_2(y_1) & \text{if } k = 1 \\ \displaystyle\int_{\Theta_k(y_1)} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, b_{k+1}(y_k) & \text{if } k \ge 2 \end{cases}. \tag{8.8}$$

It will be useful to consider a slight generalization of $h_k(y_1)$: for any non-negative, even function $f : \mathbb{R} \to [0,\infty)$ we define

$$h_k(y_1, f) := \begin{cases} b_1(y_1)\, f(y_1) & \text{if } k = 1 \\ \displaystyle\int_{\Theta_k(y_1)} P_{y_1}(\mathrm{d}y_2, \ldots, \mathrm{d}y_k)\, b_k(y_k)\, f(y_k) & \text{if } k \geq 2 \end{cases}, \tag{8.9}$$

and note that $h_k(y_1, A) = h_k(y_1)$.

The next proposition shows that $J_k(\delta; x)$ is a.n. and provides a useful auxiliary estimate. Its proof is quite tedious and is deferred to Subsection 8.3.

Proposition 8.1. *Fix $\ell \in \mathbb{N}$ and $\alpha \in (0, \frac{1}{2})$. If $\ell \geq 2$, assume that $I_j(\delta; x)$ is a.n. for $j = 1, \ldots, \ell - 1$.*

(1) *Fix any $f \in RV(\beta)$ with $0 < \beta < 1 - \ell\alpha$. Then for all $0 < \delta_0 < \kappa < 1$*

$$\int_{|y|>\delta_0 x,\, y>-\kappa x} F(x + \mathrm{d}y)\, h_\ell(y, f) \lesssim_{\delta_0,\kappa,\gamma,\ell} \frac{f(x)}{x \vee 1} \qquad (\forall x \geq 0)\,. \tag{8.10}$$

(2) *Assume that $I_\ell(\delta; x)$ is a.n. too and, moreover, $\alpha < \frac{1}{\ell+1}$. Then*

$$J_\ell(\delta; x) \text{ is a.n.} \tag{8.11}$$

We finally define $\tilde{J}_2(\delta; x) := \tilde{I}_2(\delta; x)$ and, for $k \geq 3$,

$$\tilde{J}_k(\delta; x) = \int_{|y_1| \leq \delta x} F(x + \mathrm{d}y_1)\, \tilde{h}_k(y_1)\,, \tag{8.12}$$

where we set (recall that $\tilde{b}(x, z)$ is defined in (1.22), or equivalently (5.8)):

$$\tilde{h}_k(y_1) = \int_{\Theta_{k-1}(y_1) \cap \{|y_k| \leq \overline{\gamma} |y_{k-1}|\}} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_k)\, \tilde{b}_{k+1}(y_{k-1}, y_k)\,. \tag{8.13}$$

The next result shows that $\tilde{J}_k(\delta; x)$ is a.n.. Its proof is also deferred to Subsection 8.3.

Proposition 8.2. *Fix $\ell \in \mathbb{N}$ with $\ell \geq 2$ and $0 < \alpha \leq \frac{1}{\ell+1}$. Assume that $I_j(\delta; x)$ and $\tilde{I}_j(\delta; x)$ are a.n. for $j = 1, \ldots, \ell$. Then also $\tilde{J}_\ell(\delta; x)$ is a.n..*

8.2. **Proof of Sufficiency for Theorem 1.12.** Throughout the proof we fix $\alpha \in (0, \frac{1}{3}]$ and $k = \kappa_\alpha = \lfloor 1/\alpha \rfloor - 1$, see (1.21). We stress that $k \geq 2$ and $\frac{1}{k+2} < \alpha \leq \frac{1}{k+1}$. Our goal is to prove (4.1).

We generalize (7.1), defining two sequences $Z_1, Z_2, \ldots Z_k$ and $Y_1, Y_2, \ldots, Y_k$ as follows:

$$Z_1 := \max\{X_1, \ldots, X_n\}\,, \qquad Y_1 := Z_1 - x\,,$$

and for $r \in \{2, \ldots, k\}$

$$Z_r := \begin{cases} \max\big\{\{X_j, 1 \leq j \leq n\} \setminus \{Z_j, 1 \leq j \leq r-1\}\big\} & \text{if } Y_{r-1} \leq 0, \\ \min\big\{\{X_j, 1 \leq j \leq n\} \setminus \{Z_j, 1 \leq j \leq r-1\}\big\} & \text{if } Y_{r-1} > 0. \end{cases} \tag{8.14}$$

$$Y_r := \sum_{i=1}^{r} Z_i - x \tag{8.15}$$

Intuitively, Z_r *is the largest available step towards x from $Z_1 + \ldots + Z_{r-1}$.*

In fact, we may assume that the following holds:

$$Z_r > 0 \quad \text{if } Y_{r-1} \leq 0\,, \qquad \text{while} \qquad Z_r < 0 \quad \text{if } Y_{r-1} > 0\,, \tag{8.16}$$

because, as we now show, the event that (8.16) fails to be true is negligible. This event occurs if, for some $r \le k$, either $Y_{r-1} \le 0$ and $\{X_j, 1 \le j \le n\} \setminus \{Z_j, 1 \le j \le r-1\}$ contains no positive terms or $Y_{r-1} > 0$ and this set contains no negative terms. Call $\mathcal{E}_{n,r}$ such event and recall that $I = (-h, 0]$. We first observe that $\mathrm{P}(\mathcal{E}_{n,r},\, S_n \in x + I,\, Y_{r-1} \notin I) = 0$ for all $n \ge r$ (if $Y_{r-1} > 0$ then $S_n - x \ge Y_{r-1} > 0$ on the event $\mathcal{E}_{n,r}$, and similarly if $Y_{r-1} \le -h$ then $S_n - x \le Y_{r-1} \le -h$). Next we observe that

$$\begin{aligned}\mathrm{P}(\mathcal{E}_{n,r},\, Y_{r-1} \in I) &\le \binom{n}{r-1} \mathrm{P}(S_{r-1} \in x + I,\ X_r, X_{r+1}, \ldots, X_n \le 0) \\ &\le n^{r-1}\, c^{n-(r-1)}\, \mathrm{P}(S_{r-1} \in x + I)\,,\end{aligned}$$

with $c = \mathrm{P}(X_1 \le 0) < 1$. If we set $K_r := \sum_{n=r}^{\infty} n^{r-1}\, c^{n-(r-1)} < \infty$, we can write, by (6.6),

$$\sum_{n=r}^{\infty} \mathrm{P}(\mathcal{E}_{n,r},\, S_n \in x + I) \le \sum_{n=r}^{\infty} \mathrm{P}(\mathcal{E}_{n,r},\, Y_{r-1} \in I) \le K_r\, \mathrm{P}(S_{r-1} \in x + I) \underset{x\to\infty}{=} o\big(b_1(x)\big)\,,$$

so the contribution of $\mathcal{E}_{n,r}$ to (4.1) is negligible. Henceforth we will assume that (8.16) holds.

We cover the probability space $\Omega \subseteq E_1^{(1)} \cup E_1^{(2)} \cup E_1^{(3)}$, where we recall from (7.2) that

$$E_1^{(1)} := \{Z_1 \le \gamma x\}\,, \qquad E_1^{(2)} := \{|Y_1| \le a_n\}\,, \qquad E_1^{(3)} = \{Z_1 > \gamma x,\ |Y_1| > a_n\}\,.$$

The argument to show that $\sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I, E_1^{(1)} \cup E_1^{(2)})$ is a.n. presented in Section 7 is still valid, see (7.3) and (7.6), so it remains to focus on $E_1^{(3)}$.

We introduce the constants $C_r = (\overline{\gamma})^{r-1}$ and then the events $E_r^{(1)}, E_r^{(2)}, E_r^{(3)}$ for $r \ge 2$ by

$$\begin{aligned} E_r^{(1)} = E_{r-1}^{(3)} \cap \{|Z_r| \le \gamma |Y_{r-1}|\}\,, \qquad E_r^{(2)} = E_{r-1}^{(3)} \cap \{|Y_r| \le C_r a_n\}\,, \qquad (8.17)\\ E_r^{(3)} = E_{r-1}^{(3)} \cap \{|Z_r| > \gamma |Y_{r-1}|,\ |Y_r| > C_r a_n\}\,.\end{aligned}$$

Note that we can decompose (recall that $k = \kappa_\alpha$ is fixed)

$$E_1^{(3)} \subseteq \bigcup_{r=1}^{k} E_r^{(1)} \ \cup\ \bigcup_{r=1}^{k} E_r^{(2)} \ \cup\ E_k^{(3)}\,.$$

We will show that $\sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I)$ is a.n. by estimating the contributions of $E_r^{(1)}$ and $E_r^{(2)}$, for $2 \le r \le k$, and finally the contribution of $E_k^{(3)}$. Let us note that

$$\alpha \le \tfrac{1}{k+1} < \tfrac{1}{r}\,, \qquad \text{for } r = 2, \ldots, k\,, \qquad (8.18)$$

which allows us to apply equation (8.11) from Proposition 8.1 for $\ell = r - 1$ (but not for $\ell = r$, unlike Proposition 8.2).

Remark 8.3. *We can rewrite $E_\ell^{(3)}$ more explicitly as follows:*

$$E_\ell^{(3)} = \{Z_1 > \gamma x\,,\ |Z_i| > \gamma |Y_{i-1}| \text{ for } 2 \le i \le \ell\} \cap \{|Y_i| > C_i a_n\,,\ \text{for } 1 \le i \le \ell\}\,.$$

Recalling the definition (8.6) of $\Theta_k(\cdot)$, we claim that $E_\ell^{(3)}$ can also be rewritten as

$$E_\ell^{(3)} = \{Y_1 > -\overline{\gamma} x,\ (Y_2, \ldots, Y_\ell) \in \Theta_\ell(Y_1)\} \cap \{|Y_i| > C_i a_n\,,\ \text{for } 1 \le i \le \ell\}\,. \qquad (8.19)$$

To prove the claim, we show that $|Z_i| > \gamma |Y_{i-1}|$ is equivalent to $Y_i \in \theta(Y_{i-1})$, for $2 \le i \le k$, with $\theta(\cdot)$ defined in (8.6). We recall that $Z_i = Y_i - Y_{i-1}$, see (8.15). If $Y_{i-1} > 0$, then $Z_i \le 0$ by (8.16), hence $|Z_i| > \gamma |Y_{i-1}|$ becomes $Y_{i-1} - Y_i > \gamma Y_{i-1}$, which is precisely $Y_i \in (-\infty, -\overline{\gamma} Y_{i-1}) = \theta(Y_{i-1})$. Similar arguments apply if $Y_{i-1} \le 0$, in which case $Z_i \ge 0$.

Estimate of $E_r^{(1)}$. We fix $r \in \{2, \ldots, k\}$. By exchangeability,

$$\mathrm{P}(S_n \in x+I,\, E_r^{(1)}) \le n^{r-1}\, \mathrm{P}\big((Z_1, \ldots Z_{r-1}) = (X_1, \ldots, X_{r-1}),\, S_n \in x+I,\, E_r^{(1)}\big)\,. \tag{8.20}$$

Conditionally on $(X_1, \ldots, X_{r-1}) = (z_1, \ldots, z_{r-1})$, we have $S_n = (z_1 + \ldots + z_{r-1}) + S'_{n-(r-1)}$, where we set $S'_k := X'_1 + \ldots + X'_k$ with $X'_i := X_{(r-1)+i}$. Motivated by (8.15), if we set

$$y_i := (z_1 + \ldots + z_i) - x \qquad \text{for } i = 1, \ldots, r-1\,,$$

then we can write $\{S_n \in x + I\} = \{S'_{n-(r-1)} \in -y_{r-1} + I\}$. Assume first that $y_{r-1} \le 0$, so $Z_r = M'_{n-(r-1)} := \max\{X'_i, 1 \le i \le n-(r-1)\}$. By (8.17) and (8.14), we need to evaluate

$$\mathrm{P}\Big(S'_{n-(r-1)} \in -y_{r-1} + I,\ |M'_{n-(r-1)}| \le \gamma\, |y_{r-1}|\Big)\,. \tag{8.21}$$

Since this probability is increasing in γ, applying (1.4), we get the bound

$$\lesssim \frac{1}{a_n} \left(\frac{n}{A(|y_{r-1}|)} \right)^d, \qquad \text{for all} \quad d \le \frac{1}{\gamma}\,. \tag{8.22}$$

In case $y_{r-1} > 0$, relation (8.21) holds with M'_k replaced by $(M'_k)^* := \min_{1 \le i \le k} X'_i$. Applying (1.4) to the reflected walk $(S')^* = -S'$, we see that the bound (8.22) still holds. Then, by (8.20) and $E_r^{(1)} = E_{r-1}^{(3)} \cap \{|Z_r| \le \gamma |Y_{r-1}|\}$, using (8.19) for $E_{r-1}^{(3)}$, we have the key bound

$$\mathrm{P}(S_n \in x + I,\, E_r^{(1)}) \lesssim \int\limits_{\substack{y_1 > -\overline{\gamma} x,\\ (y_2, \ldots, y_{r-1}) \in \Theta_{r-1}(y_1),\\ \tilde{y}_{r-1} \ge a_n}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1})\, \frac{n^{r+d-1}}{a_n\, A(|y_{r-1}|)^d}\,, \tag{8.23}$$

where we set

$$\tilde{y}_j := \min\{C_i^{-1}\, |y_i|, 1 \le i \le j\}\,, \qquad \text{for } j \ge 1\,. \tag{8.24}$$

(For $r = 2$ the integral in (8.23) is only over y_1, so the restriction $(y_2, \ldots, y_{r-1}) \in \Theta_{r-1}(y_1)$ and the term $P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1})$ should be ignored.)

Henceforth we fix $d \in (\frac{1}{\alpha} - r, \frac{1}{\alpha} - r + 1)$. Since $\gamma < \frac{\alpha}{1-\alpha}$ by assumption, and $r \ge 2$, we have $\frac{1}{\gamma} > \frac{1}{\alpha} - 1 \ge \frac{1}{\alpha} - r + 1 > d$, hence the constraint $d \le \frac{1}{\gamma}$ is satisfied. The sequence n^{r+d-1}/a_n is regularly varying with exponent $(d + r - 1) - \frac{1}{\alpha} > -1$, hence by (2.3)

$$\sum_{1 \le n \le A(w)} \frac{n^{r+d-1}}{a_n} \lesssim \frac{A(w)^{r+d}}{w \wedge 1} = b_{r+d}(w)\,, \qquad \forall w \ge 0\,,$$

where we recall that $b_k(\cdot)$ was defined in (1.10). Since the integral in (8.23) is restricted to $n \le A(\tilde{y}_{r-1})$, we see that

$$\begin{aligned} &\sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I,\, E_r^{(1)}) \\ &\quad \lesssim \int\limits_{\substack{y_1 > -\overline{\gamma} x\\ (y_2, \ldots, y_{r-1}) \in \Theta_{r-1}(y_1)}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1})\, \frac{b_{r+d}(\tilde{y}_{r-1} \wedge \delta x)}{A(|y_{r-1}|)^d}\,. \end{aligned} \tag{8.25}$$

We split the integral in two terms, corresponding to $|y_1| \le \delta_0 x$ and $|y_1| > \delta_0 x$. Given $\epsilon > 0$, we first show that for $\delta_0 > 0$ small enough the first term is $\le \epsilon\, b_1(x)$, for large x. We then show that for $\delta > 0$ small enough (depending on δ_0) the second term is also $\le \epsilon\, b_1(x)$, for large x. Altogether, this proves that (8.25) is a.n. and completes the estimate of $E_r^{(1)}$.

- *First term.* Since $\tilde{y}_{r-1} \le |y_{r-1}|/C_{r-1}$, see (8.24), and since $b_{r+d}(\cdot)$ is asymptotically increasing (because $r+d > \frac{1}{\alpha}$), we have

$$\frac{b_{r+d}(\tilde{y}_{r-1} \wedge \delta x)}{A(|y_{r-1}|)^d} \lesssim_r \frac{b_{r+d}(|y_{r-1}|)}{A(|y_{r-1}|)^d} = b_r(|y_{r-1}|)\,,$$

hence the contribution of $|y_1| \le \delta_0 x$ to the integral in (8.25) is bounded by

$$\int\limits_{\substack{|y_1| \le \delta_0 x \\ (y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1)}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1})\, b_r(|y_{r-1}|) =: J_{r-1}(\delta_0; x)\,,$$

see (8.7) for the definition of J_k. By Proposition 8.1 with $\ell = r-1$ (recall (8.18)), we can fix $\delta_0 > 0$ small enough so that for large x we have $J_{r-1}(\delta_0, x) \le \epsilon\, b_1(x)$.
- *Second term.* If we define $f_1(y) := 1/b_{r+d-1}(y)$, then by (8.9) we can write

$$\int\limits_{(y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1)} P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_{r-1})\, \frac{1}{A(|y_{r-1}|)^d} = h_{r-1}(y_1, f_1)\,, \qquad \forall y_1 \in \mathbb{R}\,.$$

As a consequence, the contribution of $|y_1| > \delta_0 x$ to the integral in (8.25) is at most

$$b_{r+d}(\delta x) \int\limits_{|y_1| > \delta_0 x,\, y_1 > -\overline{\gamma} x} F(x + \mathrm{d}y_1)\, h_{r-1}(y_1, f_1)\,. \tag{8.26}$$

Note that $f_1(\cdot) \in RV(\beta)$ with $\beta = 1 - (r+d-1)\alpha$. Our choice of d implies that $\frac{1}{\alpha} < r + d < \frac{1}{\alpha} + 1$, hence $0 < \beta < \alpha$. Since $\alpha < \frac{1}{r}$, see (8.18), we also have $\alpha < 1 - (r-1)\alpha$, which yields $0 < \beta < 1 - (r-1)\alpha$. By Proposition 8.1 with $f = f_1$ and $\ell = r-1$, the expression in (8.26) is $\lesssim_{\delta_0,\gamma,r} b_{r+d}(\delta x)\, \frac{f_1(x)}{x} \lesssim \frac{A(\delta x)}{x}$. Then we can fix $\delta > 0$ small (depending on δ_0) so that it is $\le \epsilon\, b_1(x)$ for large x.

Estimate of $E_r^{(2)}$. Always for $r \in \{2, \ldots, k\}$, in analogy with (8.20), we have

$$\begin{aligned} \mathrm{P}(S_n \in x + I,\, E_r^{(2)}) &\le n^r\, \mathrm{P}\big((Z_1, \ldots Z_r) = (X_1, \ldots, X_r),\, S_n \in x + I,\, E_r^{(2)}\big) \\ &\le n^r\, \mathrm{P}\big((Z_1, \ldots Z_r) = (X_1, \ldots, X_r),\, E_r^{(2)}\big) \left(\sup_{z \in \mathbb{R}} \mathrm{P}(S_{n-r} \in z + I) \right) \\ &\lesssim \frac{n^r}{a_n}\, \mathrm{P}\big((Z_1, \ldots Z_r) = (X_1, \ldots, X_r),\, E_r^{(2)}\big)\,, \end{aligned} \tag{8.27}$$

where we have applied (2.7). Since $E_r^{(2)} = E_{r-1}^{(3)} \cap \{|Y_r| \le C_r a_n\}$, by (8.19) and (8.24) we obtain

$$\mathrm{P}(S_n \in x + I,\, E_r^{(2)}) \lesssim \int\limits_{\substack{y_1 > -\overline{\gamma} x,\ (y_2,\ldots,y_{r-1}) \in \Theta_{r-1}(y_1), \\ \tilde{y}_{r-1} \ge a_n,\ |y_r| < C_r a_n}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_r)\, \frac{n^r}{a_n}\,. \tag{8.28}$$

Recalling (8.24) and (5.8), we can write

$$\sum_{1\le n\le A(\delta x)} \mathrm{P}(S_n\in x+I,\ E_r^{(2)}) \lesssim \int\limits_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\dots,y_{r-1})\in\Theta_{r-1}(y_1)\\ |y_r|\le\overline{\gamma}|y_{r-1}|}} F(x+\mathrm{d}y_1)\,P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_r) \sum_{n=A(C_r^{-1}|y_r|)}^{A(\delta x\wedge\tilde{y}_{r-1})} \frac{n^r}{a_n}$$

$$= \int\limits_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\dots,y_{r-1})\in\Theta_{r-1}(y_1)\\ |y_r|\le\overline{\gamma}|y_{r-1}|}} F(x+\mathrm{d}y_1)\,P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_r)\,\tilde{b}_{r+1}(\delta x\wedge\tilde{y}_{r-1},C_r^{-1}|y_r|)\,, \qquad (8.29)$$

where we made explicit the restriction $|y_r|\le\overline{\gamma}|y_{r-1}|$ in the domain of integration, because for $|y_r|>\overline{\gamma}|y_{r-1}|$ the integrand vanishes (since $\tilde{y}_{r-1}<C_{r-1}^{-1}|y_{r-1}|$ and $C_r=\overline{\gamma}^{r-1}$).

We split the integral (8.29) in two terms, i.e. $|y_1|\le\delta_0 x$ and $|y_1|>\delta_0 x$. First we show that, given any $\epsilon>0$, the first term is $\le \epsilon\, b_1(x)$ for $\delta_0>0$ small and x large. Then we show that the second term is $\le\epsilon\, b_1(x)$ for $\delta>0$ small (depending on δ_0) and x large.

- *First term.* Recalling (8.24), (5.8) and the definition $C_i=\overline{\gamma}^{i-1}$, we can bound

$$\tilde{b}_{r+1}(\delta x\wedge\tilde{y}_{r-1},C_r^{-1}|y_r|)\le\tilde{b}_{r+1}(C_{r-1}^{-1}|y_{r-1}|,C_{r-1}^{-1}\overline{\gamma}^{-1}|y_r|)\lesssim\tilde{b}_{r+1}(|y_{r-1}|,\overline{\gamma}^{-1}|y_r|)$$
$$\le\tilde{b}_{r+1}(|y_{r-1}|,|y_r|)\,,$$

where the last inequality holds because $\overline{\gamma}^{-1}>1$. For $|y_1|\le\delta_0 x$, when we plug this into (8.29) we obtain $\tilde{J}_r(\delta_0;x)$, see (8.12) and (8.13). By Proposition 8.2 with $\ell=r$, we can fix $\delta_0>0$ small enough so that $\tilde{J}_r(\delta_0;x)\le\epsilon\, b_1(x)$ for large x.

- *Second term.* Next we deal with $|y_1|>\delta_0 x$. Note that $\alpha(r+1)\le\alpha(k+1)\le 1$, see (8.18). We fix any $\psi\in(0,1)$, so that $\alpha(r+1-\psi)<1$. By (5.8) we can bound

$$\tilde{b}_{r+1}(\delta x\wedge\tilde{y}_{r-1},C_r^{-1}|y_r|)\lesssim A(\delta x)^{\psi}\sum_{n\ge A(C_r^{-1}|y_r|)}\frac{n^{r-\psi}}{a_n}\lesssim_r A(\delta x)^{\psi}\,b_{r+1-\psi}(|y_r|)\,,$$

where the last inequality holds by (2.4) (note that $n^{r-\psi}/a_n$ is regularly varying with index $r-\psi-\frac{1}{\alpha}<-1$). If we set $f_2(y):=A(y)^{1-\psi}$, the contribution of $|y_1|>\delta_0 x$ is

$$\lesssim A(\delta x)^{\psi}\int_{y_1>-\overline{\gamma}x,|y_1|>\delta_0 x}F(x+\mathrm{d}y_1)\,h_r(y_1,f_2).$$

Note that $f_2(y):=A(y)^{1-\psi}\in RV(\beta)$, with $\beta=\alpha(1-\psi)$, hence $0<\beta<1-r\alpha$ by our choice of ψ. We can apply point (1) in Proposition 8.1 with $\ell=r$, to get

$$\lesssim A(\delta x)^{\psi}\,\frac{f_2(x)}{x\vee 1}\underset{x\to\infty}{\sim}\delta^{\alpha\psi}\,\frac{A(x)}{x\vee 1}=\delta^{\alpha\psi}\,b_1(x)\,,$$

which is a.n..

Estimate of $E_k^{(3)}$. Finally, recalling (8.19), (8.24) and applying (1.6), we can write

$$\mathrm{P}(S_n\in x+I,\ E_k^{(3)})\lesssim n^k\,\mathrm{P}\big((Z_1,\dots Z_k)=(X_1,\dots,X_k),\,S_n\in x+I,\,E_k^{(3)}\big)$$
$$\lesssim \int\limits_{\substack{y_1>-\overline{\gamma}x,\\ (y_2,\dots,y_k)\in\Theta_k(y_1),\\ \tilde{y}_k\ge a_n}} F(x+\mathrm{d}y_1)\,P_{y_1}(\mathrm{d}y_2,\cdots,\mathrm{d}y_k)\,\frac{n^{k+1}}{a_nA(y_k)}.$$

Note that $n^{k+1}/a_n \in RV(\zeta)$ with $\zeta = k+1-\frac{1}{\alpha} > -1$, by $k = \kappa_\alpha$. Therefore, by (2.3),

$$\sum_{1 \le n \le A(\delta x)} \mathrm{P}(S_n \in x + I, E_k^{(3)}) \lesssim \int_{\substack{y_1 > -\overline{\gamma} x, \\ (y_2,\ldots,y_k) \in \Theta_k(y_1)}} F(x + \mathrm{d}y_1)\, P_{y_1}(\mathrm{d}y_2, \cdots, \mathrm{d}y_k) \frac{b_{k+2}(\delta x \wedge \tilde{y}_k)}{A(y_k)} \,. \tag{8.30}$$

We split the integral in two terms $|y_1| \le \delta_0 x$ and $|y_1| > \delta_0 x$. We recall that $\frac{1}{k+2} < \alpha \le \frac{1}{k+1}$.

- *First term.* We focus on $|y_1| \le \delta_0 x$ and distinguish two cases. First we consider $|y_k| \le \overline{\gamma}|y_{k-1}|$. By $(k+1) - \frac{1}{\alpha} > -1$ and $\tilde{y}_k \lesssim_k \min\{|y_{k-1}|, |y_k|\}$, see (8.24), we get

$$\begin{aligned} b_{k+2}(\delta x \wedge \tilde{y}_k) &\lesssim \sum_{n=1}^{A(\delta x \wedge \tilde{y}_k)} \frac{n^{k+1}}{a_n} \lesssim A(y_k) \sum_{n=1}^{A(|y_{k-1}|)} \frac{n^k}{a_n} \lesssim_\gamma A(y_k) \sum_{n=A(|y_k|)}^{A(|y_{k-1}|)} \frac{n^k}{a_n} \\ &\lesssim A(y_k)\, \tilde{b}_{k+1}(y_{k-1}, y_k) \,, \end{aligned}$$

  where the third inequality holds for $|y_k| \le \overline{\gamma}|y_{k-1}|$, and for the last inequality we recall (5.8). When we plug this bound into (8.30), with the integral restricted to $|y_1| \le \delta_0 x$ and $|y_k| \le \overline{\gamma}|y_{k-1}|$, we obtain $\tilde{J}_r(\delta_0; x)$, see (8.12)-(8.13). By Proposition 8.2 with $\ell = r$, we can fix $\delta_0 > 0$ small enough so that $\tilde{J}_r(\delta_0; x) \le \epsilon\, b_1(x)$.

  Next we consider $|y_k| > \overline{\gamma}|y_{k-1}|$, hence we can bound $A(y_k) \gtrsim_\gamma A(y_{k-1})$. Since b_{k+2} is asymptotically increasing (it is regularly varying with index $(k+2)\alpha - 1 > 0$), we can also bound $b_{k+2}(\delta x \wedge \tilde{y}_k)/A(y_k) \lesssim b_{k+2}(y_{k-1})/A(y_{k-1}) = b_{k+1}(y_{k-1})$. When we plug this into (8.30), the integrand does not depend on y_k anymore, so we can integrate over y_k to get $\int_{y_k \in \theta(y_{k-1})} F(-y_{k-1} + \mathrm{d}y_k) \lesssim_\gamma 1/A(y_{k-1})$, which multiplied by $b_{k+1}(y_{k-1})$ gives $b_k(y_{k-1})$. Then the contribution of $|y_1| \le \delta_0 x$ and $|y_k| > \overline{\gamma}|y_{k-1}|$ to (8.30) is bounded by $J_{k-1}(\delta_0; x)$, see (8.7), which is a.n. by Proposition 8.1.

- *Second term.* To deal with $\{|y_1| > \delta_0 x\}$, we fix $\nu \in (0,1)$ sufficiently close to 1, so that $(k+1+\nu)\alpha - 1 > 0$ (we recall that $\alpha > \frac{1}{k+2}$), which ensures that $b_{k+1+\nu}(\cdot)$ is asymptotically increasing. Then we can bound

$$\frac{b_{k+2}(\delta x \wedge \tilde{y}_k)}{A(y_k)} \lesssim \frac{b_{k+1+\nu}(\delta x) A(y_k)^{1-\nu}}{A(y_k)} = \frac{b_{k+1+\nu}(\delta x)}{A(y_k)^\nu} = b_{k+1+\nu}(\delta x)\, b_k(y_k)\, f_3(y_k) \,,$$

  where we set $f_3(y) := 1/b_{k+\nu}(y)$. Note that $\alpha(k+\nu) < 1$ (by $\alpha \le \frac{1}{k+1}$), hence $f_3 \in RV(\beta)$ with $\beta = 1 - \alpha(k+\nu)$ satisfies $0 < \beta < 1 - \alpha k$, i.e. the assumption of point (1) in Proposition 8.1 with $\ell = k$. The contribution of $\{|y_1| > \delta_0 x\}$ is then

$$\begin{aligned} &\lesssim b_{k+1+\nu}(\delta x) \int_{y_1 > -\overline{\gamma}x, |y_1| > \delta_0 x} F(x + \mathrm{d}y_1)\, h_k(y_1, f_3) \lesssim b_{k+1+\nu}(\delta x) \frac{f_3(x)}{x \vee 1} \\ &= \frac{b_{k+1+\nu}(\delta x)}{A(x)^{k+\nu}} \underset{x\to\infty}{\sim} \delta^{(k+1+\nu)\alpha - 1}\, b_1(x) \,, \end{aligned}$$

  which is a.n. and completes the proof.

8.3. **Technical proofs.** In this subsection we are going to prove Propositions 8.1 and 8.2. We first need two preliminary results, stated in the next Propositions 8.4 and 8.5.

First an elementary observation. Recall that $g_r(y)$ is defined in (8.2). We claim that

$$\forall r \ge 2 \,,\ \forall y \in \mathbb{R}: \qquad g_r(y) \le A(y) \int_{|z| \le \overline{\gamma}|y|} F(-y + \mathrm{d}z)\, g_{r-1}(z) \,. \tag{8.31}$$

The case $r = 2$ follows immediately from (8.2)-(8.3) and (1.10) (recall that A is increasing). Similarly, for $r \geq 3$, we simply observe that $|y_r| \leq |y|$ for $(y_2, \dots, y_r) \in \Omega_r(y)$, hence

$$\begin{aligned} g_r(y) &= \int_{|y_2| \leq \overline{\gamma}|y|} F(-y + \mathrm{d}y_2) \int_{\Omega_{r-1}(y_2)} P_{y_2}(\mathrm{d}y_3, \cdots, \mathrm{d}y_r)\, b_{r+1}(y_r) \\ &\leq A(|y|) \int_{|y_2| \leq \overline{\gamma}|y|} F(-y + \mathrm{d}y_2) \int_{\Omega_{r-1}(y_2)} P_{y_2}(\mathrm{d}y_3, \cdots, \mathrm{d}y_r)\, b_r(y_r) \\ &= A(|y|) \int_{|y_2| \leq \overline{\gamma}|y|} F(-y + \mathrm{d}y_2)\, g_{r-1}(y_2)\,. \end{aligned} \tag{8.32}$$

We are ready for our first preliminary result. If $I_r(\delta; x)$ is a.n., then for $\delta > 0$ small we have $I_r(\delta; x) \lesssim b_1(x)$ for all $x \geq 0$ (recall Definition 1.3). We now show that the same bound holds when the integral in (8.1) is enlarged to $\{y_1 > -\kappa x\}$, for any fixed $\kappa < 1$.

Proposition 8.4. *Fix $r \in \mathbb{N}$ and $\alpha \in (0, \frac{1}{2})$. Assume that $I_j(\delta; x)$ is a.n. for $j = 1, \dots, r$. Then for any $0 < \kappa < 1$*

$$\int_{y > -\kappa x} F(x + \mathrm{d}y)\, g_r(y) \lesssim_{\kappa,\gamma,r} b_1(x) \qquad \forall x \geq 0\,. \tag{8.33}$$

Proof. The case $r = 1$ is easy: since $b_2 \in RV(2\alpha - 1)$ and $2\alpha - 1 < 0$, for any fixed $\delta_0 > 0$

$$\int_{|y| > \delta_0 x,\, y > -\kappa x} F(x + \mathrm{d}y)\, b_2(y) \leq \left(\sup_{|y| > \delta_0 x} b_2(y) \right) \overline{F}((1 - \kappa)x) \lesssim_{\delta_0,\kappa} \frac{b_2(x)}{A(x)} = b_1(x)\,. \tag{8.34}$$

On the other hand, the contribution to the integral of $|y| \leq \delta_0 x$ gives $I_1(\delta_0; x)$ which is $\lesssim b_1(x)$ for $\delta_0 > 0$ small enough, as we already observed, because $I_1(\delta; x)$ is a.n..

Next we fix $r \geq 2$. By induction, we can assume that (8.33) holds with r replaced by $1, 2, \dots, r - 1$ and our goal is to prove it for r.

Assume first that $y \leq 0$, say $y = -t$ with $t \geq 0$. By (8.31) and the inductive hypothesis (8.33) for $r - 1$ (since $t \geq 0$), we get

$$g_r(-t) \leq A(t) \int_{|z| \leq \overline{\gamma} t} F(t + \mathrm{d}z)\, g_{r-1}(z) \lesssim_{\gamma,r} A(t)\, b_1(t) = b_2(t) = g_1(t)\,.$$

When we plug this bound into (8.33) restricted to $y \leq 0$, we get

$$\int_{0 \leq t < \kappa x} F(x - \mathrm{d}t)\, g_r(-t) \lesssim_{\gamma,r} \int_{0 \leq t < \kappa x} F(x - \mathrm{d}t)\, g_1(t) \lesssim_{\kappa} b_1(x)\,,$$

where the last inequality holds by the inductive hypothesis (8.33) for $r = 1$.

It remains to look at the contribution of $y > 0$ in (8.33). By (8.1), the contribution of $\{0 < y \leq \delta_1 x\}$ to (8.33) is bounded by $I_r(\delta_1, x)$ which is a.n. by assumption, hence it is $\lesssim b_1(x)$ provided $\delta_1 > 0$ is small enough. It remains to focus on $\{y > \delta_1 x\}$.

We need a simple observation; let $I = (a_1, a_2)$ where $0 \le a_1 < a_2 \le \infty$ and, for $\gamma \in (0,1)$, put $I' = (\gamma a_1, (2-\gamma)a_2)$. Then for all non-negative functions $f, g : \mathbb{R} \to [0,\infty)$

$$\begin{aligned}
&\int_{y\in xI} F(x+\mathrm{d}y)\, f(y) \int_{|z|\le\overline{\gamma}y} F(-y+\mathrm{d}z)\, g(z) \\
&= \int_{y\in xI} F(x+\mathrm{d}y)\, f(y) \int_{\gamma y\le w\le(2-\gamma)y} F(-\mathrm{d}w)\, g(y-w) \\
&\le \int_{w\in xI'} F(-\mathrm{d}w) \int_{(2-\gamma)^{-1}w\le y\le\gamma^{-1}w} F(x+\mathrm{d}y)\, f(y)\, g(y-w) \\
&= \int_{w\in xI'} F(-\mathrm{d}w) \int_{-\gamma_1 w\le v\le\gamma_2 w} F(x+w+\mathrm{d}v)\, f(w+v)\, g(v),
\end{aligned} \tag{8.35}$$

where $\gamma_1 = 1-(2-\gamma)^{-1}$ and $\gamma_2 = \gamma^{-1}-1$.

Applying (8.31) together with (8.35), the contribution of $\{y > \delta_1 x\}$ to (8.33) is

$$\begin{aligned}
\int_{\delta_1 x}^{\infty} F(x+\mathrm{d}y)\, g_r(y) &\le \int_{\delta_1 x}^{\infty} F(x+\mathrm{d}y)\, A(y) \int_{-\overline{\gamma}y}^{\overline{\gamma}y} F(-y+\mathrm{d}z)\, g_{r-1}(z) \\
&= \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w) \int_{-\gamma_1 w}^{\gamma_2 w} F(x+w+\mathrm{d}v)\, A(v+w)\, g_{r-1}(v) \\
&\lesssim_\gamma \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w)\, A(w) \int_{-\gamma_1(x+w)}^{\infty} F(x+w+\mathrm{d}v)\, g_{r-1}(v)\,.
\end{aligned}$$

Applying (8.33) for $r-1$, and the fact that $b_2(\cdot)$ is asymptotically decreasing, we obtain

$$\begin{aligned}
\int_{\delta_1 x}^{\infty} F(x+\mathrm{d}y)\, g_r(y) &\lesssim \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w)\, A(w)\, b_1(x+w) \le \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w)\, b_2(x+w) \\
&\lesssim b_2(x) \int_{\gamma\delta_1 x}^{\infty} F(-\mathrm{d}w) \lesssim_{\delta_1,\gamma} b_1(x)\,. \quad \blacksquare
\end{aligned}$$

We now introduce a generalization $g_k(y,f)$ of $g_k(y)$ (in the same way as $h_k(y,f)$ generalizes $h_k(y)$, see (8.8)-(8.9)). For any non-negative, even function $f : \mathbb{R} \to [0,\infty)$ we denote by $g_k(y_1, f)$ what we get by replacing $b_{k+1}(y_k)$ by $b_k(y_k) f(y_k)$ in (8.2), that is

$$g_k(y_1, f) := \begin{cases} b_1(y_1)\, f(y_1) & \text{if } k=1 \\ \displaystyle\int_{\Omega_k(y_1)} P_{y_1}(\mathrm{d}y_2,\ldots,\mathrm{d}y_k)\, b_k(y_k)\, f(y_k) & \text{if } k\ge 2 \end{cases}. \tag{8.36}$$

In particular, $g_k(y)$ is $g_k(y, A)$.

We are going to assume that $f(|\cdot|) \in RV(\beta)$ for some $\beta > 0$, so f is asymptotically increasing and $f(w) \lesssim f(y)$ for $|w| \le |y|$. Then, in analogy with (8.31), we claim that

$$\forall r \ge 2\,,\ \forall y \in \mathbb{R}: \qquad g_r(y,f) \le f(y) \int_{|z|\le\overline{\gamma}|y|} F(-y+\mathrm{d}z)\, g_{r-1}(z)\,. \tag{8.37}$$

The case $r=2$ follows immediately by (8.36), while for $r \ge 3$ we can argue as in (8.32), replacing $b_{r+1}(y_r)$ by $b_r(y_r, f)$ and bounding $f(y_r) \lesssim f(y)$, since $|y_r| \le |y|$ on $\Omega_r(y)$.

We now state our second preliminary result, which is in the same spirit as Proposition 8.4.

Proposition 8.5. *Fix $r \in \mathbb{N}$ and $\alpha \in (0, \frac{1}{2})$. If $r \geq 2$, assume that $I_j(\delta; x)$ is a.n. for $j = 1, \ldots, r-1$. Fix any $f \in RV(\beta)$ with $0 < \beta < 1-\alpha$. Then for all $0 < \delta_0 < \kappa < 1$*

$$\int_{|y|>\delta_0 x,\, y>-\kappa x} F(x+\mathrm{d}y)\, g_r(y,f) \lesssim_{\delta_0,\kappa,\gamma,r} \frac{f(x)}{x \vee 1} \qquad \forall x \geq 0\,. \tag{8.38}$$

Proof. Since $g_1(\cdot, f) = b_1(\cdot) f(\cdot) \in RV(\alpha+\beta-1)$ is asymptotically decreasing, we have

$$\begin{aligned} \int_{|y|>\delta_0 x,\, y>-\kappa x} F(x+\mathrm{d}y)\, g_1(y,f) &\lesssim_{\delta_0} b_1(x) f(x) \int_{y>-\kappa x} F(x+\mathrm{d}y) \\ &\lesssim_{\kappa} \frac{b_1(x) f(x)}{A(x)} = \frac{f(x)}{x \vee 1}\,, \end{aligned}$$

which proves (8.38) if $r = 1$. Henceforth we assume that $r \geq 2$ and proceed by induction. Note that we can apply Proposition 8.4 with r replaced by $r-1$ (since here we assume that $I_j(\delta; x)$ is a.n. for $j = 1, \ldots, r-1$).

Assume first that $y \leq 0$, say $y = -t$ with $t \geq 0$. Then by (8.37) we can bound

$$g_r(-t, f) \leq f(t) \int_{|z| \leq \overline{\gamma} t} F(t+\mathrm{d}z)\, g_{r-1}(z) \lesssim_{\gamma} f(t)\, b_1(t)\,, \tag{8.39}$$

where for the last inequality we apply Proposition 8.4 for $r-1$ (since $t \geq 0$). Since $f(\cdot) b_1(\cdot)$ is asymptotically decreasing, the contribution of $y \leq 0$ to (8.38) is then estimated by

$$\int_{\delta_0 x < t < \kappa x} F(x-\mathrm{d}t)\, g_r(-t,f) \lesssim_{\delta_0} f(x)\, b_1(x) \int_{\delta_0 x < t < \kappa x} F(x-\mathrm{d}t) \lesssim_{\kappa} \frac{f(x)\, b_1(x)}{A(x)} = \frac{f(x)}{x \vee 1}\,.$$

It remains to control the contribution to (8.38) of $y > 0$. By (8.37) and (8.35)

$$\begin{aligned} \int_{\delta_0 x}^{\infty} F(x+\mathrm{d}y)\, g_r(y,f) &\leq \int_{\delta_0 x}^{\infty} F(x+\mathrm{d}y)\, f(y) \int_{-\overline{\gamma} y}^{\overline{\gamma} y} F(-y+\mathrm{d}z)\, g_{r-1}(z) \\ &\lesssim \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w) \int_{-\gamma_1 w}^{\gamma_2 w} F(x+w+\mathrm{d}v)\, f(w+v)\, g_{r-1}(v) \\ &\lesssim_{\gamma} \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w)\, f(w) \int_{-\gamma_1 (x+w)}^{\infty} F(x+w+\mathrm{d}v)\, g_{r-1}(v)\,. \end{aligned} \tag{8.40}$$

Applying again Proposition 8.4 for $r-1$ we get, since $f(\cdot) b_1(\cdot)$ is asymptotically decreasing,

$$\begin{aligned} \int_{\delta_0 x}^{\infty} F(x+\mathrm{d}y)\, g_\ell(y,f) &\lesssim_{\kappa,\gamma,r} \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w)\, f(x+w)\, b_1(x+w) \\ &\lesssim f(x)\, b_1(x) \int_{\gamma \delta_0 x}^{\infty} F(-\mathrm{d}w) \lesssim_{\gamma,\delta_0} \frac{f(x)}{x \vee 1}\,. \ \blacksquare \end{aligned} \tag{8.41}$$

We are finally ready to prove Propositions 8.1 and 8.2.

Proof of Proposition 8.1. We write r in place of ℓ. We assume that $I_j(\delta; x)$ is a.n. for $j = 1, \ldots, r-1$ (if $r \geq 2$). Moreover, for point (2) we also assume that $I_r(\delta; x)$ is a.n..

Recall the definitions of $h_k(y,f)$, $g_k(y,f)$, see (8.9), (8.36). We claim that

$$\forall \text{ even } f \in RV(\beta) \text{ with } 0 < \beta < 1 - r\alpha: \qquad h_r(y,f) \lesssim_{\gamma,r} \sum_{j=1}^{r} g_j(y,f)\,. \tag{8.42}$$

Then relation (8.10) follows immediately by Proposition 8.5. This proves point (1).

For point (2), we note that for $\alpha < \frac{1}{r+1}$ we can plug $f = A$ in (8.42), because $\beta = \alpha$ satisfies $\beta < 1 - r\alpha$. This gives $h_r(y) \lesssim_\gamma \sum_{j=1}^r g_j(y)$, which plugged into (8.7) shows that $J_r(\delta;x) \lesssim_\gamma \sum_{j=1}^r I_r(\delta;x)$. Since in point (2) we assume that $I_j(\delta;x)$ for $j = 1,\ldots,r$ (including $j = r$), relation (8.11) follows and completes the proof.

It remains to prove (8.42). This holds for $r = 1$, since $h_1(y,f) = g_1(y,f)$. Henceforth we fix $r \geq 2$ and we proceed by induction.

Let us first show that

$$\begin{aligned} &\forall r' = 1,\ldots,r-1\,, \qquad \forall \text{ even } f' \in RV(\beta') \text{ with } 0 < \beta' < 1 - r'\alpha\,, \\ &\forall y < 0: \qquad \int_{z \geq -\overline{\gamma}|y|} F(-y + \mathrm{d}z)\, h_{r'}(z, f') \lesssim_\gamma \sum_{i=1}^{r'+1} g_i(y, \tfrac{f'}{A})\,, \end{aligned} \tag{8.43}$$

where we stress that $y < 0$. By the inductive assumption, we can apply (8.42) with r replaced by r' (since $r' \leq r - 1$) and f replaced by f', hence

$$\int_{z \geq -\overline{\gamma}|y|} F(-y + \mathrm{d}z)\, h_{r'}(z, f') \lesssim_\gamma \sum_{j=1}^{r'} \int_{z \geq -\overline{\gamma}|y|} F(-y + \mathrm{d}z)\, g_j(z, f')\,.$$

We now split the domain of integration in the two subsets $[-\overline{\gamma}|y|, \overline{\gamma}|y|]$ and $(\overline{\gamma}|y|, \infty)$. The first subset gives $\int_{|z| \leq \overline{\gamma}|y|} F(-y + \mathrm{d}z)\, g_j(z, f') = g_{j+1}(y, \frac{f'}{A})$, by (8.36). For the second subset we can apply Proposition 8.5 (since $-y \geq 0$), getting

$$\int_{z > \overline{\gamma}|y|} F(-y + \mathrm{d}z)\, g_j(z, f') \lesssim_\gamma \frac{f'(|y|)}{|y| \vee 1} = b_1(y)\, \frac{f'(y)}{A(y)} = g_1(y, \tfrac{f'}{A})\,,$$

where we recall that $A(\cdot)$ and $f'(\cdot)$ are even functions. This completes the proof of (8.43).

We are ready to prove (8.42). Let us first consider the case $y < 0$. By (8.9) we can write

$$h_r(y, f) = \int_{y_2 \geq -\overline{\gamma}|y|} F(-y + \mathrm{d}y_2)\, h_{r-1}(y_2, Af)\,.$$

We can now apply (8.43) with $r' := r - 1$ and $f' := Af$ (because $f' \in RV(\beta')$ with $\beta' = \alpha + \beta$ which satisfies $0 < \beta' < 1 - r'\alpha$). This proves (8.42) when $y < 0$.

Next we consider the case $y \geq 0$. If we restrict the domain of integration $\Theta_r(y)$ in (8.9) to $y_2 \geq 0, y_3 \geq 0, \ldots, y_r \geq 0$, then the domain becomes $\{0 \leq y_j \leq \overline{\gamma} y_{j-1} \text{ for } 2 \leq j \leq r\}$ which is included in $\Omega_r(y)$, see (8.3). The corresponding contribution to $h_r(y,f)$ is then bounded from above by $g_r(y,f)$, see (8.36). This proves (8.42) when $y_2 \geq 0, y_3 \geq 0, \ldots, y_r \geq 0$.

It remains to estimate $h_r(y,f)$ for $y \geq 0$, when some of the coordinates $y_2, y_3, \ldots, y_r$ in the integral in (8.9) are negative. Let us define $H := \min\{j \in \{2,\ldots,r\} : y_j < 0\}$.

In the extreme case $H = r$, the corresponding contribution to $h_r(y,f)$ is, for $r \geq 3$,

$$\begin{aligned} &\int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{r-1}=0}^{\overline{\gamma} y_{r-2}} \int_{y_r = -\overline{\gamma} y_{r-1}}^{0} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_r)\, b_r(y_r) f(y_r) \\ &+ \int_{y_2=0}^{\overline{\gamma} y} \cdots \int_{y_{r-1}=0}^{\overline{\gamma} y_{r-2}} \int_{y_r=-\infty}^{-\overline{\gamma} y_{r-1}} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_r)\, b_r(y_r) f(y_r)\,. \end{aligned} \tag{8.44}$$

If $r = 2$, one should ignore the first integrals, that is we have

$$\int_{y_2=-\overline{\gamma} y}^{0} F(-y + \mathrm{d}y_2)\, b_2(y_2)\, f(y) \;+\; \int_{y_2=-\infty}^{\overline{\gamma} y} F(-y + \mathrm{d}y_2)\, b_2(y_2)\, f(y)\,. \tag{8.45}$$

The first integral in (8.44)-(8.45) is bounded by $g_r(y,f)$, because the domain of integration for $(y_2,\dots,y_r)$ is included in $\Omega_r(y)$ (recall (8.36) and (8.3)). For the second integral, we note that $b_r(\cdot)f(\cdot)\in RV(r\alpha-1+\beta)$ is asymptotically decreasing, since $r\alpha-1+\beta<0$ by assumption, hence we can bound $b_r(y_r)f(y_r)\lesssim b_r(y_{r-1})f(y_{r-1})$. Since $P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_r)=P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_{r-1})F(-y_{r-1}+\mathrm{d}y_r)$, when we integrate over $y_r\in(-\infty,-\overline{\gamma}y_{r-1}]$ we get a factor $\lesssim_\gamma 1/A(y_{r-1})$. Overall, for $r\ge 3$ we can bound (8.44) by

$$\begin{aligned}&\le g_r(y,f)+\int_{y_2=0}^{\overline{\gamma}y}\cdots\int_{y_{r-1}=0}^{\overline{\gamma}y_{r-2}} P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_{r-1})\,\frac{1}{A(y_{r-1})}\,b_r(y_{r-1})f(y_{r-1})\\&\le g_r(y,f)+g_{r-1}(y,f)\,,\end{aligned}$$

and the same bound holds also for $r=2$. This proves (8.42) when $H=r$.

Finally, if $H=j\in\{2,\dots,r-1\}$, the contribution to $h_r(y,f)$ is (recall again (8.9))

$$\begin{aligned}&\int_{y_2=0}^{\overline{\gamma}y}\cdots\int_{y_{j-1}=0}^{\overline{\gamma}y_{j-2}}\int_{y_j=-\infty}^{0} P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_j)\int_{y_{j+1}>-\overline{\gamma}|y_j|} F(-y_j+\mathrm{d}y_{j+1})\,h_{r-j}(y_{j+1},A^jf)\\&\lesssim_\gamma \sum_{i=1}^{r-j+1}\int_{y_2=0}^{\overline{\gamma}y}\cdots\int_{y_{j-1}=0}^{\overline{\gamma}y_{j-2}}\int_{y_j=-\infty}^{0} P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_j)\,g_i(y_j,A^{j-1}f)\,,\end{aligned}\tag{8.46}$$

where we have applied (8.43) with $r'=r-j$ and $f'=A^jf$ (note that $f'\in RV(\beta')$ with $\beta'=j\alpha+\beta$, which satisfies $0<\beta'<1-r'\alpha$). We split the integral over y_j in the two subsets $[-\overline{\gamma}y_{j-1},0]$ and $(-\infty,-\overline{\gamma}y_{j-1}]$.

- On the first subset, we can enlarge the domain of integration to $(y_2,\dots,y_j)\in\Omega_j(y)$, see (8.3), hence the corresponding contribution to (8.46) is

$$\begin{aligned}&\sum_{i=1}^{r-j+1}\int_{(y_2,\dots,y_j)\in\Omega_j(y)} P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_j)\,g_i(y_j,A^{j-1}f)\\&=\sum_{i=1}^{r-j+1}\int_{(y_2,\dots,y_{j+i-1})\in\Omega_{j+i-1}(y)} P_y(\mathrm{d}y_2,\cdots,\mathrm{d}y_{j+i-1})\,b_{j+i-1}(y_{j+i-1})\,f(y_{j+i-1})\\&=\sum_{i=1}^{r-j+1} g_{j+i-1}(y,f)\,,\end{aligned}$$

  by (8.36). This proves (8.43) for the first subset.
- On the second subset, we first consider a fixed $i\ge 2$: renaming $y_j=-z$, we can write

$$\int_{y_j=-\infty}^{-\overline{\gamma}y_{j-1}} F(-y_{j-1}+\mathrm{d}y_j)\,g_i(y_j,A^{j-1}f)\tag{8.47}$$

$$\begin{aligned}&=\int_{z=\overline{\gamma}y_{j-1}}^{\infty} F(-y_{j-1}-\mathrm{d}z)\int_{\Omega_i(z)} P_{-z}(\mathrm{d}y_{j+1},\cdots\mathrm{d}y_{j+i})b_i(y_{j+i},A^{j-1}f)\\&=\int_{z=\overline{\gamma}y_{j-1}}^{\infty} F(-y_{j-1}-\mathrm{d}z)\int_{|y_{j+1}|\le\overline{\gamma}z} F(z+\mathrm{d}y_{j+1})\int_{\Omega_{i-1}(y_{j+1})} P_{y_{j+1},}(\mathrm{d}y_{j+2},\cdots\mathrm{d}y_{j+i})b_{i+j-1}(y_{j+i})f(y_{j+i})\,.\end{aligned}\tag{8.48}$$

We next write $b_{i+j-1}(\cdot)f(\cdot) = \{A^{j-1}(\cdot)f(\cdot)\}b_i(\cdot)$ and then bound $A^{j-1}(y_{j+i})f(y_{j+i}) \lesssim A^{j-1}(z)f(z)$, because $A^{j-1}(\cdot)f(\cdot)$ is asymptotically increasing, to get

$$\begin{aligned}
&\lesssim_\gamma \int_{z=\overline{\gamma}y_{j-1}}^{\infty} F(-y_{j-1} - \mathrm{d}z) A(z)^{j-1} f(z) \int_{|y_{j+1}|\le\overline{\gamma}z} F(z + \mathrm{d}y_{j+1})\, g_{i-1}(y_{j+1}) \\
&\lesssim_\gamma \int_{z=\overline{\gamma}y_{j-1}}^{\infty} F(-y_{j-1} - \mathrm{d}z)\, A(z)^{j-1}\, f(z)\, b_1(z) \lesssim_\gamma b_{j-1}(y_{j-1}) f(y_{j-1}) \,,
\end{aligned}$$

where we used (8.33) and the fact that $b_1(z)f(z)A^{j-1}(z)$ is regularly varying with index $j\alpha + \beta - 1 < 0$ and so is asymptotically decreasing. This shows that

$$\int_{y_j=-\infty}^{-\overline{\gamma}y_{j-1}} F(-y_{j-1} + \mathrm{d}y_j)\, g_i(y_j, A^{j-1}f) \lesssim_\gamma b_{j-1}(y_{j-1}) f(y_{j-1}) \,,$$

and the same bound holds also for $i = 1$ (since $g_1(z, A^{j-1}f) = b_1(z)f(z)A^{j-1}(z)$, we can directly apply (8.48)). Thus the contribution of $y_j \le -\overline{\gamma}y_{j-1}$ to (8.46) is

$$\begin{aligned}
&\lesssim_\gamma \sum_{i=1}^{r-j+1} \int_{y_2=0}^{\overline{\gamma}y} \cdots \int_{y_{j-1}=0}^{\overline{\gamma}y_{j-2}} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_{j-1})\, b_{j-1}(y_{j-1}) f(y_{j-1}) \\
&\le \sum_{i=1}^{r-j+1} \int_{\Omega_{j-1}(y)} P_y(\mathrm{d}y_2, \cdots, \mathrm{d}y_{j-1})\, b_{j-1}(y_{j-1}) f(y_{j-1}) = (r - j + 1) g_{j-1}(y).
\end{aligned}$$

This completes the proof of (8.43).

∎

Proof of Proposition 8.2. We write r in place of ℓ. We assume that $\tilde{I}_j(\delta; x)$ and $I_j(\delta; x)$ are a.n. for $j = 1, \ldots, r$, with $r \ge 2$ and $\alpha \le \frac{1}{r+1}$, and we need to show that $\tilde{J}_r(\delta; x)$ is a.n..

We first give a basic estimate: from (5.8), (2.4) and (1.10), for any $\lambda \in (0, r)$ we have

$$\tilde{b}_{r+1}(y, z) \le A(y)^\lambda\, \tilde{b}_{r+1-\lambda}(y, z) \le A(y)^\lambda \sum_{m=A(z)}^{\infty} \frac{m^{(r-\lambda)}}{a_m} \lesssim A(y)^\lambda\, b_{r+1-\lambda}(z). \tag{8.49}$$

Let us prove that $\tilde{J}_r(\delta; x)$ is a.n.. For $\alpha < \frac{1}{r+1}$ we can simply apply Proposition 8.1 with $\ell = r$, because $\tilde{J}_r(\delta; x) \lesssim J_r(\delta; x)$. Indeed, by (5.8),

$$\tilde{b}_{r+1}(y_{r-1}, y_r) \lesssim \sum_{n \ge A(|y_r|)} \frac{n^r}{a_n} \lesssim \frac{A(|y_r|)^{r+1}}{|y_r| \vee 1} = b_{r+1}(y_r) \,,$$

because n^r/a_n is regularly varying with index $r - 1/\alpha < -1$.

Henceforth we fix $\alpha = \frac{1}{r+1}$. For $r = 2$ there is nothing to prove, since $\tilde{J}_2(\delta; x) = \tilde{I}_2(\delta; x)$.

We now fix $r \ge 3$. If we consider the contribution to the integrals in (8.12)-(8.13) of $y_1 \ge 0, y_2 \ge 0, \ldots, y_{r-1} \ge 0$, the domain of integration, see (8.6), reduces to

$$\{0 \le y_1 \le \delta x\} \cap \{0 \le y_i \le \overline{\gamma}y_{i-1} \text{ for } 2 \le i \le r - 1\} \cap \{|y_r| \le \overline{\gamma}y_{r-1}\} \,.$$

This contribution is bounded from above by $\tilde{I}_r(\delta; x)$, see (1.24), which is a.n. by assumption.

Next we consider the contribution to (8.12)-(8.13) coming from $y_1, \ldots, y_{r-1}$ such that $y_i < 0$ for some $1 \le i \le r - 1$. Let us define $\tilde{H} = \max\{j \in \{1, \ldots, r - 1\} : \ y_j < 0\}$. If

$\tilde{H} = r-1$, the bound (8.49) with $\lambda = r-1$ and the fact that $y_{r-1} < 0$ show that

$$\begin{aligned}&\int_{|y_r|\le\overline{\gamma}|y_{r-1}|} F(-y_{r-1}+\mathrm{d}y_r)\,\tilde{b}_{r+1}(y_{r-1},y_r)\\ &\lesssim A(y_{r-1})^{r-1}\int_{|y_r|\le\overline{\gamma}|y_{r-1}|} F(-y_{r-1}+\mathrm{d}y_r)\,b_2(y_r) \lesssim A(y_{r-1})^{r-1}b_1(y_{r-1}) = b_r(y_{r-1}),\end{aligned}$$

where the second inequality comes from Proposition 8.4. Plugging this bound into (8.13), we see that the contribution to $\tilde{h}_r(y)$ is at most $h_{r-1}(y)$ (recall (8.8)), hence the contribution to $\tilde{J}_r(\delta;x)$ is at most $J_{r-1}(\delta;x)$ (recall (8.7)), which is a.n. by Proposition 8.1 with $\ell = r-1$.

We finally consider the contribution of $\tilde{H} = r-j$ with $j \ge 2$. This means that $y_{r-j} < 0$, while $y_{r-j+1} \ge 0, \ldots, y_{r-1} \ge 0$, and the range of integration in (8.13) is a subset of

$$\Theta_{r-j}(y_1) \cap \{y_{r-j} < 0\} \cap \{y_{r-j+1} \ge 0\} \cap \{|y_{r-j+1+\ell}| \le \overline{\gamma}|y_{r-j+\ell}|, \ell = 1, \cdots j-1\}.$$

We split this into the two subsets $\{0 \le y_{r-j+1} \le \overline{\gamma}|y_{r-j}|\}$ and $\{y_{r-j+1} > \overline{\gamma}|y_{r-j}|\}$.

On the first subset $\{0 \le y_{r-j+1} \le \overline{\gamma}|y_{r-j}|\}$, we bound $\tilde{b}_{r+1}(y_{r-1}, y_r) \lesssim A(y_{r-1})^{r-j}b_{j+1}(y_r)$, by (8.49) with $\lambda = r-j$, and then $A(y_{r-1}) \lesssim A(y_{r-j})$. Recalling the definition (8.2) of $g_j(\cdot)$, we see that this part of the integral with respect to $y_{r-j+1}, \cdots, y_r$ is

$$\begin{aligned}&\lesssim A(y_{r-j})^{r-j} \int\limits_{0\le y_{r-j+1}\le\overline{\gamma}|y_{r-j}|} F(|y_{r-j}|+\mathrm{d}y_{r-j+1})\,g_j(y_{r-j+1})\\ &\le A(y_{r-j})^{r-j} \int\limits_{z\ge-\overline{\gamma}|y_{r-j}|} F(|y_{r-j}|+\mathrm{d}z)\,g_j(z)\\ &\lesssim A(y_{r-j})^{r-j}\,b_1(y_{r-j}) = b_{r-j+1}(y_{r-j}),\end{aligned}$$

where the last inequality follows by Proposition 8.4. The contribution to (8.13) is

$$\lesssim \int_{\Theta_{r-j}(y_1)\cap\{y_{r-j}<0\}} P_y(\mathrm{d}y_2,\ldots,\mathrm{d}y_{r-j})\,b_{r-j+1}(y_{r-j}) \le h_{r-j}(y_1)\,, \tag{8.50}$$

hence the contribution to $\tilde{J}_r(\delta;x)$ is $\lesssim J_{r-j}(\delta;x)$, which is a.n. by Proposition 8.1.

On the second subset $\{y_{r-j+1} > \overline{\gamma}|y_{r-j}|\}$, we bound $\tilde{b}_{r+1}(y_{r-1}, y_r) \lesssim A(y_{r-1})^{r-j+1}b_j(y_r)$, by (8.49) with $\lambda = r-j+1$, and then $A(y_{r-1}) \lesssim A(y_{r-j+1})$, getting

$$\begin{aligned}&\lesssim \int\limits_{y_{r-j+1}>\overline{\gamma}|y_{r-j}|} F(|y_{r-j}|+\mathrm{d}y_{r-j+1})\,A(y_{r-j+1})^{r-j+1} \int\limits_{|y_{r-j+2}|\le\overline{\gamma}y_{r-j+1}} F(-y_{r-j+1}+\mathrm{d}y_{r-j+2})\,g_{j-1}(y_{r-j+2})\\ &= \int\limits_{y>\overline{\gamma}|y_{r-j}|} F(|y_{r-j}|+\mathrm{d}y)\,A(y)^{r-j+1} \int\limits_{|z|\le\overline{\gamma}y} F(-y+\mathrm{d}z)\,g_{j-1}(z)\,,\end{aligned}$$

where we have set $y = y_{r-j+1}$ and $z = y_{r-j+2}$ for short. Applying (8.35), where we recall that $\gamma_1 = 1 - (2-\gamma)^{-1}$ and $\gamma_2 = \gamma^{-1} - 1$, we get

$$\begin{aligned}
&\lesssim \int\limits_{w \ge \gamma\overline{\gamma}|y_{r-j}|} F(-\mathrm{d}w) \int\limits_{-\gamma_1 w \le v \le \gamma_2 w} F(|y_{r-j}| + w + \mathrm{d}v)\, A(w+v)^{r-j+1}\, g_{j-1}(v) \\
&\lesssim_\gamma \int\limits_{w \ge \gamma\overline{\gamma}|y_{r-j}|} F(-\mathrm{d}w)\, A(w)^{r-j+1} \int\limits_{v \ge -\gamma_1(|y_{r-j}|+w)} F(|y_{r-j}| + w + \mathrm{d}v)\, g_{j-1}(v) \\
&\lesssim_\gamma \int\limits_{w \ge \gamma\overline{\gamma}|y_{r-j}|} F(-\mathrm{d}w)\, A(w)^{r-j+1}\, b_1(|y_{r-j}| + w)\,,
\end{aligned}$$

by Proposition 8.4 with $r = j-1$. Finally, this is easily bounded by

$$\int\limits_{w \ge \gamma\overline{\gamma}|y_{r-j}|} F(-\mathrm{d}w)\, b_{r-j+2}(|y_{r-j}| + w) \lesssim F(-\gamma\overline{\gamma}|y_{r-j}|)\, b_{r-j+2}(|y_{r-j}|) \lesssim b_{r-j+1}(|y_{r-j}|)\,,$$

because $b_{r-j+2}(\cdot) \in RV(\alpha(r-j+2)-1)$ is asymptotically decreasing, since $j \ge 2$ and $\alpha = \frac{1}{r+1}$. Arguing as in (8.50), we see that the contribution to $\tilde{J}_r(\delta; x)$ is $\lesssim J_{r-j+1}(\delta; x)$, which is a.n. by Proposition 8.1. This completes the proof. ∎

9. Soft results

In this section we prove Theorem 1.5, Propositions 1.7, 1.11, 1.17 and Theorem 1.18, which are corollaries of our main results.

9.1. **Proof of Theorem 1.5.** Assume that condition (1.14) holds. By (1.10) we can write

$$\sup_{1 \le z \le x} b_2(z) = \sup_{1 \le z \le x} \frac{A(z)^2}{z} \lesssim \frac{A(x)^2}{x}\,.$$

For $0 \le z \le 1$ we can also write $b_2(z) \le A(1)^2 = b_2(1) \lesssim \frac{A(x)^2}{x}$ hence by (1.13)

$$\begin{aligned}
I_1^+(\delta; x) &\lesssim \frac{A(x)^2}{x} F([x - \delta x, x]) \underset{x\to\infty}{\sim} \frac{A(x)^2}{x} \left(\frac{1}{A((1-\delta)x)} - \frac{1}{A(x)} \right) \\
&\underset{x\to\infty}{\sim} \frac{A(x)}{x} \left(\frac{1}{(1-\delta)^\alpha} - 1 \right) \underset{\delta\to 0}{=} \frac{A(x)}{x}\, O(\delta)\,.
\end{aligned}$$

This shows that $I_1^+(\delta; x)$ is a.n., hence the SRT holds by Theorem 1.4.

Next we prove the second part of Theorem 1.5: we assume that condition (1.14) is not satisfied, and we build a probability F for which the SRT fails. Since $A \in RV(\frac{1}{2})$, we can write $A(x) = \ell(x)\sqrt{x}$ where ℓ is slowly varying. By assumption, see (1.14), there is a subsequence $x_n \to \infty$ such that $\sup_{1 \le s \le x_n} \ell(s) \gg \ell(x_n)$, hence we can find $1 \le s_n \le x_n$ for which $\ell(s_n) \gg \ell(x_n)$. We have necessarily $s_n = o(x_n)$, because $\ell(s)/\ell(x_n) \to 1$ uniformly for $s \in [\epsilon x_n, x_n]$, for any fixed $\epsilon > 0$, by the uniform convergence theorem of slowly varying functions [BGT89, Theorem 1.2.1]. Summarizing:

$$x_n \to \infty\,, \qquad s_n = o(x_n)\,, \qquad \epsilon_n := \frac{\ell(x_n)}{\ell(s_n)} \to 0\,. \tag{9.1}$$

By Lemma 10.2 below, there is a probability F on $(0,\infty)$, which satisfies (1.12), such that

$$F(\{x_n\}) \geq \frac{\epsilon_n}{A(x_n)} \qquad \text{for infinitely many } n \in \mathbb{N}\,. \tag{9.2}$$

Since $A(x) = \ell(x)\sqrt{x}$, recalling (1.13), for infinitely many $n \in \mathbb{N}$ we can write

$$I_1^+(\delta; x_n + s_n) \geq \frac{A(s_n)^2}{s_n} F(\{x_n\}) \geq \ell(s_n)^2 \frac{\epsilon_n}{A(x_n)} = \frac{\ell(s_n)}{\sqrt{x_n}} = \frac{1}{\epsilon_n} \frac{A(x_n)}{x_n} \gg \frac{A(x_n+s_n)}{x_n+s_n}\,,$$

where the last inequality holds because $\epsilon_n \to 0$ and $x_n + s_n \sim x_n$, see (9.1). This shows that $I_1^+(\delta;x)$ is not a.n., hence the SRT fails, by Theorem 1.4. ∎

9.2. **Proof of Proposition 1.7.** We claim that (1.16) is equivalent to the following relation:

$$F((x-y,x]) \underset{x\to\infty}{=} O\left(\frac{1}{A(x)}\left(\frac{y}{x}\right)^{\gamma}\right) \qquad \text{for any } y = y_x \geq 1 \text{ with } y = o(x)\,. \tag{9.3}$$

It is clear that (1.16) implies (9.3). On the other hand, if (1.16) fails, there are sequences $x_n \to \infty$, $C_n \to \infty$ and $y_n \in [1, \frac{1}{2}x_n]$ such that

$$F((x_n - y_n, x_n]) > C_n \frac{1}{A(x_n)} \left(\frac{y_n}{x_n}\right)^{\gamma}. \tag{9.4}$$

By extracting subsequences, we may assume that $\frac{y_n}{x_n} \to \rho \in [0, \frac{1}{2}]$. If $\rho = 0$, then $y_n = o(x_n)$ and (9.4) contradicts (9.3). If $\rho > 0$, then (9.4) contradicts (1.12), because it yields

$$\overline{F}(\tfrac{1}{2}x_n) \geq F((x_n - y_n, x_n]) > C_n \frac{1}{A(x_n)} \big(\rho^{\gamma} + o(1)\big) \gg \frac{1}{A(\frac{1}{2}x_n)}\,.$$

We first prove that relation (9.3) for every $\gamma < 1-2\alpha$ is a necessary condition for the SRT. We can assume that $\alpha < \frac{1}{2}$, because for $\alpha = \frac{1}{2}$ we have $\gamma < 0$ and (9.3) follows by (1.12). If we restrict the integral (1.13) to $z \in [0, y)$, where $y = y_x = o(x)$, for large x we can bound $I_1^+(\delta;x) \gtrsim b_2(y)\, F((x-y,x])$ because $b_2(z) \in RV(2\alpha - 1)$ is asymptotically decreasing. If the SRT holds, $I_1^+(\delta;x)$ is a.n. by Theorem 1.4, hence $b_2(y)\, F((x-y,x]) = o(b_1(x))$, i.e.

$$F((x-y,x]) \underset{x\to\infty}{=} o\left(\frac{1}{A(x)} \frac{b_2(x)}{b_2(y)}\right), \qquad \text{for any } y = y_x \geq 1 \text{ with } y_x = o(x)\,.$$

Since $b_2 \in RV(2\alpha - 1)$, it follows by Potter's bounds (2.2) that, for any given $\gamma < 1 - 2\alpha$, we have $\frac{b_2(x)}{b_2(y)} \lesssim_{\gamma} (\frac{y}{x})^{\gamma}$, hence (9.3) holds as claimed (even with $o(\cdot)$ instead of $O(\cdot)$).

We now turn to the sufficiency part. Let F be a probability on $[0,\infty)$ which satisfies (1.12), with $\alpha \in (0, \frac{1}{2}]$, such that relation (1.16) holds for some $\gamma > 1-2\alpha$ and $x_0, C < \infty$. We prove that the SRT holds by showing that $\tilde{I}_1^+(\delta;x)$ defined in (1.17) is a.n., by Proposition 1.11. Applying (1.16) and recalling (1.17), for fixed $\delta \in (0, \frac{1}{2})$ and large x we get

$$\tilde{I}_1^+(\delta;x) \leq \frac{C}{A(x)\,x^{\gamma}} \int_1^{\delta x} \frac{A(z)^2}{z^{2-\gamma}}\,\mathrm{d}z \underset{x\to\infty}{\sim} \frac{C'}{A(x)\,x^{\gamma}} \frac{A(\delta x)^2}{(\delta x)^{1-\gamma}} \underset{x\to\infty}{\sim} C'\,\delta^{2\alpha-1+\gamma} \frac{A(x)}{x}\,,$$

where $C' := C/(2\alpha - 1 - \gamma)$ and the first asymptotic equivalence holds by (2.3), because $z \mapsto A(z)^2/z^{2-\gamma}$ is regularly varying with index $2\alpha - (2-\gamma) > -1$ (since $\gamma > 1-2\alpha$). This shows that $\tilde{I}_1^+(\delta;x)$ is a.n. and completes the proof of Proposition 1.7. ∎

9.3. **Proof of Proposition 1.11.** By (1.17) we can write

$$\begin{aligned}\tilde{I}_1^+(\delta;x) &= \int_1^{\delta x} \bigg(\int_{\mathbb{R}} \mathbb{1}_{\{y\in[0,z)\}}\, F(x-\mathrm{d}y) \bigg) \frac{b_2(z)}{z}\,\mathrm{d}z \\ &= \int_{y\in[0,\delta x)} F(x-\mathrm{d}y) \bigg(\int_{1\vee y}^{\delta x} \frac{b_2(z)}{z}\,\mathrm{d}z \bigg) .\end{aligned} \tag{9.5}$$

We recall that b_k is defined in (1.10). Assume that $\alpha < \frac{1}{2}$. Then the function $z \mapsto b_2(z)/z$ is regularly varying with index $2\alpha - 2 < -1$, hence by (2.4), for $y \geq 0$ we can write

$$\int_{1\vee y}^{\delta x} \frac{b_2(z)}{z}\,\mathrm{d}z \leq \int_{1\vee y}^{\infty} \frac{b_2(z)}{z}\,\mathrm{d}z \lesssim b_2(1\vee y) \lesssim b_2(y) ,$$

because for $0 \leq y < 1$ we have $b_2(y) \geq A(0)^2 > 0$, see §2.2. Recalling (1.13), we have shown that $\tilde{I}_1^+(\delta;x) \lesssim I_1^+(\delta;x)$ when $\alpha < \frac{1}{2}$. Then, if $I_1^+(\delta;x)$ is a.n., also $\tilde{I}_1^+(\delta;x)$ is a.n..

We now work for $\alpha \leq \frac{1}{2}$. Let us restrict the outer integral in (9.5) to $y \in [0, \frac{\delta}{2}x)$, and the inner integral to $z \in [1\vee y, 2 \vee 2y)$. For $y \geq 1$ we have

$$\int_{1\vee y}^{2\vee 2y} \frac{b_2(z)}{z}\,\mathrm{d}z = \int_{y}^{2y} \frac{b_2(z)}{z}\,\mathrm{d}z \geq \frac{b_2(y)}{2y}(2y-y) = \frac{1}{2} b_2(y) ,$$

while for $0 \leq y < 1$ we can write $\int_{1\vee y}^{2\vee 2y} \frac{b_2(z)}{z}\,\mathrm{d}z = \int_1^2 \frac{b_2(z)}{z}\,\mathrm{d}z = C \gtrsim b_2(y)$. Overall, it follows from (9.5) that $\tilde{I}_1^+(\delta;x) \gtrsim I_1^+(\frac{\delta}{2};x)$. This completes the proof. ∎

9.4. **Proof of Proposition 1.17.** Assume that both $\tilde{I}_1(\delta;x)$ and $\tilde{I}_1^*(\delta;x)$ are a.n., see (1.25) and (1.27). We first show that, for any $\eta \in (0,1)$,

$$\forall z \in \mathbb{R},\ \forall \ell \in \mathbb{N}: \qquad \int_{|y|\leq \eta|z|} F(-z+\mathrm{d}y)\, \tilde{b}_{\ell+1}(z,y) \lesssim_\eta b_\ell(z) . \tag{9.6}$$

Since $\tilde{b}_{\ell+1}(z,y) \leq A(z)^{\ell-1}\tilde{b}_2(z,y)$ and $b_\ell(z) = A(z)^{\ell-1} b_1(z)$, see (1.22) and (1.10), it is enough to prove (9.6) for $\ell = 1$. Let us fix $0 < \delta_0 < \eta$. For $|y| > \delta_0|z|$ we can bound $\tilde{b}_2(z,y) \lesssim \tilde{b}_2(z,\delta_0 z) \lesssim_{\delta_0} b_2(z)$ and $\int_{\delta_0|z|<|y|\leq\eta|z|} F(-z+\mathrm{d}y) \lesssim_{\delta_0,\eta} 1/A(z)$. It remains to prove (9.6) for $\ell = 1$ and with η replaced by an arbitrary $\delta_0 > 0$. The left hand side of (9.6) equals $\tilde{I}_1(\eta;z)$ for $z \geq 0$ and $\tilde{I}_1^*(\eta;-z)$ for $z \leq 0$ (recall (1.23)), which are a.n. by assumption, hence we can fix $\eta = \delta_0 > 0$ small enough so that the inequality (9.6) holds for $|z| > x_0$, for a suitable $x_0 \in (0,\infty)$. Finally, for $|z| \leq x_0$ both sides of (9.6) are uniformly bounded away from 0 and ∞, hence the inequality (9.6) still holds.

Observe that, for $|z| \leq \eta|w|$, we can bound $b_\ell(z) \lesssim_\eta \tilde{b}_\ell(\frac{1}{\eta}z, z) \leq \tilde{b}_\ell(w,z)$, so (9.6) yields

$$\forall z,w \in \mathbb{R} \text{ with } |z| \leq \eta|w|,\ \forall \ell \in \mathbb{N}: \qquad \int_{|y|\leq\eta|z|} F(-z+\mathrm{d}y)\, \tilde{b}_{\ell+1}(z,y) \lesssim_\eta \tilde{b}_\ell(w,z) . \tag{9.7}$$

If we plug this inequality into (1.24), we see that $\tilde{I}_2(\delta,\eta;x) \lesssim_\eta \tilde{I}_1(\delta;x)$ and, similarly, $\tilde{I}_k(\delta,\eta;x) \lesssim_\eta \tilde{I}_{k-1}(\delta,\eta;x)$ for any $k \geq 3$. Since $\tilde{I}_1(\delta;x)$ is a.n. by assumption, it follows that $\tilde{I}_k(\delta,\eta;x)$ is a.n. for any $k \geq 2$, hence the SRT holds by Theorem 1.12.

Finally, if relation (1.16) holds both for F and for F^*, the same arguments as in the proof of Proposition 1.7, see §9.2, show that both $\tilde{I}_1(\delta;x)$ and $\tilde{I}_1^*(\delta;x)$ are a.n.. ∎

9.5. **Proof of Theorem 1.18.** Since Stone's local limit theorem applies equally to Lévy processes, see [BD97, Proposition 2], an argument similar to the random walk case (see Subsection 4.1) shows that the SRT (1.30) holds if and only if $\widehat{T}(\delta;x)$ is a.n., where

$$\widehat{T}(\delta;x) := \int_0^{\delta A(x)} \mathrm{P}(X_t \in (x-h,x])\,\mathrm{d}t\,. \tag{9.8}$$

Let $J_s := X_s - X_{s-}$ be the jump of X at time $s>0$. If we write

$$X_t = X_t^{(1)} + X_t^{(2)}\,, \qquad \text{where} \qquad X_t^{(1)} := \sum_{s\le t} J_s\,\mathbb{1}_{\{|J_s|\ge 1\}}\,,$$

then $X^{(1)}$ and $X^{(2)}$ are independent Lévy processes.

- The process $X^{(1)}$ is compound Poisson: we can write $X_t^{(1)} = S_{N_{\lambda t}}$, where $N=(N_t)_{t\ge0}$ is a standard Poisson process, $S=(S_n)_{n\in\mathbb{N}_0}$ is a random walk with step distribution $\mathrm{P}(S_1\in \mathrm{d}x) = F(\mathrm{d}x)$ given in (1.29), and $\lambda = \Pi(\mathbb{R}\setminus(-1,1)) \in (0,\infty)$.
- The process $X_t^{(2)}$ can be written as $X_t^{(2)} = \sigma B_t + \mu t + M_t$, where M is the martingale formed from the compensated sum of jumps with modulus less than 1.

To complete the proof, we show that the SRT holds for the random walk S if and only if it holds for $X^{(1)}$ (step 1) if and only if it holds for X (step 2).

Step 1. Since $X_t^{(1)} = S_{N_{\lambda t}}$, we have

$$\mathrm{P}(X_t^{(1)} \in (x-h,x]) = \sum_{n\in\mathbb{N}_0} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\mathrm{P}(S_n\in(x-h,x])\,.$$

Note that $\int_0^z e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,\mathrm{d}t = \frac{1}{\lambda}\mathrm{P}(Z_{n,\lambda}\le z)$, where $Z_{n,\lambda}$ denotes a random variable with a Gamma(n,λ) distribution. Then the quantity $\widehat{T}(\delta;x) = \widehat{T}_{X^{(1)}}(\delta;x)$ for $X^{(1)}$ equals

$$\widehat{T}_{X^{(1)}}(\delta;x) = \frac{1}{\lambda}\sum_{n\in\mathbb{N}_0} \mathrm{P}(Z_{n,\lambda}\le\delta A(x))\,\mathrm{P}(S_n\in(x-h,x])\,. \tag{9.9}$$

For $n\le\lambda\delta A(x)$ we have $\mathrm{P}(Z_{n,\lambda}\le \delta A(x)) \ge \mathrm{P}(Z_{n,\lambda}\le\frac{n}{\lambda}) \to \frac12$ as $n\to\infty$, by the central limit theorem (recall that $Z_{n,\lambda}\sim\frac{1}{\lambda}(Y_1+\ldots+Y_n)$, where Y_i are i.i.d. Exp(1) random variables). Denoting by $T_S(\delta;x)$ the quantity in (4.1) for the random walk S, and restricting the sum in (9.9) to $n\le\lambda\delta A(x)$, we get

$$\widehat{T}_{X^{(1)}}(\delta;x) \gtrsim T_S(\lambda\delta;x)\,.$$

To prove a reverse inequality, we observe that for all $z\le\frac12\frac{n}{\lambda}$ we can write, for $\epsilon>0$,

$$\mathrm{P}(Z_{n,\lambda}\le z) \le e^{\epsilon\lambda z}\,\mathrm{E}[e^{-\epsilon\lambda Z_{n,\lambda}}] = \frac{e^{\epsilon\lambda z}}{(1+\epsilon)^n} \le \left(\frac{e^{\frac12\epsilon}}{1+\epsilon}\right)^n \le e^{-cn}\,,$$

where the last inequality holds with $c=c_\epsilon>0$, provided we fix $\epsilon>0$ small. Then, splitting the sum in (9.9) according to $n\le 2\lambda\delta A(x)$ and $n>2\lambda\delta A(x)$, we get

$$\widehat{T}_{X^{(1)}}(\delta;x) \le \frac{1}{\lambda}\sum_{n\le 2\lambda\delta A(x)} \mathrm{P}(S_n\in(x-h,x]) + \sum_{n>2\lambda\delta A(x)} e^{-cn} \lesssim T_S(2\lambda\delta;x) + e^{-2c\lambda\delta A(x)}\,.$$

These inequalities show that $\widehat{T}_{X^{(1)}}(\delta;x)$ is a.n. if and only if $T_S(\delta;x)$ is a.n., that is, the SRT holds for $X^{(1)}$ if and only if it holds for S.

Step 2. Assume that $X = X^{(1)} + X^{(2)}$ and the SRT holds for $X^{(1)}$, that is $\widehat{T}_{X^{(1)}}(\delta; x)$ is a.n.. Then, given $\varepsilon > 0$, there are δ_0, x_0 such that, for all $0 < \delta < \delta_0$,

$$\forall y > x_0 : \qquad \widehat{T}_{X^{(1)}}(\delta; y) = \int_0^{\delta A(y)} \mathrm{P}(X_t^{(1)} \in (y-h, y])\, \mathrm{d}t \le \varepsilon \, \frac{A(y)}{y}. \tag{9.10}$$

Let us now write

$$\begin{aligned} &\int_0^{\delta A(x)} \mathrm{P}(X_t \in (x-h, x],\, X_t^{(2)} \le x/2)\, \mathrm{d}t \\ &\qquad = \int_{-\infty}^{x/2} \mathrm{P}(X_t^{(2)} \in \mathrm{d}z) \int_0^{\delta A(x)} \mathrm{P}(X_t^{(1)} \in (x-z-h, x-z])\, \mathrm{d}t \,. \end{aligned}$$

For $z \le x/2$ we can write $A(x) \le cA(x/2)$, for any $c > 2^\alpha$ and for large x. Then the inner integral is bounded by $\widehat{T}_{X^{(1)}}(c\delta; x-z) \le \varepsilon \frac{A(x-z)}{x-z} \lesssim \varepsilon \frac{A(x)}{x}$, by (9.10). This shows that

$$\int_0^{\delta A(x)} \mathrm{P}(X_t \in (x-h, x],\, X_t^{(2)} \le x/2)\, \mathrm{d}t \,\lesssim\, \varepsilon \, \frac{A(x)}{x} \,. \tag{9.11}$$

Note that $X^{(2)}$ has finite exponential moments, because its Lévy measure $\Pi(\,\cdot \cap (-1,1))$ is compactly supported, hence $\mathrm{E}[e^{|X_t^{(2)}|}] \le \mathrm{E}[e^{X_t^{(2)}}] + \mathrm{E}[e^{-X_t^{(2)}}] \le e^{ct}$ for a suitable $c \in (0, \infty)$. This yields the exponential bound $\mathrm{P}(|X_t^{(2)}| > a) \le e^{-a}\, e^{ct}$, for all $a \ge 0$, hence

$$\begin{aligned} &\int_0^{\delta A(x)} \mathrm{P}(X_t \in (x-h, x],\, X_t^{(2)} > x/2)\, \mathrm{d}t \\ &\qquad \le \int_0^{\delta A(x)} \mathrm{P}(X_t^{(2)} > x/2)\, \mathrm{d}t \lesssim e^{-x/2}\, e^{c\delta A(x)} \underset{x\to\infty}{=} o\left(\frac{A(x)}{x}\right) . \end{aligned}$$

Together with (9.11), this shows that $\widehat{T}_X(\delta; x)$ is a.n., that is the SRT holds for X.

If the SRT holds for X, to show that it holds for $X^{(1)}$ we can repeat the previous arguments switching X and $X^{(1)}$ (no special feature of $X^{(1)}$ was used in this step). ∎

10. Counterexamples

In this section we prove Propositions 1.6 and 1.16. We first develop some useful tools.

10.1. **Preliminary tools.**

Let us describe a practical way to build counter-examples.

Remark 10.1. *Let us fix $A \in RV(\alpha)$. Let F_1 be a probability on $(0, \infty)$ which satisfies*

$$\overline{F_1}(x) \underset{x\to\infty}{\sim} \frac{2}{A(x)} , \qquad F_1((x-h, x]) \underset{x\to\infty}{=} O\bigg(\frac{1}{xA(x)}\bigg) , \quad \forall h > 0 \,. \tag{10.1}$$

(For instance, fix $n_0 \in \mathbb{N}$ such that $c_1 := \sum_{n > n_0} \frac{2\,\alpha}{n\, A(n)} < 1$ and define $F_1(\{n_0\}) := 1 - c_1$, $F_1(\{n\}) := \frac{2\,\alpha}{n\, A(n)}$ for $n \in \mathbb{N}$ with $n > n_0$.) Let F_2 be a probability on $(0, \infty)$ such that

$$\overline{F_2}(x) \underset{x\to\infty}{=} o\bigg(\frac{1}{A(x)}\bigg) . \tag{10.2}$$

If we define $F := \frac{1}{2}(F_1 + F_2)$, we obtain a new probability on $(0, \infty)$ which satisfies

$$\overline{F}(x) \underset{x\to\infty}{\sim} \frac{1}{A(x)} , \qquad F(x+I) \ge \frac{1}{2} F_2(x+I) \,. \tag{10.3}$$

Next we state a useful result. To provide motivation, note that if F satisfies (1.12), then necessarily $F(x+I) = o(\frac{1}{A(x)})$ as $x \to \infty$ (because $\overline{F}(x-h) \sim \overline{F}(x) \sim \frac{1}{A(x)}$). Interestingly, this bound can be approached as close as one wishes, in the following sense.

Lemma 10.2. *Fix two arbitrary positive positive sequences $x_n \to \infty$ and $\epsilon_n \to 0$. For any $A \in RV(\alpha)$, with $\alpha \in (0,1)$, there is a probability F on $(0,\infty)$ satisfying* (1.12) *such that*

$$F(\{x_n\}) \geq \frac{\epsilon_n}{A(x_n)} \qquad \textit{for infinitely many } n \in \mathbb{N} \,. \tag{10.4}$$

Proof. Let us fix $A \in RV(\alpha)$. By Remark 10.1, it is enough to build a probability F_2 on $(0,\infty)$, supported on the sequence $\{x_n\}_{n\in\mathbb{N}}$, which satisfies (10.2) and

$$F_2(\{x_n\}) \geq 2\frac{\epsilon_n}{A(x_n)} \quad \text{for infinitely many } n \in \mathbb{N} \,. \tag{10.5}$$

Then, if we define $F := \frac{1}{2}(F_1 + F_2)$, the proof is completed (recall (10.3)).

By assumption $x_n \to \infty$ and $\epsilon_n \to 0$, hence we can fix a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\frac{\epsilon_{n_{k+1}}}{A(x_{n_{k+1}})} \leq \frac{1}{2}\,\frac{\epsilon_{n_k}}{A(x_{n_k})} \,, \qquad \forall k \in \mathbb{N} \,. \tag{10.6}$$

This ensures that $\sum_{k\in\mathbb{N}} \frac{\epsilon_{n_k}}{A(x_{n_k})} < \infty$ (the series converges geometrically) and we fix $k_0 \in \mathbb{N}$ so that $\sum_{k \geq k_0} \frac{\epsilon_{n_k}}{A(x_{n_k})} \leq \frac{1}{2}$. We now define F_2, supported on the set $\{x_{n_k} : \; k \geq k_0\}$, by

$$F_2(\{x_{n_k}\}) := c_2\, \frac{\epsilon_{n_k}}{A(x_{n_k})} \quad \text{for } k \geq k_0 \,, \qquad \text{where} \quad c_2 := \Bigg(\sum_{k \geq k_0} \frac{\epsilon_{n_k}}{A(x_{n_k})} \Bigg)^{-1} \geq 2 \,.$$

In this way, (10.5) is satisfied. It remains to check that (10.2) holds. Given $x \in (0,\infty)$, if we set $\bar{k} := \min\{k \geq k_0 : \; x_{n_k} > x\}$, we can write

$$F_2((x,\infty)) = \sum_{k \geq \bar{k}} c_2\, \frac{\epsilon_{n_k}}{A(x_{n_k})} \leq c_2\, \frac{\epsilon_{n_{\bar{k}}}}{A(x_{n_{\bar{k}}})} \sum_{k \geq \bar{k}} \frac{1}{2^{k-\bar{k}(x)}} \leq 2\, c_2\, \frac{\epsilon_{n_{\bar{k}}}}{A(x)} \,,$$

where we used (10.6), and the last inequality holds because $x_{n_{\bar{k}}} > x$, by definition of $\bar{k}$. Since $\epsilon_n \to 0$ by assumption, and $\bar{k} \to \infty$ as $x \to \infty$, the proof is completed. ∎

10.2. **Proof of Proposition 1.6.** Let us fix $A \in RV(\alpha)$ with $\alpha \in (0, \frac{1}{2})$. By Remark 10.1, it is enough to build a probability F_2 on $(0,\infty)$ which satisfies (10.2) and moreover

$$F_2(x+I) = O\bigg(\frac{\zeta(x)}{xA(x)} \bigg) \,, \qquad I_1^+(\delta; x; F_2) \text{ is not a.n.} \,, \tag{10.7}$$

where $I_1^+(\delta;x;F_2)$ denotes the quantity $I_1^+(\delta;x)$ in (1.13) with F replaced by F_2. Once this is done, we can set $F := \frac{1}{2}(F_1+F_2)$ and the proof is completed (recall (10.3)).

By assumption $\zeta(\cdot)$ is non-decreasing with $\lim_{x\to\infty} \zeta(x) = \infty$. Let us define $x_n := 2^n$, and fix $n_0 \in \mathbb{N}$ large enough so that $\zeta(x_{n_0-1}) \geq 1$. Let us define

$$z_n := \frac{1}{2}\frac{x_n}{\zeta(x_{n-1})^{1+\theta}} \,, \qquad \text{where } \theta > 0 \text{ will be fixed later} \,. \tag{10.8}$$

Note that $z_n \leq \frac{1}{2}x_n$ for $n \geq n_0$ (because $\zeta(x_{n-1}) \geq 1$), hence the intervals $(x_n - z_n, x_n]$ are disjoint. We may also assume that $z_n \geq 1$, possibly enlarging n_0 (if we decrease $\zeta(\cdot)$ we get a stronger statement, so we can replace $\zeta(x)$ by $\min\{\zeta(x), \log x\}$, so that $z_n \to \infty$).

We define a probability F_2 supported on the set $\bigcup_{n\ge n_0}(x_n - z_n, x_n]$, with a constant density on each interval, as follows:

$$F_2(x_n - \mathrm{d}s) := c\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)}\,\mathbb{1}_{[0,z_n]}(s)\,\mathrm{d}s\,, \qquad \forall n \ge n_0\,, \tag{10.9}$$

for a suitable $c \in (0,\infty)$. We are going to show that F_2 is a finite measure, so we can fix the constant c to make it a probability. Note that

$$F_2((x_n - z_n, x_n]) = c\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)}\,z_n = \frac{c}{2\,A(x_n)\,\zeta(x_{n-1})^\theta}\,.$$

Since $A(x_n) = A(2x_{n-1}) \sim 2^\alpha A(x_{n-1})$ as $n \to \infty$, we may assume that $A(x_n) \ge 2^{\alpha/2}A(x_{n-1})$ for all $n \ge n_0 + 1$ (possibly enlarging n_0). Since $\zeta(x_{n-1})^\theta \ge \zeta(x_{n-2})^\theta$, we obtain

$$F_2((x_n - z_n, x_n]) \le 2^{-\alpha/2} F_2((x_{n-1} - z_{n-1}, x_{n-1}])\,, \qquad \forall n \ge n_0 + 1\,.$$

It follows that, for every $n \ge n_0$,

$$\sum_{m\ge n} F_2((x_m - z_m, x_m]) \le F_2((x_n - z_n, x_n]) \sum_{m\ge n} (2^{-\alpha/2})^{m-n} = C\,F_2((x_n - z_n, x_n])\,,$$

where $C := (1 - 2^{-\alpha/2})^{-1} < \infty$. This shows that F_2 is indeed a finite measure.

For all large $x \in (0,\infty)$, we have $x_{n-1} < x \le x_n$ for a unique $n \ge n_0$, hence

$$\overline{F_2}(x) \le \sum_{m\ge n} F_2((x_m - z_m, x_m]) \le C\,F_2((x_n - z_n, x_n]) = \frac{c\,C}{2\,A(x_n)\,\zeta(x_{n-1})^\theta} \underset{x\to\infty}{=} o\bigg(\frac{1}{A(x)}\bigg)\,,$$

so that (10.2) holds. Similarly, for $x_{n-1} < x \le x_n$ we can write, by (10.9),

$$F_2(x + I) = F_2((x - h, x]) \le c\,h\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)} \le c\,h\,\frac{\zeta(x)}{x\,A(x)}\,,$$

because both $\zeta(\cdot)$ and $A(\cdot)$ are non-decreasing, hence the first relation in (10.7) holds. Finally, for fixed $\delta \in (0, \frac{1}{2})$, since $z_n \le \delta x_n$ for n large enough, we have by (2.3)

$$I_1^+(\delta; x_n; F_2) = c\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)} \int_{0\le z\le z_n} \frac{A(z)^2}{z \vee 1}\,\mathrm{d}z \underset{n\to\infty}{\sim} c\,\frac{\zeta(x_{n-1})}{x_n\,A(x_n)}\,A(z_n)^2\,.$$

Recalling (10.8), we can apply Potter's bounds (2.2), since $z_n \ge 1$, to get, for any $\epsilon > 0$,

$$I_1^+(\delta; x_n; F_2) \gtrsim_\epsilon \frac{\zeta(x_{n-1})}{x_n\,A(x_n)}\,\frac{A(x_n)^2}{\zeta(x_{n-1})^{2(1+\theta)(\alpha+\epsilon)}} = \zeta(x_{n-1})^{1-2(1+\theta)(\alpha+\epsilon)}\,\frac{A(x_n)}{x_n} \gg \frac{A(x_n)}{x_n}\,,$$

where the last inequality holds provided we choose $\theta > 0$ and $\epsilon > 0$ small enough, depending only on α, so that $1-2(1+\theta)(\alpha+\epsilon) > 0$ (we recall that $\alpha < \frac{1}{2}$). This shows that $I_1^+(\delta; x_n; F_2)$ is not a.n. and completes the proof. ∎

10.3. **Proof of Proposition 1.16.** We fix $\alpha \in (0, \frac{1}{3})$ and choose for simplicity $A(x) := x^\alpha$. We are going to build a probability F on $\mathbb{R}$ which satisfies (1.2) with $p = q = 1$, such that $\tilde{I}_1(\delta; x)$ is a.n. but $\tilde{I}_2(\delta, \eta; x)$ is not a.n., for any $\eta \in (0, 1)$. It suffices to show that $I_1(\delta; x)$ is a.n. but $I_2(\delta, \eta; x)$ is not a.n., thanks to (A.4) and (A.5).

In analogy with Remark 10.1, we fix a probability F_1, *this time on the whole real line* $\mathbb{R}$, which satisfies (1.2) with $p = q = 3$ and such that $F_1((x - h, x]) = O(\frac{1}{|x|A(x)})$ as $x \to \pm\infty$. Then we define two probabilities F_2, F_3 on $(0,\infty)$ which both satisfy (10.2), and we set

$$F := \frac{1}{3}(F_1 + F_2 + F_3^*)\,, \tag{10.10}$$

where $F_3^*(A) := F_3(-A)$ is the reflection of F_3 (so that it is a probability on $(-\infty,0)$). Clearly, (1.2) holds for F with $p=q=1$. It remains to build F_2 and F_3.

We are going to define F_2 so that

$$I_1(\delta;x;F_2) \text{ is a.n.}\,, \tag{10.11}$$

(where $I_1(\delta;x;F_2)$ denotes the quantity in (1.18) with F replaced by F_2). This implies that $I_1(\delta;x) = I_1(\delta;x;F)$ is a.n., because $I_1(\delta;x;F_1)$ is clearly a.n., while F_3^* is supported on $(-\infty,0)$ and gives no contribution.

We fix a parameter $p \in (1, \frac{1}{3\alpha})$. We set $E_{n,k} := [2^n + 2^k, 2^n + 2^k + \frac{2^k}{2k^p})$ for $n \in \mathbb{N}$ with $n \geq 2$ and for $1 \leq k \leq n-1$. Note that $E_{n,k} \subseteq [2^n+2^k, 2^n+2^{k+1})$ are disjoint intervals, and moreover $\bigcup_{k=1}^{n-1} E_{n,k} \subseteq [2^n, 2^{n+1})$. We define F_2 with a density, which is constant in each interval $E_{n,k}$ (for $n \geq 2$ and $1 \leq k \leq n-1$) and zero otherwise, given by

$$F_2(2^n + 2^k + \mathrm{d}w) := \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \, \frac{1}{(2^k)^{2\alpha}} \, \mathbb{1}_{[0,\frac{2^k}{2k^p})}(w)\,\mathrm{d}w\,, \tag{10.12}$$

where $c \in (0,\infty)$ is a suitable normalizing constant and we set for short

$$\ell(n) := \log(1+n)\,. \tag{10.13}$$

Note that

$$F_2(E_{n,k}) = \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \, \frac{1}{(2^k)^{2\alpha}} \, \frac{2^k}{2k^p} = \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \, \frac{(2^k)^{1-2\alpha}}{2k^p}\,, \tag{10.14}$$

hence

$$\begin{aligned} F_2([2^n,2^{n+1})) &= \sum_{k=1}^{n-1} F_2(E_{n,k}) \leq \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \sum_{k=1}^{n-1} (2^k)^{1-2\alpha} \lesssim \frac{c\,(2^n)^{1-2\alpha}}{\ell(n)\,(2^n)^{1-\alpha}} \\ &= \frac{c}{\ell(n)\,(2^n)^{\alpha}} \underset{n\to\infty}{=} o\bigg(\frac{1}{(2^n)^\alpha}\bigg)\,. \end{aligned}$$

Note that $F_2([2^n,2^{n+1}))$ decreases exponentially fast in n, hence for $x \in [2^n,2^{n+1})$ we have $\overline{F_2}(x) \leq \overline{F_2}(2^n) \lesssim F_2([2^n,2^{n+1})) = o(1/A(x))$, which shows that (10.2) is fulfilled. It remains to check (10.11). We do this by showing that, for any $\delta < \frac{1}{4}$,

$$I_1(\delta;x;F_2) \underset{x\to\infty}{=} o\big(b_1(x)\big) = o\bigg(\frac{1}{x^{1-\alpha}}\bigg)\,. \tag{10.15}$$

This is elementary but slightly technical, and it is shown below.

Finally, we define F_3, We introduce the disjoint intervals $G_k := [2^k, 2^k + \frac{2^k}{k^p})$ for $k \geq 2$. We let F_3 have a density, constant on every G_k (for $k \geq 2$) and zero otherwise, given by

$$F_3(2^k + \mathrm{d}z) := \frac{c'}{\ell(k)} \, \frac{k^p}{(2^k)^{1+\alpha}} \, \mathbb{1}_{[0,\frac{2^k}{k^p})}(z)\,\mathrm{d}z\,, \tag{10.16}$$

where $c' \in (0,\infty)$ is a normalizing constant. Then

$$F_3([2^k,2^{k+1})) = F_3(G_k) = \frac{c'}{\ell(k)} \, \frac{k^p}{(2^k)^{1+\alpha}} \, \frac{2^k}{k^p} = \frac{c'}{\ell(k)} \, \frac{1}{(2^k)^\alpha} = o\bigg(\frac{1}{(2^k)^\alpha}\bigg)\,.$$

Then for $x \in [2^k,2^{k+1})$ we have $\overline{F_3}(x) \leq \overline{F_3}(2^k) \lesssim \overline{F_3}([2^k,2^{k+1})) = o(1/A(x))$ as $x \to \infty$, hence (10.2) holds. Given $\eta \in (0,1)$, fix $k_0 = k_0(\eta)$ large enough so that $\frac{1}{2k^p} < \eta$ for $k \geq k_0$.

Then, recalling (1.19) and (10.10), we can write

$$I_2(\delta,\eta;2^n;F) \ge \int_{0\le y\le \delta 2^n} F(2^n+\mathrm{d}y) \int_{|z|\le \eta y} F(-y+\mathrm{d}z)\, b_3(z)$$

$$\ge \frac{1}{9}\sum_{k=k_0}^{\lfloor \log_2(\delta 2^n)\rfloor} \int_{0\le w\le \frac{2^k}{2k^p}} F_2(2^n+2^k+\mathrm{d}w) \int_{0\le z\le \frac{2^k}{2k^p}} F_3\big(2^k+w+\mathrm{d}z\big)\,(1\vee z)^{3\alpha-1}\,.$$

Note that $(1\vee z)^{3\alpha-1} \ge (\frac{2^k}{2k^p})^{3\alpha-1}$ (we recall that $\alpha<\frac{1}{3}$) and by (10.16)

$$\forall 0\le w\le \tfrac{2^k}{2k^p}: \qquad F_3\big(2^k+w+[0,\tfrac{2^k}{2k^p})\big) = \frac{c'}{\ell(k)}\,\frac{k^p}{(2^k)^{1+\alpha}}\,\frac{2^k}{2k^p} \gtrsim \frac{1}{\ell(k)}\,\frac{1}{(2^k)^\alpha}\,.$$

Since $\log_2(\delta 2^n) = n+\log_2\delta$, recalling (10.14), we can write for large n

$$I_2(\delta,\eta;2^n;F) \gtrsim \sum_{k=k_0}^{n/2} F_2(E_{n,k})\,\frac{1}{\ell(k)}\,\frac{1}{(2^k)^\alpha}\left(\frac{2^k}{2k^p}\right)^{3\alpha-1} \gtrsim \frac{1}{\ell(n)\,(2^n)^{1-\alpha}} \sum_{k=k_0}^{n/2} \frac{1}{\ell(k)\,k^{3\alpha p}}\,.$$

Since we have fixed $p<\frac{1}{3\alpha}$, applying (2.3) and recalling (10.13) we finally obtain

$$I_2(\delta,\eta;2^n;F) \gtrsim \frac{n^{1-3\alpha p}}{\ell(n)^2}\,\frac{1}{(2^n)^{1-\alpha}} \gg \frac{1}{(2^n)^{1-\alpha}} = b_1(2^n)\,.$$

This shows that $I_2(\delta,\eta;x;F)$ is not a.n.. ∎

Proof of (10.15). We recall that F_2 is supported on the intervals $E_{n,k} := [2^n+2^k, 2^n+2^k+\frac{2^k}{2k^p})$ with $n\ge 2$ and $1\le k\le n-1$. Let us set $E_n := \bigcup_{k=1}^{n-1} E_{n,k} \subseteq [2^n,2^{n+1})$.

For large $x\ge 0$, we define $n\ge 2$ such that $2^n\le x<2^{n+1}$. For $\delta\in(0,\frac{1}{4})$ and large x, the interval $(x-\delta x, x+\delta x)$ can intersect at most E_n and E_{n+1} (because the rightmost point in E_{n-1} is $2^{n-1}+2^{n-2}+\frac{2^{n-2}}{2(n-2)^p} \sim \frac{3}{4}\,2^n$ as $n\to\infty$). Consequently we can write

$$I_1(\delta;x;F_2) = \int_{|y|\le\delta x} F_2(x+\mathrm{d}y)\, b_2(y) \le \int_{z\in E_n\cup E_{n+1}} F_2(\mathrm{d}z)\, b_2(z-x)\,. \tag{10.17}$$

For $z\in E_{n+1}$ we have $z\in E_{n+1,k}$ for some $1\le k\le n$, in which case $z\ge 2^{n+1}+2^k$. Since $x<2^{n+1}$, we have $|z-x| = z-x > 2^k$ which yields $b_2(z-x)\le b_2(2^k) = (2^k)^{2\alpha-1}$. Recalling (10.14), we see that the contribution of E_{n+1} to (10.17) is at most

$$\sum_{k=1}^{n} \frac{c}{\ell(n)\,(2^{n+1})^{1-\alpha}}\,\frac{(2^k)^{1-2\alpha}}{2k^p}\,(2^k)^{2\alpha-1} \lesssim \frac{1}{\ell(n)\,(2^{n+1})^{1-\alpha}} \sum_{k=1}^{\infty}\frac{1}{k^p} = o\bigg(\frac{1}{(2^{n+1})^{1-\alpha}}\bigg)\,,$$

since we chose $p>1$. This is $o\big(\frac{1}{x^{1-\alpha}}\big)$, so it is negligible for (10.15).

Then we look at the contribution of E_n to (10.17). Assume first that $2^n+2\le x<2^{n+1}$. Then we can write $2^n+2^{\tilde k}\le x<2^n+2^{\tilde k+1}$ for a unique $\tilde k\in\{1,\ldots,n-1\}$. For $z\in E_n$ we have $z\in E_{n,k}$ for some $1\le k\le n-1$. We distinguish three cases.

- If $k\le\tilde k-1$ (in particular, $\tilde k\ge 2$), then

$$|z-x| = x-z \gtrsim (2^n+2^{\tilde k}) - (2^n+2^{\tilde k-1}+\tfrac{2^{\tilde k-1}}{2(\tilde k-1)^p}) \gtrsim 2^{\tilde k}\,,$$

hence $b_2(z-x) \lesssim b_2(2^{\tilde{k}}) = (2^{\tilde{k}})^{2\alpha-1}$. By (10.14), the contribution to (10.17) is at most

$$\sum_{k=1}^{\tilde{k}-1} F_2(E_{n,k})\,(2^{\tilde{k}})^{2\alpha-1} \le \frac{c}{\ell(n)\,(2^n)^{1-\alpha}}\,(2^{\tilde{k}})^{2\alpha-1} \sum_{k=1}^{\tilde{k}-1} (2^k)^{1-2\alpha} \lesssim \frac{c}{\ell(n)\,(2^n)^{1-\alpha}}\,,$$

which is $o\big(\frac{1}{x^{1-\alpha}}\big)$, so it is negligible for (10.15).

- If $k \ge \tilde{k}+2$, then $|z-x| = z-x \gtrsim (2^n+2^k) - (2^n+2^{\tilde{k}+1}) \ge 2^k - 2^{k-1} \gtrsim 2^k$, hence $b_2(z-x) \lesssim b_2(2^k) = (2^k)^{2\alpha-1}$ and we get

$$\sum_{k=\tilde{k}+1}^{n-1} F_2(E_{n,k})\,(2^k)^{2\alpha-1} \le \frac{c}{\ell(n)\,(2^n)^{1-\alpha}} \sum_{k=\tilde{k}+1}^{\infty} \frac{1}{2k^p} \lesssim \frac{c}{\ell(n)\,(2^n)^{1-\alpha}}\,,$$

because $p > 1$, hence this contribution is also negligible for (10.15).

- If $k \in \{\tilde{k}, \tilde{k}+1\}$, then $|z-x| \le 2^{\tilde{k}+2} - 2^{\tilde{k}} = 3\cdot 2^{\tilde{k}}$. By (10.12), since the density of F_2 is larger in $E_{n,\tilde{k}}$ than in $E_{n,\tilde{k}+1}$, we see the contribution to (10.17) is at most

$$\frac{c}{\ell(n)\,(2^n)^{1-\alpha}}\,\frac{1}{(2^{\tilde{k}})^{2\alpha}} \int_{|w|\le 3\cdot 2^{\tilde{k}}} (|w|\vee 1)^{2\alpha-1}\,\mathrm{d}w \lesssim \frac{1}{\ell(n)\,(2^n)^{1-\alpha}}\,,$$

which is negligible for (10.15).

Finally, the regime $2^n \le x < 2^n+2$ is treated similarly. For $z \in E_{n,k}$, we distinguish the cases $k \ge 2$ and $k=1$. If we set $\tilde{k} := 0$, the estimates in the two cases $k \ge \tilde{k}+2$ and $k \in \{\tilde{k}, \tilde{k}+1\}$ treated above apply with no change. ∎

Appendix A. Some technical results

Let us fix a probability F on $\mathbb{R}$ which satisfies (1.12) with $\alpha \in (0, \frac{1}{2}]$ and with $p, q > 0$. The next Lemmas show some relations between the quantities I_k and $\tilde{I}_k$ defined in (1.18), (1.19) and in (1.23), (1.24), respectively. We recall that $\kappa_\alpha \in \mathbb{N}$ is defined in (1.21).

Lemma A.1. *Fix $\eta \in (0,1)$. If $\tilde{I}_k(\delta,\eta;x)$ is a.n. for $k=\kappa_\alpha$, then it is a.n. for all $k \in \mathbb{N}$.*

Lemma A.2. *Assume $\frac{1}{\alpha} \notin \mathbb{N}$ and fix $\eta \in (0,1)$. If $I_k(\delta,\eta;x)$ is a.n. for $k=\kappa_\alpha$, then it is a.n. for all $k \in \mathbb{N}$.*

Lemma A.3. *With no restriction on α, if $\tilde{I}_{\kappa_\alpha}(\delta,\eta;x)$ is a.n., then also $I_{\kappa_\alpha}(\delta,\eta;x)$ is a.n.. The reverse implication holds if $\frac{1}{\alpha} \notin \mathbb{N}$ (but not necessarily if $\frac{1}{\alpha} \in \mathbb{N}$).*

Proof of Lemma A.2. Fix $k \in \mathbb{N}$ with $k \ge 2$ and $\eta \in (0,1)$. We are going to prove the following relations:

$$\text{if } k < \tfrac{1}{\alpha} - 1: \qquad I_{k-1}(\delta,\eta;x) \lesssim_\eta I_k(\delta,\eta;x)\,, \tag{A.1}$$

$$\text{if } k > \tfrac{1}{\alpha} - 1: \qquad I_k(\delta,\eta;x) \lesssim_\eta I_{k-1}(\delta,\eta;x)\,. \tag{A.2}$$

Since we assume that $\frac{1}{\alpha} \notin \mathbb{N}$, we have $\frac{1}{\alpha} - 2 < \kappa_\alpha < \frac{1}{\alpha} - 1$. If I_{κ_α} is a.n., it follows that also $I_{\kappa_\alpha - 1}$, $I_{\kappa_\alpha-2}$, ... are a.n., by (A.1), while $I_{\kappa_\alpha+1}, I_{\kappa_\alpha+2}, \ldots$ are a.n., by (A.2).

It remains to prove (A.1)-(A.2). For $k < \frac{1}{\alpha} - 1$, the function $b_{k+1}(y)$, see (1.10), is regularly varying with index $(k+1)\alpha - 1 < 0$, hence it is asymptotically decreasing: bounding $b_{k+1}(y_k) \gtrsim b_{k+1}(\eta y_{k-1})$ for $|y_k| \le \eta|y_{k-1}|$ gives

$$\int_{|y_k|\le\eta|y_{k-1}|} F(-y_{k-1}+\mathrm{d}y_k)\,b_{k+1}(y_k) \gtrsim F(-(1-\eta)|y_{k-1}|)\,b_{k+1}(\eta y_{k-1}) \gtrsim_\eta b_k(y_{k-1})\,,$$

which plugged into (1.19) shows that $I_k(\delta,\eta;x) \gtrsim_\eta I_{k-1}(\delta,\eta;x)$, proving (A.1).

To prove (A.2), note that for $\alpha > \frac{1}{k+1}$ the function $b_{k+1}(y)$ is asymptotically increasing: by a similar argument, we get $I_k(\delta,\eta;x) \lesssim_\eta I_{k-1}(\delta,\eta;x)$, that is (A.2). ∎

Proof of Lemma A.3. We are going to prove the following inequalities between I_1 and $\tilde I_1$:

$$I_1(\tfrac{\delta}{2};x) \lesssim \tilde I_1(\delta;x)\,, \tag{A.3}$$

$$\text{if } \alpha < \tfrac12: \qquad \tilde I_1(\delta;x) \lesssim I_1(\delta;x)\,. \tag{A.4}$$

For $k \in \mathbb{N}$ with $k \ge 2$, we have the following relations between $\tilde I_k$ and I_k, I_{k-1}:

$$\max\big\{I_{k-1}(\delta,\eta;x),\, I_k(\delta,\eta;x)\big\} \lesssim_\eta \tilde I_k(\delta,\eta;x)\,, \tag{A.5}$$

$$\text{if } k \ne \tfrac{1}{\alpha}-1: \qquad \tilde I_k(\delta,\eta;x) \lesssim_\eta \max\big\{I_{k-1}(\delta,\eta;x),\, I_k(\delta,\eta;x)\big\}\,. \tag{A.6}$$

Given these relations, if $\tilde I_{\kappa_\alpha}$ is a.n., then also I_{κ_α} is a.n.: it suffices to apply (A.3) and (A.5) with $k=\kappa_\alpha$. When $\frac1\alpha \notin \mathbb{N}$, the reverse implication also holds, because we can apply (A.4) if $\kappa_\alpha = 1$ (note that $\alpha < \frac12$, since $\frac1\alpha \notin \mathbb{N}$) or (A.6) if $\kappa_\alpha > 1$.

It remains to prove (A.3)-(A.6). By (5.8), for $|y_k| \le \eta |y_{k-1}|$ with $\eta \in (0,1)$ we can write

$$\tilde b_{k+1}(y_{k-1},y_k) \ge \sum_{m=A(|y_k|)}^{A(|y_k|/\eta)} \frac{m^k}{a_m} \gtrsim \frac{A(|y_k|)^k}{(|y_k|/\eta)\vee 1}\big(A(|y_k|/\eta) - A(|y_k|)\big) \gtrsim_\eta b_{k+1}(y_k)\,, \tag{A.7}$$

The same arguments show that, for $|y| \le \frac{\delta}{2}x$, we have $\tilde b_2(\delta x, y) \ge \tilde b_2(2y,y) \gtrsim b_2(y)$. Plugging these bounds into (1.23) and (1.24) proves (A.3) and also $\tilde I_k(\delta,\eta;x) \gtrsim_\eta I_k(\delta,\eta;x)$, which is half of (A.5). For the other half, note that for $|y_k| \le \eta|y_{k-1}|$, always by (5.8),

$$\tilde b_{k+1}(y_{k-1},y_k) \gtrsim \sum_{m=A(\eta|y_{k-1}|)}^{A(|y_{k-1}|)} \frac{m^k}{a_m} \gtrsim \frac{A(\eta|y_{k-1}|)^k}{|y_{k-1}|\vee 1}\big(A(|y_{k-1}|) - A(\eta|y_{k-1}|)\big) \gtrsim_\eta b_{k+1}(y_{k-1})\,,$$

hence

$$\int_{|y_k|\le\eta|y_{k-1}|} F(-y_{k-1}+\mathrm{d}y_k)\,\tilde b_{k+1}(y_{k-1},y_k) \gtrsim_\eta b_{k+1}(y_{k-1})F(-(1-\eta)|y_{k-1}|) \gtrsim_\eta b_k(y_{k-1})\,. \tag{A.8}$$

From (1.23) we get $\tilde I_k(\delta,\eta;x) \gtrsim_\eta I_{k-1}(\delta,\eta;x)$, which completes the proof of (A.5).

Next we prove (A.4) and (A.6). We distinguish two cases.

- If $k < \frac1\alpha - 1$, the sequence m^k/a_m is regularly varying with index $k-\frac1\alpha < -1$. By (2.4), we can write

$$\tilde b_{k+1}(y_{k-1},y_k) \le \sum_{m=A(|y_k|)}^{\infty} \frac{m^k}{a_m} \lesssim \frac{A(|y_k|)^{k+1}}{|y_k|\vee 1} = b_{k+1}(y_k)\,,$$

  which yields $\tilde I_k(\delta,\eta;x) \lesssim_\eta I_k(\delta,\eta;x)$. For $k=1$, we have proved (A.4), since $k < \frac1\alpha - 1$ means precisely $\alpha < \frac12$, while for $k \ge 2$ we have proved half of (A.6).

- If $k > \frac1\alpha - 1$, with $k \ge 2$, the sequence m^k/a_m is regularly varying with index $k-\frac1\alpha > -1$ and by (2.3) we get

$$\tilde b_{k+1}(y_{k-1},y_k) \le \sum_{m=1}^{A(|y_{k-1}|)} \frac{m^k}{a_m} \lesssim \frac{A(|y_{k-1}|)^{k+1}}{|y_{k-1}|\vee 1} = b_{k+1}(y_{k-1})\,,$$

and in analogy with (A.8) we get $\tilde{I}_k(\delta,\eta;x) \lesssim_\eta I_{k-1}(\delta,\eta;x)$. Relation (A.6) is proved. The proof is completed. ∎

Proof of Lemma A.1. Fix $k \in \mathbb{N}$ with $k \geq 2$ and $\eta \in (0,1)$. In analogy with (A.1)-(A.2), we are going to prove that

$$\text{if } k \leq \tfrac{1}{\alpha} - 1: \qquad \tilde{I}_{k-1}(\delta,\eta;x) \lesssim_\eta \tilde{I}_k(\delta,\eta;x)\,, \tag{A.9}$$

$$\text{if } k > \tfrac{1}{\alpha} - 1: \qquad \tilde{I}_k(\tfrac{\delta}{2},\eta;x) \lesssim_\eta \tilde{I}_{k-1}(\delta,\eta;x)\,, \tag{A.10}$$

where $k = \frac{1}{\alpha} - 1$ is included in (A.9) (unlike (A.1)). Since we assume that $\tilde{I}_{\kappa_\alpha}$ is a.n., and since $\kappa_\alpha \leq \frac{1}{\alpha} - 1$, we can apply (A.9) iteratively to see that $\tilde{I}_{\kappa_\alpha - 1}$, $\tilde{I}_{\kappa_\alpha - 2}$, $\ldots$ are a.n.. Similarly, since $\kappa_\alpha + 1 > \frac{1}{\alpha} - 1$, relation (A.10) shows that $\tilde{I}_{\kappa_\alpha+1}$, $\tilde{I}_{\kappa_\alpha+2}$, $\ldots$ are a.n..

It remains to prove (A.9)-(A.10). By (A.1) and (A.2) we have

$$\max\big\{I_{k-1}(\delta,\eta;x),\, I_k(\delta,\eta;x)\big\} \approx_\eta \begin{cases} I_k(\delta,\eta;x) & \text{if } k < \frac{1}{\alpha} - 1 \\ I_{k-1}(\delta,\eta;x) & \text{if } k > \frac{1}{\alpha} - 1 \end{cases}. \tag{A.11}$$

Let us fix $k \leq \frac{1}{\alpha} - 1$ and assume first that $k \geq 3$. By (A.6) and (A.11) (with $k-1$ in place of k; note that $k - 1 < \frac{1}{\alpha} - 1$), and then by (A.5), we have

$$\tilde{I}_{k-1} \lesssim_\eta \max\{I_{k-2}, I_{k-1}\} \approx_\eta I_{k-1} \leq \max\{I_{k-1}, I_k\} \lesssim_\eta \tilde{I}_k\,.$$

If $k = 2$, the assumption $k \leq \frac{1}{\alpha} - 1$ means $\alpha \leq \frac{1}{3}$, hence we can apply (A.4) followed by (A.5) to get $\tilde{I}_1 \lesssim I_1 \leq \max\{I_1, I_2\} \lesssim_\eta \tilde{I}_2$. This completes the proof of (A.9).

Fix now $k > \frac{1}{\alpha} - 1$ (note that $k \geq 2$, since $\alpha \leq \frac{1}{2}$). By (A.6) and (A.11), we can write

$$\tilde{I}_k(\tfrac{\delta}{2}) \lesssim_\eta \max\{I_{k-1}(\tfrac{\delta}{2}), I_k(\tfrac{\delta}{2})\} \lesssim_\eta I_{k-1}(\tfrac{\delta}{2})\,.$$

If $k \geq 3$, we apply (A.5) with $k-1$ in place of k, to get

$$I_{k-1}(\tfrac{\delta}{2}) \leq \max\{I_{k-2}(\tfrac{\delta}{2}), I_{k-1}(\tfrac{\delta}{2})\} \lesssim_\eta \tilde{I}_{k-1}(\tfrac{\delta}{2}) \leq \tilde{I}_{k-1}(\delta)\,.$$

This yields $\tilde{I}_k(\frac{\delta}{2}) \lesssim_\eta \tilde{I}_{k-1}(\delta)$, which is precisely (A.10). If $k = 2$, we apply (A.3) to see that $I_{k-1}(\frac{\delta}{2}) = I_1(\frac{\delta}{2}) \leq \tilde{I}_1(\delta)$. This completes the proof of (A.10). ∎

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