UNIVERSALITY IN MARGINALLY RELEVANT DISORDERED SYSTEMS

FRANCESCO CARAVENNA, RONGFENG SUN, AND NIKOS ZYGOURAS

ABSTRACT. We consider disordered systems of directed polymer type, for which disorder is so-called marginally relevant. These include the usual (short-range) directed polymer model in dimension $(2 + 1)$, the long-range directed polymer model with Cauchy tails in dimension $(1 + 1)$ and the disordered pinning model with tail exponent $1/2$. We show that in a suitable weak disorder and continuum limit, the partition functions of these different models converge to a universal limit: a log-normal random field with a multi-scale correlation structure, which undergoes a phase transition as the disorder strength varies. As a by-product, we show that the solution of the two-dimensional Stochastic Heat Equation, suitably regularized, converges to the same limit.

CONTENTS

1. INTRODUCTION

Many disordered systems arise as random perturbations of a pure (or homogenous) model. Examples include the random pinning model [G07], where the pure system is a renewal process, the directed polymer model [CSY04], where the pure system is a directed random walk, the random field Ising model [B06] and the stochastic heat equation [BC95]. A

fundamental question for such systems is: *Does the addition of disorder alter the qualitative behavior of the pure model, such as its large-scale properties and/or critical exponents?*

If the answer is yes, regardless of how small is the disorder strength, then the model is called *disorder relevant*. If, on the other hand, disorder has to be strong enough to cause a qualitative change, then the model is called *disorder irrelevant*. This difference can be understood heuristically via renormalization transformations [B06, G10]: rescaling space (coarse graining) and looking at the resulting renormalized disordered system on larger and larger spatial scales, the "effective" strength of disorder will asymptotically diverge if disorder is relevant, while it will vanish if disorder is irrelevant.

Whether a model is disorder relevant or irrelevant depends crucially on the spatial dimension d and its correlation length exponent ν. A milestone in the study of disordered systems in the physics literature is the Harris criterion [H74], which asserts that if $d < 2/\nu$, then disorder is relevant, while if $d > 2/\nu$, then it is irrelevant. In the critical case $d = 2/\nu$, disorder is marginal and Harris criterion is inconclusive: disorder can be either *marginally relevant* or *marginally irrelevant* depending on the finer details of the model.

Inspired by the study of an intermediate disorder regime for directed polymers [AKQ14a], we proposed in [CSZ13] a new perspective on disorder relevance. The key observation is that, if a model is disorder relevant, then it should be possible to tune the strength of disorder down to zero (weak disorder limit) at the same time as one rescales space (continuum limit), so as to obtain a one-parameter family of *disordered continuum models*, indexed by a macroscopic disorder strength parameter $\hat{\beta} \geqslant 0$. In a sense, such continuum models interpolate between the scaling limit of the pure model ($\hat{\beta} = 0$) and the scaling limit of the original disordered model ($\hat{\beta} = \infty$), allowing one to study the onset of the effect of disorder.

The main step in the construction of such disordered continuum models is to identify their *partition functions*. In [CSZ13], we formulated general conditions on the pure model that are consistent with the Harris criterion $d < 2/\nu$ for disorder relevance, which allowed us to construct explicitly the continuum partition functions. However, the *marginally relevant* case ($d = 2/\nu$ in the Harris criterion) escapes the framework proposed in [CSZ13].

In the present work, we develop a novel approach to study the continuum limit of marginally relevant systems of directed polymer type, which include the usual short-range directed polymer model on $\mathbb{Z}^2$, the long-range directed polymer model on $\mathbb{Z}$ with Cauchy tails, and the pinning model with tail exponent $\alpha = 1/2$. We show that, surprisingly, there is a common structure among all these marginally relevant models, and their partition functions converge in distribution to the same universal limit. More precisely,

- A properly defined *replica overlap* $\mathbf{R}_N$ for each model diverges as a slowly varying function (usually a logarithm) of the polymer length $N \to \infty$.

- If the disorder strength is sent to 0 as $\beta_N = \hat{\beta}/\sqrt{\mathbf{R}_N}$ for fixed $\hat{\beta} > 0$, then the partition function has a universal limit in distribution, *irrespective of the model*:

$$Z_{N,\beta_N} \xrightarrow[N\to\infty]{d} \boldsymbol{Z}_{\hat{\beta}} \overset{d}{=} \begin{cases} \text{log-normal} & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geqslant 1 \end{cases}. \qquad (1.1)$$

- A process-level version of (1.1) also holds: for $\hat{\beta} < 1$, the family of log partition functions $\log Z_{N,\beta_N}(x)$, indexed by the starting point x of the polymer, converges to a limiting Gaussian random field with an explicit *multi-scale covariance structure*.

See Section 2 for a precise formulation of our results.

As an interesting corollary, we gain new insights on how to define the solution of the *two-dimensional* Stochastic Heat Equation (2d SHE), which is formally written as

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + \beta\,\dot{W}(t,x)\,u(t,x), \qquad u(0,\cdot) \equiv 1, \tag{1.2}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^2$, $\beta > 0$ and $\dot{W}$ is the space-time white noise.

One source of interest in the SHE is that it is connected via the Hopf-Cole transformation ($h := \log u$) to the KPZ equation, which models stochastic interface growth [KPZ86]. However, rigorously defining the solution is a serious challenge due to ill-defined terms such as $\dot{W}\,u$. In special cases, such as the one-dimensional SHE, it was shown in [BC95] that a solution can be defined by first mollifying $\dot{W}$ and then sending the mollification parameter to zero. But there was no systematic approach to make sense of singular SPDEs until recent breakthroughs by Hairer [H13, H14], through *Regularity Structures*, and by Gubinelli, Imkeller and Perkowski [GIP15], through *Paracontrolled Distributions* (see also Kupiainen [K14] for field theoretic approach). However, all these approaches fail at the *critical* dimension two for the SHE, and the singular SPDEs that can be treated so far are all known as *sub-critical* (or *super-renormalizable* in the physics literature [K14]).

It turns out that the notion of *sub-criticality* for singular SPDEs matches with the notion of disorder relevance, while *criticality* corresponds to the case where the effect of disorder is marginal. To illustrate this fact for the SHE, consider the change of variables

$$(t,x) = T_\varepsilon(\widetilde{t}, \widetilde{x}) := (\varepsilon^{-2}\widetilde{t}, \varepsilon^{-1}\widetilde{x}),$$

which for small $\varepsilon > 0$ correspond to a coarse graining space-time transformation. Looking at (1.2), it is easily seen that $\widetilde{u}(\widetilde{t}, \widetilde{x}) := u(T_\varepsilon(\widetilde{t}, \widetilde{x}))$ formally solves the SPDE

$$\frac{\partial \widetilde{u}}{\partial \widetilde{t}} = \frac{1}{2}\Delta\widetilde{u} + \beta\varepsilon^{\frac{d}{2}-1}\dot{\widetilde{W}}\widetilde{u}, \qquad \widetilde{u}(0,\cdot) \equiv 1, \tag{1.3}$$

where $\dot{\widetilde{W}}$ is a new space-time White noise obtained from $\dot{W}$ via scaling. Therefore, coarse-graining space-time for the SHE has the effect of changing the strength of the noise to $\varepsilon^{\frac{d}{2}-1}\beta$ which, as $\varepsilon \to 0$, diverges for $d = 1$ (disorder relevance), vanishes for $d \geqslant 3$ (disorder irrelevance), and remains unchanged for $d = 2$ (marginality).

Since the difficulties in studying the regularity properties of a SPDE are related to small scale divergences, it is interesting to *blow up* space-time, i.e. consider the change of variables $(t,x) = T_{\varepsilon^{-1}}(\widetilde{t}, \widetilde{x})$. This leads to a renormalized equation which is just (1.3) with ε replaced by ε^{-1}, hence blowing-up space time produces an effective noise strength which behaves reciprocally with respect to coarse-graining, i.e. vanishes as $\varepsilon \to 0$ for $d = 1$ and diverges for $d \geqslant 3$. This explains why the SHE with $d = 1$ can be analyzed by [H14, GIP15, K14].

Since the solution of the SHE can be interpreted as the partition function of a continuum directed polymer via a generalized Feynman-Kac formula [BC95], our result for the two-dimensional directed polymer implies a similar result for the 2d SHE. More precisely, if we consider the mollified 2d SHE

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \dot{W}^\varepsilon u^\varepsilon, \qquad u^\varepsilon(0,\cdot) \equiv 1, \tag{1.4}$$

where $\dot{W}^\varepsilon$ is the space-mollification of $\dot{W}$ via convolution with a smooth probability density $j_\varepsilon(x) := \varepsilon^{-2}j(x/\varepsilon)$ on $\mathbb{R}^2$, and the noise strength is scaled as $\beta_\varepsilon = \hat{\beta}\sqrt{\frac{2\pi}{\log\varepsilon^{-1}}}$ for some $\hat{\beta} > 0$, then for each $(t,x) \in (0,\infty) \times \mathbb{R}^2$, $u^\varepsilon(t,x)$ converges (as $\varepsilon \to 0$) in distribution to the same universal limit $\boldsymbol{Z}_{\hat{\beta}}$ in (1.1) as for the other marginally relevant models.

We hope that the universal structure we have uncovered among models of directed polymer type opens the door to further understanding of marginally relevant models in general, including both statistical mechanics models that are not of directed polymer type, as well as critical singular SPDEs with non-linearity. In particular, our results suggest that for marginally relevant models there is a transition in the effect of disorder on an intermediate disorder scale. Establishing this transition in general, as well as understanding the behavior of the models at and above the transition point, will be the key future challenges.

2. The models and our results

In this section we define our models of interest and state our main results. We will denote $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.1. The models. We first introduce the disorder ω. Let $\omega = (\omega_\mathsf{x})$ be a family of i.i.d. random variables, indexed by $\mathsf{X} \in \mathbb{N}$ or $\mathsf{X} = (x, n) \in \mathbb{Z}^d \times \mathbb{N}_0$, depending on the model. Probability and expectation for ω will be denoted respectively by $\mathbb{P}$ and $\mathbb{E}$. We assume that

$$\mathbb{E}[\omega_1] = 0, \qquad \mathbb{V}\mathrm{ar}[\omega_1] = 1, \qquad \exists \beta_0 > 0 : \ \lambda(\beta) := \log \mathbb{E}[e^{\beta \omega_1}] < \infty \ \ \forall |\beta| < \beta_0.$$

Next we define the class of models we consider. We fix a reference probability law P (which will typically be the law of a random walk or a renewal process) representing the "pure" model. The disordered model is then a Gibbs perturbation $\mathrm{P}^\omega_{N,\beta}$ of P, indexed by the parameters $N \in \mathbb{N}$ (polymer length), $\beta \geqslant 0$ (disorder strength) and the disorder ω:

$$\frac{\mathrm{dP}^\omega_{N,\beta}}{\mathrm{dP}}(\cdot) := \frac{e^{H^\omega_{N,\beta}(\cdot)}}{Z^\omega_{N,\beta}}$$

for a suitable Hamiltonian $H^\omega_{N,\beta}$. The normalizing constant

$$Z^\omega_{N,\beta} := \mathrm{E}\left[e^{H^\omega_{N,\beta}} \right]$$

is the *disordered partition function* and will be the focus of this paper.

Different reference laws P and Hamiltonians $H^\omega_{N,\beta}$ give rise to different models. The first class of models we will consider are *directed polymers in random environment* on $\mathbb{Z}^{d+1}$.

Definition 2.1 (Directed polymers on $\mathbb{Z}^{d+1}$). Let $S = (S_n)_{n \in \mathbb{N}_0}$ be a random walk on $\mathbb{Z}^d$ with i.i.d. increments. For $(x, t) \in \mathbb{Z}^d \times \mathbb{N}_0$ we denote by $\mathrm{P}_{x,t}$ the law of $(S_n)_{n \geqslant t}$ started at x at time t, and we denote $\mathrm{P} := \mathrm{P}_{0,0}$ for simplicity. The partition function of the *directed polymer in random environment* is defined by

$$Z^\omega_{N,\beta}(x, t) := \mathrm{E}_{x,t}\left[e^{\sum_{n=t+1}^N \left(\beta\, \omega_{(n, S_n)} - \lambda(\beta) \right)} \right] \tag{2.1}$$

with $Z^\omega_{N,\beta} := Z^\omega_{N,\beta}(0, 0)$.

We will also consider *pinning models*, which can be viewed as directed polymers on $\mathbb{Z}^{d+1}$ with disorder present only at $x = 0$ (i.e. $\omega(n, x) = 0$ for $x \neq 0$). In this case, what really matters are the return times of the random walk S to 0, which form a *renewal process*.

Definition 2.2 (Pinning models). Let $(\tau = (\tau_n)_{n \in \mathbb{N}_0}, \mathrm{P}_t)$ be a renewal process started at $t \in \mathbb{N}_0$, i.e., $\mathrm{P}_t(\tau_0 = t) = 1$ and $(\tau_n - \tau_{n-1})_{n \in \mathbb{N}}$ are i.i.d. $\mathbb{N}$-valued random variables. If $t = 0$ we write $\mathrm{P} = \mathrm{P}_0$. The partition function of the *pinning model* started at $t \in \mathbb{N}_0$ equals

$$Z^\omega_{N,\beta}(t) := \mathrm{E}_t\left[e^{\sum_{n=t+1}^N (\beta \omega_n - \lambda(\beta)) \mathbb{1}_{\{n \in \tau\}}} \right], \tag{2.2}$$

with $Z^\omega_{N,\beta} := Z^\omega_{N,\beta}(0)$, where we have identified τ with the random set $\{\tau_0, \tau_1, \ldots\} \subset \mathbb{N}_0$.

Remark 2.3. In the pinning model it is customary to have a bias parameter $h \in \mathbb{R}$, i.e. $-\lambda(\beta)$ is replaced by $-\lambda(\beta) + h$ in (2.2). In this paper we set $h = 0$ because in the regime we are interested in, the effects of β and h can be decoupled. This will be treated elsewhere.

Note that $Z^\omega_{N,\beta}$ in (2.1)–(2.2) has been normalized so that $\mathbb{E}[Z^\omega_{N,\beta}] = 1$ (due to $-\lambda(\beta)$). The key question we consider (in connection with disorder relevance) is the following:

> **Q.** *Can one tune the disorder strength $\beta = \beta_N \to 0$ as $N \to \infty$ in such a way that the partition function Z^ω_{N,β_N} converges in law to a non-degenerate random variable?*

The answer depends crucially on the random walk S and the renewal process τ. Assume that S and τ are in the domain of attraction of a stable law, with respective index $\alpha \in (0, 2]$ and $\alpha \in (0, 1)$. Informally, this means that $\mathrm{P}(|S_1| > n) \approx n^{-\alpha}$ and $\mathrm{P}(\tau_1 > n) \approx n^{-\alpha}$ (except for $\alpha = 2$, where $\mathbb{E}[|S_1|^2] < \infty$ or, more generally, $x \mapsto \mathrm{E}[|S_1|^2 \mathbb{1}_{\{|S_1| \leqslant x\}}]$ is slowly varying).

It was shown in [CSZ13] that question **Q.** has an affirmative answer for directed polymers on $\mathbb{Z}^{1+1}$ with $\alpha \in (1, 2]$ and for pinning models with $\alpha \in (1/2, 1)$, which is a manifestation of *disorder relevance*; while disorder is irrelevant for directed polymers on $\mathbb{Z}^{1+1}$ with $\alpha \in (0, 1)$ and for pinning models with $\alpha \in (0, 1/2)$. However, the *marginal cases*

(a) directed polymers on $\mathbb{Z}^{2+1}$ with $\alpha = 2$ (e.g., finite variance);

(b) directed polymers on $\mathbb{Z}^{1+1}$ with $\alpha = 1$ (e.g., Cauchy tails);

(c) pinning models with tail exponent $\alpha = 1/2$ (e.g., the renewal arising from the return times of the simple symmetric random walk on $\mathbb{Z}$ to the origin),

fall out of the scope of the method in [CSZ13].

In this paper, we develop a novel approach to answer question **Q.** affirmatively for marginally relevant models. Even though our techniques are of wider applicability, we stick for simplicity to models of type (a)–(c) above. Let us state our precise assumptions, in the form of local limit theorems, where we allow for arbitrary slowly varying function $L(\cdot)$. However, we suggest to keep in mind the basic case when $L(\cdot)$ is constant, say $L(\cdot) \equiv 1$.

Hypothesis 2.4 (Local Limit Theorem). Assume that the directed polymer in Definition 2.1 and the pinning model in Definition 2.2 satisfy the following local limit theorems, for some slowly varying function L.

(a) $[d = 2]$ *Directed polymer on $\mathbb{Z}^{2+1}$ with $\alpha = 2$ (short range).*

$$\sup_{x \in \mathbb{Z}^2} \left\{ L(n)^2 n \, \mathrm{P}(S_n = x) - g\left(\frac{x}{L(n)\sqrt{n}}\right) \right\} \xrightarrow[n \to \infty]{} 0, \qquad (2.3)$$

where $g(x) := \frac{1}{2\pi} e^{-\frac{1}{2}|x|^2}$ denotes the standard Gaussian density on $\mathbb{R}^2$.

(b) $[d = 1]$ *Directed polymer on $\mathbb{Z}^{1+1}$ with $\alpha = 1$ (long-range with Cauchy tails).*

$$\sup_{z \in \mathbb{Z}} \left\{ L(n)^2 n \, \mathrm{P}(S_n = x) - g\left(\frac{x}{L(n)^2 n}\right) \right\} \xrightarrow[n \to \infty]{} 0, \qquad (2.4)$$

where $g(x) := \frac{1}{\pi} \frac{1}{1+x^2}$ denotes the Cauchy density on $\mathbb{R}$.

(c) $[d = 0]$ *Pinning model with $\alpha = \frac{1}{2}$.*

$$L(n)\sqrt{n} \, \mathrm{P}(n \in \tau) \xrightarrow[n \to \infty]{} c \in (0, \infty). \qquad (2.5)$$

Remark 2.5. Conditions (2.3)–(2.4) hold whenever S is an aperiodic random walk on $\mathbb{Z}^d$ in the domain of attraction of the Gaussian ($d = 2$), resp. Cauchy ($d = 1$) distribution:

$$\frac{S_n}{\phi(n)} \xrightarrow[n \to \infty]{d} g(x)\,\mathrm{d}x \qquad \text{with} \qquad \phi(n) := \big(L(n)^2\, n\big)^{1/d}, \tag{2.6}$$

by Gnedenko's local limit theorem (see e.g. [BGT89, Theorem 8.4.1]).

Condition (2.5) holds whenever $\mathrm{P}(\tau_1 = n) \sim c' \frac{L(n)}{n^{3/2}}$ as $n \to \infty$ [D97, Thm. B].

Remark 2.6 (2d Simple random walk). When S is the simple symmetric random walk on $\mathbb{Z}^2$, due to periodicity, (2.3) still holds (with $L(\cdot) \equiv 1$) provided the sup is restricted to the sub-lattice $z \in \{(a, b) \in \mathbb{Z}^2 : a + b = n \pmod 2\}$ (whose cells have area 2) and $g(\cdot)$ is replaced by $2g(\cdot)$. Consequently, relation (2.8) holds with $C = 2\|g\|_2^2$. Our main results Theorems 2.8, 2.12 and 2.13 below apply with no further change.

A crucial common feature among all models (a)–(c) above concerns the so-called expected *replica overlap*, defined for a general random walk S or renewal process τ by

$$\mathbf{R}_N := \begin{cases} \mathrm{E}\Big[\sum\limits_{n=1}^N \mathbb{1}_{\{S_n = S'_n\}} \Big] = \sum\limits_{1 \leqslant n \leqslant N,\, x \in \mathbb{Z}^d} \mathrm{P}(S_n = x)^2 \\[2mm] \mathrm{E}\Big[\sum\limits_{n=1}^N \mathbb{1}_{\{n \in \tau \cap \tau'\}} \Big] = \sum\limits_{1 \leqslant n \leqslant N} \mathrm{P}(n \in \tau)^2 \end{cases}, \tag{2.7}$$

where S' and τ' are independent copies of S and τ. For models satisfying Hypothesis 2.4, a Riemann sum approximation using (2.3)–(2.5) yields

$$\mathbf{R}_N \underset{N \to \infty}{\sim} C \sum_{n=1}^N \frac{1}{L(n)^2 n}, \qquad \text{where} \qquad C = \begin{cases} \|g\|_2^2 & \text{(directed polymers)} \\ c^2 & \text{(pinning)} \end{cases}. \tag{2.8}$$

This shows that $N \mapsto \mathbf{R}_N$ is a *slowly varying function*, cf. [BGT89, Proposition 1.5.9a], a fact which plays a crucial in our analysis. Whether $\mathbf{R}_N$ stays bounded or diverges as $N \to \infty$ will determine whether disorder is relevant or irrelevant. This leads to

Definition 2.7 (Marginal overlap condition). A directed polymer or a pinning model is said to satisfy the *marginal overlap condition*, if $\mathbf{R}_N \to \infty$ as a slowly varying function when $N \to \infty$, where $\mathbf{R}_N$ is defined in (2.7).

Under Hypothesis 2.4, the *marginal overlap condition* is satisfied when $\mathbf{R}_N \to \infty$, which by (2.8) holds if $L(n)$ stays bounded, or more generally, does not grow too fast as $n \to \infty$. We suggest the reader to keep in mind the basic case $L(n) \equiv 1$, for which $\mathbf{R}_N \sim C \log N$.

Our main result, to be stated in the next subsection, is that question **Q.** has an affirmative answer for models of directed polymer type which satisfy Hypothesis 2.4 and the marginal overlap condition. This is a signature of marginal disorder relevance in the spirit of [CSZ13]. The recent results of Berger and Lacoin [BL15a, BL15b] on free energy and critical curves reinforce this picture.

2.2. Results for directed polymer and pinning models. We are now ready to state our main results: Theorem 2.8 on the convergence of partition function with a fixed starting point; Theorem 2.12 on the joint limit of partition functions with different starting points, where multi-scale correlations emerge; and Theorem 2.13 on the Gaussian fluctuations of the partition functions as a random field indexed by the starting points.

Theorem 2.8 (Limit of partition functions). *Let $Z^\omega_{N,\beta}$ be the partition function of a directed polymer or a pinning model (cf. Definitions 2.1 and 2.2). Assume that Hypothesis 2.4 holds and the replica overlap $\mathbf{R}_N$ in (2.7) and (2.8) diverges as $N \to \infty$. Then, defining*

$$\beta_N := \frac{\hat{\beta}}{\sqrt{\mathbf{R}_N}}, \qquad with \quad \hat{\beta} \in (0, \infty), \tag{2.9}$$

the following convergence in distribution holds, where $(W_t)_{t \geqslant 0}$ is a standard Brownian motion:

$$Z^\omega_{N,\beta_N} \xrightarrow[N\to\infty]{d} \mathbf{Z}_{\hat{\beta}} := \begin{cases} \exp\left(\displaystyle\int_0^1 \frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2\,t}}\,\mathrm{d}W_t - \frac{1}{2}\int_0^1 \frac{\hat{\beta}^2}{1-\hat{\beta}^2\,t}\,\mathrm{d}t \right) & if\ \hat{\beta} < 1 \\ 0 & if\ \hat{\beta} \geqslant 1 \end{cases}. \tag{2.10}$$

Moreover, for $\hat{\beta} < 1$ one has $\lim_{N\to\infty} \mathbb{E}[(Z^\omega_{N,\beta_N})^2] = \mathbb{E}[(\mathbf{Z}_{\hat{\beta}})^2]$.

Remark 2.9. Note that for $\hat{\beta} < 1$, $\mathbf{Z}_{\hat{\beta}}$ is log-normal with

$$\mathbf{Z}_{\hat{\beta}} \stackrel{d}{=} \exp\left(\sigma_{\hat{\beta}} W_1 - \frac{\sigma_{\hat{\beta}}^2}{2} \right) \qquad with \qquad \sigma_{\hat{\beta}}^2 := \log\frac{1}{1-\hat{\beta}^2}. \tag{2.11}$$

The more involved expression in (2.10) hints at an underlying multi-scale structure, which strikingly, is common to all models that satisfy Hypothesis 2.4 and the marginal overlap condition. The heuristics for this universal multi-scale structure will be explained in Sec. 3.

A key tool in the proof of Theorem 2.8 is a version of the celebrated *Fourth Moment Theorem* for polynomial chaos expansions (see Theorem 4.2 below).

Remark 2.10. During the completion of this paper, Alberts, Clark and Kocić showed in [AKS15] that for the marginally relevant directed polymer model on the diamond hierarchical lattice, with either edge or site disorder, there is also a transition for the partition function in an intermediate disorder regime with some critical value $\hat{\beta}_c$. Their proof relies on the recursive structure of the hierarchical lattice. A difference with respect to our results is that, for $\hat{\beta} \leqslant \hat{\beta}_c$, the partition function converges to 1 and has Gaussian fluctuations. It would be interesting to apply our approach to better understand the source of this difference.

Remark 2.11. The transition from a non-degenerate limit $\mathbf{Z}_{\hat{\beta}} > 0$ to a degenerate limit $\mathbf{Z}_{\hat{\beta}} = 0$ in (2.10), as $\hat{\beta}$ increases, is reminiscent of similar transitions in other models, including Gaussian Multiplicatice Chaos [RV13] and the directed polymer on $\mathbb{Z}^{d+1}$ with $d \geqslant 3$ [CSY04]. All these are transitions from *weak disorder* to *strong disorder*, where at weak disorder, the underlying Gibbs measure associated with the partition function Z^ω_{N,β_N} is qualitatively comparable to the reference measure P, which is not the case at strong disorder. For the marginally relevant models we consider, the transition from weak to strong disorder occurs on the intermediate scale of $1/\sqrt{\mathbf{R}_N}$, while for disorder irrelevant models such as the directed polymer on $\mathbb{Z}^d$ with $d \geqslant 3$, the transition occurs on scale of order 1. Another difference is that in our models, the transition point coincides with the L^2 critical point, namely that $(Z^\omega_{N,\beta_N})_{N\geqslant 1}$ in (2.10) fails to be uniformly bounded in L^2 precisely when $\hat{\beta} \geqslant \hat{\beta}_c = 1$. This is not the case for Gaussian Multiplicative Chaos with log-correlated covariance [RV13], or for directed polymers on $\mathbb{Z}^{d+1}$ with $d \geqslant 3$ [BS10].

Next we study the partition functions $Z_{N,\beta}^\omega(\mathsf{X})$ as a random field, indexed by the polymer's starting position $\mathsf{X} = (x,t) \in \mathbb{Z}^d \times \mathbb{N}_0$ with $d \in \{1,2\}$ (for directed polymers), resp. $\mathsf{X} = t \in \mathbb{N}_0$ (for pinning models). Assuming Hypothesis 2.4 with a slowly varying $L(\cdot)$ and a divergent overlap $\mathbf{R}_N$, and recalling (2.6), we define

$$\phi^\leftarrow(|x|) := \min\big\{n \in \mathbb{N}_0 : \phi(n) \geqslant |x|\big\} = \min\big\{n \in \mathbb{N}_0 : (nL(n)^2)^{1/d} \geqslant |x|\big\}. \quad (2.12)$$

By (2.6), $\phi^\leftarrow(|x|)$ is the time at which the random walk S has a fluctuation of order $|x|$. Then, for each $\mathsf{X} = (x,t) \in \mathbb{Z}^d \times \mathbb{N}_0$ with $d \in \{0,1,2\}$, we set

$$\|\mathsf{X}\| := \begin{cases} t & \text{if } d = 0 \\ t \vee \phi^\leftarrow(|x|) & \text{if } d = 1,2 \end{cases}. \quad (2.13)$$

We suggest to keep in mind the special case $L(n) \equiv 1$, for which $\|\mathsf{X}\| = t \vee |x|^d$.

Theorem 2.8 gives the limiting distribution of the individual partition functions $Z_{N,\beta_N}^\omega(\mathsf{X})$, and it is natural to ask about the joint distributions. In the special case $L(n) \equiv 1$, i.e. $\mathbf{R}_N \sim C \log N$, partition functions $Z_{N,\beta_N}^\omega(\mathsf{X})$ and $Z_{N,\beta_N}^\omega(\mathsf{X}')$ with macroscopically distant starting points $\|\mathsf{X} - \mathsf{X}'\| = N^{1+o(1)}$ become *asymptotically independent* as $N \to \infty$, while an interesting correlation structure emerges *on all intermediate scales* $\|\mathsf{X} - \mathsf{X}'\| = N^{\zeta + o(1)}$, for any $\zeta \in (0,1)$. For general slowly varying functions $L(n)$, when $\mathbf{R}_N$ is not necessarily logarithmic, intermediate scales are encoded by $\mathbf{R}_{\|\mathsf{X}-\mathsf{X}'\|}/\mathbf{R}_N = \zeta + o(1)$. This is the content of the next theorem, where we use the shorthand notation $: e^Y : = e^{Y - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y]}$.

Theorem 2.12 (Multi-scale correlations). *Let $Z_{N,\beta}^\omega(\mathsf{X})$ be the partition function of a directed polymer (or pinning) model started at $\mathsf{X} = (x,t) \in \mathbb{Z}^d \times \mathbb{N}_0$ (cf. Def. 2.1 and 2.2), such that Hypothesis 2.4 holds and the replica overlap $\mathbf{R}_N$ in (2.7)–(2.8) diverges as $N \to \infty$.*

Consider a finite collection of space-time points $(\mathsf{X}_N^{(i)})_{1 \leqslant i \leqslant r}$, such that as $N \to \infty$,

$$\forall 1 \leqslant k, l \leqslant r: \qquad \mathbf{R}_{N - t_N^{(k)}}/\mathbf{R}_N = 1 - o(1),$$
$$\mathbf{R}_{\|\mathsf{X}_N^{(k)} - \mathsf{X}_N^{(l)}\|}/\mathbf{R}_N = \zeta_{k,l} + o(1) \quad \text{for some} \quad \zeta_{k,l} \in [0,1]. \quad (2.14)$$

Then, for $\beta_N = \hat{\beta}/\sqrt{\mathbf{R}_N}$ with $\hat{\beta} \in (0,1)$, the following joint convergence in distribution holds:

$$\big(Z_{N,\beta_N}^\omega(\mathsf{X}_N^{(i)})\big)_{1 \leqslant i \leqslant r} \xrightarrow[N \to \infty]{d} \big(: e^{\mathsf{Y}_i} :\big)_{1 \leqslant i \leqslant r}, \quad (2.15)$$

where $(\mathsf{Y}_i)_{1 \leqslant i \leqslant r}$ are jointly Gaussian random variables with

$$\mathbb{E}[\mathsf{Y}_i] = 0, \qquad \mathbb{C}\mathrm{ov}[\mathsf{Y}_i, \mathsf{Y}_j] = \log \frac{1 - \hat{\beta}^2 \zeta_{i,j}}{1 - \hat{\beta}^2}. \quad (2.16)$$

Lastly, we study $Z_{N,\beta_N}^\omega(\mathsf{X})$ as a space-time random field on the macroscopic scale $\|\mathsf{X}\| \approx N$, showing that it satisfies a law of large numbers with Gaussian fluctuations. For $\mathsf{X} = (x,t) \in \mathbb{Z}^d \times \mathbb{N}_0$, we define space-time rescaled variables as follows (recall $L(\cdot)$ from Hypothesis 2.4 and $\phi(\cdot)$ from (2.6)):

$$\widehat{\mathsf{X}}_N := (\hat{x}_N, \hat{t}_N) := \left(\frac{x}{\phi(N)}, \frac{t}{N}\right), \quad (2.17)$$

where pinning models correspond to $d = 0$ and we drop x. We first observe that $\mathbf{R}_N L(N)^2 \to \infty$ as $N \to \infty$, by (2.8) and [BGT89, Prop. 1.5.9a]. We are going to show that one has

$$Z^{\omega}_{N,\beta_N}(\mathsf{X}) \approx 1 + \frac{1}{\sqrt{\mathbf{R}_N L(N)^2}} \, G\big(\widehat{\mathsf{X}}_N\big) \tag{2.18}$$

where $G(\,\cdot\,)$ is a generalized Gaussian random field on $\mathbb{R}^d \times [0,1]$, with a logarithmically divergent covariance kernel (see (2.21) below). To make (2.18) precise, we fix a continuous test function $\psi : \mathbb{R}^d \times [0,1] \to \mathbb{R}$ with compact support and define

$$J^{\psi}_N := \frac{1}{\phi(N)^d N} \sum_{\mathsf{X} \in \mathbb{Z}^d \times \mathbb{N}_0} \left\{ \sqrt{\mathbf{R}_N L(N)^2} \left(Z^{\omega}_{N,\beta_N}(\mathsf{X}) - 1 \right) \right\} \psi\big(\widehat{\mathsf{X}}_N\big) \,, \tag{2.19}$$

where the pre-factor is the correct Riemann-sum normalization, in agreement with (2.17). We can now formulate our next result.

Theorem 2.13 (Fluctuations of the rescaled field). *Let $Z^{\omega}_{N,\beta}(\mathsf{X})$ be the partition function of a directed polymer or pinning model started at $\mathsf{X} \in \mathbb{Z}^d \times \mathbb{N}_0$ (cf. Def. 2.1 and 2.2), such that Hypothesis 2.4 holds and the replica overlap $\mathbf{R}_N$ in (2.7)–(2.8) diverges as $N \to \infty$.*

Fix any continuous function $\psi : \mathbb{R}^d \times [0,1] \to \mathbb{R}$ with compact support, and let $\beta_N = \hat{\beta}/\sqrt{\mathbf{R}_N}$ with $\hat{\beta} < 1$. Then J^{ψ}_N in (2.19) converges in distribution as $N \to \infty$ to a centered Gaussian random variable $N(0, \sigma^2_{\psi})$ with variance

$$\sigma^2_{\psi} := \frac{\hat{\beta}^2}{1 - \hat{\beta}^2} \int_{(\mathbb{R}^d \times [0,1])^2} \psi(x,t) \, K\big((x,t),(x',t')\big) \, \psi(x',t') \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}x' \, \mathrm{d}t' \,, \tag{2.20}$$

where the covariance kernel is given by

$$K\big((x_1,t_1),(x_2,t_2)\big) := \begin{cases} \dfrac{1}{2} \displaystyle\int_{|t_1-t_2|}^{2-(t_1+t_2)} \dfrac{1}{s} \, g\!\left(\dfrac{x_1 - x_2}{s^{1/d}}\right) \mathrm{d}s & \text{(directed polymers)} \\[4mm] \displaystyle\int_{t_1 \vee t_2}^{1} \dfrac{c^2}{\sqrt{s - t_1}\sqrt{s - t_2}} \, \mathrm{d}s & \text{(pinning)} \end{cases} \tag{2.21}$$

Remark 2.14. Observe that the kernel K diverges logarithmically near the diagonal:

$$K\big((x_1,t_1),(x_2,t_2)\big) \sim C \log \frac{1}{|(x_1,t_1) - (x_2,t_2)|} \qquad \text{as} \quad |(x_1,t_1) - (x_2,t_2)| \to 0 \,.$$

Note that Gaussian fields with such logarithmically divergent covariance kernels have played a central role in the theory of Gaussian Multiplicative Chaos (see e.g. [RV13]).

2.3. RESULTS FOR THE 2D STOCHASTIC HEAT EQUATION. We now state the analogues of Theorems 2.8, 2.12 and 2.13 for the $2d$ SHE

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \beta u \dot{W}, \qquad u(0,x) = 1 \;\; \forall\, x \in \mathbb{R}^2. \tag{2.22}$$

To make sense of (2.22), we first mollify the space-time white noise $\dot{W}$. Let $j \in C^{\infty}_c(\mathbb{R}^2)$ be a probability density on $\mathbb{R}^2$ with $j(x) = j(-x)$, and let $J := j * j$. For $\varepsilon > 0$, let $j_{\varepsilon}(x) := \varepsilon^{-2} j(x/\varepsilon)$. The mollified noise $\dot{W}^{\varepsilon}$ is defined formally by $\dot{W}^{\varepsilon}(t,x) := \int_{\mathbb{R}^2} j_{\varepsilon}(x-y) \dot{W}(t,y) \mathrm{d}y$, so that

$$\int_{\mathbb{R} \times \mathbb{R}^2} f(t,x) \dot{W}^{\varepsilon}(t,x) \mathrm{d}t\mathrm{d}x := \int_{\mathbb{R} \times \mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(t,x) j_{\varepsilon}(y-x) \mathrm{d}x \right) \dot{W}(t,y) \mathrm{d}t\mathrm{d}y \quad \forall\, f \in L^2(\mathbb{R} \times \mathbb{R}^2).$$

For fixed x, the process $t \mapsto \int_0^t \dot{W}^\varepsilon(s,x)\,\mathrm{d}s$ is a Brownian motion with variance $\|j\|_2^2$. Then we consider the mollified equation (with Itô integration, and $\beta = \beta_\varepsilon$ possibly depending on ε)

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon u^\varepsilon \dot{W}^\varepsilon, \qquad u^\varepsilon(0,\cdot) \equiv 1, \tag{2.23}$$

whose solution admits the generalized Feynman-Kac representation [BC95, Sec. 3 and (3.22)]

$$u^\varepsilon(t,x) = \mathrm{E}_x\left[\exp\left\{\beta_\varepsilon \int_0^t \dot{W}^\varepsilon(t-s,B_s)\mathrm{d}s - \frac{1}{2}\beta_\varepsilon^2\,\mathbb{E}\left[\left(\int_0^t \dot{W}^\varepsilon(t-s,B_s)\mathrm{d}s\right)^2\right]\right\}\right], \tag{2.24}$$

where E_x is expectation w.r.t. $(B_s)_{s\geqslant 0}$, a standard Brownian motion in $\mathbb{R}^2$ with $B_0 = x$ and $\mathbb{E}$ denotes the expectation with respect to the White noise. By a time reversal in $\dot{W}^\varepsilon$, we note that $u^\varepsilon(t,x)$ has the same distribution (for fixed (t,x)) as

$$\begin{aligned}
\widetilde{u}^\varepsilon(t,x) &:= \mathrm{E}_x\left[\exp\left\{\beta_\varepsilon \int_0^t \dot{W}^\varepsilon(s,B_s)\mathrm{d}s - \frac{1}{2}\beta_\varepsilon^2\,\mathrm{E}\left[\left(\int_0^t \dot{W}^\varepsilon(s,B_s)\mathrm{d}s\right)^2\right]\right\}\right] \\
&= \mathrm{E}_x\left[\exp\left\{\beta_\varepsilon \int_0^t \int_{\mathbb{R}^2} j_\varepsilon(B_s-y)\dot{W}(s,y)\mathrm{d}s\mathrm{d}y - \frac{1}{2}\beta_\varepsilon^2 t\|j_\varepsilon\|_2^2\right\}\right] \\
&= \mathrm{E}_{\varepsilon^{-1}x}\left[\exp\left\{\beta_\varepsilon \int_0^{\varepsilon^{-2}t} \int_{\mathbb{R}^2} j(B_{\widetilde{s}}-\widetilde{y})\dot{\widetilde{W}}(\widetilde{s},\widetilde{y})\mathrm{d}\widetilde{s}\mathrm{d}\widetilde{y} - \frac{1}{2}\beta_\varepsilon^2(\varepsilon^{-2}t)\|j\|_2^2\right\}\right], \tag{2.25}
\end{aligned}$$

where in the last step we made the change of variables $(\varepsilon\widetilde{y}, \varepsilon^2\widetilde{s}) := (y,s)$, and $\dot{\widetilde{W}}(\widetilde{s},\widetilde{y})\mathrm{d}\widetilde{s}\mathrm{d}\widetilde{y} := \varepsilon^{-2}\dot{W}(\varepsilon^2\widetilde{s},\varepsilon\widetilde{y})\mathrm{d}(\varepsilon^2\widetilde{s})\mathrm{d}(\varepsilon\widetilde{y})$ is another two-dimensional space-time white noise. (One can actually extend (2.25) so that the equality in law between $u^\varepsilon(t,x)$ and $\widetilde{u}^\varepsilon(t,x)$ holds jointly for all $t \in [0,1]$ and $x \in \mathbb{R}^2$, see (9.1) below.)

Relation (2.25) suggests that we can interpret $\widetilde{u}^\varepsilon(t,x)$ as the partition function of a directed Brownian polymer in $\mathbb{R}^2$ in a white noise space-time random environment at inverse temperature β_ε, with starting point $\varepsilon^{-1}x$ and polymer length $\varepsilon^{-2}t$. A consequence of our results for the short-range directed polymer on $\mathbb{Z}^2$ is the following analogue of Theorems 2.8 and 2.12, combined into a single theorem. Let us agree that $\|\mathsf{X}\| := t \vee |x|^2$.

Theorem 2.15 (Limits of regularized solutions). *Let $u^\varepsilon(t,x)$ be the solution of the regularized 2d SHE (2.23), with $\beta_\varepsilon = \hat{\beta}\sqrt{\frac{2\pi}{\log\varepsilon^{-1}}}$ for some $\hat{\beta} \in (0,\infty)$. Following the notation in Theorem 2.12, consider a finite collection of space-time points $\mathsf{X}_\varepsilon^{(i)} = (x_\varepsilon^{(i)}, t_\varepsilon^{(i)})$, $1 \leqslant i \leqslant r$, such that as $\varepsilon \to 0$,*

$$\forall i,j \in \{1,\dots,r\}: \quad t_\varepsilon^{(i)} = \varepsilon^{o(1)}, \quad \|\mathsf{X}_\varepsilon^{(i)} - \mathsf{X}_\varepsilon^{(j)}\| = \varepsilon^{2(1-\zeta_{i,j})+o(1)} \quad \text{for some} \quad \zeta_{i,j} \in [0,1].$$

Then for $\hat{\beta} < 1$, $\left(u^\varepsilon(\mathsf{X}_\varepsilon^{(i)})\right)_{1\leqslant i\leqslant r}$ converge in joint distribution to the same limit $\left(:e^{\mathsf{Y}_i}:\right)_{1\leqslant i\leqslant r}$ as in (2.15) as $\varepsilon \to 0$, with $\mathbb{E}[(u^\varepsilon(\mathsf{X}_\varepsilon^{(i)}))^2] \to \mathbb{E}[(:e^{\mathsf{Y}_i}:)^2]$; while for $\hat{\beta} \geqslant 1$, $u^\varepsilon(\mathsf{X}_\varepsilon^{(i)}) \Rightarrow 0$.

Remark 2.16. Applying Hopf-Cole transformation to (2.23), we note that $h^\varepsilon(t,x) := \log u^\varepsilon(t,x)$ is the solution of the regularized 2d KPZ equation

$$\frac{\partial h^\varepsilon}{\partial t} = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \dot{W}^\varepsilon - \beta_\varepsilon^2 \varepsilon^{-2}\|j\|_2^2, \qquad h^\varepsilon(0,\cdot) \equiv 0. \tag{2.26}$$

where the last term $-\beta_\varepsilon^2\varepsilon^{-2}\|j\|_2^2$ is the Itô correction. Theorem 2.15 can therefore be reformulated for the 2d KPZ equation, showing that when $\hat{\beta} \in (0,1)$, the solution h^ε has pointiwse Gaussian limits as $\varepsilon \to 0$.

Here is the analogue of Theorem 2.13.

Theorem 2.17 (Fluctuations of the solution field). *Let $u^\varepsilon(t, x)$ be as in Theorem 2.15 with $\hat\beta \in (0, 1)$. Let $\psi : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}$ be continuous with compact support, and let*

$$J_\varepsilon^\psi := \sqrt{\frac{\log \varepsilon^{-1}}{2\pi}} \int_{\mathbb{R}^2 \times [0,1]} (u^\varepsilon(t, x) - 1)\, \psi(x, 1 - t)\, \mathrm{d}x \mathrm{d}t. \qquad (2.27)$$

Then J_ε^ψ converges in distribution as $\varepsilon \to 0$ to the same Gaussian random variable $N(0, \sigma_\psi^2)$ as in Theorem 2.13 for the directed polymer model on $\mathbb{Z}^{2+1}$.

Remark 2.18. For simplicity, we have formulated our results for the 2d SHE with $u^\varepsilon(0, \cdot) \equiv 1$. However, it can be easily extended to general $u(0, \cdot)$. As it will become clear in the proof (or the heuristics in Section 3), for $\hat\beta < 1$, the limit of $u^\varepsilon(t, x)$ depends only on the white noise $\dot{W}$ in an infinitesimal time window $[t - o(1), t]$ as $\varepsilon \to 0$ (for directed polymer of length N, the partition function similarly depends only on the disorder in a time window $[1, N^{1-o(1)}]$). Therefore if we set the noise to be zero in the time window $[0, t - o(1)]$, then apply the Feynman-Kac formula (2.24) first from time t to $t - o(1)$, and then to 0, then we will see that the limit of $u^\varepsilon(t, x)$ depends on the initial condition only via a factor $\mathrm{E}_x[u^\varepsilon(0, B_t)]$.

Remark 2.19. Bertini-Cancrini [BC98] showed that if in (2.23), $\beta_\varepsilon := \sqrt{\frac{2\pi}{\log \varepsilon^{-1}} + \frac{\lambda}{(\log \varepsilon^{-1})^2}}$ for some $\lambda \in \mathbb{R}$, which corresponds to a finer window around $\hat\beta = 1$ in our notation, then u^ε is tight in a suitable space of distributions, and the two-point function $\mathbb{E}[u^\varepsilon(t, x)u^\varepsilon(t, y)]$ converges to a non-trivial limit. However, they could not identify the limit of u^ε. Combined with our result that $u^\varepsilon(t, x)$ converges in probability to 0 for each $x \in \mathbb{R}^2$ when $\hat\beta = 1$, this suggests that the random measure $u^\varepsilon(t, x)\mathrm{d}x$ may have a non-trivial limit as $\varepsilon \to 0$, which is singular w.r.t. the Lebesgue measure.

Remark 2.20. We note a formal connection between the 2d SHE and Gaussian multiplicative chaos, which typically considers random measures $M_\beta(\mathrm{d}x) := e^{\beta X_x - \beta^2 \mathbb{E}[X_x^2]/2}\mathrm{d}x$ on $[0, 1]^d$ for some Gaussian field $(X_x)_{x \in [0,1]^d}$. When the covariance kernel of X is divergent on the diagonal, X is a generalized function and to define $M_\beta(\mathrm{d}x)$, one first replaces X by its mollified version X^ε and defines $M_\beta^\varepsilon(\mathrm{d}x)$ and then takes the limit $\varepsilon \to 0$ (see [RV13] for a survey). For the 2d SHE, the exponential weight in (2.24) can be seen as the analogue of $e^{\beta X_x^\varepsilon - \beta^2 \mathbb{E}[(X_x^\varepsilon)^2]/2}$ for the mollified Gaussian field X^ε, except now the Gaussian field X^ε is indexed by $C([0, t], \mathbb{R}^2)$ endowed with the Wiener measure. As $\varepsilon \to 0$, its covariance kernel $K^\varepsilon(\cdot, \cdot)$ can be seen to diverge logarithmically in probability, if it is regarded as a random variable defined on $C([0, t], \mathbb{R}^2)^2$ endowed with the product Wiener measure.

3. HEURISTICS

In this section we illustrate the core of our approach, emphasizing the main ideas and keeping the exposition at a heuristic level. In §3.1 we recall the approach developed in [CSZ13] to deal with the disorder relevant regime, then in §3.2 we explain how it fails for marginally relevant models and how does the marginal overlap condition arise.

3.1. HEURISTICS FOR DISORDER RELEVANT REGIME. For simplicity, we use the pinning model to illustrate the general approach developed in [CSZ13] to identify limits of partition functions in a suitable continuum and weak disorder limit.

We first rewrite the partition function (2.2) for $t = 0$: since $e^{x\mathbb{1}_{\{n\in\tau\}}} = 1 + (e^x - 1)\mathbb{1}_{\{n\in\tau\}}$ for all $x \in \mathbb{R}$, we get

$$Z^\omega_{N,\beta} = \mathrm{E}\left[\prod_{n=1}^{N}\left(1 + \beta\,\eta_n\,\mathbb{1}_{\{n\in\tau\}}\right)\right] \qquad \text{where} \quad \eta_n := \frac{e^{\beta\omega_n - \lambda(\beta)} - 1}{\beta}. \tag{3.1}$$

A binomial expansion of the product in (3.1) then yields (setting $n_0 = 0$)

$$Z^\omega_{N,\beta} = 1 + \sum_{k=1}^{N}\beta^k \sum_{1\leqslant n_1 < \cdots < n_k \leqslant N}\prod_{j=1}^{k}q_{n_j - n_{j-1}}\prod_{i=1}^{k}\eta_{n_i} \qquad \text{where} \quad q_n := \mathrm{P}(n\in\tau). \tag{3.2}$$

We have thus rewritten $Z^\omega_{N,\beta}$ as a multi-linear polynomial of the i.i.d. random variables $(\eta_n)_{n\in\mathbb{N}}$, sometimes called a *polynomial chaos* expansion.

Assume for simplicity that the underlying renewal process τ satisfies

$$\mathrm{P}(\tau_1 - \tau_0 = n) \sim \frac{C}{n^{1+\alpha}} \qquad \text{as } n \to \infty \tag{3.3}$$

for some $C > 0$ and $\alpha \in (0, 1)$, which implies the local limit theorem [D97, Thm. B].

$$q_n := \mathrm{P}(n\in\tau) \sim \frac{\widetilde{C}}{n^{1-\alpha}} = \frac{\alpha\sin(\pi\alpha)}{C\pi}\cdot\frac{1}{n^{1-\alpha}}. \tag{3.4}$$

Recalling (3.1), by Taylor expansion, as $\beta \to 0$,

$$\mathbb{E}[\eta_n] = 0, \qquad \mathbb{V}\mathrm{ar}[\eta_n] \simeq 1. \tag{3.5}$$

Since the "influence" of each η_n on $Z^\omega_{N,\beta}$ is small, we can apply a *Lindeberg principle* (see e.g. [CSZ13, MOO10, NPR10]) to replace $(\eta_n)_{n\in\mathbb{N}}$ by i.i.d. standard Gaussian random variables without changing the limiting distribution of $Z^\omega_{N,\beta}$ as $N \to \infty$.

Standard i.i.d. Gaussian $(\eta_n)_{n\in\mathbb{N}}$ can be defined from a white noise $W(\mathrm{d}t)$ on $[0,\infty)$, with

$$\eta_n := \sqrt{N}\int_{\frac{n}{N}}^{\frac{n+1}{N}}W(\mathrm{d}s), \qquad n \in \mathbb{N}. \tag{3.6}$$

Setting $t_i := n_i/N$ for each $i \in \mathbb{N}$, the series (3.2) then becomes a series of stochastic integrals

$$\begin{aligned}
Z^\omega_{N,\beta} &\approx 1 + \sum_{k=1}^{N}(\beta N^{\frac{1}{2}})^k \int\cdots\int_{0<t_1<\cdots<t_k<1}\prod_{j=1}^{k}q_{\lfloor Nt_j\rfloor - \lfloor Nt_{j-1}\rfloor}\prod_{i=1}^{k}W(\mathrm{d}t_i) \\
&\approx 1 + \sum_{k=1}^{N}(\widetilde{C}\beta N^{\alpha-\frac{1}{2}})^k \int\cdots\int_{0<t_1<\cdots<t_k<1}\prod_{j=1}^{k}(t_j - t_{j-1})^{\alpha-1}\prod_{i=1}^{k}W(\mathrm{d}t_i),
\end{aligned} \tag{3.7}$$

where we have applied (3.4) that $q_{Nt} \sim \widetilde{C}(Nt)^{\alpha-1}$.

In the disorder relevant regime $\alpha \in (1/2, 1)$, we note that $g(t) := t^{\alpha-1}$ is square-integrable in $t \in [0, 1]$ and the stochastic integrals in (3.7) are all well-defined. In particular, in the weak disorder limit

$$\beta_N := \frac{\hat{\beta}}{\widetilde{C}N^{\alpha-\frac{1}{2}}}, \qquad \text{with} \quad \hat{\beta} \in (0,\infty), \tag{3.8}$$

relation (3.7) suggests that as $N \to \infty$, the partition function Z^ω_{N,β_N} converges in law to

$$\boldsymbol{Z}^W_{\hat{\beta}} := 1 + \sum_{k=1}^{\infty}\hat{\beta}^k \int\cdots\int_{0<t_1<\cdots<t_k<1}\prod_{j=1}^{k}(t_j - t_{j-1})^{\alpha-1}\prod_{i=1}^{k}W(\mathrm{d}t_i). \tag{3.9}$$

The limit $\boldsymbol{Z}_{\hat{\beta}}^W$ can then be used to define a continuum disordered pinning model [CSZ14].

For the marginal case $\alpha = 1/2$, the above approach breaks down because $1/\sqrt{t}$ just fails to be square-integrable in $[0, 1]$ and the stochastic integrals in (3.7) become undefined. Nevertheless, for each $k \in \mathbb{N}$, we note that the second moment of the k-th term in (3.2) diverges as $N \to \infty$, which hints at marginal relevance of disorder.

For directed polymer models, exactly the same phenomenon appears. The approach of [CSZ13] sketched above applies to the short-range directed polymer on $\mathbb{Z}^{1+1}$ and the long-range directed polymer on $\mathbb{Z}^{1+1}$ with tail exponent $\alpha \in (1, 2)$, and breaks down exactly at the marginal cases, which include the short-range directed polymer on $\mathbb{Z}^{2+1}$ and the long-range directed polymer on $\mathbb{Z}^{1+1}$ with tail exponent $\alpha = 1$.

3.2. HEURISTICS FOR MARGINAL RELEVANT REGIME.

We now sketch the heuristics behind our proof of Theorem 2.8. Again, we use the pinning model to illustrate our approach, focusing on the marginal case where the renewal process satisfies (3.3) with $\alpha = 1/2$.

For simplicity, while retaining the key features, we assume that $(\eta_n)_{n\in\mathbb{N}}$ are i.i.d. standard normal, and in light of (3.4), we assume for simplicity that $q_n = 1/\sqrt{n}$. Then $Z_{N,\beta}^\omega$ in (3.2) simplifies to

$$Z_N = 1 + \sum_{k=1}^N \beta_N^k \sum_{1 \leqslant n_1 < \cdots < n_k \leqslant N} \prod_{j=1}^k \frac{\eta_{n_j}}{\sqrt{n_j - n_{j-1}}}. \tag{3.10}$$

The first observation, which follows from a direct calculation, is that for each $k \in \mathbb{N}$, the associated inner sum in Z_N has second moment

$$\mathrm{E}\left[\left(\sum_{1 \leqslant n_1 < \cdots < n_k \leqslant N} \prod_{j=1}^k \frac{\eta_{n_j}}{\sqrt{n_j - n_{j-1}}}\right)^2\right] = \sum_{1 \leqslant n_1 < \cdots < n_k \leqslant N} \prod_{j=1}^k \frac{1}{n_j - n_{j-1}} \sim (\log N)^k \sim \mathbf{R}_N^k,$$

where $\mathbf{R}_N$ is the expected replica overlap defined in (2.7) and satisfies the *marginal overlap condition*. This suggests that if there is a non-trivial weak disorder limit for Z_N, then we should choose $\beta_N := \hat{\beta}/\sqrt{\mathbf{R}_N}$ for some $\hat{\beta} > 0$. Furthermore, note that $\mathrm{E}[Z_N^2] \to (1 - \hat{\beta}^2)^{-1}$ for $\hat{\beta} \in (0, 1)$ and $\mathrm{E}[Z_N^2] \to \infty$ for $\hat{\beta} \geqslant 1$, with a transition occurring at $\hat{\beta}_c = 1$.

We assume from now on $\beta_N := \hat{\beta}/\sqrt{\log N} \sim \hat{\beta}/\sqrt{\mathbf{R}_N}$ in (3.10) with $\hat{\beta} \in (0, 1)$, so that

$$Z_N = 1 + \sum_{k=1}^N \hat{\beta}^k Z_N^{(k)} \quad \text{with} \quad Z_N^{(k)} := \frac{1}{(\log N)^{\frac{k}{2}}} \sum_{1 \leqslant n_1 < \cdots < n_k \leqslant N} \prod_{j=1}^k \frac{\eta_{n_j}}{\sqrt{n_j - n_{j-1}}}. \tag{3.11}$$

To prove Theorem 2.8, that Z_N converges in law to a log-normal random variable, we will identify the limit of $Z_N^{(k)}$ for each $k \in \mathbb{N}$, where an interesting structure appears. Below are the key observations.

(A) *An elementary observation.* Let $(W(t))_{t\geqslant0}$ be a standard Brownian motion on $\mathbb{R}$. For any $\delta > 0$, let $W_\delta(t) := W(\delta t)/\sqrt{\delta}$, which is another standard Brownian motion correlated with W. A simple covariance calculation then shows that as $\delta \downarrow 0$, W and W_δ become asymptotically independent. Such asymptotic independence due to separation of scales also extends to higher-dimensional white noise, which will be crucial in our analysis.

(B) *Identifying the time scale.* Next, we identify the intrinsic time scale appearing in the limit of $Z_N^{(1)}$, and $Z_N^{(k)}$ in general. Note that for any $0 \leqslant a < b \leqslant 1$, $\frac{1}{\sqrt{\log N}} \sum_{n=N^a}^{N^b} \frac{\eta_n}{\sqrt{n}}$ converges in distribution to a Gaussian random variable with mean zero and variance $b - a$.

Therefore to approximate the sum in $Z_N^{(1)}$ by a stochastic integral, we should make the change of variable $n = N^a$, which gives

$$\frac{1}{\sqrt{\log N}} \sum_{n=N^a}^{N^b} \frac{\eta_n}{\sqrt{n}} \approx \int_a^b W^{(1)}(\mathrm{d}s) \quad \forall\, 0 \leqslant a < b \leqslant 1, \tag{3.12}$$

where $W^{(1)}$ is a standard Brownian motion. In particular, $Z_N^{(1)} \approx \int_0^1 W^{(1)}(\mathrm{d}s) = W_1^{(1)}$. This indicates that the correct time scale is exponential $t \to N^t$, rather than linear $t \to Nt$.

(C) *Identifying the structure.* Finally, we identify the limit of $Z_N^{(2)}$, where the key structure already emerges. An L^2 calculation shows that as $N \to \infty$, we can relax the range of summation:

$$Z_N^{(2)} = \frac{1}{\log N} \sum_{0 \leqslant n < m \leqslant N} \frac{\eta_n\, \eta_m}{\sqrt{n}\sqrt{m-n}} \approx \frac{1}{\sqrt{\log N}} \sum_{0 \leqslant n_1 \leqslant N} \frac{\eta_{n_1}}{\sqrt{n_1}} \Big(\frac{1}{\sqrt{\log N}} \sum_{0 \leqslant n_2 \leqslant N} \frac{\eta_{n_1+n_2}}{\sqrt{n_2}} \Big). \tag{3.13}$$

Using the approximation (3.12) with $n_1 =: N^{s_1}$ and similarly for the sum over $n_2 =: N^{s_2}$,

$$Z_N^{(2)} \approx \int_0^1 \mathrm{d}W_{s_1}^{(1)} \Big(\frac{1}{\sqrt{\log N}} \sum_{0 \leqslant n_2 \leqslant N} \frac{\eta_{N^{s_1}+n_2}}{\sqrt{n_2}} \Big) \approx \int_0^1 W^{(1)}(\mathrm{d}s_1) \int_0^1 W^{(2;s_1)}(\mathrm{d}s_2), \tag{3.14}$$

where given s_1, $W^{(2;s_1)}$ is a standard Brownian motion with

$$\frac{1}{\sqrt{\log N}} \sum_{n=N^{s_1}}^{N^{s_1}+N^{s_2}} \frac{\eta_n}{\sqrt{n}} \approx W^{(2;s_1)}(s_2). \tag{3.15}$$

To understand the relation between $W_{s_1}^{(1)}$ and $W_{s_2}^{(2;s_1)}$ and make sense of the stochastic integral in (3.14), we distinguish between the cases $s_2 < s_1$ and $s_2 > s_1$.

- **Case $s_2 < s_1$:** In this case, $N^{s_2} \ll N^{s_1}$, and observation **(A)** shows that in the limit $N \to \infty$, the white noise $(W^{(2;s_1)}(\mathrm{d}s_2))_{0 \leqslant s_2 \leqslant s_1}$ becomes asymptotically independent of $(W^{(1)}(\mathrm{d}s_1))_{0 \leqslant s_1 \leqslant 1}$. Indeed, by (3.12) and (3.15), we note that the increments of $W^{(2;s_1)}$ in a small time window $[s_2, s_2 + \Delta]$ is defined from η_n with $n \in [N^{s_1} + N^{s_2}, N^{s_1} + N^{s_2+\Delta}]$, which is an infinitesimal window contained in the range of indices $[N^{s_1}, N^{s_1+\Delta}]$ used to define the increments of $W^{(1)}$ on $[s_1, s_1 + \Delta]$. In other words, the white noise $(W^{(2;s_1)}(\mathrm{d}s_2))_{0 \leqslant s_2 \leqslant s_1}$ is effectively obtained by sampling $W^{(1)}(\mathrm{d}s_1)$ in an infinitesimal window in $[s_1, s_1 + \Delta]$. A covariance calculation as in **(A)** shows that in the limit $N \to \infty$, $W^{(1)}(\mathrm{d}s_1)$ and $W^{(2;s_1)}(\mathrm{d}s_2)$ are independent for all $a, s_1 \in [0,1]$ and $b \in [0, s_1]$. Furthermore, using the Fourth Moment Theorem, it can be shown that

$$\big(\Gamma(\mathrm{d}s_1, \mathrm{d}s_2) := W^{(1)}(\mathrm{d}s_1) \cdot W^{(2;s_1)}(\mathrm{d}s_2)\big)_{0 \leqslant s_2 < s_1 \leqslant 1} \tag{3.16}$$

  is a two-dimensional white noise, independent of $(\mathrm{d}W_{s_1}^{(1)})_{0 \leqslant s_1 \leqslant 1}$.

- **Case $s_2 > s_1$:** In this case, $N^{s_1} \ll N^{s_2}$ as $N \to \infty$. Therefore, the range of indices $[N^{s_1} + N^{s_2}, N^{s_1} + N^{s_2+\Delta}]$ essentially coincide with $[N^{s_2}, N^{s_2+\Delta}]$, which are the indices of η used to define respectively the increments of $W^{(2;s_1)}$ and $W^{(1)}$ in a small window $[s_2, s_2 + \Delta]$. This implies that in the limit $N \to \infty$, we have $W^{(2;s_1)}(\mathrm{d}s_2) = W^{(1)}(\mathrm{d}s_2)$.

By the above considerations, we can now rewrite the approximation (3.14) as

$$Z_N^{(2)} \approx \int_{0 \leqslant s_2 < s_1 \leqslant 1} \Gamma(\mathrm{d}s_1, \mathrm{d}s_2) + \int_{0 \leqslant s_1 < s_2 \leqslant 1} W^{(1)}(\mathrm{d}s_1)\, \mathrm{d}W^{(1)}(\mathrm{d}s_2), \tag{3.17}$$

where the first term is a normal random variable with mean zero and variance $1/2$, independent of the second term, which can be rewritten as

$$\int_0^1 W^{(1)}(\mathrm{d}s_2) \left(\int_0^{s_2} W^{(1)}(\mathrm{d}s_1) \right) = \int_0^1 W^{(1)}(s_2) \, W^{(1)}(\mathrm{d}s_2) = \frac{(W^{(1)}(1))^2 - 1}{2}.$$

When we consider the limit of $Z_N^{(k)}$ for $k \geqslant 3$, similar separation of scales appears when we make the change of time scale $n_i = N^{s_i}$. The limit of $Z_N^{(k)}$ admits a decomposition similar to (3.17) (but more complicated), involving independent white noises of various different dimensions up to dimension k.

So far we focused on pinning models, but everything can be extended to directed polymer models, whose partition function admits a polynomial chaos expansion analogous to (3.2): see (4.1) below. Remarkably, the structure is the same as for the pinning model: if we make the the change of time variable $n_i = N^{a_i}$ and a change of space variable $z_i = x_i n_i^{1/d}$ (assuming $L(\cdot) \equiv 1$ in Hypothesis 2.4), then similar to (3.12), $Z_N^{(1)}$ can be approximated by $\int_0^1 \int_{\mathbb{R}^d} W^{(1)}(\mathrm{d}t \, \mathrm{d}x)$ for a white noise $W^{(1)}$ on $\mathbb{R}^d \times [0, \infty)$. Concerning $Z_N^{(2)}$, in analogy with (3.14), for each $s_1 > 0$ and $x_1 \in \mathbb{R}^d$, we have an independent white noise $(W^{(2;s_1,x_1)}(\mathrm{d}s_2 \, \mathrm{d}x_2))_{s_2 \in [0,s_1], x_2 \in \mathbb{R}^d}$, which is effectively obtained by sampling $W^{(1)}$ in an infinitesimal space-time window around (s_1, x_1), while $(W^{(2;s_1,x_1)}(\mathrm{d}s_2 \, \mathrm{d}x_2))_{s_2 > s_1, x_2 \in \mathbb{R}^d} = (W^{(1)}(\mathrm{d}s_2 \, \mathrm{d}x_2))_{s_2 > s_1, x_2 \in \mathbb{R}^d}$.

4. Proof steps for Theorem 2.8

Since the proof of Theorem 2.8 (for $\hat{\beta} \in (0,1)$) is long and modular, we list here the proof steps. These contain four approximations **(A1)**–**(A4)**, plus one key step **(K)** which identifies the building blocks of the limiting partition function. The local limit theorems (2.3)–(2.5) in Hypothesis 2.4 will only be used in the approximation step **(A3)**. The other steps only use the marginal overlap condition, i.e., $\mathbf{R}_N$ is a divergent slowly varying function.

The proof steps are the same for pinning $(d = 0)$ and directed polymer models $(d = 1, 2)$, so we follow a unified notation. The starting point is a polynomial chaos expansion for the partition function of directed polymers, in analogy with (3.1)-(3.2) for pinning:

$$Z_{N,\beta_N}^\omega = 1 + \sum_{k=1}^N \hat{\beta}^k Z_N^{(k)}, \qquad (\beta_N = \hat{\beta}/\sqrt{\mathbf{R}_N}),$$

$$\text{where} \qquad Z_N^{(k)} = \frac{1}{\mathbf{R}_N^{k/2}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_k \leqslant N \\ z_1, z_2, \ldots, z_k \in \mathbb{Z}^d}} \prod_{j=1}^k q_{n_j - n_{j-1}}(z_j - z_{j-1}) \prod_{i=1}^k \eta_{(n_i, z_i)}, \tag{4.1}$$

with $n_0 := 0$, $z_0 := 0$ and

$$q_n(z) := \mathrm{P}(S_n = z), \qquad \eta_{(n,z)} = \eta_{(n,z)}^{(N)} := \frac{e^{\beta_N \omega(n,z)} - 1}{\beta_N}. \tag{4.2}$$

Note that relation (4.1) applies also to the pinning model, if we view it as a directed polymer on $\mathbb{Z}^0 := \{0\}$ (cf. Hypothesis 2.4) and identify $q_n(0)$ with $q_n = \mathrm{P}(n \in \tau)$.

As a preliminary step, we can approximate Z_{N,β_N}^ω from (4.1) in L^2 (uniformly in N) by $Z_{N,\beta_N}^{\omega,M} := 1 + \sum_{k=1}^M \hat{\beta}^k Z_N^{(k)}$ if M is large, since for $\hat{\beta} \in (0,1)$,

$$\lim_{M \to \infty} \|Z_{N,\beta_N}^\omega - Z_{N,\beta_N}^{\omega,M}\|_2^2 = \lim_{M \to \infty} \sum_{k=M+1}^\infty \hat{\beta}^{2k} \|Z_N^{(k)}\|_2^2 \leqslant \lim_{M \to \infty} \sum_{k=M+1}^\infty \hat{\beta}^{2k} = 0, \tag{4.3}$$

where we used the fact that $\|Z_N^{(k)}\|_2^2 \leqslant 1$, as one checks by (4.1) and (2.7) (see (6.4) below). We can therefore focus on identifying the limit $Z_{N,\beta_N}^{\omega,M}$ as $N \to \infty$, and send $M \to \infty$ later.

Our first step is to approximate $Z_N^{(k)}$ in (4.1) as follows.

(A1) For each $k \in \mathbb{N}$, define $\widehat{Z}_N^{(k)}$ by enlarging the range of summation for $Z_N^{(k)}$ in (4.1), allowing the time increments $n_1, n_2 - n_1, \ldots, n_k - n_{k-1}$ to vary freely in $\{1, \ldots, N\}$, and show that $\|Z_N^{(k)} - \widehat{Z}_N^{(k)}\|_2^2 \to 0$ as $N \to \infty$.

Note that this allows us to replace $Z_{N,\beta_N}^{\omega,M}$ by

$$Z_{N,\beta_N}^{(\mathbf{A1})} := 1 + \sum_{k=1}^{M} \hat{\beta}^k \widehat{Z}_N^{(k)}, \quad \text{with} \quad \left\| Z_{N,\beta_N}^{\omega,M} - Z_{N,\beta_N}^{(\mathbf{A1})} \right\|_2^2 \underset{N\to\infty}{\longrightarrow} 0. \tag{4.4}$$

With the same M as before, for each $\widehat{Z}_N^{(k)}$, we then partition the range $\{1, \ldots, N\}$ for each variable $n_1, n_2 - n_1, \ldots, n_k - n_{k-1}$ into M blocks $I_1, I_2, \cdots, I_M$, defined by (with $\mathsf{t}_0 := 0$)

$$I_i := \left(\mathsf{t}_{i-1}, \mathsf{t}_i\right] \quad \text{with} \quad \mathsf{t}_i := \mathsf{t}_i^{N,M} := \min\left\{m \in \{1, \ldots, N\} \colon \mathbf{R}_m \geqslant \frac{i}{M}\mathbf{R}_N\right\}. \tag{4.5}$$

Note that for $\mathbf{R}_N = \log N$ we have $I_i = (N^{\frac{i-1}{M}}, N^{\frac{i}{M}}]$. We can then write

$$\widehat{Z}_N^{(k)} = \frac{1}{M^{\frac{k}{2}}} \sum_{1 \leqslant i_1, \ldots, i_k \leqslant M} \Theta_{i_1, \ldots, i_k}^{N;M}, \quad \text{where}$$

$$\Theta_{i_1, \ldots, i_k}^{N;M} := \left(\frac{M}{\mathbf{R}_N}\right)^{k/2} \sum_{\substack{n_1-n_0 \in I_{i_1}, n_2-n_1 \in I_{i_2}, \ldots, n_k-n_{k-1} \in I_{i_k} \\ z_1, z_2, \ldots, z_k \in \mathbb{Z}^d}} \prod_{j=1}^{k} q_{n_j-n_{j-1}}(z_j - z_{j-1}) \prod_{i=1}^{k} \eta_{(n_i,z_i)},$$

$$\tag{4.6}$$

where $(n_0, z_0) = (0,0)$.

Remark 4.1. The intervals $(I_i)_{1 \leqslant i \leqslant M}$ encode the right time scale, as explained in **(B)** in Section 3.2, because $\mathbf{R}_{\mathsf{t}_i} - \mathbf{R}_{\mathsf{t}_{i-1}} \sim \frac{1}{M}\mathbf{R}_N$. The sum over $i_1, \ldots, i_k$ in $\widehat{Z}_N^{(k)}$ in (4.6) corresponds to a discretization of the stochastic integrals that will arise in the limit $N \to \infty$.

To ensure a proper separation of scales later on, define

$$\{1..., M\}_{\sharp}^k := \{\boldsymbol{i} = (i_1, ..., i_k) \in \{1, ..., M\}^k : |i_j - i_{j'}| \geqslant 2 \text{ for all } j \neq j'\}. \tag{4.7}$$

Our second approximation shows that the contributions to $\widehat{Z}_N^{(k)}$ in (4.6) from summation indices $\boldsymbol{i} \in \{1, \ldots, M\}^k \backslash \{1, \ldots, M\}_{\sharp}^k$ is small for large M, uniformly in large N, i.e.,

(A2) $$\lim_{M\to\infty} \limsup_{N\to\infty} \left\| \sum_{\boldsymbol{i} \in \{1,...,M\}^k \backslash \{1,...,M\}_{\sharp}^k} \frac{1}{M^{\frac{k}{2}}} \Theta_{\boldsymbol{i}}^{N;M} \right\|_2^2 = 0.$$

Therefore we can restrict the sum over $\boldsymbol{i}$ in $\widehat{Z}_N^{(k)}$ to $\boldsymbol{i} \in \{1, \ldots, M\}_{\sharp}^k$. Note that this implies we can further replace $Z_{N,\beta_N}^{(\mathbf{A1})}$ in (4.4) by

$$Z_{N,\beta_N}^{(\mathbf{A2})} := 1 + \sum_{k=1}^{M} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\boldsymbol{i} \in \{1,...,M\}_{\sharp}^k} \Theta_{\boldsymbol{i}}^{N;M}, \quad \text{with} \quad \lim_{M\to\infty} \limsup_{N\to\infty} \|Z_{N,\beta_N}^{(\mathbf{A1})} - Z_{N,\beta_N}^{(\mathbf{A2})}\|_2^2 = 0. \tag{4.8}$$

We now try to identify the limit of $\Theta_{\boldsymbol{i}}^{N;M}$ as $N \to \infty$. The heuristics sketched in Section 3.2 for $Z_N^{(1)}$ and $Z_N^{(2)}$ suggest the following:

- Case $k = 1$: the family $(\Theta_i^{N;M})_{1 \leqslant i \leqslant M}$ converges in distribution to i.i.d. standard normal random variables $(\zeta_i)_{1 \leqslant i \leqslant M}$.

- Case $k = 2$: for $i_1 \leqslant i_2 - 2$, the family $\Theta_{i_1,i_2}^{N;M}$ converges in distribution to $\zeta_{i_1}\zeta_{i_2}$, while for $i_1 \geqslant i_2 + 2$, the family $\Theta_{i_1,i_2}^{N;M}$ converges in distribution to a family of i.i.d. standard normal random variables ζ_{i_1,i_2} independent of $(\zeta_i)_{1 \leqslant i \leqslant M}$.

For $k \geqslant 3$, the limit of $\Theta_{i_1,\ldots,i_k}^{N;M}$ also turns out to be a product of independent standard normal random variables $\zeta_{\cdot}$, with *one $\zeta_{\cdot}$ for each running maxima in the sequence* $(i_1,...,i_k)$. More precisely, let us say that

$$\boldsymbol{i} := (i_1, \ldots, i_k) \in \{1..., M\}^k \text{ is a } \textit{dominated sequence} \text{ if } i_1 > i_2, \ldots, i_k. \qquad (4.9)$$

Then each $\boldsymbol{i} \in \{1, \ldots, M\}_{\sharp}^k$ can be divided into consecutive dominated sequences $\boldsymbol{i}^{(1)} := (i_1, \ldots, i_{\ell_2-1})$, $\boldsymbol{i}^{(2)} := (i_{\ell_2}, \ldots, i_{\ell_3-1})$, $\ldots$, $\boldsymbol{i}^{(\mathfrak{m})} := (i_{\ell_{\mathfrak{m}}}, \ldots, i_k)$, where $i_{\ell_1} = i_1 < \cdots < i_{\ell_{\mathfrak{m}}}$ are the successive running maxima of $(i_1, \ldots, i_k)$.

Our third approximation step shows that the random variable $\Theta_{\boldsymbol{i}}^{N;M}$ in (4.6) admits the following asymptotic factorization:

(A3) For all $M, k \in \mathbb{N}$ and for each $\boldsymbol{i} := (i_1, \ldots, i_k) \in \{1..., M\}_{\sharp}^k$,

$$\lim_{N\to\infty} \left\| \Theta_{\boldsymbol{i}}^{N;M} - \Theta_{\boldsymbol{i}^{(1)}}^{N;M} \Theta_{\boldsymbol{i}^{(2)}}^{N;M} \cdots \Theta_{\boldsymbol{i}^{(\mathfrak{m})}}^{N;M} \right\|_2^2 = 0, \qquad (4.10)$$

where $(\boldsymbol{i}^{(1)}, \ldots, \boldsymbol{i}^{(\mathfrak{m})})$ is the decomposition of $\boldsymbol{i}$ into dominated sequences.

Note that this allows us to further replace $Z_{N,\beta_N}^{(\mathbf{A2})}$ in (4.8) by

$$Z_{N,\beta_N}^{(\mathbf{A3})} := 1 + \sum_{k=1}^{M} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\boldsymbol{i} \in \{1,\ldots,M\}_{\sharp}^k} \Theta_{\boldsymbol{i}^{(1)}}^{N;M} \Theta_{\boldsymbol{i}^{(2)}}^{N;M} \cdots \Theta_{\boldsymbol{i}^{(\mathfrak{m})}}^{N;M} \quad \text{with} \quad \|Z_{N,\beta_N}^{(\mathbf{A2})} - Z_{N,\beta_N}^{(\mathbf{A3})}\|_2^2 \xrightarrow[N\to\infty]{} 0. \qquad (4.11)$$

We are now reduced to identifying the limit of $\Theta_{\boldsymbol{i}}^{N;M}$ when $\boldsymbol{i}$ are dominated sequences. Denote

$$D_M := \left\{ \boldsymbol{i} \in \bigcup_{k=1}^{\infty} \{1..., M\}_{\sharp}^k : \boldsymbol{i} \text{ is a dominated sequence} \right\}. \qquad (4.12)$$

Here is the key step in the proof of Theorem 2.8:

(K) As $N \to \infty$, the family of random variables $(\Theta_{\boldsymbol{i}}^{N;M})_{\boldsymbol{i} \in D_M}$ converges in distribution to a family of i.i.d. standard normal random variables $(\zeta_{\boldsymbol{i}})_{\boldsymbol{i} \in D_M}$.

In particular, this implies that

$$Z_{N,\beta_N}^{(\mathbf{A3})} \xrightarrow[N\to\infty]{d} \boldsymbol{Z}_{\hat{\beta}}^{(M)} := 1 + \sum_{k=1}^{M} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\boldsymbol{i} \in \{1,\ldots,M\}_{\sharp}^k} \prod_{l=1}^{\mathfrak{m}(\boldsymbol{i})} \zeta_{\boldsymbol{i}^{(l)}}. \qquad (4.13)$$

The proof of Theorem 2.8 for $\hat{\beta} \in (0, 1)$ is then completed with the last step

(A4) $$\boldsymbol{Z}_{\hat{\beta}}^{(M)} \xrightarrow[M\to\infty]{d} \boldsymbol{Z}_{\hat{\beta}} = e^{\int_0^1 \frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2 t}} dW_t - \frac{1}{2} \int_0^1 \frac{\hat{\beta}^2}{1-\hat{\beta}^2 t} dt}.$$

We will prove the key step **(K)** in Section 5. The approximation steps **(A1)**–**(A4)** will be carried out in Section 6, which then implies Theorem 2.8. The main tool to prove **(K)** is a *fourth moment theorem* for polynomial chaos expansions, due to de Jong [dJ87, dJ90], Nualart and Peccati [NP05] and Nourdin, Peccati and Reinert [NPR10]. The following versions is an extension to random variables with possibly unbounded third moment, based on the Lindeberg principle proved in [CSZ13] (which extends [R79, MOO10]).

Theorem 4.2 (Fourth moment theorem). *For each $N \in \mathbb{N}$, let $(\eta_{N,t})_{t \in \mathbb{T}}$ be independent random variables with mean 0 and variance 1, indexed by a countable set $\mathbb{T}$. Assume that $(\eta_{N,t}^2)_{N \in \mathbb{N}, t \in \mathbb{T}}$ are uniformly integrable. Fix $k \in \mathbb{N}$ and $d_1, \ldots, d_k \in \mathbb{N}$. For each $1 \leqslant i \leqslant k$, let $\Phi_N^{(i)}(\eta_{N,\cdot})$ be a multi-linear polynomial in $(\eta_{N,t})_{t \in \mathbb{T}}$ of degree d_i, i.e.,*

$$\Phi_N^{(i)}(\eta_{N,\cdot}) = \sum_{I \subset \mathbb{T}, |I| = d_i} \phi_N^{(i)}(I) \prod_{t \in I} \eta_{N,t} \qquad \text{for some real-valued } \phi_N^{(i)}(\cdot).$$

Assume further that:

(i) *For all $1 \leqslant i, j \leqslant k$, $\mathbb{E}[\Phi_N^{(i)}(\zeta_{N,\cdot}) \Phi_N^{(j)}(\zeta_{N,\cdot})] \to V(i,j)$ for some matrix V as $N \to \infty$;*

(ii) *For each $1 \leqslant i \leqslant k$, $\mathbb{E}[\Phi_N^{(i)}(\xi_\cdot)^4] \to 3V(i,i)$ as $N \to \infty$, where we have replaced $(\eta_{N,t})_{t \in \mathbb{T}}$ by i.i.d. standard normal random variables $(\xi_t)_{t \in \mathbb{T}}$;*

(iii) *The maximal influence of each variable $\eta_{N,t}$ on the polynomials of degree one among $(\Phi_N^{(i)}(\eta_{N,\cdot}))_{1 \leqslant i \leqslant k}$ is asymptotically negligible, i.e., for each $1 \leqslant i \leqslant k$ with $d_i = 1$,*

$$\max_{t \in \mathbb{T}} \sum_{I \ni t, |I| = d_i} \phi_N^{(i)}(I)^2 \xrightarrow[N \to \infty]{} 0, \tag{4.14}$$

or more explicitly (since $d_i = 1$): $\max_{t \in \mathbb{T}} |\phi_N^{(i)}(\{t\})| \to 0$ as $N \to \infty$.

Then $(\Phi_N^{(i)}(\eta_{N,\cdot}))_{1 \leqslant i \leqslant k}$ converge in law to a centered Gaussian vector with covariance V.

Proof. If we replace $(\eta_{N,t})_{t \in \mathbb{T}}$ by standard Gaussians $(\xi_t)_{t \in \mathbb{T}_N}$, Theorem 4.2 holds without the need of assuming condition (iii), thanks to [NPR10, Theorem 7.6], which is a multi-dimensional extension of the fourth moment theorem [dJ90, NP05].

To justify the replacement with Gaussians, we show that the vectors $(\Phi_N^{(i)}(\eta_{N,\cdot}))_{1 \leqslant i \leqslant k}$ and $(\Phi_N^{(i)}(\zeta_\cdot))_{1 \leqslant i \leqslant k}$ have the same limit in law as $N \to \infty$. By the Crámer-Wold device, it is enough to consider a linear combination $\Phi_N = \sum_{i=1}^k c_i \Phi_N^{(i)}$, which is a multilinear polynomial with degree $d := \max_{1 \leqslant i \leqslant k} d_i$ and with variance $\sigma_N^2 := \sum_{i=1}^k c_i^2 \sum_{I \subseteq \mathbb{T}} \phi_N^{(i)}(I)^2$. By the Lindeberg principle in [CSZ13, Theorem 2.6], for any smooth and bounded $f : \mathbb{R} \to \mathbb{R}$ there is $C = C_{d,f} < \infty$ such that for every $M > 0$

$$\left| \mathbb{E}[f(\Phi_N(\eta_{N,\cdot}))] - \mathbb{E}[f(\Phi_N(\zeta_\cdot))] \right| \leqslant C \sigma_N^2 \left\{ m_2^{>M} + M^d \max_{t \in \mathbb{T}} \sqrt{\text{Inf}_t(\Phi_N)} \right\},$$

where $\qquad m_2^{>M} := \max_{X \in \bigcup_{N \in \mathbb{N}, t \in \mathbb{T}} \{\eta_{N,t}, \zeta_t\}} \mathbb{E}[X^2 \mathbb{1}_{\{|X| > M\}}], \qquad \text{Inf}_t(\Phi_N) = \sum_{i=1}^k \sum_{I \ni t} \phi_N^{(i)}(I)^2.$

By the uniform integrability assumption on $\eta_{N,t}^2$, we can fix $M > 0$ large enough so that $m_2^{>M}$ is as small as we wish. Since $\sup_{N \in \mathbb{N}} \sigma_N^2 < \infty$ by assumption (i), the proof is completed if we show that $\max_{t \in \mathbb{T}} \sqrt{\text{Inf}_t(\Phi_N)} \to 0$ as $N \to \infty$. For polynomials $\Phi_N^{(i)}$ of degree $d_i = 1$

this holds by assumption (iii), while for $d_i \geqslant 2$ it is a consequence of the fourth-moment assumption (ii), as shown in [NPR10, Proposition 1.6 and (1.9)]. $\qquad\square$

5. Proof of key step for Theorem 2.8

In this section we prove the key step **(K)** in the proof of Theorem 2.8, formulated in Section 4, which asserts that the building blocks of the chaos expansion have asymptotic Gaussian behavior. This result actually holds in great generality and is of independent interest, so it is worth stating explicitly the assumptions we need.

We work on $\mathbb{Z}^d$ for fixed $d \in \mathbb{N}_0$ (with $\mathbb{Z}^0 := \{0\}$). For every $n \in \mathbb{N}$, we fix a function $q_n(\cdot) \in L^2(\mathbb{Z}^d)$ —not necessarily a probability kernel— and we define (cf. (2.7))

$$\mathbf{R}_N := \sum_{n=1}^{N} \|q_n\|^2 = \sum_{n=1}^{N} \left(\sum_{x \in \mathbb{Z}^d} q_n(x)^2 \right). \tag{5.1}$$

In the following sections we will focus on the special cases when $q_n(x) = \mathrm{P}(S_n = x)$ or $q_n = \mathrm{P}(n \in \tau)$, with S or τ satisfying the local limit theorems in Hypothesis 2.4. However, in this section *we only need to assume that $\mathbf{R}_N$ is a slowly varying function which diverges as $N \to \infty$*. The basic case to keep in mind is $\mathbf{R}_N \sim C \log N$.

Let us fix $M \in \mathbb{N}$ and split

$$\{1, 2, \ldots, N\} = I_1 \cup I_2 \cup \cdots \cup I_M, \tag{5.2}$$

where the intervals I_i are defined by (4.5). This definition ensures that each I_i contributes equally to $\mathbf{R}_N$, since[†]

$$\sum_{m \in I_i} \|q_m\|^2 = \mathbf{R}_{\mathsf{t}_i} - \mathbf{R}_{\mathsf{t}_{i-1}} \underset{N \to \infty}{\sim} \frac{\mathbf{R}_N}{M}. \tag{5.3}$$

Definition 5.1. *Let $E^{\mathrm{fin}} := \bigcup_{k \in \mathbb{N}} E^k$ be the set of all* finite *sequences $\boldsymbol{z} = (z_1, \ldots, z_k)$ taking values in a given set E. For $\boldsymbol{z} \in E^k \subseteq E^{\mathrm{fin}}$, we denote by $|\boldsymbol{z}| = k$ the length of $\boldsymbol{z}$. We also let $\mathbb{N}_\uparrow^k$ be the subset of* increasing *sequences $\boldsymbol{n} \in \mathbb{N}^k$, i.e., $n_1 < n_2 < \ldots < n_k$, and analogously we set $\mathbb{N}_\uparrow^{\mathrm{fin}} := \bigcup_{k \in \mathbb{N}} \mathbb{N}_\uparrow^k$.*

We focus on the random variables $\Theta_{i_1,\ldots,i_k}^{N;M}$ introduced in (4.6), which can be conveniently reformulated as follows. Given $\boldsymbol{i} \in \{1, \ldots, M\}^{\mathrm{fin}}$, we define a set of increasing sequences $\boldsymbol{n} \in \mathbb{N}_\uparrow^{\mathrm{fin}}$ *that are compatible with $\boldsymbol{i}$*, denoted by $\boldsymbol{n} \prec \boldsymbol{i}$, as follows:

$$\boldsymbol{n} \prec \boldsymbol{i} \quad \Longleftrightarrow \quad |\boldsymbol{n}| = |\boldsymbol{i}| \quad \text{and} \quad n_1 - n_0 \in I_{i_1}, \ \ldots, \ n_{|\boldsymbol{i}|} - n_{|\boldsymbol{i}|-1} \in I_{i_{|\boldsymbol{i}|}}, \tag{5.4}$$

where $n_0 = 0$. We can then define

$$\Theta_{\boldsymbol{i}}^{N,M} := \left(\frac{M}{\mathbf{R}_N} \right)^{\frac{|\boldsymbol{i}|}{2}} \sum_{\boldsymbol{n} \in \mathbb{N}_\uparrow^{\mathrm{fin}}: \ \boldsymbol{n} \prec \boldsymbol{i}} Q_{\boldsymbol{n}}, \qquad \text{with} \tag{5.5}$$

$$Q_{\boldsymbol{n}} := \sum_{\boldsymbol{x} \in (\mathbb{Z}^d)^{|\boldsymbol{n}|}} \prod_{j=1}^{|\boldsymbol{n}|} q_{n_j - n_{j-1}}(x_j - x_{j-1}) \, \eta_{(n_j, x_j)} = \sum_{\boldsymbol{x} \in (\mathbb{Z}^d)^{|\boldsymbol{n}|}} q_{\boldsymbol{n}}(\boldsymbol{x}) \, \eta_{(\boldsymbol{n}, \boldsymbol{x})}, \tag{5.6}$$

[†]Note that $\mathbf{R}_n - \mathbf{R}_{n-1} = o(\mathbf{R}_n)$ as $n \to \infty$, by the slowly varying property, hence $\mathbf{R}_{\mathsf{t}_i} \sim \frac{i}{M} \mathbf{R}_N$.

where $n_0 = x_0 = 0$ and we have introduced the further abbreviations

$$q_{\boldsymbol{n}}(\boldsymbol{x}) := \prod_{j=1}^{|\boldsymbol{n}|} q_{n_j - n_{j-1}}(x_j - x_{j-1}), \qquad \eta_{(\boldsymbol{n}, \boldsymbol{x})} := \prod_{j=1}^{|\boldsymbol{n}|} \eta_{(n_j, x_j)}. \tag{5.7}$$

Here $(\eta_{(n,x)})_{(n,x) \in \mathbb{N} \times \mathbb{Z}^d}$ are independent random variables with $\mathbb{E}[\eta_{(n,x)}] = 0$, $\mathbb{V}\text{ar}[\eta_{(n,x)}] = 1$.

We allow $\eta_{(n,x)} = \eta_{(n,x)}^{(N)}$ to depend on $N \in \mathbb{N}$, as in (4.2). We only need to assume that *the squares* $((\eta_{(n,x)}^{(N)})^2)_{N \in \mathbb{N}, (n,x) \in \mathbb{N} \times \mathbb{Z}^d}$ *are uniformly integrable*. Note that this holds for (4.2), as one easily checks by showing boundedness of $\mathbb{E}[(\eta_{(n,x)}^{(N)})^4]$, see [CSZ13, eq. (6.27)].

We now state our main result, which generalizes the key step **(K)** in Section 4. Recall that the space $D_M \subseteq \{1, \dots, M\}^{\text{fin}}$ of *dominated sequences* is defined as (cf. (4.7) and (4.12)):

$$D_M := \left\{ \boldsymbol{i} \in \{1, \dots, M\}^{\text{fin}} : \quad i_1 > i_2, i_3, \dots, i_{|\boldsymbol{i}|}, \quad |i_j - i_{j'}| \geqslant 2 \; \forall j \neq j' \right\}. \tag{5.8}$$

Proposition 5.2. *Assume that* $\mathbf{R}_N$ *in (5.1) is a slowly varying function which diverges as* $N \to \infty$. *For every fixed* $M \in \mathbb{N}$, *the random variables* $(\Theta_{\boldsymbol{i}}^{N,M})_{\boldsymbol{i} \in D_M}$ *indexed by dominated sequences in* D_M, *converge jointly in law as* $N \to \infty$ *to i.i.d. standard Gaussians* $(\zeta_{\boldsymbol{i}})_{\boldsymbol{i} \in D_M}$.

Proof. We observe that, by (5.6),

$$\mathbb{E}[Q_{\boldsymbol{n}}] = 0, \qquad \mathbb{E}[Q_{\boldsymbol{n}} Q_{\boldsymbol{n}'}] = \|q_{\boldsymbol{n}}\|^2 \, \mathbb{1}_{\{\boldsymbol{n} = \boldsymbol{n}'\}}, \tag{5.9}$$

$$\text{where} \qquad \|q_{\boldsymbol{n}}\|^2 := \prod_{j=1}^{|\boldsymbol{n}|} \|q_{n_j - n_{j-1}}\|^2 = \prod_{j=1}^{|\boldsymbol{n}|} \left(\sum_{x \in \mathbb{Z}^d} q_{n_j - n_{j-1}}(x)^2 \right). \tag{5.10}$$

We stress that $\mathbb{E}[Q_{\boldsymbol{n}} Q_{\boldsymbol{n}'}] = 0$ for $\boldsymbol{n} \neq \boldsymbol{n}'$, because $\boldsymbol{n} = (n_1, \dots, n_{|\boldsymbol{n}|})$ then contains some value, say n_j, which does not appear in $\boldsymbol{n}'$ (or the other way around), so the random variables $\eta_{(n_j, x_j)}$ appearing in the product $Q_{\boldsymbol{n}} Q_{\boldsymbol{n}'}$ are unpaired and the expectation yields zero.

It is now easy to see that the random variables $\Theta_{\boldsymbol{i}}^{N,M}$, for $\boldsymbol{i} \in D_M$, are uncorrelated and have asymptotically (as $N \to \infty$ for fixed M) unit variance. In fact

$$\mathbb{E}[\Theta_{\boldsymbol{i}}^{N,M} \Theta_{\boldsymbol{i}'}^{N,M}] = 0 \qquad \forall \boldsymbol{i} \neq \boldsymbol{i}',$$

because if $\boldsymbol{n} \prec \boldsymbol{i}$ and $\boldsymbol{n}' \prec \boldsymbol{i}'$, then $\boldsymbol{n} \neq \boldsymbol{n}'$ by (5.4), and hence $\mathbb{E}[Q_{\boldsymbol{n}} Q_{\boldsymbol{n}'}] = 0$ by (5.9). Next, by (5.5) and (5.9),

$$\mathbb{E}[(\Theta_{\boldsymbol{i}}^{N,M})^2] = \left(\frac{M}{\mathbf{R}_N} \right)^{|\boldsymbol{i}|} \sum_{\boldsymbol{n} \prec \boldsymbol{i}} \|q_{\boldsymbol{n}}\|^2 = \left(\frac{M}{\mathbf{R}_N} \right)^{|\boldsymbol{i}|} \prod_{j=1}^{|\boldsymbol{i}|} \left(\sum_{m \in I(i_j)} \|q_m\|^2 \right) \xrightarrow[N \to \infty]{} 1, \tag{5.11}$$

where in the last step we used (5.3).

We now apply the multi-dimensional version of the fourth moment theorem, Theorem 4.2, to prove that $(\Theta_{\boldsymbol{i}}^{N,M})_{\boldsymbol{i} \in D_M}$ converge to i.i.d. standard Gaussians. We have just verified condition (i) in Theorem 4.2, and the influence condition (iii) on $\Theta_{\boldsymbol{i}}^{N,M}$ with $|\boldsymbol{i}| = 1$ clearly holds. It only remains to verify condition (ii), i.e., assuming that $(\eta_{(n,x)})_{(n,x) \in \mathbb{N} \times \mathbb{Z}^d}$ are i.i.d. standard normal, we need to show that

$$\lim_{N \to \infty} \mathbb{E}[(\Theta_{\boldsymbol{i}}^{N,M})^4] = 3 \qquad \forall \boldsymbol{i} \in D_M. \tag{5.12}$$

Recalling (5.5), we can write

$$\mathbb{E}\big[(\Theta_{\boldsymbol{i}}^{N;M})^4\big] = \left(\frac{M}{\mathbf{R}_N}\right)^{2|\boldsymbol{i}|} \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d} \prec \boldsymbol{i}} \mathbb{E}[Q_{\boldsymbol{a}}\,Q_{\boldsymbol{b}}\,Q_{\boldsymbol{c}}\,Q_{\boldsymbol{d}}]\,, \tag{5.13}$$

where by (5.6) and (5.7),

$$\mathbb{E}[Q_{\boldsymbol{a}}\,Q_{\boldsymbol{b}}\,Q_{\boldsymbol{c}}\,Q_{\boldsymbol{d}}] = \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}\in(\mathbb{Z}^d)^{|\boldsymbol{i}|}} q_{\boldsymbol{a}}(\boldsymbol{x})\,q_{\boldsymbol{b}}(\boldsymbol{y})\,q_{\boldsymbol{c}}(\boldsymbol{z})\,q_{\boldsymbol{d}}(\boldsymbol{w})\,\mathbb{E}\big[\eta_{(\boldsymbol{a},\boldsymbol{x})}\,\eta_{(\boldsymbol{b},\boldsymbol{y})}\,\eta_{(\boldsymbol{c},\boldsymbol{z})}\,\eta_{(\boldsymbol{d},\boldsymbol{w})}\big]. \tag{5.14}$$

Let $(\boldsymbol{a},\boldsymbol{x}) = ((a_1,x_1),(a_2,x_2),\ldots,(a_{|\boldsymbol{i}|},x_{|\boldsymbol{i}|})) \in (\mathbb{N}\times\mathbb{Z}^d)^{|\boldsymbol{i}|}$ denote the sequence of space-time points determined by $\boldsymbol{a}$ and $\boldsymbol{x}$, and let $\mathsf{p} \in \mathbb{N}$ be the number of *distinct space-time points* in the union of the four sequences $(\boldsymbol{a},\boldsymbol{x})$, $(\boldsymbol{b},\boldsymbol{y})$, $(\boldsymbol{c},\boldsymbol{z})$, $(\boldsymbol{d},\boldsymbol{w})$:

$$\mathsf{p} := \big|(\boldsymbol{a},\boldsymbol{x})\cup(\boldsymbol{b},\boldsymbol{y})\cup(\boldsymbol{c},\boldsymbol{z})\cup(\boldsymbol{d},\boldsymbol{w})\big| = \sum_{(n,r)\in\mathbb{N}\times\mathbb{Z}^d} \mathbb{1}_{\{(n,r)\in(\boldsymbol{a},\boldsymbol{x})\cup(\boldsymbol{b},\boldsymbol{y})\cup(\boldsymbol{c},\boldsymbol{z})\cup(\boldsymbol{d},\boldsymbol{w})\}}\,. \tag{5.15}$$

The first step toward (5.12) is to show that we can restrict the two sums in (5.13)-(5.14) to configurations of $(\boldsymbol{a},\boldsymbol{x}),(\boldsymbol{b},\boldsymbol{y}),(\boldsymbol{c},\boldsymbol{z}),(\boldsymbol{d},\boldsymbol{w})$ satisfying

$$\mathsf{p} = 2|\boldsymbol{i}|\,. \tag{5.16}$$

Indeed, we can rule out the two cases $\mathsf{p} > 2|\boldsymbol{i}|$ and $\mathsf{p} < 2|\boldsymbol{i}|$ as follows.

Case 1. $\mathsf{p} > 2|\boldsymbol{i}|$. Since there are $4|\boldsymbol{i}|$ space-time points (including multiplicity) in the four sequences $(\boldsymbol{a},\boldsymbol{x})$, $(\boldsymbol{b},\boldsymbol{y})$, $(\boldsymbol{c},\boldsymbol{z})$, $(\boldsymbol{d},\boldsymbol{w})$, there must be at least one space-time point, say (a_m,x_m), which will not be matched in pair with one of the elements in $(\boldsymbol{b},\boldsymbol{y})\cup(\boldsymbol{c},\boldsymbol{z})\cup(\boldsymbol{d},\boldsymbol{w})$. Then the expectation in (5.14) vanishes because $\eta_{(a_m,x_m)}$ is not paired to any other η random variable in $\eta_{(\boldsymbol{b},\boldsymbol{y})}$, $\eta_{(\boldsymbol{c},\boldsymbol{z})}$ or $\eta_{(\boldsymbol{d},\boldsymbol{w})}$ (recall (5.7)). Therefore the contribution to the sum in (5.13) is zero in this case. $\qquad\square$

Case 2. $\mathsf{p} < 2|\boldsymbol{i}|$. Recalling (5.15), for each $1 \leqslant p < 2|\boldsymbol{i}|$, set

$$\mathcal{S}_N^{(p)} := \left(\frac{M}{\mathbf{R}_N}\right)^{2|\boldsymbol{i}|} \sum_{\substack{(\boldsymbol{a},\boldsymbol{x}),(\boldsymbol{b},\boldsymbol{y}),(\boldsymbol{c},\boldsymbol{z}) \\ (\boldsymbol{d},\boldsymbol{w})\ \text{such that } \mathsf{p}=p}} q_{\boldsymbol{a}}(\boldsymbol{x})\,q_{\boldsymbol{b}}(\boldsymbol{y})\,q_{\boldsymbol{c}}(\boldsymbol{z})\,q_{\boldsymbol{d}}(\boldsymbol{w})\,\mathbb{E}\big[\eta_{(\boldsymbol{a},\boldsymbol{x})}\,\eta_{(\boldsymbol{b},\boldsymbol{y})}\,\eta_{(\boldsymbol{c},\boldsymbol{z})}\,\eta_{(\boldsymbol{d},\boldsymbol{w})}\big]. \tag{5.17}$$

It suffices to show that $\mathcal{S}_N^{(p)} \to 0$ as $N \to \infty$, for each $p < 2|\boldsymbol{i}|$.

To lighten notation, we assume that $q_n(x) \geqslant 0$ (just replace $q_n(x)$ by $|q_n(x)|$ in the following arguments). Furthermore, we first consider the simplifying case when

$$q_n(x) \leqslant 1\,. \tag{5.18}$$

We will use the fact that $\mathbb{E}\big[\eta_{(\boldsymbol{a},\boldsymbol{x})}\,\eta_{(\boldsymbol{b},\boldsymbol{y})}\,\eta_{(\boldsymbol{c},\boldsymbol{z})}\,\eta_{(\boldsymbol{d},\boldsymbol{w})}\big] = 0$ unless the individual η variables match in pairs or quadruples, since we have assumed the η's to be i.i.d. standard normals in our attempt to verify Theorem 4.2 (ii). In any event, note that

$$\big|\mathbb{E}\big[\eta_{(\boldsymbol{a},\boldsymbol{x})}\,\eta_{(\boldsymbol{b},\boldsymbol{y})}\,\eta_{(\boldsymbol{c},\boldsymbol{z})}\,\eta_{(\boldsymbol{d},\boldsymbol{w})}\big]\big| \leqslant 3. \tag{5.19}$$

If $\mathsf{p} = p$, then we can relabel $(\boldsymbol{a},\boldsymbol{x})\cup(\boldsymbol{b},\boldsymbol{y})\cup(\boldsymbol{c},\boldsymbol{z})\cup(\boldsymbol{d},\boldsymbol{w}) = \{(f_1,h_1),(f_2,h_2),\ldots,(f_p,h_p)\}$, with $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_p$, and we set $f_0 := 0$ (since (f_i,h_i) are distinct space-time points, when $f_i = f_{i+1}$ we must have $h_i \neq h_{i+1}$). The sums in (5.13) and (5.14) can then be rewritten as sums over $(f_j,h_j)_{1\leqslant j\leqslant p}$, with another sum over all admissible assignments of $(\boldsymbol{a},\boldsymbol{x}),(\boldsymbol{b},\boldsymbol{y}),(\boldsymbol{c},\boldsymbol{z}),(\boldsymbol{d},\boldsymbol{w})$ to points in $(f_j,h_j)_{1\leqslant j\leqslant p}$.

We start by summing over all admissible values of (f_p, h_p). Denoting by $m \in \{2, 4\}$ the number of space-time points in $(\boldsymbol{a}, \boldsymbol{x})$, $(\boldsymbol{b}, \boldsymbol{y})$, $(\boldsymbol{c}, \boldsymbol{z})$, $(\boldsymbol{d}, \boldsymbol{w})$ assigned to (f_p, h_p) (for $m \in \{1, 3\}$, the expectation in (5.14) vanishes). The factors in (5.14) involving (f_p, h_p) are

$$\prod_{i=1}^{m} q_{f_p - f_{r_i}}(h_p - h_{r_i})$$

for some $r_1, \ldots, r_m \in \{0, 1, \ldots, p-1\}$. Using the assumption (5.18) that $q_n \leqslant 1$, we get

$$\sum_{(f_p, h_p)} \prod_{i=1}^{m} q_{f_p - f_{r_i}}(h_p - h_{r_i}) \leqslant \sum_{(f_p, h_p)} q_{f_p - f_{r_1}}(h_p - h_{r_1})\, q_{f_p - f_{r_2}}(h_p - h_{r_2})$$

$$\leqslant \left(\sum_{(f_p, h_p)} q_{f_p - f_{r_1}}(h_p - h_{r_1})^2 \right)^{1/2} \left(\sum_{(f_p, h_p)} q_{f_p - f_{r_2}}(h_p - h_{r_2})^2 \right)^{1/2} \qquad (5.20)$$

$$\leqslant \sum_{1 \leqslant n \leqslant N} \sum_{x \in \mathbb{Z}^d} q_n^2(x) = \mathbf{R}_N \,.$$

The last inequality holds because the range of $f_p - f_{r_i}$ is contained in $\{1, \ldots, N\}$, by (5.4).

We can iterate this estimate, summing successively over (f_{p-1}, h_{p-1}), (f_{p-2}, h_{p-2}), ..., (f_1, h_1). This, together with (5.19), shows that for fixed $M \in \mathbb{N}$, as $N \to \infty$,

$$\mathcal{S}_N^{(p)} \leqslant 3\hat{\mathrm{C}} \left(\frac{M}{\mathbf{R}_N} \right)^{2|\boldsymbol{i}|} \mathbf{R}_N^p = O\!\left(\mathbf{R}_N^{p - 2|\boldsymbol{i}|} \right), \qquad (5.21)$$

where $\hat{\mathrm{C}}$ depends only on $|\boldsymbol{i}|$ and p and bounds the number of ways of assigning $(\boldsymbol{a}, \boldsymbol{x})$, $(\boldsymbol{b}, \boldsymbol{y})$, $(\boldsymbol{c}, \boldsymbol{z})$, $(\boldsymbol{d}, \boldsymbol{w})$ to $(f_\ell, h_\ell)_{1 \leqslant \ell \leqslant p}$. Since $p < 2|\boldsymbol{i}|$, relation (5.21) shows that $\mathcal{S}_N^{(p)}$ converges to zero as N tends to infinity.

We now show how to remove the assumption $q_n \leqslant 1$ in (5.18). Setting

$$\|q_n\|_\infty := \max_{x \in \mathbb{Z}^d} q_n(x)\,, \qquad (5.22)$$

the r.h.s. of (5.20) is replaced by $\mathbf{R}_N\big(\max_{1 \leqslant n \leqslant N} \|q_n\|_\infty^{m-2} \big)$. As we sum over (f_p, h_p), ..., (f_1, p_1), we collect exactly $4|\boldsymbol{i}| - 2p$ factors of $\max_{1 \leqslant n \leqslant N} \|q_n\|_\infty$. Consequently, (5.21) becomes

$$\mathcal{S}_N^{(p)} = O\!\left(\mathbf{R}_N^{p - 2|\boldsymbol{i}|} \big(\max_{1 \leqslant n \leqslant N} \|q_n\|_\infty \big)^{4|\boldsymbol{i}| - 2p} \right). \qquad (5.23)$$

However, by (5.1) and (5.22),

$$\max_{1 \leqslant n \leqslant N} \|q_n\|_\infty^2 \leqslant \max_{1 \leqslant n \leqslant N} (\mathbf{R}_n - \mathbf{R}_{n-1}) = o(\mathbf{R}_N), \qquad (5.24)$$

since $\mathbf{R}_N$ is slowly varying and divergent. Therefore $\mathcal{S}_N^{(p)} \to 0$ also in the general case. $\qquad \square$

Continuing with the proof of (5.12), we may now restrict the sums in (5.13) and (5.14) to configurations satisfying $\mathsf{p} = 2|\boldsymbol{i}|$ (recall (5.15)). This means that *the $4|\boldsymbol{i}|$ space-time points among $(\boldsymbol{a}, \boldsymbol{x})$, $(\boldsymbol{b}, \boldsymbol{y})$, $(\boldsymbol{c}, \boldsymbol{z})$, $(\boldsymbol{d}, \boldsymbol{w})$ match exactly in pairs* (i.e. coincide two by two).

As before, let $(f_i, h_i)_{1 \leqslant i \leqslant p}$, with $f_1 \leqslant f_2 \leqslant \ldots \leqslant f_p$ and $p = 2|\boldsymbol{i}|$, be the distinct space-time points occupied by $(\boldsymbol{a}, \boldsymbol{x}) \cup (\boldsymbol{b}, \boldsymbol{y}) \cup (\boldsymbol{c}, \boldsymbol{z}) \cup (\boldsymbol{d}, \boldsymbol{w})$. In principle one could have $f_i = f_{i+1}$ (necessarily with $h_i \neq h_{i+1}$), but such configurations give a negligible contribution in (5.13), because this leaves at most $p-1$ free coordinates f_j to sum over, each of which gives by (5.20) a contribution of at most $\mathbf{R}_N$ (assuming $q_n \leqslant 1$; otherwise use (5.24)), while the prefactor in (5.17) decays as $\mathbf{R}_N^{-p}$. As a consequence, we may assume that $f_1 < f_2 < \ldots < f_p$, which means that *the time points among $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ have to match exactly in pairs*.

We now make a further restriction. Let $[\boldsymbol{a}] := [a_1, a_{|\boldsymbol{i}|}] \subseteq \mathbb{R}$ be the smallest interval containing all the points in the (increasing) sequence $\boldsymbol{a} = (a_1, a_2, \ldots, a_{|\boldsymbol{i}|})$. Then $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ is a union of disjoint closed intervals (*connected components*) whose number can range from one to four. We now show that we can restrict the sum in (5.13) to configurations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ with *exactly two connected components*. We distinguish between two cases.

Case 3. *Three or four connected components.* Since $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = |\boldsymbol{d}| = |\boldsymbol{i}|$, we must have

$$|\boldsymbol{a} \cup \boldsymbol{b} \cup \boldsymbol{c} \cup \boldsymbol{d}| \geqslant 3|\boldsymbol{i}|,$$

therefore also $\mathsf{p} \geqslant 3|\boldsymbol{i}|$, cf. (5.15), which has been excluded in **Case 1**. $\square$

Case 4. *One connected component.* Similar to (5.17), it suffices to focus on

$$\hat{\mathcal{S}}_N := \left(\frac{M}{\mathbf{R}_N} \right)^{2|\boldsymbol{i}|} \sum_{\substack{(\boldsymbol{a},\boldsymbol{x}),(\boldsymbol{b},\boldsymbol{y}),(\boldsymbol{c},\boldsymbol{z}),(\boldsymbol{d},\boldsymbol{w}) \\ \text{matching in pairs and} \\ \text{forming one connected component}}} q_{\boldsymbol{a}}(\boldsymbol{x})\, q_{\boldsymbol{b}}(\boldsymbol{y})\, q_{\boldsymbol{c}}(\boldsymbol{z})\, q_{\boldsymbol{d}}(\boldsymbol{w}) \qquad (5.25)$$

and show that $\hat{\mathcal{S}}_N \to 0$ as $N \to \infty$ (note that $\mathbb{E}\big[\eta_{(\boldsymbol{a},\boldsymbol{x})}\, \eta_{(\boldsymbol{b},\boldsymbol{y})}\, \eta_{(\boldsymbol{c},\boldsymbol{z})}\, \eta_{(\boldsymbol{d},\boldsymbol{w})}\big] = 1$ because of the "matching in pairs" condition). We will show that the "one connected component" condition effectively leads to the loss of a degree of freedom in the summation.

Without loss of generality, assume that $a_1 = \min\{a_1, b_1, c_1, d_1\}$ is the smallest among all time indices in $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$. Then it has to match either b_1, c_1 or d_1. Say $a_1 = b_1 = f_1$. It follows that $c_1 = f_u$ for some $u \in \{2, \ldots, p\}$. The constraints of matching in pairs and $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ having *one connected component* imply that either $c_1 \leqslant a_{\bar{k}}$ or $c_1 \leqslant b_{\bar{k}}$ for some $\bar{k} \geqslant 2$; w.l.o.g., assume that $c_1 \leqslant a_{\bar{k}}$. Since $\boldsymbol{a} < \boldsymbol{i}$, by (4.5) and (5.4), this implies

$$f_1 = a_1 \;<\; f_u = c_1 \leqslant\; a_{\bar{k}} \leqslant\; a_1 + \mathsf{t}_{i_2} + \mathsf{t}_{i_3} + \ldots + \mathsf{t}_{i_{\bar{k}}}$$
$$\leqslant\; f_1 + (\bar{k} - 1)\, \mathsf{t}_{i_1 - 2} \;\leqslant\; f_1 + |\boldsymbol{i}|\, \mathsf{t}_{i_1 - 2}\,,$$

where the last inequality holds because $i_\ell \leqslant i_1 - 2$ for all $\ell \in \{2, \ldots, |\boldsymbol{i}|\}$, since $\boldsymbol{i}$ is a dominated sequence, cf. (5.8). Also note $f_1 = a_1 \geqslant \mathsf{t}_{i_1 - 1}$, again by (4.5) and (5.4). Therefore

$$f_1 = a_1 \geqslant \mathsf{t}_{i_1 - 1} \qquad \text{and} \qquad f_u = c_1 \in (f_1, f_1 + \bar{m}_1]\,, \qquad \text{where} \quad \bar{m}_1 := |\boldsymbol{i}|\, \mathsf{t}_{i_1 - 2}\,. \quad (5.26)$$

We can now sum (5.25) over the variables $(f_1, h_1), \ldots, (f_p, h_p)$ subject to (5.26) for some $2 \leqslant u \leqslant p$. The sum over (f_p, h_p) has already been estimated in (5.20) with $m = 2$, and is bounded by $\mathbf{R}_N$. The same bound $\mathbf{R}_N$ applies to the sum over (f_ℓ, h_ℓ) for each $\ell = p - 1, p - 2, \ldots, u + 1$. The sum over (f_u, h_u), in view of (5.26), is bounded by

$$\sum_{f_u \in (f_1, f_1 + \bar{m}_1]} \sum_{h_u \in \mathbb{Z}^d} q_{f_u - f_{r_1}}(h_u - h_{r_1})\, q_{f_u - f_{r_2}}(h_u - h_{r_2})\,,$$

for some $r_1, r_2 \in \{0, \ldots, u-1\}$. Since $f_u = c_1$ is the first index of the sequence $\boldsymbol{c}$, we have either $r_1 = 0$ or $r_2 = 0$; w.l.o.g., assume $r_1 = 0$. We then have (recall (5.7))

$$
\sum_{\substack{f_u \in (f_1, f_1 + \bar{m}_1] \\ h_u \in \mathbb{Z}^d}} q_{f_u}(h_u) \, q_{f_u - f_{r_2}}(h_u - h_{r_2})
$$

$$
\leqslant \left(\sum_{\substack{f_u \in (f_1, f_1 + \bar{m}_1] \\ h_u \in \mathbb{Z}^d}} q_{f_u}(h_u)^2 \right)^{1/2} \left(\sum_{\substack{f_u \in (f_1, f_1 + \bar{m}_1] \\ h_u \in \mathbb{Z}^d}} q_{f_u - f_{r_2}}(h_u - h_{r_2})^2 \right)^{1/2} \tag{5.27}
$$

$$
\leqslant \left(\sum_{\substack{f_1 < n \leqslant f_1 + \bar{m}_1 \\ x \in \mathbb{Z}^d}} q_n^2(x) \right)^{1/2} \left(\sum_{\substack{1 \leqslant n \leqslant N \\ x \in \mathbb{Z}^d}} q_n^2(x) \right)^{1/2} = \sqrt{\mathbf{R}_{f_1 + \bar{m}_1} - \mathbf{R}_{f_1}} \sqrt{\mathbf{R}_N} \, .
$$

Let us recall from (4.5) that $\mathsf{t}_i = \mathsf{t}_i^{N,M}$ satisfies $\mathbf{R}_{\mathsf{t}_i} \sim \frac{i}{M} \mathbf{R}_N$ as $N \to \infty$ (for fixed $M \in \mathbb{N}$). It follows that if $j < i$, then $\mathsf{t}_j = o(\mathsf{t}_i)$ as $N \to \infty$.[†] Since $\bar{m}_1 = |\boldsymbol{i}| \mathsf{t}_{i_1 - 2}$ by (5.26) while $f_1 = a_1 \geqslant \mathsf{t}_{i_1 - 1}$, it follows that $\bar{m}_1 = o(f_1)$, and hence $\mathbf{R}_{f_1 + \bar{m}_1} \sim \mathbf{R}_{f_1}$. This implies that the r.h.s. of (5.27) equals $\sqrt{o(1) \, \mathbf{R}_{f_1}} \sqrt{\mathbf{R}_N} = o(1) \, \mathbf{R}_N$ as $N \to \infty$.

We can now sum over the remaining variables (f_ℓ, h_ℓ) for $\ell = u-1, u-2, \ldots, 1$ as we did before, with each sum bounded by $\mathbf{R}_N$ as shown in (5.20), which gives

$$
\hat{\mathcal{S}}_N \leqslant \hat{\mathrm{C}} \left(\frac{M}{\mathbf{R}_N} \right)^{2|\boldsymbol{i}|} \mathbf{R}_N^p \, o(1) \, , \tag{5.28}
$$

where $\hat{\mathrm{C}}$ is again a combinatorial factor independent of N. Since $p = 2|\boldsymbol{i}|$, for any fixed $M \in \mathbb{N}$, the r.h.s. of (5.28) vanishes as $N \to \infty$. $\qquad\square$

To complete the proof of (5.12), it only remains to show that (5.12) holds if the joint sums in (5.13) and (5.14) are restricted such that $|(\boldsymbol{a}, \boldsymbol{x}) \cup (\boldsymbol{b}, \boldsymbol{y}) \cup (\boldsymbol{c}, \boldsymbol{z}) \cup (\boldsymbol{d}, \boldsymbol{w})| = 2|\boldsymbol{i}|$ and $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ contains two connected components.

Case 5. *Two connected components.* In this case, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ must coincide two by two, i.e.,

$$
\boldsymbol{a} = \boldsymbol{b} \text{ and } \boldsymbol{c} = \boldsymbol{d}, \quad \text{or} \quad \boldsymbol{a} = \boldsymbol{c} \text{ and } \boldsymbol{b} = \boldsymbol{d}, \quad \text{or} \quad \boldsymbol{a} = \boldsymbol{d} \text{ and } \boldsymbol{b} = \boldsymbol{c}. \tag{5.29}
$$

This extends further to $(\boldsymbol{a}, \boldsymbol{x})$, $(\boldsymbol{b}, \boldsymbol{y})$, $(\boldsymbol{c}, \boldsymbol{z})$, $(\boldsymbol{d}, \boldsymbol{w})$. By symmetry, each of the three cases gives the same contribution, which leads to the factor 3 in the r.h.s. of (5.12). We can thus focus on the case $(\boldsymbol{a}, \boldsymbol{x}) = (\boldsymbol{b}, \boldsymbol{y}) \neq (\boldsymbol{c}, \boldsymbol{z}) = (\boldsymbol{d}, \boldsymbol{w})$.

Restricting the sums in (5.13) and (5.14) to $(\boldsymbol{a}, \boldsymbol{x}) = (\boldsymbol{b}, \boldsymbol{y}) \neq (\boldsymbol{c}, \boldsymbol{z}) = (\boldsymbol{d}, \boldsymbol{w})$, we obtain

$$
\left(\frac{M}{\mathbf{R}_N} \right)^{2|\boldsymbol{i}|} \sum_{\substack{\boldsymbol{a}, \boldsymbol{c} < \boldsymbol{i}, \ \boldsymbol{x}, \boldsymbol{z} \in (\mathbb{Z}^d)^{|\boldsymbol{i}|} \\ [\boldsymbol{a}] \cap [\boldsymbol{c}] = \varnothing}} q_{\boldsymbol{a}}(\boldsymbol{x})^2 \, q_{\boldsymbol{c}}(\boldsymbol{z})^2 \, . \tag{5.30}
$$

Note that if we ignore the restriction $[\boldsymbol{a}] \cap [\boldsymbol{c}] = \varnothing$, then the sum factorizes and we obtain

$$
\left(\frac{M}{\mathbf{R}_N} \right)^{2|\boldsymbol{i}|} \left(\sum_{\boldsymbol{a} < \boldsymbol{i}, \ \boldsymbol{x} \in \mathbb{Z}^d} q_{\boldsymbol{a}}(\boldsymbol{x})^2 \right) \left(\sum_{\boldsymbol{c} < \boldsymbol{i}, \ \boldsymbol{z} \in \mathbb{Z}^d} q_{\boldsymbol{c}}(\boldsymbol{z})^2 \right) \xrightarrow[N \to \infty]{} 1 \tag{5.31}
$$

[†]If $\mathsf{t}_j \in [\varepsilon \, \mathsf{t}_i, \mathsf{t}_i]$ for $\varepsilon > 0$, the slowly varying property of $\mathbf{R}_N$ would yield $\mathbf{R}_{\mathsf{t}_j} \sim \mathbf{R}_{\mathsf{t}_i}$, contradicting (4.5).

by the same variance calculation as in (5.11). This proves (5.12), because the terms in (5.30) with $[\boldsymbol{a}] \cap [\boldsymbol{c}] \neq \varnothing$ are negligible by the same bounds as in **Case 4** (cf. (5.27)), where $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ contains one connected component. $\qquad\square$

6. Proof of Theorem 2.8

In this section we will first prove the approximation steps **(A1)–(A4)** outlined in Section 4, and then conclude the proof of Theorem 2.8.

Recall that the first step **(A1)** enlarges the range of summation for $Z_N^{(k)}$ in (4.1) to $1 \leqslant n_1, n_2 - n_1, \ldots, n_k - n_{k-1} \leqslant N$.

Lemma 6.1 (Approximation (A1)). *For each $k \in \mathbb{N}$, let $Z_N^{(k)}$ be as in (4.1), and let*

$$\widehat{Z}_N^{(k)} := \frac{1}{\mathbf{R}_N^{k/2}} \sum_{\substack{1 \leqslant n_1, n_2 - n_1, \ldots, n_k - n_{k-1} \leqslant N \\ z_1, z_2, \ldots, z_k \in \mathbb{Z}^d}} \prod_{j=1}^k q_{n_j - n_{j-1}}(z_j - z_{j-1}) \prod_{i=1}^k \eta_{(n_i, z_i)}. \qquad (6.1)$$

Then

$$\lim_{N \to \infty} \mathbb{E}\big[(\widehat{Z}_N^{(k)} - Z_N^{(k)})^2\big] = 0. \qquad (6.2)$$

Proof. Recall from (5.6) that for $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}_\uparrow^k := \{\boldsymbol{n} \in \mathbb{N}^k : n_1 < n_2 < \cdots < n_k\}$,

$$Q_{\boldsymbol{n}} := \sum_{\boldsymbol{x} \in (\mathbb{Z}^d)^{|\boldsymbol{n}|}} \prod_{j=1}^{|\boldsymbol{n}|} q_{n_j - n_{j-1}}(x_j - x_{j-1}) \, \eta_{(n_j, x_j)} \qquad (\text{with } n_0 = x_0 = 0). \qquad (6.3)$$

We can then write the difference

$$\widehat{Z}_N^{(k)} - Z_N^{(k)} = \frac{1}{\mathbf{R}_N^{k/2}} \sum_{\boldsymbol{n} \in \mathbb{N}_\uparrow^k} \big(\mathbb{1}_{1 \leqslant n_1 - n_0, \ldots, n_k - n_{k-1} \leqslant N} - \mathbb{1}_{0 < n_1 < \cdots < n_k \leqslant N} \big) Q_{\boldsymbol{n}}.$$

Since $\mathbb{E}[Q_{\boldsymbol{n}} Q_{\boldsymbol{n}'}] = \mathbb{1}_{\{\boldsymbol{n} = \boldsymbol{n}'\}} \|q_{\boldsymbol{n}}\|^2$ for $\boldsymbol{n}, \boldsymbol{n}' \in \mathbb{N}_\uparrow^k$ by (5.9), we observe that

$$\mathbb{E}\big[(\widehat{Z}_N^{(k)} - Z_N^{(k)})^2\big] = \mathbb{E}\big[(\widehat{Z}_N^{(k)})^2\big] - \mathbb{E}\big[(Z_N^{(k)})^2\big] - 2\mathbb{E}\big[(\widehat{Z}_N^{(k)} - Z_N^{(k)}) Z_N^{(k)}\big] = \mathbb{E}\big[(\widehat{Z}_N^{(k)})^2\big] - \mathbb{E}\big[(Z_N^{(k)})^2\big].$$

On the other hand, recalling (2.7),

$$\mathbb{E}\big[(Z_N^{(k)})^2\big] \leqslant \mathbb{E}\big[(\widehat{Z}_N^{(k)})^2\big] = \frac{1}{\mathbf{R}_N^k} \sum_{\substack{1 \leqslant n_1, n_2 - n_1, \ldots, n_k - n_{k-1} \leqslant N \\ z_1, z_2, \ldots, z_k \in \mathbb{Z}^d}} \prod_{j=1}^k q_{n_j - n_{j-1}}(z_j - z_{j-1})^2$$

$$= \frac{1}{\mathbf{R}_N^k} \bigg(\sum_{\substack{1 \leqslant n \leqslant N \\ z \in \mathbb{Z}^d}} q_n(z)^2 \bigg)^k = 1. \qquad (6.4)$$

To prove (6.2), it then suffices to show that $\liminf_{N \to \infty} \mathbb{E}\big[(Z_N^{(k)})^2\big] \geqslant 1$, which holds since

$$\mathbb{E}\big[(Z_N^{(k)})^2\big] = \frac{1}{\mathbf{R}_N^k} \sum_{\boldsymbol{n} \in \mathbb{N}_\uparrow^k} \mathbb{1}_{0 < n_1 < \cdots < n_k \leqslant N} \|q_{\boldsymbol{n}}\|^2$$

$$\geqslant \frac{1}{\mathbf{R}_N^k} \sum_{\boldsymbol{n} \in \mathbb{N}_\uparrow^k} \mathbb{1}_{1 \leqslant n_1 - n_0, \ldots, n_k - n_{n-1} \leqslant \frac{N}{k}} \|q_{\boldsymbol{n}}\|^2 = \frac{\mathbf{R}_{N/k}^k}{\mathbf{R}_N^k},$$

which tends to 1 as $N \to \infty$ by the assumption that $\mathbf{R}_N$ is slowly varying in N. $\square$

The approximation step **(A2)** in Section 4 bounds the contributions of near-diagonal terms when the summations in $\widehat{Z}_N^{(k)}$ in (6.1) are divided into blocks.

Lemma 6.2 (Approximation (A2)). *Recall from* (4.6) *the definition of the block variables*

$$\Theta_{\boldsymbol{i}}^{N;M} := \left(\frac{M}{\mathbf{R}_N} \right)^{k/2} \sum_{n_1 \in I_{i_1}, \ldots, n_k - n_{k-1} \in I_{i_k}} Q_{\boldsymbol{n}}, \qquad \boldsymbol{i} = (i_1, \ldots, i_k) \in \{1, \ldots, M\}^k, \qquad (6.5)$$

with $Q_{\boldsymbol{n}}$ as in (6.3), *and $I_i = \left(\mathsf{t}_{i-1}, \mathsf{t}_i \right]$ defined as in* (4.5) *such that $\mathbf{R}_{\mathsf{t}_i} \sim \frac{i}{M} \mathbf{R}_N$. Then*

$$\lim_{M \to \infty} \limsup_{N \to \infty} \mathbb{E}\left[\left(\sum_{\boldsymbol{i} \in \{1,\ldots,M\}^k \setminus \{1,\ldots,M\}_\sharp^k} \frac{1}{M^{\frac{k}{2}}} \Theta_{\boldsymbol{i}}^{N;M} \right)^2 \right] = 0, \qquad (6.6)$$

where $\{1, ..., M\}_\sharp^k$ was defined in (4.7), *which consists of $\boldsymbol{i}$ with $|i_j - i_{j'}| \geq 2$ for all $j \neq j'$.*

Proof. Denote $\{1, ..., M\}_*^k := \{1, ..., M\}^k \setminus \{1, ..., M\}_\sharp^k$. Note that

$$\mathbb{E}\left[\left(\sum_{\boldsymbol{i} \in \{1,\ldots,M\}_*^k} \frac{1}{M^{\frac{k}{2}}} \Theta_{\boldsymbol{i}}^{N;M} \right)^2 \right] = \frac{1}{\mathbf{R}_N^k} \sum_{\boldsymbol{i} \in \{1,\ldots,M\}_*^k} \sum_{n_1 \in I_{i_1}, \ldots, n_k - n_{k-1} \in I_{i_k}} \|q_{\boldsymbol{n}}\|^2.$$

Recall from (5.3) that $\sum_{m \in I_i} \|q_m\|^2 \sim \mathbf{R}_N/M$ as $N \to \infty$, while $\|q_{\boldsymbol{n}}\|^2 = \prod_{j=1}^{|\boldsymbol{n}|} \|q_{n_j - n_{j-1}}\|^2$, we can therefore sum $n_k, n_{k-1}, ..., n_1$ successively to obtain

$$\limsup_{N \to \infty} \mathbb{E}\left[\left(\sum_{\boldsymbol{i} \in \{1,\ldots,M\}_*^k} \frac{1}{M^{\frac{k}{2}}} \Theta_{\boldsymbol{i}}^{N;M} \right)^2 \right] \leq \sum_{\boldsymbol{i} \in \{1,\ldots,M\}_*^k} \frac{1}{M^k},$$

which tends to 0 as $M \to \infty$, since the constraint $\boldsymbol{i} \in \{1, ..., M\}^k \setminus \{1, ..., M\}_\sharp^k$ reduces the number of free indices in $\boldsymbol{i} = (i_1, \ldots, i_k)$. $\square$

The approximation step **(A3)** concerns the asymptotic factorization of $\Theta_{\boldsymbol{i}}^{N;M}$ into a product of $\Theta_{\boldsymbol{i}^{(j)}}^{N;M}$, indexed by dominated sequences $\boldsymbol{i}^{(1)}, \ldots, \boldsymbol{i}^{(\mathfrak{m})}$ forming $\boldsymbol{i}$ (cf. (4.9)). Recall that each $\boldsymbol{i} = (i_1, \ldots, i_k) \in \{1, \ldots, M\}_\sharp^k$ can be divided into $\mathfrak{m} = \mathfrak{m}(\boldsymbol{i})$ consecutive dominated sequences $\boldsymbol{i}^{(1)} := (i_1, \ldots, i_{\ell_2 - 1})$, $\boldsymbol{i}^{(2)} := (i_{\ell_2}, \ldots, i_{\ell_3 - 1})$, ..., $\boldsymbol{i}^{(\mathfrak{m})} := (i_{\ell_\mathfrak{m}}, \ldots, i_k)$, where $i_{\ell_1} = i_1 < i_{\ell_2} < \cdots < i_{\ell_\mathfrak{m}}$ are the successive running maxima of $(i_1, \ldots, i_k)$.

Lemma 6.3 (Approximation (A3)). *For each $\boldsymbol{i} = (i_1, \ldots, i_k) \in \{1..., M\}_\sharp^k$, we have*

$$\lim_{N \to \infty} \mathbb{E}\left[\left(\Theta_{\boldsymbol{i}}^{N;M} - \Theta_{\boldsymbol{i}^{(1)}}^{N;M} \Theta_{\boldsymbol{i}^{(2)}}^{N;M} \cdots \Theta_{\boldsymbol{i}^{(\mathfrak{m}(\boldsymbol{i}))}}^{N;M} \right)^2 \right] = 0, \qquad (6.7)$$

where $(\boldsymbol{i}^{(1)}, ..., \boldsymbol{i}^{(\mathfrak{m}(\boldsymbol{i}))})$ is the decomposition of $\boldsymbol{i}$ into dominated sequences.

Proof. We first prove (6.7) for $\mathfrak{m}(\boldsymbol{i}) = 2$, with $\ell_1 = 1$ and ℓ_2 denoting the indices of the two running maxima of $\boldsymbol{i}$. Recall that

$$\Theta_{i_1,\ldots,i_k}^{N;M} = \left(\frac{M}{\mathbf{R}_N} \right)^{k/2} \sum_{\substack{n_1 \in I_{i_1}, \ldots, n_k - n_{k-1} \in I_{i_k} \\ x_1, \ldots, x_k \in \mathbb{Z}^d}} \prod_{j=1}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1}) \prod_{i=1}^{k} \eta_{(n_i, x_i)}. \qquad (6.8)$$

Note that if we replace $q_{n_{\ell_2}-n_{\ell_2-1}}(x_{\ell_2}-x_{\ell_2-1})$ by $q_{n_{\ell_2}}(x_{\ell_2})$ and replace the range of summation $n_{\ell_2}-n_{\ell_2-1} \in I_{i_{\ell_2}}$ by $n_{\ell_2} \in I_{i_{\ell_2}}$, then the above expression for $\Theta_{\boldsymbol{i}}^{N;M}$ becomes that for $\Theta_{\boldsymbol{i}^{(1)}}^{N;M}\Theta_{\boldsymbol{i}^{(2)}}^{N;M}$. We will show that these replacements are justified because using that ℓ_2 is a running maximum of $\boldsymbol{i}$, one has $n_{\ell_2} \gg n_{\ell_2-1}$ and the local limit theorem of Hypothesis 2.4 can then be applied to replace $q_{n_{\ell_2}-n_{\ell_2-1}}(x_{\ell_2}-x_{\ell_2-1})$ by $q_{n_{\ell_2}}(x_{\ell_2})$.

First note that the the summands in (6.8) for $\Theta_{\boldsymbol{i}}^{N;M}$ are all orthogonal, and the dominant L^2 contribution comes from $x_1,\ldots,x_k$ with $|x_j-x_{j-1}|$ of the order

$$\phi(n_j-n_{j-1}) := ((n_j-n_{j-1})L(n_j-n_{j-1})^2)^{1/d}$$

for each $1 \leqslant j \leqslant k$. Indeed, by the local limit theorem of Hypothesis 2.4 and a Riemann sum approximation,

$$\frac{\mathbb{E}\left[\left(\sum_{|x|\leqslant K\phi(n)} q_n(x)\eta_{(n,x)}\right)^2\right]}{\mathbb{E}\left[\left(\sum_x q_n(x)\eta_{(n,x)}\right)^2\right]} = \frac{\sum_{|x|\leqslant K\phi(n)} q_n(x)^2}{\sum_x q_n(x)^2} \xrightarrow[n\to\infty]{} \frac{\int_{|x|\leqslant K} g^2(x)\mathrm{d}x}{\int g^2(x)\mathrm{d}x} \xrightarrow[K\to\infty]{} 1.$$

Therefore by choosing K large, we can approximate $\Theta_{\boldsymbol{i}}^{N;M}$ arbitrarily closely in L^2 by

$$\widetilde{\Theta}_{\boldsymbol{i}}^{N;M} = \left(\frac{M}{\mathbf{R}_N}\right)^{k/2} \sum_{\substack{n_1\in I_{i_1},\,\ldots,\,n_k-n_{k-1}\in I_{i_k} \\ |x_1|\leqslant K\phi(n_1),\ldots,|x_k-x_{k-1}|\leqslant K\phi(n_k-n_{k-1})}} \prod_{j=1}^k q_{n_j-n_{j-1}}(x_j-x_{j-1}) \prod_{i=1}^k \eta_{(n_i,x_i)}.$$

Similarly we can approximate $\Theta_{\boldsymbol{i}^{(1)}}^{N;M}\Theta_{\boldsymbol{i}^{(2)}}^{N;M}$ arbitrarily closely in L^2 by $\widetilde{\Theta}_{\boldsymbol{i}^{(1)}}^{N;M}\widetilde{\Theta}_{\boldsymbol{i}^{(2)}}^{N;M}$, which differs from $\widetilde{\Theta}_{\boldsymbol{i}}^{N;M}$ in:

- the factor $q_{n_{\ell_2}}(x_{\ell_2})$ instead of $q_{n_{\ell_2}-n_{\ell_2-1}}(x_{\ell_2}-x_{\ell_2-1})$;
- the range of summation $n_{\ell_2} \in I_{i_{\ell_2}}$ and $|x_{\ell_2}| \leqslant K\phi(n_{\ell_2})$, instead of $n_{\ell_2}-n_{\ell_2-1} \in I_{i_{\ell_2}}$ and $|x_{\ell_2}-x_{\ell_2-1}| \leqslant K\phi(n_{\ell_2}-n_{\ell_2-1})$.

We now show that these differences are negligible in L^2 contributions. By assumption,

$$n_j-n_{j-1} \in I_{i_j} = \left(\mathsf{t}_{i_j-1}, \mathsf{t}_{i_j}\right] \quad \text{for all } 1 \leqslant j \leqslant \ell_2-1,$$

where t_a is chosen with $\mathbf{R}_{\mathsf{t}_a} \sim \frac{a}{M}\mathbf{R}_N$. Since $\mathbf{R}_N$ is slowly varying and divergent, we have $\mathsf{t}_1 \ll \mathsf{t}_2 \ll \mathsf{t}_3 \ll \cdots$ as $N \to \infty$. In particular, we have the uniform bound

$$n_{\ell_2-1} = \sum_{j=1}^{\ell_2-1}(n_j-n_{j-1}) \leqslant \sum_{j=1}^{\ell_2-1}\mathsf{t}_{i_j} = O(\mathsf{t}_{i_1}) = o(\mathsf{t}_{i_{\ell_2}-1}), \tag{6.9}$$

where the last bound holds because the assumption $\boldsymbol{i} \in \{1,\ldots,M\}_\sharp^k$ and ℓ_2 being a running maximum ensures that $i_j \leqslant i_1 < i_{\ell_2}-1$ for all $1 \leqslant j \leqslant \ell_2-1$. Therefore when we switch from the range of summation in $\widetilde{\Theta}_{\boldsymbol{i}}^{N;M}$ from $n_{\ell_2} \in n_{\ell_2-1}+(\mathsf{t}_{i_{\ell_2}-1},\mathsf{t}_{i_{\ell_2}}]$ to $n_{\ell_2} \in (\mathsf{t}_{i_{\ell_2}-1},\mathsf{t}_{i_{\ell_2}}]$, the difference is negligible in L^2 as $N \to \infty$.

Similarly, we have the uniform bound

$$|x_{\ell_2-1}| \leqslant \sum_{j=1}^{\ell_2-1}|x_j-x_{j-1}| \leqslant K\sum_{j=1}^{\ell_2-1}\phi(n_j-n_{j-1}) \leqslant K\sum_{j=1}^{\ell_2-1}\phi(\mathsf{t}_{i_j}) \ll \phi(\mathsf{t}_{i_{\ell_2}-1}), \tag{6.10}$$

and when we switch the range of summation in $\widetilde{\Theta}_{\boldsymbol{i}}^{N;M}$ from $|x_{\ell_2} - x_{\ell_2-1}| \leqslant K\phi(n_{\ell_2} - n_{\ell_2-1})$ to $|x_{\ell_2}| \leqslant K\phi(n_{\ell_2})$, the difference is again negligible in L^2 as $N \to \infty$ (recall that by construction $I_{i_{\ell_2}} \ni n_{\ell_2} - n_{\ell_2-1} \geqslant \mathsf{t}_{i_{\ell_2}-1}$).

Having justified the switch of the range of summation for n_{ℓ_2} and x_{ℓ_2} in $\widetilde{\Theta}_{\boldsymbol{i}}^{N;M}$ to $n_{\ell_2} \in I_{i_{\ell_2}}$ and $|x_{\ell_2}| \leqslant K\phi(n_{\ell_2})$, we note finally that switching $q_{n_{\ell_2}-n_{\ell_2-1}}(x_{\ell_2} - x_{\ell_2-1})$ to $q_{n_{\ell_2}}(x_{\ell_2})$ also leads to a negligible difference in L^2 as $N \to \infty$, because uniformly in x_{ℓ_2-1} and n_{ℓ_2-1} with bounds as in (6.9) and (6.10), and uniformly in $n_{\ell_2} \in I_{i_{\ell_2}}$ and $|x_{\ell_2}| \leqslant K\phi(n_{\ell_2})$, we have

$$\left| \frac{q_{n_{\ell_2}-n_{\ell_2-1}}(x_{\ell_2} - x_{\ell_2-1})}{q_{n_{\ell_2}}(x_{\ell_2})} - 1 \right| \xrightarrow[N\to\infty]{} 0, \tag{6.11}$$

which follows readily from the local limit theorem for $q(\cdot)$ in Hypothesis 2.4.

This completes the proof of (6.7) when $\boldsymbol{i}$ has two running maxima. In general when $\boldsymbol{i}$ has $\mathfrak{m}$ running maxima, occurring at indices $\ell_1 = 1, \ell_2, \ldots, \ell_{\mathfrak{m}}$, the argument is the same: we just replace $q_{n_{\ell_j}-n_{\ell_j-1}}(x_{\ell_j} - x_{\ell_j-1})$ by $q_{n_{\ell_j}}(x_{\ell_j})$ and replace the range of summation $n_{\ell_j} - n_{\ell_j-1} \in I_{i_{\ell_j}}$ by $n_{\ell_j} \in I_{i_{\ell_j}}$, one j at a time. $\qquad\square$

As explained in Section 4, for a fixed $M \in \mathbb{N}$, which is the number of blocks $(I_i)_{1 \leqslant i \leqslant M}$ that partition $[1, N]$ (cf. (5.2)), the polymer partition function Z_{N,β_N}^ω (with $\beta_N = \hat{\beta}/\sqrt{\mathbf{R}_N}$ for some $\hat{\beta} < 1$) is approximated in distribution in the $N \to \infty$ limit by the random variable $\boldsymbol{Z}_{\hat{\beta}}^{(M)}$ in (4.13). The last step **(A4)** is to show that as $M \to \infty$, $\boldsymbol{Z}_{\hat{\beta}}^{(M)}$ converges to the log-normal random variable $\boldsymbol{Z}_{\hat{\beta}}$ in Theorem 2.8.

Lemma 6.4 (Step (A4)). *Let $(\zeta_{\boldsymbol{i}})_{\boldsymbol{i}\in D_M}$ be i.i.d. standard normal random variables indexed by finite dominated sequences in D_M as defined in (4.12). Let $\hat{\beta} \in (0, 1)$, and let*

$$\boldsymbol{Z}_{\hat{\beta}}^{(M)} := 1 + \sum_{k=1}^\infty \sum_{\boldsymbol{i}\in\{1,\ldots,M\}_\sharp^k} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \prod_{l=1}^{\mathfrak{m}(\boldsymbol{i})} \zeta_{\boldsymbol{i}^{(l)}}, \tag{6.12}$$

where $(\boldsymbol{i}^{(1)}, \ldots, \boldsymbol{i}^{(\mathfrak{m}(\boldsymbol{i}))})$ is the decomposition of $\boldsymbol{i}$ into dominated sequences. Then

$$\boldsymbol{Z}_{\hat{\beta}}^{(M)} \xrightarrow[M\to\infty]{d} \boldsymbol{Z}_{\hat{\beta}} = \exp\left(\int_0^1 \frac{\hat{\beta}}{\sqrt{1 - \hat{\beta}^2\, t}} \mathrm{d}W(t) - \frac{1}{2} \int_0^1 \frac{\hat{\beta}^2}{1 - \hat{\beta}^2\, t} \mathrm{d}t \right), \tag{6.13}$$

where W is a standard one dimensional Wiener process.

Proof. Grouping $\boldsymbol{i} = (i_1, \ldots, i_k)$ according to the indices of its running maxima $\ell_1 = 1 < \ell_2 < \cdots \ell_m \leqslant k$, as well as the values of the running maxima $1 \leqslant i_{\ell_1} < \cdots < i_{\ell_m} \leqslant M$, which we denote by $\boldsymbol{i} \sim (\vec{\ell}, i_{\vec{\ell}})$, we can write (with $\ell_{m+1} := k + 1$)

$$\boldsymbol{Z}_{\hat{\beta}}^{(M)} = 1 + \sum_{k=1}^\infty \sum_{m=1}^k \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\substack{1=\ell_1<\cdots<\ell_m\leqslant k \\ 1\leqslant i_{\ell_1}<i_{\ell_2}<\cdots<i_{\ell_m}\leqslant M}} \sum_{\substack{\boldsymbol{i}\in\{1,\ldots,M\}_\sharp^k \\ \boldsymbol{i}\sim(\vec{\ell},i_{\vec{\ell}})}} \prod_{j=1}^m \zeta_{(i_{\ell_j},\ldots,i_{\ell_{j+1}-1})}. \tag{6.14}$$

Let us replace the constraints $\boldsymbol{i} \in \{1, \ldots, M\}_{\sharp}^k$ and $\boldsymbol{i} \sim (\vec{\ell}, i_{\vec{\ell}})$ by sum over $(i_{\ell_j+1}, \ldots, i_{\ell_{j+1}-1}) \in \{1, \ldots, i_{\ell_j}-1\}^{\ell_{j+1}-\ell_j-1}$ for each $1 \leqslant j \leqslant m$, i.e., approximate $\boldsymbol{Z}_{\hat{\beta}}^{(M)}$ by

$$\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)} = 1 + \sum_{k=1}^{\infty} \sum_{m=1}^{k} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\substack{1=\ell_1<\cdots<\ell_m\leqslant k \\ 1\leqslant i_{\ell_1}<i_{\ell_2}<\cdots<i_{\ell_m}\leqslant M}} \sum_{\substack{(i_{\ell_j+1},\ldots,i_{\ell_{j+1}-1}) \\ \in\{1,\ldots,i_{\ell_j}-1\}^{\ell_{j+1}-\ell_j-1}}} \prod_{j=1}^{m} \zeta_{(i_{\ell_j},\ldots,i_{\ell_{j+1}-1})}, \quad (6.15)$$

where we have extended the i.i.d. family $(\zeta_{\boldsymbol{i}})_{\boldsymbol{i}\in D_M}$ to include new independent standard normals $\zeta_{(a_1,\cdots,a_r)}$ indexed by dominated sequences $(a_1, \cdots, a_r) \in \{1, \ldots, M\}^r \setminus \{1, \ldots, M\}_{\sharp}^r$.

Note that $\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)}$ contains more summands than $\boldsymbol{Z}_{\hat{\beta}}^{(M)}$, and the summands are orthogonal. A simple calculation shows that both $\|\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)}\|_2^2$ and $\|\boldsymbol{Z}_{\hat{\beta}}^{(M)}\|_2^2$ tend to $1+\sum_{k=1}^{\infty}\hat{\beta}^{2k} = (1-\hat{\beta}^2)^{-1}$ as $M \to \infty$. Therefore

$$\|\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)} - \boldsymbol{Z}_{\hat{\beta}}^{(M)}\|_2^2 \xrightarrow[M\to\infty]{} 0.$$

For $a \in \mathbb{N}$ and $r \in \mathbb{N}$, let us now denote

$$\xi_r(a) := \sum_{(a_2,\ldots,a_r)\in\{1,\ldots,a-1\}^{r-1}} \zeta_{(a,a_2,\ldots,a_r)}. \quad (6.16)$$

Denoting $r_j := \ell_{j+1} - \ell_j$ (with $\ell_{m+1} := k + 1$) and $a_j := i_{\ell_j}$, we can then rewrite $\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)}$ as

$$\begin{aligned}
\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)} &= 1 + \sum_{k=1}^{\infty} \sum_{m=1}^{k} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\substack{1=\ell_1<\cdots<\ell_m\leqslant k \\ 1\leqslant a_1<a_2<\cdots<a_m\leqslant M}} \prod_{j=1}^{m} \xi_{\ell_{j+1}-\ell_j}(a_j) \\
&= 1 + \sum_{k=1}^{\infty} \sum_{m=1}^{k} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\substack{r_1,\ldots r_m\in\mathbb{N} \\ r_1+\cdots+r_m=k}} \sum_{1\leqslant a_1<a_2<\cdots<a_m\leqslant M} \prod_{j=1}^{m} \xi_{r_j}(a_j) \\
&= 1 + \sum_{m=1}^{\infty} \sum_{r_1,\ldots,r_m\in\mathbb{N}} \sum_{1\leqslant a_1<a_2<\ldots<a_m\leqslant M} \prod_{j=1}^{m} \frac{\hat{\beta}^{r_j}}{M^{\frac{r_j}{2}}} \xi_{r_j}(a_j) \\
&= 1 + \sum_{m=1}^{\infty} \sum_{r_1,\ldots,r_m\in\mathbb{N}} \sum_{\substack{0<t_1<t_2<\ldots<t_m\leqslant 1 \\ t_1,\ldots,t_m\in\frac{1}{M}\mathbb{N}}} \prod_{j=1}^{m} \frac{\hat{\beta}^{r_j}}{M^{\frac{r_j}{2}}} \xi_{r_j}(Mt_j) \\
&= 1 + \sum_{m=1}^{\infty} \sum_{\substack{0<t_1<t_2<\ldots<t_m\leqslant 1 \\ t_1,\ldots,t_m\in\frac{1}{M}\mathbb{N}}} \prod_{j=1}^{m} \left\{ \sum_{r\in\mathbb{N}} \frac{\hat{\beta}^r}{M^{\frac{r}{2}}} \xi_r(Mt_j) \right\}, \quad (6.17)
\end{aligned}$$

where we could interchange summations because the series is L^2 convergent when $\hat{\beta} \in (0,1)$.

We note that $(\hat{\beta}/\sqrt{M})^r \xi_r(Mt)$ are independent normal random variables for different values of $r \in \mathbb{N}$ and $t \in M^{-1}\mathbb{N}$, and hence the collection of random variables

$$\Xi_{M,t} := \sum_{r\in\mathbb{N}} \frac{\hat{\beta}^r}{M^{\frac{r}{2}}} \xi_r(Mt), \qquad t \in (0,1] \cap \frac{1}{M}\mathbb{N},$$

are also independent normal with mean zero and variance

$$\mathbb{V}\text{ar}(\Xi_{M,t}) = \sum_{r \in \mathbb{N}} \frac{\hat{\beta}^{2r}}{M^r} \mathbb{V}\text{ar}(\xi_r(Mt)) = \sum_{r \in \mathbb{N}} \frac{\hat{\beta}^{2r}}{M^r}(Mt-1)^{r-1} = \frac{\hat{\beta}^2}{M} \cdot \frac{1+o(1)}{1-\hat{\beta}^2 t},$$

where $o(1)$ is uniform in $t \in [0,1]$ for a given $\hat{\beta} < 1$. Therefore we can represent $\Xi_{M,t}$ in terms of a standard Wiener process W:

$$\Xi_{M,t} = \frac{\hat{\beta}(1+o(1))}{\sqrt{1-\hat{\beta}^2 \, t}} \int_{t-\frac{1}{M}}^{t} \mathrm{d}W_s, \qquad t \in [0,1] \cap \frac{1}{M}\mathbb{N}. \tag{6.18}$$

We can then write

$$\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)} = 1 + \sum_{m=1}^{\infty} \sum_{\substack{0<t_1<t_2<\ldots<t_m\leqslant 1 \\ t_1,\ldots,t_m \in \frac{1}{M}\mathbb{N}}} \prod_{j=1}^{m} \frac{\hat{\beta}(1+o(1))}{\sqrt{1-\hat{\beta}^2 \, t_j}} \int_{t_j-\frac{1}{M}}^{t_j} \mathrm{d}W_s. \tag{6.19}$$

For $\hat{\beta} < 1$, it is easily seen that

$$\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M)} \xrightarrow[M\to\infty]{L^2} 1 + \sum_{m=1}^{\infty} \int \cdots \int_{0<t_1<\ldots<t_m<1} \prod_{j=1}^{m} \frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2 \, t_j}} \mathrm{d}W_{t_j} =: \exp\left\{ \int_0^1 \frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2 \, t}} \mathrm{d}W(t) \right\} : \, ,$$

where the last equality holds by the properties of the Wick exponential [J97, §3.2]. Since the last expression is precisely $\boldsymbol{Z}_{\hat{\beta}}$, the proof is completed. $\qquad\square$

Proof of Theorem 2.8. When $\hat{\beta} \in (0,1)$, the convergence of Z_{N,β_N}^{ω} to $\boldsymbol{Z}_{\hat{\beta}}$ follows readily from the approximation steps **(A1)–(A4)** and the key step **(K)**, as explained in Section 4. The convergence of the second moment $\mathbb{E}[(Z_{N,\beta_N}^{\omega})^2] \to \mathbb{E}[(\boldsymbol{Z}_{\hat{\beta}})^2] = \frac{1}{1-\hat{\beta}^2}$ for $\hat{\beta} \in (0,1)$ is a simple calculation, using (4.4) and $\mathbb{E}[(\widehat{Z}_N^{(k)})^2] = 1$ (recall (6.4)).

When $\hat{\beta} \geqslant 1$, the convergence in law $Z_{N,\beta_N}^{\omega} \to 0$ follows a standard argument, which we include for completeness. Note that it suffices to show that for some $\vartheta \in (0,1)$, the fractional moment $\mathbb{E}[(Z_{N,\beta_N}^{\omega})^\vartheta]$ converges to zero as $N \to \infty$.

First we show that $\mathbb{E}[(Z_{N,\beta}^{\omega})^\vartheta]$ is non-increasing in β. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{E}[(Z_{N,\beta}^{\omega})^\vartheta] = \vartheta\mathbb{E}\Bigg[\mathrm{E}\bigg[\sum_{n=1}^{N}(\omega_{n,x_n} - \lambda'(\beta)) e^{\sum_{n=1}^{N}(\beta\omega_{n,x_n}-\lambda(\beta))} \bigg] (Z_{N,\beta}^{\omega})^{\vartheta-1} \Bigg]$$

$$= \vartheta \sum_{n=1}^{N} \mathrm{E}\bigg[\mathbb{E}\Big[(\omega_{n,x_n} - \lambda'(\beta)) e^{\sum_{i=1}^{N}(\beta\omega_{i,x_i}-\lambda(\beta))} (Z_{N,\beta}^{\omega})^{\vartheta-1} \Big] \bigg]$$

$$= \vartheta \sum_{n=1}^{N} \mathrm{E}\bigg[\widetilde{\mathbb{E}}\Big[(\omega_{n,x_n} - \lambda'(\beta))(Z_{N,\beta}^{\omega})^{\vartheta-1} \Big] \bigg],$$

where we have interpreted $e^{\sum_{i=1}^{N}(\beta\omega_{i,x_i}-\lambda(\beta))}$ as a probability density for a new law $\widetilde{\mathbb{P}}$ which exponentially tilts ω_{i,x_i} for each $1 \leqslant i \leqslant N$. Note that $(\omega_{n,x_n} - \lambda'(\beta))$ is increasing in ω_{n,x_n}, while $(Z_{N,\beta}^{\omega})^{\vartheta-1}$ is decreasing in ω_{n,x_n} because $\vartheta \in (0,1)$. Therefore by the FKG inequality,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{E}[(Z_{N,\beta}^{\omega})^\vartheta] \leqslant \vartheta \sum_{n=1}^{N} \mathrm{E}\bigg[\widetilde{\mathbb{E}}[\omega_{n,x_n} - \lambda'(\beta)] \, \widetilde{\mathbb{E}}[(Z_{N,\beta}^{\omega})^{\vartheta-1}] \bigg] = 0 \, ,$$

since

$$\widetilde{\mathbb{E}}\big[\omega_{n,x_n} - \lambda'(\beta)\big] = \mathbb{E}\Big[\big(\omega_{n,x_n} - \lambda'(\beta)\big)e^{\beta\omega_{n,x_n}-\lambda(\beta)}\Big] = \frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{E}\Big[e^{\beta\omega_{n,x_n}-\lambda(\beta)}\Big] = 0.$$

We have just shown that $\mathbb{E}[(Z^\omega_{N,\hat\beta'/\sqrt{\mathbf{R}_N}})^\vartheta] \leqslant \mathbb{E}[(Z^\omega_{N,\hat\beta/\sqrt{\mathbf{R}_N}})^\vartheta]$, for any $\hat\beta < 1 \leqslant \hat\beta'$. Since $Z^\omega_{N,\hat\beta/\sqrt{\mathbf{R}_N}}$ converges in distribution to $\boldsymbol{Z}_{\hat\beta}$ when $\hat\beta < 1$, and $(Z^\omega_{N,\hat\beta/\sqrt{\mathbf{R}_N}})^\vartheta$ is uniformly integrable, because $\vartheta \in (0,1)$ and $\mathbb{E}[Z^\omega_{N,\hat\beta/\sqrt{\mathbf{R}_N}}] = 1$, by the first part of Theorem 2.8 we then have

$$\limsup_{N\to\infty}\mathbb{E}[(Z^\omega_{N,\hat\beta'/\sqrt{\mathbf{R}_N}})^\vartheta] \leqslant \limsup_{N\to\infty}\mathbb{E}[(Z^\omega_{N,\hat\beta/\sqrt{\mathbf{R}_N}})^\vartheta]$$

$$= \mathbb{E}\bigg[\exp\Big(\vartheta\int_0^1 \frac{\hat\beta}{\sqrt{1-\hat\beta^2\,t}}\mathrm{d}W(t) - \frac{\vartheta}{2}\int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2\,t}\mathrm{d}t\Big)\bigg]$$

$$= \exp\Big(\frac{\vartheta(\vartheta-1)}{2}\int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2\,t}\mathrm{d}t\Big) = \big(1-\hat\beta^2\big)^{-\frac{\vartheta(\vartheta-1)}{2}}.$$

Letting $\hat\beta \nearrow 1$ then shows that $\mathbb{E}[(Z^\omega_{N,\hat\beta'/\sqrt{\mathbf{R}_N}})^\vartheta] \to 0$ as $N \to \infty$ whenever $\hat\beta' \geqslant 1$. $\qquad\square$

7. Proof of Theorem 2.12

To prove Theorem 2.12, we first need to extend Proposition 5.2 to random variables $\Theta^{N;M}_{\boldsymbol{i}}$ which form the building blocks of partition functions $Z^\omega_{N,\beta}(\mathsf{X})$ with starting points $\mathsf{X} = (x,t)$ other than the origin. More precisely, as in (4.6), define

$$\Theta^{N;M}_{\boldsymbol{i}}(\mathsf{X}) := \Big(\frac{M}{\mathbf{R}_N}\Big)^{k/2} \sum_{\substack{n_1-n_0\in I_{i_1},\, n_2-n_1\in I_{i_2},\ldots,\, n_k-n_{k-1}\in I_{i_k} \\ z_1,z_2,\ldots,z_k\in\mathbb{Z}^d}} \prod_{j=1}^k q_{n_j-n_{j-1}}(z_j - z_{j-1}) \prod_{i=1}^k \eta_{(n_i,z_i)},$$

$$\tag{7.1}$$

except here (z_0, n_0) is defined to be X instead of the origin.

For $\mathsf{X} = (x,t) \in \mathbb{Z}^d \times \mathbb{N}_0$ with $d \in \{0,1,2\}$, recall the definition of $\|X\|$ from (2.13). We then have the following extension of Proposition 5.2.

Proposition 7.1. *Assume that Hypothesis 2.4 holds and $\mathbf{R}_N$ in (2.7)–(2.8) diverges as $N \to \infty$. For $1 \leqslant k \leqslant r$, let $\mathsf{X}^{(k)}_N = (x^{(k)}_N, t^{(k)}_N)$ be points in $\mathbb{Z}^d \times \mathbb{N}_0$, such that*

$$\forall 1 \leqslant k,l \leqslant r: \qquad \mathbf{R}_{\|\mathsf{X}^{(k)}_N - \mathsf{X}^{(l)}_N\|}/\mathbf{R}_N = \zeta_{k,l} + o(1) \quad \text{for some } \zeta_{k,l} \in [0,1]. \tag{7.2}$$

For $M \in \mathbb{N}$, let us denote by $\widetilde{D}_M$ be the set of dominated sequences $\boldsymbol{i} \in D_M$, cf. (5.8), for which i_1/M is well separated from all the $\zeta_{k,l}$ in the following sense:

$$\widetilde{D}_M := \{\boldsymbol{i} \in D_M : \; |i_1/M - \zeta_{k,l}| > 1/M \; \forall 1 \leqslant k,l \leqslant r\}. \tag{7.3}$$

Then the vector $(\Theta^{N;M}_{\boldsymbol{i}}(\mathsf{X}^{(k)}_N))_{1\leqslant k\leqslant r,\boldsymbol{i}\in\widetilde{D}_M}$ converges in law as $N \to \infty$ to a centered Gaussian vector $(\zeta^{(k)}_{\boldsymbol{i}})_{1\leqslant k\leqslant r,\,\boldsymbol{i}\in\widetilde{D}_M}$ with covariance matrix

$$\mathbb{C}\mathrm{ov}[\zeta^{(k)}_{\boldsymbol{i}}, \zeta^{(l)}_{\boldsymbol{i}'}] = \mathbb{1}_{\{\boldsymbol{i}=\boldsymbol{i}'\}}\mathbb{1}_{\{i_1/M>\zeta_{k,l}\}}. \tag{7.4}$$

Proof. The random variable $\Theta_{\boldsymbol{i}}^{N;M}((x,t))$ has the same law as $\Theta_{\boldsymbol{i}}^{N;M}((0,0))$. Therefore, the proof of Proposition 5.2 readily implies that for each $\boldsymbol{i} \in \widetilde{D}_M$ and $1 \leqslant k \leqslant r$,

$$\mathbb{V}\mathrm{ar}(\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)})) \underset{N\to\infty}{\longrightarrow} 1 \quad \text{and} \quad \mathbb{E}\big[(\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)}))^4\big] \underset{N\to\infty}{\longrightarrow} 3.$$

By the (multidimensional) Fourth Moment Theorem 4.2, it then only remains to show that

$$\mathbb{C}\mathrm{ov}(\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)}), \Theta_{\boldsymbol{i'}}^{N;M}(\mathsf{X}_N^{(l)})) \underset{N\to\infty}{\longrightarrow} \mathbb{1}_{\{\boldsymbol{i}=\boldsymbol{i'}\}}\mathbb{1}_{\{i_1/M>\zeta_{k,l}\}} \quad \forall 1 \leqslant k,l \leqslant r, \ \boldsymbol{i},\boldsymbol{i'} \in \widetilde{D}_M. \quad (7.5)$$

Note that when the dominated sequences $\boldsymbol{i}, \boldsymbol{i'}$ are different, there are unmatched η's and consequently $\mathbb{C}\mathrm{ov}(\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)}), \Theta_{\boldsymbol{i'}}^{N;M}(\mathsf{X}_N^{(l)})) = 0$; and when $\boldsymbol{i} = \boldsymbol{i'}$, we have

$$\mathbb{E}\big[\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)})\,\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(l)})\big] = \left(\frac{M}{\mathbf{R}_N}\right)^k \sum_{\substack{n_1-t_N^{(k)}\in I_{i_1} \\ n_1-t_N^{(l)}\in I_{i_1}}} \sum_{z_1\in\mathbb{Z}^d} q_{n_1-t_N^{(k)}}(z_1-x_N^{(k)}) q_{n_1-t_N^{(l)}}(z_1-x_N^{(l)})$$

$$\times \sum_{\substack{n_2-n_1\in I_{i_2},\,...,\,n_k-n_{k-1}\in I_{i_k} \\ z_2,...,z_k\in\mathbb{Z}^d}} \prod_{j=2}^k q_{n_j-n_{j-1}}(z_j-z_{j-1})^2$$

$$\sim \frac{M}{\mathbf{R}_N} \sum_{\substack{n_1-t_N^{(k)}\in I_{i_1} \\ n_1-t_N^{(l)}\in I_{i_1}}} \sum_{z_1\in\mathbb{Z}^d} q_{n_1-t_N^{(k)}}(z_1-x_N^{(k)}) q_{n_1-t_N^{(l)}}(z_1-x_N^{(l)}), \quad (7.6)$$

where in the last step we used (2.7) (recall that $q_n(x) = \mathrm{P}(S_n = x)$ and we write $f(N) \sim g(N)$ as a shorthand for $\lim_{N\to\infty} f(N)/g(N) = 1$.

We first consider the case $i_1/M > \zeta_{k,l}$, which implies $(i_1 - 1)/M > \zeta_{k,l}$ since $\boldsymbol{i} \in \widetilde{D}_M$. In this case, since $I_{i_1} \ni n_1 - t_N^{(k)} = n_1 - n_0 \geqslant \mathsf{t}_{i_1-1}$, recalling assumption (7.2) we have

$$n_1 - t_N^{(k)} \gg |t_N^{(k)} - t_N^{(l)}|.$$

By Hypothesis 2.4, the dominant contribution to (7.6) then comes from z_1 with

$$|z_1 - x_N^{(k)}| \gg |x_N^{(k)} - x_N^{(l)}| \quad \text{as } N \to \infty.$$

By the same argument as in the proof of Lemma 6.3, we can apply the local limit theorem in Hypothesis 2.4 and replace $(x_N^{(l)}, t_N^{(l)})$ in (7.6) by $(x_N^{(k)}, t_N^{(k)})$, which implies that

$$\lim_{N\to\infty} \mathbb{E}\big[\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)})\,\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(l)})\big] = \lim_{N\to\infty} \mathbb{E}\big[\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(k)})^2\big] = 1.$$

We next consider the case $i_1/M < \zeta_{k,l}$, which implies $(i_1 + 1)/M < \zeta_{k,l}$ since $\boldsymbol{i} \in \widetilde{D}_M$. By the definitions (7.2) and (2.13) of $\zeta_{k,l}$ and $\|\mathsf{X}\|$, this implies

$$\text{either} \quad \mathbf{R}_{|t_N^{(k)}-t_N^{(l)}|}/\mathbf{R}_N = \zeta_{k,l} + o(1); \quad \text{or} \quad \mathbf{R}_{\phi^{\leftarrow}(|x_N^{(k)}-x_N^{(l)}|)}/\mathbf{R}_N = \zeta_{k,l} + o(1), \quad (7.7)$$

where we recall by (2.6) and (2.12) that $\phi^{\leftarrow}(|x|) := \min\{n \in \mathbb{N}_0 : \phi(n) \geqslant |x|\}$ with $\phi(n) := (nL(n)^2)^{1/d}$. We now show that (7.7) forces either n_1 or z_1 to vary in intervals with empty intersection.

In the first case in (7.7), we have $|t_N^{(k)} - t_N^{(l)}| \gg \mathsf{t}_{i_1+1} \gg |I_{i_1}|$ as $N \to \infty$, where we recall that $I_{i_1} = (\mathsf{t}_{i_1-1}, \mathsf{t}_{i_1}]$ with $\mathbf{R}_{\mathsf{t}_{i_1}} \sim \frac{i_1}{M}\mathbf{R}_N$. Therefore the constraints $n_1 - t_N^{(k)} \in I_{i_1}$ and $n_1 - t_N^{(l)} \in I_{i_1}$ in (7.6) are incompatible and the sum equals zero.

In the second case in (7.7), we have $\phi^{\leftarrow}(|x_N^{(k)} - x_N^{(l)}|) \gg \mathsf{t}_{i_1+1}$, and hence $|x_N^{(k)} - x_N^{(l)}| \gg \phi(\mathsf{t}_{i_1+1})$. Therefore for N large, for any fixed $C > 0$

$$\left\{ z_1 \in \mathbb{Z}^d : |z_1 - x_N^{(k)}| \leqslant C\phi(\mathsf{t}_{i_1+1}) \right\} \cap \left\{ z_1 \in \mathbb{Z}^d : |z_1 - x_N^{(l)}| \leqslant C\phi(\mathsf{t}_{i_1+1}) \right\} = \varnothing.$$

By Hypothesis 2.4, uniformly in $n \in I_{i_1} = (\mathsf{t}_{i_1-1}, \mathsf{t}_{i_1}]$, the dominant contribution to $\sum_z q_n(z)$ and $\sum_z q_n^2(z)$ come from the region $|z| \leqslant C\phi(\mathsf{t}_{i_1+1})$. Partitioning the sum in (7.6) according to whether $|z_1 - x_N^{(k)}| \leqslant C\phi(\mathsf{t}_{i_1+1})$, or $|z_1 - x_N^{(l)}| \leqslant C\phi(\mathsf{t}_{i_1+1})$, or neither, it then follows that the quantity in (7.6) tends to 0 as $N \to \infty$, which concludes the proof of (7.5). $\square$

Proof of Theorem 2.12. The approximation steps **(A1)**–**(A3)** for the partition function Z_{N,β_N}^ω outlined in Section 4 (and proved in Section 6) also applies if the starting point of the polymer is different from the origin. For the step **(A1)**, in order to show that the constraint $n_0 < n_1 < \ldots < n_k \leqslant N$ can be replaced by $1 \leqslant n_1 - n_0, \ldots, n_k - n_{k-1} \leqslant N$, we need to use the assumption $\mathbf{R}_{N-t_N^{(k)}}/\mathbf{R}_N = 1 - o(1)$ in (2.14).

It follows that we can approximate the partition functions $(Z_{N,\beta_N}^\omega(\mathsf{X}_N^{(j)}))_{1\leqslant j\leqslant r}$ jointly in L^2 (with an error uniformly small in N, when M is large, cf. (4.11)) by

$$Z_{N,\beta_N}^{\mathbf{(A3)}}(\mathsf{X}_N^{(j)}) := 1 + \sum_{k=1}^{M} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\boldsymbol{i} \in \{1,\ldots,M\}_\sharp^k} \Theta_{\boldsymbol{i}^{(1)}}^{N;M}(\mathsf{X}_N^{(j)}) \Theta_{\boldsymbol{i}^{(2)}}^{N;M}(\mathsf{X}_N^{(j)}) \cdots \Theta_{\boldsymbol{i}^{(\mathrm{m})}}^{N;M}(\mathsf{X}_N^{(j)}), \quad 1 \leqslant j \leqslant r,$$

(7.8)

where we recall that $\Theta_{\boldsymbol{i}}^{N;M}(\mathsf{X}_N^{(j)})$ is defined in (7.1).

By Proposition 7.1, as $N \to \infty$, $(Z_{N,\beta_N}^{\mathbf{(A3)}}(\mathsf{X}_N^{(j)}))_{1\leqslant j\leqslant r}$ converge jointly in distribution to

$$\boldsymbol{Z}_{\hat{\beta}}^{(M,j)} := 1 + \sum_{k=1}^{M} \frac{\hat{\beta}^k}{M^{\frac{k}{2}}} \sum_{\boldsymbol{i} \in \{1,\ldots,M\}_\sharp^k} \zeta_{\boldsymbol{i}^{(1)}}^{(j)} \zeta_{\boldsymbol{i}^{(2)}}^{(j)} \cdots \zeta_{\boldsymbol{i}^{(\mathrm{m})}}^{(j)}, \qquad 1 \leqslant j \leqslant r.$$

It only remains to prove the analogue of Lemma 6.4 and show that as $M \to \infty$, $(\boldsymbol{Z}_{\hat{\beta}}^{(M,j)})_{1\leqslant j\leqslant r}$ converge jointly to the family of log-normal random variables $(: e^{\mathsf{Y}_j} :)_{1\leqslant j\leqslant r}$ in (2.15)-(2.16).

Following the same steps as in the proof of Lemma 6.4 up to the resummation procedure in (6.17), we can approximate $\boldsymbol{Z}_{\hat{\beta}}^{(M,j)}$ in L^2 (as $M \to \infty$) by

$$\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M,j)} = 1 + \sum_{m=1}^{\infty} \sum_{\substack{0<t_1<t_2<\ldots<t_m\leqslant 1 \\ t_1,\ldots,t_m \in \frac{1}{M}\mathbb{N}}} \prod_{i=1}^{m} \Xi_{M,t_i}^{(j)},$$

where

$$\Xi_{M,t}^{(j)} := \sum_{r\in\mathbb{N}} \frac{\hat{\beta}^r}{M^{\frac{r}{2}}} \, \xi_r^{(j)}(Mt) \quad \text{for } t \in [0,1] \cap \frac{1}{M}\mathbb{N}, \quad \text{and} \quad \xi_r^{(j)}(a) := \sum_{(a_2,\ldots,a_r)\in\{1,\ldots,a-1\}^{r-1}} \zeta_{(a,a_2,\ldots,a_r)}^{(j)}.$$

Similar to (6.18), we can encode the family of jointly Gaussian random variables $\Xi_{M,t}^{(j)}$ as

$$\Xi_{M,t}^{(j)} = \frac{\hat{\beta}(1+o(1))}{\sqrt{1-\hat{\beta}^2\,t}} \int_{t-\frac{1}{M}}^{t} \mathrm{d}W_s^{(j)}, \qquad t \in [0,1] \cap \frac{1}{M}\mathbb{N}, \tag{7.9}$$

where $(W^{(j)})_{1\leqslant j\leqslant r}$ is a family of correlated Brownian motions (the explicit form of the correlations will be derived in a moment). Therefore for all $1\leqslant j\leqslant r$,

$$\widehat{\boldsymbol{Z}}_{\hat{\beta}}^{(M,j)} = 1 + \sum_{m=1}^{\infty} \sum_{\substack{0<t_1<t_2<\ldots<t_m\leqslant 1 \\ t_1,\ldots,t_m\in\frac{1}{M}\mathbb{N}}} \prod_{i=1}^{m} \frac{\hat{\beta}(1+o(1))}{\sqrt{1-\hat{\beta}^2\,t_i}} \int_{t_i-\frac{1}{M}}^{t_i} \mathrm{d}W_s^{(j)} \xrightarrow[M\to\infty]{L^2} :e^{\mathsf{Y}_j}:,$$

where $\mathsf{Y}_j := \int_0^1 \frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2\,t}}\mathrm{d}W_t^{(j)}$. It now only remains to find the covariance between $(\mathsf{Y}_j)_{1\leqslant j\leqslant r}$.

Note that for each $1\leqslant k,l\leqslant r$ and $s,t\in[0,1]\cap\frac{1}{M}\mathbb{N}$, by the definition of $\xi_r^{(j)}$ and Proposition 7.1, we have

$$\mathbb{E}\big[\Xi_{M,t}^{(k)}\Xi_{M,s}^{(l)}\big] = \sum_{r\in\mathbb{N}} \frac{\hat{\beta}^{2r}}{M^r}\,\mathbb{E}\big[\xi_r^{(k)}(Mt)\,\xi_r^{(l)}(Ms)\big]$$

$$= \mathbb{1}_{\{t=s\}} \sum_{r\in\mathbb{N}} \frac{\hat{\beta}^{2r}}{M^r} \sum_{(a_2,\ldots,a_r)\in\{1,\ldots,Mt-1\}^{r-1}} \mathbb{E}\big[\zeta_{(Mt,a_2,\ldots,a_r)}^{(k)}\zeta_{(Mt,a_2,\ldots,a_r)}^{(l)}\big]$$

$$= \mathbb{1}_{\{t=s\}} \sum_{r\in\mathbb{N}} \frac{\hat{\beta}^{2r}}{M^r} \sum_{(a_2,\ldots,a_r)\in\{1,\ldots,Mt-1\}^{r-1}} \mathbb{1}_{\{t>\zeta_{k,l}\}} = \frac{\mathbb{1}_{\{t=s>\zeta_{k,l}\}}}{M} \cdot \frac{\hat{\beta}^2(1+o(1))}{1-\hat{\beta}^2 t}.$$

Therefore for all $s,t\in[0,1]\cap\frac{1}{M}\mathbb{N}$, we have

$$\mathbb{E}\Big[\int_{t-\frac{1}{M}}^t \mathrm{d}W_u^{(k)} \int_{s-\frac{1}{M}}^s \mathrm{d}W_u^{(l)}\Big] = \frac{\mathbb{1}_{\{t=s>\zeta_{k,l}\}}}{M},$$

and hence $\mathbb{E}[W^{(k)}(T)W^{(l)}(S)] = \int_{\zeta_{k,l}}^{T\wedge S}\mathrm{d}t$ for all $0\leqslant S\leqslant T\leqslant 1$. This implies that

$$\mathbb{C}\mathrm{ov}(Y_k,Y_l) = \mathbb{E}\Big[\int_0^1 \frac{\hat{\beta}\,\mathrm{d}W_t^{(k)}}{(1-\hat{\beta}^2\,t)^{\frac{1}{2}}} \int_0^1 \frac{\hat{\beta}\,\mathrm{d}W_t^{(l)}}{(1-\hat{\beta}^2\,t)^{\frac{1}{2}}}\Big] = \int_{\zeta_{k,l}}^1 \frac{\hat{\beta}^2}{1-\hat{\beta}^2 t}\mathrm{d}t = \log\frac{1-\hat{\beta}^2\zeta_{k,l}}{1-\hat{\beta}^2},$$

which concludes the proof. $\qquad\square$

8. Proof of Theorem 2.13

In this section, we prove Theorem 2.13. First we prove an analogue of Proposition 5.2, the key step **(K)** in the proof of Theorem 2.8. The difference here is that we need to average over the starting point of the partition function. As in Theorem 2.13, let $\psi:\mathbb{R}^d\times[0,1]\to\mathbb{R}$ be a continuous function with compact support. For any finite strictly increasing sequence $\boldsymbol{n}=(n_1,\ldots,n_{|\boldsymbol{n}|})$ and $0<n_0<n_1$, we then modify the definition of $Q_{\boldsymbol{n}}$ in (5.6) as follows:

$$Q_{(n_0,\boldsymbol{n})}^{\psi} := \sum_{x_0\in\mathbb{Z}^d} \sum_{\boldsymbol{x}\in(\mathbb{Z}^d)^{|\boldsymbol{n}|}} \Big(\prod_{j=1}^{|\boldsymbol{n}|} q_{n_j-n_{j-1}}(x_j-x_{j-1})\,\eta_{(n_j,x_j)}\Big)\psi(\widehat{\mathsf{X}}_N), \qquad (8.1)$$

where

$$\widehat{\mathsf{X}}_N := \Big(\frac{x_0}{\phi(N)},\frac{n_0}{N}\Big) \qquad \text{with} \quad \phi(N):=(L(N)^2 N)^{1/d}.$$

To decompose J_N^ψ in (2.19) as we decomposed the partition function in terms of the Θ's, for $\boldsymbol{i} \in \{1, \ldots, M\}^k$, we need to modify the definition of $\Theta_{\boldsymbol{i}}^{N,M}$ in (5.5) as follows:

$$\Theta_{\boldsymbol{i}}^{N,M;\psi} := \frac{L(N)}{\phi(N)^d N} \left(\frac{M}{\mathbf{R}_N}\right)^{\frac{|\boldsymbol{i}|-1}{2}} \sum_{1 \leqslant n_0 \leqslant N} \sum_{\boldsymbol{n} \prec \boldsymbol{i}} Q_{(n_0, \boldsymbol{n})}^\psi. \tag{8.2}$$

The following analogue of Proposition 5.2 is the key step in the proof of Theorem 2.13.

Proposition 8.1. *Assume that Hypothesis 2.4 holds, and $\mathbf{R}_N$ in (2.7) is a slowly varying function which diverges as $N \to \infty$. For each $M \in \mathbb{N}$ and $\psi \in C_c(\mathbb{R}^d \times [0,1])$, the random variables $(\Theta_{\boldsymbol{i}}^{N,M,\psi})_{\boldsymbol{i} \in D_M}$ converge in joint distribution to a family of independent Gaussian random variables $(\zeta_{\boldsymbol{i}}^\psi)_{\boldsymbol{i} \in D_M}$ with $\zeta_{\boldsymbol{i}}^\psi = 0$ if $i_1 < M$; and if $i_1 = M$, then $\zeta_{\boldsymbol{i}}^\psi$ has mean zero and variance*

$$V^\psi := \int_{(\mathbb{R}^d \times [0,1])^2} \psi(x,t) K((x,t),(x',t')) \psi(x',t') \mathrm{d}x \mathrm{d}t \mathrm{d}x' \mathrm{d}t', \tag{8.3}$$

where K is defined in (2.21).

Proof. The proof is similar to that of Proposition 5.2. We will only highlight the changes in the proof. For simplicity, we assume $d \neq 0$. The case $d = 0$ can be treated similarly.

First note that we can rewrite $\Theta_{\boldsymbol{i}}^{N,M;\psi}$ in the following form:

$$\Theta_{\boldsymbol{i}}^{N,M;\psi} = \frac{\left(\frac{M}{\mathbf{R}_N}\right)^{\frac{|\boldsymbol{i}|-1}{2}}}{\phi(N)^{\frac{d}{2}} N^{\frac{3}{2}}} \sum_{\boldsymbol{x} \in (\mathbb{Z}^d)^{|\boldsymbol{i}|}} \sum_{\substack{1 \leqslant n_1 \leqslant N \\ n_2 - n_1 \in I_{i_2}, \ldots, n_{|\boldsymbol{i}|} - n_{|\boldsymbol{i}|-1} \in I_{|\boldsymbol{i}|}}} q_{\boldsymbol{n}}^\psi(\boldsymbol{x}) \eta_{(\boldsymbol{n},\boldsymbol{x})}, \tag{8.4}$$

where $\eta_{(\boldsymbol{n},\boldsymbol{x})} = \prod_{j=1}^{|\boldsymbol{n}|} \eta_{(n_j, x_j)}$, and $q_{\boldsymbol{n}}^\psi(\boldsymbol{x}) := q_{i_1, n_1}^{N,\psi}(x_1) \prod_{j=2}^{|\boldsymbol{n}|} q_{n_j - n_{j-1}}(x_j - x_{j-1})$ with

$$q_{i_1, n_1}^{N,\psi}(x_1) := \sum_{\substack{x_0 \in \mathbb{Z}^d \\ n_0 \in (0, n_1) \cap (n_1 - I_{i_1})}} \psi\left(\frac{x_0}{\phi(N)}, \frac{n_0}{N}\right) q_{n_1 - n_0}(x_1 - x_0). \tag{8.5}$$

Note that the constraint $n_1 - n_0 \in I_{i_1}$ appearing in (8.2) (inside $\boldsymbol{n} \prec \boldsymbol{i}$) has been moved to (8.5). For simplicity, we will denote the summation constraints on $\boldsymbol{n}$ in (8.4) also by $\boldsymbol{n} \prec \boldsymbol{i}$. Note that we have just casted $\Theta_{\boldsymbol{i}}^{N,M;\psi}$ in the same form as $\Theta_{\boldsymbol{i}}^{N,M}$ in (5.5).

Variance Calculations. We first show that when $i_1 < M$, $\Theta_{\boldsymbol{i}}^{N,M;\psi} \to 0$ because its variance tends to 0. Note that

$$\mathrm{Var}\left(\Theta_{\boldsymbol{i}}^{N;M;\psi}\right) = \frac{\left(\frac{M}{\mathbf{R}_N}\right)^{|\boldsymbol{i}|-1}}{\phi(N)^d N^3} \sum_{\substack{\boldsymbol{n} \prec \boldsymbol{i} \\ \boldsymbol{x} \in (\mathbb{Z}^d)^{|\boldsymbol{i}|}}} (q_{i_1, n_1}^{N,\psi}(x_1))^2 \prod_{j=2}^{|\boldsymbol{i}|} q_{n_j - n_{j-1}}(x_j - x_{j-1})^2. \tag{8.6}$$

Note that given (n_1, x_1), the sum over $(n_2, x_2), ..., (n_{|\boldsymbol{i}|}, x_{|\boldsymbol{i}|})$ asymptotically equals $\left(\frac{\mathbf{R}_N}{M}\right)^{|\boldsymbol{i}|-1}$ by the same calculations as in (5.11). Therefore

$$
\mathbb{V}\mathrm{ar}\left(\Theta_{\boldsymbol{i}}^{N;M;\psi}\right) = \frac{1+o(1)}{\phi(N)^d N^3} \sum_{\substack{x_1 \in \mathbb{Z}^d \\ 1 \leqslant n_1 \leqslant N}} (q_{i_1, n_1}^{N,\psi}(x_1))^2
$$

$$
= \frac{1+o(1)}{\phi(N)^d N^3} \sum_{\substack{x_1 \in \mathbb{Z}^d \\ 1 \leqslant n_1 \leqslant N}} \sum_{\substack{1 \leqslant n_0, n_0' \in n_1 - I_{i_1} \\ x_0, x_0' \in \mathbb{Z}^d}} \psi(\hat{\mathsf{X}}_N)\psi(\hat{\mathsf{X}}_N') q_{n_1-n_0}(x_1 - x_0) q_{n_1-n_0'}(x_1 - x_0'). \quad (8.7)
$$

Since $\psi \in C_c(\mathbb{R}^d \times [0,1])$, we can choose $A > 0$ large enough such that $\mathrm{supp}(\psi) \subset [-A, A]^d \times [0,1]$. Then in (8.7), we can restrict the sums to $|x_0|_\infty, |x_0'|_\infty \leqslant A\phi(N)$. By the local limit theorem for $q_n(\cdot)$ in Hypothesis 2.4, we observe that the dominant contribution in (8.7) comes from $x_1 \in \mathbb{Z}^d$ with $|x_1|_\infty < \widetilde{A}\phi(N)$ if $\widetilde{A}$ is large enough. By first summing over (x_0, n_0) and (x_0', n_0'), we then have

$$
\mathbb{V}\mathrm{ar}\left(\Theta_{\boldsymbol{i}}^{N;M;\psi}\right) \leqslant \frac{C}{\phi(N)^d N^3} \sum_{\substack{|x_1|_\infty < \widetilde{A}\phi(N) \\ 1 \leqslant n_1 \leqslant N}} |\psi|_\infty^2 \sum_{\substack{x_0, x_0' \in \mathbb{Z}^d \\ 1 \leqslant n_0, n_0' \in n_1 - I_{i_1}}} q_{n_1-n_0}(x_1 - x_0) q_{n_1-n_0'}(x_1 - x_0')
$$

$$
= \frac{C}{\phi(N)^d N^3} \sum_{\substack{|x_1|_\infty < \widetilde{A}\phi(N) \\ 1 \leqslant n_1 \leqslant N}} \sum_{1 \leqslant n_0, n_0' \in n_1 - I_{i_1}} |\psi|_\infty^2
$$

$$
\leqslant \frac{C|\psi|_\infty^2}{\phi(N)^d N^3} \cdot N(2\widetilde{A})^d \phi(N)^d |I_{i_1}|^2 \xrightarrow[N\to\infty]{} 0, \quad (8.8)
$$

where the convergence holds because $\mathbf{R}_{\mathsf{t}_{i_1}}/\mathbf{R}_N \sim i_1/M < 1$ implies that $|I_{i_1}| \leqslant \mathsf{t}_{i_1} = o(N)$.

We next check that when $i_1 = M$, $\Theta_{\boldsymbol{i}}^{N;M;\psi}$ has the correct limiting variance. Note that because $I_{i_1} = (\mathsf{t}_{M-1}, \mathsf{t}_M] = (\mathsf{t}_{M-1}, N]$ with $\mathsf{t}_{M-1} \ll N$, by the same bound as in (8.8), we can enlarge the range of summation of n_0, n_0' in (8.7) to $1 \leqslant n_0, n_0' \leqslant n_1$ without changing the limiting variance. Moreover, by the local limit theorem for q_n in Hypothesis 2.4, we have

$$
\mathbb{V}\mathrm{ar}\left(\Theta_{\boldsymbol{i}}^{N;M;\psi}\right) = \frac{1+o(1)}{\phi(N)^d N^3} \sum_{\substack{1 \leqslant n_0, n_0' < n_1 \leqslant N \\ x_1, x_0, x_0' \in \mathbb{Z}^d}} \psi(\hat{\mathsf{X}}_N)\psi(\hat{\mathsf{X}}_N') q_{n_1-n_0}(x_1 - x_0) q_{n_1-n_0'}(x_1 - x_0')
$$

$$
= \frac{1+o(1)}{\phi(N)^d N^3} \sum_{\substack{1 \leqslant n_0, n_0' < n_1 \leqslant N \\ x_1, x_0, x_0' \in \mathbb{Z}^d}} \psi(\hat{\mathsf{X}}_N)\psi(\hat{\mathsf{X}}_N') \frac{g\left(\frac{x_1-x_0}{\phi(n_1-n_0)}\right)}{\phi(n_1-n_0)^d} \frac{g\left(\frac{x_1-x_0'}{\phi(n_1-n_0')}\right)}{\phi(n_1-n_0')^d} \quad (8.9)
$$

$$
= \frac{1+o(1)}{\phi(N)^{3d} N^3} \sum_{\substack{1 \leqslant n_0, n_0' < n_1 \leqslant N \\ x_1, x_0, x_0' \in \mathbb{Z}^d}} \psi(\hat{\mathsf{X}}_N)\psi(\hat{\mathsf{X}}_N') \frac{g\left(\frac{x_1-x_0}{\phi(N)}/\left(\frac{n_1-n_0}{N}\right)^{\frac{1}{d}}\right)}{\frac{n_1-n_0}{N}} \frac{g\left(\frac{x_1-x_0'}{\phi(N)}/\left(\frac{n_1-n_0'}{N}\right)^{\frac{1}{d}}\right)}{\frac{n_1-n_0'}{N}},
$$

where in the last equality we have replaced $\phi(n_1 - n_0)$ and $\phi(n_1 - n_0')$ respectively by $\phi(N)(\frac{n_1-n_0}{N})^{1/d}$ and $\phi(N)(\frac{n_1-n_0'}{N})^{1/d}$. This is justified when $n_1 - n_0, n_1 - n_0' > \varepsilon N$ for any fixed $\varepsilon > 0$, because $\phi(n) = (L(n)^2 n)^{1/d}$ and $L(\cdot)$ is slowly varying; while on the other hand, the contributions to (8.9) from n_0, n_0', n_1 with $n_1 - n_0 \leqslant \varepsilon N$ or $n_1 - n_0' \leqslant \varepsilon N$ can be made arbitrarily small by choosing ε small, thanks to the same estimates as in (8.8). A

Riemann sum approximation with $y := x_0/\phi(N)$, $y' := x_0'/\phi(N)$, $z := x_1/\phi(N)$, $s = n_0/N$, $s' := n_0'/N$ and $t := n_1/N$ then gives

$$
\mathbb{V}\mathrm{ar}\left(\Theta_{\boldsymbol{i}}^{N;M;\psi}\right) \underset{N\to\infty}{\longrightarrow} \int \cdots \int_{\substack{y,y',z\in\mathbb{R}^d \\ 0\leqslant s,s'<t\leqslant 1}} \psi(y,s)\psi(y',s') \frac{g\left(\frac{z-y}{(t-s)^{1/d}}\right)}{t-s} \frac{g\left(\frac{z-y'}{(t-s')^{1/d}}\right)}{t-s'} \mathrm{d}z\mathrm{d}t\mathrm{d}y\mathrm{d}y'\mathrm{d}s\mathrm{d}s'
$$

$$
= \int \cdots \int_{\substack{y,y',z\in\mathbb{R}^d \\ 0\leqslant s,s'\leqslant 1}} \psi(y,s)K((y,s),(y',s'))\psi(y',s')\mathrm{d}y\mathrm{d}y'\mathrm{d}s\mathrm{d}s' = V^\psi, \qquad (8.10)
$$

where

$$
K((y,s),(y',s')) = \int_{s\vee s'}^1 \int_{\mathbb{R}^d} \frac{g\left(\frac{z-y}{(t-s)^{1/d}}\right)}{t-s} \frac{g\left(\frac{z-y'}{(t-s')^{1/d}}\right)}{t-s'} \mathrm{d}z\mathrm{d}t
$$

$$
= \int_{s\vee s'}^1 \int_{\mathbb{R}^d} g_{t-s}(z-y)g_{t-s'}(z-y')\mathrm{d}z\mathrm{d}t = \int_{s\vee s'}^1 g_{2t-s-s'}(y-y')\mathrm{d}t
$$

$$
= \int_{s\vee s'}^1 \frac{g\left(\frac{y-y'}{(2t-s-s')^{1/d}}\right)}{2t-s-s'}\mathrm{d}t = \frac{1}{2}\int_{|s-s'|}^{2-s-s'} \frac{g\left(\frac{y-y'}{u^{1/d}}\right)}{u}\mathrm{d}u. \qquad (8.11)
$$

Here we used the fact that the transition density g_t of a Brownian motion in $\mathbb{R}^2$ (or a Cauchy process in $\mathbb{R}$) is symmetric and scaling invariant, and $g = g_1$. Note that K agrees with the kernel in (2.21), which completes the variance verification.

Fourth moment calculations. We now apply the fourth moment theorem Theorem 4.2 to prove Proposition 8.1. By the variance calculations above, it suffices to restrict our attention to $(\Theta_{\boldsymbol{i}}^{N,M;\psi})_{\boldsymbol{i}\in D_M}$ with $i_1 = M$. For distinct $\boldsymbol{i},\boldsymbol{i}' \in D_M$ with $i_1 = i_1' = M$, it is easily seen that $\mathrm{E}[\Theta_{\boldsymbol{i}}^{N,M;\psi}\Theta_{\boldsymbol{i}'}^{N,M;\psi}] = 0$. Therefore condition (i) in Theorem 4.2 is satisfied. Clearly condition (iii) also holds. It only remains to verify the fourth moment condition:

$$
\lim_{N\to\infty} \mathbb{E}\left[(\Theta_{\boldsymbol{i}}^{N,M;\psi})^4\right] = 3V^\psi \qquad \forall\, \boldsymbol{i} \in D_M \text{ with } i_1 = M, \qquad (8.12)
$$

assuming that $(\eta_{(n,x)})_{n\in\mathbb{N},x\in\mathbb{Z}^d}$ are i.i.d. standard normal.

Using the representation for $\Theta_{\boldsymbol{i}}^{N,M;\psi}$ in (8.4), we have a similar expansion of the fourth-moment as in (5.13) and (5.14):

$$
\mathbb{E}\left[(\Theta_{\boldsymbol{i}}^{N,M;\psi})^4\right] = \frac{\left(\frac{M}{\mathbf{R}_N}\right)^{2|\boldsymbol{i}|-2}}{\phi(N)^{2d}N^6} \sum_{\substack{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\prec\boldsymbol{i} \\ \boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}\in(\mathbb{Z}^d)^{|\boldsymbol{i}|}}} q_{\boldsymbol{a}}^\psi(\boldsymbol{x})q_{\boldsymbol{b}}^\psi(\boldsymbol{y})q_{\boldsymbol{c}}^\psi(\boldsymbol{z})q_{\boldsymbol{d}}^\psi(\boldsymbol{w})\,\mathbb{E}\left[\eta_{(\boldsymbol{a},\boldsymbol{x})}\eta_{(\boldsymbol{b},\boldsymbol{y})}\eta_{(\boldsymbol{c},\boldsymbol{z})}\eta_{(\boldsymbol{d},\boldsymbol{w})}\right], \quad (8.13)
$$

where $q_{\boldsymbol{n}}^\psi(\boldsymbol{x}) := q_{M,n_1}^{N,\psi}(x_1)\prod_{j=2}^{|\boldsymbol{n}|} q_{n_j-n_{j-1}}(x_j-x_{j-1})$ with $q_{M,n_1}^{N,\psi}(x_1)$ defined as in (8.5).

Note that the only difference between the expansion in (8.13) and the expansion for $\mathbb{E}\left[(\Theta_{\boldsymbol{i}}^{N,M})^4\right]$ in (5.13)–(5.14) is that the factors $q_{n_1}(x_1)$ are replaced by $q_{M,n_1}^{N,\psi}(x_1)$, and the corresponding normalizing constant $\sum_{x_1\in\mathbb{Z}^d,n_1\in I_M} q_{n_1}(x_1)^2 \sim \mathbf{R}_N/M$ is replaced by

$$
\frac{1}{V^\psi} \sum_{\substack{x_1\in\mathbb{Z}^d \\ 1\leqslant n_1\leqslant N}} (q_{M,n_1}^{N,\psi}(x_1))^2 \sim \phi(N)^d N^3, \qquad (8.14)
$$

where this asymptotic relation follows from the variance calculations in (8.7) and (8.10).

The verification of the fourth moment condition (8.12) now follows the same argument as that for $\Theta_{\boldsymbol{i}}^{N,M}$ in Section 5. Recall from (5.15) that

$$\mathsf{p} := \big|(\boldsymbol{a},\boldsymbol{x}) \cup (\boldsymbol{b},\boldsymbol{y}) \cup (\boldsymbol{c},\boldsymbol{z}) \cup (\boldsymbol{d},\boldsymbol{w})\big| = \sum_{(n,r)\in\mathbb{N}\times\mathbb{Z}^d} \mathbb{1}_{\{(n,r)\in(\boldsymbol{a},\boldsymbol{x})\cup(\boldsymbol{b},\boldsymbol{y})\cup(\boldsymbol{c},\boldsymbol{z})\cup(\boldsymbol{d},\boldsymbol{w})\}},$$

and we relabel $(\boldsymbol{a},\boldsymbol{x}) \cup (\boldsymbol{b},\boldsymbol{y}) \cup (\boldsymbol{c},\boldsymbol{z}) \cup (\boldsymbol{d},\boldsymbol{w}) = \{(f_1,h_1),(f_2,h_2),\ldots,(f_p,h_p)\}$, with $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_p$.

In the proof of Proposition 5.2, we considered 5 cases. **Case 1** with $\mathsf{p} > 2|\boldsymbol{i}|$ can be treated exactly the same way here.

For **Case 2** with $\mathsf{p} < 2|\boldsymbol{i}|$, we can follow the same arguments up to (5.20) (note that $0 \leqslant q_n(x) \leqslant 1$ under our assumptions). If there are only two factors of q and $q^{N,\psi}$ in the l.h.s. of (5.20) that involve (f_p,h_p), then we apply Cauchy-Schwartz exactly as in (5.20), which gives the desired factors of $(\mathbf{R}_N/M)^{1/2}$ or $(\phi(N)^d N^3)^{1/2}$. If there are four factors of q and $q^{N,\psi}$, then we can pick any two factors and bound the factor of q by 1, and bound the factor of $q^{N,\psi}$ by $N\|\psi\|_\infty$, since

$$q_{M,n_1}^{N,\psi}(x_1) := \sum_{\substack{x_0\in\mathbb{Z}^d \\ n_0\in(0,n_1)\cap(n_1-I_M)}} \psi\Big(\frac{x_0}{\phi(N)},\frac{n_0}{N}\Big) q_{n_1-n_0}(x_1-x_0) \leqslant N\|\psi\|_\infty.$$

Note that the pre-factor in (8.13) will be cancelled out exactly when each q contributes a factor of $(\mathbf{R}_N/M)^{1/2}$ to the sum, and each $q^{N,\psi}$ contributes a factor of $(\phi(N)^d N^3)^{1/2}$. Each replacement of q by 1 in (5.20) leads to the loss of a factor $(\mathbf{R}_N/M)^{1/2}$ in the bound for $\mathcal{S}_N^{(p)}$ in (5.21), and similarly, each replacement of $q^{N,\psi}$ by $N\|\psi\|_\infty$ leads to the loss of a factor $(\phi(N)^d N^3)^{1/2}/N\|\psi\|_\infty$. Summing successively over (f_{p-1},h_{p-1}), (f_{p-2},h_{p-2}), ..., (f_1,h_1) then gives a similar bound as in (5.21), so that the contributions in this case is negligible.

For **Case 3** where $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ consists of three or four connected components, it again reduces to **Case 1**.

For **Case 4** where $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ consists of a single connected component, we follow the same calculations up to (5.27), where we note that because $f_u = c_1$ is the first index of the sequence $\boldsymbol{c}$, the first factor in the r.h.s. of (5.27) should be replaced by

$$\Big(\sum_{x\in\mathbb{Z}^d,\,f_1<n\leqslant f_1+\bar{m}_1} (q_n^{N,\psi}(x))^2\Big)^{1/2} \leqslant \big(C\bar{m}_1(2\widetilde{A})^d\phi(N)^d|I_M|^2\big)^{1/2} = o\big((\phi(N)^d N^3)^{1/2}\big),$$

where the inequality follows the same calculations as in (8.8), and the last equality holds since $|I_M| \leqslant N$ and $\bar{m}_1 = |\boldsymbol{i}|\mathsf{t}_{M-2} = o(N)$, by its definition in (5.26). This implies a similar bound as in (5.28) and shows that this case is also negligible.

Case 5 where $[\boldsymbol{a}] \cup [\boldsymbol{b}] \cup [\boldsymbol{c}] \cup [\boldsymbol{d}]$ consists of two connected components gives the full contribution to the limiting fourth moment of $\Theta_{\boldsymbol{i}}^{N,M;\psi}$ in (8.12), and the argument is exactly the same as in the proof of Proposition 5.2. $\qquad\square$

Proof of Theorem 2.13. Recall from (2.19) that

$$J_N^\psi := \frac{1}{\phi(N)^d N} \sum_{\mathsf{X}\in\mathbb{Z}^d\times\mathbb{N}_0} \sqrt{\mathbf{R}_N L(N)^2}\,\Big(Z_{N,\beta_N}^\omega(\mathsf{X})-1\Big)\psi(\hat{\mathsf{X}}_N).$$

To prove Theorem 2.13, i.e., show that J_N^ψ converges in distribution to a Gaussian random variable with mean zero and variance given in (2.20)–(2.21), we plug in the polynomial chaos

expansion of $Z^\omega_{N,\beta_N}(\mathsf{X})$ from (4.1) (with $(x_0, n_0) := \mathsf{X}$) and rewrite J^ψ_N as

$$J^\psi_N = \frac{L(N)}{\phi(N)^d N} \sum_{k=1}^\infty \frac{\hat{\beta}^k}{\mathbf{R}_N^{(k-1)/2}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_k \leqslant N \\ x_1, x_2, \ldots, x_k \in \mathbb{Z}^d}} q^{N,\psi}_{n_1}(x_1) \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \prod_{i=1}^k \eta_{(n_i, x_i)}, \quad (8.15)$$

where

$$q^{N,\psi}_{n_1}(x_1) := \sum_{x_0 \in \mathbb{Z}^d, 0 < n_0 < n_1} \psi\Big(\frac{x_0}{\phi(N)}, \frac{n_0}{N}\Big) q_{n_1 - n_0}(x_1 - x_0). \quad (8.16)$$

The approximation steps **(A1)**–**(A3)**, described for the partition function Z^ω_{N,β_N} in Section 4 (and proved in Section 6), can also be performed for J^ψ_N with only minor differences. Similar to (4.11), we can therefore approximate J^ψ_N by

$$\hat{J}^\psi_{N,M} := \sum_{k=1}^M \frac{\hat{\beta}^k}{M^{(k-1)/2}} \sum_{\boldsymbol{i} \in \{1, \ldots, M\}^k_\sharp} \Theta^{N;M;\psi}_{\boldsymbol{i}^{(1)}} \Theta^{N;M}_{\boldsymbol{i}^{(2)}} \cdots \Theta^{N;M}_{\boldsymbol{i}^{(\mathfrak{m})}}, \quad (8.17)$$

where $\boldsymbol{i}^{(1)}, \ldots, \boldsymbol{i}^{(\mathfrak{m})}$ is the decomposition of $\boldsymbol{i}$ into dominated sequences and $\Theta^{N;M;\psi}_{\boldsymbol{i}}$ was defined in (8.2). Taking the limit $N \to \infty$ in (8.17) and applying Proposition 8.1 as done in (4.13), we obtain

$$\hat{J}^\psi_{N,M} \xrightarrow[N \to \infty]{d} \boldsymbol{J}^{(M)}_{\hat{\beta}} := \sum_{k=1}^\infty \frac{\hat{\beta}^k}{M^{\frac{k-1}{2}}} \sum_{\boldsymbol{i} \in \{1, \ldots, M\}^k_\sharp} \prod_{l=1}^{\mathfrak{m}(\boldsymbol{i})} \zeta^\psi_{\boldsymbol{i}^{(l)}} = \sum_{k=1}^\infty \frac{\hat{\beta}^k}{M^{\frac{k-1}{2}}} \sum_{\substack{\boldsymbol{i} \in \{1, \ldots, M\}^k_\sharp \\ i_1 = M}} \zeta^\psi_{\boldsymbol{i}}, \quad (8.18)$$

where we used the fact that by Proposition 8.1 only $\zeta^\psi_{\boldsymbol{i}}$ with $i_1 = M$ are non-zero. Again, by Proposition 8.1, we note that $\boldsymbol{J}^{(M)}_{\hat{\beta}}$ is a normal random variable (as a sum of independent normal variables) with mean zero and variance

$$\mathbb{V}\mathrm{ar}(\boldsymbol{J}^{(M)}_{\hat{\beta}}) = \sum_{k=1}^\infty \frac{\hat{\beta}^{2k}}{M^{k-1}} |\{\boldsymbol{i} \in \{1, \ldots, M\}^k_\sharp : i_1 = M\}| V^\psi \xrightarrow[M \to \infty]{} \frac{\hat{\beta}^2}{1 - \hat{\beta}^2} V^\psi,$$

which is exactly the variance σ^2_ψ in (2.20). Therefore $\boldsymbol{J}^{(M)}_{\hat{\beta}}$ converges to a normal random variable with mean zero and variance σ^2_ψ. This is the analogue of step **(A4)** in Section 4, which concludes the proof of Theorem 2.13. $\qquad\square$

9. Proof for the $2d$ Stochastic Heat Equation

In this section, we prove Theorems 2.15 and 2.17 for the regularized 2d Stochastic Heat Equation (2.23). The basic strategy is to compare the solution u^ε with the partition function of a directed polymer on $\mathbb{Z}^{2+1}$, so that we can apply Theorems 2.12 and 2.13.

The starting point is the Feynman-Kac representation (2.25) for $\tilde{u}^\varepsilon(t, x)$, which has the same distribution as $u^\varepsilon(t, x)$, but differs by a time reversal in the Feynman-Kac formula (2.24). We can extend this representation jointly to all $(\tilde{u}^\varepsilon(t, x))_{t \in [0,1], x \in \mathbb{R}^2}$. Namely, let

$$\tilde{u}^\varepsilon(t, x) = \mathrm{E}_{(\varepsilon^{-2}(1-t), \varepsilon^{-1}x)}\Big[\exp\Big\{\beta_\varepsilon \int_{\varepsilon^{-2}(1-t)}^{\varepsilon^{-2}} \int_{\mathbb{R}^2} j(B_s - y)\widetilde{W}(\mathrm{d}s, \mathrm{d}y) - \frac{1}{2}\beta_\varepsilon^2 \varepsilon^{-2} t \|j\|_2^2\Big\}\Big]$$

$$= \mathrm{E}_{(\varepsilon^{-2}(1-t), \varepsilon^{-1}x)}\Big[:\exp\Big\{\beta_\varepsilon \int_{\varepsilon^{-2}(1-t)}^{\varepsilon^{-2}} \int_{\mathbb{R}^2} j(B_s - y)\widetilde{W}(\mathrm{d}s, \mathrm{d}y)\Big\} : \Big], \quad (9.1)$$

where $\mathrm{E}_{(s,y)}$ denotes expectation for a Brownian motion B starting at y at time s. It is then clear that $(\widetilde{u}^\varepsilon(t,x))_{t\in[0,1],x\in\mathbb{R}^2}$ has the same distribution as $(u^\varepsilon(t,x))_{t\in[0,1],x\in\mathbb{R}^2}$.

Furthermore, by the definition of the Wick exponential $: \exp :$ [J97, §3.2], we can write

$$\widetilde{u}^\varepsilon(t,x) = 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int\cdots\int_{\varepsilon^{-2}(1-t)<t_1<\cdots<t_k<\varepsilon^{-2}} \int_{\mathbb{R}^{2k}} \mathrm{E}_{(\varepsilon^{-2}(1-t),\varepsilon^{-1}x)}\Big[\prod_{i=1}^k j(B_{t_i}-x_i)\Big]\prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i,\mathrm{d}x_i) \quad (9.2)$$

$$= 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int\cdots\int_{\varepsilon^{-2}(1-t)<t_1<\cdots<t_k<\varepsilon^{-2}} \int_{\mathbb{R}^{2k}} \Big(\int_{\mathbb{R}^{2k}}\prod_{i=1}^k p_{t_i-t_{i-1}}(y_i-y_{i-1})j(y_i-x_i)\mathrm{d}\vec{y}\Big)\prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i,\mathrm{d}x_i),$$

where $p_t(\cdot)$ is the probability density for B_t, $(t_0,y_0)=(\varepsilon^{-2}(1-t),\varepsilon^{-1}x)$, and $\mathrm{d}\vec{y}=\mathrm{d}y_1\cdots\mathrm{d}y_k$.

We are now ready to give the proofs.

Proof of Theorem 2.15. Let $\hat\beta < 1$. We will perform a series of approximations, eventually approximating $\widetilde{u}^\varepsilon(t_\varepsilon^{(i)},x_\varepsilon^{(i)})$ by $Z_{\varepsilon^{-2},\beta_\varepsilon}^\eta(\varepsilon^{-1}x_\varepsilon^{(i)},\varepsilon^{-2}(1-t_\varepsilon^{(i)}))$, $1\leqslant i\leqslant r$, the partition functions of a directed polymer model on $\mathbb{Z}^{2+1}$. Since our approximations will be carried out in L^2 on the same probability space, it suffices to consider a single term $\widetilde{u}^\varepsilon(t_\varepsilon^{(i)},x_\varepsilon^{(i)})$, or just $\widetilde{u}^\varepsilon(1,0)$.

Step 1. First we show that $\widetilde{u}^\varepsilon(1,0)$ can be approximated in L^2 if for each $k\in\mathbb{N}$, the integral over $\vec{t}:=(t_1,\ldots,t_k)$ in (9.2) is restricted to the set

$$T_{k,\varepsilon} := \{(t_1,\ldots,t_k)\in(0,\varepsilon^{-2})^k : t_i - t_{i-1} \geqslant \sqrt{\log\varepsilon^{-1}} \ \forall\, 1\leqslant i\leqslant k\}. \quad (9.3)$$

This is necessary because in the L^2 approximations that follow, $\int p_{t_i-t_{i-1}}^2(y_i-y_{i-1})\mathrm{d}y_i$ is not integrable in t_i due to the singularity when t_i is near t_{i-1}. We thus approximate $\widetilde{u}^\varepsilon(1,0)$ by

$$v^\varepsilon(1,0) := 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int_{T_{k,\varepsilon}}\int_{\mathbb{R}^{2k}}\Big(\int_{\mathbb{R}^{2k}}\prod_{i=1}^k p_{t_i-t_{i-1}}(y_i-y_{i-1})j(y_i-x_i)\mathrm{d}\vec{y}\Big)\prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i,\mathrm{d}x_i).$$
$$(9.4)$$

By Itô isometry and the orthogonality of terms in different orders of the Wiener chaos,

$$\mathbb{E}[(\widetilde{u}^\varepsilon(1,0)-v^\varepsilon(1,0))^2] = \sum_{k=1}^\infty \beta_\varepsilon^{2k}\int_{T_{k,\varepsilon}^c}\int_{\mathbb{R}^{2k}}\Big(\int_{\mathbb{R}^{2k}}\prod_{i=1}^k p_{t_i-t_{i-1}}(y_i-y_{i-1})j(y_i-x_i)\mathrm{d}\vec{y}\Big)^2\mathrm{d}\vec{x}\mathrm{d}\vec{t}.$$

Given $\vec{t}\in T_{k,\varepsilon}^c$, let $I(\vec{t}) := \{1\leqslant i\leqslant k : t_i-t_{i-1}\leqslant\sqrt{\log\varepsilon^{-1}}\}\subseteq\{1,\ldots,k\}$. We can then rewrite the integrand $\prod_{i=1}^k p_{t_i-t_{i-1}}(y_i-y_{i-1})j(y_i-x_i)$ above as

$$\Big(\prod_{i\in I(\vec{t})} p_{t_i-t_{i-1}}(y_i-y_{i-1})\prod_{i\notin I(\vec{t})} j(y_i-x_i)\Big)\times\Big(\prod_{i\notin I(\vec{t})} p_{t_i-t_{i-1}}(y_i-y_{i-1})\prod_{i\in I(\vec{t})} j(y_i-x_i)\Big).$$

Taking the first factor as a probability density for $\vec{y}$ while taking the second factor as the integrand, we can then apply Jensen's inequality to obtain the bound

$$\mathbb{E}[(\widetilde{u}^\varepsilon(1,0) - v^\varepsilon(1,0))^2] \tag{9.5}$$

$$\leqslant \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon}^c \times \mathbb{R}^{4k}} \prod_{i \notin I(\vec{t})} p_{t_i-t_{i-1}}^2(y_i - y_{i-1}) \prod_{i \in I(\vec{t})} j^2(y_i - x_i) \prod_{i \in I(\vec{t})} p_{t_i-t_{i-1}}(y_i - y_{i-1}) \prod_{i \notin I(\vec{t})} j(y_i - x_i) \mathrm{d}\vec{x}\mathrm{d}\vec{y}\mathrm{d}\vec{t},$$

$$= \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon}^c} \int_{\mathbb{R}^{2k}} \|j\|_2^{2|I(\vec{t})|} \prod_{i \notin I(\vec{t})} p_{t_i-t_{i-1}}^2(y_i - y_{i-1}) \prod_{i \in I(\vec{t})} p_{t_i-t_{i-1}}(y_i - y_{i-1}) \mathrm{d}\vec{y}\mathrm{d}\vec{t}$$

$$= \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon}^c} \int_{\mathbb{R}^{2k}} \|j\|_2^{2|I(\vec{t})|} \prod_{i \notin I(\vec{t})} \frac{e^{-\frac{|y_i-y_{i-1}|^2}{t_i-t_{i-1}}}}{4\pi^2(t_i - t_{i-1})^2} \prod_{i \in I(\vec{t})} \frac{e^{-\frac{|y_i-y_{i-1}|^2}{2(t_i-t_{i-1})}}}{2\pi(t_i - t_{i-1})} \mathrm{d}\vec{y}\mathrm{d}\vec{t}$$

$$= \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon}^c} \|j\|_2^{2|I(\vec{t})|} \prod_{i \notin I(\vec{t})} \frac{1}{4\pi(t_i - t_{i-1})} \mathrm{d}\vec{t}$$

$$\leqslant \sum_{k=1}^\infty \beta_\varepsilon^{2k} \sum_{\substack{I \neq \varnothing \\ I \subset \{1,\ldots,k\}}} \left(\|j\|_2^2 \sqrt{\log \varepsilon^{-1}}\right)^{|I|} \left(\frac{\log(\varepsilon^{-2})}{4\pi}\right)^{k-|I|}$$

$$= \sum_{k=1}^\infty \hat{\beta}^{2k} \left(\frac{2\pi}{\log \varepsilon^{-1}}\right)^k \left\{\left(\|j\|_2^2 \sqrt{\log \varepsilon^{-1}} + \frac{\log(\varepsilon^{-2})}{4\pi}\right)^k - \left(\frac{\log(\varepsilon^{-2})}{4\pi}\right)^k\right\} \xrightarrow[\varepsilon \to 0]{} 0, \tag{9.6}$$

where in the last step we used a binomial expansion and the last convergence follows from the dominated convergence theorem, since $\hat{\beta} < 1$.

Step 2. We next show that $v^\varepsilon(1,0)$ can be approximated in L^2 by $w^\varepsilon(1,0)$, where we replace $p_{t_i-t_{i-1}}(y_i - y_{i-1})$ in (9.4) by $p_{t_i-t_{i-1}}(x_i - x_{i-1})$, i.e.,

$$w^\varepsilon(1,0) := 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int_{T_{k,\varepsilon}} \int_{\mathbb{R}^{2k}} \left(\int_{\mathbb{R}^{2k}} \prod_{i=1}^k p_{t_i-t_{i-1}}(x_i - x_{i-1}) j(y_i - x_i) \mathrm{d}\vec{y}\right) \prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i, \mathrm{d}x_i)$$

$$= 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int_{T_{k,\varepsilon}} \int_{\mathbb{R}^{2k}} \prod_{i=1}^k p_{t_i-t_{i-1}}(x_i - x_{i-1}) \prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i, \mathrm{d}x_i). \tag{9.7}$$

Indeed, by Jensen

$$\mathbb{E}[(v^\varepsilon(1,0) - w^\varepsilon(1,0))^2]$$

$$= \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times \mathbb{R}^{2k}} \left\{\int_{\mathbb{R}^{2k}} \left(\prod_{i=1}^k p_{t_i-t_{i-1}}(y_i - y_{i-1}) - \prod_{i=1}^k p_{t_i-t_{i-1}}(x_i - x_{i-1})\right) \prod_{i=1}^k j(y_i - x_i) \mathrm{d}\vec{y}\right\}^2 \mathrm{d}\vec{x}\mathrm{d}\vec{t}$$

$$\leqslant \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times \mathbb{R}^{2k} \times \mathbb{R}^{2k}} \left(\prod_{i=1}^k p_{t_i-t_{i-1}}(y_i - y_{i-1}) - \prod_{i=1}^k p_{t_i-t_{i-1}}(x_i - x_{i-1})\right)^2 \prod_{i=1}^k j(y_i - x_i) \mathrm{d}\vec{y}\mathrm{d}\vec{x}\mathrm{d}\vec{t}. \tag{9.8}$$

To show that this bound goes to 0 as $\varepsilon \downarrow 0$, we will divide the integral over $\vec{y}$ into two parts.

Using $(a + b)^2 \leqslant 2(a^2 + b^2)$, we can bound (9.8) by

$$2 \sum_{k=1}^{\infty} \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times \mathbb{R}^{2k} \times \mathbb{R}^{2k}} \Big(\prod_{i=1}^k p_{t_i - t_{i-1}}^2 (y_i - y_{i-1}) + \prod_{i=1}^k p_{t_i - t_{i-1}}^2 (x_i - x_{i-1}) \Big) \prod_{i=1}^k j(y_i - x_i) \mathrm{d}\vec{y} \mathrm{d}\vec{x} \mathrm{d}\vec{t} \quad (9.9)$$

$$= 4 \sum_{k=1}^{\infty} \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times \mathbb{R}^{2k}} \prod_{i=1}^k p_{t_i - t_{i-1}}^2 (y_i - y_{i-1}) \mathrm{d}\vec{y} \mathrm{d}\vec{t} = 4 \sum_{k=1}^{\infty} \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon}} \prod_{i=1}^k \frac{1}{4\pi(t_i - t_{i-1})} \mathrm{d}\vec{t}$$

$$\leqslant \quad 4 \sum_{k=1}^{\infty} \hat{\beta}^{2k} \Big(\frac{2\pi}{\log \varepsilon^{-1}} \Big)^k \Big(\prod_{i=1}^k \frac{\log(\varepsilon^{-2})}{4\pi} \Big) \xrightarrow[\varepsilon \to 0]{} 4 \sum_{k=1}^{\infty} \hat{\beta}^{2k},$$

which is finite since $\hat{\beta} < 1$. Firstly, this calculation implies that it suffices to show that each summand on the r.h.s. of (9.8), for a fixed $k \in \mathbb{N}$, tends to 0 as $\varepsilon \to 0$. Secondly, given $\vec{t} \in T_{k,\varepsilon}$, if we restrict the integral over $\vec{x}$ and $\vec{y}$ in (9.9) to the set

$$E_k = E_k(\vec{t}) := \{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2k} \times \mathbb{R}^{2k} : |y_i - y_{i-1}| \leqslant |t_i - t_{i-1}|^{\frac{3}{4}} \ \forall 1 \leqslant i \leqslant k \}, \quad (9.10)$$

then it is easily seen that the k-th term in (9.9) still converges to the same limit as $\varepsilon \downarrow 0$, because the dominant contribution to $\int_{\mathbb{R}^2} p_t^2(y) \mathrm{d}y$ comes from $|y|$ of the order $\sqrt{t}$. Therefore if the k-th integral over $\vec{x}$ and $\vec{y}$ in (9.9) is restricted to E_k^c, then the k-th term in (9.9) tends to 0 as $\varepsilon \downarrow 0$, and the same is true for the k-th term in (9.8).

It then only remains to consider the k-th term in (9.8), where the integral over $\vec{x}$ and $\vec{y}$ is restricted to E_k. This can be bounded by

$$\beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times E_k} \Big(\prod_{i=1}^k p_{t_i - t_{i-1}}(x_i - x_{i-1}) - \prod_{i=1}^k p_{t_i - t_{i-1}}(y_i - y_{i-1}) \Big)^2 \prod_{i=1}^k j(y_i - x_i) \mathrm{d}\vec{y} \mathrm{d}\vec{x} \mathrm{d}\vec{t}$$

$$= \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times E_k} \prod_{i=1}^k p_{t_i - t_{i-1}}^2 (y_i - y_{i-1}) \Big(e^{\sum_{i=1}^k \frac{|x_i - x_{i-1}|^2 - |y_i - y_{i-1}|^2}{2(t_i - t_{i-1})}} - 1 \Big)^2 \prod_{i=1}^k j(y_i - x_i) \mathrm{d}\vec{y} \mathrm{d}\vec{x} \mathrm{d}\vec{t}$$

$$\leqslant \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times E_k} \prod_{i=1}^k p_{t_i - t_{i-1}}^2 (y_i - y_{i-1}) \Big(e^{\frac{Ck}{(\log \varepsilon^{-1})^{1/8}}} - 1 \Big)^2 \prod_{i=1}^k j(y_i - x_i) \mathrm{d}\vec{y} \mathrm{d}\vec{x} \mathrm{d}\vec{t} \xrightarrow[\varepsilon \to 0]{} 0, \quad (9.11)$$

where the convergence follows easily from the domination by (9.9), and to obtain the inequality, we argued as follows. If we fix $A > 0$ such that the support of $j(\cdot)$ is contained in a ball of radius A, the factor $j(y_i - x_i)$ in (9.11) entails $|y_i - x_i| \leqslant A$, for all $1 \leqslant i \leqslant k$. Then, writing $|a|^2 - |b|^2 = |a - b|^2 + 2|b||a - b|$, we obtain

$$\frac{|\, |x_i - x_{i-1}|^2 - |y_i - y_{i-1}|^2 \,|}{2(t_i - t_{i-1})} \leqslant \frac{(2A)^2 + 4A|y_i - y_{i-1}|}{2(t_i - t_{i-1})} \leqslant \frac{C}{(\log \varepsilon^{-1})^{\frac{1}{8}}}$$

for any $C > 0$ when $\varepsilon > 0$ is sufficiently small. Finally, we note that $(e^x - 1)^2 \leqslant (e^{|x|} - 1)^2$.

This concludes the proof that the bound in (9.8) tends to 0 as $\varepsilon \downarrow 0$, which shows that $v^\varepsilon(1, 0)$ and $w^\varepsilon(1, 0)$ have the same limiting distribution.

Step 3. We now show that $w^\varepsilon(1, 0)$ can be approximated in L^2 by $Z_{\varepsilon^{-2}, \beta_\varepsilon}^\eta = Z_{\varepsilon^{-2}, \beta_\varepsilon}^\eta(0, 0)$, the partition function of a directed polymer on $\mathbb{Z}^{2+1}$ starting at $(0, 0)$, defined on the same probability space as w^ε.

Let $(S_n)_{n \geqslant 0}$ be an irreducible aperiodic random walk on $\mathbb{Z}^2$ with n-step increment distribution $\hat{p}_n(\cdot)$, such that $S_0 = 0$ and $\mathrm{E}[S_1(i)S_1(j)] = 1_{\{i=j\}}$ for $i, j = 1, 2$, where $S_1(i)$ denotes the i-th coordinate of S_1. Recall from (4.1) that the partition function of a directed polymer model constructed from S and i.i.d. space-time disorder η, with parameter β_ε and polymer length ε^{-2}, admits the following polynomial chaos expansion:

$$Z^\eta_{\varepsilon^{-2}, \beta_\varepsilon} = 1 + \sum_{k=1}^{\varepsilon^{-2}} \beta_\varepsilon^k \sum_{\substack{1 \leqslant n_1 < \cdots < n_k \leqslant \varepsilon^{-2} \\ x_1, \ldots, x_k \in \mathbb{Z}^2}} \prod_{i=1}^k \hat{p}_{n_i - n_{i-1}}(x_i - x_{i-1}) \prod_{i=1}^k \eta_{n_i, x_i}, \qquad (9.12)$$

where $(\eta_{n,x})_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ are i.i.d. random variables with mean 0. For our purposes, we will let η be i.i.d. standard normal variables defined from the space-time white noise $\widetilde{W}$ in the chaos expansion for $w^\varepsilon(1, 0)$ in (9.7):

$$\eta_{n,x} := \int_{\Delta_{n,x}} \widetilde{W}(\mathrm{d}s, \mathrm{d}y), \qquad n \in \mathbb{N}, \ x = (x(1), x(2)) \in \mathbb{Z}^2, \qquad (9.13)$$

where $\Delta_{n,x} := [n-1, n] \times [x(1) - 1, x(1)] \times [x(2) - 1, x(2)]$.

We can then rewrite (9.12) as

$$Z^\eta_{\varepsilon^{-2}, \beta_\varepsilon} = 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int \cdots \int_{\substack{0 < t_1 < \cdots < t_k < \varepsilon^{-2}}} \int_{(\mathbb{R}^2)^k} \prod_{i=1}^k \hat{p}_{\lceil t_i \rceil - \lceil t_{i-1} \rceil}(\lceil x_i \rceil - \lceil x_{i-1} \rceil) \prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i, \mathrm{d}x_i), \quad (9.14)$$

where we set $\hat{p}_0(\cdot) \equiv 0$.

By Gnedenko's local limit theorem (see e.g. [BGT89, Theorem 8.4.1])

$$\hat{p}_n(x) = \frac{1}{2\pi n}\left(e^{-\frac{|x|^2}{2n}} + o(1)\right) = p_n(x) + o\left(\frac{1}{n}\right) \quad \text{uniformly in } x \in \mathbb{Z}^2 \text{ as } n \to \infty, \quad (9.15)$$

where we recall that $p_n(x)$ in the right hand side is the transition kernel of Brownian motion. By similar calculations as those leading to (9.6), we can restrict the integral over $(t_1, \ldots, t_k)$ to $T_{k,\varepsilon}$ as in the definition of $v^\varepsilon(1, 0)$ and $w^\varepsilon(1, 0)$ in (9.4) and (9.7), i.e., if

$$\widetilde{Z}^\eta_{\varepsilon^{-2}, \beta_\varepsilon} := 1 + \sum_{k=1}^\infty \beta_\varepsilon^k \int_{T_{k,\varepsilon} \times (\mathbb{R}^2)^k} \prod_{i=1}^k \hat{p}_{\lceil t_i \rceil - \lceil t_{i-1} \rceil}(\lceil x_i \rceil - \lceil x_{i-1} \rceil) \prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i, \mathrm{d}x_i), \qquad (9.16)$$

then $\mathbb{E}[(\widetilde{Z}^\eta_{\varepsilon^{-2}, \beta_\varepsilon} - Z^\eta_{\varepsilon^{-2}, \beta_\varepsilon})^2] \to 0$ as $\varepsilon \downarrow 0$.

We can now bound the L^2 distance between $w^\varepsilon(1, 0)$ and $\widetilde{Z}^\eta_{\varepsilon^{-2}, \beta_\varepsilon}$:

$$\mathbb{E}[(w^\varepsilon(1, 0) - \widetilde{Z}^\eta_{\varepsilon^{-2}, \beta_\varepsilon})^2]$$

$$= \sum_{k=1}^\infty \beta_\varepsilon^{2k} \int_{T_{k,\varepsilon} \times \mathbb{R}^{2k}} \left(\prod_{i=1}^k p_{t_i - t_{i-1}}(x_i - x_{i-1}) - \prod_{i=1}^k \hat{p}_{\lceil t_i \rceil - \lceil t_{i-1} \rceil}(\lceil x_i \rceil - \lceil x_{i-1} \rceil)\right)^2 \mathrm{d}\vec{x}\mathrm{d}\vec{t}, \quad (9.17)$$

and we will separate the integration over $\vec{x}$ into two sets for each $k \in \mathbb{N}$.

Given $\vec{t} \in T_{k,\varepsilon}$ and $L > 0$, let

$$E_{k,L} = E_{k,L}(\vec{t}) := \{\vec{x} \in \mathbb{R}^{2k} : |x_i - x_{i-1}| \leqslant L\sqrt{t_i - t_{i-1}} \ \forall \, 1 \leqslant i \leqslant k\}. \qquad (9.18)$$

By the same calculations as for (9.9), we note that when the integrals over $\vec{x}$ in (9.17) are restricted to $E_{k,L}^c$ for each $k \in \mathbb{N}$, the resulting series converges to a limit (as $\varepsilon \downarrow 0$) that can

be made arbitrarily small by choosing L large. On the other hand, for any fixed $L > 0$,

$$\text{uniformly in } \vec{t} \in T_{k,\varepsilon} \text{ and } \vec{x} \in E_{k,L}, \qquad 1 - \prod_{i=1}^{k} \frac{\hat{p}_{\lceil t_i \rceil - \lceil t_{i-1} \rceil}(\lceil x_i \rceil - \lceil x_{i-1} \rceil)}{p_{t_i - t_{i-1}}(x_i - x_{i-1})} \xrightarrow[\varepsilon \to 0]{} 0$$

(9.19)

by the local central limit theorem (9.15). Therefore when the integrals over $\vec{x}$ in (9.17) are restricted to $E_{k,L}$, the resulting series also tends to 0 as $\varepsilon \downarrow 0$, as in (9.11). In conclusion, the series in (9.17) tends to 0 as $\varepsilon \downarrow 0$ and we can approximate $w^\varepsilon(1,0)$ by $Z^\eta_{\varepsilon^{-2},\beta_\varepsilon}$.

Step 4. When $\hat{\beta} < 1$, we just showed that each $\widetilde{u}^\varepsilon(t_\varepsilon^{(i)}, x_\varepsilon^{(i)})$ can be approximated in L^2 by $Z^\eta_{\varepsilon^{-2},\beta_\varepsilon}(\varepsilon^{-1}x_\varepsilon^{(i)}, \varepsilon^{-2}(1 - t_\varepsilon^{(i)}))$. Identifying ε^{-2} with N, the convergence in Theorem 2.15 then follows by applying Theorem 2.12 to $Z^\eta_{\varepsilon^{-2},\beta_\varepsilon}(\varepsilon^{-1}x_\varepsilon^{(i)}, \varepsilon^{-2}(1 - t_\varepsilon^{(i)}))$ for $1 \leqslant i \leqslant r$.

For $\hat{\beta} \geqslant 1$, the proof for $u^\varepsilon(1,0) \Rightarrow 0$ as $\varepsilon \downarrow 0$ is the same as that for the pinning and directed polymer models in Theorem 2.8. Proving that for $0 < \vartheta < 1$ the quantity $\mathbb{E}\big[\big(\widetilde{u}^\varepsilon(t,0)\big)^\vartheta\big]$ is decreasing in $\hat{\beta}$ is even simpler and proceeds as follows. Note that by (2.25),

$$\widetilde{u}^\varepsilon(t,0) = \mathrm{E}_0\left[e^{\beta \iint j(B_s - y)W(\mathrm{d}s,\mathrm{d}y) - \frac{\beta^2}{2}\|j\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d)}}\right] =: \mathrm{E}_0\left[e^{\beta G^W(B) - \frac{\beta^2}{2}\mathrm{Var}[G^W(B)]}\right],$$

where, for a fix realization of $B = (B_s)_{s\geqslant 0}$, the random variable $G^W(B)$ has a centered Gaussian distribution with variance $\|j\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d)}$. For $\beta^2 = \beta_1^2 + \beta_2^2$ we can write

$$\widetilde{u}^\varepsilon(t,0) \overset{d}{=} \mathrm{E}_0\left[e^{\beta_1 G^{W_1}(B) - \frac{\beta_1^2}{2}\mathrm{Var}[G^{W_1}(B)]} e^{\beta_2 G^{W_2}(B) - \frac{\beta_2^2}{2}\mathrm{Var}[G^{W_2}(B)]}\right]$$

where W_1, W_2 are two independent space-time white noise, and we used the fact that $\beta W \overset{\text{dist}}{=} \beta_1 W_1 + \beta_2 W_2$. Using Jensen's inequality to pass the expectation w.r.t. W_2 inside the fractional root in $\mathrm{E}[(\widetilde{u}^\varepsilon(t,0))^\vartheta]$ then gives the desired monotonicity in β as well as $\hat{\beta}$, since the two differ by a constant factor. $\qquad\square$

Proof of Theorem 2.17. Since $(\widetilde{u}^\varepsilon(t,x))_{t\in[0,1],x\in\mathbb{R}^2}$ defined in (9.1) has the same distribution as $(u^\varepsilon(t,x))_{t\in[0,1],x\in\mathbb{R}^2}$, we note that J_ε^ψ in (2.27) has the same distribution as

$$\widetilde{J}_\varepsilon^\psi := \left(\frac{\log \varepsilon^{-1}}{2\pi}\right)^{1/2} \int_{\mathbb{R}^2 \times [0,1]} (\widetilde{u}^\varepsilon(t,x) - 1)\, \psi(x, 1-t)\, \mathrm{d}t\mathrm{d}x.$$

Applying the chaos expansion (9.2) with $t_0 = \varepsilon^{-2}(1-t)$ and $x_0 = y_0 = \varepsilon^{-1}x$ gives

$$\widetilde{J}_\varepsilon^\psi = \sum_{k=1}^\infty \hat{\beta}^k \varepsilon^4 \left(\frac{2\pi}{\log \varepsilon^{-1}}\right)^{\frac{k-1}{2}}$$

(9.20)

$$\times \int \cdots \int_{\substack{x_1,\ldots,x_k \in \mathbb{R}^2 \\ 0 < t_1 < \cdots < t_k < \varepsilon^{-2}}} \left(\int_{\mathbb{R}^{2k}} p_{t_1}^{\varepsilon,\psi}(y_1) \prod_{i=2}^k p_{t_i - t_{i-1}}(y_i - y_{i-1}) j(y_i - x_i)\mathrm{d}\vec{y}\right) \prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i, \mathrm{d}x_i),$$

where we have plugged in $\beta_\varepsilon = \hat{\beta}\left(\frac{2\pi}{\log \varepsilon^{-1}}\right)^{1/2}$, changed variables $(t,x) = (1 - \varepsilon^2 t_0, \varepsilon y_0)$ producing the pre-factor ε^4, and

$$p_{t_1}^{\varepsilon,\psi}(y_1) := \int_{\mathbb{R}^2 \times [0,t_1]} \psi(\varepsilon y_0, \varepsilon^2 t_0)\, p_{t_1 - t_0}(y_1 - y_0)\, \mathrm{d}y_0\mathrm{d}t_0.$$

(9.21)

Note that (9.20) has the same form as the expansion for $\widetilde{u}^\varepsilon(1,0)$ in (9.2), except that the factor $p_{t_1}(y_1)$ therein is now replaced by $p_{t_1}^{\varepsilon,\psi}(y_1)$, and a pre-factor of $\left(\frac{2\pi}{\log\varepsilon^{-1}}\right)^{1/2}$ has been replaced by ε^4. We can now carry out the same steps as in the proof of Theorem 2.15 to approximate $\widetilde{J}_\varepsilon^\psi$ by J_N^ψ for a directed polymer on $\mathbb{Z}^2$, so that Theorem 2.13 can be applied.

In **Step 1** of the proof of Theorem 2.15 we restricted the range of integration of $t_1,\ldots,t_k$ in (9.2) so that $t_i - t_{i-1} \geqslant \sqrt{\log\varepsilon^{-1}}$ for all $1 \leqslant i \leqslant k$. This is necessary because in the L^2 approximations that follow, $\int p_{t_i-t_{i-1}}^2(y_i - y_{i-1})\mathrm{d}y_i$ is not integrable in t_i due to the singularity when t_i is near t_{i-1}. There is no such singularity for $p_{t_1}^{\varepsilon,\psi}(y_1)$, and in fact,

$$\|p^{\varepsilon,\psi}\|_2^2 := \int\limits_{\mathbb{R}^2 \times [0,\varepsilon^{-2}]} (p_{t_1}^{\varepsilon,\psi}(y_1))^2 \mathrm{d}y_1\mathrm{d}t_1 \tag{9.22}$$

$$= \int\limits_{\substack{(\mathbb{R}^2)^3}} \int\limits_{\substack{t_1\in(0,\varepsilon^{-2}) \\ t_0,t_0'\in(0,t_1)}} \psi(\varepsilon y_0, \varepsilon^2 t_0)\,\psi(\varepsilon y_0', \varepsilon^2 t_0')\, p_{t_1-t_0}(y_1-y_0)\, p_{t_1-t_0'}(y_1-y_0')\,\mathrm{d}y_0\mathrm{d}t_0\mathrm{d}y_0'\mathrm{d}t_0'\mathrm{d}y_1\mathrm{d}t_1$$

can be bounded in the same way as its discrete counterpart $q_{M,n_1}^{N,\psi}(x_1)$ from (8.5) (see the variance calculations in (8.8)–(8.10) and (8.14), with $N = \varepsilon^{-2}$, $\phi(N) = \varepsilon^{-1}$), which gives $\|p^{\varepsilon,\psi}\|_2 \sim C\varepsilon^{-4}$ as $\varepsilon \to 0$. Therefore in our current setting, we need to restrict $t_1,\ldots,t_k$ to

$$\widetilde{T}_{k,\varepsilon} := \{(t_1,\ldots,t_k) \in (0,\varepsilon^{-2})^k : t_1 > t_0,\ t_i - t_{i-1} \geqslant \sqrt{\log\varepsilon^{-1}}\ \forall\, 2 \leqslant i \leqslant k\}. \tag{9.23}$$

The rest of the calculations in **Step 1** carry through once we take into account that $\|p^{\varepsilon,\psi}\|_2$ is of the order ε^{-4}, which cancels the pre-factor ε^4 in (9.20).

In **Step 2** of the proof of Theorem 2.15, we replaced $p_{t_i-t_{i-1}}(y_i-y_{i-1})$ by $p_{t_i-t_{i-1}}(x_i-x_{i-1})$ for each $1 \leqslant i \leqslant k$, using the fact that $y_i - x_i$ must lie in the support of $j(\cdot)$. The same applies here, except that we also need to replace $p_{t_1}^{\varepsilon,\psi}(y_1)$ by $p_{t_1}^{\varepsilon,\psi}(x_1)$.

More precisely, we can first apply the same calculations as in (9.8)–(9.11) to replace $p_{t_i-t_{i-1}}(y_i-y_{i-1})$ by $p_{t_i-t_{i-1}}(x_i-x_{i-1})$ for each $2 \leqslant i \leqslant k$. The only change we need to make is to redefine the set E_k in (9.10) by

$$\widetilde{E}_k := \{(\vec{x},\vec{y}) \in \mathbb{R}^{2k} \times \mathbb{R}^{2k} : |y_i - y_{i-1}| \leqslant |t_i - t_{i-1}|^{\frac{3}{4}}\ \forall\, 2 \leqslant i \leqslant k\}. \tag{9.24}$$

After making these replacements, it only remains to bound the following simpler analogue of (9.11):

$$\varepsilon^8 \left(\frac{2\pi}{\log\varepsilon^{-1}}\right)^{k-1} \int\limits_{\widetilde{T}_{k,\varepsilon} \times \widetilde{E}_k} (p_{t_1}^{\varepsilon,\psi}(y_1) - p_{t_1}^{\varepsilon,\psi}(x_1))^2 \prod_{i=2}^k p_{t_i-t_{i-1}}^2(x_i-x_{i-1}) \prod_{i=1}^k j(y_i - x_i)\mathrm{d}\vec{y}\mathrm{d}\vec{x}\mathrm{d}\vec{t}$$

$$\leqslant C\varepsilon^8 \int\limits_{\mathbb{R}^4 \times [0,\varepsilon^{-2}]} (p_{t_1}^{\varepsilon,\psi}(y_1) - p_{t_1}^{\varepsilon,\psi}(x_1))^2 j(y_1-x_1)\mathrm{d}y_1\mathrm{d}x_1\mathrm{d}t_1 \xrightarrow[\varepsilon\to0]{} 0, \tag{9.25}$$

where in the inequality, we applied the same calculations as for (9.9), and the convergence to 0 can be easily deduced from the definition of $p_t^{\varepsilon,\psi}(y)$ in (9.21).

We can therefore approximate $\widetilde{J}_\varepsilon^\psi$ in L^2 by

$$\widehat{J}_\varepsilon^\psi := \sum_{k=1}^\infty \widehat{\beta}^k \varepsilon^4 \left(\frac{2\pi}{\log\varepsilon^{-1}}\right)^{\frac{k-1}{2}} \int \cdots \int\limits_{\widetilde{T}_{k,\varepsilon} \times \mathbb{R}^{2k}} p_{t_1}^{\varepsilon,\psi}(x_1) \prod_{i=2}^k p_{t_i-t_{i-1}}(x_i-x_{i-1}) \prod_{i=1}^k \widetilde{W}(\mathrm{d}t_i,\mathrm{d}x_i). \tag{9.26}$$

Following **Step 3** in the proof of Theorem 2.15, we now introduce a directed polymer on $\mathbb{Z}^2$, where the n-step transitional kernel $\hat{p}_n(\cdot)$ of the underlying random walk S satisfies the local limit theorem in (9.15), so that Hypothesis 2.4 holds with $L(\cdot) \equiv 1$. As in (8.15), and with $N = \varepsilon^{-2}$ and $\phi(N) = (L(N)^2 N)^{1/2} = \varepsilon^{-1}$, we can define

$$J_N^\psi = \frac{L(N)}{\phi(N)^2 N} \sum_{k=1}^\infty \frac{\hat{\beta}^k}{\mathbf{R}_N^{(k-1)/2}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_k \leqslant N \\ x_1, x_2, \ldots, x_k \in \mathbb{Z}^d}} \hat{p}_{n_1}^{N,\psi}(x_1) \prod_{j=2}^k \hat{p}_{n_j - n_{j-1}}(x_j - x_{j-1}) \prod_{i=1}^k \eta_{(n_i, x_i)}$$

$$= \sum_{k=1}^\infty \hat{\beta}^k \varepsilon^4 \left(\frac{2\pi + o(1)}{\log \varepsilon^{-1}} \right)^{\frac{k-1}{2}} \sum_{\substack{1 \leqslant n_1 < \cdots < n_k \leqslant \varepsilon^{-2} \\ x_1, x_2, \ldots, x_k \in \mathbb{Z}^d}} \hat{p}_{n_1}^{N,\psi}(x_1) \prod_{j=2}^k \hat{p}_{n_j - n_{j-1}}(x_j - x_{j-1}) \prod_{i=1}^k \eta_{(n_i, x_i)} \quad (9.27)$$

where we used $\mathbf{R}_N = \mathbf{R}_{\varepsilon^{-2}} \sim \log \varepsilon^{-1}/2\pi$ by (2.8), and

$$\hat{p}_{n_1}^{N,\psi}(x_1) := \sum_{x_0 \in \mathbb{Z}^d, 0 < n_0 < n_1} \psi(\varepsilon x_0, \varepsilon^2 n_0) \, \hat{p}_{n_1 - n_0}(x_1 - x_0). \quad (9.28)$$

Note that $\hat{p}_{n_1}^{N,\psi}(x_1)$ is a discrete sum approximation of $p_{t_1}^{\varepsilon,\psi}(x_1)$ in (9.21), and if we let $(\eta_{(n,x)})_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ be Gaussian random variables defined from $\widetilde{W}$ as in (9.13), then J_N^ψ in (9.27) is just a discrete sum approximation of $\widehat{J}_\varepsilon^\psi$ in (9.26). The L^2 difference $\|J_N^\psi - \widehat{J}_\varepsilon^\psi\|_2^2$ can be shown to vanish as $\varepsilon \to 0$, similar to (9.16)–(9.19), and we will omit the details.

Similar to **Step 4** in the proof of Theorem 2.15, we can finally apply Theorem 2.13 (for the directed polymer on $\mathbb{Z}^2$) to J_N^ψ and conclude that J_N^ψ, and hence also J_ε^ψ, converge in distribution to a Gaussian random variable with mean zero and variance σ_ψ^2. $\qquad \square$

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DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, VIA COZZI 55, 20125 MILANO, ITALY
   E-mail address: `francesco.caravenna@unimib.it`

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, 119076 SINGAPORE
   E-mail address: `matsr@nus.edu.sg`

DEPARTMENT OF STATISTICS, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UK
   E-mail address: `N.Zygouras@warwick.ac.uk`