Moment asymptotics for 2d directed polymer and Stochastic Heat Equation in the critical window

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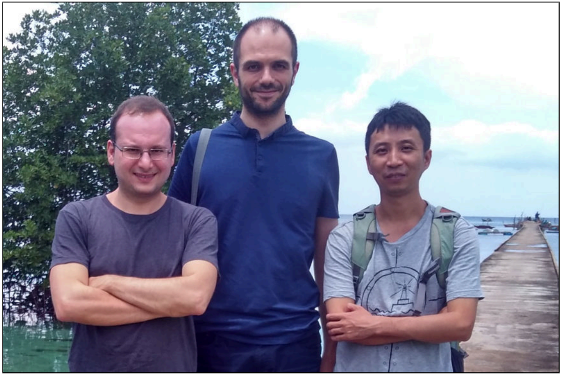
Collaborators



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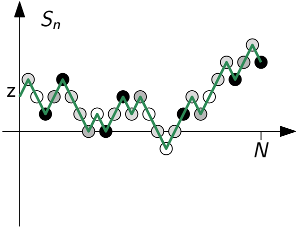
Outline

Directed Polymer in Random Environment



- $(S_n)_{n\geq 0}$ simple random walk on $\mathbb{Z}^d$
- Disorder: i.i.d. random variables $\omega(n,x)$
  zero mean, unit variance

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega(n,x)}] < \infty$$

- (-) Hamiltonian $\quad H_{N,\beta}(\omega, S) := \beta \sum_{n=1}^{N} \omega(n, S_n) - \lambda(\beta)\, N$

Partition Functions $\qquad (N \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$\mathbf{Z}_N(z) = \mathbf{E}^{\mathrm{rw}}\left[e^{H_{N,\beta}(\omega,S)} \,\middle|\, S_0 = z \right] = \frac{1}{(2d)^N} \sum_{(s_0,\ldots,s_N)\ \text{n.n.:}\ s_0=z} e^{H_{N,\beta}(\omega,s)}$$

Weak and strong disorder

For $d \geq 3$ there is weak disorder:

$$\text{for } \beta > 0 \text{ small:} \qquad \boldsymbol{Z}_N(z) \xrightarrow[N\to\infty]{\text{a.s.}} \mathcal{Z}(z) > 0$$

For $d = 1$, $d = 2$ only strong disorder:

$$\text{for any } \beta > 0: \qquad \boldsymbol{Z}_N(z) \xrightarrow[N\to\infty]{\text{a.s.}} 0 \qquad (\text{despite } \mathbb{E}[\boldsymbol{Z}_N(z)] \equiv 1)$$

Intermediate disorder regime

Can we tune $\beta = \beta_N \to 0$ to see an interesting regime ?

$$\Big(\boldsymbol{u}_N(t,x) := \boldsymbol{Z}_{Nt}(\sqrt{N}\,x)\Big)_{t\in[0,1],\ x\in\mathbb{R}^d} \xrightarrow[N\to\infty]{d} u(t,x) \ ?$$

[Motivation: properties of the underlying polymer model (Gibbs measure)]

Case $d = 1$

Theorem [Alberts, Khanin, Quastel '14]

Rescale

$$\beta_N = \frac{\hat\beta}{N^{1/4}}$$

Then

$$\boldsymbol{u}_N(t,x) \xrightarrow[N\to\infty]{d} u(t,x) \qquad \text{(up to time rev.)}$$

Solution of 1d Stochastic Heat Equation (SHE)

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \hat\beta\, \dot{W}(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases}$$

W = Gaussian white noise on $[0,1] \times \mathbb{R}$

Directed polymer and SHE

- Partition functions solve a lattice SHE

$$\boldsymbol{Z}_{N+1}(z) - \boldsymbol{Z}_N(z) \;=\; \Delta \boldsymbol{Z}_N(z) \;+\; \beta\, \tilde{\omega}(N+1, z)\, \widetilde{\boldsymbol{Z}}_N(z)$$

- Space-mollified SHE: $\qquad j_\delta(x) := \frac{1}{\delta}\, j\big(\frac{x}{\sqrt{\delta}}\big) \qquad (d=2)$

Generalized Feynman-Kac [Bertini-Cancrini '95]

$$u_\delta(t,x) \;\stackrel{d}{=}\; \mathbf{E}^{\mathrm{BM}}_{\frac{x}{\sqrt{\delta}}}\left[\exp\left\{\int_0^{\frac{t}{\delta}} \hat{\beta}\,(W * j)(\mathrm{d}s, B_s) \;-\; \tfrac{1}{2}\hat{\beta}^2\,\mathrm{d}s\right\}\right]$$

Continuum directed polymer (Wiener sausage) with $N = \frac{1}{\delta}$

$$u_\delta(t,x) \quad \text{close to} \quad \boldsymbol{u}_N(t,x) = \boldsymbol{Z}_{Nt}(\sqrt{N}x)$$

Case $d = 2$

For $d = 2$ the right scaling is

$$\beta_N \sim \sqrt{\frac{\pi}{\log N}}\,\hat{\beta}$$

[Lacoin '10] [Berger, Lacoin '15]

$$R_N := \mathbf{E}^{\mathrm{rw}}\left[\sum_{n=1}^{N} \mathbb{1}_{\{S_n = S'_n\}}\right] \sim \frac{1}{\pi}\log N$$

Logarithmic replica overlap $\iff$ disorder is marginally relevant

Problem ($t = 1$ for simplicity)

$$\boldsymbol{u}_N(x) = \boldsymbol{Z}_N(\sqrt{N}x) \xrightarrow[N\to\infty]{d} u(x)$$ random field on $\mathbb{R}^2$?

Outline

Subcritical regime $\hat{\beta} < 1$

Rescale $\qquad \beta_N \sim \sqrt{\frac{\pi}{\log N}}\,\hat{\beta} \qquad$ with $\hat{\beta} < 1$

Theorem [C., Sun, Zygouras '17]

- For every fixed $x \in \mathbb{R}^2$

$$\boldsymbol{u}_N(x) = \boldsymbol{Z}_N(\sqrt{N}x) \xrightarrow[N\to\infty]{d} \exp\left\{ N(0,\sigma^2) - \tfrac{1}{2}\sigma^2 \right\}$$

with $\sigma^2 = \log \frac{1}{1-\hat{\beta}^2}$

- For any $x \neq x' \in \mathbb{R}^2$

$\boldsymbol{u}_N(x)$ and $\boldsymbol{u}_N(x')$ become asymptotically independent (!)

[Dependence at all scales $|x - x'| = o(1)$]

A different viewpoint

$\boldsymbol{u}_N(x)$ cannot converge in a space of functions

For continuous $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ define

$$\langle \boldsymbol{u}_N \, , \phi \rangle \;:=\; \int_{\mathbb{R}^2} \boldsymbol{u}_N(x)\, \phi(x)\, \mathrm{d}x \;=\; \tfrac{1}{N} \sum_{z \in \mathbb{Z}^2} \boldsymbol{Z}_N(z)\, \phi\big(\tfrac{z}{\sqrt{N}}\big)$$

Look at $\boldsymbol{u}_N(x)$ as a random distribution on $\mathbb{R}^2$ (actually random measure)

Corollary (weak disorder)

For $\hat{\beta} < 1$ we have $\boldsymbol{u}_N(x) \xrightarrow{d} 1$ as a random measure

$$\langle \boldsymbol{u}_N \, , \phi \rangle \;\xrightarrow[N\to\infty]{d}\; \langle 1 \, , \phi \rangle = \int_{\mathbb{R}^2} \phi(x)\, \mathrm{d}x$$

Critical regime $\hat{\beta} = 1$

Consider now $\hat{\beta} = 1$, more generally the critical window

$$\beta_N = \sqrt{\frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right)} \qquad \text{with} \quad \vartheta \in \mathbb{R}$$

Key conjecture

- $\boldsymbol{u}_N(x)$ converges to a non-trivial random measure $\mathcal{U}(\mathrm{d}x)$ on $\mathbb{R}^2$

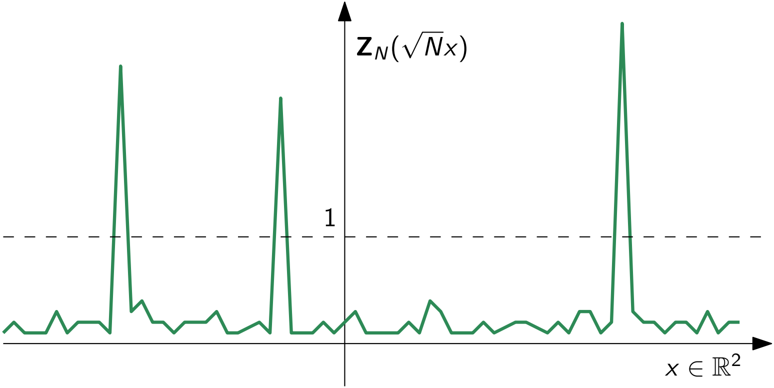
$$\langle \boldsymbol{u}_N \,,\, \phi \rangle \xrightarrow[N\to\infty]{d} \langle \mathcal{U}, \phi \rangle = \int_{\mathbb{R}^2} \phi(x)\, \mathcal{U}(\mathrm{d}x)$$

Theorem

[C., Sun, Zygouras '17]

For every fixed $x \in \mathbb{R}^2$ we have $\boldsymbol{u}_N(x) \xrightarrow[N\to\infty]{d} 0$

Heuristic picture



Second moment in the critical window

What is known

[Bertini, Cancrini '95 (on 2d SHE)]

Tightness via second moment bounds

$$\mathbb{E}\big[\langle \boldsymbol{u}_N \, , \, \phi \rangle\big] \equiv \big\langle 1, \phi \big\rangle \qquad \sup_{N \in \mathbb{N}} \mathbb{E}\big[\langle \boldsymbol{u}_N \, , \, \phi \rangle^2\big] < \infty$$

In fact $\quad \mathbb{E}\big[\langle \boldsymbol{u}_N \, , \, \phi \rangle^2\big] \xrightarrow[N \to \infty]{} \big\langle \phi, \, K\phi \big\rangle < \infty$

with explicit kernel $\quad K\big(x, x'\big) \sim C \log \dfrac{1}{|x - x'|} \qquad$ as $|x - x'| \to 0$

Corollary

Subsequential limits $\langle \boldsymbol{u}_{N_k} \, , \, \phi \rangle \xrightarrow[k \to \infty]{d} \langle \mathcal{U} \, , \, \phi \rangle \qquad$ Trivial $\mathcal{U} \equiv 1$?

Main result I. Third moment in the critical window

We determine the sharp asymptotics of third moment

Theorem [C., Sun, Zygouras '18+]

$$\lim_{N\to\infty} \mathbb{E}\big[\langle \boldsymbol{u}_N , \phi\rangle^3\big] = C(\phi) < \infty$$

- Explicit expression for $C(\phi)$ (series of multiple integrals, see below)

Corollary

Any subsequential limit $\boldsymbol{u}_{N_k} \xrightarrow{d} \mathcal{U}$ has covariance kernel $K(x, x')$

$$\rightsquigarrow \quad \mathcal{U} \not\equiv 1 \quad \text{is non-degenerate!}$$

Point-to-point partition function

So far we have considered point-to-plane partition functions $\boldsymbol{Z}_N(z)$

Point-to-point partition function

$$\boldsymbol{Z}_N(z,w) \;=\; \mathbf{E}^{\mathrm{rw}}\left[e^{H_{N,\beta}(\omega,S)}\, \mathbb{1}_{\{S_N=w\}} \,\middle|\, S_0 = z \right]$$

This correspond to the fundamental solutions of SHE (δ initial data)

We determine sharp second moment asymptotics for diffusively rescaled

$$\boldsymbol{u}_N(t;x,y) \;=\; \boldsymbol{Z}_{Nt}(\sqrt{N}x, \sqrt{N}y) \qquad\qquad t \in [0,1]\,,\; x,y \in \mathbb{R}^2$$

- Probabilistic approach (renewal) — beyond [Bertini-Cancrini '95]
- Key tool for third moment asymptotics

Main result II. Second moment for point-to-point

Critical window $\quad \beta_N = \sqrt{\frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right)} \quad$ with $\vartheta \in \mathbb{R}$

Theorem

[C., Sun, Zygouras '18+]

As $N \to \infty$

$$\mathbb{E}\big[\boldsymbol{u}_N(t; x, y)^2\big] \sim \frac{(\log N)^2}{\pi N^2}\, G_\vartheta(t)\, g_{\frac{t}{4}}(y - x)$$

- $g_t(z) = \frac{1}{2\pi t}\, e^{-\frac{|z|^2}{2t}}$ (standard Gaussian density on $\mathbb{R}^2$)
- $G_\vartheta(t) = \int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, t^{s-1}}{\Gamma(s)}\, \mathrm{d}s$ (special renewal function)

Outline

Heavy tailed renewal process

Fix $N \in \mathbb{N}$ and consider i.i.d. random variables $T_1^{(N)}, T_2^{(N)}, \ldots$ with

$$\mathrm{P}\big(T_i^{(N)} = n\big) = \frac{1}{\log N}\,\frac{1}{n} \qquad \text{for } n = 1, 2, \ldots, N$$

Consider the associated random walk (renewal process)

$$\tau_k^{(N)} \;=\; T_1^{(N)} \,+\, T_2^{(N)} \,+\, \ldots \,+\, T_k^{(N)}$$

Proposition

[C., Sun, Zygouras '18+]

$$\left(\frac{\tau^{(N)}_{\lfloor s \log N \rfloor}}{N}\right)_{s \ge 0} \xrightarrow[N\to\infty]{d} \; Y = \big(Y_s\big)_{s\ge 0}$$

Y is the subordinator (increasing Lévy process) with Lévy measure

$$\nu(\mathrm{d}t) \;=\; \frac{1}{t}\,\mathbb{1}_{(0,1)}(t)\,\mathrm{d}t$$

A special subordinator

The subordinator Y_s has an explicit density $f_s(t)$

Theorem [C., Sun, Zygouras '18+]

$$f_s(t) = \begin{cases} \dfrac{t^{s-1}\, e^{-\gamma s}}{\Gamma(s)} & \text{if } t \in (0,1) \\ \ldots & \text{if } t \geq 1 \end{cases}$$

Explicit formula for the exponentially weighted renewal density

$$G_\vartheta(t) = \int_0^\infty e^{\vartheta s}\, f_s(t)\, \mathrm{d}s = \int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, t^{s-1}}{\Gamma(s)}\, \mathrm{d}s$$

$$\sim \frac{1}{t\,(\log \frac{1}{t})^2} \quad \text{as } t \downarrow 0$$

Strong renewal theorem

Consider exponentially weighted renewal function for the random walk

$$U_N(n) = \sum_{k \geq 0} \lambda^k \, \mathrm{P}\big(\tau_k^{(N)} = n\big)$$

Renewal Theorem

[C., Sun, Zygouras '18+]

Rescale (critical window)

$$\lambda = 1 + \frac{\vartheta}{\log N} + o\big(\tfrac{1}{\log N}\big)$$

Then

$$U_N(n) \sim \frac{\log N}{N} \, G_\vartheta\big(\tfrac{n}{N}\big)$$

Space-time generalization

Random walk $(\tau_k^{(N)}, S_k^{(N)})$ in $\mathbb{N} \times \mathbb{Z}^2$

- $S_k^{(N)} = X_1^{(N)} + \ldots + X_k^{(N)}$
- $\mathrm{P}(X_1^{(N)} = x \,|\, T_1^{(N)} = n) \sim g_n(x) \qquad (n \to \infty)$

Exponentially weighted renewal function

$$\boldsymbol{U}_N(n,x) = \sum_{k \geq 0} \lambda^k \, \mathrm{P}\big(\tau_k^{(N)} = n \,,\, S_k^{(N)} = x\big)$$

Space-time Renewal Theorem [C., Sun, Zygouras '18+]

$$\boldsymbol{U}_N(n,x) \sim \frac{\log N}{N^2} \, \boldsymbol{G}_\vartheta\big(\tfrac{n}{N}, \tfrac{x}{\sqrt{n}}\big) \qquad \text{where} \quad \boldsymbol{G}_\vartheta(t,z) := G_\vartheta(t)\, g_t(z)$$

Outline

Partition function and polynomial chaos

$$
\begin{aligned}
Z_N(0) &= \mathbf{E}^{\mathrm{rw}}\left[e^{H_N(\omega,S)}\right] = \mathbf{E}^{\mathrm{rw}}\left[e^{\sum_{1\le n\le N}\sum_{x\in\mathbb{Z}^2}(\beta\omega(n,x)-\lambda(\beta))\,\mathbb{1}_{\{S_n=x\}}}\right] \\
&= \mathbf{E}^{\mathrm{rw}}\left[\prod_{1\le n\le N}\prod_{x\in\mathbb{Z}^2} e^{(\beta\omega(n,x)-\lambda(\beta))\,\mathbb{1}_{\{S_n=x\}}}\right] \\
&= \mathbf{E}^{\mathrm{rw}}\left[\prod_{1\le n\le N}\prod_{x\in\mathbb{Z}^2}\left(1+X_{n,x}\,\mathbb{1}_{\{S_n=x\}}\right)\right] \\
&= 1 + \sum_{\substack{1\le n\le N\\ x\in\mathbb{Z}^2}} \mathbf{P}^{\mathrm{rw}}(S_n=x)\,X_{n,x} \\
&\quad + \sum_{\substack{1\le n<m\le N\\ x,y\in\mathbb{Z}^2}} \mathbf{P}^{\mathrm{rw}}(S_n=x,\,S_m=y)\,X_{n,x}\,X_{m,y} + \ldots
\end{aligned}
$$

$Z_N(0)$ multi-linear polynomial of RVs $X_{n,x} := e^{\beta\omega(n,x)-\lambda(\beta)} - 1$

Polynomial chaos

$$\mathbb{E}[X_{n,x}] = 0 \qquad \mathbb{V}\mathrm{ar}[X_{n,x}] \sim \beta^2$$

Let us pretend $X_{n,x} = \beta\, Y_{n,x}$ with $\big(Y_{n,x}\big)_{n,x}$ i.i.d. $\mathcal{N}(0,1)$

Then $$\mathbf{Z}_N(0) \simeq 1 + \beta\, z_N^{(1)} + \beta^2\, z_N^{(2)} + \ldots$$

$$z_N^{(1)} := \sum_{\substack{1 \le n \le N \\ x \in \mathbb{Z}^2}} \mathrm{P}^{\mathrm{rw}}(S_n = x)\, Y_{n,x}$$

$$z_N^{(2)} := \sum_{\substack{1 \le n \le m \le N \\ x,y \in \mathbb{Z}^2}} \mathrm{P}^{\mathrm{rw}}(S_n = x,\, S_m = y)\, Y_{n,x}\, Y_{m,y}$$

The choice of β

$z_N^{(1)}$ is Gaussian with

$$\mathbb{V}\mathrm{ar}[z_N^{(1)}] = \sum_{1 \le n \le N} \sum_{x \in \mathbb{Z}^2} \mathbf{P}^{\mathrm{rw}}(S_n = x)^2 = \sum_{1 \le n \le N} \mathbf{P}^{\mathrm{rw}}(S_n = S'_n) = R_N$$

where $R_N = \mathrm{E}^{\mathrm{rw}}\left[\sum_{n=1}^{N} \mathbb{1}_{\{S_n = S'_n\}}\right]$ is the replica overlap

$$\mathbb{V}\mathrm{ar}[z_N^{(1)}] \sim \frac{1}{\pi} \sum_{1 \le n \le N} \frac{1}{n} \sim \frac{\log N}{\pi}$$

To normalize $\beta\, z_N^{(1)}$ we choose $\beta = \beta_N = \dfrac{\hat{\beta}}{\sqrt{\frac{\log N}{\pi}}}$

Sharp asymptotics

$$\mathbb{V}\mathrm{ar}\left[z_N^{(k)}\right] \sim \frac{1}{\pi^k} \sum_{0<n_1<\ldots<n_k\le N} \frac{1}{n_1} \frac{1}{n_2-n_1} \cdots \frac{1}{n_k-n_{k-1}}$$

$$= \left(\frac{\log N}{\pi}\right)^k \mathrm{P}\left(\tau_k^{(N)} \le N\right) \sim \left(\frac{\log N}{\pi}\right)^k \mathrm{P}\left(Y_{\frac{k}{\log N}} \le 1\right)$$

Variance of point-to-plane partition function

$$\mathbb{V}\mathrm{ar}\left[\boldsymbol{Z}_N(0)\right] = \sum_{k\ge1} (\beta_N)^k \, \mathbb{V}\mathrm{ar}\left[z_N^{(k)}\right] \sim \sum_{k\ge1} \left(1+\tfrac{\vartheta}{\log N}\right)^k \mathrm{P}\left(Y_{\frac{k}{\log N}} \le 1\right)$$

$$\sim (\log N) \left(\int_0^\infty e^{\vartheta s} \, \mathrm{P}(Y_s \le 1) \, \mathrm{d}s \right)$$

Similar asymptotics for point-to-point partition function $\boldsymbol{Z}_N(z,w)$

$\rightsquigarrow$ Space-time Renewal Theorems

Outline

Third moment in the critical window

$\langle \boldsymbol{u}_N , \phi \rangle = \langle \boldsymbol{Z}_N(\sqrt{N} \cdot\,) , \phi \rangle$ is multilinear polynomial of i.i.d. RVs $X_{n,x}$

$$\langle \boldsymbol{u}_N, \phi \rangle = \sum_{I \subseteq \{1,\ldots,N\} \times \mathbb{Z}^2} c(I) \prod_{(n,x) \in I} X_{n,x}$$

for suitable coefficients $c(I)$

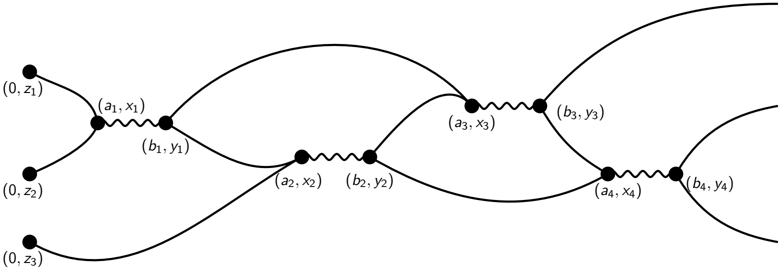
- Expand $\mathbb{E}[\langle \boldsymbol{u}_N, \phi \rangle^3]$ in 3 sums
- X's from different sums match in pairs or triples (by $\mathbb{E}[X_{n,x}] = 0$)
- Triple matchings give negligible contribution

Pairwise matching of the X's

$\rightsquigarrow$ non-trivial, yet manageable combinatorial structure

Combinatorial structure

- Three copies of RVs $X^{(1)}_{(n,x)}$, $X^{(2)}_{(n,x)}$, $X^{(3)}_{(n,x)}$ have to match in pairs
- Consecutive stretches of pairwise matchings with same labels
  E.g. first $(1,2)$, then $(1,3)$, then $(1,2)$ again ...



- Stretch $\rightsquigarrow$ Second moment of point-to-point partition function

Asymptotic third moment

Theorem

[C., Sun, Zygouras 2018+]

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(\langle \boldsymbol{u}_N, \phi\rangle - \mathbb{E}[\langle \boldsymbol{u}_N, \phi\rangle]\big)^3\Big] = C(\phi) = \sum_{m=2}^{\infty} 3\cdot 2^{m-1}\, I_m(\phi)$$

- m is the number of stretches
- $3\cdot 2^{m-1}$ is the number of choices of stretch labels

$$I_m(\phi) := \underset{\substack{0<a_1<b_1<\ldots<a_m<b_m<1 \\ x_1,y_1,x_2,y_2,\ldots,x_m,y_m\in\mathbb{R}^2}}{\int\cdots\int} \mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{y}\;\; \Phi^2_{a_1}(x_1)\,\Phi_{a_2}(x_2)$$

$$\boldsymbol{G}_\vartheta(b_1-a_1, y_1-x_1)\; g_{\frac{a_2-b_1}{2}}(x_2-y_1)\; \boldsymbol{G}_\vartheta(b_2-a_2, y_2-x_2)$$

$$\prod_{i=3}^{m} g_{\frac{a_i-b_{i-2}}{2}}(x_i-y_{i-2})\; g_{\frac{a_i-b_{i-1}}{2}}(x_i-y_{i-1})\; \boldsymbol{G}_\vartheta(b_i-a_i, y_i-x_i)$$

Conclusion

- Non trivial to prove that $C(\phi) < \infty$
- We need to show that $I_m(\phi)$ decays super-exponentially as $m \to \infty$
- We cannot factorize the integral using Hölder-type inequalities: kernels $g_t(z)$ and $\boldsymbol{G}_\vartheta(t,x)$ are barely integrable as $t \downarrow 0$
- We exploit the ordering $a_1 < b_1 < a_2 < b_2 < \dots$ in order to prove sharp recursive bounds

Work in progress

- Uniqueness of subsequential limit $\mathcal{U}$ via coarse-graining arguments
  $\rightsquigarrow$ Existence of the limit $\boldsymbol{u}_N \xrightarrow[N\to\infty]{d} \mathcal{U}$
- Properties of the limiting random measure $\mathcal{U}$
  (it looks not so close to Gaussian Multiplicative Chaos)

Thanks