THE DICKMAN SUBORDINATOR, RENEWAL THEOREMS, AND DISORDERED SYSTEMS

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Abstract. We consider the so-called Dickman subordinator, whose Lévy measure has density $\frac{1}{x}$ restricted to the interval $(0,1)$. The marginal density of this process, known as the Dickman function, appears in many areas of mathematics, from number theory to combinatorics. In this paper, we study renewal processes in the domain of attraction of the Dickman subordinator, for which we prove local renewal theorems. We then present applications to marginally relevant disordered systems, such as pinning and directed polymer models, and prove sharp second moment estimates on their partition functions.

1. Introduction and main results

1.1. Motivation. We consider the subordinator (increasing Lévy process) denoted by $Y = (Y_s)_{s\geqslant 0}$, which is pure jump with Lévy measure

$$\nu(\mathrm{d}t) := \frac{1}{t}\,\mathbb{1}_{(0,1)}(t)\,\mathrm{d}t\,. \tag{1.1}$$

Equivalently, its Laplace transform is given by

$$\mathrm{E}[e^{\lambda Y_s}] = \exp\left\{ s\int_0^1 (e^{\lambda t}-1)\,\frac{\mathrm{d}t}{t}\right\}. \tag{1.2}$$

We call Y *the Dickman subordinator* (see Remark 1.2 below). It is suggestive to view it as a "truncated 0-stable subordinator", by analogy with the well known α-stable subordinator whose Lévy measure is $\frac{1}{t^{1+\alpha}}\mathbb{1}_{(0,\infty)}(t)\,\mathrm{d}t$, for $\alpha \in (0,1)$. In our case $\alpha = 0$ and the restriction $\mathbb{1}_{(0,1)}(t)$ in (1.1) ensures that ν is a legitimate Lévy measure, i.e. $\int_{\mathbb{R}}(t^2\wedge 1)\,\nu(\mathrm{d}t) < \infty$.

Interestingly, the Dickman subordinator admits an explicit marginal density

$$f_s(t) := \frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t}\,, \qquad \text{for } s,t \in (0,\infty)\,, \tag{1.3}$$

which we recall in the following result.

Theorem 1.1 (Density of the Dickman subordinator). *For all $s \in (0,\infty)$ one has*

$$f_s(t) = \begin{cases} \dfrac{s\,t^{s-1}\,e^{-\gamma s}}{\Gamma(s+1)} & \text{for } t \in (0,1], \\[2ex] \dfrac{s\,t^{s-1}e^{-\gamma s}}{\Gamma(s+1)} - s t^{s-1}\displaystyle\int_0^{t-1} \frac{f_s(a)}{(1+a)^s}\mathrm{d}a & \text{for } t \in (1,\infty), \end{cases} \tag{1.4}$$

where $\Gamma(\cdot)$ denotes Euler's gamma function and $\gamma = -\int_0^\infty \log u\, e^{-u}\,\mathrm{d}u \simeq 0.577$ is the Euler-Mascheroni constant.

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Theorem 1.1 follows from general results about *self-decomposable Lévy processes* [Sat99]. We give the details in Appendix B, where we also present an alternative, self-contained derivation of the density $f_s(t)$, based on direct probabilistic arguments. We refer to [BKKK14] for further examples of subordinators with explicit densities.

Remark 1.2 (Dickman function and Dickman distribution). *The function*

$$\varrho(t) := e^{\gamma} f_1(t)$$

is known as the Dickman function *and plays an important role in number theory and combinatorics* [Ten95, ABT03]. *By* (1.4) *we see that* ϱ *satisfies*

$$\varrho(t) \equiv 1 \quad \text{for} \ \ t \in (0,1] \,, \qquad\qquad t\,\varrho'(t) + \varrho(t-1) = 0 \quad \text{for} \ \ t \in (1,\infty)\,, \tag{1.5}$$

which is the classical definition of the Dickman function. Examples where ϱ *emerges are:*

- *If* X_n *denotes the largest prime factor of a uniformly chosen integer in* $\{1,\ldots,n\}$, *then* $\lim_{n\to\infty} \mathrm{P}(X_n \leqslant n^t) = \varrho(1/t)$ [Dic30].
- *If* Y_n *denotes the size of the longest cycle in a uniformly chosen permutation of* n *elements, then* $\lim_{n\to\infty} \mathrm{P}(Y_n \leqslant nt) = \varrho(1/t)$ [Kin77].

Thus both $(\log X_n/\log n)$ *and* (Y_n/n) *converge in law as* $n \to \infty$ *to a random variable* L_1 *with* $\mathrm{P}(L_1 \leqslant t) = \varrho(1/t)$. *The density of* L_1 *equals* $t^{-1}\varrho(t^{-1}-1)$, *by* (1.5).

The marginal law Y_1 *of our subordinator, called the* Dickman distribution *in the literature, also arises in many contexts, from logarithmic combinatorial structures* [ABT03, Theorem 4.6] *to theoretical computer science* [HT01]. *We stress that* Y_1 *and* L_1 *are different – their laws are supported in* $(0,\infty)$ *and* $(0,1)$, *respectively – though both are related to the Dickman function: their densities are* $e^{-\gamma}\varrho(t)$ *and* $t^{-1}\varrho(t^{-1}-1)$, *respectively*

In this paper, we present a novel application of the Dickman subordinator in the context of *disordered systems*, such as pinning and directed polymer models. We will discuss the details in Section 3, but let us give here the crux of the problem in an elementary way, which can naturally arise in various other settings.

Given $\varrho, \gamma \in (0,\infty)$, let us consider the weighted series of convolutions

$$v_N := \sum_{k=1}^{\infty} \varrho^k \sum_{0<n_1<n_2<\ldots<n_k\leqslant N} \frac{1}{n_1^{\gamma}(n_2-n_1)^{\gamma}\cdots(n_k-n_{k-1})^{\gamma}}\,. \tag{1.6}$$

We are interested in the following question: for a fixed exponent $\gamma \in (0,\infty)$, *can one choose* $\varrho = \varrho_N$ *so that* v_N *converges to a non-zero and finite limit limit as* $N \to \infty$, i.e. $v_N \to v \in (0,\infty)$? The answer naturally depends on the exponent γ.

If $\gamma < 1$, we can, straightforwardly, use a Riemann sum approximation and by choosing $\varrho = \lambda N^{-1+\gamma}$, for fixed $\lambda \in (0,\infty)$, we have that v_N will converge to

$$v := \sum_{k=1}^{\infty} \lambda^k \left\{ \int\!\!\cdots\!\!\int_{0<t_1<\ldots<t_k<1} \frac{\mathrm{d}t_1\cdots\mathrm{d}t_k}{t_1^{\gamma}(t_2-t_1)^{\gamma}\cdots(t_k-t_{k-1})^{\gamma}} \right\} = \sum_{k=1}^{\infty} \lambda^k \, \frac{\Gamma(\gamma)^{k+1}}{\Gamma((k+1)\gamma)} \tag{1.7}$$

where the last equality is deduced from the normalization of the Dirichlet distribution.

If $\gamma \geqslant 1$, then, as it is readily seen, the Riemann sum approach fails, as it leads to iterated integrals which are infinite. The idea now is to express the series (1.6) as a renewal function.

The case $\gamma > 1$ is easy: we can take a small, but *fixed* $\varrho > 0$, more precisely

$$\varrho \in \left(0, \frac{1}{R}\right), \qquad \text{where} \qquad R := \sum_{n\in\mathbb{N}} \frac{1}{n^\gamma} \in (0,\infty),$$

and consider the renewal process $\tau = (\tau_k)_{k\geqslant 0}$ with inter-arrival law $\mathrm{P}(\tau_1 = n) = \frac{1}{R}\frac{1}{n^\gamma}$ for $n \in \mathbb{N}$. We can then write

$$v_N = \sum_{k=1}^{\infty} \big(\varrho R\big)^k \, \mathrm{P}(\tau_k \leqslant N) \xrightarrow[N\to\infty]{} v := \frac{\varrho R}{1-\varrho R} \in (0,\infty).$$

The case $\gamma = 1$ is more interesting[†]. This case is subtle because the normalization $R = \sum_{n\in\mathbb{N}} \frac{1}{n} = \infty$. The way around this problem is to first normalize $\frac{1}{n}$ to a probability on $\{1, 2, \ldots, N\}$. More precisely, we take

$$R_N := \sum_{n=1}^{N} \frac{1}{n} = \log N \big(1 + o(1)\big),$$

and consider the renewal process $\tau^{(N)} = (\tau_k^{(N)})_{k\geqslant 0}$ with inter-arrival law

$$\mathrm{P}\big(\tau_1^{(N)} = n\big) = \frac{1}{R_N}\frac{1}{n} \qquad \text{for } n \in \{1, 2, \ldots, N\}. \tag{1.8}$$

Note that this renewal process is a discrete analogue of the Dickman subordinator. Choosing $\varrho = \lambda/R_N$, with $\lambda < 1$, we can see, via dominated convergence, that

$$v_N = \sum_{k=1}^{\infty} \lambda^k \, \mathrm{P}(\tau_k^{(N)} \leqslant N) \xrightarrow[N\to\infty]{} v := \frac{\lambda}{1-\lambda} \in (0,\infty) \tag{1.9}$$

because $\mathrm{P}(\tau_k^{(N)} \leqslant N) \to 1$ as $N \to \infty$, for any fixed $k \in \mathbb{N}$. But when $\lambda = 1$, then $v_N \to \infty$ and then finer questions emerge, e.g., at which rate does $v_N \to \infty$? Or what happens if instead of $\mathrm{P}(\tau_k^{(N)} \leqslant N)$ we consider $\mathrm{P}(\tau_k^{(N)} = N)$ in (1.9), i.e. if we fix $n_k = N$ in (1.6)?

To answer these questions, it is necessary to explore the domain of attraction of the Dickman subordinator — to which $\tau^{(N)}$ belongs, as we show below — and to prove renewal theorems. Indeed, the left hand side of (1.9) for $\lambda = 1$ defines the *renewal measure of* $\tau^{(N)}$. Establishing results of this type is the core of our paper.

1.2. Main results. We study a class of renewal processes $\tau^{(N)}$ which generalize (1.8). Let us fix a sequence $(r(n))_{n\in\mathbb{N}}$ such that

$$r(n) := \frac{\mathsf{a}}{n}(1 + o(1)) \qquad \text{as } n \to \infty, \tag{1.10}$$

for some constant $\mathsf{a} \in (0,\infty)$, so that

$$R_N := \sum_{n=1}^{N} r(n) = \mathsf{a} \, \log N (1 + o(1)) \qquad \text{as } N \to \infty. \tag{1.11}$$

For each $N \in \mathbb{N}$, we consider i.i.d. random variables $(T_i^{(N)})_{i\in\mathbb{N}}$ with distribution

$$\mathrm{P}(T_i^{(N)} = n) := \frac{r(n)}{R_N} \, \mathbb{1}_{\{1,\ldots,N\}}(n). \tag{1.12}$$

[†]It can be called *marginal* or *critical*, due to its relations to disordered systems, see [CSZ17b] for the relevant terminology and statistical mechanics background.

(The precise value of the constant a is immaterial, since it gets simplified in (1.12).)

Let $\tau^{(N)} = (\tau_k^{(N)})_{k\in\mathbb{N}_0}$ denote the associated random walk (renewal process):

$$\tau_0^{(N)} := 0\,, \qquad \tau_k^{(N)} := \sum_{i=1}^{k} T_i^{(N)}\,. \tag{1.13}$$

We first show that $\tau^{(N)}$ is in the domain of attraction of the Dickman subordinator Y.

Proposition 1.3 (Convergence of rescaled renewal process). *The rescaled process*

$$\left(\frac{\tau^{(N)}_{\lfloor s\log N\rfloor}}{N}\right)_{s\geqslant 0}$$

converges in distribution to the Dickman subordinator $(Y_s)_{s\geqslant 0}$, as $N\to\infty$.

We then define an *exponentially weighted renewal density* $U_{N,\lambda}(n)$ for $\tau^{(N)}$, which is a local version of the quantity which appears in (1.9):

$$U_{N,\lambda}(n) := \sum_{k\geqslant 0} \lambda^k\, \mathrm{P}(\tau_k^{(N)} = n), \qquad \text{for} \quad N, n\in\mathbb{N},\ \lambda\in(0,\infty)\,. \tag{1.14}$$

We similarly define the corresponding quantity for the Dickman subordinator:

$$G_\vartheta(t) := \int_0^\infty e^{\vartheta s}\, f_s(t)\,\mathrm{d}s = \int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, s\, t^{s-1}}{\Gamma(s+1)}\,\mathrm{d}s\,, \qquad \text{for} \quad t\in(0,1]\,,\ \vartheta\in\mathbb{R}\,. \tag{1.15}$$

Our main result identifies the asymptotic behavior of the renewal density $U_{N,\lambda}(n)$ for large N and $n\leqslant N$. This is shown to be of the order $\mathrm{E}[T_1^{(N)}]^{-1} \sim (\frac{N}{\log N})^{-1}$, in analogy with the classical renewal theorem, with a sharp prefactor given by $G_\vartheta(\frac{n}{N})$.

Theorem 1.4 (Sharp renewal theorem). *Fix any $\vartheta\in\mathbb{R}$ and let $(\lambda_N)_{N\in\mathbb{N}}$ satisfy*

$$\lambda_N = 1 + \frac{\vartheta}{\log N}\big(1+o(1)\big) \qquad \textit{as } N\to\infty\,. \tag{1.16}$$

For any fixed $\delta>0$, the following relation holds as $N\to\infty$:

$$U_{N,\lambda_N}(n) = \frac{\log N}{N}\, G_\vartheta(\tfrac{n}{N})\,(1+o(1))\,, \qquad \textit{uniformly for} \ \ \delta N\leqslant n\leqslant N\,. \tag{1.17}$$

Moreover, the following uniform bound holds, for a suitable $C\in(0,\infty)$:

$$U_{N,\lambda_N}(n) \leqslant C\,\frac{\log N}{N}\, G_\vartheta(\tfrac{n}{N})\,, \qquad \forall n\leqslant N\,. \tag{1.18}$$

As anticipated, we will present an application to disordered systems in Section 3: for pinning and directed polymer models, we derive the sharp asymptotic behavior of the second moment of the partition function in the weak disorder regime (see Theorems 3.1 and 3.3).

We stress that Theorem 1.4 extends the literature on *renewal theorems in the case of infinite mean.* Typically, the cases studied in the literature correspond to renewal processes of the form $\tau_n = T_1 + \cdots + T_n$, where the i.i.d. increments $(T_i)_{i\geqslant 1}$ have law

$$\mathrm{P}(T_1 = n) = \phi(n)\, n^{-(1+\alpha)}, \tag{1.19}$$

with $\phi(\cdot)$ a slowly varying function. In case $\alpha\in(0,1]$, limit theorems for the renewal density $U(n) = \sum_{k\geqslant 1}\mathrm{P}(\tau_k = n)$ have been the subject of many works, e.g. [GL62], [E70], [D97], just to mention a few of the most notable ones. The sharpest results in this direction have been recently established in [CD16+] when $\alpha \in (0,1)$, and in [B17+] when $\alpha = 1$.

In the case of (1.19) with $\alpha = 0$, results of the sorts of Theorem 1.4 have been obtained in [NW08, N12, AB16]. However, a key difference between these references and our result is that we deal with a non-summable sequence $1/n$, hence it is necessary to consider a family of renewal processes $\tau^{(N)}$ whose law varies with $N \in \mathbb{N}$ (*triangular array*) via a suitable cutoff. This brings our renewal process out of the scope of the cited references.

We point out that it is possible to generalize our assumption (1.10), replacing the constant a by a slowly varying function $\phi(n)$ *such that* $\sum_{n\in\mathbb{N}} \phi(n)/n = \infty$. We expect that our results extend to this case with the same techniques, but we prefer to stick to the simpler assumption (1.10), which considerably simplifies notation.

Let us give an overview of the proof of Theorem 1.4 (see Section 6 for more details). In order to prove the upper bound (1.18), a key tool is the following sharp estimate on the local probability $\mathrm{P}(\tau_k^{(N)} = n)$. It suggests that the main contribution to $\{\tau_k^{(N)} = n\}$ comes from the strategy that *a single increment* $T_i^{(N)}$ *takes values close to* n.

Proposition 1.5 (Sharp local estimate). *Let us set* $\log^+(x) := (\log x)^+$. *There are constants* $C \in (0,\infty)$ *and* $c \in (0,1)$ *such that for all* $N, k \in \mathbb{N}$ *and* $n \leqslant N$ *we have*

$$\mathrm{P}\big(\tau_k^{(N)} = n\big) \;\leqslant\; C\, k\, \mathrm{P}\big(T_1^{(N)} = n\big)\; \mathrm{P}\big(T_1^{(N)} \leqslant n\big)^{k-1}\, e^{-\frac{c\,k}{\log n+1}\,\log^+ \frac{c\,k}{\log n+1}} . \tag{1.20}$$

We point out that (1.20) sharpens [AB16, eq. (1.11) in Theorem 1.1], thanks to the last term which decays super-exponentially in k. This will be essential for us, in order to counterbalance the exponential weight λ^k in the renewal density $U_{N,\lambda}(n)$, see (1.14).

In order to prove the local limit theorem (1.17), we use a strategy of independent interest: we are going to deduce it from the weak convergence in Proposition 1.3 by exploiting *recursive formulas* for the renewal densities $U_{N,\lambda}$ and G_ϑ, based on a decomposition according to the jump that straddles a fixed site; see (6.13) and (6.14). These formulas provide integral representations of the renewal densities $U_{N,\lambda}$ and G_ϑ which reduce a *local* limit behavior to an *averaged* one, thus allowing to strengthen weak convergence results to local ones.

Finally, we establish fine asymptotic properties of the continuum renewal density G_ϑ.

Proposition 1.6. *For any fixed* $\vartheta \in \mathbb{R}$, *the function* $G_\vartheta(t)$ *is continuous (actually* C^∞*) and strictly positive for* $t \in (0,1]$. *As* $t \downarrow 0$ *we have* $G_\vartheta(t) \to \infty$, *more precisely*

$$G_\vartheta(t) = \frac{1}{t(\log\frac{1}{t})^2}\left\{1 + \frac{2\vartheta}{\log\frac{1}{t}} \;+\; O\!\left(\frac{1}{(\log\frac{1}{t})^2}\right)\right\}. \tag{1.21}$$

Remark 1.7. *Our results also apply to renewal processes with a density. Fix a bounded and continuous function* $r : [0,\infty) \to (0,\infty)$ *with* $r(t) = \frac{\mathsf{a}}{t}(1 + o(1))$ *as* $t \to \infty$, *so that* $R_N := \int_0^N r(t)\,\mathrm{d}t = \mathsf{a}\,\log N(1 + o(1))$. *If we consider the renewal process* $\tau_k^{(N)}$ *in* (1.13) *with*

$$\mathrm{P}(T_i^{(N)} \in \mathrm{d}t) = \frac{r(t)}{R_N}\,\mathbb{1}_{[0,N]}(t)\,\mathrm{d}t\,,$$

then Proposition 1.3, Theorem 1.4 and Proposition 1.5 still hold, provided $\mathrm{P}\big(\tau_k^{(N)} = n\big)$ *denotes the density of* $\tau_k^{(N)}$. *The proofs can be easily adapted, replacing sums by integrals.*

1.3. Organization of the paper. In Section 2 we present multi-dimensional extensions of our main results, where we extend the subordinator and the renewal processes with a spatial component. This is guided by applications to the directed polymer model.

In Section 3 we discuss the applications of our results to disordered systems and more specifically to pinning and directed polymer models. A result of independent interest is Proposition 3.2, where we prove sharp asymptotic results on the expected number of encounters at the origin of two independent simple random walks on $\mathbb{Z}$; this also gives the expected number of encounters (anywhere) of two independent simple random walks on $\mathbb{Z}^2$.

The remaining sections 4-7 are devoted to the proofs. Appendix A contains results for disordered systems, while Appendix B is devoted to the Dickman subordinator.

2. Multidimensional extensions

We extend our subordinator Y by adding a spatial component, that for simplicity we assume to be Gaussian. More precisely, we fix a dimension $d \in \mathbb{N}$ and we let $W = (W_t)_{t \in [0,\infty)}$ denote a standard Brownian motion on $\mathbb{R}^d$. Its density is given by

$$g_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp(-\tfrac{|x|^2}{2t}) , \tag{2.1}$$

where $|x|$ is the Euclidean norm. Note that $\sqrt{\mathsf{c}}\, W_t$ has density $g_{\mathsf{c}t}(x)$, for every $\mathsf{c} \in (0,\infty)$.

Recall the definition (1.1) of the measure ν. We denote by $\boldsymbol{Y}^{\mathsf{c}} := (\boldsymbol{Y}^{\mathsf{c}}_s)_{s \geqslant 0} = (Y_s, V^{\mathsf{c}}_s)_{s \geqslant 0}$ the Lévy process on $[0,\infty) \times \mathbb{R}^d$ with zero drift, no Brownian component, and Lévy measure

$$\boldsymbol{\nu}(\mathrm{d}t, \mathrm{d}x) := \nu(\mathrm{d}t)\, g_{\mathsf{c}t}(x)\, \mathrm{d}x = \frac{\mathbb{1}_{(0,1)}(t)}{t}\, g_{\mathsf{c}t}(x)\, \mathrm{d}t\, \mathrm{d}x . \tag{2.2}$$

Equivalently, for all $\boldsymbol{\lambda} \in \mathbb{R}^{1+d}$ and $s \in [0,\infty)$,

$$\mathrm{E}[e^{\langle \boldsymbol{\lambda}, \boldsymbol{Y}^{\mathsf{c}}_s \rangle}] = \exp\left\{ s \int_{(0,1)\times\mathbb{R}^d} (e^{\langle \boldsymbol{\lambda}, (t,x) \rangle} - 1)\, \frac{g_{\mathsf{c}t}(x)}{t}\, \mathrm{d}t\, \mathrm{d}x \right\} . \tag{2.3}$$

We can identify the probability density of $\boldsymbol{Y}^{\mathsf{c}}_s$ for $s \in [0,\infty)$ as follows.

Proposition 2.1 (Density of Lévy process). *We have the following representation:*

$$(\boldsymbol{Y}^{\mathsf{c}}_s)_{s \in [0,\infty)} \quad \overset{d}{=} \quad \big((Y_s,\, \sqrt{\mathsf{c}}\, W_{Y_s}) \big)_{s \in [0,\infty)} ,$$

with W independent of Y. Consequently, $\boldsymbol{Y}^{\mathsf{c}}_s$ has probability density (recall (1.3) *and* (2.1)*)*

$$\boldsymbol{f}_s(t,x) = f_s(t)\, g_{\mathsf{c}t}(x) . \tag{2.4}$$

We now define a family of random walks in the domain of attraction of $\boldsymbol{Y}^{\mathsf{c}}$. Recall that $r(n)$ was defined in (1.10). We consider a family of probability kernels $p(n,\cdot)$ on $\mathbb{Z}^d$, indexed by $n \in \mathbb{N}$, which converge in law to $\sqrt{\mathsf{c}}\, W_1$ when rescaled diffusively. More precisely, we assume the following conditions:

$$\begin{aligned}
&\text{(i)} && \sum_{x \in \mathbb{Z}^d} x_i\, p(n,x) = 0 \quad \text{for } i = 1, \ldots, d \\
&\text{(ii)} && \sum_{x \in \mathbb{Z}^d} |x|^2\, p(n,x) = O(n) \quad \text{as } n \to \infty \\
&\text{(iii)} && \sup_{x \in \mathbb{Z}^d} \big| n^{d/2}\, p(n,x) - g_{\mathsf{c}}\big(\tfrac{x}{\sqrt{n}}\big) \big| = o(1) \quad \text{as } n \to \infty .
\end{aligned} \tag{2.5}$$

Note that $\mathsf{c} \in (0,\infty)$ is the asymptotic variance of each component. Also note that, by (iii),

$$\sup_{x\in\mathbb{Z}} p(n,x) = O\bigg(\frac{1}{n^{d/2}}\bigg) \qquad \text{as } n\to\infty\,. \tag{2.6}$$

Then we define, for every $N\in\mathbb{N}$, the i.i.d. random variables $(T_i^{(N)}, X_i^{(N)}) \in \mathbb{N}\times\mathbb{Z}^d$ by

$$\mathrm{P}\big((T_i^{(N)}, X_i^{(N)}) = (n,x)\big) := \frac{r(n)\,p(n,x)}{R_N}\,\mathbb{1}_{\{1,\ldots,N\}}(n)\,, \tag{2.7}$$

with $r(n)$, R_N as in (1.10), (1.11). Let $(\tau^{(N)}, S^{(N)})$ be the associated random walk, i.e.

$$\tau_k^{(N)} := T_1^{(N)} + \ldots + T_k^{(N)}\,, \qquad S_k^{(N)} := X_1^{(N)} + \ldots + X_k^{(N)}\,. \tag{2.8}$$

We have the following analogue of Proposition 1.3.

Proposition 2.2 (Convergence of rescaled Lévy process). *Assume that the conditions in* (2.5) *hold. The rescaled process*

$$\left(\frac{\tau^{(N)}_{\lfloor s\log N\rfloor}}{N}, \frac{S^{(N)}_{\lfloor s\log N\rfloor}}{\sqrt{N}}\right)_{s\geqslant 0}$$

converges in distribution to $(\boldsymbol{Y}_s^{\mathsf{c}} := (Y_s, V_s^{\mathsf{c}}))_{s\geqslant 0}$*, as* $N\to\infty$*.*

We finally introduce the exponentially weighted renewal density

$$\boldsymbol{U}_{N,\lambda}(n,x) := \sum_{k\geqslant 0} \lambda^k\,\mathrm{P}(\tau_k^{(N)} = n,\, S_k^{(N)} = x)\,, \tag{2.9}$$

as well as its continuum version:

$$\boldsymbol{G}_\vartheta(t,x) := \int_0^\infty e^{\vartheta s}\,\boldsymbol{f}_s(t,x)\,\mathrm{d}s = G_\vartheta(t)\,g_{\mathsf{c}t}(x) \qquad \text{for } t\in(0,1]\,,\ x\in\mathbb{R}^d\,, \tag{2.10}$$

where the second equality follows by (1.15) and Theorem 2.1. Recall (1.14) and observe that

$$\sum_{x\in\mathbb{Z}^d} \boldsymbol{U}_{N,\lambda}(n,x) = U_{N,\lambda}(n) \tag{2.11}$$

The following result is an extension of Theorem 1.4.

Theorem 2.3 (Space-time renewal theorem). *Fix any* $\vartheta\in\mathbb{R}$ *and let* $(\lambda_N)_{N\in\mathbb{N}}$ *satisfy*

$$\lambda_N = 1 + \frac{\vartheta}{\log N}\big(1+o(1)\big) \qquad \textit{as } N\to\infty\,.$$

For any fixed $\delta>0$*, the following relation holds as* $N\to\infty$*:*

$$\begin{gathered}\boldsymbol{U}_{N,\lambda_N}(n,x) \;=\; \frac{\log N}{N^{1+d/2}}\,G_\vartheta\big(\tfrac{n}{N}\big)\,g_{\mathsf{c}\frac{n}{N}}\big(\tfrac{x}{\sqrt{N}}\big)\big(1+o(1)\big)\,,\\ \textit{uniformly for}\ \ \delta N\leqslant n\leqslant N,\ |x|\leqslant \tfrac{1}{\delta}\sqrt{N}\,.\end{gathered} \tag{2.12}$$

Moreover, the following uniform bound holds, for a suitable $C\in(0,\infty)$*:*

$$\boldsymbol{U}_{N,\lambda_N}(n,x) \leqslant C\,\frac{\log N}{N}\,\frac{1}{n^{d/2}}\,G_\vartheta\big(\tfrac{n}{N}\big)\,, \qquad \forall 1\leqslant n\leqslant N\,,\ \forall x\in\mathbb{Z}^d\,. \tag{2.13}$$

The bound (2.13) is to be expected, in view of (2.12), because $\sup_{z\in\mathbb{R}^d} g_t(z) \leqslant \frac{C}{t}$. Finally, we show that the probability $\frac{\boldsymbol{U}_{N,\lambda}(n,\cdot)}{U_{N,\lambda}(n)}$ is concentrated on the diffusive scale $O(\sqrt{n})$.

Theorem 2.4. *There exists a constant $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ and $\lambda \in (0, \infty)$*

$$\sum_{x \in \mathbb{Z}^d \colon |x| > M\sqrt{n}} \frac{\boldsymbol{U}_{N,\lambda}(n, x)}{U_{N,\lambda}(n)} \leqslant \frac{C}{M^2}, \qquad \forall 1 \leqslant n \leqslant N, \ \forall M > 0. \tag{2.14}$$

3. Applications to disordered systems

In this section we discuss applications of our previous results to two marginally relevant disordered systems: the pinning model with tail exponent 1/2 and the $(2+1)$-dimensional directed polymer model. For simplicity, we focus on the case when these models are built from the simple random walk on $\mathbb{Z}$ and on $\mathbb{Z}^2$, respectively.

Both models contain *disorder*, given by a family $\omega = (\omega_i)_{i \in \mathbb{T}}$ of i.i.d. random variables; $\mathbb{T} = \mathbb{N}$ for the pinning model, $\mathbb{T} = \mathbb{N} \times \mathbb{Z}^2$ for the directed polymer model. We assume that

$$\mathbb{E}[\omega_i] = 0, \qquad \mathbb{E}[\omega_i^2] = 1, \qquad \lambda(\beta) := \log \mathbb{E}[\exp(\beta\omega_i)] < \infty \quad \forall \beta > 0. \tag{3.1}$$

An important role is played by

$$\sigma_\beta^2 := e^{\lambda(2\beta) - 2\lambda(\beta)} - 1. \tag{3.2}$$

Before presenting our results, in order to put them into context and to provide motivation, we discuss the key notion of *relevance of disorder*.

3.1. Relevance of disorder.

Both the pinning model and the directed polymer model are Gibbs measures on random walk paths, which depend on the realization of the disorder. A key question for these models, and more generally for disordered systems, is whether an arbitrarily small, but fixed amount of disorder is able to change the large scale properties of the model without disorder. When the answer is positive (resp. negative), the model is called *disorder relevant* (resp. *irrelevant*). In borderline cases, where the answer depends on finer properties, the model is called *marginally relevant* or *irrelevant.*

Important progress has been obtained in recent years in the mathematical understanding of the relevance of disorder, in particular for the pinning model, where the problem can be cast in terms of *critical point shift* (and *critical exponents*). We refer to [G10] for a detailed presentation of the key results and for the relevant literature.

The pinning model based on the simple random walk on $\mathbb{Z}$ is *marginally relevant*, as shown in [GLT10]. Sharp estimates on the critical point shift were more recently obtained in [BL18]. For the directed polymer model based on the simple random walk on $\mathbb{Z}^2$, analogous sharp results are given in [BL17], in terms of *free energy* estimates.

In [CSZ17a] we proposed a different approach to study disorder relevance: when a model is disorder relevant, it should be possible to suitably *rescale the disorder strength to zero, as the system size diverges,* and still obtain a non-trivial limiting model where disorder is present. Such an *intermediate disorder regime* had been investigated in [AKQ14a, AKQ14b] for the directed polymer model based on the simple random walk on $\mathbb{Z}$, which is disorder relevant. The starting point to build a non-trivial limiting model is to determine the scaling limits of the family of *partition functions*, which encode a great deal of information.

The scaling limits of partition functions were obtained in [CSZ17a] for several models that are disorder relevant (see also [CSZ15]). However, the case of marginally relevant models — which include the pinning model on $\mathbb{Z}$ and the directed polymer model on $\mathbb{Z}^2$ — is much more delicate. In [CSZ17b] we showed that for such models a phase transition emerges on a suitable intermediate disorder scale, and below the critical point, the family of partition functions converges to an explicit Gaussian random field (the solution of the additive stochastic heat equation, in the case of the directed polmyer on $\mathbb{Z}^2$).

In this section we focus on a suitable window around the critical point, which corresponds to a precise way of scaling down the disorder strength to zero (see (3.9) and (3.22) below). In this critical window, the partition functions are expected to converge to a non-trivial limiting random field, which has fundamental connections with singular stochastic PDEs (see the discussion in [CSZ17b]).

Our new results, described in Theorems 3.1 and 3.6 below, give sharp asymptotic estimates for the second moment of partition functions. These estimates, besides providing an important piece of information by themselves, are instrumental to investigate scaling limits. Indeed, we proved in the recent paper [CSZ18] that the family of partition functions of the directed polymer on $\mathbb{Z}^2$ admits non-trivial random field limits, whose covariance exhibits logarithmic divergence along the diagonal. This is achieved by a third moment computation on the partition function, where the second moment estimates derived here play a crucial role.

3.2. Pinning model. Let $X = (X_n)_{n\in\mathbb{N}_0}$ be the simple symmetric random walk on $\mathbb{Z}$. We set

$$u(n) := \mathrm{P}(X_{2n} = 0) = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{n}}\big(1 + o(1)\big) \qquad \text{as } n \to \infty\,. \tag{3.3}$$

Fix a sequence of i.i.d. random variables $\omega = (\omega_n)_{n\in\mathbb{N}}$, independent of X, satisfying (3.1). The (constrained) partition function of the *pinning model* is defined as follows:

$$Z_N^\beta := \mathrm{E}\Big[e^{\sum_{n=1}^{N-1}(\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{X_{2n}=0\}}}\,\mathbb{1}_{\{X_{2N}=0\}}\Big]\,, \tag{3.4}$$

where we work with X_{2n} rather than X_n to avoid periodicity issues.

Writing Z_N^β as a polynomial chaos expansion [CSZ17a] (we review the computation in Appendix A.1), we obtain the following expression for the second moment:

$$\mathbb{E}[(Z_N^\beta)^2] = \sum_{k\geqslant 1}(\sigma_\beta^2)^{k-1}\sum_{0<n_1<\ldots<n_{k-1}<n_k:=N} u(n_1)^2\,u(n_2-n_1)^2\,\cdots\,u(n_k-n_{k-1})^2\,, \tag{3.5}$$

where σ_β^2 is defined in (3.2). Let us define

$$r(n) := u(n)^2 = \frac{1}{\pi\, n}\big(1 + o(1)\big)\,, \tag{3.6}$$

$$R_N := \sum_{n=1}^N r(n) = \sum_{n=1}^N \left\{\frac{1}{2^{2n}}\binom{2n}{n}\right\}^2 = \frac{1}{\pi}\log N\,\big(1+o(1)\big)\,, \tag{3.7}$$

and denote by $(\tau_k^{(N)})_{k\in\mathbb{N}_0}$ the renewal process with increments law given by (1.12). Then, recalling (3.5) and (1.14), for every $N\in\mathbb{N}$ and $1\leqslant n\leqslant N$ we can write

$$\begin{aligned}\mathbb{E}[(Z_n^\beta)^2] &= \frac{1}{\sigma_\beta^2}\sum_{k\geqslant 1}\big(\sigma_\beta^2\,R_N\big)^k\,\mathrm{P}(\tau_k^{(N)} = n)\\ &= \frac{1}{\sigma_\beta^2}\,U_{N,\lambda}(n)\,, \qquad \text{where} \qquad \lambda := \sigma_\beta^2\,R_N\,.\end{aligned} \tag{3.8}$$

As a direct corollary of Theorem 1.4, we have the following result.

Theorem 3.1 (Second moment asymptotics for pinning model). *Let Z_N^β be the partition function of the pinning model based on the simple symmetric random walk on $\mathbb{Z}$, see* (3.4)*. Define σ_β^2 by* (3.2) *and R_N by* (3.7)*. Fix $\vartheta \in \mathbb{R}$ and rescale $\beta = \beta_N$ so that*

$$\sigma_{\beta_N}^2 = \frac{1}{R_N}\left(1 + \frac{\vartheta}{\log N}\big(1 + o(1)\big)\right) \qquad \text{as } N \to \infty \,. \tag{3.9}$$

Then, for any fixed $\delta > 0$, the following relation holds as $N \to \infty$:

$$\mathbb{E}[(Z_n^{\beta_N})^2] = \frac{(\log N)^2}{\pi\, N}\, G_\vartheta(\tfrac{n}{N})\,(1 + o(1))\,, \qquad \text{uniformly for } \ \delta N \leqslant n \leqslant N\,. \tag{3.10}$$

Moreover, the following uniform bound holds, for a suitable constant $C \in (0,\infty)$:

$$\mathbb{E}[(Z_n^{\beta_N})^2] \leqslant C\, \frac{(\log N)^2}{N}\, G_\vartheta(\tfrac{n}{N})\,, \qquad \forall 1 \leqslant n \leqslant N\,. \tag{3.11}$$

In view of (3.7), it is tempting to replace R_N by $\frac{1}{\pi}\log N$ in (3.9). However, to do this properly, a sharper asymptotic estimate on R_N as $N \to \infty$ is needed. The following result, of independent interest, is proved in Appendix A.3.

Proposition 3.2. *As $N \to \infty$*

$$R_N := \sum_{n=1}^N \left\{\frac{1}{2^{2n}}\binom{2n}{n}\right\}^2 = \frac{\log N + \alpha}{\pi} + o(1)\,, \qquad \text{with} \quad \alpha := \gamma + \log 16 - \pi\,, \tag{3.12}$$

where $\gamma = -\int_0^\infty \log u\, e^{-u}\,\mathrm{d}u \simeq 0.577$ is the Euler-Mascheroni constant.

Corollary 3.3. *Relation* (3.9) *can be rewritten as follows, with $\alpha := \gamma + \log 16 - \pi$:*

$$\sigma_{\beta_N}^2 = \frac{\pi}{\log N}\left(1 + \frac{\vartheta - \alpha}{\log N}\big(1 + o(1)\big)\right) \qquad \text{as } N \to \infty\,. \tag{3.13}$$

We stress that identifying the constant α in (3.12) is subtle, because it is a non asymptotic quantity (changing *any single term* of the sequence in brackets modifies the value of α!). To accomplish the task, in Appendix A.3 we relate α to a truly asymptotic property, i.e. the tail behavior of the first return to zero of the simple symmetric random walk on $\mathbb{Z}^2$.

Remark 3.4. *Relation* (3.13) *can be made more explicit, by expressing $\sigma_{\beta_N}^2$ in terms of β_N^2. The details are carried out in Appendix A.4.*

Remark 3.5. *If one removes the constraint $\{X_{2N} = 0\}$ from* (3.4)*, then one obtains the* free partition function $Z_N^{\beta,\mathrm{f}}$*. The asymptotic behavior of its second moment can be determined explicitly, in analogy with Theorem 3.1, see Appendix A.2.*

3.3. Directed polymer in random environment. Let $S = (S_n)_{n\in\mathbb{N}_0}$ be the simple symmetric random walk on $\mathbb{Z}^2$. We set

$$q_n(x) := \mathrm{P}(S_n = x)\,, \tag{3.14}$$

and note that, recalling the definition (3.3) of $u(n)$, we can write

$$\sum_{x\in\mathbb{Z}^2} q_n(x)^2 = \mathrm{P}(S_{2n} = 0) = \left\{\frac{1}{2^{2n}}\binom{2n}{n}\right\}^2 =: u(n)^2\,, \tag{3.15}$$

where the second equality holds because the projections of S along the two main diagonals are independent simple random walks on $\sqrt{2}\,\mathbb{Z}$.

Note that $\mathrm{Cov}[S_1^{(i)}, S_1^{(j)}] = \frac{1}{2}\mathbb{1}_{\{i=j\}}$, where $S_1^{(i)}$ is the i-th component of S_1, for $i = 1, 2$. As a consequence, $S_n/\sqrt{n}$ converges in distribution to the Gaussian law on $\mathbb{R}^2$ with density $g_{\frac{1}{2}}(\cdot)$ (recall (2.1)). The random walk S is periodic, because (n, S_n) takes values in

$$\mathbb{Z}^3_{\mathrm{even}} := \left\{z = (z_1, z_2, z_3) \in \mathbb{Z}^3 : \; z_1 + z_2 + z_3 \in 2\mathbb{Z}\right\}.$$

Then the local central limit theorem gives that, as $n \to \infty$,

$$n\, q_n(x) \;=\; g_{\frac{1}{2}}\big(\tfrac{x}{\sqrt{n}}\big)\, 2\, \mathbb{1}_{\{(n,x)\in\mathbb{Z}^3_{\mathrm{even}}\}} \;+\; o(1)\,, \qquad \text{uniformly for } x \in \mathbb{Z}^2\,, \tag{3.16}$$

where the factor 2 is due to periodicity, because the constraint $(n, x) \in \mathbb{Z}^3_{\mathrm{even}}$ restricts x in a sublattice of $\mathbb{Z}^2$ whose cells have area equal to 2.

Fix now a sequence of i.i.d. random variables $\omega = (\omega_{n,x})_{(n,x)\in\mathbb{N}\times\mathbb{Z}^2}$ satisfying (3.1), independent of S. The (constrained) partition function of the *directed polymer in random environment* is defined as follows:

$$\begin{aligned} \boldsymbol{Z}^\beta_N(x) &:= \mathrm{E}\left[e^{\sum_{n=1}^{N-1}(\beta\omega_{n,S_n} - \lambda(\beta))}\, \mathbb{1}_{\{S_N = x\}}\right] \\ &= \mathrm{E}\left[e^{\sum_{n=1}^{N-1}\sum_{z\in\mathbb{Z}^2}(\beta\omega_{n,z} - \lambda(\beta))\mathbb{1}_{\{S_n = z\}}}\, \mathbb{1}_{\{S_N = x\}}\right]. \end{aligned} \tag{3.17}$$

In analogy with (3.5) (see Appendix A.1), we have a representation for the second moment:

$$\begin{aligned} \mathbb{E}\big[\big(\boldsymbol{Z}^\beta_N(x)\big)^2\big] = \sum_{k\geqslant 1} (\sigma_\beta^2)^{k-1} \sum_{\substack{0<n_1<\ldots<n_{k-1}<n_k=N \\ x_1,\ldots,x_k\in\mathbb{Z}^2:\ x_k=x}} & q_{n_1}(x_1)^2\, q_{n_2-n_1}(x_2-x_1)^2 \cdot \\ & \cdots\, q_{n_k-n_{k-1}}(x_k - x_{k-1})^2\,. \end{aligned} \tag{3.18}$$

To apply the results in Section 2, we define for $(n, x) \in \mathbb{N} \times \mathbb{Z}^2$

$$p(n, x) := \frac{q_n(x)^2}{u(n)^2}\,, \qquad \text{where} \qquad u(n) := \frac{1}{2^{2n}}\binom{2n}{n}.$$

Note that $p(n, \cdot)$ is a probability kernel on $\mathbb{Z}^2$, by (3.15). Since $g_t(x)^2 = \frac{1}{4\pi t} g_{t/2}(x)$ (see (2.1)), it follows by (3.16) and (3.3) that, uniformly for $x \in \mathbb{Z}^2$,

$$n\, p(n, x) \;=\; g_{\frac{1}{4}}\big(\tfrac{x}{\sqrt{n}}\big)\, 2\, \mathbb{1}_{\{(n,x)\in\mathbb{Z}^3_{\mathrm{even}}\}} \;+\; o(1)\,. \tag{3.19}$$

Thus $p(n, \cdot)$ fulfills condition (iii) in (2.5) with $\mathsf{c} = \frac{1}{4}$ (the multiplicative factor 2 is a minor correction, due to periodicity). Conditions (i) and (ii) in (2.5) are also fulfilled.

Let $(\tau^{(N)}, S^{(N)}) = (\tau_k^{(N)}, S_k^{(N)})_{k\geqslant 0}$ be the random walk with increment law given by (2.7), where $r(n)$ and R_N are the same as in (3.6)-(3.7). More explicitly:

$$\mathrm{P}\big((\tau_1^{(N)}, S_1^{(N)}) = (n, x)\big) := \frac{1}{R_N}\, q_n(x)^2\, \mathbb{1}_{\{1,\ldots,N\}}(n)\,. \tag{3.20}$$

Recalling (3.18) and (2.9), we can write

$$\begin{aligned} \mathbb{E}\big[\big(\boldsymbol{Z}^\beta_n(x)\big)^2\big] &= \frac{1}{\sigma_\beta^2} \sum_{k\geqslant 1} \big(\sigma_\beta^2\, R_N\big)^k\, \mathrm{P}(\tau_k^{(N)} = n,\, S_k^{(N)} = x) \\ &= \frac{1}{\sigma_\beta^2}\, \boldsymbol{U}_{N,\lambda}(n, x)\,, \qquad \text{where} \qquad \lambda := \sigma_\beta^2\, R_N\,. \end{aligned} \tag{3.21}$$

As a corollary of Theorem 2.3, taking into account periodicity, we have the following result.

Theorem 3.6 (Second moment asymptotics for directed polymer). *Let $\boldsymbol{Z}_N^{\beta}(x)$ be the partition function of the directed polymer in random environment based on the simple symmetric random walk on $\mathbb{Z}^2$, see (3.17). Define σ_β^2 by (3.2) and R_N by (3.7). Fix $\vartheta \in \mathbb{R}$ and rescale $\beta = \beta_N$ so that*

$$\sigma_{\beta_N}^2 = \frac{1}{R_N}\left(1 + \frac{\vartheta}{\log N}\big(1 + o(1)\big)\right) \qquad \text{as } N \to \infty\,. \tag{3.22}$$

For any fixed $\delta > 0$, the following relation holds as $N \to \infty$:

$$\begin{aligned}\mathbb{E}\big[\big(\boldsymbol{Z}_n^{\beta_N}(x)\big)^2\big] &= \frac{(\log N)^2}{\pi N^2}\, G_\vartheta\big(\tfrac{n}{N}\big)\, g_{\frac{n}{4N}}\big(\tfrac{x}{\sqrt{N}}\big)\, 2\, \mathbb{1}_{\{(n,x)\in\mathbb{Z}^3_{\mathrm{even}}\}}\, (1 + o(1))\,, \\ &\text{uniformly for } \delta N \leqslant n \leqslant N,\ |x| \leqslant \tfrac{1}{\delta}\sqrt{N}\,.\end{aligned} \tag{3.23}$$

Remark 3.7. *Relation (3.22) can be equivalently rewritten as relation (3.13), as explained in Corollary 3.3. These conditions on $\sigma_{\beta_N}^2$ can be explicitly reformulated in terms of β_N^2, see Appendix A.4 for details.*

Remark 3.8. *Also for the directed polymer model we can define a* free partition function *$\boldsymbol{Z}_N^{\beta,\mathrm{f}}$, removing the constraint $\{S_{2N} = x\}$ from (3.17). The asymptotic behavior of its second moment is determined in Appendix A.2.*

4. Preliminary results

In this section we prove Propositions 1.3, 1.6, 2.1, and 2.2.

We start with Propositions 1.3 and 2.2, for which we prove convergence in the sense of finite-dimensional distributions. It is not difficult to obtain convergence in the Skorokhod topology, but we omit it for brevity, since we do not need such results.

Proof of Proposition 1.3. We recall that the renewal process $\tau_k^{(N)}$ was defined in (1.13). We set

$$Y_s^{(N)} := \frac{\tau^{(N)}_{\lfloor s \log N\rfloor}}{N}. \tag{4.1}$$

Note that the process $Y_s^{(N)}$ has independent and stationary increments (for $s \in \frac{1}{\log N}\mathbb{N}_0$), hence the convergence of its finite-dimensional distributions follows if we show that

$$Y_s^{(N)} \xrightarrow[N\to\infty]{} Y_s \quad \text{in distribution} \tag{4.2}$$

for every fixed $s \in [0, \infty)$. This could be proved by checking the convergence of Laplace transforms. We give a more direct proof, which will be useful in the proof of Proposition 2.2.

Fix $\varepsilon > 0$ and let $\Xi^{(\varepsilon)}$ be a Poisson Point Process on $[\varepsilon, 1]$ with intensity measure $s\frac{\mathrm{d}t}{t}$. More explicitly, we can write

$$\Xi^{(\varepsilon)} = \{t_i^{(\varepsilon)}\}_{i=1,\ldots,\mathcal{N}^{(\varepsilon)}}\,,$$

where the number of points $\mathcal{N}^{(\varepsilon)}$ has a Poisson distribution:

$$\mathcal{N}^{(\varepsilon)} \sim \mathrm{Pois}(\lambda^{(\varepsilon)})\,, \qquad \text{where} \qquad \lambda^{(\varepsilon)} = \int_\varepsilon^1 s\,\frac{\mathrm{d}t}{t} = s\,\log 1/\varepsilon\,, \tag{4.3}$$

while $(t_i^{(\varepsilon)})_{i\in\mathbb{N}}$ are i.i.d. random variables with law

$$\mathrm{P}(t_i^{(\varepsilon)} > x) = \frac{\int_x^1 s\,\frac{\mathrm{d}t}{t}}{\int_\varepsilon^1 s\,\frac{\mathrm{d}t}{t}} = \frac{\log x}{\log \varepsilon} \qquad \text{for} \quad x \in [\varepsilon, 1]\,. \tag{4.4}$$

We define

$$Y_s^{(\varepsilon)} := \sum_{t\in\Xi^{(\varepsilon)}} t = \sum_{i=1}^{\mathcal{N}^{(\varepsilon)}} t_i^{(\varepsilon)}\,, \tag{4.5}$$

which is a compound Poisson random variable. Its Laplace transform equals

$$\mathrm{E}[e^{-\lambda Y_s^{(\varepsilon)}}] = \exp\bigg(- s \int_\varepsilon^1 \frac{1-e^{-\lambda t}}{t}\,\mathrm{d}t\bigg)\,,$$

from which it follows that $\lim_{\varepsilon\to 0} Y_s^{(\varepsilon)} = Y^s$ in distribution (recall (1.2)).

Next we define

$$Y_s^{(N,\varepsilon)} := \frac{1}{N}\sum_{i\in I_s^{(N,\varepsilon)}} T_i^{(N)}\,, \qquad \text{where} \qquad I_s^{(N,\varepsilon)} := \big\{1 \leqslant i \leqslant \lfloor s\log N\rfloor :\ T_i^{(N)} > \varepsilon N\big\}\,. \tag{4.6}$$

Note that, by (1.10)-(1.11), for some constant $C \in (0,\infty)$ we can write

$$\begin{aligned} \mathrm{E}\big[\big|Y_s^{(N)} - Y_s^{(N,\varepsilon)}\big|\big] &= \frac{1}{N}\,\mathrm{E}\Bigg[\sum_{i\notin I_s^{(N,\varepsilon)}} T_i^{(N)}\Bigg] = \frac{\lfloor s\log N\rfloor}{N}\,\mathrm{E}\Big[T_1^{(N)}\mathbb{1}_{\{T_1^{(N)}\leqslant \varepsilon N\}}\Big] \\ &= \frac{\lfloor s\log N\rfloor}{N}\sum_{n=1}^{\lfloor \varepsilon N\rfloor} n\,\frac{r(n)}{R_N} \leqslant C\,\frac{\lfloor s\log N\rfloor}{N}\,\frac{\lfloor \varepsilon N\rfloor}{\log N} \leqslant C\,\varepsilon s\,. \end{aligned} \tag{4.7}$$

Thus $Y_s^{(N)}$ and $Y_s^{(N,\varepsilon)}$ are close in distribution for $\varepsilon > 0$ small, uniformly in $N\in\mathbb{N}$.

The proof of (4.2) will be completed if we show that $\lim_{N\to\infty} Y_s^{(N,\varepsilon)} = Y_s^{(\varepsilon)}$ in distribution, for any fixed $\varepsilon > 0$. Let us define the point process

$$\Xi^{(N,\varepsilon)} := \bigg\{ t_i^{(N,\varepsilon)} := \frac{1}{N} T_i^{(N)} :\quad i \in I_s^{(N,\varepsilon)} \bigg\}\,,$$

so that we can write

$$Y_s^{(N,\varepsilon)} := \sum_{t\in\Xi^{(N,\varepsilon)}} t = \sum_{i\in I_s^{(N,\varepsilon)}} t_i^{(N,\varepsilon)}\,.$$

It remains to show that $\Xi^{(N,\varepsilon)}$ converges in distribution to $\Xi^{(\varepsilon)}$ as $N\to\infty$ (recall (4.5)).

- The number of points $|I_s^{(N,\varepsilon)}|$ in $\Xi^{(\varepsilon)}$ has a Binomial distribution $\mathrm{Bin}(n,p)$, with
$$n = \lfloor s\log N\rfloor\,, \qquad p = \mathrm{P}(T_1^{(N)} > \varepsilon N) \sim \frac{\log 1/\varepsilon}{\log N}\,,$$
hence as $N\to\infty$ it converges in distribution to $\mathcal{N}^{(\varepsilon)} \sim \mathrm{Pois}(\lambda^{(\varepsilon)})$, see (4.3).
- Each point $t_i^{(N,\varepsilon)} \in \Xi^{(N,\varepsilon)}$ has the law of $\frac{1}{N}T_1^{(N)}$ conditioned on $T_1^{(N)} > \varepsilon N$, and it follows by (1.10)-(1.11) that as $N\to\infty$ this converges in distribution to $t_1^{(\varepsilon)}$, see (4.4).

This completes the proof of Proposition 1.3. □

Proof of Proposition 2.2. We recall that the random walk $(\tau_k^{(N)}, S_k^{(N)})$ was introduced in (2.8). We introduce the shortcut

$$\boldsymbol{Y}_s^{(N)} := (Y_s^{(N)}, V_s^{(N)}) := \left(\frac{\tau_{\lfloor s \log N \rfloor}^{(N)}}{N}, \frac{S_{\lfloor s \log N \rfloor}^{(N)}}{\sqrt{N}} \right), \quad s \geqslant 0. \tag{4.8}$$

In analogy with (4.2), it suffices to show that for every fixed $s \in [0, \infty)$

$$\boldsymbol{Y}_s^{(N)} \xrightarrow[N\to\infty]{} \boldsymbol{Y}_s := (Y_s, V_s^{\mathsf{c}}) \quad \text{in distribution}. \tag{4.9}$$

Fix $\varepsilon > 0$ and recall that $Y_s^{(\varepsilon)}$ was defined in (4.5). With Proposition 2.1 in mind, we define

$$V_s^{(\varepsilon)} := \sqrt{\mathsf{c}}\, W_{Y_s^{(\varepsilon)}}, \tag{4.10}$$

where W is an independent Brownian motion on $\mathbb{R}^d$. Since $\lim_{\varepsilon\to 0} Y_s^{(\varepsilon)} = Y_s$ in distribution, recalling Proposition 2.1 we see that for every fixed $s \in [0, \infty)$

$$\boldsymbol{Y}_s^{(\varepsilon)} := (Y_s^{(\varepsilon)}, V_s^{(\varepsilon)}) \xrightarrow[\varepsilon\to 0]{d} \boldsymbol{Y}_s = (Y_s, V_s^{\mathsf{c}}).$$

Recall the definition (4.6) of $Y_s^{(N,\varepsilon)}$ and $I_s^{(N,\varepsilon)}$. We define similarly

$$V_s^{(N,\varepsilon)} := \frac{1}{\sqrt{N}} \sum_{i \in I_s^{(N,\varepsilon)}} X_i^{(N)}. \tag{4.11}$$

We showed in (4.7) that $Y_s^{(N,\varepsilon)}$ approximates $Y_s^{(N)}$ in L^1, for $\varepsilon > 0$ small. We are now going to show that $V_s^{(N,\varepsilon)}$ approximates $V_s^{(N)}$ in L^2. Recalling (2.7), (2.5), we can write

$$\mathrm{E}\big[|X_1^{(N)}|^2 \,\big|\, T_1^{(N)} = n\big] = \sum_{x \in \mathbb{Z}^2} |x|^2 \, p(n, x) \leqslant c\, n. \tag{4.12}$$

Since conditionally on $(T_i^{(N)})_{i \notin I_s^{(N,\varepsilon)}}$, $(X_i^{(N)})_{i \notin I_s^{(N,\varepsilon)}}$ are independent with mean 0, we have

$$\begin{aligned} \mathrm{E}\big[|V_s^{(N)} - V_s^{(N,\varepsilon)}|^2\big] &= \frac{1}{N} \mathrm{E}\big[\big| \sum_{i \notin I_s^{(N,\varepsilon)}} X_i^{(N)} \big|^2\big] \\ &\leqslant \frac{c}{N} \mathrm{E}\big[\sum_{i \notin I_s^{(N,\varepsilon)}} T_i^{(N)} \big] = c\, \mathrm{E}[Y_s^{(N)} - Y_s^{(N,\varepsilon)}] \leqslant c\, C\, \varepsilon\, s, \end{aligned} \tag{4.13}$$

where we have applied (4.7). This, together with (4.7), proves that we can approximate $\boldsymbol{Y}_s^{(N)}$ by $\boldsymbol{Y}_s^{(N,\varepsilon)}$ in distribution, uniformly in N, by choosing ε small.

To complete the proof of (4.9), it remains to show that, for every fixed $\varepsilon > 0$,

$$\boldsymbol{Y}_s^{(N,\varepsilon)} := \big(Y_s^{(N,\varepsilon)}, V_s^{(N,\varepsilon)}\big) \xrightarrow[N\to\infty]{} \boldsymbol{Y}_s^{(\varepsilon)} = (Y_s^{(\varepsilon)}, V_s^{(\varepsilon)}) \quad \text{in distribution}, \tag{4.14}$$

where $V_s^{(\varepsilon)}$ was defined in (4.10). In the proof of Proposition 1.3 we showed that $\Xi^{(N,\varepsilon)}$ converges in distribution to $\Xi^{(\varepsilon)}$ as $N \to \infty$. By Skorohod's representation theorem, we can construct a coupling such that $\Xi^{(N,\varepsilon)}$ converges almost surely to $\Xi^{(\varepsilon)}$, that is the number and sizes of jumps of $Y_s^{(N,\varepsilon)}$ converge almost surely to those of $Y_s^{(\varepsilon)}$. Given a sequence of jumps of $(Y_s^{(N,\varepsilon)})_{N\in\mathbb{N}}$, say $t_{i_N}^{(N,\varepsilon)} \to t_i^{(\varepsilon)}$ for some jump $t_i^{(\varepsilon)}$ of $Y_s^{(\varepsilon)}$, we have that $X_{i_N}^{(N)}/\sqrt{N}$ converges in distribution to a centered Gaussian random variable with covariance matrix $(\mathsf{c}\, t_i^{(\varepsilon)}\, I)$, by the definition of $X_{i_N}^{(N)}$ in (2.7) and the local limit theorem in (2.5). Therefore, conditionally on all the jumps, the random variables $V_s^{(N,\varepsilon)}$ in (4.11) converges in distribution to the Gaussian law with covariance matrix

$$\sum_{i=1}^{\mathcal{N}^{(\varepsilon)}} (\mathsf{c}\, t_i^{(\varepsilon)}\, I) = \mathsf{c}\, Y_s^{(\varepsilon)}\, I\,,$$

which is precisely the law of $V_s^{(\varepsilon)} := \sqrt{\mathsf{c}}\, W_{Y_s^{(\varepsilon)}}$. This proves (4.14). □

Proof of Proposition 1.6. Note that $\mathrm{P}(Y_s \leqslant 1) = e^{-\gamma s}/\Gamma(s+1)$, by the first line of (1.4). With the change of variable $u = (\log \frac{1}{t})s$ in (1.15), we can write

$$\begin{aligned} G_\vartheta(t) &= \frac{1}{t}\int_0^\infty s\, e^{(\log t)s}\, e^{\vartheta s}\, \mathrm{P}(Y_s \leqslant 1)\, \mathrm{d}s \\ &= \frac{1}{t(\log\frac{1}{t})^2}\int_0^\infty u\, e^{-u}\, e^{\frac{\vartheta}{\log(1/t)}u}\, \mathrm{P}(Y_{u/\log(1/t)} \leqslant 1)\, \mathrm{d}u\,. \end{aligned}$$

Note that $\mathrm{P}(Y_{u/\log(1/t)} \leqslant 1) = 1 - O(\frac{1}{(\log(1/t))^2})$ as $t \downarrow 0$, for any fixed $u > 0$, by (B.7). Expanding the exponential, as $t \downarrow 0$, we obtain by dominated convergence

$$G_\vartheta(t) = \frac{1}{t(\log\frac{1}{t})^2}\left\{\int_0^\infty u\, e^{-u}\, \mathrm{d}u \;+\; \frac{\vartheta}{\log(1/t)}\int_0^\infty u^2\, e^{-u}\, \mathrm{d}u \;+\; O\left(\frac{1}{(\log(1/t))^2}\right)\right\},$$

which coincides with (1.21). □

Proof of Proposition 2.1. It suffices to compute the joint Laplace transform of $(Y_s, \sqrt{\mathsf{c}}\, W_{Y_s})$ and show that it agrees with (2.3). For $\varrho \in \mathbb{R}^2$, $s \geqslant 0$, $t > 0$, by independence of Y an W,

$$\mathrm{E}[e^{\langle \varrho, \sqrt{\mathsf{c}}\, W_{Y_s}\rangle} \,|\, Y_s = t] = \mathrm{E}[e^{\langle \varrho, \sqrt{\mathsf{c}}\, W_t\rangle}] = \mathrm{E}[e^{\sqrt{\mathsf{c}\, t}\langle \varrho, W_1\rangle}] = e^{\frac{1}{2}\mathsf{c}|\varrho|^2 t}\,.$$

Then for $\lambda \in \mathbb{R}$,

$$\mathrm{E}[e^{\lambda Y_s + \langle \varrho, \sqrt{\mathsf{c}}\, W_{Y_s}\rangle}] = \mathrm{E}[e^{(\lambda + \frac{1}{2}\mathsf{c}|\varrho|^2)Y_s}] = \exp\left\{s\int_0^1 (e^{(\lambda+\frac{1}{2}\mathsf{c}|\varrho|^2)t} - 1)\,\frac{1}{t}\,\mathrm{d}t\right\},$$

where we have applied (1.2). It remains to observe that, by explicit computation,

$$e^{(\lambda+\frac{1}{2}\mathsf{c}|\varrho|^2)t} - 1 = \int_{\mathbb{R}^2} (e^{\lambda t + \langle \varrho, x\rangle} - 1)\, g_{\mathsf{c}t}(x)\, \mathrm{d}x\,, \tag{4.15}$$

which gives (2.3). □

5. Proof of Proposition 1.5

This section is devoted to the proof of Proposition 1.5. Let us rewrite relation (1.20):

$$\mathrm{P}\big(\tau_k^{(N)} = n\big) \;\leqslant\; C\, k\, \mathrm{P}\big(T_1^{(N)} = n\big)\; \mathrm{P}\big(T_1^{(N)} \leqslant n\big)^{k-1}\, e^{-\frac{c\,k}{\log n + 1}\,\log^+ \frac{c\,k}{\log n+1}}\,. \tag{5.1}$$

The strategy, as in [AB16], is to isolate the contribution of the largest increment $T_i^{(N)}$. Our analysis is complicated by the fact that our renewal processes $\tau^{(N)}$ varies with $N \in \mathbb{N}$.

Before proving Proposition 1.5, we derive some useful consequences. We recall that the renewal process $(\tau_k^{(N)})_{k\geqslant 0}$ was defined in (1.13).

Proposition 5.1. *There are constants $C \in (0,\infty)$, $c \in (0,1)$ and, for every $\varepsilon > 0$, $N_\varepsilon \in \mathbb{N}$ such that for all $N \geqslant N_\varepsilon$, $s \in (0,\infty) \cap \frac{1}{\log N}\mathbb{N}$, $t \in (0,1] \cap \frac{1}{N}\mathbb{N}$ we have*

$$\mathrm{P}\big(\tau^{(N)}_{s \log N} = tN\big) \;\leqslant\; C\,\frac{1}{N}\,\frac{s}{t}\,t^{(1-\varepsilon)s}\,e^{-cs\,\log^+(cs)}\,. \tag{5.2}$$

Recalling that $f_s(t)$ is the density of Y_s, see (1.4), *it follows that for $N \in \mathbb{N}$ large enough*

$$\mathrm{P}\big(\tau^{(N)}_{s \log N} = tN\big) \;\leqslant\; C'\,\frac{1}{N}\,f_{cs}(t)\,. \tag{5.3}$$

Proof. Let us prove (5.3). Since $\Gamma(s+1) = e^{s(\log s - 1) + \log(\sqrt{2\pi s})}(1 + o(1))$ as $s \to \infty$, by Stirling's formula, and since $\gamma \simeq 0.577 < 1$, it follows by (1.4) that there is $c_1 > 0$ such that

$$f_s(t) \geqslant c_1\,\frac{s}{t}\,t^s\,e^{-s\,\log^+(s)}\,, \qquad \forall t \in (0,1]\,,\ \forall s \in (0,\infty)\,. \tag{5.4}$$

Then, if we choose $\varepsilon = 1 - c$ in (5.2), we see that (5.3) follows (with $C' = C/(cc_1)$).

In order to prove (5.2), let us derive some estimates. We denote by $c_1, c_2, \ldots$ generic absolute constants in $(0,\infty)$. By (1.12)-(1.11),

$$\mathrm{P}\big(T^{(N)}_1 \leqslant r\big) = \frac{R_r}{R_N} \leqslant c_1\,\frac{\log r}{\log N}\,, \qquad \forall r, N \in \mathbb{N}\,. \tag{5.5}$$

At the same time

$$\mathrm{P}\big(T^{(N)}_1 \leqslant r\big) = \frac{R_r}{R_N} = 1 - \frac{R_N - R_r}{R_N} \leqslant e^{-\frac{R_N - R_r}{R_N}}\,. \tag{5.6}$$

By (1.10), we can fix $\eta > 0$ small enough so that $\frac{R_N - R_r}{R_N} \geqslant \eta\,\frac{\log(N/r)}{\log N}$ *for all $r, N \in \mathbb{N}$ with* $r \leqslant N$. Plugging this into (5.6), we obtain a bound that will be useful later:

$$\mathrm{P}\big(T^{(N)}_1 \leqslant r\big) \leqslant \left(\frac{r}{N}\right)^{\frac{\eta}{\log N}}\,, \qquad \forall N \in \mathbb{N},\ \forall r = 1, \ldots, N\,. \tag{5.7}$$

We can sharpen this bound. For every $\varepsilon > 0$, let us show that there is $N_\varepsilon < \infty$ such that

$$\mathrm{P}\big(T^{(N)}_1 \leqslant r\big) \leqslant \left(\frac{r}{N}\right)^{\frac{1-\varepsilon}{\log N}}\,, \qquad \forall N \geqslant N_\varepsilon\,,\ \forall r = 1, 2, \ldots, N\,. \tag{5.8}$$

We first consider the range $r \leqslant N^\vartheta$, where $\vartheta := e^{-1}/c_1$. Then, by (5.5),

$$\mathrm{P}\big(T^{(N)}_1 \leqslant r\big) \leqslant \mathrm{P}\big(T^{(N)}_1 \leqslant N^\vartheta\big) \leqslant c_1\,\vartheta = e^{-1} = \big(\tfrac{1}{N}\big)^{\frac{1}{\log N}} \leqslant \big(\tfrac{r}{N}\big)^{\frac{1}{\log N}} \leqslant \big(\tfrac{r}{N}\big)^{\frac{1-\varepsilon}{\log N}}\,.$$

Next we take $r \geqslant N^\vartheta$. Then $\frac{R_N - R_r}{R_N} \geqslant (1-\varepsilon)\,\frac{\log(N/r)}{\log N}$ for N large enough, by (1.10), which plugged into (5.6) completes the proof of (5.8). We point out that the bounds (5.7), (5.8) are poor for small r, but they provide a simple and unified expression, valid for all $r = 1, \ldots, N$.

We can finally show that (5.2) follows by (5.1) (from Proposition 1.5) where we plug $k = s \log N$ and $n = tN$, for $s \in (0,\infty) \cap \frac{1}{\log N}\mathbb{N}_0$ and $t \in (0,1] \cap \frac{1}{N}\mathbb{N}$. Indeed, note that:

- by (1.12)-(1.11) we have $k\,\mathrm{P}\big(T^{(N)}_1 = n\big) \leqslant c_2\,\frac{k}{(\log N)n} = c_2\,\frac{1}{N}\,\frac{s}{t}$;
- since $\frac{k}{\log n+1} \geqslant \frac{k}{\log N+1} \geqslant c_3\,s$ for $n \leqslant N$, the last term in (5.1) matches with the corresponding term in (5.2);
- by (5.8) we have $\mathrm{P}\big(T^{(N)}_1 \leqslant n\big)^{k-1} \leqslant t^{(1-\varepsilon)s}\,t^{-\frac{1}{\log N}} \leqslant t^{(1-\varepsilon)s}\,\big(\frac{1}{N}\big)^{-\frac{1}{\log N}} = e\,t^{(1-\varepsilon)s}$, because $t \geqslant \frac{1}{N}$, hence (5.2) is deduced. □

Before starting with the proof of Proposition 1.5, we derive some large deviation estimates. We start by giving an upper bound on the upper tail $\mathrm{P}(\tau_k^{(m)} \geqslant n)$ for arbitrary $m, k, n \in \mathbb{N}$. This is a Fuk-Nagaev type inequality, see [N79, Theorem 1.1].

Lemma 5.2. *There exists a constant $C \in (1, \infty)$ such that for all $m \in \mathbb{N}$ and $s, t \in [0, \infty)$*

$$\mathrm{P}\big(\tau^{(m)}_{\lfloor s(\log m+1)\rfloor} \geqslant tm\big) \;\leqslant\; e^{-t\,\log^+(\frac{t}{Cs})}\,. \tag{5.9}$$

Proof. We are going to prove that for all $m, n, k \in \mathbb{N}$

$$\mathrm{P}\big(\tau_k^{(m)} \geqslant n\big) \;\leqslant\; \left(\frac{C\,k\,m}{n\,(\log m+1)} \wedge 1\right)^{\frac{n}{m}}, \tag{5.10}$$

which is just a rewriting of (5.9). For some $c_1 < \infty$ we have $\mathrm{E}[\tau_1^{(m)}] \leqslant c_1 \frac{m}{\log m+1}$, see (1.10)-(1.12). Since $\tau_1^{(m)} \leqslant m$, we can estimate

$$\begin{aligned}\mathrm{E}\big[e^{\lambda\tau_1^{(m)}}\big] &= 1 + \sum_{j\geqslant 1}\frac{\lambda^j}{j!}\,\mathrm{E}[(\tau_1^{(m)})^j] \leqslant 1 + \sum_{j\geqslant 1}\frac{\lambda^j}{j!}\,m^{j-1}\,\mathrm{E}[\tau_1^{(m)}] \leqslant 1 + \frac{c_1}{\log m+1}\sum_{j\geqslant 1}\frac{(\lambda m)^j}{j!}\\ &\leqslant 1 + \frac{c_1}{\log m+1}\,e^{\lambda m}\,.\end{aligned}$$

This yields, by Markov inequality, for all $\lambda \geqslant 0$,

$$\begin{aligned}\mathrm{P}\big(\tau_k^{(m)} \geqslant n\big) &\leqslant e^{-\lambda n}\,\mathrm{E}\big[e^{\lambda\tau_1^{(m)}}\big]^k = e^{-\lambda n}\,\big(1 + \tfrac{c_1}{\log m+1}e^{\lambda m}\big)^k\\ &\leqslant e^{-\lambda n}\exp\big(\tfrac{c_1\,k}{\log m+1}e^{\lambda m}\big).\end{aligned} \tag{5.11}$$

We now choose λ such that

$$\tfrac{k}{\log m+1}\,e^{\lambda m} = \tfrac{n}{m}\,, \qquad \text{that is} \qquad e^{-\lambda} = \big(\tfrac{m\,k}{n\,(\log m+1)}\big)^{\frac{1}{m}}\,.$$

If $\frac{m\,k}{n\,(\log m+1)} > 1$ relation (5.10) holds trivially, so we assume $\frac{m\,k}{n\,(\log m+1)} \leqslant 1$, so that $\lambda \geqslant 0$. This choice of λ, when plugged into (5.11), gives (5.10) with $C = e^{c_1+1}$. ☐

Remark 5.3. *Heuristically, the upper bound (5.10) corresponds to requiring that among the k increments $T_1^{(m)}, T_2^{(m)}, \ldots, T_k^{(m)}$ there are $\ell := \frac{n}{m}$ "big jumps" of size comparable to m. To be more precise, let us first recall the standard Cramer large deviations bound*

$$\mathrm{P}(\mathrm{Pois}(\lambda) > t) \leqslant e^{-t(\log\frac{t}{\lambda}-1)} = \big(\tfrac{e\lambda}{t}\big)^t\,, \qquad \forall \lambda, t > 0\,.$$

Now fix $a \in (0,1)$ and note that $\mathrm{P}(T_1^{(m)} > am) \sim p_m := \frac{c}{\log m}$ (where $c = \log\frac{1}{a}$). If we denote by $N_{k,am}$ the number of increments $T_i^{(m)}$ of size at least am, we can write

$$\mathrm{P}(N_{k,m} \geqslant \ell) = \mathrm{P}(\mathrm{Bin}(k,p_m) \geqslant \ell) \approx \mathrm{P}(\mathrm{Pois}(k\,p_m) \geqslant \ell) \leqslant \left(\frac{e\,k\,p_m}{\ell}\right)^{\ell}.$$

If we choose $\ell = \frac{n}{m}$, we obtain the same bound as in (5.10). This indicates that the strategy just outlined captures the essential contribution of the event $\{\tau_k^{(m)} \geqslant n\}$.

We complement Lemma 5.2 with a bound on the lower tail $\mathrm{P}(\tau_k^{(m)} \leqslant n)$.

Lemma 5.4. *There exists a constant $c \in (0,1)$ such that for all $m \in \mathbb{N}$ and $s, t \in [0, \infty)$*

$$\mathrm{P}\big(\tau^{(m)}_{\lfloor s(\log m+1)\rfloor} \leqslant tm\big) \;\leqslant\; e^{-c\,s\,\log^+(\frac{cs}{t})}\,. \tag{5.12}$$

Proof. We are going to prove that there exists $c \in (0,1)$ such that for all $m, n, k \in \mathbb{N}$

$$\mathrm{P}\big(\tau_k^{(m)} \leqslant n\big) \leqslant \left(\frac{n\,(\log m+1)}{c\,k\,m} \wedge 1\right)^{\frac{c\,k}{\log m+1}}, \tag{5.13}$$

which is just a rewriting of (5.12). For $\lambda \geqslant 0$ we have

$$\mathrm{P}(\tau_k^{(m)} \leqslant n) = \mathrm{P}(e^{-\lambda \tau_k^{(m)}} \geqslant e^{-\lambda n}) \leqslant e^{\lambda n}\, \mathrm{E}[e^{-\lambda T_1^{(m)}}]^k\,. \tag{5.14}$$

Next we evaluate, by (1.10)-(1.11),

$$\mathrm{E}[e^{-\lambda T_1^{(m)}}] = \sum_{n=1}^m e^{-\lambda n}\,\frac{r(n)}{R_m} = 1 - \sum_{n=1}^m (1-e^{-\lambda n})\,\frac{r(n)}{R_m} \leqslant 1 - \frac{c_1}{\log m+1}\sum_{n=1}^m \frac{1-e^{-\lambda n}}{n}\,,$$

for some $c_1 \in (0,1)$. Since the function $x \mapsto \frac{1-e^{-x}}{x}$ is decreasing for $x \geqslant 0$, we can bound

$$\mathrm{E}[e^{-\lambda T_1^{(m)}}] \leqslant 1 - \frac{c_1}{\log m+1}\int_1^{m+1} \frac{1-e^{-\lambda t}}{t}\,\mathrm{d}t = 1 - \frac{c_1}{\log m+1}\int_\lambda^{\lambda(m+1)} \frac{1-e^{-x}}{x}\,\mathrm{d}x\,.$$

We are going to fix $\frac{1}{m} \leqslant \lambda \leqslant 1$. Restricting the integration to the interval $1 \leqslant x \leqslant \lambda m$ and bounding $1-e^{-x} \geqslant (1-e^{-1})$ we obtain, for $c_2 := (1-e^{-1})c_1$,

$$\mathrm{E}[e^{-\lambda T_1^{(m)}}] \leqslant 1 - \tfrac{c_2}{\log m+1}\,\log(\lambda m) \leqslant e^{-\frac{c_2}{\log m+1}\,\log(\lambda m)} = \left(\tfrac{1}{\lambda m}\right)^{\frac{c_2}{\log m+1}}.$$

Looking back at (5.14), we obtain

$$\mathrm{P}(\tau_k^{(m)} \leqslant n) \leqslant e^{\lambda n}\left(\tfrac{1}{\lambda m}\right)^{c_2\,\frac{k}{\log m+1}}. \tag{5.15}$$

We are ready to prove (5.13). Assume first that $k \leqslant n$ and let $\lambda := \frac{k}{n\,(\log m+1)} \leqslant 1$. We may assume that $\lambda \geqslant \frac{1}{m}$, because for $\lambda m < 1$ the right hand side of (5.13) equals 1 and there is nothing to prove. We then have $\frac{1}{m} \leqslant \lambda \leqslant 1$. Plugging λ into (5.15) gives

$$\mathrm{P}(\tau_k^{(m)} \leqslant n) \leqslant \left(\frac{e^{\frac{1}{c_2}}\,n\,(\log m+1)}{km} \wedge 1\right)^{c_2\,\frac{k}{\log m+1}},$$

where we inserted "$\wedge 1$" because the left hand side is a probability. Since $x \geqslant e^{-1/x}$ for $x \geqslant 0$, in the exponent we can replace c_2 by $c := e^{-1/c_2}$, which yields (5.13).

Finally, for $k > n$ the left hand side of (5.13) vanishes, because $\tau_k^{(m)} \geqslant k$. □

Remark 5.5. *For renewal processes with a density, see Remark 1.7, the proof of Lemma 5.4 can be easily adapted, replacing sums by integrals. The only difference is that we no longer have $\tau_k^{(m)} \geqslant k$, so the case $k > n$ needs a separate treatment. To this purpose, we note that*

$$\mathrm{E}[e^{-\lambda T_1^{(m)}}] = \int_0^m e^{-\lambda t}\,\frac{r(t)}{R_m}\,\mathrm{d}t \leqslant \frac{c_0}{\log m+1}\int_0^\infty e^{-\lambda t}\,\mathrm{d}t = \frac{c_0}{\log m+1}\,\frac{1}{\lambda}\,,$$

for some $c_0 \in (1,\infty)$. If we set $\lambda = \frac{k}{n}$, by (5.14) we get

$$\mathrm{P}(\tau_k^{(m)} \leqslant n) \leqslant \left(\frac{n}{k}\right)^k \left(\frac{e\,c_0}{\log m+1}\right)^k. \tag{5.16}$$

We now give a lower bound on the right hand side of (5.13)*. We assume that the fraction therein is* $\leqslant 1$*, otherwise there is nothing to prove. Since* $c \in (0,1)$*, for* $k > n$ *we can bound*

$$\left(\frac{n}{k}\right)^{\frac{c\,k}{\log m+1}} \left(\frac{\log m+1}{c\,m}\right)^{\frac{c\,k}{\log m+1}} \geqslant \left(\frac{n}{k}\right)^{k} \left(\frac{1}{m+1}\right)^{\frac{c\,k}{\log m+1}} = \left(\frac{n}{k}\right)^{k} e^{-c\,k} \geqslant \left(\frac{n}{k}\right)^{k} e^{-k}\,.$$

This is larger than the right hand side of (5.16)*, if we take* $m \geqslant m_0 := \lfloor \exp(e^2\,c_0) \rfloor$ *(so that* $\frac{e\,c_0}{\log m+1} \leqslant e^{-1}$*). This shows that* (5.16) *holds for* $k > n$ *and* $m \geqslant m_0$*.*

It remains to consider the case $k > n$ *and* $m < m_0$*. Note that lowering* c *increases the right hand side of* (5.13)*, so we can assume that* $c \leqslant \frac{\log m_0+1}{e\,c_0\,m_0}$*. Since* $m \mapsto \frac{\log m+1}{m}$ *is decreasing for* $m \geqslant 1$*, we can bound the right hand side of* (5.13) *from below (assuming that the fraction therein is* $\leqslant 1$*) as follows, for* $k > n$ *and* $m < m_0 t$*:*

$$\left(\frac{n}{k}\,\frac{\log m_0+1}{c\,m_0}\right)^{\frac{c\,k}{\log m+1}} \geqslant \left(\frac{n}{k}\,e\,c_0\right)^{\frac{c\,k}{\log m+1}} \geqslant \left(\frac{n}{k}\,\frac{e\,c_0}{\log m+1}\right)^{\frac{c\,k}{\log m+1}}\,,$$

which is larger than the right hand side of (5.16)*. This completes the proof of* (5.13) *for renewal processes with a density, as in Remark 1.7.*

Proof of Proposition 1.5. We have to prove relation (5.1) for all $N, k, n \in \mathbb{N}$ with $n \leqslant N$.

Let us set

$$M_k^{(N)} := \max_{1\leqslant i\leqslant k} T_i^{(N)}\,,$$

and note that $\{\tau_k^{(N)} = n\} \subseteq \{M_k^{(N)} \leqslant n\}$. This yields

$$\frac{\mathrm{P}\big(\tau_k^{(N)} = n\big)}{\mathrm{P}\big(T_1^{(N)} \leqslant n\big)^k} = \mathrm{P}\big(\tau_k^{(N)} = n \,\big|\, M_k^{(N)} \leqslant n\big) = \mathrm{P}\big(\tau_k^{(n)} = n\big)\,, \tag{5.17}$$

where the last equality holds because the random variables $T_i^{(N)}$, conditioned on $\{T_i^{(N)} \leqslant n\}$, have the same law as $T_i^{(n)}$, see (1.12). Let us now divide both sides of (5.1) by $\mathrm{P}\big(T_1^{(N)} \leqslant n\big)^k$. The equality (5.17) and the observation that $\mathrm{P}(T_1^{(N)} = n)/\mathrm{P}(T_1^{(N)} \leqslant n) = \mathrm{P}(T_1^{(n)} = n)$ show that (5.1) is implied by

$$\mathrm{P}\big(\tau_k^{(n)} = n\big) \;\leqslant\; C\,k\,\frac{1}{n\,(\log n+1)}\; e^{-\frac{c\,k}{\log n+1}\,\log^+\frac{c\,k}{\log n+1}}\,. \tag{5.18}$$

Note that there is no longer dependence on N.

It remains to prove (5.18). By Lemma 5.4, more precisely by (5.13), we can bound

$$\mathrm{P}\big(\tau_k^{(n)} = n\big) \leqslant \mathrm{P}\big(\tau_k^{(n)} \leqslant n\big) \leqslant \left(\frac{\log n+1}{c\,k} \wedge 1\right)^{\frac{c\,k}{\log n+1}} = e^{-\frac{c\,k}{\log n+1}\,\log^+\frac{c\,k}{\log n+1}}\,.$$

This shows that (5.18) holds for every $k \in \mathbb{N}$ if we take $C = C(n) := n\,(\log n+1)$. Then, for any fixed $\bar{n} \in \mathbb{N}$, we can set $C := \max_{n\leqslant\bar{n}} C(n)$ and relation (5.18) holds for all $n \leqslant \bar{n}$ and $k \in \mathbb{N}$. As a consequence, it remains to prove that there is another constant $C < \infty$ such that relation (5.18) holds for all $n \geqslant \bar{n}$ and $k \in \mathbb{N}$. Note that $\bar{n} \in \mathbb{N}$ is arbitrary.

We start by estimating, for any $m \in (1, n]$ (possibly not an integer, for later convenience)

$$\begin{aligned}
&\mathrm{P}\big(\tau_k^{(n)} = n\,,\, M_k^{(n)} \in (e^{-1}m, m]\,\big) \\
&\leqslant k \sum_{r \in (e^{-1}m, m]} \mathrm{P}(T_1^{(n)} = r)\,\mathrm{P}\big(\tau_{k-1}^{(n)} = n - r\,,\, M_{k-1}^{(n)} \leqslant r\,\big) \\
&\leqslant k \max_{r \in (e^{-1}m, m]} \mathrm{P}(T_1^{(n)} = r)\;\mathrm{P}\big(T_1^{(n)} \leqslant m\,\big)^{k-1} \sum_{r \in (e^{-1}m, m]} \mathrm{P}\big(\tau_{k-1}^{(n)} = n - r \,\big|\, M_{k-1}^{(n)} \leqslant m\big)\,.
\end{aligned} \tag{5.19}$$

Since $T_i^{(n)}$ conditioned on $T_i^{(n)} \leqslant m$ is distributed as $T_i^{(m)} := T_i^{(\lfloor m \rfloor)}$, we get, by (1.12)-(1.11),

$$\begin{aligned}
&\mathrm{P}\big(\tau_k^{(n)} = n\,,\, M_k^{(n)} \in (e^{-1}m, m]\,\big) \\
&\qquad \leqslant\; c_4\, k\, \frac{1}{m\,(\log n + 1)}\;\mathrm{P}\big(T_1^{(n)} \leqslant m\,\big)^{k-1}\;\mathrm{P}\big(n - m \leqslant \tau_{k-1}^{(m)} < n - e^{-1}m\big)\,.
\end{aligned} \tag{5.20}$$

We bound $\mathrm{P}(T_1^{(n)} \leqslant m)^{k-1} \leqslant (\frac{m}{n})^{\frac{\eta(k-1)}{\log n}} \leqslant e\,(\frac{m}{n})^{\frac{\eta k}{\log n}}$, by (5.7). Choosing $m = e^{-\ell}n$ in (5.20) and summing over $0 \leqslant \ell \leqslant \log n$, we obtain the key bound

$$\begin{aligned}
&\mathrm{P}\big(\tau_k^{(n)} = n\big) \;=\; \sum_{\ell=0}^{\lfloor \log n \rfloor} \mathrm{P}\big(\tau_k^{(n)} = n\,,\, M_k^{(n)} \in (e^{-\ell-1}n, e^{-\ell}n]\,\big) \\
&\leqslant\; c_4\, k\, \frac{1}{n\,(\log n + 1)} \sum_{\ell=0}^{\lfloor \log n \rfloor} e^{\ell}\,\mathrm{P}\big(T_1^{(n)} \leqslant e^{-\ell}n\,\big)^{k-1}\,\mathrm{P}\Big((1 - e^{-\ell})n \leqslant \tau_{k-1}^{(e^{-\ell}n)} < (1 - e^{-(\ell+1)})n\Big)\,.
\end{aligned} \tag{5.21}$$

To complete the proof of (5.18), we show that, for suitable $C \in (0, \infty)$ and $c \in (0,1)$,

$$\sum_{\ell=0}^{\lfloor \log n \rfloor} e^{\ell}\,\mathrm{P}\big(T_1^{(n)} \leqslant e^{-\ell}n\,\big)^{k-1}\,\mathrm{P}\Big((1 - e^{-\ell})n \leqslant \tau_{k-1}^{(e^{-\ell}n)} < n\Big) \leqslant C\, e^{-\frac{c\,k}{\log n + 1}\,\log^+ \frac{c\,k}{\log n + 1}}\,. \tag{5.22}$$

Let $c \in (0,1)$ be the constant in Lemma 5.4. We recall that we may fix $\bar{n}$ arbitrarily and focus on $n \geqslant \bar{n}$. We fix $c' \in (0,1)$ with $c' > c$, and we choose $\bar{n}$ so that, by (5.8) with $N = n$ and $r = e^{-\ell}n$,

$$\mathrm{P}\big(T_1^{(n)} \leqslant e^{-\ell}n\,\big) \leqslant (e^{-\ell})^{\frac{c'}{\log n}} \qquad \forall n \geqslant \bar{n}\,,\ \forall \ell = 0, 1, \ldots, \lfloor \log n \rfloor\,.$$

Then (5.22) is reduced to showing that for all $n \geqslant \bar{n}$ and $k = 1, \ldots, n$

$$\sum_{\ell=0}^{\lfloor \log n \rfloor} e^{\ell}\,(e^{-\ell})^{\frac{c'(k-1)}{\log n}}\,\mathrm{P}\Big((1 - e^{-\ell})n \leqslant \tau_{k-1}^{(e^{-\ell}n)} < n\Big) \leqslant C\, e^{-\frac{c\,k}{\log n + 1}\,\log^+ \frac{c\,k}{\log n + 1}}\,. \tag{5.23}$$

We first consider the regime of $k \in \mathbb{N}$ such that

$$k > 1 + \tfrac{2}{c' - c}\,(\log n + 1)\,. \tag{5.24}$$

We use Lemma 5.4 to bound the probability in (5.23). More precisely, we apply relation (5.12) with $m = e^{-\ell}n$, $s = \frac{k-1}{\log(e^{-\ell}n)+1}$, $t = e^{\ell}$ and with $\log^+$ replaced by $\log$, to get an upper bound. Since $e^{-\ell}n \leqslant n$, we get by monotonicity

$$\begin{aligned}
\mathrm{P}\big(\tau_{k-1}^{(e^{-\ell}n)} < n\big) \;&\leqslant\; e^{-\frac{c\,(k-1)}{\log(e^{-\ell}n)+1}\,\log\left(e^{-\ell}\,\frac{c\,(k-1)}{\log(e^{-\ell}n)+1}\right)} \;\leqslant\; e^{-\frac{c\,(k-1)}{\log n + 1}\,\log\left(e^{-\ell}\,\frac{c\,(k-1)}{\log n + 1}\right)} \\
&=\; \Big\{e^{-\frac{c\,(k-1)}{\log n + 1}\,\log\frac{c\,(k-1)}{\log n + 1}}\Big\}\;\big(e^{\frac{c\,(k-1)}{\log n}}\big)^{\ell}\,.
\end{aligned} \tag{5.25}$$

Since $k-1 \geqslant \frac{k}{2}$ for $k \geqslant 2$, if we redefine $c/2$ as c, we see that the term in brackets in (5.25) matches with the right hand side of (5.23) (where we can replace $\log^+$ by log, by (5.24) and $\frac{2}{c'-c} > c$). The other term in (5.25), when inserted in the left hand side of (5.23), gives a contribution to the sum which is uniformly bounded, by (5.24):

$$\sum_{\ell=0}^{\lfloor \log n \rfloor} e^{\ell} \, (e^{-\ell})^{\frac{c'(k-1)}{\log n}} \, \left(e^{\frac{c\,(k-1)}{\log n}}\right)^{\ell} \leqslant \sum_{\ell=0}^{\infty} \left(e^{1-(c'-c)\frac{k}{\log n}}\right)^{\ell} \leqslant \sum_{\ell=0}^{\infty} e^{-\ell} < \infty \,.$$

This completes the proof of (5.23) under the assumption (5.24).

Next we consider the complementary regime of (5.24), that is

$$k \leqslant A \log n + B \,, \tag{5.26}$$

for suitably fixed constants A, B. In this case the right hand side of (5.23) is uniformly bounded from below by a positive constant. Therefore it suffices to show that

$$\sum_{\ell=1}^{\lfloor \log n \rfloor} e^{\ell} \, \mathrm{P}\Big(\tfrac{n}{2} \leqslant \tau_{k-1}^{(e^{-\ell} n)} < n \Big) \leqslant C \,, \tag{5.27}$$

where, in order to lighten notation, we removed from (5.22) the term $\ell = 0$ (which contributes at most one) and then bounded $(1-e^{-\ell})n \geqslant \frac{n}{2}$ for $\ell \geqslant 1$.

We apply Lemma 5.2 (with the constant C renamed D, to avoid confusion with (5.27)). Relation (5.9) with $m = e^{-\ell} n$, $s = \frac{k}{\log(e^{-\ell} n)+1}$, $t = \frac{1}{2} e^{\ell}$ gives

$$\mathrm{P}\big(\tau_k^{(e^{-\ell} n)} \geqslant \tfrac{n}{2}\big) \leqslant e^{-\frac{1}{2} e^{\ell} \, \log^+\left(\frac{e^{\ell}}{2D} \frac{\log n - \ell + 1}{k}\right)} = e^{-e^{\ell} \left\{ \frac{1}{2} \, \log^+\left(\frac{1}{2D} \frac{1}{x_\ell} \right) \right\}} \,, \tag{5.28}$$

where we have introduced the shorthand

$$x_\ell := \tfrac{k \, e^{-\ell}}{\log n - \ell + 1} \,. \tag{5.29}$$

For ℓ such that $x_\ell < \frac{1}{2De^2}$ the right hand side of (5.28) is at most $e^{-e^{\ell}}$. We claim that

$$x_\ell < \tfrac{1}{2De^2} \qquad \text{for all } \ell \geqslant \bar{\ell}, \text{ where} \qquad \bar{\ell} := \lfloor \log\big(4(A+B)De^2\big) \rfloor + 1 \,. \tag{5.30}$$

This completes the proof of (5.27), because the sum is at most $\sum_{\ell=1}^{\bar{\ell}} e^{\ell} + \sum_{\ell=\bar{\ell}+1}^{\infty} e^{\ell} \, e^{-e^{\ell}} < \infty$.

It remains to prove that relation (5.30) holds in regime (5.26). We recall that we may assume that n is large enough. Consider first the range $\frac{1}{2} \log n \leqslant \ell \leqslant \lfloor \log n \rfloor$: then

$$x_\ell \leqslant k \, e^{-\ell} \leqslant \tfrac{k}{\sqrt{n}} \leqslant \tfrac{A \log n + B}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0 \,,$$

hence we have $x_\ell < \frac{1}{2De^2}$ for n large enough. Consider finally the range $\ell < \frac{1}{2} \log n$: then

$$x_\ell \leqslant \tfrac{k}{\frac{1}{2}\log n} \, e^{-\ell} \leqslant \tfrac{A \log n + B}{\frac{1}{2} \log n} \, e^{-\ell} \leqslant 2(A+B) \, e^{-\bar{\ell}} \leqslant \tfrac{1}{2De^2} \,,$$

by the definition (5.30) of $\bar{\ell}$. This completes the proof. ☐

We conclude this section by extending Proposition 5.1 to the multidimensional setting. We recall that $(\tau_k^{(N)}, S_k^{(N)})$ is defined in (2.8).

Proposition 5.6. *There are constants $C \in (0,\infty)$, $c \in (0,1)$ and, for every $\varepsilon > 0$, $N_\varepsilon \in \mathbb{N}$ such that for all $N \geqslant N_\varepsilon$, $s \in (0,\infty) \cap \frac{1}{\log N}\mathbb{N}$, $t \in (0,1] \cap \frac{1}{N}\mathbb{N}$ and $x \in \frac{1}{\sqrt{N}}\mathbb{Z}^d$ we have*

$$\mathrm{P}\big(\tau_{s \log N}^{(N)} = tN \,, \; S_{s \log N}^{(N)} = x\sqrt{N}\big) \leqslant C \, \frac{1}{N^{1+\frac{d}{2}}} \, \frac{s}{t^{1+\frac{d}{2}}} \, t^{(1-\varepsilon)s} \, e^{-cs \, \log^+(cs)} \,. \tag{5.31}$$

It follows that for $N \in \mathbb{N}$ large enough

$$\mathrm{P}\big(\tau^{(N)}_{s\log N} = tN\,,\ S^{(N)}_{s\log N} = x\sqrt{N}\,\big) \;\leqslant\; C'\,\frac{1}{N}\,\frac{1}{(Nt)^{\frac{d}{2}}}\,f_{cs}(t)\,. \tag{5.32}$$

Proof. We follow closely the proof of Proposition 5.1. Relation (5.32) follows from (5.31) with $\varepsilon = 1-c$, thanks to the bound (5.4), so we focus on (5.31).

We will prove an analog of relation (5.1): for all $N,k,n \in \mathbb{N}$ with $n \leqslant N$ and for all $z \in \mathbb{Z}^d$

$$\mathrm{P}\big(\tau^{(N)}_k = n\,,\ S^{(N)}_k = z\big) \leqslant C\,\frac{k}{n^{\frac{d}{2}}}\,\mathrm{P}\big(T^{(N)}_1 = n\big)\ \mathrm{P}\big(T^{(N)}_1 \leqslant n\big)^{k-1}\,e^{-\frac{c\,k}{\log n+1}\,\log^+\frac{c\,k}{\log n+1}}. \tag{5.33}$$

Note that the only difference with respect to (5.1) is the term $n^{\frac{d}{2}}$ in the denominator.

In the proof of Proposition 5.1 we showed that (5.2) follows from (5.1). In exactly the same way, relation (5.31) follows from (5.33), by choosing $k = s\log N$, $n = Nt$, $z = x\sqrt{N}$.

It remains to prove (5.33). Arguing as in (5.17), we remove the dependence on N and it suffices to prove the following analog of (5.18): for all $n,k \in \mathbb{N}$ and for all $z \in \mathbb{Z}^d$

$$\mathrm{P}\big(\tau^{(n)}_k = n\,,\ S^{(n)}_k = z\big) \;\leqslant\; C\,\frac{k}{n^{\frac{d}{2}}}\,\frac{1}{n\,(\log n+1)}\ e^{-\frac{c\,k}{\log n+1}\,\log^+\frac{c\,k}{\log n+1}}\,. \tag{5.34}$$

To this purpose, we claim that we can modify (5.20) as follows:

$$\begin{aligned}
&\mathrm{P}\big(\tau^{(n)}_k = n\,,\ S^{(n)}_k = z\,,\ M^{(n)}_k \in (e^{-1}m, m]\,\big) \\
&\quad\leqslant c_4\,\frac{k}{m^{\frac{d}{2}}}\,\frac{1}{m\,(\log n+1)}\ \mathrm{P}\big(T^{(n)}_1 \leqslant m\big)^{k-1}\ \mathrm{P}\big(n-m \leqslant \tau^{(m)}_{k-1} < n - e^{-1}m\big)\,.
\end{aligned} \tag{5.35}$$

This is because, arguing as in (5.19), we can write

$$\begin{aligned}
&\mathrm{P}\big(\tau^{(n)}_k = n\,, S^{(n)}_k = x\,,\, M^{(n)}_k \in (e^{-1}m, m]\,\big) \\
&\leqslant k \sum_{r\in(e^{-1}m,m]\,,\,y\in\mathbb{Z}^d} \mathrm{P}(T^{(n)}_1 = r\,,\,X^{(n)}_1 = y)\,\mathrm{P}\big(\tau^{(n)}_{k-1} = n-r\,,\,S^{(n)}_{k-1} = x-y\,,\,M^{(n)}_{k-1} \leqslant r\,\big) \\
&\leqslant k\,\Big\{\max_{r\in(e^{-1}m,m]\,,\,y\in\mathbb{Z}^d} \mathrm{P}(T^{(n)}_1 = r\,,\,X^{(n)}_1 = y)\Big\}\ \mathrm{P}\big(T^{(n)}_1 \leqslant m\,\big)^{k-1} \\
&\qquad\qquad \sum_{r\in(e^{-1}m,m]} \mathrm{P}\big(\tau^{(n)}_{k-1} = n-r\,\big|\,M^{(n)}_{k-1} \leqslant m\big)\,,
\end{aligned}$$

and it follows by (2.7), (2.6) and (1.10)-(1.11) that

$$\max_{r\in(e^{-1}m,m]\,,\,y\in\mathbb{Z}^d} \mathrm{P}(T^{(n)}_1 = r\,,\,X^{(n)}_1 = y) \leqslant \frac{C}{\log n+1}\,\frac{1}{m^{1+\frac{d}{2}}}\,.$$

We can now plug $m = e^{-\ell}n$ into (5.35) and sum over $\ell = 0,1,\ldots,\lfloor\log n\rfloor$, as in (5.21). This leads to our goal (5.34), provided we prove the following analog of (5.22):

$$\sum_{\ell=0}^{\lfloor\log n\rfloor} e^{(1+\frac{d}{2})\ell}\,\mathrm{P}\big(T^{(n)}_1 \leqslant e^{-\ell}n\,\big)^{k-1}\,\mathrm{P}\Big((1-e^{-\ell})n \leqslant \tau^{(e^{-\ell}n)}_{k-1} < n\Big) \leqslant C\,e^{-\frac{c\,k}{\log n+1}\,\log^+\frac{c\,k}{\log n+1}}\,.$$

The only difference with respect to (5.22) is the term $e^{(1+\frac{d}{2})\ell}$ instead of e^{ℓ} in the sum. It is straightforward to adapt the lines following (5.22) and complete the proof. $\square$

6. Proof of Theorem 1.4

We prove separately the uniform upper bound (1.18) and the local limit theorem (1.17). For later use, we state an immediate corollary of Lemma 5.4 (with $t = 1$).

Lemma 6.1. *There is a constant $c \in (0,1)$ such that for all $N \in \mathbb{N}$, $s \in [0,\infty)$*

$$\mathrm{P}\big(\tau^{(N)}_{\lfloor s \log N \rfloor} \leqslant N\big) \leqslant e^{s - c\, s \log s} \,. \tag{6.1}$$

6.1. Proof of (1.18). Recall the definition (4.1) of $Y^{(N)}_s$. From the definition (1.14) of $U_{N,\lambda}(n)$ and the upper bound (5.3), we get for large N

$$U_{N,\lambda}(n) = \sum_{k \geqslant 0} \lambda^k \, \mathrm{P}\Big(Y^{(N)}_{\frac{k}{\log N}} = \tfrac{n}{N}\Big) \leqslant C \, \frac{\log N}{N} \left\{ \frac{1}{\log N} \sum_{k \geqslant 0} \lambda^k \, f_{c \frac{k}{\log N}}\big(\tfrac{n}{N}\big) \right\} . \tag{6.2}$$

We now choose $\lambda = \lambda_N$ as in (1.16). Then for some $A \in (0,\infty)$ we have

$$\lambda_N \leqslant 1 + A \, \frac{\vartheta}{\log N} \leqslant e^{A \frac{\vartheta}{\log N}} \,, \qquad \forall N \in \mathbb{N} \,,$$

hence

$$U_{N,\lambda_N}(n) \leqslant C \, \frac{\log N}{N} \left\{ \frac{1}{\log N} \sum_{k \geqslant 0} e^{\frac{k}{\log N} A \vartheta} \, f_{c \frac{k}{\log N}}\big(\tfrac{n}{N}\big) \right\} . \tag{6.3}$$

The bracket is a Riemann sum, which converges as $N \to \infty$ to the corresponding integral. It follows that for every $N \in \mathbb{N}$ we can write, recalling (1.15),

$$U_{N,\lambda_N}(n) \leqslant C' \, \frac{\log N}{N} \left\{ \int_0^\infty e^{s A \vartheta} \, f_{cs}\big(\tfrac{n}{N}\big) \, \mathrm{d}s \right\} = \frac{C'}{c} \, \frac{\log N}{N} \, G_{\frac{A}{c}\vartheta}\big(\tfrac{n}{N}\big) \,, \tag{6.4}$$

for some constant C'. (The fact that C' is uniform over $1 \leqslant n \leqslant N$ is proved below.)

To complete the proof of (1.18), we can replace $G_{\frac{A}{c}\vartheta}\big(\frac{n}{N}\big)$ by $G_\vartheta\big(\frac{n}{N}\big)$, possibly enlarging the constant C', because the function $t \mapsto G_\vartheta(t)$ is strictly positive, continuous and its asymptotic behavior as $t \to 0$ for different values of ϑ is comparable, by Proposition 1.6.

We finally prove the following claim: *we can bound the Riemann sum in* (6.3) *by a multiple of the coresponding integral in* (6.4), *uniformly over* $1 \leqslant n \leqslant N$. By (1.4) we can write

$$e^{sA\vartheta} \, f_{cs}(t) = \frac{1}{t} \, \exp\big((\log t + A\,\vartheta - \gamma) cs - \log \Gamma(cs) \big) \,. \tag{6.5}$$

Since $\log \Gamma(\cdot)$ is smooth and strictly convex, given any $t \in (0,\infty)$, the function $s \mapsto e^{sA\vartheta} \, f_{cs}(t)$ is increasing for $s \leqslant \bar{s}$ and decreasing for $s \geqslant \bar{s}$, where $\bar{s} = \bar{s}(t, A\,\vartheta, c)$ is characterized by

$$(\log \Gamma)'(c\bar{s}) = \log t + A\,\vartheta - \gamma \,. \tag{6.6}$$

Henceforth we fix $t = \frac{n}{N}$, with $1 \leqslant n \leqslant N$.

Let us now define $s_k := \frac{k}{\log N}$ and write

$$\frac{1}{\log N} \sum_{k \geqslant 0} e^{\frac{k}{\log N} A\,\vartheta} \, f_{c \frac{k}{\log N}}\big(\tfrac{n}{N}\big) = \sum_{k \geqslant 0} \tfrac{1}{\log N} \, e^{s_k A\,\vartheta} \, f_{cs_k}\big(\tfrac{n}{N}\big) \,. \tag{6.7}$$

If we set $\bar{k} := \max\{k \geqslant 0 : \; s_k \leqslant \bar{s}\}$, so that $s_{\bar{k}} \leqslant \bar{s} < s_{\bar{k}+1}$, we note that each term in the sum (6.7) with $k \leqslant \bar{k} - 1$ (resp. with $k \geqslant \bar{k} + 2$) can be bounded from above by the corresponding integral on the interval $[s_k, s_{k+1})$ (resp. on the interval $[s_{k-1}, s_k)$), by monotonicity of the function $s \mapsto e^{sA\vartheta} f_{cs}(t)$. For the two remaining terms, corresponding to $k = \bar{k}$ and $k = \bar{k}+1$, we replace s_k by $\bar{s}$ where the maximum is achieved. This yields

$$\frac{1}{\log N} \sum_{k \geqslant 0} e^{\frac{k}{\log N} A\,\vartheta} f_{c\frac{k}{\log N}}\big(\tfrac{n}{N}\big) \leqslant \int_0^\infty e^{sA\vartheta} f_{cs}\big(\tfrac{n}{N}\big)\,\mathrm{d}s \;+\; \tfrac{2}{\log N}\, e^{\bar{s}A\vartheta} f_{c\bar{s}}\big(\tfrac{n}{N}\big)\,. \tag{6.8}$$

It remains to deal with the last term. Recall that $s \mapsto e^{sA\vartheta} f_{cs}(\frac{n}{N})$ is maximized for $s = \bar{s}$. We will show that shifting $\bar{s}$ by $\frac{1}{\log N}$ decreases the maximum by a multiplicative constant:

$$c := \sup_{N\in\mathbb{N},\ 1\leqslant n\leqslant N} \frac{e^{\bar{s}\,A\,\vartheta} f_{c\bar{s}}(\frac{n}{N})}{e^{(\bar{s}+\frac{1}{\log N})\,A\,\vartheta} f_{c(\bar{s}+\frac{1}{\log N})}(\frac{n}{N})} \;<\; \infty\,. \tag{6.9}$$

Since $s \mapsto e^{sA\vartheta} f_{cs}(\frac{n}{N})$ is decreasing for $s \geqslant \bar{s}$, we can bound the last term in (6.8) as follows:

$$\tfrac{2}{\log N}\, e^{\bar{s}\,A\,\vartheta} f_{c\bar{s}}\big(\tfrac{n}{N}\big) \leqslant 2c \int_{\bar{s}}^{\bar{s}+\frac{1}{\log N}} e^{sA\vartheta} f_{cs}\big(\tfrac{n}{N}\big)\,\mathrm{d}s \leqslant 2c \int_0^\infty e^{sA\vartheta} f_{cs}\big(\tfrac{n}{N}\big)\,\mathrm{d}s\,,$$

which completes the proof of the claim.

It remains to prove (6.9). By the representation (6.5), the ratio in (6.9) equals

$$\begin{aligned}
&\exp\big\{-(\log\tfrac{n}{N} + A\,\vartheta - \gamma)\tfrac{c}{\log N} + \big(\log\Gamma(c\bar{s} + \tfrac{c}{\log N}) - \log\Gamma(c\bar{s})\big)\big\} \\
&\leqslant \exp\big\{O\big(\tfrac{1}{\log N}\big) + \tfrac{c}{\log N}\,(\log\Gamma)'(c\bar{s} + \tfrac{c}{\log N})\big\}\,,
\end{aligned}$$

by $1 \leqslant n \leqslant N$ and by convexity of $\log\Gamma(\cdot)$. It follows by (6.6) that $\bar{s}$ is uniformly bounded from above (indeed $\bar{s} \leqslant A\vartheta - \gamma$, because $t = \frac{n}{N} \leqslant 1$ and $(\log\Gamma)'(\cdot)$ is increasing). Then $(\log\Gamma)'(c\bar{s} + \frac{c}{\log N}) \leqslant (\log\Gamma)'(c(A\vartheta-\gamma) + \frac{c}{\log N})$ is also uniformly bounded from above. ☐

6.2. Proof of (1.17).

We organize the proof in three steps.

Step 1. We first prove an "integrated version" of (1.17). Let us define a finite measure $G_\lambda^{(N)}$ on $[0,1]$ as follows:

$$G_\lambda^{(N)}(\,\cdot\,) := \frac{1}{\log N} \sum_{n=0}^{N} U_{N,\lambda}(n)\, \delta_{\frac{n}{N}}(\,\cdot\,)\,, \tag{6.10}$$

where $\delta_t(\,\cdot\,)$ is the Dirac mass at t, and $U_{N,\lambda}(\cdot)$ is defined in (1.14). Recall also (1.15).

Lemma 6.2. *Fix $\vartheta \in \mathbb{R}$ and choose $\lambda = \lambda_N$ as in (1.16). As $N \to \infty$, the measure $G_{\lambda_N}^{(N)}$ converges weakly towards $G_\vartheta(t)\,\mathrm{d}t$, i.e. for every bounded and continuous $\phi : [0,1] \to \mathbb{R}$*

$$\int_0^1 \phi(t)\, G_{\lambda_N}^{(N)}(\mathrm{d}t) \xrightarrow[N\to\infty]{} \int_0^1 \phi(t)\, G_\vartheta(t)\,\mathrm{d}t\,. \tag{6.11}$$

Proof. Recalling the definition (1.14) of $U_{N,\lambda}(n)$, we can write

$$\begin{aligned}
\int_0^1 \phi(t)\, G_{\lambda_N}^{(N)}(\mathrm{d}t) &= \frac{1}{\log N} \sum_{n=0}^{N} U_{N,\lambda}(n)\, \phi\big(\tfrac{n}{N}\big) \\
&= \frac{1}{\log N} \sum_{k\geqslant 0} (\lambda_N)^k\, \mathrm{E}\Big[\phi\big(\tfrac{\tau_k^{(N)}}{N}\big)\, \mathbb{1}_{\{\tau_k^{(N)} \leqslant N\}}\Big] \\
&= \int_0^\infty (\lambda_N)^{\lfloor s\log N\rfloor}\, \mathrm{E}\Big[\phi\big(\tfrac{\tau_{\lfloor s\log N\rfloor}^{(N)}}{N}\big)\, \mathbb{1}_{\{\tau_{\lfloor s\log N\rfloor}^{(N)} \leqslant N\}}\Big]\,\mathrm{d}s\,.
\end{aligned} \tag{6.12}$$

Note that $\lim_{N\to\infty}(\lambda_N)^{\lfloor s\log N\rfloor} = e^{\vartheta s}$, by (1.16). Similarly, by Proposition 1.3 and the fact that Y_s is a continuous random variable,

$$\lim_{N\to\infty} \mathrm{E}\Big[\phi\big(\tfrac{\tau^{(N)}_{\lfloor s\log N\rfloor}}{N}\big)\,\mathbb{1}_{\{\tau^{(N)}_{\lfloor s\log N\rfloor}\leqslant N\}}\Big] \;=\; \mathrm{E}\big[\phi\big(Y_s\big)\,\mathbb{1}_{\{Y_s\leqslant 1\}}\big]\,.$$

Interchanging limit and integral, which we justify in a moment, we obtain from (6.12)

$$\lim_{N\to\infty}\int_0^1 \phi(t)\, G^{(N)}_{\lambda_N}(\mathrm{d}t) = \int_0^\infty e^{\vartheta s}\;\mathrm{E}\big[\phi\big(Y_s\big)\,\mathbb{1}_{\{Y_s\leqslant 1\}}\big]\,\mathrm{d}s\,.$$

If we write $\mathrm{E}\big[\phi\big(Y_s\big)\,\mathbb{1}_{\{Y_s\leqslant 1\}}\big] = \int_0^1 \phi(t)\, f_s(t)\,\mathrm{d}t$, we have proved (6.11) (recall (1.15)).

Let us finally justify that we can bring the limit inside the integral in (6.12). Since $(\lambda_N)^{\lfloor s\log N\rfloor} \leqslant e^{Cs}$ for some constant C, by (1.16), and since the function ϕ is bounded, we can apply dominated convergence on any bounded interval $s \in [0, M]$. It remains to show that the integral restricted to $s \in [M,\infty)$ is small for large M, uniformly in $N \in \mathbb{N}$. To this purpose, we use Lemma 6.1: the bound (6.1) yields

$$\|\phi\|_\infty \int_M^\infty e^{Cs}\,\mathrm{P}(\tau^{(N)}_{\lfloor s\log N\rfloor} \leqslant N)\,\mathrm{d}s \leqslant \|\phi\|_\infty \int_M^\infty e^{s(C+1-c\,\log s)}\,\mathrm{d}s\,.$$

If we take M large, so that $c\log M \geqslant C+2$, the integral is at most $\int_M^\infty e^{-s}\,\mathrm{d}s = e^{-M}$. $\square$

Step 2. We now derive convenient representation formulas for $U_{N,\lambda}(n)$ and $G_\vartheta(t)$:

$$U_{N,\lambda}(n) = \lambda \sum_{0\leqslant l<\frac{n}{2}\leqslant m\leqslant n} U_{N,\lambda}(l)\,\mathrm{P}(T^{(N)}_1 = m-l)\,U_{N,\lambda}(n-m)\,, \tag{6.13}$$

$$G_\vartheta(t) = \int_{0<u<\frac{t}{2}\leqslant v<t} G_\vartheta(u)\,\frac{1}{v-u}\,G_\vartheta(t-v)\,\mathrm{d}u\,\mathrm{d}v\,. \tag{6.14}$$

Relation (6.13) is obtained through a renewal decomposition: if we sum over the unique index $i \in \{1,\ldots,k\}$ such that $\tau^{(N)}_{i-1} < \frac{n}{2}$ while $\tau^{(N)}_i \geqslant \frac{n}{2}$, we can write

$$\begin{aligned}
\mathrm{P}(\tau^{(N)}_k = n) &= \sum_{i=1}^k \mathrm{P}\big(\tau^{(N)}_{i-1} < \tfrac{n}{2},\, \tau^{(N)}_i \geqslant \tfrac{n}{2},\, \tau^{(N)}_k = n\big) \\
&= \sum_{0\leqslant l<\frac{n}{2}\leqslant m\leqslant n}\;\sum_{i=1}^k \mathrm{P}\big(\tau^{(N)}_{i-1} = l\big)\,\mathrm{P}\big(T^{(N)}_1 = m-l\big)\,\mathrm{P}\big(\tau^{(N)}_{k-i} = n-m\big)\,.
\end{aligned}$$

Plugging this into the definition (1.14) of $U_{N,\lambda}(n)$, we obtain (6.13).

The proof of (6.14) is similar: given $s,t \in (0,\infty)$, we fix $n \in \mathbb{N}$ and sum over the unique index $i \in \{1,2,\ldots,n\}$ such that $Y_{\frac{i-1}{n}s} < \frac{t}{2}$ while $Y_{\frac{i}{n}s} \geqslant \frac{t}{2}$, to get

$$\begin{aligned}
\mathrm{P}(Y_s \in \mathrm{d}t) &= \sum_{i=1}^n \mathrm{P}\big(Y_{\frac{i-1}{n}s} < \tfrac{t}{2},\; Y_{\frac{i}{n}s} \geqslant \tfrac{t}{2},\; Y_s \in \mathrm{d}t\big) \\
&= \left(\int_{0<u<\frac{t}{2}\leqslant v<t} \left\{\sum_{i=1}^n f_{\frac{i-1}{n}s}(u)\, f_{\frac{s}{n}}(v-u)\, f_{\frac{n-i}{n}s}(t-v)\right\}\mathrm{d}u\,\mathrm{d}v\right)\mathrm{d}t\,.
\end{aligned} \tag{6.15}$$

By (1.4) we can write, for fixed $u, v \in (0,1]$ with $u<v$,

$$f_{\frac{s}{n}}(v-u) = \frac{1}{\Gamma(1+\frac{s}{n})}\,\frac{s}{n}\,(v-u)^{\frac{s}{n}-1} = \frac{s}{n}\,\frac{1}{v-u}\,\big(1+o(1)\big) \qquad \text{as } n\to\infty\,.$$

We also have the uniform upper bound $f_{\frac{s}{n}}(v-u) \leqslant C\,\frac{s}{n}\,\frac{1}{v-u}$. Then a Riemann sum approximation in (6.15) gives, for $t \in (0,1]$,

$$f_s(t) = \int_{0<u<\frac{t}{2}\leqslant v<t} \left\{ \int_0^s f_r(u)\,\frac{1}{v-u}\,f_{s-r}(t-v)\,\mathrm{d}r \right\} \mathrm{d}u\,\mathrm{d}v\,.$$

Plugging this expression in the definition (1.15) of $G_\vartheta(t)$, we obtain (6.14).

Step 3. The final step in the proof of (1.17) consists in combining formulas (6.13)-(6.14) with Lemma 6.2. First of all we note that in order to prove (1.17) uniformly for $\delta N \leqslant n \leqslant N$, it suffices to consider an arbitrary but fixed sequence $n = n_N$ such that

$$t_N := \frac{n_N}{N} \xrightarrow[N\to\infty]{} t \in (0,1]\,, \tag{6.16}$$

and prove that

$$\lim_{N\to\infty} \frac{N}{\log N}\,U_{N,\lambda_N}(n_N) \;=\; G_\vartheta(t)\,. \tag{6.17}$$

This implies (1.17), as one can prove by contradiction.

Let us prove (6.17). Recalling (6.10), we first rewrite the double sum in (6.13) as a double integral, setting $u := l/N$ and $v := m/N$, as follows (we recall that $t_N = \frac{n_N}{N}$):

$$\frac{N}{\log N}\,U_{N,\lambda_N}(n_N) = \lambda_N \int\limits_{0\leqslant u<\frac{t_N}{2}\leqslant v\leqslant t_N} G^{(N)}_{\lambda_N}\big(\mathrm{d}u\big)\;\phi^{(N)}(u,v)\;G^{(N)}_{\lambda_N}\big(t_N-\mathrm{d}v\big)\,, \tag{6.18}$$

where we set, for $0 \leqslant u < v \leqslant 1$,

$$\phi^{(N)}(u,v) := \big(N\,\log N\big)\,\mathrm{P}\big(T_1^{(N)} = \lfloor Nv\rfloor - \lfloor Nu\rfloor\big)\,.$$

Note that, by (1.12)-(1.11), we have

$$\lim_{N\to\infty} \phi^{(N)}(u,v) = \phi(u,v) := \frac{1}{v-u}\,. \tag{6.19}$$

By Lemma 6.2 and (6.16) we have the weak convergence

$$G^{(N)}_{\lambda_N}\big(\mathrm{d}u\big)\,G^{(N)}_{\lambda_N}\big(t_N-\mathrm{d}v\big) \xrightarrow[N\to\infty]{w} G_\vartheta(u)\,G_\vartheta(t-v)\,\mathrm{d}u\,\mathrm{d}v\,. \tag{6.20}$$

Since $\lambda_N \to 1$, see (1.16), by (6.19) and (6.20) it is natural to expect that the right hand side of (6.18) converges to the right hand side of (6.14). This is indeed the case, as we now show, which would complete the proof of (6.17), hence of Theorem 1.4.

We are left with justifying the convergence of the right hand side of (6.18). The delicate point is that $\phi(u,v)$ in (6.19) diverges as $v-u \downarrow 0$. Fix $\varepsilon > 0$ and consider the domain

$$D_\varepsilon := \big\{(u,v):\; v-u \geqslant \varepsilon\,t\big\}\,. \tag{6.21}$$

The convergence in (6.19) holds *uniformly over* $(u,v) \in D_\varepsilon$, and the limiting function $\frac{1}{v-u}$ is bounded and continuous on D_ε. Then, by (6.20), the integral in the right hand side of (6.18) restricted on D_ε converges to the integral in the right hand side of (6.14) restricted on D_ε.

To complete the proof, it remains to show that the integral in the right hand side of (6.18) restricted on $D_\varepsilon^c = \{v-u \leqslant \varepsilon\,t\}$ is small for $\varepsilon > 0$ small, uniformly in (large) $N \in \mathbb{N}$. By the definition (6.10) of $G^{(N)}_\lambda(\cdot)$, as well as (1.12)-(1.11), this contribution is bounded by

$$C_1 \sum_{\substack{u,v\in\frac{1}{N}\mathbb{N}_0:\\ 0\leqslant u<\frac{t_N}{2}\leqslant v\leqslant t_N\,,\; v-u\leqslant\varepsilon t}} \frac{U_{N,\lambda_N}(Nu)}{\log N}\,\frac{1}{v-u}\,\frac{U_{N,\lambda_N}(N(t_N-v))}{\log N}\,,$$

where $C_1, C_2, \ldots$ are generic constants. By the upper bound (1.18), this is at most

$$C_2 \frac{1}{N^2} \sum_{\substack{u,v \in \frac{1}{N}\mathbb{N}_0: \\ 0 \leqslant u < \frac{t_N}{2} \leqslant v \leqslant t_N \,,\ v-u \leqslant \varepsilon t}} G_\vartheta(u) \, \frac{1}{v-u} \, G_\vartheta(t_N - v) \,. \tag{6.22}$$

Since $t_N \to t$, see (6.16), we can bound this Riemann sum by the corresponding integral:

$$C_3 \int_{0<u<\frac{t}{2}\leqslant v<t \,,\ v-u\leqslant \varepsilon\, t} G_\vartheta(u) \, \frac{1}{v-u} \, G_\vartheta(t-v) \, \mathrm{d}u \, \mathrm{d}v \,.$$

Finally, if we let $\varepsilon \downarrow 0$, this integral vanishes by dominated convergence (recall (6.14)). $\square$

7. Proof of Theorems 2.4 and 2.3

We first prove Theorem 2.4, i.e. relation (2.14), which is easy. We then reduce the proof of Theorem 2.3 to that of Theorem 1.4, given in Section 6, proving separately the upper bound (2.13) and the local limit theorem (2.12).

7.1. Proof of (2.14). By (2.7) and (2.5), conditioned on the $T_i^{(N)}$'s, the random variables $X_i^{(N)}$ are independent with zero mean and $\mathrm{E}\big[|X_i^{(N)}|^2 \,\big|\, T_i^{(N)} = n_i\big] \leqslant c\, n_i$ for some $c < \infty$, see (4.12). Recalling (2.8), we then have

$$\mathrm{E}\Big[|S_k^{(N)}|^2 \,\Big|\, T_1^{(N)} = n_1, \ldots, T_k^{(N)} = n_k\Big] = \sum_{i=1}^k \mathrm{E}\Big[|X_i^{(N)}|^2 \,\Big|\, T_i^{(N)} = n_i\Big] \leqslant c\big(n_1 + \ldots + n_k\big) \,,$$

for any choice of $n_1, \ldots, n_k \in \mathbb{N}$. It follows that $\mathrm{E}\big[|S_k^{(N)}|^2 \,\big|\, \tau_k^{(N)} = n\big] \leqslant c\, n$, hence

$$\sum_{x \in \mathbb{Z}^2 :\, |x| > M\sqrt{n}} \mathrm{P}\big(\tau_k^{(N)} = n \,,\ S_k^{(N)} = x\big) = \mathrm{P}\big(\tau_k^{(N)} = n \,,\ |S_k^{(N)}| > M\sqrt{n}\big) \leqslant \frac{c}{M^2} \, \mathrm{P}\big(\tau_k^{(N)} = n\big) \,,$$

by Markov's inequality. Multiplying by λ^k and summing over k, we obtain (2.14). $\square$

7.2. Proof of (2.13). Recall the definition (4.8) of $\boldsymbol{Y}_s^{(N)}$. From the definition (2.9) of $\boldsymbol{U}_{N,\lambda}(n,x)$ and the upper bound (5.32), we get for large N

$$\boldsymbol{U}_{N,\lambda}(n,x) = \sum_{k \geqslant 0} \lambda^k \, \mathrm{P}\Big(\boldsymbol{Y}^{(N)}_{\frac{k}{\log N}} = \big(\tfrac{n}{N}, \tfrac{x}{\sqrt{N}}\big)\Big) \leqslant C \, \frac{\log N}{N} \, \frac{1}{n^{d/2}} \left\{ \frac{1}{\log N} \sum_{k \geqslant 0} \lambda^k \, f_{c\frac{k}{\log N}}\big(\tfrac{n}{N}\big) \right\} .$$

The bracket is the same as in (6.2). We showed in Subsection 6.1 that, if $\lambda = \lambda_N$ is chosen as in (1.16), the bracket is at most a constant times $G_\vartheta(\frac{n}{N})$. This proves (2.13). $\square$

7.3. Proof of (2.12). We proceed in three steps.

Step 1. We first prove an "integrated version" of (2.12). We define a finite measure $\boldsymbol{G}_\lambda^{(N)}$ on $[0,1] \times \mathbb{R}^2$ by setting

$$\boldsymbol{G}_\lambda^{(N)}(\,\cdot\,) := \frac{1}{\log N} \sum_{n=0}^N \sum_{x \in \mathbb{Z}^2} \boldsymbol{U}_{N,\lambda}(n,x) \, \delta_{(\frac{n}{N}, \frac{x}{\sqrt{N}})}(\,\cdot\,) \,, \tag{7.1}$$

where we recall that $\boldsymbol{U}_{N,\lambda}(\cdot)$ is defined in (2.9). Recall also the definition (2.10) of $\boldsymbol{G}_\vartheta(t,x)$.

Lemma 7.1. *Fix $\vartheta \in \mathbb{R}$ and choose $\lambda = \lambda_N$ as in (1.16). Then $\boldsymbol{G}^{(N)}_{\lambda_N}$ converges weakly as $N \to \infty$ towards $\boldsymbol{G}_\vartheta(t,x)\,\mathrm{d}t\,\mathrm{d}x$, i.e. for every bounded and continuous $\boldsymbol{\phi} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$*

$$\int_{(0,1)\times\mathbb{R}^2} \boldsymbol{\phi}(t,x)\,\boldsymbol{G}^{(N)}_{\lambda_N}(\mathrm{d}t,\mathrm{d}x) \xrightarrow[N\to\infty]{} \int_{(0,1)\times\mathbb{R}^2} \boldsymbol{\phi}(t,x)\,\boldsymbol{G}_\vartheta(t,x)\,\mathrm{d}t\,\mathrm{d}x\,. \tag{7.2}$$

Proof. Arguing as in (6.12), we can write

$$\int_{(0,1)\times\mathbb{R}^2} \boldsymbol{\phi}(t,x)\,\boldsymbol{G}^{(N)}_{\lambda_N}(\mathrm{d}t,\mathrm{d}x) = \int_0^\infty (\lambda_N)^{\lfloor s\log N\rfloor}\ \mathrm{E}\left[\boldsymbol{\phi}\Big(\tfrac{\tau^{(N)}_{\lfloor s\log N\rfloor}}{N}, \tfrac{S^{(N)}_{\lfloor s\log N\rfloor}}{\sqrt{N}}\Big)\,\mathbb{1}_{\{\tau^{(N)}_{\lfloor s\log N\rfloor}\leqslant N\}}\right]\mathrm{d}s\,.$$

We can exchange $\lim_{N\to\infty}$ with the integral by dominated convergence, thanks to Lemma 6.1, as shown in the proof of Lemma 6.2. Then we get, by Proposition 2.2,

$$\begin{aligned}\lim_{N\to\infty}\int_{(0,1)\times\mathbb{R}^2} \boldsymbol{\phi}(t,x)\,\boldsymbol{G}^{(N)}_{\lambda_N}(\mathrm{d}t,\mathrm{d}x) &= \int_0^\infty e^{\vartheta s}\ \mathrm{E}\left[\boldsymbol{\phi}\big(Y_s,V_s^{\mathsf{c}}\big)\,\mathbb{1}_{\{Y_s\leqslant 1\}}\right]\mathrm{d}s\\ &= \int_0^\infty e^{\vartheta s}\left(\int_{(0,1)\times\mathbb{R}^2}\boldsymbol{\phi}(t,x)\,\boldsymbol{f}_s(t,x)\,\mathrm{d}t\,\mathrm{d}x\right)\mathrm{d}s\,,\end{aligned}$$

which coincides with the right hand side of (7.2) (recall (2.10)). □

Step 2. Next we give representation formulas for $\boldsymbol{U}_{N,\lambda}(n,z)$ and $\boldsymbol{G}_\vartheta(t,x)$:

$$\boldsymbol{U}_{N,\lambda}(n,x) = \lambda \sum_{\substack{0\leqslant l<\frac{n}{2}\leqslant m\leqslant n\\ y,z\in\mathbb{Z}^2}} \boldsymbol{U}_{N,\lambda}(l,y)\,\mathrm{P}\big(T_1^{(N)} = m-l, X_1^{(N)} = z-y\big)\,\boldsymbol{U}_{N,\lambda}(n-m,x-z)\,, \tag{7.3}$$

$$\boldsymbol{G}_\vartheta(t,x) = \int\limits_{\substack{0<u<\frac{t}{2}\leqslant v<t\\ y,x\in\mathbb{R}^2}} \boldsymbol{G}_\vartheta(u,y)\,\frac{g_{\mathsf{c}(v-u)}(z-y)}{v-u}\,\boldsymbol{G}_\vartheta(t-v,x-z)\,\mathrm{d}u\,\mathrm{d}v\,. \tag{7.4}$$

These relations are proved in the same way as (6.13) and (6.14).

Step 3. We finally prove (2.12) by combining formulas (7.3)-(7.4) with Lemma 7.1. It suffices to fix arbitrary sequences $n = n_N \in \{1,\ldots,N\}$ and $x = x_N \in \mathbb{Z}^2$ such that

$$t_N := \frac{n_N}{N} \xrightarrow[N\to\infty]{} t \in (0,1]\,, \qquad w_N := \frac{x_N}{\sqrt{N}} \xrightarrow[N\to\infty]{} w \in \mathbb{R}^2\,, \tag{7.5}$$

and prove that

$$\lim_{N\to\infty} \frac{N^{1+d/2}}{\log N}\,\boldsymbol{U}_{N,\lambda_N}(n_N,w_N) = \boldsymbol{G}_\vartheta(t,w) = G_\vartheta(t)\,g_{\mathsf{c}\vartheta}(w)\,. \tag{7.6}$$

To prove (7.6), we rewrite the sums in (7.3) as integrals, recalling (7.1):

$$\begin{aligned}&\frac{N^{1+d/2}}{\log N}\,\boldsymbol{U}_{N,\lambda_N}(n_N,w_N)\\ &= \lambda_N \int\limits_{\substack{0\leqslant u<\frac{t_N}{2}\leqslant v\leqslant t_N\\ y,z\in\mathbb{R}^2}} \boldsymbol{G}^{(N)}_{\lambda_N}\big(\mathrm{d}u,\mathrm{d}y\big)\ \boldsymbol{\phi}^{(N)}(u,v;\,y,z)\ \boldsymbol{G}^{(N)}_{\lambda_N}\big(t_N-\mathrm{d}v, w_N-\mathrm{d}z\big)\,,\end{aligned} \tag{7.7}$$

where we set, for $0\leqslant u<v\leqslant 1$ and $y,z\in\mathbb{R}^2$,

$$\boldsymbol{\phi}^{(N)}(u,v;y,z) := N^{1+d/2}\ \log N\,\mathrm{P}\big(T_1^{(N)} = \lfloor Nv\rfloor - \lfloor Nu\rfloor,\ X_1^{(N)} = \lfloor\sqrt{N}z\rfloor - \lfloor\sqrt{N}y\rfloor\big)\,.$$

Note that by (2.5), (2.7) and (1.12)-(1.11) we have

$$\lim_{N\to\infty} \boldsymbol{\phi}^{(N)}(u,v;y,z) = \boldsymbol{\phi}(u,v;y,z) := \frac{g_{\mathsf{c}(v-u)}(z-y)}{v-u}\,. \tag{7.8}$$

Moreover, by Lemma 7.1 and (7.5) we have the weak convergence

$$\boldsymbol{G}^{(N)}_{\lambda_N}\big(\mathrm{d}u,\mathrm{d}y\big)\,\boldsymbol{G}^{(N)}_{\lambda_N}\big(t_N-\mathrm{d}v, w_N-\mathrm{d}z\big) \xrightarrow[N\to\infty]{w} \boldsymbol{G}_\vartheta(u,y)\,\boldsymbol{G}_\vartheta(t-v,w-z)\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}z\,. \tag{7.9}$$

Since $\lambda_N \to 1$ (see (1.16)), we expect by (7.8) and (7.9) that the right hand side of (7.7) converges to the right hand side of (7.4) as $N\to\infty$, proving our goal (7.6).

The difficulty is that the function $\boldsymbol{\phi}^{(N)}(u,v;y,z)$ converges to a function $\boldsymbol{\phi}(u,v;y,z)$ which is singular as $v-u\to 0$, see (7.8). This can be controlled as in the proof of Theorem 1.4, see the paragraphs following (6.20).

- First we fix $\varepsilon>0$ and restrict the integral in (7.7) to the domain $D_\varepsilon = \{v-u \geqslant \varepsilon\, t\}$. Here we can apply the weak convergence (7.9), because $\boldsymbol{\phi}(u,v;y,z)$ is bounded and the convergence $\boldsymbol{\phi}^{(N)}(u,v;y,z)\to\boldsymbol{\phi}(u,v;y,z)$ is uniform.
- Then we consider the contribution to the integral in (7.7) from $D_\varepsilon^c = \{v-u<\varepsilon\, t\}$. Recalling (7.1), this contribution can be written as follows:

$$\sum_{\substack{u,v\in\frac1N\mathbb{N}_0,\ y,z\in\frac{1}{\sqrt N}\mathbb{Z}^2\\ 0\leqslant u<\frac{t_N}{2}\leqslant v\leqslant t_N\,,\ v-u<\varepsilon t}} \frac{\boldsymbol{U}_{N,\lambda_N}\big(Nu,\sqrt N y\big)}{\log N}\,\boldsymbol{\phi}^{(N)}(u,v;\,y,z)\,\frac{\boldsymbol{U}_{N,\lambda_N}\big(N(t_N-v),\sqrt N(w_N-z)\big)}{\log N}\,. \tag{7.10}$$

We need to show that this is small for $\varepsilon>0$ small, uniformly in large $N\in\mathbb{N}$.

By (2.13) we can bound, uniformly in $z\in\frac{1}{\sqrt N}\mathbb{Z}^2$,

$$\frac{\boldsymbol{U}_{N,\lambda_N}\big(N(t_N-v),\sqrt N(w_N-z)\big)}{\log N} \leqslant C\,\frac{1}{N^{1+\frac d2}}\,\frac{1}{(t_N-v)^{\frac d2}}\,G_\vartheta\big(t_N-v\big)\,,$$

and note that $t_N-v\geqslant \frac{t_N}{2}-\varepsilon$. Next, by definition of $\boldsymbol{\phi}^{(N)}$ and by (1.12)-(1.11),

$$\sum_{z\in\frac{1}{\sqrt N}\mathbb{Z}^2}\boldsymbol{\phi}^{(N)}(u,v;\,y,z) = N^{1+\frac d2}\,(\log N)\,\mathrm{P}\big(T_1^{(N)} = \lfloor Nv\rfloor-\lfloor Nu\rfloor\big) \leqslant C_1\,\frac{N^{\frac d2}}{v-u}\,.$$

Finally we observe that, by (1.14), (2.9) and (1.18),

$$\sum_{y\in\frac{1}{\sqrt N}\mathbb{Z}^2}\frac{\boldsymbol{U}_{N,\lambda_N}\big(Nu,\sqrt N y\big)}{\log N} = \frac{U_{N,\lambda_N}(Nu)}{\log N} \leqslant C\,\frac1N\,G_\vartheta(u)\,.$$

These bounds show that (7.10) is bounded by a constant times

$$\frac{1}{N^2}\,\frac{1}{(\frac{t_N}{2}-\varepsilon)^{\frac d2}} \sum_{\substack{u,v\in\frac1N\mathbb{N}_0\\ 0\leqslant u<\frac{t_N}{2}\leqslant v\leqslant t_N\,,\ v-u<\varepsilon t}} G_\vartheta(u)\,\frac{1}{v-u}\,G_\vartheta(t_N-v)\,. \tag{7.11}$$

Since $t_N\to t$, we have $\frac{t_N}{2}>\frac t3$ for N large, and if we take $\varepsilon<\frac t6$ we see that the prefactor $(\frac{t_N}{2}-\varepsilon)^{-d/2}\leqslant(\frac t6)^{-d/2}$ is bounded (recall that t is fixed). The sum in (7.11) is the same as that in (6.22), which we had shown to be small for $\varepsilon>0$ small, uniformly in large $N\in\mathbb{N}$. This completes the proof. □

Appendix A. Additional results for disordered systems

In this appendix we prove some results for disordered systems, stated in Section 3.

A.1. Proof of relations (3.5) and (3.18). We recall the polynomial chaos expansion used in [CSZ17a, CSZ17b]. Let us introduce the random variables

$$\eta_i := \frac{e^{\beta\omega_i - \lambda(\beta)}}{\sigma_\beta}, \qquad \text{where} \qquad \sigma_\beta^2 := e^{\lambda(2\beta)-2\lambda(\beta)} - 1, \tag{A.1}$$

so that (η_i) are i.i.d. with zero mean and unit variance (recall (3.1)).

Recall the definition (3.4) of Z_N^β and note that we can write

$$e^{(\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{X_{2n}=0\}}} = 1 + \sigma_\beta\, \eta_n\, \mathbb{1}_{\{X_{2n}=0\}}. \tag{A.2}$$

We now write the exponential in (3.4) as a product and perform an expansion, exploiting (A.2). Recalling the definition (3.3) of $u(n)$, we obtain:

$$\begin{aligned} Z_N^\beta &= \mathrm{E}\left[\prod_{n=1}^{N-1} e^{(\beta\omega_n-\lambda(\beta))\mathbb{1}_{\{X_{2n}=0\}}}\, \mathbb{1}_{\{X_{2N}=0\}}\right] \\ &= \sum_{k=1}^{N} (\sigma_\beta)^{k-1} \sum_{0<n_1<\ldots<n_{k-1}<n_k:=N} u(n_1)\, u(n_2-n_1)\, \cdots\, u(n_k-n_{k-1}) \\ &\qquad\qquad \cdot \eta_{n_1}\, \eta_{n_2}\, \cdots\, \eta_{n_{k-1}}. \end{aligned} \tag{A.3}$$

This formula expresses Z_N^β as a multilinear polynomial of the random variables. Since the monomials for different k are orthogonal in $L^2(\mathbb{P})$, we get (3.5).

The proof of (3.18) is similar, because we can represent $\boldsymbol{Z}_N^\beta(x)$ in (3.17) as follows:

$$\begin{aligned} \boldsymbol{Z}_N^\beta(x) = \sum_{k=1}^{N} (\sigma_\beta)^{k-1} \sum_{\substack{0<n_1<\ldots<n_{k-1}<n_k:=N \\ x_1,\ldots,x_k\in\mathbb{Z}^2:\ x_k=x}} & q_{n_1}(x_1)\, q_{n_2-n_1}(x_2-x_1)\, \cdots\, q_{n_k-n_{k-1}}(x_k-x_{k-1}) \\ & \cdot \eta_{n_1,x_1}\, \eta_{n_2,x_2}\, \cdots\, \eta_{n_{k-1},x_{k-1}}. \end{aligned} \tag{A.4}$$

This completes the proof. ☐

A.2. Free partition function. For the pinning model, one can consider the *free partition function* $Z_N^{\beta,\mathrm{f}}$, in which the constraint $\{X_{2N}=0\}$ is removed from (3.4), and the sum is extended up to N:

$$Z_N^{\beta,\mathrm{f}} := \mathrm{E}\left[e^{\sum_{n=1}^N (\beta\omega_n-\lambda(\beta))\mathbb{1}_{\{X_{2n}=0\}}}\right]. \tag{A.5}$$

Then we have the following analogue of Theorem 3.1. Let us set, recalling (1.15),

$$\overline{G}_\vartheta(u) := \int_0^u G_\vartheta(t)\,\mathrm{d}t = \int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, u^s}{\Gamma(s+1)}\,\mathrm{d}s, \qquad \text{for } u\in(0,1]. \tag{A.6}$$

Proposition A.1 (Free pinning model partition function). *Rescale $\beta=\beta_N$ as in* (3.9)*. Then, for any fixed $\delta>0$, the following relation holds as $N\to\infty$:*

$$\mathbb{E}[(Z_n^{\beta_N,\mathrm{f}})^2] = (\log N)\,\overline{G}_\vartheta(\tfrac{n}{N})\,(1+o(1)), \qquad \textit{uniformly for } \ \delta N \leqslant n \leqslant N, \tag{A.7}$$

with $\overline{G}(\cdot)$ defined in (A.6)*. Moreover, the following bound holds, for a suitable $C\in(0,\infty)$:*

$$\mathbb{E}[(Z_n^{\beta_N,\mathrm{f}})^2] \leqslant C\,(\log N)\,\overline{G}_\vartheta(\tfrac{n}{N}), \qquad \forall 1\leqslant n\leqslant N. \tag{A.8}$$

Finally, since $\mathbb{E}[Z_n^{\beta_N,\mathrm{f}}] = 1$*, relations* (A.7) *and* (A.8) *holds also for* $\mathbb{V}\mathrm{ar}[Z_n^{\beta_N,\mathrm{f}}]$.

Proof. Arguing as in §A.1, one can write a decomposition for $Z_n^{\beta,\mathrm{f}}$ similar to (A.3). As a consequence, the second moment of $Z_n^{\beta,\mathrm{f}}$ is given by an expression similar to (3.5), namely

$$\mathbb{E}[(Z_n^{\beta,\mathrm{f}})^2] = 1 + \sum_{k\geqslant 1} (\sigma_\beta^2)^k \sum_{0<n_1<\ldots<n_k\leqslant n} u(n_1)^2\, u(n_2-n_1)^2 \,\cdots\, u(n_k-n_{k-1})^2\,, \tag{A.9}$$

which yields an analogue of relation (3.8):

$$\begin{aligned}\mathbb{E}[(Z_n^{\beta,\mathrm{f}})^2] &= 1 + \sum_{k\geqslant 1} \big(\sigma_\beta^2\, R_N\big)^k\, \mathrm{P}(\tau_k^{(N)} \leqslant n) = 1 + \sum_{\ell=1}^{n}\sum_{k\geqslant 1} \big(\sigma_\beta^2\, R_N\big)^k\, \mathrm{P}(\tau_k^{(N)} = \ell) \\ &= 1 + \sum_{\ell=1}^{n} U_{N,\lambda}(\ell)\,, \qquad \text{where} \qquad \lambda := \sigma_\beta^2\, R_N\,.\end{aligned}$$

It then suffices to apply (1.17) and (1.18) to get (A.7) and (A.8). □

Also for the directed polymer in random environment we can consider the *free (or point-to-plane) partition function* $\boldsymbol{Z}_N^{\beta,\mathrm{f}}$, in which the constraint $\{S_N = x\}$ is removed from (3.17), and the sum is extended up to N:

$$\boldsymbol{Z}_N^{\beta,\mathrm{f}} := \mathrm{E}\Big[e^{\sum_{n=1}^N(\beta\omega_{n,S_n}-\lambda(\beta))}\Big] = \mathrm{E}\Big[e^{\sum_{n=1}^N\sum_{z\in\mathbb{Z}^2}(\beta\omega_{n,z}-\lambda(\beta))\mathbb{1}_{\{S_n=z\}}}\Big]\,. \tag{A.10}$$

The second moment of $\boldsymbol{Z}_N^{\beta,\mathrm{f}}$ turns out to be identical to that of $Z_N^{\beta,\mathrm{f}}$ (pinning model).

Proposition A.2 (Free directed polymer partition function). *Rescale* $\beta = \beta_N$ *as in* (3.22)*. Then relations* (A.7) *and* (A.8) *hold verbatim for the free partition function* $\boldsymbol{Z}_n^{\beta_N,\mathrm{f}}$ *of the directed polymer in random environment, defined in* (A.10)*.*

Proof. Arguing as in §A.1, one can write a decomposition for $\boldsymbol{Z}_n^{\beta,\mathrm{f}}$ similar to (A.4). Then the second moment of $\boldsymbol{Z}_n^{\beta,\mathrm{f}}$ can be represented as follows:

$$\begin{aligned}\mathbb{E}\big[(\boldsymbol{Z}_n^{\beta,\mathrm{f}})^2\big] = 1 + \sum_{k\geqslant 1} (\sigma_\beta^2)^k \sum_{\substack{0<n_1<\ldots<n_k\leqslant N\\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q_{n_1}(x_1)^2\, q_{n_2-n_1}(x_2-x_1)^2 \,\cdot \\ & \cdots\, q_{n_k-n_{k-1}}(x_k-x_{k-1})^2\,.\end{aligned} \tag{A.11}$$

Since $\sum_{x\in\mathbb{Z}^2} q_n(x)^2 = u(n)^2$, see (3.15), we can sum over $x_k, x_{k-1}, \ldots, x_1$ in (A.11) to obtain precisely the same expression as in (A.9). In other words, *the free partition functions of the pinning and directed polymer models have the same second moment*:

$$\mathbb{E}\big[(\boldsymbol{Z}_n^{\beta,\mathrm{f}})^2\big] = \mathbb{E}\big[(Z_n^{\beta,\mathrm{f}})^2\big]\,.$$

This completes the proof. □

A.3. Proof of Proposition 3.2. Let $T := \min\{m \in \mathbb{N} : \ S_m = 0\}$ denote the first return time to the origin of the simple symmetric random walk on $\mathbb{Z}^2$. Let $(\xi_i)_{i\in\mathbb{N}}$ be i.i.d. random variables distributed as $T/2$. We define

$$L_N := \sum_{n=1}^{N} \mathbb{1}_{\{S_{2n}=0\}} = \max\big\{k\in\mathbb{N}_0 : \ \xi_1+\ldots+\xi_k \leqslant N\big\}\,,$$

so that, recalling (3.15) and the definition (3.12) of R_N, we can write

$$R_N = \sum_{n=1}^{N} \mathrm{P}(S_{2n} = 0) = \mathrm{E}[L_N] = \sum_{k=1}^{N} \mathrm{P}(L_N \geqslant k) = \sum_{k=1}^{N} \mathrm{P}(\xi_1 + \ldots + \xi_k \leqslant N)\,.$$

Let $(\xi_i^{(N)})_{i\in\mathbb{N}}$ be i.i.d. random variables with the law of ξ_1 conditionally on $\{\xi_1 \leqslant N\}$. Then we have the following key representation of R_N:

$$\begin{aligned} R_N &= \sum_{k=1}^{N} \mathrm{P}(\xi_1 \leqslant N)^k \, \mathrm{P}(\xi_1^{(N)} + \ldots + \xi_k^{(N)} \leqslant N) \\ &= \sum_{k=1}^{N} \mathrm{P}(\xi_1 \leqslant N)^k - \sum_{k=1}^{N} \mathrm{P}(\xi_1 \leqslant N)^k \, \mathrm{P}(\xi_1^{(N)} + \ldots + \xi_k^{(N)} > N)\,. \end{aligned} \tag{A.12}$$

We are going to show that the first sum gives the leading contribution to the right hand side of (3.12), while the second sum is negligible.

We need estimates on the law of ξ_1. By Corollary 1.2 and Remark 4 in [Uch11], we have

$$\begin{aligned} \mathrm{P}(\xi_1 = k) = \mathrm{P}(T = 2k) &= \frac{\pi}{k}\bigg(\frac{1}{(\log 16k)^2} - \frac{2\gamma}{(\log 16k)^3} + O\bigg(\frac{1}{(\log 16k)^4}\bigg)\bigg) \\ &= \frac{\pi}{k(\log k)^2} - \frac{2\pi(\gamma + \log 16)}{k(\log k)^3} + O\bigg(\frac{1}{(\log k)^4}\bigg)\,, \\ \mathrm{P}(\xi_1 \geqslant k) = \mathrm{P}(T \geqslant 2k) &= \frac{\pi}{\log k} - \frac{\pi(\gamma + \log 16)}{(\log k)^2} + O\bigg(\frac{1}{(\log k)^3}\bigg)\,, \end{aligned} \tag{A.13}$$

as $k \to \infty$, where γ is the Euler-Mascheroni constant. Then, as $N \to \infty$, we can write

$$\frac{\mathrm{P}(\xi_1 \leqslant N)}{\mathrm{P}(\xi_1 > N)} = \frac{1 - \frac{\pi}{\log N} + O(\frac{1}{(\log N)^2})}{\frac{\pi}{\log N}\big(1 - \frac{(\gamma + \log 16)}{(\log N)} + O\big(\frac{1}{(\log N)^2}\big)\big)} = \frac{\log N}{\pi} + \bigg(\frac{\gamma + \log 16}{\pi} - 1 \bigg) + o(1)\,,$$

$$\mathrm{P}(\xi_1 \leqslant N)^N = \big(1 - \tfrac{\pi}{\log N} + O(\tfrac{1}{(\log N)^2})\big)^N = e^{-\frac{\pi N}{\log N}(1 + o(1))} = o\bigg(\frac{1}{\log N}\bigg)\,.$$

From this we deduce the asymptotic behavior of the first sum in the last line of (A.12):

$$\sum_{k=1}^{N} \mathrm{P}(\xi_1 \leqslant N)^k = \frac{\mathrm{P}(\xi_1 \leqslant N)}{\mathrm{P}(\xi_1 > N)}\big(1 - \mathrm{P}(\xi_1 \leqslant N)^N\big) = \frac{\log N}{\pi} + \bigg(\frac{\gamma + \log 16}{\pi} - 1 \bigg) + o(1)\,,$$

which matches with the right hand side of (3.12). It remains to show that the second sum in the last line of (A.12) is asymptotically vanishing, i.e.

$$\lim_{N\to\infty} \varrho_N = 0\,, \qquad \text{where} \qquad \varrho_N := \sum_{k=1}^{N} \mathrm{P}(\xi_1 \leqslant N)^k \, \mathrm{P}(\xi_1^{(N)} + \ldots + \xi_k^{(N)} > N)\,. \tag{A.14}$$

Denoting by C_1, C_2 suitable absolute constants, we have by relation (A.13)

$$\mathrm{E}\Big[\xi_1^{(N)}\Big] = \frac{1}{\mathrm{P}(\xi_1 \leqslant N)} \sum_{\ell=1}^{N} \ell \, \mathrm{P}(\xi_1 = \ell) \leqslant C_1 \sum_{\ell=1}^{N} \frac{1}{(\log \ell)^2} \leqslant C_2 \frac{N}{(\log N)^2}\,, \tag{A.15}$$

hence by Markov's inequality

$$\mathrm{P}\big(\xi_1^{(N)} + \ldots + \xi_k^{(N)} > N\big) \leqslant C_2 \frac{k}{(\log N)^2}\,.$$

Since $\mathrm{P}(\xi_1 \leqslant N) \leqslant e^{-\frac{1}{\log N}}$ for large N, by (A.13), we can control the tail of ϱ_N in (A.14) by

$$\varrho_N^{>A} := \sum_{k > A\log N} \mathrm{P}(\xi_1 \leqslant N)^k \mathrm{P}(\xi_1^{(N)} + \ldots + \xi_k^{(N)} > N) \leqslant C_2 \sum_{k > A\log N} e^{-\frac{k}{\log N}} \frac{k}{(\log N)^2} .$$

By a Riemann sum approximation, the last sum converges to $\int_A^\infty x\, e^{-x}\, \mathrm{d}x = (1+A)e^{-A}$ as $N \to \infty$. In particular, for every fixed $A \in (0, \infty)$, we have shown that

$$\limsup_{N\to\infty} \varrho_N^{>A} \leqslant (1+A)e^{-A} . \tag{A.16}$$

Next we focus on the contribution $\varrho_N^{\leqslant A}$ of the terms with $k \leqslant A\log N$, i.e.

$$\begin{aligned} \varrho_N^{\leqslant A} &:= \sum_{k \leqslant A\log N} \mathrm{P}(\xi_1 \leqslant N)^k\, \mathrm{P}(\xi_1^{(N)} + \ldots + \xi_k^{(N)} > N) \\ &\leqslant (A\log N)\, \mathrm{P}(\xi_1^{(N)} + \ldots + \xi_{A\log N}^{(N)} > N) . \end{aligned} \tag{A.17}$$

We fix $\varepsilon \in (0, \frac{1}{2})$ and write

$$\xi_1^{(N)} + \ldots + \xi_k^{(N)} = \sum_{i=1}^k \xi_i^{(N)} \mathbb{1}_{\{\xi_i^{(N)} \leqslant \varepsilon^2 N\}} + \sum_{i=1}^k \xi_i^{(N)} \mathbb{1}_{\{\xi_i^{(N)} > \varepsilon^2 N\}} =: U_- + U_+ ,$$

so that we can decompose

$$\mathrm{P}(\xi_1^{(N)} + \ldots + \xi_k^{(N)} > N) \leqslant \mathrm{P}(U_- > \varepsilon N) + \mathrm{P}(U_+ > (1-\varepsilon)N) , \tag{A.18}$$

and we estimate separately each term. In analogy with (A.15) we have

$$\mathrm{E}\Big[U_-\Big] = k\, \mathrm{E}\Big[\xi_1^{(N)} \mathbb{1}_{\{\xi_1^{(N)} \leqslant \varepsilon^2 N\}}\Big] = k \sum_{\ell=1}^{\varepsilon^2 N} \frac{\ell\, \mathrm{P}(\xi_1 = \ell)}{\mathrm{P}(\xi_1 \leqslant N)} \leqslant k \sum_{\ell=1}^{\varepsilon^2 N} \frac{C_1}{(\log \ell)^2} \leqslant C_2 \frac{\varepsilon^2 N k}{(\log(\varepsilon^2 N))^2} ,$$

hence by Markov's inequality

$$\mathrm{P}\big(U_- > \varepsilon N\big) \leqslant C_2 \frac{\varepsilon k}{(\log(\varepsilon^2 N))^2} . \tag{A.19}$$

Next we observe that

$$\{U_+ > (1-\varepsilon)N\} \subseteq \Bigg(\bigcup_{i=1}^k \{\xi_i^{(N)} > (1-\varepsilon)N\} \Bigg) \cup \Bigg(\bigcup_{1 \leqslant i < j \leqslant k} \{\xi_i^{(N)} > \varepsilon^2 N, \xi_j^{(N)} > \varepsilon^2 N\} \Bigg) ,$$

because either $\xi_i^{(N)} > (1-\varepsilon)N$ for a single i, or necessarily $\xi_i^{(N)} > \varepsilon^2 N$ and $\xi_j^{(N)} > \varepsilon^2 N$ for at least two distinct $i \neq j$ (otherwise U_+ vanishes). Since for fixed $c \in (0,1)$

$$\mathrm{P}(\xi_1^{(N)} > cN) \leqslant C_1 \sum_{\ell=cN}^{N} \frac{1}{\ell\, (\log \ell)^2} \leqslant C_1 \frac{1}{(\log cN)^2} \sum_{\ell=cN}^{N} \frac{1}{\ell} \leqslant C_1 \frac{\log \frac{1}{c}}{(\log cN)^2} ,$$

it follows that

$$\mathrm{P}(U_+ > (1-\varepsilon)N) \leqslant k\, C_1 \frac{\log \frac{1}{1-\varepsilon}}{(\log((1-\varepsilon)N))^2} + \frac{k(k-1)}{2} \left[C_1 \frac{\log \frac{1}{\varepsilon^2}}{(\log(\varepsilon^2 N))^2} \right]^2 .$$

Recalling (A.17)-(A.18)-(A.19) and plugging $k = A\log N$, we get

$$\limsup_{N\to\infty} \varrho_N^{\leqslant A} \leqslant A^2 \big(C_2\, \varepsilon + C_1 \log \tfrac{1}{1-\varepsilon} \big) .$$

By (A.16), since $\varrho_N = \varrho_N^{\leqslant A} + \varrho_N^{>A}$, we obtain (A.14) by letting $\varepsilon \to 0$ and then $A \to \infty$. $\square$

A.4. Explicit asymptotics in terms of β. Relation (3.9) (equivalently (3.22)) and relation (3.13) can be rewritten more explicitly in terms of β_N. To this purpose, we need the *cumulants* κ_3, κ_4 of the distribution of ω_i (recall (3.1)), defined by

$$\lambda(\beta) = \frac{1}{2}\beta^2 + \frac{\kappa_3}{3!}\beta^3 + \frac{\kappa_4}{4!}\beta^4 + O(\beta^5) \qquad \text{as } \beta \to 0\,. \tag{A.20}$$

By direct computation $\sigma_\beta^2 = \beta^2 + \kappa_3\,\beta^3 + \big(\frac{1}{2} + \frac{7}{12}\kappa_4\big)\beta^4 + O(\beta^5)$ as $\beta \to 0$, hence

$$\sigma_\beta^2 = \varepsilon \qquad \Longrightarrow \qquad \beta^2 = \varepsilon - \kappa_3\,\varepsilon^{3/2} + \big(\tfrac{3}{2}\kappa_3^2 - \tfrac{7}{12}\kappa_4 - \tfrac{1}{2}\big)\,\varepsilon^2 + o(\varepsilon^2) \qquad \text{as } \varepsilon \to 0\,. \tag{A.21}$$

As a consequence, we can rewrite (3.13) as follows, with $\alpha := \gamma + \log 16 - \pi$:

$$\beta_N^2 = \frac{\pi}{\log N} - \frac{\kappa_3\,\pi^{3/2}}{(\log N)^{3/2}} + \frac{\pi(\vartheta - \alpha) + \pi^2\big(\frac{3}{2}\kappa_3^2 - \frac{1}{2} - \frac{7}{12}\kappa_4\big)}{(\log N)^2}\big(1 + o(1)\big)\,.$$

Appendix B. On the Dickman subordinator

Theorem 1.1 on the density of the Dickman subordinator can be deduced from general results about *self-decomposable Lévy processes*, see [Sat99, §53].

- Let us first derive (1.4) for $t \in (0,1]$. The law of Y_s satisfies the assumptions of [Sat99, Lemma 53.2] with $n = 1$, $a_1 = 1$ and $c = s$, which yields $f_s(t) = Kt^{s-1}$ for $t \in (0,1]$. To show that $K = e^{-\gamma s}/\Gamma(s)$, as in (1.4), one can apply [Sat99, Theorem 53.6] which gives $f_s(t) = (1+o(1))\kappa\, t^{s-1}/\Gamma(s)$ as $t \downarrow 0$, with $\kappa = \exp\{s(\int_0^1 \frac{e^{-x}-1}{x}\mathrm{d}x + \int_1^\infty \frac{e^{-x}}{x}\,\mathrm{d}x)\}$. The identification $\kappa = \exp\{-\gamma s\}$ follows by [GR07, Entry 8.367 (12), page 906].
- We then deduce (1.4) for $t \in (1,\infty)$. We can apply [Sat99, Theorem 51.1], which reads as follows (where $\nu(\mathrm{d}t) = \frac{s}{t}\,\mathbb{1}_{(0,1)}(t)\,\mathrm{d}t$, $\gamma_0 = 0$ and $f_s(t)$ is the density of Y_s):

$$\int_0^t y\, f_s(y)\,\mathrm{d}y = \int_0^t \left(\int_0^{t-y} f_s(u)\,\mathrm{d}u\right) y\,\frac{s}{y}\,\mathbb{1}_{(0,1)}(y)\,\mathrm{d}y\,.$$

  Differentiating with respect to t, for $t > 1$, we get $tf_s(t) = s\int_0^1 f_s(t-y)\,\mathrm{d}y$, which already shows that $f_s(t)$ can be deduced from $\{f_s(u) : u \in (t-1,t)\}$. To obtain (1.4), we further differentiate this relation (note that $f_s(\cdot) \in C^1$ on $(1,\infty)$, by [Sat99, Lemma 53.2]) to get $f_s(t) + tf_s'(t) = s\,(f_s(t) - f_s(t-1))$, which can be rewritten as $(t^{1-s}f_s(t))' = -s\,t^{-s}\,f_s(t-1)$. Integrating on $(0,t)$, since $t^{1-s}f_s(t) \to K = e^{-\gamma s}/\Gamma(s)$ as $t \downarrow 0$, we obtain $t^{1-s}f_s(t) - K = s\int_0^t \frac{f_s(u-1)}{u^s}\,\mathrm{d}u$, which coincides with the second line of (1.4) (note that $f_s(t) \equiv 0$ for $t < 0$).

This completes the proof of (1.4).[†]

We now present an alternative proof of Theorem 1.1, which exploits a key *scale invariance property* of the Dickman subordinator Y. Let M_s denote the maximal jump up to time s:

$$M_s := \max_{u\in(0,s]} \Delta Y_u\,, \qquad \text{where} \qquad \Delta Y_u := Y_u - Y_{u-} = Y_u - \lim_{\varepsilon\downarrow 0} Y_{u-\varepsilon}\,. \tag{B.1}$$

We first prove the following result.

[†] This proof was kindly provided to us by Thomas Simon.

Proposition B.1 (Scale-invariance). *Fix $s \in (0,\infty)$, $t \in (0,1)$. Conditional on all jumps of Y up to time s being smaller than t, the random variable Y_s/t has the same law as Y_s, i.e.*

$$\mathrm{P}\bigg(\frac{Y_s}{t} \in \cdot \,\bigg|\, M_s < t\bigg) = \mathrm{P}(Y_s \in \cdot)\,. \tag{B.2}$$

Proof. We use the standard representation of the Lévy process $Y = (Y_s)_{s\in[0,\infty)}$ in terms of a Poisson Point Process (PPP). Let Π be a PPP on $[0,\infty) \times (0,1)$ with intensity measure

$$\mu(\mathrm{d}x, \mathrm{d}y) := \mathrm{Leb}(\mathrm{d}x) \otimes \nu(\mathrm{d}y) = \mathrm{d}x \otimes \frac{\mathbb{1}_{(0,1)}(y)}{y}\,\mathrm{d}y\,. \tag{B.3}$$

We recall that Π is a random countable subset of $[0,\infty) \times (0,1)$, whose points we denote by (s_i, t_i). Let us define

$$\Pi^{(s,t)} := \Pi \cap ([0,s] \times (0,t))\,, \qquad Y_s^{(t)} := \sum_{(s_i,t_i)\in\Pi^{(s,t)}} t_i\,. \tag{B.4}$$

Then we can represent our Lévy process Y_s in terms of Π as follows:

$$Y_s \overset{d}{=} Y_s^{(1)}\,. \tag{B.5}$$

Let us identify Y_s with $Y_s^{(1)}$. Note that $\Delta Y_s = t \neq 0$ if and only if $(s,t) \in \Pi$, see (B.1).

On the event $\{M_s < t\} = \{\Pi \cap ([0,s] \times [t,1)) = \varnothing\}$ we have $Y_s^{(1)} = Y_s^{(t)}$, hence

$$\mathrm{P}\bigg(\frac{Y_s}{t} \in \cdot \,\bigg|\, M_s < t\bigg) = \mathrm{P}\bigg(\frac{Y_s^{(t)}}{t} \in \cdot \,\bigg|\, \Pi \cap ([0,s] \times [t,1)) = \varnothing\bigg) = \mathrm{P}\bigg(\frac{Y_s^{(t)}}{t} \in \cdot\bigg)\,,$$

because $Y_s^{(t)}$ is a function of $\Pi^{(s,t)}$, which is independent of $\Pi \cap ([0,s] \times [t,1))$, by definition of PPP. To prove our goal (B.2), it remains to show that

$$\mathrm{P}\bigg(\frac{Y_s^{(t)}}{t} \in \cdot\bigg) = \mathrm{P}\big(Y_s^{(1)} \in \cdot\big)\,.$$

By (B.4), it suffices to prove the following property: if we denote by $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ the map $(x,y) \mapsto (x, \frac{1}{t}y)$, then the random set $\phi_t(\Pi^{(s,t)})$ *has the same law as* $\Pi^{(s,1)}$.

Note that $\Pi^{(s,t)}$ is a PPP with intensity measure $\mu^{(s,t)}$ given by the original intensity measure μ restricted on $[0,s] \times (0,t)$ (see (B.3)). We also observe that the random set $\phi_t(\Pi^{(s,t)})$ is a PPP with intensity measure given by $\mu^{(s,t)} \circ \phi_t^{-1}$, i.e. the image law of $\mu^{(s,t)}$ under ϕ_t. The proof is completed by noting that ϕ_t sends $\mu^{(s,t)}$ to $\mu^{(s,1)}$, because the map $y \mapsto y/t$ sends the measure $\frac{1}{y}\,\mathbb{1}_{(0,t)}(y)\,\mathrm{d}y$ to the measure $\frac{1}{y}\,\mathbb{1}_{(0,1)}(y)\,\mathrm{d}y$. ☐

In our proof of Theorem 1.1, we will also need the following estimate. This can be deduced from [RW02, Lemma 6], but we give a direct proof in our setting.

Lemma B.2. *As $s \downarrow 0$ we have*

$$\mathrm{P}(Y_s > 1) = o(s)\,. \tag{B.6}$$

Remark B.3. *The bound* (B.6) *is an intermediate step in establishing Theorem 1.1 and it is not optimal. Indeed, it is a consequence of Theorem 1.1 that the optimal estimate is*

$$\mathrm{P}(Y_s > 1) = O(s^2) \qquad \text{as } s \downarrow 0\,, \tag{B.7}$$

because $\mathrm{P}(Y_s \leqslant 1) = e^{-\gamma s}/\Gamma(s+1)$*, by* (1.4)*, and we note that as $s \downarrow 0$ we have*

$$\Gamma(s+1) = \Gamma(1) + \Gamma'(1)s + O(s^2) = 1 - \gamma s + O(s^2)\,, \tag{B.8}$$

since $\Gamma'(1) = \int_0^\infty \log u \, e^{-u} \, \mathrm{d}u = -\gamma$. *Relation* (B.7) *then follows.*

Proof of Lemma B.2. Fix a function $\alpha_s \to \infty$ as $s \to 0$, to be determined later. Recall the definition (B.1) of $\Delta Y_u = Y_u - Y_{u-}$ and define

$$N_s := \sum_{u \in (0,s]} \mathbb{1}_{\{\Delta Y_u > \frac{1}{\alpha_s}\}} = \text{number of jumps of } Y \text{ of size} > \tfrac{1}{\alpha_s} \text{ in the interval } (0,s] \,.$$

We recall that Y only increases by jumps, that is $Y_s = \sum_{u \in (0,s]} \Delta Y_u$. We denote by $Y_s^>$ the contribution to Y_s given by jumps of size $> \frac{1}{\alpha_s}$, and $Y_s^{\leqslant} := Y_s - Y_s^>$. Then we bound

$$\mathrm{P}(Y_s > 1) \leqslant \mathrm{P}(N_s \geqslant 2) + \mathrm{P}(N_s = 1, Y_s > 1) + \mathrm{P}(N_s = 0, Y_s^{\leqslant} > 1) \tag{B.9}$$

For the first term, we note that $N_s \sim \mathrm{Pois}(\lambda_s)$ with $\lambda_s = s \int_{1/\alpha_s}^1 \frac{1}{x} \, \mathrm{d}x = s \log \alpha_s$, hence

$$\mathrm{P}(N_s \geqslant 2) = O(\lambda_s^2) = O(s^2 (\log \alpha_s)^2) \,.$$

For the third term, since $(Y_s^{\leqslant})_{s \geqslant 0}$ has Lévy measure $\frac{1}{x} \mathbb{1}_{(0, \frac{1}{\alpha_s})}(x) \, \mathrm{d}x$, we can bound

$$\mathrm{P}(Y_s^{\leqslant} > 1) \leqslant \mathrm{E}[Y_s^{\leqslant}] = s \int_0^{\frac{1}{\alpha_s}} x \, \frac{1}{x} \, \mathrm{d}x = \frac{s}{\alpha_s} \,. \tag{B.10}$$

We fix $\alpha_s = 1/s$, so that both $\mathrm{P}(N_s \geqslant 2)$ and $\mathrm{P}(Y_s^{\leqslant} > 1)$ are $O(s^{3/2})$.

It remains to estimate the second term in the right hand side of (B.9). On the event $\{N_s = 1\}$, the random variable $W := Y_s^>$ has density $\frac{1}{\log \alpha_s} \frac{1}{x} \mathbb{1}_{(\frac{1}{\alpha_s}, 1)}(x)$. Also note that $Y_s^{\leqslant}$ is independent of N_s. If we fix $\varrho_s \in (1,2)$, to be determined later, we can write

$$\begin{aligned} \mathrm{P}(N_s = 1, Y_s > 1) &\leqslant \mathrm{P}(N_s = 1, Y_s^> > \tfrac{1}{\varrho_s}) + \mathrm{P}(N_s = 1, \ Y_s^> \leqslant \tfrac{1}{\varrho_s}, \ Y_s^{\leqslant} > 1 - \tfrac{1}{\varrho_s}) \\ &\leqslant \mathrm{P}(N_s = 1) \big\{ \mathrm{P}(W > \tfrac{1}{\varrho_s}) + \mathrm{P}(Y_s^{\leqslant} > \tfrac{\varrho_s - 1}{\varrho_s}) \big\} \\ &\leqslant \lambda_s \left\{ \frac{\log \varrho_s}{\log \alpha_s} + \frac{\varrho_s}{\varrho_s - 1} \, \mathrm{E}[Y_s^{\leqslant}] \right\} , \end{aligned}$$

because $N_s \sim \mathrm{Pois}(\lambda_s)$. Since $\lambda_s = s \log \alpha_s$ and $\mathrm{E}[Y_s^{\leqslant}] = \frac{s}{\alpha_s}$, see (B.10), we get

$$\mathrm{P}(N_s = 1, Y_s > 1) \leqslant s \log \alpha_s \left\{ \frac{\log \varrho_s}{\log \alpha_s} + \frac{2s}{\alpha_s (\varrho_s - 1)} \right\} = s \log \varrho_s + \frac{\log \alpha_s}{\alpha_s} \, \frac{2s^2}{\varrho_s - 1}$$

Note that $\lim_{s \to 0} \frac{\log \alpha_s}{\alpha_s} = 0$, because we have fixed $\alpha_s = 1/s$. We now choose $\varrho_s = 1 + \sqrt{s}$ to get $\mathrm{P}(N_s = 1, Y_s > 1) = O(s^{3/2})$, which completes the proof. □

Proof of Theorem 1.1. We start proving the first line of (1.4), so we assume $t \in (0,1)$.

Recall that M_s was defined in (B.1). Plainly, we can write

$$\mathrm{P}(Y_s \leqslant t) = \mathrm{P}(Y_s \leqslant t, \, M_s < t) = \mathrm{P}(M_s < t) \, \mathrm{P}(Y_s \leqslant t \,|\, M_s < t) \,.$$

We use the PPP representation of Y_s that we introduced in the proof of Proposition B.1. In particular, if Π denotes a PPP with intensity measure μ in (B.3), we can write

$$\mathrm{P}(M_s < t) = \mathrm{P}(\Pi \cap ([0,s] \times [t,1)) = \varnothing) = e^{-\mu([0,s] \times [t,1))} = e^{-s \int_t^1 \frac{1}{y} \, \mathrm{d}y} = t^s \,.$$

For $t \in (0,1)$ we have $\mathrm{P}(Y_s \leqslant t \,|\, M_s < t) = \mathrm{P}(Y_s \leqslant 1)$, by Proposition B.1, hence

$$\mathrm{P}(Y_s \leqslant t) = t^s \, \mathrm{P}(Y_s \leqslant 1) \qquad \text{for } t \in (0,1) \,. \tag{B.11}$$

This leads to

$$f_s(t) = s \, t^{s-1} \, F_s(1) \qquad \text{for } t \in (0,1) \,, \qquad \text{where} \qquad F_s(t) := \mathrm{P}(Y_s \leqslant t). \tag{B.12}$$

It remains to identify $F_s(1)$. Since $(Y_s)_{s\geqslant 0}$ has stationary and independent increments, for any $n \in \mathbb{N}$, the density f_s is the convolution of $f_{s/n}$ with itself n times. Then for any $t \in (0,1)$ we can write, by (B.12),

$$\begin{aligned}
f_s(t) &= \int\limits_{0<t_1<\ldots<t_{n-1}<t} f_{\frac{s}{n}}(t_1)\, f_{\frac{s}{n}}(t_2-t_1) \cdots f_{\frac{s}{n}}(t-t_{n-1})\, \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} \\
&= \big(\tfrac{s}{n}\, F_{\frac{s}{n}}(1)\big)^n \int\limits_{0<t_1<\ldots<t_{n-1}<t} t_1^{\frac{s}{n}-1}\, (t_2-t_1)^{\frac{s}{n}-1} \cdots (t-t_{n-1})^{\frac{s}{n}-1}\, \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} \\
&= \big(\tfrac{s}{n}\, F_{\frac{s}{n}}(1)\big)^n\, t^{s-1} \int\limits_{0<u_1<\ldots<u_{n-1}<1} u_1^{\frac{s}{n}-1}\, (u_2-u_1)^{\frac{s}{n}-1} \cdots (1-u_{n-1})^{\frac{s}{n}-1}\, \mathrm{d}u_1 \ldots \mathrm{d}u_{n-1} \\
&= \big(\tfrac{s}{n}\, F_{\frac{s}{n}}(1)\big)^n\, t^{s-1}\, \frac{\Gamma(\frac{s}{n})^n}{\Gamma(s)} = \big(F_{\frac{s}{n}}(1)\big)^n\, t^{s-1}\, \frac{\Gamma(1+\frac{s}{n})^n}{\Gamma(s)}\,,
\end{aligned}$$

where we recognized the density of the Dirichlet distribution (with parameters n and $\frac{s}{n}$) and, in the last step, we used the property $\Gamma(1+x) = x\,\Gamma(x)$. By (B.8)

$$\Gamma(1+\tfrac{s}{n})^n \xrightarrow[n\to\infty]{} e^{-\gamma\, s}\,.$$

Since $F_u(1) = 1 - o(u)$ as $u \to 0$, by Lemma B.2, we have $\big(F_{\frac{s}{n}}(1)\big)^n \to 1$. This yields

$$f_s(t) = \lim_{n\to\infty} \big(F_{\frac{s}{n}}(1)\big)^n\, t^{s-1}\, \frac{\Gamma(1+\frac{s}{n})^n}{\Gamma(s)} = \frac{t^{s-1}\, e^{-\gamma\, s}}{\Gamma(s)} = \frac{s\, t^{s-1}\, e^{-\gamma\, s}}{\Gamma(s+1)}\,,$$

which proves the first line of (1.4).

It remains to prove the second line of (1.4). We exploit the PPP construction of Y_s, see (B.3)-(B.5). By identifying the largest jump $M_s = u$, see (B.1), we have for any $t \in (0,\infty)$

$$\begin{aligned}
\mathrm{P}(Y_s \in \mathrm{d}t) &= \int_0^{t\wedge 1} \mathrm{P}(Y_s \in \mathrm{d}t \,|\, M_s = u)\, \mathrm{P}(M_s \in \mathrm{d}u) \\
&= \int_0^{t\wedge 1} \Big\{\tfrac{1}{u}\, f_s\big(\tfrac{t-u}{u}\big)\, \mathrm{d}t\Big\} \Big\{\tfrac{s}{u}\, e^{-s\int_u^1 \frac{\mathrm{d}x}{x}}\, \mathrm{d}u\Big\} \\
&= \bigg(\int_0^{t\wedge 1} f_s\big(\tfrac{t-u}{u}\big)\, s\, u^{s-2}\, \mathrm{d}u\bigg)\, \mathrm{d}t\,.
\end{aligned} \tag{B.13}$$

The second equality holds for the following reasons.

- Y_s conditioned on $\{M_s < u\}$ has the same law as uY_s, by Proposition B.1, hence
$$\mathrm{P}(Y_s \in \mathrm{d}t \,|\, M_s = u) = \mathrm{P}(Y_s \in \mathrm{d}t - u \,|\, M_s < u) = \tfrac{1}{u}\, f_s\big(\tfrac{t-u}{u}\big)\, \mathrm{d}u\,.$$
- $\frac{s}{u}$ is the Poisson intensity of finding a jump of size u in the time interval $[0,s]$, while $e^{-s\int_u^1 \frac{\mathrm{d}x}{x}} = u^s$ is the probability that all other jumps are smaller than u, hence
$$\mathrm{P}(M_s \in \mathrm{d}u) = \mu([0,s]\times \mathrm{d}u)\, e^{-\mu([0,s]\times(u,1))} = \tfrac{s}{u}\, \mathrm{d}u\, e^{-s\int_u^1 \frac{1}{x}\,\mathrm{d}x}\,.$$

Making the change of variable $a := \frac{t-u}{u}$, we can rewrite (B.13) as

$$\begin{aligned}
f_s(t) &= s\, t^{s-1} \int_{(t-1)^+}^{\infty} \frac{f_s(a)}{(1+a)^s} \mathrm{d}a \\
&= s\, t^{s-1} \bigg(\int_0^{\infty} \frac{f_s(a)}{(1+a)^s} \mathrm{d}a - \int_0^{(t-1)^+} \frac{f_s(a)}{(1+a)^s} \mathrm{d}a\bigg).
\end{aligned} \tag{B.14}$$

For $t \in (0,1)$, the second integral equals 0, while $f_s(t) = \frac{s\, t^{s-1}\, e^{-\gamma\, s}}{\Gamma(s+1)}$ by the first line of (1.4), that we have already proved. This implies that the first integral must equal $\frac{e^{-\gamma\, s}}{\Gamma(s+1)}$. This concludes the proof of the second line of (1.4). $\square$

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