A Polymer in a Multi-Interface Medium

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Humboldt Universität zu Berlin $\sim$ November 12, 2008

References

- [CP1] F. Caravenna and N. Pétrélis
  A polymer in a multi-interface medium
  Preprint (2007), arXiv.org: 0712.3426

- [CP2] F. Caravenna and N. Pétrélis
  Depinning of a polymer in a multi-interface medium
  In preparation.

Outline

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1. Introduction
   - What is a polymer?
   - Polymers and probability

2. The model and the main results
   - Definition
   - The free energy
   - Path results

3. Techniques and ideas from the proof
   - Some heuristics
   - A renewal theory approach

What is a polymer?

A polymer is a large molecule composed of repeating smaller units, called monomers, linked together to form a chain.

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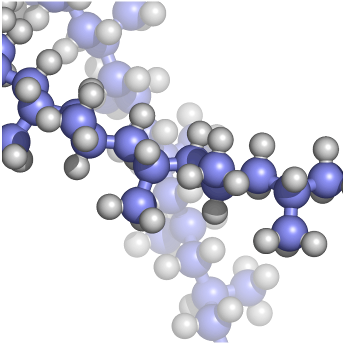

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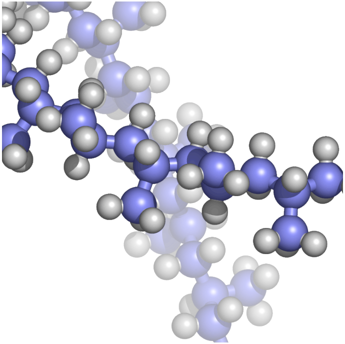

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- DNA, RNA
- Proteins
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Important research topic in chemistry, physics, biology...

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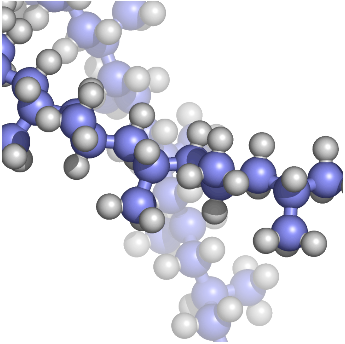

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Polymers and probability

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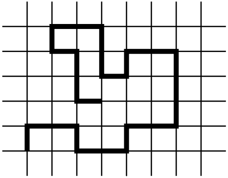
Self-avoiding walks

Simple symmetric random walk on $\mathbb{Z}^d$ conditioned to visit each site at most once $\longrightarrow$ very difficult!



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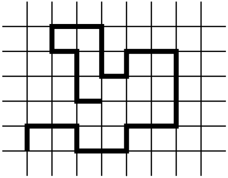
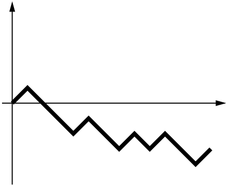


Directed walks

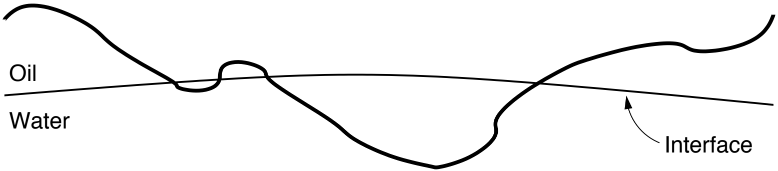
Processes with a deterministic component: (n, S_n) where S_n is the simple symmetric random walk on $\mathbb{Z}^{d-1}$

$\longrightarrow$ tractable models



Interaction with the environment

A polymer interacting with two solvents and with the interface that separates them:



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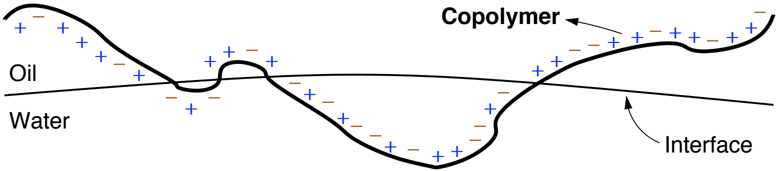

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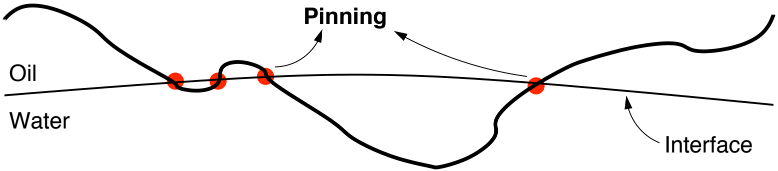

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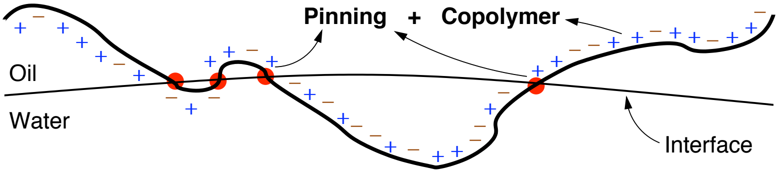

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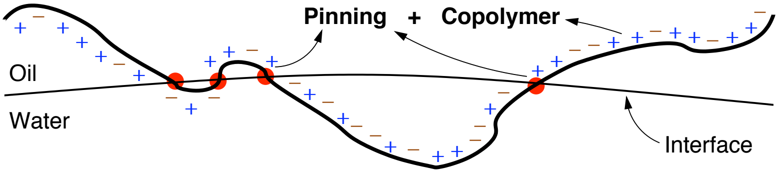

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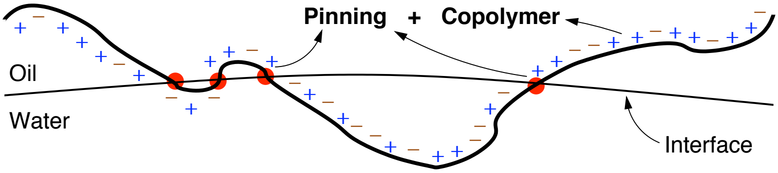

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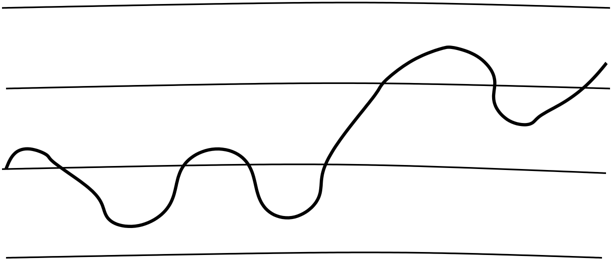
Recent results: very good comprehension (survey: [Giacomin '07])

Multi-interface media

More general environments: a multi-interface medium



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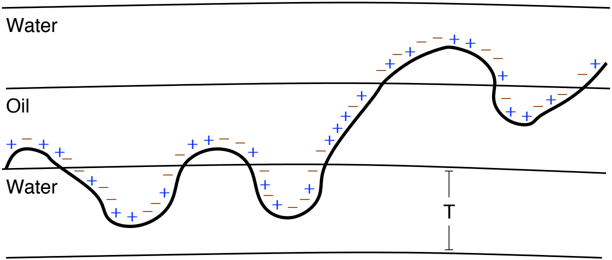

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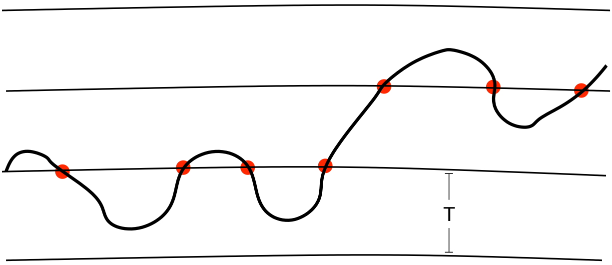

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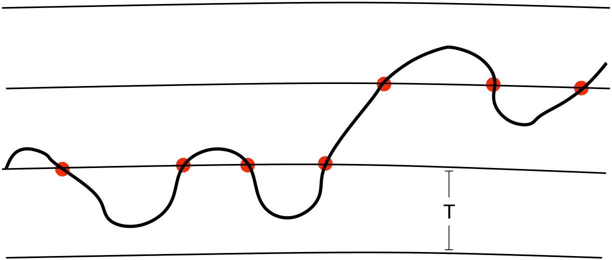

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[Copolymer in an emulsion: den Hollander, Pétrélis, Whittington, Wüthrich]

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Ingredients:

- Simple symmetric random walk $S = \{S_n\}_{n \geq 0}$ on $\mathbb{Z}$:

$$S_0 := 0\,, \qquad S_n := X_1 + \ldots + X_n\,,$$

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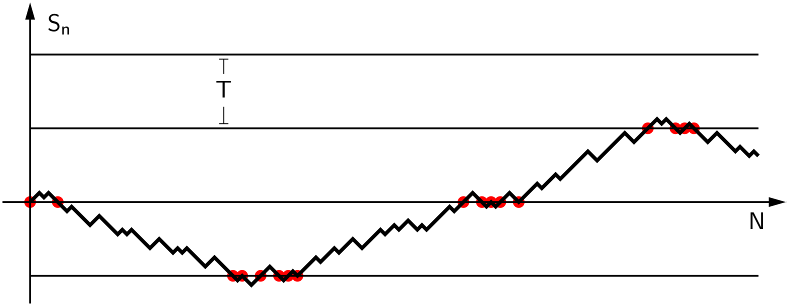

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- What is the interplay between δ and $T = \{T_N\}_N$?

The free energy

The free energy $\phi(\delta, \{T_n\}_n)$ encodes the exponential asymptotic behavior of the partition function $Z_{N,\delta}^{T_N}$ as $N \to \infty$:

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Why should one look at the free energy? Introducing the number of visits to the interfaces $L_N := \#\{i \le N : \; S_i \in T\mathbb{Z}\}$ we have

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If $\phi(\delta, \{T_n\}_n)$ is non-analytic in $\delta \in \mathbb{R}$ there is a phase transition

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- If $T_\infty < \infty$ then $\phi(\delta, T_\infty)$ is analytic: no phase transitions
- If $T_\infty = \infty$ phase transition (only) at $\delta = 0$
- Every $\{T_n\}_n \to \infty$ yields the same free energy as if $T_n \equiv \infty$ (homogeneous pinning model) $\longrightarrow$ same density of visits

Path results: the attractive case $\delta > 0$

Assume $\delta > 0$ and $T_N \to \infty$. Since $\phi'(\delta, \infty) > 0$, the polymer visits the interfaces a positive fraction of times: $L_N \sim \phi'(\delta, \infty) \cdot N$.

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$$\frac{S_N}{C_\delta\,(e^{-\frac{c_\delta}{2} T_N}\, T_N)\sqrt{N}} \Longrightarrow N(0,1)$$

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How many jumps $\widehat{L}_N$ to different interfaces? Typical size of S_N?

Not.: $S_N \asymp \alpha_N$ means S_N/α_N is tight and $\mathbf{P}_{N,\delta}^{T_N}(|S_N/\alpha_N| \geq \varepsilon) \geq \varepsilon$

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$$\frac{S_N}{T_N} \implies S_\Gamma \qquad \Gamma \sim \text{Poisson}(f(\delta, \zeta))$$

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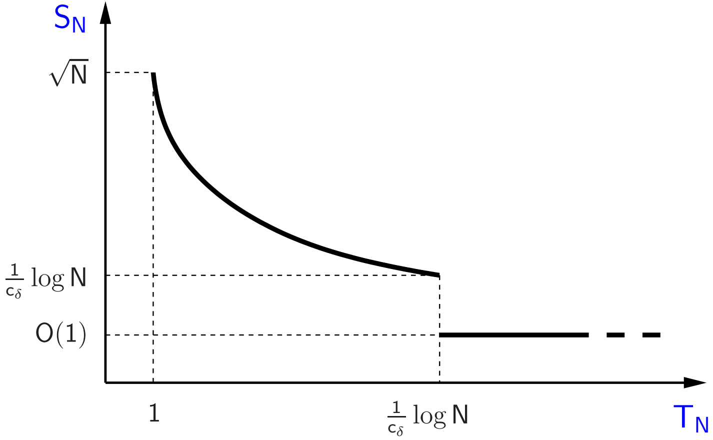

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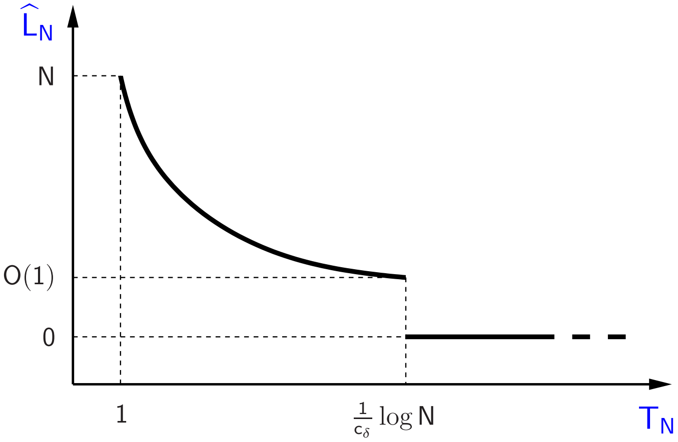

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$$c_1\, P\big(a < Z \le b\big) \;\le\; \mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{S_N}{C_\delta \sqrt{\frac{N}{T_N}}} \le b \right) \;\le\; c_2\, P\big(a < Z \le b\big)$$

with $Z \sim N(0,1)$

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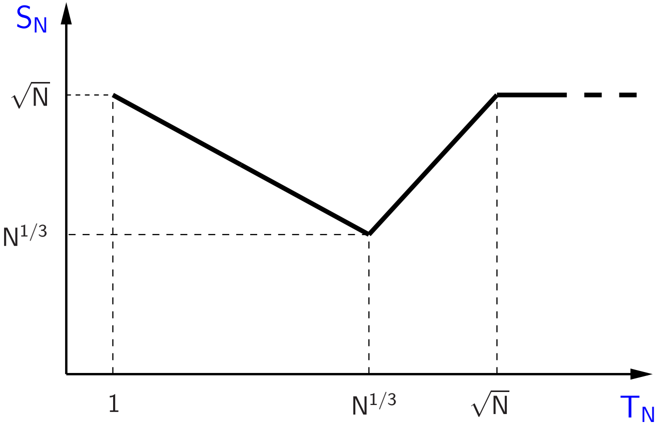

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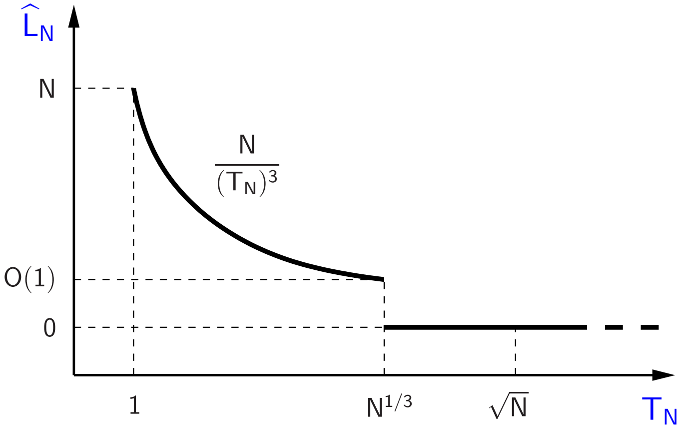

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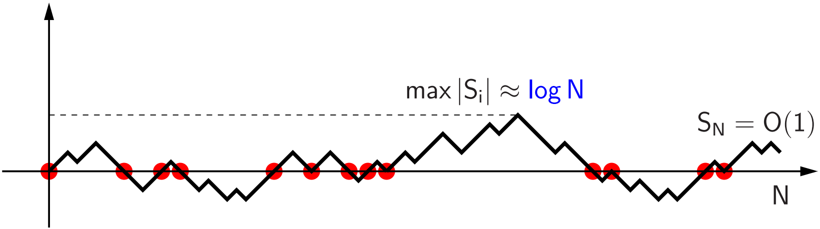

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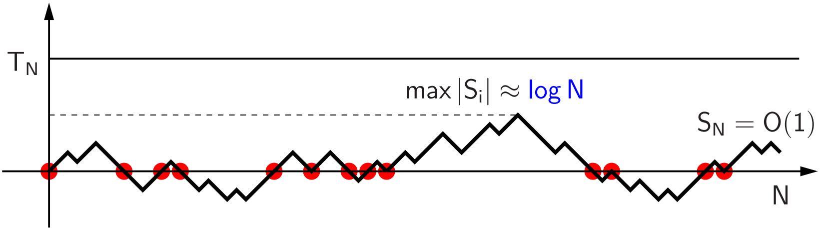

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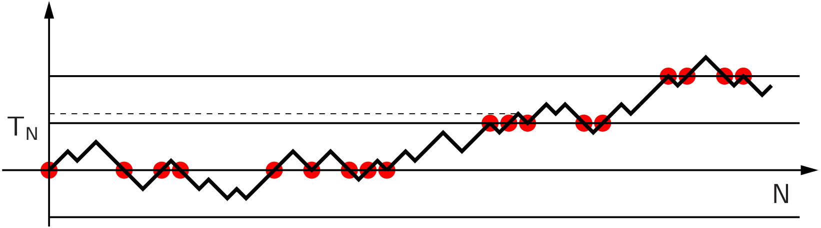

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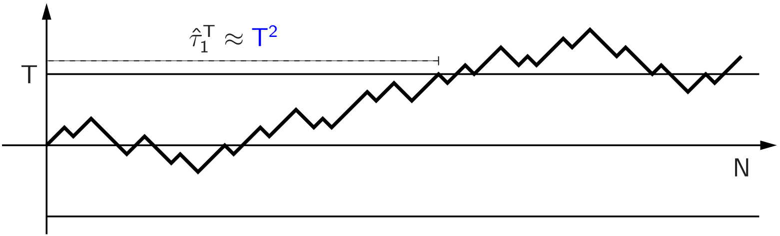

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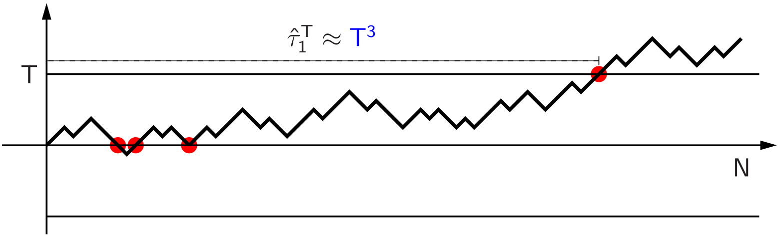

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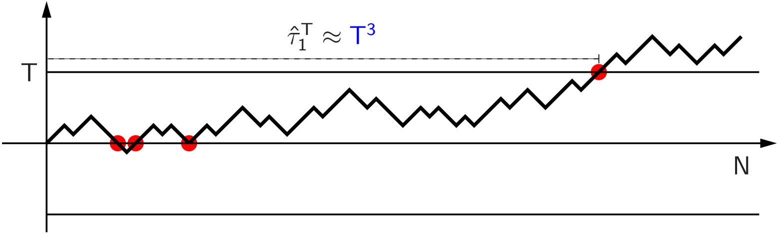

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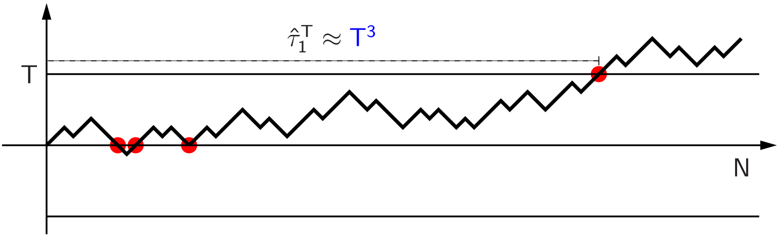

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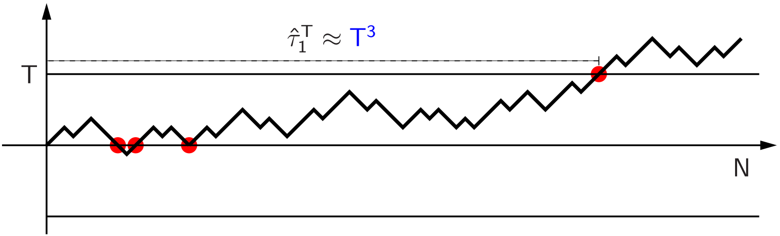

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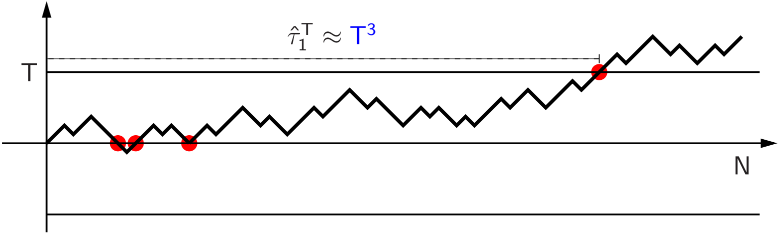

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A renewal theory approach

Let $\tau_1^T, \tau_2^T, \tau_3^T \ldots$ be the points at which S_n visits an interface

$$\tau_{k+1}^T := \inf \left\{ n > \tau_k^T : \; S_n - S_{\tau_k^T} \in \{-T, 0, T\} \right\} \qquad (T \text{ is fixed})$$

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Under the polymer measure $\mathbf{P}_{N,\delta}^T$ the process $\{\tau_n^T\}_{n\in\mathbb{N}}$ is not even time-homogeneous ... however for large N it is nearly a renewal process with a different law $\mathcal{P}_{\delta,T}$

$$K_{\delta,T}(n) \;:=\; \mathcal{P}_{\delta,T}(\tau_1^T = n) \;=\; e^{\delta}\, P(\tau_1^T = n)\, e^{-\phi(\delta,T)n}$$

Strategy of the proof

More precisely, the law of $\tau^T \cap [0, N] = \{\tau_1^T, \ldots, \tau_{L_N}^T\}$ is the same

$$\text{under} \quad \mathbf{P}_{N,\delta}^T\big(\ \cdot\ \big|\, N \in \tau^T\big) \quad \text{and} \quad \mathcal{P}_{\delta,T}\big(\ \cdot\ \big|\, N \in \tau^T\big)$$

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2. Transfer the results to $\mathcal{P}_{\delta,T_N}(\cdot \,|\, N \in \tau^{T_N})$ that is to $\mathbf{P}_{N,\delta}^{T_N}(\cdot \,|\, N \in \tau^{T_N})$ (hard part)
3. Remove the conditioning on $\{N \in \tau^{T_N}\}$

Strategy of the proof

More precisely, the law of $\tau^T \cap [0, N] = \{\tau_1^T, \ldots, \tau_{L_N}^T\}$ is the same

$$\text{under} \quad \mathbf{P}_{N,\delta}^T\big(\cdot \,\big|\, N \in \tau^T \big) \quad \text{and} \quad \mathcal{P}_{\delta,T}\big(\cdot \,\big|\, N \in \tau^T \big)$$

Important feature: $\mathcal{P}_{\delta,T}$ does not depend explicitly on N.

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Thanks.