THE WEAK COUPLING LIMIT OF DISORDERED COPOLYMER MODELS

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Abstract. A copolymer is a chain of repetitive units (*monomers*) that are almost identical, but they differ in their degree of affinity for certain solvents. This difference leads to striking phenomena when the polymer fluctuates in a non–homogeneous medium, for example made up by two solvents separated by an interface. One may observe, for instance, the localization of the polymer at the interface between the two solvents. A discrete model of such system, based on the simple symmetric random walk on $\mathbb{Z}$, has been investigated in [9], notably in the weak polymer–solvent coupling limit, where the convergence of the discrete model toward a continuum model, based on Brownian motion, has been established. This result is remarkable because it strongly suggests a universal feature of copolymer models. In this work we prove that this is indeed the case. More precisely, we determine the weak coupling limit for a general class of discrete copolymer models, obtaining as limits a one–parameter ($\alpha \in (0, 1)$) family of continuum models, based on α-stable regenerative sets.

1. Introduction

1.1. **The discrete model.** Let $S := \{S_n\}_{n=0,1,...}$ be the simple symmetric random walk on $\mathbb{Z}$, that is $S_0 = 0$ and $\{S_{n+1} - S_n\}_{n=0,1,...}$ is an IID sequence of random variables, each taking values $+1$ or -1 with probability $1/2$. If $\mathbf{P}$ is the law of S, we introduce a new probability measure $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega,\lambda,h}$ on the random walk trajectories defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \;:=\; \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} \Delta\left(S_{n-1} + S_n\right)(\omega_n + h) \right), \tag{1.1}$$

where $N \in \mathbb{N} := \{1, 2, \ldots\}$, $\lambda, h \in [0, \infty)$, we have set $\Delta(\cdot) := \mathbf{1}_{(-\infty,0)}(\cdot)$ and $\omega := \{\omega_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers. Of course $Z_{N,\omega} = Z_{N,\omega,\lambda,h}$ is the normalization constant, called *partition function* and given by

$$Z_{N,\omega} \;:=\; \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} \Delta\left(S_{n-1} + S_n\right)(\omega_n + h) \right) \right]. \tag{1.2}$$

We could have used $\Delta(S_n)$ instead of $\Delta(S_{n-1} + S_n)$, but this apparently unnatural choice actually has a nice interpretation, explained in the caption of Figure 1.

We are interested in the case when ω, called the sequence of *charges*, is chosen as a typical realization of an IID sequence (call $\mathbb{P}$ its law). We assume that ω and S are independent, so that the relevant underlying law is $\mathbb{P} \otimes \mathbf{P}$, but in reality we are interested in *quenched results*, that is, we study $\mathbf{P}_{N,\omega}$ (in the limit $N \to \infty$) for a *fixed choice* of ω. In the literature, the charge distribution is often chosen Gaussian or of binary type, for

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example $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$. We invite the reader to look at Figure 1 in order to have a quick intuitive view of what this model describes (a polymer model).

Figure 1 also schematizes an aspect of the model which is particularly relevant to us. Namely that the *Hamiltonian* of the model, *i.e.* the quantity appearing at the exponent in the right-hand side of (1.1), does not depend on the full trajectories of S, but only on the *random set* $\tau := \{n \in \mathbb{N} \cup \{0\} : S_n = 0\}$ (that we may also look at as an increasing *random sequence* $\tau =: \{\tau_0, \tau_1, \tau_2, \ldots\}$) and on the *signs* $\xi = \{\xi_j\}_{j\in\mathbb{N}}$, defined by $\xi_j := \Delta(S_n)$ for $n \in \{\tau_j + 1, \tau_{j+1} - 1\}$ (that is $\xi_j = 0$ or 1 if the j-th excursion of S is positive or negative). In fact it is easily seen that $\Delta(S_{n-1} + S_n) = \sum_{j=1}^{\infty} \xi_j \, \mathbf{1}_{(\tau_{j-1}, \tau_j]}(n)$ is a function of τ and ξ only, and this suffices to reconstruct the Hamiltonian (see (1.1)). Note that we call the variables ξ_n *signs* even if they take the values $\{0, 1\}$ instead of $\{-1, +1\}$.

Under the simple random walk law $\mathbf{P}$, the two random sequences τ and ξ are independent. Moreover, ξ is just an IID sequence of $B(1/2)$ (*i.e.*, Bernoulli of parameter 1/2) variables, while τ is a *renewal process*, that is, $\tau_0 = 0$ and $\{\tau_j - \tau_{j-1}\}_{j\in\mathbb{N}}$ is IID. Let us also point out that for every $j \in \mathbb{N}$

$$\mathbf{P}(\tau_j - \tau_{j-1} = 2n) \;=\; \mathbf{P}(\tau_1 = 2n) \stackrel{n\to\infty}{\sim} \frac{1}{2\sqrt{\pi}\, n^{3/2}}, \tag{1.3}$$

where we have introduced the notation $f(x) \sim g(x)$ for $\lim_{x\to\infty} f(x)/g(x) = 1$ (in the sequel we will also use $\sim$ to denote equality in law: for example $\omega_1 \sim \omega_2 \sim \mathcal{N}(0, 1)$).

This discussion suggests a generalized framework in which to work, that has been already introduced in [7, 17]. We start from scratch: let us consider a general renewal process $\tau = \{\tau_n\}_{n\geq 0}$ on the non-negative integers $\mathbb{N} \cup \{0\}$ such that

$$K(n) \;:=\; \mathbf{P}(\tau_1 = n) \stackrel{n\to\infty}{\sim} \frac{L(n)}{n^{1+\alpha}}, \tag{1.4}$$

where $\alpha \geq 0$ and $L : (0, \infty) \to (0, \infty)$ a *slowly varying function*, *i.e.*, a (strictly) positive measurable function such that $\lim_{x\to\infty} L(cx)/L(x) = 1$, for every $c > 0$ (see Remark 1.1 below for more details). We assume that τ is a *persistent* renewal, *i.e.*, $\mathbf{P}(\tau_1 < \infty) = \sum_{n\in\mathbb{N}} K(n) = 1$, which is equivalent to $\mathbf{P}(|\tau| = \infty) = 1$, where $|\tau|$ denotes the cardinality of τ, viewed as a (random) subset of $\mathbb{N} \cup \{0\}$. We will switch freely from looking at τ as a sequence of random variables or as a random set.

Let $\xi = \{\xi_n\}_{n\in\mathbb{N}}$ denote an IID sequence of $B(1/2)$ variables, independent of τ, that we still call signs. With the couple (τ, ξ) in our hands, we build a new sequence $\Delta = \{\Delta_n\}_{n\in\mathbb{N}}$ by setting $\Delta_n = \sum_{j=1}^{\infty} \xi_j \, \mathbf{1}_{(\tau_{j-1}, \tau_j]}(n)$, in analogy with the simple random walk case. In words, the signs Δ_n are constant between the epochs of τ and they are determined by ξ.

We are now ready to introduce the general *discrete copolymer model*, as the probability law $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega}^{\lambda,h}$ for the sequence Δ defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\Delta) \;:=\; \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} \Delta_n(\omega_n + h)\right), \tag{1.5}$$

where $N \in \mathbb{N}$, $\lambda, h \in [0, \infty)$ and $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers (a typical realization of an IID sequence, see below). The partition function $Z_{N,\omega} = Z_{N,\omega}^{\lambda,h}$ is given by

$$Z_{N,\omega} \;:=\; \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} \Delta_n(\omega_n + h)\right)\right]. \tag{1.6}$$

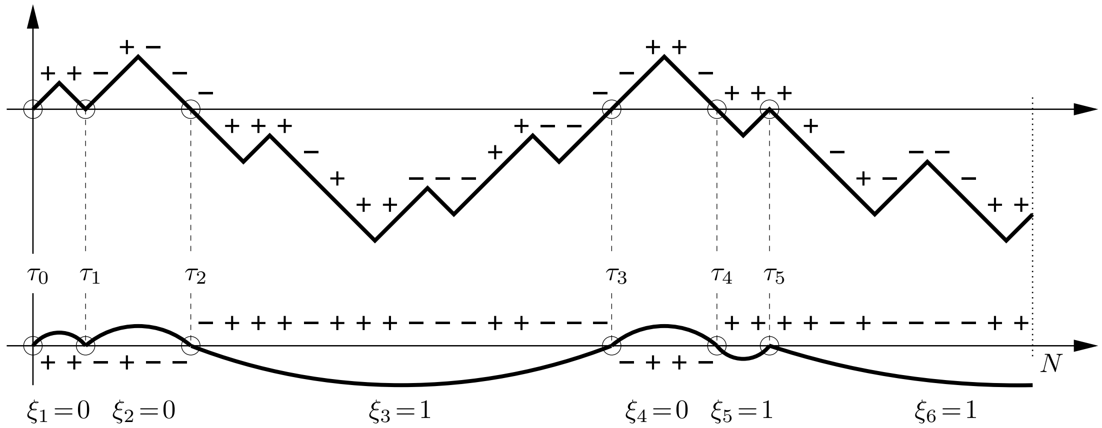

FIGURE 1. The polymer model we deal with has been introduced in the mathematical literature, see for example [24], as a modification of the law of the simple symmetric random walk $\{S_n\}_{n\geq 0}$ on $\mathbb{Z}$, with a density proportional to $\exp[\lambda \sum_{n=1}^{N}(\omega_n + h)\,\mathrm{sign}(S_n)]$ (*Boltzmann factor*). Each bond (S_{n-1}, S_n) is interpreted as a *monomer* and by definition $\mathrm{sign}(S_n)$ is the sign of (S_{n-1}, S_n), *i.e.*, it is $+1$ (resp. -1) if the monomer (S_{n-1}, S_n) lies in the upper (resp. lower) half plane. In a quicker way, $\mathrm{sign}(S_n)$ is just the sign of $S_{n-1} + S_n$. The Boltzmann factor is somewhat different from the one appearing in (1.1), but this is not a problem: in fact $\lambda \sum_{n=1}^{N}(\omega_n + h)\,\mathrm{sign}(S_n)$ can be rewritten as $-2\lambda \sum_{n=1}^{N} \Delta(S_{n-1} + S_n)(\omega_n + h) + c_N$, where $c_N := \lambda \sum_{n=1}^{N}(\omega_n + h)$ does not depend on S, therefore the quenched probability $\mathbf{P}_{N,\omega}$ is not affected by such a change. It is clear that the trajectories of the walk, that are interpreted as configurations of a *polymer chain*, have an *energetic gain* (that is, a larger Boltzmann factor) if *positively charged* monomers $[(\omega_n + h) > 0]$ lie in the upper half plane and *negatively charged* ones $[(\omega_n + h) < 0]$ lie in the lower one. The fulfillment of this requirement, even if only in a partial way, entails however an *entropic loss*: in fact the trajectories have to stick very close to the horizontal axis (the *interface*) and there are only few such random walk trajectories. The issue is precisely to understand who is the winner in this *energy-entropy competition*. The lower part of the figure stresses the fact that the Boltzmann factor does not depend on the full trajectory S, but only on the lengths and the signs of the successive excursions, described by the variables τ, ξ. In the figure it is also represented an example of the sequence of charges attached to the copolymer, in the binary case ($\omega_n \in \{-1, +1\}$).

In order to emphasize the value of α in (1.4), we will sometimes speak of a *discrete α-copolymer model*, but we stress that $\mathbf{P}_{N,\omega}$ depends on the full law $K(\cdot)$.

Note that the new model (1.5) only describes the sequence of signs Δ, while the *simple random walk model* (1.1) records the full trajectory S. However, once we project the probability law (1.1) on the variables $\Delta_n := \Delta(S_{n-1} + S_n)$, it is easy to check that the simple random walk model becomes a particular case of (1.5) and its partition function (1.2) coincides with the general one given by (1.6), provided we choose $K(\cdot)$ as the law of the first return to zero of the simple random walk (corresponding to $\alpha = \frac{1}{2}$, see (1.3) and (1.4)). As a matter of fact, since we require that $K(n) > 0$ for all large $n \in \mathbb{N}$ (*cf.* (1.4)), strictly speaking the case of the simple random walk is not covered. We stress, however, that our arguments can be adapted in a straightforward way to treat the cases in which there exists a positive integer T such that $K(n) = 0$ if $n/T \notin \mathbb{N}$ and relation (1.4) holds restricting $n \in T\mathbb{N}$ (of course $T = 2$ for the simple random walk case).

To complete the definition of the discrete copolymer model, let us state precisely our hypotheses on the *disorder* variables $\omega = \{\omega_n\}_{n\in\mathbb{N}}$. We assume that the sequence ω is IID and that ω_1 has locally finite exponential moments, that is there exists $t_0 > 0$ such that

$$\mathrm{M}(t) \;:=\; \mathbb{E}[\exp(t\omega_1)] \;<\; \infty \quad \text{for every } t \in [-t_0, t_0] \,. \tag{1.7}$$

We also fix

$$\mathbb{E}\left[\omega_1\right] \,=\, 0 \quad \text{and} \quad \mathbb{E}\left[\omega_1^2\right] \,=\, 1 \,, \tag{1.8}$$

which entails no loss of generality (it suffices to shift λ and h). In particular, these assumptions guarantee that there exists $c_0 > 0$ such that

$$\max_{t\in[-t_0,t_0]} \mathrm{M}(t) \,\le\, \exp\left(c_0\, t^2\right) . \tag{1.9}$$

Although it only keeps track of the sequence of signs Δ, we still interpret the probability law $\mathbf{P}_{N,\omega}$ defined in (1.5) as a model for an *inhomogeneous polymer* (this is the meaning of *copolymer*) that interacts with two selective solvents (the upper and lower half planes) separated by a flat interface (the horizontal axis), as it is explained in the caption of Figure 1. In particular, $\Delta_n = 0$ (resp. 1) means that the n-th monomer of the chain lies above (resp. below) the interface. To reinforce the intuition, we will sometimes describe the model in terms of full trajectories, like in the simple random walk case.

Remark 1.1. We refer to [4] for a full account on slowly varying functions. Here we just recall that the asymptotic behavior of $L(\cdot)$ is *weaker than any power*, in the sense that, as $x \to \infty$, $L(x)\, x^a$ tends to ∞ for $a > 0$ and to zero if $a < 0$. The most basic example of a slowly varying function is any positive measurable function that converges to a positive constant at infinity (in this case we say that the slowly varying function is trivial). Other important examples are positive measurable functions which behave asymptotically like the power of a logarithm, that is $L(x) \sim \log(1+x)^a$, $a \in \mathbb{R}$.

1.2. **The free energy: localization and delocalization.** This work focuses on the properties of the free energy of the discrete copolymer, defined by

$$\textsc{f}(\lambda, h) \,:=\, \lim_{N\to\infty} \textsc{f}_N(\lambda, h)\,, \qquad \text{where} \qquad \textsc{f}_N(\lambda, h) \,:=\, \frac{1}{N}\, \mathbb{E}\left[\log Z_{N,\omega}\right] . \tag{1.10}$$

The existence of such a limit follows by a standard argument, see for example [17, Ch. 4], where it is also proven that for every λ and h

$$\textsc{f}(\lambda, h) \,=\, \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}\,, \qquad \mathbb{P}(\mathrm{d}\omega)\text{-a.s. and in } L^1(\mathbb{P})\,. \tag{1.11}$$

Equations (1.10)–(1.11) are telling us that the limit in (1.11) does not depend on the (typical) realization of ω. Nonetheless it is worthwhile to stress that it does depend on $\mathbb{P}$, that is on the law of ω_1, as well as on the renewal process on which the model is built, namely on the inter-arrival law $K(\cdot)$. This should be kept in mind, even if we omit $\mathbb{P}$ and $K(\cdot)$ from the notation $\textsc{f}(\lambda, h)$.

An elementary, but crucial observation is:

$$\textsc{f}(\lambda, h) \,\ge\, 0\,, \qquad \forall \lambda, h \ge 0\,. \tag{1.12}$$

This follows simply by restricting the expectation in (1.6) to the event $\{\tau_1 > N, \xi_1 = 0\}$, on which we have $\Delta_1 = 0, \ldots, \Delta_N = 0$, hence we obtain $Z_{N,\omega} \ge \frac{1}{2}\mathbf{P}(\tau_1 > N)$ and it suffices to observe that $N^{-1} \log \mathbf{P}(\tau_1 > N)$ vanishes as $N \to \infty$, thanks to (1.4). Notice that the event $\{\tau_1 > N, \xi_1 = 0\}$ corresponds to the set of trajectories that never visit the lower half plane, therefore the right hand side of (1.12) may be viewed as the contribution to the free energy given by these trajectories. Based on this observation, it is customary to

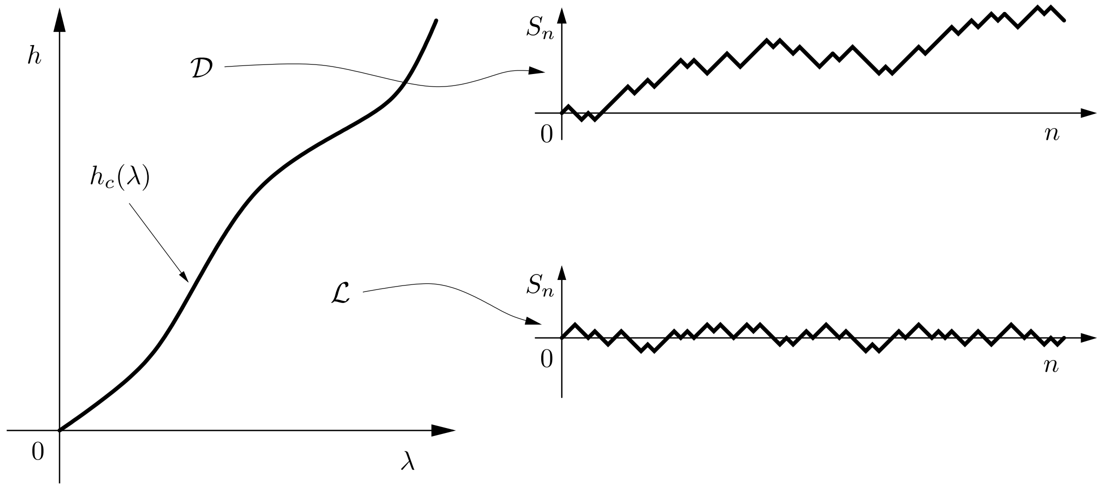

FIGURE 2. In the figure, on the left, a sketch of the phase diagram of the discrete copolymer model. The critical curve $\lambda \mapsto h_c(\lambda)$ separates the localized regime $\mathcal{L}$ from the delocalized one $\mathcal{D}$. This is a free energy characterization of the notion of (de)localization, but this characterization does correspond the to sharply different path behaviors, sketched on the right side of the figure. In a nutshell, if $(\lambda, h) \in \mathcal{L}$ then, for $N \to \infty$, the typical paths intersect the interface (the horizontal axis) with a positive density, while in the interior of $\mathcal{D}$ the path strongly prefers not to enter the lower half plane. In this work we just focus on properties of the free energy and for details on the link with path properties, including a review of the literature and open problems, we refer to [17, Ch. 7 and Ch. 8].

say that $(\lambda, h) \in \mathcal{D}$ (*delocalized regime*) if $\textsc{f}(\lambda, h) = 0$, while $(\lambda, h) \in \mathcal{L}$ (*localized regime*) if $\textsc{f}(\lambda, h) > 0$ (see also Figure 2 and its caption).

We have the following:

Theorem 1.2. *If we set* $h_c(\lambda) := \sup\{h : \textsc{f}(\lambda, h) > 0\}$*, then* $h_c(\lambda) = \inf\{h : \textsc{f}(\lambda, h) = 0\}$ *and the function* $h_c : [0, \infty) \to [0, \infty]$ *is strictly increasing and continuous as long as it is finite. Moreover we have the explicit bounds*

$$\frac{1}{2\lambda/(1+\alpha)} \log \mathrm{M}\left(-2\lambda/(1+\alpha)\right) \;\leq\; h_c(\lambda) \;\leq\; \frac{1}{2\lambda} \log \mathrm{M}\left(-2\lambda\right), \tag{1.13}$$

where the left inequality is strict when $\alpha \geq 0.801$ *(at least for* λ *small) and the right inequality is strict as soon as* $\alpha > 0$ *(for every* $\lambda < \sup\{t : \log \mathrm{M}(-2t) < \infty\}$*).*

The first part of Theorem 1.2 is proven in [9] and [6] (see also [17, Ch. 6]). In [6] one also finds the quantitative estimates (1.13), except for the strict inequalities proven in [7] (see also [25]). From (1.13) one directly extracts

$$\frac{1}{1+\alpha} \leq \liminf_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \leq \limsup_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \leq 1, \qquad \forall \alpha \geq 0. \tag{1.14}$$

For $\alpha > 0$, this result has been sharpened to

$$\max\left(\frac{1}{2}, \frac{g(\alpha)}{\sqrt{1+\alpha}}, \frac{1}{1+\alpha}\right) \leq \liminf_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \leq \limsup_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} < 1, \tag{1.15}$$

where $g(\cdot)$ is a continuous function such that $g(\alpha) = 1$ for $\alpha \geq 1$ and for which one can show that $g(\alpha)/\sqrt{1+\alpha} > 1/(1+\alpha)$ for $\alpha \geq 0.801$ (by evaluating $g(\cdot)$ numerically one can go down to $\alpha \geq 0.65$). In particular, the lower bound in (1.15) reduces to $1/2$ for $\alpha \geq 3$ and to $1/\sqrt{1+\alpha}$ for $\alpha \in [1,3]$. The bounds in (1.15) are proven in [7] and [26]. We invite the reader to look again at Figure 2.

The focus on the behavior of the *critical line* $h_c(\lambda)$ for λ small has a reason that is at the heart of this paper: our aim is to study the free energy $\textsc{f}(\lambda, h)$ of discrete copolymer models in the *weak coupling limit*, *i.e.*, when λ and h are small. We will show that the behavior of $\textsc{f}(\lambda, h)$ in this regime is captured by the exponent α appearing in (1.4), independently of the finer details of the inter-arrival law $K(\cdot)$. In particular, we prove that $h_c'(0)$ exists and that it depends only on α. In order to state these results precisely, we need to introduce a class of copolymer models in the continuum: in a suitable sense, they capture the limit of discrete copolymer models as $\lambda, h \searrow 0$.

1.3. **The continuum model: Brownian case.** E. Bolthausen and F. den Hollander introduced in [9] the *Brownian copolymer model*, whose partition function is given by

$$\widetilde{Z}_{t,\beta}^{\mathrm{BM}} := \mathbf{E}\left[\exp\left(-2\lambda \int_0^t \Delta\big(\widetilde{B}(u)\big)\big(\mathrm{d}\beta(u) + h\mathrm{d}u\big)\right)\right], \tag{1.16}$$

where once again $\lambda, h \geq 0$, $\Delta(x) := \mathbf{1}_{(-\infty,0)}(x)$ and $\widetilde{B}(\cdot)$ (the polymer), $\beta(\cdot)$ (the medium) are independent standard Brownian motions with laws $\mathbf{P}$ and $\mathbb{P}$ respectively.

The corresponding free energy $\widetilde{\textsc{f}}_{\mathrm{BM}}(\lambda, h)$ is defined as the limit as $t \to \infty$ of $\frac{1}{t}\mathbb{E}[\log \widetilde{Z}_{t,\beta}^{\mathrm{BM}}]$ and one has $\widetilde{\textsc{f}}_{\mathrm{BM}}(\lambda, h) \geq 0$ for every $\lambda, h \geq 0$, in analogy with the discrete case. Therefore, by looking at the positivity of $\widetilde{\textsc{f}}_{\mathrm{BM}}$, one can define also for the Brownian copolymer model the localized and delocalized regimes, that are separated by the critical line $\widetilde{h}_c(\lambda) := \sup\{h : \widetilde{\textsc{f}}_{\mathrm{BM}}(\lambda, h) > 0\}$. Now a real novelty comes into the game: the scaling properties of the two Brownian motions yield easily that for every $a > 0$

$$\frac{1}{a^2}\widetilde{\textsc{f}}_{\mathrm{BM}}(a\lambda, ah) = \widetilde{\textsc{f}}_{\mathrm{BM}}(\lambda, h). \tag{1.17}$$

In particular, the critical line is a straight line: $\widetilde{h}_c(\lambda) = \widetilde{\mathrm{m}}_{\mathrm{BM}}\,\lambda$, for every $\lambda \geq 0$, with

$$\widetilde{\mathrm{m}}_{\mathrm{BM}} := \sup\left\{c \geq 0 : \widetilde{\textsc{f}}_{\mathrm{BM}}(1, c) > 0\right\}. \tag{1.18}$$

We are now ready to state the main result in [9]:

Theorem 1.3. *For the simple random walk model* (1.1)*, with* ω_1 *such that* $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$*, we have*

$$\lim_{a \searrow 0} \frac{1}{a^2}\textsc{f}(a\lambda, ah) = \widetilde{\textsc{f}}_{\mathrm{BM}}(\lambda, h), \qquad \forall \lambda, h \geq 0, \tag{1.19}$$

and

$$\lim_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} = \widetilde{\mathrm{m}}_{\mathrm{BM}} \in (0, 1]. \tag{1.20}$$

The great interest of this result is that it provides a precise formulation for the fact that the Brownian copolymer model is the weak coupling scaling limit of the simple random walk copolymer model (1.1). At the same time, the fact that such a result is proven only for the simple random walk model and only for a single choice of the charges distribution appears to be a limitation. In fact, since Brownian motion is the scaling limit of many discrete processes, it is natural to guess that Theorem 1.3 should hold for a large class of discrete copolymer models and for a vast choice of charge distributions (remaining of course in the domain of attraction of the Gaussian law and adding some technical assumptions). This would show that the Brownian copolymer model has indeed a *universal* character.

In fact, Theorem 1.3 has been generalized in [18] to a large class of disorder random variables (including all bounded random variables). A further generalization has been obtained in [22], in the case when, added to the copolymer interaction, there is also a *pinning* interaction at the interface, that is an energy reward in touching the interface. We stress however that these generalizations are always for the copolymer model built over the simple random walk: going beyond the simple random walk case appears indeed to be a very delicate (albeit natural) step.

The main result of this paper is that Theorem 1.3 can be generalized to any discrete α-copolymer model with $\alpha \in (0,1)$ and to any disorder distribution satisfying (1.7)–(1.8) (see Theorem 1.5 below). For $\alpha = \frac{1}{2}$ the scaling limit is precisely the Brownian copolymer model (1.16), like in the simple random walk case, while for $\alpha \neq \frac{1}{2}$ the continuum copolymer model is defined in the next subsection. We stress from now that the scaling limit *depends only on* α: in particular, there is no dependence on the slowly varying function $L(\cdot)$ appearing in (1.4) and no dependence on $\mathbf{P}(\tau_1 = n)$ for any finite n.

1.4. **The continuum α-copolymer model.** Let us start by recalling that, for $\delta \geq 0$, *the square of δ-dimensional Bessel process* (started at 0) is the process $X = \{X_t\}_{t\geq 0}$ with values in $[0,\infty)$ that is the unique strong solution of the following equation:

$$X_t \;=\; 2\int_0^t \sqrt{X_s}\,\mathrm{d}w_s \;+\; \delta t\,, \tag{1.21}$$

where $\{w_t\}_{t\geq 0}$ is a standard Brownian motion. The *δ-dimensional Bessel process* is by definition the process $Y = \{Y_t := \sqrt{X_t}\}_{t\geq 0}$: it is a Markov process on $[0,\infty)$ that enjoys the standard Brownian scaling [23, Ch. XI, Prop. (1.10)]. We focus on the case $\delta \in (0,2)$, when a.s. the process Y visits the origin infinitely many times [23, Ch. XI, Prop. (1.5)]. We actually use the parametrization $\delta = 2(1-\alpha)$ and we then restrict to $\alpha \in (0,1)$.

It is easily checked using Itô's formula that for $\alpha = \frac{1}{2}$ (*i.e.* $\delta = 1$) the process Y has the same law as the absolute value of Brownian motion on $\mathbb{R}$. Since to define the Brownian copolymer model (1.16) we have used the full Brownian motion process, not only its absolute value, we need a modification of the Bessel process in which each excursion from zero may be either positive or negative, with the sign chosen *by fair coin tossing*. Such a process, that we denote by $\widetilde{B}^\alpha := \{\widetilde{B}^\alpha(t)\}_{t\geq 0}$, has been considered in the literature for example in [3] and is called *Walsh process of index α* (in [3] a more general case is actually considered: in their notations, our process corresponds to the choices $k = 2$, $E_1 = [0,\infty)$, $E_2 = (-\infty, 0]$ and $p_1 = p_2 = 1/2$). It is easy to see that the process $\widetilde{B}^\alpha$ inherits the Brownian scaling. We denote by $\mathbf{P}$ its law.

We are now ready to generalize the Brownian copolymer model (1.16): given $\alpha \in (0,1)$, we define the partition function of the *continuum α-copolymer model* through the formula

$$\widetilde{Z}^\alpha_{t,\beta} \;:=\; \mathbf{E}\exp\left(-2\lambda\int_0^t \Delta\big(\widetilde{B}^\alpha(u)\big)\big(\mathrm{d}\beta(u) + h\mathrm{d}u\big)\right)\,, \tag{1.22}$$

where $\beta = \{\beta(t)\}_{t\geq 0}$ always denotes a standard Brownian motion with law $\mathbb{P}$, independent of $\widetilde{B}^\alpha$, and $\Delta(x) = \mathbf{1}_{(-\infty,0)}(x)$. Since for $\alpha = \frac{1}{2}$ the process $\widetilde{B}^{1/2}$ is just a standard Brownian motion, $\widetilde{Z}^{1/2}_{t,\beta}$ coincides with $\widetilde{Z}^{\mathrm{BM}}_{t,\beta}$ defined in (1.16). For the sake of simplicity, in (1.22) we have only defined the partition function of the continuum α-copolymer model: of course, one can easily introduce the corresponding probability measure $\mathbf{P}_{t,\beta}$ on the paths of $\widetilde{B}^\alpha$, in analogy with the discrete case, but we will not need it.

Let us stress that the integral in (1.22), as well as the one in (1.16), does not really depend on the full path of the process $\widetilde{B}^\alpha$: in fact, being a function of $\Delta(\widetilde{B}^\alpha(\cdot))$, it only matters to know, for every $u \in [0,t]$, whether $B(u) < 0$ or $B(u) \geq 0$. For this reason it is natural to introduce (much like in the discrete case) the *zero level set* $\widetilde{\tau}^\alpha$ of $\widetilde{B}^\alpha(\cdot)$:

$$\widetilde{\tau}^\alpha := \big\{ s \in [0,\infty) : \ \widetilde{B}^\alpha(s) = 0 \big\} . \tag{1.23}$$

The set $\widetilde{\tau}^\alpha$ contains *almost* all the information we need, because, conditionally on $\widetilde{\tau}^\alpha$, the sign of $\widetilde{B}^\alpha$ inside each excursion is chosen just by tossing an independent fair coin. Moreover, the random set $\widetilde{\tau}^\alpha$ is a much studied object: it is in fact the *α–stable regenerative set* [23, Ch. XI, Ex. (1.25)]. Regenerative sets may be viewed as the continuum analogues of renewal processes: we discuss them in some detail in Section 2, also because it will come very handy to restate the model in terms of regenerative sets for the proofs.

The free energy for the continuum α-copolymer model is defined in close analogy to the discrete case, but proving its existence turns out to be a highly non-trivial task. For this reason, we state it as a result in its own:

Theorem 1.4. *The limit of $\frac{1}{t}\,\mathbb{E}\big[\log \widetilde{Z}^\alpha_{t,\beta}\big]$ as $t \to \infty$ exists and we call it $\widetilde{\mathrm{F}}_\alpha(\lambda,h)$. For all $\alpha \in (0,1)$ and $\lambda, h \in [0,\infty)$ we have $0 \leq \widetilde{\mathrm{F}}_\alpha(\lambda,h) < \infty$ and furthermore*

$$\lim_{t\to\infty} \frac{1}{t} \log \widetilde{Z}^\alpha_{t,\beta} \; = \; \widetilde{\mathrm{F}}_\alpha(\lambda,h) , \tag{1.24}$$

both $\mathbb{P}(\mathrm{d}\beta)$-a.s. and in $L^1(\mathbb{P})$. The function $(\lambda,h) \mapsto \mathrm{F}_\alpha(\lambda,h)$ is continuous.

Like before, the non-negativity of the free energy leads to exploiting the dichotomy $\widetilde{\mathrm{F}}_\alpha(\lambda,h) = 0$ and $\widetilde{\mathrm{F}}_\alpha(\lambda,h) > 0$ in order to define, respectively, the delocalized and localized regimes of the continuum α-copolymer model. The monotonicity of $\widetilde{\mathrm{F}}_\alpha(\lambda,\cdot)$ guarantees that if we set $\widetilde{h}^\alpha_c(\lambda) := \sup\{h \geq 0 : \widetilde{\mathrm{F}}_\alpha(\lambda,h) > 0\}$, then we also have $\widetilde{h}^\alpha_c(\lambda) := \inf\{h \geq 0 : \widetilde{\mathrm{F}}_\alpha(\lambda,h) = 0\}$. Moreover, the scaling properties of β and $\widetilde{B}^\alpha$ imply that (1.17) holds unchanged for $\widetilde{\mathrm{F}}_\alpha(\cdot,\cdot)$ so that the critical line is again a straight line: $\widetilde{h}^\alpha_c(\lambda) = \widetilde{\mathrm{m}}_\alpha \lambda$ for every $\lambda \geq 0$, with

$$\widetilde{\mathrm{m}}_\alpha := \sup\big\{ c \geq 0 : \widetilde{\mathrm{F}}_\alpha(1,c) > 0 \big\} , \tag{1.25}$$

in direct analogy with (1.18). Plainly, $\widetilde{\mathrm{m}}_{1/2} = \widetilde{\mathrm{m}}_{\mathrm{BM}}$.

1.5. **The main result.**

We can finally state the main result of this paper:

Theorem 1.5. *Consider an arbitrary discrete α-copolymer model, with $\alpha \in (0,1)$. For all $\lambda, h \geq 0$ we have*

$$\lim_{a \searrow 0} \frac{1}{a^2} \mathrm{F}(a\lambda, ah) \; = \; \widetilde{\mathrm{F}}_\alpha(\lambda,h) . \tag{1.26}$$

Moreover

$$\lim_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \; = \; \widetilde{\mathrm{m}}_\alpha . \tag{1.27}$$

Theorem 1.5 shows that the continuum α-copolymer is the universal weak interaction limit of arbitrary discrete α-copolymer models. Although the phase diagram of a discrete copolymer model does depend on the details of the inter-arrival law $K(\cdot)$, it nevertheless exhibits universal features for weak coupling. In particular, the critical line close to the origin is, to leading order, a straight line of slope $\widetilde{\mathrm{m}}_\alpha$. It is therefore clear that computing $\widetilde{\mathrm{m}}_\alpha$ or, at least, being able of improving the known bounds on $\widetilde{\mathrm{m}}_\alpha$ would mean a substantial progress in understanding the phase transition in this model. Note that, of course, given (1.27), the bounds in (1.15) are actually bounds on $\widetilde{\mathrm{m}}_\alpha$ (and they represent the state of the art on this important issue, to the the authors' knowledge).

It is remarkable that in the physical literature there is, on the one hand, quite some attention on the slope at the origin of the critical curve, see for example [14], but, on the other hand, its *universal* aspect has not been appreciated (some of the physical prediction are even in contradiction with the universality of the slope). We refer to [7, 17, 15] for overviews of the extensive physical literature on copolymer models.

We do not expect a generalization of Theorem 1.5 to $\alpha \notin (0,1)$. To be more precise, the case $\alpha = 0$ is rather particular: the critical curve is known explicitly by Theorem 1.2, the slope at the origin is universal and its value is one. The case $\alpha = 1$ with $\mathbf{E}[\tau_1] = \infty$ may still be treatable, but the associated regenerative set is the full line, so Theorem 1.5 cannot hold as stated. An even more substantial problem arises whenever $\mathbf{E}[\tau_1] < \infty$ (in particular for every $\alpha > 1$): apart from the fact that the regenerative set becomes trivial, there is a priori no reason why universality should hold. The *rationale* behind Theorem 1.5 is that at small coupling the renewal trajectories are not much perturbed by the interaction with the charges. If $\mathbf{E}[\tau_1] = \infty$, one may then hope that *long* inter-arrival gaps dominate, as they do when there is no interaction with the charges: since the statistics of long gaps depends only on the tail of $K(\cdot)$ and within long gaps the disorder can be replaced by Gaussian disorder, Theorem 1.5 is plausible. This is of course not at all the case if $\mathbf{E}[\tau_1] < \infty$.

Remark 1.6. One may imagine that (1.27) is a consequence of (1.26), but this is not true. In fact, it is easy to check that (1.26) directly implies

$$\liminf_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \;\geq\; \widetilde{\mathrm{m}}_\alpha\,, \tag{1.28}$$

but the opposite bound (for the superior limit) does not follow automatically. We obtain it as a corollary of our main technical result (Theorem 3.1).

1.6. **Outline of the paper.** We start, in Section 2, by taking a closer look at the continuum model and by giving a proof of the existence of the free energy (Theorem 1.4). Such an existence result had been overlooked in [9]. A proof was proposed in [16], in the Brownian context, giving for granted a suitable uniform boundedness property that is not straightforward (this is the issue addressed in Appendix A below). The proof that we give here therefore generalizes (from $\alpha = 1/2$ to $\alpha \in (0,1)$) and completes the proof in [16]. We follow the general scheme of the proof in [16], that is, we first define a suitably modified partition function, that falls in the realm of Kingman's super-additive ergodic Theorem [20], and then we show that such a modified partition function has the same Laplace asymptotic behavior as the original one. Roughly speaking, the modified partition function is obtained by relaxing the condition that $\widetilde{B}^\alpha(0) = 0$: one takes rather the infimum over a finite interval of starting points. If introducing such a modified partition function is a standard procedure, a straightforward application of this idea does not seem to lead far. Such an infimum procedure has to be set-up in a careful way in order to be able to perform the second step of the proof, that is stepping back to the original partition function. With respect to the proof in [16], that exploits the full path of the Brownian motion $B(\cdot)$, the one we present here is fully based on the regenerative set. Overall, establishing the existence of the continuum free energy is very much harder than the discrete counterpart case and it appears to be remarkably subtle and complex when compared to the analogous statement for *close relatives* of our model (see, *e.g.*, [11]).

In Section 3 we give the proof of our main result, Theorem 1.5, following the scheme set forth in [9] (we refer to it as the *original approach*), which is based on a four step procedure. We outline it here, in order to give an overview of the proof and to stress the points at which our arguments are more substantially novel.

(1) *Coarse graining of the renewal process.* In this step we replace the Boltzmann factor by a new, *coarser* one, which does not depend on the *short* excursions of the renewal process (in the sense that these excursions inherit the sign of a neighbor long gap). This step is technically, but not substantially different from the one in the original approach.
(2) *Switching to Gaussian charges.* The original approach exploits the well-known, and highly non-trivial, coupling result due to J. Komlós, P. Major and G. Tusnády [21]. We take instead a more direct, and more elementary, approach. In doing so we get rid of any assumption, beyond local exponential integrability, on the disorder.
(3) *From the renewal process to the regenerative set.* This is probably the crucial step. The original approach exploits heavily the underlying simple random walk and the exact formulas available for such a process. Our approach necessarily sticks to the renewal process and, in a sense, the point is showing that suitable local limit theorems (crucial here are results by R. Doney [12]) suffice to perform this step. There is however another issue that makes our general case different from the simple random walk case. In fact this step, in the original approach, is based on showing that a suitable Radon-Nikodym derivative, comparing the renewal process and the regenerative set, is uniformly bounded. In our general set-up, this Radon-Nikodym derivative is not bounded and a more careful estimate has to be carried out.
(4) *Inverse coarse graining of the regenerative set.* We are now left with a model based on the regenerative set, but depending only on the *large* excursions. We have therefore to show that putting back the dependence on the small excursions does not modify substantially the quantity we are dealing with. This is parallel to the first step: it involves estimates that are different from the ones in the original approach, because we are sticking to the regenerative set formulation and because α is not necessarily equal to $1/2$, but the difference is, essentially, just technical.

Let us finally mention that our choice of focusing on discrete copolymer models built over renewal processes leaves out another possible (and perhaps more natural) generalization of the simple random walk copolymer model (1.1): namely, the one obtained by replacing the simple random walk with a more general random walk. A general random walk crosses the interface without necessarily touching it, therefore the associated point process is a Markov renewal process [2], because one has to carry along not only the switching sign times, but also the height of the walk at these times (sometimes called the *overshoot*). This is definitely an interesting and non-trivial problem, that goes in a direction which is complementary to the one we have taken. However two remarks are in order:

(1) Symmetric random walks with IID increments in $\{-1, 0, 1\}$ touch the interface when they cross it, hence they are covered by our analysis: their weak coupling limit is the continuum $1/2$–copolymer, because $K(n) \overset{n\to\infty}{\sim} (const.)\, n^{-3/2}$ (*e.g.*, [17, App. A.5]).

(2) While one definitely expects an analog of Theorem 1.5 to hold for *rather general* random walks with increments in the domain of attraction of the normal law (with the continuum 1/2–copolymer as weak coupling limit), it is less clear what to expect when the increments of the walk are in the domain of attraction of a non Gaussian stable law. In our view, working with generalized copolymer models has, in any case, a considerable flexibility with respect to focusing on the random walk set-up.

2. A closer look at the continuum model

In this section we prove the existence of the continuum free energy $\widetilde{\mathrm{F}}_\alpha(\lambda, h)$, that is we prove Theorem 1.4. In § 2.3 we define a modified partition function, to which Kingman's super-additive ergodic theorem can be applied, and then in § 2.4 we show that this modified partition function yields the same free energy as the original one. Before starting with the proof, in § 2.1 we redefine the partition function $\widetilde{Z}^\alpha_{t,\beta}$ more directly in terms of the α-stable regenerative set $\widetilde{\tau}^\alpha$, whose basic properties are recalled in § 2.2 (cf. also the appendix § A.1). We are going to drop some dependence on α for short, writing, *e.g.*, $\widetilde{\mathrm{F}}(\lambda, h)$.

2.1. **Preliminary considerations.** As explained in § 1.4, the process $\widetilde{B}^\alpha$ is introduced just to help visualizing the copolymer, but the underlying relevant process is $\Delta(\widetilde{B}^\alpha) := \mathbf{1}_{(-\infty,0)}(\widetilde{B}^\alpha)$. So let us re-introduce $\widetilde{Z}_{t,\beta}$ more explicitly, in terms of the random set $\widetilde{\tau}^\alpha$ (*cf.* (1.23)) and of the signs of the excursions, that are sufficient to determine $\Delta(\widetilde{B}^\alpha)$.

There is no need to pass through the process $\widetilde{B}^\alpha$ to introduce $\widetilde{\tau}^\alpha$: we can define it directly as the *stable regenerative set* of index α, that is, the closure of the image of the stable subordinator of index α, *cf.* [13]. Some basic properties of regenerative sets are recalled in § 2.2 and in the appendix § A.1; we mention in particular the scale invariance property: $\widetilde{\tau}^\alpha \sim c\,\widetilde{\tau}^\alpha$, for every $c > 0$. Since $\widetilde{\tau}^\alpha$ is a closed set, we can write the open set $(\widetilde{\tau}^\alpha)^\complement = \bigcup_{n\in\mathbb{N}} I_n$ as the disjoint union of countably many (random) open intervals I_n, the connected components (*i.e.*, maximal open intervals) of $(\widetilde{\tau}^\alpha)^\complement$. Although there is no canonical way of numbering these intervals, any reasonable rule is equivalent for our purpose. As an example, one first numbers the intervals that start (*i.e.*, whose left endpoint lies) in $[0, 1)$ in decreasing order of width, obtaining $\{J^1_n\}_{n\in\mathbb{N}}$; then one does the same with the intervals that start in $[1, 2)$, getting $\{J^2_n\}_{n\in\mathbb{N}}$; and so on. Finally, one sets $I_n := J^{a_n}_{b_n}$, where $n \mapsto (a_n, b_n)$ is any fixed bijection from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$.

Let $\widetilde{\xi} = \{\widetilde{\xi}_n\}_{n\in\mathbb{N}}$ be an IID sequence of Bernoulli random variables of parameter 1/2, defined on the same probability space as $\widetilde{\tau}^\alpha$ and independent of $\widetilde{\tau}^\alpha$, that represent the signs of the excursions of $\widetilde{B}^\alpha$. We then define the process $\widetilde{\Delta}^\alpha(u) := \sum_n \widetilde{\xi}_n\, \mathbf{1}_{I_n}(u)$, which takes values in $\{0, 1\}$ and is a continuum analogue of the discrete process $\{\Delta_n\}_{n\in\mathbb{N}}$ introduced in § 1.1: $\widetilde{\Delta}^\alpha(u) = 1$ (resp. 0) means that the continuum copolymer in u is below (resp. above or on) the interface. With this definition, we have the equality in law

$$\big\{\widetilde{\Delta}^\alpha(u)\big\}_{u\geq 0} \sim \big\{\Delta\big(\widetilde{B}^\alpha(u)\big)\big\}_{u\geq 0}\,, \tag{2.1}$$

so that we can use $\widetilde{\Delta}^\alpha(\cdot)$ instead of $\Delta\big(\widetilde{B}^\alpha(\cdot)\big)$. More precisely, for $0 \leq s \leq t < \infty$ we set

$$\begin{aligned} \widetilde{\mathcal{Z}}_{s,t;\beta} \;&=\; \widetilde{\mathcal{Z}}^{\lambda,h}_{s,t;\beta} \;:=\; \mathbf{E}\left[\exp\left(\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^\alpha)\right)\right],\\ \mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^\alpha) \;&=\; \mathcal{H}^{\lambda,h}_{s,t;\beta}(\widetilde{\Delta}^\alpha) \;:=\; -2\lambda\int_s^t \widetilde{\Delta}^\alpha(u)\big(\mathrm{d}\beta(u) + h\mathrm{d}u\big)\,, \end{aligned} \tag{2.2}$$

so that the partition function $\widetilde{Z}^{\alpha}_{t,\beta}$ defined in (1.22) coincides with $\widetilde{\mathcal{Z}}_{0,t;\beta}$. For later convenience, we introduce the finite-volume free energy:

$$\widetilde{\mathrm{F}}_t(\lambda,h) \;:=\; \frac{1}{t}\mathbb{E}\left[\log \widetilde{\mathcal{Z}}_{0,t;\beta}\right]. \tag{2.3}$$

To be precise, for $\widetilde{\mathcal{Z}}_{s,t;\beta}$ and $\widetilde{\mathrm{F}}_t(\lambda,h)$ to be well defined we need to use a measurable version of $\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})$ (we build one in Remark 2.1 below).

Notice that we have the following additivity property:

$$\mathcal{H}_{r,t;\beta}(\widetilde{\Delta}^{\alpha}) \;=\; \mathcal{H}_{r,s;\beta}(\widetilde{\Delta}^{\alpha}) \;+\; \mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}), \tag{2.4}$$

for every $r<s<t$ and $\mathbf{P}\otimes\mathbb{P}$-a.e. $(\widetilde{\Delta}^{\alpha},\beta)$. Another important observation is that, for any fixed realization of $\widetilde{\Delta}^{\alpha}(\cdot)$, the process $\{\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})\}_{s,t}$ under $\mathbb{P}$ is Gaussian.

Remark 2.1. Some care is needed for definition (2.2) to make sense. The problem is that $\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})$, being a stochastic (Wiener) integral, is defined (for every fixed realization of $\widetilde{\Delta}^{\alpha}$) through an L^2 limit, hence it is not canonically defined for every β, but only $\mathbb{P}(\mathrm{d}\beta)$–a.s.. However, in order to define $\widetilde{\mathcal{Z}}_{s,t;\beta}$, we need $\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})$ to be a measurable function of $\widetilde{\Delta}^{\alpha}$, for every (or at least $\mathbb{P}$–almost every) fixed β. For this reason, we now show that it is possible to define a *version* of $\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})$ that is a measurable function of $(\beta,\widetilde{\Delta}^{\alpha},s,t,\lambda,h)$.

Let us fix a realization of the process $\{\widetilde{\Delta}^{\alpha}(u)\}_{u\in[0,\infty)}$. We build a sequence of approximating functions as follows: for $k\in\mathbb{N}$ we set

$$\widetilde{\Delta}^{\alpha}_k(u) \;:=\; \sum_{n\in\mathbb{N}:\,|I_n|\geq\frac{1}{k}} \widetilde{\xi}_n\,\mathbf{1}_{I_n}(u), \tag{2.5}$$

that is we only keep the excursion intervals of width at least $\frac{1}{k}$. Note that $\widetilde{\Delta}^{\alpha}_k(u)\to\widetilde{\Delta}^{\alpha}(u)$ as $k\to\infty$, for every $u\in\mathbb{R}^+$. By dominated convergence we then have $\widetilde{\Delta}^{\alpha}_k\to\widetilde{\Delta}^{\alpha}$ in $L^2((s,t),\mathrm{d}x)$, for all $0\leq s\leq t<\infty$, hence by the theory of Wiener integration it follows that $\lim_{k\to\infty}\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}_k)=\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})$ in $L^2(\mathrm{d}\mathbb{P})$. Note that, for any $k\in\mathbb{N}$, we have

$$\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}_k) \;=\; -2\lambda \sum_{n\in\mathbb{N}:\,|I_n|\geq\frac{1}{k}} \widetilde{\xi}_n\left(\beta_{I_n\cap(s,t)}+h\,|I_n\cap(s,t)|\right), \tag{2.6}$$

where we have set $\beta_{(a,b)}:=\beta_b-\beta_a$ and $\beta_{\emptyset}:=0$ (note that the right hand side of (2.6) is a finite sum). This shows that $\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}_k)$ is a measurable function of $(\beta,\widetilde{\Delta}^{\alpha},s,t,\lambda,h)$. Therefore, if we prove that $\lim_{k\to\infty}\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}_k)=\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha})$ $\mathbb{P}(\mathrm{d}\beta)$-a.s., we can redefine $\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}):=\liminf_{k\to\infty}\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}_k)$ and get the measurable version we are aiming at. However, for any fixed realization of $\widetilde{\Delta}^{\alpha}$, it is clear from (2.6) that $(\{\mathcal{H}_{s,t;\beta}(\widetilde{\Delta}^{\alpha}_k)\}_{k\in\mathbb{N}},\mathbb{P})$ is a supermartingale (it is a process with independent Gaussian increments of negative mean) bounded in L^2, hence it converges $\mathbb{P}(\mathrm{d}\beta)$-a.s..

2.2. **On the α-stable regenerative set.** We collect here a few basic formulas on $\widetilde{\tau}^{\alpha}$.

For $x\in\mathbb{R}$, we denote by $\mathbf{P}_x$ the law of the regenerative set started at x, that is $\mathbf{P}_x(\widetilde{\tau}^{\alpha}\in\cdot):=\mathbf{P}(\widetilde{\tau}^{\alpha}+x\in\cdot)$. Analogously, the process $\{\widetilde{\Delta}^{\alpha}(u)\}_{u\geq x}$ under $\mathbf{P}_x$ is distributed like the process $\{\widetilde{\Delta}^{\alpha}(u-x)\}_{u\geq x}$ under $\mathbf{P}=:\mathbf{P}_0$. Two variables of basic interest are

$$g_t \;=\; g_t(\widetilde{\tau}^{\alpha}) \;:=\; \sup\left\{x\in\widetilde{\tau}^{\alpha}\cap(-\infty,t]\right\}, \quad d_t \;=\; d_t(\widetilde{\tau}^{\alpha}) \;:=\; \inf\left\{x\in\widetilde{\tau}^{\alpha}\cap(t,\infty)\right\}. \tag{2.7}$$

The joint density of (g_t, d_t) under $\mathbf{P}_x$ is

$$\frac{\mathbf{P}_x\big(g_t \in \mathrm{d}a\,,\ d_t \in \mathrm{d}b\big)}{\mathrm{d}a\,\mathrm{d}b} \;=\; \frac{\alpha\,\sin(\pi\alpha)}{\pi}\,\frac{\mathbf{1}_{(x,t)}(a)\,\mathbf{1}_{(t,\infty)}(b)}{(a-x)^{1-\alpha}\,(b-a)^{1+\alpha}}\,, \tag{2.8}$$

from which we easily obtain the marginal distribution of g_t: for $y \in [x,t]$

$$G_{x,t}(y) \;:=\; \mathbf{P}_x\big(g_t \le y\big) \;=\; \frac{\sin(\pi\alpha)}{\pi}\int_x^y \frac{1}{(a-x)^{1-\alpha}\,(t-a)^{\alpha}}\,\mathrm{d}a\,. \tag{2.9}$$

Observing that $\frac{\mathrm{d}}{\mathrm{d}x}(x^\alpha/(1-x)^\alpha) = \alpha\,(x^{1-\alpha}(1-x)^{1+\alpha})^{-1}$, one obtains also the distribution of d_t: for $y \in [t,\infty)$

$$D_{x,t}(y) \;:=\; \mathbf{P}_x\big(d_t \le y\big) \;=\; \frac{\sin(\pi\alpha)}{\pi}\int_t^y \frac{(t-x)^{\alpha}}{(b-t)^{\alpha}\,(b-x)}\,\mathrm{d}b\,. \tag{2.10}$$

Let us denote by $\mathcal{F}_u$ the σ-field generated by $\widetilde{\tau}^\alpha \cap [0,u]$. The set $\widetilde{\tau}^\alpha$ enjoys the remarkable *regenerative property*, the continuum analogue of the well-known renewal property, that can be stated as follows: for every $\{\mathcal{F}_u\}_{u\ge 0}$–stopping time γ such that $\mathbf{P}(\gamma \in \widetilde{\tau}^\alpha) = 1$, the portion of $\widetilde{\tau}^\alpha$ after γ, *i.e.* the set $(\widetilde{\tau}^\alpha - \gamma) \cap [0,\infty)$, under $\mathbf{P}$ is independent of $\mathcal{F}_\gamma$ and distributed like the original set $\widetilde{\tau}^\alpha$. Analogously, the translated process $\{\widetilde{\Delta}^\alpha(\gamma+u)\}_{u\ge 0}$ is independent of $\mathcal{F}_\gamma$ and distributed like the original process $\widetilde{\Delta}^\alpha$.

2.3. **A modified partition function.** In order to apply super-additivity arguments, we introduce a modification of the partition function. We extend the Brownian motion $\beta(t)$ to negative times, setting $\beta(t) := \beta'(-t)$ for $t < 0$, where $\beta'(\cdot)$ is another standard Brownian motion independent of β, so that $\beta(t) - \beta(s) \sim \mathcal{N}(0, t-s)$ for all $s, t \in \mathbb{R}$ with $s \le t$.

Observe that $\{d_a < b\} = \{\widetilde{\tau}^\alpha \cap (a,b) \neq \emptyset\}$, where the random variable d_t has been defined in (2.7). Then for $0 \le s < t$ we set

$$\widetilde{\mathcal{Z}}^*_{s,t;\beta} \;:=\; \inf_{x\in[s-1,s]} \mathbf{E}_x\big[\exp\big(\mathcal{H}_{x,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)\big)\,,\ d_{t-1} < t\,\big]\,. \tag{2.11}$$

In words: we take the smallest free energy among polymers starting at $x \in [s-1,s]$ and coming back to the interface at some point in $(t-1,t)$. Notice that the Hamiltonian looks at the polymer only in the interval (x, d_{t-1}). Also notice that for $t < s+1$ the expression is somewhat degenerate, because for $x > t-1$ we have $d_{t-1} = x$ and therefore $\mathcal{H}_{x,d_{t-1};\beta}(\widetilde{\Delta}^\alpha) = \mathcal{H}_{x,x;\beta}(\widetilde{\Delta}^\alpha) = 0$. Therefore we may restrict the infimum over $x \in [s-1, \min\{s,t-1\}]$, and for clarity we state it explicitly:

$$\widetilde{\mathcal{Z}}^*_{s,t;\beta} \;:=\; \inf_{x\in[s-1,\min\{s,t-1\}]} \mathbf{E}_x\big[\exp\big(\mathcal{H}_{x,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)\big)\,,\ d_{t-1} < t\,\big]\,. \tag{2.12}$$

Let us stress again that $\{d_{t-1} < t\} = \{\widetilde{\tau}^\alpha \cap (t-1,t) \neq \emptyset\}$.

It is sometimes more convenient to use $\mathbf{E} = \mathbf{E}_0$ instead of $\mathbf{E}_x$. To this purpose, by a simple change of variables we have $\mathcal{H}_{x,a;\beta}(\widetilde{\Delta}^\alpha) = \mathcal{H}_{0,a-x;\theta_x\beta}(\theta_x\widetilde{\Delta}^\alpha)$, where $\theta_x f(\cdot) := f(x+\cdot)$, as it follows easily from the definition (2.2) of the Hamiltonian. Since by definition the process $\theta_x\widetilde{\Delta}^\alpha$ under $\mathbf{P}_x$ is distributed like $\widetilde{\Delta}^\alpha$ under $\mathbf{P} = \mathbf{P}_0$, we can write

$$\mathbf{E}_x\big[\exp\big(\mathcal{H}_{x,y;\beta}(\widetilde{\Delta}^\alpha)\big)\big] \;=\; \mathbf{E}\big[\exp\big(\mathcal{H}_{0,y-x;\theta_x\beta}(\widetilde{\Delta}^\alpha)\big)\big]\,. \tag{2.13}$$

Analogously, since the random variable d_{t-1} under $\mathbf{P}_x$ is distributed like $x + d_{t-1-x}$ under $\mathbf{P}$, we can rewrite the term appearing in (2.12) as

$$\mathbf{E}_x\big[\exp\big(\mathcal{H}_{x,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)\big)\,,\ d_{t-1} < t\,\big] \;=\; \mathbf{E}\big[\exp\big(\mathcal{H}_{0,d_{t-1-x};\theta_x\beta}(\widetilde{\Delta}^\alpha)\big)\,,\ d_{t-1-x} < t-x\,\big]\,. \tag{2.14}$$

These alternative expressions are very useful to get uniform bounds. In fact, if we set

$$\Theta_T(\beta, \widetilde{\Delta}^\alpha) \;:=\; \sup_{-1 \le x \le T\,,\; 0 \le y \le T+1} \big| \mathcal{H}_{0,y;\theta_x \beta}(\widetilde{\Delta}^\alpha) \big| \,, \tag{2.15}$$

from (2.12) and (2.14) we have the following upper bound:

$$\sup_{0 \le s < t \le T} \widetilde{\mathcal{Z}}^*_{s,t;\beta} \;\le\; \mathbf{E}\big[\exp\big(\Theta_T(\beta, \widetilde{\Delta}^\alpha) \big) \big] \,. \tag{2.16}$$

In a similar fashion, from the relation (2.13) we obtain the lower bound

$$\inf_{-1 \le x \le T\,,\; 0 \le y \le T+1} \mathbf{E}_x \big[\exp\big(\mathcal{H}_{x,y;\beta}(\widetilde{\Delta}^\alpha) \big) \big] \;\ge\; \mathbf{E}\big[\exp\big(-\, \Theta_T(\beta, \widetilde{\Delta}^\alpha) \big) \big] \,. \tag{2.17}$$

We finally state a very useful result, that we prove in Appendix A: for every $\eta \in (0,\infty)$ there exists $D(\eta) \in (0,\infty)$ such that

$$\mathbf{E}\big[\mathbb{E}\big[\exp\big(\eta\, \Theta_T(\beta, \widetilde{\Delta}^\alpha) \big) \big] \big] \;\le\; D(\eta)\, e^{D(\eta)\, T} \;<\; \infty \,, \qquad \text{for every } T > 0 \,. \tag{2.18}$$

2.4. **Proof of Theorem 1.4.** We start by proving the existence of the limit in (1.24) when the partition function $\widetilde{Z}^\alpha_{t,\beta} = \widetilde{\mathcal{Z}}_{0,t;\beta}$ is replaced by $\widetilde{\mathcal{Z}}^*_{0,t;\beta}$.

Proposition 2.2. *For all $\lambda, h \ge 0$, the following limit exists $\mathbb{P}(\mathrm{d}\beta)$–a.s. and in $L^1(\mathrm{d}\mathbb{P})$:*

$$\lim_{t \to \infty} \frac{1}{t} \log \widetilde{\mathcal{Z}}^*_{0,t;\beta} \;=:\; \widehat{\mathrm{F}}(\lambda, h) \,, \tag{2.19}$$

where $\widehat{\mathrm{F}}(\lambda, h)$ is finite and non-random.

Proof. We are going to check that, for all fixed $\lambda, h \ge 0$, the process $\{\log \widetilde{\mathcal{Z}}^*_{s,t;\beta}\}_{0 \le s < t < \infty}$ under $\mathbb{P}$ satisfies the four hypotheses of *Kingman's super-additive ergodic theorem*, *cf.* [20]. This entails the existence of the limit in the l.h.s. of (2.19), both $\mathbb{P}$–a.s. and in $L^1(\mathrm{d}\mathbb{P})$, as well as the fact that the limit is a function of β which is invariant under time translation $\beta(\cdot) \mapsto \theta_t \beta(\cdot) := \beta(t + \cdot)$, for every $t \ge 0$. Therefore the limit must be measurable w.r.t. the tail σ-field of $\beta(\cdot)$, hence non-random by Kolmogorov 0–1 law for Brownian motion.

The first of Kingman's conditions is that for every $k \in \mathbb{N}$, any choice of $\{(s_j, t_j)\}_{k \in \mathbb{N}}$, with $0 \le s_j < t_j$, and for every $a > 0$ we have

$$\big(\widetilde{\mathcal{Z}}^*_{s_1,t_1;\beta} \,, \ldots ,\, \widetilde{\mathcal{Z}}^*_{s_k,t_k;\beta} \big) \;\overset{d}{=}\; \big(\widetilde{\mathcal{Z}}^*_{s_1+a,t_1+a;\beta} \,, \ldots ,\, \widetilde{\mathcal{Z}}^*_{s_k+a,t_k+a;\beta} \big) \,. \tag{2.20}$$

However this is trivially true, because $\widetilde{\mathcal{Z}}^*_{s+a,t+a;\beta} = \widetilde{\mathcal{Z}}^*_{s,t;\theta_a \beta}$, as it follows from (2.12), recalling the definition of the Hamiltonian in (2.2).

The second condition is the super-additivity property: for all $0 \le r < s < t$

$$\widetilde{\mathcal{Z}}^*_{r,t;\beta} \;\ge\; \widetilde{\mathcal{Z}}^*_{r,s;\beta} \cdot \widetilde{\mathcal{Z}}^*_{s,t;\beta} \,. \tag{2.21}$$

To this purpose, for any fixed $x \in [r-1, r]$ the inclusion bound yields

$$\begin{aligned} &\mathbf{E}_x \big(\exp\big(\mathcal{H}_{x,d_{t-1};\beta} \big) \,,\; d_{t-1} < t \big) \\ &\qquad \ge\; \mathbf{E}_x \big(\exp\big(\mathcal{H}_{x,d_{s-1};\beta} \big) \exp\big(\mathcal{H}_{d_{s-1},d_{t-1};\beta} \big) \,,\; d_{s-1} < s \,,\; d_{t-1} < t \big) \,, \end{aligned} \tag{2.22}$$

where we have used the additivity of the Hamiltonian, see (2.4). We integrate over the possible values of d_{s-1} and, using the regenerative property, we rewrite the right hand side of (2.22) as follows:

$$\begin{aligned}\int_{y\in(s-1,s)} & \mathbf{E}_x\big(\exp\big(\mathcal{H}_{x,y;\beta}\big)\,,\ d_{s-1}\in \mathrm{d}y\big)\,\mathbf{E}_y\big(\exp\big(\mathcal{H}_{y,d_{t-1};\beta}\big)\,,\ d_{t-1}<t\,\big) \\ & \geq\ \mathbf{E}_x\big(\exp\big(\mathcal{H}_{x,d_{s-1};\beta}\big)\,,\ d_{s-1}<s\,\big)\cdot \widetilde{\mathcal{Z}}^*_{s,t;\beta}\,,\end{aligned} \tag{2.23}$$

where the inequality is just a consequence of taking the infimum over $y\in[s-1,s]$ and recalling the definition (2.12) of $\widetilde{\mathcal{Z}}^*_{s,t;\beta}$. Putting together the relation (2.22) and (2.23) and taking the infimum over $x\in[r-1,r]$ we have proven (2.21).

The third condition to check is

$$\sup_{t>0}\ \frac{1}{t}\,\mathbb{E}\big(\log\widetilde{\mathcal{Z}}^*_{0,t;\beta}\,\big)\ <\ \infty\,. \tag{2.24}$$

Recalling (2.12) and applying Jensen's inequality and Fubini's theorem, we can write

$$\mathbb{E}\big(\log\widetilde{\mathcal{Z}}^*_{0,t;\beta}\,\big)\ \leq\ \log\mathbf{E}\big(\mathbb{E}\big[\exp\big(\mathcal{H}_{0,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)\big)\big]\,,\ d_{t-1}<t\,\big)\,. \tag{2.25}$$

Since the Hamiltonian is a stochastic integral, *cf.* (2.2), for fixed $a<b$ and $\widetilde{\Delta}^\alpha$ we have $\mathcal{H}_{a,b;\beta}(\widetilde{\Delta}^\alpha)\sim\mathcal{N}(\mu,\sigma^2)$, where $\mu=-2\lambda h\int_a^b\widetilde{\Delta}^\alpha(u)\,\mathrm{d}u$ and $\sigma^2=4\lambda^2\int_a^b|\widetilde{\Delta}^\alpha(u)|^2\,\mathrm{d}u$. In particular $|\mu|\leq 2\lambda h(b-a)$ and $\sigma^2\leq 4\lambda^2(b-a)$, hence, on the event $\{d_{t-1}<t\}$, we have $\mathbb{E}\big[\exp\big(\mathcal{H}_{0,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)\big)\big]\leq\exp(2\lambda ht+2\lambda^2t)$, and (2.24) follows.

Finally, the fourth and last condition is that for some (hence every) $T>0$

$$\mathbb{E}\left(\sup_{0\leq s<t\leq T}\big|\log\widetilde{\mathcal{Z}}^*_{s,t;\beta}\big|\right)\ <\ \infty\,. \tag{2.26}$$

We need both a lower and an upper bound on $\widetilde{\mathcal{Z}}^*_{s,t;\beta}$. For the upper bound, directly from (2.16) we have

$$\sup_{0\leq s<t\leq T}\log\widetilde{\mathcal{Z}}^*_{s,t;\beta}\ \leq\ \log\mathbf{E}\big(\exp\big(\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big)\big)\,. \tag{2.27}$$

The lower bound is slightly more involved. The additivity of the Hamiltonian yields $\mathcal{H}_{x,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)=\mathcal{H}_{x,t-1;\beta}(\widetilde{\Delta}^\alpha)+\mathcal{H}_{t-1,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)$. Observing that $\widetilde{\Delta}^\alpha(s)$ is constant for $s\in(t-1,d_{t-1}(\widetilde{\tau}^\alpha))$, from the definition (2.2) of the Hamiltonian we can write

$$\begin{aligned}\mathcal{H}_{t-1,d_{t-1};\beta}(\widetilde{\Delta}^\alpha)\ &\geq\ -2\lambda\,|\beta_{d_{t-1}}-\beta_{t-1}|\ -\ 2\lambda h(d_{t-1}-(t-1))\\ &\geq\ -2\lambda\sup_{0\leq s<t\leq T}|\beta_t-\beta_s|\ -\ 2\lambda hT\ =:\ -C_T(\beta)\,.\end{aligned} \tag{2.28}$$

Recalling (2.12), we can therefore bound $\widetilde{\mathcal{Z}}^*_{s,t;\beta}$ from below by

$$\widetilde{\mathcal{Z}}^*_{s,t;\beta}\ \geq\ e^{-C_T(\beta)}\left(\inf_{x\in[s-1,\min\{s,t-1\}]}\mathbf{E}_x\left(\exp\big(\mathcal{H}_{x,t-1;\beta}(\widetilde{\Delta}^\alpha)\big)\;\middle|\;d_{t-1}<t\,\right)\right)\mathbf{P}_x\big(d_{t-1}<t\big)\,. \tag{2.29}$$

From (2.10) it follows easily that, for fixed T,

$$\inf_{0\leq s<t\leq T}\ \inf_{x\in[s-1,\min\{s,t-1\}]}\mathbf{P}_x\big(d_{t-1}<t\big)\ >\ 0\,. \tag{2.30}$$

Furthermore, we now show that we can replace the law $\mathbf{P}_x(\,\cdot\,|\,d_{t-1}<t\,)$ with $\mathbf{P}_x(\,\cdot\,)$ by paying a positive constant. In fact, the laws of the set $\widetilde{\tau}^\alpha\cap[x,t-1]$ under these two

probability measures are mutually absolutely continuous. The Radon-Nikodym derivative, which depends only on g_{t-1}, is computed with the help of (2.8), (2.9), (2.10) and equals

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_x(\,\cdot\,|\,d_{t-1}<t\,)}{\mathrm{d}\mathbf{P}_x(\,\cdot\,)}\,(\widetilde{\tau}^\alpha\cap[x,t-1]) \;&=\; \frac{\mathbf{P}_x(g_{t-1}\in\mathrm{d}y,\,d_{t-1}<t)}{\mathbf{P}_x(g_{t-1}\in\mathrm{d}y)\,\mathbf{P}_x(d_{t-1}<t)}\bigg|_{y=g_{t-1}} \\ &=\; \bigg(1-\frac{(t-1-g_{t-1})^\alpha}{(t-g_{t-1})^\alpha}\bigg)\cdot\frac{1}{D_{x,t-1}(t)}\,. \end{aligned} \tag{2.31}$$

Using (2.10), it is straightforward to check that, for every fixed T, the infimum of this expression over $0\le s<t\le T$ and $x\in[s-1,\min\{s,t-1\}]$ is strictly positive. Therefore, uniformly in the range of parameters, we have

$$\begin{aligned} \widetilde{\mathcal{Z}}^*_{s,t;\beta} \;&\ge\; (\mathit{const.})\,e^{-C_T(\beta)} \inf_{x\in[s-1,\min\{s,t-1\}]} \mathbf{E}_x\left(\exp\left(\mathcal{H}_{x,t-1;\beta}(\widetilde{\Delta}^\alpha)\right)\right) \\ &\ge\; (\mathit{const.})\,e^{-C_T(\beta)}\,\mathbf{E}\big(\exp\big(-\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big)\big)\,, \end{aligned} \tag{2.32}$$

where we have applied (2.17). By Jensen's inequality we then obtain

$$\inf_{0\le s<t\le T}\ \log\widetilde{\mathcal{Z}}^*_{s,t;\beta} \;\ge\; -\mathbf{E}\big(\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big) \;-\; C_T(\beta) \;+\; (\mathit{const.}')\,. \tag{2.33}$$

Putting together (2.27) and (2.33) we then get

$$\sup_{0\le s<t\le T}\ \big|\log\widetilde{\mathcal{Z}}^*_{s,t;\beta}\big| \;\le\; \log\mathbf{E}\big(\exp\big(\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big)\big) \;+\; \mathbf{E}\big(\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big) \;+\; C_T(\beta) \;+\; (\mathit{const.})\,. \tag{2.34}$$

It is clear from (2.28) that $\mathbb{E}(C_T(\beta))<\infty$, for every $T>0$. Moreover, by Jensen's inequality and (2.18) we have $\mathbb{E}\log\mathbf{E}[\exp(\Theta_T(\beta,\widetilde{\Delta}^\alpha))]\le\log\mathbf{E}[\mathbb{E}[\exp(\Theta_T(\beta,\widetilde{\Delta}^\alpha))]]<\infty$, so that $\mathbf{E}[\mathbb{E}[\Theta_T(\beta,\widetilde{\Delta}^\alpha)]]<\infty$. Therefore (2.26) is proven. □

We finally show that Proposition 2.2 still holds if we replace the modified partition function $\widetilde{\mathcal{Z}}^*_{0,t;\beta}$ with the original partition function $\widetilde{\mathcal{Z}}_{0,t;\beta}$; in particular, the free energy $\widetilde{\mathrm{F}}(\lambda,h)$ is well-defined and coincides with $\widehat{\mathrm{F}}(\lambda,h)$. We first need a technical lemma.

Lemma 2.3. *For every fixed $h\ge 0$, the function $\widehat{\mathrm{F}}(\lambda,h)$ appearing in Proposition 2.2 is a non-decreasing and continuous function of λ.*

Proof. Note that sending $\lambda\to c\lambda$ is the same as multiplying the Hamiltonian by c. By Jensen's inequality, for every $\varepsilon>0$ we have

$$\mathbf{E}_x\big(\exp\big(\mathcal{H}_{x,d_{t-1};\beta}\big)\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{1+\varepsilon} \;\le\; \mathbf{E}_x\big(\exp\big((1+\varepsilon)\mathcal{H}_{x,d_{t-1};\beta}\big)\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\,, \tag{2.35}$$

hence, taking the infimum over $x\in[-1,0]$, then $\frac{1}{t}\mathbb{E}\log(\cdot)$ and letting $t\to\infty$, we obtain $\widehat{\mathrm{F}}((1+\varepsilon)\lambda,h)\ge(1+\varepsilon)\widehat{\mathrm{F}}(\lambda,h)$. In particular, $\lambda\mapsto\widehat{\mathrm{F}}(\lambda,h)$ is non-decreasing for fixed h.

To prove the continuity, we use Hölder's inequality with $p=\frac{1}{1-\varepsilon}$ and $q=\frac{1}{\varepsilon}$, getting

$$\begin{aligned} \mathbf{E}_x\big(e^{(1+\varepsilon)\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big) \;&=\; \mathbf{E}_x\big(e^{(1-\varepsilon)\mathcal{H}_{x,d_{t-1};\beta}}\,e^{2\varepsilon\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big) \\ &\le\; \mathbf{E}_x\big(e^{\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{1-\varepsilon}\,\mathbf{E}_x\big(e^{2\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{\varepsilon}\,. \end{aligned} \tag{2.36}$$

Now observe that by (2.14) and (2.15) we can write

$$\mathbf{E}_x\big(e^{2\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{\varepsilon} \;\le\; \mathbf{E}\big(e^{2\Theta_{t+1}(\beta,\widetilde{\Delta}^\alpha)}\big)^{\varepsilon}\,. \tag{2.37}$$

Taking $\frac{1}{t}\,\mathbb{E}\,\inf_{x\in[-1,0]}\log(\cdot)$ in (2.36), applying Jensen's inequality to the last term, using (2.18) and letting $t\to\infty$ then yields

$$\widehat{\mathrm{F}}((1+\varepsilon)\lambda,h)\;\le\;(1-\varepsilon)\,\widehat{\mathrm{F}}(\lambda,h)+\varepsilon\,D(2)\,,\qquad \text{for every } \lambda,h\ge 0 \text{ and every } \varepsilon>0\,. \tag{2.38}$$

Since $\lambda\mapsto\widehat{\mathrm{F}}(\lambda,h)$ is non-decreasing, this shows that $\lambda\mapsto\widehat{\mathrm{F}}(\lambda,h)$ is continuous. □

We now pass from $\widetilde{\mathcal{Z}}^*_{0,t;\beta}$ to the original partition function $\widetilde{\mathcal{Z}}_{0,t;\beta}$ in three steps: first we remove the infimum over $x\in[-1,0]$, then we replace $\mathcal{H}_{0,d_{t-1};\beta}$ with $\mathcal{H}_{0,t-1;\beta}$ and finally we remove the event $\{d_{t-1}<t\}$. From now till the end of the proof we assume $t\ge 1$.

Step 1. It follows from the regenerative property of $\widetilde{\tau}^\alpha$ that the laws of the random set $\widetilde{\tau}^\alpha\cap[1,\infty)$ under the probabilities $\mathbf{P}=\mathbf{P}_0$ and $\mathbf{P}_x$, with $x\in[-1,0]$, are mutually absolutely continuous, with Radon-Nikodym derivative depending only on d_1, given by

$$\frac{\mathrm{d}\mathbf{P}(\widetilde{\tau}^\alpha\cap[1,\infty)\in\,\cdot\,)}{\mathrm{d}\mathbf{P}_x(\widetilde{\tau}^\alpha\cap[1,\infty)\in\,\cdot\,)}\;=\;\frac{\mathbf{P}(d_1\in\mathrm{d}z)}{\mathbf{P}_x(d_1\in\mathrm{d}z)}\bigg|_{z=d_1}\;=\;\frac{1}{(1-x)^\alpha}\,\frac{d_1}{d_1-x}\,. \tag{2.39}$$

It is clear that, uniformly on $x\in[-1,0]$, this expression is bounded from above by some constant $0<C<\infty$. Therefore, for every $\varepsilon>0$, by the Hölder inequality with $p=\frac{1+\varepsilon}{\varepsilon}$ and $q=1+\varepsilon$ we can write

$$\begin{aligned}
\mathbf{E}\big(e^{\mathcal{H}_{0,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\;&=\;\mathbf{E}\big(e^{\mathcal{H}_{0,1;\beta}+\mathcal{H}_{1,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\\
&\le\;\mathbf{E}\big(e^{\frac{1+\varepsilon}{\varepsilon}\mathcal{H}_{0,1;\beta}}\big)^{\frac{\varepsilon}{1+\varepsilon}}\,\mathbf{E}\big(e^{(1+\varepsilon)\mathcal{H}_{1,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{1/(1+\varepsilon)}\\
&\le\;\mathbf{E}\big(e^{\frac{1+\varepsilon}{\varepsilon}\mathcal{H}_{0,1;\beta}}\big)^{\frac{\varepsilon}{1+\varepsilon}}\,C^{1/(1+\varepsilon)}\,\inf_{x\in[-1,0]}\mathbf{E}_x\big(e^{(1+\varepsilon)\mathcal{H}_{1,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{1/(1+\varepsilon)}\,.
\end{aligned} \tag{2.40}$$

Analogously, again by the Hölder inequality, we have

$$\begin{aligned}
\mathbf{E}_x\big(e^{(1+\varepsilon)\mathcal{H}_{1,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\;&=\;\mathbf{E}_x\big(e^{(1+\varepsilon)(\mathcal{H}_{x,d_{t-1};\beta}-\mathcal{H}_{x,1;\beta})}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\\
&\le\;\mathbf{E}_x\big(e^{-\frac{(1+\varepsilon)^2}{\varepsilon}\mathcal{H}_{x,1;\beta}}\big)^{\frac{\varepsilon}{(1+\varepsilon)}}\,\mathbf{E}_x\big(e^{(1+\varepsilon)^2\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)^{1/(1+\varepsilon)}\,.
\end{aligned} \tag{2.41}$$

However $\mathbf{E}_x\big(e^{-\frac{(1+\varepsilon)^2}{\varepsilon}\mathcal{H}_{x,1;\beta}}\big)\le\mathbf{E}\big(e^{\frac{(1+\varepsilon)^2}{\varepsilon}\Theta_2(\beta,\widetilde{\Delta}^\alpha)}\big)$, by (2.14) and (2.15). Putting together these relations, Proposition 2.2 and (2.18), we get $\mathbb{P}(\mathrm{d}\beta)$-a.s.

$$\begin{aligned}
&\limsup_{t\to\infty}\;\frac{1}{t}\,\log\,\mathbf{E}\big(e^{\mathcal{H}_{0,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\\
&\le\;\frac{1}{(1+\varepsilon)^2}\,\limsup_{t\to\infty}\,\frac{1}{t}\,\log\,\inf_{x\in[-1,0]}\,\mathbf{E}_x\big(e^{(1+\varepsilon)^2\mathcal{H}_{x,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\;=\;\frac{\widehat{\mathrm{F}}((1+\varepsilon)^2\lambda,h)}{(1+\varepsilon)^2}\,,
\end{aligned} \tag{2.42}$$

and since $\varepsilon>0$ is arbitrary, by Lemma 2.3 the left-hand side in (2.42) does not exceed $\widehat{\mathrm{F}}(\lambda,h)$. By the definition (2.12) of $\widetilde{\mathcal{Z}}^*_{0,t;\beta}$, we have immediately an analogous lower bound for the lim inf, hence we have proven that $\mathbb{P}(\mathrm{d}\beta)$-a.s.

$$\lim_{t\to\infty}\,\frac{1}{t}\,\log\,\mathbf{E}\big(e^{\mathcal{H}_{0,d_{t-1};\beta}}\,\mathbf{1}_{\{d_{t-1}<t\}}\big)\;=\;\widehat{\mathrm{F}}(\lambda,h)\,. \tag{2.43}$$

Furthermore, the convergence holds also in $L^1(\mathbb{P})$, because the sequence in the l.h.s. is uniformly integrable, as it follows from the bounds we have obtained.

Step 2. With analogous arguments, we now show that we can replace $\mathcal{H}_{0,d_{t-1};\beta}$ with $\mathcal{H}_{0,t-1;\beta}$ in (2.43), that is, the following limit holds, $\mathbb{P}(\mathrm{d}\beta)$-a.s. and in $L^1(\mathrm{d}\mathbb{P})$:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \;=\; \widehat{\mathrm{F}}(\lambda,h)\,. \tag{2.44}$$

Since $\mathcal{H}_{0,d_{t-1};\beta} = \mathcal{H}_{0,t-1;\beta} + \mathcal{H}_{t-1,d_{t-1};\beta}$, for every $\varepsilon > 0$ we can write

$$\mathbf{E}\big(e^{(1-\varepsilon)\mathcal{H}_{0,d_{t-1};\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \;\le\; \mathbf{E}\big(e^{\frac{1-\varepsilon}{\varepsilon}\mathcal{H}_{t-1,d_{t-1};\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big)^{\varepsilon}\, \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big)^{1-\varepsilon}\,, \tag{2.45}$$

and analogously

$$\begin{aligned} &\mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \\ &\quad\le\; \mathbf{E}\big(e^{-\frac{1+\varepsilon}{\varepsilon}\mathcal{H}_{t-1,d_{t-1};\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big)^{\frac{\varepsilon}{1+\varepsilon}}\, \mathbf{E}\big(e^{(1+\varepsilon)\mathcal{H}_{0,d_{t-1};\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big)^{\frac{1}{1+\varepsilon}}\,. \end{aligned} \tag{2.46}$$

Now notice that, by the definition (2.2), since $\widetilde{\tau}^\alpha \cap (t-1, d_{t-1}) = \emptyset$, we can write

$$\big|\mathcal{H}_{t-1,d_{t-1};\beta}\big| \;\le\; 2\lambda\left(|\beta_{d_{t-1}} - \beta_{t-1}| + h(d_{t-1} - (t-1))\right), \tag{2.47}$$

from which it follows easily that $\mathbb{P}(\mathrm{d}\beta)$-a.s. and in $L^1(\mathrm{d}\mathbb{P})$

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big(e^{\gamma\,|\mathcal{H}_{t-1,d_{t-1};\beta}|} \mathbf{1}_{\{d_{t-1}<t\}}\big) \;=\; 0\,, \qquad \forall \gamma \ge 0\,. \tag{2.48}$$

From (2.45), (2.46) and (2.43) we then have $\mathbb{P}(\mathrm{d}\beta)$-a.s.

$$\begin{aligned} \frac{\widehat{\mathrm{F}}((1-\varepsilon)\lambda,h)}{1-\varepsilon} \;&\le\; \liminf_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \\ &\le\; \limsup_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \;\le\; \frac{\widehat{\mathrm{F}}((1+\varepsilon)\lambda,h)}{1+\varepsilon}\,. \end{aligned}$$

Letting $\varepsilon \to 0$ and using Lemma 2.3, we see that (2.44) holds $\mathbb{P}(\mathrm{d}\beta)$-a.s. and also in $L^1(\mathrm{d}\mathbb{P})$, thanks to the bounds (2.45), (2.46) and (2.47) that ensure the uniform integrability.

Step 3. We finally show that we can remove the indicator function $\mathbf{1}_{\{d_{t-1}<t\}}$ from equation (2.44). We have already observed that the laws of $\widetilde{\tau}^\alpha \cap [0, t-1]$ under the two probabilities $\mathbf{P}(\,\cdot\,|d_{t-1} < t)$ and $\mathbf{P}$ are mutually absolutely continuous: the corresponding Radon-Nikodym derivative $f_t = f_t(g_{t-1})$ is given by (2.31), from which we extract the bound

$$f_t(g_{t-1}) \;\ge\; 1 - \frac{(t-g_{t-1}-1)^\alpha}{(t-g_{t-1})^\alpha} \;\ge\; 1 - \frac{(t-1)^\alpha}{t^\alpha} \;\ge\; \frac{\alpha}{t}\,, \tag{2.49}$$

where the last inequality holds for large t. Therefore for large t

$$\mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \;=\; \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}}\big|d_{t-1}<t\big)\, \mathbf{P}(d_{t-1}<t) \;\ge\; \frac{\alpha}{t}\, \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}}\big)\, \mathbf{P}(d_{t-1}<t)\,, \tag{2.50}$$

and note that $\mathbf{P}(d_{t-1} < t) = G_{0,t-1}(t) \sim (const.)/t^{1-\alpha}$ as $t \to \infty$, by (2.10). Therefore

$$\mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big) \;\le\; \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}}\big) \;\le\; (const.)\, t^{2-\alpha}\, \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}} \mathbf{1}_{\{d_{t-1}<t\}}\big)\,, \tag{2.51}$$

for large t, hence by (2.44) it follows that, $\mathbb{P}(\mathrm{d}\beta)$-a.s. and in $L^1(\mathrm{d}\mathbb{P})$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big(e^{\mathcal{H}_{0,t-1;\beta}}\big) \;=\; \widehat{\mathrm{F}}(\lambda,h)\,. \tag{2.52}$$

Replacing $\frac{1}{t}$ with $\frac{1}{t-1}$ in the l.h.s. shows that the free energy $\widetilde{\mathrm{F}}(\lambda,h)$, defined as the limit in (1.24), does exist and coincides with $\widehat{\mathrm{F}}(\lambda,h)$ (we recall that $\widetilde{Z}^\alpha_{t,\beta} = \widetilde{\mathcal{Z}}_{0,t;\beta}$).

To complete the proof of Theorem 1.4, it only remains to show that the free energy $\widetilde{\mathrm{F}}(\lambda, h)$ is non-negative and continuous. By restricting, for $t > 1$, the expectation that defines $\widetilde{Z}_{0,t;\beta}$ to the event $E_t := \{d_1 > t, \widetilde{\Delta}^\alpha(\frac{t+1}{2}) = 0\} = \{d_1 > t, \widetilde{B}^\alpha(\frac{t+1}{2}) > 0\}$ and by using Jensen inequality, we have

$$\frac{1}{t}\mathbb{E}\log\widetilde{\mathcal{Z}}_{0,t;\beta} \;\geq\; \frac{1}{t}\mathbf{E}\left[\mathbb{E}[\mathcal{H}_{0,1;\beta}(\widetilde{\Delta}^\alpha)]\Big|E_t\right] + \frac{1}{t}\log\mathbf{P}(E_t) \;\geq\; -\frac{2\lambda h}{t} + \frac{1}{t}\log\mathbf{P}(E_t). \tag{2.53}$$

By (2.10) we have $\mathbf{P}(E_t) \overset{t\to\infty}{\sim} (const.)\, t^{-\alpha}$ so that the right-most side in (2.53) vanishes as $t \to \infty$ and therefore $\widetilde{\mathrm{F}}(\lambda, h) \geq 0$.

For the continuity, it is convenient to use a different parametrization. For $t > 0$ and $a, b \in \mathbb{R}$ we set

$$\mathrm{G}_t(a,b) \;:=\; \frac{1}{t}\,\mathbb{E}\left[\log\mathbf{E}\left[\exp\left(-2\int_0^t \widetilde{\Delta}^\alpha(u)\big(a\,\mathrm{d}\beta(u) + b\,\mathrm{d}u\big)\right)\right]\right]. \tag{2.54}$$

Since the argument of the exponential is a bilinear function of (a, b), it is easily checked, using Hölder's inequality, that for every fixed $t > 0$ the function $(a, b) \mapsto \mathrm{G}_t(a, b)$ is convex on $\mathbb{R}^2$. By a straightforward adaptation of the results proven in this section, the limit

$$\mathrm{G}(a,b) \;:=\; \lim_{t\to\infty}\mathrm{G}_t(a,b) \tag{2.55}$$

exists and is finite, for all $a, b \in \mathbb{R}$. For instance, for $a > 0$ and $b \geq 0$, by (2.3) and (2.2) we have $\mathrm{G}_t(a, b) = \widetilde{\mathrm{F}}_t(a, b/a)$, therefore the limit in (2.55) exists and equals $\widetilde{\mathrm{F}}(a, b/a)$; the restriction to $a > 0$ and $b \geq 0$ is however not necessary for the existence of such a limit.

Being the pointwise limit of convex functions, $\mathrm{G}(a, b)$ is convex too on $\mathbb{R}^2$, hence continuous (because finite). Therefore $\widetilde{\mathrm{F}}(\lambda, h) = \mathrm{G}(\lambda, \lambda h)$ is continuous too on $[0,\infty)\times[0,\infty)$. □

3. The proof of the main result

We fix an arbitrary value of $\alpha \in (0, 1)$ and an arbitrary discrete α-copolymer model (and we omit α in most of the notations of this section). We aim at proving an analogue of Theorem 6 in [9]. More precisely, we want to show:

Theorem 3.1. *For every choice of $\lambda > 0$ and $h > 0$, and for every choice of $\rho \in (0, 1)$ there exists $a_0 > 0$ such that for every $a \in (0, a_0]$ we have*

$$\widetilde{\mathrm{F}}\left(\frac{\lambda}{1+\rho}, \frac{h}{1-\rho}\right) \;\leq\; \frac{1}{a^2}\,\mathrm{F}(a\lambda, ah) \;\leq\; \widetilde{\mathrm{F}}\big((1+\rho)\lambda, (1-\rho)h\big). \tag{3.1}$$

Theorem 3.1 implies Theorem 1.5. In fact notice that it directly yields (1.26) when both λ and h are positive (by continuity of $\widetilde{\mathrm{F}}(\cdot,\cdot)$). If $\lambda = 0$, there is nothing to prove, because $\mathrm{F}(0, h) = \widetilde{\mathrm{F}}(0, h) = 0$. If $\lambda > 0$ and $h = 0$ instead (1.26) follows because for $h \geq 0$ we have $\mathrm{F}(\lambda, 0) - 2\lambda h \leq \mathrm{F}(\lambda, h) \leq \mathrm{F}(\lambda, 0)$ by (1.5) and (1.11), hence for every $h > 0$

$$\widetilde{\mathrm{F}}(\lambda,h) \;=\; \lim_{a\searrow 0}\frac{1}{a^2}\mathrm{F}(a\lambda, ah) \;\leq\; \liminf_{a\searrow 0}\frac{1}{a^2}\mathrm{F}(a\lambda,0) \;\leq\;$$
$$\limsup_{a\searrow 0}\frac{1}{a^2}\mathrm{F}(a\lambda,0) \;\leq\; \lim_{a\searrow 0}\frac{1}{a^2}\mathrm{F}(a\lambda,ah) + 2\lambda h \;=\; \widetilde{\mathrm{F}}(\lambda,h) + 2\lambda h \tag{3.2}$$

so that (1.26) for $h = 0$ follows by continuity of $\widetilde{\mathrm{F}}(\lambda, \cdot)$. For (1.27), in view of (1.28) it suffices to show that

$$\limsup_{\lambda\searrow 0}\frac{h_c(\lambda)}{\lambda} \;\leq\; \widetilde{\mathrm{m}}_\alpha, \tag{3.3}$$

and Theorem 3.1 does yield (3.3). In fact if $c > \widetilde{\mathrm{m}}_\alpha$, then $\widetilde{\mathrm{F}}((1+\rho)\lambda, (1-\rho)c\lambda) = 0$ for ρ sufficiently small and every $\lambda \geq 0$; the upper bound in (3.1) then yields $\mathrm{F}(a\lambda, ac\lambda) = 0$ for a small, that is $h_c(\lambda) \leq c\lambda$ for λ small, which implies (3.3).

In order to carry out the proof Theorem 3.1 it is convenient to introduce the following basic order relation.

Definition 3.2. *Let $f_{t,\varepsilon,\delta}(a,\lambda,h)$ and $g_{t,\varepsilon,\delta}(a,\lambda,h)$ be two real functions. We write $f \prec g$ if for all fixed $\lambda, h > 0$ and $\rho \in (0,1)$ there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists $a_0 = a_0(\delta, \varepsilon) > 0$ such that for every $0 < a < a_0$*

$$\limsup_{t\to\infty} f_{t,\varepsilon,\delta}(a,\lambda,h) \;\leq\; \limsup_{t\to\infty} g_{t,\varepsilon,\delta}\big(a,(1+\rho)\lambda,(1-\rho)h\big)\,. \tag{3.4}$$

The values $\delta_0, \varepsilon_0, a_0$ may also depend on λ, h, ρ. If both $f \prec g$ and $g \prec f$, we write $f \simeq g$.

Recalling the definitions (1.10) and (1.24) of the discrete and continuum finite-volume free energies $\mathrm{F}_N(\lambda,h)$ and $\widetilde{\mathrm{F}}_t(\lambda,h)$, we set

$$f^0_{t,\varepsilon,\delta}(a,\lambda,h) \;:=\; \frac{1}{a^2}\mathrm{F}_{\lfloor t/a^2\rfloor}(a\lambda, ah)\,, \qquad f^4_{t,\varepsilon,\delta}(a,\lambda,h) \;:=\; \widetilde{\mathrm{F}}_t(\lambda,h)\,, \tag{3.5}$$

(that in fact do not depend on ε, δ and on ε, δ, a). Thanks to Definition 3.2, we see immediately that proving Theorem 3.1 is equivalent to showing that $f^0 \simeq f^4$. Since the relation $\simeq$ is symmetric and transitive, we proceed by successive approximations: more precisely, we are going to prove that

$$f^0 \;\simeq\; f^1 \;\simeq\; f^2 \;\simeq\; f^3 \;\simeq\; f^4\,, \tag{3.6}$$

where $f^i = f^i_{t,\varepsilon,\delta}(a,\lambda,h)$ for $i = 1,2,3$ are suitable intermediate quantities.

The proof is divided into *four steps*, corresponding to the equivalences in (3.6). In each step we will make statements that hold when δ, ε and a are small in the sense prescribed by Definition 3.2, *i.e.*, when $0 < \delta < \delta_0$, $0 < \varepsilon < \varepsilon_0(\delta)$ and $0 < a < a_0(\delta, \varepsilon)$, for a suitable choice of δ_0, $\varepsilon_0(\cdot)$ and $a_0(\cdot,\cdot)$. For brevity, we will refer to this notion of smallness by saying that ε, δ, a are *small in the usual sense*. It is important to keep in mind that

$$t^{-1} \;\ll\; a \;\ll\; \varepsilon \;\ll\; \delta \;\ll\; 1\,. \tag{3.7}$$

At times, we will commit abuse of notation by writing $a_0(\varepsilon)$ or $a_0(\delta)$ to stress on the parameter that enters the specific computation. In order to simplify notationally the proof, we also assume that all the large numbers built with $\delta, \varepsilon, a, t$ that we encounter, such as ε/a^2, δ/ε, t/δ (hence δ/a^2, t/ε, t/a^2, ...), are integers.

Before starting with the proof, let us describe a general scheme that is common to all the four steps. The functions f^i that we consider will always be of the form

$$f^i_{t,\varepsilon,\delta}(a,\lambda,h) \;=\; \frac{1}{t}\,\mathbb{E}\,\log\,\mathbf{E}\big[\exp\big(-2a\lambda\, H^i_{t,\varepsilon,\delta}(a,h)\big)\big]\,, \tag{3.8}$$

for a suitable Hamiltonian $H^i_{t,\varepsilon,\delta}(a,h)$. Now, for $\rho \in (0,1)$, let us write

$$H^i_{t,\varepsilon,\delta}(a,h) \;=\; H^j_{t,\varepsilon,\delta}\big(a,(1-\rho)h\big) \;+\; \Delta H^{(i,j)}_{t,\varepsilon,\delta}(a,h,\rho) \tag{3.9}$$

(this relation is the definition of ΔH). Applying Hölder, Jensen and Fubini, we get

$$\begin{aligned} f^i_{t,\varepsilon,\delta}(a,\lambda,h) \;\le\; & \frac{1}{1+\rho}\, f^j_{t,\varepsilon,\delta}\big(a,(1+\rho)\lambda,(1-\rho)h\big) \\ & +\; \frac{1}{(1+\rho^{-1})t}\, \log \mathbf{E}\mathbb{E}\exp\big(-2a(1+\rho^{-1})\lambda\, \Delta H^{(i,j)}_{t,\varepsilon,\delta}(a,h,\rho)\big)\,. \end{aligned} \tag{3.10}$$

Therefore to prove $f^i \prec f^j$ it suffices to show that for every positive constant A we can choose the parameters δ, ε, a small in the usual sense such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{E}\mathbb{E}\exp\Big(-aA\, \Delta H^{(i,j)}_{t,\varepsilon,\delta}(a,h,\rho)\Big) \;\le\; 0\,. \tag{3.11}$$

Replacing $\Delta H^{(i,j)}$ by $\Delta H^{(j,i)}$ in this relation, we prove that $f^j \prec f^i$ and therefore $f^i \simeq f^j$.

3.1. **Step 1: coarse-graining of the renewal process.** We recall that by definition, see (3.5), (1.10) and (1.6), f^0 is given by

$$f^0_{t,\varepsilon,\delta}(a,\lambda,h) \;:=\; \frac{1}{a^2}\mathrm{F}_{t/a^2}(a\lambda, ah) \;=\; \frac{1}{t}\,\mathbb{E}\,\log\,\mathbf{E}\big[\exp\big(-2a\lambda\, H^0_{t,\varepsilon,\delta}(a,h)\big)\big]\,, \tag{3.12}$$

where H^0 is defined by

$$H^0_{t,\varepsilon,\delta}(a,h) \;=\; \sum_{i=1}^{t/a^2}(\omega_i + ah)\Delta_i\,. \tag{3.13}$$

The purpose of this section is to define a first intermediate approximation f^1 and to show that $f^0 \simeq f^1$, in the sense of Definition 3.2, following the general scheme (3.8)–(3.11).

We recall that the sequence $\Delta_i \in \{0,1\}$ is constant for $i \in \{\tau_j+1, \tau_j+2, \ldots, \tau_{j+1}\}$ and it is chosen by flipping a fair coin. We start by defining, for $j \in \mathbb{N}\cup\{0\}$, the basic coarse-grained blocks

$$I_j \;:=\; \big((j-1)\varepsilon/a^2, j\varepsilon/a^2\big]\,. \tag{3.14}$$

Then we set $\sigma_0 := 0$ and for $k \ge 1$

$$\sigma_k \;:=\; \inf\{j \ge \sigma_{k-1} + (\delta/\varepsilon) \,:\, \tau \cap I_j \neq \emptyset\}\,, \tag{3.15}$$

thus introducing a coarse-grained version σ of the underlying renewal τ that has a resolution of $\varepsilon/a^2 \gg 1$. We say that the block I_j is *visited* if there exists k such that $\sigma_k = j$. We stress that σ is built in such a way that if I_j is visited, we disregard the content of the next $(\delta/\varepsilon) - 1 \gg 1$ blocks, that is we dub them as *not visited* (even if they may contain renewal points). Since we are interested only in the blocks that fall inside the interval $[0, t/a^2]$, we set $m_{t/a^2} := \min\{k \,:\, \sigma_k \ge t/\varepsilon\}$. Moreover for $k \in \mathbb{N}$ we give a notation for the union of blocks between visited sites (that should be interpreted as *coarse-grained excursions*):

$$\bar I_k \;:=\; \left(\bigcup_{j=\sigma_{k-1}+1}^{\sigma_k} I_j\right) \cap \big(0, t/a^2\big]\,. \tag{3.16}$$

Note that $\bar I_k \neq \emptyset$ if and only if $k \le \sigma_{m_{t/a^2}}$; furthermore $(0, t/a^2] = \bigcup_{k=1}^{m_{t/a^2}} \bar I_k$. Each coarse-grained excursion $\bar I_k$ with $1 \le k < m_{t/a^2}$ contains exactly one visited block, namely I_{σ_k}, at its right extremity. The last coarse-grained excursion $\bar I_{m_{t/a^2}}$ may or may not end with a visited block, depending on whether $\sigma_{m_{t/a^2}} = t/\varepsilon$ or $\sigma_{m_{t/a^2}} > t/\varepsilon$.

For $1 \le k < m_{t/a^2}$ we assign a *sign* s_k to the k^{th} coarse-grained excursion by stipulating that it coincides with the *sign* just before the first renewal point in I_{σ_k} (that we call t_k, and $t_0 := 0$), that is we set $s_k := \Delta_{t_k}$. When $k = m_{t/a^2}$ we need to make a distinction: if

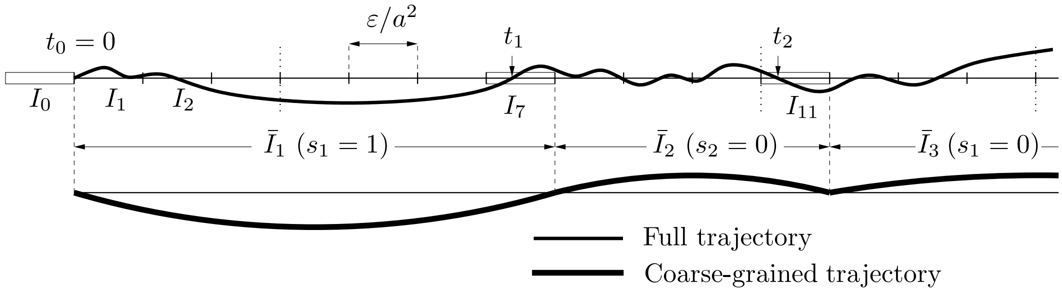

FIGURE 3. A full trajectory, on top, and the corresponding coarse-grained trajectory, below. The visited blocks are surrounded by a box and the first renewal point inside such blocks is marked by a vertical arrow: a coarse-grained excursion is everything that lies between visited blocks. One stipulates that there is a visited block to the left of the origin, containing the origin. The visited block on the right belongs to the coarse-grained excursion, while the one on the left does not. The sign of the excursion is just the sign of the full trajectory just before the vertical arrow (except possibly for the last excursion). In this example $\delta/\varepsilon = 4$, so the first three blocks to the right of a visited block (that is, up to the vertical dotted lines) cannot be visited blocks.

the coarse-grained excursion $\bar{I}_k$ ends with a visited block ($\sigma_{m_{t/a^2}} = t/\varepsilon$) we set $s_k := \Delta_{t_k}$ as before; if the coarse-grained excursion $\bar{I}_k$ is truncated ($\sigma_{m_{t/a^2}} > t/\varepsilon$) we set $s_k = \Delta_{t/a^2}$. We refer to Figure 3 for a graphical description of the quantities introduced so far.

We are now ready to introduce the first intermediate approximation f^1. According to (3.8), it suffice to define the corresponding Hamiltonian:

$$H^1_{t,\varepsilon,\delta}(a,h) := \sum_{k=1}^{m_{t/a^2}} \sum_{i\in\bar{I}_k} (\omega_i + ah) s_k = \sum_{k=1}^{m_{t/a^2}} s_k \left(Z_k(\omega) + ah|\bar{I}_k| \right) , \tag{3.17}$$

where $Z_k(\omega) := \sum_{i\in\bar{I}_k} \omega_i$. Note that we may rewrite H^0, see (3.13), as

$$H^0_{t,\varepsilon,\delta}(a,h) = \sum_{k=1}^{m_{t/a^2}} \sum_{i\in\bar{I}_k} (\omega_i + ah)\Delta_i \,. \tag{3.18}$$

Passing from H^0 to H^1 we are thus replacing the renewal τ by its coarse-grained version. Applying the general scheme (3.8)–(3.10), to prove that $f^0 \simeq f^1$ we have to establish (3.11) for $\Delta H^{(0,1)}$ and $\Delta H^{(1,0)}$, defined by

$$\begin{aligned} \Delta H^{(0,1)}_{t,\varepsilon,\delta}(a,h,\rho) &:= H^0_{t,\varepsilon,\delta}(a,h) - H^1_{t,\varepsilon,\delta}(a,(1-\rho)h) \\ &= a\rho h \sum_{i=1}^{t/a^2} \Delta_i + \sum_{k=1}^{m_{t/a^2}} \sum_{i\in\bar{I}_k} \left(\omega_i + a(1-\rho)h\right)(\Delta_i - s_k) \,, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \Delta H^{(1,0)}_{t,\varepsilon,\delta}(a,h,\rho) &:= H^1_{t,\varepsilon,\delta}(a,h) - H^0_{t,\varepsilon,\delta}(a,(1-\rho)h) \\ &= a\rho h \sum_{i=1}^{t/a^2} \Delta_i + \sum_{k=1}^{m_{t/a^2}} \sum_{i\in\bar{I}_k} (\omega_i + ah)(s_k - \Delta_i) \,. \end{aligned} \tag{3.20}$$

Formulas (3.19) and (3.20) are minimally different: in particular we are going to estimate the second term in the right-hand side by taking the absolute value. For this reason, we detail only the case of (3.19).

In order to establish (3.11) for $\Delta H^{(0,1)}$ we observe that for $a \le t_0/A^2$ (t_0 is the constant in (1.9))

$$\begin{aligned}
\mathbb{E}e^{-Aa\Delta H^{(0,1)}} &= \mathbb{E}\exp\left(-Aa^2\rho h\sum_{i=1}^{t/a^2}\Delta_i - Aa\sum_{k=1}^{m_{t/a^2}}\sum_{i\in \bar{I}_k}(\omega_i + a(1-\rho)h)(\Delta_i - s_k)\right)\\
&= \exp\left(-Aa^2\rho h\sum_{i=1}^{t/a^2}\Delta_i - Aa^2(1-\rho)h\sum_{k=1}^{m_{t/a^2}}\sum_{i\in \bar{I}_k}(\Delta_i - s_k)\right)\prod_{k=1}^{m_{t/a^2}}\prod_{i\in \bar{I}_k}\mathrm{M}\left(Aa\left(\Delta_i - s_k\right)\right)\\
&= \exp\left(-Ca^2\sum_{i=1}^{t/a^2}\Delta_i + Ba^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in \bar{I}_k}|\Delta_i - s_k|\right), \qquad (3.21)
\end{aligned}$$

where $C := A\rho h$ and $B := A(1-\rho)h + c_0A^2$ (we have used (1.9) and the fact that $|\Delta_i - s_k|^2 = |\Delta_i - s_k|$ because $|\Delta_i - s_k| \in \{0,1\}$). This shows that (3.11) is proven if we can show that for any given $B, C > 0$ we have

$$\limsup_{t\to\infty}\frac{1}{t}\log \mathbf{E}\exp\left(-Ca^2\sum_{i=1}^{t/a^2}\Delta_i + Ba^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in \bar{I}_k}|\Delta_i - s_k|\right) \le 0, \qquad (3.22)$$

for δ, ε, a small in the usual sense (recall the discussion before (3.7)).

Let us re-express (3.22) explicitly in terms of the renewal process τ and of the signs $\xi = \{\xi_j\}_{j\in\mathbb{N}}$, where $\xi_j = \Delta_{\tau_j}$. This notation has been already introduced in § 1.1: here we need also $\mathcal{N}_s := |\tau \cap [0,s]| = \min\{k \ge 1 : \tau_k > s\}$ ($s \in \mathbb{N}$). Observe that ξ is an IID sequence, as well as the sequence of the inter-arrivals $\{\eta_j := \tau_j - \tau_{j-1}\}_{j\in\mathbb{N}}$. First of all

$$\sum_{i=1}^{t/a^2}\Delta_i = \sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\xi_j\eta_j + \xi_{\mathcal{N}_{t/a^2}}\left((t/a^2) - \tau_{\mathcal{N}_{t/a^2}-1}\right) \ge \sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\xi_j\eta_j\,. \qquad (3.23)$$

For what concerns the second addendum in the exponent in (3.22), we use the fact that if $\eta_j = \tau_j - \tau_{j-1} \ge (\delta/\varepsilon)(\varepsilon/a^2) = \delta/a^2$ then necessarily the inter-arrival η_j determines a coarse-grained excursion (say, $\bar{I}_k$). This means that $\Delta_i = s_k$ for every $i \in \{\tau_{j-1}+1, \ldots, \tau_j\}$, except if $\tau_{j-1} \in I_{\sigma_{k-1}}$, in which case we cannot guarantee that $\Delta_i = s_k$ for $i \in I_{\sigma_{k-1}}$. This leads to the bound

$$\sum_{k=1}^{m_{t/a^2}}\sum_{i\in \bar{I}_k}|\Delta_i - s_k| \le \sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\eta_j\mathbf{1}_{\eta_j<\delta/a^2} + \frac{\varepsilon}{a^2}m_{t/a^2}\,. \qquad (3.24)$$

This step of the proof is therefore completed by applying the following lemma:

Lemma 3.3. *For every $B, C > 0$ we have*

$$\limsup_{t\to\infty}\frac{1}{t}\log \mathbf{E}\exp\left(Ba^2\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\eta_j\mathbf{1}_{\eta_j<\delta/a^2} + B\varepsilon m_{t/a^2} - Ca^2\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\xi_j\eta_j\right) \le 0, \qquad (3.25)$$

for δ, ε and a small in the usual sense.

Proof. Since ξ and η are independent and since ξ is an IID sequence of $B(1/2)$ variables

$$\mathbf{E}\exp\left(Ba^2\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\eta_j\mathbf{1}_{\eta_j<\delta/a^2}\,+\,B\varepsilon m_{t/a^2}\,-\,Ca^2\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\xi_j\eta_j\right)\;=$$
$$\mathbf{E}\exp\left(Ba^2\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\eta_j\mathbf{1}_{\eta_j<\delta/a^2}\,+\,B\varepsilon m_{t/a^2}\,+\,\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_j\right)\right)\right), \tag{3.26}$$

The proof now proceeds in two steps: first we will show that if δ, ε and a are small in the usual sense

$$B\varepsilon m_{t/a^2}+\frac{1}{2}\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_j\right)\right)\;\le\;B\varepsilon, \tag{3.27}$$

uniformly in η, and then that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbf{E}\left(Ba^2\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\eta_j\mathbf{1}_{\eta_j<\delta/a^2}\,+\,\frac{1}{2}\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_j\right)\right)\right)\;\le\;0. \tag{3.28}$$

For the proof of (3.27), recall first that t_k is the first contact in I_{σ_k} for $k<m_{t/a^2}$, *i.e.* $t_k:=\min\{n\in I_{\sigma_k}\,:\,n\in\tau\}$. Now let us consider the intervals $(t_{k-1},t_k]$ for $k=1,\ldots,m_{t/a^2}-1$ $(t_0:=0)$: given a value of k

(1) either in $(t_{k-1},t_k]$ there is a *long excursion*, that is there exists j^* such that $(\tau_{j^*-1},\tau_{j^*}]\subset(t_{k-1},t_k]$ with $\tau_{j^*}-\tau_{j^*-1}\ge\delta/a^2$, so that

$$B\varepsilon+\frac{1}{2}\sum_{j:\,(\tau_{j-1},\tau_j]\subset(t_{k-1},t_k]}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_j\right)\right)$$
$$\le\;B\varepsilon+\frac{1}{2}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_{j^*}\right)\right)\;\le\;B\varepsilon+\frac{1}{2}\log\left(\frac{1}{2}+\frac{1}{2}e^{-C\delta}\right)\;\le\;0\,, \tag{3.29}$$

where the last inequality holds for $\varepsilon\le\varepsilon_0(\delta)$;

(2) or in $(t_{k-1},t_k]$ there are only *short excursions*, that is $\eta_j:=\tau_j-\tau_{j-1}<\delta/a^2$ for every j such that $(\tau_{j-1},\tau_j]\subset(t_{k-1},t_k]$. In this case we bound from above $\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_j\right)\right)$ by $-\frac{1}{4}Ca^2\eta_j$ for $\delta\le\delta_0$, so that

$$B\varepsilon+\frac{1}{2}\sum_{j:\,(\tau_{j-1},\tau_j]\subset(t_{k-1},t_k]}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-Ca^2\eta_j\right)\right)\;\le\;B\varepsilon-\frac{1}{8}Ca^2(t_k-t_{k-1})\;\le\;0\,, \tag{3.30}$$

where the last inequality holds for $\varepsilon\le\varepsilon_0(\delta)$ and it follows by observing that $t_k-t_{k-1}>((\delta/\varepsilon)-1)(\varepsilon/a^2)=(\delta-\varepsilon)/a^2$.

Summing (3.29) and (3.30) from $k=1$ to $k=m_{t/a^2}-1$, we see that (3.27) holds true.

Let us therefore turn to (3.28): note that we need to estimate

$$\frac{1}{t}\log\mathbf{E}\exp\left(\sum_{j=1}^{\mathcal{N}_{t/a^2}-1}g(a^2\eta_j)\right)\quad\text{with}\quad g(x)\,:=\,Bx\mathbf{1}_{x<\delta}+\frac{1}{2}\log\left(\frac{1}{2}+\frac{1}{2}e^{-Cx}\right)\,. \tag{3.31}$$

Since $g(\cdot) \ge -\frac{1}{2}\log 2$, we can add the term $j = \mathcal{N}_{t/a^2}$ by paying at most $\sqrt{2}$, that is

$$\mathbf{E}\exp\left(\sum_{j=1}^{\mathcal{N}_{t/a^2}-1} g(a^2\eta_j)\right) \le \sqrt{2}\,\mathbf{E}\big[G_{\mathcal{N}_{t/a^2}}\big], \quad \text{where } G_n := \exp\left(\sum_{j=1}^{n} g(a^2\eta_j)\right). \tag{3.32}$$

Let us set $G_0 := 1$ and $\gamma := \mathbf{E}[\exp(g(a^2\eta_1))]$ for convenience. Since G_n is the product of n IID random variables, the process $\{G_n/\gamma^n\}_{n\ge0}$ is a martingale (with respect to the natural filtration of the sequence $\{\tau_n\}_{n\ge0}$). Assume now that $\gamma \le 1$: the process $\{G_n\}_{n\ge0}$ is a supermartingale and, since $\mathcal{N}_{t/a^2}$ is a bounded stopping time, the optional sampling theorem yields $\mathbf{E}[G_{\mathcal{N}_{t/a^2}}] \le 1$. Then from (3.32) it follows immediately that (3.28) holds, completing thus the proof.

We are left with showing that $\gamma \le 1$, that is $\mathbf{E}[\exp(g(a^2\eta_1)] \le 1$, when δ, ε and a are small in the usual sense (actually ε does not appear in this quantity). Note that

$$\mathbf{E}[\exp(g(a^2\eta_1)] - 1 = \sum_{n\in\mathbb{N}} \big[\exp(g(a^2 n)) - 1\big]\, K(n), \tag{3.33}$$

and recall that $K(n) \sim L(n)/n^{1+\alpha}$ as $n \to \infty$, with $L(\cdot)$ slowly varying at infinity. Then it follows by Riemann sum approximation that

$$\lim_{a\searrow 0} \frac{\mathbf{E}[\exp(g(a^2\eta_1)] - 1}{a^{2\alpha}L(1/a^2)} = \int_0^\infty \left[\exp\left(Bx\mathbf{1}_{x<\delta} + \frac{1}{2}\log\left(\frac{1}{2} + \frac{1}{2}e^{-Cx}\right)\right) - 1\right]\frac{\mathrm{d}x}{x^{1+\alpha}}. \tag{3.34}$$

The Riemann sum approximation is justified since $L(cn)/L(n) \to 1$ as $n \to \infty$ uniformly for c in compact sets of $(0,\infty)$ [4, Th. 1.2.1] and since for every $\epsilon > 0$ there exists $b > 0$ such that $L(n) \le bn^\epsilon$ for every n (the latter property is used to deal with very large and small values of n). A simple look at (3.34) suffices to see that the right-hand side is negative if $\delta \le \delta_0$. □

3.2. **Step 2: switching to Gaussian charges.** In this step we introduce the second intermediate approximation f^2: following (3.8), we define the corresponding Hamiltonian H^2 by

$$H^2_{t,\varepsilon,\delta}(a,h) := \sum_{k=1}^{m_{t/a^2}} s_k\left(Z_k(\widehat{\omega}) + ah|\bar{I}_k|\right), \tag{3.35}$$

where $\widehat{\omega} = \{\widehat{\omega}_i\}_{i\in\mathbb{N}}$ is an IID sequence of standard Gaussian random variables and we recall that $Z_k(\widehat{\omega}) := \sum_{i\in\bar{I}_k}\widehat{\omega}_i$. We stress that, with respect to the preceding Hamiltonian H^1, *cf.* (3.17), we have just changed the charges $\omega_i \to \widehat{\omega}_i$.

In order to apply the general scheme (3.8)–(3.11), we build the two sequences of disorder variables $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ and $\widehat{\omega} = \{\widehat{\omega}_i\}_{i\in\mathbb{N}}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is we define a *coupling*. Actually, the disorder does not appear any longer in terms of the individual charges ω_i, but it is by now summed over the coarse-grained blocks $I_j = ((j-1)\frac{\varepsilon}{a^2}, j\frac{\varepsilon}{a^2}]$, so we just need to couple the two IID sequences $\{\sum_{i\in I_j}\omega_i\}_{j\in\mathbb{N}}$ and $\{\sum_{i\in I_j}\widehat{\omega}_i\}_{j\in\mathbb{N}}$. The coupling is achieved via the standard Skorohod representation in the following way: given the IID sequence $\{\widehat{\omega}_i\}_{i\in\mathbb{N}}$ of $\mathcal{N}(0,1)$ variables, if we set $\widehat{F}(t) := \mathbb{P}(\widehat{\omega}_1 \le t)$ and $n := |I_1|$, then $\widehat{F}\left(\sum_{i\in I_j}\widehat{\omega}_i/\sqrt{n}\right) =: U_j$ is uniformly distributed over $(0,1)$. Therefore if we set $F_n(t) := \mathbb{P}(\sum_{i\in I_j}\omega_i/\sqrt{n} \le t)$ and $F_n^{-1}(s) := \inf\{t \in \mathbb{R} : F_n(t) > s\}$, that is F_n^{-1} is the generalized inverse of F_n, then the sequence $\{F_n^{-1}(U_j)\}_{j\in\mathbb{N}}$ has the same law as

$\{\sum_{i\in I_j}\omega_i/\sqrt{n}\}_{j\in\mathbb{N}}$ and we have built a coupling. For short we set $X_j^{(n)} := F_n^{-1}(U_j)$ and $Y_j := \widehat{F}^{-1}(U_j) = \sum_{i\in I_j}\widehat{\omega}_i/\sqrt{n}$. Moreover we observe that, by the Central Limit Theorem, $\lim_{n\to\infty} F_n(t) = \widehat{F}(t)$ for every $t\in\mathbb{R}$ and therefore $\lim_{n\to\infty} X_j^{(n)} = Y_j$, in $\mathbb{P}$-probability.

Lemma 3.4. *For every $C>0$*

$$\lim_{n\to\infty}\mathbb{E}\left[\exp\left(C\left|X_1^{(n)}-Y_1\right|\right)\right] \;=\; 1. \tag{3.36}$$

Proof. Since $\lim_{n\to\infty} X_1^{(n)} = Y_1$ in probability it suffices to prove that the sequence of random variables $\{\exp(C|X_1^{(n)}-Y_1|)\}_{n\in n_0+\mathbb{N}}$ is bounded in L^2 (hence uniformly integrable) for a given $n_0\in\mathbb{N}$ (we choose n_0 to be the smallest integer number larger than $16C^2/t_0^2$, with t_0 the constant in (1.9)). But this follows by the triangle and Cauchy-Schwarz inequalities

$$\sup_{n>n_0}\mathbb{E}\left[\exp\left(2C\left|X_1^{(n)}-Y_1\right|\right)\right] \le \sqrt{\left(\sup_{n>n_0}\mathbb{E}\left[\exp\left(4C\left|X_1^{(n)}\right|\right)\right]\right)\mathbb{E}\big[\exp\left(4C\left|Y_1\right|\right)\big]} < \infty\,, \tag{3.37}$$

where the second inequality follows from (1.9) and the choice of n_0, recalling that $X_1^{(n)} \sim \sum_{i=1}^n \omega_i/\sqrt{n}$ and $Y_1\sim\mathcal{N}(0,1)$. □

Let us see why Lemma 3.4 implies $f^1\simeq f^2$. First of all

$$\begin{aligned}
\min\big(H^1_{t,\varepsilon,\delta}(a,h) - H^2_{t,\varepsilon,\delta}(a,(1-\rho)h)\,,\, H^2_{t,\varepsilon,\delta}(a,h) - H^1_{t,\varepsilon,\delta}(a,(1-\rho)h)\big)& \\
\ge\; -\sum_{k=1}^{m_{t/a^2}} s_k\left|Z_k(\omega)-Z_k(\widehat{\omega})\right| \;+\; a\rho h\sum_{k=1}^{m_{t/a^2}} s_k\left|\bar{I}_k\right|& \\
\ge\; -\sum_{k=1}^{m_{t/a^2}} s_k\sum_{j=\sigma_{k-1}+1}^{\sigma_k}\left|\sum_{i\in I_j}\omega_i \;-\; \sum_{i\in I_j}\widehat{\omega}_i\right| \;+\; a\rho h\sum_{k=1}^{m_{t/a^2}} s_k\left|\bar{I}_k\right|\,,&
\end{aligned}$$

where we redefine $\sigma_{m_{t/a^2}} := t/\varepsilon$ for notational convenience (otherwise we should treat the last term $j=m_{t/a^2}$ separately). In view of (3.9)–(3.11), it suffices to show that for a,ε and δ small in the usual sense (recall the discussion before (3.7)) we have

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\log\mathbf{E}\Bigg[\exp\Bigg(-\,Aa^2\rho h\sum_{k=1}^{m_{t/a^2}} s_k\left|\bar{I}_k\right|\Bigg)\times& \\
\mathbb{E}\Bigg(\exp\Bigg(Aa\sum_{k=1}^{m_{t/a^2}} s_k\sum_{j=\sigma_{k-1}+1}^{\sigma_k}\left(\frac{\sqrt{\varepsilon}}{a}\right)\left|X_j^{(\varepsilon/a^2)}-Y_j\right|\Bigg)\Bigg)\Bigg] \;\le\; 0\,.&
\end{aligned} \tag{3.38}$$

By independence

$$\begin{aligned}
\mathbb{E}\Bigg[\exp\Bigg(Aa\sum_{k=1}^{m_{t/a^2}} s_k\sum_{j=\sigma_{k-1}+1}^{\sigma_k}\left(\frac{\sqrt{\varepsilon}}{a}\right)\left|X_j^{(\varepsilon/a^2)}-Y_j\right|\Bigg)\Bigg]& \\
=\;\prod_{k=1}^{m_{t/a^2}}\mathbb{E}\left[\exp\left(A\sqrt{\varepsilon}s_k\left|X_1^{(\varepsilon/a^2)}-Y_1\right|\right)\right]^{\sigma_k-\sigma_{k-1}}\,,&
\end{aligned} \tag{3.39}$$

and since $a^2|\bar{I}_k| = \varepsilon\,(\sigma_k - \sigma_{k-1})$ the term between square brackets in (3.38) is equal to

$$\prod_{k=1}^{m_{t/a^2}} \Big(\exp\left(-A\rho h s_k \varepsilon\right) \, \mathbb{E}\Big[\exp\Big(A\sqrt{\varepsilon} s_k \, \Big| X_1^{(\varepsilon/a^2)} - Y_1 \Big| \Big) \Big] \Big)^{\sigma_k - \sigma_{k-1}} . \tag{3.40}$$

Since $s_k \in \{0,1\}$, (3.38) is implied by

$$\exp\left(-A\rho h \varepsilon\right) \, \mathbb{E}\Big[\exp\Big(A\sqrt{\varepsilon} \, \Big| X_1^{(\varepsilon/a^2)} - Y_1 \Big| \Big) \Big] \;\leq\; 1 \,, \tag{3.41}$$

which holds for $a \leq a_0(\varepsilon)$ by Lemma 3.4. The proof of $f^1 \simeq f^2$ is complete. □

3.3. **Step 3: from the renewal process to the regenerative set.** In this crucial step we replace the discrete renewal process $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ with the continuum regenerative set $\widetilde{\tau}^\alpha$ (both processes are defined under the law $\mathbf{P}$). Recall that for the renewal process τ we have defined the coarse-grained returns $\{\sigma_k\}_{k\in\mathbb{N}}$ as well as the coarse-grained signs s_k, and $m_{t/a^2} := \inf\{k : \, \sigma_k \geq t/\varepsilon\}$. Henceforth we set $m := m_{t/a^2}$ for short and we redefine for notational convenience $\sigma_m := t/\varepsilon$ (like in the previous step).

Since $\bar{I}_k = (\frac{\varepsilon}{a^2}\sigma_{k-1}, \frac{\varepsilon}{a^2}\sigma_k]$, the second intermediate Hamiltonian H^2, *cf.* (3.35), can be rewritten as

$$H^2_{t,\varepsilon,\delta}(a,h) \;=\; \frac{1}{a} \sum_{k=1}^{m} s_k \left[\left(\sum_{\frac{\varepsilon\sigma_{k-1}}{a^2} < i \leq \frac{\varepsilon\sigma_k}{a^2}} a\widehat{\omega}_i \right) + h\varepsilon(\sigma_k - \sigma_{k-1}) \right] . \tag{3.42}$$

We now introduce the rescaled returns $\underline{\sigma}_k := \varepsilon\sigma_k$ and we let $\beta = \{\beta_t\}_{t\geq 0}$ be a standard Brownian motion, defined on the disorder probability space $(\Omega, \mathcal{F}, \mathbb{P})$. With some abuse of notation, we can redefine H^2 as

$$H^2_{t,\varepsilon,\delta}(a,h) \;=\; \frac{1}{a} \sum_{k=1}^{m} s_k \left(\beta_{\underline{\sigma}_k} - \beta_{\underline{\sigma}_{k-1}} + h(\underline{\sigma}_k - \underline{\sigma}_{k-1}) \right) , \tag{3.43}$$

which has the same law as the quantity in (3.42), hence through formula (3.8) it yields *the same* f^2. It is clear that H^2 depends on the renewal process $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ only through the vector

$$\Sigma \;:=\; (m;\, s_1, \ldots, s_m;\, \underline{\sigma}_1, \ldots, \underline{\sigma}_m) \,, \tag{3.44}$$

whose definition depends of course on $t, a, \varepsilon, \delta$.

One can define an analogous vector $\widetilde{\Sigma}$ in terms of the regenerative set $\widetilde{\tau}^\alpha$, by looking at the returns on blocks of width ε, skipping (δ/ε) blocks between successive returns. More precisely, we set $\widetilde{I}_j := ((j-1)\varepsilon, j\varepsilon]$ for $j \in \mathbb{N}$ and define

$$\widetilde{\underline{\sigma}}_0 := 0 \,, \qquad \widetilde{\underline{\sigma}}_k \;:=\; \varepsilon \cdot \inf\left\{ j \geq (\widetilde{\underline{\sigma}}_{k-1}/\varepsilon) + (\delta/\varepsilon) : \, \widetilde{\tau}^\alpha \cap \widetilde{I}_j \neq \emptyset \right\} , \quad n \in \mathbb{N} \,. \tag{3.45}$$

We then set $\widetilde{m} := \inf\{k \in \mathbb{N} : \, \widetilde{\underline{\sigma}}_k \geq t\}$ and redefine $\widetilde{\underline{\sigma}}_{\widetilde{m}} := t$. The signs $\{\widetilde{s}_k\}_{1\leq k\leq m}$ are defined in complete analogy with the discrete case, by looking at the sign $\widetilde{\Delta}^\alpha$ at the beginning of each visited block $\widetilde{I}_{\widetilde{\sigma}_k}$. We have thus completed the definition of

$$\widetilde{\Sigma} \;:=\; (\widetilde{m};\, \widetilde{s}_1, \ldots, \widetilde{s}_{\widetilde{m}};\, \widetilde{\underline{\sigma}}_1, \ldots, \widetilde{\underline{\sigma}}_{\widetilde{m}}) \,. \tag{3.46}$$

We are ready to introduce the third intermediate quantity f^3, which, in agreement with (3.8), will be defined by the corresponding Hamiltonian H^3. We replace in the right hand side of (3.43) the quantities $m, s_k, \underline{\sigma}_k$ with their continuum analogues $\widetilde{m}, \widetilde{s}_k, \widetilde{\underline{\sigma}}_k$, that is we set

$$H^3_{t,\varepsilon,\delta}(a,h) \;:=\; \frac{1}{a}\sum_{k=1}^{\widetilde{m}} \widetilde{s}_k \left(\beta_{\widetilde{\underline{\sigma}}_k} - \beta_{\widetilde{\underline{\sigma}}_{k-1}} + h(\widetilde{\underline{\sigma}}_k - \widetilde{\underline{\sigma}}_{k-1})\right) . \tag{3.47}$$

It is now convenient to modify slightly the definition (3.43) of H^2. The laws of the vectors Σ and $\widetilde{\Sigma}$ are mutually absolutely continuous (note that they are probability laws on the same finite set) and we denote by $\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}} = \frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}(\widetilde{m}; \widetilde{\underline{\sigma}}_1, \ldots, \widetilde{\underline{\sigma}}_{\widetilde{m}})$ the corresponding Radon-Nikodym derivative, which does not depend on $(\widetilde{s}_1, \ldots, \widetilde{s}_{\widetilde{m}})$: in fact, conditionally on $\widetilde{m}$, $\widetilde{\underline{\sigma}}_1, \ldots, \widetilde{\underline{\sigma}}_{\widetilde{m}}$, the signs $\widetilde{s}_1, \ldots, \widetilde{s}_m$ are IID variables that take the values $\{0, 1\}$ with equal probability, and an analogous statement holds for $s_1, \ldots, s_m$. We then redefine

$$H^2_{t,\varepsilon,\delta}(a,h) \;:=\; H^3_{t,\varepsilon,\delta}(a,h) \;-\; \frac{1}{2a\lambda}\log\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}} . \tag{3.48}$$

Note that this definition of H^2 yields *the same* f^2 as (3.43), according to (3.8).

To prove that $f^2 \simeq f^3$, we can now apply the general scheme (3.8)–(3.11). Plainly,

$$\begin{aligned} \min\Big\{ & H^2_{t,\varepsilon,\delta}(a,h) \;-\; H^3_{t,\varepsilon,\delta}(a,(1-\rho)h),\; H^3_{t,\varepsilon,\delta}(a,h) \;-\; H^2_{t,\varepsilon,\delta}(a,(1-\rho)h)\Big\} \\ & \geq\; -\frac{1}{2a\lambda}\left|\log\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}\right| \;+\; \frac{\rho h}{a}\sum_{k=1}^{\widetilde{m}} \widetilde{s}_k(\widetilde{\underline{\sigma}}_k - \widetilde{\underline{\sigma}}_{k-1}) , \end{aligned} \tag{3.49}$$

therefore, in view of (3.11), we are left with showing that for all $A, B > 0$ and for δ, ε, a small in the usual sense we have

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbf{E}\left[\exp\left(-A\sum_{k=1}^{\widetilde{m}} \widetilde{s}_k\,(\widetilde{\underline{\sigma}}_k - \widetilde{\underline{\sigma}}_{k-1}) \;+\; B\left|\log\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}\right|\right)\right] \;\leq\; 0 . \tag{3.50}$$

We have already observed that, conditionally on $\widetilde{m}, \widetilde{\underline{\sigma}}_1, \ldots, \widetilde{\underline{\sigma}}_{\widetilde{m}}$, the variables $\widetilde{s}_1, \ldots, \widetilde{s}_{\widetilde{m}}$ are IID, taking the values $\{0, 1\}$ with probability $\frac{1}{2}$ each, hence $\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}$ does not depend on these variables. Integrating over $\widetilde{s}_1, \ldots, \widetilde{s}_{\widetilde{m}}$, we can rewrite the expectation in (3.50) as

$$\mathbf{E}\left[\left(\prod_{k=1}^{\widetilde{m}}\left(\frac{1}{2} + \frac{1}{2}\exp(-A(\widetilde{\underline{\sigma}}_k - \widetilde{\underline{\sigma}}_{k-1}))\right)\right)\exp\left(B\left|\log\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}\right|\right)\right] . \tag{3.51}$$

We need some bounds on $\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}$, that are given in the following lemma (whose proof is deferred to Appendix B). Since the result we are after at this stage is for fixed $\delta > 0$, for the sake of simplicity we are going to fix $\delta = 1$: arbitrary values of δ lead to very similar estimates.

Lemma 3.5. *Fix $\delta = 1$. There exists $\kappa(\varepsilon, a) > 0$ with the property that*

$$\lim_{\varepsilon\to 0}\limsup_{a\to 0} \kappa(\varepsilon, a) \;=\; 0 , \tag{3.52}$$

such that, for all values of $\widetilde{m}$, $\widetilde{\underline{\sigma}}_1, \ldots, \widetilde{\underline{\sigma}}_{\widetilde{m}}$, the following bound holds:

$$\left|\log\frac{\mathrm{d}\Sigma}{\mathrm{d}\widetilde{\Sigma}}(\widetilde{m}; \widetilde{\underline{\sigma}}_1, \ldots, \widetilde{\underline{\sigma}}_{\widetilde{m}})\right| \;\leq\; \kappa(\varepsilon, a)\sum_{i=1}^{\widetilde{m}}\big(\log(\widetilde{\underline{\sigma}}_i - \widetilde{\underline{\sigma}}_{i-1}) + 1\big) . \tag{3.53}$$

Note that by definition $(\widetilde{\underline{\sigma}}_i - \widetilde{\underline{\sigma}}_{i-1}) \geq \delta = 1$ and therefore the right hand side of (3.53) is positive. By applying (3.53) we now see that the expression in (3.51) is bounded above by $\mathbf{E}[G_{\widetilde{m}}]$, where for $n \in \mathbb{N}$ we set

$$G_n \;:=\; \prod_{i=1}^{n} \frac{1}{2}\left(1+e^{-A(\widetilde{\underline{\sigma}}_i-\widetilde{\underline{\sigma}}_{i-1})}\right) e^{B\kappa(\varepsilon,a)}\,(\widetilde{\underline{\sigma}}_i-\widetilde{\underline{\sigma}}_{i-1})^{B\kappa(\varepsilon,a)}\,. \tag{3.54}$$

To prove (3.50), completing thus the proof that $f^2 \simeq f^3$, it therefore suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big[G_{\widetilde{m}}\big] \;\leq\; 0\,. \tag{3.55}$$

We recall that $\widetilde{m} = \inf\{k \in \mathbb{N} : \widetilde{\underline{\sigma}}_k \geq t\}$ and that we had redefined $\widetilde{\underline{\sigma}}_{\widetilde{m}} := t$ for notational convenience. It is now convenient to switch back to the natural definition (3.45) of $\widetilde{\underline{\sigma}}_{\widetilde{m}}$. This produces a minor change in $G_{\widetilde{m}}$, see (3.54): in fact, only the last factor in the product is modified, and since $(1+e^{-x}) \leq 2(1+e^{-y})$ for all $x, y \geq 0$, the *new* $G_{\widetilde{m}}$ is at most twice the *old* one. The change is therefore immaterial for the purpose of proving (3.55).

We introduce the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}\cup\{0\}}$, defined by $\mathcal{F}_n := \sigma(\widetilde{\underline{\sigma}}_0, \ldots, \widetilde{\underline{\sigma}}_n)$, and we note that $\widetilde{m}$ is a bounded stopping time for this filtration. Let us set

$$\gamma \;=\; \gamma(\varepsilon,a) \;:=\; \sup_{x\in[-\varepsilon,0]} \mathbf{E}_x\left[\frac{1}{2}\left(1+e^{-A\widetilde{\underline{\sigma}}_1}\right) e^{B\kappa(\varepsilon,a)}\,(\widetilde{\underline{\sigma}}_1)^{B\kappa(\varepsilon,a)}\right], \tag{3.56}$$

where we recall that $\mathbf{P}_x$ denotes the law of the regenerative set started at x, that is $\mathbf{P}_x(\widetilde{\tau}^\alpha \in \cdot) := \mathbf{P}(\widetilde{\tau}^\alpha + x \in \cdot)$. From (3.54) and the regenerative property of $\widetilde{\tau}^\alpha$ we obtain

$$\mathbf{E}\big[G_{n+1}\big|\mathcal{F}_n\big] \;\leq\; \gamma\, G_n\,. \tag{3.57}$$

If $\gamma \leq 1$, this relation shows that the process $\{G_n\}_{n\geq 0}$, with $G_0 := 1$, is a supermartingale. Since $\widetilde{m}$ is a bounded stopping time, from the optional sampling theorem we deduce that $\mathbf{E}[G_{\widetilde{m}}] \leq \mathbf{E}[G_0] = 1$, which clearly yields (3.55).

It only remains to show that indeed $\gamma \leq 1$, provided ε and a are small in the usual sense. Observe that $\widetilde{\underline{\sigma}}_1$, defined in (3.45), is a discretized version of the variable $d_{1-\varepsilon} = d_{1-\varepsilon}(\widetilde{\tau}^\alpha)$, defined in (2.7) (recall that $\delta = 1$): more precisely, $\widetilde{\underline{\sigma}}_1 = \varepsilon\lceil d_{1-\varepsilon}/\varepsilon\rceil$, therefore $d_{1-\varepsilon} \leq \widetilde{\underline{\sigma}}_1 \leq d_{1-\varepsilon} + \varepsilon$. Setting $\kappa := \kappa(\varepsilon, a)$ for short and applying (2.10), we obtain

$$\begin{aligned}\mathbf{E}_x\left[\frac{1}{2}\left(1+e^{-A\widetilde{\underline{\sigma}}_1}\right) e^{B\kappa}\,(\widetilde{\underline{\sigma}}_1)^{B\kappa}\right] \;&\leq\; \mathbf{E}_x\left[\frac{1}{2}\left(1+e^{-Ad_{1-\varepsilon}}\right) e^{B\kappa}\,(d_{1-\varepsilon}+\varepsilon)^{B\kappa}\right]\\ =\; \frac{\sin(\pi\alpha)}{\pi} \int_{1-\varepsilon}^{\infty} &\left[\frac{1}{2}\left(1+e^{-At}\right) e^{B\kappa}\,(t+\varepsilon)^{B\kappa}\right] \frac{((1-\varepsilon)-x)^\alpha}{(t-(1-\varepsilon))^\alpha\,(t-x)}\,\mathrm{d}t\,.\end{aligned} \tag{3.58}$$

Plainly, there exists $\kappa_0 > 0$ such that the integral in (3.58) is finite for $\kappa \in [0, \kappa_0]$, for every $x \in [-\varepsilon, 0]$, and it is in fact a *continuous function* of $(x,\kappa) \in [-\varepsilon, 0] \times [0, \kappa_0]$. Furthermore, the integral is strictly smaller than 1 for $\kappa = 0$ and every $x \in [-\varepsilon, 0]$, as it is clear from the first line of (3.58). Therefore, by continuity, there exists $\kappa_1 \in (0, \kappa_0)$ such that the integral in (3.58) is strictly smaller than one for $(x,\kappa) \in [-\varepsilon, 0] \times [0, \kappa_1]$. Looking back at (3.56), we see that indeed $\gamma \leq 1$ provided $\kappa(\varepsilon, a) \leq \kappa_1$. Thanks to (3.52), it suffices to take ε and a small in the usual sense, and the proof of $f^2 \simeq f^3$ is completed. □

3.4. **Step 4: inverse coarse-graining of the regenerative set.** This step is the close analog of step 1 (*cf.* § 3.1) in the continuum set-up, and a straightforward modification of step 4 in [9]. We will therefore be rather concise.

Recall that the function f^4 is nothing but the continuum finite-volume free energy, *cf.* (3.5), hence according to (3.8) it corresponds to the Hamiltonian (recall (2.2) and (2.3))

$$H^4_{t,\varepsilon,\delta}(a,h) := \frac{1}{a}\int_0^t \widetilde{\Delta}(u)\,(\mathrm{d}\beta(u)+h\mathrm{d}u) \;=\; \frac{1}{a}\sum_{k=1}^{\widetilde{m}}\int_{\underline{\widetilde{\sigma}}_{k-1}}^{\underline{\widetilde{\sigma}}_k} \widetilde{\Delta}(u)\,(\mathrm{d}\beta(u)+h\mathrm{d}u)\;, \tag{3.59}$$

where we have set $\widetilde{\Delta}(u) := \widetilde{\Delta}^\alpha(u)$ for short. As in the third step, we redefine $\underline{\widetilde{\sigma}}_{\widetilde{m}} := t$ for simplicity (otherwise the $k=\widetilde{m}$ term in the sum in (3.59) would require a separate notation), but we will drop this convention later.

We now rewrite $H^3_{t,\varepsilon,\delta}(a,h)$ by introducing the process

$$\widehat{\Delta}(u) := \sum_{k=1}^{\widetilde{m}} \widetilde{s}_k \mathbf{1}_{(\underline{\widetilde{\sigma}}_{k-1},\underline{\widetilde{\sigma}}_k]}(u)\,, \tag{3.60}$$

so that by (3.47) we can write

$$H^3_{t,\varepsilon,\delta}(a,h) \;=\; \frac{1}{a}\sum_{k=1}^{\widetilde{m}}\int_{\underline{\widetilde{\sigma}}_{k-1}}^{\underline{\widetilde{\sigma}}_k} \widehat{\Delta}(u)\,(\mathrm{d}\beta(u)+h\mathrm{d}u)\;. \tag{3.61}$$

Our aim is to show that $f^3 \simeq f^4$, but we prove only $f^4 \prec f^3$, since the argument for the opposite inequality is very similar. We have (recall (3.9))

$$aH^{(4,3)}_{t,\varepsilon,\delta}(a,h,\rho) \;=\; \rho h\sum_{k=1}^{\widetilde{m}}\int_{\underline{\widetilde{\sigma}}_{k-1}}^{\underline{\widetilde{\sigma}}_k} \widehat{\Delta}(u)\,\mathrm{d}u \;+\; \sum_{k=1}^{\widetilde{m}}\int_{\underline{\widetilde{\sigma}}_{k-1}}^{\underline{\widetilde{\sigma}}_k} \Big(\widetilde{\Delta}(u)-\widehat{\Delta}(u)\Big)\,(\mathrm{d}\beta(u)+h\mathrm{d}u)\,, \tag{3.62}$$

and therefore, arguing as in (3.21)–(3.22), it is sufficient to show that for every choice of A and $B>0$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbf{E}\left[\exp\left(A\int_0^t \big|\widetilde{\Delta}(u)-\widehat{\Delta}(u)\big|\mathrm{d}u \;-\; B\sum_{k=1}^{\widetilde{m}}\widetilde{s}_k\left(\underline{\widetilde{\sigma}}_k-\underline{\widetilde{\sigma}}_{k-1}\right)\right)\right] \;\le\; 0\,, \tag{3.63}$$

provided δ and ε are small in the usual sense (note that a has disappeared).

Let us now focus on the union of the excursions of $\widetilde{B}^\alpha$ whose length is shorter than δ and denote the intersection of such a set with $[0,t]$ by $J_{t,\delta}$. Then, in analogy with (3.24), we have the bound

$$\int_0^t \big|\widetilde{\Delta}(u)-\widehat{\Delta}(u)\big|\mathrm{d}u \;\le\; |J_{t,\delta}| \;+\; \widetilde{m}\,\varepsilon\,. \tag{3.64}$$

We now integrate out the $\widetilde{s}$ variables in (3.63) (recall that they are IID $B(1/2)$ variables) and observe that, since $\underline{\widetilde{\sigma}}_k-\underline{\widetilde{\sigma}}_{k-1}\ge\delta$, for every $\delta>0$ there exists ε_0 such that for $\varepsilon\le\varepsilon_0$

$$A\,\widetilde{m}\,\varepsilon \;+\; \frac{1}{2}\sum_{k=1}^{\widetilde{m}}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-B\left(\underline{\widetilde{\sigma}}_k-\underline{\widetilde{\sigma}}_{k-1}\right)\right)\right) \;\le\; 0\,. \tag{3.65}$$

Also notice that, by construction, $|J_{t,\delta}\cap(\underline{\widetilde{\sigma}}_{k-1},\underline{\widetilde{\sigma}}_k]|\le(\delta+\varepsilon)\le 2\delta$ for all $k=1,\ldots,\widetilde{m}$, hence $|J_{t,\delta}|\le 2\delta\widetilde{m}$. Therefore it remains to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbf{E}\left[\exp\left(2A\,\delta\,\widetilde{m} \;+\; \frac{1}{2}\sum_{k=1}^{\widetilde{m}}\log\left(\frac{1}{2}+\frac{1}{2}\exp\left(-B\left(\underline{\widetilde{\sigma}}_k-\underline{\widetilde{\sigma}}_{k-1}\right)\right)\right)\right)\right] \;\le\; 0\,. \tag{3.66}$$

At this point it is practical to go back to the original definition of $\widetilde{\underline{\sigma}}_{\widetilde{m}}$ (*cf.* (3.45)): this produces a change in the exponent of (3.66) which is smaller than $(\log 2)/2$ and this is irrelevant for the estimate we are after. We then rewrite (3.66) as

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big[G_{\widetilde{m}}\big] \;\le\; 0\,, \qquad \text{where} \quad G_n \;:=\; \prod_{i=1}^{n} e^{2A\delta} \sqrt{\frac{1}{2}\Big(1+e^{-B\left(\widetilde{\underline{\sigma}}_k-\widetilde{\underline{\sigma}}_{k-1}\right)}\Big)}\,. \tag{3.67}$$

Let us set

$$\gamma \;=\; \gamma(\delta,\varepsilon) \;=\; \sup_{x\in[-\varepsilon,0]} e^{2A\delta}\, \mathbf{E}_x \left[\sqrt{\frac{1}{2}\left(1+e^{-B\widetilde{\underline{\sigma}}_1}\right)}\right]\,, \tag{3.68}$$

and introduce the filtration $\{\mathcal{F}_n := \sigma(\widetilde{\underline{\sigma}}_0,\ldots,\widetilde{\underline{\sigma}}_n)\}_{n\in\mathbb{N}}$. By the regenerative property of $\widetilde{\tau}^\alpha$, we can write

$$\mathbf{E}\big[G_{n+1}\big|\mathcal{F}_n\big] \;\le\; \gamma\, G_n\,, \tag{3.69}$$

therefore if $\gamma \le 1$ the process $\{G_n\}_{n\ge 0}$, with $G_0 := 0$, is a supermartingale. Since $\widetilde{m}$ is a bounded stopping time, the optional sampling theorem yields $\mathbf{E}[G_{\widetilde{m}}] \le 1$, from which (3.67) follows. We are left with showing that $\alpha \le 1$ if δ and ε are small in the usual sense.

Recall that $d_s = d_s(\widetilde{\tau}) = \inf\{u > s :\, u \in \widetilde{\tau}^\alpha\}$ (*cf.* (2.7)) and observe that, by definition, $\widetilde{\underline{\sigma}}_1 = j\varepsilon$ if and only if $d_{\delta-\varepsilon} \in ((j-1)\varepsilon, j\varepsilon]$ (*cf.* (3.45)). Therefore we may write $\widetilde{\underline{\sigma}}_1 \ge d_{\delta-\varepsilon} \ge d_{\delta-\varepsilon+x}$ for $x \le 0$, whence

$$\mathbf{E}_x \left[\sqrt{\frac{1}{2}\left(1+e^{-B\widetilde{\underline{\sigma}}_1}\right)}\right] \;\le\; \mathbf{E}_x \left[\sqrt{\frac{1}{2}\left(1+e^{-Bd_{\delta-\varepsilon+x}}\right)}\right] \;=\; \mathbf{E}\left[\sqrt{\frac{1}{2}\left(1+e^{-Bd_{\delta-\varepsilon}}\right)}\right]\,. \tag{3.70}$$

Looking back at (3.68), we see that $\gamma \le 1$ if we show that the right hand side of (3.70) is less than $\exp(-2A\delta)$, when δ and ε are small in the usual sense. This condition can be simplified by letting $\varepsilon \searrow 0$: since $d_{\delta-\varepsilon} \to d_\delta$, $\mathbf{P}$-a.s., it suffices to show that

$$\mathbf{E}\left[\sqrt{\frac{1}{2}\left(1+\exp(-Bd_\delta)\right)}\right] \;<\; \exp(-2A\delta)\,, \qquad \text{for all } \delta > 0 \text{ small enough}\,. \tag{3.71}$$

The law of the variable d_δ is given in (2.10), hence with a change of variables we may write

$$\frac{1}{\delta}\left(1-\mathbf{E}\left[\sqrt{\frac{1}{2}\left(1+\exp(-Bd_\delta)\right)}\right]\right) \;=\; \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{1}{\delta}\left[1-\sqrt{\frac{1}{2}\left(1+\exp(-B\delta(1+v))\right)}\right]\frac{\mathrm{d}v}{v^\alpha(1+v)}\,. \tag{3.72}$$

Since the term between square brackets in the right-hand side is positive and asymptotically equivalent, as $\delta \searrow 0$, to $\delta B(1+v)/4$, Fatou's Lemma guarantees that the limit as $\delta \searrow 0$ of the expression in (3.72) is equal to $+\infty$ and this entails that (3.71) holds.

This concludes the proof of step 4 and, hence, the proof of Theorem 3.1. $\square$

Appendix A. Completing the proof of Proposition 2.2

In this section we are going to prove (2.18), that is, for every $\eta \in (0,\infty)$ there exists $D(\eta) \in (0,\infty)$ such that

$$\mathbf{E}\big[\mathbb{E}\big[\exp\big(\eta\,\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big)\big]\big] \;\le\; D(\eta)\, e^{D(\eta)\,T}\,, \qquad \text{for every } T > 0\,. \tag{A.1}$$

We first state some important estimates concerning the regenerative set $\widetilde{\tau}^\alpha$.

A.1. **Regenerative set, excursions and local time.** We recall the basic link between regenerative set and subordinators. Let $(\sigma = \{\sigma_t\}_{t\geq 0}, \mathbf{P})$ denote the stable subordinator of index α, that is the Lévy process with zero drift, zero Brownian component and with Lévy measure given by $\Pi(\mathrm{d}x) := \frac{1}{x^{1+\alpha}}\,\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x$ (we choose as usual a right-continuous version of σ). If we denote by $\Delta\sigma_t := \sigma_{t+} - \sigma_t$ the size of the jump of σ at epoch t, it is well-known that $\sigma_t = \sum_{s\in(0,t]} \Delta\sigma_s$, that is σ increases only by jumps.

We observe that σ is scale invariant: $\{\sigma_{ct}\}_{t\geq 0}$ has the same law as $\{c^{1/\alpha}\sigma_t\}_{t\geq 0}$. Let us also recall some basic estimates, *cf.* Theorems 8.2.1 and 8.2.2 in [4]:

$$\begin{aligned} \mathbf{P}(\sigma_1 > x) \;&=\; \frac{(const.)}{x^\alpha}\big(1+o(1)\big)\,, \qquad \text{as } x\to+\infty\,,\\ \mathbf{P}(\sigma_1 < x) \;&=\; \exp\bigg(-\frac{(const'.)}{x^{\alpha/(1-\alpha)}}\big(1+o(1)\big)\bigg)\,, \qquad \text{as } x\searrow 0\,. \end{aligned} \tag{A.2}$$

If we set $\mathcal{E} := [0,\infty)\times(0,\infty)$, the random set of points $\{(t,\Delta\sigma_t)\}_{t\in[0,\infty)} \cap \mathcal{E}$ (note that we only keep the positive jumps $\Delta\sigma_t > 0$) is a *Poisson random measure* (sometimes simply called *Poisson process*) on $\mathcal{E}$ with intensity measure $\mathrm{d}t\otimes\Pi(\mathrm{d}x)$, where of course $\mathrm{d}t$ denotes the Lebesgue measure. The stochastic process $\{\Delta\sigma_t\}_{t\in[0,\infty)}$ is called a *Poisson point process* on $(0,\infty)$ with intensity measure Π.

The basic link with regenerative sets is as follows: the random closed set of $[0,\infty)$ defined as the closure of the image of the process σ, that is $\overline{\{\sigma_t\}_{t\geq 0}}$, is precisely the α-stable regenerative set $\widetilde{\tau}^\alpha$ we are considering. Therefore the set of jumps $\{\Delta\sigma_t\}_{t\geq 0}$ coincides with the set of widths $\{|I_n|\}_{n\in\mathbb{N}}\cup\{0\}$ of the excursions of $\widetilde{\tau}^\alpha$.

Let us discuss an application of these results that will be useful later. If we denote by $L_t := \inf\{u\geq 0:\, \sigma_u > t\}$ the inverse of σ, known as the *local time* of $\widetilde{\tau}^\alpha$, we may write

$$\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon} \;=\; \sum_{t\in(0,L_2)} (\Delta\sigma_t)^{1-\varepsilon} \;=\; \sum_{t\in(0,L_2)} f(\Delta\sigma_t)\,, \;\; \text{where } f(x) \;:=\; x^{1-\varepsilon}\,\mathbf{1}_{[0,2]}(x)\,, \tag{A.3}$$

therefore for $\lambda > 0$ we have by Cauchy-Schwarz

$$\begin{aligned} \mathbf{E}\Bigg[\exp\Bigg(\lambda \sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon}\Bigg)\Bigg] \;&\leq\; \sum_{m\in\mathbb{N}} \mathbf{E}\Bigg[\exp\Bigg(\lambda\sum_{t\in(0,m)} f(\Delta\sigma_t)\Bigg)\,\mathbf{1}_{\{m-1<L_2\leq m\}}\Bigg]\\ &\leq\; \sum_{m\in\mathbb{N}} \sqrt{\mathbf{E}\Bigg[\exp\Bigg(2\,\lambda\sum_{t\in(0,m)} f(\Delta\sigma_t)\Bigg)\Bigg]\,\mathbf{P}\big[m-1<L_2\leq m\big]}\,. \end{aligned}$$

By the definition of L, the scale invariance of σ and (A.2), we have for some $c>0$

$$\mathbf{P}\big[m-1<L_2\leq m\big] \;\leq\; \mathbf{P}\big[\sigma_{m-1}<2\big] \;=\; \mathbf{P}\bigg[\sigma_1 < \frac{2}{(m-1)^{1/\alpha}}\bigg] \;\leq\; e^{-c\,(m-1)^{1/(1-\alpha)}}\,. \tag{A.4}$$

By Campbell's Formula for Poisson processes (*cf.* equation (3.17) in [19]) we obtain

$$\mathbf{E}\Bigg[\exp\Bigg(2\,\lambda\sum_{t\in(0,m)} f(\Delta\sigma_t)\Bigg)\Bigg] \;=\; \exp\bigg(m\int_0^\infty (e^{2\,\lambda\, f(x)}-1)\,\Pi(\mathrm{d}x)\bigg) \;=\; e^{C(\lambda)\,m}\,,$$

$$\text{where } \; C(\lambda) \;:=\; \int_0^2 \frac{e^{2\,\lambda\, x^{1-\varepsilon}}-1}{x^{1+\alpha}}\,\mathrm{d}x \;<\;\infty \;\; \text{for } \; 0<\varepsilon<1-\alpha\,. \tag{A.5}$$

From the last relations we then obtain, for some $c_1 \in (0,\infty)$,

$$\mathbf{E}\left[\exp\left(\lambda \sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon}\right)\right] \;\le\; \sum_{m\in\mathbb{N}} e^{\frac{1}{2}(C(\lambda)\, m \,-\, c\,(m-1)^{1/(1-\alpha)})} \;\le\; c_1\, e^{c_1\,(C(\lambda))^{1/\alpha}}\,, \tag{A.6}$$

where the last inequality can be checked, e.g., by approximating the sum with an integral and developing the function $e^{\frac{1}{2}[C(\lambda)x - cx^{1/(1-\alpha)}]}$ around its maximum.

Since $e^{2\lambda y} - 1 \le 2\lambda\, e^{4\lambda}\, y$ for $y \in [0,2]$, it follows from (A.5) that $C(\lambda) \le (const.)\, e^{5\,\lambda}$. By Markov's inequality we then obtain

$$\mathbf{P}\left[\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon} \;>\; x\right] \;\le\; c_1\, e^{c_1\,(C(\lambda))^{1/\alpha} \,-\, \lambda x} \;\le\; c_1\, e^{c_2\, e^{5\lambda/\alpha} \,-\, \lambda x}\,, \tag{A.7}$$

for some $c_2 \in (0,\infty)$. Optimizing over λ yields, for every $x > 0$,

$$\mathbf{P}\left[\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon} \;>\; x\right] \;\le\; \min\left\{c_1\, e^{-\frac{\alpha}{5}\, x\, [\log(\frac{\alpha}{5c_2} x) - 1]}\,,\, 1\right\} \;\le\; c_3\, e^{-c_3\, x}\,, \tag{A.8}$$

for a suitable $c_3 \in (0,\infty)$. We can finally estimate the quantity we are interested in:

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\gamma\sqrt{T}\sqrt{\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon}}\right)\right] \\
&= \int_0^\infty \mathbf{P}\left[\exp\left(\gamma\sqrt{T}\sqrt{\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon}}\right) \;>\; t\right] \mathrm{d}t \\
&= \int_0^\infty \mathbf{P}\left[\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon} \;>\; \frac{(\log t)^2}{\gamma^2 T}\right] \mathrm{d}t \\
&\le\; c_3 \int_0^\infty e^{-c_3\,(\log t)^2/(\gamma^2 T)}\, \mathrm{d}t \;=\; c_3 \int_{-\infty}^\infty e^x\, e^{-c_3\, x^2/(\gamma^2 T)}\, \mathrm{d}x \;\le\; c_4\, \gamma\sqrt{T}\, e^{c_4\, \gamma^2\, T}\,,
\end{aligned}$$

for some $c_4 \in (0,\infty)$, by a Gaussian integration. We have thus proven that, if $\varepsilon < 1 - \alpha$, there exists $c_4 \in (0,\infty)$ such that for all $\gamma, T > 0$

$$\mathbf{E}\left[\exp\left(\gamma\sqrt{T}\sqrt{\sum_{n\in\mathbb{N}:\, I_n\subseteq(0,2)} |I_n|^{1-\varepsilon}}\right)\right] \;\le\; c_4\, \gamma\sqrt{T}\, e^{c_4\, \gamma^2\, T}\,. \tag{A.9}$$

A.2. **Proof of equation (A.1).** We recall that

$$\Theta_T(\beta, \widetilde{\Delta}^\alpha) \;:=\; \sup_{-1\le x\le T\,,\; 0\le y\le T+1} \left|\mathcal{H}_{0,y;\theta_x\beta}(\widetilde{\Delta}^\alpha)\right|. \tag{A.10}$$

Recalling (2.2), we can write

$$\mathcal{H}_{0,y;\theta_x\beta}(\widetilde{\Delta}^\alpha) \;=\; -2\lambda \int_0^y \widetilde{\Delta}^\alpha(u)\, \mathrm{d}(\theta_x\beta)(u) \;-\; 2\lambda h \int_0^y \widetilde{\Delta}^\alpha(u)\, \mathrm{d}u\,,$$

and note that the second term is bounded in absolute value by $2\lambda hy$. For the purpose of proving (A.1) we may therefore focus on the first term: we set

$$\gamma_{x,y}(\beta, \widetilde{\Delta}^\alpha) \;:=\; \int_0^y \widetilde{\Delta}^\alpha(u)\,\mathrm{d}(\theta_x\beta)(u) \;=\; \int_x^{x+y} \widetilde{\Delta}^\alpha(u-x)\,\mathrm{d}\beta(u)\,, \tag{A.11}$$

$$\Gamma_T(\beta, \widetilde{\Delta}^\alpha) \;:=\; \sup_{(x,y)\in\mathcal{S}_T} \gamma_{x,y}(\beta, \widetilde{\Delta}^\alpha)\,, \qquad \text{where } \mathcal{S}_T \;:=\; [-1,T]\times[0,T+1]\,. \tag{A.12}$$

We stress that Γ_T is defined as the supremum of $\gamma_{x,y}$, not of $|\gamma_{x,y}|$. Notice however that, for fixed $\widetilde{\Delta}^\alpha$, the process $\gamma = \{\gamma_{x,y}(\beta,\widetilde{\Delta}^\alpha)\}_{x,y}$ under $\mathbb{P}$ is gaussian and centered, in particular it has the same law as $-\gamma$. Since $e^{|x|} \le e^x + e^{-x}$, we may then write

$$\mathbf{E}\Big[\mathbb{E}\big[\exp\big(\eta\,\Theta_T(\beta,\widetilde{\Delta}^\alpha)\big)\big]\Big] \;\le\; 2\,e^{2\lambda h(T+1)}\,\mathbf{E}\Big[\mathbb{E}\big[\exp\big(2\eta\lambda\,\Gamma_T(\beta,\widetilde{\Delta}^\alpha)\big)\big]\Big]\,. \tag{A.13}$$

Looking back at (A.1), we are left with showing that, for every $\eta > 0$, there exists (a possibly different) $D(\eta) \in (0,\infty)$ such that

$$\mathbf{E}\Big[\mathbb{E}\big[\exp\big(\eta\,\Gamma_T(\beta,\widetilde{\Delta}^\alpha)\big)\big]\Big] \;\le\; D(\eta)\,e^{D(\eta)\,T}\,, \qquad \forall T>0\,. \tag{A.14}$$

Let us set $\Gamma_T := \Gamma_T(\beta,\widetilde{\Delta}^\alpha)$ for short. It is convenient to split

$$\mathbf{E}\Big[\mathbb{E}\big[\exp\big(\eta\,\Gamma_T\big)\big]\Big] \;=\; \mathbf{E}\Big[\exp\big(\eta\,\mathbb{E}[\Gamma_T]\big)\cdot\mathbb{E}\big[\exp\big(\eta\,(\Gamma_T - \mathbb{E}[\Gamma_T])\big)\big]\Big]\,. \tag{A.15}$$

To prove (A.14) we use the powerful tools of the theory of continuity of gaussian processes. Let us introduce (for a fixed realization of $\widetilde{\Delta}^\alpha$) the *canonical metric* associated to the gaussian process γ, defined for $(x,y),(x',y') \in \mathcal{S}_T \;=\; [-1,T]\times[0,T+1]$ by

$$d((x,y),(x',y')) \;:=\; \sqrt{\mathbb{E}\Big[\big(\gamma_{x',y'}(\beta,\widetilde{\Delta}^\alpha) - \gamma_{x,y}(\beta,\widetilde{\Delta}^\alpha)\big)^2\Big]}\,. \tag{A.16}$$

For $\varepsilon > 0$ we define $N_T(\varepsilon) = N_{T,\widetilde{\Delta}^\alpha}(\varepsilon)$ as the least number of open balls of radius ε (in the canonical metric) needed to cover the parameter space $\mathcal{S}_T$. The quantity $\log N_T(\varepsilon)$ is called the *metric entropy* of γ. It is known [1, Corollary 4.15] that the finiteness of $\int_0^\infty \sqrt{\log N_T(\varepsilon)}\,\mathrm{d}\varepsilon$ ensures the existence of a version of the process γ which is continuous in the parameter space. Moreover, there exists a universal constant $K \in (0,\infty)$ such that

$$\mathbb{E}[\Gamma_T(\beta,\widetilde{\Delta}^\alpha)] \;\le\; K\int_0^\infty \sqrt{\log N_{T,\widetilde{\Delta}^\alpha}(\varepsilon)}\,\mathrm{d}\varepsilon\,. \tag{A.17}$$

We show below that, for $\mathbf{P}$-a.e. realization of $\widetilde{\Delta}^\alpha$, indeed $\int_0^\infty \sqrt{\log N_{T,\widetilde{\Delta}^\alpha}(\varepsilon)}\,\mathrm{d}\varepsilon < \infty$, so we may (and will) choose henceforth a continuous version of the process γ.

To estimate the right hand side of (A.15), let us denote by $\sigma_T^2 = \sigma^2_{T,\widetilde{\Delta}^\alpha}$ the maximal variance of the process γ, that is $\sigma_T^2 := \sup_{(x,y)\in\mathcal{S}_T}\mathbb{E}[\gamma_{x,y}(\beta,\widetilde{\Delta}^\alpha)^2]$. Since γ is continuous, it follows easily by Borell's inequality [1, Theorem 2.1] that

$$\mathbb{E}\big[\exp\big(\eta(\Gamma_T - \mathbb{E}[\Gamma_T])\big)\big] \;\le\; C'\,\sigma_T\,\exp\left(\frac{1}{2}\,\eta^2\,\sigma_T^2\right),$$

where $C' \in (0,\infty)$ is an absolute constant. Now observe that σ_T^2 is uniformly bounded: by (A.11) and the isometry property of the Wiener integral, since $|\widetilde{\Delta}^\alpha(\cdot)| \le 1$ we can write

$$\sigma_T^2 \;:=\; \sup_{(x,y)\in\mathcal{S}_T}\mathbb{E}\big[\gamma_{x,y}(\beta,\widetilde{\Delta}^\alpha)^2\big] \;=\; \sup_{(x,y)\in\mathcal{S}_T}\int_x^{x+y}\widetilde{\Delta}^\alpha(u-x)^2\,\mathrm{d}u \;\le\; T+1\,. \tag{A.18}$$

Looking back at (A.15) and recalling (A.17), we have proven that there exists $C \in (0,\infty)$ such that

$$\mathbf{E}\Big[\mathbb{E}\big[\exp\big(\eta\,\Gamma_T(\beta,\widetilde{\Delta}^\alpha)\big)\big]\Big] \;\le\; C\,e^{C\,\eta^2\,T}\,\mathbf{E}\bigg[\exp\bigg(K\,\eta\int_0^\infty \sqrt{\log N_{T,\widetilde{\Delta}^\alpha}(\varepsilon)}\,\mathrm{d}\varepsilon\bigg)\bigg]. \tag{A.19}$$

To complete the proof of (A.14), it remains to estimate $N_{T,\widetilde{\Delta}^\alpha}(\varepsilon)$, which requires some effort. For a fixed realization of $\widetilde{\Delta}^\alpha$, we introduce the function $\rho_T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_T(\delta) \;:=\; \sup_{(x,y),\,(x',y')\in\mathcal{S}_T:\;|(x,y)-(x',y')|\le\delta} d\big((x,y),(x',y')\big)\,, \tag{A.20}$$

where $|(x,y)-(x',y')|^2 := (x-x')^2+(y-y')^2$ denotes the Euclidean norm and we recall that the canonical metric d is defined in (A.16). Note that $\rho_T(\cdot)$ is a non-decreasing function which is eventually constant: $\rho_T(\delta) = \rho_T(\sqrt{2}(T+1))$ for every $\delta \ge \sqrt{2}(T+1)$, simply because $\sqrt{2}(T+1)$ is the diameter of the space $\mathcal{S}_T = [-1,T]\times[0,T+1]$.

Plainly, for every fixed $\delta > 0$, we can cover the square $\mathcal{S}_T$ with no more than $(\frac{T+1}{\delta}+1)^2$ open squares of side δ. Since the *Euclidean* distance between a point in a square of side δ and the center of the square is at most $\delta/\sqrt{2}$, the corresponding distance in the canonical metric is at most $\rho_T(\delta/\sqrt{2})$, by the definition of ρ_T. Therefore a square of side δ can be covered with a ball (in the canonical metric) of radius $\rho_T(\delta/\sqrt{2})$ centered at the center of the square. If we set $\varepsilon := \rho_T(\delta/\sqrt{2})$, this means that we need at most $(\frac{T+1}{\delta}+1)^2$ balls (in the canonical metric) of radius ε to cover the whole parameter space $\mathcal{S}_T$. Put otherwise, we have shown that for every $\varepsilon > 0$

$$N_T(\varepsilon) \;\le\; \left(1+\frac{T+1}{\sqrt{2}\,\rho_T^{-1}(\varepsilon)}\right)^2, \tag{A.21}$$

where ρ_T^{-1} is well-defined because ρ_T is non decreasing and *continuous*, as it will be clear below. Since $N_T(\varepsilon) = 1$ for $\varepsilon > \rho_T((T+1)/\sqrt{2})$ (we can cover $\mathcal{S}_T$ with just one ball), we obtain the estimate

$$\int_0^\infty \sqrt{\log N_T(\varepsilon)}\,\mathrm{d}\varepsilon \;\le\; \int_0^{\rho_T((T+1)/\sqrt{2})} \sqrt{2\log\left(1+\frac{T+1}{\sqrt{2}\,\rho_T^{-1}(\varepsilon)}\right)}\,\mathrm{d}\varepsilon\,.$$

By a change of variables and integrating by parts, we obtain

$$\begin{aligned}
\int_0^\infty \sqrt{\log N_T(\varepsilon)}\,\mathrm{d}\varepsilon \;&\le\; \int_0^{\frac{T+1}{\sqrt{2}}} \sqrt{2\log\left(1+\frac{T+1}{\sqrt{2}\,t}\right)}\,\mathrm{d}\rho_T(t)\\
&=\; \sqrt{2\log 2}\,\rho_T\bigg(\frac{T+1}{\sqrt{2}}\bigg) + \int_0^{\frac{T+1}{\sqrt{2}}} \frac{\rho_T(t)}{t\sqrt{2\log\left(1+\frac{T+1}{\sqrt{2}\,t}\right)}}\,\frac{T+1}{T+1+\sqrt{2}\,t}\,\mathrm{d}t\\
&\le\; \sqrt{2}\,\rho_T\bigg(\frac{T+1}{\sqrt{2}}\bigg) + \int_0^{\frac{T+1}{\sqrt{2}}} \frac{\rho_T(t)}{t\sqrt{2\log\left(1+\frac{T+1}{\sqrt{2}\,t}\right)}}\,\mathrm{d}t\,,
\end{aligned} \tag{A.22}$$

where in the integration by parts we have used the fact that, for $\mathbf{P}$-a.e. realization of $\widetilde{\Delta}^\alpha$, we have $\sqrt{2\log(1+\frac{T+1}{\sqrt{2}\,t})}\,\rho_T(t) \to 0$ as $t \to 0$, as we prove below.

To proceed with the estimates, we need to obtain bounds on ρ_T, hence we start from the definition (A.11) of $\gamma_{x,y}(\beta,\widetilde{\Delta}^\alpha)$. By the properties the Wiener integral we can write

$$\begin{aligned} d\big((x,y),(x',y')\big)^2 \;&=\; \mathbb{E}\big[\big(\gamma_{x',y'}(\beta,\widetilde{\Delta}^\alpha) - \gamma_{x,y}(\beta,\widetilde{\Delta}^\alpha)\big)^2\big] \\ &= \int_{-1}^{2T+1} \big(\widetilde{\Delta}^\alpha(u-x')\mathbf{1}_{[x',x'+y']}(u) \;-\; \widetilde{\Delta}^\alpha(u-x)\mathbf{1}_{[x,x+y]}(u)\big)^2 \,\mathrm{d}u \\ &= \int_{-1}^{2T+1} \big|\widetilde{\Delta}^\alpha(u-x')\mathbf{1}_{[x',x'+y']}(u) \;-\; \widetilde{\Delta}^\alpha(u-x)\mathbf{1}_{[x,x+y]}(u)\big| \,\mathrm{d}u\,, \end{aligned}$$

where the last equality holds simply because $\widetilde{\Delta}^\alpha(\cdot)$ takes values in $\{0,1\}$. Incidentally, this expression shows that the canonical metric $d(\cdot,\cdot)$ is continuous on $\mathcal{S}_T$ (because the translation operator is continuous in L^1). Therefore $\rho_T(\cdot)$ is a continuous function, as we stated before.

By the triangle inequality, we get for $x' \le x$

$$\begin{aligned} d\big((x,y),(x',y')\big)^2 \;&\le\; \int_{-1}^{2T+1} \widetilde{\Delta}^\alpha(u-x')\,\big|\mathbf{1}_{[x',x'+y']}(u) - \mathbf{1}_{[x,x+y]}(u)\big|\,\mathrm{d}u \\ &\qquad + \int_{-1}^{2T+1} \big|\widetilde{\Delta}^\alpha(u-x') - \widetilde{\Delta}^\alpha(u-x)\big|\,\mathbf{1}_{[x,x+y]}(u)\,\mathrm{d}u \\ &\le\; |x'-x| \,+\, |(x'+y')-(x+y)| \,+\, \int_x^{2T+1} \big|\widetilde{\Delta}^\alpha(u-x') - \widetilde{\Delta}^\alpha(u-x)\big|\,\mathrm{d}u\,. \end{aligned}$$

Recall that $\widetilde{\Delta}^\alpha(s) = \sum_{n\in\mathbb{N}} \widetilde{\xi}_n\,\mathbf{1}_{I_n}(s)$, where $\{I_n\}_{n\in\mathbb{N}}$ are the connected components of the open set $(\widetilde{\tau}^\alpha)^{\complement}$ and $\{\widetilde{\xi}_n\}_{n\in\mathbb{N}}$ are IID Bernoulli variables of parameter $1/2$. For every finite interval I we have the bound $\int_{\mathbb{R}} |\mathbf{1}_I(u-x') - \mathbf{1}_I(u-x)|\,\mathrm{d}u \le 2\min\{|I_n|, |x'-x|\}$, whence

$$\int_x^{2T+1} \big|\widetilde{\Delta}^\alpha(u-x') - \widetilde{\Delta}^\alpha(u-x)\big|\,\mathrm{d}u \;\le\; \sum_{n\in\mathbb{N}:\, I_n\cap(0,2(T+1))\neq\emptyset} \min\{|I_n|,\delta\} \tag{A.23}$$

Therefore, recalling the definition (A.20), we can write

$$\rho_T(\delta)^2 \;\le\; 3\,\delta \,+ \sum_{n\in\mathbb{N}:\, I_n\cap(0,2(T+1))\neq\emptyset} \min\{|I_n|,\delta\}\,. \tag{A.24}$$

Observe that the sum in the right hand side can be rewritten as $\delta\,N_\delta \,+\, A_\delta$, where N_δ is the *number* of excursions I_n that intersect $(0,2(T+1))$ with $|I_n| > \delta$ and A_δ is the *total area* covered by the excursions I_n that intersect $(0,2(T+1))$ with $|I_n| \le \delta$. The asymptotic behavior as $\delta \searrow 0$ of N_δ and A_δ is as follows: there exist a positive constant $c = c(\alpha)$ such that

$$\lim_{\delta\searrow 0} \delta^\alpha\,N_\delta \;=\; \lim_{\delta\searrow 0} \frac{A_\delta}{\delta^{1-\alpha}} \;=\; c\,L_{2(T+1)}\,, \qquad \mathbf{P}\text{-a.s.}\,, \tag{A.25}$$

where $\{L_t\}_{t\ge 0}$ is the local time associated to the regenerative set $\widetilde{\tau}^\alpha$ (whose definition is recalled in § A.1). The relations in (A.25) are proven in [23], *cf.* Proposition XII-(2.9) and Exercise XII-(2.14), in the special case $\alpha = \frac{1}{2}$, but the proof is easily extended to the general case. Looking back at (A.24), it follows that, for $\mathbf{P}$-a.e. realization of $\widetilde{\Delta}^\alpha$, we have $\rho_T(\delta) \sim \sqrt{2c}\,\sqrt{L_{2(T+1)}}\,\delta^{(1-\alpha)/2}$ as $\delta \searrow 0$. In particular, $\sqrt{\log(1+\frac{T+1}{\sqrt{2}\,t})}\,\rho_T(t) \to 0$ as $t \searrow 0$, a property used in the integration by parts in (A.22).

We are ready to bound the terms in the last line of (A.22). Note that the first term is easily controlled: by definition $d((x,y),(x',y')) \le 2\,\sigma_T$, hence it follows by (A.18) that

$$\sqrt{2}\,\rho_T\bigg(\frac{T+1}{\sqrt{2}}\bigg) \;\le\; 2\sqrt{2}\,\sqrt{T+1}\,. \tag{A.26}$$

Now observe that from (A.24) we have

$$\rho_T(\delta) \;\le\; F_{T+1}(\delta)\,, \qquad \text{where} \quad F_M(\delta) \;:=\; \sqrt{3\delta + \sum_{n\in\mathbb{N}\colon I_n\cap(0,2M)\neq\emptyset} \min\{|I_n|,\delta\}}\,. \tag{A.27}$$

By the scale invariance of the regenerative set $\widetilde{\tau}^\alpha$ it follows that, under $\mathbf{P}$, $\{F_M(t)\}_{t\ge0}$ has the same law as $\{\sqrt{M}\,F_1(\frac{t}{M})\}_{t\ge0}$. Therefore we can bound the second term in the last line of (A.22) as follows:

$$\int_0^{\frac{T+1}{\sqrt{2}}} \frac{\rho_T(t)}{t\sqrt{2\log\Big(1+\frac{T+1}{\sqrt{2}\,t}\Big)}}\,\mathrm{d}t \;\le\; \int_0^{\frac{T+1}{\sqrt{2}}} \frac{F_{T+1}(t)}{t\sqrt{2\log\Big(1+\frac{T+1}{\sqrt{2}\,t}\Big)}}\,\mathrm{d}t \;\overset{d}{=}\; \sqrt{T+1}\,\mathcal{M}\,, \tag{A.28}$$

where, performing the change of variable $t=(T+1)s$ in the integral, we have introduced the variable $\mathcal{M}$ defined by

$$\mathcal{M} \;:=\; \int_0^{\frac{1}{\sqrt{2}}} \frac{F_1(s)}{s\sqrt{2\log\Big(1+\frac{1}{\sqrt{2}\,s}\Big)}}\,\mathrm{d}s \;=\; \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{s}\sqrt{\frac{3s+\sum_{n\in\mathbb{N}\colon I_n\cap(0,2)\neq\emptyset}\min\{|I_n|,s\}}{2\log\Big(1+\frac{1}{\sqrt{2}\,s}\Big)}}\,\mathrm{d}s\,. \tag{A.29}$$

We can finally come back to (A.19): applying (A.22), (A.26) and (A.28) we obtain

$$\mathbf{E}\bigg[\exp\bigg(K\,\eta\int_0^\infty \sqrt{\log N_{T,\widetilde{\Delta}^\alpha}(\varepsilon)}\,\mathrm{d}\varepsilon\bigg)\bigg] \;\le\; \mathbf{E}\big[e^{K\,\eta\,\sqrt{T+1}\,(2\sqrt{2}+\mathcal{M})}\big]\,. \tag{A.30}$$

It only remains to estimate the law of $\mathcal{M}$. Let us fix an arbitrary $\varepsilon\in(0,1-\alpha)$: applying the Cauchy-Schwarz inequality, we obtain

$$\mathcal{M} \;\le\; \sqrt{\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{2\,s^{1-\varepsilon}\,\log\Big(1+\frac{1}{\sqrt{2}\,s}\Big)}\,\mathrm{d}s} \cdot \sqrt{\int_0^{\frac{1}{\sqrt{2}}}\bigg(\frac{3}{s^\varepsilon} + \sum_{n\in\mathbb{N}\colon I_n\cap(0,2)\neq\emptyset}\frac{\min\{|I_n|,s\}}{s^{1+\varepsilon}}\bigg)\mathrm{d}s}\,. \tag{A.31}$$

The first integral being finite, we may focus on the second one, in particular on the sum over the excursions $\{I_n\}_{n\in\mathbb{N}}$. Consider first the excursions such that $|I_n|\ge\frac{1}{\sqrt{2}}$, for which $\min\{|I_n|,s\}=s$: there are at most $2/(1/\sqrt{2})+1=2\sqrt{2}+1$ such excursions with $I_n\cap(0,2)\neq\emptyset$, therefore

$$\int_0^{1/\sqrt{2}} \sum_{n\in\mathbb{N}\colon I_n\cap(0,2)\neq\emptyset,\,|I_n|\ge\frac{1}{\sqrt{2}}} \frac{\min\{|I_n|,s\}}{s^{1+\varepsilon}}\mathrm{d}s \;\le\; (2\sqrt{2}+1)\int_0^{1/\sqrt{2}}\frac{1}{s^\varepsilon}\,\mathrm{d}s \;<\;\infty\,.$$

Plainly, also the last excursion $I_n \ni 2$ gives a finite contribution. It remains to consider the excursions I_n included in $(0,2)$ such that $|I_n| < \frac{1}{\sqrt{2}}$, for which we may write

$$\int_0^{\frac{1}{\sqrt{2}}} \sum_{I_n \subseteq (0,2),\, |I_n| < \frac{1}{\sqrt{2}}} \frac{\min\{|I_n|, s\}}{s^{1+\varepsilon}} \mathrm{d}s \;=\; \sum_{I_n \subseteq (0,2),\, |I_n| < \frac{1}{\sqrt{2}}} \left(\int_0^{|I_n|} \frac{1}{s^\varepsilon} \,\mathrm{d}s + \int_{|I_n|}^{\frac{1}{\sqrt{2}}} \frac{|I_n|}{s^{1+\varepsilon}} \,\mathrm{d}s \right)$$

$$= \sum_{I_n \subseteq (0,2),\, |I_n| < \frac{1}{\sqrt{2}}} \left(\frac{|I_n|^{1-\varepsilon}}{1-\varepsilon} + \frac{1}{\varepsilon} |I_n| \left(\frac{1}{|I_n|^\varepsilon} - (\sqrt{2})^\varepsilon \right) \right) \;\le\; \frac{1}{\varepsilon(1-\varepsilon)} \sum_{I_n \subseteq (0,2)} |I_n|^{1-\varepsilon} \,.$$

We have thus shown that there exist constants $0 < a, b < \infty$ (depending on ε) such that

$$\mathcal{M} \;\le\; a + b \sqrt{\sum_{n \in \mathbb{N}:\, I_n \subseteq (0,2)} |I_n|^{1-\varepsilon}} \,. \tag{A.32}$$

We can finally conclude the proof of (A.14). From (A.19), (A.30) and (A.32) it follows that equation (A.14) is proven once we show that for every $C > 0$ there exists $D = D(C) \in (0, \infty)$ such that for every $T > 0$

$$\mathbf{E}\left[\exp\left(C\sqrt{T} \sqrt{\sum_{n \in \mathbb{N}:\, I_n \subseteq (0,2)} |I_n|^{1-\varepsilon}} \right) \right] \;\le\; D \exp(D\,T) \,. \tag{A.33}$$

But this is a direct consequence of equation (A.9). □

Appendix B. Proof of Lemma 3.5

We recall that $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ and $\widetilde{\tau}^\alpha$ denote respectively the renewal process and the regenerative set, both defined under the law $\mathbf{P}$. For $x \ge 0$, we denote by $\mathbf{P}_x$ the law of the sets τ and $\widetilde{\tau}^\alpha$ started at x, that is $\mathbf{P}_x(\tau \in \cdot) := \mathbf{P}(\tau + x = \{\tau_n + x\}_{n\in\mathbb{N}} \in \cdot)$ and analogously for $\widetilde{\tau}^\alpha$. For the definition of the vectors $\Sigma := (m;\, s_1, \ldots, s_m;\, \underline{\sigma}_1, \ldots, \underline{\sigma}_m)$ and $\widetilde{\Sigma} := (\widetilde{m};\, \widetilde{s}_1, \ldots, \widetilde{s}_{\widetilde{m}};\, \underline{\widetilde{\sigma}}_1, \ldots, \underline{\widetilde{\sigma}}_{\widetilde{m}})$, we refer to Section 3.3.

In this section we fix $\delta = 1$. We have to estimate the Radon–Nikodym density $\frac{\mathrm{d}\widetilde{\Sigma}}{\mathrm{d}\Sigma}$ of the laws of $\widetilde{\Sigma}$ and Σ (which does not depend on the sign variables, see explanation between (3.47) and (3.48)), namely the quantity

$$\frac{\mathrm{d}\widetilde{\Sigma}}{\mathrm{d}\Sigma}(l;\, x_1, \ldots, x_l) \;=\; \frac{\mathbf{P}\big((\widetilde{m};\, \underline{\widetilde{\sigma}}_1, \ldots, \underline{\widetilde{\sigma}}_m) = (l;\, x_1, \ldots, x_l)\big)}{\mathbf{P}\big((m;\, \underline{\sigma}_1, \ldots, \underline{\sigma}_m) = (l;\, x_1, \ldots, x_l)\big)} \,. \tag{B.1}$$

Note that by construction $(\underline{\sigma}_{i+1} - \underline{\sigma}_i) \in [\delta, \infty) \cap \varepsilon\mathbb{N}$, and since $\delta = 1$ we assume that $x_{i+1} - x_i \in [1,\infty) \cap \varepsilon\mathbb{N}$. Using the regenerative property of $\widetilde{\tau}^\alpha$ and the renewal property of τ, the ratio in (B.1) can be estimated in terms of the probability of the first coarse-grained returns of $\widetilde{\tau}^\alpha$ and τ:

$$\prod_{i=1}^{l} c(x_i - x_{i-1}) \;\le\; \frac{\mathrm{d}\widetilde{\Sigma}}{\mathrm{d}\Sigma}(l;\, x_1, \ldots, x_l) \;\le\; \prod_{i=1}^{l} C(x_i - x_{i-1}) \,, \tag{B.2}$$

where we set for convenience $x_0 := 0$ and we have introduced, for $z \in [1, \infty) \cap \varepsilon\mathbb{N}$,

$$C(z) \;:=\; \sup_{y, \widetilde{y} \in (0,\varepsilon]} \frac{\mathbf{P}_{\widetilde{y}}\big(\inf\{u > 1 :\, u \in \widetilde{\tau}^\alpha\} \,\in\, (z, z+\varepsilon] \big)}{\mathbf{P}_{\frac{y}{a^2}}\big(\inf\{i > \frac{1}{a^2} :\, i \in \tau\} \,\in\, (\frac{z}{a^2}, j\frac{z+\varepsilon}{a^2}] \big)} \,, \tag{B.3}$$

and $c(z)$ is defined analogously, replacing the supremum (over y and $\widetilde{y}$) by the infimum (over the same variables and range). For the purpose of proving Lemma 3.5, it is actually more convenient to give a slightly different estimate than (B.2), namely

$$\exp\left(-\sum_{i=1}^{l} G(x_i - x_{i-1})\right) \;\le\; \frac{\mathrm{d}\widetilde{\Sigma}}{\mathrm{d}\Sigma}(l;\, x_1,\ldots,x_l) \;\le\; \exp\left(\sum_{i=1}^{l} G(x_i - x_{i-1})\right), \tag{B.4}$$

where $G(z) = G_{\varepsilon,a}(z)$ is defined, always for $z \in [1,\infty) \cap \varepsilon\mathbb{N}$, by

$$G_{\varepsilon,a}(z) \;:=\; \sup_{y,\widetilde{y}\in(0,\varepsilon]} \left| \log\left(\frac{\mathbf{P}_{\frac{y}{a^2}}\big(\inf\{i > \frac{1}{a^2} \,:\, i \in \tau\} \,\in\, (\frac{z}{a^2}, \frac{z+\varepsilon}{a^2}]\big)}{\mathbf{P}_{\widetilde{y}}\big(\inf\{u > 1 \,:\, u \in \widetilde{\tau}^\alpha\} \,\in\, (z, z+\varepsilon]\big)} \right) \right| . \tag{B.5}$$

Recalling the statement of Lemma 3.5, we are left with showing that

$$G_{\varepsilon,a}(z) \;\le\; \kappa(\varepsilon,a)\,\big(\log z + 1\big)\,, \qquad \text{with} \qquad \lim_{\varepsilon\to 0}\,\limsup_{a\to 0}\,\kappa(\varepsilon,a) \;=\; 0\,. \tag{B.6}$$

By the limit theory of regenerative sets [13], as $a \to 0$ the rescaled renewal process $a^2\tau = \{a^2\tau_n\}_{n\in\mathbb{N}}$ (viewed as a random closed subset of $[0,\infty)$) converges in distribution toward the regenerative set $\widetilde{\tau}^\alpha$. It follows that, for all fixed $\varepsilon \in (0,1)$, the numerator in the right hand side of (B.5) converges as $a \to 0$ toward the denominator with $\widetilde{y}$ replaced by y, for all fixed $z \in [1,\infty) \cap \varepsilon\mathbb{N}$ and $y \in (0,\varepsilon]$. In the following Lemma, proven below, we provide a quantitative control on this convergence, as a function of z and y.

Lemma B.1. *Fix $\varepsilon \in (0,1/3)$. There exists $\zeta_\varepsilon(a) > 0$ with $\lim_{a\to 0}\zeta_\varepsilon(a) = 0$ such that*

$$\big(1-\zeta_\varepsilon(a)\big)\, z^{-\zeta_\varepsilon(a)} \;\le\; \frac{\mathbf{P}_{\frac{y}{a^2}}\big(\inf\{i > \frac{1}{a^2} \,:\, i \in \tau\} \,\in\, (\frac{z}{a^2}, \frac{z+\varepsilon}{a^2}]\big)}{\mathbf{P}_{y}\big(\inf\{u > 1 \,:\, u \in \widetilde{\tau}^\alpha\} \,\in\, (z, z+\varepsilon]\big)} \;\le\; \big(1+\zeta_\varepsilon(a)\big)\, z^{\zeta_\varepsilon(a)}\,, \tag{B.7}$$

for all $a \in (0, a_0)$ (with $a_0 > 0$), $y \in [0,1/3]$ and $z \in [1,\infty) \cap \varepsilon\mathbb{N}$.

We now apply (B.7) to (B.5): since $|\log(1+x)| \le 2|x|$ for x small, for small a we obtain

$$G_{\varepsilon,a}(z) \;\le\; 2\zeta_\varepsilon(a)\big(\log z + 1\big) \;+\; \sup_{y,\widetilde{y}\in(0,\varepsilon]} \left| \log\left(\frac{\mathbf{P}_{y}\big(\inf\{u > 1 \,:\, u \in \widetilde{\tau}^\alpha\} \,\in\, (z, z+\varepsilon]\big)}{\mathbf{P}_{\widetilde{y}}\big(\inf\{u > 1 \,:\, u \in \widetilde{\tau}^\alpha\} \,\in\, (z, z+\varepsilon]\big)} \right) \right| . \tag{B.8}$$

Recalling the definition (2.7) of $d_t(\widetilde{\tau}^\alpha)$ and applying (2.10), for $z \in [1,\infty) \cap \varepsilon\mathbb{N}$ we can write

$$\mathbf{P}_{y}\big(\inf\{u > 1 \,:\, u \in \widetilde{\tau}^\alpha\} \,\in\, (z, z+\varepsilon]\big) \;=\; \frac{\sin(\pi\alpha)}{\pi} \int_z^{z+\varepsilon} \frac{(1-y)^\alpha}{(t-1)^\alpha\,(t-y)}\,\mathrm{d}t\,. \tag{B.9}$$

From this explicit expression it is easy to check that the second term in the right hand side of (B.8) vanishes as $\varepsilon \to 0$ *uniformly in* $z \in [1,\infty) \cap \varepsilon\mathbb{N}$, hence (B.6) holds true. ☐

B.1. **Proof of Lemma B.1.** We have already obtained in (B.9) an explicit expression for the denominator in (B.7). It is however more convenient to give an alternative expression: recalling again the definition (2.7) of the variable $d_t(\widetilde{\tau}^\alpha)$ and applying (2.8), we can rewrite the denominator in (B.7) as

$$I(y,z) \;:=\; \frac{\alpha\,\sin(\pi\alpha)}{\pi} \int_y^1 \mathrm{d}s \int_z^{z+\varepsilon} \mathrm{d}t\, \frac{1}{(s-y)^{1-\alpha}\,(t-s)^{1+\alpha}}\,. \tag{B.10}$$

Recalling that $K(n) := \mathbf{P}(\tau_1 = n)$ and setting $U(n) := \mathbf{P}(n \in \tau)$, we can rewrite the numerator in (B.7) using the renewal property as

$$J_a(y,z) \;:=\; \sum_{\substack{\frac{y}{a^2} \le k \le \frac{1}{a^2} \\ \frac{z}{a^2} < l \le \frac{z+\varepsilon}{a^2}}} U(k - \tfrac{y}{a^2})\, K(l-k) \;=\; \sum_{\substack{s \in [y,1] \cap a^2\mathbb{N} \\ t \in (z, z+\varepsilon] \cap a^2\mathbb{N}}} U\big(\tfrac{1}{a^2}(s-y)\big)\, K\big(\tfrac{1}{a^2}(t-s)\big)\,. \tag{B.11}$$

We now use [12, Theorem B], coupled with our basic assumption on the inter-arrival distribution (1.4), to see that

$$U(\ell) \;\overset{\ell\to\infty}{\sim}\; \frac{\alpha \sin(\pi\alpha)}{\pi}\, \frac{1}{L(\ell)\, \ell^{1-\alpha}}\,. \tag{B.12}$$

Using the asymptotic relations (1.4) and (B.12) and a Riemann sum argument (with some careful handling of the slowly varying functions, see the details below), one can check that (B.11) converges toward (B.10) as $a \to 0$, for all *fixed* $\varepsilon \in (0, 1/3)$, $z \in [1,\infty) \cap \varepsilon\mathbb{N}$ and $y \in (0, \varepsilon]$. However to obtain (B.7) a more attentive estimate is required. We set $n := 1/a^2$ for notational convenience, so that, with some abuse of notation, we can rewrite (B.11) as

$$J_n(y,z) \;:=\; \sum_{\substack{ny \le k \le n \\ nz < l \le n(z+\varepsilon)}} U(k-ny)\, K(l-k) \;=\; \sum_{\substack{s \in [y,1] \cap \frac{1}{n}\mathbb{N} \\ t \in (z,z+\varepsilon] \cap \frac{1}{n}\mathbb{N}}} U\big(n(s-y)\big)\, K\big(n(t-s)\big)\,. \tag{B.13}$$

We can now rephrase (B.7) in the following way: for every fixed $\varepsilon \in (0, 1/3)$, there exist $\zeta_\varepsilon(n) > 0$, with $\lim_{n\to\infty} \zeta_\varepsilon(n) = 0$, and $n_0 \in \mathbb{N}$ such that

$$\big(1 \,-\, \zeta_\varepsilon(n)\big)\, z^{-\zeta_\varepsilon(n)} \;\le\; \frac{J_n(y,z)}{I(y,z)} \;\le\; \big(1 \,+\, \zeta_\varepsilon(n)\big)\, z^{\zeta_\varepsilon(n)}\,, \tag{B.14}$$

for all $n \ge n_0$, $y \in [0, 1/3]$ and $z \in [1,\infty) \cap \varepsilon\mathbb{N}$. We recall that $I(y,z)$ is defined in (B.10). For convenience, we divide the rest of the proof in three steps.

Step 1. We first show that the terms in (B.13) with $k \le ny + \sqrt{n}$, that is

$$A_n(y,z) \;:=\; \sum_{\substack{ny \le k \le ny + \sqrt{n} \\ nz < l \le n(z+\varepsilon)}} U(k-ny)\, K(l-k)\,, \tag{B.15}$$

give a negligible contribution to (B.14).

By paying a positive constant, we can replace $K(\cdot)$ and $U(\cdot)$ by their asymptotic behaviors, *cf.* (1.4) and (B.12). Note that $k \le ny + \sqrt{n} \le n/2$ for large n, because $y \le 1/3$, and therefore $n(z - 1/2) \le (l-k) \le n(z + 1/3)$, because $\varepsilon \le 1/3$, for all l, k in the range of summation. We thus obtain the upper bound

$$A_n(y,z) \;\le\; C_1 \sum_{0 < h \le \sqrt{n}} \frac{1}{L(h)\, h^{1-\alpha}} \sum_{n(z-\frac{1}{2}) < m \le n(z+\frac{1}{3})} \frac{L(m)}{m^{1+\alpha}}\,, \tag{B.16}$$

for some absolute constant $C_1 > 0$. We now show that, for some absolute constant $C_2 > 0$ (not depending on z), we can write $L(m) \le C_2 L(nz)$ for every m in the range of summation. To this purpose, we recall the representation theorem of slowly varying functions:

$$L(x) \;=\; a(x)\, \exp\left(\int_1^x \frac{b(t)}{t}\, \mathrm{d}t\right)\,, \quad \text{with } \lim_{x\to\infty} a(x) \in (0,\infty) \text{ and } \lim_{x\to\infty} b(x) = 0\,, \tag{B.17}$$

see Theorem 1.3.1 in [4]. Setting $\gamma_n := \sup_{x\ge n/2} |b(x)|$, we have $\lim_{n\to\infty} \gamma_n = 0$ and for $m \in \{n(z-1/2), n(z+1/3)\}$ we can write for $z \ge 1$

$$\frac{L(m)}{L(nz)} \;\le\; \frac{a(m)}{a(nz)} \exp\left(\gamma_n \int_{n(z-\frac{1}{2})}^{n(z+\frac{1}{3})} \frac{1}{t}\,\mathrm{d}t\right) \;\le\; \frac{\sup_{k\ge \frac{n}{2}} a(k)}{\inf_{k\ge n} a(k)} \exp\left(\gamma_n \log \frac{z+\frac{1}{3}}{z-\frac{1}{2}}\right). \tag{B.18}$$

Since $z \ge 1$, it is clear that the right hand side of (B.18) is bounded from above by some absolute constant C_2 (in fact, it even converges to 1 as $n \to \infty$). From (B.16) we then obtain

$$\begin{aligned} A_n(y,z) \;&\le\; C_2\, L(nz) \sum_{0<h\le\sqrt{n}} \frac{1}{L(h)\, h^{1-\alpha}} \sum_{n(z-\frac{1}{2})<m\le n(z+\frac{1}{3})} \frac{1}{m^{1+\alpha}} \\ &\le\; C_3\, \frac{L(nz)}{n^{\alpha}\, z^{1+\alpha}} \sum_{0<h\le\sqrt{n}} \frac{1}{L(h)\, h^{1-\alpha}} \;\le\; C_4\, \frac{L(nz)}{n^{\alpha}\, z^{1+\alpha}}\, \frac{n^{\alpha/2}}{\alpha\, L(\sqrt{n})}\,, \end{aligned} \tag{B.19}$$

where C_3, C_4 are absolute positive constant and the last inequality is a classical result (Proposition 1.5.8 in [4]). Using again the representation (B.17), in analogy with (B.18), we can write

$$\frac{L(nz)}{L(\sqrt{n})} \;\le\; \frac{a(nz)}{a(\sqrt{n})} \exp\left(\gamma_n \int_{\sqrt{n}}^{nz} \frac{1}{t}\,\mathrm{d}t\right) \;\le\; C_5 \exp\left(\gamma_n \log \frac{nz}{\sqrt{n}}\right) \;=\; C_5\, n^{\gamma_n/2}\, z^{\gamma_n}\,, \tag{B.20}$$

for some absolute constant C_5. Coming back to (B.19), we have shown that there exists absolute constants C_6 and n_0 such that for all $n \ge n_0$, $z \in [1,\infty) \cap \varepsilon\mathbb{N}$ and $y \in [0, 1/3]$

$$A_n(y,z) \;\le\; \frac{C_6}{n^{(\alpha-\gamma_n)/2}}\, \frac{z^{\gamma_n}}{z^{1+\alpha}}\,. \tag{B.21}$$

Let us now look back at the integral $I(y,z)$, defined in (B.10). It is easy to check that for every fixed $\varepsilon \in (0, 1/3)$ there exists an absolute constant $C_7 = C_7(\varepsilon) > 0$ such that

$$I(y,z) \;\ge\; \frac{C_7}{z^{1+\alpha}}\,, \tag{B.22}$$

for all $y \in [0, 1/3]$ and $z \in [1,\infty) \cap \varepsilon\mathbb{N}$. If we set $\zeta'(n) := \max\{\gamma_n,\, C_6/(C_7\, n^{(\alpha-\gamma_n)/2})\}$, we have $\lim_{n\to\infty} \zeta'(n) = 0$ and from (B.21) and (B.22) we have shown that for every fixed $\varepsilon \in (0, 1/3)$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ we have

$$\frac{A_n(y,z)}{I(y,z)} \;\le\; \zeta'(n)\, z^{\zeta'(n)}\,, \tag{B.23}$$

for all $z \in [1,\infty) \cap \varepsilon\mathbb{N}$ and $y \in [0, 1/3]$. This completes the first step.

Step 2. We now consider the terms in (B.13) with $k > ny + \sqrt{n}$, or equivalently $s > y + \frac{1}{\sqrt{n}}$, that is we introduce the quantity

$$B_n(y,z) \;:=\; \sum_{\substack{s \in (y+\frac{1}{\sqrt{n}}, 1] \cap \frac{1}{n}\mathbb{N} \\ t \in (z, z+\varepsilon] \cap \frac{1}{n}\mathbb{N}}} U\big(n(s-y)\big)\, K\big(n(t-s)\big)\,, \tag{B.24}$$

and we observe that $J_n(y,z) = A_n(y,z) + B_n(y,z)$, see (B.13) and (B.15). Our aim is to prove (B.14): in view of relation (B.23), it remains to show that for every fixed $\varepsilon \in (0, 1/3)$ there exist $\zeta''(n) > 0$, with $\lim_{n\to\infty} \zeta''(n) = 0$, and $n_0 \in \mathbb{N}$ such that

$$\big(1 - \zeta''(n)\big)\, z^{-\zeta''(n)} \;\le\; \frac{B_n(y,z)}{I(y,z)} \;\le\; \big(1 + \zeta''(n)\big)\, z^{\zeta''(n)}\,, \tag{B.25}$$

for all $n \geq n_0$, $y \in [0, 1/3]$ and $z \in [1, \infty) \cap \varepsilon\mathbb{N} = \{1, 1+\varepsilon, 1+2\varepsilon, \ldots\}$. In this step we prove that (B.25) holds for $z \in [1+\varepsilon, \infty) \cap \varepsilon\mathbb{N}$, that is we exclude the case $z = 1$, that will be considered separately in the third step.

By construction, the arguments of the functions $U(\cdot)$ and $K(\cdot)$ appearing in (B.24) tend to ∞ as $n \to \infty$ uniformly in the range of summation: in fact $n(s-y) \geq \sqrt{n}$ and $n(t-s) \geq \varepsilon n$, because we assume that $z \geq 1+\varepsilon$. We can therefore replace $U(\cdot)$ and $K(\cdot)$ by their asymptotic behaviors, given in (1.4) and (B.12), by committing an asymptotically negligible error: more precisely, we can write

$$B_n(y,z) \;=\; \big(1+o(1)\big)\,\frac{C_\alpha}{n^2} \sum_{\substack{s \in (y+\frac{1}{\sqrt{n}}, 1] \cap \frac{1}{n}\mathbb{N} \\ t \in (z, z+\varepsilon] \cap \frac{1}{n}\mathbb{N}}} \left[\frac{L\big(n(s-y)\big)}{L\big(n(t-s)\big)}\right] \frac{1}{(s-y)^{1-\alpha}\,(t-s)^{1+\alpha}}\,, \tag{B.26}$$

where we set $C_\alpha := \alpha \sin(\pi\alpha)/\pi$ for short and where, here and in the sequel, $o(1)$ denotes a quantity (possibly depending on ε and varying from place to place) that vanishes as $n \to \infty$ *uniformly in* $y \in [0, 1/3]$ *and in* $z \in [1+\varepsilon, \infty) \cap \varepsilon\mathbb{N}$.

We now estimate the ratio in square brackets in the right hand side of (B.26). Recalling the representation theorem of slowly varying functions, see (B.17), uniformly for s, t in the range of summation we can write

$$\frac{L\big(n(s-y)\big)}{L\big(n(t-s)\big)} \;=\; (1+o(1))\,\exp\left(\int_{n(t-s)}^{n(s-y)} \frac{b(x)}{x}\,\mathrm{d}x\right), \tag{B.27}$$

with the convention $\int_\beta^\gamma(\ldots) := -\int_\gamma^\beta(\ldots)$ if $\beta > \gamma$. Let us set

$$\eta_n \;:=\; \sup_{x \geq \min\{\sqrt{n}, \varepsilon n\}} |b(x)|\,, \tag{B.28}$$

so that $\eta_n \to 0$ as $n \to \infty$. Uniformly for s, t in the range of summation, we can write

$$\left|\int_{n(t-s)}^{n(s-y)} \frac{b(x)}{x}\,\mathrm{d}x\right| \;\leq\; \eta_n \left|\int_{n(t-s)}^{n(s-y)} \frac{1}{x}\,\mathrm{d}x\right| \;\leq\; \eta_n\big(|\log(t-s)| \,+\, |\log(s-y)|\big)\,. \tag{B.29}$$

In the range of summation of (B.26) we have $0 < (s-y) \leq 1$, hence $|\log(s-y)| = -\log(s-y)$, and $\varepsilon \leq (t-s) \leq z+\varepsilon$, whence $|\log(t-s)| \leq -\log\varepsilon + \log(z+\varepsilon)$ (recall that $\varepsilon < 1 < z$). Coming back to (B.26), from (B.27) and (B.29) we obtain the upper bound

$$B_n(y,z) \;\leq\; \big(1+o(1)\big)\,\frac{(z+\varepsilon)^{\eta_n}}{\varepsilon^{\eta_n}} \left[\frac{C_\alpha}{n^2} \sum_{\substack{s \in (y+\frac{1}{\sqrt{n}}, 1] \cap \frac{1}{n}\mathbb{N} \\ t \in (z, z+\varepsilon] \cap \frac{1}{n}\mathbb{N}}} \frac{1}{(s-y)^{1-\alpha+\eta_n}\,(t-s)^{1+\alpha}}\right], \tag{B.30}$$

as well as the corresponding lower bound

$$B_n(y,z) \;\geq\; \big(1+o(1)\big)\,\frac{\varepsilon^{\eta_n}}{(z+\varepsilon)^{\eta_n}} \left[\frac{C_\alpha}{n^2} \sum_{\substack{s \in (y+\frac{1}{\sqrt{n}}, 1] \cap \frac{1}{n}\mathbb{N} \\ t \in (z, z+\varepsilon] \cap \frac{1}{n}\mathbb{N}}} \frac{1}{(s-y)^{1-\alpha-\eta_n}\,(t-s)^{1+\alpha}}\right]. \tag{B.31}$$

Observe that we can write $\frac{(z+\varepsilon)^{\eta_n}}{\varepsilon^{\eta_n}} = c_{\varepsilon,z,n}\, z^{\eta_n}$, with $c_{\varepsilon,z,n} = (\frac{1}{\varepsilon} + \frac{1}{z})^{\eta_n} \to 1$ as $n \to \infty$ (for fixed ε) uniformly in $z \in [1, \infty)$. We can therefore incorporate $c_{\varepsilon,z,n}$ in the $(1+o(1))$ term in (B.30) and (B.31). Recalling that we aim at proving (B.25), it remains to show that for every fixed $\varepsilon \in (0, 1/3)$ the terms in square brackets in the right hand sides of

(B.30) and (B.31), divided by the integral $I(y,z)$ defined in (B.10), converge to 1 as $n \to \infty$ uniformly in $y \in [0,1/3]$ and in $z \in [1+\varepsilon, \infty)$.

Since the summand in the right hand side of (B.30) is decreasing in t, we can replace the sum over t by an integral over a slightly shifted domain, getting the following upper bound on the term in square brackets in the right hand side of (B.30):

$$\big[\cdots\big]_{(\mathrm{B}.30)} \;\le\; \int_{z-\frac{1}{n}}^{z+\varepsilon} \left(\frac{C_\alpha}{n} \sum_{s \in (y+\frac{1}{\sqrt{n}},1] \cap \frac{1}{n}\mathbb{N}} \frac{1}{(s-y)^{1-\alpha+\eta_n}\,(t-s)^{1+\alpha}} \right) \mathrm{d}t\,. \tag{B.32}$$

By direct computation one sees that the term in the right hand side of this relation, as a function of s, is decreasing in $(0, s_0)$ and increasing in (s_0, ∞), where $s_0 = \frac{(1-\alpha+\eta_n)t+(1+\alpha)y}{2+\eta_n}$. The precise value of s_0 is actually immaterial: the important point is that each term in the sum in (B.32) can be bounded from above by an integral over $[s-\frac{1}{n}, s]$ (if $s \le s_0$) or over $[s, s+\frac{1}{n}]$ (if $s \ge s_0$). Therefore we get an upper bound replacing the sum by an integral over a slightly enlarged domain:

$$\big[\cdots\big]_{(\mathrm{B}.30)} \;\le\; \frac{\alpha\,\sin(\pi\alpha)}{\pi} \int_{y+\frac{1}{\sqrt{n}}-\frac{1}{n}}^{1+\frac{1}{n}} \mathrm{d}s \int_{z-\frac{1}{n}}^{z+\varepsilon} \mathrm{d}t\, \frac{1}{(s-y)^{1-\alpha+\eta_n}\,(t-s)^{1+\alpha}}\,. \tag{B.33}$$

With almost identical arguments one obtains the following lower bound on the term in square brackets in the right hand side of (B.31):

$$\big[\cdots\big]_{(\mathrm{B}.31)} \;\ge\; \frac{\alpha\,\sin(\pi\alpha)}{\pi} \int_{y+\frac{1}{\sqrt{n}}+\frac{1}{n}}^{1-\frac{1}{n}} \mathrm{d}s \int_{z}^{z+\varepsilon+\frac{1}{n}} \mathrm{d}t\, \frac{1}{(s-y)^{1-\alpha-\eta_n}\,(t-s)^{1+\alpha}}\,. \tag{B.34}$$

One can now check directly that, for every fixed $\varepsilon \in (0,1/3)$, the ratio between the right hand side of (B.33) and the integral $I(y,z)$ defined in (B.10) converges to 1 as $n \to \infty$, uniformly in $y \in [0,1/3]$ and in $z \in [1+\varepsilon,\infty)$. Since an analogous statement holds for the right hand side of (B.34), the second step is completed.

Step 3. To complete the proof of Lemma B.1, it only remains to prove that equation (B.25) holds true also for $z=1$. More explicitly, we have to show that as $n \to \infty$

$$\frac{B_n(y,1)}{I(y,1)} \;\longrightarrow\; 1\,, \tag{B.35}$$

uniformly in $y \in [0,1/3]$. We recall that

$$B_n(y,1) \;:=\; \sum_{\substack{s \in (y+\frac{1}{\sqrt{n}},1] \cap \frac{1}{n}\mathbb{N} \\ t \in (1,1+\varepsilon] \cap \frac{1}{n}\mathbb{N}}} U\big(n(s-y)\big)\, K\big(n(t-s)\big)\,, \tag{B.36}$$

while the integral $I(y,z)$ is defined in (B.10).

We only sketch the proof of (B.35), because the arguments are very similar to those used in the preceding steps. Note that we cannot immediately replace $K(\cdot)$ by its asymptotic behavior, because its argument $n(t-s)$ can take small values. It is therefore convenient to restrict the sum in (B.36) to $t \in (1+1/\sqrt{n}, 1+\varepsilon]$. For this restricted sum, call it $B_n'(y,1)$, one can write a formula analogous to (B.26): then, arguing as in the second step (with several simplifications), one shows that (B.35) holds true with B_n replaced by B_n'. It remains to deal with $B_n - B_n'$, that is to control the terms in (B.36) with $t \le 1+1/\sqrt{n}$. In this case one can replace $K(\cdot)$ by its asymptotic behavior by paying a positive constant:

arguing as in the first step, one can show that $(B_n(y,1) - B'_n(y,1))/I(y,1) \to 0$ as $n \to \infty$, uniformly in $y \in [0, 1/3]$. This completes the proof of (B.35) and of Lemma B.1. □

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References

[1] R. J. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, Lecture Notes, Vol. 12, Institute of Mathematical Statistics
[2] S. Asmussen, *Applied probability and queues*, Second Edition, Applications of Mathematics **51**, Springer-Verlag, New York, 2003.
[3] M. Barlow, J. Pitman and M. Yor, *Une extension multidimensionnelle de la loi de l'arc sinus*, Séminaire de Probabilités XXIII, Lecture Notes in Mathematics **1372** (1989), 294-314.
[4] N. H. Bingham, C. H. Goldie and J. L. Teugels, *Regular variation*, Cambridge University Press, 1989.
[5] M. Birkner, A. Greven and F. den Hollander, *Quenched LDP for words in a letter sequence*, arXiv.org: 0807.2611 [math.PR].
[6] T. Bodineau and G. Giacomin, *On the localization transition of random copolymers near selective interfaces*, J. Statist. Phys. **117** (2004), 17-34.
[7] T. Bodineau, G. Giacomin, H. Lacoin and F. L. Toninelli, *Copolymers at selective interfaces: new bounds on the phase diagram*, J. Statist. Phys. **132** (2008), 603-626.
[8] E. Bolthausen, F. Caravenna and B. de Tilière, *The quenched critical point of a diluted disordered polymer model*, Stoch. Process. Appl. **119** (2009), 1479-1504.
[9] E. Bolthausen and F. den Hollander, *Localization transition for a polymer near an interface*, Ann. Probab. **25** (1997), 1334-1366.
[10] F. Caravenna, G. Giacomin and M. Gubinelli, *A Numerical Approach to Copolymers at Selective Interfaces*, J. Statist. Phys. **122** (2006), 799-832.
[11] F. Comets and N. Yoshida, *Brownian directed polymers in random environment*, Commun. Math. Phys. **254** (2005), 257-287.
[12] R. A. Doney, *One-sided local large deviation and renewal theorems in the case of infinite mean*, Probab. Theory Related Fields **107** (1997), 451-465.
[13] P. J. Fitzsimmons, B. Fristedt and B. Maisonneuve, *Intersections and limits of regenerative sets*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **70** (1985), 157-173.
[14] T. Garel, D. A. Huse, S. Leibler and H. Orland, *Localization transition of random chains at interfaces*, Europhys. Lett. **8** (1989), 9-13.
[15] F. den Hollander, *Random Polymers*, École d'Été de Probabilités de Saint-Flour XXXVII–2007, Lecture Notes in Mathematics **1974**, Springer, 2009.
[16] G. Giacomin, *Localization phenomena in random polymer models*, unpublished lecture notes (available on the web-page of the author), 2004.
[17] G. Giacomin, *Random Polymer Models*, Imperial College Press, World Scientific, 2007.
[18] G. Giacomin and F. L. Toninelli, *Estimates on path delocalization for copolymers at selective interfaces*, Probab. Theory Relat. Fields **133** (2005), 464-482.
[19] J. F. C. Kingman, *Poisson processes*, Oxford University Press, 1993.
[20] J. F. C. Kingman, *Subadditive ergodic theory*, Ann. Probab. **1** (1973), 882-909.
[21] J. Komlós, P. Major and G. Tusnády, *An Approximation of Partial Sums of Independent RV's, and the Sample DF. II*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **34** (1976), 33–58.
[22] N. Pétrélis, *Copolymer at selective interfaces and pinning potentials: weak coupling limits*, Ann. Instit. H. Poincaré **45** (2009), 175-200.
[23] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Third Edition, Springer-Verlag (1999).
[24] Ya. G. Sinai, *A random walk with a random potential*, Theory Probab. Appl. **38** (1993), 382-385.
[25] F. L. Toninelli, *Disordered pinning models and copolymers: beyond annealed bounds*, Ann. Appl. Probab. **18** (2008), 1569-1587.

[26] F. L. Toninelli, *Coarse graining, fractional moments and the critical slope of random copolymers*, Electron. J. Probab. **14** (2009), 531-547.

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