QUASI-CRITICAL FLUCTUATIONS FOR $2d$ DIRECTED POLYMERS

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Abstract. We study the $2d$ directed polymer in random environment in a novel *quasi-critical regime*, which interpolates between the much studied sub-critical and critical regimes. We prove Edwards-Wilkinson fluctuations throughout the quasi-critical regime, showing that the diffusively rescaled partition functions are asymptotically Gaussian, under a rescaling which diverges arbitrarily slowly as criticality is approached. A key challenge is the lack of hypercontractivity, which we overcome deriving new sharp moment estimates.

1. Introduction

We consider the partition functions of the $2d$ directed polymer in random environment:

$$Z^{\omega}_{N,\beta}(z) := \mathrm{E}\big[\mathrm{e}^{\sum_{n=1}^{N}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\big|\, S_0 = z\big]\,, \tag{1.1}$$

where $N \in \mathbb{N}$ is the system size, $\beta \geqslant 0$ is the disorder strength, $z \in \mathbb{Z}^2$ is the starting point, and we have two independent sources of randomness:

- $S = (S_n)_{n\geqslant 0}$ is the simple random walk on $\mathbb{Z}^2$ with law P and expectation E;
- $\omega = (\omega(n,z))_{n\in\mathbb{N},\, z\in\mathbb{Z}^2}$ are i.i.d. random variables with law $\mathbb{P}$, independent of S, with

$$\mathbb{E}[\omega] = 0\,, \qquad \mathbb{E}[\omega^2] = 1\,, \qquad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega}] < \infty \quad \text{for } \beta > 0\,. \tag{1.2}$$

The factor $\lambda(\beta)$ in (1.1) has the effect to normalise the expectation:

$$\mathbb{E}\big[Z^{\omega}_{N,\beta}(z)\big] = 1\,. \tag{1.3}$$

Note that $(Z^{\omega}_{N,\beta}(z))_{z\in\mathbb{Z}^2}$ is a family of (correlated) positive random variables, depending on the random variables ω which play the role of *disorder* (or *random environment*).

In this paper we investigate the *diffusively rescaled* partition functions $Z^{\omega}_{N,\beta}(\lfloor\sqrt{N}x\rfloor)$, where $\lfloor\cdot\rfloor$ denotes the integer part. For an integrable test function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ we set

$$Z^{\omega}_{N,\beta}(\varphi) := \int_{\mathbb{R}^2} Z^{\omega}_{N,\beta}(\lfloor\sqrt{N}x\rfloor)\,\varphi(x)\,\mathrm{d}x = \frac{1}{N}\sum_{z\in\mathbb{Z}^2} Z^{\omega}_{N,\beta}(z)\,\varphi_N(z)\,, \tag{1.4}$$

where for $R > 0$ we define $\varphi_R : \mathbb{Z}^2 \to \mathbb{R}$ by

$$\varphi_R(z) := \int\limits_{[z,z+(1,1))} \varphi\big(\tfrac{y}{\sqrt{R}}\big)\,\mathrm{d}y \tag{1.5}$$

(note that $\varphi_R(z) \approx \varphi\big(\frac{z}{\sqrt{R}}\big)$ if φ is continuous). We look for the convergence in distribution of $Z^{\omega}_{N,\beta}(\varphi)$ as $N \to \infty$, under an appropriate rescaling of the disorder strength $\beta = \beta_N$.

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NOTATION. We denote by $\varphi \in C_c(\mathbb{R}^2)$ the space of functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ that are continuous and compactly supported. We write $a_N \ll b_N$, $a_N \sim b_N$, $a_N \gg b_N$ to mean that the ratio a_N/b_N converges respectively to $0, 1, \infty$ as $N \to \infty$.

1.1. The phase transition. It is known since [CSZ17b] that the partition functions undergo a *phase transition* on the scale $\beta^2 = \beta_N^2 = O(\frac{1}{\log N})$, that we now recall.

Let R_N be the *expected replica overlap* of two independent simple random walks S, S':

$$R_N := \mathrm{E}^{\otimes 2}\Bigg[\sum_{n=1}^N \mathbb{1}_{\{S_n = S'_n\}}\Bigg] = \sum_{n=1}^N \mathrm{P}(S_{2n} = 0) = \frac{\log N}{\pi} + O(1)\,, \tag{1.6}$$

see the local limit theorem (3.8). Using the more convenient parameter

$$\sigma_\beta^2 := \mathbb{V}\mathrm{ar}[\mathrm{e}^{\beta\omega - \lambda(\beta)}] = \mathrm{e}^{\lambda(2\beta) - 2\lambda(\beta)} - 1 \tag{1.7}$$

(note that $\sigma_\beta \sim \beta$ as $\beta \downarrow 0$, since $\lambda(\beta) \sim \frac{1}{2}\beta^2$), we can rescale $\beta = \beta_N$ as follows:

$$\sigma_{\beta_N}^2 = \frac{\hat\beta^2}{R_N} \sim \frac{\hat\beta^2\,\pi}{\log N}\,, \qquad \text{with} \quad \hat\beta \in (0,\infty)\,. \tag{1.8}$$

Let us recall some key results on the scaling limit of $Z_{N,\beta}^\omega(\varphi)$ from (1.4) for $\beta = \beta_N$.

- In the *sub-critical regime* $\hat\beta < 1$, after centering and rescaling by $\sqrt{\log N}$, the averaged partition function $Z_{N,\beta_N}^\omega(\varphi)$ is asymptotically Gaussian, see [CSZ17b]:†

$$\hat\beta \in (0,1): \qquad \sqrt{\log N}\,\big\{Z_{N,\beta_N}^\omega(\varphi) - \mathbb{E}[Z_{N,\beta_N}^\omega(\varphi)]\big\} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\sigma_{\varphi,\hat\beta}^2\big)\,, \tag{1.9}$$

  for an explicit limiting variance $\sigma_{\varphi,\hat\beta}^2 \in (0,\infty)$ (which *diverges* as $\hat\beta \uparrow 1$).

- In the *critical regime* $\hat\beta = 1$, actually in the *critical window* $\hat\beta^2 = 1 + O(\frac{1}{\log N})$, the averaged partition function $Z_{N,\beta_N}^\omega(\varphi)$ is asymptotically *non Gaussian*, see [CSZ23]:

$$\hat\beta = 1 + O\big(\tfrac{1}{\log N}\big): \qquad Z_{N,\beta_N}^\omega(\varphi) \xrightarrow[N\to\infty]{d} \mathscr{Z}(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\,\mathscr{Z}(\mathrm{d}x)\,, \tag{1.10}$$

  where $\mathscr{Z}(\mathrm{d}x)$ is a non-trivial random measure on $\mathbb{R}^2$ called the Stochastic Heat Flow.

Note that the sub-critical convergence (1.9) involves a rescaling factor $\sqrt{\log N}$, while *no rescaling is needed for the critical convergence* (1.10). In view of this discrepancy, it is natural to investigate the transition between these regimes.

1.2. Main result. To interpolate between the sub-critical regime $\hat\beta < 1$ and the critical regime $\hat\beta = 1$, we consider a *quasi-critical regime* in which $\hat\beta \uparrow 1$ *but slower than the critical window* $\hat\beta^2 = 1 + O(\frac{1}{\log N})$. Recalling (1.6) and (1.8), we fix $\beta = \beta_N$ such that

$$\sigma_{\beta_N}^2 = \frac{1}{R_N}\left(1 - \frac{\vartheta_N}{\log N}\right) \qquad \text{for some} \quad 1 \ll \vartheta_N \ll \log N\,. \tag{1.11}$$

(Note that $\vartheta_N = O(1)$ would correspond to the critical window, while $\vartheta_N = (1 - \hat\beta^2)\log N$ with $\hat\beta \in (0,1)$ would correspond to the sub-critical regime.)

Our main result shows that the averaged partition function $Z_{N,\beta_N}^\omega(\varphi)$ has Gaussian fluctuations *throughout the quasi-critical regime* (1.11), after centering and rescaling by the

†The result proved in [CSZ17b, Theorem 2.13] actually involves a space-time average, but the same result for the space average as in (1.4) follows by similar arguments, see [CSZ20].

factor $\sqrt{\vartheta_N}$, whose rate of divergence can be arbitrarily slow. This shows that *non-Gaussian behavior does not appear before the critical regime.* We call this result *Edwards-Wilkinson fluctuations* in view of its link with stochastic PDEs, that we discuss in Subsection 1.3.

Theorem 1.1 (Quasi-critical Edwards-Wilkinson fluctuations). *Let $Z^{\omega}_{N,\beta}(\varphi)$ denote the diffusively rescaled and averaged partition function of the 2d directed polymer model, see (1.1) and (1.4), for disorder variables ω which satisfy (1.2). Then, for $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see (1.7) and (1.11), we have the convergence in distribution*

$$\forall \varphi \in C_c(\mathbb{R}^2): \qquad \sqrt{\vartheta_N}\,\big\{Z^{\omega}_{N,\beta_N}(\varphi) - \mathbb{E}[Z^{\omega}_{N,\beta_N}(\varphi)]\big\} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\sigma^2_{\varphi}\big)\,, \tag{1.12}$$

where the limiting variance is given by

$$\sigma^2_{\varphi} := \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,K(x,x')\,\varphi(x')\,\mathrm{d}x\,\mathrm{d}x' \qquad \textit{with} \quad K(x,x') := \int_0^1 \frac{1}{2u}\,\mathrm{e}^{-\frac{|x-x'|^2}{2u}}\,\mathrm{d}u\,. \tag{1.13}$$

Our strategy to prove Theorem 1.1 is inspired by the recent paper [CC22]: we apply a Central Limit Theorem under a Lyapunov condition, which requires to estimate moments of the partition function of order higher than two (see Section 2 for a detailed explanation). A key point is [CC22] is to bound such high moments exploiting the hypercontractivity of polynomial chaos expansions in the sub-critical regime $\hat\beta < 1$. Crucially, *this fails in the quasi-critical regime* (1.11), because the main contribution to the partition function *no longer comes from a finite number of chaotic components* (see Section 3).

This is the key technical difficulty that we face in this paper, for which we need to use model-specific arguments to estimate high moments. To this purpose, we exploit and extend the strategy developed in [GQT21, CSZ23, LZ21+], deriving *novel quantitative estimates* which are essential for our approach (see Sections 4 and 5). We believe that these estimates will find several applications in future research.

1.3. Relevant context and future perspectives. The Gaussian fluctuations for $Z^{\omega}_{N,\beta}(\varphi)$ in Theorem 1.1 are closely connected to a stochastic PDE, the *Edwards-Wilkinson equation*, also known as Stochastic Heat Equation with *additive* noise:

$$\partial_t v^{(\mathsf{s},\mathsf{c})}(t,x) \;=\; \frac{\mathsf{s}}{2}\,\Delta_x v^{(\mathsf{s},\mathsf{c})}(t,x) \;+\; \mathsf{c}\,\dot W(t,x)\,, \tag{1.14}$$

where $\mathsf{s},\mathsf{c} > 0$ are fixed parameters and $\dot W(t,x)$ is space-time white noise. This equation is well-posed in any spatial dimension $d \geqslant 1$: its solution is the Gaussian process

$$v^{(\mathsf{s},\mathsf{c})}(t,x) \;=\; v^{(\mathsf{s},\mathsf{c})}(0,x) \;+\; \mathsf{c}\int_0^t \int_{\mathbb{R}^d} g_{\mathsf{s}(t-u)}(x-z)\,\dot W(u,z)\,\mathrm{d}u\,\mathrm{d}z\,,$$

where $g_t(x) := (2\pi t)^{-d/2}\,\mathrm{e}^{-\frac{|x|^2}{2t}}$ is the heat kernel on $\mathbb{R}^d$. It is known that $x \mapsto v^{(\mathsf{s},\mathsf{c})}(t,x)$ is a (random) function only for $d = 1$, while for $d \geqslant 2$ it is a genuine distribution.

Henceforth we focus on $d = 2$. The solution $v^{(\mathsf{s},\mathsf{c})}(t,\cdot)$ with initial condition $v^{(\mathsf{s},\mathsf{c})}(0,\cdot) \equiv 0$, averaged on test functions $\varphi \in C_c(\mathbb{R}^2)$, is the centered Gaussian process with covariance

$$\mathbb{E}\big[v^{(\mathsf{s},\mathsf{c})}(t,\varphi)\,v^{(\mathsf{s},\mathsf{c})}(t,\psi)\big] = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,K^{(\mathsf{s},\mathsf{c})}_t(x,y)\,\psi(y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where we set

$$K^{(\mathsf{s},\mathsf{c})}_t(x,y) := \mathsf{c}^2\int_0^t g_{2\mathsf{s}u}(x-y)\,\mathrm{d}u = \frac{\mathsf{c}^2}{2\mathsf{s}}\int_0^{2\mathsf{s}t} \frac{1}{2\pi u}\mathrm{e}^{-\frac{|x-y|^2}{2u}}\,\mathrm{d}u\,. \tag{1.15}$$

Comparing with (1.13), we can rephrase our main result (1.12): for any $\varphi \in C_c(\mathbb{R}^2)$

$$\sqrt{\vartheta_N}\,\{Z^\omega_{N,\beta_N}(\varphi) - \mathbb{E}[Z^\omega_{N,\beta_N}(\varphi)]\} \xrightarrow[N\to\infty]{d} v^{(\mathsf{s},\mathsf{c})}(1,\varphi) \qquad \text{with } \begin{cases} \mathsf{s} = \frac{1}{2}\,, \\ \mathsf{c} = \sqrt{\pi}\,. \end{cases} \tag{1.16}$$

In other term, *the diffusively rescaled partition functions in the quasi-critical regime converge, after centering and rescaling, to the solution of the Edwards-Wilkinson equation.*

Remark 1.2. *Also relation* (1.9)*, in the sub-critical regime $\hat\beta \in (0,1)$, can be rephrased as a convergence to the Edwards-Wilkinson solution $v^{(\mathsf{s},\hat{\mathsf{c}})}(1,\varphi)$ with $\hat{\mathsf{c}} = \sqrt{\pi}\,\hat\beta/\sqrt{1-\hat\beta^2}$.*

The reason why stochastic PDEs emerge naturally in the study of directed polymers is that, by the Markov property of simple random walk, the diffusively rescaled partition function $u_N(t,x) := Z^\omega_{\lfloor Nt\rfloor,\beta}(\lfloor\sqrt{N}x\rfloor)$ solves (up to a time reversal) a *discretized version* of the Stochastic Heat Equation with *multiplicative noise*:

$$\partial_t u(t,x) \;=\; \frac{1}{2}\Delta_x u(t,x) \;+\; \beta\,\dot W(t,x)\,u(t,x)\,, \tag{1.17}$$

with initial condition $u(0,x) = 1$. This gives a hint how the Edwards-Wilkinson equation (1.14) may arise in the scaling limit of directed polymer partition functions: intuitively, the singular product $\dot W(t,x)\,u(t,x)$ in (1.17) for $u(t,x) = u_N(t,x)$ converges to an independent white noise as $N \to \infty$ (see [CC22, Theorem 3.4] in the sub-critical regime).

Edwards-Wilkinson fluctuations were recently proved also for a *non-linear* Stochastic Heat Equation, see [DG22, T22+], always in the sub-critical regime. It would be interesting to extend these results in the quasi-critical regime, generalizing our Theorem 1.1.

Remark 1.3. *The multiplicative Stochastic Heat Equation* (1.17) *in the continuum is well-posed in one space dimension $d = 1$, e.g. by classical Ito-Walsh stochastic integration, but* it is ill-defined in higher dimensions $d \geqslant 2$. *For this reason, directed polymer partition functions can provide precious insight on the equation* (1.17)*. In particular, for $d = 2$, their scaling limit in the critical regime was obtained in* [CSZ23] *and called the critical* 2d *Stochastic Heat Flow, see* (1.10)*, as a natural candidate for the ill-defined solution of* (1.17)*.*

In the same spirit, the log-partition function $h_N(t,x) := \log Z^\omega_{\lfloor Nt\rfloor,\beta}(\lfloor\sqrt{N}x\rfloor)$ provides a discretized approximation for the *Kardar-Parisi-Zhang (KPZ) equation* [KPZ86]:

$$\partial_t h(t,x) \;=\; \frac{1}{2}\Delta_x h(t,x) \;+\; \frac{1}{2}|\nabla h(t,x)|^2 \;+\; \beta\,\dot W(t,x)\,,$$

with initial condition $h(0,x) = 0$. This equation too, in the continuum, is only fully understood in one space-dimension $d = 1$, via recent breakthrough techniques of regularity structures [H14] or paracontrolled distributions [GIP15, GP17]; see also [GJ14, K16]. Similar to (1.9), Edwards-Wilkinson fluctuations have been proved for $h_N(t,x)$ in the entire sub-critical regime (1.8) with $\hat\beta \in (0,1)$ [CSZ20, G20, CD20]: for $\varphi \in C_c(\mathbb{R}^2)$

$$\sqrt{\log N}\,\big\{\log Z^\omega_{N,\beta_N}(\varphi) - \mathbb{E}[\log Z^\omega_{N,\beta_N}(\varphi)]\big\} \xrightarrow[N\to\infty]{d} v^{(\mathsf{s},\hat{\mathsf{c}})}(1,\varphi)\,, \tag{1.18}$$

with s, $\hat{\mathsf{c}}$ as in Remark 1.2. This was recently extended in [NN23], which focuses on a mollification (rather than discretization) of the Stochastic Heat Equation (1.17): phrased in our setting, the results of [NN23] prove Gaussian fluctuations in the sub-critical regime for general transformations $F(Z^\omega_{N,\beta_N})$, besides $F(z) = \log z$, with general initial conditions.

It would be very interesting to extend (1.18) to the quasi-critical regime (1.11), namely to prove an analogue of our Theorem 1.18 for $\log Z^{\omega}_{N,\beta_N}(\varphi)$, which we expect to hold. A natural strategy would be to generalize the linearization procedure established in [CSZ20] to handle the logarithm. This requires estimating *negative moments* of the partition function, which is a challenge in the quasi-critical regime (since $Z^{\omega}_{N,\beta_N}(z) \to 0$ for fixed $z \in \mathbb{Z}^2$).

Local averages on *sub-diffusive scales* have also been investigated for the mollified KPZ solution in the sub-critical regime, see [C23, T23+]. Similar results can be expected for the mollified solution of the Stochastic Heat Equation (1.17), or for the directed polymer partition function, which should be obtainable in the sub-critical regime as in [CSZ17b]. It would be natural to study such local averages also in the quasi-critical regime.

We finally mention that Edwards-Wilkinson fluctuations like (1.9) and (1.18) have also been obtained in higher dimensions $d \geqslant 3$, in the so-called L^2*-weak disorder phase* where the partition function has bounded second moment [CN21, LZ22, CNN22, CCM21+], see also the previous works [MU18, GRZ18, CCM20, DGRZ20]. Unlike the two-dimensional setting, for $d \geqslant 3$ the partition function admits a non-zero limit also *beyond the* L^2*-weak disorder phase*: see [J22, J22+] for recent results in this challenging regime. It would be natural to investigate whether our approach can also be applied in higher dimensions $d \geqslant 3$, in order to prove Gaussian fluctuations *slightly beyond* the L^2-weak disorder phase.

1.4. Organization of the paper. The paper is structured as follows.

- In Section 2 we present the structure of the proof of Theorem 1.1 based on two key steps, formulated as Propositions 2.1 and 2.2.
- In Section 3 we prove Proposition 2.1.
- In Section 4 we derive upper bounds on the moments of the partition functions.
- In Section 5 we prove Proposition 2.2.
- Finally, some technical points are deferred to Appendix A.

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2. Proof of Theorem 1.1

Let us call X_N the LHS of (1.12): recalling (1.4) and (1.3), we can write

$$\begin{aligned} X_N &:= \sqrt{\vartheta_N}\,\big\{Z^{\omega}_{N,\beta_N}(\varphi) - \mathbb{E}[Z^{\omega}_{N,\beta_N}(\varphi)]\big\} \\ &= \frac{\sqrt{\vartheta_N}}{N} \sum_{z\in\mathbb{Z}^2} \big\{Z^{\omega}_{N,\beta_N}(z) - 1\big\}\,\varphi_N\big(\tfrac{z}{\sqrt{N}}\big)\,, \end{aligned} \tag{2.1}$$

with φ_N as in (1.5). In this section, we prove Theorem 1.1 via the following two main steps:

(1) we first approximate X_N in L^2 by a sum $\sum_{i=1}^M X^{(i)}_{N,M}$ of *independent* random variables, for $M = M_N \to \infty$ slowly enough;

(2) we then show that the random variables $(X^{(i)}_{N,M})_{1\leqslant i\leqslant M}$ for $M = M_N$ satisfy the assumptions of the classical *Central Limit Theorem* for triangular arrays.

2.1. First step. In order to define the random variables $X_{N,M}^{(i)}$, for $M \in \mathbb{N}$ and $1 \leqslant i \leqslant M$, we introduce a variation of (1.1), for $-\infty < A < B < \infty$:

$$Z_{(A,B],\beta}^{\omega}(z) := \mathrm{E}\big[\mathrm{e}^{\sum_{n\in(A,B]\cap\mathbb{N}}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\big|\, S_0 = z\big]\,. \tag{2.2}$$

We then define $X_{N,M}^{(i)}$ replacing $Z_{N,\beta}^{\omega}$ by $Z^{\omega}_{(\frac{i-1}{M}N,\frac{i}{M}N],\beta}$ in the definition (2.1) of X_N:

$$X_{N,M}^{(i)} = \frac{\sqrt{\vartheta_N}}{N} \sum_{z\in\mathbb{Z}^2} \big\{Z^{\omega}_{(\frac{i-1}{M}N,\frac{i}{M}N],\beta_N}(z) - 1\big\}\, \varphi_N\big(\tfrac{z}{\sqrt{N}}\big)\,. \tag{2.3}$$

Note that $Z_{(A,B],\beta}^{\omega}(z)$ only depends on $\omega(n,x)$ for $A < n \leqslant B$, moreover $\mathbb{E}[Z_{(A,B],\beta}^{\omega}(z)] = 1$. As a consequence, $X_{N,M}^{(i)}$ for $1 \leqslant i \leqslant M$ are *independent* and *centered* random variables.

The core of this first step is the following approximation result, proved in Section 3.

Proposition 2.1 (L^2 approximation). *For $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see* (1.7) *and* (1.11)*, the following relations hold for any $\varphi \in C_c(\mathbb{R}^2)$, with σ_φ^2 as in* (1.13)*:*

$$\lim_{N\to\infty} \mathbb{E}\big[X_N^2\big] = \sigma_\varphi^2\,, \qquad \forall M\in\mathbb{N}: \quad \lim_{N\to\infty} \Big\| X_N - \sum_{i=1}^{M} X_{N,M}^{(i)} \Big\|_{L^2} = 0\,. \tag{2.4}$$

From the second relation in (2.4) it follows that, for any $(M_N)_{N\in\mathbb{N}}$ with $M_N \to \infty$ slowly enough as $N \to \infty$ (see [CC22, Remark 4.2]),

$$\lim_{N\to\infty} \Big\| X_N - \sum_{i=1}^{M_N} X_{N,M_N}^{(i)} \Big\|_{L^2} = 0\,, \tag{2.5}$$

that is we approximate X_N in L^2 by a sum of independent and centered random variables. We then obtain, by the first relation in (2.4),

$$\lim_{N\to\infty} \mathbb{E}\Bigg[\bigg(\sum_{i=1}^{M_N} X_{N,M_N}^{(i)}\bigg)^2\Bigg] = \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big(X_{N,M_N}^{(i)}\big)^2\Big] = \sigma_\varphi^2\,. \tag{2.6}$$

2.2. Second step. Recalling (2.1), we can rephrase our goal (1.12) as $X_N \xrightarrow{d} \mathcal{N}(0,\sigma_\varphi^2)$. In view of (2.5), this follows if we prove the convergence in distribution

$$\sum_{i=1}^{M_N} X_{N,M_N}^{(i)} \xrightarrow[N\to\infty]{d} \mathcal{N}\big(0\,,\sigma_\varphi^2\big)\,. \tag{2.7}$$

Since $(X_{N,M_N}^{(i)})_{1\leqslant i\leqslant M_N}$ are independent and centered, we apply the classical Central Limit Theorem for triangular arrays, see e.g. [Bil95, Theorem 27.3]: since we have convergence of the variance by (2.6), it is enough to check the Lyapunov condition

$$\text{for some } p > 2: \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,M_N}^{(i)}\big|^p\Big] = 0\,. \tag{2.8}$$

This follows from the next result, proved in Section 4, where we focus on the case $p = 4$.

Proposition 2.2 (Fourth moment bound). *For $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see* (1.7) *and* (1.11)*, and for any $\varphi \in C_c(\mathbb{R}^2)$, there is a constant $C < \infty$ such that*

$$\forall M\in\mathbb{N}\,, \quad \forall 1 \leqslant i \leqslant M: \qquad \limsup_{N\to\infty} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^4\Big] \leqslant \frac{C}{M^2}\,. \tag{2.9}$$

Since the estimate (2.9) holds for any fixed M, it follows that we can let $M_N \to \infty$ slowly enough as $N \to \infty$ so that

$$\mathbb{E}\Big[\big(X_{N,M_N}^{(i)}\big)^4\Big] \leqslant \frac{2C}{M_N^2} \quad \forall i = 1, \ldots, M_N \,.$$

This shows that (2.8) holds with $p = 4$ (the sum therein is $\leqslant 2C/M_N \to 0$ as $N \to \infty$).

The proof of Theorem 1.1 is then completed once we prove Propositions 2.1 and 2.2. The next sections are devoted to these tasks.

3. Second moment bounds: proof of Proposition 2.1

In this section we prove Proposition 2.1 exploiting a polynomial chaos expansion of the partition function. We fix $(\beta_N)_{N\in\mathbb{N}}$ in the quasi-critical regime, see (1.7) and (1.11), and $\varphi \in C_c(\mathbb{R}^2)$. We denote by $C, C', \ldots$ generic constants that may vary from place to place.

3.1. Polynomial chaos expansion. The partition function admits a key polynomial chaos expansion [CSZ17a]. Let us define, for $\beta > 0$,

$$\xi_\beta(n,x) := \mathrm{e}^{\beta\omega(n,x)-\lambda(\beta)} - 1 \,, \qquad \text{for } n \in \mathbb{N}\,,\ x \in \mathbb{Z}^2 \,. \tag{3.1}$$

Recalling (1.7), we note that $(\xi_\beta(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$ are independent random variables with

$$\mathbb{E}[\xi_\beta] = 0\,, \qquad \mathbb{E}[\xi_\beta^2] = \sigma_\beta^2\,, \qquad \mathbb{E}[|\xi_\beta|^k] \leqslant C_k\,\sigma_\beta^k \quad \forall k \geqslant 3\,, \tag{3.2}$$

for some $C_k < \infty$ (for the bound on $\mathbb{E}[|\xi_\beta|^k]$ see, e.g., [CSZ17a, eq. (6.7)]).

We denote by $q_n(x)$ the random walk transition kernel:

$$q_n(x) := \mathrm{P}(S_n = x \,|\, S_0 = 0)\,. \tag{3.3}$$

Then, writing $\mathrm{e}^{\sum_n\{\beta\omega(n,x)-\lambda(\beta)\}} = \prod_n (1 + \xi_\beta(n,x))$ and expanding the product, we can write $Z^\omega_{(A,B],\beta}(z)$ in (2.2) as the following polynomial chaos expansion:

$$\begin{aligned} Z^\omega_{(A,B],\beta}(z) = 1 + \sum_{k=1}^\infty \sum_{\substack{A<n_1<\ldots<n_k\leqslant B\\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q_{n_1}(x_1 - z)\,\xi_\beta(n_1,x_1)\times \\ & \times \prod_{j=2}^k q_{n_j-n_{j-1}}(x_j - x_{j-1})\,\xi_\beta(n_j,x_j)\,, \end{aligned} \tag{3.4}$$

where we agree that the time variables $n_1 < \ldots < n_k$ are summed in the set $(A,B] \cap \mathbb{Z}$ (in particular, the seemingly infinite sum over k can be stopped at $B - A$).

Plugging (3.4) into (2.1), we obtain a corresponding polynomial chaos expansion for X_N, recall (2.1) and (1.5): if we define the averaged random walk transition kernel

$$q_n^f(x) := \sum_{z\in\mathbb{Z}^2} q_n(x - z)\,f(z)\,, \qquad \text{for } \ f : \mathbb{Z}^2 \to \mathbb{R}\,, \tag{3.5}$$

we obtain

$$X_N = \frac{\sqrt{\vartheta_N}}{N} \sum_{k=1}^\infty \sum_{\substack{0<n_1<\ldots<n_k\leqslant N\\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1)\,\xi_{\beta_N}(n_1,x_1) \prod_{j=2}^k q_{n_j-n_{j-1}}(x_j - x_{j-1})\,\xi_{\beta_N}(n_j,x_j)\,. \tag{3.6}$$

The analogous polynomial chaos expansion for the random variables $X^{(i)}_{N,M}$, see (2.3), is obtained from (3.6) restricting the sum to $\frac{i-1}{M}N < n_1 < \ldots < n_k \leqslant \frac{i}{M}N$:

$$
\begin{aligned}
X^{(i)}_{N,M} = \frac{\sqrt{\vartheta_N}}{N} \sum_{k=1}^{\infty} \sum_{\substack{\frac{i-1}{M}N < n_1 < \ldots < n_k \leqslant \frac{i}{M}N \\ x_1,\ldots,x_k \in \mathbb{Z}^2}} & q^{\varphi_N}_{n_1}(x_1)\, \xi_{\beta_N}(n_1, x_1) \times \\
& \times \prod_{j=2}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1})\, \xi_{\beta_N}(n_j, x_j)\,.
\end{aligned} \tag{3.7}
$$

Remark 3.1. *Since the random variables* $(\xi_\beta(n,x))_{n\in\mathbb{N}, x\in\mathbb{Z}^2}$ *are independent and centered, see* (3.1)*, the terms in the polynomial chaos* (3.4), (3.6), (3.7) *are orthogonal in* L^2*.*

We finally recall the local limit theorem for the simple random walk on $\mathbb{Z}^2$, see [LL10, Theorem 2.1.3]: as $n \to \infty$, uniformly for $x \in \mathbb{Z}^2$ we have†

$$
q_n(x) = \frac{1}{n/2}\Big(g\Big(\tfrac{x}{\sqrt{n/2}}\Big) + o(1)\Big)\, 2\, \mathbb{1}_{(n,x)\in\mathbb{Z}^3_{\mathrm{even}}}\,, \qquad \text{where} \qquad g(y) := \frac{\mathrm{e}^{-\frac{1}{2}|y|^2}}{2\pi}, \tag{3.8}
$$

and we set $\mathbb{Z}^3_{\mathrm{even}} := \big\{\, y = \big(y_1,\, y_2,\, y_3\big) \in \mathbb{Z}^3 \colon y_1 + y_2 + y_3 \in 2\mathbb{Z} \,\big\}$.

3.2. Proof of Proposition 2.1. Note that $\sum_{i=1}^M X^{(i)}_{N,M}$ is a polynomial chaos where all time variables $n_1 < \ldots < n_k$ belong to *one of the intervals* $(\frac{i-1}{M}N\,,\, \frac{i}{M}N]$, see (3.7). It follows that X_N is a *larger polynomial chaos* than $\sum_{i=1}^M X^{(i)}_{N,M}$, i.e. it contains more terms, hence *the difference* $X_N - \sum_{i=1}^M X^{(i)}_{N,M}$ *is orthogonal in* L^2 *to* $\sum_{i=1}^M X^{(i)}_{N,M}$ (see Remark 3.1):

$$
\Big\| X_N - \sum_{i=1}^M X^{(i)}_{N,M} \Big\|^2_{L^2} = \|X_N\|^2_{L^2} - \Big\| \sum_{i=1}^M X^{(i)}_{N,M} \Big\|^2_{L^2} = \|X_N\|^2_{L^2} - \sum_{i=1}^M \|X^{(i)}_{N,M}\|^2_{L^2}.
$$

As a consequence, to prove our goal (2.4) it is enough to show that

$$
\lim_{N\to\infty} \mathbb{E}\big[X_N^2\big] = \sigma^2_\varphi\,, \qquad \forall M \in \mathbb{N}: \quad \lim_{N\to\infty} \sum_{i=1}^M \mathbb{E}\Big[\big(X^{(i)}_{N,M}\big)^2\Big] = \sigma^2_\varphi\,, \tag{3.9}
$$

where we recall that σ^2_φ is defined in (1.13). The first relation in (3.9) follows from the second one, because $X_N = X^{(1)}_{N,1}$. Then the proof is completed by the next result. ☐

Lemma 3.2 (Quasi-critical variance). *Fix* $(\beta_N)_{N\in\mathbb{N}}$ *in the quasi-critical regime, see* (1.7) *and* (1.11)*, and* $\varphi \in C_c(\mathbb{R}^2)$*. For any* $M \in \mathbb{N}$*, the following holds for all* $i = 1, \ldots, M$*:*

$$
\lim_{N\to\infty} \mathbb{E}\big[\big(X^{(i)}_{N,M}\big)^2\big] = \sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]} := \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x')\left(\int_{\frac{i-1}{M}}^{\frac{i}{M}} \frac{1}{2u}\, \mathrm{e}^{-\frac{|x-x'|^2}{2u}}\, \mathrm{d}u\right) \mathrm{d}x\, \mathrm{d}x'\,. \tag{3.10}
$$

Proof. Let us fix $M \in \mathbb{N}$ and $1 \leqslant i \leqslant M$. We split the proof of (3.10) in the two bounds

$$
\limsup_{N\to\infty} \mathbb{E}\Big[\big(X^{(i)}_{N,M}\big)^2\Big] \leqslant \sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]} \tag{3.11}
$$

†The scaling factor in (3.8) is $n/2$ because the covariance matrix of the simple random walk on $\mathbb{Z}^2$ is $\frac{1}{2}I$, while the factor $2\mathbb{1}_{(m,z)\in\mathbb{Z}^3_{\mathrm{even}}}$ is due to periodicity.

and

$$\liminf_{N\to\infty} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \geqslant \sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]} \,. \tag{3.12}$$

We first obtain an exact expression for the second moment of $X_{N,M}^{(i)}$ by (3.7): since the random variables $\xi_\beta(n,x)$ are independent with zero mean and variance σ_β^2, we have

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \frac{\vartheta_N}{N^2} \sum_{k=1}^{\infty} (\sigma^2_{\beta_N})^k \sum_{\substack{\frac{i-1}{M}N<n_1<\ldots<n_k\leqslant\frac{i}{M}N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1)^2 \prod_{j=2}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})^2 \,.$$

We can sum the space variables $x_k, x_{k-1}, \ldots, x_2$ because $\sum_{x\in\mathbb{Z}^2} q_n(x)^2 = q_{2n}(0)$, see (3.3), while to handle the sum over x_1 we note that, recalling (3.5),

$$\sum_{x\in\mathbb{Z}^2} q_n^f(x)^2 = q_{2n}^{f,f} \qquad \text{where we set} \quad q_m^{f,f} := \sum_{z,z'\in\mathbb{Z}^2} q_m(z-z')\, f(z)\, f(z') \,. \tag{3.13}$$

We then obtain

$$\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \vartheta_N \sum_{k=1}^{\infty} (\sigma^2_{\beta_N})^k \sum_{\frac{i-1}{M}N<n_1<\ldots<n_k\leqslant\frac{i}{M}N} \frac{q_{2n_1}^{\varphi_N,\varphi_N}}{N^2} \prod_{j=2}^{k} q_{2(n_j-n_{j-1})}(0) \,. \tag{3.14}$$

We then prove the upper bound (3.11). We rename $n_1 = n$ and enlarge the sum over the other time variables $n_2, \ldots, n_k$, by letting each increment $m_j := n_j - n_{j-1}$ for $j = 2, \ldots, k$ vary in the whole interval $(0, N]$: since $\sum_{m=1}^N q_{2m}(0) = R_N$, see (1.6), we obtain

$$\begin{aligned} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] &\leqslant \vartheta_N \sum_{\frac{i-1}{M}N<n\leqslant\frac{i}{M}N} \frac{q_{2n}^{\varphi_N,\varphi_N}}{N^2} \sum_{k=1}^{\infty} (\sigma^2_{\beta_N})^k (R_N)^{k-1} \\ &= \vartheta_N \left\{ \sum_{\frac{i-1}{M}N<n\leqslant\frac{i}{M}N} \frac{q_{2n}^{\varphi_N,\varphi_N}}{N^2} \right\} \frac{\sigma^2_{\beta_N}}{1-\sigma^2_{\beta_N} R_N} \,, \end{aligned} \tag{3.15}$$

where we summed the geometric series since $\sigma^2_{\beta_N} R_N = 1 - \frac{\vartheta_N}{\log N} < 1$ for large N, by (1.11). We will prove the following Riemann sum approximation, for any given $0 \leqslant a < b \leqslant 1$:

$$\lim_{N\to\infty} \sum_{aN<n\leqslant bN} \frac{q_{2n}^{\varphi_N,\varphi_N}}{N^2} = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,\varphi(x') \left(\int_a^b \frac{1}{u}\, g\Big(\frac{x-x'}{\sqrt{u}}\Big)\, \mathrm{d}u \right) \mathrm{d}x\, \mathrm{d}x' \,, \tag{3.16}$$

where $g(y) = \frac{1}{2\pi}\, \mathrm{e}^{-\frac{1}{2}|y|^2}$ is the standard Gaussian density on $\mathbb{R}^2$, see (3.8). Plugging this into (3.15), since $1 - \sigma^2_{\beta_N} R_N = \frac{\vartheta_N}{\log N}$ and $\sigma^2_{\beta_N} \sim \frac{1}{R_N} \sim \frac{\pi}{\log N}$ as $N \to \infty$ by (1.11) and (1.6), we obtain precisely the upper bound (3.11) (note that $\pi\, \frac{1}{u}\, g(\frac{x-x'}{\sqrt{u}}) = \frac{1}{2u} \exp(-\frac{|x-x'|^2}{2u})$).

Let us now prove (3.16). This is based on the local limit theorem (3.8) as $n \to \infty$, hence the case $a = 0$ could be delicate, as the sum in (3.16) starts from $n = 1$ and, therefore, n needs not be large. For this reason, we first show that small values of n are negligible for (3.16). Since φ is compactly supported, when we plug $f = \varphi_N$ into $q_{2n}^{f,f}$, see (3.13), we can restrict the sums to $|z'| \leqslant C\sqrt{N}$, which yields the following *uniform bound*:

$$\forall m \in \mathbb{N}: \qquad |q_m^{\varphi_N,\varphi_N}| \leqslant \|\varphi\|_\infty^2 \sum_{|z'|\leqslant C\sqrt{N}} \sum_{z\in\mathbb{Z}^2} q_m(z-z') \leqslant C'\, \|\varphi\|_\infty^2\, N \,. \tag{3.17}$$

In particular, the contribution of $n \leqslant \varepsilon N$ to the LHS of (3.16) is $O(\varepsilon)$. As a consequence, it is enough to prove (3.16) when $a > 0$, which we assume henceforth.

Recalling (3.13) and applying (3.8), we can write the LHS of (3.16) as follows:

$$\sum_{aN<n\leqslant bN} \frac{q_{2n}^{\varphi_N,\varphi_N}}{N^2} = \frac{1}{N^2} \sum_{aN<n\leqslant bN} \sum_{\substack{z,z'\in\mathbb{Z}^2: \\ (n,z-z')\in\mathbb{Z}^3_{\mathrm{even}}}} \frac{2}{n}\Big(g\big(\tfrac{z-z'}{\sqrt{n}}\big) + o(1)\Big)\,\varphi\big(\tfrac{z}{\sqrt{N}}\big)\,\varphi\big(\tfrac{z'}{\sqrt{N}}\big)\,,$$

where $o(1) \to 0$ as $N \to \infty$ (because $n > aN \to \infty$ and we assume $a > 0$). The additive term $o(1)$ gives a vanishing contribution as $N \to \infty$, because we can bound $\frac{2}{n} \leqslant \frac{2}{aN}$ and $|\varphi(\cdot)| \leqslant \|\varphi\|_\infty$, and the sums contain $O(N^3)$ terms (since $|z|, |z'| \leqslant C\sqrt{N}$). Introducing the rescaled variables $u := \frac{n}{N}$ and $x := \frac{z}{\sqrt{N}}$, $x' := \frac{z'}{\sqrt{N}}$, we can then rewrite the RHS as

$$\frac{1}{N^3} \sum_{u\in(a,b]\cap\frac{\mathbb{N}}{N}} \sum_{\substack{x,x'\in\frac{\mathbb{Z}^2}{\sqrt{N}}: \\ (Nu,\sqrt{N}(x-x'))\in\mathbb{Z}^3_{\mathrm{even}}}} \frac{2}{u}\Big(g\big(\tfrac{x-x'}{\sqrt{u}}\big)\Big)\,\varphi(x)\,\varphi(x') \;+\; o(1)\,,$$

which is a Riemann sum for the integral in the RHS of (3.16). Note that the restriction $(Nu, \sqrt{N}(x-x')) \in \mathbb{Z}^3_{\mathrm{even}}$ effectively *halves* the range of the sum: indeed, for any given u and x, the sum over $x' = \frac{z'}{\sqrt{N}} \in \frac{\mathbb{Z}^2}{\sqrt{N}}$ is restricted to points $z' \in \mathbb{Z}^2$ with a fixed parity (even or odd, depending on u, x). This restriction is compensated by the multiplicative factor 2, which disappears as we let $N \to \infty$. This completes the proof of (3.16).

We finally prove the lower bound (3.12). We fix $\varepsilon > 0$ small enough and we bound the RHS of (3.14) from below as follows:

- we rename $n = n_1$ and we restrict its sum to the interval $\big(\frac{i-1}{M}N\,,\,(1-\varepsilon)\frac{i}{M}N\big]$;
- for $k \geqslant 2$, we introduce the "displacements" $m_j := n_j - n_1$ from n_1, for $j = 2, \ldots, k$, and we restrict the sum over $n_2, \ldots, n_k$ to the set $0 < m_2 < \ldots < m_k \leqslant \varepsilon \frac{i}{M} N$.

We thus obtain by (3.14)

$$\begin{aligned} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \geqslant \vartheta_N & \sum_{\frac{i-1}{M}N<n\leqslant(1-\varepsilon)\frac{i}{M}N} \frac{q_{2n}^{\varphi_N,\varphi_N}}{N^2} \times \\ & \times \Bigg(\sigma_{\beta_N}^2 + \sum_{k=2}^{\infty} (\sigma_{\beta_N}^2)^k \sum_{0<m_2<\ldots<m_k\leqslant\varepsilon\frac{i}{M}N} q_{2m_2}(0) \prod_{j=3}^{k} q_{2(m_j-m_{j-1})}(0)\Bigg). \end{aligned} \tag{3.18}$$

We now give a probabilistic interpretation to the sum over $m_2, \ldots, m_k$: following [CSZ19a] and recalling (1.6), given $N \in \mathbb{N}$ we define i.i.d. random variables $(T_i^{(N)})_{i\in\mathbb{N}}$ with distribution

$$\mathrm{P}\big(T_i^{(N)} = n\big) \;=\; \frac{q_{2n}(0)}{R_N}\,\mathbb{1}_{\{1,\ldots,N\}}(n)\,, \tag{3.19}$$

so that the second line of (3.18) can be written, renaming $\ell = k-1$, as

$$\begin{aligned} & \sigma_{\beta_N}^2\Bigg(1 \,+\, \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell\, \mathrm{P}\Big(T_1^{(N)} \,+\ldots+\, T_\ell^{(N)} \,\leqslant\, \varepsilon\,\tfrac{i}{M}N\Big)\Bigg) \\ & \quad = \sigma_{\beta_N}^2\Bigg(\frac{1}{1-\sigma_{\beta_N}^2 R_N} - \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell\, \mathrm{P}\Big(T_1^{(N)} \,+\ldots+\, T_\ell^{(N)} \,>\, \varepsilon\,\tfrac{i}{M}N\Big)\Bigg). \end{aligned} \tag{3.20}$$

Plugging this into (3.18) and recalling (3.17), we obtain

$$\begin{aligned}\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \geqslant \vartheta_N \left\{ \sum_{\frac{i-1}{M}N<n\leqslant(1-\varepsilon)\frac{i}{M}N} \frac{q_{2n}^{\varphi_N,\varphi_N}}{N^2} \right\} \frac{\sigma_{\beta_N}^2}{1-\sigma_{\beta_N}^2 R_N} \\ - \big(C' \,\|\varphi\|_\infty^2\big)\, \vartheta_N\, \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell\, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M}N\Big)\,.\end{aligned} \tag{3.21}$$

The first term in the RHS is similar to (3.15), just with $(1-\varepsilon)\frac{i}{M}$ instead of $\frac{i}{M}$, therefore *we already proved that it converges to* $\sigma^2_{\varphi,(\frac{i-1}{M},(1-\varepsilon)\frac{i}{M}]}$ *as* $N\to\infty$, see (3.16) and the following lines (recall also (3.10)). Letting $\varepsilon\downarrow 0$ after $N\to\infty$ we recover $\sigma^2_{\varphi,(\frac{i-1}{M},\frac{i}{M}]}$, hence to prove (3.12) we just need to show that the second term in the RHS of (3.21) is negligible:

$$\lim_{N\to\infty} \vartheta_N\, \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell\, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M}N\Big) = 0\,. \tag{3.22}$$

Recall that the random variables $(T_i^{(N)})_{i\in\mathbb{N}}$ are i.i.d. with distribution (3.19). Since $q_{2n}(0)\leqslant\frac{C}{n}$ by the local limit theorem (3.8), we have $\mathrm{E}[T_i^{(N)}] = \frac{1}{R_N}\sum_{n=1}^N n\, q_{2n}(0) \leqslant C\frac{N}{R_N}$ and, by Markov's inequality, we can bound

$$\mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M}N\Big) \leqslant \frac{\mathrm{E}\big[T_1^{(N)} + \ldots + T_\ell^{(N)}\big]}{\frac{\varepsilon}{M}N} \leqslant \frac{C\,\ell}{\frac{\varepsilon}{M}\, R_N}\,.$$

Since $\sum_{\ell=1}^\infty \ell\, x^\ell = \frac{x}{(1-x)^2}$, we obtain

$$\begin{aligned}\vartheta_N\, \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^\ell\, \mathrm{P}\Big(T_1^{(N)} + \ldots + T_\ell^{(N)} > \tfrac{\varepsilon}{M}N\Big) &\leqslant \vartheta_N\, \sigma_{\beta_N}^2 \frac{C}{\frac{\varepsilon}{M}\, R_N} \frac{\sigma_{\beta_N}^2 R_N}{(1-\sigma_{\beta_N}^2 R_N)^2} \\ &= \frac{C\,M}{\varepsilon} \frac{\vartheta_N\, (\sigma_{\beta_N}^2)^2}{(1-\sigma_{\beta_N}^2 R_N)^2}\,.\end{aligned}$$

Note that $1-\sigma_{\beta_N}^2 R_N = \frac{\vartheta_N}{\log N}$ and $\sigma_{\beta_N}^2 \sim \frac{1}{R_N} \sim \frac{\pi}{\log N}$ by (1.11) and (1.6), hence the last term is asymptotically equivalent to $\frac{CM}{\varepsilon}\frac{\pi^2}{\vartheta_N} \to 0$ as $N\to\infty$, since $\vartheta_N\to\infty$, see (1.11). This shows that (3.22) holds and completes the proof of Proposition 2.1. $\square$

4. General moment bounds

In this section we estimate the *moments of the partition function* $Z_{L,\beta}^\omega$ through a refinement of the operator approach from [CSZ23, Theorem 6.1] and [LZ21+, Theorem 1.3] (inspired by [GQT21]). We point out that these papers deal with the critical and sub-critical regimes, while we are interested the quasi-critical regime (1.11).

For transparency, and in view of future applications, we develop in this section a *non asymptotic approach which is independent of the regime of* β: we obtain bounds with explicit constants which hold for any given system size L and disorder strength β. Some novelties with respect to [CSZ23, LZ21+] are described in Remarks 4.4, 4.7, 4.9. These bounds will be crucially applied in Section 5 to prove Proposition 2.2.

The section is organised as follows:

- in Subsection 4.1 we give an *exact expansion* for the moments, see Theorem 4.5, in terms of suitable operators linked to the random walk and the disorder;
- Subsection 4.2 we deduce *upper bounds* for the moments, see Theorems 4.8 and 4.11, which depend on two pairs of quantities, that we call *boundary terms* and *bulk terms*;
- in Subsection 4.3 we state some basic random walk bounds needed in our analysis (we consider general symmetric random walks with sub-Gaussian tails);
- in Subsections 4.4 and 4.5 we obtain explicit estimates on the boundary terms and bulk terms, which plugged in Theorem 4.11 yield explicit bounds on the moments.

4.1. Moment expansion. The partition function $Z^{\omega}_{(A,B],\beta}(z)$ in (2.2) is called "point-to-plane", since random walk paths start at $S_0 = z$ but have no constrained endpoint. We introduce a "point-to-point" version, for simplicity when $(A, B] = (0, L]$ for $L \in \mathbb{N}$, restricting to random walk paths with a fixed endpoint $S_L = w$:

$$Z^{\omega}_{L,\beta}(z, w) := \mathrm{E}\Big[\mathrm{e}^{\sum_{n=1}^{L-1}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\, \mathbb{1}_{\{S_L=w\}}\,\Big|\, S_0 = z\Big] \tag{4.1}$$

(we stop the sum at $n = L - 1$ for later convenience).

Given two "boundary conditions" $f, g : \mathbb{Z}^2 \to \mathbb{R}$, we define the averaged version

$$\mathcal{Z}^{\omega}_{L,\beta}(f, g) := \sum_{z,w\in\mathbb{Z}^2} f(z)\, Z^{\omega}_{L,\beta}(z, w)\, g(w)\,, \tag{4.2}$$

where we use a different font to avoid confusions with the diffusively rescaled average (1.4). We focus on the *centred moments* of $Z^{\omega}_{L,\beta}(f, g)$, that we denote by

$$\mathcal{M}^{h}_{L,\beta}(f, g) := \mathbb{E}\Big[\big(\mathcal{Z}^{\omega}_{L,\beta}(f, g) - \mathbb{E}[\mathcal{Z}^{\omega}_{L,\beta}(f, g)]\big)^h\Big] \qquad \text{for } h \in \mathbb{N}\,. \tag{4.3}$$

Remark 4.1. *Recalling* (2.2)*,* (2.3) *and* (1.5)*,* (3.5)*, by translation invariance we have*

$$\mathbb{E}\left[(X^{(i)}_{N,M})^4\right] = \frac{\vartheta_N^2}{N^4}\, \mathcal{M}^4_{\frac{N}{M},\beta_N}(f_i, g)\,, \qquad \textit{where} \qquad \begin{cases} f_i(z) := q^{\varphi_N}_{\frac{i-1}{M}N}(z)\,, \\ g(w) := 1\,. \end{cases} \tag{4.4}$$

To prove Proposition 2.2, in Section 5 we will focus on $\mathcal{M}^4_{L,\beta}(f, g)$*.*

Henceforth we fix $h \in \mathbb{N}$ with $h \geqslant 2$ (the interesting case is $h \geqslant 3$). We are going to give an *exact expression* for $\mathcal{M}^h_{L,\beta}(f, g)$, see Theorem 4.5. We first need some notation.

We denote by $I \vdash \{1, \ldots, h\}$ a *partition of* $\{1, \ldots, h\}$, i.e. a family $I = \{I^1, \ldots, I^m\}$ of non empty disjoint subsets $I^j \subseteq \{1, \ldots, h\}$ with $I^1 \cup \ldots \cup I^m = \{1, \ldots, h\}$. We single out:

- the unique partition $I = * := \{\{1\}, \{2\}, \ldots, \{h\}\}$ composed by all singletons;
- the $\binom{h}{2}$ partitions of the form $I = \{\{a, b\}, \{c\} : c \neq a, c \neq b\}$, that we call *pairs*.

Example 4.2 (Cases $h = 2, 3, 4$**).** *All partitions* $I \vdash \{1, 2\}$ *are* $I = *$ *and* $I = \{\{1, 2\}\}$*.*

All partitions $I \vdash \{1, 2, 3\}$ *are* $I = *$*, three* pairs $I = \{\{a, b\}, \{c\}\}$ *and* $I = \{\{1, 2, 3\}\}$*.*

All partitions $I \vdash \{1, 2, 3, 4\}$ *are* $I = *$*, six* pairs $I = \{\{a, b\}, \{c\}, \{d\}\}$*, six* double pairs $I = \{\{a, b\}, \{c, d\}\}$*, four* triples $I = \{\{a, b, c\}, \{d\}\}$ *and the* quadruple $I = \{\{1, 2, 3, 4\}\}$*.*

Given a partition $I = \{I^1, \ldots, I^m\} \vdash \{1, \ldots, h\}$, we define for $\mathbf{x} = (x^1, \ldots, x^h) \in (\mathbb{Z}^2)^h$

$$\mathbf{x} \sim I \qquad \text{if and only if } \begin{cases} x^a = x^b & \text{if } a, b \in I^i \text{ for some } i\,, \\ x^a \neq x^b & \text{if } a \in I^i, b \in I^j \text{ for some } i \neq j \text{ with } |I^i|, |I^j| \geqslant 2\,. \end{cases} \tag{4.5}$$

For instance $\mathbf{x} \sim \{\{1,2\},\{3\},\{4\}\}$ means $x^1 = x^2$, while $\mathbf{x} \sim \{\{1,2\},\{3,4\}\}$ means $x^1 = x^2$ and $x^3 = x^4$ with $x^1 \neq x^3$. Note that $\mathbf{x} \sim *$ imposes no constraint. We also define

$$(\mathbb{Z}^2)^h_I := \big\{\mathbf{x} \in (\mathbb{Z}^2)^h : \mathbf{x} = (x^1, \ldots, x^h) \sim I\big\}\,, \tag{4.6}$$

which is essentially a copy of $(\mathbb{Z}^2)^m$ embedded in $(\mathbb{Z}^2)^h$.

A family $I_1, \ldots, I_r$ of partitions $I_i = \{I_i^1, \ldots, I_i^{m_i}\} \vdash \{1, \ldots, h\}$ is said to have *full support* if any $a \in \{1, \ldots, h\}$ belongs to some partition I_i not as a singleton, i.e. $a \in I_i^j$ with $|I_i^j| \geqslant 2$.

Example 4.3 (Full support for $h = 4$). *A single partition $I_1 \vdash \{1,2,3,4\}$ with full support is either the quadruple $I_1 = \{\{1,2,3,4\}\}$ or a double pair $I_1 = \{\{a,b\},\{c,d\}\}$. There are many families of two partitions $I_1, I_2 \vdash \{1,2,3,4\}$ with full support, for instance two non overlapping pairs such as $I_1 = \{\{1,3\},\{2\},\{4\}\}$, $I_2 = \{\{2,4\},\{1\},\{3\}\}$.*

We now introduce h-fold analogues of the random walk transition kernel (3.3) and of its averaged version (3.5): given partitions $I, J \vdash \{1, \ldots, h\}$, we define for $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$

$$\mathsf{Q}_n^{I,J}(\mathbf{z}, \mathbf{x}) := \mathbb{1}_{\{\mathbf{z} \sim I,\, \mathbf{x} \sim J\}} \prod_{i=1}^h q_n(x^i - z^i)\,, \qquad \mathsf{q}_n^{f,J}(\mathbf{x}) := \mathbb{1}_{\{\mathbf{x} \sim J\}} \prod_{i=1}^h q_n^f(x^i)\,. \tag{4.7}$$

Given $m \in \mathbb{N}_0$ and $J \vdash \{1, \ldots, h\} \neq *$, we define for $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$ the weighted *Green's kernel*

$$\mathsf{U}^J_{m,\beta}(\mathbf{z}, \mathbf{x}) := \begin{cases} \displaystyle\sum_{k=1}^\infty \mathbb{E}[\xi_\beta^J]^k \sum_{\substack{0=:n_0<n_1<\cdots<n_k:=m \\ \mathbf{y}_1,\ldots,\mathbf{y}_{k-1} \in (\mathbb{Z}^2)^h \\ \mathbf{y}_0:=\mathbf{z},\ \mathbf{y}_k:=\mathbf{x}}} \prod_{i=1}^k \mathsf{Q}^{J,J}_{n_i - n_{i-1}}(\mathbf{y}_{i-1}, \mathbf{y}_i) & \text{if } m \geqslant 1\,, \\ \mathbb{1}_{\{\mathbf{z} = \mathbf{x} \sim J\}} & \text{if } m = 0\,, \end{cases} \tag{4.8}$$

where for $J = \{J^1, \ldots, J^m\}$ with $J \neq *$ we define

$$\mathbb{E}[\xi_\beta^J] := \prod_{i:\,|J^i| \geqslant 2} \mathbb{E}[\xi_\beta^{|J^i|}]\,. \tag{4.9}$$

When J is a pair, this reduces to $\mathbb{E}[\xi_\beta^J] = \mathbb{E}[\xi_\beta^2] = \sigma_\beta^2$, see (3.2).

Remark 4.4 (On the definition of U^J). *We point out that U^J was only defined in* [CSZ23, LZ21+] *when J is a pair. Defining U^J for any partition J makes formulas simpler, as it avoids to distinguish between pairs and non-pairs in the sums* (4.12) *and* (4.18)*.*

For a pair $J = \{\{a,b\},\{c\} : c \neq a,b\}$, by Chapman-Kolmogorov we can express

$$\mathsf{U}^J_{m,\beta}(\mathbf{z}, \mathbf{x}) = U_{m,\beta}(x^a - z^a)\, \mathbb{1}_{\{x^b = x^a,\, z^b = z^a\}} \prod_{c \neq a,b} q_m(x^c - z^c)\,, \tag{4.10}$$

where we define for $x \in \mathbb{Z}^2$

$$U_{m,\beta}(x) := \sum_{k=1}^\infty (\sigma_\beta^2)^k \sum_{\substack{0=:n_0<n_1<\cdots<n_k:=m \\ x_0:=0,\, x_1,\ldots,x_{k-1} \in \mathbb{Z}^2,\, x_k:=x}} \prod_{i=1}^k q_{n_i - n_{i-1}}(x_i - x_{i-1})^2\,. \tag{4.11}$$

Given two functions $\mathsf{q}^f(\mathbf{x})$, $\mathsf{q}^g(\mathbf{x})$ and a family of matrices $\mathsf{U}_i(\mathbf{z},\mathbf{x})$, $\mathsf{Q}_i(\mathbf{z},\mathbf{x})$ for $\mathbf{x},\mathbf{z}\in\mathbb{T}$, where $\mathbb{T}$ is a countable set, we use the standard notation

$$\Big\langle \mathsf{q}^f,\, \mathsf{U}_1 \Big\{ \prod_{i=2}^r \mathsf{Q}_i\, \mathsf{U}_i \Big\} \mathsf{q}^g \Big\rangle := \sum_{\substack{\mathbf{z}_1,\ldots,\mathbf{z}_r\in\mathbb{T}\\ \mathbf{z}_1',\ldots,\mathbf{z}_r'\in\mathbb{T}}} \mathsf{q}^f(\mathbf{z}_1)\,\mathsf{U}_1(\mathbf{z}_1,\mathbf{z}_1') \Big\{ \prod_{i=2}^r \mathsf{Q}_i(\mathbf{z}_{i-1}',\mathbf{z}_i)\,\mathsf{U}_i(\mathbf{z}_i,\mathbf{z}_i') \Big\} \mathsf{q}^g(\mathbf{z}_r')\,.$$

We can now give the announced expansion for $\mathcal{M}^h_{L,\beta}(f,g)$, that we prove in Appendix A.

Theorem 4.5 (Moment expansion). *Let $\mathcal{Z}^\omega_{L,\beta}(f,g)$ be the averaged partition function in* (4.2) *with centred moments $\mathcal{M}^h_{L,\beta}(f,g)$, see* (4.3)*. For any $h\in\mathbb{N}$ with $h\geqslant 2$ we have*

$$\begin{aligned}\mathcal{M}^h_{L,\beta}(f,g) = \sum_{r=1}^{\infty} \sum_{0<n_1\leqslant m_1<\cdots<n_r\leqslant m_r<L} \sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\}\\ \text{with full support}\\ \text{and } I_i\neq I_{i-1},\ I_i\neq *\ \forall i}} \Big\{ \prod_{i=1}^r \mathbb{E}[\xi_\beta^{I_i}] \Big\} \times \\ \times \Big\langle \mathsf{q}^{f,I_1}_{n_1},\ \mathsf{U}^{I_1}_{m_1-n_1,\beta} \Big\{ \prod_{i=2}^r \mathsf{Q}^{I_{i-1},I_i}_{n_i-m_{i-1}}\ \mathsf{U}^{I_i}_{m_i-n_i,\beta} \Big\} \mathsf{q}^{g,I_r}_{L-m_r} \Big\rangle .\end{aligned} \tag{4.12}$$

Remark 4.6 (Sanity check). *In case $h=2$, the conditions $I_i\neq I_{i-1}$ and $I_i\neq *$ in* (4.12) *force $r=1$ and $I_1=\{\{1,2\}\}$. Then, recalling* (4.10)-(4.11)*, formula* (4.12) *reduces to*

$$\mathcal{M}^2_{L,\beta}(f,g) = \mathbb{V}\mathrm{ar}[\mathcal{Z}^\omega_{L,\beta}(f,g)] = \sigma_\beta^2 \sum_{\substack{0<n\leqslant m<L\\ z,x\in\mathbb{Z}^2}} q^f_n(z)\, U_{m-n,\beta}(x-z)\, q^g_{L-m}(x)\,,$$

which is a classical expansion for the variance, see e.g. [CSZ23, eq. (3.51)].

Remark 4.7 (Boundary conditions). *In* [CSZ23, LZ21+]*, the quantity $\mathsf{q}^{f,I_1}_{n_1}$ in* (4.12) *is expanded as $\mathsf{Q}^{I_1,*}_{n_1} f^{\otimes h}$ (recall* (4.7) *and* (3.5)*); similarly for $\mathsf{q}^{g,I_r}_{L-m_r}$. We keep these quantities unexpanded in order to derive tailored estimates, see Subsection 4.4, which could not be derived by simply applying operator norm bounds on $\mathsf{Q}^{I_1,*}_{n_1}$ as in* [CSZ23, LZ21+].

4.2. Moment upper bounds.

We next obtain upper bounds from (4.12). For $L\in\mathbb{N}$ we define the summed kernels

$$\widehat{\mathsf{Q}}^{I,J}_L(\mathbf{z},\mathbf{x}) := \sum_{n=1}^L \mathsf{Q}^{I,J}_n(\mathbf{z},\mathbf{x})\,, \qquad \widehat{\mathsf{q}}^{f,I}_L(\mathbf{x}) := \sum_{n=1}^L \mathsf{q}^{f,I}_n(\mathbf{x})\,. \tag{4.13}$$

Recalling (4.8) and (4.9) we set, with some abuse of notation,

$$|\mathsf{U}|^J_{m-n,\beta}(\mathbf{z},\mathbf{x}) := \mathsf{U}^J_{m-n,\beta}(\mathbf{z},\mathbf{x}) \textit{ from (4.8) with } \mathbb{E}[\xi^J_\beta] \textit{ replaced by } |\mathbb{E}[\xi^J_\beta]|\,. \tag{4.14}$$

Then, for $L\in\mathbb{N}$ and $\lambda\geqslant 0$, we define the Laplace sum

$$|\widehat{\mathsf{U}}|^J_{L,\lambda,\beta}(\mathbf{z},\mathbf{x}) := \mathbb{1}_{\{\mathbf{z}=\mathbf{x}\sim J\}} + \sum_{m=1}^L \mathrm{e}^{-\lambda m}\, |\mathsf{U}|^J_{m,\beta}(\mathbf{z},\mathbf{x})\,. \tag{4.15}$$

Finally, we introduce a *uniform bound* on the right boundary function $\mathsf{q}^{g,I_r}_{L-m_r}$ in (4.12):

$$\bar{\mathsf{q}}^{g,J}_L(\mathbf{z}) := \max_{1\leqslant n\leqslant L} \mathsf{q}^{g,J}_n(\mathbf{z})\,. \tag{4.16}$$

We can now state our first moment upper bound.

Theorem 4.8 (Moment upper bound, I). *Let $\mathcal{Z}^{\omega}_{L,\beta}(f,g)$ denote the averaged partition function in* (4.2) *with centred moment $\mathcal{M}^h_{L,\beta}(f,g)$, see* (4.3)*, for $h \in \mathbb{N}$ with $h \geqslant 2$. For any $\lambda \geqslant 0$ we have the upper bound*

$$\left|\mathcal{M}^h_{L,\beta}(f,g)\right| \leqslant \mathrm{e}^{\lambda L} \sum_{r=1}^{\infty} \Xi(r) \tag{4.17}$$

with

$$\Xi(r) := \sum_{\substack{I_1,\ldots,I_r \vdash \{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i \neq I_{i-1},\, I_i \neq * \,\forall i}} \left\{ \prod_{i=1}^{r} \left|\mathbb{E}[\xi_\beta^{I_i}]\right| \right\} \left\langle \widehat{\mathsf{q}}_L^{|f|,I_1}, |\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta} \left\{ \prod_{i=2}^{r} \widehat{\mathsf{Q}}_L^{I_{i-1},I_i} \, |\widehat{\mathsf{U}}|^{I_i}_{L,\lambda,\beta} \right\} \bar{\mathsf{q}}_L^{|g|,I_r} \right\rangle. \tag{4.18}$$

Proof. Replacing $\mathbb{E}[\xi_\beta^{I_i}]$, f, g, U in (4.12) respectively by $|\mathbb{E}[\xi_\beta^{I_i}]|$, $|f|$, $|g|$, $|\mathsf{U}|$, every term becomes non-negative. We next replace $\mathsf{q}^{|g|,I}_{L-m_r}$ by the uniform bound $\bar{\mathsf{q}}_L^{|g|,I}$ and then enlarge the sum in (4.12), allowing increments $n_i - m_{i-1}$ and $m_i - n_i$ to vary freely in $\{1,\ldots,L\}$. Plugging $1 \leqslant \mathrm{e}^{\lambda L}\,\mathrm{e}^{-\lambda m_r} \leqslant \mathrm{e}^{\lambda L}\,\mathrm{e}^{-\lambda \sum_{i=1}^{r}(m_i-n_i)}$, we obtain (4.17). □

Remark 4.9 (On the right boundary condition). *The function $\mathsf{q}^{g,I_r}_{L-m_r}$ in* (4.12) *is controlled in* [CSZ23, LZ21+] *by introducing an average over L, which forces the function g to be estimated in ℓ^∞. Our approach avoids such averaging, via the quantity $\bar{\mathsf{q}}_L^{g,J}$ from* (4.16)*: this lets us estimate the function g in ℓ^q also for $q < \infty$ (see Proposition 4.21).*

We next bound $\Xi(r)$ in (4.18), starting from the scalar product. Let us recall some functional analysis: given a countable set $\mathbb{T}$ and a function $f : \mathbb{T} \to \mathbb{R}$, we define

$$\|f\|_{\ell^p(\mathbb{T})} = \|f\|_{\ell^p} := \left(\sum_{z \in \mathbb{T}} |f(z)|^p \right)^{\frac{1}{p}} \qquad \text{for } p \in [1,\infty)\,. \tag{4.19}$$

For a linear operator $\mathsf{A} : \ell^q(\mathbb{T}) \to \ell^q(\mathbb{T}')$, with $p,q \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\mathsf{A}\|_{\ell^q \to \ell^q} := \sup_{g \neq 0} \frac{\|\mathsf{A}\,g\|_{\ell^q(\mathbb{T}')}}{\|g\|_{\ell^q(\mathbb{T})}} = \sup_{\|f\|_{\ell^p(\mathbb{T}')} \leqslant 1,\, \|g\|_{\ell^q(\mathbb{T})} \leqslant 1} \langle f, \mathsf{A}\,g \rangle\,. \tag{4.20}$$

By Hölder's inequality $|\langle g,h\rangle| \leqslant \|g\|_{\ell^p}\,\|h\|_{\ell^q}$, so the scalar product in (4.18) is bounded by

$$\|\widehat{\mathsf{q}}_L^{|f|,I_1}\|_{\ell^p} \, \||\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta}\|_{\ell^q\to\ell^q} \left\{ \prod_{i=2}^{r} \|\widehat{\mathsf{Q}}_L^{I_{i-1},I_i}\|_{\ell^q\to\ell^q} \, \||\widehat{\mathsf{U}}|^{I_i}_{L,\lambda,\beta}\|_{\ell^q\to\ell^q} \right\} \|\bar{\mathsf{q}}_L^{|g|,I_r}\|_{\ell^q}\,. \tag{4.21}$$

Remark 4.10 (Restricted ℓ^q spaces). *Due to the constraint $\mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}$ in* (4.7)*, we may regard $\widehat{\mathsf{Q}}_L^{I,J}$ as a linear operator from $\ell^q((\mathbb{Z}^2)^h_J)$ to $\ell^q((\mathbb{Z}^2)^h_I)$, see* (4.6)*. Similarly, we may view $|\widehat{\mathsf{U}}|^J_{L,\lambda,\beta}$ as a linear operator from $\ell^q((\mathbb{Z}^2)^h_J)$ to itself.*

To make the bound (4.21) more useful, we introduce a *weight* $\mathcal{W} : (\mathbb{Z}^2)^h \to (0,\infty)$, that we also identify with the diagonal operator $\mathcal{W}(\mathbf{x})\,\mathbb{1}_{\{\mathbf{x}=\mathbf{y}\}}$, so that in particular

$$\big(\mathcal{W}\,\mathsf{A}\,\tfrac{1}{\mathcal{W}}\big)(\mathbf{x},\mathbf{y}) := \mathcal{W}(\mathbf{x})\,\mathsf{A}(\mathbf{x},\mathbf{y})\,\frac{1}{\mathcal{W}(\mathbf{y})}\,.$$

Inserting $(\mathcal{W}\,\frac{1}{\mathcal{W}})$ between each pair of adjacent operators in (4.17), we improve (4.21) to

$$\begin{aligned}&\big\|\widehat{\mathsf{q}}_L^{|f|,I_1}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\,\big\|\mathcal{W}\,|\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I_1}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,\times\\&\qquad\times\bigg\{\prod_{i=2}^r\big\|\mathcal{W}\,\widehat{\mathsf{Q}}_L^{I_{i-1},I_i}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,\big\|\mathcal{W}\,|\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I_i}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\bigg\}\,\big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|g|,I_r}\big\|_{\ell^q}\,.\end{aligned}\tag{4.22}$$

In view of (4.17)-(4.18), this leads directly to our second moment upper bound.

Theorem 4.11 (Moment upper bound, II). *Let $\mathcal{Z}_{L,\beta}^{\omega}(f,g)$ be the averaged partition function in (4.2) with centred moment $\mathcal{M}_{L,\beta}^h(f,g)\leqslant \mathrm{e}^{\lambda L}\sum_{r=1}^\infty \Xi(r)$, see (4.3) and (4.17), for $\lambda\geqslant 0$ and $h\geqslant 2$. For any weight $\mathcal{W}:(\mathbb{Z}^2)^h\to(0,\infty)$ and for $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$, we have the following upper bound on $\Xi(r)$ from (4.18):*

$$\Xi(r)\;\leqslant\;\Big(\max_{I\neq *}\big\|\widehat{\mathsf{q}}_L^{|f|,I}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\Big)\Big(\max_{J\neq *}\big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|g|,J}\big\|_{\ell^q}\Big)\,\Xi^{\mathrm{bulk}}(r)\tag{4.23}$$

with

$$\Xi^{\mathrm{bulk}}(r)\;:=\sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\}\\ \text{with full support}\\ \text{and } I_i\neq I_{i-1},\ I_i\neq *\ \forall i}}\bigg\{\prod_{i=1}^r\big|\mathbb{E}[\xi_\beta^{I_i}]\big|\bigg\}\,\big(\mathsf{C}_L^{\mathsf{Q},\ell^q}\big)^{r-1}\,\big(\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^q}\big)^r\,,\tag{4.24}$$

where we set for short

$$\mathsf{C}_L^{\mathsf{Q},\ell^q}:=\max_{\substack{I,J\neq *\\ I\neq J}}\big\|\mathcal{W}\,\widehat{\mathsf{Q}}_L^{I,J}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,,\qquad \mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^q}:=\max_{I\neq *}\big\|\mathcal{W}\,|\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\,.\tag{4.25}$$

Note that the bound (4.23)-(4.24) depends on two pairs of quantities, that we call

$$\textit{boundary terms}\;\begin{cases}\big\|\widehat{\mathsf{q}}_L^{|f|,I}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\\ \big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|g|,J}\big\|_{\ell^q}\end{cases}\quad\text{and}\quad\textit{bulk terms}\;\begin{cases}\big\|\mathcal{W}\,\widehat{\mathsf{Q}}_L^{I,J}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\\ \big\|\mathcal{W}\,|\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I}\,\tfrac{1}{\mathcal{W}}\big\|_{\ell^q\to\ell^q}\end{cases}.\tag{4.26}$$

We will estimate these terms in Subsections 4.4 and 4.5 respectively, exploiting some basic random walk bounds that we collect in Subsection 4.3.

Remark 4.12 (Choice of the weight). *We will choose a weight $\mathcal{W}=\mathcal{W}_t:(\mathbb{Z}^2)^h\to(0,\infty)$ which is exponential of rate $t\geqslant 0$, that is for $\mathbf{x}=(x^1,\ldots,x^h)\in(\mathbb{Z}^2)^h$*

$$\mathcal{W}_t(\mathbf{x}):=\prod_{i=1}^h w_t(x^i)\qquad\textit{where}\qquad w_t(x):=\mathrm{e}^{-t|x|}\ \textit{for } x\in\mathbb{Z}^2\,.\tag{4.27}$$

Note that by the triangle inequality we can bound, for all $\mathbf{x},\mathbf{z}\in(\mathbb{Z}^2)^h$,

$$\frac{\mathcal{W}_t(\mathbf{z})}{\mathcal{W}_t(\mathbf{x})}\leqslant\prod_{i=1}^h\mathrm{e}^{t|z^i-x^i|}\,.\tag{4.28}$$

We will later need to consider an additional weight $\mathcal{V}_s^I$, see (4.42) below.

We finally bound the product $\big\{\prod_{i=1}^r\big|\mathbb{E}[\xi_\beta^{I_i}]\big|\big\}$ in (4.24). Recall σ_β from (1.7) and (3.2) and note that $\lim_{\beta\downarrow 0}\sigma_\beta=0$.

Proposition 4.13 (Moments of disorder). *For any $h \in \mathbb{N}$ there are $\beta_0(h) > 0$ and $C(h) < \infty$ (which depend on the disorder distribution) such that for $\beta < \beta_0(h)$ we have*

$$\left|\mathbb{E}[\xi_\beta^I]\right| \leqslant \begin{cases} \sigma_\beta^2 & \text{if } I = \{\{a,b\},\{c\}\colon c \neq a,b\} \text{ is a pair} \\ C(h)\,\sigma_\beta^3 & \text{if } I \neq * \text{ is not a pair} \end{cases} \leqslant \sigma_\beta^2 \quad \text{if } I \neq * \,. \tag{4.29}$$

Moreover

$$\text{if } I_1,\ldots,I_r \vdash \{1,\ldots,h\} \text{ have full support:} \qquad \prod_{i=1}^r \left|\mathbb{E}[\xi_\beta^{I_i}]\right| \leqslant C(h)^r \,\sigma_\beta^{\max\{2r,h\}} \,. \tag{4.30}$$

Proof. We have $|\mathbb{E}[\xi_\beta^I]| = \sigma_\beta^2$ if I is a pair, while $|\mathbb{E}[\xi_\beta^I]| = O(\sigma_\beta^3)$ if $I \neq *$ is not a pair. Indeed, if $\|I\| := \sum_{i=1}^m |I^i|\,\mathbb{1}_{\{|I^i|\geqslant 2\}}$ denotes the number of $a \in \{1,\ldots,h\}$ which are not singletons in $I = \{I^1,\ldots,I^m\} \vdash \{1,\ldots,h\}$, by (3.2) and (4.9) we have $|\mathbb{E}[\xi_\beta^I]| = O(\sigma_\beta^{\|I\|})$ (note that $\|I\| = 2$ if I is a pair while $\|I\| \geqslant 3$ if $I \neq *$ is not a pair).

Since $\lim_{\beta\downarrow 0}\sigma_\beta = 0$, we see that (4.29) holds for $\beta > 0$ for small enough, depending on h (it suffices that $\mathbb{E}[\xi_\beta^k] \leqslant \mathbb{E}[\xi_\beta^2] = \sigma_\beta^2 \leqslant 1$ for all $k \in \{3,\ldots,h\}$, see (3.2)). Finally, if $I_1,\ldots,I_r$ have full support, then each $a \in \{1,\ldots,h\}$ is a non-trivial element (i.e. not a singleton) of some partition I_i, hence $\|I_1\| + \ldots + \|I_r\| \geqslant h$ which yields $\prod_{i=1}^r \left|\mathbb{E}[\xi_\beta^{I_i}]\right| = O(\sigma_\beta^h)$. This proves (4.30) because $\prod_{i=1}^r \left|\mathbb{E}[\xi_\beta^{I_i}]\right| = O(\sigma_\beta^{2r})$ by (4.29). □

4.3. Random walk bounds. In this subsection we collect some useful random walk bounds, stated in Lemmas 4.16, 4.17 and 4.18. The proofs are deferred to Appendix B.

Instead of sticking to the simple random walk on $\mathbb{Z}^2$, we can allow for *any symmetric random walk with sub-Gaussian tails*, in the following sense.

Assumption 4.14 (Random walk). *We consider a random walk $S = (S_n)_{n\geqslant 0}$ on $\mathbb{Z}^2$ with a symmetric distribution, i.e. $q_1(x) = \mathrm{P}(S_1 = x) = q_1(-x)$ for any $x \in \mathbb{Z}^2$, and with sub-Gaussian tails, i.e. for some $c > 0$ we have, writing $x = (x^1, x^2)$,*

$$\forall t \in \mathbb{R},\ \forall a = 1,2: \qquad \mathrm{E}\big[\mathrm{e}^{t\,S_1^a}\big] = \sum_{x\in\mathbb{Z}^2} \mathrm{e}^{tx^a}\, q_1(x) \leqslant \mathrm{e}^{c\,\frac{t^2}{2}} \,. \tag{4.31}$$

Remark 4.15. *The simple random walk on $\mathbb{Z}^2$ satisfies (4.31) with $c = 1$: indeed, we can compute $\sum_{x\in\mathbb{Z}^2} \mathrm{e}^{tx^a}\, q_1(x) = \frac{1}{2}(1 + \cosh(t)) \leqslant \exp(t^2/2)$ (because $\cosh(t) \leqslant \exp(t^2/2)$).*

We derive useful bounds for the random walk transition kernel $q_n(x) = \mathrm{P}(S_n = x)$.

Lemma 4.16 (Random walk bounds). *Let Assumption 4.14 hold. There is $\mathsf{c} \in [1,\infty)$ such that for all $t \geqslant 0$ and $n \in \mathbb{N}$*

$$\forall a = 1,2: \qquad \sum_{x\in\mathbb{Z}^2} \mathrm{e}^{tx^a}\, q_n(x) \leqslant \mathrm{e}^{\mathsf{c}\frac{t^2}{2}n}\,, \qquad \sum_{x\in\mathbb{Z}^2} \mathrm{e}^{tx^a}\, \frac{q_n(x)^2}{q_{2n}(0)} \leqslant \mathrm{e}^{\mathsf{c}\frac{t^2}{2}n} \,. \tag{4.32}$$

Moreover, recalling $w_t(x) = \mathrm{e}^{-t|x|}$ from (4.27), we can bound

$$\left\|\frac{q_n}{w_t}\right\|_{\ell^1} = \sum_{x\in\mathbb{Z}^2} \mathrm{e}^{t|x|}\, q_n(x) \leqslant \mathsf{c}\,\mathrm{e}^{2\mathsf{c}t^2 n}\,, \qquad \left\|\frac{q_n}{w_t}\right\|_{\ell^\infty} = \sup_{x\in\mathbb{Z}^2}\left\{\mathrm{e}^{t|x|}\, q_n(x)\right\} \leqslant \frac{\mathsf{c}\,\mathrm{e}^{2\mathsf{c}t^2 n}}{n} \,. \tag{4.33}$$

We next extend the bounds in (4.33) to the *averaged* random walk transition kernel $q_n^f(x)$, see (3.5), for any $f : \mathbb{Z}^2 \to \mathbb{R}$. Let us agree that $a^{\frac{1}{\infty}} := 1$ for any $a > 0$.

Lemma 4.17 (Averaged random walk bounds). *Let Assumption 4.14 hold and let* c *be the constant from Lemma 4.16. For any* $t \geqslant 0$ *and* $n \in \mathbb{N}$ *we have, with* $w_t(x) = \mathrm{e}^{-t|x|}$,

$$\forall p \in [1, \infty] : \qquad \left\| \frac{q_n^f}{w_t} \right\|_{\ell^p} \leqslant \mathsf{c}\, \mathrm{e}^{2\mathsf{c}\, t^2 n} \left\| \frac{f}{w_t} \right\|_{\ell^p} , \qquad \left\| \frac{q_n^f}{w_t} \right\|_{\ell^\infty} \leqslant \frac{\mathsf{c}\, \mathrm{e}^{2\mathsf{c}\, t^2 n}}{n^{\frac{1}{p}}} \left\| \frac{f}{w_t} \right\|_{\ell^p} . \tag{4.34}$$

We finally consider the *maximal* averaged random walk transition kernel $\overline{q}_L^f : \mathbb{Z}^2 \to \mathbb{R}$:

$$\overline{q}_L^f(x) := \max_{1 \leqslant n \leqslant L} q_n^f(x) . \tag{4.35}$$

We prove a variant of Hardy-Littlewood maximal inequality, see Appendix B for the details.

Lemma 4.18 (Maximal random walk bounds). *Let Assumption 4.14 hold and let* c *be the constant from Lemma 4.16. For any* $t \geqslant 0$ *and* $L \in \mathbb{N}$ *we have, with* $w_t(x) = \mathrm{e}^{-t|x|}$,

$$\forall p \in (1, \infty] : \qquad \| \overline{q}_L^f \, w_t \|_{\ell^p} \leqslant \tfrac{p}{p-1}\, 25^{\frac{1}{p}}\, \mathsf{C} \, \| f \, w_t \|_{\ell^p} \qquad \text{with} \quad \mathsf{C} := 200\pi \, \mathsf{c}^2 \, \mathrm{e}^{4\mathsf{c}\, t^2 L} \tag{4.36}$$

(with $\frac{\infty}{\infty - 1} := 1$*).*

4.4. Boundary terms.

In this section we estimate the *boundary terms* appearing in (4.23), see (4.26). The proofs are deferred to Appendix C.

We recall that the weight $\mathcal{W}_t : (\mathbb{Z}^2)^h \to (0, \infty)$ is defined in (4.27) for $t \geqslant 0$. Our estimates contain the following constants (with c from Lemma 4.16):

$$\mathscr{C} := \mathsf{c}\, \mathrm{e}^{2\mathsf{c}\, t^2 L} , \qquad \overline{\mathscr{C}} := 5000\, \pi\, \mathsf{c}^2\, \mathrm{e}^{4\mathsf{c}\, t^2 L} , \tag{4.37}$$

where L is the "time horizon", see (4.26). We anticipate that we will take

$$t = \tfrac{1}{\sqrt{N}} \qquad \text{with} \quad N \geqslant L . \tag{4.38}$$

hence *the constants* $\mathscr{C}$ *and* $\overline{\mathscr{C}}$ *are uniformly bounded* in this regime.

We start estimating the *left boundary term* which involves $\widehat{\mathsf{q}}_L^{|f|,I}$ (see (4.13) and (4.7)). It was proved[†] in [LZ21+, Proposition 3.4], extending [CSZ23, Proposition 6.6], that for any $h \geqslant 2$ there is $C = C(h) < \infty$ such that, for any $p \in (1, \infty)$,

$$\max_{I \neq *} \left\| \widehat{\mathsf{q}}_L^{|f|,I} \, \tfrac{1}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \tfrac{p}{p-1}\, C \, L^{1 - \frac{1}{p}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^h . \tag{4.39}$$

For our goals it will be fundamental to have a *linear dependence in* L, which would amount to take $p = \infty$ in (4.39), but this is not allowed by our approach. To solve this problem, we improve the estimate (4.39), showing that for $p \in (0, \infty)$ we can still have a linear dependence in L in the RHS, provided we replace one factor $\| \frac{f}{w_t} \|_{\ell^p}$ by $\| \frac{f}{w_t} \|_{\ell^\infty}$.

[†]The factor $q = \frac{p}{p-1}$ in the RHS of (4.39), first identified in [LZ21+], is essential to allow for p which can vary with the system size L.

Proposition 4.19 (Left boundary term, I). *Recall the weights $\mathcal{W}_t$ and w_t from* (4.27). *For any $h \geqslant 2$, $t \geqslant 0$, $L \in \mathbb{N}$ we have, for any $p \in (1,\infty)$ and $\mathscr{C}$ as in* (4.37),

$$\max_{I \neq *} \Big\| \widehat{\mathsf{q}}_L^{|f|,I} \tfrac{1}{\mathcal{W}_t} \Big\|_{\ell^p} \leqslant 4\, \mathscr{C}^h\, L \left\| \frac{f}{w_t} \right\|_{\ell^\infty} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1} . \tag{4.40}$$

More generally, for any $r \in [1,\infty]$ we have (with $\frac{1}{0} := \infty$, $\frac{\infty}{\infty - 1} := 1$)

$$\max_{I \neq *} \Big\| \widehat{\mathsf{q}}_L^{|f|,I} \tfrac{1}{\mathcal{W}_t} \Big\|_{\ell^p} \leqslant 4\, \mathscr{C}^h\, \min\{\tfrac{r}{r-1}, \tfrac{p}{p-1}\}\, L^{1-\frac{1}{r}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1} . \tag{4.41}$$

We further improve the bound (4.40) through a *restricted weight* $\mathcal{V}_s^I : (\mathbb{Z}^2)^h \to (0,\infty)$, defined for a *pair* $I \vdash \{1,\ldots,h\}$ and $s \geqslant 0$ by

$$\mathcal{V}_s^I(\mathbf{x}) := w_s(x^a - x^b) = \mathrm{e}^{-s|x^a - x^b|} \qquad \text{for } I = \{\{a,b\},\{c\} \colon c \neq a,b\} . \tag{4.42}$$

Note that $\big||z^a - z^b| - |x^a - x^b|\big| \leqslant |z^a - x^a| + |x^b - z^b|$, therefore we can estimate

$$\frac{\mathcal{V}_s^I(\mathbf{z})}{\mathcal{V}_s^I(\mathbf{x})} \leqslant \mathrm{e}^{s|z^a - x^a| + s|z^b - x^b|} . \tag{4.43}$$

In analogy with (4.38), we anticipate that we will take

$$s = \tfrac{1}{\sqrt{L}} . \tag{4.44}$$

Proposition 4.20 (Left boundary term, II). *For any $h \geqslant 3$, $t \geqslant 0$, $s \in (0,1]$, $L \in \mathbb{N}$ we have, for any $p \in (1,\infty)$ and $\mathscr{C}$ as in* (4.37),

$$\max_{\substack{J \text{ pair} \\ I \neq *,\, I \not\supseteq J}} \Big\| \widehat{\mathsf{q}}_L^{|f|,I} \tfrac{\mathcal{V}_s^J}{\mathcal{W}_t} \Big\|_{\ell^p} \leqslant 36^{\frac{1}{p}}\, \mathscr{C}^h\, \frac{L}{s^{\frac{2}{p}}} \left\| \frac{f}{w_t} \right\|_{\ell^\infty}^2 \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-2} , \tag{4.45}$$

where $I \not\supseteq J$, for $I = \{I^1, \ldots, I^m\}$ and $J = \{\{a,b\},\{c\} : c \neq a,b\}$, means $I^j \not\supseteq \{a,b\}$ $\forall j$.

We next estimate the *right boundary term* which involves $\overline{\mathsf{q}}_L^{|g|,J}$, see (4.16) and (4.7), obtaining estimates analogous to (4.41) and (4.45).

Proposition 4.21 (Right boundary term). *For any $h \geqslant 2$, $t \geqslant 0$, $L \in \mathbb{N}$ we have, for any $q \in (1,\infty)$ and $\overline{\mathscr{C}}$ as in* (4.37),

$$\begin{aligned} \max_{J \neq *} \big\| \overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t \big\|_{\ell^q} &\leqslant \big(\tfrac{q}{q-1}\, \overline{\mathscr{C}} \big)^h\, \|g\, w_t\|_{\ell^{2q}}^2\, \|g\, w_t\|_{\ell^q}^{h-2} \\ &\leqslant \big(\tfrac{q}{q-1}\, \overline{\mathscr{C}} \big)^h\, \|g\, w_t\|_{\ell^\infty}\, \|g\, w_t\|_{\ell^q}^{h-1} . \end{aligned} \tag{4.46}$$

Moreover, for any $h \geqslant 3$, $s \in (0,1]$ we have, for $\overline{\mathscr{C}}$ as in (4.37),

$$\max_{\substack{I \text{ pair} \\ J \neq *,\, J \not\supseteq I}} \big\| \overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\, \mathcal{V}_s^I \big\|_{\ell^q} \leqslant \big(\tfrac{q}{q-1}\, \overline{\mathscr{C}} \big)^h\, \frac{1}{s^{\frac{2}{q}}}\, \|g\, w_t\|_{\ell^\infty}^2\, \|g\, w_t\|_{\ell^q}^{h-2} . \tag{4.47}$$

where $J \not\supseteq I$, for $J = \{J^1, \ldots, J^m\}$ and $I = \{\{a,b\},\{c\} : c \neq a,b\}$, means $J^i \not\supseteq \{a,b\}$ $\forall i$.

Remark 4.22. *We can bound* $\|g\, w_t\|_{\ell^\infty} \leqslant \|g\|_{\ell^\infty}\, \|w_t\|_{\ell^\infty}$ *and* $\|g\, w_t\|_{\ell^q} \leqslant \|g\|_{\ell^\infty}\, \|w_t\|_{\ell^q}$*. By a direct computation, see* (C.16)*, we have*

$$\|w_t\|_{\ell^\infty} = 1\,, \qquad \|w_t\|_{\ell^q} = \Bigg(\sum_{z\in\mathbb{Z}^2} \mathrm{e}^{-qt|z|} \Bigg)^{\frac{1}{q}} \leqslant \frac{36^{\frac{1}{q}}}{t^{\frac{2}{q}}}\,, \tag{4.48}$$

therefore we obtain from (4.46)

$$\max_{J\neq *} \big\| \bar{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t \big\|_{\ell^q} \leqslant \big(\tfrac{q}{q-1}\, 36^{\frac{1}{q}}\, \overline{\mathscr{C}} \big)^h\, \frac{\|g\|_{\ell^\infty}^h}{t^{\frac{2}{q}(h-1)}}\,. \tag{4.49}$$

Similarly, from (4.47) *we deduce that*

$$\max_{\substack{I \text{ pair} \\ J\neq *,\, J\not\supseteq I}} \big\| \bar{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\, \mathcal{V}_s^I \big\|_{\ell^q} \leqslant \big(\tfrac{q}{q-1}\, 36^{\frac{1}{q}}\, \overline{\mathscr{C}} \big)^h\, \frac{\|g\|_{\ell^\infty}^h}{s^{\frac{2}{q}}\, t^{\frac{2}{q}(h-2)}}\,. \tag{4.50}$$

4.5. Bulk terms.

In this section we estimate the the *bulk terms* appearing in (4.24), i.e. the constants $\mathsf{C}_L^{\mathsf{Q},\ell^q}$ and $\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^q}$ from (4.25). The proofs are also given in Appendix C.

We recall the weights $\mathcal{W}_t$ and $\mathcal{V}_s^I$, see (4.27) and (4.42). We will choose the parameters $t, s = O(\frac{1}{\sqrt{L}})$, see (4.38) and (4.44), hence *the following constants are uniformly bounded*:

$$\begin{aligned} \widehat{\mathscr{C}} &:= 4000\, \mathsf{c}^2\, \mathrm{e}^{8\mathsf{c}\, t^2 L}\,, \qquad & \widehat{\widehat{\mathscr{C}}} &:= 4000\, \mathsf{c}^2\, \mathrm{e}^{8\mathsf{c}\,(t+2s)^2 L}\,, \\ \widecheck{\mathscr{C}} &:= 2\, \mathrm{e}^{4\mathsf{c}\, t^2 L}\,, \qquad & \widecheck{\widecheck{\mathscr{C}}} &:= 2\, \mathrm{e}^{4\mathsf{c}\,(t+s)^2 L}\,. \end{aligned} \tag{4.51}$$

We first estimate the "bulk random walk term" $\mathsf{C}_L^{\mathsf{Q},\ell^q}$ which involves $\widehat{\mathsf{Q}}_L^{I,J}$, see (4.25).

Proposition 4.23 (Bulk random walk term). *For any* $h \geqslant 2$*,* $t \geqslant 0$*,* $L \in \mathbb{N}$ *we have, for any* $q \in (1,\infty)$ *and* $\widehat{\mathscr{C}}$ *from* (4.51)*,*

$$\mathsf{C}_L^{\mathsf{Q},\ell^q} := \max_{I,J\neq *,\, I\neq J} \big\| \mathcal{W}_t\, \widehat{\mathsf{Q}}_L^{I,J}\, \tfrac{1}{\mathcal{W}_t} \big\|_{\ell^q\to\ell^q} \leqslant h!\, \widehat{\mathscr{C}}^h\, q\, \tfrac{q}{q-1}\,. \tag{4.52}$$

Moreover, for $s \geqslant 0$ *and* $\widehat{\widehat{\mathscr{C}}}$ *from* (4.51)*,*

$$\max_{I,J \text{ pairs},\, I\neq J} \big\| \tfrac{\mathcal{W}_t}{\mathcal{V}_s^J}\, \widehat{\mathsf{Q}}_L^{I,J}\, \tfrac{1}{\mathcal{W}_t\, \mathcal{V}_s^I} \big\|_{\ell^q\to\ell^q} \leqslant h!\, \widehat{\widehat{\mathscr{C}}}^h\, q\, \tfrac{q}{q-1}\,. \tag{4.53}$$

(note that the weights $\mathcal{V}_s^J, \mathcal{V}_s^I$ *appear* in the denominator on both sides*).*

We next focus on the quantity $\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^q}$ in (4.25), which depends on the operator $|\widehat{\mathsf{U}}|_{L,\lambda,\beta}^I$, see (4.8) and (4.14). Recalling R_N from (1.6) and $q_n(x)$ from (3.3), we define

$$R_N^{(\lambda)} := \sum_{n=1}^N \mathrm{e}^{-\lambda n}\, q_{2n}(0)\,, \tag{4.54}$$

which reduces to R_N for $\lambda = 0$. In the next result we are going to assume that $|\mathbb{E}[\xi_\beta^I]| \leqslant \sigma_\beta^2$ for any partition $I \neq *$, which holds for $\beta > 0$ small enough (see Proposition 4.13).

Proposition 4.24 (Bulk interacting term). *Let $\beta > 0$ satisfy $\max_{I \neq *} |\mathbb{E}[\xi_\beta^I]| \leqslant \sigma_\beta^2$. For any $h \geqslant 2$, $t \geqslant 0$, $L \in \mathbb{N}$, $\lambda \geqslant 0$ such that $\sigma_\beta^2 R_L^{(\lambda)} < 1$ we have, for any $q \in (1, \infty)$ and $\breve{\mathscr{C}}$ from* (4.51)*,*

$$\mathsf{C}^{\mathsf{U}, \ell^q}_{L,\lambda,\beta} := \max_{I \neq *} \| \mathcal{W}_t \, |\widehat{\mathsf{U}}|^I_{L,\lambda,\beta} \, \tfrac{1}{\mathcal{W}_t} \|_{\ell^q \to \ell^q} \leqslant 1 + \breve{\mathscr{C}}^h \, \frac{\sigma_\beta^2 \, R_L^{(\lambda)}}{1 - \sigma_\beta^2 \, R_L^{(\lambda)}} \,. \tag{4.55}$$

Moreover, for any $s \geqslant 0$ we have, for $a \in \{+1, -1\}$and $\breve{\breve{\mathscr{C}}}$ from (4.51)*,*

$$\max_{\substack{J \, pair \\ I \neq *}} \| (\mathcal{V}_s^J)^a \, \mathcal{W}_t \, |\widehat{\mathsf{U}}|^I_{L,\lambda,\beta} \, \tfrac{1}{\mathcal{W}_t \, (\mathcal{V}_s^J)^a} \|_{\ell^q \to \ell^q} \leqslant 1 + \breve{\breve{\mathscr{C}}}^h \, \frac{\sigma_\beta^2 \, R_L^{(\lambda)}}{1 - \sigma_\beta^2 \, R_L^{(\lambda)}} \,. \tag{4.56}$$

5. Proof of Proposition 2.2

In this section we prove Proposition 2.2. The key difficulty is that our goal (2.9) involves *the (optimal)* $1/M^2$ *dependence* on the width of the time interval $(\frac{i-1}{M} N, \frac{i}{M} N]$ (recall the definition (3.7) of the random variable $X^{(i)}_{N,M}$). This requires sharp ad hoc estimates.

5.1. Setup. By formula (4.4) from Remark 4.1, for $l = 1, \ldots, M$ we can write

$$\mathbb{E}\Big[(X^{(l)}_{N,M})^4 \Big] = \frac{\vartheta_N^2}{N^4} \, \mathcal{M}^4_{L,\beta}(f, g) \tag{5.1}$$

where L, β, f, g are given as follows:

$$L = \frac{N}{M} \,, \qquad \beta = \beta_N \ \text{ in (1.11)} \,, \qquad f(\cdot) = q^{\varphi_N}_{\frac{l-1}{M} N}(\cdot) \ \text{ in (1.5)-(3.5)} \,, \qquad g(\cdot) \equiv 1 \,. \tag{5.2}$$

We can bound $\mathcal{M}^4_{\frac{N}{M}, \beta_N}(f, g)$ exploiting (4.17) for $h = 4$ and $\lambda = 0$, which yields

$$\mathbb{E}\big[(X^{(l)}_{N,M})^4 \big] \leqslant \frac{\vartheta_N^2}{N^4} \left(\Xi(1) + \Xi(2) + \sum_{r=3}^{\infty} \Xi(r) \right) , \tag{5.3}$$

where $\Xi(r)$ is defined in (4.18). We show that the only non-negligible term in (5.3) is $\Xi(2)$: more precisely, we will prove that there is $C < \infty$ such that, for any $M \in \mathbb{N}$,

$$\limsup_{N \to \infty} \frac{\vartheta_N^2}{N^4} \, \Xi(2) \leqslant \frac{C}{M^2} \,, \tag{5.4}$$

while

$$\lim_{N \to \infty} \frac{\vartheta_N^2}{N^4} \, \Xi(1) = 0 \qquad \text{and} \qquad \lim_{N \to \infty} \frac{\vartheta_N^2}{N^4} \sum_{r=3}^{\infty} \Xi(r) = 0 \,. \tag{5.5}$$

This will complete the proof of Proposition 2.2.

We estimate $\Xi(r)$ exploiting the bound (4.23)-(4.24) with the choice

$$p = q = 2 \,.$$

We need to control the *boundary terms* and the *bulk terms*, see (4.26). We recall that the weights $\mathcal{W}_t$ and $\mathcal{V}_s^I$ are defined in (4.27) and (4.42), and we fix

$$t = \tfrac{1}{\sqrt{N}} \,, \qquad s = \tfrac{1}{\sqrt{L}} = \sqrt{\tfrac{M}{N}} \,. \tag{5.6}$$

For notational lightness, we write $a \lesssim b$ whenever $a \leqslant C\, b$ for some constant $0 < C < \infty$. We also denote by $\|\varphi\|_p := (\int_{\mathbb{R}^2} \varphi(x)^p \, \mathrm{d}x)^{1/p}$ the usual L^p norm of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

5.2. Boundary terms. We estimate the *left boundary term* $\|\widehat{\mathsf{q}}_L^{|f|,I} \frac{1}{\mathcal{W}_t}\|_{\ell^2}$ applying (4.40). We recall from (5.2) that $f(\cdot) = q^{\varphi_N}_{\frac{l-1}{M}N}(\cdot)$ for $1 \leqslant l \leqslant M$. Let us estimate $\|\frac{f}{w_t}\|_{\ell^\infty}$ and $\|\frac{f}{w_t}\|_{\ell^2}$, starting from the former. By (4.34), for $l \leqslant M$ and $t = \frac{1}{\sqrt{N}}$ we have

$$\left\|\frac{f}{w_t}\right\|_{\ell^\infty} \leqslant \mathsf{c}\, \mathrm{e}^{2\mathsf{c}\, t^2 \frac{l-1}{M} N} \left\|\frac{\varphi_N}{w_t}\right\|_{\ell^\infty} \leqslant \mathsf{c}\, \mathrm{e}^{2\mathsf{c}} \left\|\frac{\varphi_N}{w_t}\right\|_{\ell^\infty}.$$

Since φ is compactly supported, say in a ball $B(0, R)$, we have that φ_N is supported in $B(0, R\sqrt{N} + \sqrt{2}) \subseteq B(0, 2R\sqrt{N})$, see (1.5). By $w_t(x) = \mathrm{e}^{-t|x|}$, we then obtain

$$\left\|\frac{\varphi_N}{w_t}\right\|_{\ell^\infty} \leqslant \mathrm{e}^{t\, 2R\sqrt{N}} \, \|\varphi_N\|_{\ell^\infty} \leqslant \mathrm{e}^{2R} \, \|\varphi\|_\infty \lesssim 1\,, \qquad \text{hence} \qquad \left\|\frac{f}{w_t}\right\|_{\ell^\infty} \lesssim 1\,, \tag{5.7}$$

because $\|\varphi_N\|_{\ell^\infty} \leqslant \|\varphi\|_\infty$. We next estimate $\|\frac{f}{w_t}\|_{\ell^2}$. By a Riemann sum approximation, we see from (1.5) that $\|\varphi_N\|_{\ell^2} \lesssim \sqrt{N}\, \|\varphi\|_2$, hence by (4.34) we obtain

$$\left\|\frac{f}{w_t}\right\|_{\ell^2} \leqslant \mathsf{c}\, \mathrm{e}^{2\mathsf{c}} \left\|\frac{\varphi_N}{w_t}\right\|_{\ell^2} \leqslant \mathsf{c}\, \mathrm{e}^{2\mathsf{c}}\, \mathrm{e}^{2R} \, \|\varphi_N\|_{\ell^2} \lesssim \sqrt{N}\,. \tag{5.8}$$

We can finally apply the estimate (4.40) for $p = 2$ and $h = 4$ to get, since $L = \frac{N}{M}$,

$$\max_{I \neq *} \left\|\widehat{\mathsf{q}}_L^{|f|,I} \tfrac{1}{\mathcal{W}_t}\right\|_{\ell^2} \leqslant 4\, \mathscr{C}^h \, L \left\|\frac{f}{w_t}\right\|_{\ell^\infty} \left\|\frac{f}{w_t}\right\|_{\ell^2}^3 \lesssim \frac{N^{\frac{5}{2}}}{M}\,. \tag{5.9}$$

We now estimate the *right boundary term* $\|\overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\|_{\ell^2}$: applying (4.49) for $q = 2$ and $h = 4$, since $g \equiv 1$ and $t = \frac{1}{\sqrt{N}}$, we obtain

$$\max_{J \neq *} \|\overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\|_{\ell^2} \leqslant \big(12\, \overline{\mathscr{C}}\big)^4 \, \frac{\|g\|_\infty^4}{t^3} \lesssim N^{\frac{3}{2}}\,. \tag{5.10}$$

Overall, we have shown that

$$\Big(\max_{I \neq *} \|\widehat{\mathsf{q}}_L^{|f|,I} \tfrac{1}{\mathcal{W}}\|_{\ell^p}\Big) \Big(\max_{J \neq *} \|\mathcal{W}\, \overline{\mathsf{q}}_L^{|g|,J}\|_{\ell^q}\Big) \lesssim \frac{N^4}{M}\,. \tag{5.11}$$

In view of (4.23), it remains to estimate $\Xi^{\mathrm{bulk}}(r)$ defined in (4.24).

5.3. Bulk terms. We next estimate the *bulk terms*, see (4.25). For the first term $\mathsf{C}_L^{\mathsf{Q},\ell^2}$, we apply directly the estimate (4.52) with $q = 2$ and $h = 4$ to get

$$\mathsf{C}_L^{\mathsf{Q},\ell^2} = \max_{I,J \neq *,\, I \neq J} \|\mathcal{W}_t\, \widehat{\mathsf{Q}}_L^{I,J} \tfrac{1}{\mathcal{W}_t}\|_{\ell^2 \to \ell^2} \leqslant 4!\, \widehat{\mathscr{C}}^4\, 4 \lesssim 1\,. \tag{5.12}$$

(Also note that $\mathsf{C}_L^{\mathsf{Q},\ell^2} \geqslant \mathcal{W}_t(0)\, \widehat{\mathsf{Q}}_L^{I,J}(0,0) \frac{1}{\mathcal{W}_t(0)} \geqslant \mathsf{Q}_2(0,0) \gtrsim 1$.)

We then focus on the second term $\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^2}$. For $L = \frac{N}{M} \leqslant N$ and $\beta = \beta_N$ as in (1.11)

$$1 - \sigma_{\beta_N}^2 R_L \geqslant 1 - \sigma_{\beta_N}^2 R_N \geqslant \frac{\vartheta_N}{\log N} > 0\,, \qquad \text{in particular} \quad \sigma_{\beta_N}^2 R_L < 1\,. \tag{5.13}$$

Then by (4.55) with $\lambda = 0$ (so that $R_N^{(\lambda)} = R_N$) we obtain, recalling that $\vartheta_N \ll \log N$,

$$\mathsf{C}^{\mathsf{U},\ell^2}_{L,\lambda,\beta} := \max_{I \neq *} \| \mathcal{W}_t \, |\widehat{\mathsf{U}}|^I_{L,\lambda,\beta} \, \tfrac{1}{\mathcal{W}_t} \|_{\ell^2 \to \ell^2} \leqslant 1 + \widetilde{\mathscr{C}}^4 \frac{\sigma^2_{\beta_N} R_L}{1 - \sigma^2_{\beta_N} R_L} \lesssim \frac{\log N}{\vartheta_N} . \tag{5.14}$$

Since $\beta_N \to 0$, we can apply (4.29) which ensures that $|\mathbb{E}[\xi^I_{\beta_N}]| \leqslant \sigma^2_{\beta_N} \leqslant \frac{1}{R_N} = O(\frac{1}{\log N})$ for any $I \neq *$ and N large, therefore there is $C < \infty$ such that

$$\Big(\max_{I \neq *} |\mathbb{E}[\xi^I_{\beta_N}]| \Big) \, \mathsf{C}^{\mathsf{Q},\ell^2}_L \, \mathsf{C}^{\mathsf{U},\ell^2}_{L,\lambda,\beta} \leqslant \frac{C}{\vartheta_N} . \tag{5.15}$$

5.4. Terms $r \geqslant 3$. We are ready to prove the second relation in (5.5), which shows that the terms $r \geqslant 3$ give a negligible contributions to $\mathbb{E}\big[(X^{(l)}_{N,M})^4\big]$, recall (5.3).

Let us denote by $c(h) \in \mathbb{N}$ the number of partitions $I \vdash \{1, \ldots, h\}$ with $I \neq *$. Then by (4.24) we have the geometric bound

$$\Xi^{\rm bulk}(r) \leqslant \big(\mathsf{C}^{\mathsf{Q},\ell^2}_L\big)^{-1} \Big\{ c(h) \Big(\max_{I \neq *} |\mathbb{E}[\xi^I_{\beta_N}]| \Big) \, \mathsf{C}^{\mathsf{Q},\ell^2}_L \, \mathsf{C}^{\mathsf{U},\ell^2}_{L,\lambda,\beta} \Big\}^r ,$$

and note that the term in brackets is $< \frac{1}{2}$ for large N, by (5.15) and $\vartheta_N \to \infty$, therefore

$$\sum_{r=3}^{\infty} \Xi^{\rm bulk}(r) \lesssim \Xi^{\rm bulk}(3) \lesssim \frac{1}{\vartheta_N^3} .$$

Applying (4.23) and (5.11), we then obtain the second relation in (5.5):

$$\frac{\vartheta_N^2}{N^4} \sum_{r=3}^{\infty} \Xi(r) \leqslant \frac{\vartheta_N^2}{M} \sum_{r=3}^{\infty} \Xi^{\rm bulk}(r) \lesssim \frac{1}{M \, \vartheta_N} \xrightarrow[N \to \infty]{} 0 .$$

Remark 5.1. *The same arguments can be applied to show that in the quasi-critical regime, the contribution of the terms $r > \lfloor \frac{h}{2} \rfloor$ for the h-th moment of $X^{(l)}_{N,M}$ is negligible as $N \to \infty$.*

5.5. Term $r = 1$. We now prove the first relation in (5.5). A partition $I \vdash \{1,2,3,4\}$ with full support is either a double pair $I = \{\{a,b\},\{c,d\}\}$ or the quadruple $I = \{1,2,3,4\}$, hence $\mathbb{E}[\xi^I_{\beta_N}] \lesssim \sigma^4_{\beta_N}$ for large N, by (4.9) and (3.2) (see also Proposition 4.13). Then, by (4.24),

$$\Xi^{\rm bulk}(1) = \sum_{\substack{I \vdash \{1,\ldots,h\} \\ \text{with full support}}} |\mathbb{E}[\xi^I_{\beta_N}]| \, \mathsf{C}^{\mathsf{U},\ell^q}_{L,\lambda,\beta} \lesssim \sigma^4_{\beta_N} \, \mathsf{C}^{\mathsf{U},\ell^2}_{L,\lambda,\beta} \lesssim \frac{1}{(\log N) \, \vartheta_N} ,$$

where we applied (5.14) and $\sigma^2_{\beta_N} \leqslant \frac{1}{R_N} = O(\frac{1}{\log N})$. Applying (4.23) and (5.11), and recalling that $\vartheta_N \ll \log N$, we obtain the first relation in (5.5):

$$\frac{\vartheta_N^2}{N^4} \, \Xi(1) \leqslant \frac{\vartheta_N^2}{M} \, \Xi^{\rm bulk}(1) \lesssim \frac{\vartheta_N}{M \, \log N} \xrightarrow[N \to \infty]{} 0 .$$

5.6. Term $r = 2$. We finally prove (5.4), which completes the proof of Proposition 2.2. We recall that $\Xi(2)$, defined by (4.18), is a sum over two partitions $I_1, I_2 \vdash \{1, \ldots, h\}$ with $I_1 \neq *$, $I_2 \neq *$ and $I_1 \neq I_2$. We then split $\Xi(2) = \Xi_{\rm pairs}(2) + \Xi_{\rm others}(2)$ where:

- $\Xi_{\rm pairs}(2)$ is the contribution to (4.18) when both I_1, I_2 are pairs;
- $\Xi_{\rm others}(2)$ is the complementary contribution when I_1 and/or I_2 is not a pair.

We first focus on $\Xi_{\text{others}}(2)$ and on the corresponding quantity $\Xi^{\text{bulk}}_{\text{others}}(2)$, see (4.24). If either I_1 or I_2 is not a pair, by Proposition 4.13 we can bound $|\mathbb{E}[\xi^{I_1}_{\beta_N}]\,\mathbb{E}[\xi^{I_2}_{\beta_N}]| \lesssim \sigma^5_{\beta_N}$, hence

$$\Xi^{\text{bulk}}_{\text{others}}(2) \lesssim \sigma^5_{\beta_N}\, \mathsf{C}^{\mathsf{Q},\ell^q}_L \, \big(\mathsf{C}^{\mathsf{U},\ell^q}_{L,\lambda,\beta}\big)^2 \lesssim \frac{1}{(\log N)^{5/2}} \left(\frac{\log N}{\vartheta_N}\right)^2 \lesssim \frac{1}{\vartheta_N^2 \sqrt{\log N}}\,,$$

where we applied (5.12), (5.14) and $\sigma^2_{\beta_N} \leqslant \frac{1}{R_N} = O(\frac{1}{\log N})$. Then, by (4.23) and (5.11),

$$\frac{\vartheta_N^2}{N^4}\,\Xi_{\text{others}}(2) \leqslant \frac{\vartheta_N^2}{M}\,\Xi^{\text{bulk}}_{\text{others}}(2) \lesssim \frac{1}{M\sqrt{\log N}} \xrightarrow[N\to\infty]{} 0\,,$$

which shows that the contribution of $\Xi_{\text{others}}(2)$ to (5.4) is negligible.

It only remains to focus on $\Xi_{\text{pairs}}(2)$: since $\mathbb{E}[\xi^I_\beta] = \sigma^2_\beta$ when I is a pair, we can write

$$\Xi_{\text{pairs}}(2) := \sum_{\substack{I_1\neq I_2\vdash\{1,\ldots,h\} \\ \text{pairs with full support}}} \sigma^4_\beta \left\langle \widehat{\mathsf{q}}^{|f|,I_1}_L\,,\, |\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta}\; \widehat{\mathsf{Q}}^{I_1,I_2}_L\; |\widehat{\mathsf{U}}|^{I_2}_{L,\lambda,\beta}\; \overline{\mathsf{q}}^{|g|,I_r}_L \right\rangle.$$

Besides inserting $\frac{1}{\mathcal{W}_t}\mathcal{W}_t$ as above, we also insert $\mathcal{V}^{I_2}_s\frac{1}{\mathcal{V}^{I_2}_s}$ on the left of $\widehat{\mathsf{Q}}^{I_1,I_2}_L$ and $|\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta}$, while we insert $\frac{1}{\mathcal{V}^{I_1}_s}\mathcal{V}^{I_1}_s$ on the right of $\widehat{\mathsf{Q}}^{I_1,I_2}_L$ and $|\widehat{\mathsf{U}}|^{I_2}_{L,\lambda,\beta}$ (recall (4.42)): we thus obtain

$$\begin{aligned} \Xi_{\text{pairs}}(2) \leqslant \sum_{\substack{I_1\neq I_2\vdash\{1,\ldots,h\} \\ \text{pairs with full support}}} & \sigma^4_\beta \left\| \widehat{\mathsf{q}}^{|f|,I_1}_L \tfrac{\mathcal{V}^{I_2}_s}{\mathcal{W}_t} \right\|_{\ell^p} \left\| \tfrac{\mathcal{W}_t}{\mathcal{V}^{I_2}_s}\, |\widehat{\mathsf{U}}|^{I_1}_{L,\lambda,\beta}\, \tfrac{\mathcal{V}^{I_2}_s}{\mathcal{W}_t} \right\|_{\ell^q\to\ell^q} \cdot \\ & \cdot \left\| \tfrac{\mathcal{W}_t}{\mathcal{V}^{I_2}_s}\, \widehat{\mathsf{Q}}^{I_1,I_2}_L\, \tfrac{1}{\mathcal{W}_t\,\mathcal{V}^{I_1}_s} \right\|_{\ell^q\to\ell^q} \left\| \mathcal{W}_t\,\mathcal{V}^{I_1}_s\, |\widehat{\mathsf{U}}|^{I_2}_{L,\lambda,\beta}\, \tfrac{1}{\mathcal{W}_t\,\mathcal{V}^{I_1}_s} \right\|_{\ell^q\to\ell^q} \left\| \mathcal{W}_t\,\mathcal{V}^{I_1}_s\, \overline{\mathsf{q}}^{|g|,I_r}_L \right\|_{\ell^q}. \end{aligned} \tag{5.16}$$

It remains to estimate these norms. Let us recall that $h = 4$, $p = q = 2$ and $t = \frac{1}{\sqrt{N}}$, $s = \frac{1}{\sqrt{L}}$, where $L = \frac{M}{N}$. We start with the boundary terms:

- applying the estimate (4.45), in view of (5.7)-(5.8), we improve the estimate (5.9):

$$\max_{\substack{I,J \text{ pairs} \\ I\neq J}} \left\| \widehat{\mathsf{q}}^{|f|,I}_L \tfrac{\mathcal{V}^J_s}{\mathcal{W}_t} \right\|_{\ell^2} \leqslant 6\,\mathscr{C}^4\,\frac{L}{s} \left\| \frac{f}{w_t} \right\|^2_{\ell^\infty} \left\| \frac{f}{w_t} \right\|^2_{\ell^2} \lesssim L^{\frac{3}{2}}\,N \lesssim \frac{N^{\frac{5}{2}}}{M^{\frac{3}{2}}}\,; \tag{5.17}$$

- applying the estimate (4.50), since $g \equiv 1$, we improve the estimate (5.10):

$$\max_{\substack{I,J \text{ pairs} \\ I\neq J}} \left\| \mathcal{W}_t\,\mathcal{V}^I_s\,\overline{\mathsf{q}}^{|g|,J}_L \right\|_{\ell^2} \leqslant \big(12\,\overline{\mathscr{C}}\big)^4\,\frac{\|g\|^4_{\ell^\infty}}{s\,t^2} \lesssim \sqrt{L}\,N \lesssim \frac{N^{\frac{3}{2}}}{\sqrt{M}}\,. \tag{5.18}$$

Overall, the product of the two boundary terms is $\lesssim \frac{N^4}{M^2}$, which improves on the previous estimates by an essential factor $\frac{1}{M}$, thanks to the use of the restricted weight $\mathcal{V}^I_s$.

We next estimate the bulk terms:

- applying (4.53) with $p = q = 2$ and $h = 4$, we obtain an analogue of (5.12):

$$\max_{\substack{I,J \text{ pairs} \\ I\neq J}} \left\| \tfrac{\mathcal{W}_t}{\mathcal{V}^J_s}\,\widehat{\mathsf{Q}}^{I,J}_L\,\tfrac{1}{\mathcal{W}_t\,\mathcal{V}^I_s} \right\|_{\ell^2\to\ell^2} \leqslant 4!\,\widehat{\widehat{\mathscr{C}}}^{\,4}\,4 \lesssim 1\,; \tag{5.19}$$

- applying (4.56) for both $a = +1$ and $a = -1$, we obtain an analogue of (5.14):

$$\max_{I,J\,\text{pairs}} \big\| (\mathcal{V}_s^J)^a \, \mathcal{W}_t \, |\widehat{\mathsf{U}}|^I_{L,\lambda,\beta} \, \tfrac{1}{\mathcal{W}_t \, (\mathcal{V}_s^J)^a} \big\|_{\ell^2 \to \ell^2} \leqslant 1 + \breve{\mathscr{C}}^4 \, \frac{\sigma_{\beta_N}^2 \, R_L}{1 - \sigma_{\beta_N}^2 \, R_L} \lesssim \frac{\log N}{\vartheta_N} \,. \tag{5.20}$$

Plugging the previous estimates into (5.16), since $\sigma_{\beta_N}^2 \leqslant \frac{1}{R_N} = O(\frac{1}{\log N})$, we finally obtain

$$\Xi_{\text{pairs}}(2) \lesssim \frac{1}{(\log N)^2} \, \frac{N^{\frac{5}{2}}}{M^{\frac{3}{2}}} \left(\frac{\log N}{\vartheta_N} \right)^2 \frac{N^{\frac{3}{2}}}{\sqrt{M}} = \frac{N^4}{M^2 \, \vartheta_N^2} \,,$$

which completes the proof of (5.4), hence of Proposition 2.2. □

Appendix A. Some technical proofs

We give the proof of Theorem 4.5. We recall that the averaged partition function $\mathcal{Z}^\omega_{L,\beta}(f,g)$ is defined in (4.1)-(4.2). In analogy with (3.4) and (3.6), by (4.1)-(4.2) we can write

$$\begin{aligned} \mathcal{Z}^\omega_{L,\beta}(f,g) - \mathbb{E}[\mathcal{Z}^\omega_{L,\beta}(f,g)] = \sum_{k=1}^\infty \sum_{\substack{0<n_1<\ldots<n_k<L \\ x_1,\ldots,x_k \in \mathbb{Z}^2}} & q^f_{n_1}(x_1) \, \xi_\beta(n_1,x_1) \times \\ & \times \left\{ \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \, \xi_\beta(n_j,x_j) \right\} q^g_{L-n_k}(x_k) \,, \end{aligned} \tag{A.1}$$

where we recall the random walk kernels (3.3) and (3.5). Recalling (4.3), we obtain

$$\begin{aligned} \mathcal{M}^h_{L,\beta}(f,g) = \mathbb{E}\Bigg[\Bigg(\sum_{k=1}^\infty \sum_{\substack{0<n_1<\ldots<n_k<L \\ x_1,\ldots,x_k \in \mathbb{Z}^2}} & q^f_{n_1}(x_1) \, \xi_\beta(n_1,x_1) \times \\ & \times \left\{ \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \, \xi_\beta(n_j,x_j) \right\} q^g_{L-n_k}(x_k) \Bigg)^h \Bigg] . \end{aligned} \tag{A.2}$$

When we expand the h-th power, we obtain a sum over h families of space-time points $A_i := \{(n^i_1, x^i_1), \ldots, (n^i_{k_i}, x^i_{k_i})\}$ for $i = 1, \ldots, h$. These points must *match at least in pairs*, i.e. any point (n^i_ℓ, x^i_ℓ) in any family A_i must coincide with at least another point (n^j_m, x^j_m) in a different family A_j for $j \neq i$, otherwise the expectation vanishes (since $\xi_\beta(n,x)$ are independent and centered). In order to handle this constraint, following [CSZ23, Theorem 6.1], we rewrite (A.2) by first *summing over the set of all space-time points*

$$A := \bigcup_{i=1}^h A_i = \bigcup_{i=1}^h \{(n^i_1, x^i_1), \ldots, (n^i_{k_i}, x^i_{k_i})\} \subseteq \mathbb{N} \times \mathbb{Z}^2$$

and then specifying *which families* each point $(n,x) \in A$ belongs to.

Let us fix the *time coordinates* $n_1 < \ldots < n_r$ of the points in A. For each such time $n \in \{n_1, \ldots, n_r\}$, we have $(n,x) \in A$ for one or more $x \in \mathbb{Z}^2$ (there are at most $h/2$ such x, by the matching constraint described above). We then make the following observations:

- if $(n,x) = (n^i_j, x^i_j)$ belongs to the family A_i, then we have in (A.2) the product of a random walk kernel "entering" (n,x) and another one "exiting" (n,x):

$$q_{n - n^i_{j-1}}(x - x^i_{j-1}) \cdot q_{n^i_{j+1} - n}(x^i_{j+1} - x) \,;$$

- if (n,x) does *not* belong to the family A_i, then we have in (A.2) a random walk kernel "jumping over time n", say $q_{n^i_j - n^i_{j-1}}(x_j - x_{j-1})$ with $n^i_{j-1} < n < n^i_j$: we can split this kernel at time n by Chapman-Kolmogorov, writing

$$q_{n^i_j - n^i_{j-1}}(x^i_j - x^i_{j-1}) = \sum_{z \in \mathbb{Z}^2} q_{n-n^i_{j-1}}(z - x^i_{j-1}) \cdot q_{n^i_j - n}(x^i_j - z)\,. \tag{A.3}$$

Then, to each time $n \in \{n_1, \ldots, n_r\}$, we can associate a vector $\mathbf{y} = (y^1, \ldots, y^h) \in (\mathbb{Z}^2)^h$ with h space coordinates, where $y^i = x$ if the family A^i contains (n,x) and $y^i = z$ from (A.3) otherwise. The constraint that a point $(n,x) \in A$ belongs to two families A^i and $A^{i'}$ means that the corresponding coordinates of the vector $\mathbf{y}$ must coincide: $y^i = y^{i'}$. In order to specify which families A^i share the same points, we assign a *partition* $I \vdash \{1, \ldots, h\}$ to each time $n \in \{n_1, \ldots, n_r\}$ and we require that $\mathbf{y} \sim I$, see (4.5).

We are now ready to provide a convenient rewriting of (A.2) by first summing over the number $r \geqslant 1$ and the time coordinates $n_1 < \ldots < n_r$, then on the corresponding space coordinates $\mathbf{y}_1, \ldots, \mathbf{y}_r$ and partitions $I_1, \ldots, I_r \vdash \{1, \ldots, h\}$ with $\mathbf{y}_i \sim I_i$. Recalling the definitions of $\mathsf{Q}^{I,J}_n$ and $\mathsf{q}^{f,J}_n$ from (4.7), we can rewrite (A.2) as follows:

$$\begin{aligned}\mathcal{M}^h_{L,\beta}(f,g) = \sum_{r=1}^{\infty} \sum_{\substack{0<n_1<\cdots<n_r<L \\ \mathbf{y}_1,\ldots,\mathbf{y}_r \in (\mathbb{Z}^2)^h}} \sum_{\substack{I_1,\ldots,I_r \vdash \{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i \neq * \ \forall i}} & \mathsf{q}^{f,I_1}_{n_1}(\mathbf{y}_1)\, \mathbb{E}[\xi^{I_1}_\beta] \times \\ & \times \left\{ \prod_{i=2}^{r} \mathsf{Q}^{I_{i-1},I_i}_{n_i - n_{i-1}}(\mathbf{y}_{i-1}, \mathbf{y}_i)\, \mathbb{E}[\xi^{I_i}_\beta] \right\} \mathsf{q}^{g,I_r}_{L-n_r}(\mathbf{y}_r)\,.\end{aligned} \tag{A.4}$$

Finally, formula (4.12) follows from (A.4) grouping together stretches of *consecutive repeated partitions*, i.e. when $I_i = J$ for consecutive indexes i. The kernel $\mathsf{U}^J_{m-n,\beta}(\mathbf{z}, \mathbf{x})$ from (4.8) does exactly this job, which leads to (4.12). □

Remark A.1. *Formula* (4.12) *still contains the product of* $\mathbb{E}[\xi^{I_i}_\beta]$ *because these factors from* (A.4) *are only partially absorbed in* $\mathsf{U}^J_{m-n,\beta}(\mathbf{z}, \mathbf{x})$*: indeed, in* (4.8) *we have* $k+1$ *points* $n_0 < n_1 < \ldots < n_k$*, but the factor* $\mathbb{E}[\xi^J_\beta]$ *therein is only raised to the power* k.

Appendix B. Random walk bounds

In this section we prove the random walk bounds from Lemmas 4.16, 4.17 and 4.18. We also prove a heat kernel bound, see Lemma B.1 below.

B.1. Proof of Lemma 4.16. We prove each of the four bounds in (4.32)-(4.33) for a different constant c (it then suffices to take the maximal value).

The first bound in (4.32) with $\mathsf{c} = c$ follows by (4.31), thanks to the independence of the increments of the random walk. This directly implies the first bound in (4.33): it suffices to estimate $\sum_{x \in \mathbb{Z}^2} \mathrm{e}^{t|x|}\, q_n(x) \leqslant \sum_{x \in \mathbb{Z}^2} \mathrm{e}^{2t|x^1|}\, q_n(x)$ (by $|x| \leqslant |x^1| + |x^2|$, Cauchy-Scwharz and symmetry) and then $\mathrm{e}^{|z|} \leqslant \mathrm{e}^{z} + \mathrm{e}^{-z}$, hence $\sum_{x \in \mathbb{Z}^2} \mathrm{e}^{t|x|}\, q_n(x) \leqslant 2\, \mathrm{e}^{2\mathsf{c}t^2 n}$.

To get the second bound in (4.33), we fix $\ell < n$ and write $q_n(x) = \sum_{y \in \mathbb{Z}^2} q_\ell(y)\, q_{n-\ell}(x-y)$ by Chapman-Kolmogorov. We next decompose the sum in the two parts $\langle y, x \rangle > \frac{1}{2}|x|^2$ and

$\langle y, x\rangle \leqslant \frac{1}{2}|x|^2$: renaming y as $x-y$ in the second part, we obtain

$$q_n(x) \leqslant \sum_{y\in\mathbb{Z}^2:\,\langle y,x\rangle\geqslant\frac{1}{2}|x|^2} \big\{q_\ell(y)\, q_{n-\ell}(x-y) + q_{n-\ell}(y)\, q_\ell(x-y)\big\}\,. \tag{B.1}$$

We can bound $q_k(x-y) \leqslant \sup_{z\in\mathbb{Z}^2} q_k(z) \leqslant \frac{\mathsf{c}}{k}$ by the local limit theorem (any random walk satysfying Assumption 4.14 is in L^2 with zero mean). We next observe that $\langle y,x\rangle \geqslant \frac{1}{2}|x|^2$ implies $|x| \leqslant 2|y|$ by Cauchy-Schwarz, therefore the first bound in (4.33) yields

$$\forall x\in\mathbb{Z}^2: \qquad \mathrm{e}^{t|x|}\, q_n(x) \leqslant \mathsf{c}\sum_{y\in\mathbb{Z}^2} \mathrm{e}^{2t|y|}\left\{\frac{q_\ell(y)}{n-\ell} + \frac{q_{n-\ell}(y)}{\ell}\right\} \leqslant \frac{2\mathsf{c}\,\mathrm{e}^{8\mathsf{c}t^2 n}}{\min\{n-\ell,\ell\}}\,.$$

If we choose $\ell = \lfloor \frac{n}{2}\rfloor$, we obtain the second bound in (4.33) renaming c.

It remains to prove the second bound in (4.32). We first note that $q_n(x)^2/q_{2n}(0) \leqslant \mathsf{c}\, q_n(x)$ for some $\mathsf{c}\in[1,\infty)$, because $q_n(x)^2 \leqslant \|q_n\|_{\ell^\infty}\, q_n(x)$ and $\|q_n\|_{\ell^\infty} \leqslant \mathsf{c}\, q_{2n}(0)$ by the local limit theorem. Since $q_n(x) = q_n(-x)$, we get

$$\begin{aligned}\sum_{x\in\mathbb{Z}^2} \mathrm{e}^{tx^a}\,\frac{q_n(x)^2}{q_{2n}(0)} - 1 &= \sum_{x\in\mathbb{Z}^2}\Big(\tfrac{\mathrm{e}^{tx^a}+\mathrm{e}^{-tx^a}}{2}-1\Big)\frac{q_n(x)^2}{q_{2n}(0)} \leqslant \mathsf{c}\sum_{x\in\mathbb{Z}^2}\Big(\tfrac{\mathrm{e}^{tx^a}+\mathrm{e}^{-tx^a}}{2}-1\Big)\, q_n(x) \\ &\leqslant \mathsf{c}\,\big(\mathrm{e}^{\mathsf{c}\frac{t^2}{2}n}-1\big) = \mathsf{c}\sum_{k=1}^\infty \tfrac{1}{k!}\,\big(\mathsf{c}\,\tfrac{t^2}{2}n\big)^k \leqslant \sum_{k=1}^\infty \tfrac{1}{k!}\,\big(\mathsf{c}^2\,\tfrac{t^2}{2}n\big)^k = \mathrm{e}^{\mathsf{c}^2\frac{t^2}{2}n}-1\,,\end{aligned}$$

which proves the second bound in (4.32) if we rename c^2 as c. □

B.2. Proof of Lemma 4.17. For any $y\in\mathbb{Z}^2$ and $p\in[1,\infty]$ we can write, recalling (3.5),

$$\frac{q_n^f(y)}{w_t(y)} = q_n^f(y)\,\mathrm{e}^{t|y|} \leqslant \sum_{z\in\mathbb{Z}^2} \mathrm{e}^{t|z|}\,|f(z)|\,\big\{\mathrm{e}^{t|y-z|}\, q_n(y-z)\big\} \leqslant \left\|\frac{f}{w_t}\right\|_{\ell^p}\left\|\frac{q_n}{w_t}\right\|_{\ell^q},$$

where $q\in[1,\infty]$ is such that $\frac{1}{p}+\frac{1}{q}=1$. Since $\|\frac{q_n}{w_t}\|_{\ell^q}^q \leqslant \|\frac{q_n}{w_t}\|_{\ell^\infty}^{q-1}\,\|\frac{q_n}{w_t}\|_{\ell^1}$, it suffices to apply the bounds in (4.33) to obtain the second bound in (4.34).

We next prove the first bound in (4.34), assuming $p\in[1,\infty)$: we have, by Jensen,

$$\begin{aligned}\left\|\frac{q_n^f}{w_t}\right\|_{\ell^p}^p &= \sum_{x\in\mathbb{Z}^2}\mathrm{e}^{tp|x|}\left|\sum_{z\in\mathbb{Z}^2} f(z)\, q_n(x-z)\right|^p \leqslant \sum_{x\in\mathbb{Z}^2}\left|\sum_{z\in\mathbb{Z}^2}\mathrm{e}^{t|z|}\, f(z)\,\mathrm{e}^{t|x-z|}\, q_n(x-z)\right|^p \\ &\leqslant \sum_{x\in\mathbb{Z}^2}\left(\sum_{z\in\mathbb{Z}^2}\big|\mathrm{e}^{t|z|}\, f(z)\big|^p\,\mathrm{e}^{t|x-z|}\, q_n(x-z)\right)\left\{\sum_{z\in\mathbb{Z}^2}\mathrm{e}^{t|x-z|}\, q_n(x-z)\right\}^{p-1},\end{aligned}$$

and the sum in the last brackets is at most $\mathsf{c}\,\mathrm{e}^{2\mathsf{c}t^2 n}$, by the first bound in (4.33). Bringing the sum over x inside the parenthesis and applying again (4.33), the proof is completed. □

B.3. Proof of Lemma 4.18. We state two key bounds, from which our goal (4.36) follows. Let $B(x,r) := \{y\in\mathbb{R}^2 : |y-x|\leqslant r\}$ denote the Euclidean ball and let $\mathcal{B}(x,r) := B(x,r)\cap\mathbb{Z}^2$ be its restriction to $\mathbb{Z}^2$. For $g:\mathbb{Z}^2\to\mathbb{R}$, define the *maximal function* $\mathcal{M}^g:\mathbb{Z}^2\to[0,\infty]$ by

$$\mathcal{M}^g(x) := \sup_{0<r<\infty}\left\{\frac{1}{|\mathcal{B}(x,r)|}\sum_{y\in\mathcal{B}(x,r)}|g(y)|\right\}. \tag{B.2}$$

Setting $\{\mathcal{M}^g > t\} := \{y \in \mathbb{Z}^2 : \mathcal{M}^g(y) > t\}$ for short, we are going to prove the following discrete version of *Hardy-Littlewood maximal inequality*:

$$\forall \lambda > 0 : \qquad |\{\mathcal{M}^g > \lambda\}| \leqslant 25 \, \frac{\|g\|_{\ell^1}}{\lambda} \, . \tag{B.3}$$

We are also going to prove the following *upper bound on* $\overline{q}_L^f$, defined in (4.35):

$$\forall L \in \mathbb{N}, \ \forall x \in \mathbb{Z}^2 : \qquad \big|\overline{q}_L^f(x) \, w_t(x)\big| \leqslant \mathsf{C} \, \big|\mathcal{M}^{f w_t}(x)\big| \qquad \text{with} \quad \mathsf{C} := 200\pi \, \mathsf{c}^2 \, \mathrm{e}^{4\mathsf{c}\, t^2 L} \, . \tag{B.4}$$

Since $\|\mathcal{M}^g\|_{\ell^\infty} \leqslant \|g\|_{\ell^\infty}$, this implies $\|\overline{q}_L^f \, w_t\|_{\ell^\infty} \leqslant \mathsf{C} \, \|f w_t\|_{\ell^\infty}$, which is our goal (4.36) for $p = \infty$. Also note that combining (B.3) and (B.4) we obtain

$$\forall \lambda > 0 : \qquad \big|\big\{ \overline{q}_L^f \, w_t > \lambda \big\}\big| \leqslant 25 \, \mathsf{C} \, \frac{\|f w_t\|_{\ell^1}}{\lambda} \, ,$$

hence our goal (4.36) for $p \in (1, \infty)$ follows by *Marcinkiewicz's Interpolation Theorem*, see e.g. [Gra14, Theorem 1.3.2 and Exercise 1.3.3(a)]. It remains to prove (B.3) and (B.4).

Proof of (B.3). We follow closely the classical proof of the Hardy-Littlewood maximal inequality, see [Gra14, Theorem 2.1.6], which is stated on $\mathbb{R}^d$ instead of $\mathbb{Z}^d$. By definition of $\mathcal{M}^g$, see (B.2), for every point $x \in \{\mathcal{M}^g > \lambda\}$ there is $r_x > 0$ such that

$$\sum_{y \in \mathcal{B}(x, r_x)} |g(y)| > \lambda \, |\mathcal{B}(x, r_x)| \, . \tag{B.5}$$

It suffices to fix any *finite* set $K \subseteq \{\mathcal{M}^g > \lambda\}$ and prove that (B.3) holds with the LHS replaced by $|K|$. From the family of balls $\mathcal{F} := \{\mathcal{B}(x, r_x) : x \in K\}$ we extract a *disjoint sub-family* $\mathcal{F}' := \{\mathcal{B}(z, r_z) : z \in K'\}$ with $K' \subseteq K$ by the greedy algorithm, see [Gra14, Lemma 2.1.5]: we first pick the ball of largest radius, then we select the ball of largest radius among the remaining ones *which do not intersect the balls that have already been picked*, and so on. By construction, if a ball $\mathcal{B}(x, r_x)$ is *not* included in $\mathcal{F}'$, then it must overlap with some ball $\mathcal{B}(z, r_z)$ of larger radius $r_z \geqslant r_x$, which implies that $\mathcal{B}(x, r_x) \subseteq \mathcal{B}(z, 3r_z)$. In other terms, tripling the radii of the balls in $\mathcal{F}'$ we cover all the balls in $\mathcal{F}$, hence

$$|K| \leqslant \sum_{z \in K'} |\mathcal{B}(z, 3r_z)| \leqslant c \sum_{z \in K'} |\mathcal{B}(z, r_z)| \leqslant \frac{c}{\lambda} \sum_{z \in K'} \sum_{y \in \mathcal{B}(z, r_z)} |g(y)| \leqslant \frac{c}{\lambda} \, \|g\|_{\ell^1} \, ,$$

where we estimated $|\mathcal{B}(z, 3r)| \leqslant c \, |\mathcal{B}(z, r)|$ (see below), we applied (B.5) and we bounded $\sum_{z \in K'} \sum_{y \in \mathcal{B}(z, r_z)} |g(y)| \leqslant \|g\|_{\ell^1}$, because the balls $\mathcal{B}(z, r_z)$ for $z \in K'$ are disjoint. To complete the proof of (B.3), we claim that we can take $c = 25$, i.e. $|\mathcal{B}(z, 3r)| \leqslant 25 \, |\mathcal{B}(z, r)|$.

Note that for $0 < r < 1$ the Euclidean ball $B(x, r)$ contains just the point x, while $B(x, 3r)$ contains at most 25 integer points, i.e. $x \pm (a, b)$ with $-2 \leqslant a, b \leqslant +2$ (all these points are inside $B(x, 3r)$ when r is close to 1). Next we note that each integer point $y = (y^1, y^2) \in B(x, r)$ is the center of a square with vertices $y^i \pm \frac{1}{2}$: the union of these squares covers the Euclidean ball $B(x, r - \frac{\sqrt{2}}{2})$ and is included in $B(x, r + \frac{\sqrt{2}}{2})$. Denoting by $m(\cdot)$ the Euclidean area (i.e. the 2-dimensional Lebesgue measure), we obtain

$$|\mathcal{B}(x, 3r)| \leqslant m\big(B(x, 3r + \tfrac{\sqrt{2}}{2})\big) = \big(3 + \tfrac{\sqrt{2}}{r}\big)^2 \, m\big(B(x, r - \tfrac{\sqrt{2}}{2})\big) \leqslant \big(3 + \tfrac{\sqrt{2}}{r}\big)^2 \, |\mathcal{B}(x, r)| \, ,$$

hence also for $r \geqslant 1$ we have $\big(3 + \frac{\sqrt{2}}{r}\big)^2 \leqslant (3 + \sqrt{2})^2 \leqslant 25$ as claimed.

Proof of (B.4). We claim that for all $1 \leqslant n \leqslant L$ and $x \in \mathbb{Z}^2$

$$q_n(x)\, \mathrm{e}^{t|x|} \leqslant \tilde{q}_n(x) := \frac{\mathsf{C}'}{n}\, \mathrm{e}^{-\frac{|x|^2}{16\mathsf{c}\, n}} \qquad \text{where} \quad \mathsf{C}' := 6\mathsf{c}\, \mathrm{e}^{4\mathsf{c}\, t^2 L}\,. \tag{B.6}$$

Indeed, we prove in Lemma B.1 below that $q_n(x) \leqslant \frac{6\mathsf{c}}{n}\, \mathrm{e}^{-\frac{|x|^2}{8\mathsf{c}\, n}}$, see (B.7), therefore

$$q_n(x)\, \mathrm{e}^{t|x|} \leqslant \frac{6\mathsf{c}}{n}\, \mathrm{e}^{t|x| - \frac{|x|^2}{8\mathsf{c}\, n}} \leqslant \frac{6\mathsf{c}}{n}\, \mathrm{e}^{-\frac{|x|^2}{16\mathsf{c}\, n}} \cdot \Big(\sup_{\gamma \geqslant 0} \mathrm{e}^{t\gamma - \frac{\gamma^2}{16\mathsf{c}\, n}} \Big) = \frac{6\mathsf{c}}{n}\, \mathrm{e}^{-\frac{|x|^2}{16\mathsf{c}\, n}}\, \mathrm{e}^{4\mathsf{c}\, t^2 n}\,,$$

which shows that (B.6) holds for $n \leqslant L$.

Let us now deduce (B.4) from (B.6). Since $\frac{w_t(x)}{w_t(z)} \leqslant \mathrm{e}^{t|x-z|}$, by (3.5) we can estimate

$$\big| q_n^f(x)\, w_t(x) \big| \leqslant \sum_{z \in \mathbb{Z}^2} |f(z)\, w_t(z)|\, q_n(x-z)\, \mathrm{e}^{t|x-z|} \leqslant \sum_{z \in \mathbb{Z}^2} |f(z)\, w_t(z)|\, \tilde{q}_n(x-z)\,,$$

hence, writing $\tilde{q}_n(y) = \int_0^\infty \mathbb{1}_{\{s \leqslant \tilde{q}_n(y)\}}\, \mathrm{d}s$, we obtain

$$\big| q_n^f(x)\, w_t(x) \big| \leqslant \int_0^\infty \mathrm{d}s \sum_{z \in \mathbb{Z}^2:\, \tilde{q}_n(x-z) \geqslant s} |f(z)\, w_t(z)|\,.$$

Since $x \mapsto \tilde{q}_n(x)$ is radially decreasing, the set $\{\tilde{q}_n(\cdot) \geqslant s\} = \big\{ z \in \mathbb{Z}^2 :\, \tilde{q}_n(x-z) \geqslant s \big\}$ is a ball $\mathcal{B}(x, r)$ of suitable radius $r = r(n, s)$. Recalling (B.3), we then obtain

$$\Big| \max_{1 \leqslant n \leqslant L} q_n^f(x)\, w_t(x) \Big| \leqslant \mathcal{M}^{f w_t}(x) \cdot \max_{1 \leqslant n \leqslant L} \int_0^\infty \mathrm{d}s\, \big| \{\tilde{q}_n(\cdot) \geqslant s\} \big| = \mathcal{M}^{f w_t}(x) \cdot \max_{1 \leqslant n \leqslant L} \|\tilde{q}_n\|_{\ell^1}\,,$$

where the equality holds because $\int_0^\infty \mathrm{d}s\, \big| \{\tilde{q}_n(\cdot) \geqslant s\} \big| = \sum_{y \in \mathbb{Z}^2} \int_0^\infty \mathrm{d}s\, \mathbb{1}_{\{s \leqslant \tilde{q}_n(y)\}} = \sum_{y \in \mathbb{Z}^2} \tilde{q}_n(y)$.

It remains to evaluate $\|\tilde{q}_n\|_{\ell^1}$: by monotonicity we can bound

$$\sum_{a \in \mathbb{Z}} \mathrm{e}^{-\frac{a^2}{16\mathsf{c}\, n}} \leqslant 1 + \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{16\mathsf{c}\, n}}\, \mathrm{d}x = 1 + \sqrt{16\pi\, \mathsf{c}\, n}\,,$$

hence writing $x = (a, b)$, so that $|x|^2 = a^2 + b^2$, we obtain

$$\|\tilde{q}_n\|_{\ell^1} = \sum_{x \in \mathbb{Z}^2} \tilde{q}_n(x) = \frac{\mathsf{C}'}{n} \Big(\sum_{a \in \mathbb{Z}} \mathrm{e}^{-\frac{a^2}{16\mathsf{c}\, n}} \Big)^2 \leqslant \mathsf{C}'\, \frac{2(1 + 16\pi\, \mathsf{c}\, n)}{n} \leqslant (2 + 32\pi)\, \mathsf{c}\, \mathsf{C}' \leqslant \mathsf{C}\,,$$

where the two last inequalities hod since $n \geqslant 1$ and $\mathsf{c} \geqslant 1$, hence $1 + 16\pi\, \mathsf{c}\, n \leqslant (1 + 16\pi)\mathsf{c}\, n$, and $2 + 32\pi \leqslant 33\pi \leqslant \frac{200}{6}\pi$, recalling the definition (B.4) of C. The proof is completed. ☐

Lemma B.1 (Heat kernel bound). *Let Assumption 4.14 hold and let* c *be the constant from Lemma 4.16. Then for every* $n \in \mathbb{N}$ *and* $x \in \mathbb{Z}^2$ *we have*

$$q_n(x) \leqslant \frac{6\mathsf{c}}{n}\, \mathrm{e}^{-\frac{|x|^2}{8\mathsf{c} n}}\,. \tag{B.7}$$

Proof. We assume that $n \geqslant 2$, since the case $n = 1$ is easier. Let us apply the formula (B.1) with $\ell = \lfloor \frac{n}{2} \rfloor$, so that $\frac{n}{3} \leqslant \ell \leqslant \frac{n}{2}$: by (4.33) (with $t = 0$) we have $q_k(x-y) \leqslant \frac{\mathsf{c}}{k} \leqslant \frac{3\mathsf{c}}{n}$ for both $k = \ell$ and $k = n - \ell$, therefore for any $\varrho \geqslant 0$

$$q_n(x) \leqslant \frac{3\mathsf{c}}{n}\, \mathrm{e}^{-\varrho|x|} \sum_{y \in \mathbb{Z}^2:\, \langle y, x\rangle \geqslant \frac{1}{2}|x|^2} \mathrm{e}^{2\varrho \langle y, \frac{x}{|x|} \rangle} \big\{ q_\ell(y) + q_{n-\ell}(y) \big\}\,, \tag{B.8}$$

where we bounded $1 \leqslant \mathrm{e}^{-\varrho|x|}\mathrm{e}^{2\varrho\langle y, \frac{x}{|x|}\rangle}$ because $\langle y, x\rangle \geqslant \frac{1}{2}|x|^2$ (with $\frac{x}{|x|} := 0$ for $x = 0$). For any $w = (w^1, w^2) \in \mathbb{R}^2$, by (4.32) and Cauchy-Schwarz we can bound

$$\sum_{y\in\mathbb{Z}^2} \mathrm{e}^{\langle y,w\rangle}\, q_\ell(y) \leqslant \sqrt{\sum_{y\in\mathbb{Z}^2} \mathrm{e}^{2y^1w^1}\, q_\ell(y) \cdot \sum_{y\in\mathbb{Z}^2} \mathrm{e}^{2y^2w^2}\, q_\ell(y)} \leqslant \mathrm{e}^{\mathsf{c}\,|w|^2\ell},$$

and similarly for $q_{n-\ell}(\cdot)$, therefore for $\max\{\ell, n-\ell\} \leqslant \frac{n}{2}$ we obtain by (B.8)

$$q_n(x) \leqslant \frac{6\mathsf{c}}{n}\, \mathrm{e}^{-\varrho|x|+2\mathsf{c}\,\varrho^2\, n}\,.$$

Optimising over ϱ leads us to choose $\varrho = \frac{|x|}{4\mathsf{c}n}$, which yields (B.7). □

Appendix C. Estimates on boundary and bulk terms

In this section we prove the estimates on the boundary terms (Propositions 4.19 and 4.20 for the left boundary, Proposition 4.21 for the right boundary) and on the bulk terms (Proposition 4.23 and Proposition 4.24).

C.1. Proof of Propositions 4.19. By the triangle inequality we can bound

$$\left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \sum_{n=1}^{L} \left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p}. \tag{C.1}$$

Writing $I = \{I^1, \ldots, I^m\}$ we can write

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p}^p = \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h} \frac{\mathsf{q}_n^{|f|,I}(\mathbf{x})^p}{\mathcal{W}_t(\mathbf{x})^p} \leqslant \prod_{j=1}^{m} \left\{ \sum_{y\in\mathbb{Z}^2} q_n^{|f|}(y)^{p|I^j|}\, \mathrm{e}^{pt|I^j|\,|y|} \right\} = \prod_{j=1}^{m} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^{p|I^j|}}^{p|I^j|}. \tag{C.2}$$

Since $\|\cdot\|_{\ell^{pk}}^{pk} \leqslant \|\cdot\|_{\ell^\infty}^{p(k-1)}\,\|\cdot\|_{\ell^p}^{p}$, from $\sum_{j=1}^m |I^j| = h$ we get (raising to $1/p$)

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{h-m} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p}^{m} \leqslant \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p}^{h-1}, \tag{C.3}$$

where the last inequality holds since $m \leqslant h-1$ for $I \neq *$. By (4.34), for any $r \in [1, \infty]$,

$$\left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty} \leqslant \frac{\mathsf{c}\,\mathrm{e}^{2\mathsf{c}\,t^2 n}}{n^{\frac{1}{r}}} \left\| \frac{f}{w_t} \right\|_{\ell^r}, \qquad \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p} \leqslant \mathsf{c}\,\mathrm{e}^{2\mathsf{c}\,t^2 n} \left\| \frac{f}{w_t} \right\|_{\ell^p}, \tag{C.4}$$

hence we obtain for $n \leqslant L$, recalling the definition of $\mathscr{C}$ in (4.41),

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \frac{\mathscr{C}^h}{n^{\frac{1}{r}}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1}. \tag{C.5}$$

Plugging this into (C.1), since $\sum_{n=1}^{L} \frac{1}{n^a} \leqslant \int_0^L \frac{1}{x^a}\,\mathrm{d}x = \frac{L^{1-a}}{1-a}$, we obtain

$$\max_{I\neq *} \left\| \frac{\widehat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \tfrac{r}{r-1}\, \mathscr{C}^h\, L^{1-\frac{1}{r}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1}, \tag{C.6}$$

which proves (4.41) for $r \geqslant p$ (so that $\min\{\frac{r}{r-1}, \frac{p}{p-1}\} = \frac{r}{r-1}$). More generally, if $r \geqslant \frac{3p}{1+2p}$, then $\frac{r}{r-1} \leqslant 3\frac{p}{p-1}$ hence (C.6) still proves (4.41).

It remains to prove (4.41) for $r \in [1, \frac{3p}{1+2p}] \subseteq [1, p)$. Let us obtain an estimate alternative to (C.5). Since $\| \cdot \|_{\ell^p}^p \leqslant \| \cdot \|_{\ell^\infty}^{p-r} \| \cdot \|_{\ell^r}^r$ for $r < p$, by (4.34) we obtain

$$\left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p} \leqslant \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{1-\frac{r}{p}} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^r}^{\frac{r}{p}} \leqslant \frac{\mathsf{c}\, \mathrm{e}^{2\mathsf{c}\, t^2 n}}{n^{\frac{1}{r}-\frac{1}{p}}} \left\| \frac{f}{w_t} \right\|_{\ell^r}, \tag{C.7}$$

which we can use to estimate one factor of $\| \frac{q_n^{|f|}}{w_t} \|_{\ell^p}$ appearing in (C.3) (recall that $h \geqslant 2$): applying again the first bound in (C.4), for $n \leqslant L$ we obtain from (C.3)

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \frac{\mathscr{C}^h}{n^\gamma} \left\| \frac{f}{w_t} \right\|_{\ell^r}^2 \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-2} \qquad \text{with} \quad \gamma := \tfrac{2}{r} - \tfrac{1}{p} = \tfrac{1}{r} + \tfrac{p-r}{pr} \,. \tag{C.8}$$

The RHS of (C.8) is smaller than the RHS of (C.5) if and only if

$$\frac{1}{n^\gamma} \left\| \frac{f}{w_t} \right\|_{\ell^r} < \frac{1}{n^{\frac{1}{r}}} \left\| \frac{f}{w_t} \right\|_{\ell^p} \qquad \Longleftrightarrow \qquad n > \tilde{n} := \left(\frac{\| \frac{f}{w_t} \|_{\ell^r}}{\| \frac{f}{w_t} \|_{\ell^p}} \right)^{\frac{pr}{p-r}} . \tag{C.9}$$

Note that for $r \in [1, \frac{3p}{1+2p}]$ we have $\gamma > 1$, indeed $\gamma - 1 \geqslant \frac{2(1+2p)}{3p} - \frac{1+p}{p} = \frac{p-1}{3p}$. Then $\sum_{n > \tilde{n}}^\infty \frac{1}{n^\gamma} \leqslant \int_{\tilde{n}}^\infty \frac{1}{x^\gamma}\, \mathrm{d}x = \frac{1}{\gamma-1}\, \tilde{n}^{1-\gamma} \leqslant \frac{3p}{p-1}\, \tilde{n}^{1-\gamma}$, hence by (C.8) we can bound

$$\sum_{n > \tilde{n}} \left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \frac{3p}{p-1}\, \mathscr{C}^h\, \tilde{n}^{1-\gamma} \left\| \frac{f}{w_t} \right\|_{\ell^r}^2 \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-2} = \frac{3p}{p-1}\, \mathscr{C}^h \left\| \frac{f}{w_t} \right\|_{\ell^r}^{\frac{r(p-1)}{p-r}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h - \frac{r(p-1)}{p-r}} ,$$

where the equality follows by the definitions of $\tilde{n}$ in (C.9) and γ in (C.8). For the contribution of $n \leqslant \tilde{n}$, the previous bound (C.5) with $r = p$ yields, as in (C.6),

$$\sum_{n=1}^{\tilde{n}} \left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^2} \leqslant \frac{p}{p-1}\, \mathscr{C}^h\, \tilde{n}^{1-\frac{1}{p}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h} = \frac{p}{p-1}\, \mathscr{C}^h \left\| \frac{f}{w_t} \right\|_{\ell^r}^{\frac{r(p-1)}{p-r}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h - \frac{r(p-1)}{p-r}} ,$$

having used the definition of $\tilde{n}$ in (C.9). Overall, see (C.1), for $r \in [1, \frac{3p}{1+2p}]$ we have

$$\max_{I \neq *} \left\| \frac{\hat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \underbrace{\frac{4p}{p-1}\, \mathscr{C}^h \left\| \frac{f}{w_t} \right\|_{\ell^r}^{\frac{1}{\alpha}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-\frac{1}{\alpha}}}_{A} \qquad \text{with } \alpha := \tfrac{p-r}{r(p-1)} \in (0,1] \,. \tag{C.10}$$

At the same time, we can apply again the previous bound (C.6) with $r = p$ to estimate

$$\max_{I \neq *} \left\| \frac{\hat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \underbrace{\frac{p}{p-1}\, \mathscr{C}^h\, L^{1-\frac{1}{p}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h}}_{B} . \tag{C.11}$$

Combining these bounds we get $\max_{I \neq *} \| \frac{\hat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \|_{\ell^2} \leqslant A^\alpha B^{1-\alpha}$, hence

$$\forall r \in [1, \tfrac{3p}{1+2p}] : \qquad \max_{I \neq *} \left\| \frac{\hat{\mathsf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \frac{4p}{p-1}\, \mathscr{C}^h\, L^{1-\frac{1}{r}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1} ,$$

which coincides with our goal (4.41), since $\min\{\frac{r}{r-1}, \frac{p}{p-1}\} = \frac{p}{p-1}$ for $r < p$. $\square$

C.2. Proof of Proposition 4.20. We follow the proof of Proposition 4.19. By the triangle inequality, as in (C.1), it is enough to show that

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^p} \leqslant \frac{36^{\frac{1}{p}} \mathscr{C}^h}{s^{2/p}} \left\| \frac{f}{w_t} \right\|_{\ell^\infty}^2 \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-2} . \tag{C.12}$$

We assume for ease of notation that $J = \{\{1,2\},\{3\},\ldots,\{h\}\}$. Let us fix a partition $I = \{I^1,\ldots,I^m\}$ such that $I \not\supseteq J$, say $1 \in I^1$ and $2 \in I^2$. In analogy with (C.2), we have

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^p}^p \leqslant \widehat{\Sigma}_n^{(1,2)} \cdot \prod_{j=3}^m \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^{p|I^j|}}^{p|I^j|} . \tag{C.13}$$

where

$$\widehat{\Sigma}_n^{(1,2)} := \sum_{y^1,y^2 \in \mathbb{Z}^2} \big(q_n^{|f|}(y^1)\, \mathrm{e}^{t|y^1|}\big)^{p|I^1|} \big(q_n^{|f|}(y^2)\, \mathrm{e}^{t|y^2|}\big)^{p|I^2|} \mathrm{e}^{-ps|y^1-y^2|} . \tag{C.14}$$

By a uniform bound, we can estimate

$$\begin{aligned} \widehat{\Sigma}_n^{(1,2)} &\leqslant \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{p|I^1|} \sum_{y^1,y^2 \in \mathbb{Z}^2} \left(\frac{q_n^{|f|}(y^2)}{w_t(y^2)} \right)^{p|I^2|} \mathrm{e}^{-ps|y^1-y^2|} \\ &= \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{p|I^1|} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p}^{p|I^2|} \left(\sum_{y \in \mathbb{Z}^2} \mathrm{e}^{-ps|y|} \right) . \end{aligned} \tag{C.15}$$

Since $2|z| \geqslant |z^1| + |z^2|$ for $z = (z^1,z^2) \in \mathbb{Z}^2$ and $1 - \mathrm{e}^{-x} \geqslant \frac{2}{3}x$ for $0 \leqslant x \leqslant \frac{1}{2}$, we can bound

$$\sum_{z \in \mathbb{Z}^2} \mathrm{e}^{-ps|z|} \leqslant \sum_{z \in \mathbb{Z}^2} \mathrm{e}^{-s|z|} \leqslant \left(\sum_{x \in \mathbb{Z}} \mathrm{e}^{-s\frac{|x|}{2}} \right)^2 \leqslant \left(\frac{2}{1-\mathrm{e}^{-\frac{s}{2}}} \right)^2 \leqslant \frac{36}{s^2} . \tag{C.16}$$

Plugging these estimates into (C.13) and bounding $\|\cdot\|_{\ell^{pk}}^{pk} \leqslant \|\cdot\|_{\ell^\infty}^{p(k-1)} \|\cdot\|_{\ell^p}^p$, since $\sum_{j=1}^m |I^j| = h$ and $m \leqslant h-1$, we obtain (raising to $1/p$)

$$\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^p} \leqslant \frac{36^{\frac{1}{p}}}{s^{2/p}} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{h-m+1} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p}^{m-1} \leqslant \frac{36^{\frac{1}{p}}}{s^{2/p}} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{2} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p}^{h-2} .$$

Applying the estimates in (C.4), we obtain (C.12). $\square$

C.3. Proof of Proposition 4.21. The second line of (4.46) follows by the first line because $\|\cdot\|_{\ell^{2q}}^2 \leqslant \|\cdot\|_{\ell^\infty} \|\cdot\|_{\ell^q}$. Let us prove the first line of (4.46). Writing $J = \{J^1,\ldots,J^m\}$ and arguing as in (C.2), we can write

$$\|\overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\|_{\ell^q}^q = \sum_{\mathbf{x} \in (\mathbb{Z}^2)^h} \overline{\mathsf{q}}_L^{|g|,J}(\mathbf{x})^q\, \mathcal{W}_t(\mathbf{x})^q \leqslant \prod_{j=1}^m \left\{ \sum_{y \in \mathbb{Z}^2} \big(\overline{q}_L^{|g|}(y)\, w_t(y)\big)^{q|J^j|} \right\} = \prod_{j=1}^m \|\overline{q}_L^{|g|}\, w_t\|_{\ell^{q|J^j|}}^{q|J^j|} ,$$

where $\overline{q}_L^{|g|}(y) := \max_{1 \leqslant n \leqslant L} q_n^{|g|}(y)$, see (4.35). Since $J \neq *$, we have $|J^j| \geqslant 2$ for at least one j, say for $j = 1$, hence for $k = |J^1|$ we bound $\|\cdot\|_{\ell^{qk}}^{qk} \leqslant \|\cdot\|_{\ell^\infty}^{q(k-2)} \|\cdot\|_{\ell^{2q}}^{2q}$, while for all other $k = |J^j| \geqslant 1$ we simply bound $\|\cdot\|_{\ell^{qk}}^{qk} \leqslant \|\cdot\|_{\ell^\infty}^{q(k-1)} \|\cdot\|_{\ell^q}^q$. Since $\sum_{j=1}^m |J^j| = h$, we obtain

$$\|\overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\|_{\ell^q}^q \leqslant \|\overline{q}_L^{|g|}\, w_t\|_{\ell^{2q}}^{2q} \|\overline{q}_L^{|g|}\, w_t\|_{\ell^q}^{q(m-1)} \|\overline{q}_L^{|g|}\, w_t\|_{\ell^\infty}^{q(h-m-1)} \leqslant \|\overline{q}_L^{|g|}\, w_t\|_{\ell^{2q}}^{2q} \|\overline{q}_L^{|g|}\, w_t\|_{\ell^q}^{q(h-2)} ,$$

because $m \leqslant h-1$ for $J \neq *$. In order to obtain the first line of (4.46), it suffices to apply the estimate (4.36), where we can bound $\frac{2q}{2q-1}\, 25^{\frac{1}{2q}}\, \mathsf{C} \leqslant \frac{q}{q-1}\, 25^{\frac{1}{q}}\, \mathsf{C} \leqslant \frac{q}{q-1}\, 25\, \mathsf{C} = \frac{q}{q-1}\overline{\mathscr{C}}$.

We next prove (4.47). We may assume that $I = \{\{1,2\},\{3\},\ldots,\{h\}\}$. Let us fix a partition $J = \{J^1,\ldots,J^m\}$ with $J \not\supseteq I$, say $1 \in J^1$ and $2 \in J^2$. In analogy with (C.13), we can write

$$\max_{\substack{J\neq *\\ J\not\supseteq I}} \big\|\overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\, \mathcal{V}_s^I\big\|_{\ell^q}^q \leqslant \overline{\Sigma}_M^{(1,2)} \cdot \prod_{j=3}^m \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^{q|J^j|}}^{q|J^j|},$$

where, as in (C.14)-(C.15), we have

$$\begin{aligned}\overline{\Sigma}_M^{(1,2)} &:= \sum_{y^1,y^2\in\mathbb{Z}^2} \big(\overline{q}_L^{|g|}(y^1)\, w_t(y^1)\big)^{q|J^1|} \big(\overline{q}_L^{|g|}(y^2)\, w_t(y^2)\big)^{q|J^2|}\, w_s(y^1-y^2)^q \\ &\leqslant \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^\infty}^{q|J^1|} \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^q}^{q|J^2|} \big\|w_s\big\|_{\ell^q}^q .\end{aligned}$$

Bounding $\big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^{q|J^j|}}^{q|J^j|} \leqslant \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^\infty}^{q(|J^j|-1)} \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^q}^q$ for $j \geqslant 2$, we then obtain

$$\begin{aligned}\max_{\substack{J\neq *\\ J\not\supseteq I}} \big\|\overline{\mathsf{q}}_L^{|g|,J}\, \mathcal{W}_t\big\|_{\ell^q}^q &\leqslant \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^\infty}^{q(h-m+1)} \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^q}^{q(m-1)} \big\|w_s\big\|_{\ell^q}^q \\ &\leqslant \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^\infty}^{2q} \big\|\overline{q}_L^{|g|}\, w_t\big\|_{\ell^q}^{q(h-2)} \big\|w_s\big\|_{\ell^q}^q ,\end{aligned}$$

because $\sum_{j=1}^m |J^j| = h$ and $m \leqslant h-1$ for $J \neq *$. We conclude applying (4.36) and (4.48). $\square$

C.4. Proof of Proposition 4.23. Let us set for short $p := \frac{q}{q-1}$ (so that $\frac{1}{p} + \frac{1}{q} = 1$). We are going to use a key functional inequality from [CSZ23, Lemma 6.8], in the improved version from [LZ21+, eq. (3.21) in the proof of Proposition 3.3]:

$$\sum_{\mathbf{z}\in(\mathbb{Z}^2)^h_I,\, \mathbf{x}\in(\mathbb{Z}^2)^h_J} \frac{f(\mathbf{z})\, g(\mathbf{x})}{\big(1+|\mathbf{x}-\mathbf{z}|^2\big)^{h-1}} \leqslant C_1\, p\, q\, \|f\|_{\ell^p}\, \|g\|_{\ell^q} \qquad \text{where} \quad C_1 := 2^{2h}(1+\pi)^h . \tag{C.17}$$

(The value of C_1 is extracted from [LZ21+, proof of Proposition 3.3] where $C_1 \leqslant 2^{3h+1}(\frac{c}{2})^{h-1}pq$ with $c \leqslant 1+\pi$ from [LZ21+, proof of Lemma A.1], hence $C_1 \leqslant 2^{2h+2}\,(1+\pi)^{h-1}$.)

We show below the following bound on $\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z},\mathbf{x}) = \sum_{n=1}^L \prod_{i=1}^h q_n(x^i - z^i)$:

$$\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z},\mathbf{x}) \leqslant \frac{C_2\, \mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{16\mathsf{c}\,L}}}{\big(1+|\mathbf{x}-\mathbf{z}|^2\big)^{h-1}} \qquad \text{where} \quad C_2 := h!\,(200\,\mathsf{c}^2)^h . \tag{C.18}$$

Recalling (4.28), since $\widehat{\mathsf{Q}}_L^{I,J}(\mathbf{z},\mathbf{x}) = \widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z},\mathbf{x})\, \mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}$, see (4.7)-(4.13), we obtain

$$\big(\mathcal{W}_t\, \widehat{\mathsf{Q}}_L^{I,J}\, \tfrac{1}{\mathcal{W}_t}\big)(\mathbf{z},\mathbf{x}) \leqslant \frac{C_2\, \mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}}{\big(1+|\mathbf{x}-\mathbf{z}|^2\big)^{h-1}} \prod_{i=1}^h \mathrm{e}^{t|z^i-x^i| - \frac{|z^i-x^i|^2}{16\mathsf{c}\,L}} \leqslant \frac{C_2\, \mathrm{e}^{8\mathsf{c}ht^2L}\, \mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}}{\big(1+|\mathbf{x}-\mathbf{z}|^2\big)^{h-1}} ,$$

because $\max_{a\in\mathbb{R}}\{ta - \frac{a^2}{16\mathsf{c}\,L}\} = 8\mathsf{c}\, t^2 L$. Applying (C.17), get (4.52) since $800(1+\pi) \leqslant 4000$.

We next prove (4.53). Let I, J be pairs, say $I = \{\{a,b\},\{c\}\colon c \neq a, c \neq b\}$ and $J = \{\{\tilde a,\tilde b\},\{c\}\colon c \neq \tilde a, c \neq \tilde b\}$. For $\mathbf{z} \sim I$ and $\mathbf{x} \sim J$ we have $z^a = z^b$, hence

$$\frac{1}{\mathcal{V}_s^I(\mathbf{x})} \leqslant \mathrm{e}^{s|x^a-x^b|} \leqslant \mathrm{e}^{s\{|x^a-z^a|+|z^a-z^b|+|z^b-x^b|\}} = \mathrm{e}^{s|x^a-z^a|}\, \mathrm{e}^{s|z^b-x^b|} ,$$

and similarly $\frac{1}{\mathcal{V}_s^J(\mathbf{z})} \leqslant \mathrm{e}^{s|x^{\tilde{a}}-z^{\tilde{a}}|}\,\mathrm{e}^{s|z^{\tilde{b}}-x^{\tilde{b}}|}$. Arguing as above, we obtain (4.53):

$$\begin{aligned}\Big(\tfrac{\mathcal{W}_t}{\mathcal{V}_s^J}\,\widehat{\mathsf{Q}}_L^{I,J}\,\tfrac{1}{\mathcal{W}_t\,\mathcal{V}_s^I}\Big)(\mathbf{z},\mathbf{x}) &\leqslant \frac{C_2\,\mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}}{(1+|\mathbf{x}-\mathbf{z}|^2)^{h-1}}\prod_{i=1}^h \mathrm{e}^{(t+2s)|z^i-x^i|-\frac{1}{16\mathsf{c}\,L}|z^i-x^i|^2} \\ &\leqslant \frac{C_2\,\mathrm{e}^{8\mathsf{c}h(t+2s)^2L}\,\mathbb{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}}{(1+|\mathbf{x}-\mathbf{z}|^2)^{h-1}}\,.\end{aligned}$$

Let us prove (C.18). By the bound $q_n(x) \leqslant \frac{6\mathsf{c}}{n}\,\mathrm{e}^{-\frac{|x|^2}{8\mathsf{c}\,n}}$ proved in Lemma B.1 we obtain

$$\mathsf{Q}_n^{*,*}(\mathbf{z},\mathbf{x}) = \prod_{i=1}^h q_n(x^i - z^i) \leqslant \frac{(6\mathsf{c})^h}{n^h}\,\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{8\mathsf{c}\,n}}\,,$$

hence for $\mathbf{x}=\mathbf{z}$ we get $\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{x},\mathbf{x}) = \sum_{n=1}^L \mathsf{Q}_n^{*,*}(\mathbf{z},\mathbf{x}) \leqslant (6\mathsf{c})^h\sum_{n=1}^\infty \frac{1}{n^2} = (6\mathsf{c})^h\,\frac{\pi^2}{6} \leqslant 2\,(6\mathsf{c})^h$ which is compatible with (C.18). We next assume that $\mathbf{x}\neq\mathbf{z}$: note that for $A = \frac{|\mathbf{x}-\mathbf{z}|^2}{8\mathsf{c}} > 0$

$$\sum_{n=1}^L \frac{\mathrm{e}^{-\frac{A}{n}}}{n^h} \leqslant \frac{\mathrm{e}^{-\frac{A}{2L}}}{A^{h-1}}\left\{\frac{1}{A}\sum_{n=1}^\infty \varphi\big(\tfrac{n}{A}\big)\right\} \qquad \text{where} \qquad \varphi(t) := \frac{\mathrm{e}^{-\frac{1}{2t}}}{t^h}\,.$$

Since $\varphi(\cdot)$ is unimodal, we can bound $\frac{1}{A}\sum_{n=1}^\infty \varphi\big(\frac{n}{A}\big) \leqslant \int_0^\infty \varphi(t)\,\mathrm{d}t + \frac{1}{A}\|\varphi\|_\infty$ and note that $\int_0^\infty \varphi(t) = 2^{h-1}\int_0^\infty s^{h-2}\,\mathrm{e}^{-s}\,\mathrm{d}s = 2^{h-1}\,(h-2)!$ while $\|\varphi\|_\infty = (2h)^h\,\mathrm{e}^{-h} \leqslant 2^h h!/\sqrt{2\pi h} \leqslant \frac{1}{2}2^h h!$, therefore for $A\geqslant 1$ we get $\frac{1}{A}\sum_{n=1}^\infty\varphi\big(\frac{n}{A}\big) \leqslant 2^h h!$. Overall, recalling (4.13), we have for $\mathbf{x}\neq\mathbf{z}$

$$\widehat{\mathsf{Q}}_L^{*,*}(\mathbf{z},\mathbf{x}) \leqslant \sum_{n=1}^L \mathsf{Q}_n^{*,*}(\mathbf{z},\mathbf{x}) \leqslant \frac{(48\,\mathsf{c}^2)^h\,\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{16\mathsf{c}\,L}}}{|\mathbf{x}-\mathbf{z}|^{2(h-1)}}\,2^h\,h! \leqslant \frac{h!\,(200\,\mathsf{c}^2)^h\,\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{16\mathsf{c}\,L}}}{(1+|\mathbf{x}-\mathbf{z}|^2)^{h-1}}\,,$$

where we last bounded $|\mathbf{x}-\mathbf{z}|^2 \geqslant \frac{1}{2}(1+|\mathbf{x}-\mathbf{z}|^2)$ for $\mathbf{x}\neq\mathbf{z}$. We have proved (C.18). ☐

C.5. Proof of Proposition 4.24. Let us define $p := \frac{q}{q-1}$ so that $\frac{1}{p}+\frac{1}{q} = 1$. Since

$$\|\mathsf{A}\|_{\ell^q\to\ell^q} := \sup_{\mathsf{f},\mathsf{g}:\ \|\mathsf{f}\|_{\ell^p}\leqslant 1,\,\|\mathsf{g}\|_{\ell^q}\leqslant 1}\ \sum_{\mathbf{z},\mathbf{x}\in(\mathbb{Z}^2)_I^h} \mathsf{f}(\mathbf{z})\,\mathsf{A}(\mathbf{z},\mathbf{x})\,\mathsf{g}(\mathbf{x})\,,$$

we can bound $\sum_{\mathbf{z},\mathbf{x}}\mathsf{f}(\mathbf{z})\,|\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x})\,\mathsf{g}(\mathbf{x}) \leqslant \big(\sum_{\mathbf{z},\mathbf{x}}\mathsf{f}(\mathbf{z})^p\,|\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x})\big)^{1/p}\big(\sum_{\mathbf{z},\mathbf{x}}|\widehat{\mathsf{U}}|^I(\mathbf{z},\mathbf{x})\,\mathsf{g}(\mathbf{x})^q\big)^{1/q}$ by Cauchy-Schwarz, hence we obtain

$$\|\mathsf{A}\|_{\ell^q\to\ell^q} \leqslant \max\left\{\sup_{\mathbf{z}\in(\mathbb{Z}^2)_I^h}\ \sum_{\mathbf{x}\in(\mathbb{Z}^2)_I^h}\mathsf{A}(\mathbf{z},\mathbf{x})\,,\ \sup_{\mathbf{x}\in(\mathbb{Z}^2)_I^h}\ \sum_{\mathbf{z}\in(\mathbb{Z}^2)_I^h}\mathsf{A}(\mathbf{z},\mathbf{x})\right\}. \tag{C.19}$$

We will prove (4.55) and (4.56) exploiting this bound.

We recall that $U_{n,\beta}(x)$ is defined in (4.11) and we define

$$U_{n,\beta} := \sum_{x\in\mathbb{Z}^2} U_{n,\beta}(x) = \sum_{k=1}^\infty (\sigma_\beta^2)^k \sum_{0=:n_0<n_1<\cdots<n_k:=n}\ \prod_{i=1}^k q_{2(n_i-n_{i-1})}(0)\,. \tag{C.20}$$

When we sum $U_{n,\beta}$ for $n = 1, \ldots, L$, if we enlarge the sum range in (C.20) by letting each increment $m_i := n_i - n_{i-1}$ vary freely in $\{1, \ldots, M\}$, recalling (4.54) we obtain

$$\sum_{n=1}^{L} \mathrm{e}^{-\lambda n}\, U_{n,\beta} \leqslant \sum_{k=1}^{\infty} (\sigma_\beta^2)^k \Bigg(\sum_{m=1}^{L} \mathrm{e}^{-\lambda m}\, q_{2m}(0) \Bigg)^k = \sum_{k=1}^{\infty} (\sigma_\beta^2\, R_L^{(\lambda)})^k = \frac{\sigma_\beta^2\, R_L^{(\lambda)}}{1 - \sigma_\beta^2\, R_L^{(\lambda)}} \,. \tag{C.21}$$

We next estimate the exponential spatial moments of $U_{n,\beta}(x)$. Pluggin the second bound from (4.32) into (4.11), writing $x = (x^1, x^2)$ and $x^a = \sum_{i=1}^{k} (x_i^a - x_{i-1}^a)$, we obtain

$$\forall a = 1, 2: \qquad \sum_{x \in \mathbb{Z}^2} \mathrm{e}^{t x^a}\, U_{n,\beta}(x) \leqslant \mathrm{e}^{\mathsf{c}\, \frac{t^2}{2} n}\, U_{n,\beta} \,.$$

From this, by $|x| \leqslant |x^1| + |x^2|$, Cauchy-Schwarz and $\mathrm{e}^{t|x^a|} \leqslant \mathrm{e}^{t x^a} + \mathrm{e}^{-t x^a}$, we deduce that

$$\sum_{x \in \mathbb{Z}^2} \mathrm{e}^{t|x|}\, U_{n,\beta}(x) \leqslant 2\, \mathrm{e}^{2\mathsf{c}\, t^2 n}\, U_{n,\beta} \,. \tag{C.22}$$

We now fix a partition $I = \{I^1, \ldots, I^m\} \neq *$ and a *pair* $J = \{\{a, b\}, \{c\} : c \neq a, b\}$. Our goal is to prove (4.56), which also yields (4.55) for $s = 0$. By (4.28) and (4.43) we have the following rough bound, for any $a \in \{-1, +1\}$:

$$\frac{\mathcal{W}_t(\mathbf{z})\, \mathcal{V}_s^J(\mathbf{z})^a}{\mathcal{W}_t(\mathbf{x})\, \mathcal{V}_s^J(\mathbf{x})^a} \leqslant \mathrm{e}^{2(t+s)|x^a - z^a|} \prod_{c \neq a,b} \mathrm{e}^{(t+s)|x^c - z^c|} \,. \tag{C.23}$$

We may order $|I^1| \geqslant |I^2| \geqslant \ldots \geqslant |I^m|$, so that $|I^1| \geqslant 2$. Given $\mathbf{z}, \mathbf{x} \in (\mathbb{Z}^2)_I^h$, denoting by x^{I^j} the common value of x^a for $a \in I^j$, by (4.7) we can write

$$\mathsf{Q}_n^{I,I}(\mathbf{z}, \mathbf{x}) \;=\; q_n(x^{I^1} - z^{I^1})^{|I^1|} \prod_{j=2}^{m} q_n(x^{I^j} - z^{I^j})^{|I^j|} \;\leqslant\; q_n(x^{I^1} - z^{I^1})^2 \prod_{j=2}^{m} q_n(x^{I^j} - z^{I^j}) \,,$$

because $q_n(\cdot) \leqslant 1$. Since $|\mathbb{E}[\xi_\beta^I]| \leqslant \sigma_\beta^2$ by assumption, from (4.8) we can bound

$$|\mathsf{U}|_{n,\beta}^I(\mathbf{z}, \mathbf{x}) \leqslant U_{n,\beta}(x^{I^1} - z^{I^1}) \prod_{j=2}^{m} q_n(x^{I^j} - z^{I^j}) \,,$$

therefore by (C.22), (C.23) and the first bound in (4.33) we obtain

$$\sum_{\mathbf{x} \in (\mathbb{Z}^2)_I^h} \Bigg(|\mathsf{U}|_{n,\beta}^I(\mathbf{z}, \mathbf{x})\, \frac{\mathcal{W}_t(\mathbf{z})\, \mathcal{V}_s(\mathbf{z})^a}{\mathcal{W}_t(\mathbf{x})\, \mathcal{V}_s(\mathbf{x})^a} \Bigg) \leqslant 2^h\, \mathrm{e}^{4h\mathsf{c}\,(t+s)^2 n}\, U_{n,\beta} \,, \tag{C.24}$$

which yields, recalling (4.15),

$$\sup_{\mathbf{z} \in (\mathbb{Z}^2)_I^h} \sum_{\mathbf{x} \in (\mathbb{Z}^2)_I^h} |\hat{\mathsf{U}}|_{L,\lambda,\beta}^J(\mathbf{z}, \mathbf{x})\, \frac{\mathcal{W}_t(\mathbf{z})\, \mathcal{V}_s(\mathbf{z})^a}{\mathcal{W}_t(\mathbf{x})\, \mathcal{V}_s(\mathbf{x})^a} \leqslant 1 + 2^h\, \mathrm{e}^{4h\mathsf{c}\,(t+s)^2 L} \sum_{n=1}^{L} \mathrm{e}^{-\lambda n}\, U_{n,\beta} \,, \tag{C.25}$$

and the same holds exchanging $\mathbf{x}$ and $\mathbf{z}$ by symmetry (note that the bound (C.23) is symmetric in $\mathbf{x} \leftrightarrow \mathbf{z}$). Recalling (C.19) and (C.21), we obtain (4.56) (hence (4.55)). $\square$

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