THE ASYMPTOTIC SMILE OF A MULTISCALING STOCHASTIC VOLATILITY MODEL

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Abstract. We consider a stochastic volatility model which captures relevant stylized facts of financial series, including the multiscaling of moments. Using large deviations techniques, we determine the asymptotic shape of the implied volatility surface in *any* regime of small maturity $t \to 0$ or extreme log-strike $|\kappa| \to \infty$ (with bounded maturity). Even if the price has continuous paths, we show that out-of-the-money implied volatility diverges for small maturity, producing a very pronounced smile. When $|\kappa|$ is much larger than t, the implied volatility is asymptotically an explicit function of just the ratio (κ/t).

We consider a stochastic volatility model with continuous paths, introduced in [ACDP12], in which the volatility jumps at the arrival times of a Poisson process. Despite the few parameters, this model captures relevant stylized facts of financial series, such as the crossover in the log-return distribution from power-law (small time) to Gaussian (large time), the clustering of volatility and the so-called *multiscaling of moments*. It was recently shown in [DP14] that the latter phenomenon happens for a wide class of models, in which the volatility is driven by a Lévy subordinator under a super-linear mean-reversion.

In this paper we stick to the basic model introduced in [ACDP12], focusing on pricing. Our main results are *explicit* asymptotic formulas for the price of European options and for the corresponding implied volatility, cf. Theorems 2.2 and 3.3. Let us summarize some of the highlights, referring to §2.3 for a more detailed discussion.

- We allow for any regime of either extreme log-strike $|\kappa| \to \infty$ (with arbitrary bounded maturity t, possibly varying with κ) or small maturity $t \downarrow 0$ (with arbitrary log-strike κ, possibly varying with t). This flexibility yields uniform estimates for the implied volatility surface $\sigma_{\mathrm{imp}}(\kappa, t)$ in *open regions* of the plane (κ, t), cf. Corollary 2.4.
- Out-of-the-money implied volatility *diverges* for small maturity, i.e. $\sigma_{\mathrm{imp}}(\kappa, t) \to \infty$ as $t \downarrow 0$ for any $\kappa \neq 0$, while $\sigma_{\mathrm{imp}}(0, t) \to \sigma_0 < \infty$ (see Figure 2). This shows that stochastic volatility models *without jumps in the price* can produce very steep skews for the small-time volatility smile, cf. [Gat06, Chapter 5, "Why jumps are needed"]. What lies behind this phenomenon is the asymptotic emergence of heavy tails in the small-time distribution of the volatility, where jumps are present. Interestingly, the same mechanism is responsible for the multiscaling of moments.
- We obtain the asymptotic expression $\sigma_{\mathrm{imp}}(\kappa, t) \sim f(\kappa/t)$, for an explicit function $f(\cdot)$ of *just the ratio* (κ/t), whenever $|\kappa|$ is much larger than t (including, in particular, the regimes $t \downarrow 0$ for fixed $\kappa \neq 0$, and $|\kappa| \to \infty$ for fixed $t > 0$). In §2.3 we provide a heuristic explanation for this interesting phenomenon, which is shared by different models without moment explosion.

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Since the moment generating function of our model admits no closed formula, the core of the paper is constituted by *tail estimates* for the log-price distribution, cf. Theorem 3.1. These are based on large deviations bounds for suitable functionals of a Poisson process, which might be of independent interest (see Corollary 4.2 and Remark 4.3). From these estimates, we derive asymptotic formulas for option price and implied volatility using the general approach in [CC14], that we summarize in §6.1 and §7.1.

The paper is organized as follows.

- In Section 1 we define the model and we set up some notation.
- In Sections 2 and 3 we present our main asymptotic results on implied volatility, option price and tail probability, with a general discussion in §2.3.
- In Section 4 we prove some key moment estimates, which are the cornerstone of our approach, together with some large deviations results for the Poisson process.
- The following sections 5, 6 and 7 contain the proof of our main results concerning tail probability, option price and implied volatility, respectively.
- Finally, some technical results have been deferred to the Appendix A.

1. The model

In §1.1 we recall the definition of the process $(Y_t)_{t\geq 0}$, introduced in [ACDP12], for the de-trended log-price of a financial asset under the historical measure. In §1.2 we describe its evolution under the risk-neutral measure (switching notation to $(X_t)_{t\geq 0}$ for clarity) and in §1.3 we define the price of a call option and the related implied volatility.

1.1. **The historical measure.** We fix four real parameters $0 < D < \frac{1}{2}$, $V > 0$, $\lambda > 0$ and $\tau_0 < 0$, whose meaning is discussed in a moment. We consider a stochastic volatility model $(Y_t)_{t\geq 0}$, with $Y_0 := 0$, defined by

$$\mathrm{d}Y_t = \sigma_t \, \mathrm{d}B_t \,, \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion and $(\sigma_t)_{t\geq 0}$ is an *independent* process, built as follows: denoting by $(N_t)_{t\geq 0}$ a Poisson process of intensity λ (independent of $(B_t)_{t\geq 0}$) with jump times $0 < \tau_1 < \tau_2 < \ldots$, we set

$$\sigma_t := \mathsf{c} \, \frac{\sqrt{2D}}{(t - \tau_{N_t})^{\frac{1}{2} - D}} \,, \qquad \text{where} \qquad \mathsf{c} := \frac{\lambda^{D - \frac{1}{2}} \, V}{\sqrt{\Gamma(2D+1)}} \,, \tag{1.2}$$

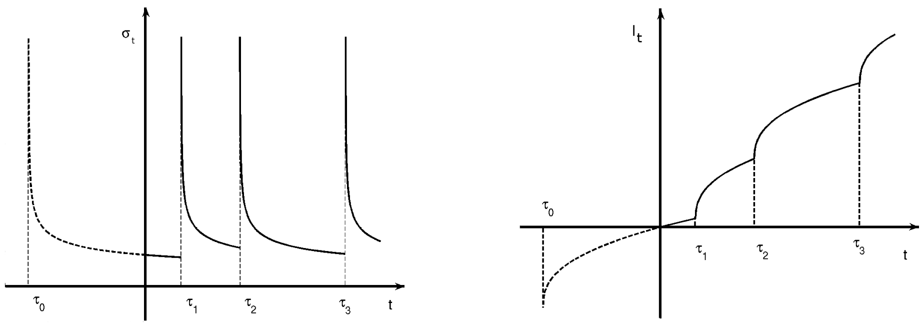
and $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x$ is Euler's gamma function. Note that $\tau_{N_t} = \max\{\tau_k : \ \tau_k \leq t\}$ is the last jump time of the Poisson process before t, hence the volatility σ_t *diverges* at the jump times of the Poisson process, which can be thought as shocks in the market. We refer to Figure 1 for a graphical representation.

We can now describe the meaning of the parameters:

- $\lambda \in (0, \infty)$ represents the average frequency of shocks;
- $D \in (0, \frac{1}{2})$ tunes the decay exponent of the volatility after a shock;
- $V \in (0, \infty)$ represents the *large-time volatility*,† because (see Appendix A.1)

$$V = \lim_{t\to\infty} \sqrt{\mathrm{E}[\sigma_t^2]} \,; \tag{1.3}$$

†The constant c in (1.2) was called σ in [ACDP12] and used as a parameter in place of V (note that c and V are proportional). Our preference for V is due to its direct meaning as large-time volatility, by (1.3).



(A) *Volatility* $(\sigma_t)_{t\geq 0}$ (B) *Time change* $(I_t)_{t\geq 0}$

FIGURE 1. Paths of the *time change* and of the *spot volatility process*

- $\tau_0 \in (-\infty, 0)$ tunes the *initial volatility* σ_0, since

$$\sigma_0 = \mathsf{c}\,\sqrt{2D}\,(-\tau_0)^{D-\frac{1}{2}} = \frac{\lambda^{D-\frac{1}{2}}\,V}{\sqrt{\Gamma(2D)}}\,(-\tau_0)^{D-\frac{1}{2}}\,. \tag{1.4}$$

Given this correspondence, *one can use* σ_0 *as a parameter instead of* τ_0.†

As discussed in [ACDP12], the process $(Y_t)_{t\geq 0}$ in (1.1) can be represented as a *time-changed Brownian motion*: more precisely,

$$Y_t := W_{I_t}\,, \qquad \text{with} \qquad I_t := \int_0^t \sigma_s^2\,\mathrm{d}s\,, \tag{1.5}$$

where $(W_t)_{t\geq 0}$ is another Brownian motion, independent of $(I_t)_{t\geq 0}$. It follows by (1.2) that for $t \in [\tau_k, \tau_{k+1}]$ one has $I_t - I_{\tau_k} = \mathsf{c}^2(t-\tau_k)^{2D}$, cf. (1.2), hence

$$I_t := \mathsf{c}^2\left\{(t-\tau_{N_t})^{2D} - (-\tau_0)^{2D} + \sum_{k=1}^{N_t}(\tau_k - \tau_{k-1})^{2D}\right\}, \tag{1.6}$$

with the convention that the sum in (1.6) is zero when $N_t = 0$ (see Figure 1).

Remark 1.1. In the limiting case $D = \frac{1}{2}$ one has $\sigma_t = V$ and $I_t = V^2 t$, hence our model reduces to Brownian motion with constant volatility: $Y_t = VB_t = W_{V^2 t}$. We exclude this case from our analysis just because it has to be treated separately in the proofs.

1.2. **The risk-neutral measure.** We are going to consider a natural risk-neutral measure, under which the price $(S_t)_{t\geq 0}$ evolves according to the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = \sigma_t\,\mathrm{d}B_t\,, \tag{1.7}$$

where σ_t is the process defined in (1.2). As a matter of fact, there is a one-parameter class of equivalent martingale measures which allow to modify the value of the parameter $\lambda \in (0,\infty)$ freely (see Appendix A.2). Here we assume to have fixed that parameter, and still call it λ.

†We point out that in [ACDP12] the parameter $-\tau_0$ was chosen randomly, as an independent $Exp(\lambda)$ random variable (just like τ_1, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, ...). With this choice, the process $(t - \tau_{N_t})_{t\geq 0}$ becomes *stationary* (with $Exp(\lambda)$ one-time marginal distributions), hence the volatility $(\sigma_t)_{t\geq 0}$ is a stationary process too, by (1.2). In our context, it is more natural to have a fixed value for the initial volatility.

Let us denote by $(X_t)_{t\geq 0}$ the *log-price* process under the risk-neutral measure:

$$X_t := \log S_t\,,$$

with $X_0 = 0$, i.e. $S_0 = 1$. It follows by (1.7) that $\mathrm{d}X_t = \sigma_t\,\mathrm{d}B_t - \frac{1}{2}\sigma_t^2\,\mathrm{d}t$, hence

$$X_t = W_{I_t} - \frac{1}{2}I_t\,, \tag{1.8}$$

where the process $(I_t)_{t\geq 0}$, cf. (1.5)-(1.6), is independent of the Brownian motion $(W_t)_{t\geq 0}$. As a consequence, the price $(S_t)_{t\geq 0}$ is a *time-changed geometric Brownian motion*:

$$S_t = e^{X_t} = e^{W_{I_t} - \frac{1}{2}I_t}\,. \tag{1.9}$$

Representations (1.8), (1.9) are so useful that we can take them as definitions of our model.

Definition 1.2. *The log-price* $(X_t)_{t\geq 0}$ *and price* $(S_t)_{t\geq 0}$ *processes, under the risk-neutral measure, evolve according to* (1.8) *and* (1.9) *respectively, with* $(I_t)_{t\geq 0}$ *defined in* (1.6).

1.3. **Option price and implied volatility.** The price of a (normalized) European call option, with log-strike $\kappa \in \mathbb{R}$ and maturity $t \geq 0$, under our model is

$$c(\kappa, t) := \mathrm{E}[(S_t - e^{\kappa})^+] = \mathrm{E}[(e^{X_t} - e^{\kappa})^+]\,. \tag{1.10}$$

We recall that, for a given volatility parameter $\sigma \in (0,\infty)$, the Black&Scholes price of a European call option equals $\mathsf{C_{BS}}(\kappa, \sigma\sqrt{t})$, where

$$\mathsf{C_{BS}}(\kappa, v) := \mathrm{E}[(e^{W_{v^2} - \frac{1}{2}v^2} - e^{\kappa})^+] = \begin{cases} (1-e^{\kappa})^+ & \text{if } v = 0\,, \\ \Phi(d_1) - e^{\kappa}\Phi(d_2) & \text{if } v > 0\,, \end{cases} \tag{1.11}$$

where

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}}\,\mathrm{d}t\,, \qquad d_1 := -\frac{\kappa}{v} + \frac{v}{2}\,, \qquad d_2 := -\frac{\kappa}{v} - \frac{v}{2}\,. \tag{1.12}$$

Since $\Phi(-x) = 1 - \Phi(x)$, the following symmetry relation holds:

$$\mathsf{C_{BS}}(-\kappa, v) = 1 - e^{-\kappa} + e^{-\kappa}\mathsf{C_{BS}}(\kappa, v)\,. \tag{1.13}$$

Definition 1.3. *For* $t > 0$ *and* $\kappa \in \mathbb{R}$*, the* implied volatility $\sigma_{\mathrm{imp}}(\kappa, t)$ *of our model is the unique value of* $\sigma \in (0,\infty)$ *such that* $c(\kappa, t)$ *in* (1.10) *equals* $\mathsf{C_{BS}}(\kappa, \sigma\sqrt{t})$*, that is*

$$c(\kappa, t) = \mathsf{C_{BS}}(\kappa, \sigma_{\mathrm{imp}}(\kappa, t)\sqrt{t})\,. \tag{1.14}$$

Recalling (1.9), since $(I_t)_{t\geq 0}$ is independent of $(W_t)_{t\geq 0}$, the call price $c(\kappa, t)$ in (1.10) enjoys the representation

$$c(\kappa, t) = \mathrm{E}\left[\mathsf{C_{BS}}(\kappa, v)\big|_{v=\sqrt{I_t}}\right]\,, \tag{1.15}$$

known as *Hull-White formula* [HW87]. As a consequence, the symmetry relation (1.13) transfers from Black&Scholes to our model:

$$c(-\kappa, t) = 1 - e^{-\kappa} + e^{-\kappa}c(\kappa, t)\,. \tag{1.16}$$

Looking at (1.14), it follows that *the implied volatility of our model is symmetric in* κ:

$$\sigma_{\mathrm{imp}}(-\kappa, t) = \sigma_{\mathrm{imp}}(\kappa, t)\,. \tag{1.17}$$

As a consequence, in the sequel we focus on the regime $\kappa \geq 0$.

Remark 1.4. Properties (1.15)-(1.16)-(1.17) hold for *any* stochastic volatility model (1.7) for which the volatility $(\sigma_t)_{t\geq 0}$ is independent of the Brownian motion $(B_t)_{t\geq 0}$, because any such model enjoys the representation (1.9), with $(I_t)_{t\geq 0}$ defined as in (1.5) (cf. [RT96]).

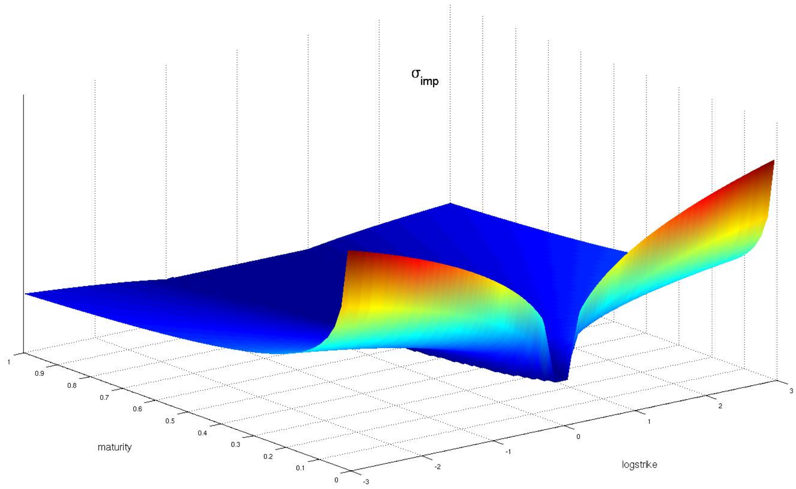


FIGURE 2. Asymptotic behavior of $\sigma_{\mathrm{imp}}(\kappa, t)$.

2. Main results: implied volatility

In this section we present our main results on the asymptotic behavior of the implied volatility $\sigma_{\mathrm{imp}}(\kappa, t)$ of our model. We allow for a variety of regimes with bounded maturity. More precisely, we consider an *arbitrary family of values of* (κ, t) such that

$$\text{either } t \to \bar{t} \in (0, \infty) \text{ and } \kappa \to \infty\,, \qquad \text{or } t \to 0 \text{ with arbitrary } \kappa \geq 0\,. \tag{2.1}$$

Allowing for both sequences $((\kappa_n, t_n))_{n\in\mathbb{N}}$ and functions $((\kappa_s, t_s))_{s\in[0,\infty)}$, we omit subscripts.

We agree with the conventions $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We are going to use the following asymptotic notations, for positive functions f, g:

$$f \sim g, \quad f \ll g, \quad f \gg g \qquad \Longleftrightarrow \qquad \frac{f}{g} \to 1, \quad \frac{f}{g} \to 0, \quad \frac{f}{g} \to \infty \quad \text{respectively}, \tag{2.2}$$

$$f \asymp g \qquad \Longleftrightarrow \qquad \log f \sim \log g\,, \qquad \text{i.e.} \quad \frac{\log f}{\log g} \to 1\,. \tag{2.3}$$

When discussing heuristics, we will sometimes use the informal notation $f \approx g$.

2.1. **Auxiliary functions.** We introduce two functions $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 : (0, \infty) \to (0, \infty)$ by

$$\boldsymbol{\kappa}_1(t) := \sqrt{t}\sqrt{\log\left(1 + \frac{1}{t}\right)}\,, \qquad \boldsymbol{\kappa}_2(t) := t^D\sqrt{\log\left(1 + \frac{1}{t}\right)}\,, \tag{2.4}$$

which will act as boundaries for κ, separating different asymptotic regimes as $t \to 0$. We write $\log(1+\frac{1}{t})$ instead of $\log\frac{1}{t}$ just for convenience, so that $\boldsymbol{\kappa}_1(t)$ and $\boldsymbol{\kappa}_2(t)$ are well-defined for all $t > 0$, but of course $\log(1 + \frac{1}{t}) \sim \log\frac{1}{t}$ as $t \to 0$. Note that $\boldsymbol{\kappa}_1(t) < \boldsymbol{\kappa}_2(t)$ for $t < 1$, because $D < \frac{1}{2}$ by assumption.

We also define an auxiliary function $\mathsf{f} : (0, \infty) \to \mathbb{R}$ by

$$\mathsf{f}(a) := \min_{m\in\mathbb{N}_0} \mathsf{f}_m(a)\,, \qquad \text{with} \qquad \mathsf{f}_m(a) := m + \frac{a^2}{2\mathsf{c}^2 m^{1-2D}}\,, \tag{2.5}$$

where c is the constant in (1.2). Since $D < \frac{1}{2}$, one can actually restrict the minimum in (2.5) to $m \geq 1$ and the minimization can be performed explicitly (see Appendix A.3). In particular, the function f is *continuous and strictly increasing* and satisfies

$$\mathsf{f}(a) \sim \begin{cases} 1 + \dfrac{a^2}{2\mathsf{c}^2} & \text{if } a \downarrow 0 \\ \Big((1-D)^{\frac{1/2-D}{1-D}} \mathsf{C} \Big) \, a^{\frac{1}{1-D}} & \text{if } a \uparrow \infty \end{cases}, \qquad \text{with} \quad \mathsf{C} := \frac{(1-D)^{\frac{1/2}{1-D}}}{(\frac{1}{2}-D)^{\frac{1/2-D}{1-D}}} \, \frac{1}{\mathsf{c}^{\frac{1}{1-D}}} \,. \tag{2.6}$$

Remark 2.1. In the limiting case $D = \frac{1}{2}$ (where we adopt the convention $0^0 := 1$), the minimum in (2.5) is trivially attained for $m = 0$, so that $\mathsf{f}(a) = \mathsf{f}_0(a) = \frac{a^2}{2\mathsf{c}^2}$.

2.2. **Implied volatility.** The next theorem, proved in Section 7, is our main result. It provides a *complete asymptotic picture* of the implied volatility in any regime (2.1) of small maturity and/or large strike (see Figure 2). The corresponding asymptotic results for the tail probability $\mathrm{P}(X_t > \kappa)$ and for the option price $c(\kappa, t)$ are presented in Section 3.

Theorem 2.2 (Implied volatility)**.** *Consider a family of values of (κ, t) with $\kappa \geq 0$, $t > 0$.*

(a) *If $t \to \bar{t} \in (0, \infty)$ and $\kappa \to \infty$, or if $t \to 0$ and $\kappa \gg \boldsymbol{\kappa}_2(t)$ (e.g., $\kappa \to \bar{\kappa} \in (0, \infty]$), the following relation holds:*

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \frac{1}{\sqrt{2\mathsf{C}}} \left(\frac{\frac{\kappa}{t}}{\sqrt{\log \frac{\kappa}{t}}} \right)^{\frac{1/2-D}{1-D}} , \tag{2.7}$$

where C is the constant defined in (2.6)*.*

(b) *If $t \to 0$ and $\kappa \sim a\, \boldsymbol{\kappa}_2(t)$, for some $a \in (0, \infty)$,*

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \left\{ \frac{1}{\sqrt{2\, \mathsf{f}(a)}} \right\} \frac{\kappa}{\boldsymbol{\kappa}_1(t)} \,, \tag{2.8}$$

where the function $\mathsf{f}(\cdot)$ is defined in (2.5)*.*

(c) *If $t \to 0$ and $(\sqrt{2D+1}\, \sigma_0)\, \boldsymbol{\kappa}_1(t) \leq \kappa \ll \boldsymbol{\kappa}_2(t)$, where σ_0 is defined in* (1.4)*,*

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \left\{ \frac{1}{\sqrt{2\big(D + 1 - \frac{\log \kappa}{\log t}\big)}} \right\} \frac{\kappa}{\boldsymbol{\kappa}_1(t)} \,, \tag{2.9}$$

and note that $\frac{1}{2} \leq \frac{\log \kappa}{\log t} \leq D$ for κ in the range under consideration.

(d) *Finally, if $t \to 0$ and $0 \leq \kappa \leq (\sqrt{2D+1}\, \sigma_0)\, \boldsymbol{\kappa}_1(t)$,*

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \sigma_0 \,. \tag{2.10}$$

Let us give a qualitative description of Theorem 2.2. Recall (2.3), (2.4) and note that $\boldsymbol{\kappa}_1(t) \asymp \sqrt{t}$ and $\boldsymbol{\kappa}_2(t) \asymp t^D$. If we fix $t > 0$ small and increase $\kappa \in [0, \infty)$, we can describe the implied volatility $\sigma_{\mathrm{imp}}(\kappa, t)$ as follows (cf. Figure 2):

- $\sigma_{\mathrm{imp}}(\kappa, t) \approx \sigma_0$ is roughly constant for $0 \leq \kappa \lessapprox \sqrt{t}$, cf. (2.10);
- then $\sigma_{\mathrm{imp}}(\kappa, t) \approx \kappa / \sqrt{t}$ grows linearly for $\sqrt{t} \lessapprox \kappa \lessapprox t^D$, cf. (2.9);
- then $\sigma_{\mathrm{imp}}(\kappa, t) \approx (\kappa/t)^\gamma$ grows sublinearly for $\kappa \gtrapprox t^D$, cf. (2.7), with an exponent $\gamma = \frac{1/2-D}{1-D}$ that can take any value in $(0, \frac{1}{2})$ depending on D.

We stress that formula $\sigma_{\mathrm{imp}}(\kappa, t) \approx (\kappa/t)^\gamma$ holds also as $t \downarrow 0$ for fixed $\kappa > 0$.

Remark 2.3. It is interesting to observe that the four relations (2.7), (2.8), (2.9) and (2.10) match at the boundaries of the respective intervals of applicability:

- by the asymptotics in (2.6), relation (2.8) reduces to (2.7) as $a \to \infty$ (note that $\log \frac{1}{t} \sim \frac{1}{1-D} \log \frac{\kappa}{t}$ for $\kappa \sim a\boldsymbol{\kappa}_2(t)$), while it reduces to (2.9) as $a \to 0$;
- relation (2.9) coincides with (2.10) for $\kappa = (\sigma_0\sqrt{2D+1})\,\boldsymbol{\kappa}_1(t)$.

Also note that in the limiting case $D = \frac{1}{2}$ one has $\sigma_0 = \mathsf{c} = V$, cf. (1.2) and (1.4), and moreover $\mathsf{f}(a) = \frac{a^2}{2\mathsf{c}^2}$, cf. Remark 2.1. As a consequence, relations (2.7), (2.8) and (2.10) reduce to $\sigma_{\mathrm{imp}}(\kappa, t) \sim V$, in perfect agreement with the fact that for $D = \frac{1}{2}$ our model becomes Black&Scholes model with constant volatility V, cf. Remark 1.1.[†]

2.3. **Discussion.** We conclude this section with a more detailed discussion of Theorem 2.2, highlighting the most relevant points and outlining further directions of research.

Joint volatility surface asymptotics. In Theorem 2.2 we allow for arbitrary families of (κ, t), besides the usual regimes $\kappa \to \infty$ for fixed t, or $t \downarrow 0$ for fixed κ. Interestingly, this flexibility yields *uniform estimates on the implied volatility surface in open regions of the plane*, as we now show. Recalling (2.4), for $T, M \in (0, \infty)$ we define the region

$$\mathcal{A}_{T,M} := \left\{ (\kappa, t) \in \mathbb{R}^2 :\ 0 < t < T,\ \kappa > M\boldsymbol{\kappa}_2(t) \right\}.$$

Corollary 2.4 (Joint surface asymptotics)**.** *Fix $T > 0$. For every $\varepsilon > 0$ there exists $M = M(T, \varepsilon) > 0$ such that for all $(\kappa, t) \in \mathcal{A}_{T,M}$*

$$\frac{1-\varepsilon}{\sqrt{2\mathsf{C}}} \left(\frac{\frac{\kappa}{t}}{\sqrt{\log \frac{\kappa}{t}}} \right)^{\frac{1/2-D}{1-D}} \le \sigma_{\mathrm{imp}}(\kappa, t) \le \frac{1+\varepsilon}{\sqrt{2\mathsf{C}}} \left(\frac{\frac{\kappa}{t}}{\sqrt{\log \frac{\kappa}{t}}} \right)^{\frac{1/2-D}{1-D}} . \tag{2.11}$$

Proof. By contradiction, assume that for some $T, \varepsilon > 0$ and for every $M \in \mathbb{N}$ one can find $(\kappa, t) \in \mathcal{A}_{T,M}$ such that relation (2.11) fails. We can then extract subsequences $M_n \to \infty$ and $(\kappa_n, t_n) \in \mathcal{A}_{T,M_n}$ such that $t_n \to \bar{t} \in [0, T]$. The subsequence $((\kappa_n, t_n))_{n\in\mathbb{N}}$ satisfies the assumptions of part (a) in Theorem 2.2: if $\bar{t} > 0$, then $\kappa_n \to \infty$, while if $\bar{t} = 0$, then $\kappa_n \gg \boldsymbol{\kappa}_2(t_n)$, because $(\kappa_n, t_n) \in \mathcal{A}_{T,M_n}$ and $M_n \to \infty$. However, relation (2.7) fails by construction, contradicting Theorem 2.2. □

Small-maturity divergence of implied volatility. Relation (2.7) shows that, for fixed $\kappa > 0$, the implied volatility *diverges* as $t \downarrow 0$, producing a very steep smile for small maturity. This is typical for models with jumps in the price [AL12], but remarkably our stochastic volatility model has *continuous paths.* What lies behind this phenomenon is the very same mechanism that produces the multiscaling of moments [ACDP12], i.e., the fact that the volatility σ_t has *approximate heavy tails* as $t \downarrow 0$.

In order to give a heuristic explanation, we anticipate that, under mild assumptions, option price and tail probability are linked by $c(\kappa, t) \asymp \mathrm{P}(X_t > \kappa)$ as $t \downarrow 0$ for fixed $\kappa > 0$ (see Theorem 6.2 below). In the Black&Scholes case $\mathsf{C}_{\mathsf{BS}}(\kappa, \sigma\sqrt{t}) \asymp \exp(-\kappa^2/(2\sigma^2 t))$, hence by Definition 1.3 it follows that implied volatility and tail probability are linked by

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t\,(-\log \mathrm{P}(X_t > \kappa))}} . \tag{2.12}$$

[†]Note that relation (2.9) does not apply for $D = \frac{1}{2}$, because in this case $\boldsymbol{\kappa}_1(t) = \boldsymbol{\kappa}_2(t)$ and consequently there is no κ for which $(\sigma_0\sqrt{2D+1})\,\boldsymbol{\kappa}_1(t) \le \kappa \ll \boldsymbol{\kappa}_2(t)$.

This relation shows that $\sigma_{\mathrm{imp}}(\kappa, t)$ stays bounded as $t \downarrow 0$ when $-\log \mathrm{P}(X_t > \kappa) \sim C/t$ for some $C = C(\kappa) \in (0, \infty)$, as in the Heston model [JFL12].

This is *not* the case for our model, where $-\log \mathrm{P}(X_t > \kappa) \ll 1/t$. The reason is that as $t \downarrow 0$, by (1.8), the distribution of $X_t \approx W_{I_t}$ is approximately Gaussian with random variance $I_t = \int_0^t \sigma_s^2 \, \mathrm{d}s$, hence $\mathrm{P}(X_t > \kappa) \asymp \mathrm{E}[\exp(-\kappa^2/(2I_t))]$. Although $\mathrm{E}[I_t] \approx t$, the point is that I_t can take with non negligible probability *atypically large values*, as large as $t^{D/(1-D)} \gg t$, leading to $\mathrm{P}(X_t > \kappa) \approx \exp(-C/t^{D/(1-D)})$. Plugged into (2.12), this estimate explains the t-dependence in (2.7), apart from logarithmic factors (we refer to relation (3.2) below for a more precise estimate).

On a "universal" asymptotic relation. In the regime when (2.7) holds, the implied volatility $\sigma_{\mathrm{imp}}(\kappa, t)$ is asymptotically a function $f(\kappa/t)$ of just the ratio (κ/t). This feature appears to be shared by *different models without moment explosion* (with the function $f(\cdot)$ depending on the model). For instance, in Carr-Wu's finite moment logstable model [CW04], as shown in [CC14, Theorem 2.1],

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim B_\alpha \left(\frac{\kappa}{t}\right)^{-\frac{2-\alpha}{2(\alpha-1)}} \quad \text{for} \quad \kappa \gg t^{1/\alpha} \,,$$

where B_α is an explicit constant. Another example is provided by Merton's jump diffusion model [M76] for which, extending [BF09], we showed in [CC14, formula (2.35)] that

$$\sigma_{\mathrm{imp}}^2(\kappa, t) \sim \frac{\delta}{2\sqrt{2}} \frac{\frac{\kappa}{t}}{\sqrt{\log \frac{\kappa}{t}}} \quad \text{for} \quad \kappa \gg t \,.$$

To understand heuristically the source of this phenomenon, note that $\sigma_{\mathrm{imp}}(\kappa, t) \sim f(\kappa/t)$ means in particular that $\sigma_{\mathrm{imp}}(2\kappa, 2t) \sim \sigma_{\mathrm{imp}}(\kappa, t)$, which by (2.12) translates into

$$\mathrm{P}(X_{2t} > 2\kappa) \asymp \mathrm{P}(X_t > \kappa)^2 \,. \tag{2.13}$$

If the log-price increments are *approximately stationary*, in the sense that $\mathrm{P}(X_t > \kappa) \asymp \mathrm{P}(X_{2t} - X_t > \kappa)$, the previous relation can be rewritten more expressively as

$$\mathrm{P}(X_{2t} > 2\kappa) \asymp \mathrm{P}(X_t > \kappa) \, \mathrm{P}(X_{2t} - X_t > \kappa) \,. \tag{2.14}$$

This says, heuristically, that the most likely way to produce the event $\{X_{2t} > 2\kappa\}$ is through the events $\{X_t > \kappa\}$ and $\{X_{2t} - X_t > \kappa\}$, which are approximately independent.

Relation (2.13) holds indeed for our model, see (3.2) below, as well as for Carr-Wu and Merton models, cf. [CC14, (2-15)-(2.17) and (2.34)], in the regime when κ is much larger than t (how much depending on the model). On the other hand, relation (2.13) typically *fails* for models with moment explosion, such as the Heston model, for which the implied volatility $\sigma_{\mathrm{imp}}(\kappa, t)$ is *not* asymptotically a function of just the ratio (κ/t), cf. [CC14, §2.3].

Further directions of research. The tail probability asymptotics in Theorem 3.1 below include the regime $t \to \infty$, which is however excluded for the implied volatility asymptotics in Theorem 2.2 (and for the option price asymptotics in Theorem 3.3 below). This is because we rely on the approach in [CC14], recalled in §6.1 below, which assumes that the maturity is bounded from above, but extension to unbounded maturity are certainly possible. For general results in the regime $t \to \infty$, we refer to [Te09].

It should also be stressed that our model has a symmetric smile $\sigma_{\mathrm{imp}}(-\kappa, t) = \sigma_{\mathrm{imp}}(\kappa, t)$, a limitation shared by all stochastic volatility models with independent volatility (recall Remark 1.4). To produce an asymmetry, one should correlate the volatility with the price (leverage effect). In the framework of our model, this can be obtained e.g. introducing *jumps in the price* correlated to those of the volatility. This possibility is investigated in [C15].

3. Main result: tail probability and option price

In this section we present explicit asymptotic estimates for the option price $c(\kappa, t)$ and for the tail probability $\mathrm{P}(X_t > \kappa)$ of our model. Before starting, we note that the following convergence in distribution follows from relations (1.8) and (1.6) (see §5.1):

$$\frac{X_t}{\sqrt{t}} \xrightarrow[t \downarrow 0]{d} \sigma_0 W_1 , \tag{3.1}$$

where σ_0 is the constant in (1.4)

3.1. **Tail probability.** For families of (κ, t) satisfying (2.1), we distinguish the regime of *typical deviations*, when $\mathrm{P}(X_t > \kappa)$ is bounded away from zero, from the regime of *atypical deviations*, when $\mathrm{P}(X_t > \kappa) \to 0$. The former regime corresponds to $t \to 0$ with $\kappa = O(\sqrt{t})$ and the (strictly positive) limit of $\mathrm{P}(X_t > \kappa)$ can be easily computed, by (3.1).

On the other hand, the regime of atypical deviations $\mathrm{P}(X_t > \kappa) \to 0$ includes $t \to 0$ with $\kappa \gg \sqrt{t}$ and $t \to \bar{t} \in (0, \infty)$ with $\kappa \to \infty$, and also $t \to \infty$ with $\kappa \gg t$ (not included in (2.1)). In all these cases we determine an asymptotic equivalent of $\log \mathrm{P}(X_t > \kappa)$ which, remarkably, is sharp enough to get the estimates on the implied volatility in Theorem 2.2. We refer to §6.1-§7.1 for more details, where we summarize the general results of [CC14] linking tail probability, option price and implied volatility.

The following theorem, on the asymptotic behavior of $\mathrm{P}(X_t > \kappa)$, is proved in Section 5. Note that items (a), (b) and (c) correspond to atypical deviations, while the last item (d) corresponds to typical deviations. We recall that $\boldsymbol{\kappa}_1(\cdot)$ and $\boldsymbol{\kappa}_2(\cdot)$ are defined in (2.4).

Theorem 3.1 (Tail probability). *Consider a family of values of (κ, t) with $\kappa \geq 0$, $t > 0$.*

(a) *If $t \to \infty$ and $\kappa \gg t$, or if $t \to \bar{t} \in (0, \infty)$ and $\kappa \to \infty$, or if $t \to 0$ and $\kappa \gg \boldsymbol{\kappa}_2(t)$,*

$$\log \mathrm{P}(X_t > \kappa) \sim -\mathsf{C} \left(\frac{\kappa}{t^D} \right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1/2-D}{1-D}} , \tag{3.2}$$

where the constant C is defined in (2.6).

(b) *If $t \to 0$ and $(\sqrt{2}\,\sigma_0)\, \boldsymbol{\kappa}_1(t) \leq \kappa \leq M\, \boldsymbol{\kappa}_2(t)$, for some $M < \infty$,*

$$\log \mathrm{P}(X_t > \kappa) \sim t^{\mathsf{f}\left(\frac{\kappa}{\boldsymbol{\kappa}_2(t)}\right)} = -\mathsf{f}\left(\frac{\kappa}{\boldsymbol{\kappa}_2(t)} \right) \log \frac{1}{t} , \tag{3.3}$$

where $\mathsf{f}(\cdot)$ is defined in (2.5).

(c) *If $t \to 0$ and $\sqrt{t} \ll \kappa \leq (\sqrt{2}\,\sigma_0)\, \boldsymbol{\kappa}_1(t)$,*

$$\log \mathrm{P}(X_t > \kappa) \sim -\frac{\kappa^2}{2\sigma_0^2\, t} \sim -\frac{1}{2\sigma_0^2} \left(\frac{\kappa}{\boldsymbol{\kappa}_1(t)} \right)^2 \log \frac{1}{t} . \tag{3.4}$$

(d) *Finally, if $t \to 0$ and $\kappa \sim a\sqrt{t}$ for some $a \in [0, \infty)$,*

$$\mathrm{P}(X_t > \kappa) \to 1 - \Phi\left(\frac{a}{\sigma_0} \right) > 0 , \tag{3.5}$$

where $\Phi(\cdot)$ is the distribution function of a standard Gaussian, cf. (1.12).

Remark 3.2. Observe that item (b) in Theorem 3.1 can be made more explicit:

- if $t \to 0$ and $\kappa \sim a\, \boldsymbol{\kappa}_2(t)$, for some $a \in (0, \infty)$,

$$\log \mathrm{P}(X_t > \kappa) \sim -\mathsf{f}(a) \log \frac{1}{t} ; \tag{3.6}$$

- if $t \to 0$ and $(\sqrt{2}\,\sigma_0)\,\boldsymbol{\kappa}_1(t) \le \kappa \ll \boldsymbol{\kappa}_2(t)$,

$$\log \mathrm{P}(X_t > \kappa) \sim -\log \frac{1}{t}\,, \tag{3.7}$$

because $\mathsf{f}(0) = 1$ by (2.6).

It is even possible to gather items (a) and (b) in Theorem 3.1: defining

$$g(\kappa, t) := \log\left(1 + \tfrac{1}{t}\right) + \log(1+\kappa)\,,$$

the following relation holds if $\kappa \to \infty$ with bounded t, or if $t \to 0$ with $\kappa \ge (\sqrt{2}\,\sigma_0)\,\sqrt{t}$:

$$\log \mathrm{P}(X_t > \kappa) \sim -\mathsf{f}\left(\frac{\kappa}{t^D\sqrt{g(\kappa,t)}}\right) g(\kappa,t)\,. \tag{3.8}$$

This follows by the asymptotics in (2.6), observing that $g(\kappa, t) \sim \log \frac{1}{t}$ if $t \to 0$ and $\kappa \to 0$, while $g(\kappa,t) \sim \log \frac{1}{t} + \log \kappa = \log \frac{\kappa}{t}$ if both $t \to 0$ and $\kappa \to \infty$.

3.2. **Option price.** We finally turn to the option price $c(\kappa, t)$. As we discuss in §7.1, sharp estimates on the implied volatility, such as in Theorem 2.2, can be derived from the asymptotic behavior of $\log c(\kappa,t)$ if κ is bounded away from zero, or from the asymptotic behavior of $\log(c(\kappa,t)/\kappa)$ if $\kappa \to 0$. For this reason, in the next theorem (proved in Section 6) we give the asymptotic behavior of $\log c(\kappa,t)$ and $\log(c(\kappa,t)/\kappa)$, expressed in terms of the tail probability $\mathrm{P}(X_t > \kappa)$ (whose asymptotic behavior can be read from Theorem 3.1).

Theorem 3.3 (Option price)**.** *Consider a family of values of (κ, t) with $\kappa \ge 0$, $t > 0$.*

(a) *If $t \to \bar{t} \in (0, \infty)$ and $\kappa \to \infty$, or if $t \to 0$ and $\kappa \to \bar{\kappa} \in (0, \infty]$,*

$$\log c(\kappa, t) \sim \log \mathrm{P}(X_t > \kappa)\,. \tag{3.9}$$

(b) *If $t \to 0$ and $\kappa \to 0$ with $\kappa \gg \sqrt{t}$, excluding the "anomalous regime" of the next item,*

$$\log\left(c(\kappa,t)/\kappa\right) \sim \log \mathrm{P}(X_t > \kappa)\,. \tag{3.10}$$

(c) *If $t \to 0$ and $(\sqrt{2D+1}\,\sigma_0)\,\boldsymbol{\kappa}_1(t) \le \kappa \ll \boldsymbol{\kappa}_2(t)$ ("anomalous regime"),*

$$\log\left(c(\kappa,t)/\kappa\right) \sim -\left(D + 1 - \frac{\log \kappa}{\log t}\right) \log \frac{1}{t}\,, \tag{3.11}$$

and note that $\frac{1}{2} \le \frac{\log \kappa}{\log t} \le D$ for κ in the range under consideration.

(d) *If $t \to 0$ and $\kappa \sim a\sqrt{t}$ for some $a \in (0, \infty)$,*

$$\frac{c(\kappa,t)}{\kappa} \to D\left(\frac{a}{\sigma_0}\right), \qquad \textit{with} \qquad D(x) := \frac{\varphi(x)}{x} - \Phi(-x)\,, \tag{3.12}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the density and distribution function of a standard Gaussian.

(e) *Finally, if $t \to 0$ and $\kappa \ll \sqrt{t}$ (including $\kappa = 0$),*

$$c(\kappa, t) \sim \frac{\sigma_0}{\sqrt{2\pi}}\,\sqrt{t}\,. \tag{3.13}$$

4. Key large deviations estimates

In this section we prove the following crucial estimate on the exponential moments of the time-change process I_t, defined in (1.6). As we show in the next Section 5, this will be the key to the proof of relation (2.7) in Theorem 2.2.

Proposition 4.1. *Fix a family of values of (b,t) with $b>0$, $t>0$ such that*

$$b \to \infty \qquad \text{and} \qquad \frac{b}{\frac{1}{t^{2D}} \log(1+\frac{1}{t})} \to \infty \,. \tag{4.1}$$

(The second relation in (4.1) is only relevant when $t \downarrow 0$, because if $t \to \bar{t} \in (0,\infty]$ it follows by the first one.) Then the following asymptotic relation holds:

$$\log \mathrm{E}[e^{bI_t}] \sim \widetilde{C}\, t\, b^{\frac{1}{2D}} (\log b)^{\frac{2D-1}{2D}} \,, \qquad \text{with} \qquad \widetilde{C} = \mathsf{c}^{\frac{1}{D}} (2D)^{\frac{1}{2D}} (1-2D)^{\frac{1-2D}{2D}} \,. \tag{4.2}$$

From this one can easily derive Large Deviations estimates for the right tail of I_t.

Corollary 4.2. *Consider a family of values of (κ,t) with $\kappa>0$, $t>0$ such that*

$$\kappa \gg \max\{t, t^{2D}\} \,. \tag{4.3}$$

(More explicitly, extracting subsequence, one may assume that either $t \to 0$ and $\kappa \gg t^{2D}$, or $t \to \bar{t} \in (0,\infty)$ and $\kappa \to \infty$, or $t \to \infty$ and $\kappa \gg t$.) Then the following relation holds:

$$\log \mathrm{P}(I_t > \kappa) \sim -\frac{1}{1-2D} \left(\frac{\kappa}{\mathsf{c}^2\, t^{2D}}\right)^{\frac{1}{1-2D}} \left(\log \frac{\kappa}{t}\right) . \tag{4.4}$$

Remark 4.3. Recalling (1.6), the time-change process I_t can be seen as a natural additive functional of the inter-arrival times $\tau_k - \tau_{k-1}$ of a Poisson process:

$$I_t = g(t-\tau_{N_t}) - g(-\tau_0) + \sum_{k=1}^{N_t} g(\tau_k - \tau_{k-1}) \,, \tag{4.5}$$

with the choice $g(x) := \mathsf{c}^2 x^{2D}$. Remarkably, Proposition 4.1 and Corollary 4.2 continue to hold for a *wide class of functions* $g(\cdot)$, as it is clear from the proofs: what really matters is the asymptotic behavior $g(x) \sim \mathsf{c}^2 x^{2D}$ as $x \downarrow 0$. (Also note that the value of τ_0 in (4.5) plays no role in Proposition 4.1 and Corollary 4.2, so one can set $\tau_0 = 0$.)

Proof of Corollary 4.2 (sketch). The proof is completely analogous to that of Theorem 5.1 in Section 5, to which we refer for more details. Let us set

$$\gamma_{\kappa,t} = \left(\frac{\kappa}{t^{2D}}\right)^{\frac{1}{1-2D}} \left(\log \frac{\kappa}{t}\right) , \qquad \text{so that} \qquad \frac{\gamma_{\kappa,t}}{\kappa} = \left(\frac{\kappa}{t}\right)^{\frac{2D}{1-2D}} \left(\log \frac{\kappa}{t}\right) . \tag{4.6}$$

By (4.3), the family (t,b) with $b := \frac{\gamma_{\kappa,t}}{\kappa}$ satisfies (4.1). Then (4.2) yields, for $\alpha \geq 0$,

$$\log \mathrm{E}\left(\exp\left(\alpha\, \gamma_{\kappa,t} \frac{I_t}{\kappa}\right)\right) \sim \Lambda(\alpha)\, \gamma_{\kappa,t} \,, \qquad \text{where} \qquad \Lambda(\alpha) := \widetilde{C} \left(\frac{1-2D}{2D}\right)^{\frac{1-2D}{2D}} \alpha^{\frac{1}{2D}} \,,$$

with $\widetilde{C}$ defined in (4.2). By the Gärtner-Ellis Theorem [DZ98, Theorem 2.3.6],[†] we get

$$\log \mathrm{P}\left(\frac{I_t}{\kappa} > x\right) \sim -\gamma_{\kappa,t}\, I(x) \,, \tag{4.7}$$

[†] In principle one should compute $\Lambda(\alpha)$ for all $\alpha \in \mathbb{R}$ in order to apply the Gärtner-Ellis Theorem, which yields a full Large Deviations Principle. However, being interested in the right-tail behavior, cf. (4.4), it is enough to focus on $\alpha \geq 0$, as it is clear from the proof in [DZ98, Theorem 2.3.6].

where $I(\cdot)$ is the Fenchel-Legendre transform of $\Lambda(\cdot)$, i.e. (for $x \geq 0$)

$$I(x) := \sup_{\alpha\in\mathbb{R}} \{\alpha x - \Lambda(\alpha)\} = \big(\bar\alpha x - \Lambda(\bar\alpha)\big)\Big|_{\bar\alpha = \frac{2D}{1-2D}\left(\frac{2D}{\widetilde C}x\right)^{\frac{2D}{1-2D}}}$$

$$= \frac{2D}{1-2D}\left[\left(\frac{2D}{\widetilde C}\right)^{\frac{2D}{1-2D}} - \widetilde C\left(\frac{2D}{\widetilde C}\right)^{\frac{1}{1-2D}}\right] x^{\frac{1}{1-2D}} = \left(\frac{2D\,x}{\widetilde C^{2D}}\right)^{\frac{1}{1-2D}} = \frac{1}{1-2D}\left(\frac{x}{\mathsf{c}^2}\right)^{\frac{1}{1-2D}}.$$

Setting $x = 1$ in (4.7) yields (4.4). □

4.1. **Preliminary results.** We start with a useful upper bound on I_t (defined in (1.6)).

Lemma 4.4. *For all $t \geq 0$ the following upper bound holds:*

$$I_t \leq \sigma_0^2\, t \,+\, \mathsf{c}^2 N_t^{1-2D} t^{2D}\,, \tag{4.8}$$

where the constants σ_0 and c are defined in (1.4) *and* (1.2).

Proof. Since $(a+b)^{2D} - b^{2D} \leq 2D\, b^{2D-1}\, a$ for all $a, b > 0$ by concavity (recall that $D < \frac{1}{2}$), on the event $\{N_t = 0\}$ we can write, recalling (1.6) and (1.4),

$$I_t = \mathsf{c}^2\big\{(t-\tau_0)^{2D} - (-\tau_0)^{2D}\big\} \leq \mathsf{c}^2\, 2D\,(-\tau_0)^{2D-1}\, t \,=\sigma_0^2\, t\,, \tag{4.9}$$

proving (4.8). Analogously, on the event $\{N_t \geq 1\} = \{0 \leq \tau_1 \leq t\}$ we have

$$\begin{aligned} I_t &:= \mathsf{c}^2\left\{(\tau_1-\tau_0)^{2D} - (-\tau_0)^{2D} + \sum_{k=2}^{N_t}(\tau_k - \tau_{k-1})^{2D} + (t-\tau_{N_t})^{2D}\right\} \\ &\leq \mathsf{c}^2\left\{2D(-\tau_0)^{2D-1} t + \sum_{k=2}^{N_t}(\tau_k - \tau_{k-1})^{2D} + (t-\tau_{N_t})^{2D}\right\}. \end{aligned} \tag{4.10}$$

For all $\ell \in \mathbb{N}$ and $x_1, \ldots, x_\ell \in \mathbb{R}$, it follows by Hölder's inequality with $p := \frac{1}{2D}$ that

$$\sum_{k=1}^{\ell} x_k^{2D} \leq \left(\sum_{k=1}^{\ell}(x_k^{2D})^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\ell} 1\right)^{1-\frac{1}{p}} = \left(\sum_{k=1}^{\ell} x_k\right)^{2D} \ell^{1-2D}\,. \tag{4.11}$$

Choosing $\ell = N_t$ and $x_1 = \tau_2 - \tau_1$, $x_k = (\tau_{k+1} - \tau_k)$ for $2 \leq k \leq \ell - 1$ and $x_\ell = (t - \tau_{\ell-1})$, since $\sum_{k=1}^{\ell} x_k = t - \tau_1 \leq t$, we get from (4.10)

$$I_t \leq \mathsf{c}^2\left(2D(-\tau_0)^{2D-1} t + N_t^{1-2D} t^{2D}\right) = \sigma_0^2\, t \,+\, \mathsf{c}^2 N_t^{1-2D} t^{2D}\,,$$

completing the proof of (4.8). □

We now link the exponential moments of I_t to those of the log-price X_t.

Lemma 4.5 (No moment explosion)**.** *For every $t \in [0,\infty)$ and $p \in \mathbb{R}$ one has*

$$\mathrm{E}\left[e^{pX_t}\right] = \mathrm{E}\left[e^{\frac{1}{2}p(p-1)I_t}\right] < \infty\,. \tag{4.12}$$

Proof. By the definition (1.8) of X_t, the independence of I and W gives

$$\mathrm{E}\left[e^{pX_t}\right] = \mathrm{E}\left[e^{p(W_{I_t} - \frac{1}{2}I_t)}\right] = \mathrm{E}\left[e^{p(\sqrt{I_t}W_1 - \frac{1}{2}I_t)}\right] = \mathrm{E}\left[e^{\frac{1}{2}(p\sqrt{I_t})^2 - \frac{1}{2}pI_t}\right] = \mathrm{E}\left[e^{\frac{1}{2}p(p-1)I_t}\right],$$

which proves the equality in (4.12). Applying the upper bound (4.8) yields

$$\mathrm{E}\left[e^{\frac{1}{2}p(p-1)I_t}\right] \leq \mathrm{E}\left[e^{\frac{1}{2}p(p-1)(\sigma_0^2\, t + \mathsf{c}^2 N_t^{1-2D} t^{2D})}\right] = \mathrm{E}\left[e^{c_1 t + c_2\, t^{2D}\, N_t^{1-2D}}\right] \leq \mathrm{E}\left[e^{c_1 t + c_2\, t^{2D}\, N_t}\right],$$

for suitable $c_1, c_2 \in (0,\infty)$ depending on p and on the parameters of the model. The right hand side is finite because $N_t \sim \mathit{Pois}(\lambda t)$ has finite exponential moments of all orders. □

4.2. **Proof of Proposition 4.1.** Let us set

$$B_{t,b} = t\, b^{\frac{1}{2D}} \, (\log b)^{\frac{2D-1}{2D}} \,. \tag{4.13}$$

We are going to show that (4.2) holds by proving separately upper and lower bounds, i.e.

$$\limsup \frac{1}{B_{t,b}} \log \mathrm{E}[e^{bI_t}] \le \widetilde{C} \,, \qquad \liminf \frac{1}{B_{t,b}} \log \mathrm{E}[e^{bI_t}] \ge \widetilde{C} \,. \tag{4.14}$$

We start with the upper bound and we split the proof in steps.

Step 1. Preliminary upper bound. The upper bound (4.8) on I_t yields

$$\mathrm{E}\left[e^{bI_t}\right] = \sum_{j=0}^{\infty} \mathrm{E}[e^{bI_t} | N_t = j]\, \mathrm{P}(N_t = j) \le e^{\sigma_0^2\, tb} \sum_{j=0}^{\infty} e^{\mathsf{c}^2 t^{2D} b\, j^{1-2D}} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \,.$$

Since $j! \sim j^j e^{-j} \sqrt{2\pi j}$ as $j \uparrow \infty$, there is $c_1 \in (0,\infty)$ such that $j! \ge \frac{1}{c_1} j^j e^{-j}$ for all $j \in \mathbb{N}_0$. Bounding $e^{-\lambda t} \le 1$, we obtain

$$\mathrm{E}\left[e^{bI_t}\right] \le c_1\, e^{\sigma_0^2\, tb} \sum_{j=0}^{\infty} e^{\mathsf{c}^2 t^{2D} b\, j^{1-2D}} \frac{(\lambda t)^j}{j^j e^{-j}} = c_1\, e^{\sigma_0^2\, tb} \sum_{j=0}^{\infty} e^{f(j)} \,, \tag{4.15}$$

where for $x \in [0,\infty)$ we set

$$f(x) = f_{t,b}(x) := \mathsf{c}^2 \left(t^{2D} b\right) x^{1-2D} - x \left(\log \frac{x}{\lambda t} - 1 \right) , \tag{4.16}$$

with the convention $0 \log 0 = 0$. Note that

$$f'(x) = (1-2D)\mathsf{c}^2 b \left(\frac{x}{t}\right)^{-2D} - \log\left(\frac{x}{t}\right) + \log \lambda \,, \tag{4.17}$$

hence $f'(x)$ is continuous and strictly decreasing on $(0,\infty)$, with $\lim_{x\downarrow 0} f'(x) = +\infty$ and $\lim_{x\uparrow\infty} f'(x) = -\infty$. As a consequence, there is a unique $\bar{x}_{t,b} \in (0,\infty)$ with $f'(\bar{x}_{t,b}) = 0$ and the function $f(x)$ attains its global maximum on $[0,\infty)$ at the point $x = \bar{x}_{t,b}$:

$$\max_{x\in[0,\infty)} f(x) = f(\bar{x}_{t,b}) \,. \tag{4.18}$$

We are going to show that the leading contribution to the sum in (4.15) is given by a *single term* $e^{f(j)}$, for $j \asymp \bar{x}_{t,b}$. We first need asymptotic estimates on $\bar{x}_{t,b}$ and $f(\bar{x}_{t,b})$.

Step 2. Estimates on $\bar{x}_{t,b}$ and $f(\bar{x}_{t,b})$. We first prove that

$$\bar{x}_{t,b} \to \infty \,, \qquad \frac{\bar{x}_{t,b}}{t} \to \infty \,, \tag{4.19}$$

by showing that for any fixed $M \in (0,\infty)$ one has $\bar{x}_{t,b} > M$ and $\bar{x}_{t,b}/t > M$ eventually. Since $b \to \infty$ by assumption (4.1), uniformly for x such that $(x/t) \in [0,M]$ we have

$$f'(x) \ge (1-2D)\mathsf{c}^2 b\, M^{-2D} - \log M + \log \lambda =: C_1 b + C_2 \to \infty \,.$$

Recalling that $\bar{x}_{t,b}$ is the solution of $f'(x) = 0$, it follows that $(\bar{x}_{t,b}/t) > M$ eventually. Likewise, uniformly for x such that $x \in [0,M]$, by assumption (4.1) we can write

$$f'(x) \ge (1-2D)\mathsf{c}^2 b \left(\frac{M}{t}\right)^{-2D} - \log\left(\frac{M}{t}\right) + \log \lambda =: C_1\, t^{2D}\, b - \log \frac{1}{t} + C_2 \to \infty \,,$$

hence $\bar{x}_{t,b} > M$ eventually, completing the proof of (4.19).

Next we prove that $\bar{x}_{t,b}$ has the following asymptotic behavior:

$$\bar{x}_{t,b} \sim \big(2D(1-2D)\mathsf{c}^2\big)^{\frac{1}{2D}} \left(\frac{t^{2D}\, b}{\log b}\right)^{\frac{1}{2D}}, \tag{4.20}$$

arguing as follows. Recalling (4.17), the equation $f'(\bar{x}_{t,b}) = 0$ can be rewritten as

$$\frac{\bar{x}_{t,b}}{t} = \left(\frac{(1-2D)\mathsf{c}^2 b}{\log \frac{\bar{x}_{t,b}}{t} - \log \lambda}\right)^{\frac{1}{2D}} \sim \left(\frac{(1-2D)\mathsf{c}^2\, b}{\log \frac{\bar{x}_{t,b}}{t}}\right)^{\frac{1}{2D}}, \tag{4.21}$$

because $\bar{x}_{t,b}/t \to \infty$ by (4.19). Inverting (4.21) and using again $\bar{x}_{t,b}/t \to \infty$ gives

$$\log \frac{\bar{x}_{t,b}}{t} \sim (1-2D)\mathsf{c}^2\, b \left(\frac{\bar{x}_{t,b}}{t}\right)^{-2D} = o(b)\,, \tag{4.22}$$

and we recall that $b \to \infty$ by assumption (4.1). Taking log in (4.21) gives

$$\log \frac{\bar{x}_{t,b}}{t} \sim \frac{1}{2D}\Big\{ \log[(1-2D)\mathsf{c}^2] + \log b - \log\big(\log \tfrac{\bar{x}_{t,b}}{t}\big)\Big\} \sim \frac{1}{2D} \log b\,, \tag{4.23}$$

having used (4.22). Plugging (4.23) into (4.21) gives precisely (4.20).

Looking back at (4.16), we obtain the asymptotic behavior of $f(\bar{x}_{t,b})$: by (4.19) and (4.22)

$$\begin{aligned} f(\bar{x}_{t,b}) &= \mathsf{c}^2\,(t^{2D}b)\,\bar{x}_{t,b}^{1-2D} - \bar{x}_{t,b}\log\frac{\bar{x}_{t,b}}{t}\big(1+o(1)\big) \\ &= \mathsf{c}^2\,(t^{2D}b)\bar{x}_{t,b}^{1-2D} - \bar{x}_{t,b}\frac{(1-2D)\mathsf{c}^2\,(t^{2D}b)}{\bar{x}_{t,b}^{2D}}\big(1+o(1)\big) \\ &= 2D\,\mathsf{c}^2\,(t^{2D}b)\,\bar{x}_{t,b}^{1-2D}\big(1+o(1)\big)\,, \end{aligned} \tag{4.24}$$

hence applying (4.20), and recalling the definition of $B_{t,b}$ and $\widetilde{C}$ in (4.13) and (4.2),

$$f(\bar{x}_{t,b}) \sim (2D)^{\frac{1}{2D}}\,(1-2D)^{\frac{1}{2D}-1}\,\mathsf{c}^{\frac{1}{D}}\,\frac{t\,b^{\frac{1}{2D}}}{(\log b)^{\frac{1}{2D}-1}} = \widetilde{C}\,B_{t,b}\,. \tag{4.25}$$

Step 3. Completing the upper bound. We can finally come back to (4.15). Henceforth we set $\bar{x} := \bar{x}_{t,b}$ to lighten notation. We control $f(x)$ for $x \ge 2\bar{x}$ as follows: since $f'(\cdot)$ is strictly decreasing, and $f(2\bar{x}) \le f(\bar{x})$ by (4.18),

$$f(x) = f(2\bar{x}) + \int_{2\bar{x}}^{x} f'(s)\mathrm{d}s \le f(\bar{x}) + f'(2\bar{x})(x-2\bar{x})\,.$$

Observe that $f'(2\bar{x}) = -|f'(2\bar{x})| < 0$, hence

$$\sum_{j\ge 2\bar{x}} e^{f(j)} \le e^{f(\bar{x})} \sum_{j\ge 2\bar{x}} e^{-|f'(2\bar{x})|(j-2\bar{x})} = \frac{e^{f(\bar{x})}}{1-e^{-|f'(2\bar{x})|}}. \tag{4.26}$$

By (4.17), recalling that $f'(\bar{x}) = 0$, we can write

$$f'(2\bar{x}) = f'(2\bar{x}) - 2^{-2D} f'(\bar{x}) = 2^{-2D}\log\left(\frac{\bar{x}}{t}\right) - \log\left(\frac{2\bar{x}}{t}\right) \to -\infty\,,$$

because $\bar{x}/t \to \infty$ by (4.19). Then $1 - e^{-|f'(2\bar{x})|} > \frac{1}{2}$ eventually and (4.26) yields

$$\sum_{j\ge 2\bar{x}} e^{f(j)} \le 2\,e^{f(\bar{x})}\,. \tag{4.27}$$

The initial part of the sum can be simply bounded by

$$\sum_{0\le j<2\bar{x}} e^{f(j)} \le (2\bar{x}+1)\, e^{f(\bar{x})} . \tag{4.28}$$

Looking back at (4.15), we can finally write

$$\log \mathrm{E}\left[e^{bI_t}\right] \le \log c_1 + \sigma_0^2\, b\, t + \log(2\bar{x}+3) + f(\bar{x}) . \tag{4.29}$$

Comparing (4.20) and (4.25), we see that $\bar{x} = O(f(\bar{x})/\log b) = o(f(\bar{x}))$, because $b \to \infty$, hence $\log(2\bar{x}+3) = o(\bar{x}) = o(f(\bar{x}))$. Again by (4.25) we have $bt = o(\bar{x}) = o(f(\bar{x}))$, because $D < \frac{1}{2}$. This means that the first three terms in the right hand side of (4.29) are negligible compared to $f(\bar{x})$, and since $f(\bar{x}) \sim \widetilde{C}\, B_{t,b}$ by relation (4.24), we obtain

$$\limsup \frac{1}{B_{t,b}} \log \mathrm{E}\left[e^{bI_t}\right] \le \widetilde{C} ,$$

proving the desired upper bound in (4.14).

Step 4. Lower bound. By (1.6), since $(\tau_1 - \tau_0)^{2D} \ge (-\tau_0)^{2D}$, we have the following lower bound on I_t on the event $\{N_t \ge 1\}$:

$$I_t \ge \mathsf{c}^2 \left\{ (t-\tau_{N_t})^{2D} + \sum_{k=2}^{N_t} (\tau_k - \tau_{k-1})^{2D} \right\} . \tag{4.30}$$

To match the upper bound, note that Hölder's inequality (4.11) becomes an equality when all the terms $x_k = \tau_k - \tau_{k-1}$ are equal. We can make this approximately true introducing for $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$ the event A_m defined by

$$A_m := \left\{ \tau_1 < \varepsilon \frac{t}{m} \right\} \cap \bigcap_{i=2}^{m} \left\{ [(i-1)-\varepsilon]\frac{t}{m} < \tau_i < [(i-1)+\varepsilon]\frac{t}{m} \right\} \cap \left\{ \tau_{m+1} > t \right\} , \tag{4.31}$$

which ensures that $N_t = m$ and $\tau_k - \tau_{k-1} \ge (1-2\varepsilon)\frac{t}{m}$ for $2 \le k \le m$ and $t - \tau_m \ge (1-2\varepsilon)\frac{t}{m}$. In particular, recalling (4.30), on the event A_m we have the lower bound

$$I_t \ge \mathsf{c}^2\, m\big((1-2\varepsilon)\tfrac{t}{m}\big)^{2D} = (1-2\varepsilon)^{2D}\, \mathsf{c}^2\, m^{1-2D} t^{2D} . \tag{4.32}$$

Since $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$ are i.i.d. $Exp(\lambda)$ random variables, and on the event A_m one has $\tau_k - \tau_{k-1} \le (1+2\varepsilon)\frac{t}{m}$ for $2 \le k \le m+1$, a direct estimate on the densities yields

$$\mathrm{P}(A_m) \ge \big(\lambda e^{-\lambda(1+2\varepsilon)\frac{t}{m}}\big)^m \big(\varepsilon \tfrac{t}{m}\big)^m e^{-\lambda(1+2\varepsilon)\frac{t}{m}} = e^{-(1+2\varepsilon)(1+\frac{1}{m})\lambda t} \frac{(\varepsilon\lambda t)^m}{m^m} , \tag{4.33}$$

hence by (4.32)

$$\mathrm{E}\left[e^{bI_t}\right] \ge \mathrm{E}\left[e^{bI_t} 1_{A_m}\right] \ge e^{(1-2\varepsilon)^{2D} \mathsf{c}^2\, (t^{2D} b)\, m^{1-2D}} \mathrm{P}(A_m) \ge e^{\widetilde{f}(m)} \tag{4.34}$$

where we define $\widetilde{f}(x)$, for $x \ge 0$ by

$$\widetilde{f}(x) = \widetilde{f}_{t,b,\varepsilon}(x) := (1-2\varepsilon)^{2D}\, \mathsf{c}^2\, (t^{2D} b)\, x^{1-2D} - x \log \frac{x}{\varepsilon\lambda t} - (1+2\varepsilon)(1+\tfrac{1}{m})\lambda t .$$

Note that $\widetilde{f}(x)$ resembles $f(x)$, cf. (4.16). Since the leading contribution to the upper bound was given by $e^{f(\bar{x})}$, where $\bar{x} = \bar{x}_{b,t}$ is the maximizer of $f(\cdot)$, it is natural to choose $m = \lfloor \bar{x} \rfloor$ in the lower bound (4.34). Since $\bar{x} \to \infty$ and $t \ll \bar{x}$, cf. (4.19), we have

$$\widetilde{f}(\lfloor \bar{x} \rfloor) \sim .\widetilde{f}(\bar{x}) \sim (1-2\varepsilon)^{2D}\, \mathsf{c}^2\, (t^{2D} b)\, \bar{x}^{1-2D} - \bar{x}\, \log \frac{\bar{x}}{t} \big(1 + o(1)\big) ,$$

and recalling (4.24)-(4.25) we obtain

$$\widetilde{f}(\lfloor \bar{x} \rfloor) \sim f(\bar{x}) - \big[1-(1-2\varepsilon)^{2D}\big] \mathsf{c}^2\, (t^{2D} b)\, \bar{x}^{1-2D} \sim \left[1 - \frac{1-(1-2\varepsilon)^{2D}}{2D}\right] \widetilde{C}\, B_{t,b}\,,$$

which coupled to (4.34) yields

$$\liminf \frac{1}{B_{t,b}} \log \mathrm{E}\left[e^{bI_t}\right] \geq \left[1 - \frac{1-(1-2\varepsilon)^{2D}}{2D}\right] \widetilde{C}\,.$$

Letting $\varepsilon \to 0$ we obtain the desired lower bound in (4.14), completing the proof. □

5. Proof of Theorem 3.1 (tail probability)

In this section we prove relation (3.1) and Theorem 3.1.

5.1. **Proof of relation** (3.1) **and of Theorem 3.1, part** (d)**.** For any $t \geq 0$, by (1.8)

$$X_t \overset{d}{=} \sqrt{I_t}\, W_1 - \frac{1}{2} I_t\,.$$

Since $I_0 = 0$, a.s. one has $I_t/t = (I_t - I_0)/t \to I_0' = \sigma_0^2$ as $t \downarrow 0$, cf. (1.6)-(1.4). Then

$$\frac{X_t}{\sqrt{t}} \overset{d}{=} \sqrt{\frac{I_t}{t}}\, W_1 - \frac{1}{2}\sqrt{t}\frac{I_t}{t} \xrightarrow[t\downarrow 0]{\text{a.s.}} \sigma_0\, W_1\,,$$

proving relation (3.1). Relation (3.5) follows from (3.1), proving part (d) in Theorem 3.1.

5.2. **Proof of Theorem 3.1, part** (a)**.** Recall the definition of $\boldsymbol{\kappa}_1(t)$ and $\boldsymbol{\kappa}_2(t)$ in (2.4). Let us fix a family of (κ, t) with $\kappa > 0$, $t > 0$ as in item (a) of Theorem 3.1, i.e.

$$\begin{cases} \text{either} \quad t \to \infty \quad \text{and} \quad \dfrac{\kappa}{t} \to \infty\,, \\[2ex] \text{or} \quad t \to \bar{t} \in (0,\infty) \quad \text{and} \quad \kappa \to \infty\,, \\[2ex] \text{or} \quad t \to 0 \quad \text{and} \quad \dfrac{\kappa}{t^D \sqrt{\log \frac{1}{t}}} \to \infty\,. \end{cases} \tag{5.1}$$

We are going to prove the following result, which is stronger than our goal (3.2).

Theorem 5.1. *For any family of values of* (κ, t) *satisfying* (5.1)*, the random variables* $\frac{X_t}{\kappa}$ *satisfy the* large deviations principle *with rate* $\alpha_{t,\kappa}$ *and good rate function* $I(\cdot)$ *given by*

$$\alpha_{t,\kappa} := \left(\frac{\kappa}{t^D}\right)^{\frac{1}{1-D}} \left(\log\frac{\kappa}{t}\right)^{\frac{1/2-D}{1-D}}, \qquad I(x) := \mathsf{C}\, |x|^{\frac{1}{1-D}} \tag{5.2}$$

where C *is defined in* (2.6)*. This means that for every Borel set* $A \subseteq \mathbb{R}$

$$-\inf_{x \in \mathring{A}} I(x) \leq \liminf \frac{1}{\alpha_{t,\kappa}} \log \mathrm{P}\left(\frac{X_t}{\kappa} \in A\right) \leq \limsup \frac{1}{\alpha_{t,\kappa}} \log \mathrm{P}\left(\frac{X_t}{\kappa} \in A\right) \leq -\inf_{x \in \overline{A}} I(x)\,,$$

where $\mathring{A}$ *and* $\overline{A}$ *denote respectively the interior and the closure of* A*. In particular, choosing* $A = (1,\infty)$*, relation* (3.2) *in Theorem 3.1 holds.*

Proof. We are going to show that, with $\alpha_{t,\kappa}$ as in (5.2), the following limit exists for $\beta \in \mathbb{R}$:

$$\Lambda(\beta) := \lim \frac{1}{\alpha_{t,\kappa}} \log \mathrm{E}[e^{\beta \alpha_{t,\kappa} \frac{X_t}{\kappa}}] , \tag{5.3}$$

where $\Lambda : \mathbb{R} \to \mathbb{R}$ is everywhere finite and continuously differentiable. By the Gärtner-Ellis Theorem [DZ98, Theorem 2.3.6], it follows that $\frac{X_t}{\kappa}$ satisfies a LDP with good rate $\alpha_{t,\kappa}$ and with rate function $I(\cdot)$ given by the Fenchel-Legendre transform of $\Lambda(\cdot)$, i.e.

$$I(x) = \sup_{\beta \in \mathbb{R}} \left\{ \beta x - \Lambda(\beta) \right\} . \tag{5.4}$$

The proof is thus reduced to computing $\Lambda(\beta)$ and then showing that $I(x)$ coincides with the one given in (5.2). Recalling (4.12), the determination of $\Lambda(\beta)$ in (5.3) is reduced to the asymptotic behaviour of exponential moments of I_t. This is possible by Proposition 4.1.

Fix a family of values of (κ, t) satisfying (5.1) and note that $\alpha_{t,\kappa}$ in (5.2) satisfies

$$\alpha_{t,\kappa} \to \infty , \qquad \frac{\alpha_{t,\kappa}}{\kappa} = \left(\frac{\kappa}{t} \right)^{\frac{D}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1/2-D}{1-D}} \to \infty .$$

For fixed $\beta \in \mathbb{R} \setminus \{0\}$ we set

$$b = b_{t,\kappa} := \frac{1}{2} \beta \frac{\alpha_{t,\kappa}}{\kappa} \left(\beta \frac{\alpha_{t,\kappa}}{\kappa} - 1 \right) \sim \frac{\beta^2}{2} \left(\frac{\alpha_{t,\kappa}}{\kappa} \right)^2 \to \infty . \tag{5.5}$$

In order to check the second condition in (4.1), note that if $t \to 0$

$$\frac{b}{\frac{1}{t^{2D}} \log \frac{1}{t}} \sim \frac{\beta^2}{2} \left(\frac{\kappa}{t^D \sqrt{\log \frac{1}{t}}} \right)^{\frac{2D}{1-D}} \left(\frac{\log \frac{\kappa}{t}}{\log \frac{1}{t}} \right)^{\frac{1-2D}{1-D}} \to \infty ,$$

again by (5.1). Applying (4.2), by (4.12) and (5.5) we get

$$\begin{aligned} \log \mathrm{E}\left[e^{\beta \alpha_{t,\kappa} \frac{X_t}{\kappa}} \right] &= \log \mathrm{E}[e^{b_{t,k} I_t}] \sim \widetilde{C}\, t\, b_{t,k}^{\frac{1}{2D}} (\log b_{t,k})^{\frac{2D-1}{2D}} \\ &\sim \widetilde{C}\, t \left(\frac{\beta^2}{2} \right)^{\frac{1}{2D}} \left(\frac{\kappa}{t} \right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1-2D}{2D(1-D)}} \left(\frac{2D}{1-D} \log \frac{\kappa}{t} \right)^{\frac{2D-1}{2D}} \\ &= \mathsf{c}^{\frac{1}{D}}\, D \left(\frac{(1-2D)(1-D)}{2} \right)^{\frac{1-2D}{2D}} |\beta|^{\frac{1}{D}}\, \alpha_{t,\kappa} , \end{aligned}$$

where in the last step we have used the definitions (5.2), (4.2) of $\alpha_{t,\kappa}$ and $\widetilde{C}$.

This shows that the limit (5.3) exists with

$$\Lambda(\beta) = \hat{C}\, |\beta|^{\frac{1}{D}} , \qquad \text{and} \qquad \hat{C} = \mathsf{c}^{\frac{1}{D}}\, D \left(\frac{(1-2D)(1-D)}{2} \right)^{\frac{1-2D}{2D}} .$$

To determine the rate function $I(x)$ in (5.4) we have to maximize over $\beta \in \mathbb{R}$ the function

$$h(\beta) := \beta x - \Lambda(\beta) .$$

Since $h'(\beta) = x - \Lambda'(\beta) = x - \frac{1}{D} \hat{C} \operatorname{sign}(\beta) |\beta|^{\frac{1}{D}-1}$, the only solution to $h'(\bar{\beta}) = 0$ is

$$\bar{\beta} = \bar{\beta}_x = \operatorname{sign}(x) \left(\frac{D|x|}{\hat{C}} \right)^{\frac{D}{1-D}}$$

and consequently

$$I(x) = h(\bar\beta_x) = \bar\beta_x\, x - \Lambda(\bar\beta_x) = |x|^{\frac{1}{1-D}} \left(\frac{D}{\hat C}\right)^{\frac{D}{1-D}} (1-D) = \mathsf{C}\, |x|^{\frac{1}{1-D}} \,,$$

where C is the constant defined in (2.6). Having shown that $I(x)$ coincides with the one given in (5.2), the proof of Theorem 5.1 is completed. □

5.3. **Technical interlude.** Let us give some estimates on $\mathrm{P}(X_t > \kappa | N_t = m)$. Recall the definition (1.6) of the time-change process I_t. On the event $\{N_t = 0\}$ we have

$$I_t = (t - \tau_0)^{2D} - (-\tau_0)^{2D} \underset{t\downarrow 0}{\sim} \sigma_0^2\, t \,,$$

where σ_0^2 is defined in (1.4). Consequently, by the definition (1.8) of X_t,

$$\begin{aligned} \mathrm{P}(X_t > \kappa | N_t = 0) &= \mathrm{P}\left(W_1 > \frac{\kappa}{\sqrt{I_t}} + \frac{1}{2}\sqrt{I_t} \,\middle|\, N_t = 0 \right) \\ &= 1 - \Phi\bigg(\frac{\kappa}{\sigma_0 \sqrt{t}} \big(1 + o(1)\big) \bigg) = \exp\left(- \frac{\kappa^2}{2\,\sigma_0^2\, t} \big(1 + o(1)\big) \right) \\ &= \exp\left(- \frac{1}{2\,\sigma_0^2} \left(\frac{\kappa}{\boldsymbol{\kappa}_1(t)} \right)^2 \log \frac{1}{t} \big(1 + o(1)\big) \right), \end{aligned} \tag{5.6}$$

where $\Phi(z) = \mathrm{P}(W_1 \le z)$ and we have used the standard estimate $\log(1 - \Phi(z)) \sim -\frac{1}{2} z^2$ as $z \to \infty$ together with the definition (2.4) of $\boldsymbol{\kappa}_1(t)$ and $\log(1 + \frac{1}{t}) \sim \log \frac{1}{t}$.

Let us now consider the event $\{N_t = m\}$ with $m \ge 1$: applying the bound (4.8) we obtain

$$I_t \le \sigma_0^2\, t \,+\, \mathsf{c}^2 N_t^{1-2D} t^{2D} = \sigma_0^2\, t \,+\, \mathsf{c}^2 m^{1-2D} t^{2D} \underset{t\downarrow 0}{\sim} \mathsf{c}^2 m^{1-2D} t^{2D} \,,$$

hence, in analogy with (5.6), we get the upper bound

$$\begin{aligned} \mathrm{P}(X_t > \kappa | N_t = m) &\le 1 - \Phi\bigg(\frac{\kappa}{\sqrt{\mathsf{c}^2 t^D m^{\frac{1}{2} - D}}} \big(1 + o(1)\big) \bigg) \\ &= \exp\left(- \frac{1}{2\mathsf{c}^2 m^{1-2D}} \left(\frac{\kappa}{\boldsymbol{\kappa}_2(t)} \right)^2 \log \frac{1}{t} \big(1 + o(1)\big) \right), \end{aligned} \tag{5.7}$$

having used the definition (2.4) of $\boldsymbol{\kappa}_2(t)$ and $\log(1 + \frac{1}{t}) \sim \log \frac{1}{t}$.

For a lower bound, we argue as in the proof of Proposition 4.1: for any $\varepsilon > 0$, on the event $A_m \subseteq \{N_t = m\}$ defined in (4.31), with $m \ge 1$, one has the lower bounds (4.32) on I_t and (4.33) on $\mathrm{P}(A_m)$. Then, using $(1 + \frac{1}{m}) \le 2$,

$$\begin{aligned} \mathrm{P}(X_t > \kappa | N_t = m) &\ge \mathrm{P}(X_t > \kappa | A_m) \frac{\mathrm{P}(A_m)}{\mathrm{P}(N_t = m)} \\ &\ge \left(1 - \Phi\bigg(\frac{\kappa}{(1 - 2\varepsilon)^D\, \mathsf{c}\, m^{\frac{1}{2} - D} t^D} \big(1 + o(1)\big) \bigg) \right) e^{-[(1+2\varepsilon)(1 + \frac{1}{m}) - 1]\lambda t} \varepsilon^m \frac{m!}{m^m} \\ &\ge \exp\left(- \frac{1}{2(1 - 2\varepsilon)^{2D} \mathsf{c}^2 m^{1-2D}} \left(\frac{\kappa}{\boldsymbol{\kappa}_2(t)} \right)^2 \log \frac{1}{t} \big(1 + o(1)\big) \right) e^{-(1+4\varepsilon)\lambda t} \varepsilon^m \frac{m!}{m^m} \,. \end{aligned} \tag{5.8}$$

5.4. **Proof of Theorem 3.1, part** (c)**.** Fix a family of values of (κ, t) with

$$t \to 0\,, \qquad \sqrt{t} \ll \kappa \le (\sqrt{2}\,\sigma_0)\,\boldsymbol{\kappa}_1(t)\,. \tag{5.9}$$

If we define

$$\varrho_{\kappa,t} := \frac{\kappa}{\boldsymbol{\kappa}_1(t)}\,, \tag{5.10}$$

we can rewrite (5.6) as

$$\mathrm{P}(X_t > \kappa | N_t = 0) = e^{-\frac{\varrho_{\kappa,t}^2}{2\sigma_0^2}\log\frac{1}{t}(1+o(1))} = t^{\frac{\varrho_{\kappa,t}^2}{2\sigma_0^2}+o(1)}\,. \tag{5.11}$$

Since $N_t \sim Pois(\lambda t)$, for every $M \in \mathbb{N}_0$

$$\mathrm{P}(N_t \ge M+1) = \sum_{k=M+1}^{\infty} e^{-\lambda t}\,\frac{(\lambda t)^k}{k!} \le (\lambda t)^{M+1}\,,$$

hence as $t \to 0$ we can write

$$\mathrm{P}(X_t > \kappa) = \sum_{m=0}^{M} \mathrm{P}(X_t > \kappa | N_t = m) e^{-\lambda t}\frac{(\lambda t)^m}{m!} + O(t^{M+1})\,. \tag{5.12}$$

Since the last $O(t^{M+1})$ term is non-negative, this relation for $M = 0$ coupled to (5.11) gives

$$t^{\frac{\varrho_{\kappa,t}^2}{2\sigma_0^2}+o(1)} \le \mathrm{P}(X_t > \kappa) \le t^{\frac{\varrho_{\kappa,t}^2}{2\sigma_0^2}+o(1)} + O(t)\,. \tag{5.13}$$

By assumption (5.9) one has $\varrho_{\kappa,t} \le \sqrt{2\sigma_0}$, hence we can write

$$O(t) = t^{1+o(1)} \le t^{\frac{\varrho_{\kappa,t}^2}{2\sigma_0^2}+o(1)}\,,$$

which by (5.13) yields

$$\lim \frac{\log \mathrm{P}(X_t > \kappa)}{\frac{\varrho_{\kappa,t}^2}{2\,\sigma_0^2}\log\frac{1}{t}} = -1\,.$$

This shows that relation (3.4) holds, completing the proof of part (c) of Theorem 3.1. ☐

5.5. **Proof of Theorem 3.1, part** (b)**.** Following Remark 3.2, we split this case in two:

- first we consider a family of values of (κ, t) with
$$t \to 0\,, \qquad (\sqrt{2}\,\sigma_0)\,\boldsymbol{\kappa}_1(t) \le \kappa \ll \boldsymbol{\kappa}_2(t)\,, \tag{5.14}$$
and our goal is to prove (3.7);
- afterwards we will consider the regime
$$\kappa \sim a\,\boldsymbol{\kappa}_2(t)\,, \qquad \text{for some} \quad a \in (0, \infty)\,. \tag{5.15}$$
and our goal is to prove (3.6).

By a subsequence argument, these cases prove relation (3.3) and hence part (b).

Let us assume (5.14). Recalling (5.10), we have $\varrho_{\kappa,t} \ge \sqrt{2}\sigma_0$, so that by (5.11)

$$\mathrm{P}(X_t > \kappa | N_t = 0) \le t^{1+o(1)}\,. \tag{5.16}$$

Since $\kappa/\boldsymbol{\kappa}_2(t) \to 0$ by (5.14), relation (5.7) for $m = 1$ gives

$$\mathrm{P}(X_t > \kappa | N_t = 1) \le e^{-o(1)\log\frac{1}{t}} = t^{o(1)}\,.$$

Then relation (5.12) for $M = 1$, bounding $e^{-\lambda t} \le 1$, yields

$$\mathrm{P}(X_t > \kappa) \le t^{1+o(1)} + \lambda t \, t^{o(1)} + O(t^2) = t^{1+o(1)} ,$$

where the $o(1)$ changes from place to place. This proves "half" of our goal (3.7), namely

$$\limsup \frac{\log \mathrm{P}(X_t > \kappa)}{\log \frac{1}{t}} \le -1 . \tag{5.17}$$

Next we apply relation (5.8) for $m = 1$: since $t \to 0$ and $k/\boldsymbol{\kappa}_2(t) = o(1)$ by (5.14), we get

$$\mathrm{P}(X_t > \kappa | N_t = 1) \ge e^{-o(1) \log \frac{1}{t}} \, e^{-(1+4\varepsilon)\lambda t} \, \varepsilon = t^{o(1)} .$$

Then, recalling that $\mathrm{P}(N_t = 1) = e^{-\lambda t} \lambda t \sim \lambda t$,

$$\mathrm{P}(X_t > \kappa) \ge \mathrm{P}(X_t > \kappa | N_t = 1) \mathrm{P}(N_t = 1) \ge t^{1+o(1)} ,$$

which yields

$$\liminf \frac{\log \mathrm{P}(X_t > \kappa)}{\log \frac{1}{t}} \ge -1 .$$

Together with (5.17), this completes the proof of relation (3.7) under assumption (5.14).

Next we assume (5.15). By (5.6) we have

$$\mathrm{P}(X_t > \kappa | N_t = 0) \le \exp\bigg(- \frac{a^2}{2\,\sigma_0^2} \frac{1}{t^{1-2D}} \log \frac{1}{t} \big(1 + o(1)\big) \bigg) = o(t^{M+1}) ,$$

for *any* fixed $M \in \mathbb{N}$. As a consequence, relation (5.12) together with the upper bounds (5.16) and (5.7) yields, for every fixed $M \in \mathbb{N}$,

$$\begin{aligned} \mathrm{P}(X_t > \kappa) &\le \sum_{m=1}^{M} \exp\bigg(- \frac{a^2}{2\mathsf{c}^2 m^{1-2D}} \log \frac{1}{t} \big(1 + o(1)\big) \bigg) (\lambda t)^m + O(t^{M+1}) \\ &\le M \max_{m \in \{1,\dots,M\}} t^{\frac{a^2}{2\mathsf{c}^2 m^{1-2D}} + m + o(1)} + O(t^{M+1}) \\ &\le M t^{\mathsf{f}(a) + o(1)} + O(t^{M+1}) , \end{aligned}$$

where $\mathsf{f}(\cdot)$ is defined in (2.5). If we fix M large enough, so that $M + 1 > \mathsf{f}(a)$, the term $O(t^{M+1})$ gives a negligible contribution and we obtain

$$\limsup \frac{\log \mathrm{P}(X_t > \kappa)}{\log \frac{1}{t}} \le -\mathsf{f}(a) , \tag{5.18}$$

which is "half" of relation (3.6). To prove the corresponding lower bound, let $\bar{m} = \bar{m}_a \in \mathbb{N}$ be the value for which the minimum in the definition (2.5) of $\mathsf{f}(a)$ is attained, i.e.

$$\mathsf{f}(a) = \mathsf{f}_{\bar{m}}(a) = \frac{a^2}{2\mathsf{c}^2 \bar{m}^{1-2D}} + \bar{m} . \tag{5.19}$$

Recalling (5.15), the lower bound (5.8) for $m = \bar{m}$ gives

$$\begin{aligned} \mathrm{P}(X_t > \kappa | N_t = \bar{m}) &\ge \exp\bigg(- \frac{a^2}{2(1-2\varepsilon)^{2D} \mathsf{c}^2 \bar{m}^{1-2D}} \log \frac{1}{t} \big(1 + o(1)\big) \bigg) e^{-(1+4\varepsilon)\lambda t} \varepsilon^{\bar{m}} \frac{\bar{m}!}{\bar{m}^{\bar{m}}} \\ &\sim t^{\frac{a^2}{2(1-2\varepsilon)^{2D} \mathsf{c}^2 \bar{m}^{1-2D}} + o(1)} (const.) \end{aligned}$$

where (*const.*) denotes a constant depending on ε and $\bar{m}$, which can be absorbed in the $o(1)$ term in the exponent. Since $\mathrm{P}(N_t = \bar{m}) \geq (const.')t^{\bar{m}}$, we get

$$\mathrm{P}(X_t > \kappa) \geq \mathrm{P}(X_t > \kappa | N_t = \bar{m})\mathrm{P}(N_t = \bar{m}) = t^{\frac{a^2}{2(1-2\varepsilon)^{2D}\mathsf{c}^2\bar{m}^{1-2D}} + \bar{m} + o(1)},$$

hence

$$\liminf \frac{\log \mathrm{P}(X_t > \kappa)}{\log \frac{1}{t}} \geq -\left(\frac{a^2}{2(1-2\varepsilon)^{2D}\mathsf{c}^2\bar{m}^{1-2D}} + \bar{m}\right).$$

Since $\varepsilon > 0$ is arbitrary, we can let $\varepsilon \to 0$ in this relation and the right hand side becomes $-\mathsf{f}(a)$, by (5.19). Recalling (5.18), we have completed the proof of relation (3.6). $\square$

6. Proof of Theorem 3.3 (option price)

In this section we prove Theorem 3.3, or more precisely we derive it from Theorem 3.1 (which is proved in Section 5). This is based on the results recently obtained in [CC14] that link tail probability and option price asymptotics, that we now summarize.

6.1. **From tail probability to option price.** In this subsection $(X_t)_{t\geq 0}$ denotes a generic stochastic process, representing the risk-neutral log-price, such that $(e^{X_t})_{t\geq 0}$ is a martingale. In order to determine the asymptotic behavior of the call price $c(\kappa, t) = \mathrm{E}[(e^{X_t} - e^{\kappa})^+]$ along a given family of values of (κ, t) with $\kappa > 0$, $t > 0$, we need some assumptions. We start with the regime of *atypical deviations*, i.e. we consider a family of (κ, t) such that $\mathrm{P}(X_t > \kappa) \to 0$.

Hypothesis 6.1. *Along the family of (κ, t) under consideration, one has $\mathrm{P}(X_t > \kappa) \to 0$ and for every fixed $\varrho \in [1, \infty)$ the following limit exists in $[0, +\infty]$:*

$$I_+(\varrho) := \lim \frac{\log \mathrm{P}(X_t > \varrho\kappa)}{\log \mathrm{P}(X_t > \kappa)}, \qquad \text{and moreover} \qquad \lim_{\varrho \downarrow 1} I_+(\varrho) = 1. \tag{6.1}$$

We also need to formulate some moment conditions. The first condition is

$$\forall \eta \in (0, \infty): \qquad \limsup \mathrm{E}[e^{(1+\eta)X_t}] < \infty, \tag{6.2}$$

where the limit is taken along the given family of (κ, t) (however, only t enters in (6.2)). Note that if t is bounded from above, say $t \leq T$, it suffices to require that

$$\forall \eta \in (0, \infty): \qquad \mathrm{E}[e^{(1+\eta)X_T}] < \infty, \tag{6.3}$$

because $(e^{(1+\eta)X_t})_{t\geq 0}$ is a submartingale and consequently $\mathrm{E}[e^{(1+\eta)X_t}] \leq \mathrm{E}[e^{(1+\eta)X_T}]$. The second moment condition, to be applied when $t \to 0$ and $\kappa \to 0$, is

$$\exists C \in (0, \infty): \qquad \mathrm{E}[e^{2X_t}] \leq 1 + C\kappa^2. \tag{6.4}$$

(We have stated the moment assumptions (6.2) and (6.4) in a form that is enough for our purposes, but they can actually be weakened, as we showed in [CC14].)

The next theorem, taken from [CC14, Theorem 1.5], links the tail probability $\mathrm{P}(X_t > \kappa)$ and the option price $c(\kappa, t)$ in the regime of atypical deviations, generalizing [BF09].

Theorem 6.2. *Consider a risk-neutral log-price $(X_t)_{t\geq 0}$ and a family of values of (κ, t) with $\kappa > 0$, $t > 0$ such that Hypothesis 6.1 is satisfied.*

- *In case $\liminf \kappa > 0$ and $\limsup t < \infty$, if the moment condition (6.2) hold, then*

$$\log c(\kappa, t) \sim \log \mathrm{P}(X_t > \kappa) + \kappa. \tag{6.5}$$

- *In case $\kappa \to 0$ and $t \to 0$, if the moment condition* (6.4) *holds, and if in addition*

$$\lim_{\varrho \to +\infty} I_+(\varrho) = +\infty \,, \tag{6.6}$$

then

$$\log \big(c(\kappa, t)/\kappa \big) \sim \log \mathrm{P}(X_t > \kappa) \,. \tag{6.7}$$

Next we discuss the case of *typical deviations*, i.e. we consider a family of values of (κ, t) such that $\kappa \to 0$, $t \to 0$ in such a way that $\mathrm{P}(X_t > \kappa)$ is bounded away from zero. In this case we assume the convergence in distribution of X_t, suitably rescaled, as $t \to 0$.

Hypothesis 6.3. *There is a positive function $(\gamma_t)_{t>0}$ with $\lim_{t\downarrow 0} \gamma_t = 0$ such that X_t/γ_t converges in law as $t \downarrow 0$ to some random variable Y:*

$$\frac{X_t}{\gamma_t} \xrightarrow[t\downarrow 0]{d} Y \,. \tag{6.8}$$

The next result is [CC14, Theorem 1.11].

Theorem 6.4. *Assume that Hypothesis 6.3 is satisfied, and moreover the moment condition* (6.4) *holds with $\kappa = \gamma_t$, i.e.*

$$\exists C \in (0, \infty) : \quad \mathrm{E}\left[e^{2X_t}\right] < 1 + C\gamma_t^2 \,. \tag{6.9}$$

Consider a family of values of (κ, t) such that $t \to 0$ and $\kappa \sim a\gamma_t$, with $a \in [0, \infty)$ (in case $a = 0$, we mean $\kappa = o(\gamma_t)$). Then, assuming that $\mathrm{P}(Y > a) > 0$, one has

$$c(\kappa, t) \sim \gamma_t \, \mathrm{E}[(Y - a)^+] \,. \tag{6.10}$$

6.2. **Proof of Theorem 3.3, part** (a)**.** We fix a family of values of (κ, t) such that either $t \to \bar{t} \in (0, \infty)$ and $\kappa \to \infty$, or $t \to 0$ and $\kappa \to \bar{\kappa} \in (0, \infty]$. Let us check the assumptions of Theorem 6.2. Relation (3.2) shows that for all $\varrho \geq 1$

$$\lim \frac{\log \mathrm{P}(X_t > \varrho\kappa)}{\log \mathrm{P}(X_t > \kappa)} = \varrho^{\frac{1}{1-D}} \,, \tag{6.11}$$

hence Hypothesis 6.1 is satisfied with $I_+(\varrho) := \varrho^{\frac{1}{1-D}}$. The moment condition (6.2) is implied by (6.3), which holds for all $T \in (0, \infty)$, by Lemma 4.5. By Theorem 6.2, relation (6.5) holds. However, since $-\log \mathrm{P}(X_t > \kappa)/\kappa \to \infty$ by (3.2) (note that $\frac{1}{1-D} > 1$), relation (6.5) yields

$$\log c(\kappa, t) \sim \log \mathrm{P}(X_t > \kappa) \,, \tag{6.12}$$

which is precisely relation (3.9). This completes the proof of part (a) of Theorem 3.3. $\square$

6.3. **Proof of Theorem 3.3, part** (b)**.** Let us fix a family of values of (κ, t) with $t \to 0$ and $\kappa \to 0$, such that $\kappa \gg \sqrt{t}$, excluding the regime $(\sqrt{2D+1}\,\sigma_0)\,\boldsymbol{\kappa}_1(t) \leq \kappa \ll \boldsymbol{\kappa}_2(t)$ of part (c). By a subsequence argument, it suffices to consider separately the following regimes:

(i) $\sqrt{t} \ll \kappa \ll \boldsymbol{\kappa}_1(t)$;

(ii) $\kappa \sim a\,\boldsymbol{\kappa}_1(t)$ with $a \in (0, \sqrt{2D+1}\sigma_0]$;

(iii) $\kappa \sim a\,\boldsymbol{\kappa}_2(t)$ with $a \in (0, \infty)$;

(iv) $\kappa \gg \boldsymbol{\kappa}_2(t)$.

We start checking Hypothesis 6.1 in regimes (i), (iii) and (iv) (the regime (ii) will be considered later). In regime (iv), relation (3.2) holds, cf. part (a) in Theorem 3.1, hence (6.11) applies again and $I_+(\varrho) = \varrho^{\frac{1}{1-D}}$ (recall (6.1)). In regime (i), by relation (3.4),

$$I_+(\varrho) := \lim \frac{\log \mathrm{P}(X_t > \varrho\kappa)}{\log \mathrm{P}(X_t > \kappa)} = \varrho^2 . \tag{6.13}$$

Finally, in regime (iii), by (3.3) (or equivalently (3.6)),

$$I_+(\varrho) := \lim \frac{\log \mathrm{P}(X_t > \varrho\kappa)}{\log \mathrm{P}(X_t > \kappa)} = \frac{\mathsf{f}(\varrho a)}{\mathsf{f}(a)} . \tag{6.14}$$

In all cases, Hypothesis 6.1 and relation (6.6) are satisfied. As we show in a moment, also the moment condition (6.4) is satisfied. Having checked all the assumptions of Theorem 6.2 (recall that $t \to 0$ and $\kappa \to 0$), relation (6.7) holds. This coincides with our goal (3.10), completing the proof of part (b) of Theorem 3.3 in regimes (i), (iii) and (iv).

It remains to check the moment condition (6.4) in regimes (i), (iii) and (iv). Since $\kappa \gg \sqrt{t}$ in all these regimes, this follows immediately from the next Lemma.

Lemma 6.5. *There exists a constant $C \in (0, \infty)$ such that*

$$\mathrm{E}\left[e^{2X_t}\right] \le 1 + C\,t , \qquad \forall 0 \le t \le 1 .$$

Proof. By the equality in (4.12) and the upper bound (4.8), we can write

$$\mathrm{E}[e^{2X_t}] = \mathrm{E}[e^{I_t}] \le e^{\sigma_0^2 t}\, \mathrm{E}[e^{\mathsf{c}^2 t^{2D} N_t^{1-2D}}] . \tag{6.15}$$

Next observe that, by Cauchy-Schwarz inequality and $\mathrm{P}(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$,

$$\begin{aligned} \mathrm{E}[e^{\mathsf{c}^2 t^{2D} N_t^{1-2D}}] &= \mathrm{P}(N_t = 0) + e^{\mathsf{c}^2 t^{2D}} \mathrm{P}(N_t = 1) + \mathrm{E}[e^{\mathsf{c}^2 t^{2D} N_t^{1-2D}} 1_{\{N_t \ge 2\}}] \\ &\le 1 + e^{\mathsf{c}^2 t^{2D}} \lambda t + \sqrt{\mathrm{E}[e^{2\mathsf{c}^2 t^{2D} N_t^{1-2D}}] \mathrm{P}(N_t \ge 2)} . \end{aligned} \tag{6.16}$$

Note that $\mathrm{P}(N_t \ge 2) = 1 - e^{-\lambda t}(1 + \lambda t) = \frac{1}{2}(\lambda t)^2 + o(t^2)$ as $t \downarrow 0$. For all $0 \le t \le 1$ we can write $\mathrm{E}[e^{2\mathsf{c}^2 t^{2D} N_t^{1-2D}}] \le \mathrm{E}[e^{2\mathsf{c}^2 N_1^{1-2D}}] =: c_1 < \infty$, and $e^{\mathsf{c}^2 t^{2D}} \le e^{\mathsf{c}^2}$, hence (6.16) yields

$$\mathrm{E}[e^{\mathsf{c}^2 t^{2D} N_t^{1-2D}}] \le 1 + e^{\mathsf{c}^2} \lambda t + \sqrt{\frac{c_1 \lambda^2}{2} \left(t^2 + o(t^2)\right)} \le 1 + c_2\, t ,$$

for some $c_2 < \infty$. Consequently, by (6.15),

$$\mathrm{E}[e^{2X_t}] \le e^{\sigma_0^2 t}\big(1 + c_2\, t\big) = \big(1 + \sigma_0^2\, t + o(t)\big)\big(1 + c_2\, t\big) \le 1 + Ct ,$$

for some $C < \infty$. □

We are left with considering regime (ii), i.e. we fix a family of (κ, t) such that

$$t \to 0 \qquad \text{and} \qquad \kappa \sim a\, \boldsymbol{\kappa}_1(t) , \quad \text{for some} \quad a \in (0, \sqrt{2D+1}\, \sigma_0] . \tag{6.17}$$

In this regime the assumptions of Theorem 3.3 are *not* verified, hence we proceed by bare hands estimates. Our goal is to prove (3.10) which, recalling (3.4), can be rewritten as

$$\log\big(c(\kappa, t)/\kappa\big) \sim -\frac{a^2}{2\sigma_0^2} \log \frac{1}{t} . \tag{6.18}$$

We prove separately upper and lower bounds for this relation.

Let us set

$$k' := \sqrt{2}\, \sigma_0\, \boldsymbol{\kappa}_1(t) , \qquad k'' := B\, \boldsymbol{\kappa}_2(t) , \tag{6.19}$$

for fixed $B \in (0,\infty)$, chosen later. Noting that $\kappa < \kappa' < \kappa''$, since $D < \frac{1}{2}$, we can write

$$\begin{aligned} c(\kappa,t) &= \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{X_t>\kappa\}}\right] \\ &= \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{\kappa<X_t\le\kappa'\}}\right] + \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{\kappa'<X_t\le\kappa''\}}\right] + \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{X_t>\kappa''\}}\right] \\ &= (1)+(2)+(3)\,. \end{aligned} \tag{6.20}$$

By Fubini's theorem, for $\kappa \ge 0$ and $0 \le a < b$,

$$\begin{aligned} \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{a<X_t\le b\}}\right] &= \mathrm{E}\left[\left(\int_\kappa^\infty e^x\, 1_{\{x<X_t\}}\,\mathrm{d}x\right)1_{\{a<X_t\le b\}}\right] \\ &= \int_\kappa^b e^x\,\mathrm{P}(\max\{a,x\}<X_t\le b)\,\mathrm{d}x \\ &\le (e^b-1)\,\mathrm{P}(X_t>\max\{a,\kappa\})\,, \end{aligned} \tag{6.21}$$

hence

$$(1) = \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{\kappa<X_t\le\kappa'\}}\right] \le (e^{\kappa'}-1)\mathrm{P}(X_t>\kappa) \sim \kappa'\,\mathrm{P}(X_t>\kappa)\,, \tag{6.22}$$

because $\kappa' \to 0$. Note that, by (6.17) and (3.4),

$$\log \mathrm{P}(X_t>\kappa) \sim -\frac{a^2}{2\sigma_0^2}\log\frac{1}{t}\,, \tag{6.23}$$

and since $\frac{\kappa'}{\kappa} \sim \frac{\sqrt{2}\sigma_0}{a} = (\mathit{const.})$, recall (6.19), it follows by (6.22) that

$$\log\frac{(1)}{\kappa} \le \log\frac{\kappa'}{\kappa} + \log\mathrm{P}(X_t>\kappa) = -\frac{a^2}{2\sigma_0^2}\log\frac{1}{t}\big(1+o(1)\big)\,.$$

In a similar way, always using (6.21), since $\kappa < \kappa'$ and $\kappa'' \to 0$,

$$(2) = \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{\kappa'<X_t\le\kappa''\}}\right] \le (e^{\kappa''}-1)\mathrm{P}(X_t>\kappa') \sim \kappa''\,\mathrm{P}(X_t>\kappa')\,. \tag{6.24}$$

Again by (6.23) with $a = \sqrt{2}\sigma_0$, noting that $\frac{\kappa}{\kappa''} \sim \frac{a}{B}(\frac{1}{t})^{D-\frac{1}{2}}$, we can write

$$\log\frac{(2)}{\kappa} \le -\big(1+o(1)\big)\log\frac{1}{t} - \log\frac{\kappa}{\kappa''} \le -\left(D+\frac{1}{2}+o(1)\right)\log\frac{1}{t}\,.$$

Finally, by Cauchy-Schwarz inequality

$$(3) = \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{X_t>\kappa''\}}\right] \le \kappa\sqrt{\mathrm{E}\left[\left(\frac{e^{X_t}-e^{\kappa}}{\kappa}\right)^2\right]\mathrm{P}(X_t>\kappa'')}\,. \tag{6.25}$$

By Lemma 6.5 and $\mathrm{E}[e^{X_t}] = 1$ (recall that $(e^{X_t})_{t\ge0}$ is a martingale) we have

$$\mathrm{E}\left[\left(\frac{e^{X_t}-e^{\kappa}}{\kappa}\right)^2\right] = \frac{\mathrm{E}[e^{2X_t}]-2e^{\kappa}+e^{2\kappa}}{\kappa^2} \le \frac{1+Ct-2+e^{2\kappa}}{\kappa^2} = \frac{Ct}{\kappa^2} + \frac{e^{2\kappa}-1}{\kappa^2} \to 0\,,$$

because $\kappa \to 0$ and $\kappa/\sqrt{t} \to \infty$, by (6.17) and the definition (2.4) of $\boldsymbol{\kappa}_1(t)$. In particular, for some constant $C' < \infty$ we have

$$(3) \le \kappa\sqrt{C'\,\mathrm{P}(X_t>\kappa'')}\,.$$

Recalling (3.6), it follows that

$$\log\frac{(3)}{\kappa} \le -\big(1+o(1)\big)\frac{1}{2}\mathsf{f}(B)\log\frac{1}{t}\,. \tag{6.26}$$

Since $\log(a+b+c) \le \log 3 + \max\{\log a, \log b, \log c\}$, we obtain by (6.20)

$$\log \frac{c(\kappa,t)}{\kappa} \le -\big(1+o(1)\big) \min\left\{\frac{a^2}{2\sigma_0^2}, D+\frac{1}{2}, \frac{\mathsf{f}(B)}{2}\right\} \log\frac{1}{t}\,. \tag{6.27}$$

Let us choose $B > 0$ large enough, so that $\frac{\mathsf{f}(B)}{2} > D + \frac{1}{2}$. We now use the assumption $a \le \sqrt{2D+1}\sigma_0$, cf. (6.17), which implies

$$\log \frac{c(\kappa,t)}{\kappa} \le -\big(1+o(1)\big) \frac{a^2}{2\sigma_0^2} \log\frac{1}{t}\,, \tag{6.28}$$

which is "half" of our goal (6.18).

In order to obtain the corresponding lower bound, we observe that for every $\hat\kappa > \kappa$

$$\begin{aligned} c(\kappa,t) &= \mathrm{E}\left[\left(e^{X_t}-e^{\kappa}\right) 1_{\{X_t>\kappa\}}\right] \ge \mathrm{E}\left[\left(e^{X_t}-e^{\kappa}\right) 1_{\{X_t>\hat\kappa\}}\right] \ge (e^{\hat\kappa}-e^{\kappa})\mathrm{P}(X_t>\hat\kappa) \\ &\ge (\hat\kappa-\kappa)\mathrm{P}(X_t>\hat\kappa). \end{aligned} \tag{6.29}$$

Always for κ as in (6.17), choosing $\hat\kappa = (1+\varepsilon)\kappa$ gives, recalling (6.23),

$$\log \frac{c(\kappa,t)}{\kappa} \ge \log\varepsilon + \log \mathrm{P}(X_t > (1+\varepsilon)\kappa) = -(1+\varepsilon)^2 \frac{a^2}{2\sigma_0^2} \log\frac{1}{t}\big(1+o(1)\big)\,. \tag{6.30}$$

This shows that, along the given family of values of (κ, t),

$$\liminf \frac{\frac{c(\kappa,t)}{\kappa}}{\log\frac{1}{t}} \ge -(1+\varepsilon)^2 \frac{a^2}{2\sigma_0^2}\,.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\log \frac{c(\kappa,t)}{\kappa} \ge -\big(1+o(1)\big) \frac{a^2}{2\sigma_0^2} \log\frac{1}{t}\,. \tag{6.31}$$

Together with (6.28), this completes the proof of (6.18) and of part (b) of Theorem 3.3. ☐

6.4. **Proof of Theorem 3.3, part** (c)**.** Let us fix a family of values of (κ, t) with

$$t \to 0 \qquad \text{and} \qquad \sqrt{2D+1}\,\sigma_0\,\boldsymbol{\kappa}_1(t) \le \kappa \ll \boldsymbol{\kappa}_2(t)\,. \tag{6.32}$$

Our goal is to prove (3.11), that we can rewrite equivalently as

$$\log\big(c(\kappa,t)/\kappa\big) \sim -\log\frac{1}{t} - \log\frac{\kappa}{t^D}\,. \tag{6.33}$$

We are going to prove upper and lower bounds for this relation.

Consider first the subregime of (6.32) given by $\kappa \le \sqrt{2}\,\sigma_0\boldsymbol{\kappa}_1(t)$, so assume (without loss of generality, by extracting a subsequence) that $\kappa \sim a\,\boldsymbol{\kappa}_1(t)$ with $a \in [\sqrt{2D+1}\,\sigma_0, \sqrt{2}\,\sigma_0]$. Note that all the steps from (6.19) until (6.27) can be applied verbatim. However, this time $a \ge \sqrt{2D+1}\,\sigma_0$, hence $\frac{a^2}{2\sigma_0^2} \ge D + \frac{1}{2}$ and instead of relation (6.28) we get

$$\log \frac{c(\kappa,t)}{\kappa} \le -\big(1+o(1)\big)\left(D+\frac{1}{2}\right) \log\frac{1}{t}\,, \tag{6.34}$$

which is "half" of our goal (3.11), equivalently (6.33) (note that $\frac{\log\kappa}{\log t} \sim \frac{1}{2}$ in this subregime).

Next we consider the subregime of (6.32) when $\kappa > \sqrt{2}\,\sigma_0\boldsymbol{\kappa}_1(t)$. Defining $\kappa'' := B\,\boldsymbol{\kappa}_2(t)$ as in (6.19), we modify (6.20) as follows:

$$c(\kappa,t) = \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{\kappa<X_t\le\kappa''\}}\right] + \mathrm{E}\left[(e^{X_t}-e^{\kappa})1_{\{X_t>\kappa''\}}\right] =: (A) + (B)\,. \tag{6.35}$$

Applying (6.21), we estimate the first term as follows, since $\kappa'' \to 0$:

$$(A) = \mathrm{E}\left[(e^{X_t} - e^{\kappa}) 1_{\{\kappa < X_t \le \kappa''\}}\right] \le (e^{\kappa''} - 1)\, \mathrm{P}(X_t > \kappa) \sim \kappa''\, \mathrm{P}(X_t > \kappa)\,.$$

Since $\kappa/\boldsymbol{\kappa}_2(t) \to 0$ under (6.32), relation (3.3) yields $\log \mathrm{P}(X_t > \kappa) \sim -(1 + o(1)) \log \frac{1}{t}$, because $\mathsf{f}(0+) = 1$ by (2.6). Moreover $\log(\kappa''/\kappa) \sim \log(t^D/\kappa)$ by definition of $\boldsymbol{\kappa}_2(t)$, hence

$$\log \frac{(A)}{\kappa} \le \log \frac{\kappa''}{\kappa} + \log \mathrm{P}(X_t > \kappa) \le -\big(1 + o(1)\big) \bigg(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \bigg)\,.$$

The term (B) in (6.35) coincides with term (3) in (6.25), hence by (6.26)

$$\log \frac{(B)}{\kappa} \le -\big(1 + o(1)\big) \frac{\mathsf{f}(B)}{2} \log \frac{1}{t} \le -\big(1 + o(1)\big) \frac{\mathsf{f}(B)}{2} \bigg(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \bigg)\,,$$

where the second inequality holds just because $\kappa \le t^D$ by (6.32), hence $\log \frac{\kappa}{t^D} \le 0$. Choosing B large enough, so that $\mathsf{f}(B) > 2$, the inequality $\log(a + b) \le \log 2 + \log \max\{a, b\}$ yields

$$\log \frac{c(\kappa, t)}{\kappa} \le -\big(1 + o(1)\big) \bigg(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \bigg)\,. \tag{6.36}$$

We have thus proved "half" of our goal (6.33).

We finally turn to the lower bound, for which we do not need to distinguish subregimes, but we work in the general regime (6.32). We are going to apply (6.29) with $\hat{\kappa} = \varepsilon \boldsymbol{\kappa}_2(t)$. Recalling that $\log \mathrm{P}(X_t > \varepsilon \boldsymbol{\kappa}_2(t)) \sim -\mathsf{f}(\varepsilon) \log \frac{1}{t}$ by (3.6), and moreover

$$\log \frac{\hat{\kappa} - \kappa}{\kappa} \sim \log \left(\frac{\varepsilon \boldsymbol{\kappa}_2(t)}{\kappa} - 1 \right) \sim \log \frac{t^D}{\kappa}\,,$$

relation (6.29) gives

$$\log \frac{c(\kappa, t)}{\kappa} \ge -\big(1 + o(1)\big) \bigg(\mathsf{f}(\varepsilon) \log \frac{1}{t} + \log \frac{\kappa}{t^D} \bigg)\,. \tag{6.37}$$

Since $\varepsilon > 0$ is arbitrary and $\lim_{\varepsilon \downarrow 0} \mathsf{f}(\varepsilon) = \mathsf{f}(0) = 1$, cf. (2.5), we have shown that

$$\log \frac{c(\kappa, t)}{\kappa} \ge -\big(1 + o(1)\big) \bigg(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \bigg)\,.$$

Together with (6.34) and (6.36), this completes the proof of our goal (6.33). □

6.5. **Proof of Theorem 3.3, parts** (d) **and** (e)**.** By (3.1), Hypothesis 6.3 is satisfied with $\gamma = \sqrt{t}$ and $Y = \sigma_0 W_1$, while the moment condition (6.9) is verified by Lemma 6.5. We can then apply relation (6.10) in Theorem 6.4, which for $\kappa \sim a\sqrt{t}$ yields

$$\begin{aligned} c(\kappa, t) &\sim \sqrt{t}\, \sigma_0\, \mathrm{E}\left[\left(W_1 - \frac{a}{\sigma_0} \right)^+ \right] = \sqrt{t}\, \sigma_0 \left[\int_{\frac{a}{\sigma_0}}^{\infty} x\, \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, \mathrm{d}x - \frac{a}{\sigma_0} \int_{\frac{a}{\sigma_0}}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, \mathrm{d}x \right] \\ &= \sqrt{t}\, \sigma_0 \left(\frac{e^{-\frac{a^2}{2\sigma_0^2}}}{\sqrt{2\pi}} - \frac{a}{\sigma_0} \left(1 - \Phi\left(\frac{a}{\sigma_0} \right) \right) \right) = \sqrt{t}\, \sigma_0 \left(\varphi\left(\frac{a}{\sigma_0} \right) - \frac{a}{\sigma_0} \Phi\left(-\frac{a}{\sigma_0} \right) \right). \end{aligned}$$

For $a > 0$ this coincides with (3.12), while for $a = 0$ it coincides with (3.13). □

7. Proof of Theorem 2.2 (implied volatility)

In this section we prove Theorem 2.2, or more precisely we derive it from Theorem 3.3 (which is proved in Section 6). The proof is based on the results recently obtained in [CC14] that link option price and implied volatility asymptotics, that we now summarize.

7.1. **From option price to implied volatility.** As we showed in [CC14], whenever the option price $c(\kappa, t)$ vanishes, its asymptotic behavior determines that of the corresponding implied volatility through *explicit universal formulas*. Let us define the function

$$D(z) := \frac{1}{z}\varphi(z) - \Phi(-z), \qquad \forall z > 0\,, \tag{7.1}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the density and distribution function of a standard Gaussian. Since $D : (0,\infty) \to (0,\infty)$ is a smooth and strictly decreasing, its inverse $D^{-1} : (0,\infty) \to (0,\infty)$ is also smooth, strictly decreasing and has the following asymptotic behavior [CC14, §3.1]:

$$D^{-1}(y) \sim \sqrt{2\,(-\log y)} \quad \text{ as } y \downarrow 0\,, \qquad D^{-1}(y) \sim \frac{1}{\sqrt{2\pi}}\frac{1}{y} \quad \text{ as } y \uparrow \infty\,. \tag{7.2}$$

The next result, taken from [CC14, Theorem 1.1], links option price and implied volatility in a model independent way. We point out that related results have appeared in [BF09, L04, RR09, G10, GL14] and we refer to [CC14] for a discussion.

Theorem 7.1. *Consider a family of values of (κ, t) with $\kappa \ge 0$, $t > 0$, such that $c(\kappa, t) \to 0$.*

- *In case* $\liminf \kappa > 0$*, one has*

$$\sigma_{\text{imp}}(\kappa, t) \sim \left(\sqrt{\frac{-\log c(\kappa,t)}{\kappa} + 1} - \sqrt{\frac{-\log c(\kappa,t)}{\kappa}} \right) \sqrt{\frac{2\kappa}{t}}\,. \tag{7.3}$$

- *In case $\kappa \to 0$, with $\kappa > 0$, one has*

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{1}{D^{-1}\left(\frac{c(\kappa,t)}{\kappa}\right)} \frac{\kappa}{\sqrt{t}}\,. \tag{7.4}$$

- *In case $\kappa = 0$, one has*

$$\sigma_{\text{imp}}(0, t) \sim \sqrt{2\pi}\, \frac{c(0,t)}{\sqrt{t}}\,. \tag{7.5}$$

Remark 7.2. Whenever $\frac{-\log c(\kappa,t)}{\kappa} \to \infty$, formula (7.3) simplifies to

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log c(\kappa,t))}}\,. \tag{7.6}$$

Analogously, by (7.2), formula (7.4) can be made more explicit as follows:

$$\sigma_{\text{imp}}(\kappa, t) \sim \begin{cases} \dfrac{\kappa}{\sqrt{2t(-\log(c(\kappa,t)/\kappa))}} & \text{if } \dfrac{c(\kappa,t)}{\kappa} \to 0\,; \\[2ex] \dfrac{\kappa}{D^{-1}(a)\sqrt{t}} & \text{if } \dfrac{c(\kappa,t)}{\kappa} \to a \in (0,\infty)\,; \\[2ex] \sqrt{2\pi}\,\dfrac{c(\kappa,t)}{\sqrt{t}} & \text{if } \dfrac{c(\kappa,t)}{\kappa} \to \infty \;\text{ or if }\; \kappa = 0\,. \end{cases} \tag{7.7}$$

7.2. **Proof of Theorem 2.2, part** (a)**.** Consider a family of values of (κ, t) such that either $t \to \bar{t} \in (0, \infty)$ and $\kappa \to \infty$, or $t \to 0$ and $\kappa \gg \boldsymbol{\kappa}_2(t)$. We consider two subregimes:

(i) either $t \to \bar{t} \in (0, \infty)$ and $\kappa \to \infty$, or $t \to 0$ and $\kappa \to \bar{\kappa} \in (0, \infty]$;

(ii) both $t \to 0$ and $\kappa \to 0$ with $\kappa \gg \boldsymbol{\kappa}_2(t)$.

Our goal is to prove that in both subregimes relation (2.7) holds.

We start with subregime (i). By Theorems 3.3 and 3.1, relations (3.9) and (3.2) give

$$\log c(\kappa, t) \sim -\log \mathrm{P}(X_t > \kappa) \sim -\mathsf{C}\left(\frac{\kappa}{t^D}\right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t}\right)^{\frac{1/2-D}{1-D}} . \tag{7.8}$$

Next we apply Theorem 7.1: since $\liminf \kappa > 0$ in this subregime, recalling Remark 7.2, relation (7.6) holds, because $|\log c(\kappa, t)| \gg |\log \kappa|$ by (7.8). Then we get

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t\,(-\log c(\kappa, t))}} \sim \frac{1}{\sqrt{2\mathsf{C}}} \left(\frac{\frac{\kappa}{t}}{\sqrt{\log \frac{\kappa}{t}}}\right)^{\frac{1/2-D}{1-D}} , \tag{7.9}$$

which is precisely our goal (2.7).

Next we consider subregime (ii). Again by Theorems 3.3 and 3.1, relations (3.10) and (3.2) show that $-\log(c(\kappa, t)/\kappa)$ is asymptotically equivalent to the right hand side of (7.8). By Theorem 7.1 we can apply relation (7.4), which by Remark 7.2 reduces to the first line of (7.7). In analogy with (7.9), we obtain again our goal (2.7). $\square$

7.3. **Proof of Theorem 2.2, part** (b)**.** Next we consider a family of values of (κ, t) with $t \to 0$ and $\kappa \sim a\boldsymbol{\kappa}_2(t)$ for some $a \in (0, \infty)$, and our goal is to prove (2.8). By Theorems 3.3 and 3.1, relations (3.10) and (3.3) (cf. also (3.6)) yield

$$\log\big(c(\kappa, t)/\kappa\big) \sim -\log \mathrm{P}(X_t > \kappa) \sim -\mathsf{f}(a) \log \frac{1}{t} .$$

By Theorem 7.1 and Remark 7.2, recalling the definition (2.4) of $\boldsymbol{\kappa}_1(t)$, relation (7.7) gives

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log(c(\kappa, t)/\kappa))}} \sim \left\{\frac{1}{\sqrt{2\mathsf{f}(a)}}\right\} \frac{\kappa}{\boldsymbol{\kappa}_1(t)} ,$$

proving our goal (2.8). $\square$

7.4. **Proof of Theorem 2.2, part** (c)**.** Next we consider a family of values of (κ, t) with $t \to 0$ and $(\sqrt{2D+1}\,\sigma_0)\,\boldsymbol{\kappa}_1(t) \le \kappa \ll \boldsymbol{\kappa}_2(t)$, and our goal is to prove (2.9). Plugging relation (3.11) from Theorem 3.3 into the first line of relation (7.7) (recall Theorem 7.1 and Remark 7.2), by the definition (2.4) of $\boldsymbol{\kappa}_1(t)$ we obtain

$$\sigma_{\mathrm{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t\,(-\log(c(\kappa, t)/\kappa))}} \sim \left\{\frac{1}{\sqrt{2\big(D+1-\frac{\log \kappa}{\log t}\big)}}\right\} \frac{\kappa}{\boldsymbol{\kappa}_1(t)} ,$$

proving our goal (2.9). $\square$

7.5. **Proof of Theorem 2.2, part** (d)**.** Next we consider a family of values of (κ, t) with $t \to 0$ and $0 \le \kappa \le \sqrt{2D+1}\,\sigma_0\,\boldsymbol{\kappa}_1(t)$, and our goal is to prove (2.10), i.e. $\sigma_{\mathrm{imp}}(\kappa, t) \sim \sigma_0$.

First we consider the case of typical deviations, i.e. when $\kappa \sim a\sqrt{t}$ for some $a \in [0, \infty)$. In case $a > 0$, relation (3.12) from Theorem 3.3 gives

$$\frac{c(\kappa, t)}{\kappa} \to D\left(\frac{a}{\sigma_0}\right) \sim D\left(\frac{\kappa}{\sigma_0\sqrt{t}}\right) ,$$

which plugged into relation (7.4) from Theorem 7.1 yields our goal $\sigma_{\mathrm{imp}}(\kappa,t) \sim \sigma_0$. In case $a = 0$, i.e. if $\kappa = o(\sqrt{t})$, relation (3.13) from Theorem 3.3 gives $c(\kappa,t) \sim \frac{\sigma_0}{\sqrt{2\pi}}\sqrt{t}$, hence $c(\kappa,t)/\kappa \to \infty$. We can thus apply relation (7.4) from Theorem 7.1, in the simplified form given by the third line of (7.7) (recall Remark 7.2), getting our goal $\sigma_{\mathrm{imp}}(\kappa,t) \sim \sigma_0$.†

Next we consider the case of atypical deviations, i.e. when $\kappa \gg \sqrt{t}$. By Theorems 3.3 and 3.1, relations (3.10) and (3.4) yield

$$\log\big(c(\kappa,t)/\kappa\big) \sim -\log \mathrm{P}(X_t > \kappa) \sim -\frac{\kappa^2}{2\sigma_0^2 t}\,.$$

By Theorem 7.1 and Remark 7.2, since $\kappa \to 0$, the first line of relation (7.7) gives

$$\sigma_{\mathrm{imp}}(\kappa,t) \sim \frac{\kappa}{\sqrt{2t\,(-\log(c(\kappa,t)/\kappa))}} \sim \sigma_0\,,$$

proving our goal (2.10). The proof of Theorem 2.2 is completed. □

Appendix A. Miscellanea

A.1. **Proof of relation** (1.3)**.** We recall that $(N_t)_{t\geq 0}$ denotes a Poisson process of intensity λ, with jump times $\tau_1, \tau_2, \ldots$, while $\tau_0 \in (-\infty, 0)$ is a fixed parameter. The random variable τ_{N_t} represents the last jump time prior to t.

It is well-known that the random variable $t - \tau_{N_t}$, conditionally on the event $\{N_t \geq 1\}$, is distributed like an exponential random variable $Y \sim \mathit{Exp}(\lambda)$ conditionally on $\{Y \leq t\}$. As a consequence, the following equality in distribution holds:

$$(t - \tau_{N_t}) \stackrel{d}{=} Y\, 1_{\{Y\leq t\}} + (t + |\tau_0|)\, 1_{\{Y>t\}}\,.$$

It follows easily that as $t \to \infty$ the random variable $t - \tau_{N_t}$ converges to Y in distribution. Moreover, for every $\alpha \in (0,1)$ we have

$$\begin{aligned} \mathrm{E}\left[\frac{1}{(t-\tau_{N_t})^\alpha}\right] &= \mathrm{E}\left[\frac{1}{Y^\alpha}\,\middle|\, Y \leq t\right](1 - e^{-\lambda t}) + \frac{1}{(t+|\tau_0|)^\alpha}\, e^{-\lambda t} \\ &\xrightarrow[t\to\infty]{} \mathrm{E}\left[\frac{1}{Y^\alpha}\right] = \int_0^\infty \frac{1}{y^\alpha}\lambda\, e^{-\lambda y}\,\mathrm{d}y = \lambda^\alpha\,\Gamma(1-\alpha)\,. \end{aligned}$$

Choosing $\alpha = 1 - 2D$ and recalling (1.2), we obtain $\lim_{t\to\infty} \mathrm{E}[\sigma_t^2] = V^2$, proving (1.3). □

A.2. **Martingale measures.** Let $(Y_t)_{t\geq 0}$ be the martingale in (1.1), i.e. $\mathrm{d}Y_t = \sigma_t\,\mathrm{d}B_t$, which represents the detrended log-price under the historical measure. We recall that $(\sigma_t)_{t\geq 0}$ is the process defined in (1.2), where $\tau_0 \in (-\infty,0)$ is a parameter and $(\tau_k)_{k\geq 1}$ are the jumps of a Poisson process $(N_t)_{t\geq 0}$ of intensity λ, independent of the Brownian motion $(B_t)_{t\geq 0}$.

For $\widetilde{\lambda} \in (0,\infty)$ and $T \in (0,\infty)$, define the equivalent probability measure $\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}$ by

$$\frac{\mathrm{d}\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}}{\mathrm{d}\mathrm{P}} := e^{-\int_0^T \frac{\sigma_s}{2}\,\mathrm{d}B_s - \frac{1}{2}\int_0^T (\frac{\sigma_s}{2})^2\,\mathrm{d}s} \cdot e^{(\log\frac{\widetilde{\lambda}}{\lambda})N_T - (\widetilde{\lambda}-\lambda)T} =: R_1 \cdot R_2\,. \tag{A.1}$$

Note that R_2 is the Radon-Nikodym derivative (on the time interval $[0,T]$) of the law of a Poisson process of intensity $\widetilde{\lambda}$ with respect to that of intensity λ. Denoting by $\mathcal{G}$ the σ-algebra generated by $(N_t)_{t\in[0,T]}$, the volatility $(\sigma_t)_{t\in[0,T]}$ is a $\mathcal{G}$-measurable process. Conditionally on $\mathcal{G}$, the trajectories $t \mapsto \sigma_t$ are thus *deterministic*, hence the random variable

†If $\kappa = 0$ one should apply relation (7.5), rather than (7.4), from Theorem 7.1, which however coincides with the the third line of (7.7), so the conclusion is the same.

$\int_0^T \frac{\sigma_s}{2}\,\mathrm{d}B_s$ is Gaussian with zero mean and variance $\int_0^T (\frac{\sigma_s}{2})^2 \mathrm{d}s < \infty$ (by (1.2), since $D < \frac{1}{2}$). Recalling the definition (A.1) of R_1, it follows immediately that $\mathrm{E}[R_1|\mathcal{G}] = 1$.

The previous observations show that (A.1) defines indeed a probability $\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}$, since

$$\mathrm{E}[R_1 R_2] = \mathrm{E}[\mathrm{E}[R_1|\mathcal{G}]R_2] = \mathrm{E}[R_2] = 1\,,$$

and $(N_t)_{t\in[0,T]}$ under $\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}$ is a Poisson process with intensity $\widetilde{\lambda}$. Moreover, the process

$$\widetilde{B}_t := B_t + \int_0^t \frac{\sigma_s}{2}\,\mathrm{d}s\,, \qquad \text{i.e.} \qquad \mathrm{d}\widetilde{B}_t := \mathrm{d}B_t + \frac{\sigma_t}{2}\,\mathrm{d}t\,, \tag{A.2}$$

is a Brownian motion under the conditional law $\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}(\,\cdot\,|\mathcal{G})$, by Girsanov's theorem. The fact that the distribution of $(\widetilde{B}_t)_{t\in[0,T]}$ conditionally on $\mathcal{G}$ does not depend on $\mathcal{G}$ (it is the Wiener measure), means that $(\widetilde{B}_t)_{t\in[0,T]}$ is independent of $\mathcal{G}$, i.e. of $(N_t)_{t\in[0,T]}$.

Summarizing: under $\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}$ the process $(\widetilde{B}_t)_{t\in[0,T]}$ in (A.2) is a Brownian motion and $(N_t)_{t\in[0,T]}$ is an independent Poisson process of intensity $\widetilde{\lambda}$. Rewriting (1.1) as

$$\mathrm{d}Y_t = \sigma_t\,\mathrm{d}\widetilde{B}_t - \frac{1}{2}\sigma_t^2\,\mathrm{d}t\,,$$

by Ito's formula the process $(S_t := e^{Y_t})_{t\in[0,T]}$ solves the stochastic differential equation

$$\mathrm{d}S_t = S_t\,\mathrm{d}Y_t + \frac{1}{2}S_t\,\mathrm{d}\langle Y\rangle_t = S_t\,\mathrm{d}Y_t + \frac{1}{2}S_t\,\sigma_t^2\,\mathrm{d}t = \sigma_t\,S_t\,\mathrm{d}\widetilde{B}_t\,. \tag{A.3}$$

We have thus shown that under $\widetilde{\mathrm{P}}_{\widetilde{\lambda},T}$ the price $(S_t)_{t\in[0,T]}$ evolves according to (1.7) (where the Brownian motion $\widetilde{B}_t$ has been renamed B_t), with the process $(\sigma_t)_{t\in[0,T]}$ still defined by (1.2), except that the Poisson process $(N_t)_{t\in[0,T]}$ has now intensity $\widetilde{\lambda}$.

A.3. **A minimization problem.** Let us recall from (2.5) the definition of $\mathsf{f} : (0,\infty) \to \mathbb{R}$:

$$\mathsf{f}(a) := \min_{m\in\mathbb{N}_0} \mathsf{f}_m(a)\,, \qquad \text{with} \qquad \mathsf{f}_m(a) := m + \frac{a^2}{2\mathsf{c}^2 m^{1-2D}}\,. \tag{A.4}$$

We also recall that, since $D < \frac{1}{2}$, we can restrict the minimum to $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

For fixed $a \in (0,\infty)$, if we minimize $\mathsf{f}_m(a)$ over $m \in (0,\infty)$, rather than over $m \in \mathbb{N}$, the global minimum is attained at the unique $\widetilde{m}_a \in (0,\infty)$ with $\frac{\partial}{\partial m}\mathsf{f}_m(a)|_{m=\widetilde{m}_a} = 0$, i.e.

$$\widetilde{m}_a = \left(\sqrt{\tfrac{1}{2} - D}\,\frac{a}{\mathsf{c}}\right)^{\frac{1}{1-D}}\,.$$

Since $m \mapsto \mathsf{f}_m(a)$ is decreasing on $(0, \widetilde{m}_a)$ and increasing on $(\widetilde{m}_a, \infty)$, it follows that

$$\mathsf{f}(a) = \min\left\{\mathsf{f}_{\lfloor \widetilde{m}_a \rfloor}(a), \mathsf{f}_{\lceil \widetilde{m}_a \rceil}(a)\right\}, \tag{A.5}$$

where $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \le x\}$ and $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \ge x\}$ denote the lower and upper integer part of x, respectively. In particular, if $\widetilde{m}_a = k \in \mathbb{N}$ is an integer, i.e. if

$$a = \hat{a}_k := \frac{\mathsf{c}}{\sqrt{\frac{1}{2} - D}}\kappa^{1-D}\,,$$

then $\mathsf{f}(a) = \mathsf{f}_k(a)$. Next we observe that for $a \in (\hat{a}_k, \hat{a}_{k+1})$ one has $\widetilde{m}_a \in (k, k+1)$, hence $\mathsf{f}(a) = \min\{\mathsf{f}_k(a), \mathsf{f}_{k+1}(a)\}$ by (A.5). By direct computation, one has

$$\mathsf{f}_k(a) \le \mathsf{f}_{k+1}(a) \qquad \Longleftrightarrow \qquad a \le x_k := \frac{\mathsf{c}}{\sqrt{\frac{1}{2k^{1-2D}} - \frac{1}{2(k+1)^{1-2D}}}}\,.$$

(Note that $\hat{a}_k < x_k < \hat{a}_{k+1}$, by convexity of $z \mapsto z^{-(1-2D)}$, and $x_k \sim \hat{a}_k$ as $\kappa \to \infty$.) Setting $x_0 := 0$ for convenience, the previous considerations show that

$$\mathsf{f}(a) = \mathsf{f}_k(a) \qquad \text{for all } a \in [x_{k-1}, x_k) \text{ and } k \in \mathbb{N}\,. \tag{A.6}$$

Since $\mathsf{f}_k(x_k) = \mathsf{f}_{k+1}(x_k)$ by construction, the function f is continuous and strictly increasing (but it is *not* convex, as one can check). The asymptotics in (2.6) follow easily by (A.6) and (A.4), which yield $\mathsf{f}(a) \sim \mathsf{f}_{\widetilde{m}_a}(a)$ as $a \to \infty$.

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