Scaling and Universality in Probability

Francesco Caravenna

Università degli Studi di Milano-Bicocca

Leiden $\sim$ March 13, 2014

Overview

A more expressive (but less fancy) title would be

Convergence of Discrete Probability Models
to a Universal Continuum Limit

This is a key topic of classical and modern probability theory

Overview

A more expressive (but less fancy) title would be

Convergence of Discrete Probability Models
to a Universal Continuum Limit

This is a key topic of classical and modern probability theory

I will present a (limited) selection of representative results, in order to convey the main ideas and give the flavor of the subject

Outline

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of "measurable" subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space.

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of "measurable" subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space. We will be "concrete":

(Metric space E, "Borel σ-algebra", Probability μ)

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of "measurable" subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space. We will be "concrete":

(Metric space E, Probability μ)

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of “measurable” subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space. We will be “concrete”:

(Metric space E, Probability μ)

- Integral $\int_E \varphi \, \mathrm{d}\mu$ for bounded and continuous $\varphi : E \to \mathbb{R}$

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of "measurable" subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space. We will be "concrete":

(Metric space E, Probability μ)

- Integral $\int_E \varphi \, \mathrm{d}\mu$ for bounded and continuous $\varphi : E \to \mathbb{R}$
- Discrete probability $\mu = \sum_i p_i \, \delta_{x_i}$ with $x_i \in E$, $p_i \in [0,1]$

Reminders (I). Probability spaces

Fix a set Ω. A probability P is a map from subsets of Ω to $[0,1]$ s.t.

$$\mathrm{P}(\Omega) = 1\,, \qquad \mathrm{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathrm{P}(A_i) \quad \text{for disjoint } A_i$$

[P is only defined on a subclass (σ-algebra) $\mathcal{A}$ of "measurable" subsets of Ω]

$(\Omega, \mathcal{A}, \mathrm{P})$ is an abstract probability space. We will be "concrete":

(Metric space E, Probability μ)

- Integral $\int_E \varphi \, \mathrm{d}\mu$ for bounded and continuous $\varphi : E \to \mathbb{R}$
- Discrete probability $\mu = \sum_i p_i \, \delta_{x_i}$ with $x_i \in E$, $p_i \in [0,1]$

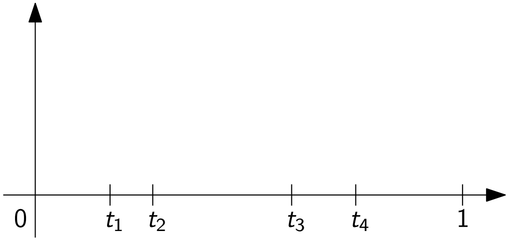
$$\int_E \varphi \, \mathrm{d}\mu := \sum_i p_i \, \varphi(x_i)$$

Riemann sums and integral on $[0, 1]$

Riemann sums and integral on $[0, 1]$

- Partition $\underline{t} = (t_0, t_1, \ldots, t_k)$ of $[0, 1]$

$$0 = t_0 < t_1 < \ldots < t_k = 1 \qquad (k \in \mathbb{N})$$



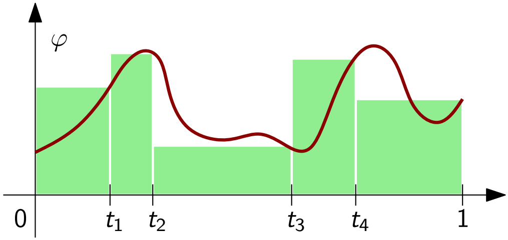
Riemann sums and integral on $[0,1]$

- Partition $\underline{t} = (t_0, t_1, \dots, t_k)$ of $[0,1]$

$$0 = t_0 < t_1 < \ldots < t_k = 1 \qquad (k \in \mathbb{N})$$

- Riemann sum of a function $\varphi : [0,1] \to \mathbb{R}$ relative to $\underline{t}$

$$R(\varphi, \underline{t}) := \sum_{i=1}^{k} \varphi(t_i)\,(t_i - t_{i-1})$$



Riemann sums and integral on $[0,1]$

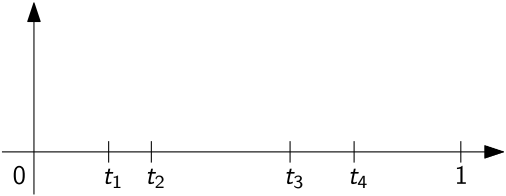
- Partition $\underline{t} = (t_0, t_1, \ldots, t_k)$ of $[0,1]$
$$0 = t_0 < t_1 < \ldots < t_k = 1 \qquad (k \in \mathbb{N})$$
- Riemann sum of a function $\varphi : [0,1] \to \mathbb{R}$ relative to $\underline{t}$
$$R(\varphi, \underline{t}) := \sum_{i=1}^{k} \varphi(t_i)\,(t_i - t_{i-1})$$

Theorem

Let $\underline{t}^{(n)}$ be partitions with
$$\operatorname{mesh}(\underline{t}^{(n)}) := \max_{1 \le i \le k_n} \left(t_i^{(n)} - t_{i-1}^{(n)}\right) \xrightarrow[n\to\infty]{} 0$$
If $\varphi : [0,1] \to \mathbb{R}$ is continuous, then
$$R(\varphi, \underline{t}^{(n)}) \xrightarrow[n\to\infty]{} \int_0^1 \varphi(x)\,\mathrm{d}x$$

A probabilistic reformulation

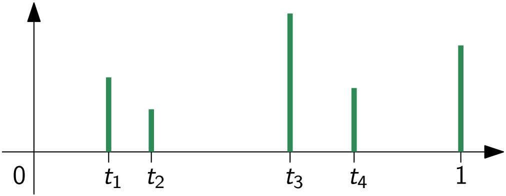
Partition $\underline{t} = (t_0, t_1, \ldots, t_k) \quad \rightsquigarrow \quad$ discrete probability $\mu_{\underline{t}}$ on $[0, 1]$



A probabilistic reformulation

Partition $\underline{t} = \big(t_0, t_1, \ldots, t_k\big) \quad \rightsquigarrow \quad$ discrete probability $\mu_{\underline{t}}$ on $[0,1]$

$$\mu_{\underline{t}}(\cdot) := \sum_{i=1}^{k} p_i \, \delta_{t_i}(\cdot) \qquad \text{where} \qquad p_i := t_i - t_{i-1}$$



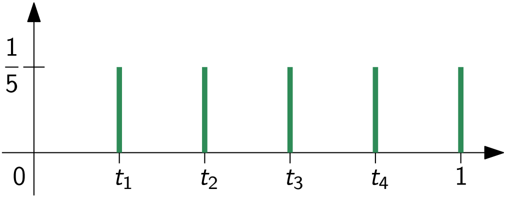
A probabilistic reformulation

Partition $\underline{t} = (t_0, t_1, \ldots, t_k) \quad \rightsquigarrow \quad$ discrete probability $\mu_{\underline{t}}$ on $[0,1]$

$$\mu_{\underline{t}}(\cdot) := \sum_{i=1}^{k} p_i \, \delta_{t_i}(\cdot) \qquad \text{where} \qquad p_i := t_i - t_{i-1}$$

Uniform partition

$\underline{t} = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1) \quad \rightsquigarrow \quad \mu_{\underline{t}} =$ uniform probability on $\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$



A probabilistic reformulation

Key observation: Riemann sum is . . .

$$R(\varphi, \underline{t}) = \sum_{i=1}^{k} \varphi(t_i)\, p_i$$

A probabilistic reformulation

Key observation: Riemann sum is . . . integral w.r.t. $\mu_{\underline{t}}$

$$R(\varphi, \underline{t}) = \sum_{i=1}^{k} \varphi(t_i)\, p_i = \int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}}$$

A probabilistic reformulation

Key observation: Riemann sum is . . . integral w.r.t. $\mu_{\underline{t}}$

$$R(\varphi, \underline{t}) = \sum_{i=1}^{k} \varphi(t_i)\, p_i = \int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}}$$

Theorem

If $\mathrm{mesh}(\underline{t}^{(n)}) \to 0$ and $\varphi : [0,1] \to \mathbb{R}$ is continuous, then

$$\int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}^{(n)}} \xrightarrow[n\to\infty]{} \int_{[0,1]} \varphi \, \mathrm{d}\boldsymbol{\lambda} \qquad (\star)$$

with $\boldsymbol{\lambda} :=$ Lebesgue measure (probability) on $[0,1]$

A probabilistic reformulation

Key observation: Riemann sum is ... integral w.r.t. $\mu_{\underline{t}}$

$$R(\varphi, \underline{t}) = \sum_{i=1}^{k} \varphi(t_i)\, p_i = \int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}}$$

Theorem

If $\mathrm{mesh}(\underline{t}^{(n)}) \to 0$ and $\varphi : [0,1] \to \mathbb{R}$ is continuous, then

$$\int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}^{(n)}} \xrightarrow[n\to\infty]{} \int_{[0,1]} \varphi \, \mathrm{d}\boldsymbol{\lambda} \qquad (\star)$$

with $\boldsymbol{\lambda}$:= Lebesgue measure (probability) on $[0,1]$

- Relation $(\star)$ is a convergence of $\mu_{\underline{t}^{(n)}}$ toward $\boldsymbol{\lambda}$ (Scaling Limit)

A probabilistic reformulation

Key observation: Riemann sum is ... integral w.r.t. $\mu_{\underline{t}}$

$$R(\varphi, \underline{t}) = \sum_{i=1}^{k} \varphi(t_i)\, p_i = \int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}}$$

Theorem

If $\operatorname{mesh}(\underline{t}^{(n)}) \to 0$ and $\varphi : [0,1] \to \mathbb{R}$ is continuous, then

$$\int_{[0,1]} \varphi \, \mathrm{d}\mu_{\underline{t}^{(n)}} \xrightarrow[n\to\infty]{} \int_{[0,1]} \varphi \, \mathrm{d}\boldsymbol{\lambda} \qquad (\star)$$

with $\boldsymbol{\lambda}$:= Lebesgue measure (probability) on $[0,1]$

- Relation $(\star)$ is a convergence of $\mu_{\underline{t}^{(n)}}$ toward $\boldsymbol{\lambda}$ (Scaling Limit)
- Universality: the limit $\boldsymbol{\lambda}$ is the same, for any choice of $\underline{t}^{(n)}$

Weak convergence

- E is a Polish space (complete separable metric space), e.g.

$$[0,1]\,, \qquad C([0,1]) := \{\text{continuous } f : [0,1] \to \mathbb{R}\}\,, \qquad \ldots$$

Weak convergence

- E is a Polish space (complete separable metric space), e.g.

$$[0,1]\,, \qquad C([0,1]) := \{\text{continuous } f : [0,1] \to \mathbb{R}\}\,, \qquad \ldots$$

- $(\mu_n)_{n\in\mathbb{N}}$, μ are probabilities on E

Weak convergence

- E is a Polish space (complete separable metric space), e.g.

$$[0,1]\,, \qquad C([0,1]) := \{\text{continuous } f:[0,1]\to\mathbb{R}\}\,, \qquad \ldots$$

- $(\mu_n)_{n\in\mathbb{N}}$, $\boldsymbol{\mu}$ are probabilities on E

Definition (weak convergence of probabilities)

We say that μ_n converges weakly to $\boldsymbol{\mu}$ (notation $\mu_n \Rightarrow \boldsymbol{\mu}$) if

$$\int_E \varphi \, \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_E \varphi \, \mathrm{d}\boldsymbol{\mu}$$

for every $\varphi \in C_b(E) := \{\text{continuous and bounded } \varphi : E \to \mathbb{R}\}$

Weak convergence

- E is a Polish space (complete separable metric space), e.g.

$$[0,1]\,, \qquad C([0,1]) := \{\text{continuous } f : [0,1] \to \mathbb{R}\}\,, \qquad \ldots$$

- $(\mu_n)_{n\in\mathbb{N}}$, $\boldsymbol{\mu}$ are probabilities on E

Definition (weak convergence of probabilities)

We say that μ_n converges weakly to $\boldsymbol{\mu}$ (notation $\mu_n \Rightarrow \boldsymbol{\mu}$) if

$$\int_E \varphi \, \mathrm{d}\mu_n \xrightarrow[n\to\infty]{} \int_E \varphi \, \mathrm{d}\boldsymbol{\mu}$$

for every $\varphi \in C_b(E) := \{\text{continuous and bounded } \varphi : E \to \mathbb{R}\}$

[Analysts call this weak-$*$ convergence; note that $\mu_n, \boldsymbol{\mu} \in C_b(E)^*$]

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

$\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$?

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$

Example

$\mu_n =$ uniform probability on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$

$\mu_n \Rightarrow \boldsymbol{\lambda}$ (Lebesgue)

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$

Example

$\mu_n =$ uniform probability on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$ $\qquad A := \mathbb{Q} \cap [0,1]$

$\mu_n \Rightarrow \boldsymbol{\lambda}$ (Lebesgue) but $1 = \mu_n(A) \not\to \boldsymbol{\lambda}(A) = 0$

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$

Example

$\mu_n =$ uniform probability on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$ $\qquad A := \mathbb{Q} \cap [0,1]$

$\mu_n \Rightarrow \boldsymbol{\lambda}$ (Lebesgue) but $1 = \mu_n(A) \not\to \boldsymbol{\lambda}(A) = 0$

- Weak convergence means $\mu_n(A) \to \boldsymbol{\mu}(A)$ for "nice" $A \subseteq E$

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$

Example

$\mu_n =$ uniform probability on $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$ $\qquad A := \mathbb{Q} \cap [0,1]$

$\mu_n \Rightarrow \boldsymbol{\lambda}$ (Lebesgue) $\qquad$ but $\qquad 1 = \mu_n(A) \not\to \boldsymbol{\lambda}(A) = 0$

- Weak convergence means $\mu_n(A) \to \boldsymbol{\mu}(A)$ for "nice" $A \subseteq E$

Theorem

$\mu_n \Rightarrow \boldsymbol{\mu}$ iff $\mu_n(A) \to \boldsymbol{\mu}(A)$ $\forall$ meas. $A \subseteq E$ with $\boldsymbol{\mu}(\partial A) = 0$

A useful reformulation of $\mu_n \Rightarrow \boldsymbol{\mu}$

- $\mu_n \Rightarrow \boldsymbol{\mu}$ does not imply $\mu_n(A) \to \boldsymbol{\mu}(A)$ for all meas. $A \subseteq E$

Example

$\mu_n =$ uniform probability on $\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ $\qquad A := \mathbb{Q} \cap [0,1]$

$\mu_n \Rightarrow \boldsymbol{\lambda}$ (Lebesgue) $\quad$ but $\quad 1 = \mu_n(A) \not\to \boldsymbol{\lambda}(A) = 0$

- Weak convergence means $\mu_n(A) \to \boldsymbol{\mu}(A)$ for "nice" $A \subseteq E$

Theorem

$\mu_n \Rightarrow \boldsymbol{\mu}$ iff $\mu_n(A) \to \boldsymbol{\mu}(A)$ $\forall$ meas. $A \subseteq E$ with $\boldsymbol{\mu}(\partial A) = 0$

- Weak convergence links measurable and topological structures

Rest of the talk

Three interesting examples of weak convergence, leading to

- Brownian motion
- Schramm-Löwner Evolution (SLE)
- Continuum disordered pinning models

Rest of the talk

Three interesting examples of weak convergence, leading to

- Brownian motion
- Schramm-Löwner Evolution (SLE)
- Continuum disordered pinning models

Common mathematical structure

- A Polish space E

Rest of the talk

Three interesting examples of weak convergence, leading to

- Brownian motion
- Schramm-Löwner Evolution (SLE)
- Continuum disordered pinning models

Common mathematical structure

- A Polish space E
- A sequence of discrete probabilities μ_n (easy) on E

Rest of the talk

Three interesting examples of weak convergence, leading to

- Brownian motion
- Schramm-Löwner Evolution (SLE)
- Continuum disordered pinning models

Common mathematical structure

- A Polish space E
- A sequence of discrete probabilities μ_n (easy) on E
- A "continuum" probability $\boldsymbol{\mu}$ (difficult!) such that $\mu_n \Rightarrow \boldsymbol{\mu}$

Outline

From random walk to Brownian motion

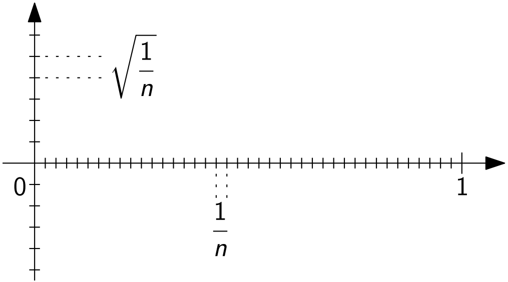
- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\} \quad (\text{with } \|\cdot\|_\infty)$
- $E_n := \Big\{ \qquad \Big\} \subseteq C([0,1])$

From random walk to Brownian motion

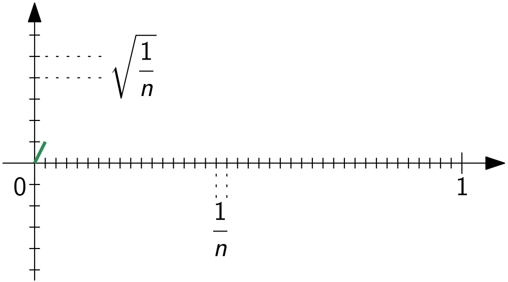
- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \{ \qquad \} \subseteq C([0,1])$



Case $n = 40$

From random walk to Brownian motion

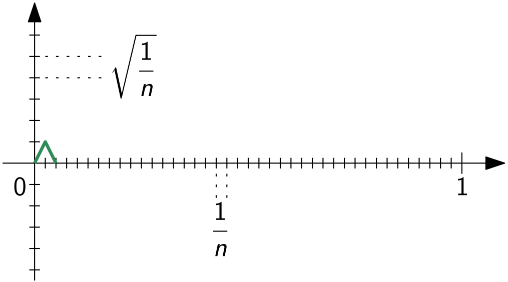
- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \{ \qquad \} \subseteq C([0,1])$



Case $n = 40$

From random walk to Brownian motion

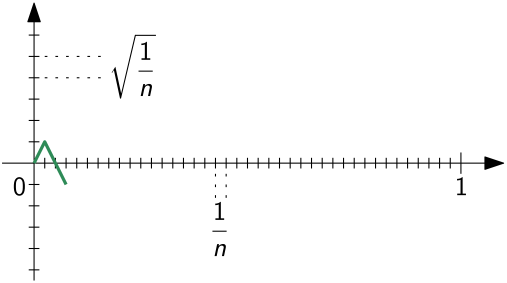
- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{ \qquad \Big\} \subseteq C([0,1])$



Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{ \qquad \Big\} \subseteq C([0,1])$

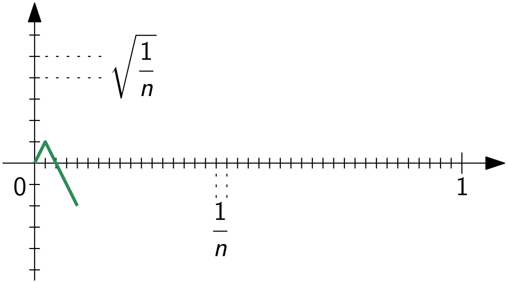


Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{$

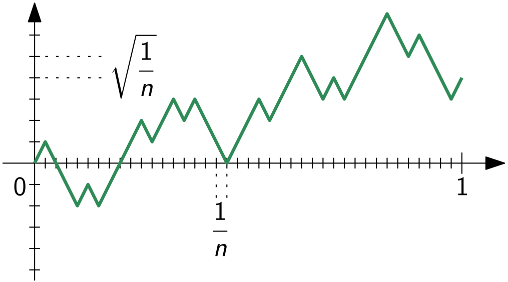
$\Big\} \subseteq C([0,1])$



Case $n = 40$

From random walk to Brownian motion

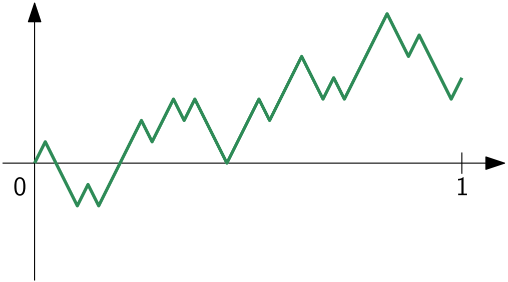
- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Bigg\{ \qquad \Bigg\} \subseteq C([0,1])$



Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{$ piecewise linear $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f\left(\frac{i+1}{n}\right) = f\left(\frac{i}{n}\right) \pm \sqrt{\frac{1}{n}} \Big\} \subseteq C([0,1])$

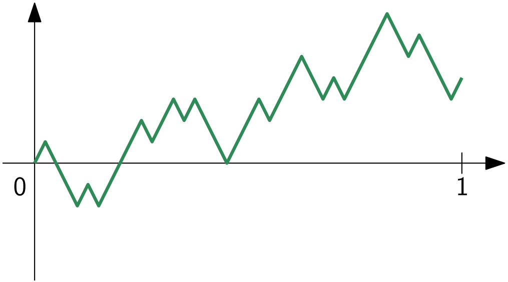


Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \left\{ \text{piecewise linear } f : [0,1] \to \mathbb{R} \text{ with } f(0) = 0 \text{ and } f\left(\tfrac{i+1}{n}\right) = f\left(\tfrac{i}{n}\right) \pm \sqrt{\tfrac{1}{n}} \right\} \subseteq C([0,1])$

  $|E_n| = 2^n$

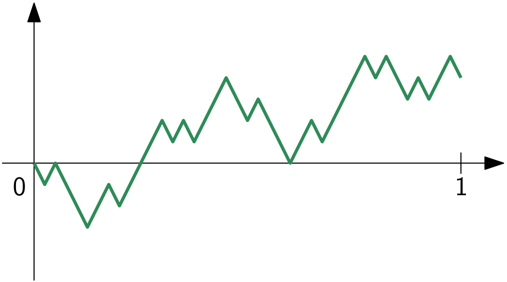


Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{$ piecewise linear $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f\left(\frac{i+1}{n}\right) = f\left(\frac{i}{n}\right) \pm \sqrt{\frac{1}{n}} \Big\} \subseteq C([0,1])$

  $|E_n| = 2^n$

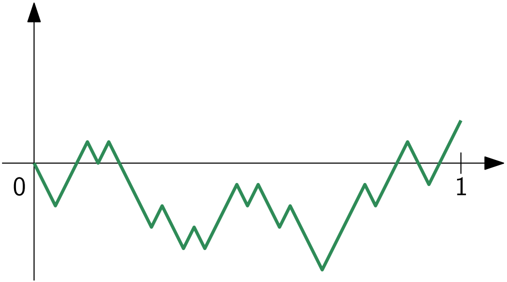


Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \Big\{$ piecewise linear $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f\big(\frac{i+1}{n}\big) = f\big(\frac{i}{n}\big) \pm \sqrt{\frac{1}{n}} \Big\} \subseteq C([0,1])$

  $|E_n| = 2^n$



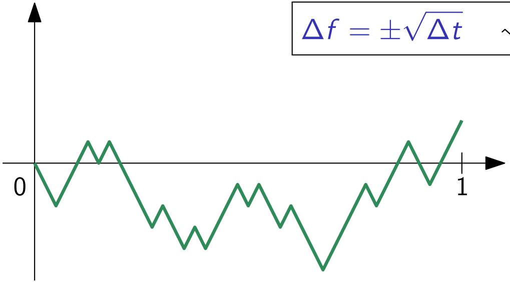
Case $n = 40$

From random walk to Brownian motion

- $E := C([0,1]) = \{\text{continuous } f : [0,1] \to \mathbb{R}\}$ (with $\|\cdot\|_\infty$)
- $E_n := \left\{ \text{piecewise linear } f : [0,1] \to \mathbb{R} \text{ with } f(0) = 0 \text{ and } f\left(\frac{i+1}{n}\right) = f\left(\frac{i}{n}\right) \pm \sqrt{\frac{1}{n}} \right\} \subseteq C([0,1])$

  $|E_n| = 2^n$

$$\boxed{\Delta f = \pm\sqrt{\Delta t} \quad \rightsquigarrow \quad \text{slope}(f) = \pm\sqrt{n}}$$



Case $n = 40$

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \, \delta_f(\cdot)$$

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \, \delta_f(\cdot)$$

Theorem (Donsker)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on $C([0,1])$: $\mu_n \Rightarrow \boldsymbol{\mu}$

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \, \delta_f(\cdot)$$

Theorem (Donsker)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on $C([0,1])$: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ on $C([0,1])$ is called Wiener measure

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \delta_f(\cdot)$$

Theorem (Donsker)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on $C([0,1])$: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ on $C([0,1])$ is called Wiener measure

- Deep result!

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \, \delta_f(\cdot)$$

Theorem (Donsker)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on $C([0,1])$: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ on $C([0,1])$ is called Wiener measure

- Deep result!
- Wiener measure is the law of Brownian motion

From random walk to Brownian motion

Let μ_n be the probability on $C([0,1])$ which is uniform on E_n:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \, \delta_f(\cdot)$$

Theorem (Donsker)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on $C([0,1])$: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ on $C([0,1])$ is called Wiener measure

- Deep result!
- Wiener measure is the law of Brownian motion
- Wiener measure is a "natural" probability on $C([0,1])$ (like Lebesgue for $[0,1]$)

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

- X describes a random element of E

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

The law (or distribution) μ_X of X is a probability on E

$$\mu_X(A) = \mathrm{P}(X^{-1}(A)) = \mathrm{P}(X \in A) \qquad \text{for } A \subseteq E$$

- X describes a random element of E

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

The law (or distribution) μ_X of X is a probability on E

$$\mu_X(A) = \mathrm{P}(X^{-1}(A)) = \mathrm{P}(X \in A) \qquad \text{for } A \subseteq E$$

- X describes a random element of E
- μ_X describes the values taken by X and the resp. probabilities

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

The law (or distribution) μ_X of X is a probability on E

$$\mu_X(A) = \mathrm{P}(X^{-1}(A)) = \mathrm{P}(X \in A) \qquad \text{for } A \subseteq E$$

- X describes a random element of E
- μ_X describes the values taken by X and the resp. probabilities

Instead of a probability μ on E, it is often convenient to work with a random variable X with law μ

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

The law (or distribution) μ_X of X is a probability on E

$$\mu_X(A) = \mathrm{P}(X^{-1}(A)) = \mathrm{P}(X \in A) \qquad \text{for } A \subseteq E$$

- X describes a random element of E
- μ_X describes the values taken by X and the resp. probabilities

Instead of a probability μ on E, it is often convenient to work with a random variable X with law μ

When $E = C([0,1])$, a r.v. X is a stochastic process

Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$

[where $(\Omega, \mathcal{A}, \mathrm{P})$ is some abstract probability space]

The law (or distribution) μ_X of X is a probability on E

$$\mu_X(A) = \mathrm{P}(X^{-1}(A)) = \mathrm{P}(X \in A) \qquad \text{for } A \subseteq E$$

- X describes a random element of E
- μ_X describes the values taken by X and the resp. probabilities

Instead of a probability μ on E, it is often convenient to work with a random variable X with law μ

When $E = C([0,1])$, a r.v. $X = (X_t)_{t\in[0,1]}$ is a stochastic process

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$\mathrm{P}(Y_i = +1) = \mathrm{P}(Y_i = -1) = \frac{1}{2}$$

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$\mathrm{P}(Y_i = +1) = \mathrm{P}(Y_i = -1) = \frac{1}{2}$$

Simple random walk: $S_0 := 0 \qquad S_n := Y_1 + Y_2 + \ldots + Y_n$

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$\mathrm{P}(Y_i = +1) = \mathrm{P}(Y_i = -1) = \frac{1}{2}$$

Simple random walk: $\quad S_0 := 0 \qquad S_n := Y_1 + Y_2 + \ldots + Y_n$

Diffusive rescaling: space $\propto \sqrt{\text{time}}$

$$X^{(n)}(t) := \text{linear interpol. of } \frac{S_{nt}}{\sqrt{n}} \qquad t \in [0,1]$$

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$\mathrm{P}(Y_i = +1) = \mathrm{P}(Y_i = -1) = \frac{1}{2}$$

Simple random walk: $S_0 := 0 \qquad S_n := Y_1 + Y_2 + \ldots + Y_n$

Diffusive rescaling: space $\propto \sqrt{\text{time}}$

$$X^{(n)}(t) := \text{linear interpol. of } \frac{S_{nt}}{\sqrt{n}} \qquad t \in [0,1]$$

The law of $X^{(n)}$ (r.v. in $C([0,1])$) is μ_n uniform probab. on E_n

Simple random walk

Let us build a stochastic process $X^{(n)}$ with law μ_n

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$P(Y_i = +1) = P(Y_i = -1) = \frac{1}{2}$$

Simple random walk: $S_0 := 0 \qquad S_n := Y_1 + Y_2 + \ldots + Y_n$

Diffusive rescaling: space $\propto \sqrt{\text{time}}$

$$X^{(n)}(t) := \text{linear interpol. of } \frac{S_{nt}}{\sqrt{n}} \qquad t \in [0,1]$$

The law of $X^{(n)}$ (r.v. in $C([0,1])$) is μ_n uniform probab. on E_n

Donsker: The law of simple random walk, diffusively rescaled, converges weakly to the law of Brownian motion

General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

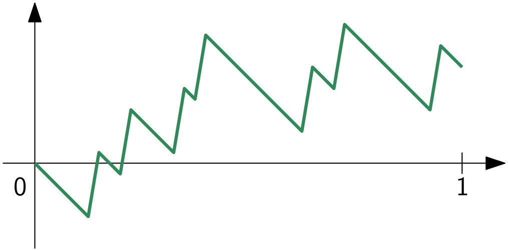
Define random walk S_n and its diffusive rescaling $X^{(n)}(t)$ as before

General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

Define random walk S_n and its diffusive rescaling $X^{(n)}(t)$ as before

E.g. $\quad \mathrm{P}(Y_i = +2) = \frac{1}{3}, \quad \mathrm{P}(Y_i = -1) = \frac{2}{3}$

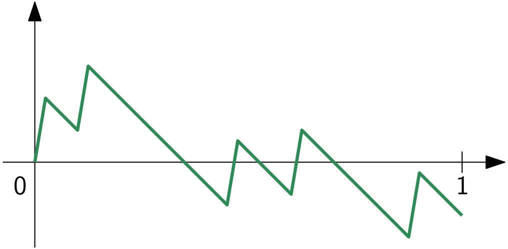


General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

Define random walk S_n and its diffusive rescaling $X^{(n)}(t)$ as before

$$\text{E.g.} \qquad P(Y_i = +2) = \frac{1}{3}, \quad P(Y_i = -1) = \frac{2}{3}$$

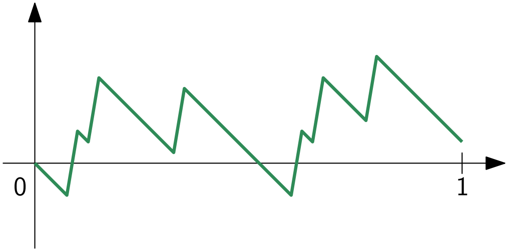


General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

Define random walk S_n and its diffusive rescaling $X^{(n)}(t)$ as before

$$\text{E.g.} \qquad \mathrm{P}(Y_i = +2) = \frac{1}{3}, \quad \mathrm{P}(Y_i = -1) = \frac{2}{3}$$

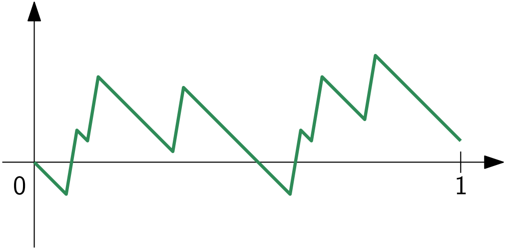


General random walks

Instead of coin tossing, take independent random variables Y_i with a generic law, with zero mean and finite variance (say 1)

Define random walk S_n and its diffusive rescaling $X^{(n)}(t)$ as before

$$\text{E.g.} \qquad \mathrm{P}(Y_i = +2) = \frac{1}{3}, \quad \mathrm{P}(Y_i = -1) = \frac{2}{3}$$



The law μ_n of $X^{(n)}$ is a (non uniform!) probability on $C([0,1])$

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} := \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} := \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality: $\mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} \ := \ \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality: $\mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Example (Feller I, Chapter III)

- $T_+(f) := \text{Leb}\{t \in [0,1] : f(t) > 0\}$
  $= \{\text{amount of time in which } f > 0\}$

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} := \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality: $\quad \mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Example (Feller I, Chapter III)

- $T_+(f) := \mathrm{Leb}\{t \in [0,1] : f(t) > 0\}$
  $= \{\text{amount of time in which } f > 0\}$
- $A := \big\{ f : \; T_+(f) \geq 0.95 \;\text{ or }\; T_+(f) \leq 0.05 \big\} \subseteq C([0,1])$

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} := \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality: $\mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Example (Feller I, Chapter III)

- $T_+(f) := \mathrm{Leb}\{t \in [0,1] : f(t) > 0\}$
  $= \{\text{amount of time in which } f > 0\}$
- $A := \big\{ f : \; T_+(f) \geq 0.95 \;\text{ or }\; T_+(f) \leq 0.05 \big\} \subseteq C([0,1])$

Then $\mu_n(A) \to \boldsymbol{\mu}(A)$

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} \;:=\; \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality: $\quad \mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Example (Feller I, Chapter III)

- $T_+(f) := \text{Leb}\{t \in [0,1] : f(t) > 0\}$
  $\qquad\qquad = \{\text{amount of time in which } f > 0\}$
- $A := \big\{ f : \; T_+(f) \geq 0.95 \;\text{ or }\; T_+(f) \leq 0.05 \big\} \subseteq C([0,1])$

Then $\mu_n(A) \to \boldsymbol{\mu}(A) \simeq \mathbf{0.29}$. Random walk has a chance of 29% of spending more than 95% of its time on the same side of the axis!

Universality of Brownian motion

Theorem (Donsker)

$$\mu_n \Rightarrow \boldsymbol{\mu} := \text{Wiener measure}$$

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

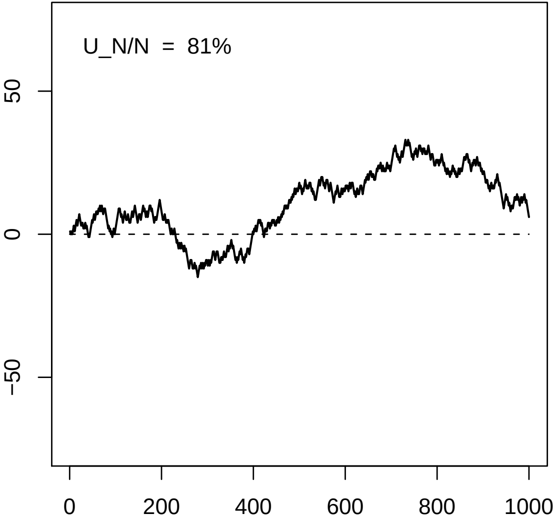
Universality: $\mu_n(A) \longrightarrow \boldsymbol{\mu}(A) \quad \forall A \subseteq C([0,1])$ with $\boldsymbol{\mu}(\partial A) = 0$

Example (Feller I, Chapter III)

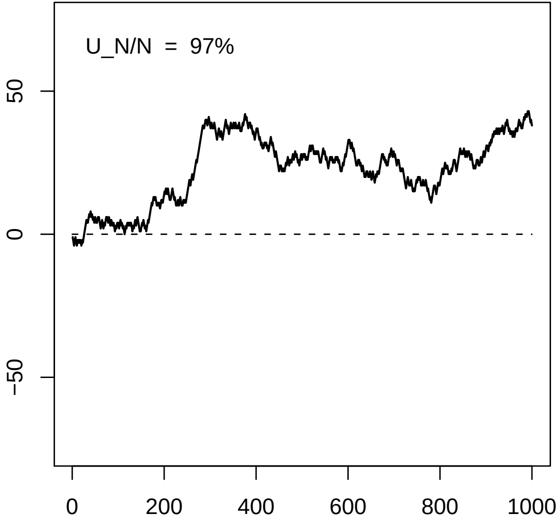
- $T_+(f) := \mathrm{Leb}\{t \in [0,1] : f(t) > 0\}$
  $= \{\text{amount of time in which } f > 0\}$
- $A := \{f : \; T_+(f) \geq 0.95 \text{ or } T_+(f) \leq 0.05\} \subseteq C([0,1])$

Then $\mu_n(A) \to \boldsymbol{\mu}(A) \simeq \mathbf{0.29}$. Random walk has a chance of 13% of spending more than 99% of its time on the same side of the axis!

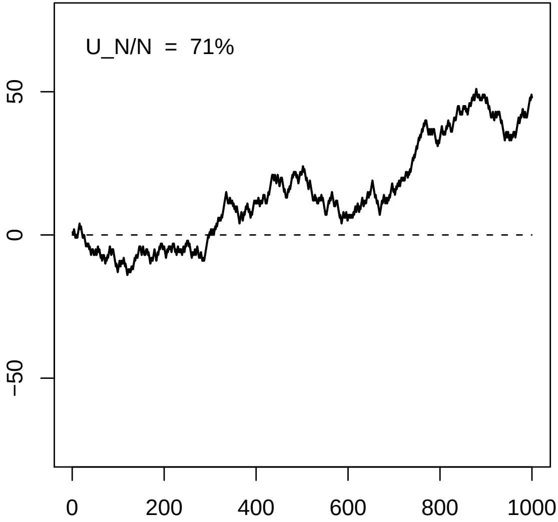
Some sample paths of the SRW



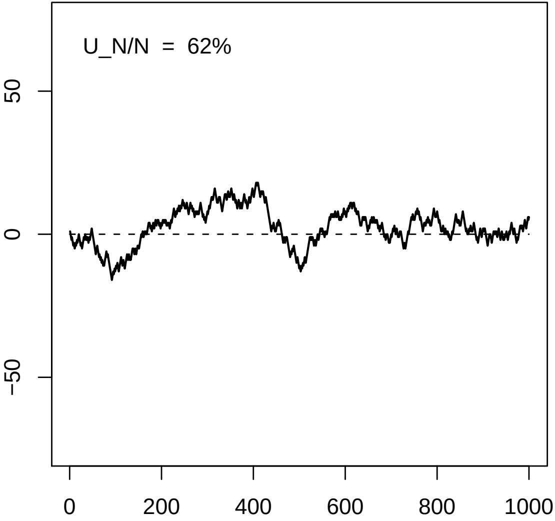
Some sample paths of the SRW



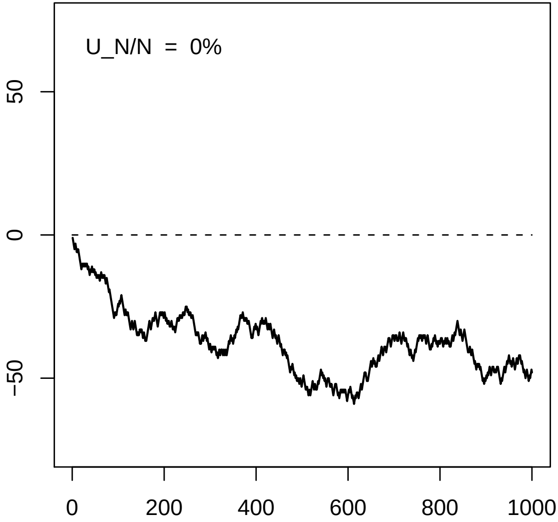
Some sample paths of the SRW



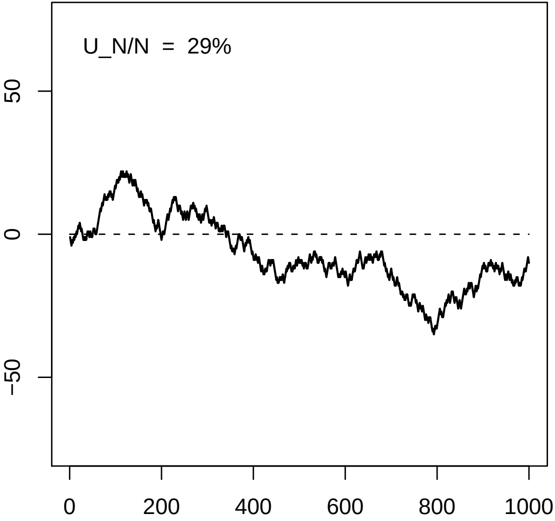
Some sample paths of the SRW



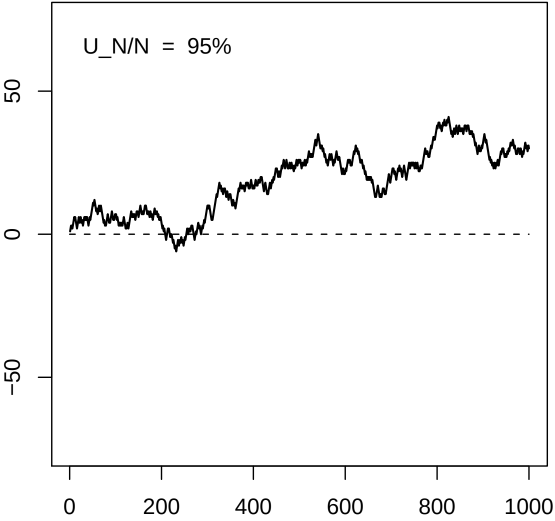
Some sample paths of the SRW



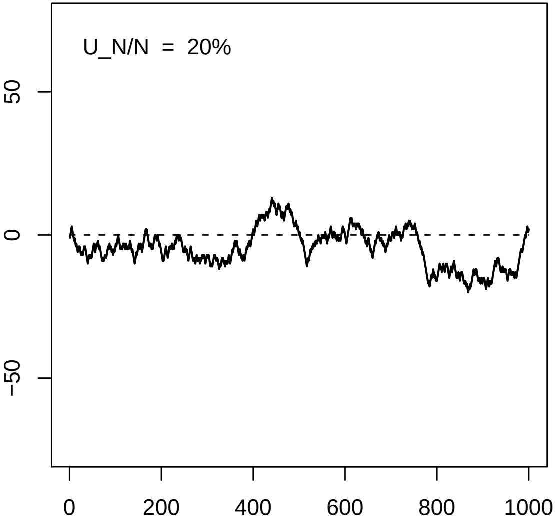
Some sample paths of the SRW



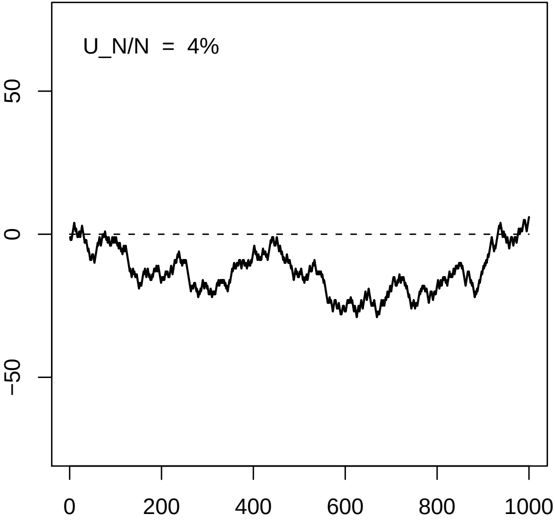
Some sample paths of the SRW



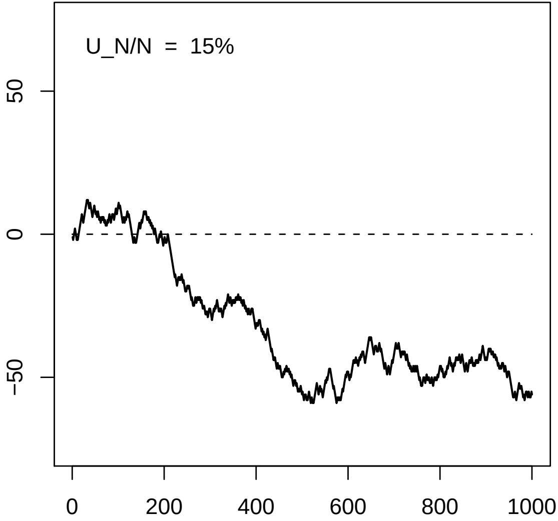
Some sample paths of the SRW



Some sample paths of the SRW



Some sample paths of the SRW



Outline

1. Weak Convergence of Probability Measures

2. Brownian Motion

3. A glimpse of SLE

4. Scaling Limits in presence of Disorder

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

We present an instance of SLE, which emerges as the scaling limit of percolation (spatial version of coin tossing)

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

We present an instance of SLE, which emerges as the scaling limit of percolation (spatial version of coin tossing)

Fix a simply connected Jordan domain $D \subseteq \mathbb{R}^2$ and $A, B \in \partial D$

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

We present an instance of SLE, which emerges as the scaling limit of percolation (spatial version of coin tossing)

Fix a simply connected Jordan domain $D \subseteq \mathbb{R}^2$ and $A, B \in \partial D$

$$
\begin{aligned}
E &:= \left\{ \text{continuous } f : [0,1] \to \overline{D} \text{ with } f(0) = A,\ f(1) = B \right\} \\
&= \left\{ \text{curves in } \overline{D} \text{ joining } A \text{ to } B \right\} \qquad [\|\cdot\|_\infty \text{ norm, up to reparam.}]
\end{aligned}
$$

A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

We present an instance of SLE, which emerges as the scaling limit of percolation (spatial version of coin tossing)

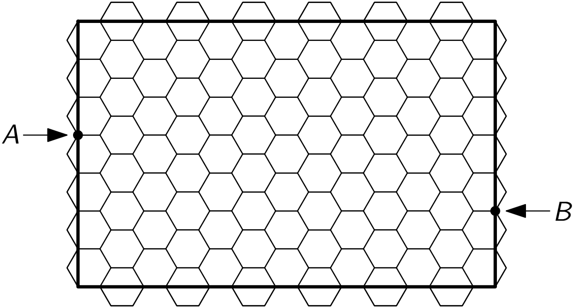
Fix a simply connected Jordan domain $D \subseteq \mathbb{R}^2$ and $A, B \in \partial D$

$$E := \left\{\text{continuous } f : [0,1] \to \overline{D} \text{ with } f(0) = A,\ f(1) = B\right\}$$
$$= \left\{\text{curves in } \overline{D} \text{ joining } A \text{ to } B\right\} \qquad [\|\cdot\|_\infty \text{ norm, up to reparam.}]$$

We now introduce discrete probabilities μ_n on E

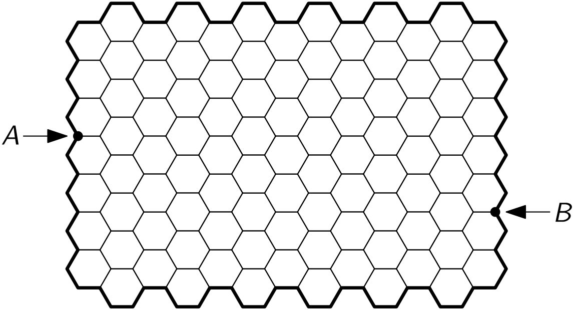
1. The rescaled hexagonal lattice

1. The rescaled hexagonal lattice



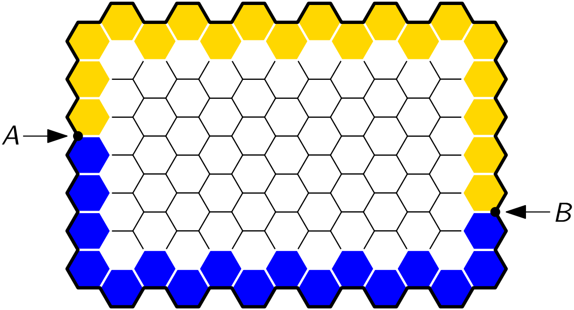
- Fix $n \in \mathbb{N}$ and consider the hexagonal lattice of side $\frac{1}{n}$

1. The rescaled hexagonal lattice



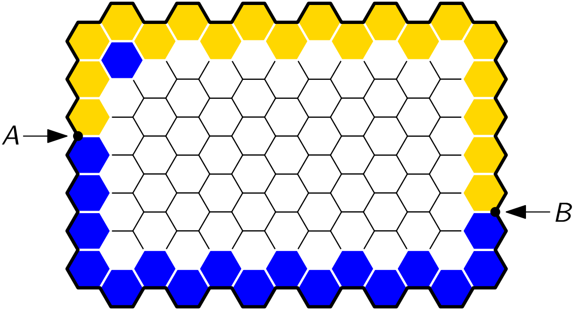
- Fix $n \in \mathbb{N}$ and consider the hexagonal lattice of side $\frac{1}{n}$
- Approximate ∂D with a closed loop in the lattice

2. Percolation



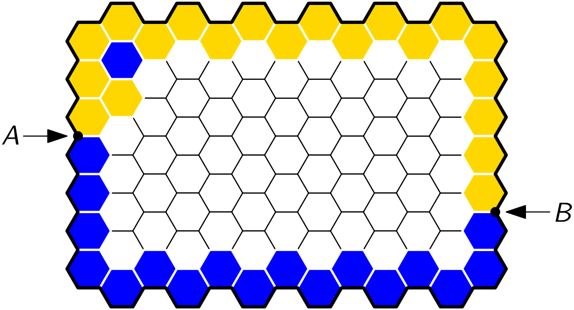
- Boundary hexagons colored yellow (A to B) and blue (B to A)

2. Percolation



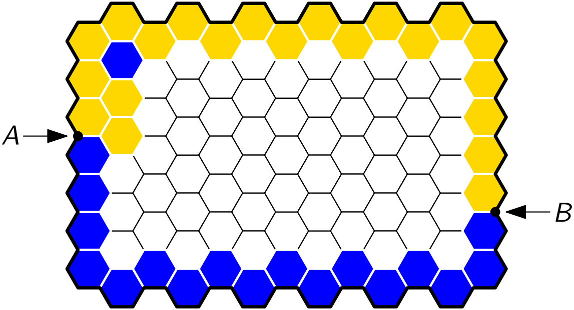
- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)

2. Percolation



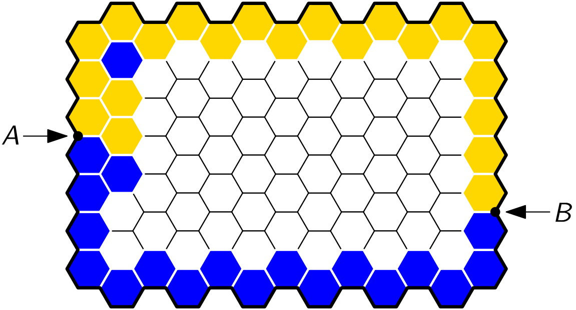
- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)

2. Percolation



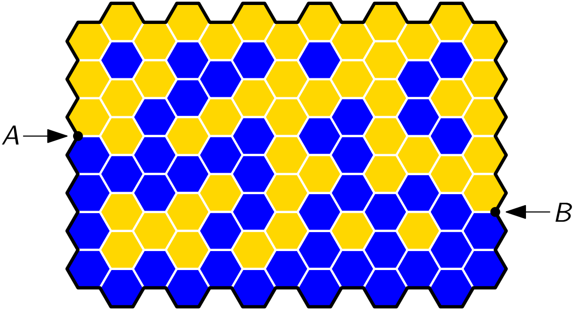
- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)

2. Percolation



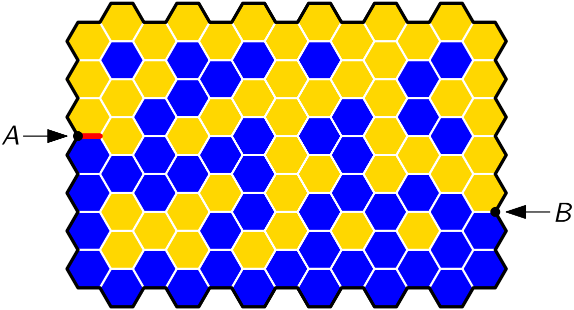
- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)

2. Percolation



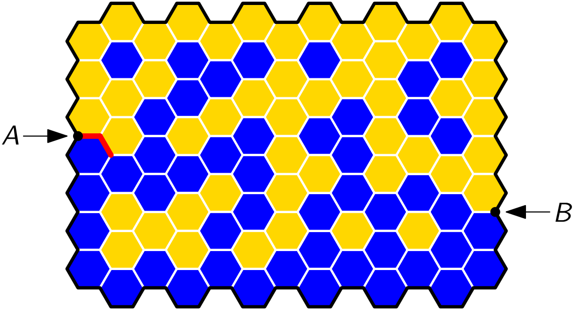
- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)

3. The exploration path



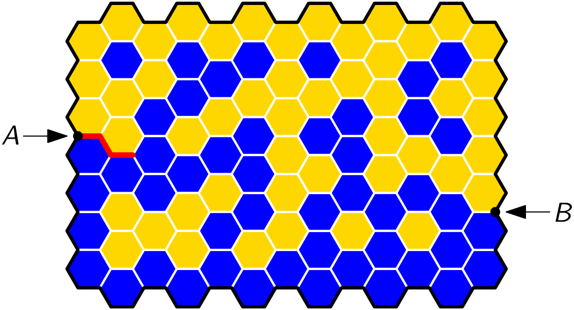
- Exploration path: start from A and follow the boundary between yellow and blue hexagons, eventually leading to B

3. The exploration path



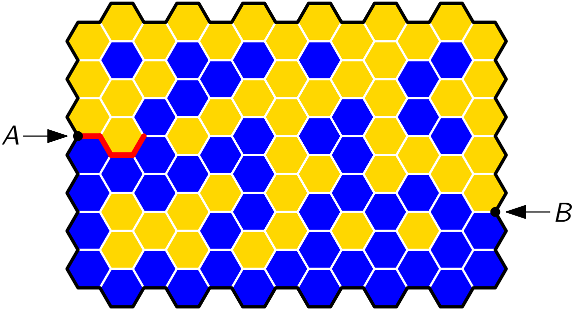
- Exploration path: start from A and follow the boundary between yellow and blue hexagons, eventually leading to B

3. The exploration path



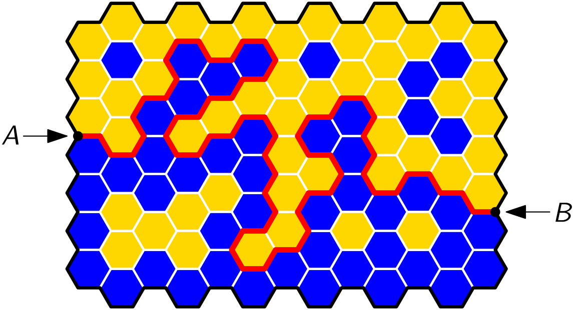
- Exploration path: start from A and follow the boundary between yellow and blue hexagons, eventually leading to B

3. The exploration path



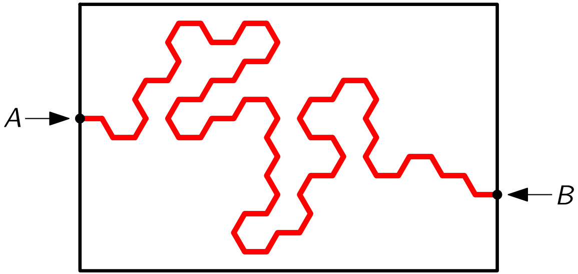
- Exploration path: start from A and follow the boundary between yellow and blue hexagons, eventually leading to B

3. The exploration path



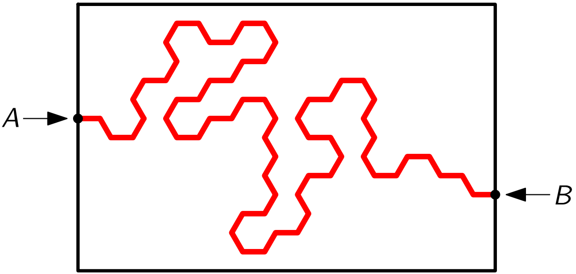
- Exploration path: start from A and follow the boundary between yellow and blue hexagons, eventually leading to B

4. The law μ_n



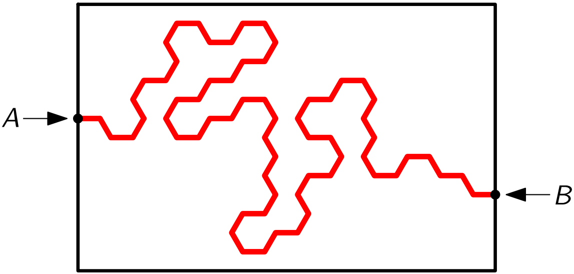
- Forgetting the colors, the exploration path is an element of E (continuous curve $A \to B$)

4. The law μ_n



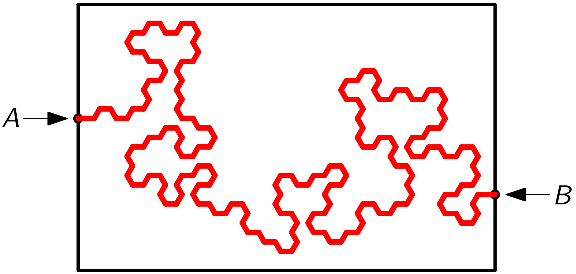
- Forgetting the colors, the exploration path is an element of E (continuous curve $A \to B$)
- It is a random element of E (determined by coin tossing)

4. The law μ_n



- Forgetting the colors, the exploration path is an element of E (continuous curve $A \to B$)
- It is a random element of E (determined by coin tossing)
- Its law μ_n is a discrete probability on E $\left(\frac{1}{n} = \text{lattice mesh}\right)$

4. The law μ_n



- Forgetting the colors, the exploration path is an element of E (continuous curve $A \to B$)
- It is a random element of E (determined by coin tossing)
- Its law μ_n is a discrete probability on E $\;(\frac{1}{n} = \text{lattice mesh})$

Scaling limit of the exploration path

Fix a (simply connected) Jordan domain D and points $A, B \in \partial D$

$$E := \left\{ \text{curves in } \overline{D} \text{ joining } A \text{ to } B \right\}$$

Scaling limit of the exploration path

Fix a (simply connected) Jordan domain D and points $A, B \in \partial D$

$$E := \left\{ \text{curves in } \overline{D} \text{ joining } A \text{ to } B \right\}$$

Theorem (Schramm; Smirnov; Camia & Newman)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on E: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ is the law of (the trace of) SLE(6)

Scaling limit of the exploration path

Fix a (simply connected) Jordan domain D and points $A, B \in \partial D$

$$E := \{\text{curves in } \overline{D} \text{ joining } A \text{ to } B\}$$

Theorem (Schramm; Smirnov; Camia & Newman)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on E: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ is the law of (the trace of) SLE(6)

- Extremely challenging!

Scaling limit of the exploration path

Fix a (simply connected) Jordan domain D and points $A, B \in \partial D$

$$E := \{\text{curves in } \overline{D} \text{ joining } A \text{ to } B\}$$

Theorem (Schramm; Smirnov; Camia & Newman)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on E: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ is the law of (the trace of) SLE(6)

- Extremely challenging!
- Universality? Independence of the lattice (conjecture)

Scaling limit of the exploration path

Fix a (simply connected) Jordan domain D and points $A, B \in \partial D$

$$E := \{\text{curves in } \overline{D} \text{ joining } A \text{ to } B\}$$

Theorem (Schramm; Smirnov; Camia & Newman)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly on E: $\mu_n \Rightarrow \boldsymbol{\mu}$

The limiting probability $\boldsymbol{\mu}$ is the law of (the trace of) SLE(6)

- Extremely challenging!
- Universality? Independence of the lattice (conjecture)
- Conformal Invariance. For another Jordan domain D'

$$\boldsymbol{\mu}_{D';A',B'} = \phi_{\#}(\boldsymbol{\mu}_{D;A,B})$$

where $\phi : D \to D'$ is conformal with $\phi(A) = A'$, $\phi(B) = B'$

Outline

1. Weak Convergence of Probability Measures

2. Brownian Motion

3. A glimpse of SLE

4. Scaling Limits in presence of Disorder

From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

From simple to Bessel random walk

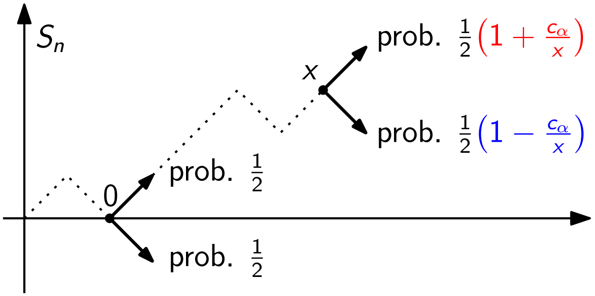
The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0,1)$ and define the α-Bessel random walk as follows:

From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0,1)$ and define the α-Bessel random walk as follows:

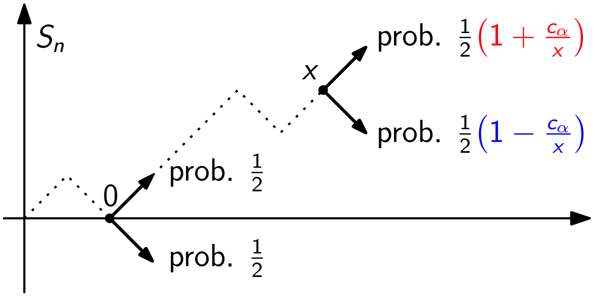


$\boxed{c_\alpha := \frac{1}{2} - \alpha}$

From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0,1)$ and define the α-Bessel random walk as follows:



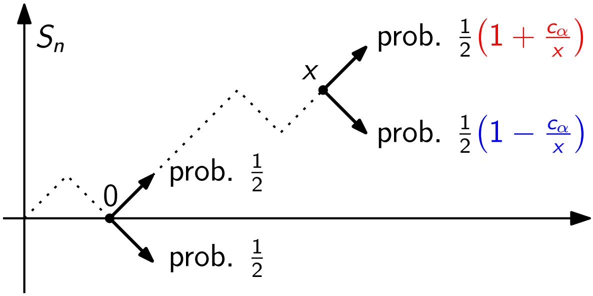
$$\boxed{c_\alpha := \frac{1}{2} - \alpha}$$

- ($\alpha = \frac{1}{2}$) no drift ($c_\alpha = 0$) $\rightsquigarrow$ simple random walk

From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0,1)$ and define the α-Bessel random walk as follows:



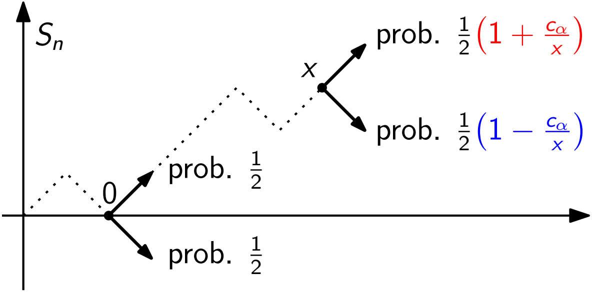
$$\boxed{c_\alpha := \tfrac{1}{2} - \alpha}$$

- $(\alpha = \frac{1}{2})$ no drift $(c_\alpha = 0)$ $\rightsquigarrow$ simple random walk
- $(\alpha < \frac{1}{2})$ drift away from the origin $(c_\alpha > 0)$

From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0,1)$ and define the α-Bessel random walk as follows:



$$\boxed{c_\alpha := \frac{1}{2} - \alpha}$$

- $(\alpha = \frac{1}{2})$ no drift $(c_\alpha = 0)$ $\rightsquigarrow$ simple random walk
- $(\alpha < \frac{1}{2})$ drift away from the origin $(c_\alpha > 0)$
- $(\alpha > \frac{1}{2})$ drift toward the origin $(c_\alpha < 0)$

Diffusively rescaled α-Bessel RW

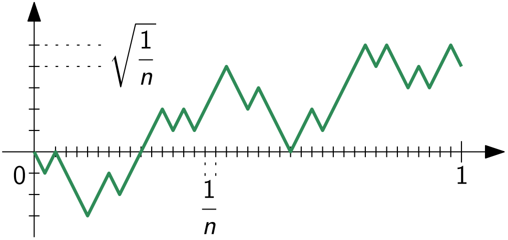
Definition

$\mu_{n,\alpha}$:= law of diffusively rescaled α-Bessel RW

Diffusively rescaled α-Bessel RW

Definition

$\mu_{n,\alpha}$:= law of diffusively rescaled α-Bessel RW

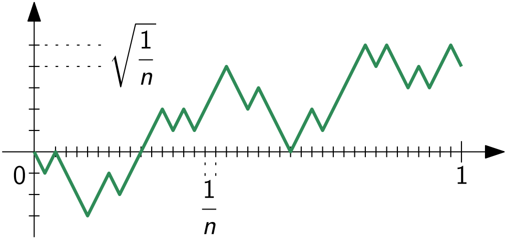


Discrete probability on $E_n \subseteq C([0,1])$

Diffusively rescaled α-Bessel RW

Definition

$\mu_{n,\alpha}$:= law of diffusively rescaled α-Bessel RW



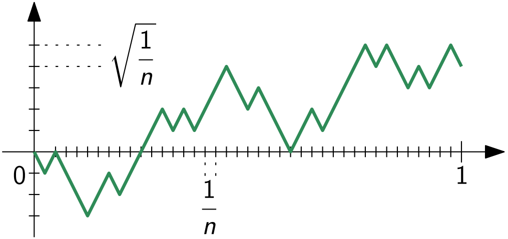
Discrete probability on $E_n \subseteq C([0,1])$

Not uniform for $\alpha \neq \frac{1}{2}$

Diffusively rescaled α-Bessel RW

Definition

$\mu_{n,\alpha}$:= law of diffusively rescaled α-Bessel RW



Discrete probability on $E_n \subseteq C([0,1])$

Not uniform for $\alpha \neq \frac{1}{2}$

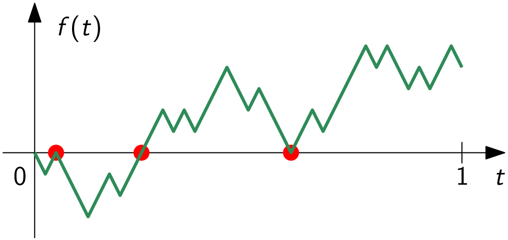
Theorem (Extension of Donsker)

$\forall \alpha \in (0,1)$, $\mu_{n,\alpha}$ converges weakly on $C([0,1])$: $\mu_{n,\alpha} \Rightarrow \boldsymbol{\mu}_\alpha$

$\big[\, \boldsymbol{\mu}_\alpha$:= law of "α-Bessel process" (Brownian motion for $\alpha = \frac{1}{2}$) $\big]$

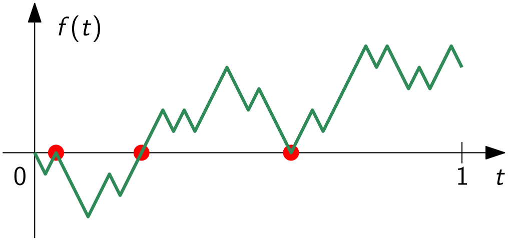
The disordered pinning model

Idea: reward/penalize α-Bessel RW $\mu_{n,\alpha}$ each time it visits zero



The disordered pinning model

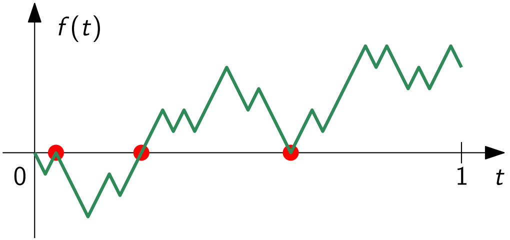
Idea: reward/penalize α-Bessel RW $\mu_{n,\alpha}$ each time it visits zero



- Fix a real sequence $\omega = (\omega_i)_{i \in \mathbb{N}}$ (charges attached to $t = \frac{i}{n}$)

The disordered pinning model

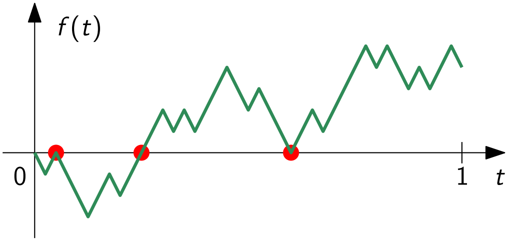
Idea: reward/penalize α-Bessel RW $\mu_{n,\alpha}$ each time it visits zero



- Fix a real sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ (charges attached to $t = \frac{i}{n}$)
- Total charge (energy) of a path $H_n^\omega(f) := \sum_{i=1}^n \omega_i \, \mathbb{1}_{\{f(\frac{i}{n})=0\}}$

The disordered pinning model

Idea: reward/penalize α-Bessel RW $\mu_{n,\alpha}$ each time it visits zero



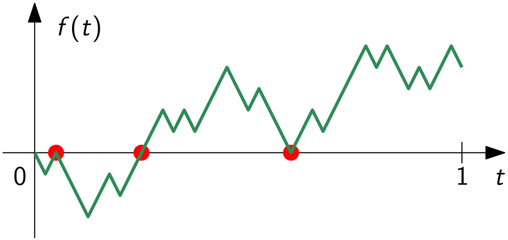
- Fix a real sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ (charges attached to $t = \frac{i}{n}$)
- Total charge (energy) of a path $H_n^\omega(f) := \sum_{i=1}^n \omega_i \, \mathbb{1}_{\{f(\frac{i}{n})=0\}}$

Disordered pinning model $\mu_{n,\alpha}^\omega$ (Gibbs measure)

$$\mu_{n,\alpha}^\omega(f) := \qquad e^{H_n^\omega(f)} \, \mu_{n,\alpha}(f) \,, \qquad \forall f \in E_n$$

The disordered pinning model

Idea: reward/penalize α-Bessel RW $\mu_{n,\alpha}$ each time it visits zero



- Fix a real sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ (charges attached to $t = \frac{i}{n}$)
- Total charge (energy) of a path $H_n^\omega(f) := \sum_{i=1}^n \omega_i \, \mathbb{1}_{\{f(\frac{i}{n})=0\}}$

Disordered pinning model $\mu_{n,\alpha}^\omega$ (Gibbs measure)

$$\mu_{n,\alpha}^\omega(f) := \frac{1}{\text{(normaliz.)}} \, e^{H_n^\omega(f)} \, \mu_{n,\alpha}(f) \,, \qquad \forall f \in E_n$$

The disordered pinning model

$\mu_{n,\alpha}^{\omega}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω ?

The disordered pinning model

$\mu_{n,\alpha}^{\omega}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω? In a random way!

$(\omega_i)_{i\in\mathbb{N}}$ independent $\mathcal{N}(h,\beta^2)$ [mean $h \in \mathbb{R}$, variance $\beta^2 > 0$]

The disordered pinning model

$\mu_{n,\alpha}^{\omega}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω ? In a random way!

$(\omega_i)_{i\in\mathbb{N}}$ independent $\mathcal{N}(h,\beta^2)$ [mean $h \in \mathbb{R}$, variance $\beta^2 > 0$]

Disordered systems: two sources of randomness!

- First we sample a typical ω, called (quenched) disorder

The disordered pinning model

$\mu^{\omega}_{n,\alpha}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω ? In a random way!

$(\omega_i)_{i\in\mathbb{N}}$ independent $\mathcal{N}(h,\beta^2)$ [mean $h \in \mathbb{R}$, variance $\beta^2 > 0$]

Disordered systems: two sources of randomness!

- First we sample a typical ω, called (quenched) disorder
- Then we have a probability $\mu^{\omega}_{n,\alpha}$ on the space E_n of RW paths

The disordered pinning model

$\mu_{n,\alpha}^{\omega}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω? In a random way!

$(\omega_i)_{i\in\mathbb{N}}$ independent $\mathcal{N}(h, \beta^2)$ [mean $h \in \mathbb{R}$, variance $\beta^2 > 0$]

Disordered systems: two sources of randomness!

- First we sample a typical ω, called (quenched) disorder
- Then we have a probability $\mu_{n,\alpha}^{\omega}$ on the space E_n of RW paths

The disordered pinning model $\mu_{n,\alpha}^{\omega}$ is a random probability on E_n

[i.e. a random variable $\omega \mapsto \mu_{n,\alpha}^{\omega}$ taking values in $\mathcal{M}_1(E_n)$]

The disordered pinning model

$\mu_{n,\alpha}^{\omega}$ is a probability on $C([0,1])$ that depends on the sequence ω

How to choose the charges ω? In a random way!

$(\omega_i)_{i\in\mathbb{N}}$ independent $\mathcal{N}(h,\beta^2)$ [mean $h \in \mathbb{R}$, variance $\beta^2 > 0$]

Disordered systems: two sources of randomness!

- First we sample a typical ω, called (quenched) disorder
- Then we have a probability $\mu_{n,\alpha}^{\omega}$ on the space E_n of RW paths

The disordered pinning model $\mu_{n,\alpha}^{\omega}$ is a random probability on E_n

[i.e. a random variable $\omega \mapsto \mu_{n,\alpha}^{\omega}$ taking values in $\mathcal{M}_1(E_n)$]

Weak convergence of $\mu_{n,\alpha}^{\omega}$ [of its law] to some random probab. $\boldsymbol{\mu}_{\alpha}^{\omega}$?

Scaling limits of disordered pinning model

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

Scaling limits of disordered pinning model

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

- $(\alpha < \frac{1}{2})$ Disorder disappears in the scaling limit!

$$\mu_{n,\alpha}^{\omega} \Rightarrow \boldsymbol{\mu}_{\alpha} \quad \text{law of } \alpha\text{-Bessel process} \quad (\text{as if } \omega \equiv 0)$$

Scaling limits of disordered pinning model

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

- $(\alpha < \frac{1}{2})$ Disorder disappears in the scaling limit!

  $\mu_{n,\alpha}^{\omega} \Rightarrow \boldsymbol{\mu}_{\alpha}$ law of α-Bessel process (as if $\omega \equiv 0$)

- $(\alpha > \frac{1}{2})$ Disorder survives in the scaling limit!

  $\mu_{n,\alpha}^{\omega} \Rightarrow \boldsymbol{\mu}_{\alpha}^{\omega}$ truly random probability on $C([0,1])$

Scaling limits of disordered pinning model

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

- $(\alpha < \frac{1}{2})$ Disorder disappears in the scaling limit!

  $\mu^{\omega}_{n,\alpha} \Rightarrow \boldsymbol{\mu}_{\alpha}$ law of α-Bessel process (as if $\omega \equiv 0$)

- $(\alpha > \frac{1}{2})$ Disorder survives in the scaling limit!

  $\mu^{\omega}_{n,\alpha} \Rightarrow \boldsymbol{\mu}^{\omega}_{\alpha}$ truly random probability on $C([0,1])$

Recall that $\mu^{\omega}_{n,\alpha} \ll \mu_{n,\alpha}$ for every $n \in \mathbb{N}$ (Gibbs measure)

Scaling limits of disordered pinning model

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

- ($\alpha < \frac{1}{2}$) Disorder disappears in the scaling limit!

  $\mu_{n,\alpha}^{\omega} \Rightarrow \boldsymbol{\mu}_{\alpha}$ law of α-Bessel process (as if $\omega \equiv 0$)

- ($\alpha > \frac{1}{2}$) Disorder survives in the scaling limit!

  $\mu_{n,\alpha}^{\omega} \Rightarrow \boldsymbol{\mu}_{\alpha}^{\omega}$ truly random probability on $C([0,1])$

Recall that $\mu_{n,\alpha}^{\omega} \ll \mu_{n,\alpha}$ for every $n \in \mathbb{N}$ (Gibbs measure)

However $\boldsymbol{\mu}_{\alpha}^{\omega} \not\ll \boldsymbol{\mu}_{\alpha}$ for a.e. ω! (no continuum Gibbs meassure)

Scaling limits of disordered pinning model

Theorem (F. Caravenna, R. Sun, N. Zygouras)

Rescale suitably β, h (disorder mean and variance) and let $n \to \infty$

- $(\alpha < \frac{1}{2})$ Disorder disappears in the scaling limit!

  $\mu^{\omega}_{n,\alpha} \Rightarrow \boldsymbol{\mu}_\alpha$ law of α-Bessel process (as if $\omega \equiv 0$)

- $(\alpha > \frac{1}{2})$ Disorder survives in the scaling limit!

  $\mu^{\omega}_{n,\alpha} \Rightarrow \boldsymbol{\mu}^{\omega}_\alpha$ truly random probability on $C([0,1])$

  Recall that $\mu^{\omega}_{n,\alpha} \ll \mu_{n,\alpha}$ for every $n \in \mathbb{N}$ (Gibbs measure)

  However $\boldsymbol{\mu}^{\omega}_\alpha \not\ll \boldsymbol{\mu}_\alpha$ for a.e. ω ! (no continuum Gibbs meassure)

- $(\alpha = \frac{1}{2})$ Work in progress. . .

Thanks

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if μ_n^ω, μ^ω are random probabilities on E?

[$\omega \in \Omega$ probability space]

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if μ_n^ω, μ^ω are random probabilities on E?

[$\omega \in \Omega$ probability space]

- The space $\tilde{E} := \mathcal{M}_1(E)$ is also Polish

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if μ_n^ω, μ^ω are random probabilities on E?

[$\omega \in \Omega$ probability space]

- The space $\tilde{E} := \mathcal{M}_1(E)$ is also Polish
- Random probabilities μ_n^ω, μ^ω are $\tilde{E}$-valued random variables

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if μ_n^ω, μ^ω are random probabilities on E?

[$\omega \in \Omega$ probability space]

- The space $\tilde{E} := \mathcal{M}_1(E)$ is also Polish
- Random probabilities μ_n^ω, μ^ω are $\tilde{E}$-valued random variables
- Their laws are probabilities on $\tilde{E}$: weak convergence applies!

Weak convergence in presence of disorder

- E is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on E
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if μ_n^ω, μ^ω are random probabilities on E?

$[\omega \in \Omega$ probability space$]$

- The space $\tilde{E} := \mathcal{M}_1(E)$ is also Polish
- Random probabilities μ_n^ω, μ^ω are $\tilde{E}$-valued random variables
- Their laws are probabilities on $\tilde{E}$: weak convergence applies!

We still write $\mu_n^\omega \Rightarrow \mu^\omega$ for this convergence

(heuristics/intuition analogous to the non-disordered case)