THE CRITICAL CURVES OF THE RANDOM PINNING AND COPOLYMER MODELS AT WEAK COUPLING

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ABSTRACT. We study random pinning and copolymer models, when the return distribution of the underlying renewal process has a polynomial tail with finite mean. We compute the asymptotic behavior of the critical curves of the models in the weak coupling regime, showing that it is universal. This proves a conjecture of Bolthausen, den Hollander and Opoku for copolymer models [8], which we also extend to pinning models.

1. INTRODUCTION

The presence of disorder can drastically change the statistical mechanical properties of a physical system and alter the nature of its phase transitions, leading to new phenomena. Using a random walk to model a polymer chain, the effect of disorder on random walk models has been of particular interest recently, giving rise to many random polymer models [25, 20]. In this work we focus on two important classes of random polymer models, the so-called pinning and copolymer models.

- In the pinning model, the disorder is attached to a defect line, which can either attract or repel the random walk path. As the temperature varies, a localization-delocalization phase transition takes place: for sufficiently low temperatures, the random walk is absorbed at the defect line, while at high temperature it wanders away. The origins of such a model can be traced back to studies of wetting phenomena [18, 17] or localization of flux line in superconducting vortex arrays [28]. We also mention the interesting phenomenon of DNA denaturation, cf. [29].

- In the copolymer model, disorder is distributed along the random walk path, which meanders between two solvents separated by a flat interface. Each step of the random walk path can be regarded as a monomer, and the disorder attached to the monomer determines whether it prefers one solvent or the other. Also for this model, a sharp localization-delocalization phenomenon is observed: the typical random walk paths are either very close to the interface (in order to place most monomers in their preferred solvents) or very far from it, according to the temperature. The origins of the copolymer model in this context can be traced back to [19].

The purpose of this paper is to investigate the phase diagram of both models in the weak coupling regime, in the case when the excursions of the random walk away from the defect line (or interface) have a power-law tail with finite mean.

1.1. **Review of the models.** We first recall the definition of the random pinning and copolymer models. For a general overview, we refer to [25, 20, 21, 12].

The polymer chain is modeled by a Markov chain $S = \{S_n\}_{n \geq 0}$ on $\mathbb{Z}$ with $S_0 = 0$, that will be called *the walk*. Probability and expectation for S will be denoted respectively by P and E. We denote by $\tau := \{\tau_n\}_{n \geq 0}$, with $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$, the sequence of random times in which the walk visits 0, so that τ is a *renewal process* with $\tau_0 = 0$. We assume that τ is *non-terminating*, that is $P(\tau_1 < \infty) = 1$, and that

$$K(n) := P(\tau_1 = n) = \frac{\varphi(n)}{n^{1+\alpha}}, \quad \forall n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.1}$$

where $\alpha \in [0, +\infty)$ and $\varphi : (0, \infty) \to (0, \infty)$ is a slowly varying function. In this paper *we focus on the case $\alpha > 1$, for which the mean return time is finite*:

$$\mu := \mathrm{E}[\tau_1] \in (1, \infty). \tag{1.2}$$

Remark 1.1. Many interesting examples have periodicity issues, that is, there exists $\mathrm{T} \in \mathbb{N}$ such that $K(n) = 0$ if $n \notin \mathrm{T}\mathbb{N}$. For instance, the simple symmetric random walk on $\mathbb{Z}$ satisfies (1.1) for $n \in 2\mathbb{N}$, with $\alpha = 1/2$ and $\varphi(\cdot)$ converging asymptotically to a constant. However, we shall assume for simplicity that $K(n) > 0$ for every $n \in \mathbb{N}$. Everything can be easily extended to the periodic case, at the expense of some cumbersome notation.

Remark 1.2. As it will be clear, in our framework the fundamental object is the renewal process τ, and *there is no need to refer to the Markov chain S*. However, let us mention that, for any $\alpha > 0$ and any slowly varying function φ, a nearest-neighbor Markov chain S on $\mathbb{Z}$ with Bessel-like drift can be constructed, which satisfies assumption (1.1) asymptotically, that is $K(n) \sim \varphi(n)/n^{1+\alpha}$ as $n \to \infty$, cf. [2].

The disorder is modeled by a sequence $\omega := \{\omega_n\}_{n \geq 1}$ of i.i.d. real random variables. Probability and expectation for ω will be denoted respectively by $\mathbb{P}$ and $\mathbb{E}$. We assume that

$$M(t) := \mathbb{E}[e^{t\omega_1}] < \infty \quad \forall |t| < t_0, \text{ with } t_0 > 0, \qquad \mathbb{E}[\omega_1] = 0, \qquad \mathbb{V}\mathrm{ar}(\omega_1) = 1. \tag{1.3}$$

We will often be making use of the log-moment generating function, that is

$$\Lambda(t) := \log M(t) = \frac{1}{2} t^2 + o(t^2) \quad \text{as } t \to 0. \tag{1.4}$$

Given a realization of the disorder ω, the random *pinning model* is defined by a Gibbs transform of the law P of the renewal process τ (or, if one wishes, of the walk S):

$$\mathrm{dP}_{N,\beta,h}^{\mathrm{pin},\omega} := \frac{1}{Z_{N,\beta,h}^{\mathrm{pin},\omega}} e^{\sum_{n=1}^{N}(\beta\omega_n - h_a^{\mathrm{pin}}(\beta) + h)\mathbf{1}_{\{n \in \tau\}}} \mathrm{dP}, \tag{1.5}$$

where $\{n \in \tau\}$ is a shorthand for $\bigcup_{k \in \mathbb{N}} \{\tau_k = n\}$, the event that the renewal process τ visits n (which corresponds to $\{S_n = 0\}$, referring to the walk S). The parameter $\beta \geq 0$ is the coupling constant (or inverse temperature), $h \in \mathbb{R}$ adds a bias to the disorder, and

$$h_a^{\mathrm{pin}}(\beta) := \Lambda(\beta) = \frac{\beta^2}{2} + o(\beta^2) \qquad (\text{as } \beta \downarrow 0) \tag{1.6}$$

is the *annealed* critical point, the significance of which will be discussed later. The normalizing constant

$$Z_{N,\beta,h}^{\mathrm{pin},\omega} := \mathrm{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n - h_a^{\mathrm{pin}}(\beta) + h)\mathbf{1}_{\{n \in \tau\}}}\right] \tag{1.7}$$

is called the *partition function*. We will also consider the *constrained* partition function

$$Z_{N,\beta,h}^{\mathrm{pin,c},\omega} = \mathrm{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n - h_a^{\mathrm{pin}}(\beta) + h)\mathbf{1}_{\{n \in \tau\}}} \mathbf{1}_{\{N \in \tau\}}\right]. \tag{1.8}$$

In order to define the copolymer model, one traditionally works with nearest-neighbor walks S that make symmetric excursions in the positive or negative half-plane. The signs of the excursions are then i.i.d. symmetric $\{\pm 1\}$-valued random variables, independent of τ. In our framework, it is actually simpler to proceed as in [12]. Without making any reference to the walk S, under the law P we introduce a sequence of i.i.d. symmetric ± 1 random variables $\varepsilon := \{\varepsilon_n\}_{n \geq 1}$, independent of the renewal process τ. Intuitively, the variable ε_n models the sign of the excursion of the walk during the renewal interval (τ_{n-1}, τ_n) (even when this interval has length 1). Given the disorder ω, the *copolymer model* is then defined via a similar Gibbs transform of the law P:

$$\mathrm{dP}_{N,\lambda,h}^{\mathrm{cop},\omega} := \frac{1}{Z_{N,\lambda,h}^{\mathrm{cop},\omega}} \, e^{-2\lambda \sum_{n=1}^{N} (\omega_n + h_a^{\mathrm{cop}}(\lambda) - h) \mathbf{1}_{\{\varepsilon_n = -1\}}} \mathrm{dP}, \qquad (1.9)$$

where $\lambda \geq 0$ is the coupling constant, $h \in \mathbb{R}$ and $Z_{N,\lambda,h}^{\mathrm{cop},\omega}$ have the same interpretation as in the pinning model, and

$$h_a^{\mathrm{cop}}(\lambda) := \frac{1}{2\lambda} \Lambda(-2\lambda) = \lambda + o(\lambda) \quad (\text{as } \lambda \downarrow 0) \qquad (1.10)$$

is the corresponding *annealed* critical point. The constrained partition function for the copolymer model is given by

$$
\begin{aligned}
Z_{N,\lambda,h}^{\mathrm{cop,c},\omega} &= \mathrm{E}\Big[e^{-2\lambda \sum_{n=1}^{N} (\omega_n + h_a^{\mathrm{cop}}(\lambda) - h) \mathbf{1}_{\{\varepsilon_n = -1\}}} \mathbf{1}_{\{N \in \tau\}} \Big] \\
&= \mathrm{E}\Bigg[\prod_{j=1}^{|\tau \cap (0,N]|} \left(\frac{1 + e^{-2\lambda \sum_{n=\tau_{j-1}+1}^{\tau_j} (\omega_n + h_a^{\mathrm{cop}}(\lambda) - h)}}{2} \right) \mathbf{1}_{\{N \in \tau\}} \Bigg],
\end{aligned}
\qquad (1.11)
$$

where $|\tau \cap (0,N]| = \max\{k \geq 0 : \tau_k \leq N\}$ is the number of renewal points that appear before N, and we have integrated the excursions signs.

Remark 1.3. The different parametrization of the copolymer model, as compared to the pinning one, is to conform with most of the existing literature. To recover the pinning parametrization, it suffices to replace ω by $-\omega$, 2λ by β and $2\lambda h$ by h.

Many statistical properties of the models can be captured through the (quenched) *free energies*, which are defined by

$$
\begin{aligned}
\mathrm{F}^{\mathrm{pin}}(\beta, h) &:= \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta,h}^{\mathrm{pin},\omega} = \lim_{N \to \infty} \frac{1}{N} \, \mathbb{E} \log Z_{N,\beta,h}^{\mathrm{pin},\omega}, \\
\mathrm{F}^{\mathrm{cop}}(\lambda, h) &:= \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\beta,h}^{\mathrm{cop},\omega} = \lim_{N \to \infty} \frac{1}{N} \, \mathbb{E} \log Z_{N,\lambda,h}^{\mathrm{cop},\omega},
\end{aligned}
\qquad (1.12)
$$

where the limits exist $\mathbb{P}$-a.s. and remain unchanged if we replace the partition functions by their constrained counterparts (see [20, Ch. 4]). From the definition of the partition functions, by restricting the expectation E to the event $\{\tau_1 > N\}$ (that is $\{S_n > 0$ for $n = 1, 2, ..., N\}$ in the walk interpretation) and observing that $\log \mathrm{P}(\tau_1 > N) = O(\log N)$, by (1.1), it follows that the free energies are nonnegative. A (quenched) localization-delocalization transition can be determined from the critical curves

$$h_c^{\mathrm{pin}}(\beta) := \sup\{h \colon \mathrm{F}^{\mathrm{pin}}(\beta, h) = 0\} \quad \text{and} \quad h_c^{\mathrm{cop}}(\lambda) := \sup\{h \colon \mathrm{F}^{\mathrm{cop}}(\lambda, h) = 0\}. \qquad (1.13)$$

The meaning of these critical curves is that, when h exceeds the critical value, the free energy is strictly positive and the polymer puts a positive fraction of its monomers at $\{0\}$, in the pinning model (resp. in $\{0, -1, -2, ...\}$, in the copolymer model); on the other hand, these fractions equal 0 when h is below the critical value. We refer to [20, 25] for details.

The effect of the disorder is best seen through comparison of the (quenched) models with their *annealed* counterparts. In particular, the annealed free energies are defined by

$$\mathrm{F}_a^{\mathrm{pin}}(\beta, h) := \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} Z_{N,\beta,h}^{\mathrm{pin},\omega} \quad \text{and} \quad \mathrm{F}_a^{\mathrm{cop}}(\lambda, h) := \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} Z_{N,\beta,h}^{\mathrm{cop},\omega} .$$

The annealed models are exactly solvable and their phase diagrams can be completely determined (see e.g. [20]). In particular, the critical value of h, above which the annealed free energy is strictly positive, is $h = 0$, both for the pinning and copolymer model. This is simply because we have subtracted the "true" annealed critical points $h_a^{\mathrm{pin}}(\beta)$ and $h_a^{\mathrm{cop}}(\beta)$ from h in the definition of our models, for later convenience. As a consequence, a key signature that disorder alters the nature of the phase transition is that the quenched critical curve $h_c^{\mathrm{pin}}(\beta)$ (resp. $h_c^{\mathrm{cop}}(\lambda)$) is *strictly positive* for any small β (resp. λ). In this case, disorder is said to be *relevant*; otherwise, it is said to be *irrelevant*.

It has been shown in [31] that for the copolymer model disorder is relevant for every $\alpha > 0$, regardless of the underlying renewal process τ (satisfying the above mentioned assumptions). On the other hand, for the pinning model, it is known that disorder is irrelevant when $\alpha < 1/2$ or when $\alpha = 1/2$ and $\sum_{n \geq 1} 1/n\varphi(n)^2 < \infty$ (cf. [1, 30, 14, 26]), relevant when $\alpha > 1/2$, and believed to be also relevant (almost confirmed in [3, 4, 22, 23]) when $\alpha = 1/2$ and $\sum_{n \geq 1} 1/n\varphi(n)^2 = \infty$. See [21] for an overview.

1.2. **The main results.** A fundamental problem for random pinning and copolymer models, when disorder is relevant, is the asymptotic behavior of the critical curves in the *weak coupling regime* $\beta, \lambda \downarrow 0$. The interest of this question lies in the belief that such asymptotic behavior should be *universal*, i.e., not depend too much on the fine details of the model.

For the copolymer model, the behavior of the critical curve $h_c^{\mathrm{cop}}(\lambda)$ for small λ has been investigated extensively. In the seminal paper [7], Bolthausen and den Hollander investigated the special copolymer model in which $S = \{S_n\}_{n \geq 1}$ is the simple symmetric random walk on $\mathbb{Z}$ and the disorder variables ω_n are $\{\pm 1\}$-valued and symmetric. They were able to show that the slope of the critical curve $\lim_{\lambda \downarrow 0} h_c^{\mathrm{cop}}(\lambda)/\lambda$ exists and coincides with the critical point of a *continuum copolymer model* (in which the walk S is replaced by a Brownian motion and the disorder sequence ω is replaced by white noise). This result was recently extended by Caravenna and Giacomin [10]: for the general class of copolymer models that we consider in this paper, in the case $\alpha \in (0, 1)$, it was shown that the slope of the critical curve exists and is a *universal quantity*, namely it is the critical point of a suitable α-continuum copolymer model. In particular, the slope *depends only on α* and not on finer details of the renewal process τ and disorder ω. For consistency with the literature, we define (recall (1.10))

$$m_\alpha := \lim_{\lambda \downarrow 0} \frac{h_a^{\mathrm{cop}}(\lambda) - h_c^{\mathrm{cop}}(\lambda)}{\lambda} = 1 - \lim_{\lambda \downarrow 0} \frac{h_c^{\mathrm{cop}}(\lambda)}{\lambda} . \tag{1.14}$$

The precise value of m_α, in particular for $\alpha = 1/2$, has been a matter of a long debate. It was conjectured by Monthus in [27], on the ground of non-rigorous renormalisation arguments, that $m_{1/2} = 2/3$, and a generalization of the same argument yields the conjecture $m_\alpha = 1/(\alpha + 1)$. The rigorous lower bound $m_\alpha \geq 1/(\alpha + 1)$, for every $\alpha \geq 0$, was proved by Bodineau and Giacomin in [5]. Very recently, it was shown by Bolthausen, den Hollander and Opoku [8] that this lower bound is strict for every $\alpha \in (0, \infty)$, thus ruling out Monthus' conjecture (see also [6] for earlier, partial results, and [11] for a related numerical study).

In this work, we focus on the case $\alpha > 1$. Let us stress that this case was not considered in [10], because no non-trivial continuum model is expected to exist, due to the finite mean of

the underlying renewal process. This consideration might even cast doubts on the (existence and) universality of the limit in (1.14). However, it was recently proved in [8] that

$$\liminf_{\lambda \downarrow 0} \frac{h_a^{\mathrm{cop}}(\lambda) - h_c^{\mathrm{cop}}(\lambda)}{\lambda} \geq \frac{2 + \alpha}{2(1 + \alpha)}, \qquad \forall \alpha \in (1, \infty). \tag{1.15}$$

(The "critical curve" in [8] corresponds to $h_a^{\mathrm{cop}}(\lambda) - h_c^{\mathrm{cop}}(\lambda)$ in our notation; furthermore, our exponent α is what they call $\alpha - 1$, hence the right hand side in (1.15) reads $\frac{1+\alpha}{2\alpha}$ in [8].)

The universal lower bound (1.15), depending only on α, led naturally to the conjecture [8] that m_α exists also for $\alpha > 1$ and coincides with the right hand side of (1.15). Our first main result proves this conjecture, establishing in particular the universality of the slope.

Theorem 1.4. *For any copolymer model defined as above, with $\alpha > 1$, the limit in* (1.14) *exists and equals $m_\alpha = \frac{2+\alpha}{2(1+\alpha)}$. Equivalently,*

$$\lim_{\lambda \downarrow 0} \frac{h_c^{\mathrm{cop}}(\lambda)}{\lambda} = \frac{\alpha}{2(1 + \alpha)}. \tag{1.16}$$

Remark 1.5. In a work in progress [13], the partition function of the copolymer model under weak coupling is shown to converge, for every $\alpha > 1$, to an explicit "trivial" continuum limit, the exponential of a Brownian motion with drift, *which carries no dependence on α*. In particular, the continuum limit of the partition function gives no information on the slope of the critical curve, which is in stark contrast to the case $\alpha \in (0, 1)$.

For the random pinning model with $\alpha > 1$, rough upper and lower bounds of the order β^2 are known for the critical curve $h_c^{\mathrm{pin}}(\beta)$, cf. [3, 16]. (The quadratic, rather than linear, behavior is simply due to the different way the parameters (β, h) and (λ, h) appear in the two models, cf. Remark 1.3). We sharpen these earlier results by establishing the following analogue of Theorem 1.4.

Theorem 1.6. *For any random pinning model defined as above, with $\alpha > 1$, we have*

$$\lim_{\beta \downarrow 0} \frac{h_c^{\mathrm{pin}}(\beta)}{\beta^2} = \frac{\alpha}{1 + \alpha} \frac{1}{2\mu}, \tag{1.17}$$

where $\mu := \mathrm{E}[\tau_1]$.

Thus, the asymptotic behavior of the critical curve of the random pinning model with $\alpha > 1$ is also universal, in the sense that it depends only on the exponent α and on the mean μ of the underlying renewal process, and not on the finer details of the renewal process or the disorder distribution.

Remark 1.7. For the random pinning model with $\alpha > 1$, it is also shown in [13] that the partition function under weak coupling converges, in the continuum limit, to the exponential of a Brownian motion with drift, which depends on μ *but not on $\alpha > 1$*. As a consequence, the continuum limit gives no information on the asymptotic behavior (1.17).

The fact that we can prove the same type of result for the random pinning and copolymer models is not unexpected for $\alpha > 1$. In fact, when the underlying renewal process has finite mean, there is typically a positive fraction of monomers interacting with the disorder, for both models, since $\mathrm{P}(n \in \tau)$ and $\mathrm{P}(\varepsilon_n = -1)$ (recall (1.5) and (1.9)) tend to a positive constant as $n \to \infty$ ($1/\mu$ and $1/2$, respectively). Also, the annealed free energies of both models are proportional to the square of the coupling constants β, respectively λ, if the bias h is also scaled properly (proportional to β^2 for the pinning model and to λ for the

copolymer model). This fact is then reflected in the (annealed) correlation length of the system, which plays a central role in the coarse graining analysis we will carry out later.

When $\alpha < 1$, the above analogies between the random pinning and copolymer models break down. In [3] it was shown that for $\alpha \in (1/2, 1)$, $h_c^{\mathrm{pin}}(\beta)$ satisfies the bounds

$$c^{-1}\beta^{\frac{2\alpha}{2\alpha-1}}\psi(\beta^{-2}) \leq h_c^{\mathrm{pin}}(\beta) \leq c\beta^{\frac{2\alpha}{2\alpha-1}}\psi(\beta^{-2})$$

for some $c > 0$ and a slowly varying function $\psi(\cdot)$, related in an explicit way to $\varphi(\cdot)$. It is then reasonable to conjecture that, in analogy with the universality of the slope of the critical curve for the copolymer model, the limit

$$\lim_{\beta \downarrow 0} \frac{h_c^{\mathrm{pin}}(\beta)}{\beta^{\frac{2\alpha}{2\alpha-1}}\psi(\beta^{-2})} \qquad (1.18)$$

exists and is also universal. The method we use in this paper falls short in answering this question. However, the random pinning model under weak coupling does admit a continuum limit when $\alpha \in (1/2, 1)$, which is currently under construction in [13]. This gives hope to prove the existence and universality of the limit in (1.18).

1.3. **Organization and main ideas.** We present the proof of Theorem 1.6, concerning the random pinning model, in Sections 2 (lower bound) and 3 (upper bound). The proof of Theorem 1.4, concerning the copolymer model, follows the same line of arguments —in fact, the upper bound is significantly easier in this case— so we only sketch the proofs and highlight the differences in Sections 4 and 5.

Remark 1.8. The upper bound on $h_c^{\mathrm{cop}}(\lambda)$ in relation (1.16) for the copolymer is the same as the lower bound (1.15), which was established in [8] as an application of a quenched large deviations principle, developed by the authors and their collaborators. Here we present an alternative and self-contained proof, which is remarkably short (see Section 5).

The proof of the lower bound on $h_c(\cdot)$ is based on a refinement of the "fractional moment and coarse graining" method, developed in [16, 23], see [21] for an overview. The standard application of this moment method makes use of a change of measure, which via the use of Hölder's inequality gives rise to an energy-entropy balance. In [16, 23], the entropy factor can be bounded by an arbitrary constant, while the energy factor can be made arbitrarily small. Our refinement requires optimizing this energy-entropy balance, which is crucial in obtaining the precise constants. We also need a refinement in the coarse graining procedure. In the standard application, the polymer only needs to place a positive fraction of monomers at the interface in each visited coarse-grained block, while in our case, we need to ensure that this positive fraction is in fact close to 1. For this step, $\alpha > 1$ plays a crucial role.

The upper bound on $h_c(\cdot)$ makes use of the following smoothing inequality (the distinction between the pinning and copolymer, as usual, is simply due to their different parametrization).

Theorem 1.9. *Recall relation (1.3). For the pinning model, for every $0 < \beta < t_0/2$ and $|\delta| < \beta t_0/2$, there exists $A_{\beta,\delta} \in (0,\infty)$ with $\lim_{(\beta,\delta)\to(0,0)} A_{\beta,\delta} = 1$, such that*

$$0 \leq \mathrm{F}^{\mathrm{pin}}(\beta, h_c^{\mathrm{pin}}(\beta) + \delta) \leq \frac{1+\alpha}{2}A_{\beta,\delta}\frac{\delta^2}{\beta^2}. \qquad (1.19)$$

For the copolymer model, for every $0 < \lambda < t_0/4$ and $|\delta| < t_0/2$, there exists $A_{\lambda,\delta} \in (0,\infty)$ with $\lim_{(\lambda,\delta)\to(0,0)} A_{\lambda,\delta} = 1$, such that

$$0 \leq \mathrm{F}^{\mathrm{cop}}(\lambda, h_c^{\mathrm{cop}}(\lambda) + \delta) \leq \frac{1+\alpha}{2}A_{\lambda,\delta}\delta^2. \qquad (1.20)$$

The smoothing inequality was first proved in [24], without the precision on the constant and under more restrictive assumptions on the disorder. In the case of Gaussian disorder, it appears in [20] with the right constant $(1+\alpha)/2$, cf. Theorem 5.6 and Remar 5.7 therein. The general statements we use here are proved in [9]. We remark that the precise (asymptotic) constant $(1+\alpha)/2$ is crucial in obtaining the exact limits of $h_c^{\text{cop}}(\lambda)/\lambda$ an $h_c^{\text{pin}}(\beta)/\beta^2$.

The idea to prove the upper bound is to couple the smoothing inequality with a rough linear (but quantitative) lower bound on the free energies. More precisely, we prove that for every $c \in \mathbb{R}$,

$$\liminf_{\beta \downarrow 0} \frac{\mathrm{F}^{\text{pin}}(\beta, c\beta^2)}{\beta^2} \geq \frac{1}{\mu}\left[c - \frac{1}{2\mu}\right] \quad \text{and} \quad \liminf_{\lambda \downarrow 0} \frac{\mathrm{F}^{\text{cop}}(\lambda, c\lambda)}{\lambda^2} \geq c - \frac{1}{2}. \tag{1.21}$$

Remarkably, enforcing the compatibility of these inequalities with the corresponding smoothing inequalities (1.19) and (1.20) leads to the sharp upper bound on the critical curves.

Finally, a remark on notations. To ease the reading, we will drop the superscripts pin and cop from our notation for the free energy, partition function, and critical curve. This should not lead to confusion, since pinning and copolymer models are treated in separate sections. Moreover, we will refrain from using the integer parts, that is, instead of $\lfloor x \rfloor$ we simply write x. It will be clear from the context when the integer part of x is used.

2. On the Pinning Model: Lower Bound

As already mentioned, we use the "fractional moment and coarse graining" method (see [21]), but with several crucial refinements, that we now explain.

2.1. **The general strategy.** To obtain a lower bound on the critical curve $h_c(\beta)$, it suffices to prove $\mathrm{F}(\beta, h) = 0$ for suitably chosen h as a function of β. This is further reduced to showing that for some $\zeta \in (0, 1)$, we have

$$\liminf_{N \to \infty} \mathbb{E}\big[\big(Z_{N,\beta,h}^{\text{c},\omega}\big)^{\zeta}\big] < \infty. \tag{2.1}$$

Indeed, note that

$$\mathrm{F}(\beta, h) = \liminf_{N \to \infty} \frac{1}{N} \mathbb{E}\big[\log Z_{N,\beta,h}^{\text{c},\omega}\big] = \liminf_{N \to \infty} \frac{1}{N\zeta} \mathbb{E}\big[\log \big(Z_{N,\beta,h}^{\text{c},\omega}\big)^{\zeta}\big]$$
$$\leq \liminf_{N \to \infty} \frac{1}{N\zeta} \log \mathbb{E}\big[\big(Z_{N,\beta,h}^{\text{c},\omega}\big)^{\zeta}\big] = 0.$$

To obtain (2.1), we employ a coarse-graining scheme. The idea is to divide the system into (large) finite blocks, each one being of size k, the *correlation length* of the annealed model, proportional to $1/\beta^2$. We estimate the partition functions on different blocks separately, and then "glue" these estimates together through a coarse-graining procedure.

We first estimate the partition function of a system of size k, the coarse-graining length scale. Let $\widetilde{\mathbb{P}}_{-\delta,k}$ denote the law under which $\{\omega_i\}_{1 \leq i \leq k}$ are i.i.d. with density

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{-\delta,k}}{\mathrm{d}\mathbb{P}} = \prod_{i=1}^{k} e^{-\delta\omega_i - \Lambda(-\delta)}, \tag{2.2}$$

which is an exponential tilting of the law of $\{\omega_i\}_{1 \leq i \leq k}$. We then apply the standard change of measure trick: by Hölder's inequality, for any $\zeta \in (0, 1)$

$$\mathbb{E}\big[\big(Z_{k,\beta,h}^{\text{c},\omega}\big)^{\zeta}\big] = \widetilde{\mathbb{E}}_{-\delta,k}\left[\big(Z_{k,\beta,h}^{\text{c},\omega}\big)^{\zeta} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_{-\delta,k}}\right] \leq \widetilde{\mathbb{E}}_{-\delta,k}\big[Z_{k,\beta,h}^{\text{c},\omega}\big]^{\zeta} \widetilde{\mathbb{E}}_{-\delta,k}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_{-\delta,k}}\right)^{\frac{1}{1-\zeta}}\right]^{1-\zeta}. \tag{2.3}$$

The second factor is easily computed:

$$
\widetilde{\mathbb{E}}_{-\delta,k}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_{-\delta,k}}\right)^{\frac{1}{1-\zeta}}\right]^{1-\zeta} = \mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_{-\delta,k}}\right)^{\frac{\zeta}{1-\zeta}}\right]^{1-\zeta} = \mathbb{E}\left[\prod_{i=1}^{k} e^{\frac{\zeta}{1-\zeta}\delta\omega_i + \frac{\zeta}{1-\zeta}\Lambda(-\delta)}\right]^{1-\zeta}
$$

$$
= e^{(1-\zeta)k\left[\Lambda\left(\frac{\zeta}{1-\zeta}\delta\right)+\frac{\zeta}{1-\zeta}\Lambda(-\delta)\right]}. \tag{2.4}
$$

Using (1.8) and recalling (1.6), the first factor in (2.3) can also be computed:

$$
\widetilde{\mathbb{E}}_{-\delta,k}\left[Z_{k,\beta,h}^{\mathrm{c},\omega}\right] = \mathrm{E}\left[e^{(\Lambda(\beta-\delta)-\Lambda(\beta)-\Lambda(-\delta)+h)\,|\tau\cap[1,k]|}\,1_{\{k\in\tau\}}\right]. \tag{2.5}
$$

Since $\tau_{|\tau\cap[1,k]|} \le k \le \tau_{|\tau\cap[1,k]|+1}$, it follows by the strong law of large numbers that

$$
\frac{|\tau\cap[1,k]|}{k} \xrightarrow[k\to\infty]{} \frac{1}{\mu} \qquad \text{P-a.s..} \tag{2.6}
$$

Since $\mathrm{P}(k\in\tau) \to \frac{1}{\mu} > 0$ as $k\to\infty$, by the renewal theorem, we have the convergence in distribution

$$
\frac{|\tau\cap[1,k]|}{k} \xrightarrow[k\to\infty]{\mathrm{d}} \frac{1}{\mu} \qquad \text{under } \mathrm{P}(\cdot\,|\,k\in\tau). \tag{2.7}
$$

We now parametrize everything in terms of β. Let us set

$$
h_\beta = c\beta^2, \quad \delta_\beta = a\beta, \quad k_\beta = \frac{t}{\beta^2}, \qquad \text{for } c,a,t\in(0,\infty) \text{ with } a>c.
$$

As $\beta\downarrow 0$, we have $k_\beta\to\infty$ and, recalling (1.4),

$$
(\Lambda(\beta-\delta_\beta)-\Lambda(\beta)-\Lambda(-\delta_\beta)+h_\beta)\frac{k_\beta}{\mu} \sim (h_\beta-\beta\delta_\beta)\frac{k_\beta}{\mu} \longrightarrow (c-a)\frac{t}{\mu} \in (-\infty,0).
$$

Together with the fact that $\mathrm{P}(k\in\tau)\to\frac{1}{\mu}$ as $k\to\infty$, it follows from (2.5) and (2.7) that

$$
\lim_{\beta\downarrow 0} \widetilde{\mathbb{E}}_{-\delta_\beta,k_\beta}\left[Z_{k_\beta,\beta,h_\beta}^{\mathrm{c},\omega}\right] = \frac{1}{\mu}e^{(c-a)\frac{t}{\mu}}, \tag{2.8}
$$

hence, by (2.3) and (2.4),

$$
\limsup_{\beta\downarrow 0} \mathbb{E}\left[\left(Z_{k_\beta,\beta,h_\beta}^{\mathrm{c},\omega}\right)^\zeta\right] \le \frac{1}{\mu^\zeta}e^{\frac{\zeta}{\mu}(c-a)t}e^{\frac{a^2}{2}\frac{\zeta}{1-\zeta}t} \qquad \forall a>c. \tag{2.9}
$$

Note that the exponent is a polynomial of second degree in a: optimizing over a yields

$$
a = \frac{1-\zeta}{\mu}, \tag{2.10}
$$

and one gets the basic estimate

$$
\limsup_{\beta\downarrow 0} \mathbb{E}\left[\left(Z_{k_\beta,\beta,h_\beta}^{\mathrm{c},\omega}\right)^\zeta\right] \le \frac{1}{\mu^\zeta}\exp\left\{\frac{\zeta}{\mu}\left(c-\frac{1-\zeta}{2\mu}\right)t\right\}. \tag{2.11}
$$

To feed this estimate into the coarse graining scheme and obtain (2.1), we need to make the right hand side arbitrarily small. This can be accomplished by choosing t large enough, *provided $c < (1-\zeta)/(2\mu)$*, or equivalently, $h/\beta^2 < (1-\zeta)/(2\mu)$. As we will see later, the coarse graining scheme works only if $\zeta > 1/(1+\alpha)$, which leads to $h/\beta^2 < \alpha/(2(1+\alpha)\mu)$ and thus to the sharp lower bound (1.17) on $h_c(\beta)$.

Note that the bound (2.11) is derived via a *subtle balance between the cost of changing the measure and the annealed partition function under the changed measure*, i.e., the two factors in the right hand side (2.3). This is in contrast to [21, Proposition 7.1], where:

- the cost of changing the measure is only required to be an arbitrary fixed constant for each coarse graining block;
- the annealed partition function under the changed measure is small over any interval whose length exceeds a δ-proportion of the coarse graining block.

In our case, the cost of changing the measure needs to be estimated sharply; furthermore, in order to balance this cost and get (2.11), *we need to average the partition function under the changed measure over an interval whose length is close to the full coarse graining block.* Fortunately such configurations can be shown to give the dominant contribution in the case $\alpha > 1$. We stress that this is *not* the case when $\alpha < 1$.

2.2. **Proof of the lower bound.** We divide the proof into several steps.

STEP 1. Let us first set up the proper framework. To prove the lower bound for $h_c(\beta)$ in (1.17), we show that for every $\varepsilon > 0$ small enough there exists $\beta_0 = \beta_0(\varepsilon) \in (0, \infty)$ such that for every $\beta \in (0, \beta_0)$ we have

$$\mathrm{F}\big(\beta,\, c_\varepsilon \beta^2\big) = 0\,, \qquad \text{where} \qquad c_\varepsilon := (1 - \varepsilon)\frac{\alpha}{1 + \alpha}\frac{1}{2\mu}\,. \tag{2.12}$$

As explained in Section 2.1, it suffices to show that there exists $\zeta = \zeta_\varepsilon \in (0, 1)$ such that

$$\liminf_{N \to \infty} \mathbb{E}\big[\big(Z_{N,\beta,c_\varepsilon\beta^2}^{\mathrm{c},\omega}\big)^\zeta\big] < \infty. \tag{2.13}$$

Henceforth let $\varepsilon \in (0, 1)$ be fixed. We then set

$$\zeta_\varepsilon := \frac{1}{1 + \alpha} + \frac{\varepsilon}{2}\frac{\alpha}{1 + \alpha} \in \big(\tfrac{1}{1+\alpha}, 1\big)\,. \tag{2.14}$$

For later convenience, we assume that ε is small enough so that $(1 + \alpha - \frac{\varepsilon}{2})\zeta_\varepsilon > 1$ and also $\alpha - \varepsilon/2 > 1$ (which is possible, since $\alpha > 1$). We set the coarse graining length scale to be

$$k = k_{\beta,\varepsilon} = t_\varepsilon/\beta^2 \tag{2.15}$$

for some $t_\varepsilon \in (0, \infty)$, which *depends only on* ε and will be fixed at the end of the proof.

Recall from (2.2) the exponentially tilted law $\widetilde{\mathbb{P}} := \widetilde{\mathbb{P}}_{-\delta,k}$. We will use it with $k = k_{\beta,\varepsilon}$ and

$$\delta = \delta_{\beta,\varepsilon} := a_\varepsilon \beta := \frac{1 - \zeta_\varepsilon}{\mu}\beta, \tag{2.16}$$

where the choice of a_ε is (a posteriori) optimal, recall (2.10). Note that $c_\varepsilon \leq (1 - \zeta_\varepsilon)\frac{1}{2\mu}$ by (2.12) and (2.14), hence $a_\varepsilon \geq 2c_\varepsilon$ by (2.16). In particular, we stress that

$$a_\varepsilon > c_\varepsilon\,, \tag{2.17}$$

a relation that will be used several times in the sequel.

Now we recall the crucial relation (2.8), which can be rewritten in our current setting as

$$\lim_{\beta \downarrow 0} \widetilde{\mathbb{E}}\big[Z_{t_\varepsilon\beta^{-2},\beta,c_\varepsilon\beta^2}^{\mathrm{c},\omega}\big] = \frac{1}{\mu}e^{-\frac{1}{\mu}(a_\varepsilon - c_\varepsilon)t_\varepsilon}\,. \tag{2.18}$$

The convergence (2.18) is actually *uniform* when t_ε varies in a compact subset of $(0, \infty)$. In particular, there exists $\beta_1(\varepsilon) > 0$ such that for all $\beta \in (0, \beta_1(\varepsilon))$ and all $n \in \mathbb{N}$ with $(1 - \varepsilon/5)t_\varepsilon\beta^{-2} \leq n \leq t_\varepsilon\beta^{-2}$, we have

$$\frac{1 - \varepsilon}{\mu}e^{-\frac{1}{\mu}(a_\varepsilon - c_\varepsilon)\beta^2 n} \leq \widetilde{\mathbb{E}}\big[Z_{n,\beta,c_\varepsilon\beta^2}^{\mathrm{c},\omega}\big] \leq \frac{1 + \varepsilon}{\mu}e^{-\frac{1}{\mu}(a_\varepsilon - c_\varepsilon)\beta^2 n}\,. \tag{2.19}$$

This uniform bound follows from the convergence in (2.7), because the functions $x \mapsto e^{-Cx}$ are uniformly bounded and uniformly Lipschitz, if C ranges over a bounded set. Note that the upper and lower bounds in (2.19) are bounded away from 0 and ∞ (for a fixed $\varepsilon > 0$), because $(1 - \frac{\varepsilon}{5})t_\varepsilon \leq \beta^2 n \leq t_\varepsilon$.

STEP 2. We now develop the coarse graining scheme. The system size N will be a multiple of the coarse graining length scale: $N = mk = mt_\varepsilon\beta^{-2}$ for some $m \in \mathbb{N}$. We then partition $\{1, \ldots, N\}$ into m blocks $B_1, \ldots, B_m$ of size $k = t_\varepsilon\beta^{-2}$, defined by

$$B_i := \big\{ (i-1)k + 1, \ldots, ik \big\} \subseteq \{1, \ldots, N\},$$

so that the macroscopic (coarse-grained) "configuration space" is $\{1, \ldots, m\}$. A macroscopic configuration is a subset $J \subseteq \{1, \ldots, m\}$. By a decomposition according to which blocks are visited by the renewal process (we call these blocks occupied), we can then write

$$Z^{c,\omega}_{N,\beta,c_\varepsilon\beta^2} = \sum_{J \subseteq \{1,\ldots,m\}:\ m \in J} \hat{Z}_J$$

where for $J = \{j_1, \ldots, j_\ell\}$, with $1 \leq j_1 < j_2 < \ldots < j_\ell = m$ and $\ell = |J|$,

$$\hat{Z}_J := \sum_{\substack{d_1,f_1 \in B_{j_1} \\ d_1 \leq f_1}} \cdots \sum_{\substack{d_{\ell-1},f_{\ell-1} \in B_{j_{\ell-1}} \\ d_{\ell-1} \leq f_{\ell-1}}} \sum_{d_\ell \in B_{j_\ell} = B_m} \left(\prod_{i=1}^{\ell} K(d_i - f_{i-1}) z_{d_i} Z_{d_i,f_i} \right), \qquad (2.20)$$

where we set $f_0 := 0$ and $f_\ell := N = mk$, and for all $d < f \in \mathbb{N}$,

$$z_d := e^{\beta\omega_d - \Lambda(\beta) + c_\varepsilon\beta^2}, \qquad Z_{d,f} := Z^{c,\vartheta^d\omega}_{f-d,\beta,c_\varepsilon\beta^2},$$

with $\vartheta^d\omega := \{(\vartheta^d\omega)_n\}_{n \in \mathbb{N}} = \{\omega_{n+d}\}_{n \in \mathbb{N}}$ defined as a shift of the disorder ω. Since $\zeta_\varepsilon < 1$, one has that $(a + b)^{\zeta_\varepsilon} \leq a^{\zeta_\varepsilon} + b^{\zeta_\varepsilon}$, for all $a, b \geq 0$, and consequently

$$\mathbb{E}\big[(Z^{c,\omega}_{N,\beta,c_\varepsilon\beta^2})^{\zeta_\varepsilon} \big] \leq \sum_{J \subseteq \{1,\ldots,m\}:\ m \in J} \mathbb{E}\big[(\hat{Z}_J)^{\zeta_\varepsilon} \big]. \qquad (2.21)$$

To bound $\mathbb{E}\big[(\hat{Z}_J)^{\zeta_\varepsilon} \big]$, we apply the change of measure as in (2.2). Let $\widetilde{\mathbb{P}}_J$ be the law of the disorder obtained from $\mathbb{P}$, where independently for each $n \in \bigcup_{i \in J} B_i$, the law of ω_n is tilted with density $e^{-\delta\omega_n - \Lambda(-\delta)}$, with $\delta = a_\varepsilon\beta$ as chosen in (2.16). Then by the same argument as in (2.3), we have

$$\mathbb{E}\big[(\hat{Z}_J)^{\zeta_\varepsilon} \big] \leq \widetilde{\mathbb{E}}_J\big[\hat{Z}_J \big]^{\zeta_\varepsilon} \widetilde{\mathbb{E}}_J\left[\left(\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}_J} \right)^{\frac{1}{1-\zeta_\varepsilon}} \right]^{1-\zeta_\varepsilon}. \qquad (2.22)$$

To bound the second factor, note that for $J = \{j_1, \ldots, j_\ell\}$, the same calculation as in (2.4) gives (recall that $k = t_\varepsilon\beta^{-2}$ and $\delta = a_\varepsilon\beta$)

$$\widetilde{\mathbb{E}}_J\left[\left(\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}_J} \right)^{\frac{1}{1-\zeta_\varepsilon}} \right]^{1-\zeta_\varepsilon} = \mathbb{E}\left[\left(\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}_J} \right)^{\frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}} \right]^{1-\zeta_\varepsilon} = \mathbb{E}\left[\prod_{i=1}^{\ell} \prod_{n \in B_{j_i}} e^{\frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}\delta\omega_n + \frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}\Lambda(-\delta)} \right]^{1-\zeta_\varepsilon}$$

$$= e^{(1-\zeta_\varepsilon)\left[\Lambda(\frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}\delta) + \frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}\Lambda(-\delta) \right]k\ell} < e^{(1+\frac{\varepsilon}{20})\frac{a_\varepsilon^2}{2}\frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}t_\varepsilon|J|}, \qquad (2.23)$$

where the last inequality holds, by (1.4), for β small enough, say $\beta \in (0, \beta_2(\varepsilon))$, for some $\beta_2(\varepsilon) > 0$. To bound the first factor in (2.22), recall that $a_\varepsilon > c_\varepsilon$ and note that, for every $n \in \bigcup_{i \in J} B_i$, for $\beta < \beta_2(\varepsilon)$ we have

$$\widetilde{\mathbb{E}}_J[z_d] = e^{\Lambda(\beta-\delta) - \Lambda(\beta) - \Lambda(-\delta) + c_\varepsilon\beta^2} = e^{-(a_\varepsilon - c_\varepsilon)\beta^2 + o(\beta^2)} < 1, \qquad (2.24)$$

provided $\beta_2(\varepsilon)$ is chosen small enough. Furthermore, for $d, f \in B_i$ for any $i \in J$,

$$\widetilde{\mathbb{E}}_J[Z_{d,f}] = \widetilde{u}(f-d), \qquad \text{where} \qquad \widetilde{u}(n) := \widetilde{\mathbb{E}}\big[Z_{n,\beta,c_\varepsilon\beta^2}^{\mathrm{c},\omega}\big]. \qquad (2.25)$$

Therefore from (2.20), we obtain

$$\widetilde{\mathbb{E}}_J[\hat{Z}_J] \leq \sum_{\substack{d_1,f_1\in B_{j_1} \\ d_1\leq f_1}} \cdots \sum_{\substack{d_{\ell-1},f_{\ell-1}\in B_{j_{\ell-1}} \\ d_{\ell-1}\leq f_{\ell-1}}} \sum_{d_\ell\in B_{j_\ell}=B_m} \left(\prod_{i=1}^{\ell} K(d_i-f_{i-1})\widetilde{u}(f_i-d_i) \right). \qquad (2.26)$$

This expression is nice because it would be the probability of a renewal event, if $\widetilde{u}$ were replaced by

$$u(n) := \mathrm{P}(n \in \tau).$$

We are going to make this replacement in the next step, but first we need some estimates.

Since $u(n) \to 1/\mu$ as $n \to \infty$ by the renewal theorem, we can choose $C_1 \in (0,\infty)$ such that

$$\frac{1}{C_1\mu} \leq u(n) \leq \frac{C_1}{\mu} \qquad \forall n \in \mathbb{N}. \qquad (2.27)$$

Furthermore, by relation (1.1) and the fact that slowly varying functions are asymptotically dominated by any polynomial, it follows that there exists $C_{2,\varepsilon} \in (0,\infty)$ be such that

$$K(n) \leq \frac{C_{2,\varepsilon}}{n^{1+\alpha-\frac{\varepsilon}{2}}}, \qquad \forall n \in \mathbb{N}. \qquad (2.28)$$

Finally, since $0 \leq |\tau \cap [1,n]| \leq k$ for every $n \in \{1,\ldots,k=t_\varepsilon\beta^{-2}\}$, recalling (2.5) we obtain

$$e^{-2(a_\varepsilon-c_\varepsilon)t_\varepsilon}u(n) \leq \widetilde{u}(n) = \widetilde{\mathbb{E}}\big[Z_{n,\beta,c_\varepsilon\beta^2}^{\mathrm{c},\omega}\big] = \mathrm{E}\Big[e^{\left(-(a_\varepsilon-c_\varepsilon)\beta^2+o(\beta^2)\right)|\tau\cap[1,n]|}\mathbf{1}_{\{n\in\tau\}}\Big] \\ \leq u(n) \leq 1, \qquad (2.29)$$

for $\beta \in (0,\beta_2(\varepsilon))$, because $a_\varepsilon > c_\varepsilon$ (recall (2.24)).

STEP 3. We now replace $\widetilde{u}(\cdot)$ in (2.26) by a suitable multiple of $u(\cdot)$. However, this is only possible for occupied blocks that are surrounded by occupied blocks. The blocks with unoccupied neighboring blocks have to be dealt with in a different way.

Let us be more precise. Fix i such that both $j_i \in J$ and $j_i - 1 \in J \cup \{0\}$. Then we claim that the terms in (2.26) with $|d_i - f_{i-1}| \leq \frac{\varepsilon}{10}k$ give the main contribution. Indeed, setting $\overline{f}_{i-1} = (i-1)k$, $\overline{d}_i = (i-1)k+1$,

$$\widetilde{u}(\overline{f}_{i-1}-\overline{d}_{i-1}) K(\overline{d}_i-\overline{f}_{i-1}) \widetilde{u}(\overline{f}_i-\overline{d}_i) \geq \big(e^{-2(a_\varepsilon-c_\varepsilon)t_\varepsilon}\frac{1}{C_1\mu}\big)^2 K(1), \qquad (2.30)$$

where we used the lower bounds in (2.29) and (2.27). Using instead the upper bound in (2.29) that $\widetilde{u}(\cdot) \leq 1$, together with (2.28), yields

$$\sum_{\substack{f_{i-1}\in B_{j_{i-1}},\, d_i\in B_{j_i} \\ |d_i-f_{i-1}|>\frac{\varepsilon}{10}k}} \widetilde{u}(f_{i-1}-d_{i-1}) K(d_i-f_{i-1}) \widetilde{u}(f_i-d_i) \leq \frac{k^2\,C_{2,\varepsilon}}{(\frac{\varepsilon}{10}k)^{1+\alpha-\frac{\varepsilon}{2}}} = \frac{10^{1+\alpha}\,C_{2,\varepsilon}}{\varepsilon^{1+\alpha}\,k^{\alpha-\frac{\varepsilon}{2}-1}}.$$

$$(2.31)$$

Recall that ε is chosen small enough, so that $\alpha - \frac{\varepsilon}{2} > 1$, and $k = t_\varepsilon\beta^{-2} \to \infty$ as $\beta \downarrow 0$. Therefore, we can find $\beta_3(\varepsilon) \in (0,\infty)$ such that for every $\beta \in (0,\beta_3(\varepsilon))$, the contribution of the terms in (2.26) with $|d_i - f_{i-1}| > \frac{\varepsilon}{10}k$ is smaller than the contribution of the terms with $|d_i - f_{i-1}| \leq \frac{\varepsilon}{10}k$ (comparing (2.30) and (2.31)).

To summarize: when $\beta < \beta_3(\varepsilon)$, the right hand side of (2.26) can be bounded from above by restricting the sum to $|d_i - f_{i-1}| \le \frac{\varepsilon}{10}k$ for every i such that both $j_i \in J$ and $j_i - 1 \in J$, provided one introduces a multiplicative factor of 2 for each such i.

Let us now set

$$\overset{\circ}{J} := \left\{ j \in J : \ j - 1 \in J \cup \{0\} \text{ and } j + 1 \in J \cup \{m + 1\} \right\}.$$

If $j \in \overset{\circ}{J}$, say $j = j_i$ for some $1 \le i \le |J| = l$, then we have restricted the summation in (2.26) to both $|d_i - f_{i-1}| \le \frac{\varepsilon}{10}k$ and $|d_{i+1} - f_i| \le \frac{\varepsilon}{10}k$, which yields $f_i - d_i > (1 - \frac{\varepsilon}{5})k$. Recalling (2.25), (2.19) and (2.27), we can then bound

$$\widetilde{u}(f_i - d_i) \le D_\varepsilon u(f_i - d_i), \qquad \text{where} \qquad D_\varepsilon := C_1(1 + \varepsilon) e^{-\frac{1}{\mu}(a_\varepsilon - c_\varepsilon)(1 - \frac{\varepsilon}{5})t_\varepsilon}. \qquad (2.32)$$

This is the crucial replacement, after which we can remove the restrictions $|d_i - f_{i-1}| \le \frac{\varepsilon}{10}k$ in (2.26) to get an upper bound. (Note that D_ε can be made arbitrarily small by choosing t_ε large, because $a_\varepsilon > c_\varepsilon$, so we may assume henceforth that $D_\varepsilon < 1$.)

It only remains to deal with the terms $j \in J \setminus \overset{\circ}{J}$, i.e. the occupied blocks that have at least one neighboring block which is unoccupied. For these blocks, we replace $\widetilde{u}(f_i - d_i)$ by $u(f_i - d_i)$, thanks to (2.29). Gathering the above considerations, we can upgrade (2.26) to

$$\widetilde{\mathbb{E}}_J\big[\hat{Z}_J\big] \le 2^{|J|}(D_\varepsilon)^{|\overset{\circ}{J}|} \sum_{\substack{d_1, f_1 \in B_{j_1} \\ d_1 \le f_1}} \cdots \sum_{\substack{d_{\ell-1}, f_{\ell-1} \in B_{j_{\ell-1}} \\ d_{\ell-1} \le f_{\ell-1}}} \sum_{d_\ell \in B_{j_\ell} = B_m} \left(\prod_{i=1}^{\ell} K(d_i - f_{i-1}) u(f_i - d_i) \right), \qquad (2.33)$$

where we note that the summation is now the probability of a renewal event.

STEP 4. We now deal with the gaps between occupied blocks. Let $i \in \{1, \dots, \ell\}$ be such that $j_i \in J$ but $j_i - 1 \notin J \cup \{0\}$, that is $j_i - j_{i-1} \ge 2$. Since $d_i \in B_{j_i}$ and $f_{i-1} \in B_{j_{i-1}}$, we have $d_i - f_{i-1} \ge (j_i - j_{i-1} - 1)k$. Then it follows from (2.28) that

$$K(d_i - f_{i-1}) \le \frac{C_{2,\varepsilon}}{k^{1+\alpha-\frac{\varepsilon}{2}}} \frac{1}{(j_i - j_{i-1} - 1)^{1+\alpha-\frac{\varepsilon}{2}}} \le \frac{2^{1+\alpha} C_{2,\varepsilon}}{k^{1+\alpha-\frac{\varepsilon}{2}}} \frac{1}{(j_i - j_{i-1})^{1+\alpha-\frac{\varepsilon}{2}}}, \qquad (2.34)$$

where the last inequality holds because $n - 1 \ge \frac{n}{2}$ for $n \ge 2$. Furthermore, by (2.27),

$$u(f_{i-1} - d_{i-1}) \le C_1^2 \, u(\overline{f}_{i-1} - d_{i-1}), \qquad u(f_i - d_i) \le C_1^2 \, u(f_i - \overline{d}_i), \qquad (2.35)$$

where we recall that $\overline{f}_{i-1} = (i-1)k$ and $\overline{d}_i = (i-1)k + 1$ denote respectively the last point of the block B_{i-1} and the first point of the block B_i.

We can now insert the bounds (2.34), (2.35) into (2.33), starting with the smallest $i \in \{1, \dots, \ell\}$ such that $j_i - j_{i-1} \ge 2$ (if any), and then proceeding in increasing order. (When there are two consecutive gaps, that is, when both $j_i - j_{i-1} \ge 2$ and $j_{i-1} - j_{i-2} \ge 2$, the first bound in (2.35) becomes $u(f_{i-1} - \overline{d}_{i-1}) \le C_1^2 \, u(\overline{f}_{i-1} - \overline{d}_{i-1})$, because we have already replaced d_{i-1} by $\overline{d}_{i-1}$ in the previous step.) In this way, we eliminate all the terms in (2.33) that depend on f_{i-1} and d_i, and the double sum over f_{i-1} and d_i can be removed by introducing a multiplicative factor k^2.

Having eliminated the gaps, we are left with "clusters of consecutive occupied blocks": more precisely, the surviving sums in (2.33) are those over variables f_{j-1}, d_j with $a < j \le b$, for every maximal interval $\{a, \dots, b\} \subseteq \{1, \dots, \ell\}$. These sums can be factorized and each such interval gives a contribution equals to the probability (hence bounded by 1) that the

renewal process visits a clusters of consecutive occupied blocks. Therefore

$$\widetilde{\mathbb{E}}_J\big[\hat{Z}_J\big] \;\leq\; 2^{|J|}\,(D_\varepsilon)^{|\mathring{J}|} \prod_{i\in\{1,\dots,\ell\}:\ j_i-j_{i-1}\geq 2} C_1^4\,k^2\,\frac{2^{1+\alpha}\,C_{2,\varepsilon}}{k^{1+\alpha-\frac{\varepsilon}{2}}}\,\frac{1}{(j_i-j_{i-1})^{1+\alpha-\frac{\varepsilon}{2}}}$$
$$= \;2^{|J|}\,(D_\varepsilon)^{|\mathring{J}|}\left(\frac{2^{1+\alpha}\,C_1^4\,C_{2,\varepsilon}}{k^{\alpha-\frac{\varepsilon}{2}-1}}\right)^{|\{i\in\{1,\dots,\ell\}:\ j_i-j_{i-1}\geq 2\}|}\prod_{i\in\{1,\dots,\ell\}}\frac{1}{(j_i-j_{i-1})^{1+\alpha-\frac{\varepsilon}{2}}}\,. \tag{2.36}$$

Next observe that

$$|J\setminus \mathring{J}| \;=\; |J|-|\mathring{J}| \;\leq\; 2\,|\{i\in\{1,\dots,\ell\}:\ j_i-j_{i-1}\geq 2\}|\,, \tag{2.37}$$

since each point in $J\setminus \mathring{J}$ is either the starting point or ending point of a gap, i.e., a pair $\{j_{i-1},j_i\}$ with $j_i-j_{i-1}\geq 2$. Since $\alpha-\frac{\varepsilon}{2}>1$, by our choice of ε, and $k=t_\varepsilon\beta^{-2}\to\infty$ as $\beta\downarrow 0$, there exits $\beta_4(\varepsilon)\in(0,\infty)$ such that for every $\beta\in(0,\beta_4(\varepsilon))$, we have

$$\frac{2^{1+\alpha}\,C_1^4\,C_{2,\varepsilon}}{k^{\alpha-\frac{\varepsilon}{2}-1}} \;\leq\; (D_\varepsilon)^2\,.$$

Since $D_\varepsilon<1$, it follows from (2.36) and (2.37) that

$$\widetilde{\mathbb{E}}_J\big[\hat{Z}_J\big] \;\leq\; (2D_\varepsilon)^{|J|}\prod_{i\in\{1,\dots,\ell\}}\frac{1}{(j_i-j_{i-1})^{1+\alpha-\frac{\varepsilon}{2}}}\,. \tag{2.38}$$

STEP 5. We now conclude the proof. Looking back at (2.21), (2.22), (2.23) and (2.38), we can write

$$\mathbb{E}\big[(Z_{N,\beta,c_\varepsilon\beta^2}^{c,\omega})^{\zeta_\varepsilon}\big] \;\leq\; \sum_{J\subseteq\{1,\dots,m\}:\ m\in J}\left(\prod_{i\in\{1,\dots,\ell\}}\frac{G_\varepsilon}{(j_i-j_{i-1})^{(1+\alpha-\frac{\varepsilon}{2})\zeta_\varepsilon}}\right), \tag{2.39}$$

where, recalling the definition (2.32) of D_ε, we have set

$$G_\varepsilon \;:=\; (2D_\varepsilon)^{\zeta_\varepsilon}\,e^{(1+\frac{\varepsilon}{20})\frac{a_\varepsilon^2}{2}\frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}t_\varepsilon} \;=\; \big(2(1+\varepsilon)C_1\big)^{\zeta_\varepsilon}\,e^{\frac{\zeta_\varepsilon}{\mu}(a_\varepsilon-c_\varepsilon)\frac{\varepsilon}{5}t_\varepsilon}\,e^{\big\{\frac{\zeta_\varepsilon}{\mu}(c_\varepsilon-a_\varepsilon)+(1+\frac{\varepsilon}{20})\frac{a_\varepsilon^2}{2}\frac{\zeta_\varepsilon}{1-\zeta_\varepsilon}\big\}t_\varepsilon}\,.$$

Let us now replace the value of $a_\varepsilon=\frac{1-\zeta_\varepsilon}{\mu}$ that we fixed in (2.16) (recall from (2.10) that this value is optimal in minimizing the second exponential, if we neglect the term $\varepsilon/20$), getting

$$G_\varepsilon \;=\; \big(2(1+\varepsilon)C_1\big)^{\zeta_\varepsilon}\,e^{\frac{\zeta_\varepsilon}{\mu}(\frac{1-\zeta_\varepsilon}{\mu}-c_\varepsilon)\frac{\varepsilon}{5}t_\varepsilon}\,e^{\frac{\zeta_\varepsilon}{\mu}(c_\varepsilon-(1-\frac{\varepsilon}{20})\frac{1-\zeta_\varepsilon}{2\mu})t_\varepsilon}\,.$$

We now substitute in the value of $c_\varepsilon=(1-\varepsilon)\frac{\alpha}{1+\alpha}\frac{1}{2\mu}$ set in (2.12), and substitute inside the parentheses in the exponential the value of $\zeta_\varepsilon=\frac{1}{1+\alpha}+\frac{\varepsilon}{2}\frac{\alpha}{1+\alpha}$ set in (2.14), which gives

$$G_\varepsilon \;<\; \big(2(1+\varepsilon)C_1\big)^{\zeta_\varepsilon}\,e^{\frac{\zeta_\varepsilon}{\mu}(\frac{\alpha}{1+\alpha}\frac{1}{2\mu})\frac{\varepsilon}{5}t_\varepsilon}\,e^{-\frac{\zeta_\varepsilon}{\mu}\frac{9\varepsilon}{20}\frac{\alpha}{1+\alpha}\frac{1}{2\mu}t_\varepsilon} \;=\; \big(2(1+\varepsilon)C_1\big)^{\zeta_\varepsilon}\,e^{-\frac{\zeta_\varepsilon}{\mu}\frac{\alpha}{1+\alpha}\frac{1}{2\mu}\frac{\varepsilon}{4}t_\varepsilon}\,.$$

We are ready for the final step: by the definition (2.14) of ζ_ε, and the fact that ε has been fixed small enough, we have $(1+\alpha-\frac{\varepsilon}{2})\zeta_\varepsilon>1$. Since the upper bound for G_ε vanishes as $t_\varepsilon\to+\infty$, we can fix $t_\varepsilon\in(0,\infty)$ large enough, depending only on ε, such that

$$\sum_{n=1}^\infty \frac{G_\varepsilon}{n^{(1+\alpha-\frac{\varepsilon}{2})\zeta_\varepsilon}} \;<\; 1\,.$$

The right hand side of (2.39) is then smaller than one, because it can be recognized as the probability of visiting m for a renewal process, with return distribution given by $\widetilde{K}(n):=G_\varepsilon/n^{(1+\alpha-\frac{\varepsilon}{2})\zeta_\varepsilon}$ and $\widetilde{K}(\infty)=1-\sum_{n\in\mathbb{N}}\widetilde{K}(n)>0$. In conclusion, we have shown

that for any $\varepsilon > 0$ small enough, we can find $\beta_0(\varepsilon) := \min\{\beta_1(\varepsilon), \beta_2(\varepsilon), \beta_3(\varepsilon), \beta_4(\varepsilon)\} \in (0, \infty)$ and $t_\varepsilon \in (0, \infty)$, such that for all $\beta \in (0, \beta_0(\varepsilon))$

$$\mathbb{E}\big[\big(Z^{c,\omega}_{mt_\varepsilon \beta^{-2}, \beta, c_\varepsilon \beta^2}\big)^{\zeta_\varepsilon}\big] \leq 1 \qquad \forall\, m \in \mathbb{N},$$

where c_ε was defined in (2.12). This establishes (2.13) and concludes the proof.

3. ON THE PINNING MODEL: UPPER BOUND

3.1. **A lower bound on the free energy.** The strategy of the proof has been outlined in Section 1.3. First we prove the lower bound on the free energy of the pinning model stated in (1.21), which we restate here as a lemma.

Lemma 3.1. *For every $c \in \mathbb{R}$*

$$\liminf_{\beta \downarrow 0} \frac{\mathrm{F}(\beta, c\beta^2)}{\beta^2} \geq \frac{1}{\mu}\left[c - \frac{1}{2\mu}\right]. \tag{3.1}$$

Proof. A naive lower bound on the free energy is to apply Jensen's inequality, interchanging the log in (1.12) with the expectation $\mathbf{E}$ over the renewal process that appear in the partition function (recall (1.7)). However, this only leads to a trivial bound, as the expression in the exponential in (1.7) is a linear function of the disorder ω. To get a better bound, before applying Jensen we perform a partial integration over a subset of the renewal points, obtaining a "coarse-grained Hamiltonian" that is no longer linear in ω. This has certain analogies with Theorem 5.2 in [20]. The details are as follows.

For $q \in \mathbb{N}$ and $\ell \geq q$, we define $H^{(q)}_{\ell,\omega}$ to be the free energy of the constrained model of size ℓ conditioned to have exactly q returns:

$$H^{(q)}_{\ell,\omega} := \log \mathrm{E}\big[e^{\sum_{n=1}^{\ell}(\beta\omega_n - \Lambda(\beta) + h)\mathbf{1}_{\{n\in\tau\}}}\big|\tau_q = \ell\big].$$

Let $\tau^{(q)} := \{\tau^{(q)}_n\}_{n\in\mathbb{N}_0}$ with $\tau^{(q)}_n := \tau_{nq}$, which is a renewal process that keeps one in every q renewal points in τ. We also set

$$\mathcal{N}^{(q)}_N := \max\{n \in \mathbb{N}_0 : \tau^{(q)}_n \leq N\} = |\tau^{(q)} \cap [1, N]|.$$

By requiring $N \in \tau^{(q)}$ and taking conditional expectation w.r.t. $\tau^{(q)}$, we obtain

$$Z_{N,\omega} \geq \mathrm{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n - \Lambda(\beta)+h)\mathbf{1}_{\{n\in\tau\}}}\mathbf{1}_{\{N\in\tau^{(q)}\}}\right] = \mathrm{E}\left[\exp\left(\sum_{j=1}^{\mathcal{N}^{(q)}_N} H^{(q)}_{\tau^{(q)}_j - \tau^{(q)}_{j-1}, \vartheta^{\tau^{(q)}_{j-1}}\omega}\right)\mathbf{1}_{\{N\in\tau^{(q)}\}}\right],$$

where $\vartheta^i\omega = \{\omega_{n+i}\}_{n\in\mathbb{N}}$ defines a shift of the disorder ω.

Since $\mathrm{E}[\tau^{(q)}_1] = q\mu$ and $\tau^{(q)}_{\mathcal{N}^{(q)}_N} \leq N \leq \tau^{(q)}_{\mathcal{N}^{(q)}_N + 1}$, by the strong law of large numbers

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{\mathcal{N}^{(q)}_N} f(\tau^{(q)}_j - \tau^{(q)}_{j-1}) = \frac{1}{q\mu}\mathrm{E}\big[f(\tau^{(q)}_1)\big], \qquad \mathrm{P}\text{-a.s. and in } L^1(\mathrm{dP}), \tag{3.2}$$

for every function $f : \mathbb{N} \to \mathbb{R}$ such that $f(\tau^{(q)}_1) \in L^1(\mathrm{dP})$. Since $\mathrm{P}(N \in \tau^{(q)}) \to \frac{1}{q\mu} > 0$ as $N \to \infty$, by the renewal theorem, it is not difficult to deduce that

$$\lim_{N\to\infty} \mathrm{E}\left[\frac{1}{N}\sum_{j=1}^{\mathcal{N}^{(q)}_N} f(\tau^{(q)}_j - \tau^{(q)}_{j-1})\,\bigg|\, N \in \tau^{(q)}\right] = \frac{1}{q\mu}\mathrm{E}\big[f(\tau^{(q)}_1)\big]. \tag{3.3}$$

We are going to apply this to $f(\ell) := \mathbb{E}[H^{(q)}_{\ell,\omega}]$. Recalling (1.12), by Jensen's inequality we get

$$\mathrm{F}(\beta, h) \geq \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}\mathrm{E}\left[\sum_{j=1}^{\mathcal{N}^{(q)}_N} H^{(q)}_{\tau^{(q)}_j - \tau^{(q)}_{j-1}, \vartheta^{\tau^{(q)}_{j-1}} \omega} \middle| N \in \tau^{(q)} \right] = \frac{1}{q\mu} \mathbb{E}\mathrm{E}\left[H^{(q)}_{\tau^{(q)}_1, \omega} \right]$$

$$= \frac{1}{q\mu} \sum_{N=1}^{\infty} \mathrm{P}(\tau_q = N) \, \mathbb{E}\left[\log \mathrm{E}\left[e^{\sum_{n=1}^N (\beta\omega_n - \Lambda(\beta) + h)\mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] \right] \tag{3.4}$$

$$= \frac{h - \Lambda(\beta)}{\mu} + \frac{1}{q\mu} \sum_{N=1}^{\infty} \mathrm{P}(\tau_q = N) \, \mathbb{E}\left[\log \mathrm{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] \right].$$

The rest of this section is devoted to studying this lower bound as $\beta, h \downarrow 0$.

Let us denote

$$u_{N,q}(n) := \mathrm{P}(n \in \tau | \tau_q = N), \qquad u_{N,q}(n,m) := \mathrm{P}(n \in \tau, \, m \in \tau | \tau_q = N). \tag{3.5}$$

By Jensen's inequality,

$$\log \mathrm{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] - \sum_{n=1}^N \beta\omega_n u_{N,q}(n) \geq 0. \tag{3.6}$$

Therefore we can apply Fatou's Lemma in (3.4) to obtain

$$\liminf_{\beta \downarrow 0} \frac{\mathrm{F}(\beta, c\beta^2)}{\beta^2} \tag{3.7}$$

$$= \frac{c - \frac{1}{2}}{\mu} + \liminf_{\beta \downarrow 0} \frac{1}{q\mu} \sum_{N=1}^{\infty} \mathrm{P}(\tau_q = N) \, \mathbb{E}\left[\frac{\log \mathrm{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] - \sum_{n=1}^N \beta\omega_n u_{N,q}(n)}{\beta^2} \right]$$

$$\geq \frac{c - \frac{1}{2}}{\mu} + \frac{1}{q\mu} \sum_{N=1}^{\infty} \mathrm{P}(\tau_q = N) \, \mathbb{E}\left[\liminf_{\beta \downarrow 0} \frac{\log \mathrm{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] - \sum_{n=1}^N \beta\omega_n u_{N,q}(n)}{\beta^2} \right].$$

By Taylor expansion, for fixed disorder ω and as $\beta \downarrow 0$, we have

$$\mathrm{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] = 1 + \beta \sum_{n=1}^N \omega_n u_{N,q}(n) + \frac{1}{2}\beta^2 \sum_{m,n=1}^N \omega_m \omega_n u_{N,q}(m,n) + o(\beta^2).$$

Since $\log(1 + x) = x - \frac{1}{2}x^2 + o(x^2)$ as $x \downarrow 0$, we obtain

$$\mathbb{E}\left[\liminf_{\beta \downarrow 0} \frac{\log \mathrm{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n \in \tau\}}} \middle| \tau_q = N \right] - \sum_{n=1}^N \beta\omega_n u_{N,q}(n)}{\beta^2} \right]$$

$$= \mathbb{E}\left[\frac{1}{2} \sum_{m,n=1}^N \omega_m \omega_n \big(u_{N,q}(m,n) - u_{N,q}(m) u_{N,q}(n) \big) \right]$$

$$= \frac{1}{2} \sum_{n=1}^N (u_{N,q}(n) - u_{N,q}(n)^2) = \frac{q}{2} - \frac{1}{2} \sum_{n=1}^N u_{N,q}(n)^2, \quad (3.8)$$

where the last equality holds, by (3.5), because $\sum_{n=1}^N \mathbf{1}_{\{n \in \tau\}} = q$ on the event $\{\tau_q = N\}$.

Note that

$$\sum_{n=1}^N u_{N,q}(n)^2 = \mathrm{E}\big[|\tau \cap \widetilde{\tau} \cap (0, N]| \,\big|\, \tau_q = \widetilde{\tau}_q = N \big], \tag{3.9}$$

where $\widetilde{\tau}$ is an independent copy of τ. Intuitively, since each renewal process $\tau, \widetilde{\tau}$ has mean return time μ, the expression in (3.9) should be of the order q/μ. In order to prove it, we fix $\eta > 0$. Decomposing the right hand side in (3.9) according to whether $|\tau \cap \widetilde{\tau} \cap (0, N]| \leq (1 + \eta)N/\mu^2$ or not, and noting that $|\tau \cap (0, N]| \, \mathbf{1}_{\{\tau_q = N\}} = q$, we obtain

$$
\sum_{n=1}^{N} u_{N,q}(n)^2 \;\leq\; \frac{(1+\eta)N}{\mu^2} + q \frac{\mathrm{P}\big(|\tau \cap \widetilde{\tau} \cap (0, N]| > (1+\eta)N/\mu^2, \tau_q = \widetilde{\tau}_q = N\big)}{\mathrm{P}(\tau_q = \widetilde{\tau}_q = N)}
$$

$$
\leq\; \frac{(1+\eta)N}{\mu^2} + q \frac{\sqrt{\mathrm{P}\big(|\tau \cap \widetilde{\tau} \cap (0, N]| > (1+\eta)N/\mu^2\big) \cdot \mathrm{P}(\tau_q = N)}}{\mathrm{P}(\tau_q = N)^2} \,,
$$

where we used Cauchy-Schwarz inequality for the second inequality. We note that $\tau \cap \widetilde{\tau}$ is a renewal process with finite mean μ^2. Therefore, by a standard Cramer large deviation estimate [15, Theorem 2.2.3], there exist $C_\eta \in (0, \infty)$ such that for all $q \in \mathbb{N}$ large enough

$$
\max_{N \in ((1-\eta)q\mu, (1+\eta)q\mu)} \mathrm{P}\big(|\tau \cap \widetilde{\tau} \cap (0, N]| > (1+\eta)N/\mu^2\big) \leq e^{-C_\eta \, q},
$$

and hence, uniformly in $N \in ((1 - \eta)q\mu, (1 + \eta)q\mu)$, we have

$$
\sum_{n=1}^{N} u_{N,q}(n)^2 \leq q \frac{(1+\eta)^2}{\mu} + q \frac{e^{-\frac{1}{2}C_\eta \, q}}{\mathrm{P}(\tau_q = N)}. \tag{3.10}
$$

We finally plug the bound (3.10) into (3.8), and then into (3.7) (note that the denominator $\mathrm{P}(\tau_q = N)$ in (3.10) gets simplified). Restricting the summation to $N \in ((1-\eta)q\mu, (1+\eta)q\mu)$, thanks to (3.6), we obtain, for q large enough,

$$
\liminf_{\beta \downarrow 0} \frac{\mathrm{F}(\beta, c\beta^2)}{\beta^2} \geq \frac{c - \frac{1}{2}}{\mu} + \frac{1}{2\mu} \Big(1 - \frac{(1+\eta)^2}{\mu} \Big) p_\eta(q) - \frac{1}{2\mu} (2\eta q\mu) \, e^{-\frac{1}{2}C_\eta \, q},
$$

where $p_\eta(q) := \mathrm{P}\big(\tau_q \in ((1-\eta)q\mu, (1+\eta)q\mu)\big) \to 1$ as $q \to \infty$ by the law of large numbers. First letting $q \uparrow \infty$ and then letting $\eta \downarrow 0$ gives the desired bound (3.1). $\qquad \square$

3.2. **Completing the proof.** Recall the smoothing inequality (1.19)

$$
0 \leq \mathrm{F}^{\mathrm{pin}}(\beta, h) \leq \frac{1+\alpha}{2} A_{\beta, h - h_c(\beta)} \frac{(h - h_c(\beta))^2}{\beta^2} \,. \tag{3.11}
$$

Setting $h = c\beta^2$, we then have

$$
\liminf_{\beta \downarrow 0} \frac{\mathrm{F}(\beta, c\beta^2)}{\beta^2} \leq \frac{1+\alpha}{2} \Big[c - \Big(\limsup_{\beta \downarrow 0} \frac{h_c(\beta)}{\beta^2} \Big) \Big]^2. \tag{3.12}
$$

Combining (3.12) with (3.1) gives

$$
\frac{1+\alpha}{2} \Big[c - \Big(\limsup_{\beta \downarrow 0} \frac{h_c(\beta)}{\beta^2} \Big) \Big]^2 \geq \frac{1}{\mu} \Big(c - \frac{1}{2\mu} \Big) \qquad \forall \, c \in \mathbb{R}.
$$

We can rewrite this inequality as

$$
A c^2 + B c + C \geq 0 \qquad \forall \, c \in \mathbb{R},
$$

with

$$
A = \frac{1+\alpha}{2}, \quad B = -(1+\alpha) \limsup_{\beta \downarrow 0} \frac{h_c(\beta)}{\beta^2} - \frac{1}{\mu}, \quad C = \frac{1+\alpha}{2} \Big(\limsup_{\beta \downarrow 0} \frac{h_c(\beta)}{\beta^2} \Big)^2 + \frac{1}{2\mu^2}.
$$

(Note that the $\limsup$ is finite, because $0 \le h_c(\beta) \le \Lambda(\beta) = \frac{1}{2}\beta^2 + o(\beta^2)$ for every $\beta \ge 0$, cf. [20, Proposition 5.1].) Then we must have $B^2 - 4AC \le 0$ and this readily leads to

$$\limsup_{\beta \downarrow 0} \frac{h_c(\beta)}{\beta^2} \le \frac{1}{2\mu}\frac{\alpha}{1+\alpha},$$

which is precisely the upper bound in (1.17).

4. On the Copolymer Model: Lower Bound

We now consider the copolymer model, with constrained partition function

$$Z_{N,\lambda,h}^{\mathrm{cop,c},\omega} = \mathrm{E}\Big[e^{-2\lambda \sum_{n=1}^{N}(\omega_n + h_a(\lambda) - h)\mathbf{1}_{\{\varepsilon_n = -1\}}}\mathbf{1}_{\{N \in \tau\}}\Big].$$

where we recall that $h_a(\lambda) = h_a^{\mathrm{cop}}(\lambda) = (2\lambda)^{-1}\Lambda(-2\lambda)$, cf. (1.10). The method and steps are the same as for the pinning model, discussed in Section 2, with only minor differences. In fact, replacing ω by $-\omega$, 2λ by β and $2\lambda h$ by h casts the copolymer partition function in exactly the same form as the random pinning model, the only difference being $\mathbf{1}_{\{n \in \tau\}}$ in the pinning partition function replaced by $\mathbf{1}_{\{\varepsilon_n = -1\}}$, cf. (1.8).

Relations (2.5), (2.6) and (2.7) still hold with $|\tau \cap [1,k]|$ replaced by $\sum_{n=1}^{k} \mathbf{1}_{\{\varepsilon_n = -1\}}$ and μ replaced by 2. Following the same procedure, it suffices to show

$$\liminf_{N \to \infty} \mathbb{E}\big[\big(Z_{N,\lambda,h}^{\mathrm{c},\omega}\big)^\zeta\big] < \infty \tag{4.1}$$

for

$$h = h_\varepsilon := c_\varepsilon \lambda := (1-\varepsilon)\frac{\alpha}{2(\alpha+1)}\lambda \qquad \text{and} \qquad \zeta = \zeta_\varepsilon := \frac{1}{1+\alpha} + \frac{\varepsilon}{2}\frac{\alpha}{1+\alpha}. \tag{4.2}$$

The only major difference in the calculation is in (2.20). In the copolymer case, this is replaced by

$$\hat{Z}_J := \sum_{\substack{d_1,f_1 \in B_{j_1} \\ d_1 \le f_1}} \cdots \sum_{\substack{d_{\ell-1},f_{\ell-1} \in B_{j_{\ell-1}} \\ d_{\ell-1} \le f_{\ell-1}}} \sum_{d_\ell \in B_{j_\ell} = B_m} \left(\prod_{i=1}^{\ell} K(d_i - f_{i-1}) z_{f_{i-1},d_i}^{\mathrm{cop}} Z_{d_i,f_i} \right), \tag{4.3}$$

where $z_{f_{i-1},d_i}^{\mathrm{cop}} := z_{(f_{i-1},d_i]}^{\mathrm{cop}}$, with

$$z_I^{\mathrm{cop}} := \frac{1 + e^{-2\lambda \sum_{n \in I \cap \mathbb{N}}(\omega_n + h_a(\lambda) - h)}}{2} \qquad \text{for any } I \subset (0,\infty).$$

Recalling the notation $\overline{f}_{i-1} := j_{i-1}k$ and $\overline{d}_i := (j_i - 1)k$, with $k = t_\varepsilon \beta^{-2} = t_\varepsilon \lambda^{-2}/4$, we use that (see [31, (3.16)])

$$z_{f_{i-1},d_i}^{\mathrm{cop}} \le 2\, z_{(f_{i-1},\overline{f}_{i-1}] \cup (\overline{d}_i,d_i]}^{\mathrm{cop}}\, z_{f_{i-1},\overline{d}_i}^{\mathrm{cop}}. \tag{4.4}$$

Following (2.21) we have that

$$\mathbb{E}\big[\big(Z_{N,\lambda,c_\varepsilon\lambda}^{\mathrm{c},\omega}\big)^{\zeta_\varepsilon}\big] \le \sum_{J \subseteq \{1,\ldots,m\}:\ m \in J} \mathbb{E}\big[\big(\hat{Z}_J\big)^{\zeta_\varepsilon}\big], \tag{4.5}$$

for ζ_ε chosen in (4.2) (the same as in (2.14)). Substituting (4.4) into (4.3), we have

$$\mathbb{E}\big[\big(\hat{Z}_J\big)^{\zeta_\varepsilon}\big] \le 2^\ell \prod_{i=1}^{\ell} \mathbb{E}\left[\left(z_{\overline{f}_{i-1},\overline{d}_i}^{\mathrm{cop}}\right)^{\zeta_\varepsilon}\right] \mathbb{E}\big[\big(\breve{Z}_J\big)^{\zeta_\varepsilon}\big],$$

where

$$\breve{Z}_J := \sum_{\substack{d_1,f_1 \in B_{j_1} \\ d_1 \leq f_1}} \cdots \sum_{\substack{d_{\ell-1},f_{\ell-1} \in B_{j_{\ell-1}} \\ d_{\ell-1} \leq f_{\ell-1}}} \sum_{d_\ell \in B_{j_\ell} = B_m} \left(\prod_{i=1}^\ell K(d_i - f_{i-1}) z^{\mathrm{cop}}_{(f_{i-1},\overline{f}_{i-1}] \cup (\overline{d}_i, d_i]} Z_{d_i,f_i} \right).$$

To proceed further, one needs to note that $(a+b)^{\zeta_\varepsilon} \leq a^{\zeta_\varepsilon} + b^{\zeta_\varepsilon}$, for all $a, b \geq 0$, hence

$$\mathbb{E}\left[\left(z^{\mathrm{cop}}_{\overline{f}_{i-1},\overline{d}_i} \right)^{\zeta_\varepsilon}\right] \leq \frac{1}{2^{\zeta_\varepsilon}} \left(1 + \mathbb{E}\left[e^{-2\lambda\zeta_\varepsilon \sum_{n=\overline{f}_{i-1}+1}^{\overline{d}_i}(\omega_n + h_a(\lambda) - h)} \right] \right)$$

$$= \frac{1}{2^{\zeta_\varepsilon}} \left(1 + \mathbb{E}\left[e^{(\overline{d}_i - \overline{f}_{i-1})[\Lambda(-2\lambda\zeta_\varepsilon) - \zeta_\varepsilon \Lambda(-2\lambda) - 2\lambda\zeta_\varepsilon h]} \right] \right) \leq \frac{1}{2^{\zeta_\varepsilon}}(1+1) = 2^{1-\zeta_\varepsilon},$$

where the last inequality holds because $\Lambda(\zeta_\varepsilon x) \geq \zeta_\varepsilon \Lambda(x)$ for every $x \in \mathbb{R}$, by convexity, and $h = h_\varepsilon \geq 0$, cf. (4.2). Finally, let $\widetilde{\mathbb{P}}_J$ be the law of the disorder obtained from $\mathbb{P}$, where independently for each $n \in \bigcup_{i \in J} B_i$, the law of ω_n is tilted with density $e^{\delta\omega_n - \Lambda(\delta)}$, with

$$\delta := a_\varepsilon \beta := (1 - \zeta_\varepsilon)\lambda,$$

cf. (2.16). In complete analogy with (2.24), we have

$$\widetilde{\mathbb{E}}_J\left[z^{\mathrm{cop}}_{(f_{i-1},\overline{f}_{i-1}] \cup (\overline{d}_i,d_i]} \right] \leq 1.$$

The rest of the proof then proceeds exactly as in the analysis of the pinning model.

5. On the Copolymer Model: Upper Bound

The proof goes along the very same lines as for the pinning model, cf. Section 3. In fact, the analogue of the lower bound (3.1) on the free energy is much simpler for the copolymer.

Lemma 5.1. *For every $c \in \mathbb{R}$*

$$\liminf_{\lambda \downarrow 0} \frac{\mathrm{F}(\lambda, c\lambda)}{\lambda^2} \geq c - \frac{1}{2}. \tag{5.1}$$

Proof. A direct application of Jensen's inequality is sufficient. Let

$$\mathcal{N}_N := \max\{n \in \mathbb{N}_0 \colon \tau_n \leq N\} = |\tau \cap [1,N]|.$$

Recalling (1.12) and (1.11), in analogy with (3.3) we obtain

$$\mathrm{F}(\lambda, h) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \mathrm{E}\left[\prod_{j=1}^{\mathcal{N}_N} \frac{1 + e^{-2\lambda \sum_{n=\tau_{j-1}+1}^{\tau_j}(\omega_n + h_a(\lambda) - h)}}{2} \,\bigg|\, N \in \tau \right]$$

$$\geq \frac{1}{\mu} \mathbb{E}\mathrm{E}\left[\log \frac{1 + e^{-2\lambda \sum_{n=1}^{\tau_1}(\omega_n + h_a(\lambda) - h)}}{2} \right]$$

$$= \frac{1}{\mu} \sum_{N=1}^\infty K(N)\, \mathbb{E}\left[\log \frac{1 + e^{-2\lambda \sum_{n=1}^{N}(\omega_n + h_a(\lambda) - h)}}{2} + \lambda \sum_{n=1}^{N}(\omega_n + h_a(\lambda) - h) \right]$$

$$\quad - \lambda(h_a(\lambda) - h),$$

where the term $\lambda \sum_{n=1}^{N}(\omega_n + h_a(\lambda) - h)$ is inserted to ensure that the expression inside the expectation is nonnegative, by Jensen's inequality. We can then apply Fatou's Lemma, analogously to (3.7): recalling (1.10), a simple Taylor expansion yields

$$\liminf_{\lambda \downarrow 0} \frac{\mathrm{F}(\lambda, c\lambda)}{\lambda^2} \geq c - \frac{1}{2},$$

completing the proof. $\qquad\square$

Coupling the lower bound (5.1) with the smoothing inequality (1.20) for the copolymer model, exactly as we did for the pinning model in Section 3.2, we obtain

$$\limsup_{\lambda \downarrow 0} \frac{h_c(\lambda)}{\lambda} \leq \frac{\alpha}{2(1+\alpha)},$$

which completes the proof of Theorem 1.4.

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