

# A RENEWAL THEORY APPROACH TO PERIODIC COPOLYMERS WITH ADSORPTION

FRANCESCO CARAVENNA, GIAMBATTISTA GIACOMIN, AND LORENZO ZAMBOTTI

**ABSTRACT.** We consider a general model of a heterogeneous polymer chain fluctuating in the proximity of an interface between two selective solvents. The heterogeneous character of the model comes from the fact that the *monomer units* interact with the solvents and with the interface according to some *charges* that they carry. The charges repeat themselves along the chain in a periodic fashion. The main question on this model is whether the polymer remains tightly close to the interface, a phenomenon called *localization*, or there is a marked preference for one of the two solvents yielding thus a *delocalization* phenomenon.

In this paper we present an approach that yields sharp estimates on the partition function of the model in all regimes (localized, delocalized and critical). This in turn allows to get a precise pathwise description of the polymer measure, obtaining the full *scaling limits* of the model. A key point is the closeness of the polymer measure to suitable *Markov renewal processes* and Markov renewal theory is precisely one of the central mathematical tools of our analysis.

2000 *Mathematics Subject Classification:* 60K35, 82B41, 82B44

*Keywords:* Random Walks, Renewal Theory, Markov Renewal Theory, Scaling limits, Polymer models, Wetting Models.

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Two motivating models.** Let  $S := \{S_n\}_{n=0,1,\dots}$  be a random walk,  $S_0 = 0$  and  $S_n = \sum_{j=1}^n X_j$ , with IID symmetric increments taking values in  $\{-1, 0, +1\}$ . Hence the law of the walk is identified by  $p := \mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1)$ , and we assume that  $p \in (0, 1/2)$ . The case  $p = 1/2$  can be treated in an analogous way but requires some notational care because of the periodicity of the walk. We also consider a sequence  $\omega := \{\omega_n\}_{n \in \mathbb{N} = \{1, 2, \dots\}}$  of real numbers with the property that  $\omega_n = \omega_{n+T}$  for some  $T \in \mathbb{N}$  and for every  $n$ : we denote by  $T(\omega)$  the minimal value of  $T$ .

Consider the following two families of modifications of the law of the walk, both indexed by a parameter  $N \in \mathbb{N}$ :

*Pinning and wetting models.* For  $\lambda \geq 0$  consider the probability measure  $\mathbf{P}_{N,\omega}$  defined by

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(S) \propto \exp\left(\lambda \sum_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}}\right). \quad (1.1)$$

The walk receives a *pinning reward*, which may be negative or positive, each time it visits the origin. By considering the directed walk viewpoint, that is  $\{(n, S_n)\}_n$ , one may interpret this model in terms of a directed linear chain receiving an energetic contribution

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*Date:* February 12, 2007.

when it touches an interface. The main question is whether for large  $N$  the typical paths of  $\mathbf{P}_{N,\omega}$  are rather attracted or repelled by the interface.

There is an extensive literature on periodic pinning and wetting models, the majority of which is restricted to the  $T = 2$  case, we mention for example [13, 26], see [16] for further discussion and references.

*Copolymer near a selective interface.* Much in the same way we introduce

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(S) \propto \exp\left(\lambda \sum_{n=1}^N \omega_n \operatorname{sign}(S_n)\right), \quad (1.2)$$

where if  $S_n = 0$  we set  $\operatorname{sign}(S_n) := \operatorname{sign}(S_{n-1}) \mathbf{1}_{\{S_{n-1} \neq 0\}}$ . This convention for defining  $\operatorname{sign}(0)$ , that will be kept throughout the paper, means simply that  $\operatorname{sign}(S_n) = +1, 0, -1$  according to whether the bond joining  $S_{n-1}$  and  $S_n$  lies above, on, or below the  $x$ -axis.

Also in this case we take a directed walk viewpoint and then  $\mathbf{P}_{N,\omega}$  may be interpreted as a polymeric chain in which the monomer units, the bonds of the walk, are charged. An interface, the  $x$ -axis, separates two solvents, say oil above and water below: positively charged monomers are hydrophobic and negatively charged ones are instead hydrophilic. In this case one expects a competition between three possible scenarios: polymer preferring water, preferring oil or undecided between the two and choosing to fluctuate in the proximity of the interface.

We select [24, 29] from the physical literature on periodic copolymers, keeping however in mind that periodic copolymer modeling has a central role in applied chemistry and material science.

**1.2. A general model.** We point out that the models presented in § 1.1 are particular examples of the polymer measure with Hamiltonian

$$\mathcal{H}_N(S) = \sum_{i=\pm 1} \sum_{n=1}^N \omega_n^{(i)} \mathbf{1}_{\{\operatorname{sign}(S_n)=i\}} + \sum_{n=1}^N \omega_n^{(0)} \mathbf{1}_{\{S_n=0\}} + \sum_{n=1}^N \tilde{\omega}_n^{(0)} \mathbf{1}_{\{\operatorname{sign}(S_n)=0\}}, \quad (1.3)$$

where  $\omega^{(\pm 1)}$ ,  $\omega^{(0)}$  and  $\tilde{\omega}^{(0)}$  are periodic sequences of real numbers. Observe that, by our conventions on  $\operatorname{sign}(0)$ , the last term gives an energetic contribution (of pinning/depinning type) to the bonds lying on the interface. We use the shorthand  $\omega$  for the four periodic sequences appearing in (1.3), and we will use  $T = T(\omega)$  to denote the smallest common period of the sequences. We will refer to  $\omega$  as to the *charges* of our system.

Besides being a natural model, generalizing and interpolating between pinning and copolymer models, the general model we consider is the one considered at several instances, see e.g. [30] and references therein.

Starting from the Hamiltonian (1.3), for  $a = c$  (*constrained*) or  $a = f$  (*free*) we introduce the polymer measure  $\mathbf{P}_{N,\omega}^a$  on  $\mathbb{Z}^N$ , defined by

$$\frac{d\mathbf{P}_{N,\omega}^a}{d\mathbf{P}}(S) = \frac{\exp(\mathcal{H}_N(S))}{\tilde{Z}_{N,\omega}^a} (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}}), \quad (1.4)$$

where  $\tilde{Z}_{N,\omega}^a := \mathbf{E}[\exp(\mathcal{H}_N)(\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}})]$  is the *partition function*, that is the normalization constant. Observe that the polymer measure  $\mathbf{P}_{N,\omega}^a$  is invariant under the joint transformation  $S \rightarrow -S$ ,  $\omega^{(+1)} \rightarrow \omega^{(-1)}$ , hence by symmetry we may (and will)

assume that

$$h_\omega := \frac{1}{T(\omega)} \sum_{n=1}^{T(\omega)} (\omega_n^{(+)} - \omega_n^{(-)}) \geq 0. \quad (1.5)$$

We also set  $\mathbb{S} := \mathbb{Z}/(T\mathbb{Z})$  and for  $\beta \in \mathbb{S}$  we write equivalently  $[n] = \beta$  or  $n \in \beta$ . Notice that the charges  $\omega_n$  are functions of  $[n]$ , and we can write  $\omega_{[n]} := \omega_n$ .

**1.3. The free energy viewpoint.** The standard statistical mechanics approach leads naturally to consider the free energy of the model, that is the limit as  $N \rightarrow \infty$  of  $(1/N) \log \tilde{Z}_{N,\omega}^a$ . It is however practical to observe that we can add to the Hamiltonian  $\mathcal{H}_N$  a term which is constant with respect to  $S$  without changing the polymer measure. Namely if we set

$$\mathcal{H}'_N(S) := \mathcal{H}_N(S) - \sum_{n=1}^N \omega_n^{(+1)},$$

which amounts to sending  $\omega_n^{(+1)} \rightarrow 0$ ,  $\omega_n^{(-1)} \rightarrow (\omega_n^{(-1)} - \omega_n^{(+1)})$  and  $\tilde{\omega}_n^{(0)} \rightarrow (\tilde{\omega}_n^{(0)} - \omega_n^{(+1)})$ , we can write

$$\frac{d\mathbf{P}_{N,\omega}^a}{d\mathbf{P}}(S) = \frac{\exp(\mathcal{H}'_N(S))}{Z_{N,\omega}^a} (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}}), \quad (1.6)$$

where  $Z_{N,\omega}^a$  is a new partition function given by

$$Z_{N,\omega}^a = \mathbf{E} \left[ \exp(\mathcal{H}'_N) (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}}) \right] = \tilde{Z}_{N,\omega}^a \cdot e^{-\sum_{n=1}^N \omega_n^{(+1)}}. \quad (1.7)$$

At this point we define the free energy:

$$F_\omega := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,\omega}^c. \quad (1.8)$$

A proof of the existence of such a limit goes through standard super-additive arguments, as well as the fact that the superscript  $c$  could be changed to  $f$  and the result does not change (see e.g. [16], but a complete proof, without using super-additivity, is given below).

The principle that the free energy contains the crucial information on the large  $N$  behavior of the system is certainly not violated in this context. In order to clarify this point let us first observe that  $F_\omega \geq 0$  for every  $\omega$ . This follows by noticing that the energetic contribution to the trajectories that stay positive and come back to zero for the first time at epoch  $N$  is just  $\omega_N^{(0)}$ , hence by (1.7)

$$\frac{1}{N} \log Z_{N,\omega}^c \geq \frac{\omega_N^{(0)}}{N} + \frac{1}{N} \log \mathbf{P}(S_n > 0, n = 1, \dots, N-1, S_N = 0) \xrightarrow{N \rightarrow \infty} 0, \quad (1.9)$$

where we have simply used the fact that the distribution of the first return to zero of  $S$  is sub-exponential (see (2.2) for a much sharper estimate). This suggests a natural dichotomy and, inspired by (1.9), we give the following definition:

**Definition 1.1.** *The polymer chain defined by (1.4) is said to be:*

- *localized if  $F_\omega > 0$ ;*
- *delocalized if  $F_\omega = 0$ .*

A priori one is certainly not totally convinced by such a definition. Localization, as well as delocalization, should be given in terms of path properties of the process: it is quite clear that the energy  $\mathcal{H}'_N(S)$  of trajectories  $S$  which do not come back very often (i.e. not in a positively recurrent fashion) to the interface will be either negative or  $o(N)$ , but this

is far from being a convincing statement of localization. An analogous observation can be made for delocalized polymer chains.

Nonetheless, with a few exceptions, much of the literature focuses on free energy estimates. For example in [5] one can find the analysis of the free energy of a subset of the class of models we are considering here and in § 1.7 of the same work it is argued that some (weak) path statements of localization and delocalization can be extracted from the free energy. We will come back with a review of the existing literature after we have stated our main results, but we anticipate from now that our purpose is going well beyond free energy estimates.

One of the main results in [5] is a formula for  $F_\omega$ , obtained via Large Deviations arguments. We will not give the precise expression now, the reader can find it in § 3.2 below, but we point out that this formula is proven here using arguments that are more elementary and these arguments yield at the same time much stronger estimates. More precisely, there exists a positive parameter  $\delta^\omega$ , which is given explicitly and analyzed in detail in § 2.1, that determines the *precise asymptotic behavior* of the partition function (the link between  $\delta_\omega$  and  $F_\omega$  will be clarified right after the statement):

**Theorem 1.2 (Sharp asymptotic estimates).** *Fix  $\eta \in \mathbb{S}$  and consider the asymptotic behavior of  $Z_{N,\omega}^c$  as  $N \rightarrow \infty$  along  $[N] = \eta$ . Then:*

- (1) *If  $\delta^\omega < 1$  then  $Z_{N,\omega}^c \sim C_{\omega,\eta}^< / N^{3/2}$ ;*
- (2) *If  $\delta^\omega = 1$  then  $Z_{N,\omega}^c \sim C_{\omega,\eta}^= / N^{1/2}$ ;*
- (3) *If  $\delta^\omega > 1$  then  $F_\omega > 0$  and  $Z_{N,\omega}^c \sim C_{\omega,\eta}^> \exp(F_\omega N)$ ,*

where the quantities  $F_\omega$ ,  $C_{\omega,\eta}^>$ ,  $C_{\omega,\eta}^<$  and  $C_{\omega,\eta}^=$  are given explicitly in Section 3.

Of course by  $a_N \sim b_N$  we mean  $a_N/b_N \rightarrow 1$ . Note that Theorem 1.2 implies the existence of the limit in (1.8) and that  $F_\omega = 0$  exactly when  $\delta^\omega \leq 1$ , but we stress that in our arguments we do not rely on (1.8) to define  $F_\omega$ . We also point out that analogous asymptotic estimates can be obtained for the free partition function, see Proposition 3.2.

It is rather natural to think that from such precise estimates one can extract detailed information on the limit behaviors of the system. This is correct, notably we can consider

- (1) Infinite volume limits, that is weak limits of  $\mathbf{P}_{N,\omega}^a$  as a measure on  $\mathbb{R}^N$ ;
- (2) Scaling limits, that is limits in law of the process  $S$ , suitably rescaled, under  $\mathbf{P}_{N,\omega}^a$ .

Here we will focus only on (2): the case (1) is considered in [7].

A word of explanation on the fact that there appears to be two types of delocalized polymer chains: the ones with  $\delta^\omega = 1$  and the ones with  $\delta^\omega < 1$ . As we will see, these two cases exhibit substantially different path behavior (even if both display distinctive features of delocalized paths, notably a vanishing density of visits at the interface). As it will be clear, in the case  $\delta^\omega < 1$  the system is *strictly* delocalized in the sense that a small perturbation in the charges leaves  $\delta^\omega < 1$  (as a matter of fact, for charges of a fixed period the mapping  $\omega \mapsto \delta^\omega$  is continuous), while  $\delta^\omega$  is rather a borderline, or *critical*, case.

**1.4. The scaling limits.** The main results of this paper concern the *diffusive rescaling* of the polymer measure  $\mathbf{P}_{N,\omega}^a$ . More precisely, let us define the map  $X^N : \mathbb{R}^N \mapsto C([0, 1])$ :

$$X_t^N(x) = \frac{x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} + (Nt - \lfloor Nt \rfloor) \frac{x_{\lfloor Nt \rfloor + 1} - x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}}, \quad t \in [0, 1],$$

where  $\lfloor \cdot \rfloor$  denotes the integer part of  $\cdot$  and  $\sigma^2 := 2p$  is the variance of  $X_1$  under the original random walk measure  $\mathbf{P}$ . Notice that  $X_t^N(x)$  is nothing but the linear interpolation

of  $\{x_{\lfloor Nt \rfloor}/(\sigma\sqrt{N})\}_{t \in \frac{\mathbb{N}}{N} \cap [0,1]}$ . For  $a = f, c$  we set:

$$Q_{N,\omega}^a := \mathbf{P}_{N,\omega}^a \circ (X^N)^{-1}.$$

Then  $Q_{N,\omega}^a$  is a measure on  $C([0, 1])$ , the space of real continuous functions defined on the interval  $[0, 1]$ , and we want to study the behavior as  $N \rightarrow \infty$  of this sequence of measures.

We start by fixing a notation for the following standard processes:

- the Brownian motion  $\{B_\tau\}_{\tau \in [0,1]}$ ;
- the Brownian bridge  $\{\beta_\tau\}_{\tau \in [0,1]}$  between 0 and 0;
- the Brownian motion *conditioned to stay non-negative on  $[0, 1]$*  or, more precisely, the Brownian meander  $\{m_\tau\}_{\tau \in [0,1]}$ , cf. [28], and its modification by a random flip  $\{m_\tau^{(p)}\}_{\tau \in [0,1]}$ , defined as  $m^{(p)} = \sigma m$ , where  $\mathbb{P}(\sigma = 1) = 1 - \mathbb{P}(\sigma = -1) = p \in [0, 1]$  and  $(m, \sigma)$  are independent;
- the Brownian bridge *conditioned to stay non-negative on  $[0, 1]$*  or, more precisely, the normalized Brownian excursion  $\{e_\tau\}_{\tau \in [0,1]}$ , also known as the Bessel bridge of dimension 3 between 0 and 0, see [28]. For  $p \in [0, 1]$ ,  $\{e_\tau^{(p)}\}_{\tau \in [0,1]}$  is the flipped excursion defined in analogy with  $m^{(p)}$ ;
- the skew Brownian motion  $\{B_\tau^{(p)}\}_{\tau \in [0,1]}$  and the skew Brownian bridge  $\{\beta_\tau^{(p)}\}_{\tau \in [0,1]}$  of parameter  $p$ , cf. [28] (the definition is recalled in Remark 1.5 below).

Finally, we introduce a last process, labeled by two parameters  $p, q \in [0, 1]$ : consider a random variable  $U \mapsto [0, 1]$  with the arcsine law:  $\mathbb{P}(U \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}$ , and processes  $\beta^{(p)}, m^{(q)}$  as defined above, with  $(U, \beta^{(p)}, m^{(q)})$  independent triple. Then we denote by  $\{B_\tau^{(p,q)}\}_{\tau \in [0,1]}$  the process defined by:

$$B_\tau^{(p,q)} := \begin{cases} \sqrt{U} \beta_{\frac{\tau}{U}}^{(p)} & \text{if } \tau \leq U \\ \sqrt{1-U} m_{\frac{\tau-U}{1-U}}^{(q)} & \text{if } \tau > U \end{cases}.$$

Then we have the following Theorem, which is the main result of this paper:

**Theorem 1.3 (Scaling limits).** *For every  $\eta \in \mathbb{S}$ , if  $N \rightarrow \infty$  along  $[N] = \eta$ , then the sequence of measures  $\{Q_{N,\omega}^a\}$  on  $C([0, 1])$  converges weakly. More precisely:*

- (1) *for  $\delta^\omega < 1$  (strictly delocalized regime):  $Q_{N,\omega}^c$  converges to the law of  $e^{(p_\omega, \eta)}$  and  $Q_{N,\omega}^f$  converges to the law of  $m^{(p_\omega, \eta)}$ , for some parameters  $p_{\omega,\eta}^a \in [0, 1]$ ,  $a = f, c$ .*
- (2) *for  $\delta^\omega = 1$  (critical regime):  $Q_{N,\omega}^c$  converges to the law of  $\beta^{(p_\omega)}$  and  $Q_{N,\omega}^f$  converges to the law of  $B^{(p_\omega, q_\omega, \eta)}$ , for some parameters  $p_\omega, q_{\omega,\eta} \in [0, 1]$ .*
- (3) *for  $\delta^\omega > 1$  (localized regime):  $Q_{N,\omega}^a$  converges as  $N \rightarrow \infty$  to the measure concentrated on the constant function taking the value zero (no need of  $[N] = \eta$ ).*

The exact values of the parameters  $p_{\omega,\eta}^a$ ,  $p_\omega$  and  $q_{\omega,\eta}$  are given in (5.5), (5.7), (5.17) and (5.19). See also the Remarks 5.3, 5.4, 5.7 and 5.8.

**Remark 1.4.** It is natural to wonder why the results for  $\delta^\omega \leq 1$  may depend on  $[N] \in \mathbb{S}$ . First of all we stress that only in very particular cases there is effectively a dependence on  $\eta$  and we characterize these instances precisely, see § 2.3. In particular there is no dependence on  $[N]$  for the two motivating models (the pinning and the copolymer one) described in § 1.1, and more generally if  $h_\omega > 0$ . However this dependence on the boundary

condition phenomenon is not a pathology, but it is a sign of the presence of *first order phase transitions* in this class of models. Nonetheless the phenomenon is somewhat surprising since the model is one dimensional. This issue, that is naturally clarified when dealing with the infinite volume limit of the model, is treated in [7].

**Remark 1.5. (Skew Brownian motion).** We recall that  $B^{(p)}$ , respectively  $\beta^{(p)}$ , is a process such that  $|B^{(p)}| = |B|$ , resp.  $|\beta^{(p)}| = |\beta|$ , in distribution, but in which the sign of each excursion is chosen to be +1 (resp. -1) with probability  $p$  (resp.  $1 - p$ ) instead of  $1/2$ . Observe that for  $p = 1$  we have  $B^{(1)} = |B|$ ,  $\beta^{(1)} = |\beta|$ ,  $m^{(1)} = m$  and  $e^{(1)} = e$  in distribution. Moreover  $B^{(1/2)} = B$  and  $\beta^{(1/2)} = \beta$  in distribution. Notice also that the process  $B^{(p,q)}$  differs from the  $p$ -skew Brownian motion  $B^{(p)}$  only for the last excursion in  $[0, 1]$ , whose sign is +1 with probability  $q$  instead of  $p$ .

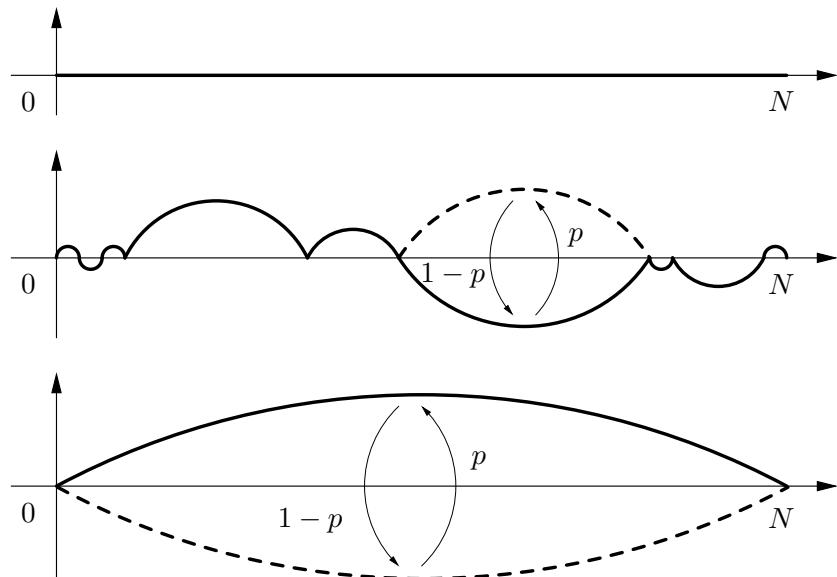


FIGURE 1. A schematic view of the scaling limits for the constrained endpoint case. While in the localized regime, image on top, on large scale the polymer cannot be distinguished from the interface, in the strictly delocalized regime, bottom image, the visits to the interface are only a few and all close to the endpoints (the sign of the excursion is obtained by flipping one biased coin). In between there is the critical case: the zeros of the limiting process coincide with the zero of a Brownian bridge, as found for the homogeneous wetting case [18, 9, 6], but this time the signs of the excursions vary and they are chosen by flipping independent biased coin. Of course this suggests that the trajectories in the localized cased should be analyzed without rescaling (this is done in [7]).

**1.5. Motivations and a look at the literature.** From an applied viewpoint, the interest in periodic models of the type we consider appears to be at least two-fold:

- (1) On the one hand periodic models are often (e.g. [13, 24]) motivated as caricatures of the *quenched disordered* models, like the ones in which the charges are a typical realization of a sequence of independent random variables (e.g. [1, 4, 16, 30] and

references therein). In this respect, periodic models may be viewed as *weakly inhomogeneous*, and the approximation of *strongly inhomogeneous* quenched models with periodic ones, in the limit of large period, sets forth very interesting and challenging questions. We believe that, if the precise description of the periodic case that we have obtained in this work highlights the limitations of periodic modeling for strongly inhomogeneous systems (compare in particular the anomalous decay of quenched partition functions along subsequences pointed out in [17, section 4] and our Th. 1.2), it is at the same time an essential step toward understanding the large period limit, and the method we use in this paper may allow a generalization that yields information on this limit.

- (2) One the other hand, as already mentioned above, periodic models are absolutely natural and of direct relevance for application, for example when dealing with *molecularly engineered polymers* (see [25, 29] for a sample of the theoretical physics literature, but the applied literature is extremely vast).

From a mathematical standpoint, our work may be viewed as a further step in the direction of

- (a) extending to the periodic setting precise path estimates obtained for homogeneous models;
- (b) clarifying the link between the free energy characterization and the path characterization of the different regimes.

With reference to (a), we point out the novelty with respect to the works on homogeneous models [24, 18, 9, 6]. Although the basic role of renewal theory techniques to get the crucial estimates had been emphasized already in [9, 6], we stress that the underlying key renewal processes that appear in our inhomogeneous context are not standard renewals, but rather *Markov renewal processes*, cf. [2]. Understanding the algebraic structure leading to this type of renewals is one of the central points of our work, see § 3.1.

We also point out that the Markov renewal processes appearing in the critical regime have step distributions with *infinite mean*. Even for ordinary renewal process, the exact asymptotic behavior of the Green function in the case of infinite mean has been a long-standing problem, cf. [15] and [21], which has been solved only recently by R. A. Doney in [10]. The extension of this result to the framework of Markov renewal theory, that we consider here in the case of finite-state modulating chain, presents some additional problems (see Remark 3.1 and Appendix A) and, to our knowledge, has not been considered in the literature. In Section 5 we also give an extension to our Markov-renewal situation of the beautiful theory of convergence of regenerative sets developed in [12].

A final observation is that, like in [6], the estimates we get here are really sharp in all regimes and our method goes well beyond the case of random walks with jumps  $\pm 1$  and 0, to which we restrict for the sake of conciseness.

With reference to (b), we point out that in the models we consider there is a variety of delocalized path behaviors, which are not captured by the free energy. This is suggestive also in view of progressing in the understanding of the delocalized phase in the quenched models [17].

### 1.6. Outline of the paper.

The exposition is organized as follows:

- In Section 2 we define the basic parameter  $\delta^\omega$  and we analyze the dependence on the boundary condition  $[N] = \eta$  of our results.
- In Section 3 we clarify the link of our models with Markov renewal theory and we obtain the asymptotic behavior of  $Z_{N,\omega}^c$  and  $Z_{N,\omega}^f$ , proving Theorem 1.2.

- In Section 4 we present a basic splitting of the polymer measure into zero level set and excursions and we point out the importance of the partition function.
- In Section 5 we compute the scaling limits of  $\mathbf{P}_{N,\omega}^a$ , proving Theorem 1.3.
- Finally, the Appendices contain the proof of some technical results.

## 2. A CLOSER LOOK INTO THE MAIN RESULTS

**2.1. The order parameter  $\delta^\omega$ .** A remarkable feature of our results, see Theorem 1.2 and Theorem 1.3, is the fact that the properties of the polymer measure are essentially encoded in one single parameter  $\delta^\omega$ , that can be regarded as the *order parameter* of our models. This subsection is devoted to defining this parameter, but before we need some preliminary notation.

We start with the law of the first return to zero of the original walk:

$$\tau_1 := \inf\{n > 0 : S_n = 0\} \quad K(n) := \mathbf{P}(\tau_1 = n). \quad (2.1)$$

It is a classical result [11, Ch. XII.7] that

$$\exists \lim_{n \rightarrow \infty} n^{3/2} K(n) =: c_K \in (0, \infty). \quad (2.2)$$

The key observation is that, by the  $T$ -periodicity of the charges  $\omega$  and by the definition (1.5) of  $h_\omega$ , we can define a  $\mathbb{S} \times \mathbb{S}$  matrix  $\Sigma_{\alpha,\beta}$  by the following relation:

$$\sum_{n=n_1+1}^{n_2} (\omega_n^{(-1)} - \omega_n^{(+1)}) = -(n_2 - n_1) h_\omega + \Sigma_{[n_1], [n_2]}. \quad (2.3)$$

Thus we have decomposed the above sum into a drift term and a fluctuating term, where the latter has the remarkable property of depending on  $n_1$  and  $n_2$  only through their equivalence classes  $[n_1]$  and  $[n_2]$  in  $\mathbb{S}$ . Now we can define three basic objects:

- for  $\alpha, \beta \in \mathbb{S}$  and  $\ell \in \mathbb{N}$  we set

$$\Phi_{\alpha,\beta}^\omega(\ell) := \begin{cases} \omega_\beta^{(0)} + (\tilde{\omega}_\beta^{(0)} - \omega_\beta^{(+1)}) & \text{if } \ell = 1, \ell \in \beta - \alpha \\ \omega_\beta^{(0)} + \log \left[ \frac{1}{2} \left( 1 + \exp(-\ell h_\omega + \Sigma_{\alpha,\beta}) \right) \right] & \text{if } \ell > 1, \ell \in \beta - \alpha ; \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

- for  $x \in \mathbb{N}$  we introduce the  $\mathbb{S} \times \mathbb{S}$  matrix  $M_{\alpha,\beta}^\omega(x)$  defined by

$$M_{\alpha,\beta}^\omega(x) := e^{\Phi_{\alpha,\beta}^\omega(x)} K(x) \mathbf{1}_{(x \in \beta - \alpha)}; \quad (2.5)$$

- summing the entries of  $M^\omega$  over  $x$  we get a  $\mathbb{S} \times \mathbb{S}$  matrix that we call  $B^\omega$ :

$$B_{\alpha,\beta}^\omega := \sum_{x \in \mathbb{N}} M_{\alpha,\beta}^\omega(x). \quad (2.6)$$

The meaning of these quantities will emerge clearly in the next subsection. For the moment we stress that they are *explicit functions* of the charges  $\omega$  and of the law of the underlying random walk (to lighten the notation, the  $\omega$ -dependence of these quantities will be often dropped in the following).

Observe that  $B_{\alpha,\beta}$  is a finite dimensional matrix with positive entries, hence the Perron–Frobenius Theorem (see e.g. [2]) entails that  $B_{\alpha,\beta}$  has a unique real positive eigenvalue, called the Perron–Frobenius eigenvalue, with the property that it is a simple root of the

characteristic polynomial and that it coincides with the spectral radius of the matrix. This is exactly our order parameter:

$$\delta^\omega := \text{Perron-Frobenius eigenvalue of } B^\omega. \quad (2.7)$$

**2.2. A random walk excursions viewpoint.** In this subsection we are going to see that the quantities defined in (2.4) and (2.5) emerge in a natural way from the algebraic structure of the constrained partition function  $Z_{N,\omega}^c$ . Let us look back to our Hamiltonian (1.3): its specificity comes from the fact that it can be decomposed in an efficient way by considering the return times to the origin of  $S$ . More precisely we define for  $j \in \mathbb{N}$

$$\tau_0 = 0 \quad \tau_{j+1} = \inf\{n > \tau_j : S_n = 0\},$$

and we set  $\iota_N = \sup\{k : \tau_k \leq N\}$ . We also set  $T_j = \tau_j - \tau_{j-1}$  and of course  $\{T_j\}_{j=1,2,\dots}$  is, under  $\mathbf{P}$ , an IID sequence. By conditioning on  $\tau$  and integrating on the up-down symmetry of the random walk excursions one easily obtains the following expression for the constrained partition function:

$$\begin{aligned} Z_{N,\omega}^c &= \mathbf{E} \left[ \prod_{j=1}^{\iota_N} \exp(\Phi_{[\tau_{j-1}], [\tau_j]}^\omega(\tau_j - \tau_{j-1})); \tau_{\iota_N} = N \right] \\ &= \sum_{k=1}^N \sum_{\substack{t_0, \dots, t_k \in \mathbb{N} \\ 0 =: t_0 < t_1 < \dots < t_k := N}} \prod_{j=1}^k M_{[t_{j-1}], [t_j]}^\omega(t_j - t_{j-1}), \end{aligned} \quad (2.8)$$

where we have used the quantities introduced in (2.4) and (2.5). This formula shows in particular that the partition function  $Z_{N,\omega}^c$  is a function of the entries of  $M^\omega$ .

We stress that the algebraic form of (2.8) is of crucial importance: it will be analyzed in detail and exploited in Section 3 and will be the key to the proof of Theorem 1.2.

**2.3. The regime  $\omega \in \mathcal{P}$ .** In this subsection we look more closely at the dependence of our main result Theorem 1.3 on the boundary condition  $[N] = \eta$ . It is convenient to introduce the subset  $\mathcal{P}$  of charges defined by:

$$\mathcal{P} := \{\omega : \delta^\omega \leq 1, h_\omega = 0, \exists \alpha, \beta : \Sigma_{\alpha, \beta} \neq 0\}, \quad (2.9)$$

where we recall that  $h_\omega$  and  $\Sigma_{\alpha, \beta}$  have been defined respectively in (1.5) and (2.3).

The basic observation is that if  $\omega \notin \mathcal{P}$  the constants  $p_{\omega, \eta}^c, p_{\omega, \eta}^f, p_\omega$  and  $q_{\omega, \eta}$  actually have no dependence on  $\eta$  and they take all the same value, namely 1 if  $h_\omega > 0$  and  $1/2$  if  $h_\omega = 0$  (see the Remarks 5.3, 5.4, 5.7 and 5.8). Then the results in Theorem 1.3 for  $\delta^\omega \leq 1$  may be strengthened in the following way:

**Proposition 2.1.** *If  $\omega \notin \mathcal{P}$ , then the sequence of measures  $\{Q_{N,\omega}^a\}$  on  $C([0, 1])$  converges weakly as  $N \rightarrow \infty$ . In particular, setting  $p_\omega := 1$  if  $h_\omega > 0$  and  $p_\omega := \frac{1}{2}$  if  $h_\omega = 0$ :*

- (1) *for  $\delta^\omega < 1$  (strictly delocalized regime),  $Q_{N,\omega}^f$  converges to the law of  $m^{(p_\omega)}$  and  $Q_{N,\omega}^c$  converges to the law of  $e^{(p_\omega)}$*
- (2) *for  $\delta^\omega = 1$  (critical regime),  $Q_{N,\omega}^f$  converges to the law of  $B^{(p_\omega)}$  and  $Q_{N,\omega}^c$  converges to the law of  $\beta^{(p_\omega)}$ .*

This stronger form of the scaling limits holds in particular for the two motivating models of § 1.1, the pinning and the copolymer models, for which  $\omega$  never belongs to  $\mathcal{P}$ . This is clear for the pinning case, where by definition  $\Sigma \equiv 0$ , while the copolymer model with

$h_\omega = 0$  has always  $\delta^\omega > 1$ , as we prove in Appendix B. However we stress that there do exist charges  $\omega$  (necessarily belonging to  $\mathcal{P}$ ) for which there is indeed a dependence on  $[N] = \eta$  in the delocalized and critical scaling limits. This interesting phenomenon may be understood in statistical mechanics terms and is analyzed in detail in [7].

### 3. SHARP ASYMPTOTIC BEHAVIOR FOR THE PARTITION FUNCTION

In this section we are going to derive the precise asymptotic behavior of  $Z_{N,\omega}^c$  and  $Z_{N,\omega}^f$ , in particular proving Theorem 1.2. The key observation is that the study of the partition function for the models we are considering can be set into the framework of the theory of *Markov renewal processes*, see [2, Ch. VII.4].

**3.1. A link with Markov Renewal Theory.** The starting point of our analysis is equation (2.8). Let us call a function  $\mathbb{N} \times \mathbb{S} \times \mathbb{S} \ni (x, \alpha, \beta) \mapsto F_{\alpha,\beta}(x) \geq 0$  a *kernel*. For fixed  $x \in \mathbb{N}$ ,  $F_{\cdot,\cdot}(x)$  is a  $\mathbb{S} \times \mathbb{S}$  matrix with non-negative entries. Given two kernels  $F$  and  $G$  we define their convolution  $F * G$  as the kernel defined by

$$(F * G)_{\alpha,\beta}(x) := \sum_{y \in \mathbb{N}} \sum_{\gamma \in \mathbb{S}} F_{\alpha,\gamma}(y) G_{\gamma,\beta}(x - y) = \sum_{y \in \mathbb{N}} [F(y) \cdot G(x - y)]_{\alpha,\beta}, \quad (3.1)$$

where  $\cdot$  denotes matrix product. Then, since by construction  $M_{\alpha,\beta}(x) = 0$  if  $[x] \neq \beta - \alpha$ , we can write (2.8) in the following way:

$$Z_{N,\omega}^c = \sum_{k=1}^N \sum_{\substack{t_1, \dots, t_k \in \mathbb{N} \\ 0 < t_1 < \dots < t_k := N}} [M(t_1) \cdot \dots \cdot M(N - t_{k-1})]_{0,[N]} = \sum_{k=1}^{\infty} [M^{*k}]_{[0],[N]}(N), \quad (3.2)$$

where  $F^{*n}$  denotes the  $n$ -fold convolution of a kernel  $F$  with itself (the  $n = 0$  case is by definition  $[F^{*0}]_{\alpha,\beta}(x) := \mathbf{1}_{(\beta=\alpha)} \mathbf{1}_{(x=0)}$ ). In view of (3.2), we introduce the kernel

$$\mathcal{Z}_{\alpha,\beta}(n) := \sum_{k=1}^{\infty} [M^{*k}]_{\alpha,\beta}(n), \quad (3.3)$$

so that  $Z_{N,\omega}^c = \mathcal{Z}_{[0],[N]}(N)$  and more generally  $Z_{N-k,\theta_k \omega}^c = \mathcal{Z}_{[k],[N]}(N - k)$ ,  $k \leq N$ , where we have introduced the shift operator for  $k \in \mathbb{N}$ :

$$\theta_k : \mathbb{R}^{\mathbb{S}} \mapsto \mathbb{R}^{\mathbb{S}}, \quad (\theta_k \zeta)_\beta := \zeta_{[k]+\beta}, \quad \beta \in \mathbb{S}.$$

Our goal is to determine the asymptotic behavior as  $N \rightarrow \infty$  of the kernel  $\mathcal{Z}_{\alpha,\beta}(N)$  and hence of the partition function  $Z_{N,\omega}^c$ . To this aim, we introduce an important transformation of the kernel  $M$  that exploits the algebraic structure of (3.3): we suppose that  $\delta^\omega \geq 1$  (the case  $\delta^\omega < 1$  requires a different procedure) and we set for  $b \geq 0$  (cf. [2, Th. 4.6])

$$A_{\alpha,\beta}^b(x) := M_{\alpha,\beta}(x) e^{-bx}.$$

Let us denote by  $\Delta(b)$  the Perron–Frobenius eigenvalue of the matrix  $\sum_x A_{\alpha,\beta}^b(x)$ . As the entries of this matrix are analytic and nonincreasing functions of  $b$ ,  $\Delta(b)$  is analytic and nonincreasing too, hence strictly decreasing because  $\Delta(0) = \delta^\omega \geq 1$  and  $\Delta(\infty) = 0$ . Therefore there exists a single value  $F_\omega \geq 0$  such that  $\Delta(F_\omega) = 1$ , and we denote by  $\{\zeta_\alpha\}_\alpha$ ,  $\{\xi_\alpha\}_\alpha$  the Perron–Frobenius left and right eigenvectors of  $\sum_x A_{\alpha,\beta}^{F_\omega}(x)$ , chosen to have (strictly) positive components and normalized in such a way that  $\sum_\alpha \zeta_\alpha \xi_\alpha = 1$  (the remaining degree of freedom in the normalization is immaterial). Then we set

$$\Gamma_{\alpha,\beta}(x) := M_{\alpha,\beta}(x) e^{-F_\omega x} \frac{\xi_\beta}{\xi_\alpha}, \quad (3.4)$$

and observe that we can rewrite (3.2) as

$$\mathcal{Z}_{\alpha,\beta}(n) = \exp(F_\omega n) \frac{\xi_\alpha}{\xi_\beta} U_{\alpha,\beta}(n), \quad \text{where} \quad U_{\alpha,\beta}(n) := \sum_{k=1}^{\infty} [\Gamma^{*k}]_{\alpha,\beta}(n). \quad (3.5)$$

The kernel  $U_{\alpha,\beta}(n)$  has a basic probabilistic interpretation that we now describe. Notice first that by construction we have  $\sum_{\beta,x} \Gamma_{\alpha,\beta}(x) = 1$ , i.e.  $\Gamma$  is a *semi-Markov kernel*, cf. [2]. Then we can define a Markov chain  $\{(J_k, T_k)\}$  on  $\mathbb{S} \times \mathbb{N}$  by:

$$\mathbb{P}[(J_{k+1}, T_{k+1}) = (\beta, x) \mid (J_k, T_k) = (\alpha, y)] = \Gamma_{\alpha,\beta}(x), \quad (3.6)$$

and we denote by  $\mathbb{P}_\alpha$  be the law of  $\{(J_k, T_k)\}$  with starting point  $J_0 = \alpha$  (the value of  $T_0$  plays no role). The probabilistic meaning of  $U_{\alpha,\beta}(x)$  is then given by:

$$U_{\alpha,\beta}(n) = \sum_{k=1}^{\infty} \mathbb{P}_\alpha(T_1 + \dots + T_k = n, J_k = \beta). \quad (3.7)$$

We point out that the process  $\{\tau_k\}_{k \geq 0}$  defined by  $\tau_0 := 0$  and  $\tau_k := T_1 + \dots + T_k$  under the law  $\mathbb{P}_\alpha$  is what is called a (discrete) *Markov-renewal process*, cf. [2]. This provides a generalization of classical renewal processes, since the increments  $\{T_k\}$  are not IID but they are rather governed by the process  $\{J_k\}$  in the way prescribed by (3.6). The process  $\{J_k\}$  is called the *modulating chain* and it is indeed a genuine Markov chain on  $\mathbb{S}$ , with transition kernel  $\sum_{x \in \mathbb{N}} \Gamma_{\alpha,\beta}(x)$ , while in general the process  $\{T_k\}$  is *not* a Markov chain.

One can view  $\tau = \{\tau_n\}$  as a (random) subset of  $\mathbb{N}$ . More generally it is convenient to introduce the subset

$$\tau^\beta := \bigcup_{k \geq 0: J_k = \beta} \{\tau_k\}, \quad \beta \in \mathbb{S}, \quad (3.8)$$

so that equation (3.7) can be rewritten as

$$U_{\alpha,\beta}(n) = \mathbb{P}_\alpha(n \in \tau^\beta). \quad (3.9)$$

This shows that the kernel  $U_{\alpha,\beta}(n)$  is really an extension of the Green function of a classical renewal process. In analogy with the classical case, the asymptotic behavior of  $U_{\alpha,\beta}(n)$  is sharply linked to the asymptotic behavior of the kernel  $\Gamma$ , i.e. of  $M$ . To this aim, we notice that our setting is an *heavy-tailed* one: more precisely for every  $\alpha, \beta \in \mathbb{S}$ , by (2.2), (2.5) and (2.4) we have

$$\lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} M_{\alpha,\beta}(x) = L_{\alpha,\beta} := \begin{cases} c_K \frac{1}{2} \left(1 + \exp(\Sigma_{\alpha,\beta})\right) \exp(\omega_\beta^{(0)}) & \text{if } h_\omega = 0 \\ c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) & \text{if } h_\omega > 0 \end{cases}. \quad (3.10)$$

The rest of this section is devoted to finding the asymptotic behavior of  $U_{\alpha,\beta}(n)$  and hence of  $\mathcal{Z}_{\alpha,\beta}(n)$ , proving in particular Theorem 1.2. For convenience we consider the three regimes separately.

**3.2. The localized regime ( $\delta^\omega > 1$ ).** If  $\delta^\omega > 1$  then necessarily  $F_\omega > 0$ . Notice that  $\sum_x \Gamma_{\alpha,\beta}(x) > 0$ , so that in particular the modulating chain  $\{J_k\}$  is irreducible. The unique invariant measure  $\{\nu_\alpha\}_\alpha$  is easily seen to be equal to  $\{\zeta_\alpha \xi_\alpha\}_\alpha$ .

Let us compute the *mean*  $\mu$  of the semi-Markov kernel  $\Gamma$ :

$$\mu := \sum_{\alpha,\beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} x \nu_\alpha \Gamma_{\alpha,\beta}(x) = \sum_{\alpha,\beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} x e^{-F_\omega x} \zeta_\alpha M_{\alpha,\beta}(x) \xi_\beta = -\frac{\partial \Delta}{\partial b} \Big|_{b=F_\omega} \in (0, \infty)$$

(for the last equality see for example [5, Lemma 2.1]). Then we can apply the Markov Renewal Theorem, cf. [2, Th. VII.4.3], that in our periodic setting gives

$$\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} U_{\alpha,\beta}(x) = T \frac{\nu_\beta}{\mu}. \quad (3.11)$$

Then by (3.5) we obtain the desired asymptotic behavior:

$$\mathcal{Z}_{\alpha,\beta}(x) \sim \xi_\alpha \zeta_\beta \frac{T}{\mu} \exp(F_\omega x) \quad x \rightarrow \infty, \quad [x] = \beta - \alpha, \quad (3.12)$$

and for  $\alpha = [0]$  and  $\beta = \eta$  we have proven part (3) of Theorem 1.2, with  $C_{\omega,\eta}^> = \xi_0 \zeta_\eta T / \mu$ .

**3.3. The critical case ( $\delta^\omega = 1$ ).** In this case  $F_\omega = 0$  and equation (3.4) reduces to

$$\Gamma_{\alpha,\beta}(x) = M_{\alpha,\beta}(x) \frac{\xi_\beta}{\xi_\alpha}. \quad (3.13)$$

The random set  $\tau^\beta$  introduced in (3.8) can be written as the union  $\tau^\beta = \bigcup_{k \geq 0} \{\tau_k^\beta\}$ , where the points  $\{\tau_k^\beta\}_{k \geq 0}$  are taken in increasing order, and we set  $T_k^\beta := \tau_{k+1}^\beta - \tau_k^\beta$  for  $k \geq 0$ . Notice that the increments  $\{T_k^\beta\}$  correspond to sums of the variables  $\{T_i\}$  between the visits of the chain  $\{J_k\}$  to the state  $\beta$ : for instance we have

$$T_1^\beta = T_{\kappa+1} + \dots + T_\ell, \quad \kappa := \inf\{k \geq 0 : J_k = \beta\}, \quad \ell := \inf\{k > \kappa : J_k = \beta\}.$$

Equation (3.6) then yields that  $\{T_k^\beta\}_{k \geq 0}$  is an independent sequence under  $\mathbb{P}_\alpha$ , and that the variables  $T_k^\beta$  have for  $k \geq 1$  the same distribution  $q^\beta(n) := \mathbb{P}_\alpha(T_1^\beta = n)$  that actually does not depend on  $\alpha$ . The variable  $T_0^\beta$  in general has a different law  $q^{(\alpha;\beta)}(n) := \mathbb{P}_\alpha(T_0^\beta = n)$ .

These considerations yield the following crucial observation: for fixed  $\alpha$  and  $\beta$ , the process  $\{\tau_k^\beta\}_{k \geq 0}$  under  $\mathbb{P}_\alpha$  is a (delayed) classical renewal process, with typical step distribution  $q^\beta(\cdot)$  and initial step distribution  $q^{(\alpha;\beta)}(\cdot)$ . By (3.9),  $U_{\alpha,\beta}(n)$  is nothing but the Green function (or renewal mass function) of this process: more explicitly we can write

$$U_{\alpha,\beta}(x) = \left( q^{(\alpha;\beta)} * \sum_{n=0}^{\infty} (q^\beta)^{*n} \right)(x). \quad (3.14)$$

Of course  $q^{(\alpha;\beta)}$  plays no role for the asymptotic behavior of  $U_{\alpha,\beta}(x)$ . The key point is rather the precise asymptotic behavior of  $q^\beta(x)$  as  $x \rightarrow \infty$ ,  $x \in [0]$ , which is given by

$$q^\beta(x) \sim \frac{c_\beta}{x^{3/2}}, \quad \text{where} \quad c_\beta := \frac{1}{\zeta_\beta \xi_\beta} \sum_{\alpha,\gamma} \zeta_\alpha L_{\alpha,\gamma} \xi_\gamma > 0, \quad (3.15)$$

as it is proven in detail in Appendix A. Then the asymptotic behavior of (3.14) follows by a result of Doney, cf. [10, Th. B]:

$$U_{\alpha,\beta}(x) \sim \frac{T^2}{2\pi c_\beta} \frac{1}{\sqrt{x}} \quad x \rightarrow \infty, \quad [x] = \beta - \alpha, \quad (3.16)$$

(the factor  $T^2$  is due to our periodic setting). Combining equations (3.5), (3.15) and (3.16) we finally get the asymptotic behavior of  $\mathcal{Z}_{\alpha,\beta}(x)$ :

$$\mathcal{Z}_{\alpha,\beta}(x) \sim \frac{T^2}{2\pi} \frac{\xi_\alpha \zeta_\beta}{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} \frac{1}{\sqrt{x}} \quad x \rightarrow \infty, \quad [x] = \beta - \alpha. \quad (3.17)$$

Taking  $\alpha = [0]$  and  $\beta = \eta$ , we have the proof of part (2) of Theorem 1.2.

**Remark 3.1.** We point out that formula (3.15) is quite non-trivial. First, the asymptotic behavior  $x^{-3/2}$  of the law of the variables  $T_1^\beta$  is the same as that of the  $T_i$ , although  $T_1^\beta$  is the sum of a random number of the non-independent variables ( $T_i$ ). Second, the computation of the prefactor  $c_\beta$  is by no means an obvious task (we stress that the precise value of  $c_\beta$  is crucial in the proof of Proposition 5.5 below).

**3.4. The strictly delocalized case ( $\delta^\omega < 1$ ).** We prove that the asymptotic behavior of  $\mathcal{Z}_{\alpha,\beta}(x)$  when  $\delta^\omega < 1$  is given by

$$\mathcal{Z}_{\alpha,\beta}(x) \sim \left( [(1-B)^{-1}L(1-B)^{-1}]_{\alpha,\beta} \right) \frac{1}{x^{3/2}} \quad x \rightarrow \infty, \quad [x] = \beta - \alpha, \quad (3.18)$$

where the matrices  $L$  and  $B$  have been defined in (3.10) and (2.6). In particular, taking  $\alpha = [0]$  and  $\beta = \eta$ , (3.18) proves part (1) of Theorem 1.2 with

$$C_{\omega,\eta}^< := [(1-B)^{-1}L(1-B)^{-1}]_{0,\eta}.$$

To start with, it is easily checked by induction that for every  $n \in \mathbb{N}$

$$\sum_{x \in \mathbb{N}} [M^{*n}]_{\alpha,\beta}(x) = [B^n]_{\alpha,\beta}. \quad (3.19)$$

Next we claim that, by (3.10), for every  $\alpha, \beta \in \mathbb{S}$

$$\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} [M^{*k}]_{\alpha,\beta}(x) = \sum_{i=0}^{k-1} [B^i \cdot L \cdot B^{(k-1)-i}]_{\alpha,\beta}. \quad (3.20)$$

We proceed by induction on  $k$ . The  $k = 1$  case is just equation (3.10), and we have that

$$M^{*(n+1)}(x) = \sum_{y=1}^{x/2} \left( M(y) \cdot M^{*n}(x-y) + M(x-y) \cdot M^{*n}(y) \right)$$

(strictly speaking this formula is true only when  $x$  is even, however the odd  $x$  case is analogous). By the inductive hypothesis equation (3.20) holds for every  $k \leq n$ , and in particular this implies that  $\{x^{3/2}[M^{*k}]_{\alpha,\beta}(x)\}_{x \in \mathbb{N}}$  is a bounded sequence. Therefore we can apply Dominated Convergence, and using (3.19) we get

$$\begin{aligned} \exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} [M^{*(n+1)}]_{\alpha,\beta}(x) &= \sum_{\gamma} \left( B_{\alpha,\gamma} \sum_{i=0}^{n-1} [B^i \cdot L \cdot B^{(n-1)-i}]_{\gamma,\beta} + L_{\alpha,\gamma} [B^{*n}]_{\gamma,\beta} \right) \\ &= \sum_{i=0}^n [B^i \cdot L \cdot B^{n-i}]_{\alpha,\beta}. \end{aligned}$$

Our purpose is to apply the asymptotic result (3.20) to the terms of (3.5) and we need a bound to apply Dominated Convergence. What we are going to show is that

$$x^{3/2} [M^{*k}]_{\alpha,\beta}(x) \leq C k^3 [B^k]_{\alpha,\beta} \quad (3.21)$$

for some positive constant  $C$  and for all  $\alpha, \beta \in \mathbb{S}$  and  $x, k \in \mathbb{N}$ . We proceed again by induction: for the  $k = 1$  case, thanks to (3.10), it is possible to find  $C$  such that (3.21) holds true (this fixes  $C$  once for all). Now assuming that (3.21) holds for all  $k < n$  we

show that it does also for  $k = n$  (we suppose for simplicity that  $n = 2m$  is even, the odd  $n$  case being analogous). Then we have (assuming that also  $x$  is even for simplicity)

$$\begin{aligned} x^{3/2} [M^{*2m}]_{\alpha,\beta}(x) &= 2 \sum_{y=1}^{x/2} \sum_{\gamma \in \mathbb{S}} [M^{*m}]_{\alpha,\gamma}(y) x^{3/2} [M^{*m}]_{\gamma,\beta}(x-y) \\ &\leq 2 \cdot 2^{3/2} C m^3 \sum_{y=1}^{x/2} \sum_{\gamma \in \mathbb{S}} [M^{*m}]_{\alpha,\gamma}(y) [B^m]_{\gamma,\beta} \leq C (2m)^3 [B^{2m}]_{\alpha,\beta}, \end{aligned}$$

where we have applied (3.19), and (3.21) is proven.

The r.h.s. of (3.21) is summable in  $k$  because the matrix  $B$  has spectral radius  $\delta^\omega < 1$ . We can thus apply Dominated Convergence to (3.5) using (3.20) and we get (3.18) by:

$$\lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} \mathcal{Z}_{\alpha,\beta}(x) = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} [B^i \cdot L \cdot B^{(k-1)-i}]_{\alpha,\beta} = [(1-B)^{-1} \cdot L \cdot (1-B)^{-1}]_{\alpha,\beta}.$$

This concludes the proof of Theorem 1.3.  $\square$

**3.5. The free partition function.** We want now to compute the asymptotic behavior of the free partition function. In particular we have:

**Proposition 3.2 (Sharp asymptotic estimates, free case).** *As  $N \rightarrow \infty$ ,  $[N] = \eta$ , we have:*

- (1) *for  $\delta^\omega < 1$  (strictly delocalized regime)  $Z_{N,\omega}^f \sim C_{\omega,\eta}^{<,\text{f}} / N^{1/2}$  ;*
- (2) *for  $\delta^\omega = 1$  (critical regime)  $Z_{N,\omega}^f \sim C_{\omega,\eta}^{=,\text{f}}$ ;*
- (3) *for  $\delta^\omega > 1$  (localized regime)  $Z_{N,\omega}^f \sim C_{\omega,\eta}^{>,\text{f}} \exp(F_\omega N)$ .*

where  $C_{\omega,\eta}^{>,\text{f}}$ ,  $C_{\omega,\eta}^{<,\text{f}}$  and  $C_{\omega,\eta}^{=,\text{f}}$  are explicit positive constants, depending on  $\omega$  and  $\eta$ .

**Proof.** Conditioning on the last zero of  $S$  before epoch  $N$ , we have the useful formula

$$Z_{N,\omega}^f = \sum_{t=0}^N Z_{t,\omega}^c P(N-t) \exp(\tilde{\Phi}_{[t],[N]}(N-t)). \quad (3.22)$$

where  $P(n) := \mathbf{P}(\tau_1 > n) = \sum_{k=n+1}^{\infty} K(k)$  and:

$$\tilde{\Phi}_{\alpha,\beta}(\ell) := \log \left[ \frac{1}{2} \left( 1 + \exp(-\ell h_\omega + \Sigma_{\alpha,\beta}) \right) \right] \mathbf{1}_{(\ell > 1)} \mathbf{1}_{(\ell \in \beta - \alpha)}, \quad (3.23)$$

which differs from  $\Phi$  in not having the terms of interaction with the interface, cf. (2.4).

Since also the asymptotic behavior of  $P(\ell) \exp(\tilde{\Phi}_{\alpha,\beta}(\ell))$  will be needed, we set:

$$\tilde{L}_{\alpha,\beta} := \lim_{\ell \rightarrow \infty, \ell \in \beta - \alpha} \sqrt{\ell} P(\ell) e^{\tilde{\Phi}_{\alpha,\beta}(\ell)} = \begin{cases} c_K (1 + \exp(\Sigma_{\alpha,\beta})) & \text{if } h_\omega = 0 \\ c_K & \text{if } h_\omega > 0 \end{cases}, \quad (3.24)$$

as it follows easily from (3.23) and from the fact that  $P(\ell) \sim 2 c_K / \sqrt{\ell}$  as  $\ell \rightarrow \infty$ . For the rest of the proof we consider the different regimes separately.

*The strictly delocalized case.* Notice that:

$$N^{1/2} Z_{N-k, \theta_k \omega}^f = \sum_{t=0}^{N-k} \mathcal{Z}_{[k], [t+k]}(t) N^{1/2} P(N-k-t) \exp\left(\tilde{\Phi}_{[t+k], [N]}(N-k-t)\right).$$

Then by (3.24) we obtain

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} N^{1/2} Z_{N-k, \theta_k \omega}^f = \sum_{t=0}^{\infty} \mathcal{Z}_{[k], [t+k]}(t) \tilde{L}_{[t+k], \eta} = [(1-B)^{-1} \tilde{L}]_{[k], \eta} \quad (3.25)$$

since

$$\sum_{t=0}^{\infty} \mathcal{Z}_{\alpha, \gamma}(t) = \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} M_{\alpha, \gamma}^{*k}(t) = \sum_{k=0}^{\infty} B_{\alpha, \gamma}^{*k} = [(I-B)^{-1}]_{\alpha, \gamma}. \quad (3.26)$$

*The critical case.* For  $N \in \eta$  and  $k \leq N$ :

$$Z_{N-k, \theta_k \omega}^f = \sum_{\gamma} \sum_{t=0}^{N-k} \mathcal{Z}_{[k], \gamma}(t) P(N-k-t) \exp\left(\tilde{\Phi}_{\gamma, \eta}(N-k-t)\right).$$

By the previous results and using (3.24) we obtain that for every  $k \in \mathbb{N}$

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} Z_{N-k, \theta_k \omega}^f = \xi_{[k]} \frac{T}{2\pi} \frac{\langle \zeta, \tilde{L}_{\cdot, \eta} \rangle}{\langle \zeta, L\xi \rangle} \int_0^1 \frac{dt}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} = \xi_{[k]} \frac{T}{2} \frac{\langle \zeta, \tilde{L}_{\cdot, \eta} \rangle}{\langle \zeta, L\xi \rangle}, \quad (3.27)$$

where we denote the canonical scalar product in  $\mathbb{R}^{\mathbb{S}}$  by  $\langle \cdot, \cdot \rangle$ :

$$\langle \varphi, \psi \rangle := \sum_{\alpha \in \mathbb{S}} \varphi_{\alpha} \psi_{\alpha}, \quad \varphi, \psi \in \mathbb{R}^{\mathbb{S}}.$$

*The localized case.* By (3.22):

$$\begin{aligned} e^{-F_{\omega} N} Z_{N-k, \theta_k \omega}^f &= e^{-F_{\omega} N} \sum_{t=0}^{N-k} \mathcal{Z}_{[k], [N-t]}(N-k-t) P(t) \exp\left(\tilde{\Phi}_{[N-t], [N]}(t)\right) \\ &= e^{-F_{\omega} k} \sum_{\gamma \in \mathbb{S}} \sum_{t=0}^{N-k} e^{-F_{\omega} t} P(t) \left[ \exp\left(\tilde{\Phi}_{\gamma, [N]}(t)\right) e^{-F_{\omega} (N-k-t)} \mathcal{Z}_{[k], \gamma}(N-k-t) \right]. \end{aligned}$$

Since by (3.12) the expression in brackets converges as  $N \rightarrow \infty$  and  $N \in [t] + \gamma$ , we obtain

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} e^{-F_{\omega} N} Z_{N-k, \theta_k \omega}^f = \xi_{[k]} e^{-F_{\omega} k} \left( \frac{T}{\mu} \sum_{\gamma \in \mathbb{S}} \sum_{t=0}^{\infty} e^{-F_{\omega} t} P(t) \exp\left(\tilde{\Phi}_{\gamma, \eta}(t)\right) \zeta_{\gamma} \right).$$

Observe that the term in parenthesis is just a function of  $\eta$ . □

#### 4. A PRELIMINARY ANALYSIS OF THE POLYMER MEASURE

In this section we give some preliminary material which will be used in section 5 for the proof of the scaling limits of our models. We are going to show that the core of the polymer measure is encoded in its zero level set and that the law of the latter is expressed in terms of the partition function. This explains the crucial importance retained by the partition function for the study of  $\mathbf{P}_{N, \omega}^a$ .

We start giving a very useful decomposition of  $\mathbf{P}_{N, \omega}^a$ . The intuitive idea is that a path  $(S_n)_{n \leq N}$  can be split into two main ingredients:

- the family  $(\tau_k)_{k=0,1,\dots}$  of returns to zero of  $S$  (defined in § 2.2);
- the family of excursions from zero  $(S_{i+\tau_{k-1}} : 0 \leq i \leq \tau_k - \tau_{k-1})_{k=1,2,\dots}$

Moreover, since each excursion can be either positive or negative, it is also useful to consider separately the signs of the excursions  $\sigma_k := \text{sign}(S_{\tau_{k-1}+1})$  and the absolute values  $(e_k(i) := |S_{i+\tau_{k-1}}| : i = 0, \dots, \tau_k - \tau_{k-1})$ . Observe that these are trivial for an excursion with length 1: in fact if  $\tau_k = \tau_{k-1} + 1$  then  $\sigma_k = 0$  and  $e_k(0) = e_k(1) = 0$ .

Let us first consider the returns  $(\tau_k)_{k \leq \iota_N}$  under  $\mathbf{P}_{N,\omega}^a$ , where we recall the definition  $\iota_N = \sup\{k : \tau_k \leq N\}$ . The law of this process can be viewed as a probability measure  $p_{N,\omega}^a$  on the class of subsets of  $\{1, \dots, N\}$ : indeed for  $A \subseteq \{1, \dots, N\}$ , writing

$$A = \{t_1, \dots, t_{|A|}\}, \quad 0 =: t_0 < t_1 < \dots < t_{|A|} \leq N, \quad (4.1)$$

we can set

$$p_{N,\omega}^a(A) := \mathbf{P}_{N,\omega}^a(\tau_i = t_i, i \leq \iota_N). \quad (4.2)$$

Then from the very definition (1.6) of  $\mathbf{P}_{N,\omega}^a$  and from the strong Markov property of  $\mathbf{P}$ , we have the following basic

**Lemma 4.1.** *With the notation (4.1), for  $A \subseteq \{1, \dots, N\}$ : if  $a = c$ ,  $p_{N,\omega}^c(A) \neq 0$  if and only if  $t_{|A|} = N$ , and in this case:*

$$p_{N,\omega}^c(A) = \frac{1}{Z_{N,\omega}^c} \prod_{i=1}^{|A|} M_{[t_{i-1}], [t_i]}(t_i - t_{i-1}), \quad (4.3)$$

while for  $a = f$ :

$$p_{N,\omega}^f(A) = \frac{1}{Z_{N,\omega}^f} \left[ \prod_{i=1}^{|A|} M_{[t_{i-1}], [t_i]}(t_i - t_{i-1}) \right] P(N - t_{|A|}) \exp \left( \tilde{\Phi}_{[t_{|A|}], [N]}(N - t_{|A|}) \right). \quad (4.4)$$

Thus the law of the zero level set is explicitly given in terms of the kernel  $M_{\alpha,\beta}(n)$  and of the partition function  $Z_{N,\omega}^a$ . The following two lemmas (that follow again from the definition (1.6)) show that, conditionally on the zero level set, the signs are independent and the excursions are just the excursions of the unperturbed random walk  $S$  under  $\mathbf{P}$ . This shows that the zero level set is indeed the core of the polymer measure  $\mathbf{P}_{N,\omega}^a$ .

**Lemma 4.2.** *Conditionally on  $\{\iota_N, (\tau_j)_{j \leq \iota_N}\}$ , under  $\mathbf{P}_{N,\omega}^a$  the signs  $(\sigma_k)_{k \leq \iota_N+1}$  form an independent family. For  $k \leq \iota_N$ , the conditional law of  $\sigma_k$  is specified by:*

- if  $\tau_k = 1 + \tau_{k-1}$ , then  $\sigma_k = 0$ ;
- if  $\tau_k > 1 + \tau_{k-1}$ , then  $\sigma_k$  can take the two values  $\pm 1$  with

$$\mathbf{P}_{N,\omega}^a(\sigma_k = +1 \mid \iota_N, (\tau_j)_{j \leq \iota_N}) = \frac{1}{1 + \exp \left\{ -(\tau_k - \tau_{k-1}) h_\omega + \Sigma_{[\tau_{k-1}], [\tau_k]} \right\}}. \quad (4.5)$$

For  $a = f$ , when  $\tau_{\iota_N} < N$  there is a last incomplete excursion in the interval  $\{0, \dots, N\}$ , whose sign  $\sigma_{\iota_N+1}$  is also specified by (4.5), provided we set  $\tau_{\iota_N+1} := N$ .

**Lemma 4.3.** *Conditionally on  $\{\iota_N, (\tau_j)_{j \leq \iota_N}, (\sigma_j)_{j \leq \iota_N+1}\}$ , the excursions  $(e_k(\cdot))_{k \leq \iota_N+1}$  form an independent family under  $\mathbf{P}_{N,\omega}^a$ . For  $k \leq \iota_N$ , the conditional law of  $e_k(\cdot)$  on the event  $\{\tau_{k-1} = \ell_0, \tau_k = \ell_1\}$  is specified for  $f = (f_i)_{i=0, \dots, \ell_1 - \ell_0}$  by*

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a(e_k(\cdot) = f \mid \iota_N, (\tau_j)_{j \leq \iota_N}, (\sigma_j)_{j \leq \iota_N+1}) \\ &= \mathbf{P}(S_i = f_i : i = 0, \dots, \ell_1 - \ell_0 \mid S_i > 0 : i = 1, \dots, \ell_1 - \ell_0 - 1, S_{\ell_1 - \ell_0} = 0). \end{aligned} \quad (4.6)$$

For  $a = f$ , when  $\tau_{\iota_N} < N$  the conditional law on the event  $\{\tau_{\iota_N} = \ell < N\}$  of the last incomplete excursion  $e_{\iota_N+1}(\cdot)$  is specified for  $f = (f_i)_{i=0,\dots,N-\ell}$  by

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a \left( e_{\iota_N+1}(\cdot) = f \mid \iota_N, (\tau_j)_{j \leq \iota_N}, (\sigma_j)_{j \leq \iota_N+1} \right) \\ &= \mathbf{P} \left( S_i = f_i : i = 0, \dots, N - \ell \mid S_i > 0 : i = 1, \dots, N - \ell \right). \end{aligned} \quad (4.7)$$

We stress that Lemmas 4.1, 4.2 and 4.3 fully characterize the polymer measure  $\mathbf{P}_{N,\omega}^a$ . It is worth stressing that, conditionally on  $(\tau_k)_{k \in \mathbb{N}}$ , the joint distribution of  $(\sigma_j, e_j)_{j \leq \iota_N}$  does not depend on  $N$ . In this sense, all the  $N$ -dependence is contained in the law  $p_{N,\omega}^a$  of the zero level set. This fact will be exploited in the next section.

## 5. PROOF OF THEOREM 1.3 AND PROPOSITION 2.1

In this section we show that the measures  $\mathbf{P}_{N,\omega}^a$  converge under Brownian rescaling, proving Theorem 1.3 and Proposition 2.1. The results and proofs follow closely those of [9] and we shall refer to this paper for several technical lemmas; for the tightness of  $(Q_{N,\omega}^a)_{N \in \mathbb{N}}$  in  $C([0, 1])$ , we refer to [8].

**Lemma 5.1.** *For any  $\omega$  and  $a = c, f$  the sequence  $(Q_{N,\omega}^a)_{N \in \mathbb{N}}$  is tight in  $C([0, 1])$ .*

From now on, we consider separately the three regimes  $\delta^\omega > 1$ ,  $\delta^\omega < 1$  and  $\delta^\omega = 1$ .

**5.1. The localized regime ( $\delta^\omega > 1$ ).** We prove point (3) of Theorem 1.3. By Lemma 5.1 it is enough to prove that  $\mathbf{P}_{N,\omega}^a(|X_t^N| > \varepsilon) \rightarrow 0$  for all  $\varepsilon > 0$  and  $t \in [0, 1]$  and one can obtain this estimate explicitly. We point out however that in this regime one can avoid using the compactness lemma and one can obtain a stronger result by elementary means: observe that for any  $k, n \in \mathbb{N}$  such that  $n > 1$  and  $k + n \leq N$ , we have

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a(S_k = S_{k+n} = 0, S_{k+i} \neq 0 \text{ for } i = 1, \dots, n-1) \\ & \leq \frac{\frac{1}{2} \left( 1 + \exp \left( \sum_{i=1}^n (\omega_{k+i}^{(-1)} - \omega_{k+i}^{(+1)}) \right) \right)}{Z_{n,\theta_k \omega}^c} =: \hat{K}_k(n), \end{aligned} \quad (5.1)$$

and this holds both for  $a = c$  and  $a = f$ . Inequality (5.1) is obtained by using the Markov property of  $S$  both in the numerator and the denominator of the expression (1.6) defining  $\mathbf{P}_{N,\omega}^a(\cdot)$  after having bounded  $Z_{N,\omega}^a$  from below by inserting the event  $S_k = S_{k+n} = 0$ . Of course  $\lim_{n \rightarrow \infty} (1/n) \log \hat{K}_k(n) = -F_\omega$  uniformly in  $k$  (notice that  $\hat{K}_{k+T}(n) = \hat{K}_k(n)$ ). Therefore if we fix  $\varepsilon > 0$  by the union bound we obtain (we recall that  $\{\tau_j\}_j$  and  $\iota_N$  were defined in Section 2.2) for some  $c > 0$ :

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a \left( \max_{j=1,2,\dots,\iota_N} \tau_j - \tau_{j-1} > (1 + \varepsilon) \log N / F_\omega \right) \\ & \leq \sum_{k \leq N - (1 + \varepsilon) \log N / F_\omega} \sum_{n > (1 + \varepsilon) \log N / F_\omega} \hat{K}_k(n) \\ & \leq N \sum_{n > (1 + \varepsilon) \log N / F_\omega} \max_{k=0,\dots,T-1} \hat{K}_k(n) \leq \frac{c}{N^\varepsilon}. \end{aligned}$$

Let us start with the constrained case: notice that  $\mathbf{P}_{N,\omega}^c(dS)$ -a.s. we have  $\tau_{\iota_N} = N$  and hence  $\max_{j \leq \iota_N} \tau_j - \tau_{j-1} \geq \max_{n=1,\dots,N} |S_n|$ , since  $|S_{n+1} - S_n| \leq 1$ . Then we immediately

obtain that for any  $C > 1/\mathcal{F}_\omega$

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c \left( \max_{n=1,\dots,N} |S_n| > C \log N \right) = 0, \quad (5.2)$$

which is of course a much stronger statement than the scaling limit of point (3) of Theorem 1.3. If we consider instead the measure  $\mathbf{P}_{N,\omega}^f$ , the length of the last excursion has to be taken into account too: however, an argument very close to the one used in (5.1) yields also that the last excursion is exponentially bounded (with the same exponent) and the proof of point (3) of Theorem 1.3 is complete.  $\square$

**5.2. The strictly delocalized regime ( $\delta^\omega < 1$ ).** We prove point (1) of Theorem 1.3 and Proposition 2.1. We set for  $t \in \{1, \dots, N\}$ :

$$D_t := \inf\{k = 1, \dots, N : k > t, S_k = 0\}, \quad G_t := \sup\{k = 1, \dots, N : k \leq t, S_k = 0\}.$$

The following result shows that in the strictly delocalized regime, as  $N \rightarrow \infty$ , the visits to zero under  $\mathbf{P}_{N,\omega}^a$  tend to be very few and concentrated at a finite distance from the origin if  $a = f$  and from 0 or  $N$  if  $a = c$ .

**Lemma 5.2.** *If  $\delta^\omega < 1$  there exists a constant  $C > 0$  such that for all  $L > 0$ :*

$$\limsup_{N \rightarrow \infty} \left[ \mathbf{P}_{N,\omega}^f(G_N \geq L) + \mathbf{P}_{N,\omega}^c(G_{N/2} \geq L) + \mathbf{P}_{N,\omega}^c(D_{N/2} \leq N - L) \right] \leq CL^{-1/2}.$$

**Proof.** We consider e.g.  $\mathbf{P}_{N,\omega}^c(G_{N/2} \geq L)$ : using Lemma 4.1 to write down this probability, recalling the definition (3.5) of the kernel  $\mathcal{Z}_{\alpha,\beta}(n)$  and using (3.18), we obtain

$$\begin{aligned} \mathbf{P}_{N,\omega}^c(G_{N/2} \geq L) &= \sum_{x=L}^{\lfloor N/2 \rfloor} \mathcal{Z}_{0,[x]}(x) \sum_{z=\lfloor N/2 \rfloor+1}^N \frac{M_{[x],[z]}(z-x) \mathcal{Z}_{[z],[N]}(N-z)}{\mathcal{Z}_{0,[N]}(N)} \\ &\leq C_1 N^{3/2} \sum_{x=L}^{\lfloor N/2 \rfloor} x^{-3/2} \sum_{z=\lfloor N/2 \rfloor+1}^N (z-x)^{-3/2} (N+1-z)^{-3/2} \leq C_2 L^{-1/2}, \end{aligned}$$

for some positive constants  $C_1$  and  $C_2$ , and the proof is completed.  $\square$

*The signs.* Since the zeros are concentrated near the boundary, to complete the proof it is enough to argue as in the proof of Theorem 9 in [9]. More precisely, by Lemma 5.2 for large  $N$  the typical paths of  $\mathbf{P}_{N,\omega}^a$  are essentially made up of one big excursion, whose absolute value converges in law to the Brownian excursion  $\{e_t\}_{t \in [0,1]}$  for  $a = c$  and to the Brownian meander  $\{m_t\}_{t \in [0,1]}$  for  $a = f$  by standard invariance principles, cf. [19] and [3]. Therefore to complete the proof we only have to show that there exists the limit (as  $N \rightarrow \infty$  along  $[N] = \eta$ ) of the probability that the process (away from  $\{0, 1\}$ ) lives in the upper half plane. In the general case we have different limits depending on the sequence  $[N] = \eta$  and on  $a = f, c$ , while if  $\omega \notin \mathcal{P}$  all such limits coincide.

We start with the constrained case: given Lemma 5.2, it is sufficient to show that

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^c(S_{N/2} > 0) =: p_{\omega,\eta}^c. \quad (5.3)$$

Formula (5.3) follows from the fact that

$$\mathbf{P}_{N,\omega}^c(S_{N/2} > 0) = \sum_{\alpha,\beta \in \mathbb{S}} \sum_{x < N/2} \sum_{y > N/2} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,[N]}(N-y)}{\mathcal{Z}_{0,[N]}(N)},$$

where for all  $z \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{S}$  we set:

$$\rho_{\alpha,\beta}^+(z) := \frac{1}{1 + \exp(-z h_\omega + \Sigma_{\alpha,\beta})}, \quad (5.4)$$

cf. (4.5). By Dominated Convergence and by (3.10) and (3.26) we get:

$$\begin{aligned} \exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} N^{3/2} \sum_{x < N/2} \sum_{y > N/2} \mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,\eta}(N-y) \\ = [(1-B)^{-1}]_{0,\alpha} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) [(1-B)^{-1}]_{\beta,\eta}. \end{aligned}$$

Then by (3.18) it follows that (5.3) holds true, with

$$p_{\omega,\eta}^c := \frac{\sum_{\alpha,\beta} [(1-B)^{-1}]_{0,\alpha} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) [(1-B)^{-1}]_{\beta,\eta}}{[(1-B)^{-1} L (1-B)^{-1}]_{0,\eta}}. \quad (5.5)$$

**Remark 5.3.** Observe that by (3.10):

- if  $h_\omega > 0$  then in (5.5) the denominator is equal to the numerator, so that  $p_{\omega,\eta}^c = 1$  for all  $\eta$ .
- if  $h_\omega = 0$  and  $\Sigma \equiv 0$  then in (5.5) the denominator is equal to twice the numerator, so that  $p_{\omega,\eta}^c = 1/2$  for all  $\eta$ .
- in the remaining case, i.e. if  $\omega \in \mathcal{P}$ , in general  $p_{\omega,\eta}^c$  depends on  $\eta$ .

Now let us consider the free case. This time it is sufficient to show that

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^f(S_N > 0) =: p_{\omega,\eta}^f. \quad (5.6)$$

However we can write

$$\mathbf{P}_{N,\omega}^f(S_N > 0) = \sum_{\alpha} \sum_{x < N} \frac{\mathcal{Z}_{0,\alpha}(x) \cdot \frac{1}{2} P(N-k)}{Z_{N,\omega}^f},$$

and using (3.22), (3.26) and (3.24) we obtain that (5.6) holds with

$$p_{\omega,\eta}^f = \frac{\sum_{\alpha} [(1-B)^{-1}]_{0,\alpha} c_K}{[(1-B)^{-1} \tilde{L}]_{0,\eta}}. \quad (5.7)$$

**Remark 5.4.** Again, observe that by (3.24):

- if  $h_\omega > 0$  then in (5.7) the denominator is equal to the numerator and  $p_{\omega,\eta}^f = 1$  for all  $\eta$ .
- if  $h_\omega = 0$  and  $\Sigma \equiv 0$  then in (5.7) the denominator is equal to twice the numerator, so that  $p_{\omega,\eta}^f = 1/2$  for all  $\eta$ .
- in the remaining case, i.e. if  $\omega \in \mathcal{P}$ , in general  $p_{\omega,\eta}^f$  depends on  $\eta$  and is different from  $p_{\omega,\eta}^c$ .

The proof of point (1) of Theorem 1.3 and Proposition 2.1 is then concluded.  $\square$

**5.3. The critical regime ( $\delta^\omega = 1$ ).** We prove point (2) of Theorem 1.3 and Proposition 2.1. As in the previous section, we first determine the asymptotic behavior of the zero level set and then we pass to the study of the signs of the excursions.

*The zero level set.* We introduce the random closed subset  $\mathcal{A}_N^a$  of  $[0, 1]$ , describing the zero set of the polymer of size  $N$  rescaled by a factor  $1/N$ :

$$\mathbb{P}(\mathcal{A}_N^a = A/N) = p_{N,\omega}^a(A), \quad A \subseteq \{0, \dots, N\},$$

recall (4.2). Let us denote by  $\mathcal{F}$  the class of all closed subsets of  $\mathbb{R}^+ := [0, +\infty)$ . We are going to put on  $\mathcal{F}$  a topological and measurable structure, so that we can view the law of  $\mathcal{A}_N^a$  as a probability measure on  $\mathcal{F}$  and we can study the weak convergence of  $\mathcal{A}_N^a$ .

We endow  $\mathcal{F}$  with the topology of Matheron, cf. [22] and [12, section 3], which is a metrizable topology. To define it, to a closed subset  $F \subseteq \mathbb{R}^+$  we associate the compact nonempty subset  $\tilde{F}$  of the interval  $[0, \pi/2]$  defined by  $\tilde{F} := \arctan(F \cup \{+\infty\})$ . Then the metric  $\rho(\cdot, \cdot)$  we take on  $\mathcal{F}$  is

$$\rho(F, F') := \max \left\{ \sup_{t \in \tilde{F}} d(t, \tilde{F}'), \sup_{t' \in \tilde{F}'} d(t', \tilde{F}) \right\} \quad F, F' \in \mathcal{F}, \quad (5.8)$$

where  $d(s, A) := \inf\{|t - s|, t \in A\}$  is the standard distance between a point and a set. The r.h.s. of (5.8) is the so-called Hausdorff metric between the compact nonempty sets  $\tilde{F}, \tilde{F}'$ .

Thus by definition a sequence  $\{F_n\}_n \subset \mathcal{F}$  converges to  $F \in \mathcal{F}$  if and only if  $\rho(F_n, F) \rightarrow 0$ . This is equivalent to requiring that for each open set  $G$  and each compact  $K \subset \mathbb{R}^+$

$$\begin{aligned} F \cap G \neq \emptyset &\implies F_n \cap G \neq \emptyset \text{ eventually,} \\ F \cap K = \emptyset &\implies F_n \cap K = \emptyset \text{ eventually.} \end{aligned} \quad (5.9)$$

Another necessary and sufficient condition for  $F_n \rightarrow F$  is that  $d(t, F_n) \rightarrow d(t, F)$  for every  $t \in \mathbb{R}^+$ .

This topology makes  $\mathcal{F}$  a separable and compact metric space [22, Th. 1-2-1], in particular a Polish space. Endowing  $\mathcal{F}$  with the Borel  $\sigma$ -field, we have that the space  $\mathcal{M}_1(\mathcal{F})$  of probability measures on  $\mathcal{F}$  is compact with the topology of weak convergence.

The crucial result is the convergence in distribution as  $N \rightarrow \infty$  of the random set  $\mathcal{A}_N^a$  towards the zero set of a Brownian motion for  $a = f$  or of a Brownian bridge for  $a = c$ .

**Proposition 5.5.** *If  $\delta^\omega = 1$  then as  $N \rightarrow \infty$*

$$\mathcal{A}_N^f \implies \{t \in [0, 1] : B(t) = 0\}, \quad (5.10)$$

$$\mathcal{A}_N^c \implies \{t \in [0, 1] : \beta(t) = 0\}. \quad (5.11)$$

The proof of Proposition 5.5 is achieved comparing the law of  $\mathcal{A}_N^f$  and  $\mathcal{A}_N^c$  with the law of a random set  $\mathcal{R}_N$  defined as follows. With the notation introduced in §3.1, we introduce the rescaled random set  $\mathcal{R}_N$ :

$$\mathcal{R}_N := \text{range}\{\tau_i/N, i \geq 0\} = \tau/N \subset \mathbb{R}^+$$

under  $\mathbb{P}_{[0]}$ . Notice that for any  $A = \{t_1, \dots, t_{|A|}\} \subset \{1, \dots, N\}$  we have (setting  $t_0 := 0$ ):

$$\mathbb{P}_{[0]}(\tau \cap \{1, \dots, N\} = A) = \left[ \prod_{i=1}^{|A|} M_{[t_{i-1}], [t_i]}(t_i - t_{i-1}) \right] Q_{t_{|A|}}(N - t_{|A|}) \frac{\xi_{[t_{|A|}]} \xi_{[0]}}{\xi_{[0]}}, \quad (5.12)$$

where  $Q_\alpha(t) := \sum_\beta \sum_{s=t+1}^\infty \Gamma_{\alpha,\beta}(s)$ .

The key step to prove Proposition 5.5 is given by the following Lemma, whose proof uses the theory of regenerative sets and their connection with subordinators, see [12].

**Lemma 5.6.** *The random set  $\mathcal{R}_N$  converges in distribution to  $\{t \geq 0 : B(t) = 0\}$ .*

**Proof.** Recalling the definition (3.8) of  $\tau^\beta$ , we introduce the random set

$$\mathcal{R}_N^\beta := \text{range} \{ \tau_k/N : k \geq 0, J_k = \beta \} = \tau^\beta/N, \quad \beta \in \mathbb{S}$$

under  $\mathbb{P}_{[0]}$ . Notice that  $\mathcal{R}_N = \cup_\beta \mathcal{R}_N^\beta$ . We divide the rest of the proof in two steps.

*Step 1.* This is the main step: we prove that the law of  $\mathcal{R}_N^\beta$  converges to the law of  $\{t \geq 0 : B(t) = 0\}$ . For this we follow the proof of Lemma 5 in [9]. Let  $\{P(t)\}_{t \geq 0}$  be a Poisson process with rate  $\gamma > 0$ , independent of  $(T_i^\beta)_{i \geq 0}$ . Then  $\sigma_t = [T_1^\beta + \dots + T_{P(t)}^\beta]/N$  is a non decreasing right-continuous process with independent stationary increments and  $\sigma_0 = 0$ , that is  $\sigma = (\sigma_t)_{t \geq 0}$  is a subordinator. By the standard theory of Lévy processes, the law of  $\sigma$  is characterized by the Laplace transform of its one-time distributions:

$$\mathbb{E}[\exp(-\lambda\sigma_t)] = \exp(-t\phi_N(\lambda)), \quad \lambda \geq 0, t \geq 0,$$

for a suitable function  $\phi_N : [0, \infty) \mapsto [0, \infty)$ , called Lévy exponent, which has a canonical representation, the Lévy–Khintchin formula (see e.g. (1.15) in [12]):

$$\phi_N(\lambda) = \int_{(0, \infty)} (1 - e^{-\lambda s}) \gamma \mathbb{P}(T_1^\beta/N \in ds) = \gamma \sum_{n=1}^{\infty} (1 - \exp(-\lambda n/N)) q^\beta(n).$$

We denote the closed range  $\{\sigma_t : t \geq 0\}$  of the subordinator  $\sigma$  by  $\widehat{\mathcal{R}}_N^\beta$ . Then, following [12],  $\widehat{\mathcal{R}}_N^\beta$  is a regenerative set. Moreover  $\mathcal{R}_N^\beta = T_0^\beta/N + \widehat{\mathcal{R}}_N^\beta$ .

Notice now that the law of the regenerative set  $\widehat{\mathcal{R}}_N^\beta$  is invariant under the change of time scale  $\sigma_t \rightarrow \sigma_{ct}$ , for  $c > 0$ , and in particular independent of  $\gamma > 0$ . Since  $\phi_N \rightarrow c\phi_N$  under this change of scale, we can fix  $\gamma = \gamma_N$  such that  $\phi_N(1) = 1$  and this will be implicitly assumed from now on. Then, by Proposition (1.14) of [12], the law of  $\widehat{\mathcal{R}}_N^\beta$  is uniquely determined by  $\phi_N$ .

The asymptotic behavior of  $q^\beta$  given in (3.15) yields easily  $\phi_N(\lambda) \rightarrow \lambda^{1/2} =: \Phi_{BM}(\lambda)$  as  $N \rightarrow \infty$ . It is now a matter of applying the result in [12, §3] to obtain that  $\widehat{\mathcal{R}}_N^\beta$  converges in law to the regenerative set corresponding to  $\Phi_{BM}$ . However by direct computation one obtains that the latter is nothing but the zero level set of a Brownian motion, therefore  $\widehat{\mathcal{R}}_N^\beta \Rightarrow \{t \in [0, 1] : B(t) = 0\}$ . From the fact that  $T_0^\beta/N \rightarrow 0$  a.s., the same weak convergence for  $\mathcal{R}_N^\beta$  follows immediately.

*Step 2.* Notice that  $\mathcal{R}_N = \cup_\beta \mathcal{R}_N^\beta$  is the union of non independent sets. Therefore, although we know that each  $\mathcal{R}_N^\beta$  converges in law to  $\{t \geq 0 : B(t) = 0\}$ , it is not trivial that  $\mathcal{R}_N$  converges to the same limit. We start showing that for every positive  $t \geq 0$ , the distance between the first point in  $\mathcal{R}_N^\alpha$  after  $t$  and the first point in  $\mathcal{R}_N^\beta$  after  $t$  converges to zero in probability. More precisely, for any closed set  $F \subset [0, \infty)$  we set:

$$d_t(F) := \inf(F \cap (t, \infty)). \quad (5.13)$$

and we claim that for all  $\alpha, \beta \in \mathbb{S}$  and  $t \geq 0$ ,  $|d_t(\mathcal{R}_N^\alpha) - d_t(\mathcal{R}_N^\beta)| \rightarrow 0$  in probability.

Recalling (3.14) and the notation introduced there, we can write for all  $\epsilon > 0$ :

$$\mathbb{P}_{[0]} \left( d_t(\mathcal{R}_N^\alpha) \geq d_t(\mathcal{R}_N^\beta) + \epsilon \right) = \sum_{\gamma} \sum_{y=0}^{\lfloor Nt \rfloor} U_{0,\gamma}(y) \sum_{z=\lfloor Nt \rfloor-y+1}^{\infty} q^{(\gamma;\beta)}(z) \sum_{w=\lfloor N\epsilon \rfloor}^{\infty} q^{(\beta;\alpha)}(w).$$

Arguing as in the proof of (3.15), it is easy to obtain the bound  $q^{(\beta;\alpha)}(w) \leq C_1 w^{-3/2}$  and by (3.16) we have  $U_{0,\gamma}(y) \leq C_2 y^{-1/2}$ , with  $C_1, C_2$  positive constants. Therefore

$$\mathbb{P}_{[0]} \left( d_t(\mathcal{R}_N^\alpha) \geq d_t(\mathcal{R}_N^\beta) + \epsilon \right) \leq \frac{C_3}{N^{1/2}} \left( \int_0^{t/T} dy \int_{(t-y)/T}^\infty dz \int_{\epsilon/T}^\infty dw \frac{1}{y^{1/2} z^{3/2} w^{3/2}} \right)$$

for some positive constant  $C_3$ , having used the convergence of the Riemann sums to the corresponding integral. The same computations can be performed exchanging  $\alpha$  with  $\beta$ , hence the claim is proven.

Now notice that  $d_t(\mathcal{R}_N) = \min_{\alpha \in \mathbb{S}} d_t(\mathcal{R}_N^\alpha)$ , and since  $\mathbb{S}$  is a finite set we have that also  $|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^\beta)| \rightarrow 0$  in probability for any fixed  $\beta \in \mathbb{S}$ . Since we already know that  $\mathcal{R}_N^\beta$  converges weakly to the law of  $\{t \geq 0 : B(t) = 0\}$ , the analogous statement for  $\mathcal{R}_N$  follows by standard arguments. More precisely, let us look at  $(\mathcal{R}_N, \mathcal{R}_N^\beta)$  as a random element of the space  $\mathcal{F} \times \mathcal{F}$ : by the compactness of  $\mathcal{F}$  it suffices to take any convergent subsequence  $(\mathcal{R}_{k_n}, \mathcal{R}_{k_n}^\beta) \Rightarrow (\mathfrak{B}, \mathfrak{C})$  and to show that  $\mathbb{P}(\mathfrak{B} \neq \mathfrak{C}) = 0$ . However we can write

$$\{\mathfrak{B} \neq \mathfrak{C}\} = \bigcup_{t \in \mathbb{Q}^+} \bigcup_{n \in \mathbb{N}} \{|d_t(\mathfrak{B}) - d_t(\mathfrak{C})| > 1/n\},$$

by the right-continuity of  $t \mapsto d_t$ , and by the Portmanteau Theorem we have

$$\mathbb{P}_{[0]} (|d_t(\mathfrak{B}) - d_t(\mathfrak{C})| > 1/n) \leq \limsup_{N \rightarrow \infty} \mathbb{P}_{[0]} (|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^\beta)| > 1/n) = 0,$$

because  $|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^\beta)| \rightarrow 0$  in probability.  $\square$

**Proof of Proposition 5.5: equation (5.11).** First, we compute the Radon-Nykodim derivative  $f_N^c$  of the law of  $\mathcal{A}_N^c \cap [0, 1/2]$  with respect to the law of  $\mathcal{R}_N^{1/2} := \mathcal{R}_N \cap [0, 1/2]$ , using (4.3) and (5.12). For  $F = \{t_1/N, \dots, t_k/N\} \subset [0, 1/2]$  with  $0 =: t_0 < t_1 < \dots < t_k$  integer numbers, the value of  $f_N^c$  at  $\mathcal{R}_N^{1/2} = F$  depends only on  $g_{1/2}(F)$  and is given by:

$$f_N^c(g_{1/2}(F)) = f_N^c(t_k/N) = \frac{\sum_{n=N/2}^N M_{[t_k], [n]}(n - t_k) \mathcal{Z}_{[n], [N]}(N - n)}{\mathcal{Z}_{0, [N]}(N) Q_{[t_k]}(N/2 - t_k)} \frac{\xi_0}{\xi_{[t_k]}},$$

where for any closed set  $F \subset [0, \infty)$  we set:

$$g_t(F) := \sup(F \cap [0, t]). \quad (5.14)$$

By (3.17), for all  $\varepsilon > 0$  and uniformly in  $g \in [0, 1/2 - \varepsilon]$ :

$$f_N^c(g) \sim \frac{[L\xi]_{[Ng]} \frac{T^2}{2\pi} \frac{\zeta_{[N]}}{\langle \zeta, L\xi \rangle} T^{-1} \int_0^{1/2} y^{-1/2} (1 - y - g)^{-3/2} dy}{\frac{T^2}{2\pi} \frac{\xi_0 \zeta_{[N]}}{\langle \zeta, L\xi \rangle} T^{-1} [L\xi]_{[Ng]} / \xi_{[Ng]} 2(1/2 - g)^{-1/2}} \frac{\xi_0}{\xi_{[Ng]}} = \frac{\sqrt{1/2}}{1 - g} =: r(g).$$

If  $\Psi$  is a bounded continuous functional on  $\mathcal{F}$  such that  $\Psi(F) = \Psi(F \cap [0, 1/2])$  for all  $F \in \mathcal{F}$ , then, setting  $Z_B := \{t \in [0, 1] : B(t) = 0\}$  and  $Z_\beta := \{t \in [0, 1] : \beta(t) = 0\}$ , we get:

$$\mathbb{E}[\Psi(Z_\beta)] = \mathbb{E} [\Psi(Z_B) r(g_{1/2}(Z_B))],$$

see formula (49) in [9]. By Lemma 5.6 and by the asymptotic behavior of  $f_N^c$  we obtain

$$\mathbb{E}[\Psi(\mathcal{A}_N^c)] = \mathbb{E} [\Psi(\mathcal{R}_N^{1/2}) f_N^c(g_{1/2}(\mathcal{R}_N^{1/2}))] \xrightarrow{N \rightarrow \infty} \mathbb{E} [\Psi(Z_B) r(g_{1/2}(Z_B))] = \mathbb{E}[\Psi(Z_\beta)],$$

i.e.  $\mathcal{A}_N^c \cap [0, 1/2]$  converges to  $Z_\beta \cap [0, 1/2]$ . Notice now that the distribution of the random set  $\{1 - t : t \in \mathcal{A}_N^c \cap [1/2, 1]\}$  under  $\mathbf{P}_{N,\omega}^c$  is the same as the distribution of  $\mathcal{A}_N^c \cap [0, 1/2]$  under  $\mathbf{P}_{N,\overline{\omega}}^c$ , where  $\overline{\omega}_{[i]} := \omega_{[N-i]}$ . Therefore we obtain that  $\mathcal{A}_N^c \cap [1/2, 1]$  converges to  $Z_\beta \cap [0, 1/2]$  and the proof is complete.

**Proof of Proposition 5.5: equation (5.10).** By conditioning on the last zero, from (4.3) and (4.4) we see that if  $\Psi$  is a bounded continuous functional on  $\mathcal{F}$  then

$$\mathbb{E} \left[ \Psi(\mathcal{A}_N^f) \right] = \sum_{k=0}^N \mathbb{E} \left[ \Psi \left( \frac{k}{N} \mathcal{A}_k^c \right) \right] \frac{Z_{k,\omega}^c}{Z_{N,\omega}^f} P(N-k) \exp \left( \tilde{\Phi}_{[k],[N]}(N-k) \right).$$

We denote by  $\beta^t$  a Brownian bridge over the interval  $[0, t]$ , i.e. a Brownian motion over  $[0, t]$  conditioned to be 0 at time  $t$ , and we set  $Z_{\beta^t} := \{s \in [0, t] : \beta^t(s) = 0\} \stackrel{d}{=} tZ_\beta$ . By (5.11) it follows that if  $k/N \rightarrow t$  then the random set  $\frac{k}{N} \mathcal{A}_k^c$  converges in distribution to  $Z_{\beta^t}$ . Then, applying (3.17) and (3.27), we obtain as  $N \rightarrow \infty$  along  $[N] = \eta$ :

$$\begin{aligned} \mathbb{E} [\Psi(\mathcal{A}_k^c)] &= \sum_{k=0}^N \sum_{\gamma} 1_{(k \in \gamma)} \mathbb{E} \left[ \Psi \left( \frac{k}{N} \mathcal{A}_k^c \right) \right] \frac{Z_{k,\omega}^c}{Z_{N,\omega}^f} P(N-k) \exp \left( \tilde{\Phi}_{\gamma,\eta}(N-k) \right) \\ &\longrightarrow \int_0^1 \mathbb{E}[\Psi(Z_{\beta^t})] \frac{1}{\pi t^{\frac{1}{2}} (1-t)^{\frac{1}{2}}} dt \cdot \sum_{\gamma} \frac{1}{T} \frac{T^2}{2} \frac{\xi_0 \zeta_{\gamma}}{\langle \zeta, L\xi \rangle} \frac{\tilde{L}_{\gamma,\eta}}{\xi_0 \frac{T}{2} \frac{\langle \zeta, \tilde{L}_{\cdot,\eta} \rangle}{\langle \zeta, L\xi \rangle}} = \mathbb{E}[\Psi(Z_B)]. \end{aligned}$$

Since the result does not depend on the subsequence  $[N] = \eta$ , we have indeed proven that  $\mathcal{A}_N^f$  converges in distribution to  $Z_B$ .  $\square$

*The signs.* In order to conclude the proof of part (2) of Theorem 1.3 and Proposition 2.1 in the critical case ( $\delta^\omega = 1$ ) we follow closely the proof given in Section 8 of [9]. Having already proven the convergence of the zero level set, we only have to *paste* the excursions (recall Lemmas 4.2 and 4.3). The weak convergence under diffusive rescaling of  $e_k(\cdot)$  for  $k \leq \iota_N$  towards the Brownian excursion  $e(\cdot)$  and of the last excursion  $e_{\iota_N+1}(\cdot)$  for  $a = f$  towards the Brownian meander  $m(\cdot)$  has been proved in [19] and, respectively, in [3]. Then it only remains to focus on the signs.

We start with the constrained case: we are going to show that for all  $t \in (0, 1)$

$$\exists \lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c(S_{\lfloor tN \rfloor} > 0) =: p_\omega, \quad (5.15)$$

and the limit is independent of  $t$ . We point out that actually we should fix the extremities of the excursion embracing  $t$ , that is we should rather prove that

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c \left( S_{\lfloor tN \rfloor} > 0 \mid G_{\lfloor tN \rfloor}/N \in (a - \varepsilon, a), D_{\lfloor tN \rfloor}/N \in (b, b + \varepsilon) \right) = p_\omega, \quad (5.16)$$

for  $a < t < b$  and  $\varepsilon > 0$  (recall the definition of  $G_t$  and  $D_t$  in § 5.2), but in order to lighten the exposition we will stick to (5.15), since proving (5.16) requires only minor changes.

We have, recalling (5.4):

$$\mathbf{P}_{N,\omega}^c(S_{\lfloor tN \rfloor} > 0) = \sum_{\alpha, \beta} \sum_{x < \lfloor tN \rfloor} \sum_{y > \lfloor tN \rfloor} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,[N]}(N-y)}{\mathcal{Z}_{0,[N]}(N)}.$$

By Dominated Convergence and by (3.17):

$$\begin{aligned} \exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} N^{1/2} \sum_{x < \lfloor tN \rfloor} \sum_{y > \lfloor tN \rfloor} \mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,\eta}(N-y) \\ = \frac{1}{T^2} \int_0^t dx \int_t^1 dy [x(y-x)^3(1-y)]^{-\frac{1}{2}} \left( \frac{T^2}{2\pi} \right)^2 \frac{\xi_0 \zeta_\alpha \xi_\beta \zeta_\eta}{\langle \zeta, L\xi \rangle^2} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) \end{aligned}$$

see (3.10). We obtain that (5.15) holds true with

$$p_\omega := \frac{\sum_{\alpha,\beta} \zeta_\alpha c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) \xi_\beta}{\langle \zeta, L\xi \rangle} \quad (5.17)$$

**Remark 5.7.** Observe the following: by (3.10),

- if  $h_\omega > 0$  then in (5.17) the denominator is equal to the numerator, so that  $p_\omega = 1$ .
- if  $h_\omega = 0$  and  $\Sigma \equiv 0$  then in (5.17) the denominator is equal to twice the numerator, so that  $p_\omega = 1/2$ .

Now let us consider the free case. We are going to show that for all  $t \in (0, 1]$ :

$$\lim_{\substack{N \rightarrow \infty \\ [N] = \eta}} \mathbf{P}_{N,\omega}^f(S_{[tN]} > 0) = \left(1 - \frac{2 \arcsin \sqrt{t}}{\pi}\right) p_\omega + \frac{2 \arcsin \sqrt{t}}{\pi} q_{\omega,\eta} =: p_{\omega,\eta}^f(t), \quad (5.18)$$

where  $p_\omega$  is the same as above, see (5.17), while  $q_{\omega,\eta}$  is defined in (5.19) below. We stress again that we should actually fix the values of  $G_{[tN]}$  and  $D_{[tN]}$  like in (5.16), proving that the limiting probability is either  $p_\omega$  or  $q_{\omega,\eta}$  according to whether  $D_{[tN]} \leq N$  or  $D_{[tN]} > N$ , but this will be clear from the steps below. Formula (5.18) follows from the fact that

$$\begin{aligned} \mathbf{P}_{N,\omega}^f(S_{[tN]} > 0) &= \sum_{\alpha,\beta} \sum_{x < [tN]} \sum_{y > [tN]} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) Z_{N-y,\theta_{[y]}\omega}^f}{Z_{N,\omega}^f} \\ &\quad + \sum_{\alpha} \sum_{x < [tN]} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,[N]}^+(N-x) P(N-x) \exp(\tilde{\Phi}_{[x],[N]}(N-x))}{Z_{N,\omega}^f}. \end{aligned}$$

Letting  $N \rightarrow \infty$  with  $[N] = \eta$ , by (3.27) the first term in the r.h.s. converges to:

$$\begin{aligned} &\int_0^t \frac{dx}{x^{\frac{1}{2}}} \int_t^1 \frac{dy}{(y-x)^{\frac{3}{2}}} \sum_{\alpha,\beta} \frac{1}{T^2} \frac{T^2 \xi_0 \zeta_\alpha}{2\pi \langle \zeta, L\xi \rangle} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) \frac{\xi_\beta \frac{T}{2} \langle \zeta, \tilde{L}_{\cdot,\eta} \rangle}{\langle \zeta, L\xi \rangle} \cdot \frac{\langle \zeta, L\xi \rangle}{\xi_0 \frac{T}{2} \langle \zeta, \tilde{L}_{\cdot,\eta} \rangle} \\ &= \left(1 - \frac{2 \arcsin \sqrt{t}}{\pi}\right) \cdot p_\omega \end{aligned}$$

while the second term converges to

$$\int_0^t \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \frac{1}{T} \sum_{\alpha} \frac{T^2 \xi_0 \zeta_\alpha}{2\pi \langle \zeta, L\xi \rangle} c_K \cdot \frac{\langle \zeta, L\xi \rangle}{\xi_0 \frac{T}{2} \langle \zeta, \tilde{L}_{\cdot,\eta} \rangle} = \frac{2 \arcsin \sqrt{t}}{\pi} \cdot \frac{c_K \sum_\gamma \zeta_\gamma}{\langle \zeta, \tilde{L}_{\cdot,\eta} \rangle}.$$

Therefore we obtain (5.18) with:

$$q_{\omega,\eta} = \frac{c_K \sum_\gamma \zeta_\gamma}{\langle \zeta, \tilde{L}_{\cdot,\eta} \rangle}. \quad (5.19)$$

**Remark 5.8.** We observe that, by (3.24):

- if  $h_\omega > 0$ , or if  $h_\omega = 0$  and  $\Sigma \equiv 0$ , then  $p_{\omega,\eta}^f(t) = q_{\omega,\eta} = p_\omega$  for all  $t$  and  $\eta$
- in the remaining case, i.e. if  $\omega \in \mathcal{P}$ , in general  $p_{\omega,\eta}^f(t)$  depends on  $t$  and  $\eta$ .

Now that we have proven the convergence of the probabilities of the signs of the excursion, in order to conclude the proof of point (2) of Theorem 1.3 and Proposition 2.1, it is enough to use the excursion theory of Brownian motion: for the details we refer to the proof of Theorem 11 in [9].  $\square$

## APPENDIX A. AN ASYMPTOTIC RESULT

In this appendix we are going to prove that equation (3.15) holds true, but we need first some preliminary notation and results.

Given an irreducible  $T \times T$  matrix  $Q_{\alpha,\beta}$  with nonnegative entries, its Perron–Frobenius eigenvalue (= spectral radius) will be denoted by  $\mathbf{Z} = \mathbf{Z}(Q)$  and the corresponding left and right eigenvectors (with any normalization) will be denoted by  $\{\zeta_\alpha\}, \{\xi_\alpha\}$ . We recall that  $\zeta_\alpha, \xi_\alpha > 0$ . Being a simple root of the characteristic polynomial,  $\mathbf{Z}(Q)$  is an analytic function of the entries of  $Q$ , and

$$\frac{\partial \mathbf{Z}}{\partial Q_{\alpha,\beta}} = \frac{\zeta_\alpha \xi_\beta}{\langle \zeta, \xi \rangle}. \quad (\text{A.1})$$

Hence  $\mathbf{Z}(Q)$  is a strictly increasing function of each of the entries of  $Q$ .

Now, let  $Q$  denote the transition matrix of an irreducible, positive recurrent Markov chain, and let us introduce the matrix  $Q^{(\gamma)}$  and the vector  $\delta^{(\gamma)}$ , defined by

$$[Q^{(\gamma)}]_{\alpha,\beta} := Q_{\alpha,\beta} \mathbf{1}_{(\beta \neq \gamma)} \quad [\delta^{(\gamma)}]_\alpha := \mathbf{1}_{(\alpha=\gamma)}.$$

By monotonicity,  $\mathbf{Z}(Q^{(\gamma)}) < \mathbf{Z}(Q) = 1$  for all  $\gamma$ . Then we can define the geometric series

$$(1 - Q^{(\gamma)})^{-1} := \sum_{k=0}^{\infty} (Q^{(\gamma)})^k.$$

The interesting point is that, for every fixed  $\gamma$ , the vector  $\alpha \mapsto [(1 - Q^{(\gamma)})^{-1}]_{\gamma,\alpha}$  is (a multiple of) the left Perron–Frobenius eigenvector of the matrix  $Q$ . Similarly the vector  $\alpha \mapsto [(1 - Q^{(\gamma)})^{-1} \cdot Q]_{\alpha,\gamma}$  is (a multiple of) the right Perron–Frobenius eigenvector of  $Q$ . More precisely we have

$$[(1 - Q^{(\gamma)})^{-1}]_{\gamma,\alpha} = \frac{\nu_\alpha}{\nu_\gamma} \quad [(1 - Q^{(\gamma)})^{-1} \cdot Q]_{\alpha,\gamma} = 1, \quad (\text{A.2})$$

where  $\{\nu_\alpha\}_\alpha$  is the unique invariant law of the chain, that is  $\sum_\alpha \nu_\alpha Q_{\alpha,\beta} = \nu_\beta$  and  $\sum_\alpha \nu_\alpha = 1$ . Equation (A.2) can be proved by exploiting its probabilistic interpretation in terms of expected number of visits to state  $\alpha$  before the first return to site  $\gamma$ , see [2, section I.3].

Next we turn to our main problem. We recall for convenience the notations introduced in § 3.1 and § 3.3. The process  $\{\tau_k\}_{k \geq 0}$  where  $\tau_0 = 0$  and  $\tau_k = T_1 + \dots + T_k$  is a Markov–renewal process associated to the semi–Markov kernel  $\Gamma_{\alpha,\gamma}(n)$  (defined in (3.13)) and  $\{J_k\}_{k \geq 0}$  is its modulating chain. We denote by  $\mathbb{P}_\beta$  the law of  $\{(J_k, \tau_k)\}_{k \geq 0}$  with starting point  $J_0 = \beta$  and we set  $\ell := \inf\{k > 0 : J_k = \beta\}$ . Then  $q^\beta(x)$  denotes the law of  $\tau_\ell$  under  $\mathbb{P}_\beta$  and we want to determine its asymptotic behavior.

We anticipate that the notations are necessarily quite involved, but the basic idea is simple. By the periodic structure of the kernel  $\Gamma$  it follows that  $q^\beta(x)$  is zero if  $[x] \neq 0$ . On the other hand, when  $[x] = [0]$ , by summing over the possible values of the index  $\ell$  and using equation (3.6) we obtain

$$\begin{aligned} q^\beta(x) &= \mathbb{P}_\beta(\tau_1 = x, J_1 = \beta) + \sum_{k=1}^{\infty} \mathbb{P}_\beta(J_i \neq \beta : 1 \leq i \leq k, J_{k+1} = \beta, \tau_{k+1} = x) \\ &= \sum_{k=0}^{\infty} \left( (\Gamma^{(\beta)})^{*k} * \Gamma \right)_{\beta,\beta}(x), \end{aligned} \quad (\text{A.3})$$

where we have introduced the kernel  $\Gamma_{\alpha,\gamma}^{(\beta)}(x) := \Gamma_{\alpha,\gamma}(x)\mathbf{1}_{(\gamma \neq \beta)}$  that gives the law of the steps with index  $k < \ell$ . Looking at (A.3), we set  $V_{\alpha,\gamma}^{(\beta)}(x) := \sum_{k=0}^{\infty} ((\Gamma^{(\beta)})^{*k})_{\alpha,\gamma}(x)$  and we can write

$$q^\beta(x) = (V^{(\beta)} * \Gamma)_{\beta,\beta}(x) = \sum_{\gamma \in \mathbb{S}} \sum_{y=0}^{x-1} V_{\beta,\gamma}^{(\beta)}(y) \Gamma_{\gamma,\beta}(x-y). \quad (\text{A.4})$$

The asymptotic behavior of  $q^\beta(x)$  can be extracted from the above expression. To this aim, we need to know both the asymptotic behavior as  $n \rightarrow \infty$  and the sum over  $n \in \mathbb{N}$  of the two kernels  $\Gamma_{\gamma,\beta}(n)$  and  $V_{\beta,\gamma}^{(\beta)}(n)$  appearing in the r.h.s.

- By (3.13) and (3.10), as  $n \rightarrow \infty$  along  $[n] = \beta - \gamma$  we have

$$\Gamma_{\gamma,\beta}(n) \sim \frac{\widehat{L}_{\gamma,\beta}}{n^{3/2}} \quad \text{where} \quad \widehat{L}_{\gamma,\beta} := L_{\gamma,\beta} \frac{\xi_\beta}{\xi_\gamma}. \quad (\text{A.5})$$

Moreover, the sum over  $n \in \mathbb{N}$  gives

$$\sum_{n \in \mathbb{N}} \Gamma_{\gamma,\beta}(n) = B_{\gamma,\beta} \frac{\xi_\beta}{\xi_\gamma} =: \widehat{B}_{\gamma,\beta}. \quad (\text{A.6})$$

- For the asymptotic behavior of the kernel  $V^{(\beta)} := \sum_{k=0}^{\infty} (\Gamma^{(\beta)})^{*k}$  we can apply the theory developed in § 3.4 for the case  $\delta^\omega < 1$ , because the matrix  $\sum_{x \in \mathbb{N}} \Gamma_{\alpha,\gamma}^{(\beta)}(x)$  is just  $[\widehat{B}^{(\beta)}]_{\alpha,\gamma}$  by (A.6) (we recall the convention  $[Q^{(\beta)}]_{\alpha,\gamma} := Q_{\alpha,\gamma} \mathbf{1}_{(\gamma \neq \beta)}$  for any matrix  $Q$ ) which has Perron–Frobenius eigenvalue strictly smaller than 1. Since

$$\Gamma_{\alpha,\gamma}^{(\beta)}(n) \sim \frac{[\widehat{L}^{(\beta)}]_{\alpha,\gamma}}{n^{3/2}} \quad n \rightarrow \infty, \quad [n] = \gamma - \alpha,$$

we can apply (3.18) to get the asymptotic behavior as  $n \rightarrow \infty$ ,  $[n] = \alpha - \gamma$ :

$$V_{\alpha,\gamma}^{(\beta)}(n) \sim \left( [(1 - \widehat{B}^{(\beta)})^{-1} \widehat{L}^{(\beta)} (1 - \widehat{B}^{(\beta)})^{-1}]_{\alpha,\gamma} \right) \frac{1}{n^{3/2}}. \quad (\text{A.7})$$

On the other hand, for the sum over  $n \in \mathbb{N}$  an analog of (3.19) yields

$$\sum_{n \in \mathbb{N}} V_{\alpha,\gamma}^{(\beta)}(n) = \sum_{k=0}^{\infty} [(\widehat{B}^{(\beta)})^k]_{\alpha,\gamma} = [(1 - \widehat{B}^{(\beta)})^{-1}]_{\alpha,\gamma}. \quad (\text{A.8})$$

As equations (A.5) and (A.7) show, both kernels  $V^{(\beta)}$  and  $\Gamma$  have a  $n^{-3/2}$  tail. Then from (A.4) it follows that as  $x \rightarrow \infty$  along  $[x] = 0$

$$q^\beta(x) \sim \sum_{\gamma \in \mathbb{S}} \left\{ \left( \sum_{n \in \mathbb{N}} V_{\beta,\gamma}^{(\beta)}(n) \right) \Gamma_{\gamma,\beta}(x) + V_{\beta,\gamma}^{(\beta)}(x) \left( \sum_{n \in \mathbb{N}} \Gamma_{\gamma,\beta}(n) \right) \right\}.$$

Now it suffices to apply (A.8), (A.5), (A.7) and (A.6) to see that indeed  $q^\beta(x) \sim c_\beta / x^{3/2}$  as  $x \rightarrow \infty$  along  $[x] = 0$ , where the positive constant  $c_\beta$  is given by

$$c_\beta = \left[ (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L} \right]_{\beta,\beta} + \left[ (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L}^{(\beta)} \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \right]_{\beta,\beta}.$$

Using the fact that  $[(1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B}]_{\beta,\beta} = 1$ , which follows from (A.2) applied to the matrix  $Q = \widehat{B}$ , we can rewrite the above expression as

$$c_\beta = \left[ (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L} \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \right]_{\beta,\beta} = \frac{1}{\nu_\beta} \sum_{\alpha, \gamma \in \mathbb{S}} \nu_\alpha \widehat{L}_{\alpha,\gamma},$$

where  $\{\nu_\alpha\}_\alpha$  is the invariant measure of the matrix  $\widehat{B}$  and the second equality follows again from (A.2). However from (A.6) it is easily seen that  $\{\nu_\alpha\} = \{\zeta_\alpha \xi_\alpha\}$ , and recalling the definition (A.5) of  $\widehat{L}$  we finally obtain the expression for  $c_\beta$  given in equation (3.15):

$$c_\beta = \frac{1}{\zeta_\beta \xi_\beta} \sum_{\alpha, \gamma} \zeta_\alpha L_{\alpha,\gamma} \xi_\gamma. \quad (\text{A.9})$$

## APPENDIX B. A LOCALIZATION ARGUMENT

Let us give a proof that for the *copolymer near a selective interface* model, described in § 1.1, the charge  $\omega$  never belongs to  $\mathcal{P}$  (see (2.9) for the definition of  $\mathcal{P}$ ). More precisely, we are going to show that if  $h_\omega = 0$  and  $\Sigma \not\equiv 0$  then  $\delta^\omega > 1$ , that is the periodic copolymer with zero-mean, nontrivial charges is always localized. As a matter of fact this is an immediate consequence of the estimates on the critical line obtained in [5]. However we want to give here an explicit proof, both because it is more direct and because the model studied in [5] is built over the simple random walk measure, corresponding to  $p = 1/2$  with the language of Section 1, while we consider the case  $p < 1/2$ .

We recall that, by (A.1), the Perron-Frobenius eigenvalue  $Z(Q)$  of an irreducible matrix  $Q$  is increasing in the entries of  $Q$ . We also point out a result proved by Kingman [20]: if the matrix  $Q = Q(t)$  is a function of a real parameter  $t$  such that all the entries  $Q_{\alpha,\beta}(t)$  are log-convex functions of  $t$  (that is  $t \mapsto \log Q_{\alpha,\beta}(t)$  is convex for all  $\alpha, \beta$ ), then also  $t \mapsto Z(Q(t))$  is a log-convex function of  $t$ .

Next we come to the *copolymer near a selective interface* model: with reference to the general Hamiltonian (1.3), we are assuming that  $\omega_n^{(0)} = \tilde{\omega}_n^{(0)} = 0$  and  $h_\omega = 0$  (where  $h_\omega$  was defined in (1.5)). In this case the integrated Hamiltonian  $\Phi_{\alpha,\beta}(\ell)$ , see (2.4), is given by

$$\Phi_{\alpha,\beta}(\ell) = \begin{cases} 0 & \text{if } \ell = 1 \text{ or } \ell \notin \beta - \alpha \\ \log \left[ \frac{1}{2} \left( 1 + \exp(\Sigma_{\alpha,\beta}) \right) \right] & \text{if } \ell > 1 \text{ and } \ell \in \beta - \alpha \end{cases}.$$

We recall that the law of the first return to zero of the original walk is denoted by  $K(\cdot)$ , see (2.1), and we introduce the function  $q : \mathbb{S} \rightarrow \mathbb{R}^+$  defined by

$$q(\gamma) := \sum_{x \in \mathbb{N}, [x] = \gamma} K(x)$$

(notice that  $\sum_\gamma q(\gamma) = 1$ ). Then the matrix  $B_{\alpha,\beta}$  defined by (2.6) becomes

$$B_{\alpha,\beta} = \begin{cases} \frac{1}{2} \left( 1 + \exp(\Sigma_{\alpha,\beta}) \right) q(\beta - \alpha) & \text{if } \beta - \alpha \neq [1] \\ K([1]) + \frac{1}{2} \left( 1 + \exp(\Sigma_{\alpha,\alpha+[1]}) \right) \cdot (q([1]) - K([1])) & \text{if } \beta - \alpha = [1] \end{cases} \quad (\text{B.1})$$

By (2.7), to prove localization we have to show that the Perron–Frobenius eigenvalue of the matrix  $(B_{\alpha,\beta})$  is strictly greater than 1, that is  $Z(B) > 1$ . Applying the elementary

convexity inequality  $(1 + \exp(x))/2 \geq \exp(x/2)$  to (B.1) we get

$$B_{\alpha,\beta} \geq \tilde{B}_{\alpha,\beta} := \begin{cases} \exp(\Sigma_{\alpha,\beta}/2) q(\beta - \alpha) & \text{if } \beta - \alpha \neq [1] \\ K(1) + \exp(\Sigma_{\alpha,\alpha+[1]}/2) \cdot (q([1]) - K(1)) & \text{if } \beta - \alpha = [1] \end{cases}. \quad (\text{B.2})$$

By hypothesis  $\Sigma_{\alpha_0,\beta_0} \neq 0$  for some  $\alpha_0, \beta_0$ , therefore the inequality above is strict for  $\alpha = \alpha_0, \beta = \beta_0$ . We have already observed that the P–F eigenvalue is a strictly increasing function of the entries of the matrix, hence  $Z(B) > Z(\tilde{B})$ . Therefore it only remains to show that  $Z(\tilde{B}) \geq 1$ , and the proof will be completed.

Again an elementary convexity inequality applied to the second line of (B.2) yields

$$\tilde{B}_{\alpha,\beta} \geq \hat{B}_{\alpha,\beta} := \exp(c(\beta - \alpha) \Sigma_{\alpha,\beta}/2) \cdot q(\beta - \alpha) \quad (\text{B.3})$$

where

$$c(\gamma) := \begin{cases} 1 & \text{if } \gamma \neq [1] \\ \frac{q([1]) - K(1)}{q([1])} & \text{if } \gamma = [1] \end{cases}.$$

We are going to prove that  $Z(\hat{B}) \geq 1$ . Observe that setting  $v_\alpha := \Sigma_{[0],\alpha}$  we can write

$$\Sigma_{\alpha,\beta} = \Sigma_{[0],\beta} - \Sigma_{[0],\alpha} = v_\beta - v_\alpha.$$

Then we make a similarity transformation via the matrix  $L_{\alpha,\beta} := \exp(v_\beta/2) \mathbf{1}_{(\beta=\alpha)}$ , getting

$$\begin{aligned} C_{\alpha,\beta} &:= [L \cdot \hat{B} \cdot L^{-1}]_{\alpha,\beta} = \exp((c(\beta - \alpha) - 1)\Sigma_{\alpha,\beta}/2) \cdot q(\beta - \alpha) \\ &= \exp(d\Sigma_{\alpha,\alpha+[1]} \mathbf{1}_{(\beta-\alpha=1)}) \cdot q(\beta - \alpha), \end{aligned}$$

where we have introduced the constant  $d := -K(1)/(2q([1]))$ . Of course  $Z(\hat{B}) = Z(C)$ . Also notice that by the very definition of  $\Sigma_{\alpha,\beta}$  we have  $\Sigma_{\alpha,\alpha+[1]} = \omega_{\alpha+[1]}^{(-1)} - \omega_{\alpha+[1]}^{(+1)}$ , hence the hypothesis  $h_\omega = 0$  yields  $\sum_{\alpha \in \mathbb{S}} (\Sigma_{\alpha,\alpha+[1]}) = 0$ .

Thus we are finally left with showing that  $Z(C) \geq 1$  where  $C_{\alpha,\beta}$  is an  $\mathbb{S} \times \mathbb{S}$  matrix of the form

$$C_{\alpha,\beta} = \exp(w_\alpha \mathbf{1}_{(\beta-\alpha=1)}) \cdot q(\beta - \alpha) \quad \text{where} \quad \sum_{\alpha} w_\alpha = 0 \quad \sum_{\gamma} q(\gamma) = 1.$$

To this end, we introduce an interpolation matrix

$$C_{\alpha,\beta}(t) := \exp(t \cdot w_\alpha \mathbf{1}_{(\beta-\alpha=1)}) \cdot q(\beta - \alpha),$$

defined for  $t \in \mathbb{R}$ , and notice that  $C(1) = C$ . Let us denote by  $\eta(t) := Z(C(t))$  the Perron–Frobenius eigenvalue of  $C(t)$ : as the entries of  $C(t)$  are log–convex functions of  $t$ , it follows that also  $\eta(t)$  is log–convex, therefore in particular convex. Moreover  $\eta(0) = 1$  (the matrix  $C(0)$  is bistochastic) and using (A.1) one easily checks that  $\frac{d}{dt}\eta(t)|_{t=0} = 0$ . Since clearly  $\eta(t) \geq 0$  for all  $t \in \mathbb{R}$ , by convexity it follows that indeed  $\eta(t) \geq 1$  for all  $t \in \mathbb{R}$ , and the proof is complete.

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DIPARTIMENTO DI MATEMATICA PURA E APPLICATA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY

*E-mail address:* francesco.caravenna@math.unipd.it

LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES (CNRS U.M.R. 7599) AND UNIVERSITÉ PARIS 7 – DENIS DIDEROT, U.F.R. MATHEMATIQUES, CASE 7012, 2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE

*E-mail address:* giacomin@math.jussieu.fr

LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES (CNRS U.M.R. 7599) AND UNIVERSITÉ PARIS 6 – PIERRE ET MARIE CURIE, U.F.R. MATHEMATIQUES, CASE 188, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE

*E-mail address:* zambotti@ccr.jussieu.fr