Enhanced Schauder Estimates for families of distributions

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Based on joint works with

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References

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  Hairer's multilevel Schauder estimates without Regularity Structures

- [CZ 20] F.C., L. Zambotti EMS Surv. Math. Sci. 7 (2020), 207-251
  Hairer's Reconstruction Theorem without Regularity Structures

- [H 14] M. Hairer Invent. Math. 198 (2014), 269-504
  A theory of regularity structures

- [OW 19] F. Otto, H. Weber Arch. Ration. Mech. Anal. 232 (2019), 873-950
  Quasilinear SPDEs via rough paths

- [ZC 23] P. Zorin-Kranich Rev. Mat. Iberoam. 39 (2023), 1233-1246
  The reconstruction theorem in quasinormed spaces

Summary

Classical Schauder Estimates on $\mathbb{R}^d$: $\beta > 0$ $\alpha \in \mathbb{R}$

$$f \in \mathcal{C}^{\alpha} \qquad K(x) \simeq \frac{1}{|x|^{d-\beta}} \qquad \Rightarrow \qquad K * f \in \mathcal{C}^{\alpha+\beta}$$

α- HÖLDER DISTRIBUTION SINGULAR KERNEL IMPROVED REGULARITY

Enhanced Schauder Estimates: $f \rightsquigarrow F = (F_x)_{x \in \mathbb{R}^d}$

[BCZ 23+]

GERM

= FAMILY OF DISTRIBUTIONS

Inspired by M. Hairer MULTI-LEVEL SCHAUDER ESTIMATES [H 14]

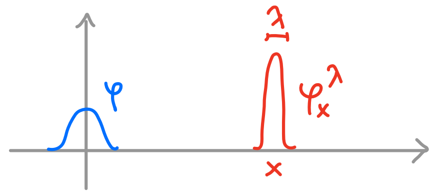
DISTRIBUTIONS

$$\mathcal{D} = \{\text{test functions}\} = \{\varphi : \mathbb{R}^d \to \mathbb{R},\ \varphi \in C_c^\infty\}$$

$$\mathcal{D}' = \{\text{distributions}\} = \{f : \mathcal{D} \to \mathbb{R} \text{ linear \& "continuous"}\}$$

$$|f(\varphi)| \lesssim \|\varphi\|_{C^r} = \max_{|k| \le r} \|\partial^k \varphi\|_\infty \quad \text{loc. unif.}$$

Scaled test function: $\varphi_x^\lambda(z) := \dfrac{1}{\lambda^d}\, \varphi\left(\dfrac{z-x}{\lambda}\right)$



"Local regularity at x" of $f \in \mathcal{D}'$: $\quad \lambda \mapsto f(\varphi_x^\lambda) \quad$ for $\lambda \downarrow 0$

HÖLDER SPACES $\mathcal{C}^\alpha$

$\alpha \in \mathbb{R}$

- $\alpha < 0$: $\quad \mathcal{C}^\alpha := \{ f \in \mathcal{D}' : |f(\varphi_x^\lambda)| \lesssim \lambda^\alpha \}$ $\quad \lambda \in (0,1]$

- $\alpha > 0$: $\quad \alpha \in (0,1)$ $\quad \mathcal{C}^\alpha := \{ f$ functions : $|f(x) - f(y)| \lesssim |x-y|^\alpha \}$

  $\alpha \geq 1$ $\quad \mathcal{C}^\alpha := \{ f$ differentiable s.t. $\nabla f \in \mathcal{C}^{\alpha-1} \}$

$$\mathcal{C}^\alpha = \{ f \in \mathcal{D}' : |f(\varphi_x^\lambda)| \lesssim \lambda^\alpha \text{ for } \varphi \text{ s.t. } \int P(x)\,\varphi(x)\,dx = 0 \}$$

$P(x)$: POLYNOMIAL OF DEGREE $\leq \alpha$

- $\partial_{x_i} : \mathcal{C}^\alpha \to \mathcal{C}^{\alpha-1}$

Singular Kernels

For simplicity, we consider translation invariant Kernels

$$K(x,y) = K(x-y)$$

Assumption: $K(\cdot)$ compactly supported & singular at zero

$$|K(z)| \lesssim \frac{1}{|z|^{d-\beta}} \mathbb{1}_{\{|z| \leq \rho\}} \qquad \text{for some } \beta, \rho > 0$$

$$|\partial^{\ell} K(z)| \lesssim \frac{1}{|z|^{d-\beta+|\ell|}} \mathbb{1}_{\{|z| \leq \rho\}} \qquad \text{for } \underbrace{\text{all}}_{|\ell| \leq c \text{ is enough}} \ \ell \in \mathbb{N}_0^d$$

CLASSICAL SCHAUDER ESTIMATES

Convolution $\quad (K * f)(\cdot) := \int_{\mathbb{R}^d} f(\cdot - z)\, K(z)\, dz \quad \in \mathcal{D}'$

WELL-DEFINED FOR ALL $f \in \mathcal{D}'$

If $K(\cdot)$ is smooth (non-singular) $\Rightarrow \quad K * f \in C^\infty$

If $K(\cdot)$ is $(d-\beta)$-singular, it improves regularity by $\beta > 0$.

Theorem (SCHAUDER). $\quad f \in \mathcal{C}^\alpha, \quad \alpha \in \mathbb{R}$

$$\Rightarrow \quad K * f \in \mathcal{C}^{\alpha+\beta}$$

Example: Additive Heat Equation

Given $\xi \in \mathcal{D}'(\mathbb{R}^{1+d})$ with $\xi(t,\cdot) \equiv 0$ for $t \leq 0$, we solve

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \xi(t,x) \\ u(t,\cdot) \equiv 0 \quad \text{for } t \leq 0 \end{cases}$$

$$\iff \quad u = \text{"}(\partial_t - \Delta_x)^{-1}\xi\text{"} = K * \xi \qquad K(t,x) = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} \mathbb{1}_{\{t>0\}}$$

$$\lesssim \frac{1}{(\sqrt{t}+|x|)^d}$$

$$d \to d+2-2 \to \beta$$

Parabolic metric $\quad |(t,x)|_{par} := \sqrt{t} + |x|$

$$\xi \in \mathcal{C}^{\alpha}_{par} \Rightarrow u = K * \xi \in \mathcal{C}^{\alpha+2}_{par}$$

GERMS

PDEs with nonlinearities & noise are much harder, e.g.

(KPZ) $$\partial_t h(t,x) = \frac{1}{2}\Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \xi(t,x)$$

Regularity Structures: Taylor-like expansions around every $z=(t,x)$:

$$h(\cdot) \simeq \underbrace{h(z) + \{u(\cdot)-u(z)\} + \ldots + \text{POLYNOMIAL}}_{F_z(\cdot)\in\mathcal{D}'} \qquad (\text{CLOSE TO } z)$$

Def. We call GERM a family $F=(F_z)_{z\in\mathbb{R}^d} \subseteq \mathcal{D}'$.

Homogeneity and Coherence

[CZ 20]

$r \in \mathbb{N}_0 \qquad \bar{\alpha}, \alpha, \gamma \in \mathbb{R} \qquad \alpha \leq \gamma$

$$\mathcal{B}^r := \left\{ \varphi \in C_c^\infty : \ \operatorname{supp}(\varphi) \subseteq B(0,1), \quad \|\varphi\|_{C^r} := \max_{|\ell| \leq r} \|\partial^\ell \varphi\|_\infty \leq 1 \right\}$$

Homogeneous germs:

$$\mathcal{G}^{\bar{\alpha}}_{\mathrm{HOM}} := \left\{ F = (F_x)_{x \in \mathbb{R}^d} : \quad |F_x(\varphi_x^\lambda)| \lesssim \lambda^{\bar{\alpha}} \quad \begin{array}{l} \text{unif. for } x \in \text{compacts} \\ \lambda \in (0,1], \ \varphi \in \mathcal{B}^r \end{array} \right\}$$

Coherent germs:

$$\mathcal{G}^{\alpha,\gamma}_{\mathrm{COH}} := \left\{ F = (F_x)_{x \in \mathbb{R}^d} : \quad |(F_y - F_x)(\varphi_x^\lambda)| \lesssim \lambda^{\alpha} (|y-x| + \lambda)^{\gamma - \alpha} \quad {}'' \right\}$$

THE RECONSTRUCTION THEOREM

[CZ 20] [ZC 23]
[H 14] [OW 19]

$\bar{\alpha}, \alpha \leq \underset{\neq 0}{\gamma} \in \mathbb{R}$

Any coherent germ $F = (F_x) \in \mathcal{G}^{\alpha,\gamma}_{\mathrm{COH}}$ admits a distribution

$$f := \mathcal{R}F \in \mathcal{D}'$$

"RECONSTRUCTION OF F"

(unique iff $\gamma > 0$) which is "locally well approximated by F":

$$|(F_x - f)(\varphi_x^\lambda)| \lesssim \lambda^\gamma$$

I.E. $(F_x - f)_x \in \mathcal{G}^{\gamma}_{\mathrm{HOM}}$

- $F = (F_x) \longmapsto f = \mathcal{R}F$ is linear
- Homogeneous germs $F = (F_x) \in \mathcal{G}^{\bar{\alpha}}_{\mathrm{HOM}} \rightsquigarrow f = \mathcal{R}F \in \mathcal{C}^{\bar{\alpha}}$

CONVOLUTIONS OF GERMS

Coherent germs $F = (F_x)$ are enriched descriptions of $f = RF \in \mathcal{D}'$

Can we "lift the convolution" $K * f$ to the space of coherent germs?

$$\begin{array}{ccccc} F = (F_x) \in \mathcal{G}^{\alpha,\gamma}_{\mathrm{COH}} & \xrightarrow{\mathcal{K}} & \mathcal{G}^{?}_{\mathrm{COH}} & & \\ \downarrow R & & \downarrow R & & \\ f \in \mathcal{D}' & \xrightarrow{K} & \mathcal{D}' & \ni & K * f = R(\mathcal{K}F) \end{array}$$

Naive guess $\mathcal{K}F = (K * F_x)_{x \in \mathbb{R}^d}$ does not work (not coherent)

Schauder Estimates for Germs

[BCZ 23+]

Right definition: $(\mathcal{K} F)_x := K * F_x - \mathcal{T}_x^{\gamma+\beta}(K * \{F_x - RF\})$

"TAYLOR POLYNOMIAL"

$$\mathcal{T}_x^{\alpha}(f) := \sum_{|\ell|<\alpha} \frac{\partial^{\ell} f(x)}{\ell!} (\cdot - x)^{\ell}$$

WELL-DEFINED

"POINTWISE DERIVATIVES"

Theorem 1 (SCHAUDER FOR GERMS)

$\alpha+\beta \neq 0, \quad \gamma+\beta \notin \mathbb{N}_0$

$\mathcal{K} : \mathcal{G}_{COH}^{\alpha,\gamma} \longrightarrow \mathcal{G}_{COH}^{(\alpha+\beta)\wedge 0,\, \gamma+\beta}$ satisfies $R(\mathcal{K}F) = K * (RF)$

$\cap\, \mathcal{G}_{HOM}^{\bar{\alpha}} \longrightarrow \cap\, \mathcal{G}_{HOM}^{(\bar{\alpha}+\beta)\wedge 0}$

(I.E. THE DIAGRAM COMMUTES)

Models and Modelled Distributions

In Regularity Structures germs are decomposed along a basis:

$$F_x(\cdot) = \langle f, \Pi \rangle_x(\cdot) := \sum_{\substack{i \in I}} f^i(x)\, \Pi^i_x(\cdot)$$

$\hookrightarrow$ FINITE SET

Definition (MODEL) $\Pi = (\Pi^i)_{i \in I}$ with $\Pi^i = (\Pi^i_x)_{x \in \mathbb{R}}$ satisfies

- $|\Pi^i_x(\varphi^\lambda_x)| \lesssim \lambda^{\alpha_i}$ for some homogeneities $(\alpha_i)_{i \in I}$
- $\Pi^i_y = \sum_{j \in I} \Pi^j_x \, \Gamma^{ji}_{xy}$ for some coefficients $(\Gamma^{ji}_{xy})_{i,j \in I}$

Note: In RS further properties are imposed: [H14]

- GROUP PROPERTY: $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$
- TRIANGULAR STRUCTURE: $\Gamma_{xy}^{ii} = 1$, $\Gamma_{xy}^{ji} = 0$ if $j \neq i$, $\alpha_j \geq \alpha_i$
- ANALYTIC BOUND: $|\Gamma_{xy}^{ji}| \lesssim |x-y|^{\alpha_i - \alpha_j}$

Set $\bar{\alpha} := \min_{i \in I} \alpha_i$ and fix $\gamma > \max_{i \in I} \alpha_i$.

To ensure that $F = \langle f, \Pi \rangle$ is coherent, we impose:

Definition (γ- MODELLED DISTRIBUTION) $f = (f^i)_{i \in I}$ satisfies

- $|f^i(x)| \lesssim 1$
- $\left| f^i(x) - \sum_{j \in I} \Gamma_{xy}^{ij} f^j(y) \right| \lesssim |x-y|^{\gamma - \alpha_i}$

Multi-Level Schauder Estimates

[BCZ 23+]

$$\left\{\begin{array}{ll}\Pi & \text{model} \\ f & \gamma\text{-modelled distrib.}\end{array}\right. \Rightarrow \quad F = \langle f, \Pi \rangle \in \mathcal{C}^{\bar{\alpha},\gamma}_{\mathrm{COH}} \cap \mathcal{C}^{\bar{\alpha}}_{\mathrm{HOM}}$$

Can we lift the operator $\mathcal{K}$ on the space of models and modelled distributions?

Theorem 2 (Multi-Level Schauder) $\mathcal{K}\langle f, \Pi \rangle = \langle \hat{f}, \hat{\Pi} \rangle$ for

- extended model $\hat{\Pi}$
- extended $(\gamma+\beta)$-modelled distrib. $\hat{f}$

$\left.\right\}$ indexed by $\hat{I} := I \cup \{\ell \in \mathbb{N}_0^d : |\ell| < \gamma+\beta\}$

EXTENDED MODEL AND MODELLED DISTRIBUTION

Extended $\hat{\Pi} = (\hat{\Pi}^{i})_{i \in \hat{I}}$ and $\hat{f} = (\hat{f}^{i})_{i \in \hat{I}}$ indexed by

$$\hat{I} := I \cup \underbrace{\mathrm{POLY}(\gamma+\beta)}_{\{\ell \in \mathbb{N}_0^d \,:\, |\ell| < \gamma+\beta\}}$$

- $$\hat{\Pi}^{i}_{x} = \begin{cases} K * \Pi^{i}_{x} - \mathcal{T}^{\alpha_i+\beta}_{x}\left(K * \Pi^{i}_{x}\right) & \text{if } i \in I \\ (\cdot - x)^{i} & \text{if } i \in \mathrm{POLY}(\gamma+\beta) \end{cases}$$

- $$\hat{f}^{i}(x) = \begin{cases} f^{i}(x) & \text{if } i \in I \\ \sum\limits_{j : \alpha_j+\beta > |i|} f^{j}(x)\, \partial^{i}(K * \Pi^{j}_{x})(x) - \partial^{i}(K * \{\langle f, \Pi \rangle_x - R\langle f, \Pi \rangle\})(x) & \text{if } i \in \mathrm{POLY}(\gamma+\beta) \end{cases}$$

Conclusion

We prove enhanced Schauder Estimates with minimal assumptions

- Non translation invariant Kernels $K(x,y)$ which need not annihilate polynomials
- We allow $\gamma < 0$ $\rightsquigarrow$ non unique reconstruction $\mathcal{R}F$
- We do not require extra properties of Γ_{xy}^{ij}
- We separate COHERENCE and HOMOGENEITY $\rightsquigarrow$ results of independent interest

Grazie !