A Polymer in a Multi-Interface Medium

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LPMA $\sim$ December 8, 2009

References

- [CP1] F. Caravenna and N. Pétrélis
  A polymer in a multi-interface medium
  AAP (2009)

- [CP2] F. Caravenna and N. Pétrélis
  Depinning of a polymer in a multi-interface medium
  EJP (2009)

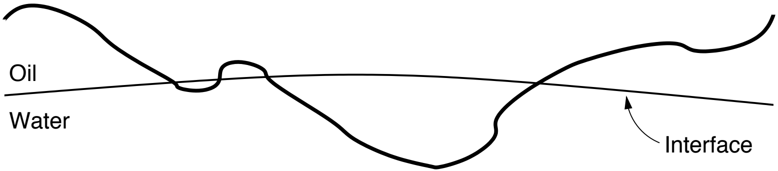
Outline

1. Introduction and motivations
   - Polymer models
2. The model and the main results
   - Definition of the model
   - The free energy
   - Path results
3. Techniques and ideas from the proof
   - Some heuristics
   - A renewal theory approach

Outline

Copolymer and pinning at a single interface

A polymer interacting with two solvents and with the interface that separates them:



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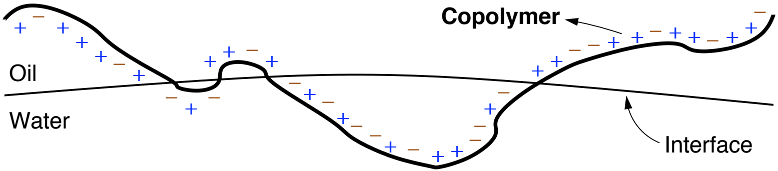

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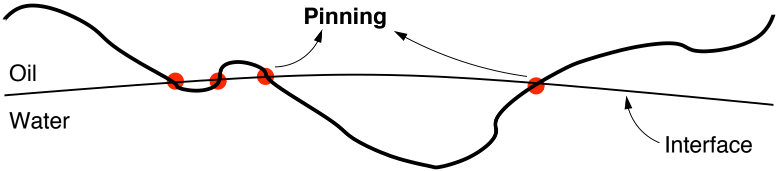

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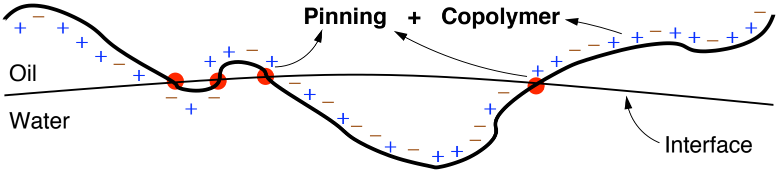

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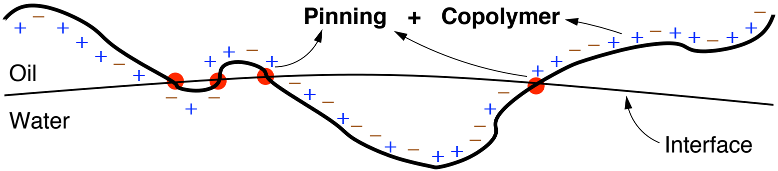

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Localization vs. delocalization? Phase transitions?

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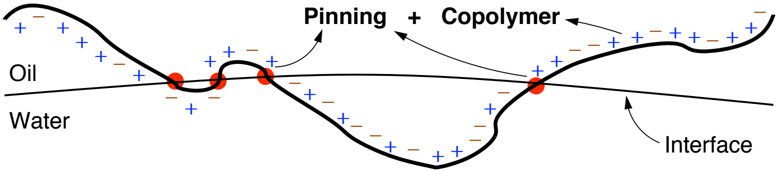

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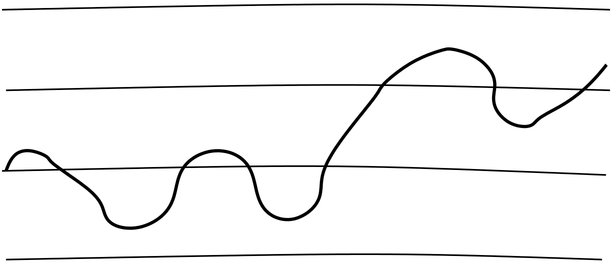
Localization vs. delocalization? Phase transitions?

Recent results: very good comprehension (survey: [Giacomin '07])

Multi-interface media

More general environments: a multi-interface medium



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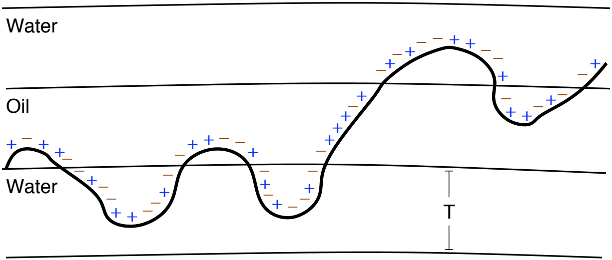

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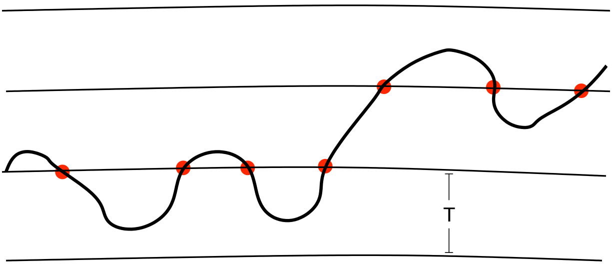

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- We focus on the pinning case. Homogeneous interaction (attractive or repulsive), general $T_N \longrightarrow$ Path behavior ?

Polymer in a slit

Recent physical literature:

Polymer confined between two walls and interacting with them

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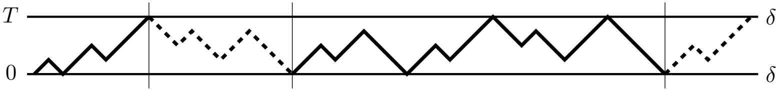

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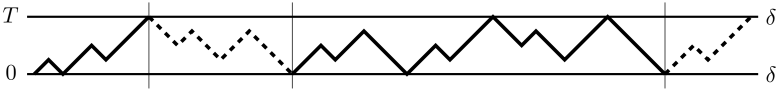

Attraction/repulsion of interfaces by polymers

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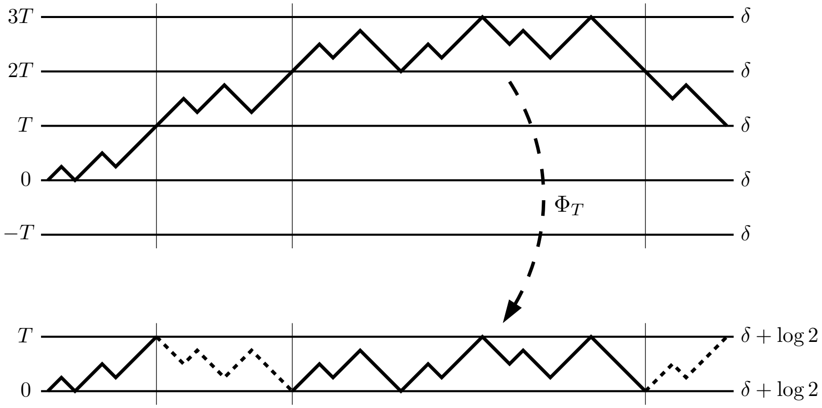

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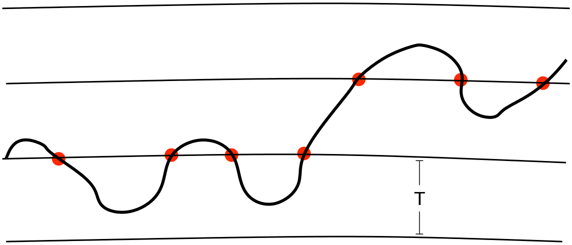
Recall the situation we want to model



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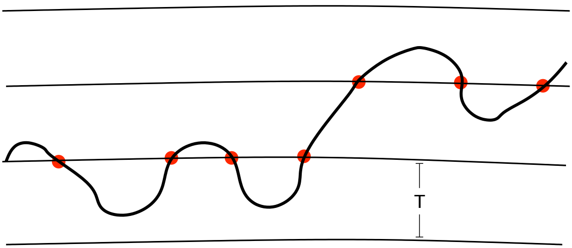

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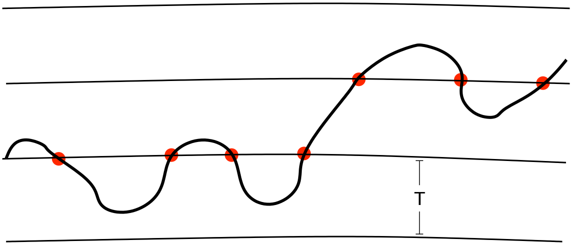

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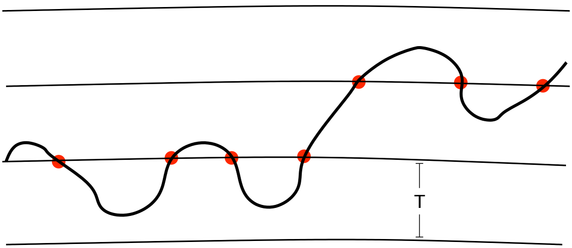

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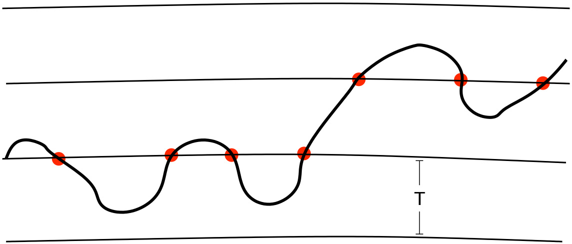

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Random polymer model: probability measure $\mathbf{P}_{N,\delta}^{T}$ on paths

- $(1+1)$-dimensionale model: $\{(i, S_i)\}_{i \geq 0}$
- $\mathbf{P}_{N,\delta}^{T}$ absolutely continuous w.r.t. SRW $\{S_i\}_{i \geq 0}$

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Ingredients of $\mathbf{P}_{N,\delta}^{T}$:

- Simple symmetric random walk $S = \{S_n\}_{n\geq 0}$ on $\mathbb{Z}$:

$$S_0 := 0\,, \qquad S_n := X_1 + \ldots + X_n\,,$$

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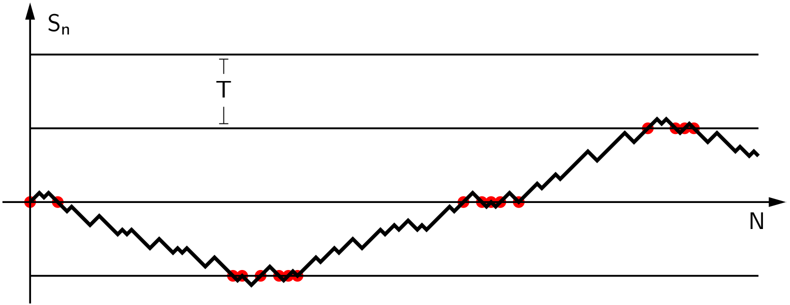

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Penalisation of the simple random walk

The free energy

The free energy $\phi(\delta, \{T_n\}_n)$ encodes the exponential asymptotic behavior of the normalization constant $Z_{N,\delta}^{T_N}$ (partition function)

$$\phi(\delta, \{T_n\}_n) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\delta}^{T_N} \qquad \text{(super-additivity)}$$

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$\phi(\delta, \{T_n\}_n)$ non-analytic at δ $\longleftrightarrow$ phase transition at δ

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Theorem ([CP1])

$$\phi(\delta, \{T_n\}_n) = \phi(\delta, T_\infty) = \begin{cases} \left(Q_{T_\infty}\right)^{-1}(e^{-\delta}) & \text{if } T_\infty < +\infty \\ \left(Q_\infty\right)^{-1}(e^{-\delta} \wedge 1) & \text{if } T_\infty = +\infty \end{cases}$$

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- $\phi(\delta, T_\infty)$ is analytic on $\mathbb{R}$: no phase transitions
- $\phi'(\delta, T_\infty) > 0$ for every $\delta \in \mathbb{R}$: positive density of contacts

$$L_N \sim \phi'(\delta, T_\infty) \cdot N \qquad \text{(diffusive scaling of } S_N)$$

The free energy: results

If $T_N \to \infty$

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- Same path behavior? NO!
- If $\delta < 0$ then $Z_{N,\delta}^{T_N} = \exp(o(N))$. In fact

$$Z_{N,\delta}^{T_N} \approx \frac{(const.)}{N^{3/2}} \, f\left(\frac{N}{T_N^2}\right) g\left(\frac{N}{T_N^3}\right),$$

improving known results for the polymer in a slit.

Path results: the attractive case $\delta > 0$

Assume $\delta > 0$ and $T_N \to \infty$. The polymer visits the interfaces a positive fraction of times: $L_N \sim \phi'(\delta, \infty) \cdot N \qquad (\phi'(\delta, \infty) > 0)$

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$$\frac{S_N}{C_\delta \, (e^{-\frac{c_\delta}{2} T_N} T_N)\sqrt{N}} \Longrightarrow N(0,1)$$

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$$\frac{S_N}{T_N} \Longrightarrow S_\Gamma \qquad \Gamma \sim \mathrm{Poisson}(f(\delta, \zeta))$$

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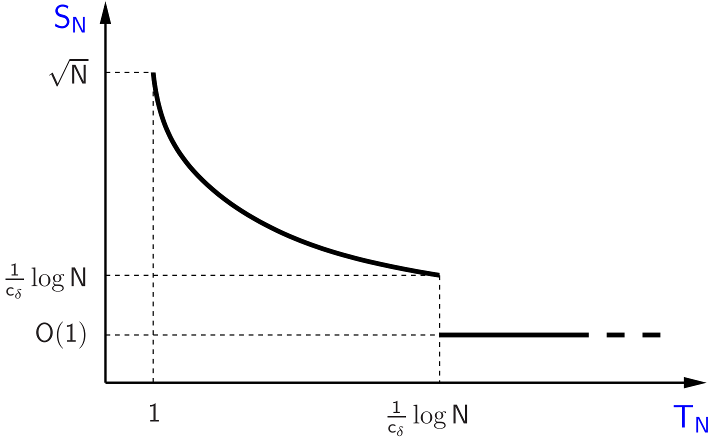

- Sub-diffusive scaling ($T_N \to \infty$)
- Transition at $T_N \approx \log N$

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$$c_1\, P\big(a < Z \le b\big) \;\le\; \mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{S_N}{C_\delta \sqrt{\frac{N}{T_N}}} \le b\right) \;\le\; c_2\, P\big(a < Z \le b\big)$$

with $Z \sim N(0,1)$

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- If $T_N \sim (const.) N^{1/3} \longrightarrow O(1)$ visited interfaces.
- If $T_N \gg N^{1/3} \longrightarrow 1$ visited interface, $L_N = O(1)$.

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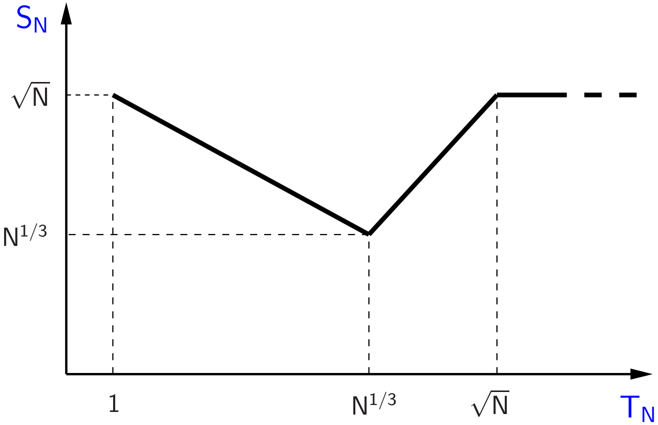

- Sub-diffusive if $1 \ll T_N \ll \sqrt{N}$
- Transitions $T_N \approx N^{1/3}, \sqrt{N}$

Outline

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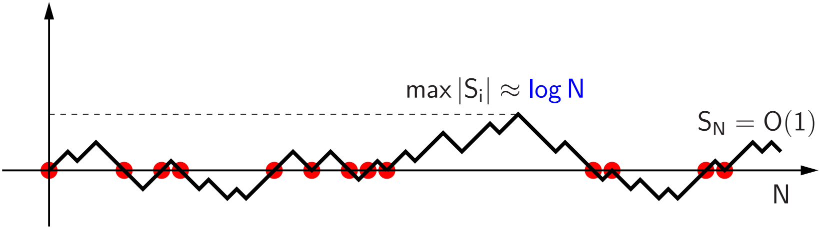

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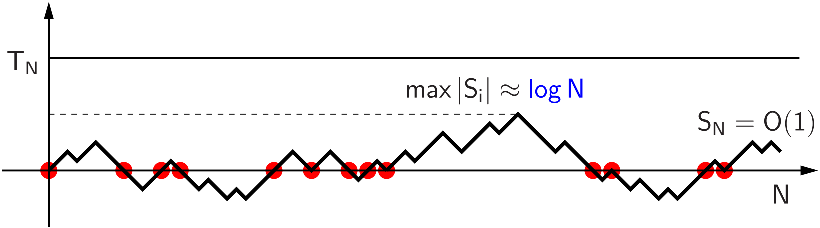

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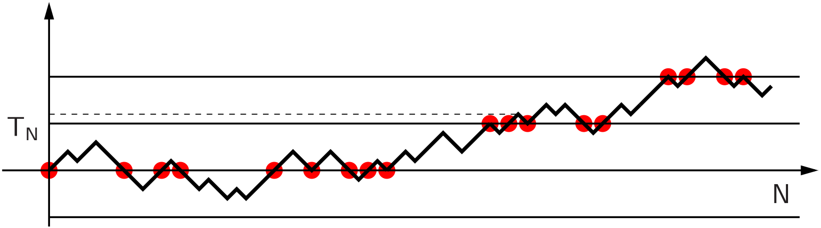

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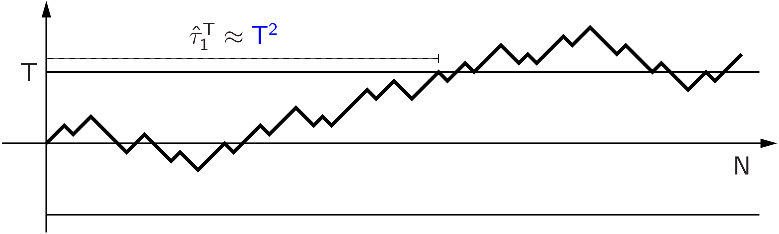

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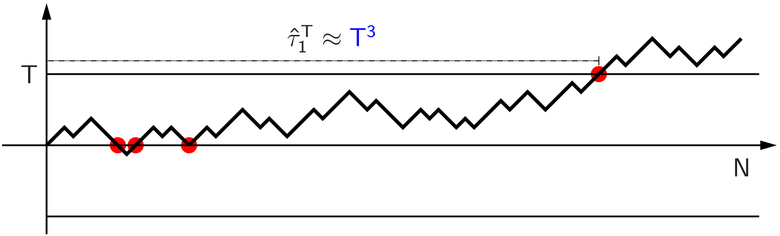

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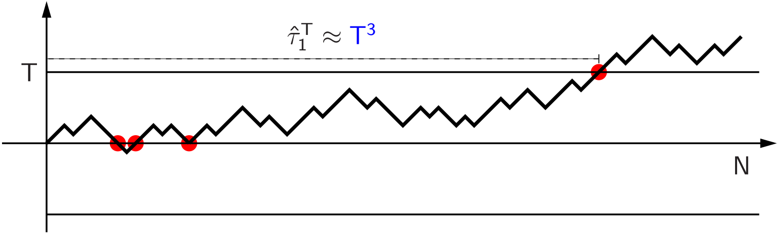

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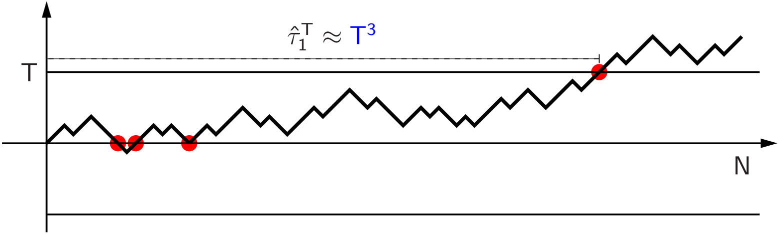

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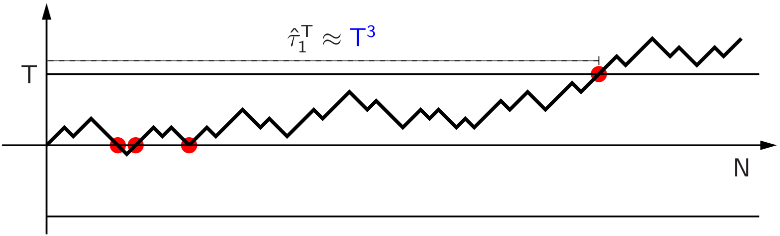

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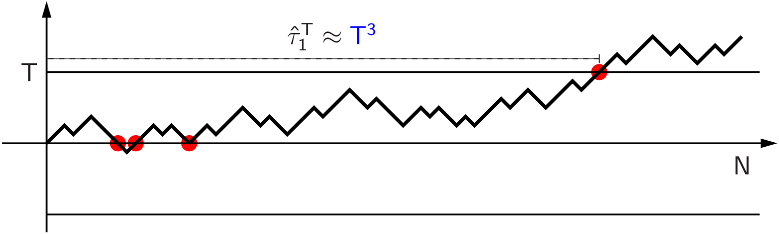

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A renewal theory approach

Let $\tau_1^T, \tau_2^T, \tau_3^T \ldots$ be the points at which S_n visits an interface

$$\tau_{k+1}^T := \inf \left\{ n > \tau_k^T : \; S_n - S_{\tau_k^T} \in \{-T, 0, T\} \right\} \qquad (T \text{ is fixed})$$

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Strategy of the proof

For fixed T, the law of $\tau^T \cap [0, N] = \{\tau_1^T, \ldots, \tau_{L_N}^T\}$ is the same

under $\quad \mathbf{P}_{N,\delta}^T\big(\,\cdot\,\big|\, N \in \tau^T\big) \quad$ and $\quad \mathcal{P}_{\delta,T}\big(\,\cdot\,\big|\, N \in \tau^T\big)$

- $\mathcal{P}_{\delta,T}$ does not depend explicitly on N
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- Uniform renewal theorems

Merci.