Scaling and Multiscaling in Financial Indexes: a Simple Model

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joint work with Alessandro Andreoli (Padova),
Paolo Dai Pra (Padova) and Gustavo Posta (Politecnico di Milano)

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Outline

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Black & Scholes Model

Black & Scholes model for the price S_t of a stock price or index:

$$\mathrm{d}S_t = S_t \, (r \, \mathrm{d}t + \sigma \, \mathrm{d}W_t)$$

- σ (the volatility) and r (the interest rate) are constant
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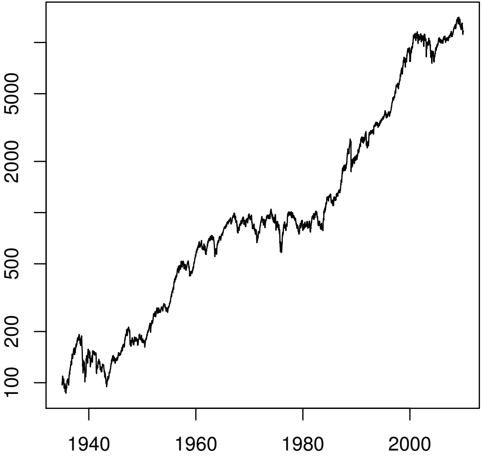
Basic example: Dow Jones Industrial Average (DJIA).

DJIA Time Series (1935-2009)

Exponential growth of the DJIA [log plot]:



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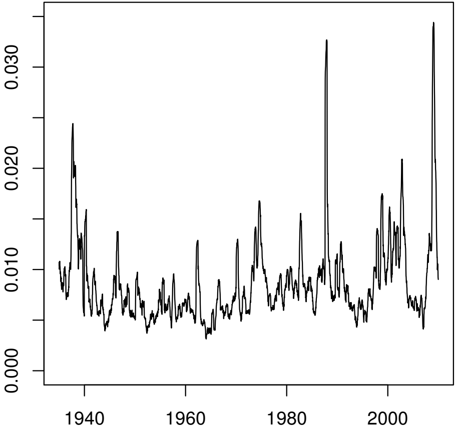
Empirical volatility



Local standard deviation of log-returns in a window of 100 days

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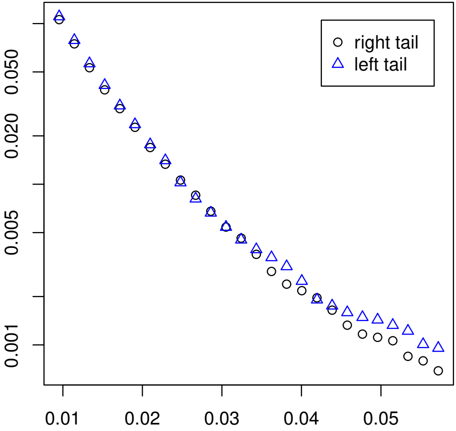
DJIA Time Series (1935-2009)

Tails of daily log-return distribution [log plot]



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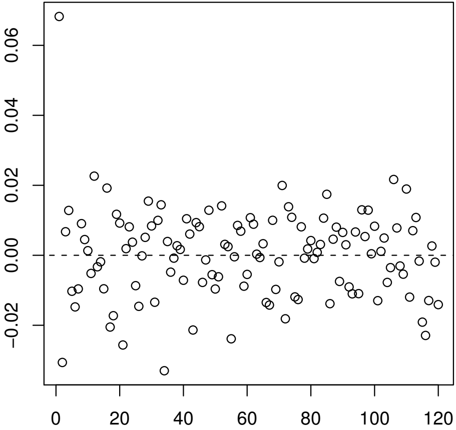
DJIA Time Series (1935-2009)

Decorrelation of daily log-returns over 1–120 days



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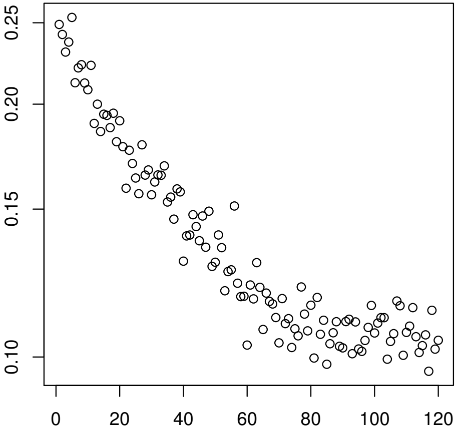
- The increments $(X_{t+h} - X_t)$, called log-returns, have a distribution with tails heavier than Gaussian.
- Log-returns corresponding to disjoint time intervals are uncorrelated… but not independent!
  The correlation between $|X_{t+h} - X_t|$ and $|X_{s+h} - X_s|$, called volatility autocorrelation, has a slow decay in $|t-s|$, up to moderate values of $|t-s|$ (clustering of volatility).

DJIA Time Series (1935-2009)

Volatility autocorrelation over 1–120 days [log plot]



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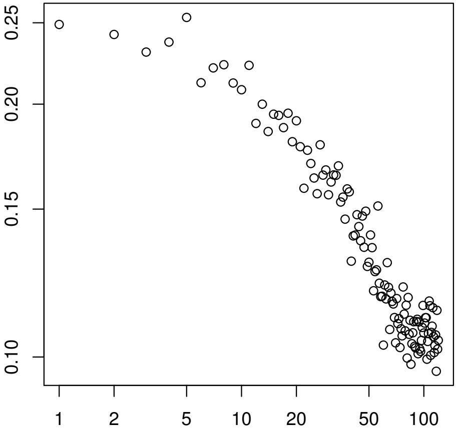
Volatility autocorrelation over 1–120 days [log-log plot]



Some Alternative Models

Continuous time: the constant volatility σ is replaced by a stochastic process $(\sigma_t)_{t\geq 0}$ $\longrightarrow$ stochastic volatility model

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Example: generalized Ornstein-Uhlenbeck processes

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Discrete time: autoregressive models such as the GARCH are used:

$$\varepsilon_t = \sigma_t \, z_t \,, \qquad \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \varepsilon_{t-1}^2$$

where $\varepsilon_t := X_{t+1} - X_t$ and $(z_t)_{t\in\mathbb{N}}$ are i.i.d. $N(0,1)$.

Scaling Properties

More recently, some striking scaling properties of stock indexes of developed markets have been emphasized.

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- Diffusive scaling of log-regurns
- Multiscaling (or anomalous scaling) of moments

Diffusive Scaling of Log-Returns

Denote by $\hat{p}_h(\cdot)$ the empirical distribution of the log-return over h days, for an observed time series $(x_t)_{1 \le t \le T}$ of the detrended log-index X:

$$\hat{p}_h(\cdot) := \frac{1}{T-h} \sum_{t=1}^{T-h} \delta_{x_{t+h}-x_t}(\cdot)\,,$$

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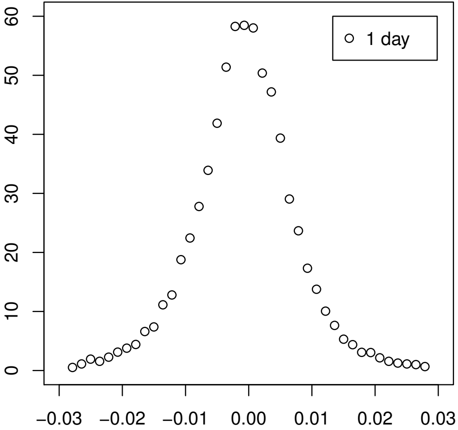
$$X_{t+h} - X_t \overset{d}{\approx} \sqrt{h}\,(X_{t+1} - X_t) \quad \rightarrow \quad \hat{p}_h(\mathrm{d}r) \simeq \frac{1}{\sqrt{h}}\, g\left(\frac{r}{\sqrt{h}}\right)\mathrm{d}r$$

where g is a non-Gaussian density.

DJIA Time Series (1935-2009)

Rescaled empirical density of log-returns (1 day)



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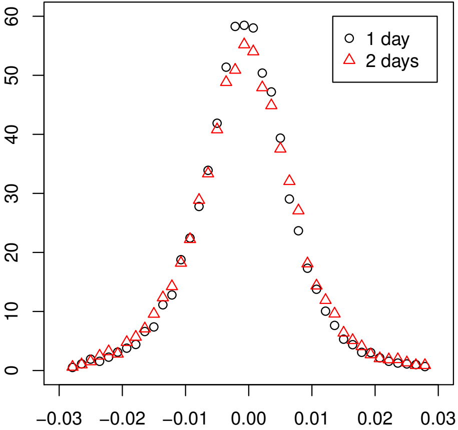
Rescaled empirical density of log-returns (1-2 days)



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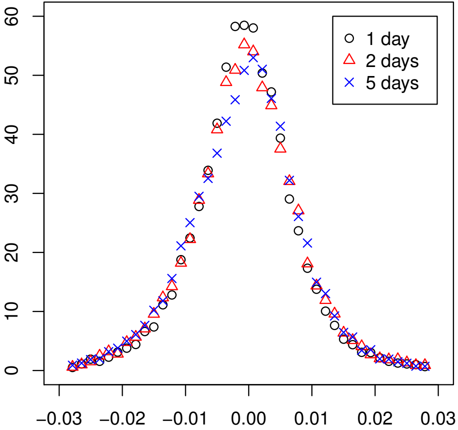
Rescaled empirical density of log-returns (1-2-5 days)



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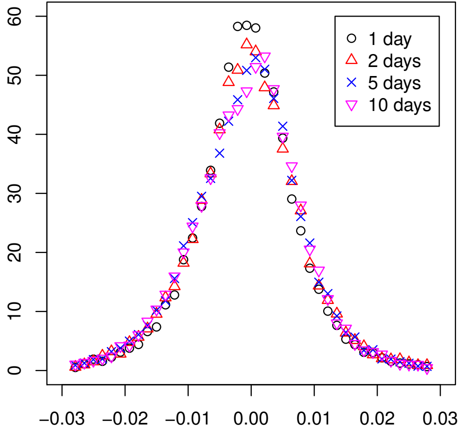
Rescaled empirical density of log-returns (1-2-5-10 days)



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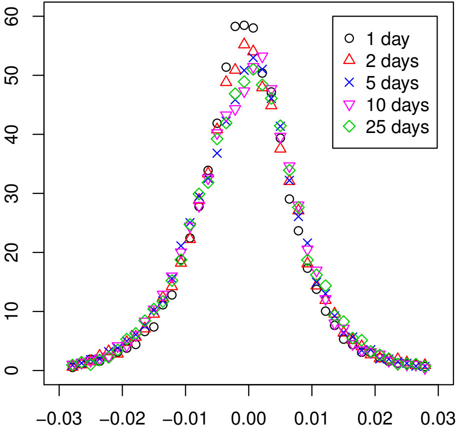
Rescaled empirical density of log-returns (1-2-5-10-25 days)



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Multiscaling of Moments

Consider the empirical *q-th moment* of the log-return over h days:

$$\hat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q \, \hat{p}_h(\mathrm{d}r)$$

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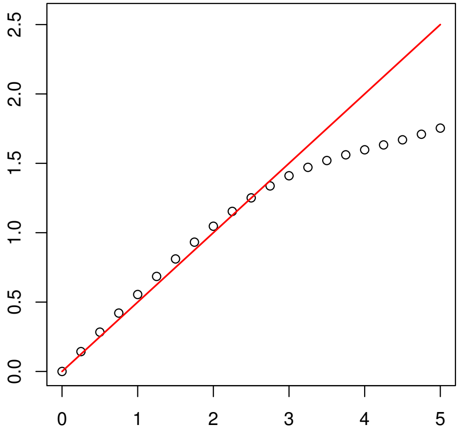
For $q > q^*$ we have the anomalous scaling (or multiscaling)

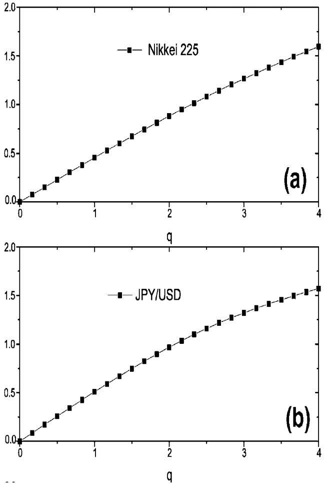
$$\hat{m}_q(h) \approx h^{A(q)}\,, \qquad \text{with} \quad A(q) < \frac{q}{2}\,.$$

DJIA Time Series (1935-2009)

Scaling exponent $A(q)$ (linear regression of $\log \hat{m}_q(h)$ vs. $\log h$)



Other Data Series (from [Di Matteo, Aste & Dacorogna, 2005])



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Baldovin & Stella standpoint: scaling properties should primarily guide the construction of the model.

Baldovin & Stella's Model

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Assume g is symmetric and let g^* be its Fourier transform.

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Let $(Y_t)_{t\geq 0}$ be the process with finite dimensional densities

$$p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = h\left(\frac{x_1}{\sqrt{t_1}}, \frac{x_2 - x_1}{\sqrt{t_2 - t_1}}, \ldots, \frac{x_n - x_{n-1}}{\sqrt{t_n - t_{n-1}}}\right),$$

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where $h : \mathbb{R}^n \to \mathbb{R}$ has Fourier transform h^* given by

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- Is the definition well-posed? Conditions on g.

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Apart from this issue, there is still no multiscaling of moments. This is solved introducing a time inhomogeneity in the model.

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Fix a (periodic) sequence of epochs $0 < \tau_1 < \tau_2 < \cdots < \tau_n \uparrow +\infty$ and a parameter $0 < D \leq 1/2$. Define a new process $(X_t)_{t \geq 0}$ by

$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1)\,,$$

$$X_t := Y_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \qquad \text{for } t \in [\tau_n, \tau_{n+1})\,.$$

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$$X_t := Y_{t^{2D}} \qquad \text{for } t \in [0, \tau_1)\,,$$

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- Increments are amplified immediately after the times $(\tau_n)_{n \geq 1}$ and then progressively damped out.
- Interpretation: $(\tau_n)_{n \geq 1}$ linked to "shocks" in the market.

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- Define a simple model capturing the essential features of Baldovin & Stella's construction.
- Easy to describe and to simulate.
- Rigorous proofs of the mentioned stylized facts.

Outline

Our Model

Fix $D \in (0, 1/2]$, $\lambda \in (0, \infty)$ and a probability ν on $(0, \infty)$ (the law of σ). These are our "parameters".

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- an i.i.d. sequence of r.v.s $\Sigma = (\sigma_n)_{n \geq 0}$ with marginal law ν.

For $t \geq 0$ we set

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\},$$

so that $\tau_{i(t)}$ is the location of the last point in $\mathcal{T}$ before t.

Our Model

Our process

Define the auxiliary process $I = (I_t)_{t \geq 0}$ by

$$I_t := \sigma_{i(t)}^2 \left(t - \tau_{i(t)}\right)^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 \left(\tau_k - \tau_{k-1}\right)^{2D} - \sigma_0^2 \left(-\tau_0\right)^{2D}.$$

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- $(I_t)_{t \geq 0}$ is a strictly increasing process with absolutely continuous paths, independent of the BM $(W_t)_{t \geq 0}$.
- Thus our model $X = (X_t)_{t \geq 0}$ may be viewed as an independent random time change of a Brownian motion.

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- The process X is a zero-mean, square-integrable martingale, provided $E(\sigma^2) = \int \sigma^2 \nu(d\sigma) < \infty$.
- $E\left[|X_t|^q\right] < +\infty$ iff $E\left(\sigma^q\right) < +\infty$.

Outline

Approximate Diffusive Scaling

Theorem

As $h \downarrow 0$ we have the convergence in distribution

$$\frac{(X_{t+h} - X_t)}{\sqrt{h}} \xrightarrow[h\downarrow 0]{d} f(x)\,dx\,,$$

where $f(\cdot)$ is an explicit mixture of Gaussian densities.

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$$f(x) = \int_0^\infty \nu(d\sigma) \int_0^\infty ds\, \lambda e^{-\lambda s} \, \frac{s^{1/2-D}}{\sigma\sqrt{4D\pi}} \, \exp\left(-\frac{s^{1-2D} x^2}{4D\sigma^2} \right) .$$

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Heavy tails of f related to multiscaling of $E\left[|X_{t+h} - X_t|^q\right]$.

Multiscaling of Moments

Theorem

Let $q > 0$, *and assume* $E(\sigma^q) := \int \sigma^q \nu(d\sigma) < +\infty$.
The moment $m_q(h) := E(|X_{t+h} - X_t|^q) = E(|X_h|^q)$ *is finite and has the following asymptotic behavior as* $h \downarrow 0$*:*

$$m_q(h) \sim \begin{cases} C_q\, h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q\, h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q\, h^{Dq+1} & \text{if } q > q^* \end{cases}, \qquad \text{where} \quad q^* := \frac{1}{(\frac{1}{2} - D)}.$$

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- C_q explicit function of D, λ and $E(\sigma^q)$ (used in estimation)
- We can write $m_q(h) \approx h^{A(q)}$ with scaling exponent $A(q)$

$$A(q) := \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} q/2 & \text{if } q \le q^* \\ Dq + 1 & \text{if } q \ge q^* \end{cases}.$$

Decay of Correlations

Theorem

The correlations of the absolute values of the increments of the process X have the following asymptotic behavior as $h \downarrow 0$:

$$\lim_{h\downarrow 0} \rho(|X_{s+h} - X_s|, |X_{t+h} - X_t|)$$

$$=: \rho(t-s) = \frac{2}{\pi \, Var(\sigma \, |W_1| \, S^{D-1/2})} \, e^{-\lambda|t-s|} \, \phi(\lambda|t-s|) \, .$$

where

$$\phi(x) := Cov\big(\sigma \, S^{D-1/2} \, , \, \sigma \, (S+x)^{D-1/2}\big)$$

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- The function $\phi(\cdot)$ has a slower than exponential decay.

Outline

1. Introduction: the Black & Scholes Model
2. Beyond Black & Scholes
3. Our Model
4. Main Results
5. Estimation and Simulations
6. Conclusions

Estimation of the Parameters

The parameters of our model are D, λ and the law of σ, that we want to estimate on the DJIA time series (1935–2009).

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2. Constants C_1 and C_2 functions of D, λ, $E(\sigma)$ and $E(\sigma^2)$:

$$C_1 = \frac{2}{\sqrt{\pi}} \sqrt{D} \, \Gamma(\tfrac{1}{2} + D) \, E(\sigma) \, \lambda^{1/2-D} \qquad C_2 = 2D \, \Gamma(2D) \, E(\sigma^2) \, \lambda^{1-2D} .$$

Estimation of the Parameters

3. Volatility autocorrelation $\rho(t)$ function of D, λ, $E(\sigma)$, $E(\sigma^2)$:

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$$\log \widehat{m}_q(h) \sim \widehat{A}(q) \, (\log h) + \log \widehat{C}_q \qquad \widehat{m}_q(h) := \frac{1}{T-h} \sum_{i=1}^{T-h} |x_{i+h} - x_i|^q$$

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Estimation of the Parameters

Loss function: $(T = 40)$

$$L(D, \lambda, E(\sigma), E(\sigma^2)) = \frac{1}{2}\left\{\left(\frac{\widehat{C}_1}{C_1} - 1\right)^2 + \left(\frac{\widehat{C}_2}{C_2} - 1\right)^2\right\} + \int_0^5 \left(\frac{\widehat{A}(q)}{A(q)} - 1\right)^2 \frac{\mathrm{d}q}{5} + \sum_{t=1}^{400} \frac{e^{-t/T}}{\sum_{s=1}^{400} e^{-s/T}} \left(\frac{\widehat{\rho}(t)}{\rho(t)} - 1\right)^2$$

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Estimator: minimization constrained on $E(\sigma^2) \geq E(\sigma)^2$.

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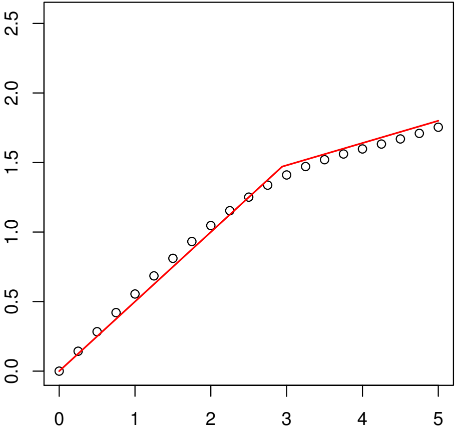
$$(\widehat{D}\,, \widehat{\lambda}\,, \widehat{E(\sigma)}\,, \widehat{E(\sigma^2)}) = \arg\min L(D, \lambda, E(\sigma), E(\sigma^2))$$

$$\widehat{D} \simeq 0.16 \quad \widehat{\lambda} \simeq 0.00097 \quad \widehat{E(\sigma)} \simeq 0.108 \quad \widehat{E(\sigma^2)} \simeq \left(\widehat{E(\sigma)}\right)^2$$

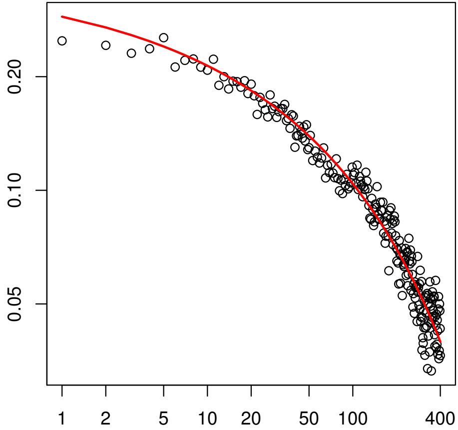
DJIA Time Series (1935-2009)

Empirical (circles) and theoretical (line) scaling exponent $A(q)$



DJIA Time Series (1935-2009)

Empirical (circles) and theoretical (line) volatility autocorrelation [log plot]



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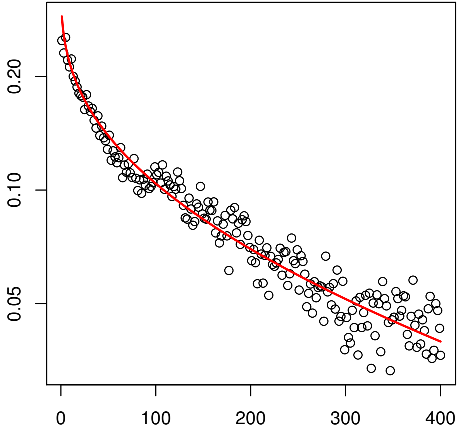
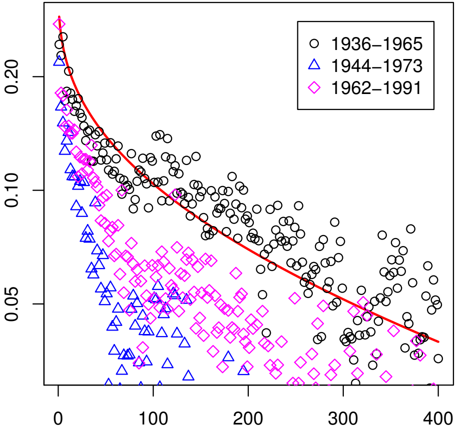


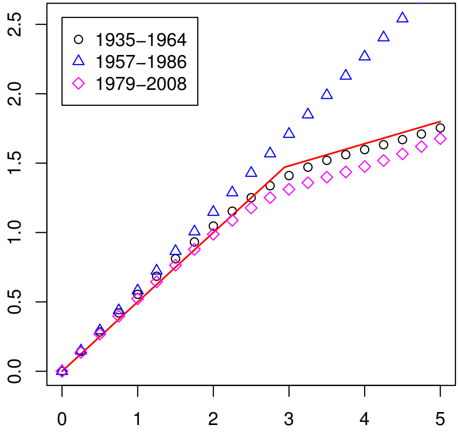
DJIA Time Series (1935-2009)

Volatility autocorrelation over sub-periods of 30 years [log plot]



DJIA Time Series (1935-2009)

Empirical scaling exponent $A(q)$ over sub-periods of 30 years.



Estimation of the Law of σ

The estimated values give $E(\sigma^2) \simeq E(\sigma)^2 \;\rightarrow\; Var(\sigma) \simeq 0$

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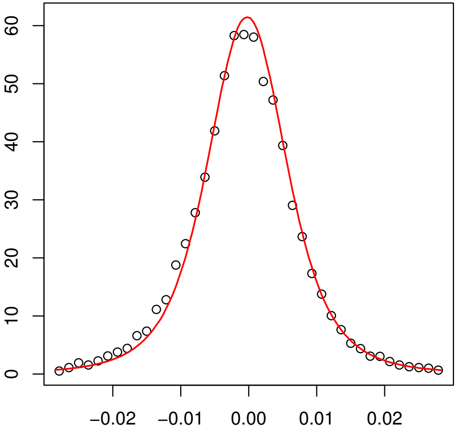
The agreement is remarkably good (both bulk and tails).

DJIA Time Series (1935-2009)

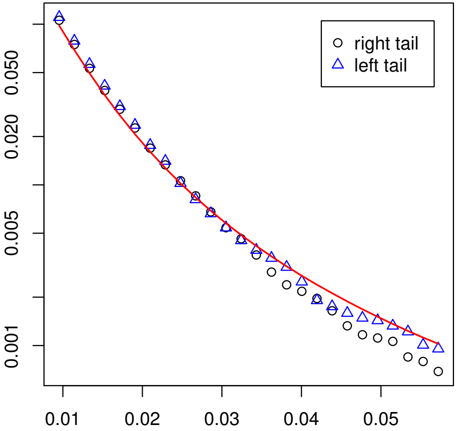
Empirical (circles) and theoretical (line) distribution of daily log return



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to 3 st. dev.

DJIA Time Series (1935-2009)

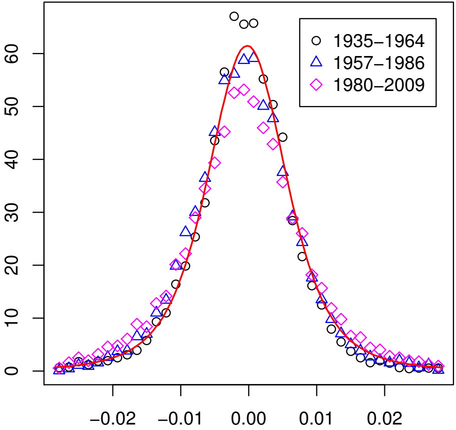
Empirical and theoretical tails of daily log return [log plot]



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: 1 to 6 st. dev.

DJIA Time Series (1935-2009)

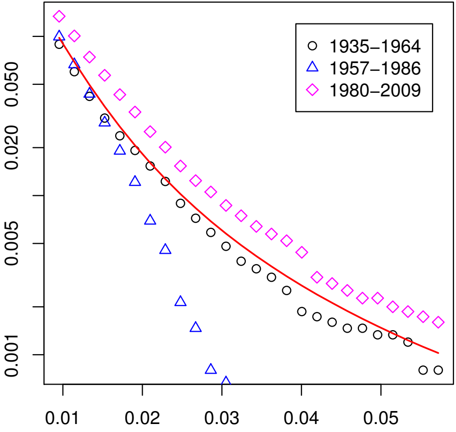
Variability of the distribution in sub-periods of 30 years



Daily log-return standard deviation ≈ 0.01 $\longrightarrow$ Range: -3 to 3 st. dev.

DJIA Time Series (1935-2009)

Variability of the left tail in sub-periods of 30 years



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The law of the log-returns is completely determined by the t^{2D} time scaling at the points of the Poisson process $\mathcal{T}$.

Variability of estimators

We have noticed that a considerable variability in different time windows is present for the multiscaling of moments, for the decay of correlations and for the empirical distribution (bulk and tails).

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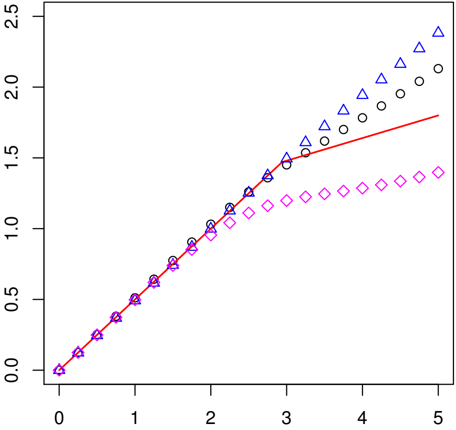
A significant variability is present also in our model.

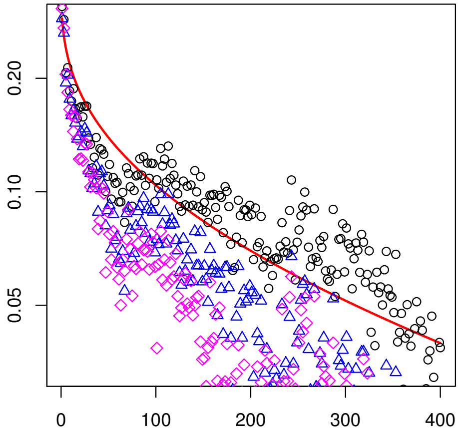
Simulated Data (75 years)

Simulated scaling exponent of our model over sub-periods of 30 years



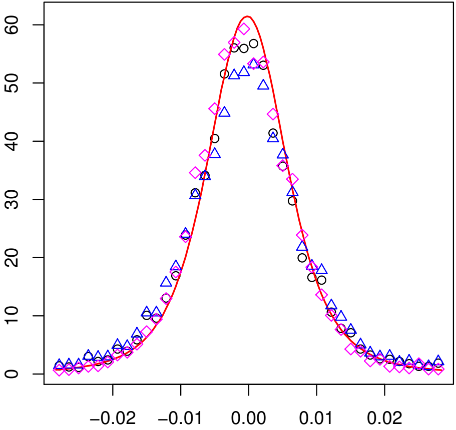
Simulated Data (75 years)

Simulated volatility autocorrelation of our model over sub-periods of 30 years



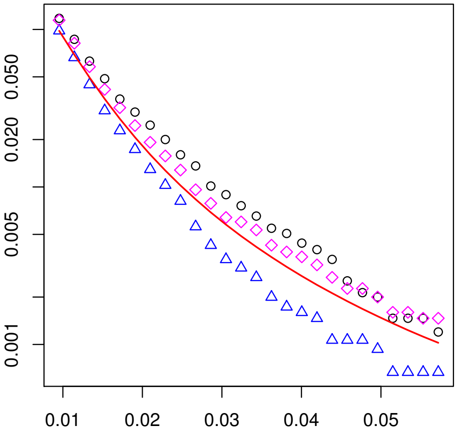
Simulated Data (75 years)

Simulated distribution of our model over sub-periods of 30 years



Simulated Data (75 years)

Simulated tails of our model over sub-periods of 30 years

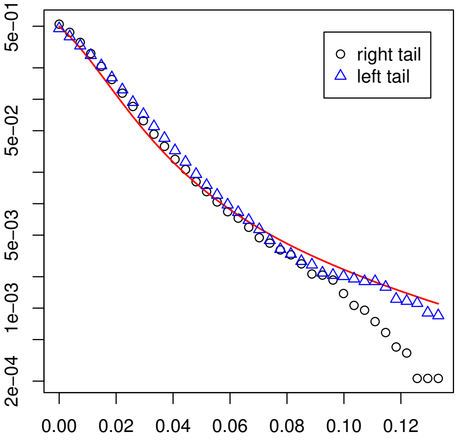


Other observables

Is everything going as expected?

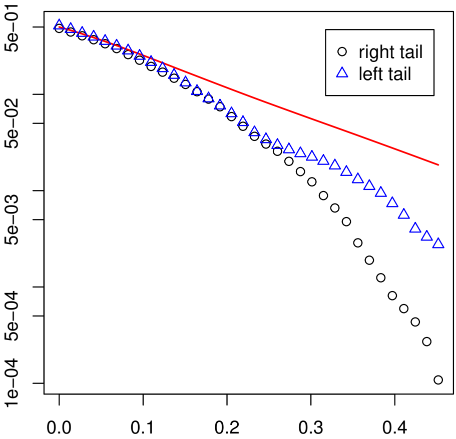
DJIA Time Series (1935-2009)

Empirical and theoretical tails of 5-day log return [log plot]



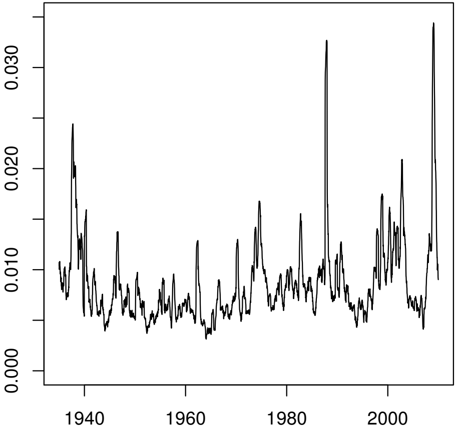
DJIA Time Series (1935-2009)

Empirical and theoretical tails of 400-day log return [log plot]



DJIA Time Series (1935-2009)

Empirical volatility



Local standard deviation of log-returns in a window of 100 days

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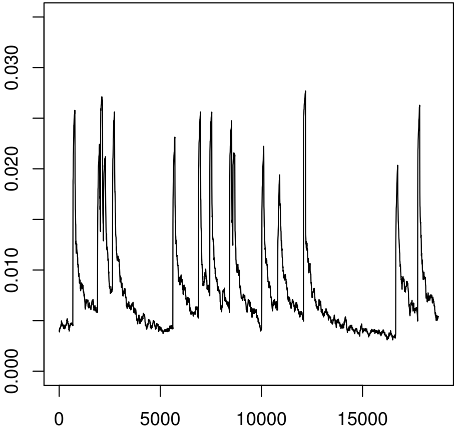

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Outline

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Next steps:

- Solve specific problems by using this model: pricing of options, portfolio management, ...

Thanks.