Universality in Marginally Relevant Disordered Systems

(joint work with R. Sun and N. Zygouras)

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Overview

We consider disordered systems that are marginally relevant

- Directed Polymers in Random Environment with $d = 2$
- Disordered Pinning Models with $\alpha = 1/2$ (tail exponent)

- Stochastic Heat Equation (SHE) with $d = 2$
- Directed Polymer with Cauchy tails with $d = 1$ $[\mathrm{P}(S_1 > \pm n) \sim \frac{c}{n}]$

We present some results on the the scaling limits of these models
The key focus will be on partition functions

For simplicity, we only give statements for Directed Polymer
(but switch to Pinning for proofs)

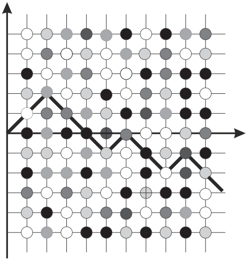
Outline

1. Directed Polymer

2. Main Results

3. Sketch of the proof

Directed Polymer in Random Environment



- Reference Model: random walk $(S_n)_{n \ge 0}$ on $\mathbb{Z}^d$ (zero mean, finite variance, law $\mathbf{P}^{\mathrm{ref}}$)
- Disorder: i.i.d. $(\omega_{n,x})_{n \in \mathbb{N}_0,\, x \in \mathbb{Z}^d}$

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta \omega_{n,x}}] < \infty \qquad (\beta > 0 \text{ small})$$

(zero mean, unit variance, law $\mathbb{P}$)

Directed polymer in random environment

$$\mathbf{P}_N^{\omega}(S) = \frac{1}{Z_N^{\omega}} \, e^{\sum_{n=1}^{N} (\beta \omega_{n,S_n} - \lambda(\beta))} \, \mathbf{P}^{\mathrm{ref}}(S)$$

RW paths in corridors of large $\omega > 0$ have high probability (energy gain)

. . . but such paths are few! (entropy loss) $\rightsquigarrow$ Who wins?

Disorder Relevance

- $[d \geq 3,\ \ \beta > 0$ small$]$ $\qquad \mathbf{P}_N^\omega$ "similar" to $\mathbf{P}^{\mathrm{ref}}$ $\qquad$ (entropy wins)

$$\mathbf{P}_N^\omega\left(\frac{S_N}{\sqrt{N}} \in \cdot\right) \xrightarrow[N\to\infty]{d} \mathcal{N}(0,1) \qquad \mathbb{P}(\mathrm{d}\omega)\text{-a.s.}$$

i.e. the same under $\mathbf{P}^{\mathrm{ref}}$ $\qquad$ [Imbrie, Spencer 1988] [Bolthausen 1989]

- $[d \leq 2,\ \$ any $\beta > 0]$ $\qquad \mathbf{P}_N^\omega$ "different" from $\mathbf{P}^{\mathrm{ref}}$ $\qquad$ (energy wins)

$$\max_{x\in\mathbb{Z}^d} \mathbf{P}_N^\omega(S_N = x) \geq c > 0 \qquad \mathbb{P}(\mathrm{d}\omega)\text{-a.s.}$$

unlike $\mathbf{P}^{\mathrm{ref}}(S_N = x) = O\left(\frac{1}{\sqrt{N}}\right) = o(1)$ $\qquad$ [Carmona, Hu 2002]

[Comets, Shiga, Yoshida 2003]

(Conj. super-diffusivity!) $\qquad$ [Vargas 2007]

Disorder is "irrelevant" for $d \geq 3$, while it is "relevant" for $d \leq 2$

Partition function

The normalizing constant Z_N^ω is called partition function

$$Z_N^\omega = \mathbf{E}^{\mathrm{ref}}\left[e^{\sum_{n=1}^N (\beta\omega_{n,S_n} - \lambda(\beta))}\right] = \mathbf{E}^{\mathrm{ref}}\left[e^{\beta\sum_{n=1}^N \omega_{n,S_n}}\right] e^{-\lambda(\beta)N}$$

Martingale argument

$$\mathbb{P}(\mathrm{d}\omega)\text{-a.s.} \qquad \lim_{N\to\infty} Z_N^\omega = Z_\infty^\omega$$

Note that for $\beta = 0$ (trivially) $Z_\infty^\omega = 1$

- $[d \geq 3]$ $\exists\, \beta_c > 0$ such that $Z_\infty^\omega \begin{cases} > 0 & \text{if } \beta \leq \beta_c \text{ (weak disorder)} \\ = 0 & \text{if } \beta > \beta_c \text{ (strong disorder)} \end{cases}$
- $[d \leq 2]$ $Z_\infty^\omega = 0$ for all $\beta > 0$ (strong disorder), i.e. $\beta_c = 0$

Intermediate disorder regime

Case $d \leq 2$: any fixed disorder strength $\beta > 0$, no matter how small, has dramatic effects in the large scale regime $N \to \infty$

Intermediate disorder regime?

Can we tune $\beta = \beta_N \to 0$ as $N \to \infty$ and see interesting effects ?

Find the (vanishing) window for β in which "strong disorder emerges"

Case $d = 1$ [Alberts, Khanin, Quastel '14]

If $\beta_N = \dfrac{\hat{\beta}}{N^{1/4}}$ then $\mathbf{Z}_N^\omega \xrightarrow[N\to\infty]{d} \mathcal{Z}^W$ where the limit is random

$\mathcal{Z}^W > 0$ for all $\hat{\beta} > 0$ $\quad \lim_{\hat{\beta}\to 0} \mathcal{Z}^W = 1 \quad \lim_{\hat{\beta}\to\infty} \mathcal{Z}^W = 0$

Strong disorder emerges on the scale $\beta \propto N^{-1/4}$ (no "phase transition")

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Marginal relevance

(Marginal) relevance holds at the critical dimension $d = 2$

Logarithmic replica overlap

$$R_N := \mathbf{E}^{\mathrm{ref}}\left[\sum_{n=1}^{N} \mathbb{1}_{\{S_n = S'_n\}}\right] = \sum_{n=1}^{N} \sum_{x \in \mathbb{Z}^2} \mathrm{P}(S_n = x)^2 \sim c \log N$$

The same scaling $R_N \sim c \log N$ happens for:

- random walks on $\mathbb{Z}$ with Cauchy tails
- renewal processes with tail exponent $1/2$ ($\rightsquigarrow$ Pinning Model)

The next result applies to any of these models

Main result 1: log-normality

Theorem 1. [C., Sun, Zygouras '15]

Rescaling $\beta := \dfrac{\hat\beta}{\sqrt{c \log N}}$ the partition function converges in distribution

$$Z_N^{\omega} \xrightarrow[N\to\infty]{d} \mathcal{Z}^{W} = \begin{cases} \text{log-normal} & \text{if } \hat\beta < 1 \\ 0 & \text{if } \hat\beta \geq 1 \end{cases}$$

For $\hat\beta < 1$:

$$\mathcal{Z}^{W} = \exp\left\{ \int_0^1 \frac{\hat\beta}{\sqrt{1-\hat\beta^2 t}} \, \mathrm{d}W_t - \frac{1}{2}\int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2 t} \, \mathrm{d}t \right\}$$

$$\stackrel{d}{=} \exp\left\{ \sigma_{\hat\beta} W_1 - \frac{1}{2}\sigma_{\hat\beta}^2 \right\} \qquad \text{with} \quad \sigma_{\hat\beta} = \log \frac{1}{1-\hat\beta^2}$$

Some comments

- Universality across models. All marginally relevant Directed Polymer and Pinning models exhibit the same limit in distribution for $\boldsymbol{Z}_N^\omega$

  The same also for the 2d Stochastic Heat Equation (see below)

- Phase transition (weak to strong disorder) at scale $\beta = \dfrac{\hat{\beta}}{\sqrt{c \log N}}$

  with finite critical point $\hat{\beta}_c = 1$

  The same intermediate disorder scale appears in [Lacoin '10] and [Berger, Lacoin '15] where they obtain free energy estimates

$\hat{\beta}_c = 1$ (threshold for $\boldsymbol{Z}_N^\omega \to 0$) corresponds to $\mathbb{E}[(\boldsymbol{Z}_N^\omega)^2] \to \infty$

(unlike Directed Polymer for $d \geq 3$ or Gaussian Multiplicative Chaos)

- Key challenge. Identify the limit for $\hat{\beta}_c = 1$ (work in progress)

Main result 2: different starting points

Let $Z_N^\omega(x)$ be the partition function for the RW starting at $x\sqrt{N}$

$$Z_N^\omega(x) = \mathbf{E}^{\mathrm{ref}}\left[e^{H^\omega(S)} \,\middle|\, S_0 = x\sqrt{N} \right] \qquad (x \in \mathbb{R}^2)$$

We look at $Z_N^\omega(x)$ as a random field on $\mathbb{R}^2$. Scaling limit?

Henceforth we fix $\beta = \frac{\hat{\beta}}{\sqrt{c \log N}}$ with $\hat{\beta} < 1$ ("weak disorder" regime)

Theorem 2a. [C., Sun, Zygouras '15]

$Z_N^\omega(x)$ and $Z_N^\omega(x')$ are asymptotically independent for $x \neq x'$

Corollary

$$\langle Z_N^\omega \,,\, \phi \rangle \xrightarrow[N\to\infty]{\mathbb{P}} \langle 1 \,,\, \phi \rangle \qquad \text{for every } \phi \in C_0(\mathbb{R}^2)$$

Main result 2: multi-scale correlations

Dependence in $\boldsymbol{Z}_N^\omega(x)$, $\boldsymbol{Z}_N^\omega(x')$ at which scale of $|x - x'|$? All scales!

Introduce the shorthand $:e^Y: \,=\, e^{Y - \frac{1}{2}\mathbb{V}\mathrm{ar}[Y]}$

Theorem 2b. [C., Sun, Zygouras '15]

Fix $x = x_N$, $x' = x'_N$ such that

$$|x - x'| \asymp (\sqrt{N}\,)^{\zeta - 1} \qquad\qquad \zeta \in [0,1]$$

Then

$$\left(\boldsymbol{Z}_N^\omega(x)\,,\, \boldsymbol{Z}_N^\omega(x')\right) \xrightarrow[N\to\infty]{d} \left(\,:e^Y:\,,\; :e^{Y'}:\,\right)$$

where Y, Y' are joint $\mathcal{N}(0, \sigma_{\hat\beta}^2)$ with

$$\mathbb{C}\mathrm{ov}\left[Y, Y'\right] = \log \frac{1 - \zeta\hat\beta^2}{1 - \hat\beta^2}$$

Main result 2: diffusivity

Similar results can be given for the point to point partition function

This yields result on the polymer endpoint distribution (always for $\hat{\beta} < 1$)

Diffusivity

- Central Limit Theorem

$$\mathbf{P}_N^{\omega}\left(\frac{S_N}{\sqrt{N}} \in \cdot\right) \xrightarrow[N\to\infty]{d} N(0,1) \qquad \text{in } \mathbb{P}(\mathrm{d}\omega)\text{-probability}$$

- Local Limit Theorem with random corrections

$$\sqrt{N}\, \mathbf{P}_N^{\omega}\left(S_N = \lfloor x\sqrt{N}\rfloor\right) \xrightarrow[N\to\infty]{d} \left(: e^{Y_x} : \right) \frac{e^{-|x|^2/2}}{2\pi}$$

Main result 3: fluctuations

We have see that $Z_N^\omega(x) \xrightarrow[N\to\infty]{\mathbb{P}} 1$ (as a Schwartz distribution on $\mathbb{R}^2$)

This can be viewed as a LLN. Here is the corresponding CLT.

Theorem 3. [C., Sun, Zygouras '15]

$$Z_N^\omega(x) \overset{d}{\approx} 1 + \frac{1}{\sqrt{\log N}} G(x) \qquad \text{in } \mathcal{S}'$$

where $G(x)$ is a generalized Gaussian field on $\mathbb{R}^2$ with

$$\mathbb{C}\mathrm{ov}\left[G(x), G(x')\right] \sim C \log \frac{1}{|x-x'|}$$

More precisely

$$\left\langle \sqrt{\log N}\,\left(Z_N^\omega - 1\right), \phi \right\rangle \xrightarrow[N\to\infty]{d} \langle G, \phi \rangle \qquad \forall\, \phi \in C_0(\mathbb{R}^2)$$

What about $\hat\beta = 1$?

For $\hat\beta = 1$: $Z_N^\omega(x) \to 0$ in law $\mathbb{V}\mathrm{ar}[Z_N^\omega(x)] \to \infty$

However, covariances are finite for $x \neq x'$: [Bertini, Cancrini '95]

$$\mathbb{C}\mathrm{ov}[Z_N^\omega(x)\,,\, Z_N^\omega(x')] \xrightarrow[N\to\infty]{} K(x, x') < \infty$$

$$\text{where} \qquad K(x, x') \sim C \log \tfrac{1}{|x-x'|}$$

It follows that $\mathbb{V}\mathrm{ar}\big[\langle Z_N^\omega\,,\,\phi\rangle\big] \to (\phi, K\phi) < \infty$

Conjecture

For $\hat\beta = 1$ the partition function $Z_N^\omega(x)$ has a non-trivial limit in law (in the space of Schwartz distributions on $\mathbb{R}^2$)

The 2d Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \beta\, \dot{W}(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2$$

where $W(\mathrm{d}t, \mathrm{d}x)$ is (space-time) white noise on $[0,\infty) \times \mathbb{R}^2$

Space mollification: $$W_\delta(\mathrm{d}t, x) := \int_{y \in \mathbb{R}^2} \frac{1}{\delta}\, j\big(\tfrac{x-y}{\sqrt{\delta}}\big)\, W(\mathrm{d}t, \mathrm{d}y)$$

Generalized Feynman-Kac Formula [Bertini, Cancrini '95]

$$u_\delta(t,x) \overset{d}{=} \mathrm{E}_{\frac{x}{\sqrt{\delta}}}\left[\exp\left\{ \int_0^{\frac{t}{\delta}} \big(\beta\, W_1(\mathrm{d}s, B_s) \;-\; \tfrac{1}{2}\beta^2\, \mathrm{d}s\big) \right\}\right]$$

By "soft arguments" $u_\delta(t,x) \overset{d}{\approx} Z_N^\omega$ for $N = \frac{t}{\delta}$ (2d Directed Polymer)

Theorems 1, 2, 3 apply verbatim to $u_\delta(t,x)$

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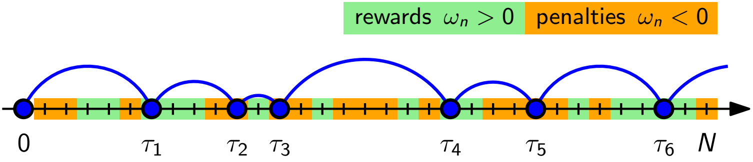
Disordered pinning model



Renewal process $\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \ldots\} \subseteq \mathbb{N}_0$

$$\mathbf{P}^{\mathrm{ref}}(\tau_{i+1} - \tau_i = n) \sim \frac{C}{n^{1+\alpha}}, \qquad C > 0, \quad \alpha \in (0,1)$$

$$\mathbf{P}^{\mathrm{ref}}(n \in \tau) \sim \frac{c}{n^{1-\alpha}} \qquad \text{(heavy-tailed renewal theory)}$$

$$\mathbf{P}_N^\omega(\mathrm{d}\tau) := \frac{1}{Z_N^\omega} e^{\sum_{n=1}^N (\beta\omega_n - \lambda(\beta) + h) \mathbb{1}_{\{n \in \tau\}}} \mathbf{P}^{\mathrm{ref}}(\mathrm{d}\tau)$$

[Alexander, Berger, Derrida, Giacomin, Lacoin, Toninelli, Zygouras, ...]

Partition function

Pinning model is disorder irrelevant if $\alpha < \frac{1}{2}$, disorder relevant if $\alpha > \frac{1}{2}$.

For $\alpha = \frac{1}{2}$ it is marginally relevant (analogous to 2d Directed Polymer)

Let us look at the partition function:

$$
\begin{aligned}
Z_N^\omega &= \mathbf{E}^{\mathrm{ref}}\left[e^{\sum_{n=1}^N (\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right] \\
&= \mathbf{E}^{\mathrm{ref}}\left[\prod_{n=1}^N e^{(\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right] = \mathbf{E}^{\mathrm{ref}}\left[\prod_{n=1}^N \left(1 + X_n \mathbb{1}_{\{n\in\tau\}}\right)\right] \\
&= 1 + \sum_{n=1}^N \mathbf{P}^{\mathrm{ref}}(n\in\tau) X_n + \sum_{0<n<m\le N} \mathbf{P}^{\mathrm{ref}}(n\in\tau,\, m\in\tau) X_n X_m + \ldots
\end{aligned}
$$

where $X_n = e^{\beta\omega_n - \lambda(\beta)} - 1$

Z_N^ω is multi-linear polynomial of X_n's! (decoupled τ and ω)

Partition function

$$X_n \approx \beta\, Y_n \quad \text{with} \quad Y_n \sim \mathcal{N}(0,1) \qquad\qquad \mathrm{P}(n \in \tau) \approx n^{-(1-\alpha)}$$

$$Z_N^\omega = 1 + \beta \sum_{0<n\le N} \frac{Y_n}{n^{1-\alpha}} + \beta^2 \sum_{0<n<m\le N} \frac{Y_n\, Y_m}{n^{1-\alpha}\,(m-n)^{1-\alpha}} + \ldots$$

$$= 1 + \frac{\beta}{N^{1-\alpha}} \sum_{t\in(0,1]\cap\frac{\mathbb{Z}}{N}} \frac{Y_t}{t^{1-\alpha}} + \left(\frac{\beta}{N^{1-\alpha}}\right)^2 \sum_{s<t\in(0,1]\cap\frac{\mathbb{Z}}{N}} \frac{Y_s\, Y_t}{s^{1-\alpha}\,(t-s)^{1-\alpha}} \ldots$$

Write $Y_t = N^{\frac{1}{2}}(W_{t+\frac{1}{N}} - W_t)$ (stochastic Riemann sum) and rescale

$$\beta = \frac{\hat{\beta}}{N^{\alpha-\frac{1}{2}}}$$

We now assume that $\alpha > \frac{1}{2}$ (so that $\beta \to 0$) and we get:

Partition function

$$Z_N^\omega \xrightarrow[N\to\infty]{d} 1 + \hat\beta \int_0^1 \frac{\mathrm{d}W_t}{t^{1-\alpha}} + \hat\beta^2 \int_{0<s<t<1} \frac{\mathrm{d}W_s\,\mathrm{d}W_t}{s^{1-\alpha}\,(t-s)^{1-\alpha}} + \ldots$$

$$= \sum_{k=0}^{\infty} \hat\beta^k \int_{0<t_1<\ldots<t_k<1} \frac{\mathrm{d}W_{t_1}\cdots\mathrm{d}W_{t_k}}{t_1^{1-\alpha}\,(t_2-t_1)^{1-\alpha}\cdots(t_k-t_{k-1})^{1-\alpha}}$$

where the series converges in L^2, for any $\hat\beta \in (0,\infty)$

What happens for $\alpha = \frac{1}{2}$? Integrals ill-defined since $\frac{1}{\sqrt{t}} \notin L^2_{\mathrm{loc}}$

The marginal regime $\alpha = \frac{1}{2}$

$$Z_N^{\omega} = 1 + \beta \sum_{0<n\le N} \frac{Y_n}{\sqrt{n}} + \beta^2 \sum_{0<n<m\le N} \frac{Y_n\, Y_m}{\sqrt{n}\sqrt{m-n}} + \ldots$$

$$= 1 + \beta \quad \Sigma_1 \quad + \beta^2 \quad \Sigma_2 \quad + \ldots$$

Goal: find the joint limit in distribution of all these sums

Σ_1 is easy: by the CLT

$$\Sigma_1 \approx \mathcal{N}(0, \sigma^2) \qquad \text{where} \qquad \sigma^2 = \beta^2 \sum_{0<n\le N} \frac{1}{n} \sim \beta^2 \log N$$

We then rescale $\boxed{\beta = \beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}}}$ Other terms converge?

The right time scale

$$\beta\Sigma_1 = \frac{1}{\sqrt{\log N}} \sum_{0<n\le N} \frac{Y_n}{\sqrt{n}} \approx \int_{1<t<N} \frac{W(\mathrm{d}t)}{\sqrt{t(\log N)}} = \int_{0<a<1} B(\mathrm{d}a) = B_1$$

Time change $\quad t = f(a) := N^a \;$ for $\; 0 < a < 1$

$$\frac{W(\mathrm{d}f(a))}{\sqrt{f'(a)}} \overset{d}{=} B(\mathrm{d}a) \quad \text{for another Brownian motion } B$$

$$\beta^2\Sigma_2 = \frac{1}{\log N} \sum_{0<m<n\le N} \frac{Y_m\, Y_n}{\sqrt{m}\,\sqrt{n-m}} \approx \frac{1}{\log N} \int_{1<s<t<N} \frac{W(\mathrm{d}s)\, W(\mathrm{d}t)}{\sqrt{s}\,\sqrt{t-s}}$$

$$\approx \frac{1}{\log N} \int_{\substack{1<s<N \\ 1<u<N}} \frac{W(\mathrm{d}s)\, W(s+\mathrm{d}u)}{\sqrt{s}\,\sqrt{u}}$$

$$\approx \int_{\substack{0<a<1 \\ 0<b<1}} \frac{W(\mathrm{d}N^a)}{\sqrt{N^a(\log N)}} \frac{W(N^a + \mathrm{d}N^b)}{\sqrt{N^b\,(\log N)}} \approx \int_{\substack{0<a<1 \\ 0<b<1}} B(\mathrm{d}a)\, \tilde{B}_a(\mathrm{d}b)$$

Decoupling of scales

$$B(\mathrm{d}a) \propto W(\mathrm{d}N^a) \qquad \tilde{B}_a(\mathrm{d}b) \propto W(N^a + \mathrm{d}N^b)$$

- Case $a < b$. Then $N^a \ll N^b$ hence $W(N^a + \mathrm{d}N^b) \approx W(\mathrm{d}N^b)$

  This means that $\tilde{B}_a(\mathrm{d}b) \approx B(\mathrm{d}b)$ hence

$$\beta^2 \Sigma_2^{<} \approx \int_{0<a<b<1} B(\mathrm{d}a)\, B(\mathrm{d}b) = \frac{(B_1)^2 - 1}{2}$$

- Case $a > b$. Then $N^a \gg N^b$ hence $W(N^a + \mathrm{d}N^b)$ is approx. independent of $W(\mathrm{d}N^b)$. It follows that

$$\beta^2 \Sigma_2^{>} = \int_{0<b<a<1} B(\mathrm{d}a)\, \tilde{B}_a(\mathrm{d}b) \approx \int_{0<b<a<1} \Gamma(\mathrm{d}a, \mathrm{d}b)$$

  where Γ is a two-dimensional white noise!

Wrapping up

We have shown that

$$\beta\Sigma_1 \xrightarrow[N\to\infty]{d} B_1 \qquad \beta^2\Sigma_2 \xrightarrow[N\to\infty]{d} \frac{(B_1)^2-1}{2} + \mathcal{N}(0, \tfrac{1}{2})$$

For $\beta^3\Sigma_3$ there is a similar decomposition

$$\beta^3\Sigma_3 \approx \int_{\substack{0<a<1 \\ 0<b<1 \\ 0<c<1}} B(\mathrm{d}a)\, \tilde{B}_a(\mathrm{d}b)\, \hat{B}_{a,b}(\mathrm{d}c) \qquad (\text{"correlated" BMs})$$

The running maxima of (a, b, c) determine the limit of $\beta^3\Sigma_3$

- Decompose $(a_1, \ldots, a_k)$ in subsequences corresp. to running maxima
- Each subsequence $(a_i, \ldots, a_j)$, of length $d = j - 1 + i$, gets an independent d-dim. white noise $\Gamma_d(\mathrm{d}a_i, \ldots, \mathrm{d}a_j)$
- Integrate white noises and re-sum the series $\rightsquigarrow$ log-normal

Fourth moment theorem

4th Moment Theorem [de Jong 90] [Nualart, Peccati, Reinert 10]

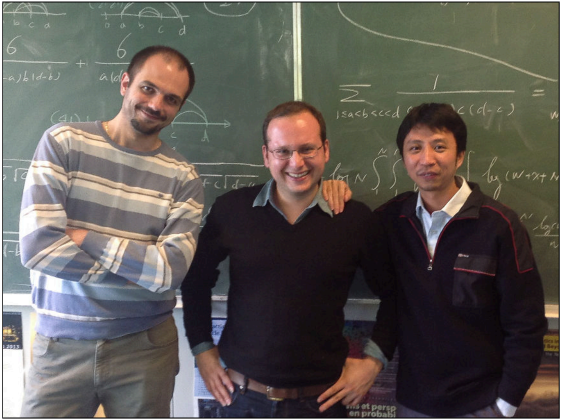
Consider a multi-linear polynomial $Y_N = \sum_{|I|=\ell} \psi_N(I) \prod_{i \in I} Y_i$

which is homogeneous (of degree ℓ)

- $\max_i \psi_N(i) \xrightarrow[N\to\infty]{} 0$ (in case $\ell = 1$) [Small influences!]
- $\mathbb{E}[(Y_N)^2] \xrightarrow[N\to\infty]{} \sigma^2$
- $\mathbb{E}[(Y_N)^4] \xrightarrow[N\to\infty]{} 3\,\sigma^4$

Then $$Y_N \xrightarrow[N\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

Thanks



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)