The 2D Stochastic Heat Equation and related critical models

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LPSM, Paris $\sim$ 6 March 2025

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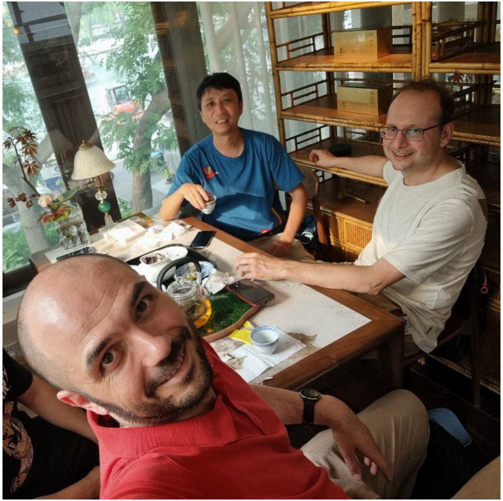
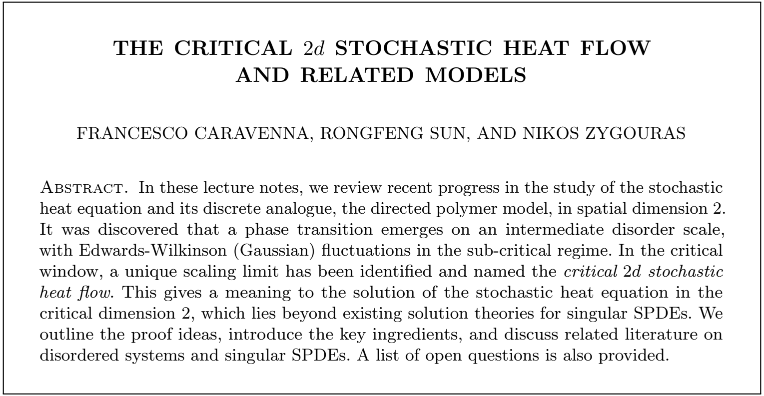


Lecture notes



THE CRITICAL $2d$ STOCHASTIC HEAT FLOW AND RELATED MODELS

FRANCESCO CARAVENNA, RONGFENG SUN, AND NIKOS ZYGOURAS

Abstract. In these lecture notes, we review recent progress in the study of the stochastic heat equation and its discrete analogue, the directed polymer model, in spatial dimension 2. It was discovered that a phase transition emerges on an intermediate disorder scale, with Edwards-Wilkinson (Gaussian) fluctuations in the sub-critical regime. In the critical window, a unique scaling limit has been identified and named the *critical* $2d$ *stochastic heat flow*. This gives a meaning to the solution of the stochastic heat equation in the critical dimension 2, which lies beyond existing solution theories for singular SPDEs. We outline the proof ideas, introduce the key ingredients, and discuss related literature on disordered systems and singular SPDEs. A list of open questions is also provided.

https://arxiv.org/abs/2412.10311

In a nuthshell

Stochastic Heat Equation (SHE)

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} u(t,x)} + \beta\, \xi(t,x)\, u(t,x) \qquad t \geq 0,\ x \in \mathbb{R}^d$$

Singular random potential $\xi(t,x)$ "space-time white noise"

Main result

We construct a natural candidate solution of SHE in space dimension $d = 2$

called the critical 2D Stochastic Heat Flow (SHF)

Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

- Fundamental PDE $+$ universal random potential $\xi(t,x)$

  white noise $=$ "continuum" i.i.d. random variables

- KPZ equation [Kardar–Parisi–Zhang *PRL* 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x) \qquad \text{(KPZ)}$$

Cole–Hopf transformation $h(t,x) = \log u(t,x)$

Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ u(t,x) expected $\begin{cases} \text{non-smooth function} & d = 1 \\ \text{genuine distribution} & d \geq 2 \end{cases}$

Product $\xi(t,x)\, u(t,x)$ unclear: no classical space to solve SHE (as a PDE)

Stochastic integral for $d = 1$ [Ito/Walsh, Da Prato–Zabczyk]

SHE solution $u(t,x) > 0$ starting from $u(0,\cdot) \geq 0$ $\rightsquigarrow$ "KPZ solution" $h(t,x) = \log u(t,x)$

The role of dimension

Revolution in 2010s: **robust solution theories** for **sub-critical SPDEs**

[Hairer *Invent. Math.* 14] [Gubinelli–Imkeller–Perkowski *Forum Math Pi* 15] [. . .]

SHE and KPZ: robust theories apply only for $d = 1$

Space-time blow-up $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$

$$\partial_t \tilde{u}(t,x) = \Delta_x \tilde{u}(t,x) + \varepsilon^{\frac{2-d}{2}} \beta \, \tilde{\xi}(t,x) \, \tilde{u}(t,x)$$

$$\text{as } \varepsilon \downarrow 0 \text{ the noise strength } \begin{cases} \text{vanishes} & d < 2 \quad \text{sub-critical} \\ \text{unchanged} & d = 2 \quad \textbf{critical} \\ \text{diverges} & d > 2 \quad \text{super-critical} \end{cases}$$

What can we do?

We focus on the critical dimension $d = 2$

Regularized noise $\xi_N(t,x)$ $\rightsquigarrow$ well-defined solution $u_N(t,x)$

(discretization, mollification, Fourier cutoff, ...)

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, \xi_N(t,x)\, u_N(t,x) \\ u_N(t,0) \equiv 1 \quad \text{(for simplicity)} \end{cases} \tag{reg-SHE}$$

Fix $\xi_N(t,x) \xrightarrow[N\to\infty]{} \xi(t,x)$

Does $u_N(t,x)$ converge to an interesting limit?

Which notion of convergence?

Do not expect pointwise convergence $\qquad \varphi : \mathbb{R}^2 \to \mathbb{R}$

Space-average $\qquad u_N(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, \mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\varphi) \qquad (?)$

$\iff$ random measure on $\mathbb{R}^2 \qquad \underbrace{u_N(t,x)}_{\geq 0 \;\; \mathbb{E}[\cdot]=1} \mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\mathrm{d}x) \qquad (?)$

Convergence? No interesting limit! [Berger–C.–Turchi 25+]

$\forall$ density $\varphi \in C_c(\mathbb{R}^2)$, $\int \varphi = 1$

$$u_N(t,\varphi) \xrightarrow[N\to\infty]{\forall \beta > 0} 0 \qquad u_N(t,\varphi) \xrightarrow[N\to\infty]{\beta = 0} 1$$

The need for renormalization

We take $\quad \beta = \beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right)$

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta_N \, \xi_N(t,x) \, u_N(t,x) \qquad \text{(reg-SHE)}$$

Formally $\quad \beta_N \, \xi_N(t,x) \, u_N(t,x) \longrightarrow 0 \quad \ldots$ $\qquad$ *but actually not!*

- $\mathbb{V}\mathrm{ar}\big[u_N(t,\varphi) \big] \xrightarrow[N\to\infty]{} K_t^{\vartheta}(\varphi,\varphi) > 0$ [Bertini–Cancrini *J Phys A* 98]
- Convergence of higher moments [C.S.Z. *CMP* 19] [Gu–Quastel–Tsai *PMP* 21]
- Convergence in law of $u_N(t,\varphi) \iff u_N(t,x)\,\mathrm{d}x$?

Main result

Theorem [C.S.Z. *Invent. Math.* 23]

Take $$\beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$$ for some $\vartheta \in \mathbb{R}$

Then u_N converges in law to a unique and non-trivial limit $\mathscr{U}^{\vartheta}$

$$u_N = \big(u_N(t,x)\,\mathrm{d}x\big)_{t\geq 0} \xrightarrow[N\to\infty]{d} \mathscr{U}^{\vartheta} = \big(\mathscr{U}^{\vartheta}(t,\mathrm{d}x)\big)_{t\geq 0}$$

$\mathscr{U}^{\vartheta}$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of measures on $\mathbb{R}^2$

SHF and Stochastic Heat Equation

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad \text{(initial condition 1 at time 0)}$$

We actually build the SHF as a two-parameter space-time process

$$\Big(\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x)\Big)_{0 \le s \le t < \infty} \qquad \text{(starting at time } s \text{ from } \mathrm{d}y\text{)}$$

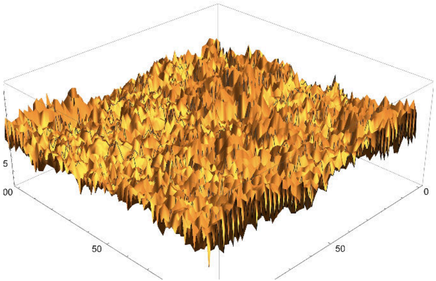
Why "flow"? Chapman-Kolmogorov for $s < t < u$ [Clark–Mian 24+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y; u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y; \underbrace{t, \mathrm{d}x)\, \mathscr{U}^{\vartheta}(t, \mathrm{d}x}_{\substack{\text{non-trivial "product"} \\ \text{of measures}}}; u, \mathrm{d}z)$$

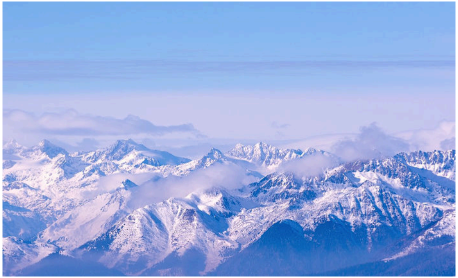
How does the SHF look like?

We can efficiently simulate the SHF via $u_N(t,x)$ time $O(N^2)$

Some not-so-randomly picked realizations [M. Mucciconi, N. Zygouras]



KPZ $\approx \log u_N(t,x)$



Alps, Italy/France (vecteezy.com)

Properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. 25+]

  "not a function"

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)

  "*barely* not a function"

- $\mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)] = \mathrm{d}x \qquad \mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)\,\mathscr{U}^{\vartheta}(t, \mathrm{d}y)] = \underbrace{K^{\vartheta}(t, x-y)}_{\approx \log|x-y|^{-1}}\, \mathrm{d}x\, \mathrm{d}y$

- Formulas for higher moments [C.S.Z. *CMP* 19] [Gu–Quastel–Tsai *PMP* 21]

- Scaling covariance $\quad a^{-1}\,\mathscr{U}^{\vartheta}(a\,t, \mathrm{d}(\sqrt{a}\,x)) \overset{d}{=} \mathscr{U}^{\vartheta + \log a}(t, \mathrm{d}x)$

Properties of the SHF

Theorem [Berger–C.–Turchi 25+]

$$\mathscr{U}_1^{\vartheta}(B(0,1)) \xrightarrow[\vartheta \to +\infty]{d} 0 \qquad \Longleftrightarrow \qquad \frac{\mathscr{U}_t^{\vartheta}(B(0,\sqrt{t}))}{t} \xrightarrow[t \to \infty]{d} 0$$

"strong disorder" "superdiffusivity"

Theorem [C.–Donadini 25+]

$$(\xi_N, u_N) \xrightarrow[N \to \infty]{d} (\xi, \mathscr{U}^{\vartheta}) \qquad \text{with } \xi \text{ and } \mathscr{U}^{\vartheta} \text{ independent (!)}$$

Axiomatic characterization

Recently Li-Cheng Tsai provided an axiomatic characterization of the SHF

Theorem [Tsai 24+]

Let $\mathscr{Z} = \big(\mathscr{Z}_{s,t}(\mathrm{d}x, \mathrm{d}y)\big)_{s \le t}$ be a stochastic process on $\mathscr{M}_+(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying

- continuity of $(s,t) \mapsto \mathscr{Z}_{s,t}$
- independence of $\mathscr{Z}_{s,t}$ and $\mathscr{Z}_{t,u}$ $\qquad \forall s < t < u$
- convolution $\mathscr{Z}_{s,u} = \mathscr{Z}_{s,t} * \mathscr{Z}_{t,u}$ (Chapman-Kolmogorov) $\qquad \forall s < t < u$
- moments $\mathbb{E}\big[\prod_{i=1}^n \mathscr{Z}_{s,t}(\varphi_i, \psi_i)\big]$ for $n = 1, 2, 3, 4$ coincide with those of $\mathscr{U}^\vartheta$

Then $\mathscr{Z}$ has the same distribution as the SHF $\mathscr{U}^\vartheta$

Gaussian Multiplicative Chaos?

Much studied class of random measures: Gaussian Multiplicative Chaos (GMC)

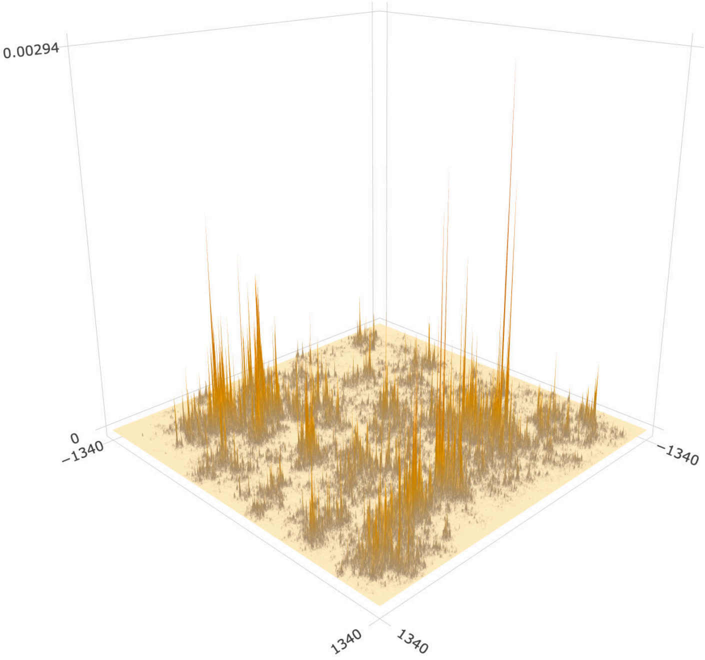
$$\mathscr{M}(\mathrm{d}x) = \text{“}\, \mathrm{e}^{X(x) - \frac{1}{2}\mathbb{V}\mathrm{ar}[X(x)]}\, \mathrm{d}x \,\text{”} \qquad X(\cdot) \text{ generalized Gaussian field}$$

Comparison with (non-usual) GMC: $\mathscr{U}^{\vartheta}$ log-correlated $\rightsquigarrow$ X log-log-correlated

Theorem [C.S.Z. *AoP* 23]

The critical 2d Stochastic Heat Flow $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is not a GMC

Simulations: zooming in $\mathscr{U}^{\vartheta}(1, \mathrm{d}x) \approx u_N(1, x)$ for $N = 50\,000$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0
−1340
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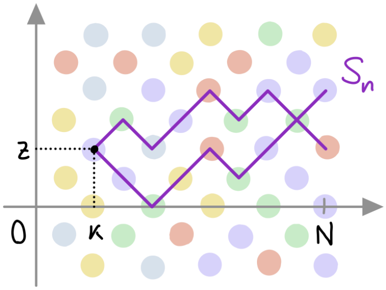
Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n, x) \sim \mathcal{N}(0, 1)$
- $H(S, \omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \sim \mathcal{N}(0, N-k)$



Partition Functions

$(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z_{N,\beta}^{\omega}(k, z) = \mathrm{E}\left[\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Partition functions and SHE

Diff. rescaled partition functions $=$ discretized SHE solutions

$$Z^{\omega}_{N,\beta}\big(N(1-t),\sqrt{N}x\big) \;=\; u_N(t,x) \qquad \text{(time rev.)}$$

Partition functions solve a difference equation: with $\xi_N \approx \boldsymbol{\omega}$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}_{\beta_{\mathrm{SHE}}} \; \xi_N(t,x)\, u_N(t,x) \\ u_N(0,x) \;\equiv\; 1 \end{cases} \qquad \text{(reg-SHE)}$$

Discrete analogue of Feynman-Kac

$$u_N(t,x) \;\approx\; \mathrm{E}\Big[\mathrm{e}^{\beta_{\mathrm{SHE}} \int_{1-t}^{1} \xi(s,B_s) - \frac{1}{2}\beta_{\mathrm{SHE}}^2\, t} \,\Big|\, B_{1-t} = x \Big]$$

Techniques

Existence of subsequential limits $u_{N_k}(t,x)\,\mathrm{d}x \to \mathscr{U}^{\vartheta}(t,\mathrm{d}x)$ is easy. Uniqueness?

Cauchy sequence

$$u_N(t,x)\,\mathrm{d}x \approx u_M(t,x)\,\mathrm{d}x \qquad \text{for large } N,\ M$$

- Coarse-graining (tool: sharp second moment computations)
- Lindeberg priciple (tool: higher moment bounds)

Probabilistic handle on the SHE solution $u_N(t,x)$ via directed polymers

Correlation of $u_N(t,x)$ $\rightsquigarrow$ overlap of random walks

Summarizing

Singular Stochastic PDEs such as SHE and KPZ

closely linked to Directed Polymers

(discretized solutions $\leftrightsquigarrow$ partition functions)

Role of dimension d and disorder strength β

Critical dimension $d = 2$ + critcal disorder strength $\beta = \beta_N$

Scaling limit of partition functions $\rightsquigarrow$ Stochastic Heat Flow $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$

Many features are understood, but several questions are still open

Future challenges

- Finer regularity properties of the SHF
- Universality
- SHF as a Markov process (state space, martingale problem, ...)
- Black noise / sensitivity features cf. [Himwich–Parekh 24+]
- Critical $2d$ KPZ? "How to take log of $\mathscr{U}^{\vartheta}$?"

Related models

Stochastic Heat Equation with Lévy noise [Berger–Chong–Lacoin *CMP* 23]

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \mathbb{P}(\xi > t) \approx t^{-\alpha}$$

Well-posedness ($+$ intermittency) under optimal assumptions

Anisotropic KPZ equation [Cannizzaro–Erhard–Toninelli *CPAM* 23, *Duke* 23]

$$\partial_t h(t,x) \;=\; \Delta_x h(t,x) + \beta \left\{ (\partial_{x_1} h(t,x))^2 - (\partial_{x_2} h(t,x))^2 \right\} + \xi(t,x)$$

Fixed $\beta > 0 \quad \rightsquigarrow \quad$ logarithmic superdiffusivity

Merci