**Università degli studi di Padova**

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# A multivariate model for financial indexes subject to volatility shocks

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# Chapter 1

# Introduction

In recent years, development of stochastic models for time series and studies on statistical data coming from financial indexes have been deeply linked. This thesis is based on the article [1], and deals with the model presented there both from theoretical point of view and through numerical simulation and comparison with data coming from real time series. We focus in particular on cross-asset correlations.

The most basic model, lying behind results on financial markets such as the celebrated Black and Scholes formula, is based on the following diffusion

$$dX_t = \sigma dW_t. \tag{1.1}$$

$X_t$ represents the logarithm of the detrended price of a financial index, and its evolution is assumed to be driven by a standard Brownian motion $W = (W_t)_{t\geq 0}$. The constant $\sigma$ is called volatility. Despite the wide popularity of this model and its frequent use in applications, it has been pointed out that it is not consistent with several empirical facts, common to a wide set of financial assets; these nontrivial statistical properties are called *stylized facts*. These facts are usually formulated in terms of qualitative properties of returns of the time serie under examination, but they are very difficult to explain through a model which is simple and analytically tractable. Nevertheless, their so widespread presence in price series leads to think of a common source, and a very interesting problem is the attempt to give a simple mathematical explanation to this properties, shared by indexes that a priori could display very different behaviors, such as stock prices and exchange rates.

We now enumerate some of these stylized facts. We refer again to [1], and to [3] for a more detailed description.

1. The volatility is not constant; in particular, it may display high peaks, that can be interpreted as shocks in the market.

2. The empirical distribution of the increments $X_{t+h} - X_t$ of log-prices

(the log returns) has tails heavier than Gaussian (power law or Pareto-like tail)

3. Log-returns corresponding to disjoint time-intervals are uncorrelated, but not independent. The autocorrelaton function of absolute returns $\rho(t)$ has a slow decay in the time lag $t$. This phenomenon is called *volatility clustering.*

4. Denote with $p_h$ the empirical distribution of the (detrended) log-returns, relative to a fixed time length $h$. For $h$ within a suitable time scale, this *scaling property* approximately holds:

$$p_h(dr) \approx \frac{1}{\sqrt{h}} g\left(\frac{r}{\sqrt{h}}\right) dr \tag{1.2}$$

where $g$ is a probability density with tails heavier than Gaussian. For bigger $h$ instead, returns distribution looks more and more like a normal distribution.

5. Consider the $q-th$ empirical moment of the distribution of $h$-returns:

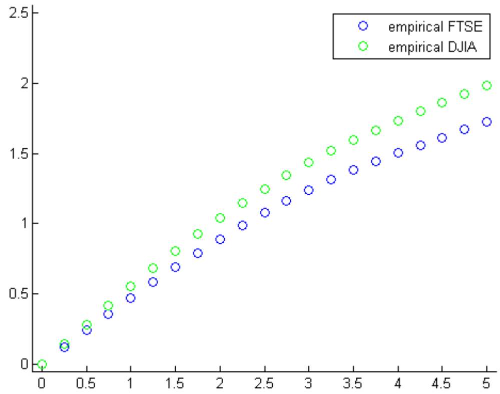
$$m_q(h) = \frac{1}{T-h}\sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q p_h(dr). \tag{1.3}$$

From formula (1.2), changing variables $r/\sqrt{h} = y$, it would be natural to guess that $m_q(h) \approx h^{q/2}$. This is indeed what one observes for moments of small order ($q < \overline{q}$, $\overline{q}$ depending on the index). For moments of higher order, a different scaling relation $h^{A(q)}$ holds, for some $A(q) < q/2$ (see Figure 1.1). This is the so called *multiscaling of moments.*

Why should properties of series with very different origins be so similar? A variety of models can be found in the literature, displaying one or the other feature (stochastic volatility models, ARCH, GARCH), and in recent years very refined models have been proposed (e.g. FIGARCH) capable to show a behavior which agrees with all of this facts (see [4], [5], [6]). Although models with sufficiently many parameters can be adapted to data, the soundness of the procedure of statistical inference may be very weak. Our aim is to give a simpler and more natural mathematical explanation of the arising of such widespread phenomena.

In this work we present the model for time series proposed in [1], with the purpose of understanding the behavior of cross-asset correlations of time series evolving following this model, and understanding how close this behavior is to what we can see in real series. After a brief, general presentation of the model and its properties in Chapter 2, we develop in Chapter 3 the

Figure 1.1: Multiscaling exponent




theoretical part on cross-asset correlations of the bivariate model, under different hypothesis on the processes involved. We will see how the main role in decay of correlations is played by the process of shocks of the market.

In Chapter 4 we try to understand how close is our modelization to real data. Starting from the DJIA and FTSE time series of the last 27 years, we find a very good agreement between data and univariate model, if we consider separately the two indexes. We then try to validate the bivariate model on these two real time series.

To deal with this matter, being cross-asset correlations heavily related to the process of shocks in the market, we will have to develop an algorithm to estimate past crisis. We find that the majority of the crisis estimated by the algorithm in DJIA and FTSE are shared by the two indexes, making us think of two processes of shocks with a common part, that we try to model trough the union of Poisson point processes.

With this choice for process of crisis, applying the theory developed in Chapter 3, we find an excellent agreement also between the predicted and the empirical decay of cross-asset correlations. With respect to this, we also observe a property that we believe had never been detected before in real series: cross-asset correlations and autocorrelations of the single indexes decay qualitatively and quantitatively in the same fashion, as predicted by the theory on our model (see Chapter 3 and 4).

# Chapter 2

# Definition and properties

In this chapter we describe the model presented in [1], and state properties and results related to the stylized facts presented in the introduction. For proofs of all the statements we refer to [1] as well. A different proof of Theorem 3 on volatility autocorrelation will be given in the following Chapter 3 on the bivariate model, as a particular case of the result on correlations between different such processes (see Remark 2). A statistical analysis and considerations on model and real data are postponed to Chapter 4.

## 2.1 Definition of the model

Given two real numbers $D \in (0, 1/2]$, $\lambda \in (0, \infty)$, the model is defined upon three sources of randomness, defined on a probability space $(\Omega, \mathcal{F}, P)$:

- a Brownian motion $W = (W_t)_{t \geq 0}$
- $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ a Poisson point process on $\mathbb{R}$ of rate $\lambda$;
- a sequence $\Sigma = (\sigma_n)_{n \in \mathbb{N}}$ of i.i.d. positive random variables with marginal law $\nu$. We denote by $\sigma$ a variable such that $\sigma \sim \nu$;

Let's recall that a point process $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ is said to be a Poisson point process of rate $\lambda$ if its increments $(\tau_n - \tau_{n-1})_{n \in \mathbb{Z}}$ are independent exponential variables of parameter $\lambda$. Then the mean distance between consecutive points of $\mathcal{T}$ is $1/\lambda$.

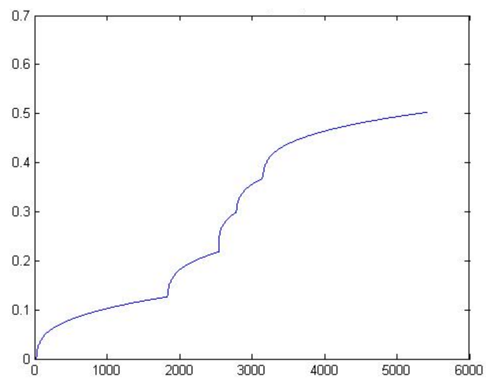
We suppose $W$, $\mathcal{T}$, $\Sigma$ are independent. By convention we label the points of $\mathcal{T}$ so that $\tau_0 < 0 < \tau_1$. We are going to define a process $X_t$ just for $t \geq 0$, and for this purpose we will actually need only the points of $\mathcal{T} \cap [\tau_0, \infty)$, that is $(\tau_n)$ for $n \geq 0$. We'll see that taking also a negative point $(\tau_0)$ is necessary to have a stationary process. By the memoryless property, the random variables $-\tau_0, \tau_1, (\tau_n - \tau_{n-1})_{n>1}$ are independent and identically distributed $Exp(\lambda)$. This will provide us an easy way to obtain numerical simulation.

We come now to the definition of our model for financial time series, a stochastic process $X = (X_t)_{t\geq 0}$ built as follows:

$$
\begin{aligned}
&X_t = \sigma_0 \left( W_{(t-\tau_0)^{2D}} - W^{2D}_{(-\tau_0)} \right), \qquad \text{for } t \in [0, \tau_1] \\
&X_t = X_{\tau_n} + \sigma_n \left( W_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} - W_{\sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \right), \\
&\text{for } t \in [\tau_n, \tau_{n+1}], n \geq 1
\end{aligned}
\tag{2.1}
$$

How to read this definition? An interpretation is the introduction of a time inhomogeneity $t \to t^{2D}$ at times $\tau_n$, and a renewal of the volatility $\sigma_n$. This construction could well represent a financial time series, where at "random" times there are shocks in the market, modeled by our Poisson point process. The reaction of the market is an acceleration of the dynamics immediately after the shock, and a gradual slowing down at later times, until a new shock accelerate again the dynamics. This behavior is due to the shape of the function $t \to t^{2D}$, $D \in (0, 1/2]$, which is steep for $t$ close to 0 and bends down for increasing $t$. The random variables $\sigma_n$ represent inhomogeneity in the impact of crisis.

Figure 2.1: Time inhomogeneity




The translations invariance of the law of $\tau$ produces, as we will see, stationary increments for the process $X$.

Also note that for $D = 1/2$ the model reduces to a simple random volatility model $dX_t = \sigma_t dW_t$, where $\sigma_t = \Sigma_{k=0}^{\infty} \sigma_k \mathbf{1}_{[}\tau_k, \tau_{k+1})(t)$ is a (random) piecewise constant process.

We now give another definition of the process $X$. These two definitions are equivalent in law, but the latter is more convenient for proving the properties of the model. For $t \geq 0$, define

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = \#\{\mathcal{T} \cap (0, t]\}$$

$i(t)$ is the number of positive times in the Poisson process before $t$, so that $\tau_{i(t)}$ is the location of the last point in $\mathcal{T}$ before $t$. We introduce the process $I = (I_t)_{t\geq 0}$

$$I_t = \sigma_{i(t)}^2 (t - \tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 (\tau_k - \tau_{k-1})^{2D} - \sigma_0^2 (-\tau_0)^{2D} \tag{2.2}$$

where we agree that the sum in the right hand side is 0 if $i(t) = 0$. Now define the process

$$X_t = W_{I(t)} \tag{2.3}$$

We easily see that the two definitions (2.1) and (2.3) are equivalent because of the scale invariance of Brownian motion. Observe that $I$ is a strictly increasing process with absolutely continuous paths, and it is independent of the Brownian motion $W$. Thus this model may be viewed as an independent random time change of a Brownian motion.

## 2.2 Main properties

We are now going to state some properties of the process $X$:

**Proposition 1** (Basic Properties)**.** *Let $X$ be the process defined above in 2.1 and 2.3; then the following assertions are satisfied:*

1. *$X$ has stationary increments.*
2. *$X$ can be represented as a stochastic volatility process:*

$$dX_t = v_t dB_t \tag{2.4}$$

*where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. $B$ and $v = (v_t)_{t\geq 0}$ are defined by*

$$\begin{aligned} B_t &:= \int_0^{I_t} \frac{1}{\sqrt{I'(I^{-1(u)})}} dW_u \\ v_t &:= \sqrt{I'(t)} = \sqrt{2D}\sigma_{i(t)}(t - \tau_{i(t)})^{D-\frac{1}{2}} \end{aligned} \tag{2.5}$$

*where $I'(s) = \frac{dI}{ds}(s)$. Note that, for $D < 1/2$, the volatility $v_t$ has singularities at the random times $\tau_n$.*

3. *$X$ is a zero-mean, square-integrable martingale (provided $E(\sigma^2) < 1$).*
4. *The distributions of the increments of $X$ is ergodic.*
5. *$E(|X_t|^q) < \infty$ for some (and hence any) $t > 0 \Leftrightarrow E(\sigma^q) < \infty$*

*Remark* 1. Property 4 implies, in particular, that if we have a measurable function $F : \mathbb{R}^k \to \mathbb{R}$, for every $\delta > 0$, $k \in \mathbb{N}$, and for any choice of the positive intervals $(a_1, b_1), ..., (a_k, b_k)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} F(X_{n\delta+b_1} - X_{n\delta+a_1}, ..., X_{n\delta+b_k} - X_{n\delta+a_k}) = \\ = E\left[F(X_{+b_1} - X_{a_1}, ..., X_{b_k} - X_{a_k})\right] \tag{2.6}$$

almost surely, provided the expectation appearing in the right hand side is well defined. That is, if we have a long empirical time series of the process, averaging a function of the increments over a long time period we get a value approximately close to its expected value. Thanks to this property, it is meaningful to compare theoretical results and simulated data coming from the model with real data series, when these series have a considerable length. For the initial estimation of the parameters of the model, we will mainly focus on multiscaling of moments and correlation of increments of the process $X$, then we will see a very good agreement also for the theoretical and empirical distribution of increments (see Chapter 4).

Note, from Property 5 and from the fact that for most real financial indexes there is empirical evidence that $E(|X_t|^q) < \infty$ for all $q > 0$, that the main interest of our model is when $\sigma$ has finite moments of all orders.

We are now ready to state some results, important because they establish a link between our model and the stylized fact presented in the introduction; that is, in the limit for $h \downarrow 0$, the process $X$ defined in 2.1 and 2.3 is consistent with important facts empirically detected in many (financial) real time series, namely: diffusive scaling of returns, multiscaling of moments, slow decay of volatility autocorrelation.

**Theorem 1** (Diffusive scaling)**.** *We have that in distribution, for $h \downarrow 0$,*

$$\frac{X_{t+h} - X_t}{\sqrt{h}} \longrightarrow f(x)dx \tag{2.7}$$

*where $f$ is a mixture of centered Gaussian densities*

$$f(x) = \int_0^\infty \nu(d\sigma) \int_0^\infty dt\, \lambda e^{-\lambda t} \frac{t^{1/2-D}}{\sigma\sqrt{4D\pi}} exp\left(-\frac{t^{1-2D}x^2}{4D\sigma^2}\right).$$

Let's compute the $q - th$ moment of $x$ under the probability density $f$; using the change of variables $y = \frac{x\, t^{1/2-D}}{(2D)^{1/2}\sigma}$, and denoting with $N$ a standard normal variable

$$\begin{aligned}
&E_f(|x|^q) = \int_{\mathbb{R}} |x|^q f(x)dx = \\
&\int_{\mathbb{R}} dx \int_0^\infty \nu(d\sigma) \int_0^\infty dt\, |x|^q\, \lambda e^{-\lambda t} \tfrac{t^{1/2-D}}{\sigma\sqrt{4D\pi}} exp\left(-\tfrac{t^{1-2D}x^2}{4D\sigma^2}\right) = \\
&\int_{\mathbb{R}} dy \int_0^\infty \nu(d\sigma) \int_0^\infty dt\, \tfrac{(2D)^{1/2}\sigma}{t^{1/2-D}} \left|\tfrac{y(2D)^{1/2}\sigma}{t^{1/2-D}}\right|^q \lambda e^{-\lambda t} \tfrac{t^{1/2-D}}{\sigma\sqrt{4D\pi}} exp\left(-\tfrac{y^2}{2}\right) = \\
&(2D)^{q/2} \int_{\mathbb{R}} \tfrac{|y|^q}{\sqrt{2\pi}} exp\left(-\tfrac{y^2}{2}\right) dy \int_0^\infty |\sigma|^q \nu(d\sigma) \int_0^\infty \lambda e^{-\lambda t} t^{(D-1/2)q} dt = \\
&(2D)^{q/2} E(|N|^q) E(\sigma^q) \int_0^\infty \lambda e^{-\lambda s} s^{(D-1/2)q} ds
\end{aligned}$$

which is not finite if $q \geq q^* = (1/2 - D)^{-1}$. Therefore the density $f$ has always polynomial tails, because $E_f(|x|^q) = \infty$ for $q \geq q^*$ (and this does not depend on the distribution of $\sigma$). Property 5 says that if $\sigma$ has finite moments of all orders, so does $X_{t+h} - X_t$. Then the function $f$, which describes the asymptotic law, for $h \downarrow 0$, of $\frac{X_{t+h}-X_t}{\sqrt{h}}$, has a different tail behavior from the density of $X_{t+h} - X_t$, for fixed $h$.

This feature of $f$ is linked to another property of our model: the multiscaling of moments. Let's define the $q - th$ moment of the log returns, at time scale $h$:

$$m_q(h) := E\left(|X_{t+h} - X_t|^q\right) = E\left(|X_h|^q\right) \tag{2.8}$$

the last equality holding for the ergodicity of increments. Because of the scaling properties (2.7), we would expect $m_q(h)$ to approximate in some sense $h^{\frac{q}{2}} \int x^q f(x)dx = C_q h^{\frac{q}{2}}$, for $h \downarrow 0$. This is actually true for $q < q^* = (1/2 - D)^{-1}$, that is, for $q$ such that the $q - th$ moment of the limit distribution is finite. For $q \geq q^* = (1/2 - D)^{-1}$, the $q - th$ moment of the limit distribution is not finite, and an analogous reasoning cannot be applied. It turns out instead that a faster scaling holds, namely $m_q(h) \approx h^{Dq+1}$. This transition in the scaling of $m_q(h)$ is known as multiscaling of moments, a property empirically detected in many time series, in particular in financial series; we will deepen the relations between this empirical facts and our model in chapter 4. Let's now state precisely this result:

**Theorem 2** (Multiscaling of moments)**.** *For $q > 0$ such that $E(\sigma^q)$ is finite, the $q - th$ moment of log returns $m_q(h)$ is finite and has the following asymptotic behavior as $h \downarrow 0$:*

$$m_q(h) \sim \begin{cases} C_q h^{\frac{q}{2}}, & if\, q < q^* \\ C_q h^{\frac{q}{2}} \log\left(\dfrac{1}{h}\right), & if\, q = q^* \\ C_q h^{Dq+1}, & if\, q > q^* \end{cases} \tag{2.9}$$

*Constants $C_q \in (0, \infty)$ are*

$$C_q = \begin{cases} E(|W_1|^q)E(\sigma^q)\lambda^{\frac{q}{q^*}}(2D)^{\frac{q}{2}}\Gamma(1 - q/q^*), & if\, q < q^* \\ E(|W_1|^q)E(\sigma^q)\lambda(2D)^{\frac{q}{2}}, & if\, q = q^* \\ E(|W_1|^q)E(\sigma^q)\lambda\left[\displaystyle\int_0^\infty ((1+x)^{2D} - x^{2D})^{\frac{q}{2}}dx + \frac{1}{Dq+1}\right], & if\, q > q^* \end{cases}$$

*where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ denotes Euler's Gamma function.*
*As a consequence, the scaling exponent $A(q)$ is*

$$A(q) = \lim_{h\downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} \dfrac{q}{2} & if\, q \leq q^* \\ Dq + 1 & if\, q \geq q^* \end{cases} \tag{2.10}$$

So in this model the multiscaling exponent is a piecewise linear function of $q$. We will use this explicit formulas in 4.1, when we will deal with the problem of estimating parameters to fit real data.

We are now going to state the last theoretical result of this chapter. It concerns the volatility autocorrelation of the process $X$, that is the correlations of absolute values of returns at a given time distance (scale). Recall that the correlation coefficient of two random variables $X$ and $Y$ is

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

For the process $X$, introduce $\xi = (\xi_t)_{t\geq 0}$, the process of absolute values of increments, for $h$ fixed: $\xi_t = |X_{t+h} - X_t|$. Then the volatility autocorrelation of $X$ is

$$\begin{aligned}\rho(t-s) = \rho(\xi_s,\xi_t) &= \frac{Cov(\xi_s,\xi_t)}{\sqrt{Var(\xi_s)Var(\xi_t)}} \\ &= \frac{Cov(|X_{s+h}-X_s|,|X_{t+h}-X_t|)}{\sqrt{Var(|X_{s+h}-X_s|)Var(|X_{t+h}-X_t|)}}\end{aligned} \tag{2.11}$$

Indeed, being the process stationary, the quantity we have defined above depends just on the time difference $t-s$. Let's state our result, concerning as above the asymptotic behavior as $h \downarrow 0$.

**Theorem 3** (Volatility autocorrelaton). *Suppose $E(\sigma^\delta)$ is finite for some $\delta > 2$. For $t \geq 0$,*

$$\lim_{h\downarrow 0} \rho(t) = \frac{2}{\pi} \frac{Cov\left(\sigma S^{D-1/2}, \sigma(\lambda t + S)^{D-1/2}\right) e^{-\lambda t}}{Var(\sigma|N|S^{D-1/2})} \tag{2.12}$$

*where $S$ is an exponential variable with parameter 1, $N$ is a standard normal variable, they are mutually independent and independent from $\sigma$.*

From this theorem we get information about the decay of volatility autocorrelation of our process. Note that

$$\begin{aligned}&Cov\left(\sigma S^{D-1/2}, \sigma(x+S)^{D-1/2}\right) = \\ &E\left(\sigma S^{D-1/2}\sigma(x+S)^{D-1/2}\right) - E\left(\sigma S^{D-1/2}\right)E\left(\sigma(x+S)^{D-1/2}\right) = \\ &E(\sigma^2)E\left(S^{D-1/2}(x+S)^{D-1/2}\right) - E(\sigma)^2 E\left(S^{D-1/2}\right)E\left((x+S)^{D-1/2}\right) = \\ &Var(\sigma)(E\left(S^{D-1/2}(x+S)^{D-1/2}\right)) + E(\sigma)^2 Cov\left(S^{D-1/2},(x+S)^{D-1/2}\right)\end{aligned}$$

Note that as $x \longrightarrow \infty$,

$$\begin{aligned}E\left(S^{D-1/2}(x+S)^{D-1/2}\right) &\approx x^{D-1/2} \\ Cov\left(S^{D-1/2},(x+S)^{D-1/2}\right) &\approx x^{D-3/2};\end{aligned}$$

Recall that $\lambda$ is the rate of the Poisson process. We have that that for $t >> 1/\lambda$, that is for time scales greater then the mean time distance between the "crisis" $\tau_n$, correlations of our process decay exponentially fast, because of the term $e^{-\lambda t}$. For smaller times, $t = O(1/\lambda)$, a relevant contribution is given by $Cov\left(\sigma S^{D-1/2}, \sigma(\lambda t + S)^{D-1/2}\right)$, and then the decay of volatility autocorrelation is between polynomial and exponential.

# Chapter 3

# The bivariate model

In this chapter we want to study the correlation of returns of a bivariate version $(X, Y) = (X_t, Y_t)_{t\geq 0}$ of the model defined in Chapter 2, driven by two independent Brownian Motions, with different hypothesis on the two Poisson processes of "shocks" and, therefore, on the time changes related to $X$ and $Y$. We see that the relevant quantity to establish the correlation coefficients is the covariance of the square roots of the two time changes, and we derive an explicit expression for covariance and correlations in the bivariate case. We then apply the same reasoning to derive in a different way the already known result concerning the volatility autocorrelation when we consider the returns of the same process $X$ at different times, as the time scale goes to 0. From these results we can easily see how the correlation decays with time difference, in both cases of returns of different processes and returns of the same process at different times.

## 3.1 Definition of the bivariate model

We are going to base the definition of the model upon the following quantities:

- two Brownian motions $W^X = \left(W_t^X\right)_{t\geq 0}$ and $W^Y = \left(W_t^Y\right)_{t\geq 0}$;
- two positive constants $D^X$ and $D^Y$;
- two Poisson point processes on $\mathbb{R}$: $\mathcal{T}^X = (\tau_n^X)_{n\in\mathbb{Z}}$ and $\mathcal{T}^Y = (\tau_n^Y)_{n\in\mathbb{Z}}$, of rates respectively $\lambda^X$ and $\lambda^Y$;
- two sequences of i.i.d. positive random variables, $\Sigma^X = (\sigma_n^X)_{n\in\mathbb{N}}$ and $\Sigma^Y = (\sigma_n^Y)_{n\in\mathbb{N}}$, with marginal law respectively $\nu^X$ and $\nu^Y$.

We suppose the independence of $W^X$, $\mathcal{T}^X$ and $\Sigma^X$, so that the $X$, the first component of $(X, Y)$, will be the same kind of process as the one defined in Chapter 2, and for the same reason we assume the independence of $W^Y$, $\mathcal{T}^Y$

and $\Sigma^Y$. We also suppose the independence of $W^X$ and $(W^Y, \mathcal{T}^Y, \Sigma^Y)$ and, viceversa, the independence of $W^Y$ and $(W^X, \mathcal{T}^X, \Sigma^X)$.

We are now going to state two other assumptions on the joint process; the idea lying behind these hypothesis is that we need to specify a certain relation between the time changes $I^X$ and $I^Y$, whereas there was no need to do so in the univariate case, because all the information needed was contained in the natural definition of the process of crisis using the Poisson point process and a sequence of i.i.d. random variables, independent of the Poisson process as well. In fact, in the bivariate case, the assumptions we need to make are more difficult to formalize; on the other hand they are a bit more general, and we expect them to be satisfied by any process we want to use to model the crisis of a financial index.

These hypothesis are:

1. We need a first hypothesis to ensure a certain kind of time invariance for the bivariate process. What we will actually need is that

$$Cov(|X_{s+h} - X_s|, |Y_{t+h} - Y_t|) = Cov(|X_h - X_0|, |Y_{t-s+h} - Y_{t-s}|)$$

and later we will see that a sufficient condition for this is the stationarity of the increments of the joint process of crisis $(I^X, I^Y)$. This is reasonable assumption on the process of crisis, and it was satisfied in the univariate case because of the properties of the Poisson process $\mathcal{T}$. In the bivariate case we will more formally request the following property. Denote with $\delta_{(\cdot)}$ the Dirac function; for any $I = [a, b]$, consider the sequences of the couples $(\tau_k^X - a, \sigma_k^X)$ and $(\tau_k^Y - a, \sigma_k^Y)$ for $\tau_k^X$ and $\tau_k^Y$ varying in $I$; for any $a$, $b \geq 0$ we define the function $Z_{[a,b]}$ as

$$Z_{[a,b]} := \left( \sum_{k:\tau_k^X \in [a,b]} \delta_{(\tau_k^X - a, \sigma_k^X)}, \sum_{k:\tau_k^Y \in [a,b]} \delta_{(\tau_k^Y - a, \sigma_k^Y)} \right),$$

a function of $\mathcal{M}^2([0, a-b], \mathbb{R})$. We are going to assume that the distribution of the random variable $Z_{[a,b]}$ depend just on the time difference $a - b$, and do not depend on $a$, $b$ itself.

2. Given a Borel set $I \subset \mathbb{R}$, we let $\mathcal{G}_I^X$ denote the $\sigma$-algebra generated by the family of random variables

$$\left( \tau_k^X \mathbf{1}_{\{\tau_k^X \in I\}}, \sigma_k^X \mathbf{1}_{\{\tau_k^X \in I\}} \right)_{k \geq 0}.$$

Informally, $\mathcal{G}_I^X$ may be viewed as the $\sigma$-algebra generated by the variables $\tau_k^X$, $\sigma_k^X$ for the values of $k$ such that $\tau_k^X \in I$. Analogously we define $\mathcal{G}_I^Y$. We then define $\mathcal{G}_I$ as the generated $\sigma$-algebra $\mathcal{G}_I^X \vee \mathcal{G}_I^Y$. We suppose that for disjoint time Borel sets $I$, $I'$, the $\sigma$-algebras $\mathcal{G}_I$

and $\mathcal{G}_{I'}$ are independent. In the univariate case this fact was a consequence of the properties of the Poisson process and of the independence of $(\sigma_k)_{k\geq 0}$. In the multivariate case this is not true, and we have to request explicitly this property, because we are in a more general setting. This hypothesis, in any case, is quite natural: we are supposing that what happens to the process of crisis in a certain time segment is not related to what happens to the process of crisis in a disjoint time segment (a weaker version of the loss of memory property).

We will require something more in some examples and when we will have to validate this model on empirical data. In these examples these hypothesis will be satisfied; the verification is left to the reader.

We now can define $X$ and $Y$ as

$$X_t = W^X_{I^X_t}, \quad Y_t = W^Y_{I^Y_t}$$

where the random time changes $I^X_t$ and $I^Y_t$ are defined as in 2.2. The following is an example in which the situation above is realized.

**Example 1.** We can obtain a model where the jumps are different but not independent using the union of Poisson processes:

- $\mathcal{T}^i$, $i = 1, 2, 3$ are independent Poisson point processes with intensities $\lambda_i$, $i = 1, 2, 3$;
- $\mathcal{T}^X = \mathcal{T}^1 \cup \mathcal{T}^2$, $\mathcal{T}^Y = \mathcal{T}^1 \cup \mathcal{T}^3$;
- let $\tau \in \mathcal{T}^2$( resp $\mathcal{T}^3$): the corresponding $\sigma$ is drawn from $\nu^X$ (resp $\nu^Y$), independently of $W^X$, $W^Y$, $\mathcal{T}^1$, $\mathcal{T}^2$, $\mathcal{T}^3$ and of all other $\sigma$s;
- let $\tau \in \mathcal{T}^1$: the corresponding $(\sigma^X, \sigma^Y)$ is drawn from a distribution $\tilde{\nu}$ with marginals $\nu^X$ and $\nu^Y$, independently of $W^X$, $W^Y$, $\mathcal{T}^1$, $\mathcal{T}^2$, $\mathcal{T}^3$ and of all other $\sigma$s.

In modeling financial indexes, correlation between $\nu^X$ and $\nu^Y$ could represent how we expect the reaction to a crisis of indexes $X$ and $Y$ to be related.

This specification of the model is the one we will use for the comparison with the real time series of FTSE and DJIA; in Chapter 4 we will see that this way of taking the process of shocks in the market is supported by empirical observations.

## 3.2 Covariance and correlations of log-returns

For a give time scale h,

$$\xi_t = |X_{t+h} - X_t|, \quad \eta_t = |Y_{t+h} - Y_t|,$$

are the absolute values of the returns of $X$ and $Y$ at time $t$. We are interested in the correlations between these two variables, and we start from computing their covariance. In fact, we are now going to state a result on the asymptotic behavior of the covariance of log-returns as the time scale goes to 0.

**Theorem 4** (Covariance of absolute log-returns)**.** *Let the process* $(X,Y)$ *be defined as above. Then, for any* $t \geq s \geq 0$*, the following holds:*

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{Cov(\xi_s, \eta_t)}{h} = \lim_{h\downarrow 0} \frac{Cov(\xi_0, \eta_{t-s})}{h} = \\ &\frac{4\sqrt{D^X D^Y}}{\pi} Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_0^Y(t-s-\tau_0^Y)^{D^Y-1/2}\right) e^{-\lambda^Y(t-s)} \end{aligned} \tag{3.1}$$

*Proof.* We start the computations on $Cov(\xi_s, \eta_t)$ writing more explicitly the quantities involved:

$$\begin{aligned} &Cov(\xi_s, \eta_t) = E(\xi_s \eta_t) - E\xi_s E\eta_t = \\ &E(|X_{s+h} - X_s||Y_{t+h} - Y_t|) - E|X_{s+h} - X_s|E|Y_{t+h} - Y_t| = \\ &E\left(|W_1^X|\sqrt{I_{s+h}^X - I_s^X}|W_1^Y|\sqrt{I_{t+h}^Y - I_t^Y}\right) - \\ &E\left(|W_1^X|\sqrt{I_{s+h}^X - I_s^X}\right) E\left(|W_1^Y|\sqrt{I_{t+h}^Y - I_t^Y}\right) \end{aligned}$$

We are supposing $W^X$ and $W^Y$ are independent Brownian motions, and both are independent of the time changes, so

$$\begin{aligned} &Cov(\xi_s, \eta_t) = \\ &E(|W_1^X||W_1^Y|)E\left(\sqrt{I_{s+h}^X - I_s^X}\sqrt{I_{t+h}^Y - I_t^Y}\right) - \\ &E|W_1^X|E|W_1^Y|E\left(\sqrt{I_{s+h}^X - I_s^X}\right) E\left(\sqrt{I_{t+h}^Y - I_t^Y}\right) = \\ &E(|W_1|)^2 Cov\left(\sqrt{I_{s+h}^X - I_s^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right) = \\ &\tfrac{2}{\pi} Cov\left(\sqrt{I_{s+h}^X - I_s^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right) \end{aligned}$$

Hypothesis 1 implies the stationarity of the increments of $(I^X, I^Y)$, therefore

$$Cov\left(\sqrt{I_{s+h}^X - I_s^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right) = Cov\left(\sqrt{I_h^X}, \sqrt{I_{t-s+h}^Y - I_{t-s}^Y}\right)$$

and to prove our result it is enough to compute $Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)$ for $t \geq 0$. We have that the covariance of the absolute values of the returns actually depends just on $Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)$, where $t$ is the time difference; the next step is to understand the behavior of this covariance as $h \downarrow 0$.

$$I_h = \sigma_{i(h)}^2 (h - \tau_{i(h)})^{2D} + \sum_{k=1}^{i(h)} \sigma_{k-1}^2 (\tau_k - \tau_{k-1})^{2D} - \sigma_0^2(-\tau_0)^{2D}$$

Almost surely, for $h$ small enough, $i(h) = i(0) = 0$, because $i(\cdot)$ is an increasing function,

$$\{0 < i(h), \forall h > 0\} = \{\tau_1 = 0\}$$

and $P(\tau_1 > 0) = 1$; so the sum in the right hand vanishes and a.s.

$$\lim_{h\downarrow 0} \frac{I_h}{h} = \lim_{h\downarrow 0} \frac{\sigma^2_{i(h)}(h-\tau_{i(h)})^{2D} - \sigma_0^2(-\tau_0)^{2D}}{h} = \lim_{h\downarrow 0} \frac{\sigma_0^2(h-\tau_0)^{2D} - \sigma_0^2(-\tau_0)^{2D}}{h} = 2D\sigma_0^2(-\tau_0)^{2D-1}$$

Analogously, being $i(\cdot)$ increasing and right continuous, and being $P(\tau_{i(t)+1} > 0) = 1$, we have $i(t+h) = i(t)$ for $h$ small enough, a.s.; then

$$\lim_{h\downarrow 0} \frac{I_{t+h} - I_t}{h} = \lim_{h\downarrow 0} \frac{\sigma^2_{i(t+h)}(t+h-\tau_{i(t+h)})^{2D} - \sigma^2_{i(t)}(t-\tau_{i(t)})^{2D}}{h} = \lim_{h\downarrow 0} \frac{\sigma^2_{i(t)}(t+h-\tau_{i(t)})^{2D} - \sigma^2_{i(t)}(t-\tau_{i(t)})^{2D}}{h} = 2D\sigma^2_{i(t)}(t-\tau_{i(t)})^{2D-1}$$

We were interested in the behavior of $Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)$ in the limit as $h$ goes to 0. By bilinearity of covariance,

$$\frac{Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)}{h} = Cov\left(\sqrt{\frac{I_h^X}{h}}, \sqrt{\frac{I_{t+h}^Y - I_t^Y}{h}}\right)$$

and we have just seen the limit behavior for $h \downarrow 0$ of the arguments of the covariance in the right hand side. So the question now is: "Can we take the limit inside the covariance?"

We will prove that this is possible because of the uniform integrability of the families

$$\left\{\frac{I_h^X}{h} : h \in (0,1]\right\}, \quad \left\{\frac{I_{t+h}^Y - I_t^Y}{h} : h \in (0,1]\right\}.$$

(See subsection (4) for proof and details).

Then we have:

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)}{h} = \\
&Cov\left(\lim_{h\downarrow 0}\sqrt{\frac{I_h^X}{h}}, \lim_{h\downarrow 0}\sqrt{\frac{I_{t+h}^Y - I_t^Y}{h}}\right) = \\
&Cov\left(\sqrt{2D^X(\sigma_0^X)^2(-\tau_0^X)^{2D^X-1}}, \sqrt{2D^Y(\sigma^Y_{i^Y(t)})^2(t-\tau^Y_{i^Y(t)})^{2D^Y-1}}\right) = \\
&2\sqrt{D^X D^Y} Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma^Y_{i^Y(t)}(t-\tau^Y_{i^Y(t)})^{D^Y-1/2}\right).
\end{aligned} \tag{3.2}$$

Suppose $t = 0$ in

$$Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_{i^Y(t)}^Y(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\right);$$

a simple substitution leads us to

$$Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_0^Y(-\tau_0^Y)^{D^Y-1/2}\right). \tag{3.3}$$

Now take $t > 0$ in

$$Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_{i^Y(t)}^Y(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\right)$$

We can obtain a better representation of this quantity multiplying the right term in the covariance by the characteristic function of $\{i^Y(t) = 0\}$ plus the characteristic function of its complement:

$$\begin{aligned}
&Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_{i^Y(t)}^Y(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)=0\}}\right) + \\
&Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_{i^Y(t)}^Y(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)>0\}}\right)
\end{aligned}$$

The second covariance is 0 because $\sigma_0^X(-\tau_0^X)^{D^X-1/2}$ is $\mathcal{G}_0^X$ measurable, where $\mathcal{G}_0^X = \sigma(\sigma_0^X, \tau_0^X)$; $\sigma_{i^Y(t)}^Y(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)>0\}}$ is $\mathcal{G}_{>0}^Y$ measurable, where $\mathcal{G}_{>0}^Y = \sigma(\sigma_k^Y, \tau_k^Y : k > 0)$, and $\mathcal{G}_0^X$, $\mathcal{G}_{>0}^Y$ are mutually independent (because of hypothesis 2). So, using the fact that $\mathbf{1}_{\{i^Y(t)=0\}}$ is $\mathcal{G}_{>0}^Y$ measurable, because so is $\mathbf{1}_{\{i^Y(t)>0\}}$, we have

$$\begin{aligned}
&Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_{i^Y(t)}^Y(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\right) = \\
&E\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}\sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)=0\}}\right) \\
&-E\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}\right)E\left(\sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\mathbf{1}_{\{i(t)^Y=0\}}\right) = \\
&E\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}\sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\right)E\left(\mathbf{1}_{\{i^Y(t)=0\}}\right) \\
&-E\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}\right)E\left(\sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\right)E\left(\mathbf{1}_{\{i(t)^Y=0\}}\right) = \\
&Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\right)P(i^Y(t)=0) = \\
&Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\right)e^{-\lambda^Y t}
\end{aligned}$$

Note that for $t = 0$ in the expression above we obtain exactly (3.3). In other words, this expression holds for $t \geq 0$, and our theorem is proved.

□

From this theorem we obtain an asymptotic evaluation for correlations between log-returns, when the time scale goes to 0. Recall that the correlation coefficient between $\xi_0$ and $\eta_t$ is defined as

$$\rho(\xi_0, \eta_t) = \rho(|X_h|, |Y_{t+h} - Y_t|) = \frac{Cov(\xi_0, \eta_t)}{\sqrt{Var(\xi_0)Var(\eta_t)}}$$

and, form [1], the fact that for the kind of process we are studying

$$\lim_{h\downarrow 0}\frac{1}{h}Var(|X_{t+h}-X_t|)=2D\lambda^{1-2D}Var(\sigma|N|S^{D-1/2}).$$

where $S$ is an exponential variable of parameter 1, $N$ is a standard normal variable and they are mutually independent and independent from all the random variables involved.

**Corollary 1** (Decay of cross-asset correlations)**.** *For the process* $(X,Y)$ *defined above, for any* $t\geq s\geq 0$*, the following expression holds as* $h\downarrow 0$*:*

$$\lim_{h\downarrow 0}\rho(\xi_s,\eta_t)=\lim_{h\downarrow 0}\rho(\xi_0,\eta_{t-s})=$$
$$\frac{2}{\pi}\frac{e^{-\lambda^Y(t-s)}}{(\lambda^X)^{1/2-D^X}(\lambda^Y)^{1/2-D^Y}}\frac{Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2},\sigma_0^Y(t-s-\tau_0^Y)^{D^Y-1/2}\right)}{\sqrt{Var(\sigma^X|N|S^{D^X-1/2})Var(\sigma^Y|N|S^{D^Y-1/2})}} \tag{3.4}$$

*where with* $S$ *we denote an exponential variable of parameter* 1 *and with* $N$ *a standard normal variable, and they are mutually independent and both independent from all the other random variables.*

*Proof.* It is sufficient to substitute in the definition of the correlation coefficient the expression 3.1 for the covariance and use the already known result on variance, and simplify some terms.

□

This general formula can be simplified if we make other assumptions on the process. We can go back to Example 1 and make some more explicit computations.

**Example 2.** In this particular case we can obtain a numerical evaluation of

$$Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2},\sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\right)e^{-\lambda^Y t}$$

simulating $J_i$, $i=1,2,3$, exponential variables of parameters $\lambda_i$, $i=1,2,3$, taking $-\tau_0^X=\min(J_1,J_2)$, $-\tau_0^Y=\min(J_1,J_3)$, $\sigma_0^X$ and $\sigma_0^Y$ drawn as specified above. We can also rewrite it as

$$\left(\lambda^X\right)^{1/2-D^X}\left(\lambda^Y\right)^{1/2-D^Y}Cov\left(\sigma_0^X(S^X)^{D^X-1/2},\sigma_0^Y(\lambda^Y t+S^Y)^{D^Y-1/2}\right)e^{-\lambda^Y t}$$

where $S^X=\min\{S^{1,X},S^2\}$, $S^Y=\min\{S^{1,Y},S^3\}$, and $S^{\cdot}$ are exponential variables with parameters:
$S^{1,X}\sim\frac{\lambda_1}{\lambda_1+\lambda_2}$, $S^2\sim\frac{\lambda_2}{\lambda_1+\lambda_2}$, $S^{1,Y}\sim\frac{\lambda_1}{\lambda_1+\lambda_3}$, $S^3\sim\frac{\lambda_3}{\lambda_1+\lambda_3}$
and $S^{1,X}(\lambda_1+\lambda_3)=S^{1,Y}(\lambda_1+\lambda_2)$.
Note that $S^X$ and $S^Y$ are correlated exponential variables of parameter 1,

and that in this case $\lambda^X = \lambda_1 + \lambda_2$ and $\lambda^Y = \lambda_1 + \lambda_3$, so we can simplify the expression and obtain

$$\lim_{h\downarrow 0} \rho(\xi_0, \eta_t) =$$

$$\frac{2}{\pi} \frac{Cov\left(\sigma^X (S^X)^{D^X-1/2}, \sigma^Y (\lambda^Y t + S^Y)^{D^Y-1/2}\right)}{\sqrt{Var(\sigma^X |N| S^{D^X-1/2}) Var(\sigma^Y |N| S^{D^Y-1/2})}} e^{-\lambda^Y t} \tag{3.5}$$

In Chapter 4 we will use this specific expression to model the decay of correlations between FTSE and DJIA. Because our estimate of the volatilities $\sigma^X$ and $\sigma^Y$ will be in both cases as constant variables, we will actually use the expression,

$$\frac{2}{\pi} \frac{Cov\left((S^X)^{D^X-1/2}, (\lambda^Y t + S^Y)^{D^Y-1/2}\right)}{\sqrt{Var(|N| S^{D^X-1/2}) Var(|N| S^{D^Y-1/2})}} e^{-\lambda^Y t} \tag{3.6}$$

and so it is not necessary to make more hypothesis on the joint distribution $\tilde{\nu}$ of $\sigma^X$ and $\sigma^Y$ in the common shocks.

*Remark* 2. The method used above to derive a formula for the covariances of two different processes can be used also to derive the covariance at different times for a single process, $X = W_{I_t}$. Indeed, taking $X = Y$, we can repeat the same deductions with the only difference that

$$\begin{aligned}
&Cov(|X_h|, |X_{t+h} - X_t|) = Cov(\xi_0, \xi_t) = \\
&E(|W_1||\widehat{W_1}|) E\left(\sqrt{I_h}\sqrt{I_{t+h} - I_t}\right) - \\
&E|W_1| E|\widehat{W_1}| E\left(\sqrt{I_h}\right) E\left(\sqrt{I_{t+h} - I_t}\right) = \\
&E(|W_1|)^2 Cov\left(\sqrt{I_h}, \sqrt{I_{t+h} - I_t}\right) = \tfrac{2}{\pi} Cov\left(\sqrt{I_h}, \sqrt{I_{t+h} - I_t}\right)
\end{aligned}$$

not because $W^X$ and $W^Y$ are independent Brownian motions, being $W^X = W^Y =: W$, but because $W_1$ and $\widehat{W_1}$ come from disjoint time intervals, if we suppose $t > h$, and the Brownian motion $W$ has independent increments. So, proceeding as in the case $X \neq Y$, we obtain the formula

$$\lim_{h\downarrow 0} \frac{Cov\left(\sqrt{I_h}, \sqrt{I_{t+h} - I_t}\right)}{h} = Cov\left(\sigma_0 (-\tau_0)^{D-1/2}, \sigma_{i(t)} (t - \tau_{i(t)})^{D-1/2}\right)$$

Now the only meaningful case is $t > 0$, and multipling the right term in the covariance by the characteristic function of $\{i(t) = 0\}$ plus the characteristic function of its complement we get, trough analogous computations,

$$Cov\left(\sigma_0 (-\tau_0)^{D-1/2}, \sigma_0 (t - \tau_0)^{D-1/2}\right) e^{-\lambda t}$$

and

$$\frac{2}{\pi} \frac{Cov\left(\sigma S^{D-1/2}, \sigma(\lambda t + S)^{D-1/2}\right) e^{-\lambda t}}{Var(\sigma |N| S^{D-1/2})},$$

the result of Theorem 3, which is proved in [1].

*Remark* 3. Consider now the following hypothesis on X and Y:

- $D^X = D^Y =: D$
- $\tau^X = \tau^Y =: \tau$
- $\sigma^X = \sigma^Y =: \sigma$
- $W^X$ independent from $W^Y$, and both independent from $D$, $\tau$, $\sigma$

We are in the hypothesis of Corollary 1; in this case $I^X = I^Y =: I$, and (3.4) becomes

$$\lim_{h\downarrow 0} \rho(\xi_0, \eta_t) = \frac{2}{\pi} \frac{Cov\left(\sigma S^{D-1/2}, \sigma(\lambda t + S)^{D-1/2}\right) e^{-\lambda t}}{Var(\sigma|N|S^{D-1/2})},$$

which is exactly the same expression for the autocorrelation coefficients of Remark 2. We actually got it trough analogous computations, with the only difference that the independence of the normal variables comes from independence of Brownian motions in the case of two different processes, from the independence of increments of a single Brownian motion in the case of a single process. This fact is quite surprising but an analysis of real data (comparison between the DJIA and the FTSE 100 indexes) suggests that this property is very close to what we see in financial markets. In any case, also if we consider $I^X = I^Y$ an excessive hypothesis on the two processes, because it would mean to modify the two Brownian motions with exactly the same time change, an empirical data analysis on the "shocks" of the DJIA and FTSE 100 indexes suggests that $I^X$ and $I^Y$ should be very strongly related, having in particular many common jumps. We will discuss in detail the agreement of these results with empirical data in Sections 4.2 and 4.3.

## 3.3 Proof of uniform integrability

This is the most technical section of this chapter, and it is devoted to prove that

$$\lim_{h\downarrow 0} \frac{Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)}{h} = Cov\left(\lim_{h\downarrow 0} \sqrt{\frac{I_h^X}{h}}, \lim_{h\downarrow 0} \sqrt{\frac{I_{t+h}^Y - I_t^Y}{h}}\right).$$

This will conclude the proof of the statements above.
Let's write it in a formal

**Proposition 2.** *Consider a stochastic process*

$$I_t = \sigma_{i(t)}^2 (t - \tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 (\tau_k - \tau_{k-1})^{2D} - \sigma_0^2 (-\tau_0)^{2D}$$

*where*

- *$D \in (0, 1/2]$*
- *$\mathcal{T} = (\tau_n)_{n\in\mathbb{Z}}$ is a Poisson Process on $\mathbb{R}$ with intensity $\lambda$;*
- *$\Sigma = (\sigma_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. positive random variables with marginal law $\nu$;*
- *$i(t) = \sup\{n \geq 0 : \tau_n \leq t\}$.*

*Let $I^X$ and $I^Y$ be two stochastic processes defined this way, depending on $D^X, \lambda^X, \nu^X$ and $D^Y, \lambda^Y, \nu^Y$. Than at any time $t \geq 0$,*

$$\lim_{h\downarrow 0} Cov\left(\sqrt{\frac{I^X_h}{h}}, \sqrt{\frac{I^Y_{t+h} - I^Y_t}{h}}\right) = Cov\left(\lim_{h\downarrow 0}\sqrt{\frac{I^X_h}{h}}, \lim_{h\downarrow 0}\sqrt{\frac{I^Y_{t+h} - I^Y_t}{h}}\right).$$

This will follow from the fact that

$$\left\{\frac{I_h}{h} : h \in (0, 1]\right\}$$

is uniformly bounded in $L^\delta$ for some $\delta > 1$.

$$\frac{I_h}{h}, \quad \frac{I_{t+h} - I_t}{h}$$

have the same distribution, because of time invariance, then also the latter family for fixed $t$, $h \in (0, 1]$, is uniformly bounded in $L^\delta$ for some $\delta > 1$. This does not depend on the parameters of the process, so it'll be true both for $I^X$ and $I^Y$.

$$\left(\sqrt{\frac{I^X_h}{h}}\sqrt{\frac{I^Y_{t+h}}{h}}\right)^\delta \leq \left(\frac{1}{2}\right)^\delta \left(\frac{I^X_h}{h} + \frac{I^Y_{t+h}}{h}\right)^\delta \leq \frac{1}{2}\left(\left(\frac{I^X_h}{h}\right)^\delta + \left(\frac{I^Y_{t+h}}{h}\right)^\delta\right),$$

where the second inequality comes from

$$(a + b)^\delta \geq 2^{\delta-1}(a^\delta + b^\delta), \quad \forall a, b \geq 0.$$

So also the expectation of quantity on the left hand side is uniformly bounded.

$$\sqrt{\frac{I^X_h}{h}}\sqrt{\frac{I^Y_{t+h}}{h}}$$

converges a.s. (see section 1), and it is a uniformly bounded in $L^\delta$, so it is also a uniformly integrable family for $h \leq 1$. a.s. convergence of a uniformly integrable family implies $L^1$ convergence (all of this is explained in

[2], Chapter 13).

$$\sqrt{\frac{I_h^X}{h}}, \quad \sqrt{\frac{I_{t+h}^Y}{h}}$$

are uniformly integrable for $h \leq 1$, because they are uniformly bounded families in $L^{2\delta}$, and so they converge $L^1$ for the same reasons. Then

$$\begin{aligned}
&\lim_{h\downarrow 0} Cov\left(\sqrt{\tfrac{I_h^X}{h}}, \sqrt{\tfrac{I_{t+h}^Y - I_t^Y}{h}}\right) = \\
&\lim_{h\downarrow 0} E\left(\sqrt{\tfrac{I_h^X}{h}}\sqrt{\tfrac{I_{t+h}^Y - I_t^Y}{h}}\right) + \lim_{h\downarrow 0} E\left(\sqrt{\tfrac{I_h^X}{h}}\right) \lim_{h\downarrow 0} E\left(\sqrt{\tfrac{I_{t+h}^Y - I_t^Y}{h}}\right) = \\
&E\left(\lim_{h\downarrow 0}\sqrt{\tfrac{I_h^X}{h}}\sqrt{\tfrac{I_{t+h}^Y - I_t^Y}{h}}\right) + E\left(\lim_{h\downarrow 0}\sqrt{\tfrac{I_h^X}{h}}\right) E\left(\lim_{h\downarrow 0}\sqrt{\tfrac{I_{t+h}^Y - I_t^Y}{h}}\right) = \\
&Cov\left(\lim_{h\downarrow 0}\sqrt{\tfrac{I_h^X}{h}}, \lim_{h\downarrow 0}\sqrt{\tfrac{I_{t+h}^Y - I_t^Y}{h}}\right).
\end{aligned}$$

The only step we still need is now the following lemma, and then the proof we'll be completed.

**Lemma 1.** *The class of random variables*

$$\left\{\frac{I_h^X}{h} : h \in (0,1]\right\}$$

*is uniformly integrable.*

*Proof.* This means that for given $\epsilon > 0$, there exist $K \in [0, \infty)$ such that

$$E(|I_h^X/h|; |I_h^X/h| > K) < \epsilon, \quad \forall h \in (0, h_0]$$

A sufficient condition for the Uniform Integrability property is the boundedness of the family $\left\{\frac{I_h^X}{h} : h \in (0, h_0]\right\}$ in $L^p$ for some $p > 1$. See [2] for details.

$$I_t = \sigma_{i(t)}^2 (t - \tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 (\tau_k - \tau_{k-1})^{2D} - \sigma_0^2(-\tau_0)^{2D}$$

Let's estimate $E(I_t^\delta)$

$$E(I_t^\delta) = E(I_t^\delta | i(t) = 0) P(i(t) = 0) + \sum_{k=1}^{\infty} E(I_t^\delta | i(t) = k) P(i(t) = k)$$

Conditioning on $i(t) = 0$,

$$I_t = \sigma_0^2 (t - \tau_0)^{2D} - \sigma_0^2(-\tau_0)^{2D} \leq 2D\sigma_0^2(-\tau_0)^{2D-1} t$$

in a right neighborhood of $t = 0$. This is because

- $\sigma_0^2(t-\tau_0)^{2D} - \sigma_0^2(-\tau_0)^{2D}|_{t=0} = 0$
- $\frac{dI_t}{dt}|_{t=0} = 2D\sigma_0^2(-\tau_0)^{2D-1}$
- $\sigma_0^2(t-\tau_0)^{2D} - \sigma_0^2(-\tau_0)^{2D}$ is concave in t.

So

$$E(I_t^\delta|i(t)=0) \leq (2D)^\delta E\left(\sigma_0^{2\delta}\right) E\left((-\tau_0)^{\delta(2D-1)}\right) t^\delta \leq C^0 t^\delta$$

because $-\tau_0$ is an random variable with exponential distribution.
$E\left((-\tau_0)^{\delta(2D-1)}\right)$ is finite if $\delta(2D-1) > -1$. Indeed in $x^{\delta(2D-1)}e^{-\lambda x}$ is integrable for $x \to \infty$, and also for $x \downarrow 0$ if $\delta(2D-1) > -1$.
Because of the choice of $D$ we know that $-1 < (2D-1) < 0$; therefore choosing $\delta > 1$ but small enough, the condition $\delta(2D-1) > -1$ will be satisfied.

Conditioning on $i(t) = k$, $k \geq 1$

$$\begin{aligned}
I_t &= \sigma_k^2(t-\tau_k)^{2D} + \textstyle\sum_{j=1}^k \sigma_{j-1}^2(\tau_j - \tau_{j-1})^{2D} - \sigma_0^2(-\tau_0)^{2D} \\
&= \sigma_k^2(t-\tau_k)^{2D} + \textstyle\sum_{j=2}^k \sigma_{j-1}^2(\tau_j - \tau_{j-1})^{2D} + \sigma_0^2(\tau_1-\tau_0)^{2D} - \sigma_0^2(-\tau_0)^{2D} \\
&\leq \sigma_k^2(t-\tau_k)^{2D} + \textstyle\sum_{j=2}^k \sigma_{j-1}^2(\tau_j - \tau_{j-1})^{2D} + \sigma_0^2(t-\tau_0)^{2D} - \sigma_0^2(-\tau_0)^{2D} \\
&\leq \sigma_k^2(t-\tau_k)^{2D} + \textstyle\sum_{j=2}^k \sigma_{j-1}^2(\tau_j - \tau_{j-1})^{2D} + 2D\sigma_0^2(-\tau_0)^{2D-1}t
\end{aligned}$$

and, using Jensen inequality and the fact that $2D < 1$,

$$\begin{aligned}
&\sigma_k^2(t-\tau_k)^{2D} + \textstyle\sum_{j=2}^k \sigma_{j-1}^2(\tau_j - \tau_{j-1})^{2D} \leq k\left(\frac{\sigma_k^{\frac{1}{D}}(t-\tau_k) + \sum_{j=2}^k \sigma_{j-1}^{\frac{1}{D}}(\tau_j-\tau_{j-1})}{k}\right)^{2D} \\
&\leq k \max_{j=1:k}(\sigma_j^2)\left(\frac{(t-\tau_k)+\sum_{j=2}^k(\tau_j-\tau_{j-1})}{k}\right)^{2D} \leq \max_{j=1:k}(\sigma_j^2)\, k\left(\tfrac{t}{k}\right)^{2D}
\end{aligned}$$

Then

$$I_t \leq \max_{l=0:k}(\sigma_l^2)\left(2D(-\tau_0)^{2D-1}t + k\left(\frac{t}{k}\right)^{2D}\right)$$

and

$$\begin{aligned}
&E(I_t^\delta|i(t)=k)P(i(t)=k) \\
&\leq E\left[\max_{l=0:k}(\sigma_l^2)^\delta\left(2D(-\tau_0)^{2D-1}t + k\left(\tfrac{t}{k}\right)^{2D}\right)^\delta\right]\frac{(\lambda t)^k}{k!} \\
&\leq E\left[\max_{l=0:k}(\sigma_l^{2\delta})\right]2^{\delta-1}\left((2D)^\delta E\left[(-\tau_0)^{(2D-1)\delta}\right]t^\delta\frac{(\lambda t)^k}{k!} + k^\delta\left(\tfrac{t}{k}\right)^{2D\delta}\frac{(\lambda t)^k}{k!}\right)
\end{aligned}$$

We can estimate the maximum of a finite number of positive random variables with their sum

$$\max_{l=0:k}(\sigma_l^{2\delta}) \leq \sum_{l=0}^k \sigma_l^{2\delta}$$

and then pass to the expected values

$$E\left[\max_{l=0:k}(\sigma_l^{2\delta})\right] \leq E\left[\sum_{l=0}^{k}(\sigma_l^{2\delta})\right] \leq (k+1)E\left[\sigma^{2\delta}\right] =: f(k)$$

We are proving uniform integrability for $t$ positive but small, so let's suppose $t \leq 1$. For suitable positive constants $C_1$ and $C_2$,

$$\begin{aligned} &E(I_t^\delta|i(t)=k)P(i(t)=k) \leq f(k)C_1\tfrac{\lambda^k}{k!}t^\delta + f(k)C_2k^\delta\left(\tfrac{t}{k}\right)^{2D\delta}\tfrac{(\lambda t)^k}{k!} \\ &\leq f(k)C_1\tfrac{\lambda^k}{k!}t^\delta + f(k)C_2k^{\delta(1-2D)}\tfrac{\lambda^k}{k!}t^{1+2D\delta} \end{aligned}$$

$\delta$ is such that $\delta < 1+2D\delta$; indeed this inequality is equivalent to $\delta(2D-1) > -1$, and we needed our $\delta$ to satisfy this inequality in a previous step. Therefore $t^{1+2D\delta} \leq t^\delta$, and then

$$E(I_t^\delta|i(t)=k)P(i(t)=k) \leq f(k)\left(C_1 + C_2k^{\delta(1-2D)}\right)\frac{\lambda^k}{k!}t^\delta$$

For a suitable $\delta > 1$, for any $t \in (0,1]$

$$E(I_t^\delta) \leq \left[C_0 + \sum_{k=1}^{\infty} f(k)C_3k^{\delta(1-2D)}\frac{\lambda^k}{k!}\right]t^\delta \leq C_4t^\delta$$

where $C_3$ and $C_4$ are positive constants. This is because $f(k) = (k+1)E\left[\sigma^{2\delta}\right]$, and we are supposing $\sigma^{2\delta}$ integrable ($E(\sigma^q) < \infty$ for every $q > 0$, see proposition 14); therefore $\sum_{k=1}^{\infty} f(k)C_3k^{\delta(1-2D)}\frac{\lambda^k}{k!}$ is convergent, and so

$$E(I_t^\delta) \leq C_4t^\delta$$

We have now proved that $\left\{\frac{I_t^X}{t} : t \in (0,1]\right\} \subset L^\delta$, for some $\delta > 1$. So this family of random variables is uniformly integrable and the lemma has been proved.

□

# Chapter 4

# Model and real data

In [1], Section 3 is entirely devoted to a numerical comparison between the model and the time serie of the daily opening prices of the Down Jones Industrial Average (DJIA) from Jan 2, 1935 to Dec 21, 2009, that is 18849 daily data from a period of 75 years. In this thesis we want to focus our attention to cross-asset correlations, so we need at least two indexes. We will analyze the DJIA and the FTSE 100 (Financial Times Stock Exchange) time series, again daily opening prices but from Apr 02, 1984 to Apr 04, 2011, because of the shorter length of FTSE. So we focus on a period of 27 years, for 6822 daily data.

We have used the software MATLAB for data analysis, simulations and plots contained in this pages. The DJIA and FTSE time series can be downloaded at `http://finance.yahoo.com`.

## 4.1 Estimation of parameters

The first problem we have to face in comparing our model with real data is the estimation of the parameters: $D \in (0, 1/2]$, $\lambda \in (0, \infty)$, and the probability distribution $\nu$, the marginal law of $\Sigma = (\sigma_n)_{n \in \mathbb{N}}$. We denote by $\sigma$ a variable such that $\sigma \sim \nu$. In principle this belongs to an infinite dimensional space, so for now we focus on the moments $E(\sigma)$ and $E(\sigma^2)$. Recall that we can obtain from our data a realization of $\sigma$ each time we have a jump in the Poisson process. We'll see from our estimate of the rate $\lambda$ that we can expect $\sim O(10)$ jumps from our empirical series, so it is pointless to give a more precise description of $\sigma$ than the one given by the first and second moment, with a so small sample size.

We want to find the values for these parameters such that the predictions of the model are the closest to what we see in real data. For this purpose, we have to find some significant quantities (taking into account interesting features related to stylized facts), and focus on them for this comparison. We'll then perform a least square minimization. So we consider:

- the multiscaling coefficients $C_1$ and $C_2$;
- the multiscaling exponent $A(q)$;
- the volatility autocorrelation function $\rho(t)$.

Our empirical data are two discrete time series of length $N = 6822$. In the beginning we consider the two series separately. We fix now some notation for FTSE index time series, but the same notation could represent any discrete time series. So the description of the procedure we are giving will work for DJIA as well.

$s = (s_i)_{0 \leq i \leq N}$ is a series of empirical data (price of FTSE index);

$x = (x_i)_{0 \leq i \leq N}$ is the series of detrended log-prices; more precisely

$$x_i = log(s_i) - \overline{d}(i), \text{ where } \overline{d}(i) = \frac{1}{250} \sum_{k=i-250}^{i-1} \log(s_i)$$

$\overline{d}(i)$ is the local rate of linear growth, obtained as the mean log-price on the previous 250 days. Other reasonable choices for $d(i)$ are possible, but they affect the analysis only in a minor way.

### 4.1.1 Computing significant quantities

In this paragraph we see how to compute $C_1$, $C_2$, $A(q)$ and $\rho(t)$ from every choice of the parameters $D$, $\lambda$, $E(\sigma)$ and $E(\sigma^2)$, and how to evaluate the corresponding empirical quantities. From (2.10) we know that

$$A(q) = \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} \frac{q}{2} & if\, q \leq q^* \\ Dq + 1 & if\, q \geq q^* \end{cases} \tag{4.1}$$

therefore we can obtain $A$ very easily (it actually depends just on $D$).

An expression for the multiscaling constants $C_1$ and $C_2$ can be obtained as well, from Theorem 2 for $q = 1$ and $q = 2$. Since $q^* = (1/2 - D)^{-1} > 2$ (because $0 \leq D \leq 1/2$), $q < q^*$ and we have

$$\begin{aligned} C_1 &= \frac{2\sqrt{D}\Gamma(1/2 + D)E(\sigma)\lambda^{1/2-D}}{\sqrt{\pi}} \\ C_2 &= 2D\,\Gamma(2D)E(\sigma^2)\lambda^{1-2D}. \end{aligned} \tag{4.2}$$

We want now to define the corresponding empirical quantities; it requires some care because we have series in discrete time (daily sample), and the theoretical quantities above are defined for $h \downarrow 0$, where $h$ is the time scale of the return. We first evaluate the empirical moment of order $q$ over $h$ days:

$$m_q(h) = \frac{1}{N-h} \sum_{i=1}^{N-h} |x_{i+h} - x_i|^q$$

By 2.9, the relation $\log \widehat{m}_q(h) \sim A(q) \log h + \log C_q$ should hold for $h$ small. Indeed, plotting $(\log \widehat{m}_q(h))$ versus $(\log h)$ one finds an approximately linear behavior for $h$ and $q$ positive and small ($h = 1, 2, 3, 4, 5$ and $0 < q \lesssim 5$), whereas linearity is gradually lost as $h$ or $q$ increases. Motivated by this observation, for every fixed $q \in \{0.25, 0.5, 0.75, ..., 5\}$, we estimate $A(q)$ and $C_q$ by a standard linear regression of $(\log \widehat{m}_q(h))$ versus $(\log h)$, for $h = 1, 2, 3, 4, 5$. We call the values we have found $\widehat{A}(q)$ and $\widehat{C}_q$.

For what concerns volatility autocorrelation, that is

$$\rho(t) = Corr(|X_h|, |X_{t+h} - X_t|),$$

we have an explicit representation for the limit as $h \downarrow 0$ of theoretical values given by Theorem 3:

$$\lim_{h \downarrow 0} \rho(t) = \frac{2}{\pi} \frac{Cov\left(\sigma S^{D-1/2}, \sigma(\lambda t + S)^{D-1/2}\right) e^{-\lambda t}}{Var(\sigma |N| S^{D-1/2})} \tag{4.3}$$

where $S$ is an exponential variable with parameter 1, $N$ is a standard normal variable, they are mutually independent and independent of $\sigma$. Although the expression is not explicit, it can be easily evaluated numerically (this part actually will take a quite long time to be done when we'll have to perform the numerical minimization). Also note that $\rho(t)$ depends on $D$, $\lambda$ and just the first and second moment of $\sigma$, $E(\sigma)$ and $E(\sigma^2)$. For our procedure, this is necessary since we limit our description of the distribution $\sigma$ to the two parameters $E(\sigma)$ and $E(\sigma^2)$, and from these we have to compute $A$, $C_1$, $C_2$ and $\rho$.

The analogous empirical quantity is the sample correlation coefficient for absolute returns over $h$ days:

$$\rho_h(t) = Corr(|x_{\cdot+h} - x_\cdot|, |x_{\cdot+t+h} - x_{\cdot+t}|).$$

We cannot take the limit for $h \downarrow 0$, so we are going to compare $\rho(t)$ with $\widehat{\rho}_h(t)$ for $h = 1$ day.

### 4.1.2 Loss functional

At this stage we have to define a *loss functional* which represents the distance between empirical and theoretical quantities, and we do it in the (weighted) least squares sense. We'll then minimize it to find the "best" choice for $D$, $\lambda$, $E(\sigma)$, $E(\sigma^2)$.

$$\begin{aligned}\mathcal{L}\left(D,\lambda,E(\sigma),E(\sigma^2)\right) &= \frac{1}{2}\left[\left(\frac{\widehat{C}_1}{C_1}-1\right)^2+\left(\frac{\widehat{C}_2}{C_2}-1\right)^2\right]+\\ \frac{1}{20}\sum_{k=1}^{20}\left(\frac{\widehat{A}(k/4)}{A(k/4)}-1\right)^2 &+ \sum_{n=1}^{400}\frac{e^{-n/T}}{\sum_{m=1}^{400}e^{-m/T}}\left(\frac{\widehat{\rho}(n)}{\rho_1(n)}-1\right)^2\end{aligned} \tag{4.4}$$

*Remark* 4. It is important at this stage to point out that, although the the algorithm for estimation of parameters is rigorous, and the loss function we use is precisely defined and comes from a clear motivation, the precise quantities returned by the algorithm (our estimates) are subject to a certain level of arbitrariness. For instance, in determining the empirical quantities $A$ and $C$, we have performed a linear regression for $q \leq 5$. This is for sure a reasonable choice, but others were possible as well. We also have to chose the weights in (4.4), and what we did is not the only reasonable choice. In particular we have to choose the constant $T$, which controls a discount factor for the correlations (bigger $T$ means bigger weights for long term correlations). In this thesis, after various numerical experiments, we have decided to stick on $T = 200$, whereas in [1] the choice was $T = 40$. This clearly depends on the importance one gives to different quantities to estimate, and it would be interesting to formalize such considerations in a mathematical way, but we are going to focus on other issues in this work. In any case, we do not think that the attempt to give very precise estimates could be very fruitful, being the set of empirical data too small for this purpose. E.g., the estimate of $\lambda$ and $\sigma$ substantially depends on the realizations of the Poisson process; after many numerical minimizations, with different parameters and different methods, we can assert that the order of $\lambda$ is for sure $O(10^{-3})$. This implies that in our set of around 7000 daily sampled empirical data, we should have $\sim 10$ shocks in the market, and therefore the same number of realizations of $\sigma$. Clearly, there is no hope to get a precise estimate for $\lambda$ and $\sigma$ from this very small sample; therefore we have to keep in mind that only rough estimates are possible.

We can now define the estimator $\left(\overline{D},\overline{\lambda},\overline{E(\sigma)},\overline{E(\sigma^2)}\right)$ of the parameters

$$D\in(0,1/2],\quad \lambda\in(0,\infty),\quad E(\sigma)\in(0,\infty),\quad E(\sigma^2)\in(E(\sigma)^2,\infty)$$

as the point of the domain of feasible solutions which minimizes $\mathcal{L}$:

$$\left(\overline{D},\overline{\lambda},\overline{E(\sigma)},\overline{E(\sigma^2)}\right) = \arg\min \mathcal{L}\left(D,\lambda,E(\sigma),E(\sigma^2)\right)$$

We have to add the constraint $E(\sigma^2) \in (E(\sigma)^2,\infty)$ because for every random variable $Var(\sigma) = E(\sigma^2) - E(\sigma)^2 \geq 0$, so this condition is necessary if we want our minimizer to make sense for our definition of the model.

### 4.1.3 Results

A numerical minimization (using MATLAB), together with analytical considerations on the local behavior of the functional lead us to the following estimates for the parameters for the FTSE series:

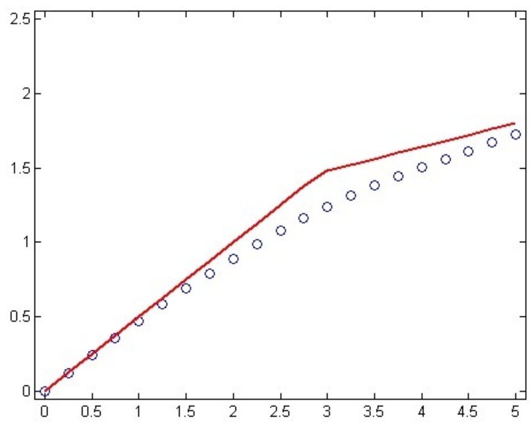
$$\overline{D} \approx 0.16; \quad \overline{\lambda} \approx 0.0018; \quad \overline{\sigma} \approx 0.11\,(constant); \tag{4.5}$$

We estimate the random variable $\sigma$ with a constant because with sigma possibly random we don't find a value for $\mathcal{L}$ significantly lower than $\mathcal{L}\left(\overline{D}, \overline{\lambda}, \overline{\sigma}, \overline{\sigma}^2\right)$, so the estimated variance of $\sigma$ is 0.

With this parameters, the significant quantities we have considered before are the following:

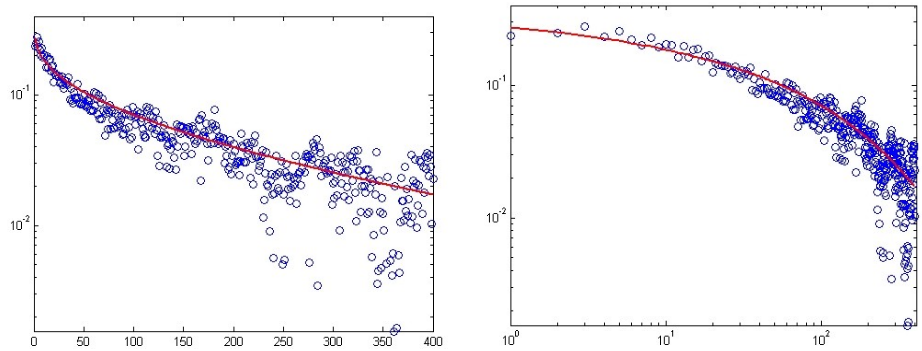
- for what concerns the multiscaling constants we find $\overline{C}_1 = 0.00791$, $\overline{C}_2 = 1.47 * 10^{-4}$, whereas the empirical values are $\widehat{C}_1 = 0.00829$, $\widehat{C}_2 = 1.44 * 10^{-4}$;
- the theoretical multiscaling exponent $\overline{A}(q)$ is represented by the red continuous line, the empirical multiscaling exponent $\widehat{A}(q)$ by the blue dotted line in Figure 4.1
- the theoretical volatility autocorrelation function $\overline{\rho}(t)$ is represented by the red continuous line, the empirical volatility autocorrelation function $\widehat{\rho}_1(q)$ by the blue dots line in Figure 4.2.

Figure 4.1: FTSE multiscaling exponent




Our estimate for the multiscaling exponent looks smoothed out by the empirical curve. The decay of volatility autocorrelation is between polynomial and exponential, and fits very well empirical data considering the fact that they are quite widespread. We can observe a excellent agreement for

Figure 4.2: FTSE volatility autocorrelation




(a) log plot (b) loglog plot

what concerns the volatility autocorrelation, a slightly worse fitting for the multiscaling exponent. Different choices of the weights would have given the opposite situation, but some heuristic considerations (i.e. the stability of the estimates) lead us to stick to this choice. In any case, we can conclude that there is a good agreement between data and model.

We now switch to the daily opening prices DJIA series from from Apr 02, 1984 to Apr 04, 2011. As we said before, we are interested in a comparison between the two indexes and in their correlation, but before doing that we have to analyze this series as we did for the FTSE. So we apply again the procedure described above, setting $s = (s_i)_{0\leq i\leq N}$ equal to the DJIA series, and then going on in analogous way. We obtain the following estimates for the DJIA series:

$$\overline{D} \approx 0.14; \quad \overline{\lambda} \approx 0.0013; \quad \overline{\sigma} \approx 0.135; \tag{4.6}$$

because, again, we don't find a variance significantly different from 0 for $\sigma$.

- for what concerns the multiscaling constants we estimate $\overline{C}_1 = 0.00732$, $\overline{C}_2 = 1.37*10^{-4}$, whereas the empirical values are $\widehat{C}_1 = 0.00758$, $\widehat{C}_2 = 1.30*10^{-4}$;
- the theoretical multiscaling exponent $\overline{A}(q)$ is represented by the yellow continuous line, the empirical multiscaling exponent $\widehat{A}(q)$ by the black dots in Figure 4.3;
- the theoretical volatility autocorrelation function $\overline{\rho}(t)$ is represented by the yellow continuous line, the empirical volatility autocorrelation function $\widehat{\rho}_1(q)$ by the black dots in Figure 4.4.

Again, the prediction of the model is very close to real data.

We are now going to compare the theoretical distribution of log returns for our model $p_t(\cdot) = P(X_t \in \cdot) = P(X_{n+t} - X_n \in \cdot)$ for $t = 1$ day,

Figure 4.3: DJIA multiscaling exponent

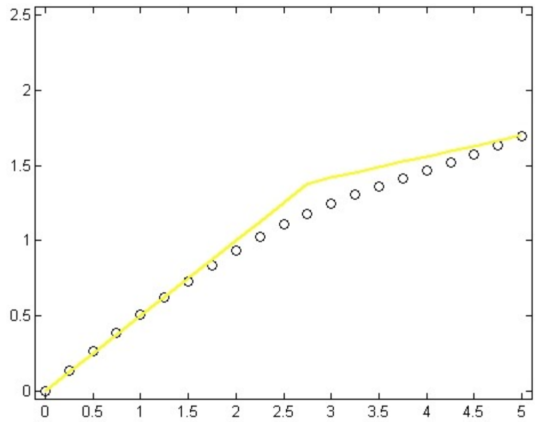
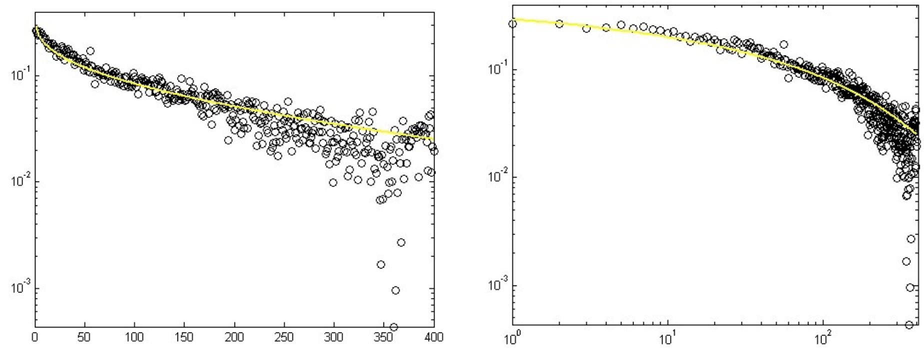



Figure 4.4: DJIA volatility autocorrelation




(a) log plot (b) loglog plot

with the analogous empirical quantity. We do not have an explicit analytic expression for $p_t$, but we can easily obtain it via Monte Carlo simulations. From empirical and simulated data, we get the empirical distribution $\widehat{p}_t$ and the distribution predicted by our model $\overline{p}_t$ by:

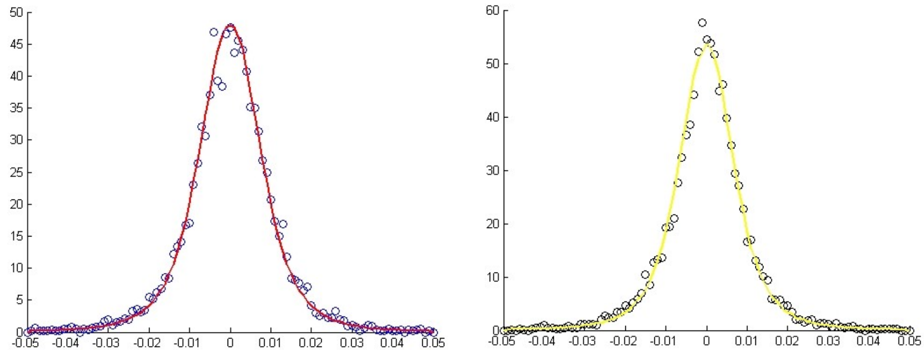
$$p_t = \frac{1}{N+1-t} \sum_{i=0}^{N-t} \delta_{x_{i+t}-x_i}(\cdot)$$

where we denote by $\delta$ the Dirac function. Figure (4.5) represent the bulks of the distributions (densities in the range [-0.05,0.05]). As before, the blue dotted line is the empirical FTSE distribution, the red continuous line is the theoretical FTSE distribution, the black dotted line is the empirical DJIA distribution, the yellow continuous line is the theoretical DJIA distribution. In Figure (4.6) we plot the integrated tails of $p(\cdot)$, that is the functions

$z \longrightarrow P(X_1 > z)$ (right tail) and $z \longrightarrow P(X_1 < -z)$ (left tail).

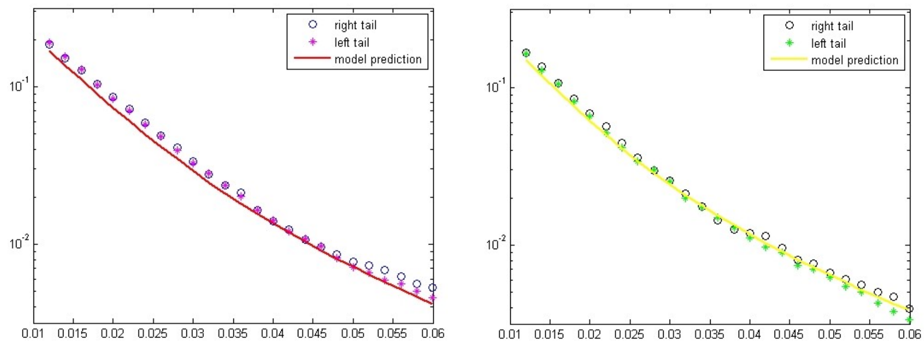
The agreement between theoretical and empirical distribution is remarkably good in both cases, especially if one considers that this curves are a tests *a posteriori*, and no parameter has been estimated using these curves.

Figure 4.5: Distributions of log returns



(a) FTSE (b) DJIA

Figure 4.6: Integrated tails




(a) FTSE (b) DJIA

## 4.2 Finding past shocks

Suppose again $s$ is the time series we are modeling; $s = (s_i)_{0 \le i \le N}$ are our empirical data. If we have reasons to think that the model explained above provides a good description of the behavior of the real process, it is interesting to have an estimate of the past realizations of the Poisson process. In other words, if we suppose our series is sampled from a process in continuous time, and we denote with 0 the time of the first and with $N$ the time of the last observation, we would like to have an estimate of $\tau_k$, $k = 0, ..., i(N)$.

This can be an important point when checking the possible cross-asset correlations of two such processes; we have seen in chapter 3 that, in the most general case of two processes

$$X_t = W^X_{I^X_t}, \quad Y_t = W^Y_{I^Y_t}$$

where we suppose the independence of the Brownian Motions $W^X$ and $W^Y$, without additional hypothesis on the other sources of alea, (3.4) gives us the expression

$$\lim_{h\downarrow 0} \rho(\xi_0, \eta_t) =$$

$$\frac{2}{\pi} \frac{e^{-\lambda^Y t}}{(\lambda^X)^{1/2-D^X}(\lambda^Y)^{1/2-D^Y}} \frac{Cov\left(\sigma_0^X(-\tau_0^X)^{D^X-1/2}, \sigma_0^Y(t-\tau_0^Y)^{D^Y-1/2}\right)}{\sqrt{Var(\sigma^X|N|S^{D^X-1/2})Var(\sigma^Y|N|S^{D^Y-1/2})}}$$

With stronger hypothesis on the processes, especially on $\mathcal{T}^X$ and $\mathcal{T}^Y$, the expression can be more explicit. Therefore, it is interesting to estimate realizations of jumps in the past, in order to make some considerations on their relation and on the possible properties we are going to assume for jump times of FTSE and DJIA. In the following section we give an algorithm which does this job, and use it for estimates of shock times of the real series we are studying.

### 4.2.1 Heuristic motivation

The idea for our algorithm to find past $\tau$s comes from property 2 in Proposition 1, which asserts that the process $X$ can be represented as a stochastic volatility process:

$$dX_t = v_t dB_t \tag{4.7}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. $B$ and $v = (v_t)_{t\geq 0}$ are defined by

$$\begin{aligned} B_t &:= \int_0^{I_t} \frac{1}{\sqrt{I'(I^{-1(u)})}} dW_u \\ v_t &:= \sqrt{I'(t)} = \sqrt{2D}\sigma_{i(t)}(t-\tau_{i(t)})^{D-\frac{1}{2}} \end{aligned} \tag{4.8}$$

where $I'(s) = \frac{dI}{ds}(s)$. For $D < 1/2$, the volatility $v_t$ has singularities (peaks) at the random times $\tau_n$. Motivated by this observation, we expect some kind of empirical variance to attain its maximum at shock times. Recall that our process could be defined by

$$X_t = W_{I(t)}$$

where

$$I_t = \sigma^2_{i(t)}(t-\tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} \sigma^2_{k-1}(\tau_k - \tau_{k-1})^{2D} - \sigma^2_0(-\tau_0)^{2D}$$

Our data are in discrete time; for a fixed time $T$, we consider the empirical mean variance in the last $t$ days

$$\widehat{V}(t) = \widehat{V}_T(t) := \frac{1}{t} \sum_{s=T-t}^{T} (x_{s+1} - x_s)^2. \tag{4.9}$$

If $t$ has a suitable magnitude ($\sim O(1000)$), we can think that the value of $\widehat{V}(t)$ is close to its expected value, conditioned to a realization of $\mathcal{T}$ and $\Sigma$; that is, we are averaging just on the Brownian motion $W$. In fact, the mean time distance between shock times is $\sim O(1000)$, so arithmetic average on a time length of the same order can be taken as an approximation of an average for fixed shock times, at the same time being 1000 increments sufficient to average the Brownian motion. Recall that conditioning on $\mathcal{T}$ and $\Sigma$ is the same as conditioning on the process $I$. So let's see what would be the correspondent theoretical quantity:

$$E(V(t)|\mathcal{T},\Sigma) = E\left(\frac{1}{t}\sum_{s=T-t}^{T}(X_{s+1}-X_s)^2\middle|\,\mathcal{T},\Sigma\right)$$
$$\frac{1}{t}\sum_{s=T-t}^{T} E\left((X_{s+1}-X_s)^2\middle|\,\mathcal{T},\Sigma\right) = \frac{1}{t}\sum_{s=T-t}^{T} E\left((W_{I_{s+1}}-W_{I_s})^2\middle|\,\mathcal{T},\Sigma\right) \tag{4.10}$$

For a fixed realization of $I$, $W_{I_{s+1}} - W_{I_s} \sim N(0, I_{s+1} - I_s)$; we get

$$E\left((W_{I_{s+1}}-W_{I_s})^2\middle|\,\mathcal{T},\Sigma\right) = I_{s+1} - I_s$$

Therefore

$$E(V(t)|\mathcal{T},\Sigma) = \frac{I_{T+1} - I_{T-t}}{t} \approx \frac{I_T - I_{T-t}}{t}.$$

From this considerations and the following proposition we see the motivation lying behind our algorithm.

**Proposition 3.** *Let $I = (I_t)_{t\geq 0}$ the random time change of our process, i.e.*

$$I_t = \sigma_{i(t)}^2 (t - \tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 (\tau_k - \tau_{k-1})^{2D} - \sigma_0^2(-\tau_0)^{2D}$$

*where $\mathcal{T} = (\tau_n)_{n\in\mathbb{Z}}$ is a Poisson point process and $\Sigma = (\sigma_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. positive random variables. Fix a time $T > 0$, and suppose $M$ is a positive constant bigger than $T - \tau_{i(T)}$. Then, conditioning on a realization of $\mathcal{T}$ and $\Sigma$, for $t$ varying in $(0, M)$, one of these two conditions is always realized:*

1. *$F_T(t) = \frac{I_T - I_{T-t}}{t}$ attains its maximum at $m := T - \tau_{i(T)}$*

2. *for $v \in (0, M-m)$ denote with $\Delta\tau(v)$ the distance between the points of the Poisson process immediately before and after $\tau_{i(T)} - v$. Then $\exists\, v \in (0, M-m)$ such that $m \geq C(\Sigma)\Delta\tau(v)$, where $C(\Sigma)$ is a constant depending on the realizations of $\sigma_{i(T-M)}, ..., \sigma_{i(T)}$.*

This lead us to think that $\widehat{V}_T(t)$ is likely to attain its maximum at $t = m$ (or very close to it), at least for $m$ not to big, i.e. for $T$ not too far from $\tau_{i(T)}$, the last shock before $T$. Indeed, for $t$ not to small,

$$\widehat{V}(t) \sim E(V(t)|\mathcal{T}, \Sigma) = F_T(t),$$

and the proposition above states that $F_T(t)$ attains its maximum at $m$ for $T$ close enough to $\tau_{i(T)}$. The fact that maxima in $t$ of $\widehat{V}_T(t)$, for different $T$s, are attained many times at the same point $\tilde{n}$, gives us an indication of the presence of a realization of $\mathcal{T}$ in $\tilde{n}$. Our algorithm computes $\widehat{V}_T(t)$ for $t$ running from $\sim 20$ to $\sim 2000$, so it is very unlikely to find a very big distance between jumps that could hide the last realization of $\mathcal{T}$ (recall this distance is distributed as an exponential variable). We repeat this procedure for $T = N$, $T = N-1$, $T = N-2$ and so on, finding each time the maximum point for $t$ running from $\sim 20$ to $\sim 2000$ of $\widehat{V}_T(t)$. In the end, we check which days were the most frequent maxima. Not surprisingly, as seen from Figure 4.7, the maxima are very concentrated in few days; therefore, we take those days as the estimates for $\tau_k$, $k = 0, ..., i(N)$.

Here we give a proof of the proposition, and in the next section a detailed description of the algorithm.

*Proof.* Consider $F_T(t) = \frac{I_T - I_{T-t}}{t}$. Define $m := T - \tau_{i(T)}$. For $t \in (0, m)$, $F_T(\cdot) = \sigma^2_{i(T)}((T - \tau_{i(T)})^{2D} - (T - t - \tau_{i(t)})^{2D})$ is increasing:

$$F'_T(t) = \sigma^2_{i(T)} \frac{2D(T - t - \tau_{i(t)})^{2D-1} t - (I_T - I_{T-t})}{t^2} > 0$$

because $I_s$ is strictly concave as a function of $s$ when it does not cross jump times, and therefore $I_s - I_r < \left.\frac{dI_t}{dt}\right|_{t=r} (s - r)$. So the maximum point of $F_T(\cdot)$ for $t \in (0, m]$ is $m$.

We want now some conditions under which

$$\forall\, u \in (m, M), \quad F_T(u) < F_T(m).$$

It is worthy at this stage to consider $F$ as a function of two variables, $F(T, t)$.

$$\begin{aligned} F(T,u) < F(T,m) \quad &\Leftrightarrow \quad \frac{I_T - I_{T-u}}{u} < \frac{I_T - I_{T-m}}{m} \\ \Leftrightarrow \quad & uI_{T-m} - mI_{T-u} < I_T(u-m) \\ \Leftrightarrow \quad & m(I_{T-m} - I_{T-u}) < (I_T - I_{T-m})(u-m) \\ \Leftrightarrow \quad & F(T-m, u-m) < F(T,m) \end{aligned} \tag{4.11}$$

We have chosen $m$ such that

$$F_T(m) = \frac{I_T - I_{\tau_{i(T)}}}{m} = \frac{\sigma^2_{i(T)} m^{2D}}{m}$$

whereas $F(T-m, u-m) = F(\tau_{i(T)}, u-m)$ and $u \in (m, M)$; so we are now looking for conditions under which

$$F(\tau_{i(T)}, v) < F(T, m) \quad \forall\, v \in (0, M-m).$$

Suppose that this is not true, i.e.

$$F(\tau_{i(T)}, v) \geq F(T, m) \quad \exists\, v \in (0, M-m).$$

Using Jensen inequality, defining $l := \#A$, where

$$A := \{n \in \mathbb{N} : \tau_{i(T)} - v < \tau_n < \tau_{i(T)}\}, \quad \tilde{A} = A \cup \{i(\tau_{i(T)} - v)\}$$

we get

$$\begin{aligned}
&F(\tau_{i(T)}, v) = \frac{I_{\tau_{i(T)}} - I_{\tau_{i(T)}-v}}{v} \leq \\
&\frac{1}{v} \sum_{k \in A} \sigma_k^2 (\tau_{k+1} - \tau_k)^{2D} + \frac{\sigma^2_{i(\tau_{i(T)}-v)} \left( I_{\tau_{i(\tau_{i(T)}-v)+1}} - I_{\tau_{i(T)}-v} \right)}{v} \leq \\
&\frac{1}{v} \max_{k \in A} \left(\sigma_k^2\right) \left[ \sum_{k \in A} (\tau_{k+1} - \tau_k)^{2D} \right] + \sigma^2_{i(\tau_{i(T)}-v)} (\tau_{i(\tau_{i(T)}-v)+1} - \tau_{i(\tau_{i(T)}-v)})^{2D-1} \leq \\
&\frac{1}{v} \max_{k \in A} \left(\sigma_k^2\right) l \left[ \frac{\sum_{k \in A} (\tau_{k+1} - \tau_k)}{l} \right]^{2D} + \sigma^2_{i(\tau_{i(T)}-v)} (\Delta\tau(v))^{2D-1} \leq \\
&2 \max_{k \in \tilde{A}} \left(\sigma_k^2\right) \max \left[ \left(\frac{l}{v}\right), \frac{1}{\Delta\tau(v)} \right]^{1-2D}
\end{aligned} \tag{4.12}$$

where we have defined

$$\Delta\tau(v) := \tau_{i(\tau_{i(T)}-v)+1} - \tau_{i(\tau_{i(T)}-v)}$$

that is, $\Delta\tau(v)$ is the distance between the points of the Poisson process before and after $\tau_{i(T)} - v$. Recall that $l/v$ is the mean distance between points of $\mathcal{T}$ between $(\tau_{i(T)})$ and $\tau_{i(T)} - v$. For every $v \in (0, M-m)$, we can find $\hat{v} \in (0, M-m)$ such that $l/v \leq \Delta\tau(\hat{v})$. Therefore, a sufficient condition to be

$$F_T(u) < F_T(m), \quad \forall\, u \in (m, M)$$

is that, $\forall\, v \in (0, M-m)$,

$$2 \max_{k \in \tilde{A}} \left(\sigma_k^2\right) \max \left[ \left(\frac{l}{v}\right), \frac{1}{\Delta\tau(v)} \right]^{1-2D} < \sigma^2_{i(T)} m^{2D-1}$$

or (equivalent)

$$m < C(\sigma)\Delta\tau(v), \quad \forall\, v \in (0, M - m)$$

where $C(\Sigma)$ is a constant depending on the realizations of $\sigma_{i(T-M)}, ..., \sigma_{i(T)}$. In our cases, with $\sigma$ constant variable, $C(\Sigma) = 2^{1/(2D-1)}$. □

### 4.2.2 The algorithm

In this subsection we describe the algorithm we use to estimate the shock times in the past from $s = (s_i)_{0\leq i\leq N}$, the discrete time series of empirical data. We can think of it as a sample of a continuous time process, and of the (discrete) shock times we will find as an approximation of real shocks happened in the past. $x = (x_i)_{0\leq i\leq N}$ is the series of detrended log-prices; the algorithm starts from $x_N$ and goes back in the past, computing for every time index $i$ the mean empirical variance of the increments in the last $i$ days. We do it for $i$ in a time window of $l = 2000$ days, that is for $i = 1, ..., l = 2000$, and then we look for the index $I$ at which the maximum of the variance is attained (the maximum mean variance in the last $l$ days.) This is an indication of a peak in the variance at time $N - I$, or at least close to $N - I$. To avoid possible noise-effects due to big increments in the last few days, when looking for the maximum we ignore the last 22 days ($i = 1, ..., 22$). Indeed, for a short period fluctuations due to Brownian motion may be too large.

At this point we shift one day back in time, and we repeat the procedure. If the maximum is attained at the same day as before, we'll have a stronger indication that that day a shock in the market happened, in our modelization of the stochastic process. We shift again one day back, and so on until we reach the beginning in time of our empirical series.

We now choose which days are the best estimate for past shocks. If for some days the maximum of variance has been attained many times, we have good reasons to consider that day a shock time. In fact, what happens doing this analysis is that maxima of variance concentrate on a small set of days, supporting the validity of the method for the estimation. When doing this, we often observe two or more very close days at which maxima are realized many times (close means less than 20 days of distance). This, more than being due to very close shocks, is likely to be a consequence of the discrepancy between the model and the real process; in fact, in real markets, shocks and increments of volatility are often a consequence of an information (or a set of informations) coming from outside or inside the market. We try to model this feature with a discontinuity of the volatility at a fixed time, but in reality this phenomenon could be spread in more days, or influencing different assets composing the aggregated index at slightly different times.

We give below the MatLab code which implements the algorithm and some results.

### 4.2.3 MatLab implementation and results

*Function empirical variance*:

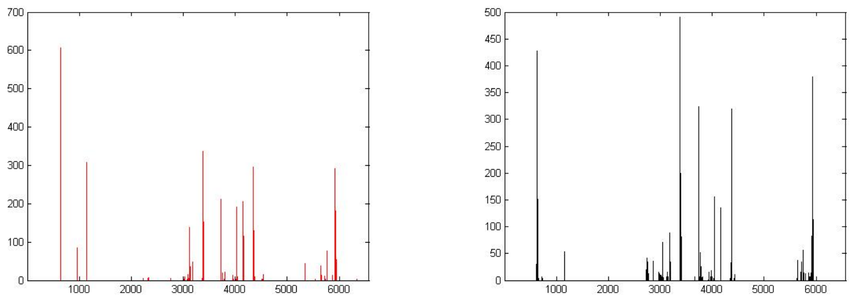
```
%input: vector x of empirical data supposed
%time increasing
%
%output: vector y, where y(i) is the mean
%empirical variance of x in the last i days

function y = emp_var(x)
  y = [];
  y(1) = (x(end)-x(end-1))^2;
  for i = 2:length(x)-1
    y(i) = (y(i-1)*(i-1)+(x(end+1-i)-x(end-i))^2)/i;
  end
end
```

*Function returning the days of realization of the maxima*:

```
%input: vector x of empirical data supposed
%time increasing
%
%output: vector z where z(i) is the likely
%distance from N of the first shock time before i
%
%l=length of the time window where we are
%looking for the peak in the variance
%ignore=number of days we skip from T to
%avoid local maxima due to recent fluctuations

function z = tau_finder(x)

  l=2000;
  ignore=22;

  for i = 1:length(x)-l
    v = emp_var(x(end+1-l-i:end+1-i));
    [C,I] = max(v(ignore+1:end));
    y(i) = I+ignore;
    z(i) = i+y(i);
  end

end
```

The output of the function *tau finder* is a vector $z$ where $z(i)$ is the likely distance from $N$ of the first shock time before $i$. To have the estimated time indexes of past $\tau$s is sufficient now to choose the most frequent values for $N - z(i)$ (you can use the function *mode*). In Figure 4.7 you can see the graphical evidence that maxima are concentrated on a small set of days for both FTSE and DJIA, supporting the validity of the method.

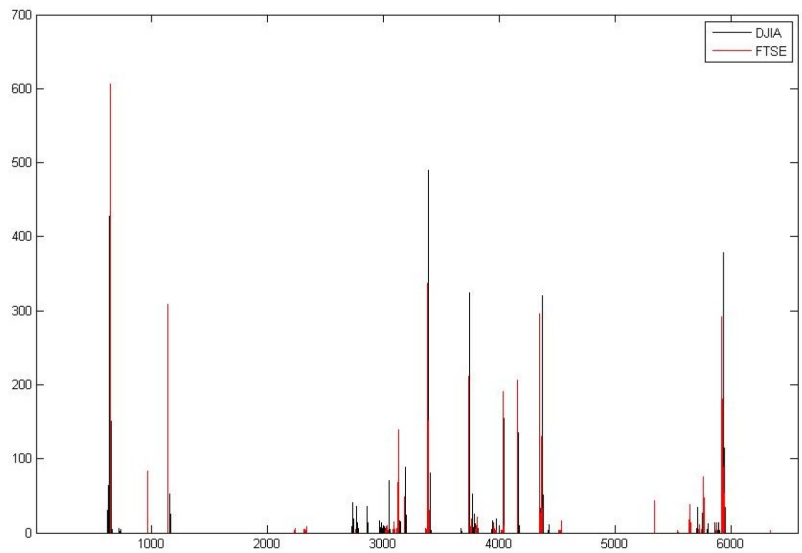
Figure 4.7: Shock times; x-axis: increasing time index; y-axis: y(i)=number of times the maximum variance is realized at i




(a) FTSE shock times (b) DJIA shock times

It is natural at this point to wonder if there is a relation between the shocks in the two indexes, and a straightforward experiment is to try to superimpose the two graphics (see Figure 4.8). What we get is a clear indication that the shock times of the two series are almost coincident, only the magnitude (or evidence) being different and having very few shocks which are present just in one of the two indexes.

Figure 4.8: Common jumps: overlap of Figures 4.7 (a) and (b)




In table 4.1 we show our estimated dates of shock times. To decide which dates to choose, we have to decide how many realizations of the maximum variance at a certain day (or close to it) are necessary to decide to consider that date a "crisis". Our choice is the following, it is based on heuristic considerations and, as we already pointed out in Remark 4, we do not believe that considering this issue from a much more formal point of view could be very fruitful.

FTSE: we have considered an indication of a shock in the market the realization of 84 or more maxima of the variance. We do not have considered an indication in this direction the realization of 21 or less maxima.
In between these values, there are just two dates left, for which we will consider questionable the presence of a shock; they are:
the 22/10/97, with 48 occurrences;
the 12/05/07, with 43 occurrences.

DJIA: we have considered an indication of a shock in the market the realization of 84 or more maxima of the variance. We do not have considered an indication in this direction the realization of 25 or less maxima.
In between these values, there are just three dates left, for which we will consider questionable the presence of a shock; they are:
the 24/07/07, with 57 occurrences;
the 01/07/96, with 54 occurrences;
the 11/10/89, with 53 occurrences.

In table 4.1 we have also marked with the word "sparse" crisis for which the days of realization of maxima of the variance are more widespread (on an interval of $\sim 50$ days).

## 4.3 Application to cross-asset correlation

In this section we are going to apply results and considerations presented in the previous chapters to the cross asset-correlations of DJIA and FTSE. We are going to compare empirical quantities with predictions of our model, and we will show some surprising results.

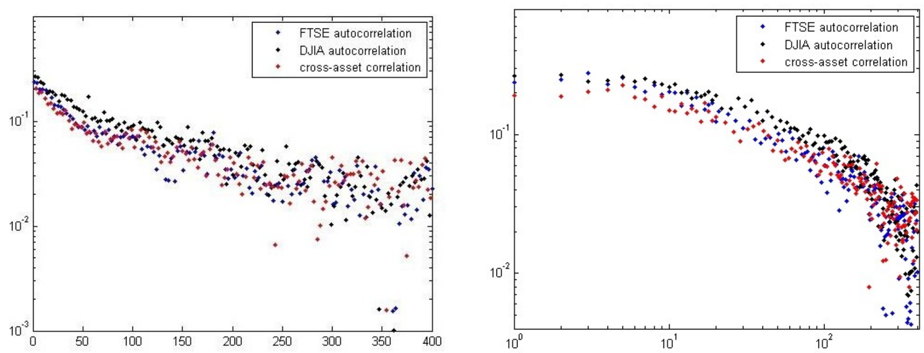
We have seen above that our estimates for jumps in FTSE and DJIA are strictly related. The occurrence of a jump for an index comes very often together with the occurrence of a jump for the other index, with a difference of a few days (less than one month).

As a consequence, a first idea to try a rough modelization of cross asset correlations is to suppose $\mathcal{T}^f$, jump process for FTSE, and $\mathcal{T}^d$, jump process for DJIA, to be the same process. But if this is true from Remark 3, and from the fact that $D$ and $\sigma$ are very similar for FTSE and DJIA, we would expect the decay of volatility autocorrelation in the DJIA, the decay of volatility autocorrelation in the FTSE and the decay of cross-asset correlation of absolute returns to display a similar behavior. In fact, this is exactly what happens if we plot these quantities (see Figure 4.9); to our knowledge, this property had never been detected before.

Table 4.1: Estimated dates of shock times

| *FTSE* | *DJIA* |
|---|---|
| 14/10/87 | 15/09/87 |
| 24/01/89 | |
| 26/09/89 | 11/10/89 (questionable, 53) |
| | 09/01/96 |
| | 01/07/96 (questionable, 54) |
| | 13/03/97 (sparse) |
| 08/08/97 | |
| 22/10/97 (questionable, 48) | 16/10/97 |
| 04/08/98 | 31/07/98 |
| 30/12/99 | 04/01/00 |
| 09/03/01 | 09/03/01 |
| 06/09/01 | 06/09/01 |
| 12/06/02 | 05/07/02 |
| 12/05/07 (questionable, 43) | |
| 24/07/07 | 24/07/07 (questionable, 57) |
| 15/01/08 | 04/01/08 (sparse) |
| 03/09/08 | 15/09/08 |

Figure 4.9: Comparison of empirical correlations




(a) log plot; one point out of three is plotted

(b) loglog plot; for $t \geq 20$, one point out of three is plotted

Under this rough hypothesis our estimate for cross-asset correlations is therefore our prediction for the decay of volatility autocorrelation in FTSE or DJIA, or a mean between the two. But we can do much better with a slightly more precise description, using the specification of the model given in Examples 1 and 2. Jump process for FTSE and jump process for DJIA are almost coincident, but there are some shocks which are present in one of the two processes but not in the other. This let us think that the best way to model the joint process is to take the jumps as $\mathcal{T}^f$, jump process for FTSE, and $\mathcal{T}^d$, jump process for DJIA, where

- $\mathcal{T}^i$, $i = 1, 2, 3$ are independent Poisson point processes with intensities $\lambda_i$, $i = 1, 2, 3$;
- $\mathcal{T}^f = \mathcal{T}^1 \cup \mathcal{T}^2$, $\mathcal{T}^d = \mathcal{T}^1 \cup \mathcal{T}^3$;
- $\sigma^f$ and $\sigma^d$ are constant variables, and so we do not need to specify their joint distribution.

From the previous estimates on the relevant quantities of parameters we have

$$\overline{D}_f \approx 0.16; \quad \overline{\lambda}_f \approx 0.0018; \quad \overline{\sigma}_f \approx 0.11; \tag{4.13}$$

for the FTSE and

$$\overline{D}_d \approx 0.14; \quad \overline{\lambda}_d \approx 0.0013; \quad \overline{\sigma}_d \approx 0.135; \tag{4.14}$$

for the DJIA. So what we have to do is estimate $\lambda_i$, $i = 1, 2, 3$, under the conditions

$$\begin{aligned} &\lambda_1 + \lambda_2 = \overline{\lambda}_f = 0.0018; \\ &\lambda_1 + \lambda_3 = \overline{\lambda}_f = 0.0013; \\ &\lambda_i \geq 0 \, \forall i. \end{aligned} \tag{4.15}$$

Note that because of these constraints there is only one degree of freedom in the set of feasible $\lambda s$, and in fact this set is a segment in $\mathbb{R}^3$.

Define $\widehat{\gamma}_h(t)$ as the empirical correlation coefficient over h days:

$$\widehat{\gamma}_h(t) = Corr(|x^f_{\cdot+h} - x^f_{\cdot}|, |x^d_{\cdot+t+h} - x^d_{\cdot+t}|).$$

where $x^f$ and $x^d$ are the FTSE and DJIA series of detrended log returns.

From (3.6), we know that in this case the explicit limit expression for cross-asset correlations is

$$\gamma(t) = \frac{2}{\pi} \frac{Cov\left(\sigma^f (S^f)^{D^f - 1/2}, \sigma^d (\lambda^d t + S^d)^{D^d - 1/2}\right)}{\sqrt{Var(\sigma^f |N| S^{D^f - 1/2}) Var(\sigma^d |N| S^{D^d - 1/2})}} e^{-\lambda^d t}$$

where $S^f = \min\{S^{1,f}, S^2\}$, $S^d = \min\{S^{1,d}, S^3\}$, and $S^{\cdot}$ are exponential variables with parameters:
$S^{1,f} \sim \frac{\lambda_1}{\lambda_1+\lambda_2}$, $S^2 \sim \frac{\lambda_2}{\lambda_1+\lambda_2}$, $S^{1,d} \sim \frac{\lambda_1}{\lambda_1+\lambda_3}$, $S^3 \sim \frac{\lambda_3}{\lambda_1+\lambda_3}$
and $S^{1,f}(\lambda_1 + \lambda_3) = S^{1,d}(\lambda_1 + \lambda_2)$.

So the only quantities still not determined are $\lambda_i$, $i = 1, 2, 3$. For any choice, we have a theoretical decay of $\gamma(t)$; although we don't have a completely explicit expression, we can easily obtain a numerical evaluation in the same fashion as for the evaluation of the volatility in (4.3). This way we have our estimates for $\lambda_i$, $i = 1, 2, 3$, and we can define a loss functional

$$\mathcal{J}\left(\lambda_1, \lambda_2, \lambda_3\right) = \sum_{n=1}^{400} \frac{e^{-n/T}}{\sum_{m=1}^{400} e^{-m/T}} \left(\frac{\widehat{\gamma}(n)}{\gamma_1(n)} - 1\right)^2 \tag{4.16}$$

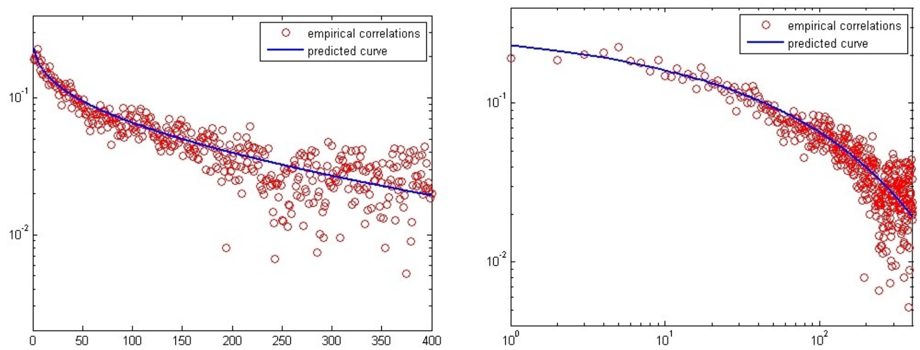
where $T = 40$, and minimize it subject to 4.15.

In this case, the numerical minimization is very easy because the space of feasible solutions is a segment (one dimensional) and we find that at the minimum the values are:

$$\lambda_1 = 0.0012; \quad \lambda_2 = 0.0006; \quad \lambda_3 = 0.0001.$$

In Figure 4.10 we can see a very good agreement of the prediction of our model for this choice of the parameters and the empirical decay for what concerns the cross-asset correlation coefficients, $t = 1, .., 400$ days.

Figure 4.10: FTSE and DJIA cross-asset correlations




(a) log plot (b) loglog plot

As the fact that $\lambda_3$ is that small suggests, an almost indistinguishable plot could be obtained supposing $\lambda_3 = 0$. This means supposing that shocks for DJIA are generated by a Poisson process, and shocks for FTSE are given by the same process plus some additional ones, given by a sparser and independent Poisson process. These estimates, due to the small sample size, are too rough to allow more quantitative considerations. In any case, if we want to see a reason for the situation above, we can suppose that shocks in the DJIA index always determine a shock in the FTSE index, whereas it is possible to see a shock in the FTSE which does not imply a significant increment in the empirical variance of DJIA.

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