An invitation to FK percolation

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Statistical Mechanics and Stochastic PDEs

CIME Summer School

Cetraro, Italy

- **Bernoulli percolation: an appetiser.**
  (Definition, ordering, phase transition, non-triviality of p_c, sharpness)

- Introduction to FK-percolation.
  (Definition, FKG, ordering, infinite volume measures, phase transition)

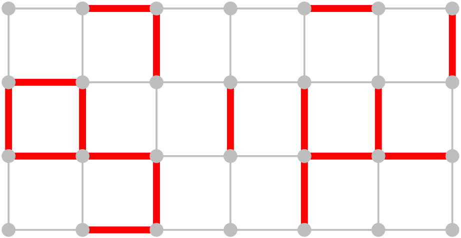
- "Quadrichotomy" theorem.
  (sharpness for 2D FK-perco, types of the phase transition, $p_c = p_{\mathrm{sd}}$)

- Continuity/discontinuity of phase transition via 6V.
  (BKW correspondence, 6V free energy via Bethe ansatz, continuous phase trans. iff $1 \leq q \leq 4$, 6V height funct: localisation/delocalisation)

- Rotational invariance for critical measure ($1 \leq q \leq 4$).
  (FK on isoradial graphs, track exchanges, universality of FK across isoradial graphs, rotational invariance).

Definition

Setting: $\mathbb{Z}^d = (V, E)$; $p \in [0, 1]$.



$\mathbb{P}_p$: edges are **open** with probability p
closed with probability $1 - p$ **independently**.

Random configuration $\omega \in \{0, 1\}^E$.

Question: geometry of **large** connected components (or clusters)

Increasing coupling

Goal: produce samples $\omega_p \sim \mathbb{P}_p$ for all p so that

$$\omega_p(e) \leq \omega_{p'}(e) \quad \text{for all } e \in E \text{ and } p < p'$$

(we say $\omega_p \leq \omega_{p'}$)

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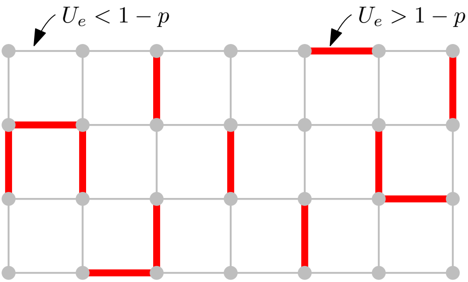

Let $(U_e)_{e \in E}$ be i.i.d. uniforms on $[0, 1]$.

$$\omega_p(e) = \begin{cases} 0 & \text{if } U_e \leq 1 - p \\ 1 & \text{if } U_e > 1 - p, \end{cases} \qquad \text{for all } e \in E \text{ and } p \in [0, 1]$$

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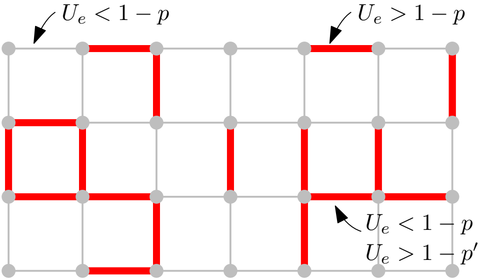

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Phase transition

Question: does there exist an infinite cluster under $\mathbb{P}_p$?

$$\theta(p) := \mathbb{P}_p[0 \text{ is in an infinite cluster}] = \mathbb{P}_p[0 \leftrightarrow \infty].$$

Since $\mathbb{P}_p$ is translation invariant and ergodic

$$\theta(p) = 0 \Rightarrow \mathbb{P}_p[\text{there exists an infinite cluster}] = 0$$
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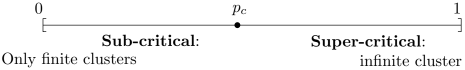
Recall $p \mapsto \theta(p)$ is increasing. Define the *critical point*

$$p_c = p_c(\mathbb{Z}^d) = \sup\{p \in [0,1] : \theta(p) = 0\}.$$

Questions of interest



- Is p_c non-trivial $(0 < p_c < 1)$?

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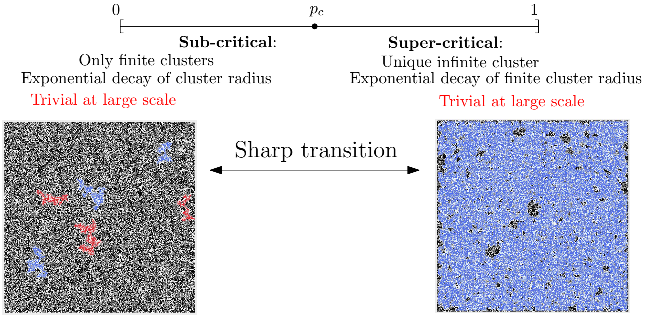

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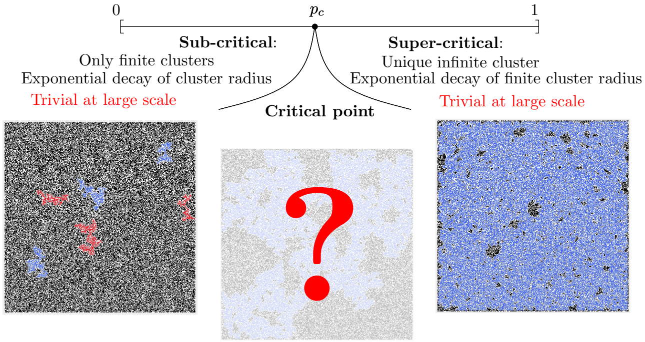

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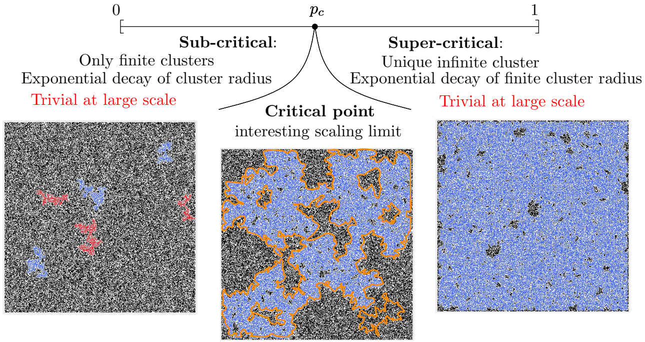

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- Sharpness of phase transition: exponential decay away from p_c.
- Continuity of phase transition (existence of critical phase)
- Behaviour at criticality

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Proof: Peierls' argument

$$\begin{aligned}
\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] &\leq \mathbb{P}_p[\exists \gamma \in A_n \text{ formed only of open edges}] \\
&\leq \mathbb{E}_p[\#\{\gamma \in A_n \text{ formed only of open edges}\}] \\
&= \sum_{\gamma \in A_n} \mathbb{P}_p[\gamma \text{ is formed only of open edges}\}] \\
&= |A_n| \cdot p^n \xrightarrow[n\to\infty]{} 0 \qquad \text{if } p < \tfrac{1}{2d-1},
\end{aligned}$$

where $A_n = \{\text{simple paths of len } n \text{ from } 0\}$; $\quad |A_n| \leq 2d \cdot (2d-1)^{n-1}$.

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$\square$

Remark: for $p < \frac{1}{2d-1}$ we actually have

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$$

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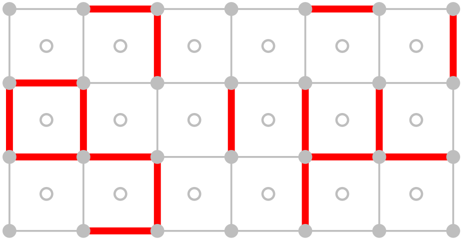

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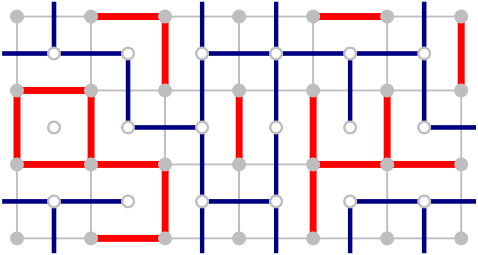

- Primal clusters are surrounded by dual circuits
- If $\omega \sim \mathbb{P}_p$, the dual $\omega^* \sim \mathbb{P}_{1-p}$.

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$$\begin{aligned}
\mathbb{P}_p[\Lambda_r \not\leftrightarrow \infty] &\leq \mathbb{P}_p[\exists \text{ dual circuit around } \Lambda_r \text{ in } \omega^*] \\
&\leq \sum_{k \geq r} \mathbb{P}_p[(k+\tfrac{1}{2}, \tfrac{1}{2}) \text{ connected in } \omega^* \text{ to distance } k] \\
&\leq \sum_{k \geq r} \mathbb{P}_{1-p}[0 \leftrightarrow \partial\Lambda_k] \qquad\qquad \text{if } p > 1 - \tfrac{1}{3} \\
&\leq \sum_{k \geq r} e^{-ck} < 1 \qquad\qquad \text{for } r \text{ sufficiently large}
\end{aligned}$$

Thus $\mathbb{P}_p[\Lambda_r \leftrightarrow \infty] > 0$, so $\theta(p) > 0$. $\square$

Sharp phase transition of Bernoulli percolation on $\mathbb{Z}^d$

Theorem (Menshikov 1986; Aizenman & Barsky 1987):
For all $d \geq 1$ *and all* $p < p_c(\mathbb{Z}^d)$, *there exists* $c > 0$ *such that*

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Theorem (Russo's formula):
For A *an increasing event*

$$\frac{d\mathbb{P}_p[A]}{dp} = \sum_e \mathbb{P}_p\big[\omega \cup \{e\} \in A \text{ but } \omega \setminus \{e\} \notin A\big]$$

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Sharp phase transition: proof (Duminil-Copin, Tassion 2016)

Fix p; set $$\varphi_p(S) := \sum_{x \in \partial S} \mathbb{P}_p\big[0 \xleftrightarrow{S} x\big].$$

(a) exists S with $\varphi_p(S) \leq 1/2$

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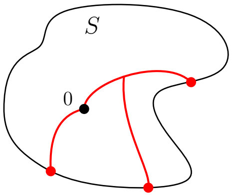

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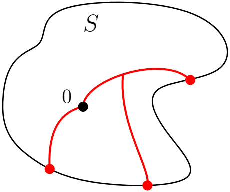

X_1 = points on ∂S connected to 0

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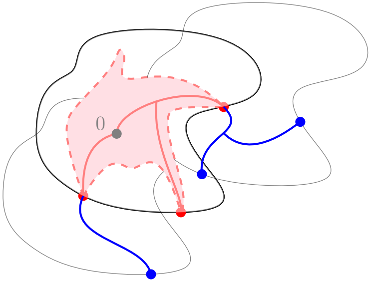

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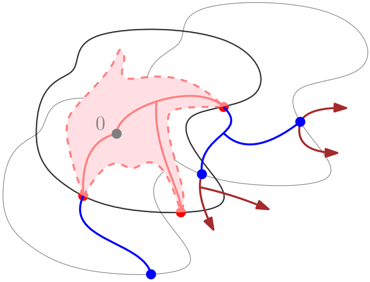

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$(X_k)_k$ subcrit. branching process

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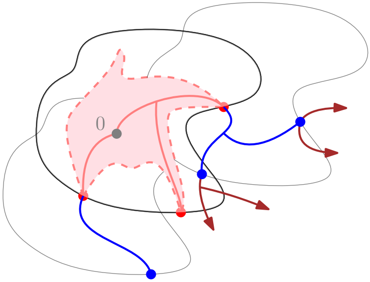

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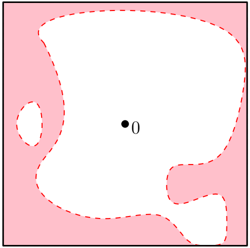
If 0 not connected to $\partial\Lambda_n \ldots$,

$$\mathbb{E}_p\big[\#\text{pivotals} \,\big|\, 0 \not\leftrightarrow \partial\Lambda_n\big] \ge \inf_S \varphi_p(S).$$

$\ldots$ by conditioning on the cluster of $\partial\Lambda_n$.



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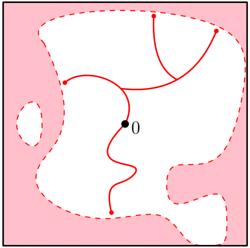

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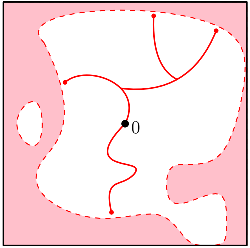

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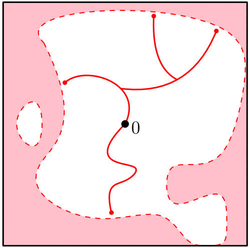

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$$p_c = \sup\Big\{p \in [0,1] \,:\, \inf_S \varphi_p(S) \le 1/2\Big\}.$$

- Bernoulli percolation: an appetiser.
  (Definition, ordering, phase transition, non-triviality of p_c, sharpness)

- **Introduction to FK-percolation.**
  (Definition, FKG, ordering, infinite volume measures, phase transition)

- "Quadrichotomy" theorem.
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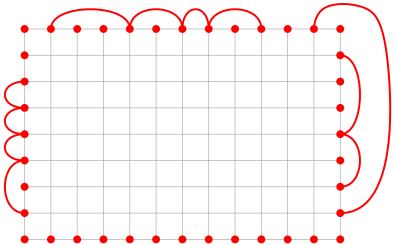
Defintion in finite volume

- **Setting:** G finite subgraph of $\mathbb{Z}^2$; configurations $\omega \in \{0,1\}^{E(G)}$.
- **Boundary:** $\partial G = \{v \in V(G) :$ with neighbour outside of $G\}$;
  boundary conditions ξ = partition of ∂G.
  $\xi = 0$ nothing wired (free b.c.) $\quad$ $\xi = 1$ all wired (wired b.c.).
- **Measure:** $p \in [0,1]$, $q \geq 1$,

$$\phi^{\xi}_{G,p,q}[\omega] = \frac{1}{Z^{\xi}_{G,p,q}} p^{\#\text{open edges}}(1-p)^{\#\text{closed edges}} q^{\#\text{clusters (with } \xi)}$$



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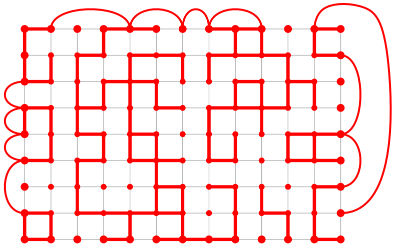

Defintion in finite volume

- **Setting:** G finite subgraph of $\mathbb{Z}^2$; configurations $\omega \in \{0,1\}^{E(G)}$.
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Spatial Markov Property

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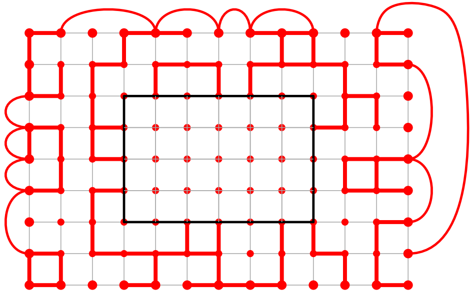

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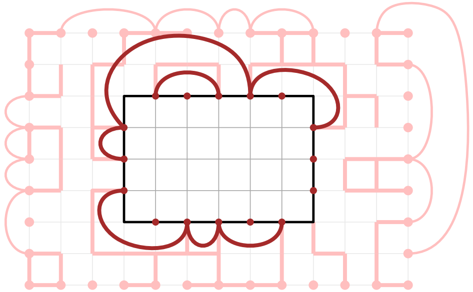

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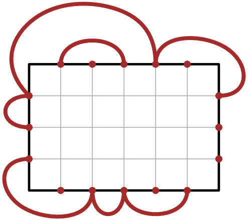

Ordering of measures: generalities

Two defs of **stochastic monotonicity**: measures μ, ν on $\{0,1\}^E$:

(a) $\mu \leq_{\mathrm{st}} \nu$ if they may be coupled increasingly;

(b) $\mu \leq_{\mathrm{st}} \nu$ if $\mu[A] \leq \nu[A]$ for all increasing event A.

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Moreover, it suffices to check conditions for $\omega \Delta \omega' = \{e, f\}$.

Monotonicity properties for FK-percolation $q \geq 1$

- **FKG / positive association:** for A, B increasing events

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- **Bounded by percolation:** $\mathbb{P}_{\frac{p}{p+(1-p)q}} \leq_{\text{st}} \phi^{\xi}_{G,p,q} \leq_{\text{st}} \mathbb{P}_p$

Infinite volume measures; phase transition

- **Free energy:** $f(p,q) = \lim_N \frac{1}{N^d} \log Z^{\xi}_{\Lambda_N,p,q}$.

- **Thermodynamic lim:** $\phi^0_{p,q} = \lim_{n\to\infty} \phi^0_{\Lambda_n,p,q}$ and $\phi^1_{p,q} = \lim_{n\to\infty} \phi^1_{\Lambda_n,p,q}$

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- **Phase transition:** $p_c = p_c(q) = \sup\{p : \phi^0_{p,q}[0 \leftrightarrow \infty] = 0\}$.

  $\phi^1_{p,q}[0 \leftrightarrow \infty] = 0$ for $p < p_c$ $\qquad$ $\phi^0_{p,q}[0 \leftrightarrow \infty] > 0$ for $p > p_c$

An invitation to FK percolation

Ioan Manolescu

University of Fribourg

11-15 Sep 2023
Statistical Mechanics and Stochastic PDEs
CIME Summer School
Cetraro, Italy

Lecture 2

Recall from lecture 1

- G finite subgraph of $\mathbb{Z}^2$; ξ = boundary conditions; $p \in [0,1]$, $q \geq 1$, ξ = partition of ∂G.

$$\phi^{\xi}_{G,p,q}[\omega] = \frac{1}{Z^{\xi}_{G,p,q}} p^{\#\text{open edges}} (1-p)^{\#\text{closed edges}} q^{\#\text{clusters (with } \xi)}$$

- Monotonicity in p and ξ; positive association.
- Infinite volume measures defined by taking limits:

$$\phi^0_{p,q} = \lim_{n\to\infty} \phi^0_{\Lambda_n,p,q} \qquad \text{and} \qquad \phi^1_{p,q} = \lim_{n\to\infty} \phi^1_{\Lambda_n,p,q}$$

Translation invariant, ergodic measures on $\{0,1\}^{E(\mathbb{Z}^2)}$

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$$\phi^1_{p,q}[0 \leftrightarrow \infty] = 0 \text{ for } p < p_c \qquad \phi^0_{p,q}[0 \leftrightarrow \infty] > 0 \text{ for } p > p_c$$

Bound on boundary influence

Proposition: $\phi^1_{p,q}[0 \leftrightarrow \infty] = 0 \implies \phi^0_{p,q} = \phi^1_{p,q}$

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Proof: Explore the cluster of the boundary of Λ_N to show that

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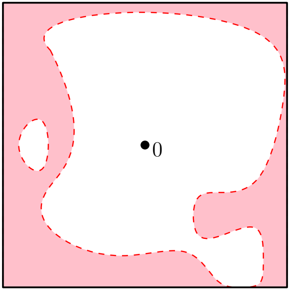


- Bernoulli percolation: an appetiser.
  (Definition, ordering, phase transition, non-triviality of p_c, sharpness)

- Introduction to FK-percolation.
  (Definition, FKG, ordering, infinite volume measures, phase transition)

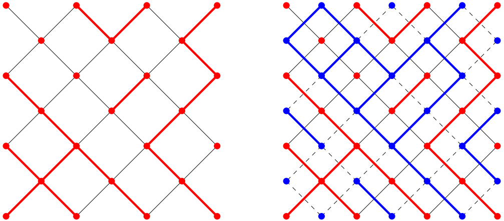
- **“Quadrichotomy” theorem.**
  (sharpness for 2D FK-perco, types of the phase transition, $p_c = p_{\mathrm{sd}}$)

- Continuity/discontinuity of phase transition via 6V.
  (BKW correspondence, 6V free energy via Bethe ansatz, continuous phase trans. iff $1 \leq q \leq 4$, 6V height funct: localisation/delocalisation)

- Rotational invariance for critical measure ($1 \leq q \leq 4$).
  (FK on isoradial graphs, track exchanges, universality of FK across isoradial graphs, rotational invariance).

!!! From now one $d = 2$!!!

Dual; auto-duality in $2d$



ω $\qquad\qquad$ ω^*

Duality: if $\omega \sim \phi^{\xi}_{G,p,q}$, then $\omega^* \sim \phi^{\xi^*}_{G^*,p^*,q}$

$$\frac{p^*}{1-p^*}\frac{p}{1-p} = q.$$

(boundary conditions: $\xi = 0 \Rightarrow \xi^* = 1$). In particular $(\phi^1_{p,q})^* = \phi^0_{p,q}$.

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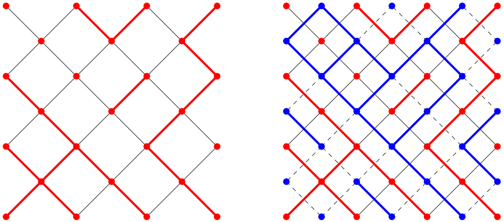

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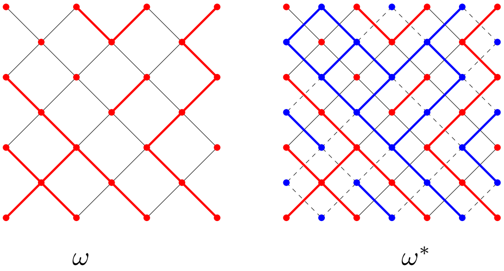

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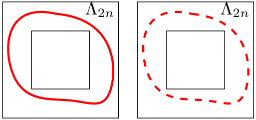
Conjecture: $p_c = p_{\text{sd}}$

Quadrichotomy result

$H_n = \{\exists$ open circuit in $\Lambda_{2n} \setminus \Lambda_n\}$.

$H_n^* =$ same for dual.



Theorem (Duminil-Copin, Tassion '19; also D-C, Sidoravicius, T '15)

Fix $q \geq 1$ and $p \in [0,1]$. Then exactly one occurs (for all n):

(a) $\phi^1_{\Lambda_n}[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$;

(b) $\phi^0_{\Lambda_{2n}\setminus\Lambda_n}[H_n] \geq c$ *and* $\phi^1_{\Lambda_{2n}\setminus\Lambda_n}[H_n^*] \geq c$;

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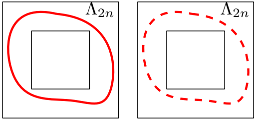

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Consequence: $p_c = p_{\mathrm{sd}} = \dfrac{\sqrt{q}}{1+\sqrt{q}}$ (and sharp phase transition).

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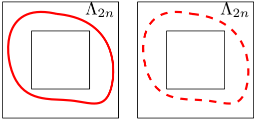

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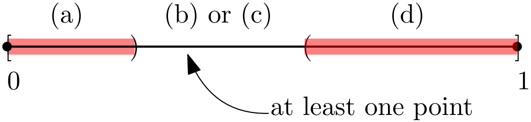

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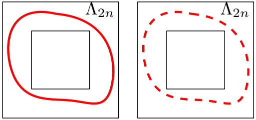

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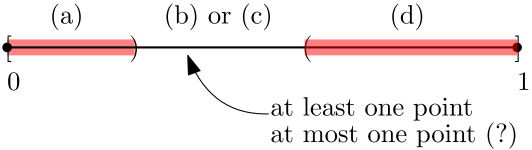

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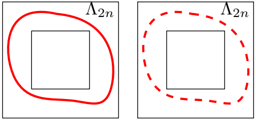


Theorem (Duminil-Copin, Tassion '19; also D-C, Sidoravicius, T '15)

Fix $q \geq 1$ and $p \in [0,1]$. Then exactly one occurs (for all n):

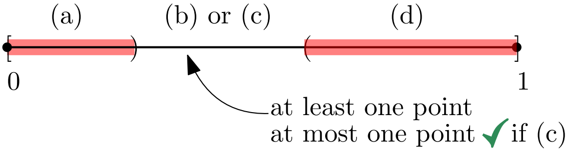
(a) $\phi^1_{\Lambda_n}[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}$; *(sub-crit)*

(b) $\phi^0_{\Lambda_{2n}\setminus\Lambda_n}[H_n] \geq c$ *and* $\phi^1_{\Lambda_{2n}\setminus\Lambda_n}[H_n^*] \geq c$; *(RSW)*

(c) $\phi^0[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}$ *and* $\phi^1[0 \overset{*}{\longleftrightarrow} \partial\Lambda_n] \leq e^{-cn}$; *(discont.)*

(d) $\phi^0_{\Lambda_n}[0 \overset{*}{\longleftrightarrow} \partial\Lambda_n] \leq e^{-cn}$. *(super-crit)*

for some $c = c(p,q) > 0$. Furthermore (a) and (d) occur on open sets.



Consequence for phase transition: (b) at one point

Assume **(b) occurs at some** $p_0 \in (0.1)$.

Theorem: For A increasing

$$\frac{d}{dp}\phi_{G,p}[A] \asymp \sum_{e \in \mathbb{E}} \phi_{G,p}[A \,|\, \omega(e) = 1] - \phi_{G,p}[A \,|\, \omega(e) = 0] = \sum_{e \in \mathbb{E}} I_e(A)$$

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Theorem: $\dfrac{d\phi_{G,p}[A]}{dp} \geq c\,\phi_{G,p}[A](1-\phi_{G,p}[A])\log\Big(\dfrac{1}{\max_e I_e(A)}\Big).$

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Proof: Due to (b), for all $p \leq p_0$, under $\phi^1_{\Lambda_{3n},p}$,

$$\max_e I_e(H_n) \leq \phi^1_{\Lambda_n}[0 \leftrightarrow \partial\Lambda_n] \leq n^{-c}$$

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We conclude $\qquad \phi^1_{\Lambda_{3n},p}[H_n] \xrightarrow[n\to\infty]{} 0 \qquad$ for all $p < p_0$,

Which implies (a) for $p < p_0$. $\square$

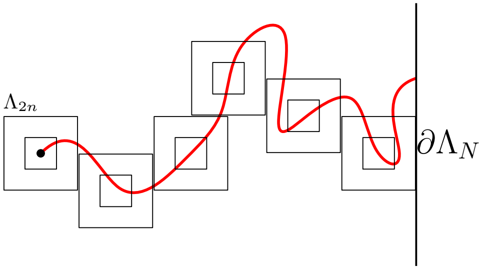
Finite-size criterion for (a) and (d)

Proposition: there exists some $\delta > 0$ s.t.

$$\big(\exists n \text{ s.t. } \phi^1_{\Lambda_{2n}}[H_n^*] > 1-\delta\big) \Leftrightarrow \big(\exists c > 0 \text{ s.t. } \phi^1_{\Lambda_N}[0 \leftrightarrow \partial\Lambda_N] \le e^{-cN} \ \ \forall N\big).$$

Proof: If $0 \leftrightarrow \partial\Lambda_N$, there exists a path of $\lfloor N/4n \rfloor$ disjoint translates of $\Lambda_{2n} \setminus \Lambda_n$ by points of $(n\mathbb{Z})^2$ for which H_n^* fails.

Nb. of paths $\le C^{\lfloor N/4n \rfloor}$; proba for one path $\le \delta^{\lfloor N/4n \rfloor}$ (Peierls arg)



Finite-size criterion for (a) and (d)

Proposition: there exists some $\delta > 0$ s.t.

$$\big(\exists n \text{ s.t. } \phi^1_{\Lambda_{2n}}[\Lambda_n \leftrightarrow \partial\Lambda_{2n}] < \delta\big) \Leftrightarrow \big(\exists c > 0 \text{ s.t. } \phi^1_{\Lambda_N}[0 \leftrightarrow \partial\Lambda_N] \leq e^{-cN} \;\; \forall N\big).$$

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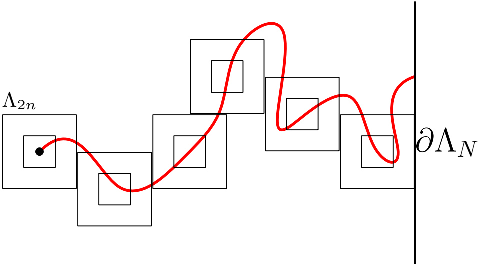

Distinguishing (b) from (c)

Fix p so that (a) and (d) fail. Then $\phi^1[H_n] \geq c$ and $u_n^* := \phi^0[H_n^*] \geq c$.

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$$u_n := \phi^0_{\Lambda_{4n}}[H_n] \qquad\qquad u_n^* := \phi^1_{\Lambda_{4n}\setminus\Lambda_n}[H_n^*]$$

(b) $u_n, u_n^* > c > 0$ for all n

(c) $u_n, u_n^* \leq e^{-cn}$ for all n.

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$$u_{10n} \leq C u_n^2 \qquad\qquad \text{(renormalisation inequality)}$$

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(b) $u_n, u_n^* > c > 0$ for all n (c) $u_n, u_n^* \leq e^{-cn}$ for all n.

$$u_{C\,kn} \leq C u_n^2 (C\,u_n)^k \quad \forall k \geq 1 \qquad\qquad \text{(direct route)}$$

This suffices to prove the *dichotomy* between (b) and (c)

(easy: $u_n \geq C(u_n^*)^{1/C}$ and $u_n^* \geq C u_n^{1/C}$)

Distinguishing (b) from (c)

Goal $u_{C\,kn} \leq C u_n^2 (C\, u_n)^k$

Assume $p < p_{\text{sd}}$

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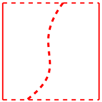
Fact: $\phi\Big[$  $\Big] > c$

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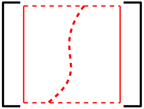
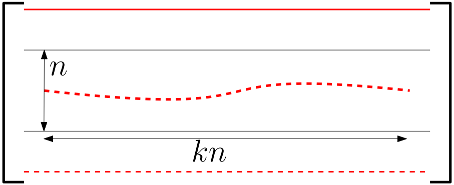
Fact: ϕ  $> c$

Thm (RSW): ϕ 

$> c^k$

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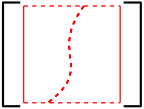
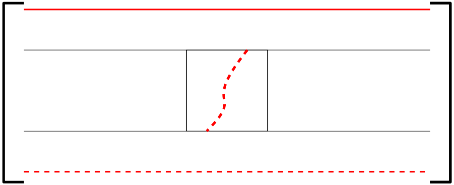
Fact: ϕ  $> c$

Thm (RSW): ϕ  $> c$

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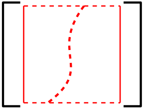
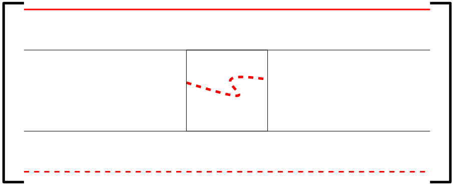
Fact: ϕ  $> c$

Thm (RSW): ϕ  $> c$

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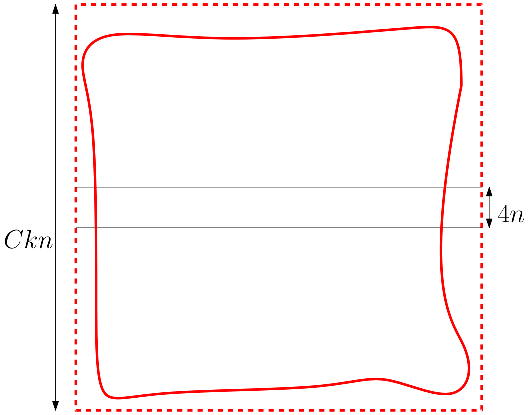

$u_{Ckn} \leq$ probability of above

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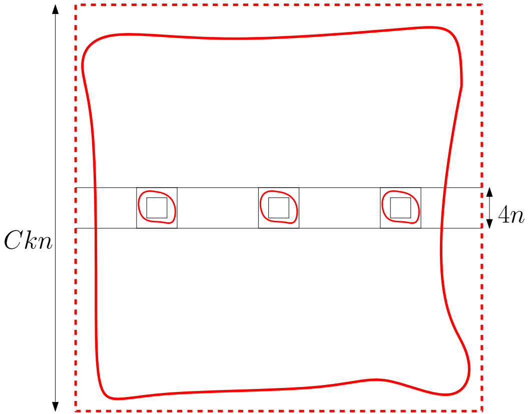

$c_1^k\, u_{Ckn} \leq$ probability of above

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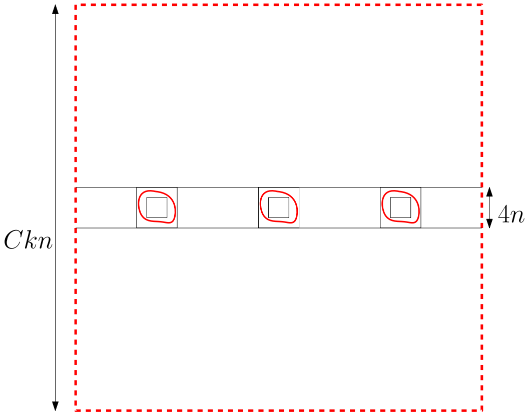
Goal $u_{C\,kn} \leq C u_n^2 (C\,u_n)^k$

Assume $p < p_{\mathrm{sd}}$

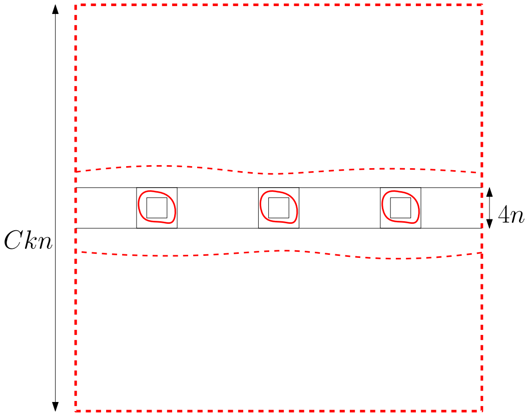


$c_2^k\, c_1^k\, u_{Ckn} \leq$ probability of above

Distinguishing (b) from (c)

Goal $u_{C\,kn} \le C u_n^2 (C\,u_n)^k$

Assume $p < p_{\mathrm{sd}}$



$c_3^k\, c_2^k\, c_1^k\, u_{Ckn} \le$ probability of above

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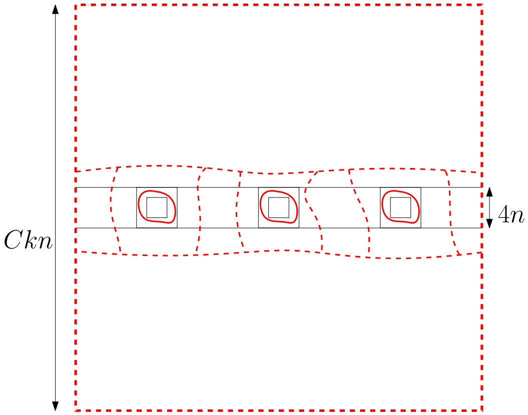

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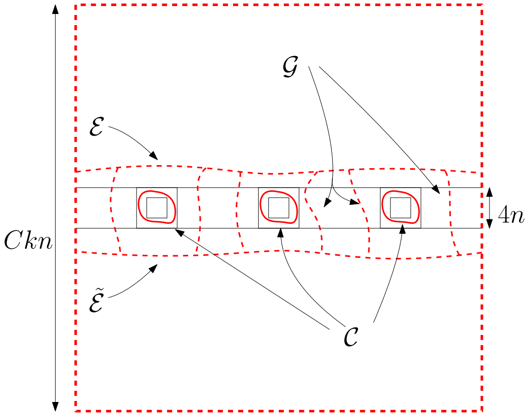


$c_3^k\, c_2^k\, c_1^k\, u_{Ckn} \leq$ probability of above $\leq u_n^C$

- Bernoulli percolation: an appetiser.
  (Definition, ordering, phase transition, non-triviality of p_c, sharpness)
- Introduction to FK-percolation.
  (Definition, FKG, ordering, infinite volume measures, phase transition)
- "Quadrichotomy" theorem.
  (sharpness for 2D FK-perco, types of the phase transition, $p_c = p_{\mathrm{sd}}$)
- **Continuity/discontinuity of phase trans via 6V.**
  (BKW correspondence, 6V free energy via Bethe ansatz, continuous phase trans. iff $1 \leq q \leq 4$, 6V height funct: localisation/delocalisation)
- Rotational invariance for critical measure ($1 \leq q \leq 4$).
  (FK on isoradial graphs, track exchanges, universality of FK across isoradial graphs, rotational invariance).

Main result

Theorem

- *For* $1 \leq q \leq 4$ *the phase transition of FK on* $\mathbb{Z}^2$ *is* **continuous**

  ($\phi^0 = \phi^1$*, RSW, expected conformally invariant scaling limit)*

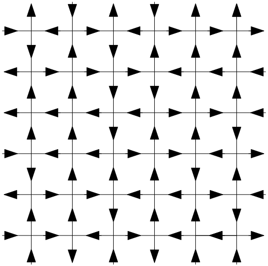
- *For* $q > 4$ *the phase transition of FK on* $\mathbb{Z}^2$ *is* **discontinuous**

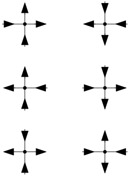
  ($\phi^0 \neq \phi^1$*, exponential decay in* ϕ^0*, no critical phase).*

References:

- Duminil-Copin, Sidoravicius, Tassion (2017);
- Duminil-Copin, Gagnebin,Harel, M., Tassion (2021);
- Ray, Spinka (2020);
- Lammers, Glazman (2023).

Six vertex on the tours





$a = 1$

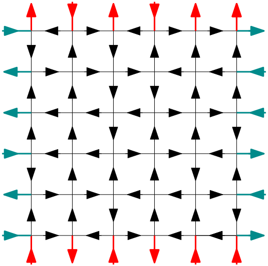
$b = 1$

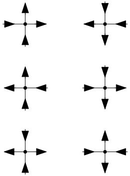
$c > 0$

$(\Delta = 1 - c^2/2)$.

$$w_{6V}(\vec{\omega}) = c^{\#\text{type } c \text{ vertices}}$$

Six vertex on the tours





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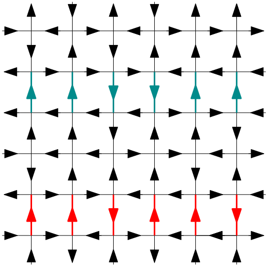

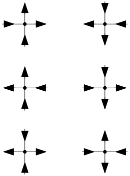

$a = 1$

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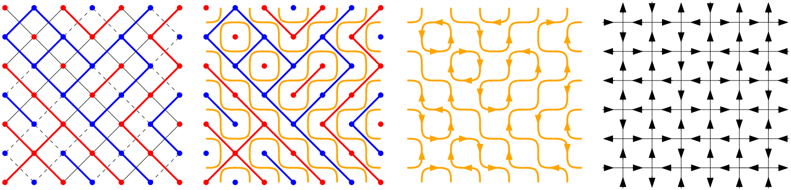
$(\Delta = 1 - c^2/2)$.

$$w_{6V}(\vec{\omega}) = c^{\#\text{type } c \text{ vertices}}$$

Ice rule $\Rightarrow$ same number of up-arrows on each row.

$$Z_{6V}^{(k)}(\mathbb{T}_{N,M}) = \sum_{\substack{\vec{\omega} \\ \frac{N}{2}+k \text{ up-arrow}}} w_{6V}(\vec{\omega}) \qquad f_{6V}(\alpha) = \lim_{N} \lim_{M} \frac{1}{NM} \log Z^{(\alpha N)}(\mathbb{T}_{N,M})$$

BKW correspondence $(p = p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}})$



$$\sqrt{q} = e^{\lambda} + e^{-\lambda} \qquad \text{and} \qquad c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}.$$

Prop: $$f_{\mathrm{FK}} = f_{6V} + \tfrac{1}{4}\log q + \log(1+\sqrt{q});$$

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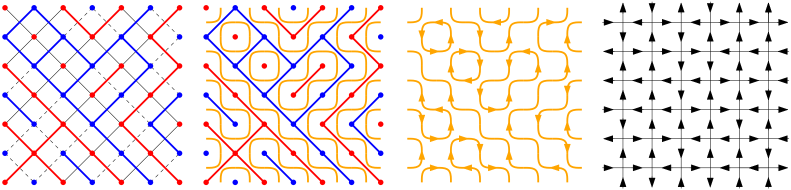

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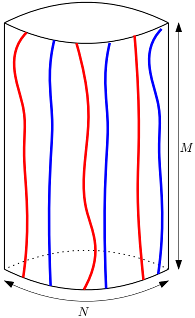
$$\phi_{\mathbb{T}^{\times}_{N,M}}[\mathcal{E}_{\alpha N}]^{\frac{1}{NM}} = \exp\big(f_{6V}(\alpha) - f_{6V}(0) + o(1)\big)$$



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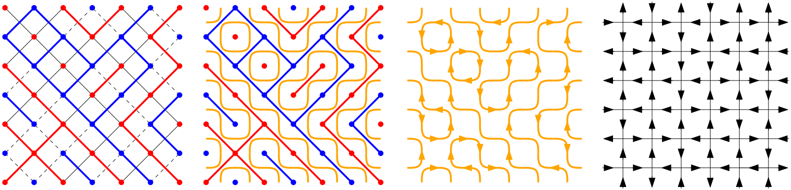

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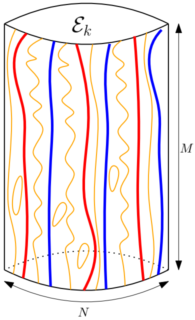

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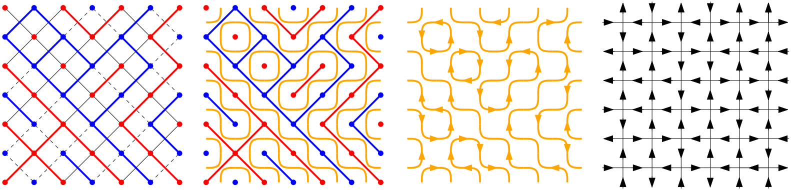

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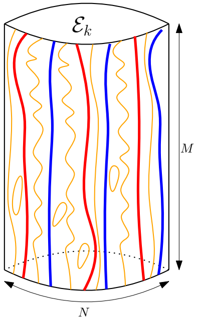
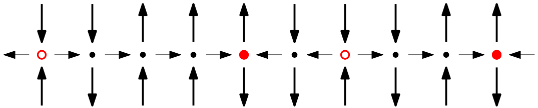
Prop: $$f_{\mathrm{FK}} = f_{6V} + \tfrac{1}{4}\log q + \log(1+\sqrt{q});$$

$$\phi_{\mathbb{T}^{\times}_{N,M}}[\mathcal{E}_{\alpha N}]^{\frac{1}{NM}} = \exp\big(f_{6V}(\alpha) - f_{6V}(0) + o(1)\big)$$

$$\begin{cases} \geq c\alpha & \text{(c): discontinuity} \\ \leq c\alpha^2 & \text{(b): RSW} \end{cases}$$



Transfer matrix, relations to $f_{6V}(\alpha)$



$$V(\psi,\psi') = \sum_{\substack{\text{possible}\\ \text{completions}}} c^{\#\text{type } c \text{ vertices}} \qquad \forall \psi,\psi' \in \{\pm 1\}^L = \{\uparrow,\downarrow\}^L.$$

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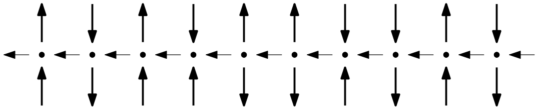

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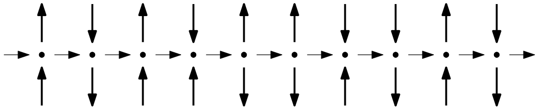
Transfer matrix, relations to $f_{6V}(\alpha)$



$$V(\psi,\psi') = \sum_{\substack{\text{possible}\\ \text{completions}}} c^{\#\text{type } c \text{ vertices}} \qquad \forall \psi,\psi' \in \{\pm 1\}^L = \{\uparrow,\downarrow\}^L.$$

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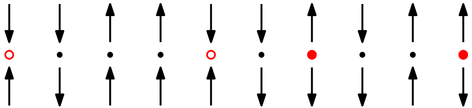

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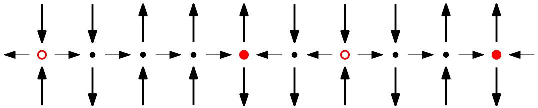

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Conservation of arrows $\Rightarrow$ V split in blocks $V^{(k)}_{k=-N/2,\dots,N/2}$

k = excess number of up-arrows.

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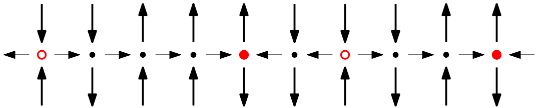

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Conservation of arrows $\Rightarrow$ V split in blocks $V^{(k)}_{k=-N/2,\dots,N/2}$

k = excess number of up-arrows.

$V^{(k)}$ positive entries & irreducible $\Rightarrow$ $\Lambda^{(k)}(N)$ main eigenvalue.

$$Z^{(k)}(\mathbb{T}_{N,M}) = \mathrm{Tr}(V^{(k)})^M = (\Lambda^{(k)}(N))^M (1 + O(e^{-\epsilon M}))$$

Bethe ansatz for six-vertex

$$\mu := \begin{cases} \arccos(-\Delta) & \text{if } c \leq 2 \\ = 0 & \text{if } c > 2 \end{cases} \quad \text{and} \quad \mathcal{D} := (-\pi + \mu, \pi - \mu).$$

$$\Theta : \mathcal{D}^2 \to \mathbb{R} \text{ define by } \exp(-i\Theta(x, y)) = e^{i(x-y)} \cdot \frac{e^{-ix} + e^{iy} - 2\Delta}{e^{-iy} + e^{ix} - 2\Delta}.$$

$$L(z) := 1 + \frac{c^2 z}{1 - z}, \qquad M(z) := 1 - \frac{c^2}{1 - z}, \quad I_j = j - \tfrac{n+1}{2} \text{ for } j = 1, \ldots, n.$$

Fix $n = N/2 - k$. Let $(p_1, p_2, \ldots, p_n) \in \mathcal{D}^n$ be distinct and satisfy the equations

$$N p_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k), \qquad \forall j \in \{1, \ldots, n\}. \tag{BE}$$

Then, the vector ψ given by $\psi(\vec{x}) := \sum_{\sigma \in \mathfrak{S}_n} A_\sigma \prod_{k=1}^{n} \exp\left(i p_{\sigma(k)} x_k\right)$, where

$A_\sigma := \varepsilon(\sigma) \prod_{1 \leq k < \ell \leq n} e^{i p_{\sigma(k)}} \left(e^{-\mathrm{i} p_{\sigma(k)}} + e^{\mathrm{i} p_{\sigma(\ell)}} - 2\Delta\right)$, satisfies $V\psi = \Lambda\psi$, where

$$\Lambda = \begin{cases} \prod_{j=1}^{n} L(e^{ip_j}) + \prod_{j=1}^{n} M(e^{ip_j}) & \text{if } p_1, \ldots, p_n \text{ are non zero,} \\ \left[2 + c^2(N-1) + c^2 \sum_{j \neq \ell} \partial_1 \Theta(0, p_j)\right] \cdot \prod_{j \neq \ell} M(e^{ip_j}) & \text{if } p_\ell = 0 \text{ for some } \ell. \end{cases}$$

Bethe ansatz: condensation

If $$N p_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k), \qquad \forall j \in \{1, \ldots, n\}.$$

then $$\Lambda = \prod_{j=1}^{n} L(e^{ip_j}) + \prod_{j=1}^{n} M(e^{ip_j}) \text{ is an eigenvalue.}$$

Suppose $\frac{1}{N} \sum_i \delta_{p_i} \to \rho\, dx$ (**condensation**). Then

ρ satisfies $$2\pi\rho(x) = 1 + \int_{\mathcal{D}} \partial_x \Theta(x, y) \rho(y) dy \qquad \forall x \in \mathcal{D}$$

eigenvalue: $$\frac{1}{N} \log \Lambda = \int_{\mathcal{D}} \log \left| M\left(e^{ix}\right) \right| \rho(x) dx + o(1).$$

Questions:
- existence of solutions, condensation?
- is the resulting eigenvalue the main one?
- computation of ρ and Λ.

Results obtained via Bethe ansatz

We manage to estimate $\Lambda^{(\alpha N)}(N)$ and find:

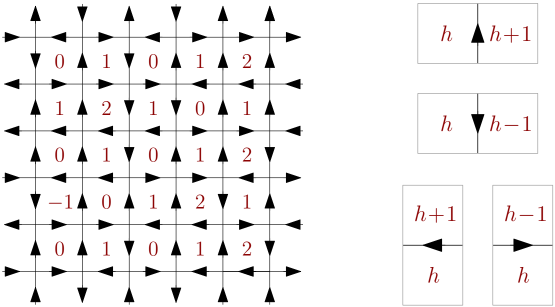
$$f_{6V}(\alpha) - f_{6V}(0) = \lim_N \tfrac{1}{N} \log \frac{\Lambda^{(\alpha N)}(N)}{\Lambda^{(0)}(N)} = \begin{cases} -C\,\alpha + o(\alpha) & \text{if } c > 2, \\ -C\,\alpha^2 + o(\alpha^2) & \text{if } 0 < c \leq 2. \end{cases}$$

$$\phi_{\mathbb{T}^\times_{L,M}}[\mathcal{E}_{\alpha N}] = \begin{cases} \exp\big(-C\,\alpha\, NM(1 + o(1))\big) & \text{if } c > 2 \\ \exp\big(-C\,\alpha^2\, NM(1 + o(1))\big) & \text{if } 0 < c \leq 2 \end{cases}$$

Conclusion: Continuous phase transition (RSW) for $c \leq 2 \Leftrightarrow q \leq 4$

Discontinuous (exp decay in ϕ^0) for $c > 2 \Leftrightarrow q > 4$

Height function of 6V



$c \geq 1 \Rightarrow$ FKG for both h and $|h|$.

For "flat" boundary conditions:

- $c > 2 \Rightarrow$ localised regime (∃ limit for h, not shift-invariant)
- $1 \leq c \leq 2 \Rightarrow$ delocalised (log variance, limit for ∇h, GFF lim?)

An invitation to FK percolation

Ioan Manolescu

University of Fribourg

11-15 Sep 2023
Statistical Mechanics and Stochastic PDEs
CIME Summer School
Cetraro, Italy

Lecture 3

Recall from lecture 2

- FK-percolation on $\mathbb{Z}^2$; sharp phase transition.

- Critical point $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ (duality)

- Discontinuous phase transition if $q > 4$:
  $\phi^0_{p_c,q} \neq \phi^1_{p_c,q}$, finite correlation length at $p_c \dots$

- **Continuous phase transition if** $q \in [1,4]$**:** unique infinite volume measure $\phi^0_{p_c,q} = \phi^1_{p_c,q}$,
  - Algebraic decay at p_c,
  - No infinite cluster at p_c: $\phi_{p_c,q}[0 \leftrightarrow \infty] = 0$.
  - **RSW property** $\phi^0_{\Lambda_{2n}\setminus\Lambda_n,p_c,q}[H_n] \geq c$ and $\phi^1_{\Lambda_{2n}\setminus\Lambda_n,p_c,q}[H^*_n] \geq c$.

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!!! Fix $1 \leq q \leq 4$ and $p = p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$!!!

- Bernoulli percolation: an appetiser.

  (Definition, ordering, phase transition, non-triviality of p_c, sharpness)

- Introduction to FK-percolation.

  (Definition, FKG, ordering, infinite volume measures, phase transition)

- "Quadrichotomy" theorem.

  (sharpness for 2D FK-perco, types of the phase transition, $p_c = p_{\mathrm{sd}}$)

- Continuity/discontinuity of phase transition via 6V.

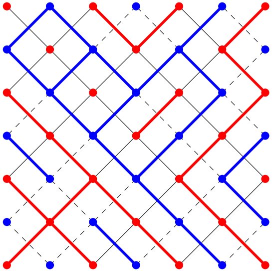
  (BKW correspondence, 6V free energy via Bethe ansatz, continuous phase trans. iff $1 \leq q \leq 4$, 6V height funct: localisation/delocalisation)

- **Rotational inv. for critical measure** $(1 \leq q \leq 4)$

  (FK on isoradial graphs, track exchanges, universality of FK across isoradial graphs, rotational invariance).

!!! Fix $1 \leq q \leq 4$ and "p critical" !!!

Major conjecture



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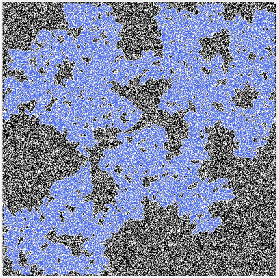

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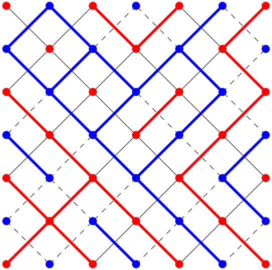

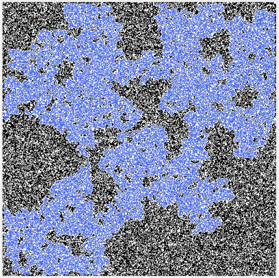

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RSW:

- finite number of macroscopic clusters;
- fractal behaviour
- algebraic decay of connection probabilities
- algebraic mixing

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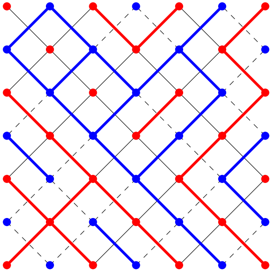

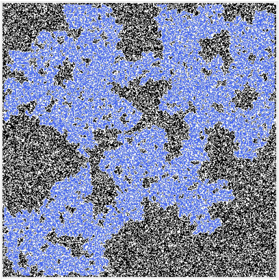

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- finite number of macroscopic clusters;
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Conjectures:

- Existence of scaling limit.
- **Conformal invariance scaling limit.**
- Scaling limit depends on q.
- **Universality.**

Critical exponents

One arm: $\phi_{p_c}[0 \leftrightarrow \partial\Lambda_n] = n^{-\boldsymbol{\xi_1}+o(1)}$ as $n \to \infty$,

Two point function: $\phi_{p_c}[0 \leftrightarrow x] = |x|^{-\boldsymbol{\eta}+o(1)}$ as $|x| \to \infty$,

Cluster volume: $\phi_{p_c}[|\mathsf{C}| \geq n] = n^{-\boldsymbol{\zeta}+o(1)}$ as $n \to \infty$,

Free energy: $f''(p) = |p - p_c|^{-\boldsymbol{\alpha}+o(1)}$ as $p \to p_c$,

Percolation probability: $\theta(p) := \phi_p[0 \leftrightarrow \infty] = (p - p_c)^{\boldsymbol{\beta}+o(1)}$ as $p \searrow p_c$,

Susceptibility: $\chi(p) := \phi_p[|\mathsf{C}|\mathbf{1}_{\{|\mathsf{C}|<\infty\}}] = |p - p_c|^{-\boldsymbol{\gamma}+o(1)}$ as $p \to p_c$,

Correlation length: $\xi(p) = |p - p_c|^{-\boldsymbol{\nu}+o(1)}$ as $p \to p_c$.

where $\xi(p)^{-1} := \lim_{n\to\infty} -\frac{1}{n} \log \phi_p[0 \leftrightarrow \partial\Lambda_n \text{ but } 0 \not\leftrightarrow \infty]$.

Scaling relations

One arm:	$\pi_1(n) := \phi_{p_c}[0 \longleftrightarrow \partial\Lambda_n] = n^{-\xi_1+o(1)}$	as $n \to \infty$,
Two point function:	$\phi_{p_c}[0 \leftrightarrow x] = \lvert x\rvert^{-\eta+o(1)}$	as $\lvert x\rvert \to \infty$,
Cluster volume:	$\phi_{p_c}[\lvert\mathsf{C}\rvert \geq n] = n^{-\zeta+o(1)}$	as $n \to \infty$,
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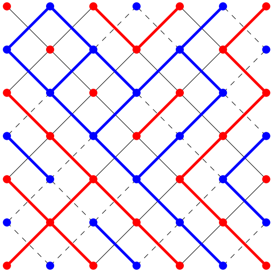
Four arms (q = 1):	$\pi_4(n) := \phi_{p_c}[\text{four arms from } 0 \text{ to } \partial\Lambda_n] = n^{-\xi_4+o(1)}$	as $n \to \infty$,
Influence ($q > 1$):	$\Delta_{p_c}(n) := \phi^1_{\Lambda_n,p_c}[e \text{ open}] - \phi^0_{\Lambda_n,p_c}[e \text{ open}] = n^{-\iota+o(1)}$	as $n \to \infty$.

Scaling relations:

$$\eta = 2\xi_1, \qquad \textbf{(R1)}$$
$$\zeta = \xi_1/(2-\xi_1) \qquad \textbf{(R2)}$$
$$\beta = \nu\xi_1 \qquad \textbf{(R3)}$$
$$\gamma = (2-2\xi_1)\nu \qquad \textbf{(R4)}$$

$$\alpha = 2 - 2\nu \qquad \textbf{(R6)}$$
$$\nu = \begin{cases} 1/(2-\xi_4) & (q=1) \\ 1/(2-\iota) & (q>1) \end{cases}. \qquad \textbf{(R0)}$$

Loop representation & topology



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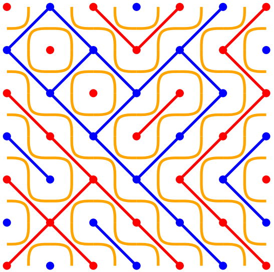

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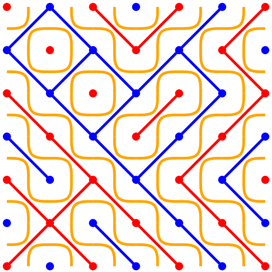

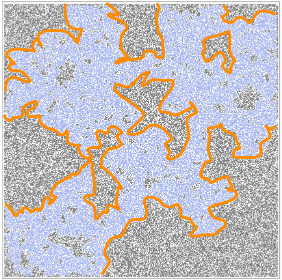

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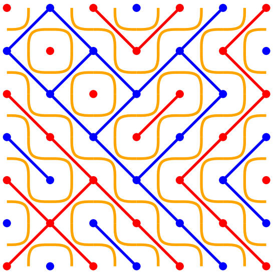

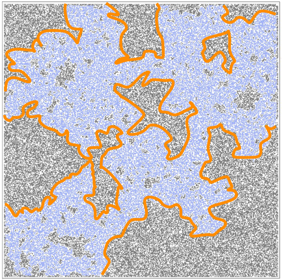

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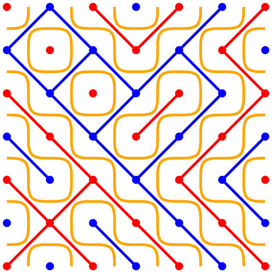

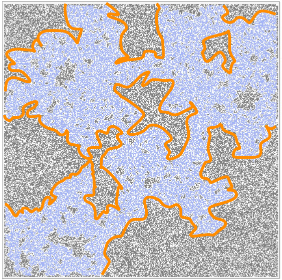

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$$d(\mathcal{F}, \mathcal{F}') \leq \epsilon$$

if for each loop $\gamma \in \mathcal{F}$ with
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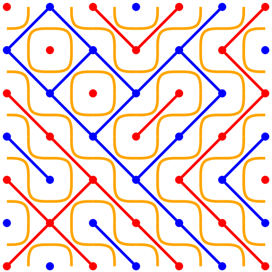

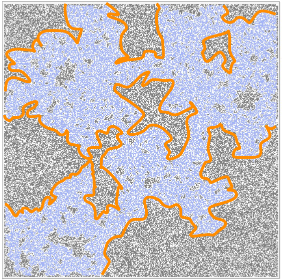

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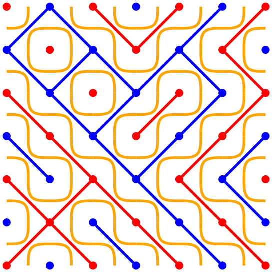

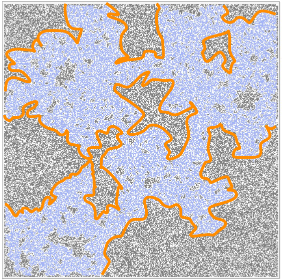

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- **Scaling limit conj:** $\lim_{\delta \to 0} \phi_{\delta\mathbb{Z}^2, p_c, q} = CLE_\kappa$

Results

[Duminil-Copin, Kozlowski, Krachun, M.,Oulamara '20]

Theorem (Rotational invariance)

Fix $\alpha \in (0, \pi)$. *Then*

$$\mathbf{d}\big(\phi_{p_c,\delta\mathbb{Z}^2}, \phi_{p_c,e^{i\alpha}\delta\mathbb{Z}^2}\big) \xrightarrow[\delta\to 0]{} 0.$$

"Any sub-sequential limit of $\phi_{\delta\mathbb{Z}^2}$ *is invariant under rotations."*

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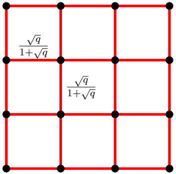

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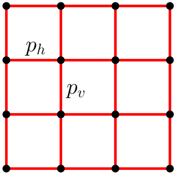

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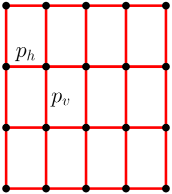
where $T = T_{p_v,p_h}$ *is an explicit linear deformation (given by isoradiality).*



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Theorem (Rotational invariance)

Fix $\alpha \in (0, \pi)$*. Then*

$$\mathbf{d}\big(\phi_{p_c,\delta\mathbb{Z}^2}, \phi_{p_c,e^{i\alpha}\delta\mathbb{Z}^2}\big) \xrightarrow[\delta\to 0]{} 0.$$

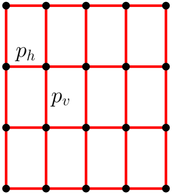
"Any sub-sequential limit of $\phi_{\delta\mathbb{Z}^2}$ *is invariant under rotations."*

Theorem (Universality)

Fix $p_v, p_h \in (0,1)$ *with* $\frac{p_v}{1-p_v}\frac{p_h}{1-p_h} = q$*. Then*

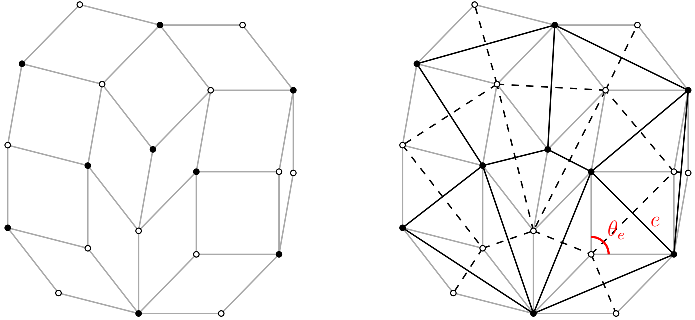
$$\mathbf{d}\big(\phi_{p_c,\delta\mathbb{Z}^2}, \phi_{p_v,p_h,T\delta\mathbb{Z}^2}\big) \xrightarrow[\delta\to 0]{} 0,$$

where $T = T_{p_v,p_h}$ *is an explicit linear deformation (given by isoradiality).*



"Universality across inhomogeneous percolation when properly embedded."

Isoradial graphs (Duffin '68, Baxter '78, Kenyon '02)

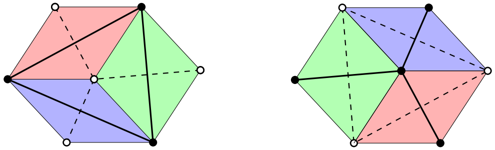


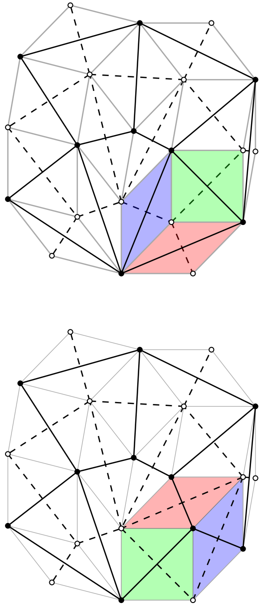
G isoradial $\Longrightarrow \phi_G$ canonical percolation measure with parameters

$$y_e := \frac{p_e}{1-p_e} = \sqrt{q}\,\frac{\sin(r(\pi-\theta_e))}{\sin(r\theta_e)},$$

with $r = \frac{1}{\pi}\arccos(\sqrt{q}/2)$

Star-triangle transformation





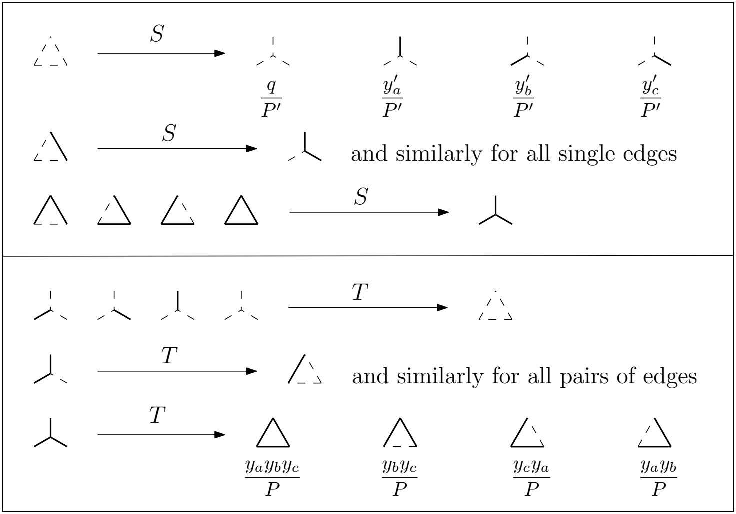
Laws of connections are preserved if

$$y_a y_b y_c + y_a y_b + y_b y_c + y_c y_a = q.$$

where $y_u = \frac{p_u}{1-p_u}$.

!!! **This holds for isoradial weights** !!!

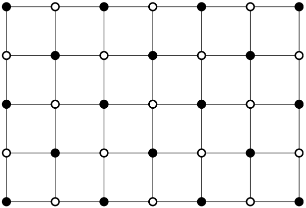
Star-triangle transformation: coupling



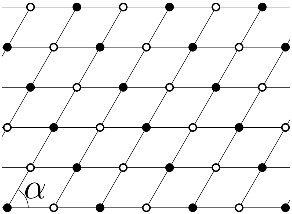
where $y_u = \frac{p_u}{1-p_u}$ and $y'_u = q/y_u$.

Universality among isoradial square lattices

$\mathbb{L}_{\pi/2}$
$\downarrow$
$\phi_{\mathbb{L}_{\pi/2}}$



$\mathbb{L}_\alpha$
$\downarrow$
$\phi_{\mathbb{L}_\alpha}$



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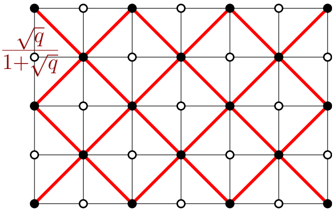

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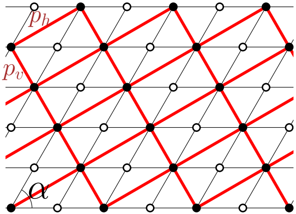

where $\quad \dfrac{p_v}{1-p_v} = \sqrt{q}\,\dfrac{\sin(r(\pi-\theta))}{\sin(r\theta)} \quad$ and $r = \frac{1}{\pi}\arccos(\sqrt{q}/2)$

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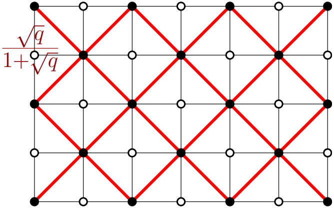

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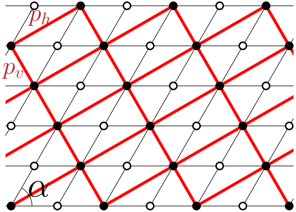

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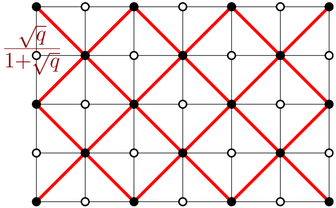
Theorem (Grimmett, M. '13; Duminil-Copin, M., Li '18)

Fix $\alpha \in (0,\pi)$. Then $\phi_{\mathbb{L}_\alpha}$ exhibits RSW-type bounds.

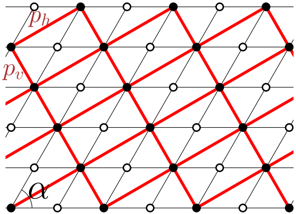
"Universality of RSW"

Universality among isoradial square lattices

$\mathbb{L}_{\pi/2}$
$\downarrow$
$\phi_{\mathbb{L}_{\pi/2}}$



$\mathbb{L}_\alpha$
$\downarrow$
$\phi_{\mathbb{L}_\alpha}$



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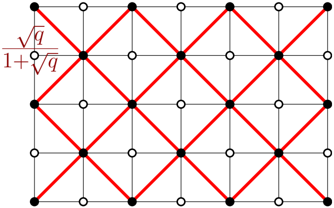
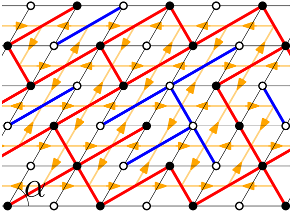
"Universality of scaling limit."

Universality among isoradial square lattices

$\mathbb{L}_{\pi/2}$
$\downarrow$
$\phi_{\mathbb{L}_{\pi/2}}$



$\mathbb{L}_\alpha$
$\downarrow$
$\phi_{\mathbb{L}_\alpha}$



where $\quad \dfrac{p_v}{1-p_v} = \sqrt{q}\,\dfrac{\sin(r(\pi-\theta))}{\sin(r\theta)} \quad$ and $r = \frac{1}{\pi}\arccos(\sqrt{q}/2)$

$$a = \frac{\sin\left(\lambda\frac{\alpha}{\pi}\right)}{\sin(\lambda/2)}; \quad b = \frac{\sin\left(\lambda(1-\frac{\alpha}{\pi})\right)}{\sin(\lambda/2)}; \quad c = 2\cos(\lambda/2); \quad \text{where } \lambda = \arccos(\sqrt{q}/2).$$

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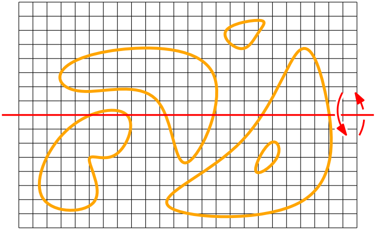
"Universality of scaling limit."

Rotational invariance from universality



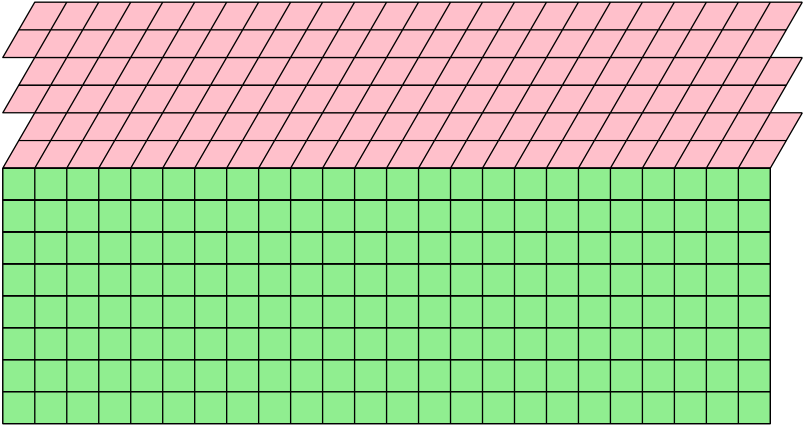
$\phi_{\mathbb{L}(\pi/2)}$ invariant under horizontal reflection σ_0

$\phi_{\mathbb{L}(\alpha)}$ invariant under reflection σ_α w.r.t. $\mathbb{R}e^{i\alpha/2}$

Asymptotically same (universality)$\Rightarrow$

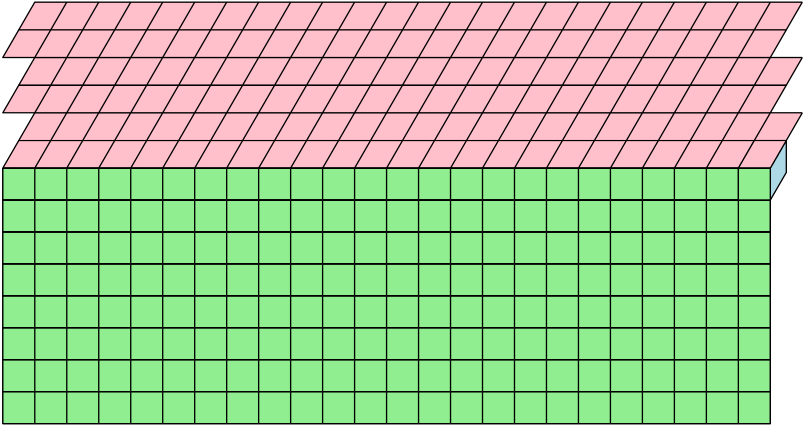
$\phi_{\mathbb{L}(\pi/2)}$ asymptotically invariant under $\sigma_\alpha \circ \sigma_0 =$ rotation by α.

Towards universality: idea



$\mathbb{L}^{(0)}$

Towards universality: idea



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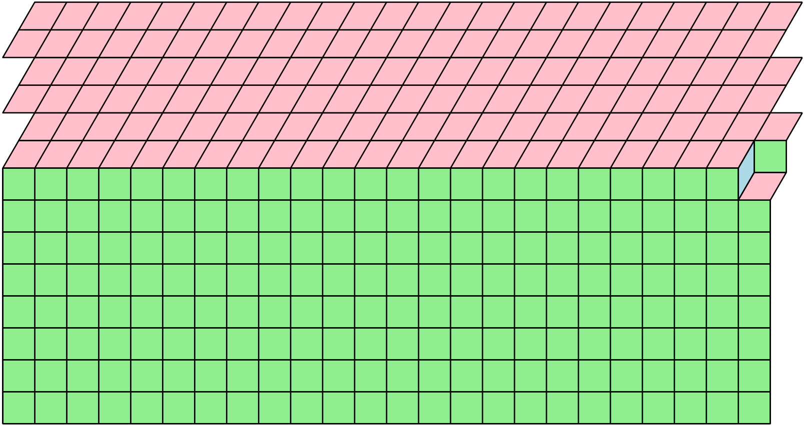
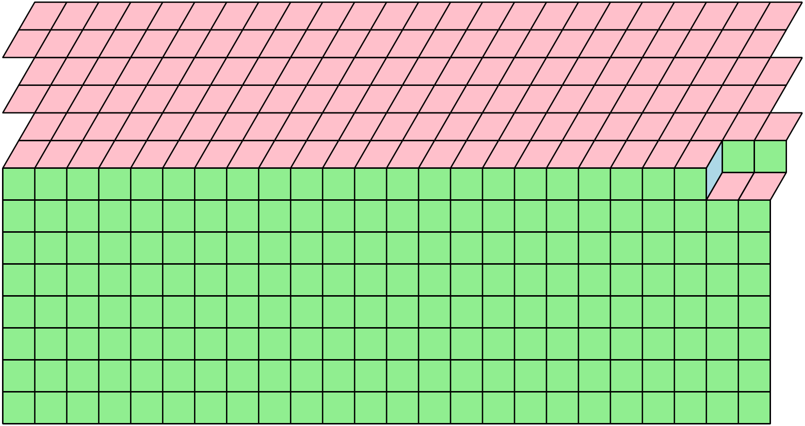
Towards universality: idea



$\mathbb{L}^{(0)}$

Towards universality: idea



$\mathbb{L}^{(0)}$

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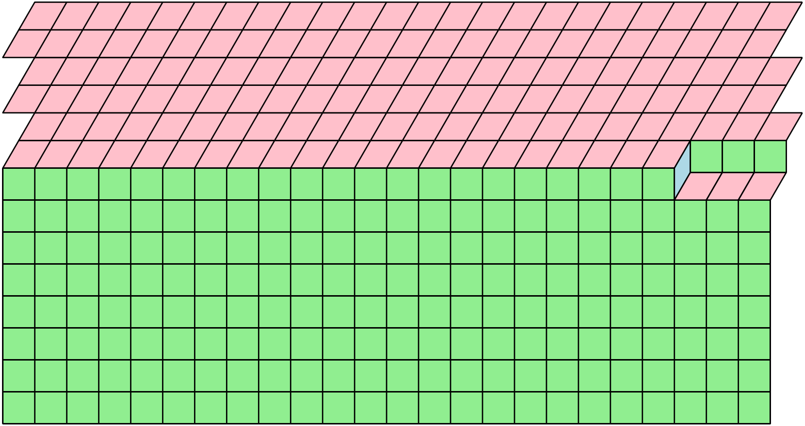
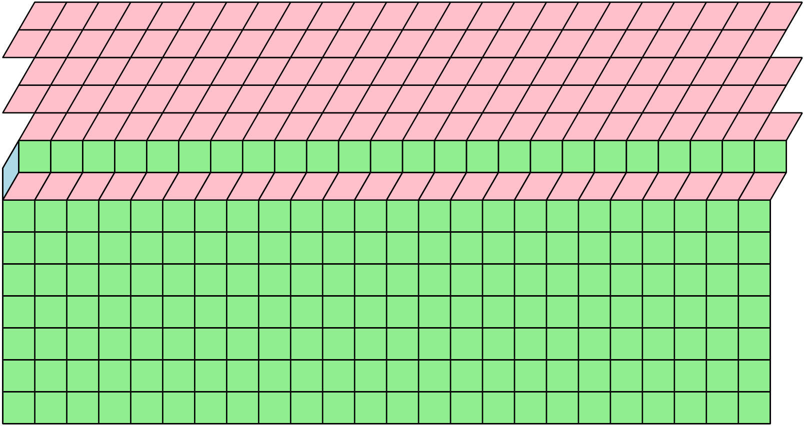


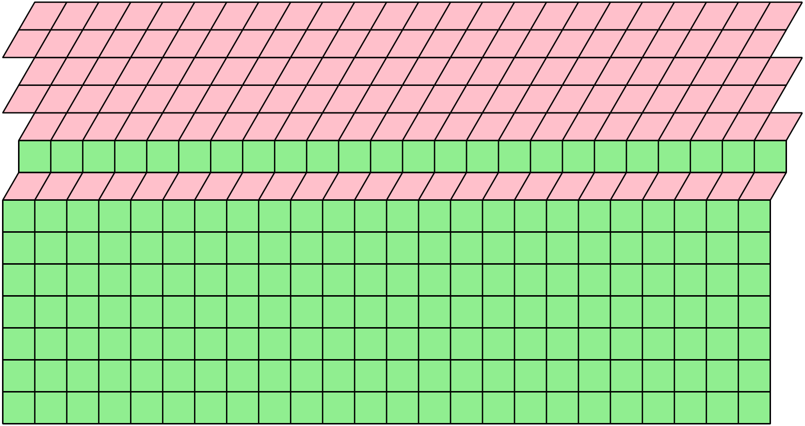
$\mathbb{L}^{(0)}$

Towards universality: idea



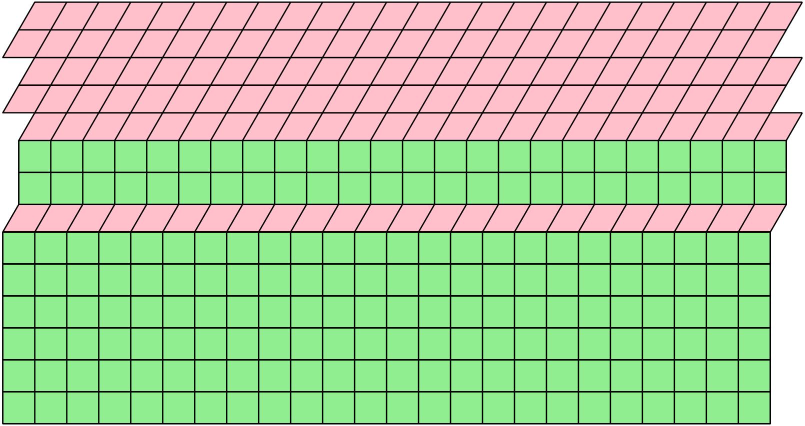
$\mathbb{L}^{(0)}$

Towards universality: idea



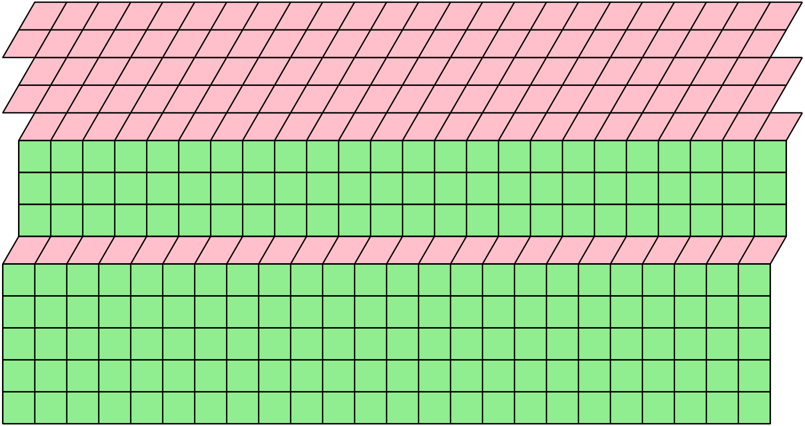
$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)}$

Towards universality: idea



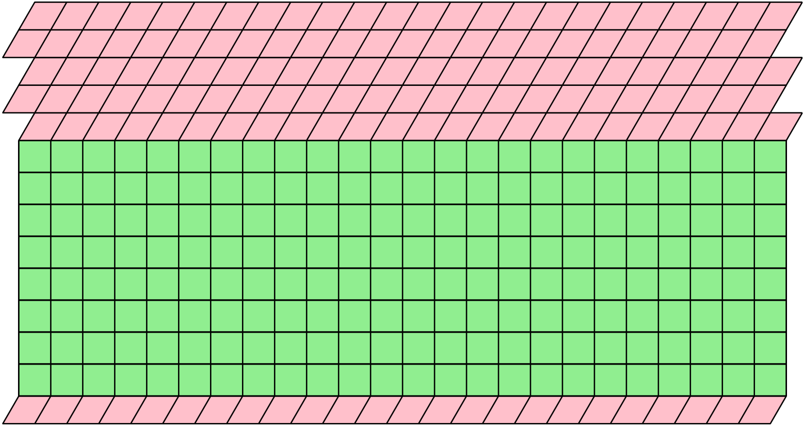
$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)}$

Towards universality: idea



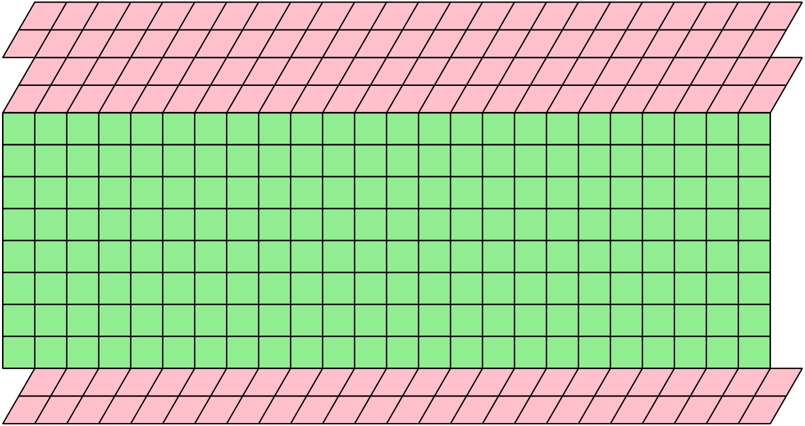
$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)}$

Towards universality: idea



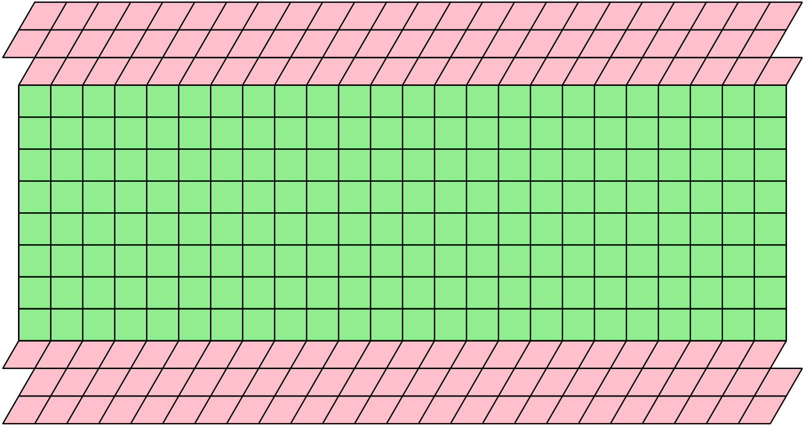
$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \cdots \longrightarrow \mathbb{L}^{(N)}$$

Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)}$$

Towards universality: idea



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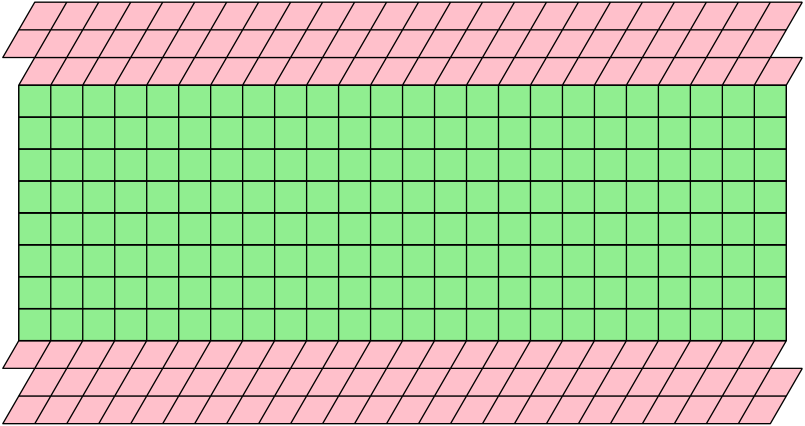
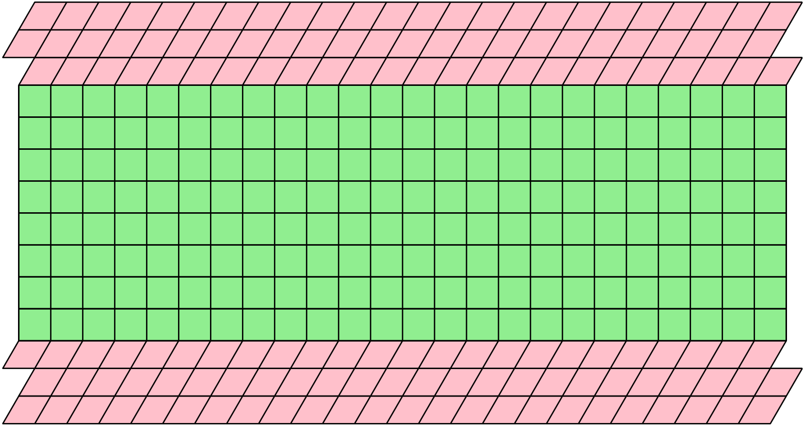
Towards universality: idea



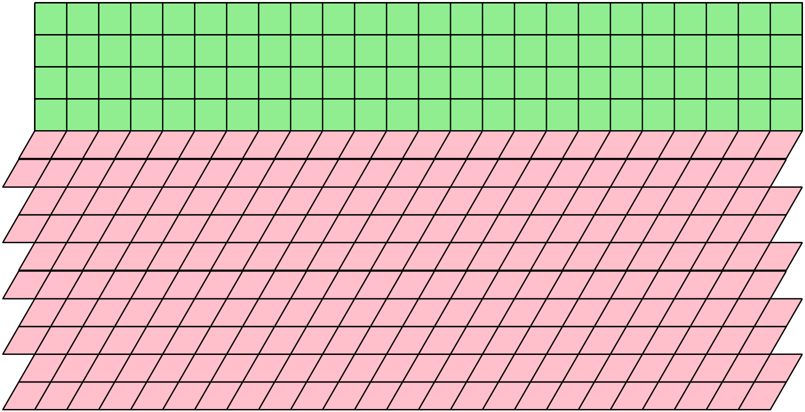
$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \dots \longrightarrow \mathbb{L}^{(N)} \dots \longrightarrow \mathbb{L}^{(2N)}$$

Towards universality: idea



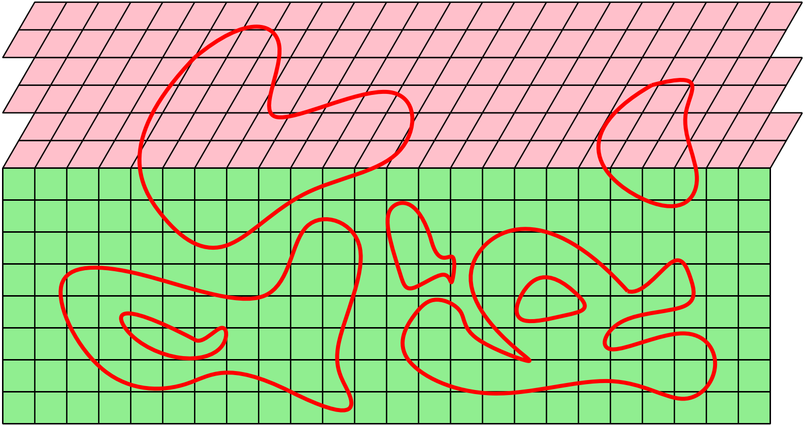
$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \dots \longrightarrow \mathbb{L}^{(N)} \dots \longrightarrow \mathbb{L}^{(2N)}$$

Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \dots \longrightarrow \mathbb{L}^{(N)} \dots \longrightarrow \mathbb{L}^{(2N)} \dots \longrightarrow \mathbb{L}^{(N^2)}$$

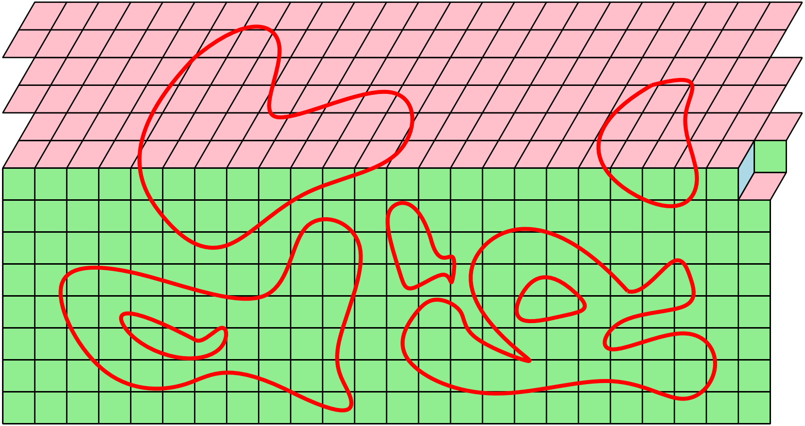
Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)} \ldots \longrightarrow \mathbb{L}^{(N^2)}$$

$\omega^{(0)}$

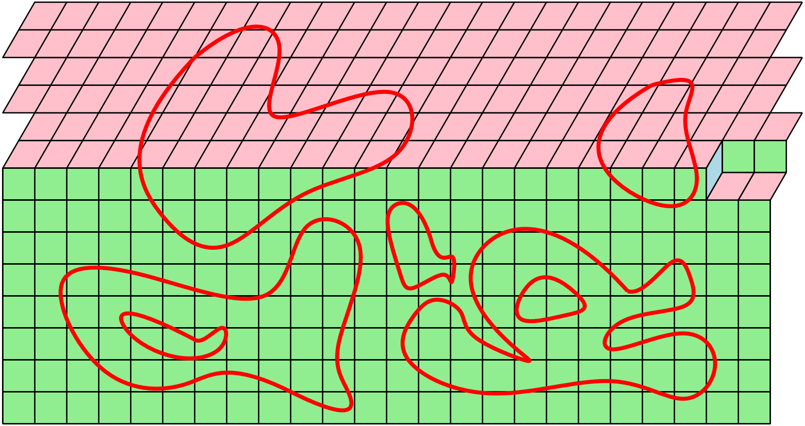
Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)} \ldots \longrightarrow \mathbb{L}^{(N^2)}$$

$\omega^{(0)}$

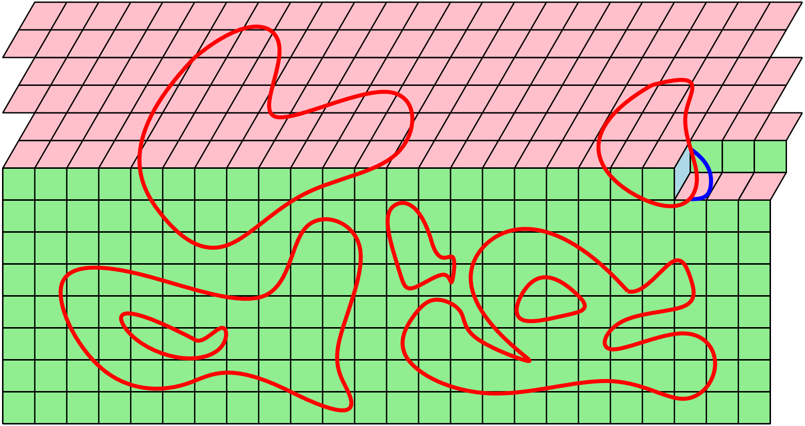
Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)} \ldots \longrightarrow \mathbb{L}^{(N^2)}$$

$\omega^{(0)}$

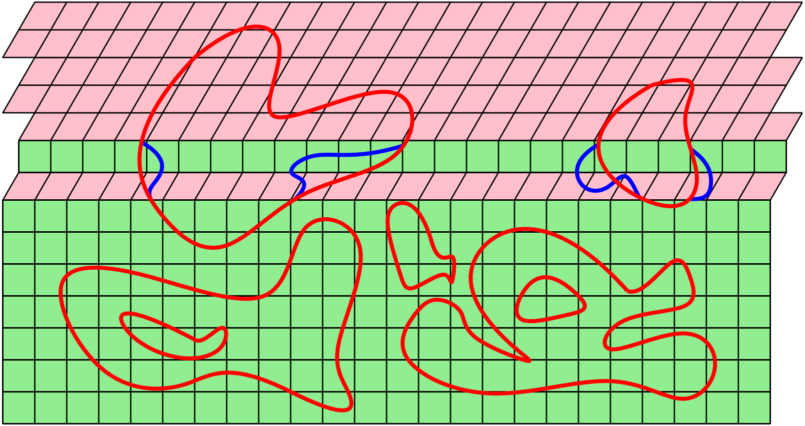
Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)} \ldots \longrightarrow \mathbb{L}^{(N^2)}$$

$\omega^{(0)}$

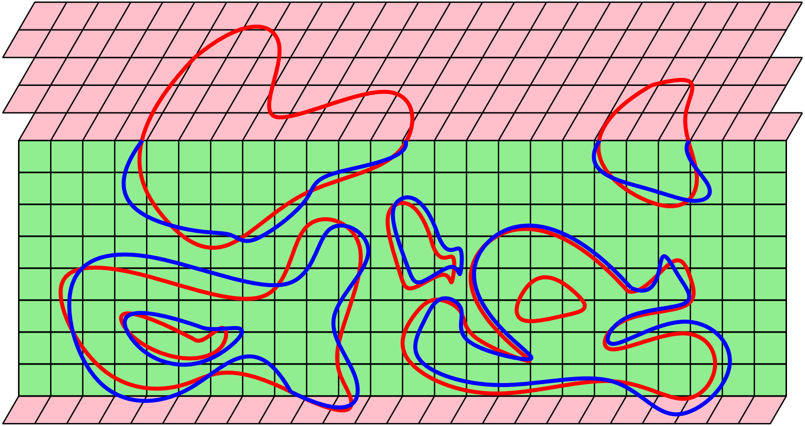
Towards universality: idea



$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)} \ldots \longrightarrow \mathbb{L}^{(N^2)}$

$\omega^{(0)} \longrightarrow \omega^{(1)}$

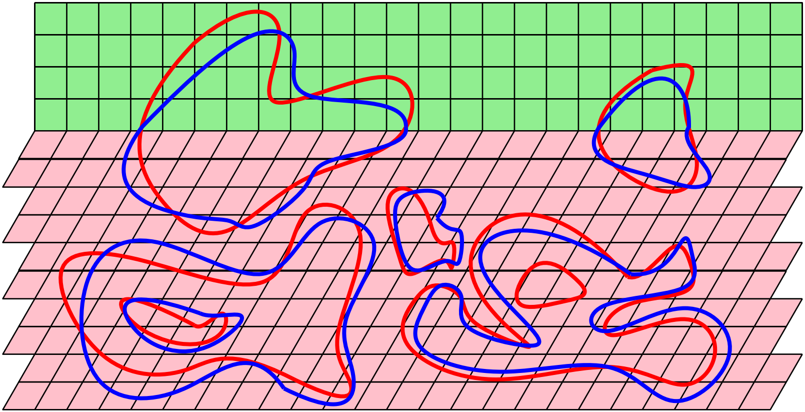
Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \ldots \longrightarrow \mathbb{L}^{(N)} \ldots \longrightarrow \mathbb{L}^{(2N)} \ldots \longrightarrow \mathbb{L}^{(N^2)}$$

$$\omega^{(0)} \longrightarrow \omega^{(1)} \longrightarrow \omega^{(2)} \ldots \longrightarrow \omega^{(N)}$$

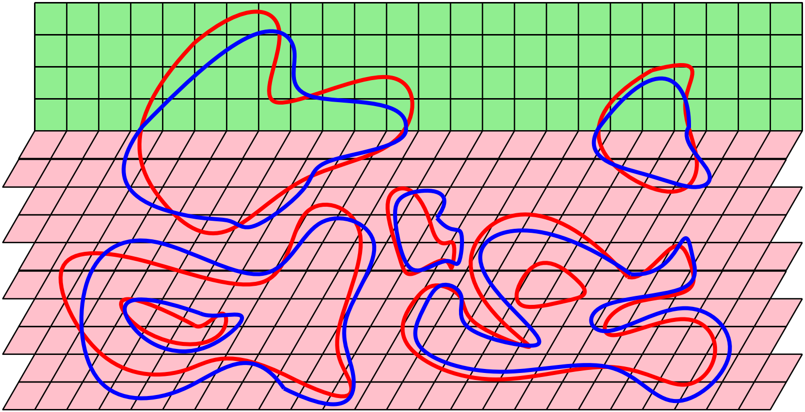
Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \dots \longrightarrow \mathbb{L}^{(N)} \dots \longrightarrow \mathbb{L}^{(2N)} \dots \longrightarrow \mathbb{L}^{(N^2)}$$

$$\omega^{(0)} \longrightarrow \omega^{(1)} \longrightarrow \omega^{(2)} \dots \longrightarrow \omega^{(N)} \quad \dots \quad \dots \quad \longrightarrow \omega^{(N^2)}$$

Towards universality: idea



$$\mathbb{L}^{(0)} \longrightarrow \mathbb{L}^{(1)} \longrightarrow \mathbb{L}^{(2)} \cdots \longrightarrow \mathbb{L}^{(N)} \cdots \longrightarrow \mathbb{L}^{(2N)} \cdots \longrightarrow \mathbb{L}^{(N^2)}$$
$$\omega^{(0)} \longrightarrow \omega^{(1)} \longrightarrow \omega^{(2)} \cdots \longrightarrow \omega^{(N)} \quad \cdots \quad \cdots \quad \longrightarrow \omega^{(N^2)}$$

i.i.d. centred changes $\Rightarrow O(\sqrt{N})$ differences $\Rightarrow$ universality

Simulation

[by Jhih-Huang Li]

Towards universality: problems

- Encoding geometry of loops?

- Are successive transformations i.i.d./ ergodic?

- Is the drift (i.e. expected change) 0?

Towards universality: problems

- Encoding geometry of loops?

  Keep track of extrema (tops) of large clusters;

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  Keep track of extrema (tops) of large clusters;

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  Don't worry about it (resample at each stage)

- ☐ Is the drift (i.e. expected change) 0?

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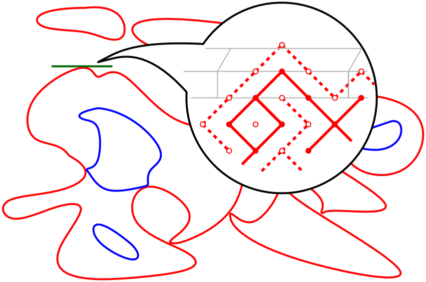
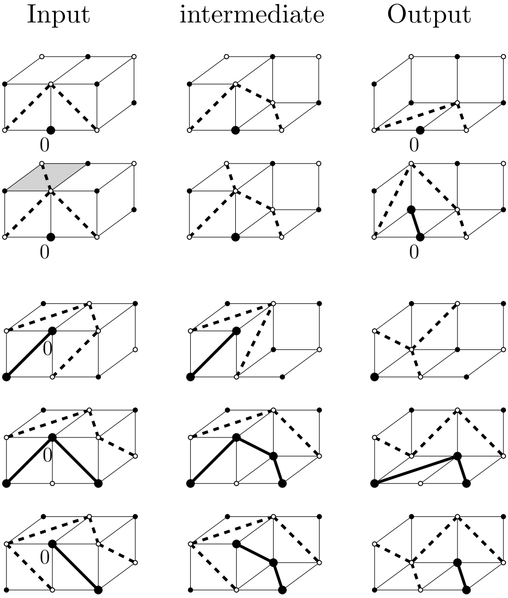


Drift is well defined (for top)



Increment

0

$\sin\alpha$

-1

$\sin\alpha - 1$

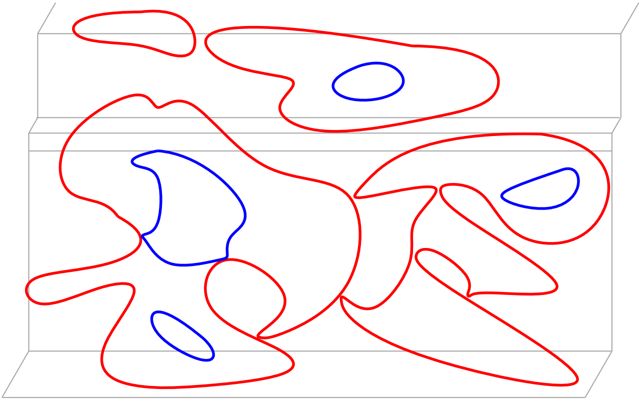
$\sin\alpha - 1$

Encoding geometry of large loops

Fix $\epsilon < \eta < 1$.

Extrema of clusters $\geq \epsilon$ don't move $\Rightarrow$ larger loops conserve shape

Clusters $\geq \epsilon$ move with drift $\Rightarrow$ larger loops affected linearly.



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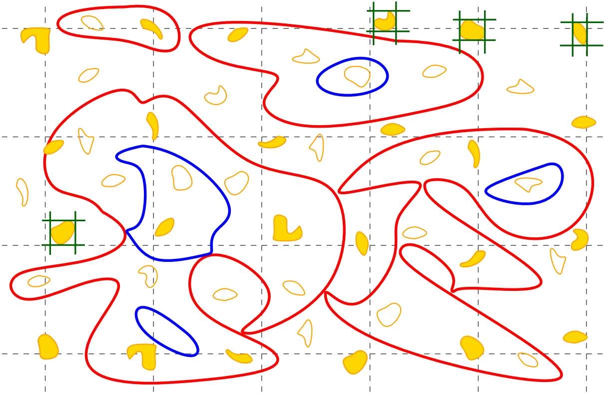


Towards universality: problems

- ☑ How to encode the geometry of loops?

  Keep track of extrema (tops) of large clusters;

- ☑ Are successive transformations i.i.d./ ergodic?

  The can be made to be i.i.d.

Law of large numbers for cluster movement!

Towards universality: problems

☑ How to encode the geometry of loops?

Keep track of extrema (tops) of large clusters;

☑ Are successive transformations i.i.d./ ergodic?

The can be made to be i.i.d.

Law of large numbers for cluster movement!

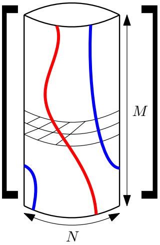
☐ Is the drift zero? $v_{\text{Top}}(\frac{\pi}{2}, \alpha)$, $v_{\text{Right}}(\frac{\pi}{2}, \alpha)$

Option 1: exact integrability (via 6V and Bethe Ansatz)

Option 2: use symmetries of model

Zero drift via six vertex

$$\phi_{\mathbf{L}(\alpha)}$$



$$\sim \frac{Z_{6V}^{(1)}(M,N)}{Z_{6V}^{(0)}(M,N)} \sim \left(\frac{\Lambda^{(1)}(N)}{\Lambda^{(0)}(N)}\right)^M \sim e^{-c(\alpha)\frac{M}{N}}$$

weights $a = \frac{\sin\left(\lambda\frac{\alpha}{\pi}\right)}{\sin(\lambda/2)}$; $b = \frac{\sin\left(\lambda(1-\frac{\alpha}{\pi})\right)}{\sin(\lambda/2)}$; $c = 2\cos(\lambda/2)$;

$(\lambda = \arccos(\sqrt{q}/2))$.

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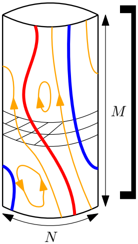
$\phi_{\mathbf{L}(\alpha)}\Big[$ 

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Fact: $c(\alpha) = C_\Delta \cdot \sin(\alpha)$ (when M denotes the number or rows)

$c(\alpha)$ computed explicitly using Bethe Ansatz

$\phi_{\mathbb{L}(\alpha)}[$ primal/dual vertical interface on $\mathbb{T}_{M,N}]$
only depends on the **euclidian** height of $\mathbb{T}_{M,N}$

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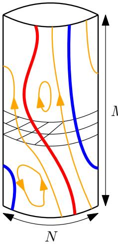
$$\phi_{\mathbf{L}(\alpha)}\Bigg[$$ 

$$\Bigg] \sim \frac{Z^{(1)}_{6V}(M,N)}{Z^{(0)}_{6V}(M,N)} \sim \left(\frac{\Lambda^{(1)}(N)}{\Lambda^{(0)}(N)}\right)^M \sim e^{-c(\alpha)\frac{M}{N}}$$

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Implies $v_{\text{Top}}(\pi/2,\alpha) = 0$**.**

Towards universality: problems

- How to encode the geometry of loops?

  Keep track of extrema (tops) of large clusters;

  If these don't move much, then nothing moves much!

- Are successive transformations i.i.d./ ergodic?

  Don't worry about it (resample at each stage)

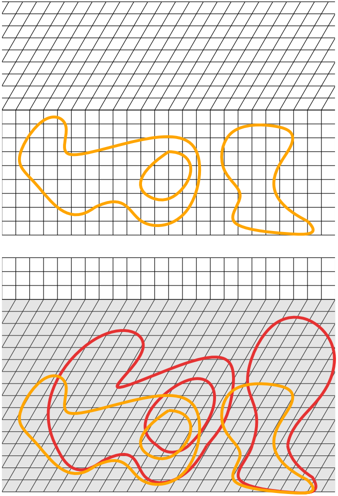
- Is the drift (i.e. expected change) 0?

  $v_{\text{Top}} = \mathbb{E}[\Delta_{\text{Top}}]$ is well defined. Show it's 0:

  ~~Option 1: exact integrability (via 6V and Bethe Ansatz)~~

  Option 2: use symmetries of model

What if drift $\neq 0$?



Theorem

For any $\alpha \in (0, \pi)$ *there exists* $A_{\alpha,\pi/2}$ *linear s.t.*

$$\mathbf{d}\big(\phi_{\delta\mathbb{L}(\pi/2)}, \phi_{\delta\mathbb{L}(\alpha)} \circ A_{\alpha,\pi/2}\big) \xrightarrow[\delta\to 0]{} 0.$$

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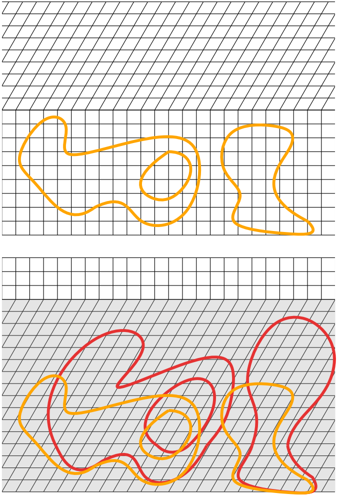

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Corollary

For any α, $\phi_{\delta\mathbb{L}(\pi/2)}$ is asymptotically invariant under $A_{\alpha,\pi/2} S_\alpha A^{-1}_{\alpha,\pi/2}$:

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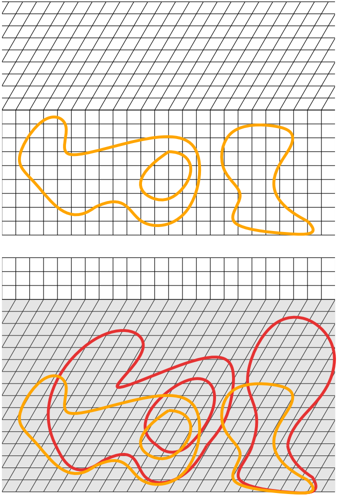

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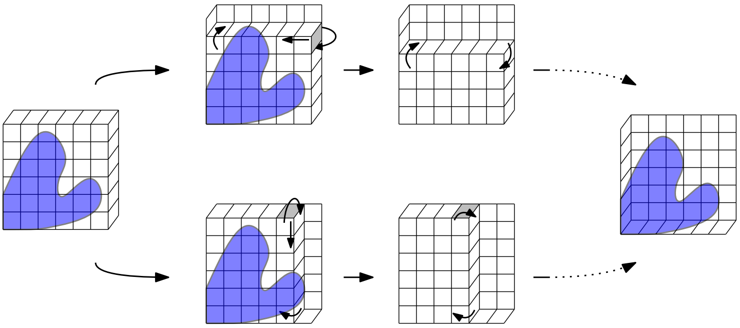
Fact: The group generated by $\{A_{\alpha,\pi/2} S_\alpha A^{-1}_{\alpha,\pi/2} : \alpha \in (0,\pi)\}$ contains **all rotations**.

And now back to universality...

$$v_{\mathrm{Right}}(\pi/2, \alpha) = v_{\mathrm{Top}}(\pi/2, \pi/2 + \alpha) = v_{\mathrm{Top}}(\pi/2, \pi/2 - \alpha).$$



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- By symmetry, $\quad v_{\text{Top}}(\alpha, \alpha/2) = v_{\text{Right}}(\alpha, \alpha/2).$
- $\phi_{\mathbb{L}(\alpha/2)}$ invariant under $S_{\alpha/2}$

Lemma *For all $\alpha \in (0, \pi)$, if $\phi_{\mathbb{L}(\alpha)}$ is asymptotically rotationally invariant, then $A_{\alpha,\alpha/2} = \mathrm{id}$.*

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Lemma *For all $\alpha \in (0,\pi)$, if $\phi_{\mathbb{L}(\alpha)}$ is asymptotically rotationally invariant, then $A_{\alpha,\alpha/2} = \mathrm{id}$.*

Set $\mathcal{A}$ the set of angles $\alpha \in (0,\pi)$ such that

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Known: • $\pi/2 \in \mathcal{A}$

• $\alpha \in \mathcal{A} \Rightarrow \phi_{\mathbb{L}(\alpha)}$ rot. invariant $\Rightarrow \alpha/2 \in \mathcal{A}$

• $\alpha \in \mathcal{A} \Rightarrow \pi - \alpha \in \mathcal{A}$

Conclusion: $\mathcal{A}$ dense in $[0, \pi] \Rightarrow$ universality

Thank you!