

Pinning and Wetting Transition for (1+1)-Dimensional Fields with Laplacian Interaction

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Workshop on Gradient Models and Elasticity

University of Warwick, June 11th, 2008

References

- ▶ [CD1] F. Caravenna and J.-D. Deuschel
Pinning and wetting transition for (1+1)-dimensional fields with Laplacian interaction, Ann. Probab. (to appear)

- ▶ [CD2] F. Caravenna and J.-D. Deuschel
Scaling limits of (1+1)-dimensional pinning models with Laplacian interaction, preprint (2008).

Outline

1. The Models

Introduction

Wetting and pinning models

2. Free Energy Results

The free energy

The phase transition

The disordered case

3. Path Results

Path results

Refined critical scaling limit

4. Sketch of the Proof

Integrated random walk

Markov renewal theory

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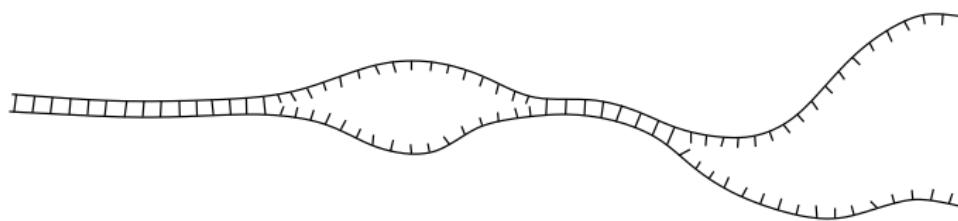
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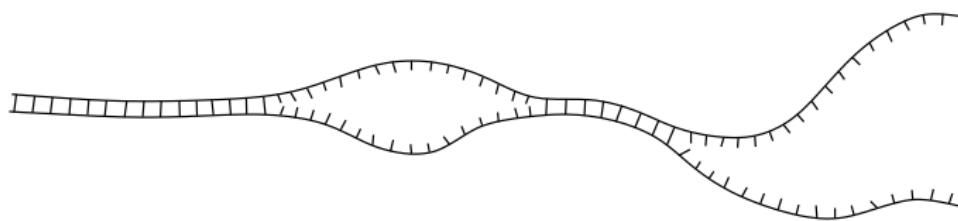
Some motivations

DNA denaturation transition at high temperature

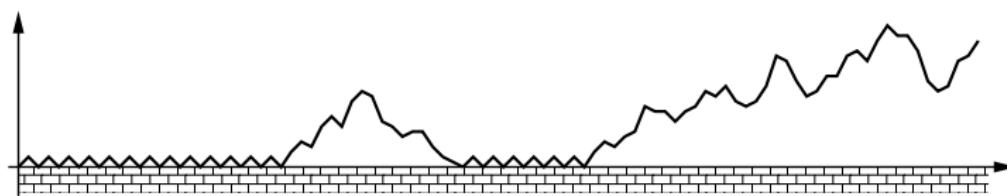


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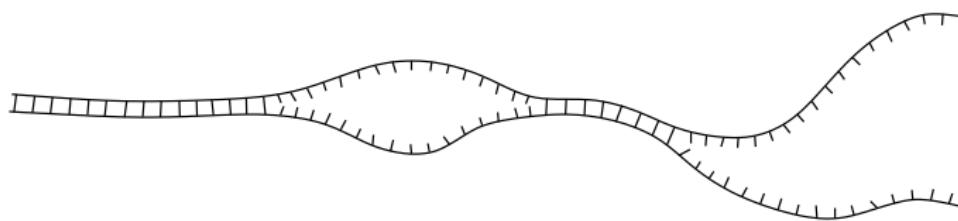


(1+1)-dimensional model: field above an impenetrable wall

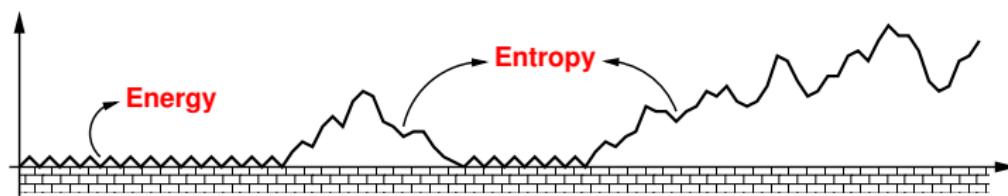


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(1+1)-dimensional model: field above an impenetrable wall



The general wetting model

The field $\varphi = \{\varphi_i\}_{1 \leq i \leq N}$ in the free case:

$$\mathbb{P}_{0,N}^w(d\varphi_1, \dots, d\varphi_N) := \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}_{0,N}^w} \prod_{i=1}^N d\varphi_i^+$$

- ▶ $d\varphi_i^+$ is the Lebesgue measure on $[0, \infty)$
- ▶ $\mathcal{H}_N(\varphi)$ describes the structure of the chain (to be specified)
- ▶ $\mathcal{Z}_{0,N}^w$ is the normalization constant (partition function)

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- ▶ $\delta_0(\cdot)$ is the Dirac mass at zero
- ▶ $\varepsilon \geq 0$ is the strength of the pinning interaction

The general pinning model

Analogous to the wetting case but without repulsion: $d\varphi_i^+ \rightarrow d\varphi_i$

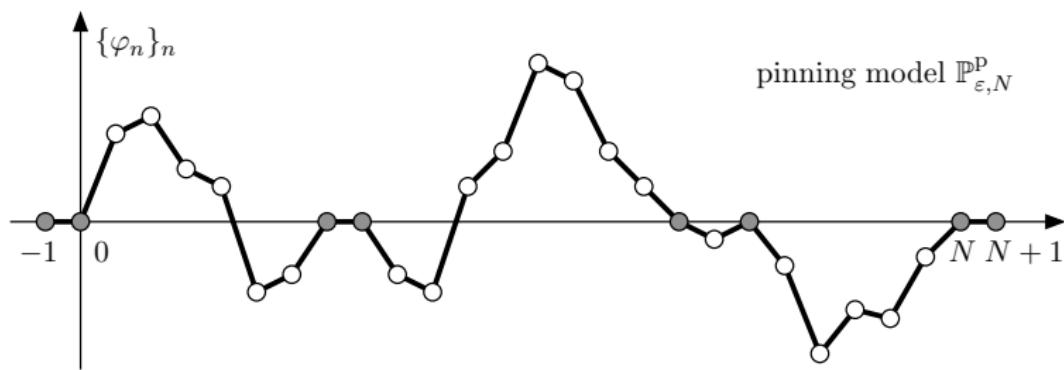
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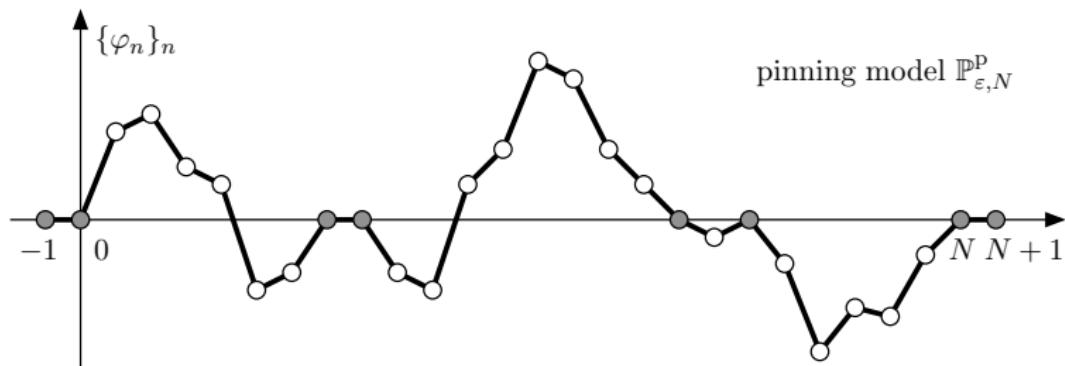
$$\mathbb{P}_{\varepsilon, N}^p(d\varphi_1, \dots, d\varphi_N) := \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}_{\varepsilon, N}^p} \prod_{i=1}^N (d\varphi_i + \varepsilon \cdot \delta_0(d\varphi_i))$$

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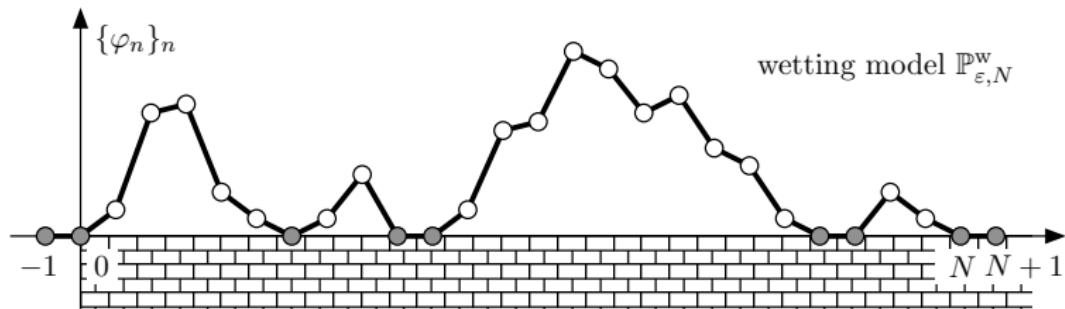
Pinning VS Wetting (+ boundary conditions)



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pinning model $\mathbb{P}_{\varepsilon,N}^P$



wetting model $\mathbb{P}_{\varepsilon,N}^W$

Some questions

Once $\mathcal{H}_N(\varphi)$ is chosen and $\varepsilon \geq 0$ is fixed:

- ▶ What are the properties of $\mathbb{P}_{\varepsilon,N}^{\text{P}}$ and $\mathbb{P}_{\varepsilon,N}^{\text{W}}$ for large N ?

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How to choose $\mathcal{H}_N(\varphi)$?

The choice of $\mathcal{H}_N(\varphi)$

The simplest choice is the gradient case:

$$\mathcal{H}_N(\varphi) := \sum_{i=1}^N V(\nabla \varphi_i), \quad \nabla \varphi_i := \varphi_i - \varphi_{i-1},$$

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+ regularity (see later). **Gaussian case:** $V(x) \propto x^2$

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[Itozaki, Yoshida SPA 01] [Deuschel, Giacomin, Zambotti PTRF 05]
 [Caravenna, Giacomin, Zambotti EJP 06]

The choice of $\mathcal{H}_N(\varphi)$

We rather consider the Laplacian case:

$$\mathcal{H}_N(\varphi) := \sum_{i=0}^N V(\Delta\varphi_i)$$

$$\Delta\varphi_i := \nabla\varphi_{i+1} - \nabla\varphi_i = \varphi_{i+1} + \varphi_{i-1} - 2\varphi_i$$

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- ▶ [Semi-flexible polymers](#): Δ favors affine configurations

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Interpretation of the free case $\varepsilon = 0$:

- ▶ $\mathbb{P}_{0,N}^{\text{P}}$ is (the bridge of) the integral of a random walk
- ▶ $\mathbb{P}_{0,N}^{\text{w}}$ is further conditioned to stay ≥ 0

Laplacian interaction in $(d + 1)$ -dimension

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Henceforth we study $\mathbb{P}_{\varepsilon, N}^{\text{P}}$ and $\mathbb{P}_{\varepsilon, N}^{\text{W}}$ with [Laplacian interaction](#) and boundary conditions $\varphi_{-1} = \varphi_0 = \varphi_N = \varphi_{N+1} = 0$

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Assumptions on V :

$$\int_{\mathbb{R}} e^{-V(x)} dx = 1, \quad \int_{\mathbb{R}} x e^{-V(x)} dx = 0, \quad \int_{\mathbb{R}} x^2 e^{-V(x)} dx = 1$$

+ regularity: $x \mapsto e^{-V(x)}$ continuous and $V(0) < +\infty$.

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How to define localization and delocalization?

Recall the [partition function](#): (zero boundary conditions)

$$\mathcal{Z}_{\varepsilon, N}^a = \int_{\Omega_N^a} e^{-\mathcal{H}_N(\varphi)} \prod_{i=1}^{N-1} (d\varphi_i + \varepsilon \delta_0(d\varphi_i))$$

where $a \in \{p, w\}$ and $\Omega_N^p = \mathbb{R}^{N-1}$ while $\Omega_N^w = [0, \infty)^{N-1}$

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$$F^a(\varepsilon) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Z}_{\varepsilon, N}^a \quad (\text{super-additivity})$$

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Basic observation: $F^a(\varepsilon) \geq F^a(0) = 0$ for all $\varepsilon \geq 0$ and $a \in \{p, w\}$

$$\mathcal{Z}_{\varepsilon, N}^a \geq \mathcal{Z}_{0, N}^a \approx N^{-c} \quad (c > 0)$$

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 - ▶ if $F^a(\varepsilon_c^a + h) \geq C h$ [**1st order trans.**] $\varepsilon = \varepsilon_c^a$ may be **localized** (phase coexistence, dependence of boundary conditions)

The phase transition

Theorem ([CD1])

Both $\mathbb{P}_{\varepsilon, N}^p$ and $\mathbb{P}_{\varepsilon, N}^w$ undergo a non-trivial phase transition:

$$0 < \varepsilon_c^p < \varepsilon_c^w < \infty$$

and $F^a(\varepsilon)$ is analytic on $[0, \varepsilon_c^a) \cup (\varepsilon_c^a, \infty)$. ([variational formula](#))

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$$C_1 \frac{h}{\log \frac{1}{h}} \leq F^p(\varepsilon_c^p + h) \leq o(h)$$

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- In the wetting model the transition is of 1st order:

$$F^w(\varepsilon_c^w + h) \sim C_2 h \quad [\ell_N \sim D N, \quad D > 0]$$

The gradient case

Differences in the gradient case

- ▶ the transition is non-trivial only in the wetting model:

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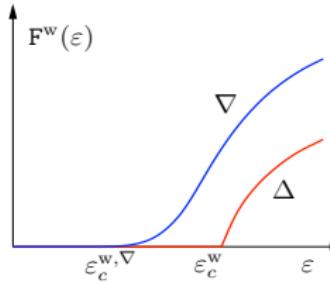
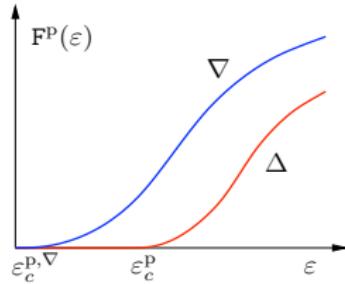
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▶ skip

A look at the disordered case

Disordered version of our model: $(d\varphi_i^p = d\varphi_i \text{ and } d\varphi_i^w = d\varphi_i^+)$

$$\mathbb{P}_{\varepsilon, \beta, \omega, N}^a(d\varphi_1, \dots, d\varphi_N) := \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}_{\varepsilon, \beta, \omega, N}^a} \prod_{i=1}^N (d\varphi_i^a + \varepsilon e^{\beta \omega_i} \delta_0(d\varphi_i))$$

where $\beta \geq 0$ and $\{\omega_i\}_{i \in \mathbb{N}}$ are IID $\mathcal{N}(0, 1)$ (law P indep. \mathbb{P}^a).

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Quenched free energy

$$F^a(\varepsilon, \beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Z}_{\varepsilon, \beta, \omega, N}^a \geq 0$$

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Localization: $F^a(\varepsilon, \beta) > 0 \iff \varepsilon > \varepsilon_c^a(\beta)$ (critical line)

Smoothing effect of disorder

What is the behavior of $\varepsilon_c^a(\beta)$ for small β ? ($\varepsilon_c^a = \varepsilon_c^a(0)$)

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Very general proof: rare-stretches in ω (Large Deviations)

► proof

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Some deeper questions

We have established the existence of a phase transition:

$$\ell_N = \begin{cases} o(N) & \text{if } \varepsilon < \varepsilon_c^a \\ \sim D \cdot N & \text{if } \varepsilon > \varepsilon_c^a \end{cases}$$

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Yes in the **pinning case** and under additional assumptions on $V(\cdot)$:

- ▶ **symmetry**: $V(x) = V(-x)$ for every $x \in \mathbb{R}$
- ▶ **uniform strict convexity**: $\exists \gamma > 0$ s. t. $V(x) - \gamma \frac{x^2}{2}$ is convex
- ▶ **regularity**: $x \mapsto e^{-V(x)}$ is continuous and $V(0) < \infty$

$$\int_{\mathbb{R}} e^{-V(x)} dx = 1 \quad \int_{\mathbb{R}} x^2 e^{-V(x)} dx = 1$$

A closer look at the typical paths [CD2]

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Scaling Limits

We rescale and interpolate linearly the field: for $t \in [0, 1]$

$$\hat{\varphi}_N(t) := \frac{\varphi_{\lfloor Nt \rfloor}}{N^{3/2}} + (Nt - \lfloor Nt \rfloor) \frac{\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}}{N^{3/2}}$$

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Let $\{B_t\}_{t \in [0,1]}$ standard BM, $I_t := \int_0^t B_s \, ds$ integrated BM

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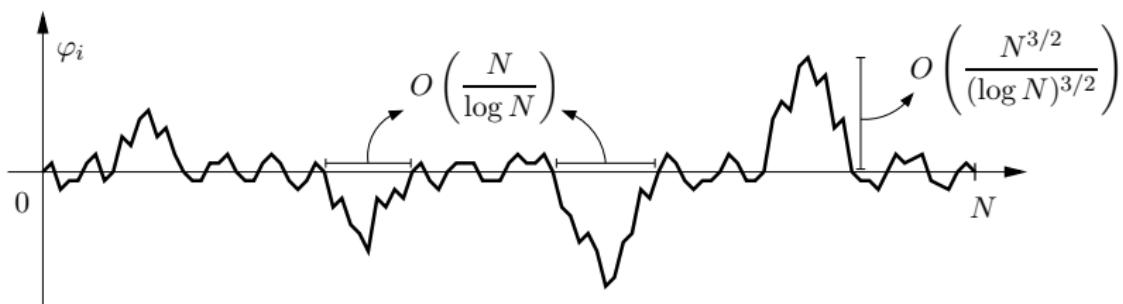
Theorem (Scaling Limits [CD2])

The rescaled field $\{\hat{\varphi}_N(t)\}_{t \in [0,1]}$ under $\mathbb{P}_{\varepsilon, N}^p$ converges in distribution on $C([0, 1])$ as $N \rightarrow \infty$, for every $\varepsilon \geq 0$. The limit is

- ▶ If $\varepsilon < \varepsilon_c^p$, the law of $\{\hat{I}_t\}_{t \in [0,1]}$
- ▶ If $\varepsilon = \varepsilon_c^p$ or $\varepsilon > \varepsilon_c^p$, the law concentrated on $f(t) \equiv 0$

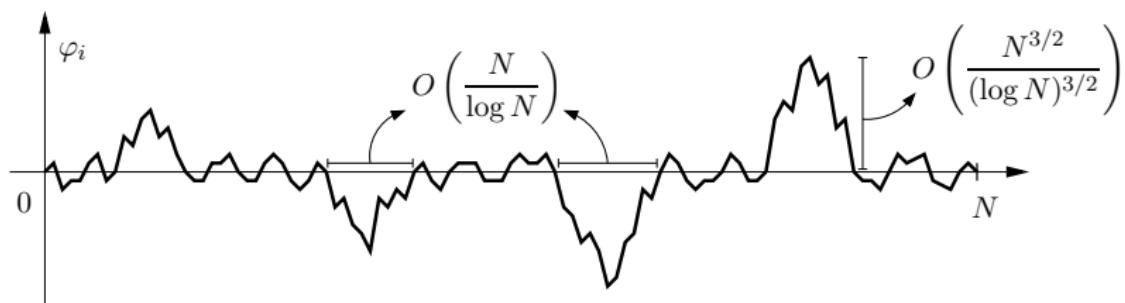
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For $\varepsilon = \varepsilon_c^p$ the field has very large fluctuations ($\approx \frac{N^{3/2}}{(\log N)^c}$).



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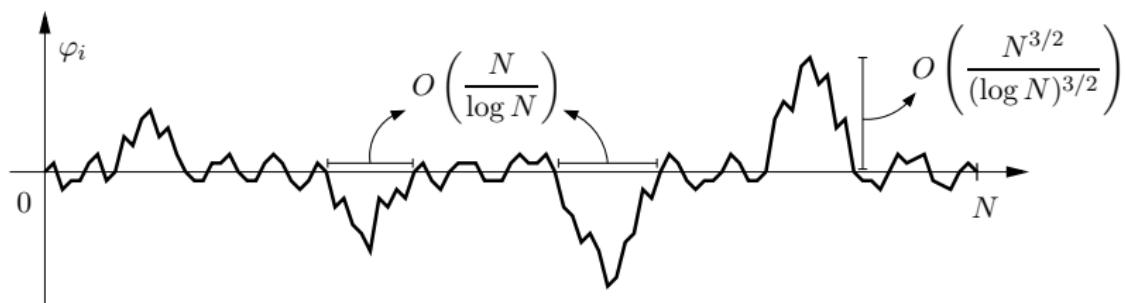
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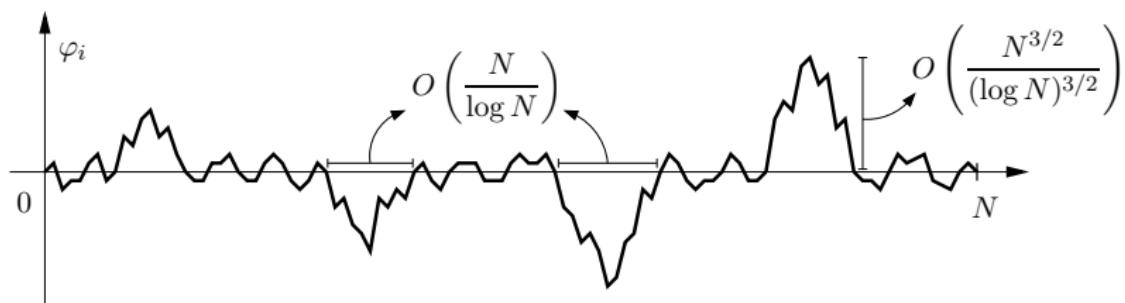
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the set $\frac{1}{N} \{i \in \{1, \dots, N\} : \varphi_i = 0\}$ becomes dense in $[0, 1]$

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Alternative idea: look at the field in a distributional sense

The critical regime

Introduce the random measure (finite, signed) on $[0, 1]$

$$\mu_N(dt) := \frac{(\log N)^{5/2}}{N^{3/2}} \varphi_{\lfloor Nt \rfloor} dt = \tilde{\varphi}_N(t) dt$$

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Let $\{L_t\}_{t \in [0,1]}$ be the stable symmetric Lévy process of index $\frac{2}{5}$

$$\begin{array}{lll} 0 \text{ drift} & 0 \text{ Brownian component} & \Pi(dx) = c|x|^{-7/5} dx \end{array}$$

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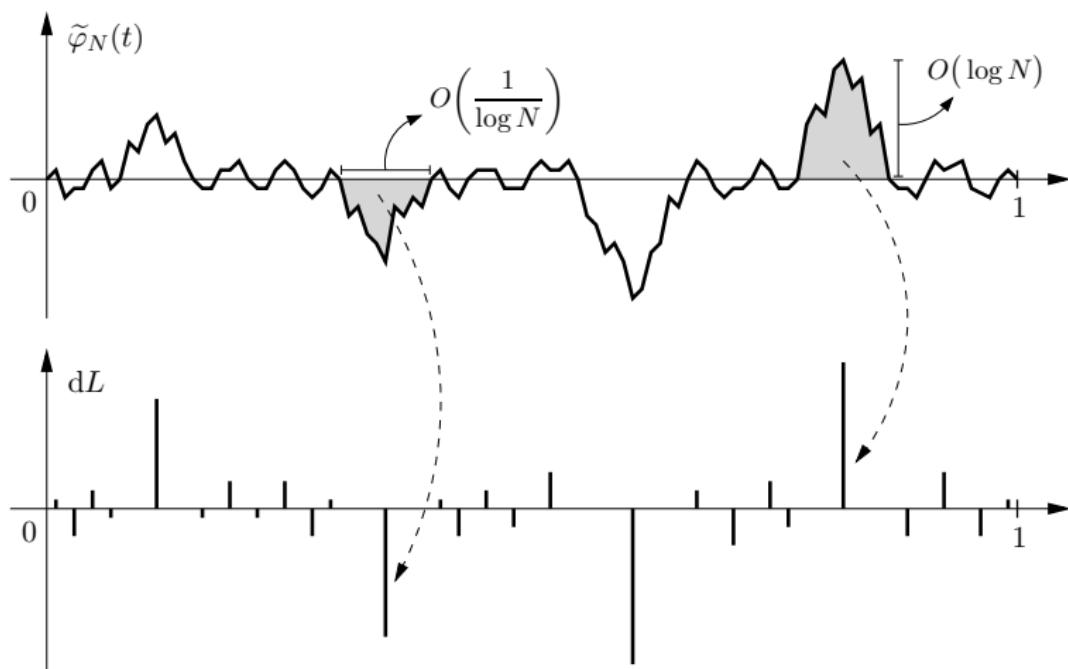
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Theorem ([CD2])

μ_N under $\mathbb{P}_{\varepsilon_c^p, N}^p$ converges in distribution as $N \rightarrow \infty$ toward dL .

The critical regime



disorder

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A random walk viewpoint ($\varepsilon = 0$)

Let $\{X_i\}_{i \in \mathbb{N}}$ be IID random variables with law

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The free case $\varepsilon = 0$

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- ▶ Under $\mathbb{P}_{0,N}^w$ the same, under the further conditioning $\{Z_1 \geq 0, \dots, Z_N \geq 0\}$

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Once we know τ, J , the whole field $\{\varphi_i\}_i$ is reconstructed by pasting independent excursions from $\{Z_i\}_i$ (cond. to stay ≥ 0)

The law of the excursions

Pinning case: good control (Donsker's inv. pr. + LLT)

$$\left\{ \frac{Z_{\langle Nt \rangle}}{N^{3/2}} \right\}_{t \in [0,1]} \text{ condit. on } (Y_N, Z_N) = (0,0) \implies \{\hat{I}_t\}_{t \in [0,1]}$$

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$$\mathbf{P}(Z_1 \geq 0, \dots, Z_N \geq 0) \approx ?$$

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Entropic repulsion

$$\mathbf{P}(Z_1 \geq 0, \dots, Z_N \geq 0) \approx N^{-1/4} \quad [\text{Sinai (SRW)}]$$

$$\mathbf{P}(Z_1 \geq 0, \dots, Z_N \geq 0 \mid Y_N = 0, Z_N = 0) \approx N^{-1/2} \quad [\text{conj.}]$$

Markov renewal processes

Given a (sub-)probability kernel $K_{x,dy}(n)$:

$$\int_{y \in \mathbb{R}} \sum_{n \in \mathbb{N}} K_{x,dy}(n) = c \leq 1, \quad \forall x \in \mathbb{R}$$

we build the Markov renewal process τ with modulating chain J :

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The law of (τ, J) conditionally on $\{N, N + 1\} \subseteq \tau$ is

$$\mathcal{P}(\tau_i = t_i, J_i \in dy_i \mid \{N, N + 1\} \subseteq \tau) = \frac{1}{C_N} \prod_i K_{y_{i-1},dy_i}(t_i - t_{i-1})$$

with $C_N = \mathcal{P}(\{N, N + 1\} \subseteq \tau)$.

The law of the contact set

Consider the following kernels: for $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$

$$G_{x,dy}^p(n) := \varepsilon \frac{\mathbf{P}_x(Z_{n-1} \in dy, Z_n \in dz)}{dz} \Big|_{z=0}$$

$$G_{x,dy}^w(n) := \varepsilon \frac{\mathbf{P}_x(Z_i \geq 0, i \leq n, Z_{n-1} \in dy, Z_n \in dz)}{dz} \Big|_{z=0}$$

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Reminds of [Markov renewal theory](#)

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We exploit the **invariance properties**: for every F , $v(y)$

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If we determine F , $v(\cdot)$ such that

$$K_{x, dy}(n) := G_{x, dy}^a(n) e^{-F \cdot n} \frac{v(y)}{v(x)}$$

is a **probability kernel**, we have the crucial relation

$$\mathbb{P}_{\varepsilon, N}^a (\tau_i = t_i, J_i \in dy_i) = \mathcal{P}(\tau_i = t_i, J_i \in dy_i \mid \{N, N+1\} \subseteq \tau)$$

A Perron-Frobenius problem

It turns out that:

- ▶ F is the solution of the equation

$$\text{spectral radius of } \left(\sum_{n \in \mathbb{N}} G_{x,dy}^a(n) e^{-F \cdot n} \right)_{x,y} = 1$$

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- ▶ In fact $F = F^a(\varepsilon)$ is the free energy
- ▶ $v(\cdot)$ is the principal eigenfunction:

$$\int_{y \in \mathbb{R}} \left(\sum_{n \in \mathbb{N}} G_{x,dy}^a(n) e^{-F \cdot n} \right)_{x,y} v(y) = v(x)$$