On the 2d KPZ and Stochastic Heat Equation

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Overview

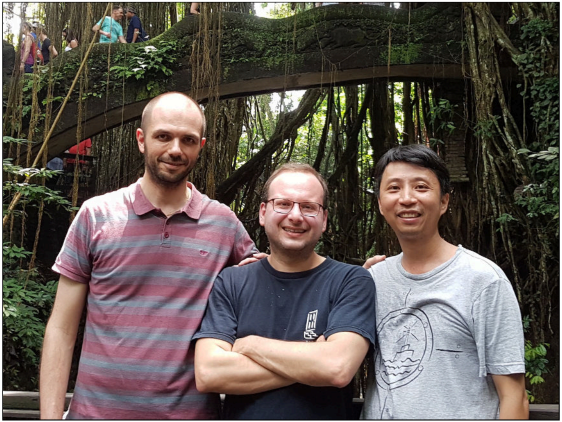
I will talk about two stochastic PDEs on $\mathbb{R}^d$ (mainly $d = 2$)

- Kardar-Parisi-Zhang Equation (KPZ)
- Multiplicative Stochastic Heat Equation (SHE)

- Very interesting, yet ill-defined equations!
- GOAL: find a "reasonable solution"

Stochastic Analysis $\leftrightsquigarrow$ Statistical Mechanics

Collaborators



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

References

- [CSZ 17]
  Universality in marginally relevant disordered systems
  Ann. Appl. Probab. 2017

- [CSZ 18a]
  On the moments of the (2+1)-dimensional directed polymer and Stochastic Heat Equation in the critical window
  to appear in Commun. Math. Phys.

- [CSZ 18b]
  The two-dimensional KPZ equation in the entire subcritical regime
  arXiv, Dec 2018

($d = 2$) [Bertini Cancrini 98]
[Chatterjee Dunlap 18] [Gu 18] [Gu Quastel Tsai 19]

($d \geq 3$) [Magnen Unterberger 18] [Gu Ryzhik Zeitouni 18]
[Dunlap Gu Ryzhik Zeitouni 19] [Comets Cosco Mukherjee 18 19a 19b]

White noise

What is (space-time) white noise $\xi = \xi(t,x)$ on $\mathbb{R}^{1+d}$?

- A centered Gaussian field with

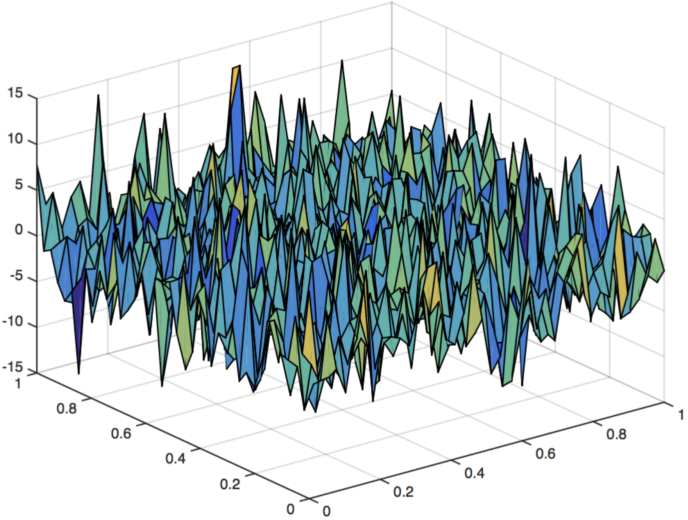
$$\mathrm{Cov}[\,\xi(t,x), \xi(t',x')\,] = \delta(t-t')\,\delta(x-x')$$

- A random (Schwartz) distribution such that

$$\langle \phi, \xi \rangle \;=\; \int_{\mathbb{R}^{1+d}} \phi(t,x)\, \xi(t,x)\, \mathrm{d}t\, \mathrm{d}x \;\sim\; N(0, \|\phi\|_{L^2}^2)$$

- The scaling limit of i.i.d. $N(0,1)$ random variables indexed by $\mathbb{Z}^{1+d}$

White noise



The simplest stochastic PDE

Additive Stochastic Heat Equation $(t > 0, \quad x \in \mathbb{R}^d)$

$$\partial_t v \; = \; \tfrac{1}{2} \, \Delta_x v \; + \; c \, \xi \qquad \text{(EW)}$$

also known as Edwards-Wilkinson equation $(c \in \mathbb{R})$

- Well-posed (pathwise) in any dimension d
- Explicit solution $v = c \, g * \xi$

$$v(t,x) = c \int_0^t \int_{\mathbb{R}^2} g_{t-s}(x-y) \, \xi(s,y) \, \mathrm{d}s \, \mathrm{d}y \qquad g_t(x) = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}}$$

v is a function for $d = 1$, a genuine distribution for $d \geq 2$

- v is Gaussian process with explicit covariance

The KPZ equation

KPZ [Kardar Parisi Zhang PRL'86]

$$\partial_t h = \tfrac{1}{2}\Delta_x h + \tfrac{1}{2}|\nabla_x h|^2 + \beta\,\xi \qquad \text{(KPZ)}$$

Model for random interface growth

$h = h(t,x) =$ interface height at time $t \geq 0$, space $x \in \mathbb{R}^d$

$\xi = \xi(t,x) =$ space-time white noise $\qquad \beta > 0$ noise strength

$|\nabla_x h|^2$ ill-defined $\qquad (\nabla_x h(t,\cdot)$ exp. distribution$)$

For smooth ξ we can linearize KPZ by Cole-Hopf transformation

$$u(t,x) := e^{h(t,x)}$$

The multiplicative Stochastic Heat Equation (SHE)

SHE $(t > 0, \ x \in \mathbb{R}^d)$

$$\partial_t u = \tfrac{1}{2} \Delta_x u + \beta\, u\, \xi \qquad \text{(SHE)}$$

Product $u\,\xi$ ill-defined (non smooth function $\times$ distribution)

- $(d = 1)$ Well-posed by Ito integration [Walsh 80's]
- $(d = 1)$ SHE and KPZ are now well-understood in a robust sense Regularity Structures (Hairer) or Paracontrolled Distributions (GIP)
- $(d \geq 2)$ No general theory. How to "find a solution"?

(KPZ) and (SHE) in dimensions $d \geq 2$

Mollify the white noise $\xi(t,x)$ in space on scale $\varepsilon > 0$

$$\xi^\varepsilon(t,\cdot) := \xi(t,\cdot) * \varrho_\varepsilon$$

$\varrho_\varepsilon(x) := \varepsilon^{-d}\varrho(\varepsilon^{-1}x)$ probability density $\varrho \in C_c^\infty(\mathbb{R}^d)$

- Replace ξ by ξ^ε $\rightsquigarrow$ solutions $h^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ well-defined (Ito)
- Do $h^\varepsilon(t,x)$ and $u^\varepsilon(t,x)$ have a limit as $\varepsilon \downarrow 0$?
- Disorder strength $\beta = \beta_\varepsilon$ needs to be renormalized

Mollified equations

Mollified SHE

$$\begin{cases} \partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, u^\varepsilon \, \xi^\varepsilon \\ u^\varepsilon(0,\cdot) \equiv 1 \end{cases} \qquad (\varepsilon\text{-SHE})$$

Set $h^\varepsilon(t,x) := \log u^\varepsilon(t,x)$ (Cole-Hopf). By Ito's formula

Mollified KPZ

$$\begin{cases} \partial_t h^\varepsilon = \frac{1}{2}\Delta h^\varepsilon + \frac{1}{2}|\nabla h^\varepsilon|^2 + \beta_\varepsilon \, \xi^\varepsilon - C_\varepsilon \\ h^\varepsilon(0,\cdot) \equiv 0 \end{cases} \qquad (\varepsilon\text{-KPZ})$$

where $C_\varepsilon = \beta_\varepsilon^2 \, \varepsilon^{-d} \, \|\varrho\|_{L^2}^2$

Key problem

Can we choose $\beta_\varepsilon \in (0,\infty)$ so that
$u^\varepsilon(t,x)$ and $h^\varepsilon(t,x)$ have a limit as $\varepsilon \downarrow 0$?

YES $(\ldots)$

$$\beta_\varepsilon = \begin{cases} \hat\beta \text{ (fixed)} & d=1 \\ \hat\beta \, \dfrac{1}{\sqrt{\log \varepsilon^{-1}}} & d=2 \\ \hat\beta \, \varepsilon^{\frac{d-2}{2}} & d \geq 3 \end{cases} \qquad \hat\beta \in (0,\infty)$$

Note that $\beta_\varepsilon \to 0$ for $d=2$ and $d \geq 3$

Main result I. Phase transition and LLN

Space dimension $d = 2$ $\qquad \beta_\varepsilon = \frac{\hat\beta}{\sqrt{\log \varepsilon^{-1}}}$ $\qquad \hat\beta \in (0, \infty)$

Theorem (SHE marginal laws) [CSZ 17]

Phase transition with critical value $\hat\beta_c = \sqrt{2\pi}$ ("weak to strong disorder")

- Fix $t > 0$, $x \in \mathbb{R}^2$:

$$u^\varepsilon(t,x) \xrightarrow[\varepsilon\downarrow 0]{d} \begin{cases} \exp\left(\sigma_{\hat\beta}\, Z - \frac{1}{2}\,\sigma_{\hat\beta}^2\right) & \text{if } \hat\beta < \sqrt{2\pi} \\ 0 & \text{if } \hat\beta \geq \sqrt{2\pi} \end{cases}$$

where

$$Z \sim N(0,1) \qquad \sigma_{\hat\beta}^2 := \log \frac{2\pi}{2\pi - \hat\beta^2}$$

Sub-critical regime $\hat{\beta} < \sqrt{2\pi}$

- Fix $\hat{\beta} < \sqrt{2\pi}$. For distinct $x_1, \ldots, x_n \in \mathbb{R}^2$

$$u^\varepsilon(t, x_i) \quad \text{become asymptotically independent as } \varepsilon \downarrow 0$$

Look at $u^\varepsilon(t, \cdot)$ as a random distribution on $\mathbb{R}^2$ $\qquad (\mathbb{E}[u^\varepsilon(t,x)] \equiv 1)$

Corollary (Law of large numbers)

$$u^\varepsilon(t, \cdot) \xrightarrow[\varepsilon \downarrow 0]{d} 1 \qquad \text{as a distribution}$$

$$\forall \phi \in C_c(\mathbb{R}^2): \qquad \int_{\mathbb{R}^2} u^\varepsilon(t,x)\, \phi(x)\, \mathrm{d}x \xrightarrow[\varepsilon \downarrow 0]{d} \int_{\mathbb{R}^2} \phi(x)\, \mathrm{d}x$$

Analogous results for the KPZ solution $h^\varepsilon(t,x) = \log u^\varepsilon(t,x)$

Main result II. Fluctuations for SHE

Recall that $\beta_\varepsilon = \dfrac{\hat\beta}{\sqrt{\log \varepsilon^{-1}}}$ sub-critical $\hat\beta \in (0, \sqrt{2\pi})$

Rescaled SHE solution $\mathcal{U}^\varepsilon(t,x) := \dfrac{1}{\beta_\varepsilon}\big(u^\varepsilon(t,x) - 1\big)$

Theorem (EW fluctuations for SHE) [CSZ 17]

$$\mathcal{U}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon \downarrow 0]{d} v(t,\cdot) \qquad \text{as a distribution}$$

v = solution of Edwards-Wilkinson equation

$$\partial_t v = \tfrac{1}{2}\Delta_x v + c\,\xi \qquad \text{where} \qquad c = \sqrt{\tfrac{2\pi}{2\pi - \hat\beta^2}} > 1 \tag{EW}$$

$$\partial_t \mathcal{U}^\varepsilon = \tfrac{1}{2}\Delta_x \mathcal{U}^\varepsilon + (1 + \beta_\varepsilon \mathcal{U}^\varepsilon)\,\xi^\varepsilon$$

From SHE to KPZ?

Key tools

1. SHE solution $u^\varepsilon(t,x)$ has explicit Wiener-Chaos expansion
2. 4th Moment Theorem for Gaussianity [Nourdin, Peccati, Reinert]

Problem: Wiener Chaos not available for KPZ sol. $h^\varepsilon(t,x) = \log u^\varepsilon(t,x)$

Naive idea: Since $u^\varepsilon(t,x) \to 1$ (as a distribution), by Taylor expansion

$$h^\varepsilon(t,x) \approx \big(u^\varepsilon(t,x) - 1\big) \ ?$$

NO, because $u^\varepsilon(t,x)$ is not close to 1 pointwise

We can correct Taylor expansion and deduce fluctuations of h^ε from u^ε

Main result III. Sub-critical fluctuations for KPZ

Recall that $\beta_\varepsilon = \dfrac{\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ sub-critical $\hat{\beta} \in (0, \sqrt{2\pi})$

Rescaled KPZ solution $\mathcal{H}^\varepsilon(t,x) := \dfrac{1}{\beta_\varepsilon}\big(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\big)$

Theorem (EW fluctuations for KPZ) [CSZ 18b]

$$\mathcal{H}^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{d} v(t,\cdot) \qquad \text{as a distribution}$$

$v =$ solution of Edwards-Wilkinson equation

$$\partial_t v = \tfrac{1}{2}\Delta_x v + c\,\xi \qquad \text{where} \qquad c = \sqrt{\frac{2\pi}{2\pi - \hat{\beta}^2}} > 1 \qquad \text{(EW)}$$

Same as for SHE (but different centering)

Alternative proof by [Gu 18] (only for small $\hat{\beta}$)

A variation on KPZ

Last result was motivated by a paper of Chatterjee and Dunlap [CD 18] who considered a variation of KPZ

$$\partial_t \tilde{h}^\varepsilon = \tfrac{1}{2}\Delta_x \tilde{h}^\varepsilon + \tfrac{1}{2}\,\beta_\varepsilon\,|\nabla_x \tilde{h}^\varepsilon|^2 + \xi^\varepsilon$$

The same $\beta_\varepsilon = \dfrac{\hat{\beta}}{\sqrt{\log \varepsilon^{-1}}}$ now multiplies the non-linearity instead of ξ^ε

Theorem [Chatterjee Dunlap 18]

For $\hat{\beta}$ sufficiently small, the centered solution $\tilde{h}^\varepsilon(t,\cdot) - \mathbb{E}[\tilde{h}^\varepsilon]$ admits subsequential limits in law as $\varepsilon \downarrow 0$ (as a random distribution on $\mathbb{R}^2$)

Any limit is not the solution of (EW) with $c = 1$
(what one would get simply removing the non-linearity)

Relation with our results

Recall "our" KPZ : $\partial_t h^\varepsilon = \frac{1}{2}\Delta_x h^\varepsilon + \frac{1}{2}|\nabla_x h^\varepsilon|^2 + \beta_\varepsilon\, \xi^\varepsilon - C_\varepsilon$

Scaling relation

$$\tilde{h}^\varepsilon(t,x) - \mathbb{E}[\tilde{h}^\varepsilon] = \frac{1}{\beta_\varepsilon}\left(h^\varepsilon(t,x) - \mathbb{E}[h^\varepsilon]\right) = \mathcal{H}^\varepsilon(t,x)$$

Theorem [CSZ 18b]

For every sub-critical $\hat{\beta} < 1$, the centered solution $\tilde{h}^\varepsilon(t,\cdot) - \mathbb{E}[\tilde{h}^\varepsilon]$ admits a unique limit in law as $\varepsilon \downarrow 0$ (as a random distribution on $\mathbb{R}^2$)

The limit is the solution of (EW) with $c = \frac{1}{\sqrt{1-\hat{\beta}^2}} > 1$

Grazie!

The critical regime

What about the critical point $\hat{\beta} = \sqrt{2\pi}$?

More generally, critical window [Bertini Cancrini 98]

$$\beta_\varepsilon = \sqrt{\frac{2\pi}{\log \varepsilon^{-1}} \left(1 + \frac{\vartheta}{\log \varepsilon^{-1}}\right)} \qquad \text{with} \quad \vartheta \in \mathbb{R}$$

Nothing is known for KPZ $h^\varepsilon(t,x)$, some progress for SHE $u^\varepsilon(t,x)$

Key conjecture

$u^\varepsilon(t,\cdot)$ has a limit $\mathcal{U}(t,\cdot)$ for $\varepsilon \downarrow 0$, as a random distribution on $\mathbb{R}^2$

$$\langle u^\varepsilon(t,\cdot), \phi \rangle := \int_{\mathbb{R}^2} u^\varepsilon(t,x)\, \phi(x)\, \mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \int_{\mathbb{R}^2} \mathcal{U}(t,x)\, \phi(x)\, \mathrm{d}x$$

(actually a random measure, since $u^\varepsilon \geq 0$)

Second moment in the critical window

What is known [Bertini Cancrini 98]

Tightness via second moment bounds

$$\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle\big] \equiv \langle 1,\phi\rangle \qquad \sup_{\varepsilon>0} \mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] < \infty$$

More precisely $\mathbb{E}\big[\langle u^\varepsilon(t,\cdot),\phi\rangle^2\big] \xrightarrow[\varepsilon\downarrow 0]{} \langle \phi,\, K\phi\rangle < \infty$

Explicit kernel $K(x,x') \sim C \log \frac{1}{|x-x'|}$ as $|x-x'| \to 0$

Corollary

$$\exists \text{ subsequential limits } \langle u^{\varepsilon_k}(t,\cdot),\phi\rangle \xrightarrow[k\to\infty]{d} \langle \mathcal{U}\,,\,\phi\rangle$$

Can the limit be trivial $\mathcal{U}(t,\cdot) \equiv 1$?

Main result III. Third moment in the critical window

We compute the sharp asymptotics of third moment

Theorem [CSZ 18a]

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left[\langle u^\varepsilon(t,\cdot), \phi\rangle^3\right] = C(\phi) < \infty$$

Corollary

Any subsequential limit $u^{\varepsilon_k}(t,\cdot) \xrightarrow{d} \mathcal{U}(t,\cdot)$ has covariance $K(x,x')$

$$\rightsquigarrow \quad \mathcal{U}(t,\cdot) \not\equiv 1 \quad \text{is non-degenerate!}$$

Very recently, [Gu Quastel Tsai 19] improved our result showing that all moments converge to a finite limit (not only 3rd moment)