Renewal Theory, Disordered Systems, and Stochastic PDEs

Francesco Caravenna
Università degli Studi di Milano-Bicocca

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A journey through complex systems

A workshop in honor of Paolo Dai Pra's 60th birthday

To start with

I first met Paolo in 2003 (almost TWENTY years ago)

I was set to meet my PhD supervisor Giambattista (for the first time!)
and Paolo let us use his office in Padova

I came back to Padova in 2006 where I stayed until 2010
During those beautiful years I met many of the people that are now here

Paolo introduced me to the world of academia
I started teaching side to side by him
We wrote together a book and a research paper

He even helped me with hiking :-)

Grazie, Paolo!

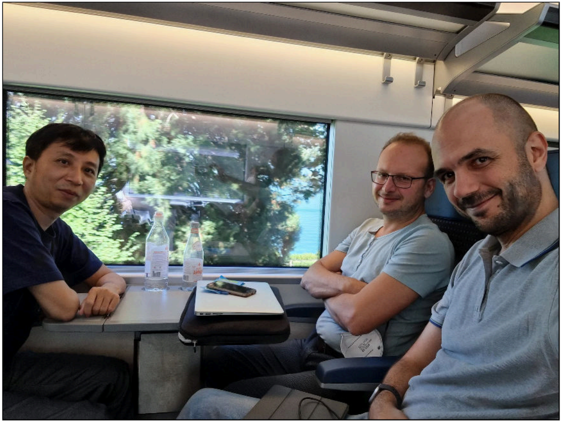
Overview

1. Renewal Theorem for ultra-heavy tailed renewal processes
2. Disordered Systems: the Directed Polymer in Random Environment
3. Stochastic PDEs: the $2d$ Stochastic Heat Equation

Main references:

- [CSZ19] *The Dickman subordinator, renewal theorems, and disordered systems*, EJP (2019)
- [CSZ21] *The Critical 2d Stochastic Heat Flow*, arXiv (2021)

Based on joint works with



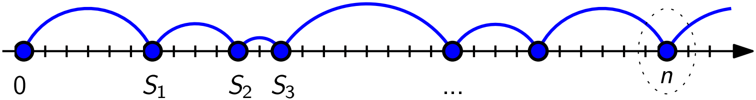
Rongfeng Sun (NUS) and Nikos Zygouras (Warwick)

Outline

Renewal process

Random walk $S_k := X_1 + X_2 + \ldots X_k$ with positive increments

(X_i) i.i.d. $\qquad X_i \in \mathbb{N} = \{1, 2, \ldots\}$ $\qquad$ aperiodic



Renewal function $\qquad u(n) := \mathrm{P}(S \text{ visits } n) = \sum_{k \geq 0} \mathrm{P}(S_k = n)$

Renewal Theorem (Erdos, Feller, Pollard 1949)

$$\lim_{n \to \infty} u(n) = \frac{1}{\mathrm{E}[X]} \qquad \text{also when } \mathrm{E}[X] = \infty$$

Heavy tails

When $\mathrm{E}[X] = \infty$ we have $u_n \to 0$. At which rate?

Tail Assumption

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} \frac{\ell_n}{n^\alpha} \qquad 0 < \alpha < 1 \qquad \ell_{\cdot} \text{ slowly varying}$$

Theorem

[Garsia, Lamperti 1962] [Doney 1997]

$$u(n) \underset{n\to\infty}{\sim} \frac{c}{\mathrm{E}[X \wedge n]} = \frac{c_\alpha}{\ell_n \, n^{1-\alpha}} \qquad \text{with} \quad c_\alpha := \frac{\sin \pi\alpha}{\pi}$$

$+$ local assumption for $\alpha \leq \frac{1}{2}$: $\qquad \mathrm{P}(X = n) \leq C \, \dfrac{\ell_n}{n^{1+\alpha}}$

Necessary and sufficient conditions are known [Caravenna, Doney 19]

Ultra-heavy tails

We now focus on the extreme case $\alpha = 0$

$$\mathrm{P}(X = n) = p_n \sim \frac{1}{n}$$

This makes sense via truncation at scale N

$$\mathrm{P}(X^{(N)} = n) = \frac{p_n \, \mathbb{1}_{\{1 \le n \le N\}}}{p_1 + \ldots + p_N} \sim \frac{1}{n} \, \frac{\mathbb{1}_{\{1 \le n \le N\}}}{\log N}$$

Triangular array of renewal processes

$$S_k^{(N)} = X_1^{(N)} + \ldots + X_k^{(N)}$$

Renewal function (exponentially weighted)

$$u^{(N)}(n) = \mathrm{P}(S^{(N)} \text{ visits } n) = \sum_{k \ge 0} \left(1 + \tfrac{\vartheta}{\log N}\right)^k \mathrm{P}(S_k^{(N)} = n)$$

Strong Renewal Theorem

Since $\mathrm{E}[X^{(N)}] \sim \dfrac{N}{\log N}$ we expect $u^{(N)}(n) \approx \dfrac{\log N}{N}$ as $n \approx N \to \infty$

Theorem [CSZ19]

$$u^{(N)}(n) \sim \frac{\log N}{N} \, G_\vartheta\left(\tfrac{n}{N}\right) \qquad \text{uniformly for } \delta N \le n \le N$$

where $$G_\vartheta(t) := \int_0^\infty \frac{\mathrm{e}^{(\vartheta-\gamma)s} \, s \, t^{s-1}}{\Gamma(s+1)} \, \mathrm{d}s$$

- Renewal process $S^{(N)} = (S_k^{(N)})_{k\in\mathbb{N}}$ (suitably rescaled) $\underset{N\to\infty}{\longrightarrow}$ Lévy process $Y = (Y_s)_{s\ge 0}$ "Dickman subordinator"
- $G_\vartheta(t)$ is the renewal function of Y

Outline

1. Renewal Theory

2. Disordered Systems

3. Stochastic PDEs

Directed Polymer in Random Environment

Disordered model in statistical mechanics

"random walk interacting with a random medium" (Gibbs)

Introduced in the 1980s to describe interfaces in Ising model

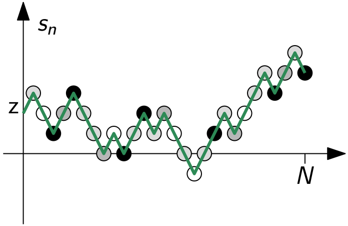
[Imbrie, Spencer JSP 88] [Bolthausen CMP 89]

A stream of mathematical research in the last 25 years

- Localization phenomena
- Super-diffusivity
- KPZ universality class

St. Flour 2016 Lecture notes by Francis Comets

Partition Functions



- $s = (s_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- $\omega(n,z)$ independent $\mathcal{N}(0,1)$ (disorder)
- $H_N(s,\omega) := \sum_{n=1}^{N} \omega(n, s_n) \qquad \sim \mathcal{N}(0,N)$

Directed Polymer Partition Functions $\quad (N \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z(N,z) := \mathrm{E}\left[e^{\beta H_N(s,\omega)}\right] \frac{\mathrm{E}\left[e^{\beta H_N(s,\omega)}\right]}{e^{\frac{1}{2}\beta^2 N}} = \frac{1}{(2d)^N} \sum_{\substack{s=(s_0,\ldots,s_N) \\ \text{s.r.w. path with } s_0=z}} e^{\beta H_N(s,\omega)} \frac{e^{\beta H_N(s,\omega)}}{e^{\frac{1}{2}\beta^2 N}}$$

Hidden (but deep) connection to the renewal function $u^{(N)}(n)$!

Moments

The random variables $\big(Z(N,z)\big)_{z\in\mathbb{Z}^d}$ depend on disorder ω

- They are stationary with unit mean:

$$\mathbb{E}\big[Z(N,z)\big] = 1$$

- They are not independent with explicit covariance:

$$\mathbb{C}\mathrm{ov}\left[Z(N,z), Z(N,z')\right] \sim \sum_{1\le\ell\le N} \beta^2\, q(2\ell, z-z')\cdot v(N-\ell)$$

$q(n,z) := \mathrm{P}(s_n = z)$ is the random walk transition kernel

$$v(n) = 1 + \sum_{1\le\ell\le n} \beta^2\, q(2\ell,0) + \sum_{1\le\ell<m\le n} \beta^2\, q(2\ell,0)\, \beta^2\, q(2(m-\ell),0) + \ldots$$

Renewal theory

Now fix $d = 2$. We look closely at

$$v(n) = 1 + \sum_{1 \le \ell \le n} \beta^2\, q(2\ell, 0) + \sum_{1 \le \ell < m \le n} \beta^2\, q(2\ell, 0)\, \beta^2\, q(2(m-\ell), 0) + \ldots$$

- Local CLT: $q(2\ell, 0) \sim \frac{1}{\pi}\frac{1}{\ell}$
- Critical rescaling: $\beta^2 = \frac{\pi}{\log N}$

Renewal theory interpretation: $\mathrm{P}(X^{(N)} = \ell) := \beta^2\, q(2\ell, 0)$

$$v(n) = 1 + \mathrm{P}(S_1^{(N)} \le n) + \mathrm{P}(S_2^{(N)} \le n) + \ldots = \sum_{m=0}^{m} u^{(N)}(m)$$

The renewal function $u^{(N)}(\cdot)$ sheds light on directed polymers

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Stochastic Heat Equation

Singular stochastic PDE on $\mathbb{R}^d$

$$\partial_t u(t,x) = \Delta u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

$u(0,x) \equiv 1$ for simplicity

$\xi(t,x) =$ space-time white noise

ξ is very irregular $\rightsquigarrow$ product $u\,\xi$ is classically ill-defined

- $(d = 1)$ Well-posed via stochastic integration (Ito-Walsh 1980s)
  Also pathwise, via Regularity Structures or Paracontrolled Calculus
- $(d \geq 2)$ No solution theory

Critical $2d$ Stochastic Heat Equation

Now fix $d = 2$. We can regularize the noise

- mollification in space: $\xi_\delta = \xi * g_\delta \qquad \delta > 0$
- discretization in space-time: $\xi(t,x) \rightsquigarrow \omega(\frac{n}{N}, \frac{z}{\sqrt{N}}) \quad$ i.i.d. $\mathcal{N}(0,1)$

$\rightsquigarrow$ (SHE) becomes a difference equation on the rescaled lattice $\frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}}{\sqrt{N}}$

The solution $u(t,x)$ of the discretized SHE is...

... the rescaled Directed Polymer Partition Function $Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)$!

(dicrete Feynman-Kac formula)

Does $Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)$ admit a non-trivial limit as $N \to \infty$?

Yes, but

- We must look at $Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)$ as a distribution in x

$$Z(\lfloor tN \rfloor, \varphi) := \int_{\mathbb{R}^2} Z(\lfloor tN \rfloor, \lfloor x\sqrt{N} \rfloor)\, \varphi(x)\, \mathrm{d}x \qquad\qquad \varphi \in C_c(\mathbb{R}^2)$$

- We need to critically rescale $\beta^2 \sim \dfrac{\pi}{\log N}\left(1 + \dfrac{\vartheta}{\log N}\right)$

  This ensures convergence of the (mean and) variance of $Z(\lfloor tN \rfloor, \varphi)$

  $\rightsquigarrow$ renewal theory interpretation

Main result

Theorem [CSZ21]

With the critical rescaling

$$\beta^2 = \frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \text{for } \vartheta \in \mathbb{R}$$

we have the joint convergence in distribution over $t \geq 0$, $\varphi \in C_c(\mathbb{R}^2)$

$$Z(tN, \varphi) \xrightarrow[N\to\infty]{d} \mathscr{Z}(t, \varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}(t, \mathrm{d}x)$$

The limiting process $\mathscr{Z}(t, \mathrm{d}x)$ is called critical $2d$ Stochastic Heat Flow

$\rightsquigarrow$ It is the natural candidate solution of the critical $2d$ (SHE)

Conclusions

Renewal Theory is a beautiful research area, classical and recent

It is also a remarkably useful tool for many different models, including some Disorder Systems and Stochastic PDEs

We constructed the critical $2d$ Stochastic Heat Flow $(\mathscr{Z}(t, \mathrm{d}x))_{t \geq 0}$ as a natural candidate solution for the critical $2d$ (SHE)

It is a universal stochastic process of random measures on $\mathbb{R}^2$
Several properties are known, but many features are still open...

Buon compleanno, Paolo!

The Dickman subordinator

Our renewal process $S^{(N)}$ is attracted to a pure jump Lévy process Y

$$\left(\frac{S^{(N)}_{\lfloor s \log N \rfloor}}{N}\right)_{s \geq 0} \xrightarrow[N\to\infty]{d} Y = (Y_s)_{s\geq 0}$$

called the Dickman subordinator

- Lévy measure $$\nu^Y(\mathrm{d}t) := \frac{1}{t}\,\mathbb{1}_{(0,1)}(t)\,\mathrm{d}t$$
- Explicit density $$\frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t} = \frac{\mathrm{e}^{-\gamma s}\, s\, t^{s-1}}{\Gamma(s+1)} \quad \text{for } t \in (0,1)$$

$G_\vartheta(t)$ is the (exponentially weighted) renewal function of Y

$$G_\vartheta(t) = \int_0^\infty \mathrm{e}^{\vartheta s}\, \frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t}\,\mathrm{d}s$$

Polynomial chaos

- Simple random walk kernel on $\mathbb{Z}^2$ $\qquad q(n,z) = \mathrm{P}(s_n = z)$
- New i.i.d. centred random variables $\qquad \tilde{\omega}(n,z) := \dfrac{e^{\beta\omega(n,x) - \frac{1}{2}\beta^2} - 1}{\beta}$

Polynomial chaos

Equivalent rewriting of the partition function

$$Z(N,z) = 1 + \beta \sum_{\substack{1 \le \ell \le N \\ x \in \mathbb{Z}^2}} q(\ell,x)\,\tilde{\omega}(\ell,x)$$
$$+\beta^2 \sum_{\substack{1 \le \ell < m \le N \\ x,y \in \mathbb{Z}^2}} q(\ell,x)\,q(m-\ell,y-x)\,\tilde{\omega}(\ell,x)\,\tilde{\omega}(m,y) + \ldots$$

Variance

Scaling limit of the variance

$$\mathbb{V}\mathrm{ar}[Z(N,\varphi)] \approx \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, K_N(x,y)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y$$

Explicit kernel

$$K_N(x,y) = \beta^2 \sum_{1\le m<n\le N} \mathrm{P}(s_m = \sqrt{N}(x-y)) \cdot u^{(N)}_{n-m}$$

where $u^{(N)}$ = renewal function of ultra-heavy tailed renewal process

$$\lim_{N\to\infty} K_N(x,y) = \pi \iint_{0<s<t<1} \underbrace{g_s(x-y)}_{\text{heat kernel on } \mathbb{R}^2} \cdot \underbrace{G_\vartheta(t-s)}_{\substack{\text{renewal function of the} \\ \text{Dickman subordinator}}} \, \mathrm{d}s\, \mathrm{d}t$$