Gaussian Limits for 2d Directed Polymers, SHE and KPZ

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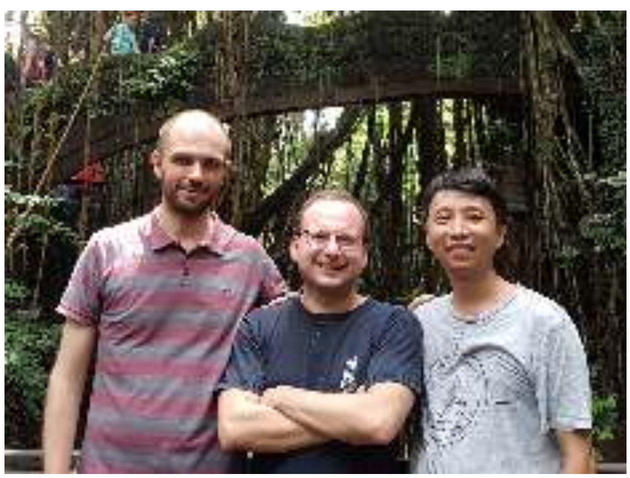


Joint with Nikos Zygouras and Rongfeng Sun

Overview

1. Introduction

   D.P. partition functions Z, SHE and KPZ

2. The 2d case: main results

   (logarithmic divergences & marginal relevance)

3. Proof ideas and techniques.

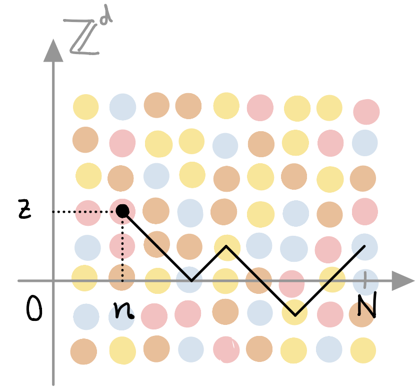
   - Polynomial chaos & 4^{th} moment theorems
   - Edwards-Wilkinson fluctuations for averaged Z
   - Edwards-Wilkinson fluctuations for averaged $\log Z$

4. Key tool: concentration of measure

1. Introduction

Directed Polymers in random environment

- $(S_i)_{i \geq 0}$ simple random walk on $\mathbb{Z}^d$



- $(\omega(i,y))_{i\geq 0,\, y\in\mathbb{Z}^d}$ i.i.d. rv's (disorder)

$$\mathbb{E}[\omega] = 0 \qquad \mathbb{V}\mathrm{AR}[\omega] = 1$$

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}] < \infty$$

- Partition Functions indexed by $\underset{\substack{\uparrow\\ \text{TIME HORIZON}}}{N \in \mathbb{N}}$, $\underset{\substack{\uparrow\\ \text{STARTING POSITION}}}{(n,z) \in \mathbb{N}_0 \times \mathbb{Z}^d}$, $\underset{\substack{\uparrow\\ \text{DISORDER STRENGTH}}}{\beta \geq 0}$

$$Z_N^\omega(n,z) := \underset{\text{RW}}{\mathrm{E}}\left[e^{\sum_{i=n+1}^{N} \{\beta\omega(i,S_i) - \lambda(\beta)\}} \,\middle|\, S_n = z \right]$$

- $\left(Z_N^\omega(n,z)\right)_{(n,z)\in\mathbb{Z}^d}$ is a family of random variables (w.r.t. ω)
  - Stationary in z
  - $Z_N^\omega(n,z) \geq 0$
  - $\mathbb{E}[Z_N^\omega(n,z)] \equiv 1$ (due to $-\lambda(\beta)$)

TIGHTNESS OF 1-DIM. MARGINALS $\leftarrow$ (last two items)

Key problem Scaling limit of $Z^\omega_{N,\beta}(\cdot,\cdot)$

Can we tune $\beta = \beta_N \geq 0$ so that

$$\Big(Z^\omega_N(tN, x\sqrt{N}) \Big)_{t\in[0,1],\, x\in\mathbb{R}^d} \xrightarrow{d} \Big(\mathcal{Z}(t,x) \Big)_{t\in[0,1],\, x\in\mathbb{R}^d} \;?$$

DIFFUSIVE RESCALING — "INTERESTING" RANDOM FIELD

(INTEGER PART $\lfloor tN \rfloor$, $\lfloor x\sqrt{N} \rfloor$)

(d = 1) Yes! [Alberts, Khanin, Quastel 14] [CSZ 17]

→ Rongfeng's lectures

- $\beta_N = \dfrac{\hat\beta}{N^{1/4}}$ with $\hat\beta \in [0,\infty)$
- $\mathcal{Z}(t,x) = U(1-t,x)$ with $U(t,x)$ solution of SHE

1d Stochastic Heat Equation:

$$\begin{cases} \partial_t U = \frac{1}{2}\, \partial_{xx} U + \sqrt{2}\,\hat\beta\, U \cdot \xi & (t>0) \\ U(0,x) \equiv 1 \end{cases}$$

ξ: (SPACE-TIME) WHITE NOISE ON $\mathbb{R}^{1+1}$

- OK for any $\hat\beta \in [0,\infty)$ (no phase transition

  ↳ intermediate regime)

- $\mathcal{Z}(t,x)$ random continuous function

  ↳ No longer true for $d \geq 2$: $\mathcal{Z}$ expected to be a distribution

Stochastic Heat Equation (SHE)

Consider a slightly modified partition function:

$$\tilde{Z}_N^\omega(n,z) := E\left[e^{\sum_{i=n}^{N} \{\beta\omega(i,S_i) - \lambda(\beta)\}} \,\middle|\, S_n = z \right]$$

$$= \underbrace{e^{\beta\omega(n,z)-\lambda(\beta)}}_{\text{IMMATERIAL AS } \beta\to 0} Z_N^\omega(n,z)$$

Exercise: $\tilde{Z} = \tilde{Z}_N^\omega$ solves the difference equation

$$\tilde{Z}(n,z) - \tilde{Z}(n+1,z)$$

$$= \underbrace{\frac{1}{2d}\sum_{y\sim z}\left(\tilde{Z}(n+1,y) - \tilde{Z}(n+1,z)\right)}_{\text{LATTICE LAPLACIAN}} + \eta(n,z)\cdot Z(n,z)$$

where

$$\boxed{\eta(n,z) := e^{\beta\omega(n,z)-\lambda(\beta)} - 1}$$

$$\mathbb{E}[\eta(n,z)] = 0, \qquad \mathbb{V}\text{AR}[\eta(n,z)] = e^{\lambda(2\beta)-2\lambda(\beta)} - 1 \sim \beta^2 \quad \text{as } \beta\downarrow 0$$

Diffusive rescaling: $\quad (n,z) = (tN, x\sqrt{N}) \qquad t\geq 0,\ x\in\mathbb{R}^d$

$$-\partial_t \tilde{Z}(tN, x\sqrt{N}) = \frac{1}{2}\Delta_x \tilde{Z}(tN, x\sqrt{N}) + N\,\eta(tN, x\sqrt{N}) \cdot Z(tN, x\sqrt{N})$$

INCREMENTAL RATIOS
WITH $\delta t = \frac{1}{N}$, $\delta x = \frac{1}{\sqrt{N}}$

- If $\xi(t,x) :=$ SPACE-TIME WHITE NOISE on $\mathbb{R}^{1+d}$, then

$$\xi^{(N)}(t,x) := \frac{\int_{[t,t+\frac{1}{N})\times[x,x+\frac{1}{\sqrt{N}})^d} \xi(t',x')\,dt'\,dx'}{\frac{1}{N}\cdot\left(\frac{1}{\sqrt{N}}\right)^d} \sim \mathcal{N}(0, N^{1+\frac{d}{2}})$$

- We replace $\underbrace{N\,\eta(tN, x\sqrt{N})}_{\mathbb{V}\mathrm{AR} \sim N^2\beta^2}$ by $\beta^{SHE}\,\xi^{(N)}(t,x)$:

$=$

$$\boxed{\beta^{SHE} = \beta^{POLYMER} \cdot N^{\frac{2-d}{4}}}$$

- If $\beta = \beta_N \to 0$, then $Z(n,z) \sim \tilde{Z}(n,z) = e^{\beta\,\omega(n,z) - \lambda(\beta)}\,Z(n,z)$

Summarizing : rescaled partition functions

$$\boxed{\mathcal{Z}^{(N)}(t,x) := Z_N^\beta(tN, x\sqrt{N})}$$

approximately satisfy a discretized SHE : (backward)

$$\begin{cases} -\partial_t \mathcal{Z}^{(N)} = \frac{1}{2}\Delta \mathcal{Z}^{(N)} + \underbrace{\beta N^{\frac{2-d}{4}}}_{\beta^{SHE}} \cdot \xi^{(N)} \cdot \mathcal{Z}^{(N)} & (0 \leqslant t < 1) \\ \mathcal{Z}^{(N)}(1,x) \equiv 1 & \text{(final condition)} \end{cases}$$

$$\xi^{(N)}(t,x) := \text{white noise averaged on } [t, t+\tfrac{1}{N}) \times [x, x+\tfrac{1}{\sqrt{N}})^d$$

Key message

Directed polymer partition functions are a natural regularization of the SHE solution, via discretization (rather than mollification) of the noise.

$$\xi^{\varepsilon}(t,x) := \big(\xi(t,\cdot) * \rho_\varepsilon\big)(x) \qquad \left[\rho_\varepsilon(\cdot) := \varepsilon^{-d}\rho(\varepsilon^{-1}\cdot),\ \rho \in C_c^\infty\right]$$

Mollified SHE:

Rongfeng's lecture

$$\begin{cases} \partial_t u^{(\varepsilon)} = \frac{1}{2}\Delta u^{(\varepsilon)} + \sqrt{2}\, \underbrace{\beta\, \varepsilon^{\frac{d-2}{2}}}_{\beta^{SHE}} \cdot \xi^{\varepsilon} \cdot u^{(\varepsilon)} & (t>0) \\ u^{(\varepsilon)}(0,x) \equiv 1 \end{cases}$$

PERIODICITY OF SRW

Meta-Theorem

We have $\mathcal{Z}^{(N)}(t,x) \overset{d}{\approx} u^{(\varepsilon)}(1-t,x)$ with the correspondence

$$\varepsilon = \frac{1}{\sqrt{N}}, \qquad \beta^{SHE} = \beta N^{\frac{2-d}{4}} = \beta\, \varepsilon^{\frac{d-2}{2}}.$$

Precise meaning: every result we can prove for $\mathcal{Z}^{(N)}$

we can also prove it for $u^{(\varepsilon)}$!

We approximate $u^{(\varepsilon)}$ by $\mathcal{Z}^{(N)}$ (this way round!)
through their chaos expansions ($\to$ Rongfeng's lectures)

This can be tedious, but involves no conceptual difficulty.

KPZ equation

Let $u^{(\varepsilon)}$ solve the mollified SHE. (well-posed, by Ito-Walsh stochastic integral)

Then, by Ito formula,

$$\boxed{h^{(\varepsilon)}(t,x) := \log u^{(\varepsilon)}(t,x)}$$

solves the KPZ equation:

PERIODICITY OF SRW

$$\begin{cases} \partial_t h^{(\varepsilon)} = \frac{1}{2} \Delta h^{(\varepsilon)} + \frac{1}{2} |\nabla h^{(\varepsilon)}|^2 + \sqrt{2}\, \underbrace{\beta\, \varepsilon^{\frac{d-2}{2}}}_{\beta^{SHE} = \beta^{KPZ}} \xi^{\varepsilon} - 2 \underbrace{\|\rho\|_{L^2}^2\, \varepsilon^{-2}}_{\text{"ITO CORRECTION"}} \\ h^{(\varepsilon)}(0,x) \equiv 0 \end{cases}$$

In the framework of directed polymers, we consider the

"discretized KPZ solution"

$$\log \mathcal{Z}^{(N)}(t,x) = \log Z_N^{\beta}(tN, x\sqrt{N})$$

Meta-Theorem

We have $\log \mathcal{Z}^{(N)}(t,x) \overset{d}{\approx} h^{(\varepsilon)}(t,x)$ with the correspondence

$$\varepsilon = \frac{1}{\sqrt{N}}, \qquad \beta^{KPZ} = \beta N^{\frac{2-d}{4}} = \beta \varepsilon^{\frac{d-2}{2}}.$$

Concrete statement: Every result we can prove for $\log \mathcal{Z}^{(N)}$, we can also prove it for $h^{(\varepsilon)}$.

In the sequel, we will mainly focus on directed polymers.

2. THE 2d CASE: MAIN RESULTS

Henceforth we set $d=2$.

Key problem Scaling limit of $Z^{\omega}_{N,\beta}(\cdot,\cdot)$

Can we tune $\beta=\beta_N \geq 0$ so that

$$\left(Z^{\omega}_{N}(tN, x\sqrt{N})\right)_{t\in[0,1],\, x\in\mathbb{R}^2} \xrightarrow{d} \left(\mathcal{Z}(t,x)\right)_{t\in[0,1],\, x\in\mathbb{R}^2} \; ?$$

DIFFUSIVE RESCALING

"INTERESTING" RANDOM FIELD

(UP TO SUITABLE SCALING)

$$\left(\log Z^{\omega}_{N}(tN, x\sqrt{N})\right)_{t\in[0,1],\, x\in\mathbb{R}^2} \xrightarrow{d} \left(\mathcal{H}(t,x)\right)_{t\in[0,1],\, x\in\mathbb{R}^2} \; ?$$

When $d=1$ we need to choose $\beta \sim \dfrac{\hat{\beta}}{N^{1/4}}$.

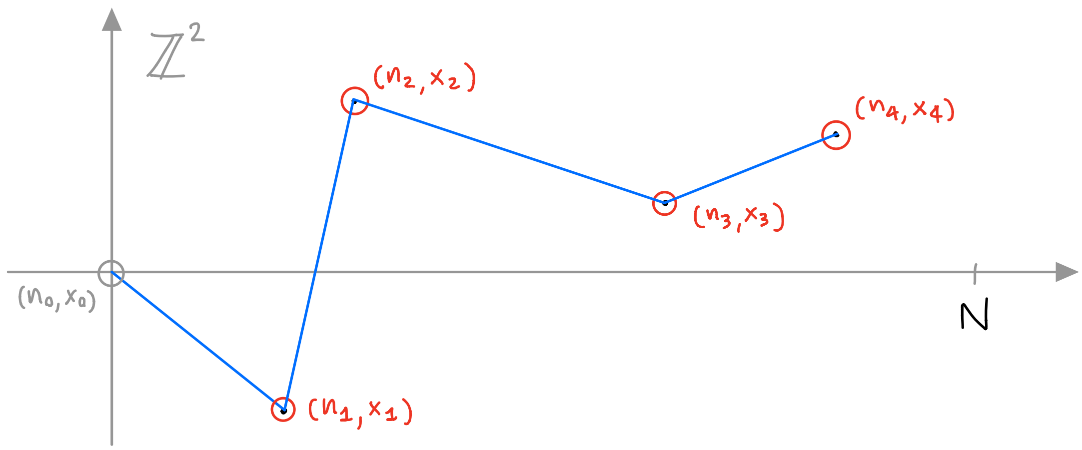
When $d=2$ the right choice is logarithmic.

Lemma Asymptotic variance

For $\beta=\beta_N \sim \dfrac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$, with $\hat{\beta}\in[0,\infty)$, we have

$$(t<1) \qquad \lim_{N\to\infty} \mathrm{VAR}\left[Z^{\omega}_{N}(tN, x\sqrt{N})\right] = \begin{cases} \dfrac{\hat{\beta}^2}{1-\hat{\beta}^2} & \text{if } \hat{\beta}<1 \\ \infty & \text{if } \hat{\beta}\geq 1 \end{cases}$$

<u>Proof.</u> Polynomial chaos expansion ($t=0$, $x=0$ for simplicity)



$$Z_N^{\omega}(0,0) = 1 + \sum_{k=1}^{\overset{\infty}{\cancel{N}}} \sum_{\substack{0 = n_0 < n_1 < \dots < n_k \leq N \\ x_0 = 0,\ x_1, \dots, x_k \in \mathbb{Z}^2}} \left(\prod_{i=1}^{k} q_{n_i - n_{i-1}}(x_i - x_{i-1}) \right) \prod_{i=1}^{k} \eta(n_i, x_i)$$

- $\eta(n,x) := e^{\beta\omega(n,x) - \lambda(\beta)} - 1$ $\quad \begin{cases} \mathbb{E}[\eta(n,x)] = 0 \\ \mathbb{V}\mathrm{AR}[\eta(n,x)] =: \sigma^2 \sim \beta^2 \end{cases}$

- $q_n(x) := P(S_n = x) \sim \frac{1}{n}\, g_{\frac{1}{2}}\!\left(\frac{x}{\sqrt{n}}\right) \cdot 2\,\mathbb{1}_{\{(n,x) \in \mathbb{Z}^3_{\text{EVEN}}\}}$

with $g_t(z)$ heat Kernel on $\mathbb{R}^2$:

$$g_t(z) := \frac{1}{2\pi t}\, e^{-\frac{|z|^2}{2t}}$$

Key computations:

- $$\sum_{x\in\mathbb{Z}^2} q_n(x)^2 = \sum_{x\in\mathbb{Z}^2} q_n(x)\, q_n(-x) = q_{2n}(0) \sim \frac{1}{\pi n} \qquad (n\to\infty)$$

- $$R_N := \sum_{n=1}^{N} \sum_{x\in\mathbb{Z}^2} q_n(x)^2 \sim \sum_{n=1}^{N} \frac{1}{\pi n} \boxed{\sim \frac{\log N}{\pi}} + O(1)$$

We can call R_N expected replica overlap:

INDEPENDENT SRW's

$$R_N = \sum_{n=1}^{N} \sum_{x\in\mathbb{Z}^2} \underbrace{P(S_n = x)^2}_{P(S_n=x,\, S'_n=x)} = \sum_{n=1}^{N} P(S_n = S'_n)$$

$$= \boxed{E\left[\sum_{n=1}^{N} \mathbb{1}_{\{S_n = S'_n\}}\right]}$$

Divergence of R_N is a signature of disorder relevance

(→ see Rongfeng's lectures).

The logarithmic divergence means marginal relevance.

- In $d=1$ we have $R_N \sim c\sqrt{N}$ ⟶ disorder relevance
- In $d\geq 3$ we have $R_N \to c < \infty$ ⟶ disorder irrelevance

Back to polynomial chaos:

$$Z_N^\beta = 1 + \sum_{k=1}^{N} \sum_{\substack{0=n_0<n_1<\dots<n_k\leq N \\ x_0=0,\, x_1,\dots,x_k\in\mathbb{Z}^2}} \left(\prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1}) \right) \prod_{i=1}^{k} \eta(n_i,x_i)$$

Chaos of different degrees k are orthogonal in L^2, because the $\eta(n,x)$'s are centered and independent.

Moreover $\mathbb{V}\text{AR}[\eta(n,x)] =: \sigma^2 \sim \beta^2$ as $\beta \downarrow 0$. Then

$$\mathbb{V}\text{AR}[Z_N^\omega] \sim \sum_{k=1}^{N} (\beta^2)^k \sum_{\substack{0=n_0<n_1<\dots<n_k\leq N \\ x_0=0,\, x_1,\dots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})^2$$

$$= \sum_{k=1}^{N} (\beta^2)^k \sum_{0=n_0<n_1<\dots<n_k\leq N} \prod_{i=1}^{k} \Bigg(\underbrace{\sum_{x\in\mathbb{Z}^2} q_{n_i-n_{i-1}}(x)^2}_{\sim \frac{1}{\pi(n_i-n_{i-1})}} \Bigg)$$

$$\sim \sum_{k=1}^{N} \left(\frac{\beta^2}{\pi}\right)^k \sum_{0=n_0<n_1<\dots<n_k\leq N} \prod_{i=1}^{k} \frac{1}{n_i-n_{i-1}}$$

We now enlarge the sum, allowing each increment n_i-n_{i-1} to range freely in $\{1,2,\dots,N\}$ $\longrightarrow$ UPPER BOUND:

$$\mathbb{V}\mathrm{AR}\left[Z_N^{\omega}\right] \lesssim \sum_{k=1}^{N}\left(\frac{\beta^2}{\pi}\right)^k \prod_{i=1}^{k}\left(\sum_{\underset{= n_i - n_{i-1}}{n=1}}^{N} \frac{1}{n}\right)$$

$$\sim \sum_{k=1}^{N}\left(\frac{\beta^2}{\pi}\right)^k (\log N)^k$$

If we fix $\beta \sim \dfrac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$, we then obtain

$$\mathbb{V}\mathrm{AR}\left[Z_N^{\omega}\right] \lesssim \sum_{k=1}^{N}(\hat{\beta}^2)^k \sim \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \qquad \text{ok!}$$

For a matching LOWER BOUND, we truncate the sum to $k \leq K$, then we restrict each increment to $1 \leq n_i - n_{i-1} \leq \dfrac{N}{K}$.

$$\mathbb{V}\mathrm{AR}\left[Z_N^{\omega}\right] \gtrsim \sum_{k=1}^{K}\left(\frac{\beta^2}{\pi}\right)^k \prod_{i=1}^{k}\left(\sum_{\underset{= n_i - n_{i-1}}{n=1}}^{N/K} \frac{1}{n}\right)$$

$$\sim \sum_{k=1}^{K}\left(\frac{\beta^2}{\pi}\right)^k \Big(\underbrace{\log \frac{N}{K}}_{\sim \log N \text{ as } N\to\infty}\Big)^k$$

hence $\mathbb{V}\mathrm{AR}\left[Z_N^{\omega}\right] \gtrsim \displaystyle\sum_{k=1}^{K}(\hat{\beta}^2)^k \underset{K\to\infty}{\sim} \frac{\hat{\beta}^2}{1-\hat{\beta}^2}$ $\square$

Henceforth we work in the "SUBCRITICAL REGIME"

$$\boxed{\beta = \beta_N \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{with } \hat{\beta} \in [0,1)}$$

Partition functions $Z_N^{\beta}(tN, x\sqrt{N})$ have bounded variance.

We have a pretty complete understanding of this regime.

Remark. The "CRITICAL REGIME" is

$$\hat{\beta} = 1 + O\left(\frac{1}{\log N}\right)$$

"CRITICAL WINDOW"

This is more challenging! (→ Nikos' lectures)

We know that the variance of Z_N^{ω} diverges for $\hat{\beta} = 1$.

A finer analysis shows that

$$\mathbb{V}\mathrm{AR}\left[Z_N^{\omega} \right] \sim c \log N$$

(→ Nikos' lectures)

Main results

Theorem 1. (CSZ 17) "Naive" scaling limits

Set $\beta = \beta_N \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} \in [0,\infty]$. Then

$$\left(Z_N^{\omega}(tN, x\sqrt{N})\right)_{t\in[0,1),\, x\in\mathbb{R}^2} \xrightarrow[\text{FINITE DIM. DISTRIBUTIONS}]{d} \left(\mathcal{Z}(t,x)\right)_{t\in[0,1),\, x\in\mathbb{R}^2}$$

- SUBCRITICAL REGIME $\hat{\beta} < 1$: log-normal marginals

$$\boxed{\mathcal{Z}(t,x) \overset{d}{=} e^{\sigma Y - \frac{1}{2}\sigma^2}} \qquad Y \sim \mathcal{N}(0,1), \quad \sigma^2 = \log \frac{\hat{\beta}^2}{1-\hat{\beta}^2}$$

- CRITICAL OR SUPER-CRITICAL REGIME $\hat{\beta} \geq 1$:

$$\mathcal{Z}(t,x) \equiv 0 \quad (!) \qquad \left(\text{despite } \mathbb{E}[Z_N^{\omega}] \equiv 1\right)$$

- $\forall \hat{\beta}$: $\mathcal{Z}(t,x)$ are i.i.d. for all $t\in[0,1),\, x\in\mathbb{R}^2$ (!!!)

We will understand what is going on!

Let us spell out the marginal scaling limit: for fixed (t,x)

$$Z_N^\omega(tN, x\sqrt{N}) \xrightarrow{d} \begin{cases} e^{\sigma Y - \frac{1}{2}\sigma^2} & (\hat{\beta} < 1) \\ 0 & (\hat{\beta} \geq 1) \end{cases}$$

with $\quad Y \sim N(0,1), \quad \sigma^2 = \dfrac{\hat{\beta}^2}{1-\hat{\beta}^2}$

Equivalently, for "KPZ":

$$\log Z_N^\omega(tN, x\sqrt{N}) \xrightarrow{d} \begin{cases} \sigma Y - \frac{1}{2}\sigma^2 & (\hat{\beta} < 1) \\ -\infty & (\hat{\beta} \geq 1) \end{cases}$$

Remarks

- Phase transition on the scale $\beta \sim \dfrac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with

  explicit critical point $\hat{\beta}_c = 1$ $\quad (= L^2 \text{ critical point})$

- "Weak to strong disorder": $Z_N \xrightarrow{d} \mathcal{Z} > 0$ vs. $Z_N \xrightarrow{d} 0$

- $d \geq 3$: transition for $\beta \sim O(1)$ $\quad (\beta < \beta_c$ vs. $\beta > \beta_c)$

  Critical point unknown $\neq L^2$ critical point

- $d = 1$: no phase transition

Key insight. In the subcritical regime $\hat\beta < 1$, the partition function $Z_N^\omega(tN, x\sqrt{N})$ effectively depends on the disorder $\omega(n,z)$ in a "tiny" neighborhood of $(tN, x\sqrt{N})$:

$$(n,z) \in [tN, tN + o(N)) \times [x\sqrt{N}, x\sqrt{N} \pm o(\sqrt{N}))^d$$

This explains the asymptotic independence of $Z_N^\omega(tN, x\sqrt{N})$ for distinct space-time points (t,x).

A quantitative formulation is provided by the following:

Lemma. Subcritical covariances

Set $\beta = \beta_N \sim \frac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat\beta \in [0,1)$. Then

$$\mathrm{Cov}\left[Z_N^\omega(tN, x\sqrt{N}), Z_N^\omega(t'N, x'\sqrt{N})\right] \sim \frac{\hat\beta^2}{1-\hat\beta^2} \, \frac{K((t,x),(t',x'))}{\log N}$$

with

$$K((t,x),(t',x')) := \int_{|t-t'|}^{(1-t)+(1-t')} g_s(|x-x'|)\, ds$$

as $|(t,x)-(t',x')| \to 0$

$$\sim \log \frac{1}{|t-t'| + |x-x'|^2}$$

Corollary Law of large numbers

Set $\beta = \beta_N \sim \dfrac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}$, $\hat\beta \in [0,1)$. Fix $\varphi \in C_c([0,1]\times\mathbb{R}^2)$.

$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, Z_N^\omega(tN, x\sqrt{N})\, dt\, dx \xrightarrow{d} \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, dt\, dx$$

i.e. $Z_N^\omega(tN, x\sqrt{N}) \xrightarrow{d} 1$ as a distribution on $[0,1]\times\mathbb{R}^2$

With more work, we can show that, with $\sigma^2 = \dfrac{\hat\beta^2}{1-\hat\beta^2}$,

$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, \log Z_N^\omega(tN, x\sqrt{N})\, dt\, dx \xrightarrow{d} -\frac{1}{2}\sigma^2 \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, dt\, dx$$

i.e. $\log Z_N^\omega(tN, x\sqrt{N}) \xrightarrow{d} -\frac{1}{2}\sigma^2 \quad \sim \mathbb{E}[\log Z_N^\omega]$

as a distribution on $[0,1]\times\mathbb{R}^2$

→ Having a LLN, it is natural to look for a CLT.

Theorem 2. (CSZ 17)

Edwards-Wilkinson fluctuations for Z

Set $\beta = \beta_N \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$, $\hat{\beta} \in (0,1)$. Fix $\varphi \in C_c([0,1]\times\mathbb{R}^2)$.

$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x) \, \frac{\sqrt{\log N}}{\hat{\beta}} \left(Z_N^{\omega}(tN, x\sqrt{N}) - 1 \right) dt\, dx$$

$$\xrightarrow{d} \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, v(1-t,x)\, dt\, dx$$

where $(v(t,x))_{t\geq 0, x\in\mathbb{R}^2}$ is a gen. Gaussian fileld

$$\mathbb{C}\mathrm{ov}\left[v(t,x), v(t',x') \right] = \gamma \, K((t,x),(t',x'))$$

$$\gamma = \frac{1}{1-\hat{\beta}^2}$$

$(v(t,x))_{t\geq 0, x\in\mathbb{R}^2}$ is the solution of the E-W equation:

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \gamma \, \xi \\ v(0,x) \equiv 0 \end{cases}$$

Theorem 3. (CSZ 20) Edwards-Wilkinson fluctuations for $\log Z$

Set $\beta = \beta_N \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$, $\hat{\beta} \in (0,1)$. Fix $\varphi \in C_c([0,1]\times\mathbb{R}^2)$.

$$\sim -\tfrac{1}{2}\sigma^2$$

$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, \frac{\sqrt{\log N}}{\hat{\beta}} \left(\log Z_N^\omega(tN, x\sqrt{N}) - \mathbb{E}\left[\log Z_N^\omega\right]\right) dt\, dx$$

$$\xrightarrow{d} \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\, v(1-t,x)\, dt\, dx$$

where $(v(t,x))_{t\geq 0, x\in\mathbb{R}^2}$ is a gen. Gaussian fileld

$$\mathbb{C}\mathrm{ov}\left[v(t,x), v(t',x')\right] = \gamma\, K((t,x),(t',x'))$$

$$\gamma = \frac{1}{1-\hat{\beta}^2}$$

$(v(t,x))_{t\geq 0, x\in\mathbb{R}^2}$ is the solution of the E-W equation:

$$\begin{cases} \partial_t v = \frac{1}{2}\Delta v + \gamma\,\xi \\ v(0,x) \equiv 0 \end{cases}$$

3. Proof ideas and techniques

Let us describe the main ideas and techniques in the proof of the results that we have presented.

Polynomial chaos & 4-th moment theorems

We prove convergence in distribution exploiting the polynomial chaos expansion of the partition function.

In dimension $d=1$, the situation is conceptually simple:

- every term of order $k \in \mathbb{N}$ in the polynomial chaos expansion of the partition function Z_N^ω converges in distribution to a corresponding term of order k in a Wiener chaos expansion (which defines the continuum partition function).

[$d=1$: "disorder relevant" case, see Rongfeng's lectures]

Example (pinning model). For $\alpha \in (0, \frac{1}{2})$ consider

$$Z_N := \beta \sum_{n=1}^{N} \frac{\omega_n}{n^\alpha} + \beta^2 \sum_{1 \leq n < m \leq N} \frac{\omega_n \omega_m}{n^\alpha (m-n)^\alpha}$$

$$= \left(\beta N^{\frac{1}{2}-\alpha}\right) \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{\omega_n}{\left(\frac{n}{N}\right)^\alpha} + \left(\beta N^{\frac{1}{2}-\alpha}\right)^2 \frac{1}{(\sqrt{N})^2} \sum_{1 \leq n < m \leq N} \frac{\omega_n \omega_m}{\left(\frac{n}{N}\right)^\alpha \left(\frac{m}{N} - \frac{n}{N}\right)^\alpha}$$

If we scale $\beta = \beta_N \sim \dfrac{\hat{\beta}}{N^{\frac{1}{2}-\alpha}}$, then $Z_N \xrightarrow{d} \mathcal{Z}$ with

$\xi(t)\,dt$ (WHITE NOISE $\xi = \dot{W}$) $\xi(s)\,\xi(t)\,ds\,dt$

$$\mathcal{Z} := \underbrace{\hat{\beta} \int_0^1 \frac{1}{t^\alpha}\, dW_t}_{\mathcal{Z}_1 \in 1^{ST} \text{ WIENER CHAOS}} + \underbrace{\hat{\beta}^2 \iint_{0<s<t<1} \frac{1}{s^\alpha (t-s)^\alpha}\, dW_s\, dW_t}_{\mathcal{Z}_2 \in 2^{ND} \text{ WIENER CHAOS}}$$

- $\mathcal{Z}_1$ is Gaussian, while $\mathcal{Z}_2$ is non Gaussian.
- $\mathcal{Z}_2$ is not independent of $\mathcal{Z}_1$.

Key features of space dimension $d=1$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{Partition function } Z_N^\omega} \arrow[r, dashed, "\text{is a function of}"] \arrow[d] & \boxed{\text{Discrete disorder } (\omega(n,z))} \arrow[d] \\
\boxed{\mathcal{Z} \text{ (cont. part. funct.)}} \arrow[r, dashed, "\text{is a function of}", "\text{(Wiener chaos)}"'] & \boxed{\text{White noise } (\xi(t,x)\,dt\,dx)}
\end{tikzcd}$$

Converge in distribution as $N \to \infty$

(under diffusive rescaling)

Crucially, in $d=2$ the bottom arrow "$\dashrightarrow$" breaks down!

E.g. for $\hat{\beta}<1$:

$$Z_N^\omega(0,0) \xrightarrow{d} \mathcal{Z} \qquad \text{[log-normal f.d.d.]}$$

$$\sqrt{\log N}\left(Z_N^\omega(tN, x\sqrt{N})-1\right)dt\,dx \xrightarrow{d} \mathscr{V} \qquad \text{[EW for } Z\text{]}$$

$$\sqrt{\log N}\left(\log Z_N^\omega(tN, x\sqrt{N})-\mathbb{E}[\cdots]\right)dt\,dx \xrightarrow{d} \tilde{\mathscr{V}} \qquad \text{[EW for } \log Z\text{]}$$

Limits $\mathcal{Z}$, $\mathscr{V}$, $\tilde{\mathscr{V}}$ no longer functions of white noise $\xi(t,x)\,dt\,dx$

(= scaling limit of discrete disorder $\omega(n,z)$)

They also depend on extra randomness: independent white noises $\xi'(t,x)\,dt\,dx$ (higher order features of $\omega(n,z)$)

Example (toy model). Consider the random variable

$$Z_N := \underbrace{\frac{1}{\sqrt{N}}\sum_{n=1}^N \omega_n}_{1^{\text{ST}}\text{ CHAOS}} + \underbrace{\frac{1}{\sqrt{N}}\sum_{n=1}^N \omega_n\omega_{n+1}}_{2^{\text{ND}}\text{ CHAOS}}$$

$(N\to\infty)$ $\Big\downarrow d$

More gen. $\displaystyle \frac{1}{\sqrt{N\cdot M}}\sum_{n=1}^N\sum_{m=n+1}^{n+M}\omega_n\omega_m \quad (M \ll N)$

$$\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_1' \quad \text{independent } \mathcal{N}(0,1)$$

$$\int_0^1 \xi(t)\,dt + \int_0^1 \xi'(t)\,dt \quad \text{indep. white noises}$$

Example (pinning model). Consider the random variable

$$Z_N := \underbrace{\beta \sum_{n=1}^{N} \frac{\omega_n}{\sqrt{n}}}_{Z_{N,1}} + \underbrace{\beta^2 \sum_{1 \le n < m \le N} \frac{\omega_n \omega_m}{\sqrt{n}\sqrt{m-n}}}_{Z_{N,2}}$$

Choose $\beta = \beta_N \sim \dfrac{\hat\beta}{\sqrt{\log N}}$, then $\left(\mathbb{E}[Z_{N,1}^2] \sim \mathbb{E}[Z_{N,2}^2] \sim \text{const.}\right)$

independent $\mathcal{N}(0,1)$

$$Z_N \xrightarrow{d} \hat\beta\, \mathcal{Z}_1 + \hat\beta^2 \left\{ \frac{(\mathcal{Z}_1)^2 - 1}{2} + \frac{\mathcal{Z}_1'}{\sqrt{2}} \right\}$$

$$\int_0^1 \xi(t)\,dt \qquad \iint_{0<t<u<1} \xi(t)\,\xi(u)\,dt\,du \qquad \int_0^1 \xi'(t)\,dt$$

- $Z_{N,1} \xrightarrow{d} \mathcal{Z}_1 \sim \mathcal{N}(0, \hat\beta^2) = \hat\beta \int_0^1 \xi(t)\,dt$

- $$Z_{N,2} = \frac{\hat\beta^2}{\log N} \Bigg\{ \underbrace{\sum_{\substack{1 \le n < m \le N \\ m-n > n}}}_{Z_{N,2}^{>}} + \underbrace{\sum_{\substack{1 \le n < m \le N \\ m-n \le n}}}_{Z_{N,2}^{\le}} \Bigg\} \frac{\omega_n \omega_m}{\sqrt{n}\sqrt{m-n}}$$

$$Z_{N,2}^{>} \longrightarrow \frac{(\mathcal{Z}_1)^2 - 1}{2} \qquad Z_{N,2}^{\le} \longrightarrow \frac{\mathcal{Z}_1'}{\sqrt{2}}$$

points m and n are "close together"

The previous example is very similar to directed polymers:

$$Z_N^\omega = 1 + \beta \sum_{\substack{1 \le n \le N \\ x \in \mathbb{Z}^2}} q_n(x)\, \omega(n,x) + \beta^2 \sum_{\substack{1 \le n < m \le N \\ x,y \in \mathbb{Z}^2}} q_n(x)\, q_{m-n}(y-x)\, \omega(n,x)\, \omega(m,y) + \dots$$

In [CSZ 17] we identified a hierarchy of

independent white noises $\left(\xi^{(i)}(t,x)\, dt\, dt\right)_{i \in \mathbb{N}}$

= scaling limits of suitable subsets ("dominated sequences") of the polynomial chaos expansion of Z_N.

(= generalizations of $Z_{N,2}^{\le}$ and $Z_{N,2}^{>}$ in the Example)

These independent white noises are the "basic bricks" to prove our Gaussian results for $\hat{\beta} < 1$; see [CSZ 17], [CSZ 20].

We will present here an alternative simpler approach. However, let us mention the key tool that we used to prove Gaussianity:

$\boxed{\text{4}^{\text{TH}}\ \text{MOMENT THEOREM for polynomial chaos}}$

[de Jong 87, 90] [Nualart, Peccati '05] [Nourdin, Peccati, Reinert '10]

We state a version that is useful for our needs.
Let $(\omega(n,x))_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ be i.i.d., centered, in L^2.

Theorem — 4TH MOMENT FOR POLYNOMIAL CHAOS

Let X_N be a polynomial chaos of fixed order $k \in \mathbb{N}$

$$X_N = \sum_{\substack{0 < n_1 < \dots < n_k \leq N \\ x_1, \dots, x_k \in \mathbb{Z}^2}} C_N(n_1, \dots, n_k; x_1, \dots, x_k) \prod_{i=1}^{k} \omega(n_i, x_i)$$

such that $\mathbb{E}[X_N^2] \to \sigma^2 < \infty$ as $N \to \infty$.

If $k=1$, also assume that the maximal influence vanishes:

$$\max_{0 < n \leq N,\ x \in \mathbb{Z}^2} C_N(n, x) \longrightarrow 0$$

If $\mathbb{E}[X_N^4] \to 3(\sigma^2)^2$, then

$$X_N \xrightarrow{d} \mathcal{X} \sim \mathcal{N}(0, \sigma^2)$$

Extension to vectors. In particular, if $\mathbb{C}\mathrm{ov}[X_N, X'_N] \to 0$,

$$(X_N, X'_N) \xrightarrow{d} (\mathcal{X}, \mathcal{X}') \text{ independent.}$$

Exercise. Prove the results stated in the last two Examples (toy model & pinning) using the 4th moment theorem.

Edwards - Wilkinson fluctuations for Z

Fix $\varphi \in C_c([0,1]\times\mathbb{R}^2)$ and define

$$X_N := \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\,\sqrt{\log N}\,\Big(Z_N^\omega(tN, x\sqrt{N}) - 1\Big)\,dt\,dx$$

Theorem (E-W). Set $\beta = \beta_N \sim \dfrac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}$, $\hat\beta \in [0,1)$. Then

$$X_N \xrightarrow{d} \mathcal{N}(0,\sigma^2) \quad \text{for a suitable } \sigma^2 = \sigma^2_\varphi \quad \text{(explicit!)}$$

Proof. We exploit the polynomial chaos expansion of X_N

(inherited from Z_N^ω)

$$X_N = \sum_{A \subseteq \{1,\dots,N\}\times\mathbb{Z}^2} \overset{\substack{\mathbb{R}\\ \cup\!\!\!\shortmid}}{c_N(A)} \cdot \overset{\substack{\prod_{(n,x)\in A}\eta(n,x)\\ \|}}{\eta(A)}$$

$$A = \{\underset{\substack{\downarrow\\ \ell(A)}}{(n_1,x_1)}, \dots, \underset{\substack{\downarrow\\ r(A)}}{(n_k,x_k)}\}$$

We can then write the second moment

$$\mathbb{E}[X_N^2] = \sum_{\ell=1}^{N} \sum_{r=\ell+1}^{N} \sum_{\substack{A \subseteq \{1,\dots,N\} \times \mathbb{Z}^2 \\ A = \{(\ell,x_1),\dots,(r,x_k)\}}} c_N(A)^2 \qquad (\star)$$

We know from covariance asymptotics that $\mathbb{E}[X_N^2] \to \sigma^2$.

The key technical input for the sequel is the following:

Lemma. For any $\varepsilon > 0$, the main contribution to $\mathbb{E}[X_N^2]$ in $(\star)$ comes from $\ell \leq r \leq \ell + \varepsilon N$: (points "close together")

$$\lim_{N \to \infty} \mathbb{E}\left[X_N^2 \, \mathbb{1}_{\{r > \ell + \varepsilon N\}} \right] = 0 \qquad (1)$$

Moreover, the contribution to $\mathbb{E}[X_N^2]$ of $|\ell - \ell_0| \leq \varepsilon N$ is negligible as $\varepsilon \downarrow 0$, uniformly in ℓ_0 and N:

$$\lim_{\varepsilon \downarrow 0} \sup_{N} \sup_{\ell_0} \mathbb{E}\left[X_N^2 \, \mathbb{1}_{\{|\ell - \ell_0| \leq \varepsilon N\}} \right] = 0 \qquad (2)$$

We proceed with the proof of the Theorem (assuming the Lemma).

Fix $M \in \mathbb{N}$. (cutoff: $\frac{1}{M} = \varepsilon$ in the Lemma)

Partition the sum over ℓ in $(\star)$ in M equal contributions:

$$\sum_{\ell=\tau_0=1}^{\tau_1} (\cdots) \simeq \frac{\sigma^2}{M} \qquad \sum_{\ell=\tau_1+1}^{\tau_2} (\cdots) \simeq \frac{\sigma^2}{M} \qquad \cdots \qquad \sum_{\ell=\tau_{M-1}+1}^{\tau_M=N} (\cdots) \simeq \frac{\sigma^2}{M}$$

For suitable $\tau_i = \tau_i^{M,N}$.

Then we decompose the random variable X_N as follows:

$$X_N = \left(X_{N,1} + \ldots + X_{N,i} + \ldots + X_{N,M}\right) + R_{N,M}$$

remainder

$$X_{N,i} = \sum_{A \subseteq (\tau_{i-1}, \tau_i] \times \mathbb{Z}^2} c_N(A) \cdot \eta(A)$$

The random variables $(X_{N,i})_{i=1,\ldots,M}$ are independent!

Moreover, by (1) and (2) in the Lemma:

- Remainder is negligible: $\lim_{N\to\infty,\, M\to\infty} R_{N,M} = 0$ in L^2.

- Each $X_{N,i}$ has $\mathbb{E}[X_{N,i}] = 0$, $\mathbb{E}[X_{N,i}^2] \simeq \frac{\sigma^2}{M}$

We can apply the Feller-Lindeberg CLT for triangular arrays:

$$\left(X_{N,1} + \ldots + X_{N,M}\right) \xrightarrow[N\to\infty,\ M\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

provided Lindberg's condition is satisfied:

$$\forall \varepsilon > 0: \quad \sum_{i=1}^{M} \mathbb{E}\left[X_{N,i}^2 \, \mathbb{1}_{\{|X_{N,i}| > \varepsilon\}} \right] \xrightarrow[N\to\infty,\ M\to\infty]{} 0 .$$

This follows from Lyapunov's condition:

$$\exists\, p > 2: \quad \sum_{i=1}^{M} \mathbb{E}\left[|X_{N,i}|^p \right] \xrightarrow[N\to\infty,\ M\to\infty]{} 0,$$

which holds by the hypercontractivity of polynomial chaos:
(see Rongfeng's lectures)

$$\mathbb{E}\left[|X_{N,i}|^p\right] \leq \mathbb{E}\left[X_{N,i}^2 \right]^{\frac{p}{2}} \lesssim \frac{1}{M^{\frac{p}{2}}} \ll \frac{1}{M} \qquad (p>2)$$

Warning: to work in the entire subcritical regime $\hat{\beta} \in [0,1)$, we need the sharp form of the hypercontractivity bound:

$$p = p(\hat{\beta}) \downarrow 2 \qquad \text{as } \hat{\beta} \uparrow 1$$

In conclusion, we have proved that

$$X_N = \left(X_{N,1} + \ldots + X_{N,M} \right) + R_{N,M} \xrightarrow[N\to\infty,\ M\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

(We actually take $M = M_N \to \infty$ slowly as $N \to \infty$) $\square$

Proof of the Lemma (sketch). We focus on the simplification

$$X_N^{(TOY)} := \frac{\sqrt{\log N}}{N} \sum_{\substack{0<\ell\le r\le N \\ x,y\in\mathbb{Z}^2}} e^{-\frac{|x|^2}{2N}}\, \eta(\ell,x)\, q_{r-\ell}(y-x)\, \eta(r,y)$$

(test with φ)

Since $\mathbb{E}[\eta(n,z)]=0$, $\mathbb{E}[\eta(n,z)^2] \sim \beta^2 \sim \frac{\hat\beta^2\pi}{\log N}$, we have

$$\mathbb{E}\left[\left(X_N^{(TOY)}\right)^2\right] = \frac{(\log N)^{-1}}{N^2} \sum_{0<\ell\le r\le N} \underbrace{\left(\sum_{x\in\mathbb{Z}^2} e^{-\frac{|x|^2}{N}}\right)}_{\simeq N} \underbrace{\left(\sum_{y'\in\mathbb{Z}^2} q_{r-\ell}(y')^2\right)}_{\simeq \frac{1}{r-\ell}}$$

$$\simeq \frac{1}{N(\log N)} \underbrace{\sum_{\ell=1}^{N} \sum_{r=\ell+1}^{N} \frac{1}{r-\ell}}_{\simeq N\cdot\log N} \simeq 1$$

Key observation: the main contribution comes from $r-\ell = o(N)$

$$\text{e.g.} \quad \sum_{\ell=1}^{N} \sum_{r=\ell+1}^{\ell+\frac{N}{\log N}} \frac{1}{r-\ell} \simeq N\cdot \overbrace{\log\frac{N}{\log N}}^{(\log N - \cancel{\log\log} N)} \sim N\log N$$

The right scale for the harmonic series is N^α with $\alpha\in(0,1)$:

$$\sum_{n=1}^{N} \frac{1}{n} \sim \log N \qquad \sum_{n=1}^{N^\alpha} \frac{1}{n} \sim \log N^\alpha = \alpha(\log N)$$

$\square$

Edwards - Wilkinson fluctuations for $\log Z$

Fix $\varphi \in C_c([0,1]\times\mathbb{R}^2)$ and define

$$Y_N := \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\,\sqrt{\log N}\left(\log Z_N^\omega(tN, x\sqrt{N}) - \mathbb{E}\left[\log Z_N^\omega\right]\right) dt\,dx$$

Theorem (E-W). Set $\beta = \beta_N \sim \frac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}$, $\hat\beta \in [0,1)$. Then

$$Y_N \xrightarrow{d} \mathcal{N}(0,\sigma^2) \quad \text{for a suitable } \sigma^2 = \sigma^2_\varphi \quad \text{(explicit!)}$$

Key difficulty: no polynomial chaos expansion for $\log Z$.

Solution: suitable "linearization of $\log Z$":

$$\left(\log Z_N^\omega(tN, x\sqrt{N}) - \mathbb{E}\left[\log Z_N^\omega\right]\right) dt\,dx \simeq \left(Z_N^\omega(tN, x\sqrt{N}) - 1\right) dt\,dx$$

$\Rightarrow$ {E-W for $\log Z$} corollary of {E-W for Z}

- Only true as measures in (t,x) (i.e. testing against φ)
- False for fixed (t,x): $Z_N^\omega(tN, x\sqrt{N}) \simeq e^{\mathcal{N}(0,\sigma^2) - \frac{1}{2}\sigma^2}$

Proof of the "linearization".

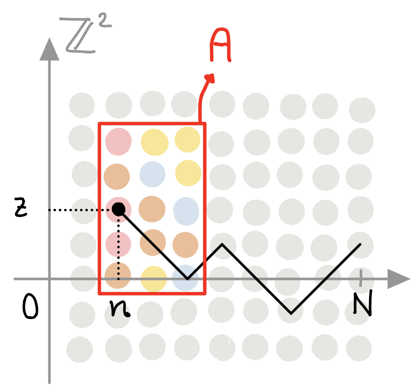
Set $\beta = \beta_N \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$, $\hat{\beta} \in [0,1)$.

Step 1: Pointwise approximation of $Z_N^\omega(tN, x\sqrt{N})$

for fixed (t,x).

Given $A \subseteq \mathbb{N} \times \mathbb{Z}^2$, define

Z_A^ω := partition function which only samples disorder from A



We choose a suitable $A = A(tN, x\sqrt{N})$ such that

$$Z_N^\omega(tN, x\sqrt{N}) = Z_A^\omega + \hat{Z}_A^\omega$$

"SMALL" (to be explained)

Then we write

$$\log Z_N^\omega(tN, x\sqrt{N}) = \log Z_A^\omega + \log\left(1 + \frac{\hat{Z}_A^\omega}{Z_A^\omega}\right)$$

$$= \log Z_A^\omega + \frac{\hat{Z}_A^\omega}{Z_A^\omega} + \underbrace{R_N(tN, x\sqrt{N})}_{\text{REMAINDER}}$$

We prove that $\hat{Z}^{\omega}_{A}$ is "small", i.e. $R_N(tN, x\sqrt{N})$ is "small", in the following precise sense.

Lemma 1. We can choose a box of "negligible size":

$$A = A_N(t,x) = [tN, tN + o(N)) \times [x\sqrt{N} - o\sqrt{N}, x\sqrt{N} + o(\sqrt{N}))^2$$

so that

$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x) \sqrt{\log N} \left\{ R_N(tN, x\sqrt{N}) - \mathbb{E}[R_N] \right\} dt\, dx \xrightarrow[N\to\infty]{L^2} 0$$

Step 2: Taking care of $\log Z_A$

The box A has negligible size on the diffusive scale

$\Rightarrow$ $\log Z_A$ gives no contribution to the correlations!

Lemma 2.

$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x) \sqrt{\log N} \left\{ \log Z_A - \mathbb{E}[\log Z_A] \right\} dt\, dx \xrightarrow[N\to\infty]{L^2} 0$$

Step 3: Dealing with the ratio $\dfrac{\hat{Z}_A^\omega}{Z_A^\omega}$

Recall that

$$Y_N := \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\,\sqrt{\log N}\,\Big(\log Z_N^\omega(tN, x\sqrt{N}) - \mathbb{E}\big[\log Z_N^\omega\big]\Big)\,dt\,dx$$

In view of ☺ and by Lemmas 1 and 2, we have

$$Y_N \overset{L^2}{\simeq} \tilde{Y}_N := \int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\,\sqrt{\log N}\,\left(\frac{\hat{Z}_A^\omega}{Z_A^\omega} - \cancel{\mathbb{E}[\dots]}\right)dt\,dx$$

It turns out that $\dfrac{\hat{Z}_A^\omega}{Z_A^\omega} \simeq \left(Z_{A^c}^\omega - 1\right)$, in the following sense.

Lemma 3.

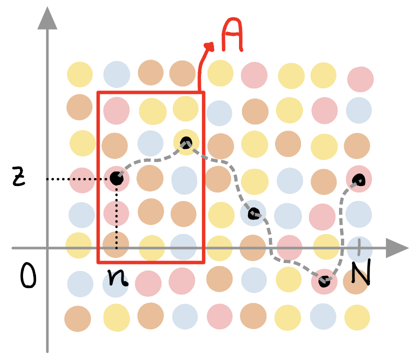
$$\int_{[0,1]\times\mathbb{R}^2} \varphi(t,x)\,\sqrt{\log N}\,\left\{\frac{\hat{Z}_A^\omega}{Z_A^\omega} - \left(Z_{A^c}^\omega - 1\right)\right\}dt\,dx \xrightarrow[N\to\infty]{L^2} 0$$

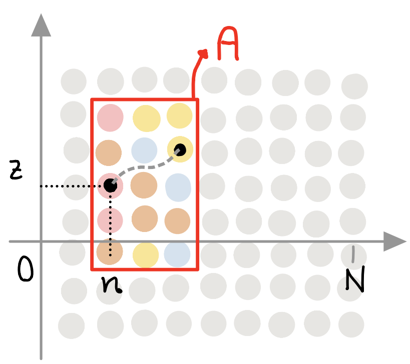
Finally, we show that $Z_{A^c}^\omega \simeq Z_N^\omega$ (A has negligible size) $\square$

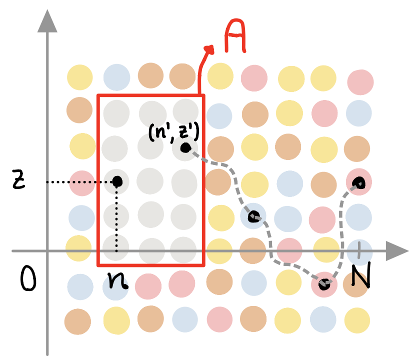
Idea of the proof We can represent

$$\hat{Z}^{\omega}_{A} := Z^{\omega}_{N}(n,z) - Z^{\omega}_{A}$$

samples at least one disorder variable from A^c





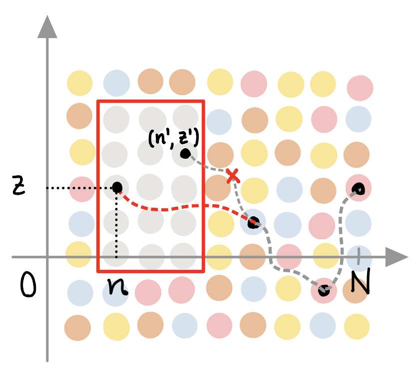


$$= Z^{\omega}_{A} \times \left(Z^{\omega}_{A^c}(n',z') - 1 \right)$$

$$\simeq Z^{\omega}_{A} \times \left(Z^{\omega}_{A^c}(n,z) - 1 \right)$$

$$\Downarrow$$

$$\boxed{\frac{\hat{Z}^{\omega}_{A}}{Z^{\omega}_{A}} \simeq \left(Z^{\omega}_{A^c} - 1 \right)}$$



□

4. Key tool: Concentration of measure

To prove E-W fluctuations for $\log Z$, we need to deal with

the ratio $\dfrac{\hat{Z}_A^\omega}{Z_A^\omega}$

The previous Lemmas 1, 2, 3 involve L^2 approximation

We need bounds on negative moments of partition function:

$$\mathbb{E}\left[\frac{1}{(Z_A^\omega)^p}\right] = \mathbb{E}\left[e^{-p\,(\log Z_A^\omega)}\right]$$

which require bounding the left tail of $\log Z_A^\omega$.

This can be achieved through concentration of measure.

Assumption on disorder. $\exists\ \gamma > 1$ and $c_1, c_2 \in (0,\infty)$ s.t.

$$\text{©} \qquad \mathbb{P}\left(\left|g(\omega_1, \ldots, \omega_n) - M_g\right| \geq t\right) \leqslant c_1\, e^{-\frac{t^\gamma}{c_2}} \qquad \forall t \geq 0$$

M_g: MEDIAN (OR MEAN $\mathbb{E}[g(\omega_1, \ldots, \omega_n)]$)

for every $n \in \mathbb{N}$ and all $g : \mathbb{R}^n \to \mathbb{R}$ convex and 1-Lipschitz.

Remark: Known to hold under reasonable assumptions on $\omega = \omega_1$

- BOUNDED ω's (with $\gamma = 2$) [Talagrand]

- DENSITY $f_\omega(\cdot) = e^{-V(\cdot) + U(\cdot)}$ such that

  $V(\cdot)$ is uniformly strictly convex, $U(\cdot)$ is bounded

  (with $\gamma = 2$) [Brascamp-Lieb inequality]

  In particular: OK if ω IS GAUSSIAN (even non convex g's)
  [Borell] [Sudakov, Tsirelson]

- STRETCHED EXPONENTIAL $f_\omega(t) = c' e^{-c|t|^\gamma}$, $\gamma \in [1,2)$

See the book [Ledoux 01].

Ideally, we want to apply the concentration of measure © to

$$g_N(\omega_A) := \log Z_A^\omega = \log E\left[e^{\sum_{(n,z)\in A} \{\beta\, \omega(n,S_n) - \lambda(\beta)\}} \right]$$

$(\omega(n,z))_{(n,z)\in A}$

for a bounded set $A \subseteq \{1, \dots, N\} \times \mathbb{Z}^2$ (with $\beta \sim \frac{\hat\beta}{\sqrt{\log N}}$, $\hat\beta \in [0,1)$).

- The function $g_N(\cdot)$ is convex (by Hölder).
- It is C_N-Lipschitz, but with $C_N \to \infty$ as $N \to \infty$!

However, the local Lipschitz constant

$$C_N(\omega) := |\nabla g_N(\omega)| \qquad \left(C_N = \sup_\omega C_N(\omega) \right)$$

turns out to be tight as $N \to \infty$:

$$\forall \varepsilon > 0 \ \exists K < \infty : \quad \mathbb{P}\left(|\nabla g_N(\omega)| \leq K \right) > 1 - \varepsilon \quad \forall N \in \mathbb{N}$$

This is enough to apply a concentration inequality for the left tail of locally Lipschitz functions:

$$\mathbb{P}\left(g(\omega) \leq a - t \right) \cdot \mathbb{P}\left(g(\omega) \geq a, \ |\nabla g(\omega)| \leq K \right) \leq c_1 \, e^{-\frac{(t/K)^\gamma}{c_2}}$$

We can give a uniform lower bound on the second factor:

$$\forall \hat{\beta} < 1 \ \exists a, K : \quad \inf_{N \in \mathbb{N}} \mathbb{P}\left(g_N(\omega) \geq a, \ |\nabla g_N(\omega)| \leq K \right) = \vartheta > 0$$

(Proof: Paley-Zygmund inequality + positive moment bounds.)

This yields the desired bound on the left lail of $\log Z$.

Theorem. Set $\beta \sim \frac{\hat{\beta}}{\sqrt{\log N}}$, $\hat{\beta} \in [0,1)$.

There is $c = c(\hat{\beta}) < \infty$ such that, $\forall\, A \subseteq \{1,\dots,N\} \times \mathbb{Z}^2$,

$$\mathbb{P}\left(\log Z_A^{\omega} \leqslant -t\right) \leqslant c\, e^{-\frac{t^{\gamma}}{c}}, \qquad \forall t \geqslant 0$$

As a consequence, negative moments are bounded:

$$\forall p \in (0,\infty): \quad \sup_{N \in \mathbb{N}} \ \sup_{A \subseteq \{1,\dots,N\} \times \mathbb{Z}^2} \mathbb{E}\left[\frac{1}{|Z_A^{\omega}|^p}\right] < \infty .$$

Thanks!

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