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Thesis

Portfolio allocation and monitoring under volatility shocks

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Introduction

Portfolio allocation is the problem of finding an investment strategy that is the best under a certain criterion. One of the most famous criteria is given by the mean-variance approach, introduced by Markovitz [4]. This approach quantifies the risk by using the variance, so investors seek the maximum return after specifying an acceptable risk level, or vice versa, they may specify a target return and minimize the risk. This approach has then been studied in detail and many publications have been written on the subject (see [5]).

In addition to the problem of allocation, it is possible to study monitoring strategies; these help the investor to decide if, at a certain date before the allocation time horizon, it is better to hold the position or to rebalance the portfolio. In this work we analyze two different monitoring strategies; the first one is based on the construction of a confidence interval for the evolution of the index. The second has been inspired by the paper by Xun Li and Xun Yu Zhou [3] and consists in finding a lower bound for the value of the portfolio at a certain date.

Another important topic in finance is to find a model that describes the evolution of an asset, typically a stock price or a financial index. One of the most famous one is the Black and Scholes model, which corresponds to a geometric Brownian motion with constant coefficients. However, this model is not compatible with some important characteristics observed in well-known financial indexes; we will use here a recently developed model that has been designed to catch these features [1].

The goal of the present work is to use the mentioned model to deal with the problem of portfolio allocation. We will analyze a scenario in which it is only possible to invest in a risk-free asset and in the index that the model reproduces, which is in this work the S&P500. The investor is an agent who invests using a mean-variance criterion with a monthly time horizon. Moreover, by using two monitoring methods we will analyze the evolution over time of the position and possible strategies to test whether the market behaves differently from what we expected.

In chapter 1 we describe the model and its main features.

In chapter 2 we deal with the problems of calibration of the model to market data, simulation of trajectories using the model, optimal allocation and monitoring of the evolution of the index.

In chapter 3 we present some numerical results that we obtained using the methodologies discussed in section 2.

1 The Model

A financial model has recently been developed to study and reproduce some characteristics of famous financial indexes, like the Dow Jones Industrial Average. The basic model, which led to the famous Black and Scholes formula, assumes that the logarithm X_t of the price of the underlying index, after subtracting the trend, is given by

$$dX_t = \sigma dW_t$$

where σ (the volatility) is constant and W_t is a standard Brownian motion. Despite its success, this model is not compatible with some facts empirically observed in various series. These characteristics are the following:

- the volatility is not constant: in particular, it can have peaks that can be interpreted as shocks in the market;
- the empirical distribution of the log returns $X_{t+h}-X_t$ has tails heavier than Gaussian;
- log-returns that correspond to disjoint time periods are incorrelated but not independents: in fact, if they were independent and $t > s$, the correlation between $|X_{t+h} - X_t|$ and $|X_{s+h} - X_s|$ would be zero, however a slow decay with respect to $|t-s|$ is observed, until moderates value of $|t - s|$. This phenomenon is called clustering of volatility.

We here summarize the essential features of the model, as described in [1]; given two real numbers $D \in (0, 1/2]$, $\lambda \in (0, +\infty)$ and a probability ν on $(0, +\infty)$, the model is defined by the following sources of alea :

- a standard Brownian motion $W = (W_t)_{t\geq 0}$
- a Poisson point process $\mathcal{T} = (\tau_n)_{n\in\mathbb{Z}}$ with intensity λ
- a sequence $\Sigma = (\sigma_n)_{n\geq 0}$ of positive iid random variables. The law of each variable is given by ν.

We assume that $W, \mathcal{T}, \Sigma$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that they are independent. We indicate the points of $\mathcal{T}$ so that $\tau_0 < 0 < \tau_1$. Given t $\in [0, \tau_1]$, we define

$$X_t := \sigma_0 \left(W_{(t-\tau_0)^{2D}} - W_{(-\tau_0)^{2D}} \right)$$

whereas for $t \in [\tau_n, \tau_{n+1}]$, $n \geq 1$, we define

$$X_t := X_{\tau_n} + \sigma_n \left(W_{(t-\tau_n)^{2D} + \sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} - W_{\sum_{k=1}^{n} (\tau_k - \tau_{k-1})^{2D}} \right)$$

In other words, at the random instants τ_n the time inhomogeneity $t \mapsto t^{2D}$ and the volatility is randomly updated: $\sigma_{n-1} \rightsquigarrow \sigma_n$. A possible financial interpretation of this mechanism is that jumps in the volatility corresponds to shocks in the market. The reaction of the market is not homogeneous in time: if $D < 1/2$, the dynamics is fast immediately after the shock, and tends to slow down later, until a new jump occurs.

It is possible to give another definition of the model X; for $t \geq 0$, define

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = card\{\mathcal{T} \cap (0,t]\}$$

so that $\tau_{i(t)}$ is the location of the last point in $\mathcal{T}$ before t. Now we introduce the process I_t by

$$I_t := \sigma_{i(t)}^2 \left(t - \tau_{i(t)}\right)^{2D} + \sum_{k=1}^{i(t)} \sigma_{k-1}^2 \left(\tau_k - \tau_{k-1}\right)^{2D} - \sigma_0^2 \left(-\tau_0\right)^{2D}$$

with the agreement that the sum in the right hand side is zero if $i(t) = 0$. We can then redefine our basic process $X = (X_t)_{t \geq 0}$ according to the following

Lemma 1: *Let (X_t) and I_t as above; then*

$$X_t := W_{I_t}$$

Proof. Using the recursive definition of the process X, we have

$$X_t = \sigma_{i(t)} \left(W_{(t-\tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} (\tau_k - \tau_{k-1})^{2D}} - W_{\sum_{k=1}^{i(t)} (\tau_k - \tau_{k-1})^{2D}} \right) +$$

$$\sum_{i=1}^{i(t)} \sigma_{i-1} \left(W_{\sum_{k=1}^{i} (\tau_k - \tau_{k-1})^{2D}} - W_{\sum_{k=1}^{i-1} (\tau_k - \tau_{k-1})^{2D}} \right) - \sigma_0 W_{(-\tau_0)^{2D}}$$

with the agreement that $\sum_{k=1}^{i-1} (\tau_k - \tau_{k-1})^{2D} = 0$ if $i = 1$; so we have

$$(X_t | \sigma, \mathcal{T}) = \sigma_{i(t)} W_{(t-\tau_{i(t)})^{2D}} + \sum_{i=1}^{i(t)} \sigma_{i-1} W_{(\tau_k - \tau_{k-1})^{2D}} - \sigma_0 W_{(-\tau_0)^{2D}}$$

Each Wiener in the previous sum is independent to the others; moreover, conditionally on σ and $\mathcal{T}$,

$$Var(X_t) = I_t$$

This shows that, conditionally on σ and $\mathcal{T}$,

$$X_t \sim W_{I_t}$$

Therefore, if A is a real borelian set and $\mathcal{G}$ is the σ-field generated by σ and $\mathcal{T}$, we have

$$\mathbb{P}(X_t \in A) = \mathbb{E}(\mathbb{P}(X_t \in A | \mathcal{G})) = \mathbb{E}(\mathbb{P}(W_{I_t} \in A | \mathcal{G})) = \mathbb{P}(W_{I_t} \in A)$$

$\square$

Main results

First of all, we report two simple observations

PROPOSITION 1: *The following properties hold:*

1. *The process X has stationary increments.*

2. *The process X conditioned to $\mathcal{T}$ and σ is markovian.*

Proof. For what concerns the first property, see [1]; for the Markov property, we fix an instant t_0 and we introduce $Y_t = X_{t+t_0} - X_{t_0}$; we can now observe that

$$I_{t+t_0} - I_{t_0} = \sigma^2_{i(t+t_0)} \left(t + t_0 - \tau_{i(t+t_0)}\right)^{2D} +$$

$$\sum_{k=i(t_0)+1}^{i(t+t_0)} \sigma^2_{k-1} \left(\tau_k - \tau_{k-1}\right)^{2D} - \sigma^2_{i(t_0)} \left(t_0 - \tau_{i(t_0)}\right)^{2D}$$

So we have

$$d\mathbb{P}(Y_t|\mathcal{T},\sigma) = d\mathbb{P}\left(W_{I_{t+t_0}} - W_{I_{t_0}}|\mathcal{T},\sigma\right) = d\mathbb{P}\left(W_{I_{t+t_0}-I_{t_0}}|\mathcal{T},\sigma\right)$$

Conditionally on $\mathcal{T}$ and σ, all the quantities in $I_{t+t_0} - I_{t_0}$ are known, so

$$(Y_t|\mathcal{T},\sigma) \sim \mathcal{N}(0, I_{t+t_0} - I_{t_0})$$

and it is independent from $\mathcal{F}_{t_0}$. □

Now let's consider the time series of an index $(s_i)_{1\leq i\leq T}$ over a period of $T >> 1$ days and we denote by p_h the empirical distribution of the log-returns (after subtracting the trend) corresponding to the period of the last h days before T:

$$p_h(.) := \frac{1}{T-h}\sum_{i=1}^{T-h} \delta_{x_{i+h}-x_i}(.), \qquad x_i := log(s_i) - \overline{d_i},$$

where $\overline{d_i}$ is the estimate of the local rate of linear growth of $log(s_i)$, which can be approximated by $\frac{1}{M}\sum_{K=i-M}^{i-1} log(s_k)$, with an appropriate choiche of M; on the other hand, $\delta_x(.)$ represents the Dirac measure concentrated at $x \in \mathbb{R}$. The statistical analysis of various indexes, like the Dow Jones Industrial Average or the Nikkei 255, shows that, for h within a suitable time scale, p_h obeys approximately a diffusive scaling relation

$$p_h(dr) \simeq \frac{1}{\sqrt{h}} g\left(\frac{r}{\sqrt{h}}\right) dr \tag{1.1}$$

where g is a probability density with tail heavier than Gaussian. If one considers the q-th empirical moment $m_q(h)$, defined by

$$m_q(h) := \frac{1}{T-h}\sum_{i=1}^{T-h} |x_{i+h} - x_i|^q = \int |r|^q p_h(dr)$$

from relation 1.1 it is natural to guess that $m_q(h)$ should scale as $h^{q/2}$. This is indeed what one observes for moments of small order$q \leq \overline{q}$, for a certain $\overline{q}$. However, for moments of higher order $q > \overline{q}$, the different scaling relation $h^{A(q)}$, with $A(q) < q/2$, takes place. This is the so-called multiscaling of moments. One of the main results which has been obtained in [1] is therefore the following

THEOREM 1 (Multiscaling of moments): *Let $q > 0$ and assume $\mathbb{E}(\sigma^q) < +\infty$. Then the quantity $m_q(h) := \mathbb{E}(|X_{t+h} - X_t|^q) = \mathbb{E}(|X_h|^q)$ is finite and has the following asymptotic behaviour $h \downarrow 0$:*

$$m_q(h) \sim \begin{cases} C_q h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q h^{\frac{q}{2}} log(\frac{1}{h}) & \text{if } q = q^* \\ C_q h^{Dq+1} & \text{if } q > q^* \end{cases} \qquad \text{where } q^* := \frac{1}{\frac{1}{2} - D}$$

The constant $C_q \in (0, +\infty)$ is given by

$$C_q(h) := \begin{cases} \mathbb{E}(|W_1|^q)\mathbb{E}(\sigma^q)\lambda^{q/q^*}(2D)^{q/2}\Gamma(1 - q/q^*) & \text{if } q < q^* \\ \mathbb{E}(|W_1|^q)\mathbb{E}(\sigma^q)\lambda(2D)^{q/2} & \text{if } q = q^* \\ \mathbb{E}(|W_1|^q)\lambda[\int_0^{+\infty}((1+x)^{2D} - x^{2D})^{\frac{q}{2}} dx + \frac{1}{Dq+1}] & \text{if } q > q^* \end{cases} \tag{1.2}$$

where $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx$ denotes Euler's Gamma function and W_1 a standard normal.

COROLLARY 1: *The following relation holds true*

$$A(q) := \lim_{h \downarrow 0} \frac{log(m_q(h))}{log(h)} = \begin{cases} q/2 & \text{if } q \leq q^* \\ Dq + 1 & \text{if } q > q^* \end{cases} \tag{1.3}$$

Proof. First we consider the case $q < q^*$; then we have

$$\frac{\log(m_q(h))}{\log(h)} = \frac{\log(C_q h^{\frac{q}{2}} \log(\frac{1}{h}))}{\log(h)} = \frac{\log(C_q) + \frac{q}{2}\log(h)}{\log(h)} \longrightarrow \frac{q}{2}$$

For what concernes the case $q = q^*$,

$$\frac{\log(m_{q^*}(h))}{\log(h)} = \frac{\log(C_{q^*} h^{\frac{q^*}{2}} log(\frac{1}{h}))}{\log(h)} = \frac{\log(C_{q^*}) + \frac{q^*}{2}\log(h) + \log(\log(\frac{1}{h}))}{\log(h)} \longrightarrow$$

$$\frac{q^*}{2} = Dq^* + 1$$

Finally, for $q > q^*$,

$$\frac{\log(m_q(h))}{\log(h)} = \frac{C_q h^{Dq+1}}{\log(h)} = \frac{\log(C_q) + (Dq+1)\log(h)}{\log(h)} \longrightarrow Dq + 1$$

$\square$

As regards volatility, the following result holds true.

Theorem 2 (Volatility autocorrelation): *Assume that $\mathbb{E}(\sigma^2) < +\infty$. The correlation of the increments of the process X has the following asymptotic behaviour as $h \downarrow 0$:*

$$\lim_{h\downarrow 0} \rho(|X_{s+h}-X_s|, |X_{t+h}-X_t|) = \rho(t-s) := \frac{2}{\pi Var(\sigma|W_1|S^{D-1/2})} e^{-\lambda|t-s|}\phi(\lambda|t-s|) \tag{1.4}$$

where $S \sim Exp(1)$ is independent of σ and W_1, and

$$\phi(x) := Cov(\sigma S^{D-1/2}, \sigma(S+x)^{D-1/2})$$

This shows that the volatility autocorrelation decays exponentially fast for time scales greater than the mean distance $1/\lambda$ between the epochs τ_k. For shorter time scales, a relevant contribution is given by the function ϕ. Note that we can write

$$\phi(x) = Var(\sigma)\mathbb{E}(S^{D-1/2}(S+x)^{D-1/2}) + \mathbb{E}(\sigma)^2 Cov(S^{D-1/2}, (S+x)^{D-1/2})$$

As $x \to +\infty$, the two terms in the right hand side has the following behaviour

$$\mathbb{E}(S^{D-1/2}(S+x)^{D-1/2}) \approx x^{D-1/2}, \qquad Cov(S^{D-1/2}, (S+x)^{D-1/2}) \approx x^{D-3/2}$$

2 Calibration, simulation, allocation and monitoring

In this chapter we will analyse the calibration of the model to the historical data, the simulation of trajectories of the model to make a forecast of the evolution of the index, the problem of the optimal allocation and, finally, the monitoring of the portfolio position. We will consider the following scenario: we are at a certain date, that we indicate with t and we want to invest with a time horizon of one month, i.e. up to $t+22$ (22 working days).

2.1 Model calibration

After downloading the S&P500 time series (adjusted closure) from 3rd January 1950 to the chosen date t, we proceed to the calibration of the model to historical data. We denote the S&P500 time series by $(s_i)_{0\leq i\leq N}$, whereas the corresponding detrended log-S&P500 time series will be denoted by $(x_i)_{0\leq i\leq N}$. The idea here is to compare some empirical quantities with the corresponding theoretical predictions; more precisely, we are going to consider the following quantities:

- the theoretical exponent $A(q)$, which has been defined in (1.3)
- the multiscaling constants C_1 and C_2, that are given by (1.2) for $q=1$ and $q=2$
- the volatility autocorrelation (1.4)

All these quantities are function of the parameters of the model D, λ, σ, so we estimate these parameters comparing those theoretical quantities with the corresponding empirical estimates.

For what concerns C_1, C_2, because $q^* = (\frac{1}{2}-D)^{-1} > 2$, we can write more explicitly

$$C_1 = \frac{2\sqrt{D}\Gamma(\frac{1}{2}+D)\mathbb{E}(\sigma)\lambda^{1/2-D}}{\sqrt{\pi}}, \qquad C_2 = 2D\Gamma(2D)\mathbb{E}(\sigma^2)\lambda^{1-2D}$$

Defining the corresponding empirical quantities requires some care because the time series is in discrete time so no limit $h \downarrow 0$ is possible. First of all, we evaluate the empirical q-moment $\hat{m}_q(h)$, i.e.

$$\hat{m}_q(h) := \frac{1}{N+1-h}\sum_{i=0}^{N-h} |x_{i+h}-x_i|^q$$

By Corollary 1.3, we have that the relation $log(\hat{m}_q(h)) \sim A(q)log(h) + log(C_q)$ should hold for h small. From a standard linear regression of

$log(\hat{m_q}(h))$ versus $log(h)$ for $h = 1, 2, 3, 4, 5$, we therefore determine the empirical values of $A(q)$ and C_q, that we call $\hat{A}(q)$ and $\hat{C}_q$.

For what concerns the theoretical volatility autocorrelation, Theorem 2 and the stationarity of the increments of the process X yield

$$\rho(t) := \lim_{h \downarrow 0} \rho(|X_h|, |X_{t-h} - X_t|) = \frac{2}{\pi Var(\sigma|W_1|S^{D-1/2})} e^{-\lambda t}\phi(\lambda t)$$

For the analogous empirical quantity, we define the empirical volatility autocorrelation $\hat{\rho_h}(t)$ over h days as the sample correlation coefficient of the two sequences $(|x_{i+h} - x_i|)_{0 \le i \le N-h-t}$ and $(|x_{i+h+t} - x_{i+t}|)_{0 \le i \le N-h-t}$. Since no $h \downarrow 0$ limit can be taken on discrete data, we are going to compare $\rho(t)$ with $\hat{\rho_h}(t)$ for $h = 1$.

We can now define the following loss function, which evaluates the distance between the theoretical quantities $C_1, C_2, A(q), \rho(t)$ and the corresponding empirical quantities

$$L\left(D, \lambda, \mathbb{E}(\sigma), \mathbb{E}(\sigma^2)\right) = \frac{1}{2}\left\{ a_1 \left(\frac{\hat{C}_1}{C_1} - 1\right)^2 + a_2 \left(\frac{\hat{C}_2}{C_2} - 1\right)^2 \right\} +$$

$$\frac{a_3}{20} \sum_{k=1}^{20} \left(\frac{\hat{A}(k/4)}{A(k/4)} - 1\right)^2 + a_4 \sum_{n=1}^{400} \frac{e^{-n/T}}{\left(\sum_{m=1}^{400} e^{-m/T}\right)} \left(\frac{\hat{\rho}_1(n)}{\rho(n)} - 1\right)^2$$

The terms a_1, a_2, a_3, a_4, T are parameters which have here been set to $1, 1, 1, 20, 40$ respectively; we decide to raise the a_4 parameter to weigh the impact of correlations more.

Because of the complicated structure of the loss function, it is not possible to minimize it leaving all the four variables free. We therefore proceed in the following way: we can observe that the term

$$\sum_{k=1}^{n} \left(\frac{\hat{A}(k/4)}{A(k/4)} - 1\right)^2$$

depends only on D. This term represents the fitting of the parameter D for what concerns the multiscaling; we can therefore estimate lower and upper bounds for it by means of qualitative considerations. First of all, we plot a graph of the empirical multiscaling obtained from historical data; then we compare it with the theoretical multiscaling expected by the model.

Let $D_1, D_2 \in (0, 1/2]$, $D_1 < D_2$; we denote by $A_{D_1}(q)$ and $A_{D_2}(q)$ the corresponding multiscaling quantities. We can now notice that, for all $q > 0$, $A_{D_1}(q) \le A_{D_2}(q)$; in fact, we observe that $q^*_{D_1} = \frac{1}{\frac{1}{2} - D_1} < \frac{1}{\frac{1}{2} - D_2} = q^*_{D_2}$, therefore using (1.3) we have

$$\begin{cases} A_{D_1}(q) = q/2 = A_{D_2}(q) & 0 < q \le q^*_{D_1} \\ A_{D_1}(q) = D_1 q + 1 < q/2 = A_{D_2}(q) & q^*_{D_1} < q \le q^*_{D_2} \\ A_{D_1}(q) = D_1 q + 1 < D_2 q + 1 = A_{D_2}(q) & q^*_{D_2} > q \end{cases}$$

Therefore we can try to increase the values of D until we find appropriate bounds; an example is shown in figure 2.2.

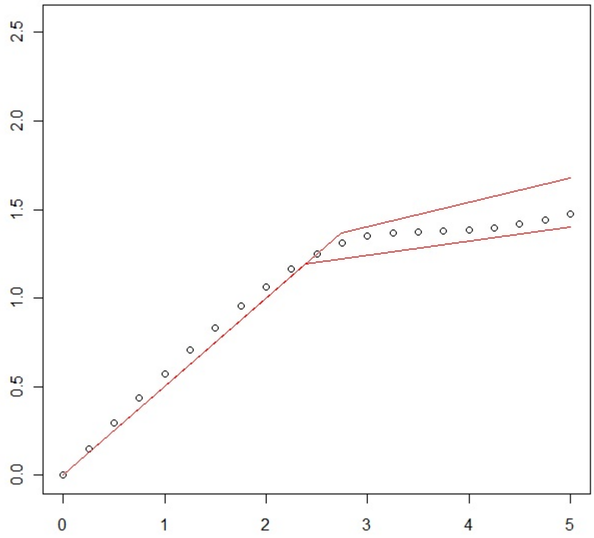

Figure 2.1: Multiscaling for the S&P500 from 3rd January 1950 to 31st December 1999 with the upper and lower bounds for values of D of 0.08 and 0.135 .

After that, we discretize the D interval we have found into several points, here in number of 10, and we minimize over the other three variables on the grid chosen for D. Furthermore, as regards λ, since we are considering a very long time series (50 years of observations), it seems logical to consider as a lower bound for the time between two shocks a value of about a year, because we are interested in large scale shocks. We will then impose the constraint $\lambda \leq 1/200$.

Using the program Mathematica we proceed to the numerical implementation and we find estimates for the four parameters of the model. For example, calibrating till the 31st December 1999, we find the values

$$\hat{D} \simeq 0.0763333, \quad \hat{\lambda} \simeq 0.000611612, \quad \mathbb{E}(\hat{\sigma}) \simeq 0.23555, \quad \mathbb{E}(\hat{\sigma^2}) \simeq 0.0554834$$

As in [1], also here σ can be considered as constant, since

$$\mathbb{E}(\sigma^2) - \mathbb{E}(\sigma)^2 = 0.00002315$$

and also

$$\frac{\sqrt{Var(\sigma)}}{\mathbb{E}(\sigma)} = 9.82 \times 10^{-5}$$

2.1.1 Last shock time estimate

Another important thing to estimate is the time of the last shock occurred in the market; to do so we can use the following method.

Suppose we are in t and the last shock occurred in τ_n, the penultimate in τ_{n-1}, and so on. Consider the function

$$\mathbb{E}\left\{ \frac{1}{l} \sum_{i=1}^{l} (X_{t-i+1} - X_{t-i})^2 \, | \tau, \sigma \right\} := f(l)$$

We treat first the case $0 < l \leq t - \tau_n$; in this interval the function has the followin expression

$$f(l) = \frac{\sigma^2}{l}\left[(t-\tau_n)^{2D} - (t-\tau_n-l)^{2D}\right]$$

We observe that f is continuous and increasing; in fact, if we introduce the function $g(l) = -(t-\tau_n-l)^{2D}$ we can rewrite f in the following way

$$f(l) = \sigma^2 \frac{g(l)-g(0)}{l}$$

so that f is increasing because it is the difference quotient of a convex function.

Now we consider the interval $t-\tau_n \leq l \leq t-\tau_{n-1}$; here the expression to study is

$$f(l) = \frac{\sigma^2}{l}\left[(t-\tau_n)^{2D} + (\tau_n-\tau_{n-1})^{2D} - (t-l-\tau_{n-1})^{2D}\right]$$

Its derivative is

$$f'(l) = \frac{\sigma^2}{l^2}\left[2Dl(t-l-\tau_{n-1})^{2D-1} + (t-l-\tau_{n-1})^{2D} - (t-\tau_n)^{2D} - (\tau_n-\tau_{n-1})^{2D}\right]$$

which, evaluated in $l = t-\tau_n$, equals

$$f'(t-\tau_n) = \frac{\sigma^2}{(t-\tau_n)^2}\left[2D(t-\tau_n)(\tau_n-\tau_{n-1})^{2D-1} - (t-\tau_n)^{2D}\right]$$

So $f'(t-\tau_n) < 0$ if and only if $\tau_{n-1} \leq \tau_n - \frac{t-\tau_n}{2D^{\frac{1}{2D-1}}}$; since the function $D \mapsto 2D^{\frac{1}{2D-1}}$ has a minimum in the interval $0 < D \leq .5$ at the point $D = 0.186682$ with function value 29.122, $f'(t-\tau_n) < 0$ if $\tau_{n-1} \leq \tau_n - \frac{t-\tau_n}{29.122}$, which is equivalent to

$$t-\tau_n \leq 29.122(\tau_n-\tau_{n-1})$$

Doing a crude approximation, such as $\tau_n - \tau_{n-1} \simeq 1/\lambda$, we can rewrite the condition in the following way

$$t-\tau_n \leq \frac{29.122}{\lambda}$$

Since we assumed that $\lambda \leq 1/200$, is sufficient that

$$t-\tau_n \leq \frac{29}{\frac{1}{200}} = 5800 \leq \frac{29.122}{\lambda}$$

which is a very reasonable condition and almost always true given the estimates obtained with simulations.

If this condition is met, the function has a local maximum in $l = t-\tau_n$; recalling that N indicates the length of the time series $(x_i)_{0\leq i\leq N}$, we can therefore apply an algorithm that evaluates the empirical quantity

$$\frac{1}{l}\sum_{i=1}^{l}(x_{N-i+1}-x_{N-i})^2 := \hat{f}_N(l)$$

for $l = 1, \dots, L$ with L properly chosen. We obtain graphs like the one in figure 2.2. The estimate of the last shock time is given by

$$\hat{\tau}_N^* := t - \left(\underset{1 \le l \le L}{\operatorname{argmax}} \quad \hat{f}_N(l) \right) + 1$$

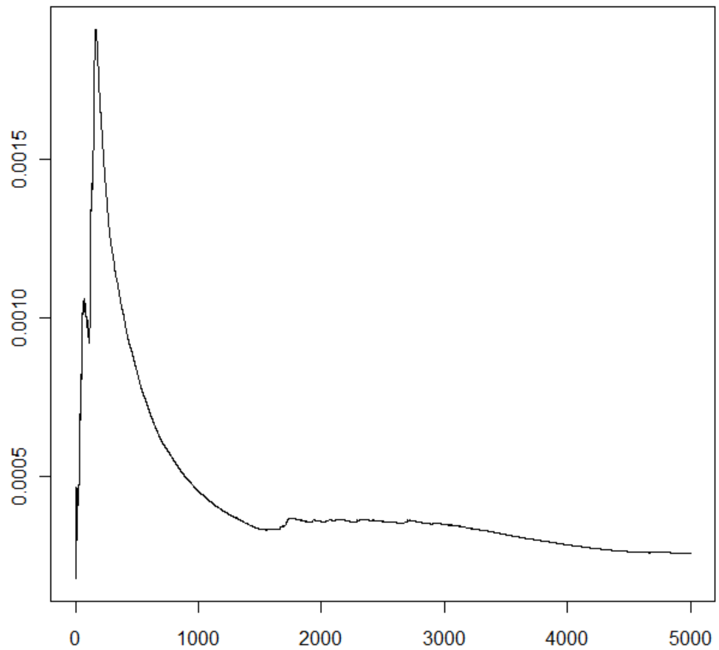

Figure 2.2: Plot of the quantity $\hat{f}_N(l)$ for $l = 1, \dots, 5000$ for the S&P500 time series from 3rd January 1950 to 2nd July 2010. The peak correponds to the estimate of the last shock $\hat{\tau}_N^*$ for the day 7/10/2008; notice that it is near to the default of the Lehman Brothers bank that occured on 15th September 2008.

We use the following approach to test the consistency of the method; first of all, we estimate the last shock as described beofore, then we use the same method going back along the time series using a moving window. More precisely, we choose an appropriate time interval ΔN, then we consider the empirical quantity $\hat{f}_N(l)$ with $l = 1, \dots, L$, next $\hat{f}_{N-\Delta N}(l)$, later $\hat{f}_{N-2\Delta N}(l)$, and so on, and finally we compare the quantities we obtained. In the figure 2.3 an example of this method is shown.

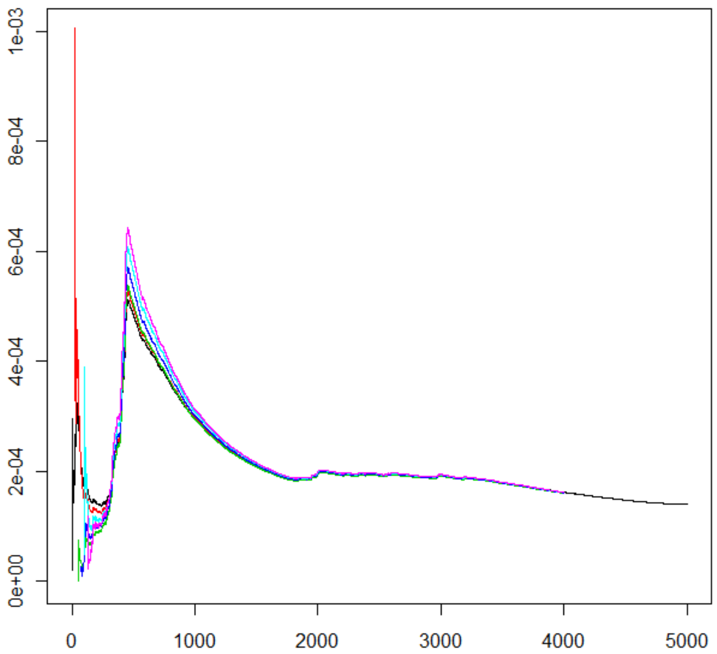

Figure 2.3: Plot of the quantitie s$\hat{f}_N(l), \hat{f}_{N-25}(l), \ldots, \hat{f}_{N-150}(l)$ for $l = 1, \ldots, 5000$. We can notice that there is a peak that is recognised as a shock by all the instances of the method, giving a strong signal of shock in that point.

Another way to test the method is the following; we take the time series and we use the estimation method going back in the time series by one value at a time, i.e. we calculate, for $i = 0, \ldots, I$, with I properly chosen, the quantity $\hat{\tau}^*_{N-i}$. Then we plot the estimates obtained and we get results similar to the one in figure 2.4.

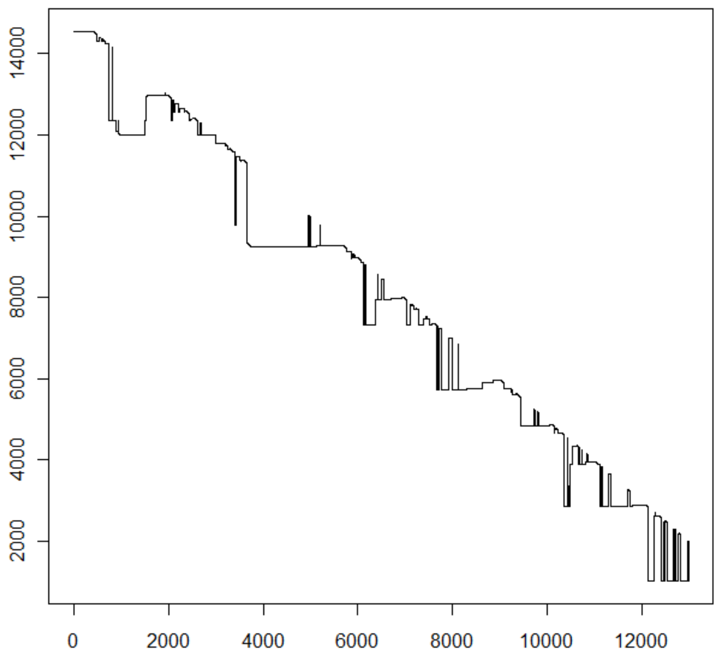

Figure 2.4: Plot of the quantity $\hat{\tau}^*_{N-i}$ with $i = 0, \ldots, N - 5000$ for the S&P500 time series from 3rd January 1950 to 2 July 2010. At the steps we have large-scale shocks. Notice the presence of noise at the vertical lines.

These two tests show a good level of consistency of the method; in fact, the results shown in the figures 2.3 and 2.4 give clear indications of the instants in which the shocks occurred. The method represented by figure 2.4 is appropriate to study the shock on the whole time series while the one in figure 2.3 is more useful for the estimate of the last shock. For our goals, we would use more this last estimate method.

2.2 Simulation

After having calibrated the model, it is possible to simulate trajectories to make a forecast of the market evolution. We take a time horizon of $N = 22$ days, i.e. a month; we have

$$I_{t+h} - I_t = \sigma^2_{i(t+h)} \left(t + h - \tau_{i(t+h)}\right)^{2D} + \sum_{k=i(t)+1}^{i(t+h)} \sigma^2_{k-1} \left(\tau_k - \tau_{k-1}\right)^{2D} - \sigma^2_{i(t)} \left(t - \tau_{i(t)}\right)^{2D}$$

and

$$I_h = \sigma^2_{i(h)} \left(h - \tau_{i(h)}\right)^{2D} + \sum_{k=1}^{i(h)} \sigma^2_{k-1} \left(\tau_k - \tau_{k-1}\right)^{2D} - \sigma^2_0 \left(-\tau_0\right)^{2D}$$

If we introduce $\mathcal{T}^t := \mathcal{T} - t$ and we denote the points in $\mathcal{T}^t$ by $\tau^t := \tau - t$ and $i^t(h) := i(h+t)$, we have

$$I_{t+h} - I_t = \sigma^2_{i^t(h)} \left(h - \tau^t_{i^t(h)}\right)^{2D} +$$

$$\sum_{k=i^t(0)+1}^{i^t(h)} \sigma^2_{k-1} \left(\tau^t_k - \tau^t_{k-1}\right)^{2D} - \sigma^2_{i^t(0)} \left(-\tau^t_{i^t(0)}\right)^{2D}$$

Now, as reported in [1], this shows that $I_{t+h} - I_t$ and I_h are the same function of the processes characterized by the sequences $\mathcal{T}^t$ and $\mathcal{T}$ and because of the fact that $\mathcal{T}^t$ and $\mathcal{T}$ have the same distribution, also $I_{t+h} - I_t$ and I_h have the same distribution. Moreover, the distribution of $I_{t+h} - I_t$ conditional to $\tau_{i(t)} = \overline{\tau}$ and σ is equivalent to the distribution of I_h conditional to $\tau_0 = \overline{\tau} - t$ and σ. We can therefore state that

$$d\mathbb{P}(X_{t+h} - X_t | \tau_{i(t)} = \overline{\tau}, \sigma) = d\mathbb{P}(X_h | \tau_0 = \overline{\tau} - t, \sigma)$$

To simulate a trajectory like $X_{t+1}, \dots, X_{t+22}$ we can simulate a trajectory $X_1, \dots, X_{22}$ with the condition $\tau_0 = \hat{\tau}^*_N - t$ and add the value X_t. During the simulations we would construct 1000000 trajectories this way.

After that, we obtain the value of the index in the following way. Since X_t represents the detrended logarithm of the value of the index, we have already seen that empirically the trend is estimated with a sample mean of the last M values of the logarithm of the underying index; following the choice made in [1], we take $M = 250$, so we have

$$X_t = \log(S_t) - \frac{1}{250} \left(\log(S_{t-250} + \dots + \log(S_{t-1})\right) \tag{2.1}$$

and also

$$X_{t+1} = \log(S_{t+1}) - \frac{1}{250} \left(\log(S_{t-249} + \dots + \log(S_t)\right)$$

Therefore

$$X_{t+1} - X_t = \log(S_{t+1}) - \log(S_t) - \frac{1}{250} \left(\log(S_t) - \log(S_{t-250})\right)$$

so

$$S_{t+1} = S_t \exp(X_{t+1} - X_t) \sqrt[250]{\frac{S_t}{S_{t-250}}} \tag{2.2}$$

In the figures we reported two trajectories simulated this way.

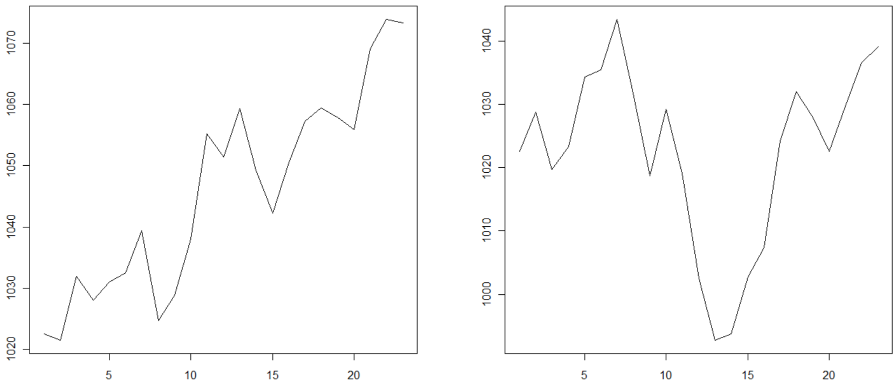

Figure 2.5: Two trajectories that simulates a possible evolution of the S&P500 index for the 22 days followin the 2nd July 2010.

After simulating the trajectories, we can construct a 80 % probability confidence interval for the evolution of the index. This is obtained by simply taking as lower bound the 10 % quantile and as upper bound the 90 % quantile.

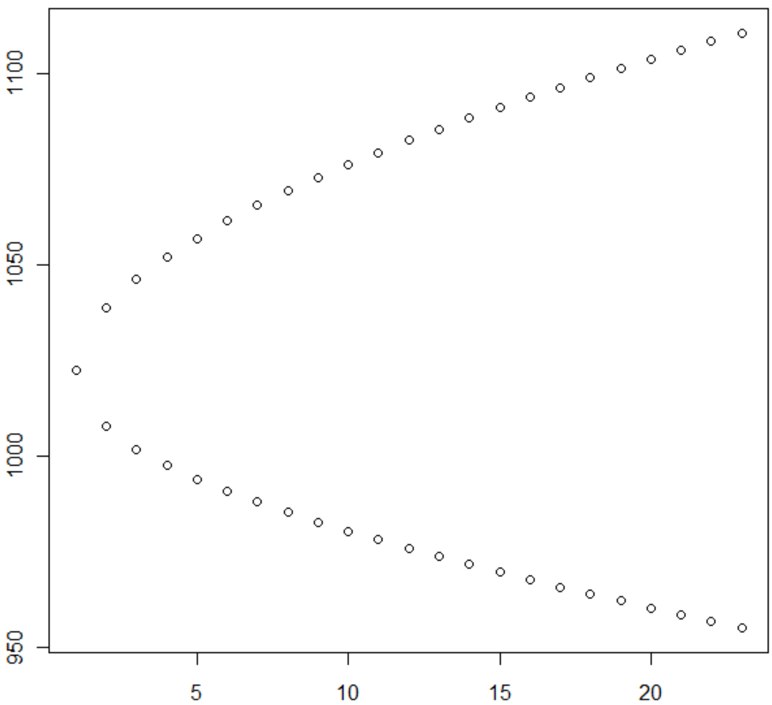

Figure 2.6: 80 % probability confidence interval for the evolution of the S&P500 index for the 22 days following the 2nd July 2010.

2.3 Allocation

Suppose that the agent invests by using a mean-variance approach, i.e. his or her optimal allocation is given by the solution of the problem

$$\textbf{Problem 1} \qquad \begin{aligned} &\min \frac{1}{2}\alpha^2 Var(1+\mu) \\ &\alpha\mathbb{E}(1+\mu) + (1-\alpha)(1+r_{22}) \geq \rho \end{aligned}$$

where α is the percentage of wealth invested in the risky asset, ρ is the target return and μ is defined by

$$S_{t+22} = S_t(1+\mu)$$

In this case the model is uniperiodal, with a 22-days period. In this scenario, we can calculate the optimal α explicitly.

LEMMA 2: *The optimal percentage α^* of risky asset is given by*

$$\alpha^* = \begin{cases} 0 & \text{if } \rho \leq 1 + r_{22} \\ \frac{\rho-(1+r_{22})}{\mathbb{E}(1+\mu)-(1+r_{22})} & \text{if } \rho > 1 + r_{22} \end{cases} \tag{2.3}$$

Proof. If $\rho \leq 1 + r_{22}$, then if we put $\alpha = 0$ in the minimization problem, we have

$$\frac{1}{2}\alpha^2 Var(1+\mu) = 0 \qquad (1+r_{22}) \geq \rho$$

so the constraint is satisfied and the objective function reaches its minimum.

On the contrary, if $\rho > 1 + r_{22}$, we notice that α satisfies the constraint in the problem if and only if

$$\alpha\left[\mathbb{E}(1+\mu) - (1+r_{22})\right] \geq \rho - (1+r_{22})$$

which gives

$$\alpha \begin{cases} \geq \frac{\rho-(1+r_{22})}{\mathbb{E}(1+\mu)-(1+r_{22})} & \text{if} \quad \mathbb{E}(1+\mu) - (1+r_{22}) > 0 \\ \leq \frac{\rho-(1+r_{22})}{\mathbb{E}(1+\mu)-(1+r_{22})} & \text{if} \quad \mathbb{E}(1+\mu) - (1+r_{22}) < 0 \end{cases}$$

We should choose α^* that minimizes $|\alpha|$ to minimize α^2 in the objective function; therefore

$$\alpha^* = \begin{cases} \frac{\rho-(1+r_{22})}{\mathbb{E}(1+\mu)-(1+r_{22})} & \text{if} \quad \mathbb{E}(1+\mu) - (1+r_{22}) > 0 \\ \frac{\rho-(1+r_{22})}{\mathbb{E}(1+\mu)-(1+r_{22})} & \text{if} \quad \mathbb{E}(1+\mu) - (1+r_{22}) < 0 \end{cases}$$

which gives the conclusion. □

Notice that we need to know $\mathbb{E}(1+\mu)$ in formula (2.3) in order to obtain α^*; this can be obtained numerically by the simulation of the trajectories and later we will also provide a formula for it.

We therefore first have to find μ; to do so, we will use the following fact.

LEMMA 3: *Recalling the definition of X_t in (2.1), the following formula holds true*

$$S_{t+n} = S_t^{\left(\frac{251}{250}\right)^n} \left\{ \prod_{i=1}^{n} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{n-i}} \right\} \left\{ \prod_{i=1}^{n} \exp(X_{t+i} - X_{t+i-1})^{\left(\frac{251}{250}\right)^{n-i}} \right\}$$

Proof. We proceed by induction; recall from 2.2 that

$$S_{t+1} = S_t \exp(X_{t+1} - X_t) \sqrt[250]{\frac{S_t}{S_{t-250}}} \tag{2.4}$$

that we can be re-written as

$$S_{t+1} = S_t^{\frac{251}{250}} S_{t-250}^{\frac{-1}{250}} \exp(X_{t+1} - X_t)$$

so the result is valid for $n = 1$. Now suppose that the formula is valid for $n-1$; then we have

$$\begin{aligned}
S_{t+n} &= S_{t+n-1} \exp(X_{t+n} - X_{t+n-1}) \sqrt[250]{\frac{S_{t+n}}{S_{t-250+n}}} \\
&= S_t^{\left(\frac{251}{250}\right)^{n-1}} \left\{ \prod_{i=1}^{n-1} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{n-1-i}} \right\} \left\{ \prod_{i=1}^{n-1} \exp(X_{t+i} - X_{t+i-1})^{\left(\frac{251}{250}\right)^{n-1-i}} \right\} \\
&\quad \exp(X_{t+n} - X_{t+n-1}) \sqrt[250]{\frac{S_{t+n-1}}{S_{t-250+n-1}}} \\
&= S_t^{\left(\frac{251}{250}\right)^{n-1}} \left\{ \prod_{i=1}^{n-1} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{n-1-i}} \right\} \left\{ \prod_{i=1}^{n-1} \exp(X_{t+i} - X_{t+i-1})^{\left(\frac{251}{250}\right)^{n-1-i}} \right\} \\
&\quad \exp(X_{t+n} - X_{t+n-1}) \left(S_{t-250+n-1}\right)^{-\frac{1}{250}} \\
&\quad \left(S_t^{\left(\frac{251}{250}\right)^{n-1}} \left\{ \prod_{i=1}^{n-1} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{n-1-i}} \right\} \left\{ \prod_{i=1}^{n-1} \exp(X_{t+i} - X_{t+i-1})^{\left(\frac{251}{250}\right)^{n-1-i}} \right\} \right)^{\frac{1}{250}} \\
&= S_{t+n} = S_t^{\left(\frac{251}{250}\right)^{n}} \left\{ \prod_{i=1}^{n} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{n-i}} \right\} \left\{ \prod_{i=1}^{n} \exp(X_{t+i} - X_{t+i-1})^{\left(\frac{251}{250}\right)^{n-i}} \right\}
\end{aligned}$$

□

Another useful result is the following

Proposition 2: *The following formula holds true*

$$\begin{aligned}
\log(1+\mu) &= \left[\left(\frac{251}{250}\right)^{22} - 1 \right] \log(S_t) \\
&+ \sum_{i=1}^{22} \frac{-1}{250} \left(\frac{251}{250}\right)^{22-i} \log(S_{t-250+i-1}) \\
&+ \sum_{i=1}^{22} \left(\frac{251}{250}\right)^{22-i} (X_{t+i} - X_{t+i-1})
\end{aligned}$$

Proof. Using Lemma 3, we have

$$S_{t+n} = S_t \left[S_t^{\left(\frac{251}{250}\right)^{n} - 1} \left\{ \prod_{i=1}^{n} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{n-i}} \right\} \left\{ \prod_{i=1}^{n} \exp(X_{t+i} - X_{t+i-1})^{\left(\frac{251}{250}\right)^{n-i}} \right\} \right]$$

Therefore

$$1+\mu = S_t^{\left(\frac{251}{250}\right)^{22}-1}\left\{\prod_{i=1}^{22} S_{t-250+i-1}^{\frac{-1}{250}\left(\frac{251}{250}\right)^{22-i}}\right\}\left\{\prod_{i=1}^{22}\exp(X_{t+i}-X_{t+i-1})^{\left(\frac{251}{250}\right)^{22-i}}\right\}$$

from which the result is easily derived. □

We can now state this result

THEOREM 3: *The distibution of* $\log(1+\mu)$ *is given by*

$$f(z) := \sum_{k_1=0}^{+\infty}\cdots\sum_{k_{22}=0}^{+\infty} e^{-22\lambda}\left(\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\right)\int_{A(k_1,\dots,k_{22})}\frac{\exp\left(-\frac{1}{2}\frac{g(z)^2}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\,d\boldsymbol{\tau}$$

where, we recall, λ *is the intensity of the Poisson point process* τ*,*

$$A(k_1,\dots,k_{22}) := \{\boldsymbol{\tau} = (\tau_{1,1},\dots,\tau_{1,k_1},\dots,\tau_{22,1},\dots,\tau_{22,k_{22}}) : \forall\ 1\le i\le 22,$$
$$t+i-1 \le \tau_{i,1} < \cdots < \tau_{i,k_i} \le t+i\}$$

with the agreement that if $k_i = 0$ *for some* i*, there are no items of the type* $\tau_{i,j}$ *in the expression;*

$$g(z) := z - \left[\left(\frac{251}{250}\right)^{22} - 1\right]\log(S_t) - \sum_{i=1}^{22}\frac{-1}{250}\left(\frac{251}{250}\right)^{22-i}\log(S_{t-250+i-1})$$

and finally

$$\sigma_{i,k_i}^2 := \left(\frac{251}{250}\right)^{2(22-i)}\sigma^2\left[(t+1-\tau_{i,k_i})^{2D} + \sum_{j=0}^{k_i-1}(\tau_{i,k_i-j}-\tau_{i,k_i-j-1})^{2D} - (t+i-1-\tau_{i,0})^{2D}\right]$$

with $\tau_{i,0} := \max(\mathcal{T}\cap(-\infty, t+i-1])$ *and the agreement that if* $k_i = 0$ *we have* $\tau_{i,k_i} = \tau_{i,0}$ *and the sum is not considered.*

Proof. Suppose that we know the parameter σ and all the shocks that will occur on the interval $[t, t+22]$. Using the symbols introduced in the Theorem, we have, for each $\boldsymbol{\tau}\in A(k_1,\dots,k_{22})$,

$$t \le \tau_{1,1} < \cdots < \tau_{1,k_1} < t+1 < \cdots < t+21 < \tau_{22,1} < \cdots < \tau_{22,k_{22}} \le t+22$$

Under this assumptions, the distribution of

$$\left(\frac{251}{250}\right)^{22-i}(X_{t+i}-X_{t+i-1})$$

is a normal with fixed variance. In fact, recalling Lemma 1, we have that conditionally on σ and $\mathcal{T}$, X_t is a Brownian motion. So, using again Lemma 1, we can see that

$$X_{t+i}-X_{t+i-1} \sim \mathcal{N}\left(0, \sigma^2\left[(t+1-\tau_{i,k_i})^{2D} + \sum_{j=0}^{k_i-1}(\tau_{i,k_i-j}-\tau_{i,k_i-j-1})^{2D} - (t+i-1-\tau_{i,0})^{2D}\right]\right)$$

so that

$$\left(\frac{251}{250}\right)^{22-i}(X_{t+i}-X_{t+i-1}) \sim \mathcal{N}(0,\sigma^2_{i,k_i})$$

Continuing to suppose that σ and the shock times are known, the process also has the conditional independence property by Proposition 1, so we can write

$$\sum_{i=1}^{22}\left(\frac{251}{250}\right)^{22-i}(X_{t+i}-X_{t+i-1}) \sim \mathcal{N}\left(0,\sum_{i=1}^{22}\sigma^2_{i,k_i}\right)$$

We can therefore obtain the distribution of

$$\sum_{i=1}^{22}\left(\frac{251}{250}\right)^{22-i}(X_{t+i}-X_{t+i-1})$$

conditional to the number of jumps in the selected period and their positions. To obtain the distribution without conditioning it will be sufficient to integrate over all possible values, i.e.

$$d\mathbb{P}\left(\sum_{i=1}^{22}\left(\frac{251}{250}\right)^{22-i}(X_{t+i}-X_{t+i-1})\right)(x) =$$

$$\sum_{k_1=0}^{+\infty}\cdots\sum_{k_{22}=0}^{+\infty} e^{-22\lambda}\left(\prod_{i=1}^{22}\frac{\lambda^{k_1}}{k_i!}\right)\int_{A(k_1,\ldots,k_{22})}\frac{\exp\left(-\frac{1}{2}\frac{x^2}{\sum_{i=1}^{22}\sigma^2_{i,k_i}}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma^2_{i,k_i}\right)}}\,d\boldsymbol{\tau}$$

Using Proposition 2 we can see that, for all $A \in \mathcal{B}(\mathbb{R})$,

$$\Pr\left(\log(1+\mu)\in A\right) =$$

$$\Pr\left(\sum_{i=1}^{22}\left(\frac{251}{250}\right)^{22-i}(X_{t+i}-X_{t+i-1}) \in A - \left[\left(\frac{251}{250}\right)^{22}-1\right]\log(S_t) + \sum_{i=1}^{22}\frac{-1}{250}\left(\frac{251}{250}\right)^{22-i}\log(S_{t-250+i-1})\right)$$

So the density of $\log(1+\mu)$ evaluated in z is the density, that we have just obtained, and evaluated in

$$z - \left[\left(\frac{251}{250}\right)^{22}-1\right]\log(S_t) + \sum_{i=1}^{22}\frac{-1}{250}\left(\frac{251}{250}\right)^{22-i}\log(S_{t-250+i-1})$$

□

We can now calculate more explicitly the quantities $\mathbb{E}(1+\mu)$ and $\mathbb{E}\left((1+\mu)^2\right)$.

Proposition 3: *The following formulas hold true*

$$\mathbb{E}(1+\mu) = e^{-22\lambda}e^{\beta}\sum_{k_1,\ldots,k_{22}=0}^{+\infty}\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\int_{A(k_1,\ldots,k_{22})}\exp\left(\frac{1}{2}\sum_{j=1}^{22}\sigma^2_{i,k_i}\right)d\boldsymbol{\tau}$$

$$\mathbb{E}\left((1+\mu)^2\right) = e^{-22\lambda}e^{2\beta}\sum_{k_1,\ldots,k_{22}=0}^{+\infty}\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\int_{A(k_1,\ldots,k_{22})}\exp\left(2\sum_{j=1}^{22}\sigma^2_{i,k_i}\right)d\boldsymbol{\tau}$$

and

$$\beta := \left[\left(\frac{251}{250}\right)^{22} - 1\right] \log(S_t) + \sum_{i=1}^{22} \frac{-1}{250} \left(\frac{251}{250}\right)^{22-i} \log(S_{t-250+i-1})$$

i.e., β is definied so that $g(z) = z - \beta$.

Proof. We first calculate $\mathbb{E}(1+\mu)$, recalling that we denoted by f the density of $\log(1+\mu)$ (remember that $1+\mu > 0$);

$$\mathbb{E}(1+\mu) = \int_{-\infty}^{+\infty} e^z f(z)\, dz =$$

$$\int_{-\infty}^{+\infty} e^z e^{-22\lambda} \sum_{k_1,\dots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\dots,k_{22})} \frac{\exp\left(-\frac{1}{2}\frac{(z-\beta)^2}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\, d\boldsymbol{\tau}\, dz =$$

$$e^{-22\lambda} \sum_{k_1,\dots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\dots,k_{22})} \int_{-\infty}^{+\infty} e^z \frac{\exp\left(-\frac{1}{2}\frac{(z-\beta)^2}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\, dz\, d\boldsymbol{\tau} =$$

$$e^{-22\lambda} \sum_{k_1,\dots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\dots,k_{22})} \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2}\frac{(z-\beta)^2-2(\sum_{i=1}^{22}\sigma_{i,k_i}^2)z}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\, dz\, d\boldsymbol{\tau} =$$

Notice that

$$\int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2}\frac{(z-\beta)^2-2(\sum_{i=1}^{22}\sigma_{i,k_i}^2)z}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\, dz =$$

$$\exp\left(\frac{1}{2}\sum_{i=1}^{22}\sigma_{i,k_i}^2 + \beta\right) \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2}\frac{(z-(\beta+\sum_{i=1}^{22}\sigma_{i,k_i}^2))^2}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\, dz =$$

$$\exp\left(\frac{1}{2}\sum_{i=1}^{22}\sigma_{i,k_i}^2 + \beta\right)$$

So that we have

$$e^{-22\lambda} \sum_{k_1,\dots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\dots,k_{22})} \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2}\frac{(z-\beta)^2-2(\sum_{i=1}^{22}\sigma_{i,k_i}^2)z}{\sum_{i=1}^{22}\sigma_{i,k_i}^2}\right)}{\sqrt{2\pi\left(\sum_{i=1}^{22}\sigma_{i,k_i}^2\right)}}\, dz\, d\boldsymbol{\tau} =$$

$$e^{-22\lambda} \sum_{k_1,\dots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\dots,k_{22})} \exp\left(\frac{1}{2}\sum_{i=1}^{22}\sigma_{i,k_i}^2 + \beta\right) d\boldsymbol{\tau} =$$

$$e^{-22\lambda} e^{\beta} \sum_{k_1,\dots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\dots,k_{22})} \exp\left(\frac{1}{2}\sum_{i=1}^{22}\sigma_{i,k_i}^2\right) d\boldsymbol{\tau}$$

The calculations for $\mathbb{E}\left((1+\mu)^2\right)$ are similar. $\square$

Proposition 4: *For what concerns the problem of mean-variance allocation, this model is equivalent to a geometric Brownian motion with constant coefficients, given for $t \in [t, t+22]$ by*

$$dS_t = S_t(a\ dt + b\ dW_t)$$

with

$$a = \frac{1}{22}\left(-22\lambda + \beta + L_1\right)$$

$$b = \sqrt{\frac{1}{22}\left(22\lambda + L_2 - 2L_1\right)}$$

where

$$L_1 = \log\left\{\sum_{k_1,\ldots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\ldots,k_{22})} \exp\left(\frac{1}{2}\sum_{j=1}^{22}\sigma_{i,k_i}^2\right)\ d\boldsymbol{\tau}\right\}$$

$$L_2 = \log\left\{\sum_{k_1,\ldots,k_{22}=0}^{+\infty} \prod_{i=1}^{22} \frac{\lambda^{k_i}}{k_i!} \int_{A(k_1,\ldots,k_{22})} \exp\left(2\sum_{j=1}^{22}\sigma_{i,k_i}^2\right)\ d\boldsymbol{\tau}\right\}$$

Proof. The solution of the SDE of the geometric Brownian motion is given by

$$S_t = S_0 \exp\left[\left(a - \frac{1}{2}b^2\right)t + bW_t\right]$$

so that

$$S_t = S_0 e^N$$

where $N \sim \mathcal{N}(m, \Sigma^2)$, with $m = \left(a - \frac{1}{2}b^2\right)t$, $\Sigma^2 = b^2 t$. However, we also have

$$S_t = S_0(1+\mu)$$

so

$$1 + \mu = e^N$$

and

$$\mathbb{E}(1+\mu) = \mathbb{E}(e^N) = e^{m+\frac{1}{2}\Sigma^2}$$

$$\mathbb{E}\left\{(1+\mu)^2\right\} = \mathbb{E}(e^{2N}) = e^{2m+2\Sigma^2}$$

We can therefore find m and Σ^2 solving this system

$$\begin{cases} e^{m+\frac{1}{2}\Sigma^2} = e^{-22\lambda}e^{\beta}\sum_{k_1,\ldots,k_{22}=0}^{+\infty}\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\int_{A(k_1,\ldots,k_{22})}\exp\left(\frac{1}{2}\sum_{j=1}^{22}\sigma_{i,k_i}^2\right)\ d\boldsymbol{\tau} \\ e^{2m+2\Sigma^2} = e^{-22\lambda}e^{2\beta}\sum_{k_1,\ldots,k_{22}=0}^{+\infty}\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\int_{A(k_1,\ldots,k_{22})}\exp\left(2\sum_{j=1}^{22}\sigma_{i,k_i}^2\right)\ d\boldsymbol{\tau} \end{cases}$$

$$\begin{cases} m + \frac{1}{2}\Sigma^2 = -22\lambda + \beta + \log\left\{\sum_{k_1,\ldots,k_{22}=0}^{+\infty}\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\int_{A(k_1,\ldots,k_{22})}\exp\left(\frac{1}{2}\sum_{j=1}^{22}\sigma_{i,k_i}^2\right)\ d\boldsymbol{\tau}\right\} \\ 2m + 2\Sigma^2 = -22\lambda + 2\beta + \log\left\{\sum_{k_1,\ldots,k_{22}=0}^{+\infty}\prod_{i=1}^{22}\frac{\lambda^{k_i}}{k_i!}\int_{A(k_1,\ldots,k_{22})}\exp\left(2\sum_{j=1}^{22}\sigma_{i,k_i}^2\right)\ d\boldsymbol{\tau}\right\} \end{cases}$$

$$\begin{cases} m + \frac{1}{2}\Sigma^2 = -22\lambda + \beta + L_1 \\ 2m + 2\Sigma^2 = -22\lambda + 2\beta + L_2 \end{cases}$$

This is a linear system with solution

$$\begin{cases} m = -33\lambda + \beta + 2L_1 - \frac{1}{2}L_2 \\ \Sigma^2 = 22\lambda + L_2 - 2L_1 \end{cases}$$

so

$$\begin{cases} 22\left(a - \frac{1}{2}b^2\right) = -33\lambda + \beta + 2L_1 + \frac{1}{2}L_2 \\ 22b^2 = 22\lambda + L_2 - 2L_1 \end{cases}$$

which gives the conclusion. □

2.4 Monitoring

We now treat the problem of monitoring over time the evolution of the index in which we invested. We may consider two approaches.

2.4.1 Monitoring of the index value

The first method is to check whether the price of the index stays inside a given confidence interval; if so, we keep the position until the expiration, otherwise we calibrate again the model adding the historical data that we observed until the index has exited the confidence interval; then we simulate again new trajectories and obtain a new optimal allocation for the previous expiration date.

Two are the problems with this approach: in the first place, adding few values to a time series that has thousands of values does not change very much the parameters; on the other hand, if we are near a shock, the confidence interval we obtain is meaningless because of the high volatility.

To solve these problems first of all we consider the parameters of the model as constant, because the contribution of few days does not change very much the parameters. Moreover, we implement a double daily check: not only do we make sure that the index stays inside the confidence interval, but we also update daily the last shock estimate. The following scenarios are possible

- the index stays inside the confidence interval and the last shock estimate does not markedly change during the entire period; in this case we do not change the allocation for the entire month duration;
- the index exits from the confidence interval but the last shock estimate does not change; we simply simulate again new trajectories, we generate a new confidence interval and we calculate again the optimum allocation for the same expiration date; if the wealth is sufficient to reach the target return by simply putting everything into the risk-free asset, we can choose to do so;
- we estimate that a new shock has occurred; here it does not make much sense to generate the confidence interval again because it becomes meaningless due to the high volatility. It is better to continue reallocating daily the position to handle the high volatility.

The second approach is explained in the following section.

2.4.2 Monitoring of the portfolio value

Suppose that we use the mean-variance strategy previously described. Then, at time $t+n$ the portfolio's value, that we indicate by V_n, is given by

$$V_n = \alpha(1+\mu_n) + (1-\alpha)(1+r_n)$$

where μ_n is defined by

$$S_{t+n} = S_t(1+\mu_n)$$

and r_n is the n-days risk-free rate starting from time t.

First we state the following result

LEMMA 4: *Assume that $\rho > 1 + r_{22}$; then the following formula holds true, for all $n \leq 22$;*

$$\mathbb{P}\left(V_n \geq \rho\frac{1+r_n}{1+r_{22}}\right) = \mathbb{P}\left(1+\mu_n \geq \mathbb{E}(1+\mu)\frac{1+r_n}{1+r_{22}}\right)$$

if $\mathbb{E}(1+\mu) - (1+r_{22}) > 0$, *otherwise*

$$\mathbb{P}\left(V_n \geq \rho\frac{1+r_n}{1+r_{22}}\right) = \mathbb{P}\left(1+\mu_n \leq \mathbb{E}(1+\mu)\frac{1+r_n}{1+r_{22}}\right)$$

Proof.

$$\mathbb{P}\left(V_n \geq \rho\frac{1+r_n}{1+r_{22}}\right) = \mathbb{P}\left(\alpha(1+\mu_n) + (1-\alpha)(1+r_n) \geq \rho\frac{1+r_n}{1+r_{22}}\right) =$$

$$\mathbb{P}\left(\frac{\rho-(1+r_{22})}{\mathbb{E}(1+\mu)-(1+r_{22})}(1+\mu_n) + \frac{\mathbb{E}(1+\mu)-\rho}{\mathbb{E}(1+\mu)-(1+r_{22})}(1+r_n) \geq \rho\frac{1+r_n}{1+r_{22}}\right) =$$

$$\mathbb{P}\left(\frac{(1+r_{22})[\rho-(1+r_{22})](1+\mu_n) + [\mathbb{E}(1+\mu)-\rho](1+r_{22})(1+r_n) - \rho(1+r_n)(\mathbb{E}(1+\mu)-(1+r_{22})}{(1+r_{22})[\mathbb{E}(1+\mu)-(1+r_{22})]} \geq 0\right) =$$

$$\mathbb{P}\left(\frac{(1+\mu_n)(1+r_{22})[\rho-(1+r_{22})] - (1+r_n)[\rho-(1+r_{22})]\mathbb{E}(1+\mu)}{(1+r_{22})[\mathbb{E}(1+\mu)-(1+r_{22})]} \geq 0\right) =$$

$$\mathbb{P}\left(\frac{(1+\mu_n)(1+r_{22}) - (1+r_n)\mathbb{E}(1+\mu)}{\mathbb{E}(1+\mu)-(1+r_{22})} \geq 0\right)$$

from which we have the conclusion. □

Finally, we recall the following Lemma from [2] (see Exercise 3.11, page 48 there);

LEMMA 5: *Let X be a random variable with $\mathbb{E}(X^2) = 1$ and $\mathbb{E}(|X|) \geq a > 0$; then*

$$\mathbb{P}(|X| \geq \lambda a) \geq (1-\lambda)^2 a^2$$

for all $\lambda \in [0,1]$.

We can now state the following

PROPOSITION 5: *Given* $n \leq 22$, *let* $m_n := \mathbb{E}(1+\mu_n)$, $\Sigma_n^2 := \mathbb{E}((1+\mu_n)^2)$ *and define*

$$X_n := \frac{1+\mu_n}{\Sigma_n} > 0$$

then, we have

$$\mathbb{P}(V_n \geq l) \geq (1-\lambda)^2 \left(\frac{m_n}{\Sigma_n}\right)^2$$

where $\lambda \in [0,1]$ *and* l *satisfies*

$$l \leq \alpha\lambda m_n + (1-\alpha)(1+r_n)$$

Proof. We have

$$\begin{cases} \mathbb{E}(X_n^2) = 1 \\ \mathbb{E}(|X_n|) = \mathbb{E}(X_n) = \frac{\mathbb{E}(1+\mu_n)}{\Sigma_n} = \frac{m_n}{\Sigma_n} \end{cases}$$

We can therefore apply Lemma 5 obtaining

$$\mathbb{P}\left\{\frac{1+\mu_n}{\Sigma_n} \geq \lambda\frac{m_n}{\Sigma_n}\right\} \geq (1-\lambda)^2 \left(\frac{m_n}{\Sigma_n}\right)^2$$

which is equivalent to

$$\mathbb{P}\left\{1+\mu_n \geq \lambda m_n\right\} \geq (1-\lambda)^2 \left(\frac{m_n}{\Sigma_n}\right)^2 \tag{2.5}$$

We now want to obtain a particular l so that $\mathbb{P}(V_n \geq l)$ is sufficiently large. We have the following

$$\begin{aligned} \mathbb{P}(V_n \geq l) &= \mathbb{P}(\alpha[(1+\mu_n) - (1+r_n)] + (1+r_n) \geq l) \\ &= \mathbb{P}((1+\mu_n) - (1+r_n) \geq \frac{1}{\alpha}(l - (1+r_n)) \\ &= \mathbb{P}((1+\mu_n) \geq \frac{1}{\alpha}(l - (1+r_n)) + (1+r_n) \\ &= \mathbb{P}\left((1+\mu_n) \geq \frac{l-(1-\alpha)(1+r_n)}{\alpha}\right) \end{aligned} \tag{2.6}$$

We now choose a particular λ so that the right hand side in (2.5) is sufficently large; using (2.5) and (2.6) the right hand side in (2.5) will give us a lower bound for the quantity $\mathbb{P}(V_n \geq l)$ if

$$\lambda m_n \geq \frac{l-(1-\alpha)(1+r_n)}{\alpha}$$

i.e.

$$l \leq \alpha\lambda m_n + (1-\alpha)(1+r_n)$$

□

3 Numerical results

In this section we present the numerical results that we have obtained; the results are about the optimal allocation and its monitoring. We will also introduce a benchmark portfolio to compare its performance to those obtained with our strategy. As previously mentioned, we have calibrated the model using historical data from 3rd January 1950 to 31st December 1999, so we can use the data from 3rd January 2000 to test the model and the strategies discussed above.

3.1 Allocation

We now explain in detail how we obtain the optimal allocation. Suppose that we are in time t and our time horizon is $t + 22$.

- First of all, we simulate many trajectories (1 million) for the evolution of the index from $t + 1$ to $t + 22$, using a Monte-Carlo simulation.
- Then we use these trajectories to obtain an 80 % empirical confidence interval for the evolution of the index.
- Next we use the trajectories to estimate $\mathbb{E}(1 + \mu)$. We do so by calculating the empirical mean of the 22-days returns of the trajectories.
- Finally, we calculate the optimal allocation as in Lemma 2, using the estimate of $\mathbb{E}(1 + \mu)$ that we have just obtained. We introduce a constraint for the allocation, i.e. $\alpha \in [-1, 2]$, to avoid extreme portfolios. We fix a target return of 5 %, i.e. $\rho = 1.01$.

Then we use the historical data of the value of the index on $t+1, \ldots, t+22$ and of the risk-free interest rate (using the 3-Month Treasury Bill Secondary Market Interest Rate that can be found at `http://research.stlouisfed.org/fred2/series/DTB3?cid=116`) to calculate the value of the portfolio on these dates.

3.2 Monitoring

The investment strategy that we are implementing is not a standard buy-and-hold strategy, because we monitor the portfolio using the two approaches discussed in section 2.4.

3.2.1 First method

First of all, at time t we calculate the last shock time estimate for all the dates $t+1, \ldots, t+22$; in other words, we obtain the last shock time estimate for $t + i$ as if we were now in $t + i$, for $i = 0, \ldots, 21$. Obviously, this can be done only if we already know the values of S_{t+i}, for $i = 1, \dots, 21$; we can do this because we are testing the strategy on historical data. If we were applying the strategy without knowing the future evolution of the index we could only know the present last shock time estimate, and we would have to update it every day. Then, at time $t+i$, with $i = 1, \dots, 21$, one of the following scenarios is possible.

- If the value of the index is inside the given confidence interval and the last shock time estimate is not significantly different from the one that we had in $t+i-1$, we hold the position.
- If the value of the index is outside the confidence interval and the last shock time estimate is not significantly different from the one that we had in $t+i-1$, we calculate again the optimal allocation and a new confidence interval as described in Section 3.1. If the wealth is sufficient to reach the target return by simply putting everything into the risk-free asset, we do so; in this case there is no need to update the confidence interval.
- If the last shock time estimate is significantly different from the one that we had in $t+i-1$ we calculate the optimal allocation each day until $t+22$.

3.2.2 Second method

This method uses the results obtained in Section 2.4.2; it consists of the following steps.

- At time t, after having simulated the trajectories for the evolution of the index, we calculate, for $n = 1, \dots, 21$, the quantity

$$\left(\frac{m_n}{\Sigma_n}\right)^2 \tag{3.1}$$

  where m_n is the empirical mean of the n-days returns of the simulated trajectories and Σ_n is the empirical second moment of the n-days returns.

- Then, deciding by analogy to [3] for an 80 % level, for $n = 1, \dots, 21$, we find λ_n such that

$$(1-\lambda_n)^2 \left(\frac{m_n}{\Sigma_n}\right)^2 = 0.8$$

  i.e.

$$\lambda_n = 1 - \sqrt{\frac{4}{5}\frac{\Sigma_n}{m_n}}$$

  Recall that, according to Lemma 5, λ_n should be in $[0,1]$, therefore we should have

$$\frac{\Sigma_n}{m_n} < \frac{5}{4}$$

- Next we calculate, for $n = 1, \ldots, 21$, the quantity

$$l_n = \alpha \lambda_n m_n + (1 - \alpha)(1 + r_n)$$

Therefore, according to Proposition 5, it follows that, for $n = 1, \ldots, 21$,

$$\mathbb{P}(V_n \geq l_n) \geq 0.8 \tag{3.2}$$

If at a certain date $t + i$ we have that $V_i < l_i$ we calculate again the optimal allocation from that point onwards; we also update the values l_n for $n > i$. If the wealth is sufficient to reach the target return by simply putting everything into the risk-free asset, we do so; in this case there is no need to update the values l_n.

3.3 Benchmark

We use a benchmark to compare the results obtained with our strategy. The benchmark portfolio is characterized by a buy-and-hold strategy; the percentage of risky asset is calculated in t with formula (2.3), but in this case, the term $\mathbb{E}(1+\mu)$ is not estimated using the simulated trajectories but as the mean of monthly returns on disjoint intervals calculated on historical data. In other words, it is given by

$$\widehat{\mathbb{E}(1+\mu)} := \frac{1}{i^*} \sum_{i=1}^{i^*} \frac{s_{22i+1}}{s_{22(i-1)+1}}$$

where $(s_i)_{1 \leq i \leq t}$ is the time series of the value of the index and $i^* = \sup\{n > 0 : 22n + 1 \leq t\}$. We introduce again a constraint for the optimal percentage of risky asset: $\alpha \in [-1, 2]$.

3.4 Results

3.4.1 First monitoring method

In this section we present the results obtained using the first monitoring method. Here, by " strategy " we mean the allocation strategy as described in section 3.1 and by " strategy portfolio " the corresponding portfolio. The results are shown using tables and plots; the first column of each table represents the time; the second contains the values of the benchmark portfolio, whereas the third contains the values of the strategy portfolio. The fourth and the fifth columns contain the optimal percentage of risky asset of the benchmark and strategy portfolio respectively. The sixth and the seventh columns represent respectively the lower and upper bound of the 80 % confidence interval for the index value. The eight column gives the value of the index and the last column reports the last shock time estimate. We will start from t that equals 31st December 1999.

	bench-p	strat-p	bench α^*	strat α^*	lower b	upper b	index	LSTE
t	100	100	1.28	0.4762057			1469.25	t-342
t+1	98.76	99.55	1.28	0.4762057	1455.19	1485.36	1455.22	t-342
t+2	93.88	97.75	1.28	-1	1449.87	1492.62	1399.42	t-342
t+3	94.11	97.60	1.28	-1	1385.71	1414.33	1402.11	t-342
t+4	94.22	97.55	1.28	-1	1380.33	1420.93	1403.45	t-342
t+5	97.54	94.93	1.28	-1	1376.33	1426.22	1441.47	t-342
t+6	98.94	93.91	1.28	-1	1427.74	1457.09	1457.60	t-342
t+7	97.27	95.17	1.28	-1	1443.82	1473.44	1438.56	t-342
t+8	96.72	95.63	1.28	-1	1425.01	1454.22	1432.25	t-342
t+9	98.23	94.52	1.28	-1	1419.80	1461.29	1449.68	t-342
t+10	99.58	93.53	1.28	-1	1415.95	1466.89	1465.15	t-342
t+11	98.70	94.23	1.28	-1	1412.76	1471.79	1455.14	t-342
t+12	98.76	94.22	1.28	-1	1410.19	1476.34	1455.90	t-342
t+13	97.85	94.94	1.28	-1	1407.94	1480.61	1455.57	t-342
t+14	97.48	95.26	1.28	-1	1405.85	1484.56	1441.36	t-342
t+15	93.99	97.93	1.28	-1	1403.96	1488.32	1401.53	t-342
t+16	94.73	97.38	1.28	-1	1388.17	1416.37	1410.03	t-342
t+17	94.21	97.83	1.28	-1	1382.92	1422.92	1404.09	t-342
t+18	93.72	98.26	1.28	-1	1378.98	1428.01	1398.56	t-342
t+19	90.36	100.98	1.28	0	1375.72	1432.50	1360.16	t-342
t+20	93.35	101.00	1.28	0	1372.93	1436.60	1394.46	t-342
t+21	94.64	101.02	1.28	0	1370.91	1440.40	1409.28	t-342
t+22	94.62	101.04					1409.12	

Table 3.1: Numerical results obtained from t that equals 31st December 1999. With bench-p and strat-p we mean benchmark portfolio and strategy portfolio respectively. Moreover, bench α^* and strat α^* indicate the optimal allocation for the benchmark portfolio and for the strategy portfolio respectively. Lower b and upper b are the lower and the upper bound given by the confidence interval, index is the value of the index and LSTE means last shock time estimate.

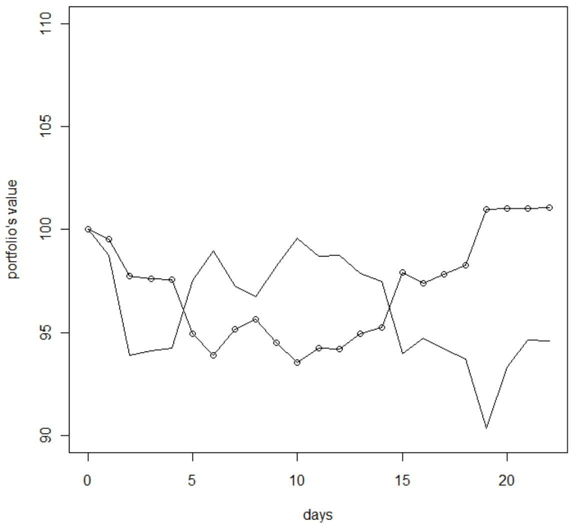

Figure 3.1: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

We can see that the strategy portfolio significantly overperforms the benchmark portfolio. Actually the allocations suggested by the two methods are quite different. We should notice that the index exits from the confidence interval several times, in fact there are two rebalancing of the strategy portfolio.

Here we present the result obtained when t equals 31st December 2005.

	bench-p	strat-p	bench α^*	strat α^*	lower b	upper b	index	LSTE
t	100	100	1.53	2			1248.29	t-864
t+1	94.31	92.58	1.53	2	1240.17	1256.95	1202.08	t-864
t+2	92.58	90.40	1.53	2	1194.098	1210.26	1188.05	t-864
t+3	92.05	89.73	1.53	2	1180.10	1196.05	1183.74	t-864
t+4	92.55	90.34	1.53	2	1176.79	1199.44	1187.89	t-864
t+5	92.33	90.07	1.53	2	1174.25	1202.04	1186.19	t-864
t+6	92.82	90.67	1.53	2	1172.07	1204.26	1190.25	t-864
t+7	91.92	89.55	1.53	2	1170.17	1206.24	1182.99	t-864
t+8	92.49	90.25	1.53	2	1168.45	1208.08	1187.70	t-864
t+9	91.22	88.68	1.53	2	1166.85	1209.77	1177.45	t-864
t+10	92.08	89.74	1.53	2	1165.30	1211.29	1184.52	t-864
t+11	93.48	91.47	1.53	2	1163.90	1212.82	1195.98	t-864
t+12	92.08	89.73	1.53	2	1162.56	1214.30	1184.63	t-864
t+13	90.94	88.31	1.53	2	1161.35	1215.79	1175.41	t-864
t+14	90.00	87.15	1.53	2	1160.20	1217.19	1167.87	t-864
t+15	89.49	86.51	1.53	2	1159.05	1218.56	1163.75	t-864
t+16	90.05	87.20	1.53	2	1157.94	1219.90	1168.41	t-864
t+17	90.74	88.05	1.53	2	1156.83	1221.16	1174.07	t-864
t+18	90.79	88.11	1.53	2	1155.78	1222.45	1174.55	t-864
t+19	90.39	87.61	1.53	2	1154.68	1223.61	1171.36	t-864
t+20	91.60	89.10	1.53	2	1153.68	1224.73	1181.27	t-864
t+21	92.59	90.32	1.53	2	1152.60	1225.84	1189.41	t-864
t+22	93.04	90.88					1193.19	

Table 3.2: Numerical results obtained from t that equals 31st December 2005.

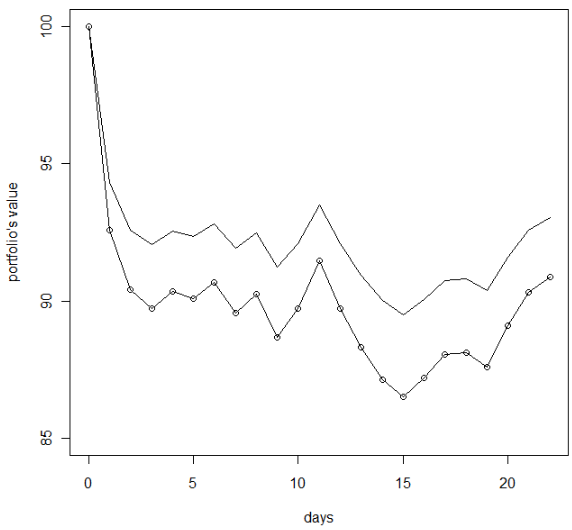

Figure 3.2: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

This should be a low volatility period, due to the distance of the last shock. Both the strategy portfolio and the benchmark portfolio do not manage to reach the target return. Actually the strategy portfolio underperforms the benchmark portfolio. In fact, both of them are long on the risky asset, but the strategy portfolio has a greater exposure.

Now we will analyze the period characterized by the Lehman Brothers bank bankruptcy; here t is 29th August 2008.

	bench-p	strat-p	bench α^*	strat α^*	lower b	upper b	index	LSTE
t	100	100	1.42	-0.59			1282.83	t-260
t+1	99.41	100.25	1.42	-0.59	1267.80	1296.47	1277.58	t-260
t+2	99.12	100.38	1.42	-0.59	1261.21	1301.80	1274.98	t-260
t+3	94.86	102.16	1.42	0	1255.92	1305.77	1236.83	t-260
t+4	95.47	102.17	1.42	0	1251.45	1309.14	1242.31	t-260
t+5	98.31	102.18	1.42	0	1247.41	1311.98	1267.79	t-260
t+6	93.48	102.18	1.42	0	1243.62	1314.50	1224.51	t-260
t+7	94.32	102.19	1.42	0	1240.08	1316.78	1232.04	t-260
t+8	96.21	102.20	1.42	0	1236.73	1318.78	1249.05	t-260
t+9	96.50	102.20	1.42	0	1233.48	1320.68	1251.70	t-260
t+10	89.93	102.21	1.42	0	1230.42	1322.40	1192.70	t+10
t+11	92.25	102.22	1.42	0	1227.26	1323.89	1213.60	t+10
t+12	85.88	102.22	1.42	0	1224.17	1325.35	1156.39	t+10
t+13	91.46	102.23	1.42	0	1221.23	1326.63	1206.51	t+10
t+14	96.87	102.24	1.42	0	1218.35	1327.87	1255.08	t+10
t+15	91.52	102.24	1.42	0	1215.52	1329.01	1207.09	t+10
t+16	89.41	102.24	1.42	0	1212.78	1330.21	1188.22	t+10
t+17	89.15	102.26	1.42	0	1210.02	1331.19	1185.87	t+10
t+18	91.74	102.26	1.42	0	1207.39	1332.24	1209.18	t+10
t+19	92.19	102.27	1.42	0	1204.78	1333.21	1213.27	t+10
t+20	80.29	102.27	1.42	0	1202.06	1333.95	1106.42	t+10
t+21	86.96	102.28	1.42	0	1199.33	1334.84	1166.36	t+10
t+22	86.37	102.29	1.42	0			1161.06	

Table 3.3: Numerical results obtained from t that equals 28th August 2008. Notice that the confidence interval was never updated.

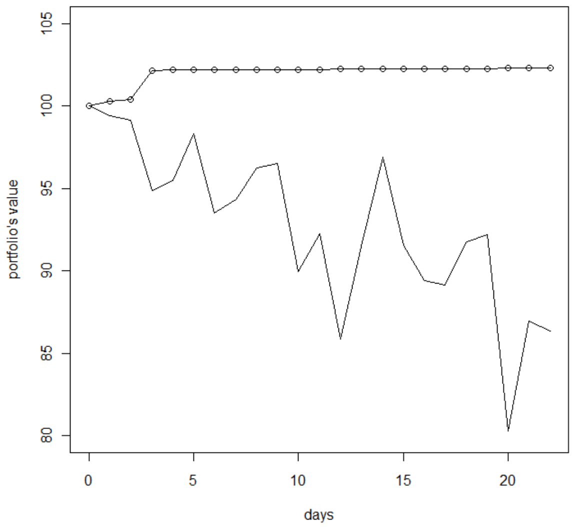

Figure 3.3: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

Here we can notice several things. First of all, there is a change in the last time shock estimate. In fact, we have identified a shock in $t+10$ which corresponds to the 15th September 2008, the date in which the Lehman Brothers bank has declared bankruptcy. From that time on, we have a daily rebalancing of the strategy portfolio and we do not calculate the confidence interval as it is insignificant because of the high volatility. On the other hand, we can observe that here the monitoring signals have been very useful, since they have permitted the strategy portfolio to reach a "safe" wealth; in fact, from time $t+2$ there is no exposure on the risky asset and the target return is acquired in $t+22$. This fact has permitted the strategy portfolio not to undergo the bad effects of the high volatility of this period. Finally, observe that the benchmark portfolio is long on the risky asset with leverage; this led to a nearly 15 % loss.

Here we present the result obtained when t equals the 2nd November 2009.

	bench-p	strat-p	bench α^*	strat α^*	lower b	upper b	index	LSTE
t	100	100	1.40	0.55			1042.88	t-267
t+1	100.34	100.13	1.40	0.55	1031.55	1054.63	1045.41	t-267
t+2	100.48	100.19	1.40	0.55	1027.14	1059.88	1046.50	t-267
t+3	103.19	101.26	1.40	0	1024.05	1064.27	1066.63	t-267
t+4	103.55	101.26	1.40	0	1021.44	1067.98	1069.30	t-267
t+5	106.75	101.26	1.40	0	1019.26	1071.44	1093.08	t-267
t+6	106.74	101.26	1.40	0	1017.40	1074.72	1093.01	t-267
t+7	107.48	101.26	1.40	0	1015.92	1078.03	1098.51	t-267
t+8	105.97	101.26	1.40	0	1014.30	1080.88	1087.24	t-267
t+9	106.81	101.26	1.40	0	1013.03	1083.85	1093.48	t-267
t+10	108.94	101.26	1.40	0	1011.95	1086.84	1109.30	t-267
t+11	109.07	101.26	1.40	0	1010.90	1089.68	1110.32	t-267
t+12	109.00	101.26	1.40	0	1010.19	1092.73	1109.80	t-267
t+13	107.00	101.26	1.40	0	1009.87	1096.02	1094.90	t-267
t+14	106.52	101.26	1.40	0	1009.24	1099.03	1091.38	t-267
t+15	108.52	101.26	1.40	0	1008.57	1101.73	1106.24	t-267
t+16	108.45	101.26	1.40	0	1007.79	1104.40	1105.65	t-267
t+17	109.12	101.26	1.40	0	1006.99	1106.82	1110.63	t-267
t+18	106.54	101.26	1.40	0	1006.17	1109.24	1091.49	t-267
t+19	107.10	101.26	1.40	0	1005.78	1112.03	1095.63	t-267
t+20	108.88	101.26	1.40	0	1005.26	1114.49	1108.86	t-267
t+21	108.93	101.26	1.40	0	1004.67	1116.92	1109.24	t-267
t+22	107.67	101.26					1099.92	

Table 3.4: Numerical results obtained from t that equals the 2nd November 2009.

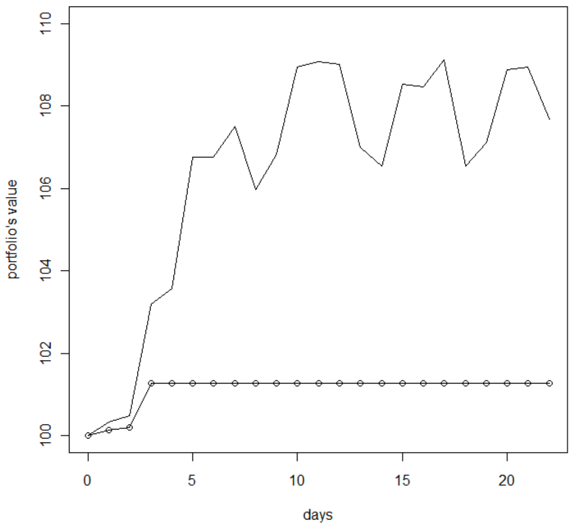

Figure 3.4: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

The strategy portfolio succeeded in reaching the target return by investing only in the risk-free asset from time $t+3$; the benchmark portfolio obtained a better performance.

3.4.2 Second monitoring method

Here we present the results obtained with the second monitoring method. We use again tables and plots to report the data. Here the lower bound column represents the lower bound for the value of the portfolio l_n as in formula (3.2).

	bench-p	strat-p	bench α^*	strat α^*	lower b	LSTE
t	100	100	1.28	0.4805732		t-342
t+1	98.76	99.55	1.28	0.4805732	98.31	t-342
t+2	93.88	97.73	1.28	-1	99.16	t-342
t+3	94.11	97.58	1.28	-1	100.57	t-342
t+4	94.22	97.53	1.28	-1	105.88	t-342
t+5	97.54	94.93	1.28	-1	100.56	t-342
t+6	98.94	93.90	1.28	-1	100.05	t-342
t+7	97.27	95.17	1.28	-1	99.28	t-342
t+8	96.72	95.62	1.28	-1	101.08	t-342
t+9	98.23	94.50	1.28	-1	99.09	t-342
t+10	99.58	93.53	1.28	-1	102.95	t-342
t+11	98.70	94.20	1.28	-1	100.09	t-342
t+12	98.76	94.19	1.28	-1	100.13	t-342
t+13	97.85	94.90	1.28	-1	100.74	t-342
t+14	97.48	95.21	1.28	-1	101.09	t-342
t+15	93.99	97.88	1.28	-1	101.13	t-342
t+16	94.73	97.33	1.28	-1	102.60	t-342
t+17	94.21	97.78	1.28	-1	102.23	t-342
t+18	93.72	98.20	1.28	-1	101.54	t-342
t+19	90.36	100.94	1.28	0	105.06	t-342
t+20	93.35	100.96	1.28	0	103.14	t-342
t+21	94.64	100.98	1.28	0	101.91	t-342
t+22	94.62	101.00				

Table 3.5: Numerical results obtained from t that equals 31st December 1999.

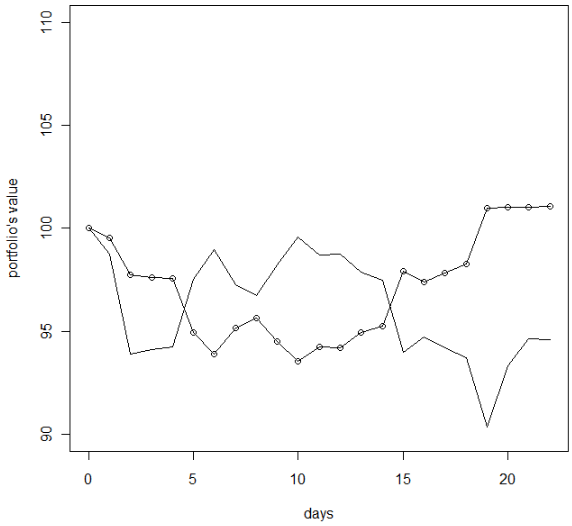

Figure 3.5: Plot of the values of the two portfolios, the line is the benchmark portfolio and the other is the strategy portfolio.

We can notice that the strategy obtained is nearly the same as the one that we obtained using the other monitoring method, so the results are nearly identical.

Here we present the result obtained when t equals 31st December 2005.

	bench-p	strat-p	bench α^*	strat α^*	lower b	LSTE
t	100	100	1.53	2		t-864
t+1	94.31	92.58	1.53	2	100.30	t-864
t+2	92.58	90.40	1.53	2	92.37	t-864
t+3	92.05	89.73	1.53	2	101.94	t-864
t+4	92.55	90.34	1.53	2	101.86	t-864
t+5	92.33	90.07	1.53	2	90.65	t-864
t+6	92.82	90.67	1.53	2	97.36	t-864
t+7	91.92	89.55	1.53	2	98.55	t-864
t+8	92.49	90.25	1.53	2	104.42	t-864
t+9	91.22	88.68	1.53	2	63.21	t-864
t+10	92.08	89.74	1.53	2	66.56	t-864
t+11	93.48	91.47	1.53	2	62.94	t-864
t+12	92.08	89.73	1.53	2	66.31	t-864
t+13	90.94	88.31	1.53	2	162.20	t-864
t+14	90.00	87.15	1.53	2	66.44	t-864
t+15	89.49	86.51	1.53	2	67.79	t-864
t+16	90.05	87.20	1.53	2	69.88	t-864
t+17	90.74	88.05	1.53	2	70.21	t-864
t+18	90.79	88.11	1.53	2	63.23	t-864
t+19	90.39	87.61	1.53	2	69.98	t-864
t+20	91.60	89.10	1.53	2	69.45	t-864
t+21	92.59	90.32	1.53	2	66.21	t-864
t+22	93.04	90.88				

Table 3.6: Numerical results obtained from t that equals 31st December 2005.

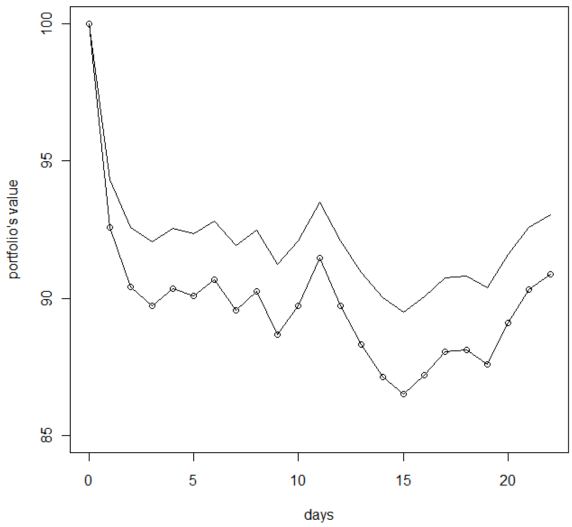

Figure 3.6: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

The results are identical to those obtained with the first monitoring method. Notice that there are several days in which the value of the portfolio is below the lower bound suggested by the second monitoring method, but there is no rebalancing.

Now we will analyze the period characterized by the Lehman Brothers bank bankruptcy; here t is 29th August 2008.

	bench-p	strat-p	bench α^*	strat α^*	lower b	LSTE
t	100	100	1.42	-0.59		t-260
t+1	99.41	100.25	1.42	2	100.67	t-260
t+2	99.12	99.84	1.42	2	101.89	t-260
t+3	94.86	93.85	1.42	2	99.86	t-260
t+4	95.47	94.68	1.42	2	92.12	t-260
t+5	98.31	98.54	1.42	2	99.37	t-260
t+6	93.48	91.81	1.42	2	92.49	t-260
t+7	94.32	92.93	1.42	2	91.65	t-260
t+8	96.21	95.47	1.42	2	95.52	t-260
t+9	96.50	95.87	1.42	2	97.30	t-260
t+10	89.93	86.83	1.42	-1	91.43	t+10
t+11	92.25	85.32	1.42	-1	113.39	t+10
t+12	85.88	89.35	1.42	2	87.23	t+10
t+13	91.46	97.09	1.42	2	78.73	t+10
t+14	96.87	104.90	1.42	0	65.45	t+10
t+15	91.52	104.91	1.42	0	50.29	t+10
t+16	89.41	104.92	1.42	0	61.91	t+10
t+17	89.15	104.92	1.42	0	74.75	t+10
t+18	91.74	104.93	1.42	0	67.43	t+10
t+19	92.19	104.94	1.42	0	73.28	t+10
t+20	80.29	104.94	1.42	0	82.01	t+10
t+21	86.96	104.95	1.42	0	96.15	t+10
t+22	86.37	104.96	1.42	0		

Table 3.7: Numerical results obtained from t that equals 28th August 2008. Notice that the confidence interval was never updated.

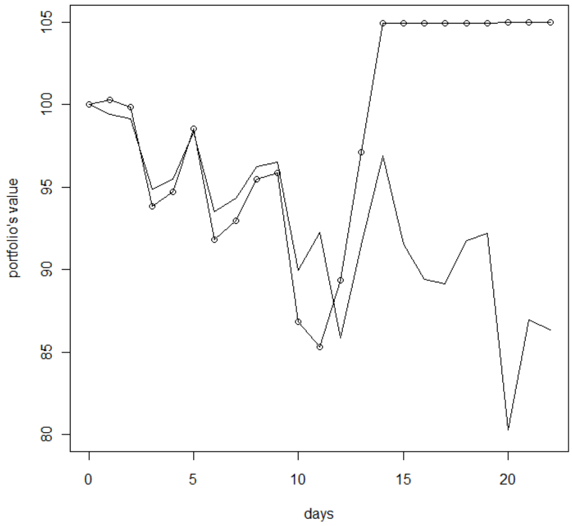

Figure 3.7: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

With the second monitoring the strategy portfolio obtains worse performance than the one with the first monitoring. However, it still overperform the benchmark portfolio.

Here we present the result obtained when t equals the 2nd November 2009.

	bench-p	strat-p	bench α^*	strat α^*	lower b	LSTE
t	100	100	1.40	0.55		t-267
t+1	100.34	100.13	1.40	-1	100.46	t-267
t+2	100.48	100.03	1.40	-1	104.33	t-267
t+3	103.19	98.10	1.40	-1	101.19	t-267
t+4	103.55	97.86	1.40	-1	101.01	t-267
t+5	106.75	95.68	1.40	-1	105.22	t-267
t+6	106.74	95.69	1.40	-1	103.26	t-267
t+7	107.48	95.21	1.40	-1	106.65	t-267
t+8	105.97	96.18	1.40	-1	100.67	t-267
t+9	106.81	95.63	1.40	-1	103.00	t-267
t+10	108.94	94.25	1.40	-1	98.11	t-267
t+11	109.07	94.16	1.40	-1	98.34	t-267
t+12	109.00	94.21	1.40	-1	100.85	t-267
t+13	107.00	95.47	1.40	-1	102.16	t-267
t+14	106.52	95.78	1.40	-1	106.64	t-267
t+15	108.52	94.47	1.40	-1	100.20	t-267
t+16	108.45	94.53	1.40	-1	100.13	t-267
t+17	109.12	94.10	1.40	-1	101.87	t-267
t+18	106.54	95.72	1.40	-1	103.60	t-267
t+19	107.10	95.36	1.40	-1	99.24	t-267
t+20	108.88	94.21	1.40	-1	99.78	t-267
t+21	108.93	94.18	1.40	-1	100.00	t-267
t+22	107.67	94.97				

Table 3.8: Numerical results obtained from t that equals the 2nd November 2009.

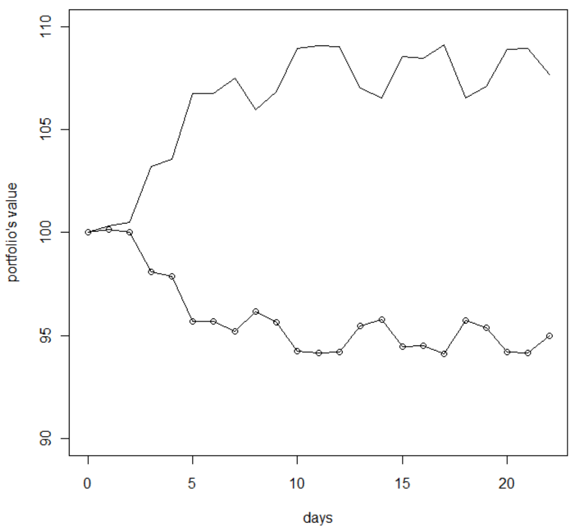

Figure 3.8: Plot of the values of the two portfolios, the plain line is the benchmark portfolio and the dotted line is the strategy portfolio.

Here the strategy portfolio did not manage to reach the target return, because, even if there are a lot of monitoring signal, the strategy suggests to short the risky asset and this turns out to be a bad choice.

3.5 Conclusions and possible extensions

3.5.1 Remarks

Now we state the main remarks that we can draw from the numerical results.

- The possibility of investing everything in the risk-free asset when there is a monitoring signal has been very useful. In fact, it guarantees that the target return is reached and it also keeps the portfolio volatility low.
- The confidence interval is not significant both in periods of high and low volatility; in fact, if the volatility is large it is too wide, whereas in periods of low volatility it is too narrow.
- When a monitoring signal is received and there is a rebalancing of the portfolio, the strategy often completely changes from long to short and vice versa. This fact implies that the forecast of the future trend of the index has a low dependency on historical data, otherwise we would observe a gradual change.

3.5.2 Comments and possible extensions

The first of the previous remarks is connected with the idea expressed by the 80 % rule ([3]). In fact, consider the discounted target return at time n, i.e. the quantity

$$\rho\frac{1+r_n}{1+r_{22}}$$

where ρ is the 22-days target return whereas r_n and r_{22} are the n-days and 22-days risk-free interest rates from t respectively. So, if we have to reach the target return ρ at time $t+22$, there should be a time $t+n$ with $1 \leq n \leq 21$ such that

$$V_n \geq \rho\frac{1+r_n}{1+r_{22}}$$

Therefore, if we receive a monitoring signal, it makes sense to check whether the current portfolio value is above the discounted target return or not.

The excessive change in amplitude of the confidence interval is explained by the fact that the model generates a very high volatility when the shock was recent; on the contrary, when we are far from the last shock the volatility declines too much. This fact seems in contrast with the very good fitting of the model concerning the volatility (see [1]). However, we could argue that on a long-range scale, the effects of the alternation of these very high - very low volatility periods offset each other. Conversely, in a specific period, these volatility jumps seem too large. This could be an indication about the fact that this model is more effective on long periods, where these effects compensate; therefore, it could be a good idea to choose a longer time horizon, for example one year. It would also be possible to force a minimum value for the volatility, so that it does not decline too much; this possibility has already been considered by the authors of [1] [1].

The third remark requires some care. We want now to point out the fact that the model does not estimate efficiently the trend of the index. To demonstrate this, we will use the following figure.

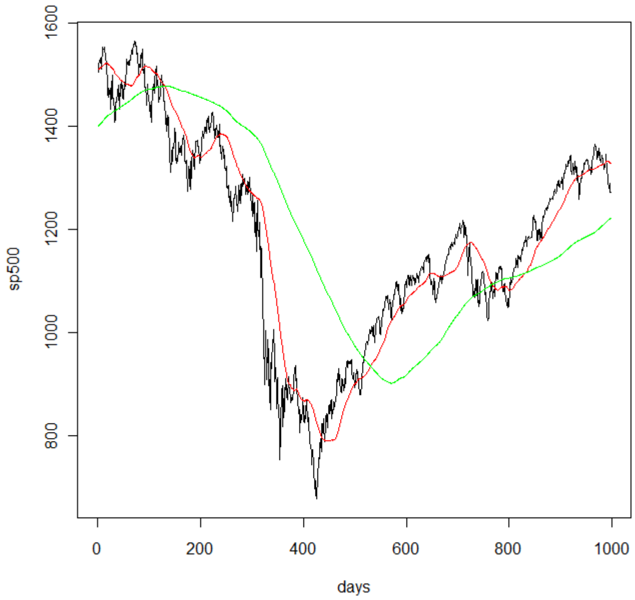

Figure 3.9: Plot of the S&P500 index from 3rd July 2006 to 13th June 2011. The SMA(50) and the SMA(250) are also plotted, respectively in red and green.

We can observe that in August 2009 there is a strong buy signal since the SMA(50) (Simple Moving Average over 50 days) crosses above the SMA(250) (Simple Moving Average over 250 days). In fact, the SMA(n) at time t is defined by the formula

$$SMA(n) = \frac{1}{n}\left(S_t + \cdots + S_{t-(n-1)}\right)$$

So the SMA(50) and the SMA(250) are mid-term and long-term trend indicators respectively. Therefore, when the SMA(50) crosses above the SMA(250) there is a buy signal because the SMA(250) acts as a support. It is also clear from the plot that the trend is bullish from May 2009 onwards. Now we simulate trajectories from t that equals 10th August 2009 to $t+22$ and we calculate the mean of the increments of these trajectories. This quantity should be the model estimate of the monthly trend of the index. In other words, we simply estimate $\mathbb{E}(1+\mu)$ as described in section 3.1. The result is the following

$$\widehat{\mathbb{E}(1+\mu)} = 0.9804$$

[1] Verbal communication by A. Andreoli.

Therefore, the model does not recognize the bullish trend. To explain this fact, recall the recursive formula (2.2) for the value of the index

$$S_{t+1} = S_t \exp(X_{t+1} - X_t) \sqrt[250]{\frac{S_t}{S_{t-250}}}$$

This formula is composed of three factors; the first one is the current value of the index. The second one is the fluctuation given by the model and the third one is the factor that affects the trend. We can notice that this term only compares two particular values of the index: the current one and the one in $t - 250$, so we can not expect that it is very precise.

Because of the fact that we can not observe (X_t) in the real world, we need a procedure to obtain (X_t) from (S_t), which is observable. In this work, we used the formula (2.1) to do so. The formula is the following

$$X_t = \log(S_t) - \frac{1}{250} \left(\log(S_{t-250} + \cdots + \log(S_{t-1})\right)$$

Hence, inverting this formula we have been able to obtain (S_t) from (X_t); the term $\sqrt[250]{\frac{S_t}{S_{t-250}}}$ is a side effect of this inverting procedure. Actually, the model itself does not provide any information about the index trend, since the model simulates the logarithm of the price of the underlying index, **after subtracting the trend**. Once again, the trend factor in (2.2) is only a side-effect of the procedure we have used to detrend the logarithm of the value of the index. Actually, (X_t) is a model for the volatility only.

Now the reason should be clear why the trend estimate is not precise. The value of the index in t that equals 10th August 2009 is clearly below the value of the index of one year before. So the factor $\sqrt[250]{\frac{S_t}{S_{t-250}}}$ is smaller than one and the trend estimate is bearish. An adjustment is thus needed, since the factor gives bearish signals on the lows and bullish signals on the highs. On the contrary, a good investment strategy should buy on the lows and sell on the highs.

A possible solution could be the following: as was mentioned at the beginning of chapter 1, the basic Black and Scholes model assumes that

$$dX_t = \sigma \; dW_t$$

so, instead of a geometric Brownian motion model like

$$dS_t = S_t(a_t \; dt + \sigma \; dW_t)$$

we could use a model like

$$dS_t = S_t(a_t \; dt + dX_t)$$

The Brownian motion term is replaced by (X_t), which is a very good model for the volatility; moreover, there is an additional term that should catch the information about the trend and give more consistent forecasts.

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