

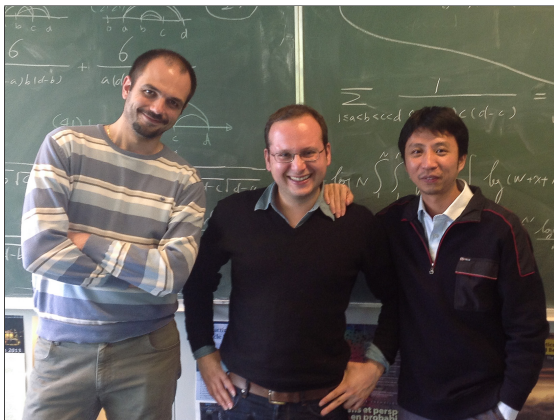
Polynomial Chaos and Scaling Limits of Disordered Systems

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Coworkers



Joint work with Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

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Very general framework, illustrated by 3 concrete examples

1. Disordered pinning models (Pinning)
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Inspired by recent work of Alberts, Quastel and Khanin on DPRE

Outline

1. Disordered Systems and their Scaling Limits
2. Main Results (I): Partition Function
3. Main Results (II): Continuum Disordered Pinning Model
4. Further Developments

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Lattice $\Omega \subseteq \mathbb{R}^d$ “spins” $\sigma = (\sigma_x)_{x \in \Omega} \in \{0, 1\}^\Omega$ or $\{-1, +1\}^\Omega$

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Disordered law

Random Gibbs measure on spin configurations σ , indexed by disorder ω

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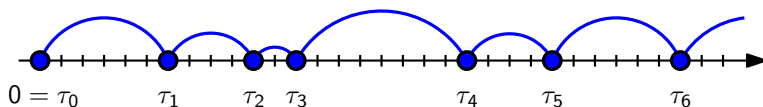
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$$\mathbf{P}_{\Omega, \lambda, h}^\omega(\sigma) := \frac{1}{Z_{\Omega, \lambda, h}^\omega} \exp \left(\sum_{x \in \Omega} (\lambda\omega_x + h)\sigma_x \right) \mathbf{P}_\Omega^{\text{ref}}(\sigma)$$

Partition function $Z_{\Omega, \lambda, h}^\omega = \mathbf{E}_\Omega^{\text{ref}}[e^{\sum_{x \in \Omega} (\lambda\omega_x + h)\sigma_x}]$

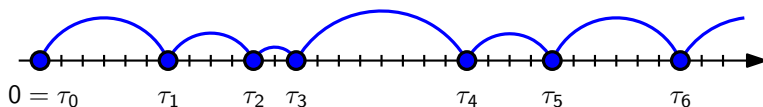
1. Disordered pinning model



Reference law: **renewal process** $\tau = \{0 = \tau_0 < \tau_1 < \tau_2 < \dots\} \subseteq \mathbb{N}_0$

$$\mathbf{P}^{\text{ref}}((\tau_{i+1} - \tau_i) = n) \sim \frac{C}{n^{1+\alpha}}, \quad \text{tail exponent } \alpha \in (0, 1)$$

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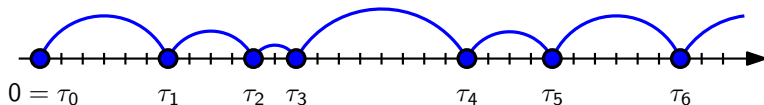


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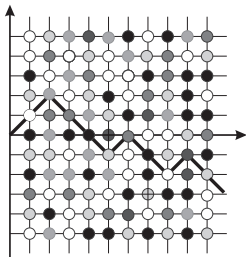
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2. Directed polymer in random environment

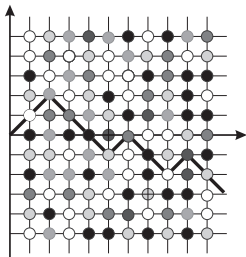


Reference law: symmetric **random walk**
 $X = (X_n)_{n \geq 0}$ on \mathbb{Z} , in the domain of attraction
 of a **stable Lévy process** with index $\alpha \in (0, 2]$

$$\mathbf{Var}^{\text{ref}}(X_1) < \infty \text{ if } \alpha = 2$$

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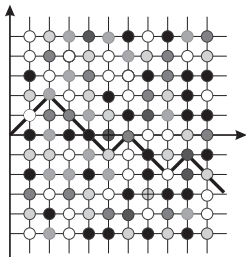


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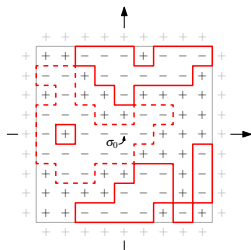
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Reference law: **critical 2d Ising model** with “+” boundary conditions



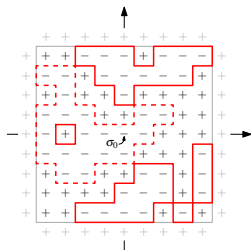
Lattice $\Omega := \{-N, \dots, N\} \times \{-N, \dots, N\}$

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Common feature: reference law $\mathbf{P}_\Omega^{\text{ref}}$ admits a **non-trivial continuum limit**

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Fix $\Omega \subset \mathbb{R}^d$ bounded open with smooth boundary, and consider the lattice

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Does the **disordered model** $\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega$ admit a non-trivial continuum limit?

A direct approach?

Recall the definition of the (discrete) disordered law:

$$\mathbf{P}_{\Omega_\delta, \lambda, h}^{\omega}(\mathrm{d}\sigma) \propto \exp\left(\sum_{x \in \Omega_\delta} (\lambda \omega_x + h) \sigma_x\right) \mathbf{P}_{\Omega_\delta}^{\mathrm{ref}}(\mathrm{d}\sigma)$$

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Difficulty is substantial: $\mathcal{P}_{\Omega, \lambda, h}^\omega$ can be **singular** w.r.t. $\mathcal{P}_{\Omega}^{\text{ref}}$!

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The partition function

The disordered system $\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega$ is a difficult object (a **random probability**)

Let us be less ambitious and focus on the **partition function**

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YES, for Pinning and DPRE (and hopefully for Ising too)

Assumptions

k -point function $\mathbf{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]$ defined on $(\Omega_\delta)^k \rightsquigarrow$ extended on Ω^k

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$\forall k \in \mathbb{N}$.

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$\forall k \in \mathbb{N}$. Furthermore $\sum_{k \in \mathbb{N}} \|\psi_\Omega^{(k)}\|_{L^2(\Omega^k)}^2 < \infty$

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$$\exists \gamma > 0 : \quad \frac{\mathbf{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]}{(\delta^\gamma)^k} \xrightarrow[\delta \downarrow 0]{\text{in } L^2(\Omega^k)} \psi_\Omega^{(k)}(x_1, \dots, x_k) \quad (*)$$

$$\forall k \in \mathbb{N}. \quad \text{Furthermore } \sum_{k \in \mathbb{N}} \|\psi_\Omega^{(k)}\|_{L^2(\Omega^k)}^2 < \infty$$

Pointwise convergence in $(*)$ leads to $\psi_\Omega^{(k)}(x_1, \dots, x_k) \approx |x_i - x_j|^{-\gamma}$

Assumptions

k -point function $\mathbf{E}_{\Omega_\delta}^{\text{ref}}[\sigma_{x_1} \cdots \sigma_{x_k}]$ defined on $(\Omega_\delta)^k \rightsquigarrow$ extended on Ω^k

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L^2 convergence then requires that

$$\gamma < \frac{d}{2}$$

Main result (I): partition function

Theorem [C., Sun, Zygouras '13]

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Wiener chaos expansion (converging in L^{2-})

- Case $\sigma_x \in \{-1, 1\}$. The same, up to minor modifications (cf. below)

Motivating models: Pinning and DPRE

- **Pinning.** Dimension $d = 1$, exponent $\gamma = 1 - \alpha$,

$$\psi_{\Omega}^{(k)}(x_1, \dots, x_k) = \frac{c^k}{x_1^{1-\alpha} (x_2 - x_1)^{1-\alpha} \dots (x_k - x_{k-1})^{1-\alpha}}$$

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These restrictions are not technical, but substantial (physical)!

Motivating models: Ising

Pointwise convergence of k -point function, with exponent $\gamma = \frac{1}{8}$, toward

$$\psi_{\Omega}^{(k)}(x_1, \dots, x_k) \text{ conformally covariant,}$$

was proved in [Chelkak, Hongler, Izyurov '12].

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Recall that we consider **random field 2d Ising model** at the critical point, with external field $(\lambda \omega_x + h)_{x \in \Omega_\delta}$

We fix continuous functions $\hat{\lambda} : \overline{\Omega} \rightarrow (0, \infty)$ and $\hat{h} : \overline{\Omega} \rightarrow \mathbb{R}$ and set

$$\lambda = \hat{\lambda}(x) \delta^{7/8} \quad h = \hat{h}(x) \delta^{15/8}$$

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Theorem [C., Sun, Zygouras '13]

As $\delta \downarrow 0$ one has the convergence in law

$$e^{-\frac{1}{2}\|\hat{\lambda}\|_2^2} \delta^{-1/4} Z_{\Omega_\delta, \lambda, h}^{\text{red}} \implies Z_{\Omega; \hat{\lambda}, \hat{h}}^{\text{red}}$$

where $W(dx)$ is **white noise on \mathbb{R}^d** and

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Conformal covariance: if $\phi : \tilde{\Omega} \rightarrow \Omega$ is a conformal map,

$$Z_{\Omega; \hat{\lambda}, \hat{h}}^W \stackrel{\text{dist.}}{=} Z_{\tilde{\Omega}; \tilde{\lambda}, \tilde{h}}^W$$

where $\tilde{\lambda}(x) := |\phi'(x)|^{7/8} \hat{\lambda}(\phi(x))$ and $\tilde{h}(x) := |\phi'(x)|^{15/8} \hat{h}(\phi(x))$

Sketch of the proof

1. **Linearization.** Since $\sigma_x \in \{0, 1\}$, every function of σ_x is linear

$$Z_{\Omega_\delta, \lambda, h}^\omega = \mathbf{E}_{\Omega_\delta}^{\text{ref}} \left[\prod_{x \in \Omega_\delta} e^{(\lambda \omega_x + h) \sigma_x} \right] = \mathbf{E}_{\Omega_\delta}^{\text{ref}} \left[\prod_{x \in \Omega_\delta} (1 + \epsilon_x \sigma_x) \right]$$

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Partition function is a **multilinear polynomial** of the random variables ϵ_x , with coefficient given by the k -point functions of \mathbf{P}^{ref}

Sketch of the proof

3. Lindeberg principle, extending [Mossel, O'Donnell, Oleszkiewicz '10]

The law of a multilinear polynomial is insensitive toward the distribution of the ϵ_x (keeping same mean and variance) \rightsquigarrow independent Gaussians

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white noise W integrated on cell $\Delta_x := (x - \frac{\delta}{2}, x + \frac{\delta}{2})^d$

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4. Wiener chaos expansion. Plugging the assumption

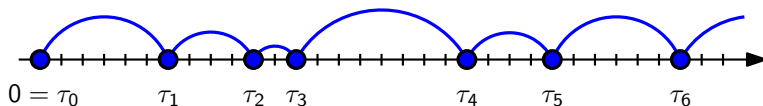
$$\mathbf{E}_{\Omega_\delta}^{\text{ref}} [\sigma_{x_1} \cdots \sigma_{x_k}] \simeq (\delta^\gamma)^k \psi_\Omega^{(k)}(x_1, \dots, x_k)$$

yields a Wiener chaos expansion with $\hat{\lambda} = \lambda \delta^{\gamma - \frac{d}{2}}$ and $\hat{h} = h' \delta^{\gamma - d}$ □

Outline

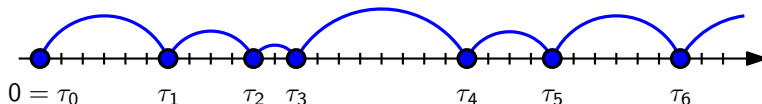
1. Disordered Systems and their Scaling Limits
2. Main Results (I): Partition Function
3. Main Results (II): Continuum Disordered Pinning Model
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Back to pinning models



$\tau = \{\tau_0 < \tau_1 < \tau_2 < \dots\}$ random element of $E := \{\text{closed subsets of } \mathbb{R}\}$

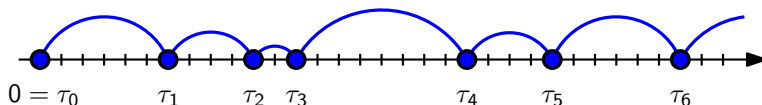
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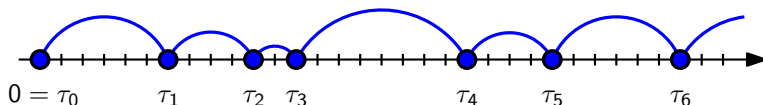


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What happens for the disordered model $\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega$? ($\Omega = (0, 1)$)

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Restrict $\alpha \in (\frac{1}{2}, 1)$. Fix $\hat{\lambda} > 0$, $\hat{h} \in \mathbb{R}$ and set

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Theorem (existence and universality of the CDPM)

As $\delta \downarrow 0$, the rescaled discrete set $(\delta\tau, \mathbf{P}_{\Omega_\delta, \lambda, h}^\omega)$ converges in distribution on E to a universal random closed set $(\tau, \mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W)$, called CDPM

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The CDPM has any a.s. property of the α -stable regenerative set \mathcal{P}^{ref}

$$\mathcal{A} \subseteq E, \quad \mathcal{P}^{\text{ref}}(\mathcal{A}) = 1 \quad \implies \quad \mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W(\mathcal{A}) = 1, \quad \mathbb{P}(\text{d}W)\text{-a.s.}$$

Example: $\mathcal{A} = \{A \subseteq \mathbb{R} : \text{Hausdorff dim. of } A = \alpha\}$

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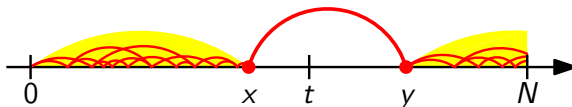
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Theorem (singularity)

The CDPM $\mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W$ law is singular w.r.t. \mathcal{P}^{ref} for \mathbb{P} -a.e. W

Construction strategy

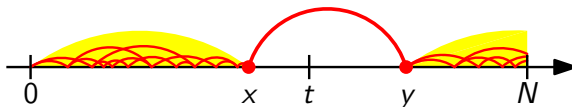
Macroscopic observables (**finite-dimensional distributions**) expressed using partition functions with suitable boundary conditions



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Scaling limit (at the process level) of $(Z_{x,y}^{\text{cond}}, Z_{x,y})_{0 \leq x < y \leq N} \rightsquigarrow$
 Definition of CDPM via “finite-dimensional distributions”

The same can be done for DPPE, cf. [Alberts, Khanin, Quastel '12]

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Analogous procedure for Ising?

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Possible alternative approach: define continuum disordered law $\mathcal{P}_{\Omega, \hat{\lambda}, \hat{h}}^W$ assigning its k -point function $\mathcal{E}_{\Omega, \hat{\lambda}, \hat{h}}^W[\sigma_{x_1} \cdots \sigma_{x_k}]$?

A generalization of our theorem about the scaling limit of partition functions yields the corresponding **scaling limit of correlations**:

$$E_{\Omega_\delta, \lambda, h}^\omega[\sigma_{x_1} \cdots \sigma_{x_k}] \xrightarrow[\delta \downarrow 0]{d} \mathcal{E}_{\Omega, \hat{\lambda}, \hat{h}}^W[\sigma_{x_1} \cdots \sigma_{x_k}] := \text{Wiener chaos expansion}$$

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Disorder relevance vs. irrelevance

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Why the restriction $\alpha > \frac{1}{2}$ for pinning? [And $\alpha \in (1, 2]$ for DPRE]

- ▶ The regime $\alpha < \frac{1}{2}$ is **disorder-irrelevant** for pinning models

If $\lambda > 0$ is small, the disordered model $\mathbf{P}_{\Omega_\delta, \lambda, h}^\omega$ has **same** properties (e.g. critical exponents) as the **non-disordered** model ($\lambda = 0$)

Disorder relevance vs. irrelevance

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Our restriction involving L^2 convergence of k -point function ($\gamma < \frac{d}{2}$) matches with **Harris criterion** $\nu < \frac{2}{d}$ for disorder relevance

(ν correlation length exponent $\rightsquigarrow \nu = \frac{1}{d-\gamma}$)

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Continuum partition function $\mathbf{Z}_{\Omega, \hat{\lambda}, \hat{h}}^W \rightsquigarrow$ continuum free energy

$$\mathbf{F}(\hat{\lambda}, \hat{h}) := \lim_{\Omega \uparrow \mathbb{R}^d} \frac{1}{\text{Leb}(\Omega)} \log \mathbf{Z}_{\Omega, \hat{\lambda}, \hat{h}}^W$$

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Conjecture

$$\lim_{h \downarrow 0} \frac{\langle \sigma_0 \rangle_{\hat{\lambda} h^{\frac{7}{15}}, h}}{h^{\frac{1}{15}}} = \frac{\partial \mathbf{F}}{\partial h}(\hat{\lambda}, 1) \quad \text{refining [Camia, Garban, Newman '12]}$$

Thanks