Regular coupling of probability measures on spaces of distributions;

Francesco De Vecchi

Università degli Studi di Pavia

based on a joint work with Massimiliano Gubinelli and Mattia Turra

The Mathematics of Subjective Probability,
September 10th 2025, Milano

Thanks to **Osterwalder-Schrader** formulation of (constructive) quantum field theory, in order to build a (bosonic) quantum field we need to define a **Gibbs measure** on the space $\mathcal{S}(\mathbb{R}^d)$ heuristically given by the expression

$$\mu(\mathrm{d}\varphi) = {}''\frac{e^{-S(\varphi)}}{Z}\mathcal{D}\varphi''$$

In the case of bosonic scalar particle

$$\begin{aligned} S(\varphi) &= S_{\mathrm{free}}(\varphi) + S_{\mathrm{inf}}(\varphi) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \varphi(x)(-\Delta + m^2)\varphi(x)\mathrm{d}x + \int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x. \end{aligned}$$

When $V = 0$, the measure

$$\mu_{\text{free}}(\mathrm{d}\varphi) = {}''\frac{e^{-S_{\text{free}}(\varphi)}}{Z}\mathcal{D}\varphi''$$

is the Gaussian measure with covariance $(-\Delta + m^2)^{-1}$.

Interacting case: The measure becomes

$$\mu_{\text{int}}(\mathrm{d}\varphi) = {}''\exp\left(-\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x\right)''\mu_{\text{free}}(\mathrm{d}\varphi),$$

Two difficulties:

- for $d \geq 1$ the support of μ_{free} is not contained in a space of functions (generally $\operatorname{supp}(\mu_{\text{free}}) \subset H_\ell^{\frac{-d+1}{2}-\varepsilon}(\mathbb{R}^2)$) (*continuum limit*),
- for $d \geq 0$ the support of μ is not contained in a space of "bounded at infinity" objects and so the integral $\int_{\mathbb{R}^d} V(\varphi(x))\mathrm{d}x$ is infinite (*infinite volume limit*).

We try to define μ_{int} as a (weak) limit of a sequence of approximate measures, e.g.

$$\mu_{\mathrm{int}} = \lim_{\varepsilon \to 0} \mu_{\mathrm{int}}^{\varepsilon} := \exp\left(- \int_{\mathbb{R}^d} f_{\varepsilon}(x) V_{\varepsilon}(g_{\varepsilon} * (\varphi)(x)) \, \mathrm{d}x \right) \mu_{\mathrm{free}}(\mathrm{d}\varphi)$$

where g_{ε} is a smooth mollifier, $f_{\varepsilon} \to 1$ is a (positive) regular function with compact support, and $V_{\varepsilon}: \mathbb{R} \to \mathbb{R}$ is a regular function which is a modification of the potential V depending on ε.

The previous definition depends, in principle, on the regularization f_{ε}, g_{ε} and V_{ε} used to define the regularized measure $\mu_{\mathrm{int}}^{\varepsilon}$. This dependence is particularly important in the case of infinite volume limit. We want to study a definition of the measure μ_{int} not (directly) depending on a specific approximations $\mu_{\mathrm{int}}^{\varepsilon}$ but only on the action S (or some renormalized version of S).

There is an other possible characterization of the measure $\mu = e^{-S(\varphi)}\mathcal{D}\varphi$. Indeed, exploiting the alleged translation invariance of $\mathcal{D}\varphi$, we get the equality

$$0 = \int \frac{\delta}{\delta\varphi}(e^{-S(\varphi)}F(\varphi))\mathcal{D}\varphi$$

which is equivalent to the **integration by parts formula**

$$\int \frac{\delta S(\varphi)}{\delta\varphi}F(\varphi)\mu(\mathrm{d}\varphi) = \int \frac{\delta F(\varphi)}{\delta\varphi}\mu(\mathrm{d}\varphi).$$

This equation implies a infinite set of conditions which Schwinger functions S_N must satisfy: they are usually called **Dyson-Schwinger equations**. In probabilistic terms Dyson-Schwinger equations are a form of integration by parts formula.

The naif form of Dyson-Schwinger equations is often not well defined, since it is necessary to incorporate the renormalization procedure. For this reason we introduce the renormalized action

$$S_\varepsilon(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}(|\nabla_x \varphi|^2 + m^2|\varphi|^2)\ \mathrm{d}x + \int_{\mathbb{R}^d} f_\varepsilon(x) V_\varepsilon((g_\varepsilon * \varphi)(x))\ \mathrm{d}x.$$

In the interacting case the integration by parts formula becomes a condition satisfied in the limit $\varepsilon \to 0$, namely we say that the measure μ

$$\lim_{\varepsilon \to 0} \int \frac{\delta S_\varepsilon(\varphi)}{\delta \varphi} F(\varphi) \mu(\mathrm{d}\varphi) = \int \frac{\delta F(\varphi)}{\delta \varphi} \mu(\mathrm{d}\varphi).$$

More formally, if E is a Banach space where $\frac{\delta S_\varepsilon}{\delta \varphi}$ is well defined and $\mathcal{M}$ is a family of probability measure on E we define:

Definition **(D-Gubinelli-Turra, 2022)** *We say that a measure $\nu \in \mathcal{M}$ satisfies the **integration by parts formula with respect to** $\left(\frac{\delta S_\varepsilon}{\delta \varphi}\right)_{\varepsilon > 0}$ **and** $\mathcal{M}$ **(IBPF)** if, for any $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\int_E \left\langle \frac{\delta F(\varphi)}{\delta \varphi}, f \right\rangle \nu(\mathrm{d}\varphi) = \lim_{\varepsilon \to 0} \int_E F(\varphi) \left\langle \frac{\delta S_\varepsilon}{\delta \varphi}, f \right\rangle \nu(\mathrm{d}\varphi), \qquad \text{for any } F \in \mathrm{Cyl}_E^b.$$

With the previous definition we can ask the following questions:

1. Does the IBPF equation have solutions?
2. Are the solutions to equation IBPF unique?
3. If $\mu_{\mathrm{int}} = \lim_{\varepsilon \to 0} \mu_{\mathrm{int}}^\varepsilon$ exists, is it a solution to IBPF?

Related problems: FPK equation

If we consider the second order operator given by

$$\begin{aligned}\mathcal{L}^{\varepsilon}(F)(\varphi) &= \frac{1}{2}\mathrm{Tr}_{L^2}\left(\frac{\delta^2 F}{\delta\varphi^2}(\varphi)\right) - \left\langle \frac{\delta S_\varepsilon}{\delta\varphi}, \frac{\delta F}{\delta\varphi}\right\rangle_{L^2} \\ &= \frac{1}{2}\mathrm{Tr}_{L^2}\left(\frac{\delta^2 F}{\delta\varphi^2}(\varphi)\right) \\ &\quad - \int \big(-\Delta\varphi + m^2\varphi + g_\varepsilon * (f_\varepsilon V_\varepsilon'(g_\varepsilon * \varphi))\big)(x)\frac{\delta F}{\delta\varphi}(x)\ \mathrm{d}x,\end{aligned}$$

we can reformulate the integration by parts problem through a second order (symmetric) equation $\mathcal{L}^{\varepsilon,*}(\nu)=0$. Indeed we have that

$$\mathcal{L}^{\varepsilon}(\varphi(x)F) - \varphi(x)\,\mathcal{L}^{\varepsilon}(F) = -\frac{\delta S_\varepsilon(\varphi)}{\delta\varphi}(x)\,F(\varphi) + \frac{\delta F(\varphi)}{\delta\varphi}(x).$$

Definition *The probability measure $\nu \in \mathcal{M}$ ($\mathcal{M}$ is a subset of Borel measures on E) is a* ***symmetric solution to the Fokker-Planck-Kolmogorov equation (FPK equation)*** *associated with the stochastic quantization operator $\{\mathcal{L}^\varepsilon\}_{\varepsilon > 0}$ if for any $F, G \in \mathrm{Cyl}_E^b$*

$$\lim_{\varepsilon \to 0} \int_E \mathcal{L}^\varepsilon(F)(\varphi)\nu(\mathrm{d}\varphi) = 0 \tag{1}$$

$$\lim_{\varepsilon \to 0} \left(\int_E G(\varphi)\mathcal{L}^\varepsilon(F)(\varphi)\nu(\mathrm{d}\varphi) - \int_E \mathcal{L}^\varepsilon(G)(\varphi)F(\varphi)\nu(\mathrm{d}\varphi) \right) = 0.$$

Taking very mild assumptions on the space E and on the set of measures $\mathcal{M}$ (in particular requiring that $\|\varphi\|_E \in L^p(\nu)$ for some $p \geqslant 1$ big enough and for any $\nu \in \mathcal{M}$), it is possible to prove that ν is a symmetric solution to FPK equation if and only if ν is a solution to the IBPF.

The study of solutions to (stationary) FPK equation is a classical topic in stochastic analysis.

When E is finite dimensional the research is very well developed, and existence and uniqueness for symmetric (and nonsymmetric) solutions has been proved with very weak assumptions on the drift $\frac{\delta S_\varepsilon(\varphi)}{\delta\varphi}$ (see, e.g., the book [Bogachev-Krylov-Röckner-Shaposhnikov, 2015/2022] and references within). In the infinite dimensional setting there are fewer results, mainly for the non-renormalized case $\frac{\delta S_\varepsilon}{\delta\varphi} = \frac{\delta S}{\delta\varphi}$ (incomplete list):

- The existence for infinite dimensional FPK equation (even if non-singular) using Lyapunov functions by [Kirillov, 1994] (on non-renormalized equation with $V(x) = \cos(\beta x)$) and by [Bogachev-Röckner, 2001];
- The problem of uniqueness of equation with (singular in another sense) dissipative drift was considered by [Bogachev-DaPrato-Röckner, 2009] (see also [Röckner-Zhu-Zhu, 2014]).

We focus on a case where it is (partially) possible to answer to the previous questions. Namely, we consider the case where $d=2$ and

$$V(z)=\exp(\beta z), \quad V_\varepsilon(z)=\exp\left(\beta z-\frac{\beta^2}{2}C_\varepsilon\right),$$

where $C_\varepsilon=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{|\hat{g}_\varepsilon(k)|^2}{|k|^2+m^2}\mathrm{d}k\sim\log\left(\frac{1}{\varepsilon}\right)$, $\beta>0$, and $d=1$ (namely $\mathbb{R}^d=\mathbb{R}^2$).

This model, studied firstly by [Høegh-Krohn,1971] and [Albeverio-Høegh-Krohn,1973], has important applications to Liouville quantum gravity.

The stochastic quantization of this model has been studied by [Garban,2020], [Hoshino-Kawabi-Kusuoka, 2021 and 2020], [Oh-Robert-Wang, 2021], [Oh-Robert-Tzvetkov-Wang, 2020] (on compact manifolds), and by [Albeverio-D-Gubinelli, 2021] and [Barashkov-D, 2025] on $\mathbb{R}^2$.

The first problem in solving the previous equation is that the limit

$$\lim_{\varepsilon \longrightarrow 0} \frac{\delta S_\varepsilon}{\delta \varphi}(\varphi)$$

does not exist with respect to a generic measure $\nu \in \mathcal{P}_{\mathrm{Radon}}(E)$.

A simple class of measure for which the limit is defined is **the set of absolutely continuous measure w.r.t. $\mu_{\mathbf{free}}$.** Due to the ergodicity of μ_{int} with respect to space/time translations it cannot be absolutely continuous w.r.t. μ_{free}.

The idea is to extend the Da Prato-Debussche trick from the singular SPDEs to the study of measures on infinite dimensional spaces.

Regular coupling of probability measures

Idea: Space of measures close to the GFF but not necessarily absolutely continuous to it. Let A be a convex cone in E with stronger norm than E and consider

$$d_A^p(\nu_1, \nu_2) = \inf_{\sigma \in \Gamma(\nu_1, \nu_2)} \int \|\varphi_1 - \varphi_2\|_A^p \sigma(\mathrm{d}\varphi_1, \mathrm{d}\varphi_2)$$

for some $p > 0$. Then we define the set of possible solution $\mathcal{M}_A := \{\nu \in \mathcal{P}_{\mathrm{Randon}}(E) : d_A^p(\nu, \mu_{\mathrm{free}}) < +\infty\}$.

If $\nu \in \mathcal{M}_A$ it means that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two random variables $X : \Omega \to E$ and $Y : \Omega \to A \subset E$ such that

$$\mu_{\mathrm{free}} = \mathrm{Law}_{\mathbb{P}}(X), \quad \nu = \mathrm{Law}_{\mathbb{P}}(X + Y).$$

Let $E = \mathcal{C}_\ell^{-\delta}(\mathbb{R}^2)$ and consider

$$A = H^1(\mathbb{R}^2) \text{ with its natural norm.}$$

In this case $\mathcal{M}_A^p$ is closely related to the set of absolutely continuous measures w.r.t. μ_{free}.

- [Üstunel-Zakai,2000] proved that if ν is equivalent to μ_{free} there exists a coupling in $H^1(\mathbb{R}^2)$;
- [Feyel-Üstunel,2003], [Fang-Shao-Sturm,2010], [Bogachev-Kolesnikov,2013] proved and used that $\{\mathcal{H}(\nu, \mu_{\mathrm{free}})\} \subset \mathcal{M}_A^2$;
- In [Cavalletti,2012] the space $\mathcal{M}_A^1 \subset \{\nu \text{a. c. w.r.t } \mu_{\mathrm{free}}\}$ is studied proving using optimal transport.

Regular couplings and QFT

- In [Barashkov-Gubinelli,2021] φ^4_2, $A = H^1_\ell(\mathbb{R}^2)$ and $\exp(\beta\varphi)_2$, $\beta^2 < 8\pi$, $A = H^1_\ell(\mathbb{R}^2)$
- [Bauerschmidt-Hofstetter,2022] $\cos(\beta\varphi)_2$, $\beta^2 < 6\pi$, $A = \mathcal{C}^{2-\frac{\beta^2}{4\pi}}(\mathbb{T}^2)$, [Barashkov-Gunaratnam-Hofstetter,2023] $P(\varphi)_2$, and $A = H^{2-\varepsilon}(\mathbb{T}^2)$, [Hofstetter-Zeitouni,2025] $\exp(\beta\varphi)_2$, $\beta^2 < 8\pi$, $A = \mathcal{C}^{2-\eta(\beta)-\varepsilon}(\mathbb{T}^2)$ where $\eta(\beta) = \frac{2\beta}{\sqrt{2\pi}} - \frac{\beta^2}{4\pi}$;
- [Barashkov,2022] $\cos(\beta\varphi)_2$, $\beta^2 < 4\pi$, $A = L^\infty_\ell(\mathbb{R}^2)$, [Gubinelli-Meyer,2024] $\cos(\beta\varphi)_2$, $\beta^2 < 6\pi$, $A = B^{2-\frac{\beta^2}{4\pi}-\varepsilon}_{p,p,\ell}(\mathbb{R}^2)$;
- [Barashkov-D,2025] coupling for $\cosh(\beta\varphi)_2$ for $\beta^2 < 4\pi$ and $A = H^1_\ell(\mathbb{R}^2)$.
- [Gubinelli-Hofmanova,2021] coupling for φ^4_3 measure in $A = \mathcal{C}^{\frac{1}{2}-\varepsilon}_\ell(\mathbb{R}^3)$;
- In [Barashkov-D-Zachhuber,2023] Gaussian Gibbs measure for Anderson Hamiltonian $H^{\mathrm{And}} = -\Delta + \xi - \infty$, $A = H^{1-\varepsilon}(\mathbb{T}^2)$;

Application to integration by parts

In the specific case of exponential models we consider

$$E = B_X + B_Y$$

where $B_X = \mathcal{C}_\ell^{-\delta}(\mathbb{R}^2)$ and

$$B_Y = B^{s(\beta)}_{p(\beta),p(\beta),\ell}(\mathbb{R}^2) \cap \{\text{negative functions}\}$$

where $0 < s(\beta) < 1$ and $p(\beta) \in (1, +\infty)$ are suitable numbers (depending on β).

We recall that the operator $\mathcal{L}^\varepsilon$ reads as follows:

$$\begin{aligned}\mathcal{L}^\varepsilon(F)(\varphi) \;&=\; \frac{1}{2}\mathrm{Tr}_{L^2}\Big(\frac{\delta^2 F}{\delta\varphi^2}(\varphi)\Big) - \int_{\mathbb{R}^d}(-\Delta\varphi + m^2\varphi)\,\frac{\delta F}{\delta\varphi}(x)\,\mathrm{d}x \\ &\quad -\beta\int_{\mathbb{R}^d} g_\varepsilon * \Big(f_\varepsilon \exp\Big(\beta g_\varepsilon * \varphi - \frac{\beta^2}{2}C_\varepsilon\Big)\Big)(x)\,\frac{\delta F}{\delta\varphi}(x)\,\mathrm{d}x.\end{aligned}$$

Lemma *For any $\nu \in \mathcal{M}_{B_Y}$ there exists a Borel probability measure σ on $B_X \times B_Y$ for which*

$$P_{X,*}(\sigma) = \mu_{\mathrm{free}}, \quad P_{X+Y,*}(\mu) = \nu$$

with $P_X, P_{X+Y}\colon B_X \times B_Y \to E$ s.t. $P_X(X,Y) = X$, $P_{X+Y}(X,Y) = X + Y$.

We call the space of measures with the previous properties $\mathcal{M}_{B_X \times B_Y}$.

The previous lemma permit to lift the FPK equation from the space E to the space $B_X \times B_Y$, indeed defining

$$\begin{aligned}\mathcal{L}^{\varepsilon}_{X,Y}(G)(X,Y) &= \mathrm{Tr}_{L^2}\Big(\frac{\delta^2 G}{\delta X^2}(X,Y)\Big) - \int_{\mathbb{R}^2}(-\Delta+m^2)X(x)\frac{\delta G}{\delta X}(x)\,\mathrm{d}x\\ &\quad -\int_{\mathbb{R}^2}(-\Delta+m^2)Y(x)\frac{\delta G}{\delta Y}(x)\,\mathrm{d}x\\ &\quad -\int_{\mathbb{R}^2}\beta g_\varepsilon * \Big(f_\varepsilon \exp\Big(\beta g_\varepsilon * (X+Y) - \frac{\beta^2}{2}C_\varepsilon\Big)\Big)\frac{\delta G}{\delta Y}(x)\,\mathrm{d}x.\end{aligned}$$

If $F\colon \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ is regular enough, taking $\varphi = X+Y$, then

$$\mathcal{L}^{\varepsilon}(F)(X+Y) = \mathcal{L}^{\varepsilon}_{X,Y}(F(X+Y)).$$

The introduction of the operator $\mathcal{L}^{\varepsilon}_{X,Y}$ is based on the Da Prato-Debussche trick for studying singular SPDEs.

The main advantage of considering the operator $\mathcal{L}^{\varepsilon}_{X,Y}$ is the following.

Proposition *Suppose* $\beta^2 < 8\pi$, *then there are* $p > 1$, $s, \ell, \ell', \delta > 0$ *such that, taking*

$$B_X = C^{-\delta}_{\ell'}(\mathbb{R}^2), \quad B_Y = B^{s-\delta}_{p,p,\ell}(\mathbb{R}^2) \cap \{negative\ functions\},$$

we have a set $\tilde{\Omega}_X \subset B_X$, *with* $\mu_{\mathrm{free}}(\tilde{\Omega}_X) = 1$, *for which there exists an operator* $\mathcal{L}_{X,Y}$ *such that, for every smooth, bounded cylinder function* $G\colon B_X \times B_Y \to \mathbb{R}$, *we have*

$$\lim_{\varepsilon \to 0} \mathcal{L}^{\varepsilon}_{X,Y}(G)(X,Y) = \mathcal{L}_{X,Y}(G)(X,Y)$$

for every $(X,Y) \in P_X^{-1}(\tilde{\Omega}_X)$.

The proof is based on the fact that

$$\exp\Big(\beta g_\varepsilon * X - \frac{\beta^2}{2} C_\varepsilon\Big) \rightarrow :\exp(\beta X):$$

the Wick exponential of the Gaussian field X, and the convergence is ν_{free}-a.e. The convergence is in distributions (as positive measure) and in the Besov space

$$B^{-\frac{\beta^2}{4\pi}(r-1)-\delta}_{r,r,\ell''}(\mathbb{R}^2), \qquad \beta^2 r < 8\pi, r \geqslant 1.$$

This means that

$$\begin{aligned}\mathcal{L}_{X,Y}(G)(X,Y) &= \mathrm{Tr}_{L^2}\Big(\frac{\delta^2 G}{\delta X^2}(X,Y)\Big) - \int_{\mathbb{R}^2} (-\Delta + m^2) X(x) \frac{\delta G}{\delta X}(x)\, \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} ((-\Delta + m^2) Y + \beta :\exp(\beta X): e^{\beta Y}) \frac{\delta G}{\delta Y}(x)\, \mathrm{d}x.\end{aligned}$$

Theorem (D-Gubinelli-Turra, 2022) *For any $\beta^2 < 8\pi$, there is a measure $\nu \in \mathcal{M}_{B_Y}$ which is solution to the IBPF associated with $\frac{\delta S_\varepsilon}{\delta \varphi}$.*

Remark: In the paper we prove a stronger result: there is a probability measure $\mu \in \mathcal{P}_{\mathrm{Radon}}(B_X \times B_Y)$ such that $\mu \in \mathcal{M}_{B_X \times B_Y}$, i.e.

$$P_{X,*}(\mu) = \nu_{\mathrm{free}}, \quad P_{X+Y,*}(\mu) = \nu \in \mathcal{M}_{B_Y}, \qquad \text{and}$$

$$\int_{B_X \times B_Y} \mathcal{L}_{X,Y}(G)(X,Y)\mu(\mathrm{d}X, \mathrm{d}Y) = 0, \quad G \in \mathrm{Cyl}^b_{B_X \times B_Y}$$

$$\int_{B_X \times B_Y} (\mathcal{L}_{X,Y}(F)H - \mathcal{L}_{X,Y}(H)F)(X+Y)\mu(\mathrm{d}X, \mathrm{d}Y) = 0, \quad F, H \in \mathrm{Cyl}^b_E$$

The existence is based on the presence of the Lyapunov functions

$$\mathcal{L}^{\varepsilon}_{X,Y}(V_1(X,Y)) \leqslant -V_2(X,Y) + V_3(X),$$

where $V_1, V_2 \colon B_Y \times B_Y \rightarrow \mathbb{R}_+$, V_2 has compact sublevels, and $V_3 \colon B_X \rightarrow \mathbb{R}_+$

$$\int_{B_X} V_3(X)\nu_{\text{free}}(\mathrm{d}X) < +\infty$$

In particular,

$$\begin{aligned}
V_1(X,Y) &= \|X\|^p_{B^{-s'}_{p,p,\ell}} + \|Y\|^p_{B^{s}_{p,p,\ell}}, \\
V_2(X,Y) &= (1-\sigma)\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} + C_\sigma \|Y\|^p_{B^{s+2/p}_{p,p,\ell}}, \\
V_3(X) &= \frac{1}{\sigma}\big(C + \|{:}\exp(\beta X){:}\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}\big).
\end{aligned}$$

Consider $\tilde{\gamma} := 3 - 2\sqrt{2} \approx 0.172$.

Theorem (D-Gubinelli-Turra, 2022) *Suppose that $\beta^2 < 4\pi\tilde{\gamma}$.*

Then, the solution to the IBPF problem w.r.t. $\left(\frac{\delta S_\varepsilon}{\delta \varphi}\right)_\varepsilon$ and $\mathcal{M}_{B_Y}$ is unique.

Remark: We also get a uniqueness result for a slightly more restrictive formulation (namely the FPK equation associated with the operator $\mathcal{L}_{X,Y}$) of the IBPF problem in the regime $\alpha^2 < 4\pi\gamma$, where $\gamma \approx 0.55$.

The proof is essentially based on the study of the (classical) solutions to the resolvent equation

$$(\mathcal{L}^\varepsilon_{X,Y}(G^\lambda_\varepsilon) - \lambda G^\lambda_\varepsilon)(X,Y) = F(X,Y).$$

Proposition *Let $F \in \mathrm{Cyl}_{B_X \times B_Y}$ be a function with compact support in Fourier variables, then there exists a classical solution $G_\varepsilon^\lambda \in C^2(B_X \times B_Y)$ to the resolvent equation. This solution has the following properties:*

i. There exists $\varepsilon_0 > 0$ such that, for every $\mu_1, \mu_2 \in \mathcal{M}_{B_X \times B_Y}$ and for every $\varsigma \in (0,1)$, there are two constants $C_{\mu_1,\mu_2,\varsigma} > 0$ and $K > 0$ such that

$$\lambda \left| \int G_\varepsilon^\lambda \, (\mathrm{d}\mu_1 - \mathrm{d}\mu_2) \right| \lesssim \varsigma + \frac{\lambda}{\lambda + K} C_{\mu_1,\mu_2,\varsigma},$$

where the constant included in the symbol $\lesssim$ does not depend on λ, μ_1, μ_2, ε or ς.

ii. If $\beta^2 < 4\pi\tilde{\gamma}$, then there exists $q > 1$ such that, for every measurable $\mathcal{K}\colon B_X \times B_Y \to B^{s(\beta)}_{q,q,\ell/2}$ and every $\mu \in \mathcal{M}_{B_X \times B_Y}$, we have

$$\int |\langle \nabla_Y G_\varepsilon^\lambda, \mathcal{K} \rangle| \, \mathrm{d}\mu \lesssim_\lambda \left(\int \|\mathcal{K}\|^q_{B^{s(\beta)}_{q,q,\ell/2}} \, \mathrm{d}\mu \right)^{1/q},$$

uniformly in $\varepsilon > 0$.

Idea of proof of the Theorem: Suppose that μ_1 and μ_2 are two solutions then

$$\begin{aligned}
& \int F(\mathrm{d}\mu_1 - \mathrm{d}\mu_2) = \\
= \quad & -\int (\lambda G_\varepsilon^\lambda - \mathcal{L}_{X,Y}^\varepsilon(G_\varepsilon^\lambda) + \mathcal{L}_{X,Y}(G_\varepsilon^\lambda))(\mathrm{d}\mu_1 - \mathrm{d}\mu_2) \\
= \quad & \lambda \int G_\varepsilon^\lambda(\mathrm{d}\mu_2 - \mathrm{d}\mu_1) + \int \langle \beta\,(:e^{\beta X}\!: e^{\beta Y} - f_\varepsilon : e^{\beta g_\varepsilon * X}\!: e^{\beta g_\varepsilon * Y}), \nabla_Y G_\varepsilon^\lambda \rangle(\mathrm{d}\mu_2 - \mathrm{d}\mu_1) \\
\lesssim_{\varepsilon \to 0} \quad & \varsigma + \frac{\lambda}{\lambda + K} C_{\mu_1,\mu_2,\varsigma} \to \varsigma \quad \text{as } \lambda \to 0,
\end{aligned}$$

where we used that $F = \lambda G_\varepsilon^\lambda - \mathcal{L}_{X,Y}^\varepsilon(G_\varepsilon^\lambda)$ and that (since G_ε^λ is regular enough)

$$\int \mathcal{L}_{X,Y}(G_\varepsilon^\lambda)\mathrm{d}\mu_1 = \int \mathcal{L}_{X,Y}(G_\varepsilon^\lambda)\mathrm{d}\mu_2 = 0$$

being μ_1 and μ_2 solution to the FPK equation associated with $\mathcal{L}_{X,Y}$.

Idea of proof of the Proposition (resolvent equation): Consider the system

$$\partial_t X_t^\varepsilon = -(-\Delta + m^2) X_t^\varepsilon + \xi_t, \tag{2}$$

$$\partial_t Y_t^\varepsilon = -(-\Delta + m^2) Y_t^\varepsilon - f_\varepsilon : e^{\beta g_\varepsilon * X_t^\varepsilon} : e^{\beta g_\varepsilon * Y_t^\varepsilon}, \tag{3}$$

then

$$G_\varepsilon^\lambda(X, Y) = \mathbb{E}_{X_0^\varepsilon = X, Y_0^\varepsilon = Y}\left[\int_0^\infty e^{-\lambda t} F(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t\right].$$

From which, at least formally, we get

$$\nabla_Y G_\varepsilon^\lambda(X, Y) = \mathbb{E}_{X_0^\varepsilon = X, Y_0^\varepsilon = Y}\left[\int_0^\infty e^{-\lambda t} \mathrm{d}F(X_t^\varepsilon, Y_t^\varepsilon)[\nabla_{Y_0} X_t^\varepsilon, \nabla_{Y_0} Y_t^\varepsilon] \mathrm{d}t\right]$$

etc.

It is essential to study the derivatives of the flow with respect to the initial conditions. For example $\nabla_{Y_0} Y_t^{\varepsilon}[h]$ which solves the following SPDE

$$(\partial_t - \Delta + m^2 + \beta^2 f_{\varepsilon} \!:\! e^{\beta g_{\varepsilon} * X} \!:\! e^{\beta g_{\varepsilon} * Y} g_{\varepsilon} *) \nabla_{Y_0} Y_t^{\varepsilon}[h] = 0, \quad \nabla_{Y_0} Y_0^{\varepsilon}[h] = h.$$

$$(\partial_t - \Delta + m^2 + V_{\varepsilon}''(\varphi)) \nabla_{Y_0} Y_t^{\varepsilon}[h] = 0$$

From the previous equation we are able to get some apriori estimates

$$\| \tilde{g}_{\varepsilon} * \nabla_{Y_0} Y_t^{\varepsilon}[h] \|_{L^2_{\ell}(\mathbb{R}^2)} \lesssim e^{-kt} \| h \|_{L^2_{\ell}(\mathbb{R}^2)},$$

and

$$\| \tilde{g}_{\varepsilon} * (\nabla_{Y_0} Y_t^{\varepsilon}[h] - e^{(-\Delta + m^2) t} h) \|_{L^2_{\ell}(\mathbb{R}^2)} \lesssim \\ \lesssim (1+t)^{\sigma} \| f_{\varepsilon} \!:\! e^{\alpha (g_{\varepsilon} * X_s)} \!:\! \|_{L^{r_1}_{\ell_1}(\mathbb{R}_+, B^{s_1}_{r_1, r_1, \ell_1})} \| h \|_{B^{s_2}_{r_2, r_2, \ell_2}}.$$

Open problems

- Study the Wesserstein space $\mathcal{M}_A$ (for different A), its relation with absolutely continuous measures and its optimal transport properties;
- Generalize the various analytical construction of operators theory (like resolvent, core etc.) for operators defined on the "dual spaces" of $\mathcal{M}_A$ (this problem is related with relation between invariance and infinitesimal invariance for SPDEs);
- Measures with additional structure (i.e. $\varphi^4_{4-\varepsilon}$ measures);
- Use the coupling for studying properties of the measures (like Markovianity (see [Barashkov-Gunaratnam,2025]), or more generally definition of Gibbs measures (on the continuum) in Dobrushin-Lanford-Ruelle approach).

Thank you for the attention!