Local large deviations and the strong renewal theorem

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(Picture taken from MFO website)

Overview

Necessary and sufficient conditions
for the Strong Renewal Theorem (SRT)

Problem dates back to Garsia and Lamperti (1962)

First focus on renewal processes (random walks with positive increments)
then on general random walks (much harder!)

Key tool: local large deviation estimate of independent interest

For simplicity we stick to the lattice case, but everything applies to non-lattice distributions

Outline

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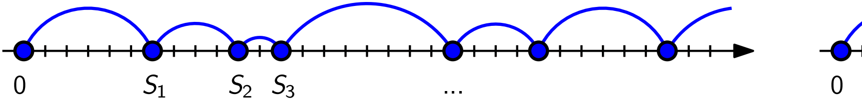
The classical renewal theorem

Random walk $S_k := X_1 + X_2 + \dots X_k$ with positive increments

(X_i) i.i.d. $\qquad X_i \in \mathbb{N} = \{1, 2, \dots\}$ $\qquad$ aperiodic



Renewal (Green) function $\qquad u(n) := \mathrm{P}(S \text{ visits } n) = \sum_{k \geq 0} \mathrm{P}(S_k = n)$

Renewal Theorem (Erdos, Feller, Pollard 1949)

$$\lim_{n \to \infty} u(n) = \frac{1}{\mathrm{E}[X]} = 0 \;\text{ if } \mathrm{E}[X] = \infty$$

The case of infinite mean

When $\mathrm{E}[X] = \infty$, at which rate $u(n) \to 0$?

Define the truncated mean

$$m(n) := \mathrm{E}[X \wedge n] = \mathrm{E}[\min\{X, n\}]$$

Guess (SRT) $\qquad u(n) \underset{n\to\infty}{\sim} \frac{c}{m(n)} \qquad$ for some $c \in (0, \infty)$

Assumptions are needed! But there is hope

Lemma (Erickson 1973)

$$\frac{n}{m(n)} \;\le\; \sum_{i=1}^{n} u(i) \;\le\; 2\,\frac{n}{m(n)}$$

Asymptotically stable renewal processes

Tail Assumption

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} \frac{L(n)}{n^\alpha} \qquad \alpha \in (0,1) \qquad L(\cdot) \text{ slowly varying}$$

Explicit computation $\quad m(n) := (1-\alpha)\,\mathrm{E}[X \wedge n] \underset{n\to\infty}{\sim} L(n)\, n^{1-\alpha}$

Theorem (Garsia, Lamperti 1962) (Erickson 1970)

$$\liminf_{n\to\infty} \frac{u(n)}{1/m(n)} = c_\alpha := \frac{\sin \pi\alpha}{\pi}$$

- If $\alpha > \frac{1}{2}$ lim inf can be replaced by lim (the SRT holds)
- If $\alpha \leq \frac{1}{2}$ there are counterexamples!

Sufficient conditions for the SRT

Local Assumption

$$\mathrm{P}(X = n) \le C \, \frac{L(n)}{n^{1+\alpha}}$$

Theorem (Doney 1997) (Williamson 1968) (Vatutin, Topchii 2013)

Under Local Assumption, the SRT holds also for $\alpha \le \frac{1}{2}$

$$u(n) = \mathrm{P}(S \text{ visits } n) \underset{n\to\infty}{\sim} \frac{c_\alpha}{m(n)} = \frac{c_\alpha}{L(n)\, n^{1-\alpha}}$$

Sufficient conditions weaker than the Local Assumption (integral criteria) were given more recently in (Chi 2013, 2015)

What can go wrong?

Without Local Assumption, we only know that

$$\mathrm{P}(X = n) = o\big(\mathrm{P}(X \geq n)\big) = o(1)\,\frac{L(n)}{n^{\alpha}}$$

where $o(1)$ can be as slow as we wish (along subsequences)

$$\mathrm{P}(S \text{ visits } n) \geq \mathrm{P}(X_1 = n) \quad \text{can be } \gg \frac{1}{L(n)\,n^{1-\alpha}} \quad \text{if } \alpha \leq \frac{1}{2}$$

The SRT fails because a single step (more generally, few steps) can give "atypical" contribution to $\mathrm{P}(S \text{ visits } n)$

We must find optimal conditions on X to rule this out

Necessary and sufficient conditions

Define a weighted sum of $\mathrm{P}(X = n), \mathrm{P}(X = n-1), \ldots, \mathrm{P}(X = n - \delta n)$

$$I^+(\delta; n) := \sum_{r=1}^{\delta n} w_r \, \mathrm{P}(X = n - r) \qquad w_r := \frac{r^{2\alpha-1}}{L(r)^2}$$

Definition

We say that $I^+(\delta; n)$ is asymptotically negligible (a.n.) iff

$$I^+(\delta; n) \ll \frac{1}{m(n)} \qquad \text{as } n \to \infty\,,\ \delta \to 0$$

More precisely:

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{I^+(\delta; n)}{1/m(n)} = 0$$

SRT for renewal processes

Recall the Tail Assumption

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} \frac{L(n)}{n^\alpha} \qquad \alpha \in (0,1) \qquad L(\cdot) \text{ slowly varying}$$

Theorem (SRT for renewal processes) (Caravenna, Doney 2016)

- If $\alpha > \frac{1}{2}$ the SRT holds with no further assumption
- If $\alpha \le \frac{1}{2}$ the SRT holds if and only if $I^+(\delta; n)$ is a.n.

The weighted sum $I^+(\delta; n)$ is explicit, but slightly involved

We give a practical sufficient condition in terms of probability of intervals

A nearly optimal condition

Corollary

For $\alpha \le \frac{1}{2}$, a sufficient condition for the SRT is

$$\mathrm{P}(n < X \le n + r) \le C \, \frac{L(n)}{n^\alpha} \left(\frac{r}{n}\right)^\gamma \qquad \forall r = 1, \ldots, n \qquad (\star)$$

for some $C < \infty$ and $\gamma > 1 - 2\alpha$

$$\mathrm{P}(X \le n + r \,|\, X > n) \le C \left(\frac{r}{n}\right)^\gamma \qquad (\star)$$

Condition $(\star)$ is actually nearly optimal

Corollary

A necessary condition for the SRT is that $(\star)$ holds for every $\gamma < 1 - 2\alpha$

Outline

General random walks

Random walk $S_k := X_1 + X_2 + \ldots X_k$ on $\mathbb{Z}$

(X_i) i.i.d. $\qquad X_i \in \mathbb{Z} \qquad$ aperiodic

Tail Assumption

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} p\,\frac{L(n)}{n^\alpha} \qquad \mathrm{P}(X < -n) \underset{n\to\infty}{\sim} q\,\frac{L(n)}{n^\alpha}$$

$p > 0, \;\; q \geq 0 \qquad \alpha \in (0,1) \qquad L(\cdot)$ slowly varying

We use the shorthand

$$m(n) := L(n)\, n^{1-\alpha}$$

which is $\simeq \mathrm{E}[|X| \wedge n]$

A first renewal theorem

Renewal (Green) function $$u(n) := \sum_{k \geq 0} \mathrm{P}(S_k = n)$$

Theorem (Williamson 1968) (Erickson 1971)

$$\liminf_{n \to \infty} \frac{u(n)}{1/m(n)} = c_{\alpha,p,q}$$

- If $\alpha > \frac{1}{2}$ lim inf can be replaced by lim (the SRT holds)
- If $\alpha \leq \frac{1}{2}$ there are counterexamples

The Local Assumption is again sufficient condition for the SRT

$$\mathrm{P}(X = n) \leq C \, \frac{L(n)}{|n|^{1+\alpha}}$$

A first condition

Extend the previously introduced weighted sum (note the $|r|$)

$$I_1(\delta; n) := \sum_{|r|=1}^{\delta n} w_r \, \mathrm{P}(X = n + r) \qquad w_r := \frac{|r|^{2\alpha-1}}{L(|r|)^2}$$

$I_1(\delta; n)$ depends only on X^+, but SRT may also depend on X^-

Proposition

For random walks, a sufficient (not necessary) condition for the SRT is

$$I_1(\delta; n) \quad \text{and} \quad I_1(\delta; -n) \quad \text{are both a.n.}$$

In particular, a sufficient condition is that $(\star)$ holds for both n and $-n$

In general, assuming that $I_1(\delta; -n)$ is a.n. is too strong

SRT for random walks

For $k \geq 2$ define a k-tuple weighted sum, depending on $\delta, \eta \in (0,1)$

$$I_k(\delta, \eta; n) := \sum_{\substack{|r_1| \leq \delta n \\ |r_j| \leq \eta |r_{j-1}| \text{ for } 2 \leq j \leq r}} \mathrm{P}(X = n + r_1)\, \mathrm{P}(X = -r_1 + r_2) \cdots$$

$$\cdot\, \mathrm{P}(X = -r_{k-1} + r_k)\, \frac{|r_k|^{(k+1)\alpha - 1}}{L(|r_k|)^{k+1}}$$

Theorem (SRT for random walks) (Caravenna, Doney 2016)

Consider a random walk satisfying the Tail Assumption with $p, q > 0$

- If $\alpha > \frac{1}{2}$ the SRT holds with no further assumption
- If $\frac{1}{3} < \alpha < \frac{1}{2}$ the SRT holds if and only if $I_1(\delta; n)$ is a.n.
- Given $k \geq 2$, if $\frac{1}{k+2} < \alpha < \frac{1}{k+1}$ the SRT holds if and only if $I_k(\delta, \eta; n)$ is a.n., for every fixed $\eta \in (0,1)$

Further considerations

- Necessary and sufficient conditions available also for $\alpha = \frac{1}{k}$, $k \in \mathbb{N}$, in terms of slightly modified quantities $\tilde{I}_k(\delta, \eta; n)$

- Given $\alpha \in (0,1)$ let $k \in \mathbb{N}$ be such that $\frac{1}{k+2} < \alpha < \frac{1}{k+1}$

  SRT holds $\iff$ I_k is a.n. $\iff$ $I_{k'}$ is a.n. $\forall k' \in \mathbb{N}$

- There are examples where $I_1(\delta; x)$ is a.n. but $I_2(\delta, \eta; x)$ is not a.n.

- When $q = 0$ the previous conditions are sufficient, but we don't expect them to be necessary

Outline

Asymptotically stable random walks

Consider a random walk $S_k = X_1 + \ldots + X_k$ on $\mathbb{Z}$

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} p\,\frac{L(n)}{n^\alpha} \qquad \mathrm{P}(X < -n) \underset{n\to\infty}{\sim} q\,\frac{L(n)}{n^\alpha}$$

We allow both $\alpha \in (0,1)$ and $\alpha \in (1,2)$ (with $\mathrm{E}[X] = 0$)

We know that $\dfrac{S_k}{a_k} \xrightarrow[k\to\infty]{d} Y$ α-stable law $\qquad \left(a_k \sim \tilde{L}(k)\, k^{\frac{1}{\alpha}}\right)$

Gnedenko LLT

$$\mathrm{P}(S_k = n) = \frac{1}{a_k}\left(\varphi\left(\frac{n}{a_k}\right) + o(1)\right) = \frac{o(1)}{a_k} \quad \text{for } n \gg a_k$$

We improve on this bound, giving a quantitative estimate of $o(1)$

Local large deviations

Theorem (Local Large Deviations) (Caravenna, Doney 2016)

$$\mathrm{P}(S_k = n) \le \frac{C}{a_k} \, \frac{k\, L(n)}{n^\alpha}$$

For every $\gamma \in (0,1]$

$$\mathrm{P}\big(S_k = n,\ \max\{X_1, \ldots, X_k\} \le \gamma n\big) \le \frac{C_\gamma}{a_k} \left(\frac{k\, L(n)}{n^\alpha} \right)^{\lceil \frac{1}{\gamma} \rceil} \qquad (\square)$$

- Improves on Gnedenko's bound with no extra assumptions
- Key tool for the SRT
- $(\square)$ is a local version of the Fuk-Nagaev inequality
- Extended to the Cauchy case $\alpha = 1$ in (Berger 2017)

Outline

1. SRT for Renewal Processes

2. SRT for Random Walks

3. Local Large Deviations

4. Proof of the SRT

Probabilistic proof: strategy

Goal: SRT

$$u(n) \underset{n\to\infty}{\sim} \frac{c_\alpha}{m(n)} = \frac{c_\alpha}{L(n)\, n^{1-\alpha}}$$

$$u(n) = \sum_{k\geq 0} \mathrm{P}(S_k = n) = \underbrace{\sum_{k \leq \delta \frac{n^\alpha}{L(n)}}}_{(1)} + \underbrace{\sum_{\delta \frac{n^\alpha}{L(n)} < k \leq \frac{1}{\delta} \frac{n^\alpha}{L(n)}}}_{(2)} + \underbrace{\sum_{k > \frac{1}{\delta} \frac{n^\alpha}{L(n)}}}_{(3)}$$

- (3) is always negligible by Gnedenko's LLT $\mathrm{P}(S_k = n) \leq \frac{C}{a_k}$
- (2) is a Riemann sum approximation: by $\mathrm{P}(S_k = n) \sim \frac{1}{a_k}\varphi(\frac{n}{a_k})$ $\rightsquigarrow (\ldots) \rightsquigarrow$ it gives full contribution to SRT as $n \to \infty$, $\delta \to 0$

SRT holds $\iff$ $(1) \ll \frac{1}{m(n)}$ is asymptotically negligible

Necessity: SRT $\Rightarrow$ $I^+(\delta; n)$ is a.n.

$$
\begin{aligned}
\mathrm{P}(S_k = n) &\geq \mathrm{P}(S_k = n,\ \exists! X_i > C a_k) \\
&= (\ldots) \\
&\geq c \sum_{1 \leq r \leq \delta n} \mathrm{P}(X = n - r)\, \frac{k}{a_k}\, \mathbb{1}_{\{\frac{r^\alpha}{L(r)} < k \leq 2 \frac{r^\alpha}{L(r)}\}}
\end{aligned}
$$

Then

$$
\begin{aligned}
(1) &= \sum_{k \leq \delta \frac{n^\alpha}{L(n)}} \mathrm{P}(S_k = n) \\
&\geq c \sum_{1 \leq r \leq \delta n} \mathrm{P}(X = n - r) \sum_{\frac{r^\alpha}{L(r)} < k \leq 2 \frac{r^\alpha}{L(r)}} \frac{k}{a_k} \\
&\geq c \sum_{1 \leq r \leq \delta n} \mathrm{P}(X = n - r)\, \frac{r^{2\alpha - 1}}{L(r)^2} \ = \ I^+(\delta; n)
\end{aligned}
$$

Sufficiency: $I^+(\delta; n)$ is a.n. $\Rightarrow$ SRT

Fix suitable $\gamma \in (0,1)$. Using local large deviation

$$\mathrm{P}(S_k = n,\ \max\{X_1,\ldots,X_k\} \le \gamma n) \quad \text{is negligible}$$

$$\mathrm{P}(S_k = n,\ \max\{X_1,\ldots,X_k\} > \gamma n) \le k \sum_{r \le (1-\gamma)n} \mathrm{P}(X = n-r)\,\mathrm{P}(S_{k-1} = r)$$

$$\text{Then} \qquad (1) \le \sum_{r \le (1-\gamma)n} \mathrm{P}(X = n-r) \sum_{k \le \delta \frac{n^\alpha}{L(n)}} k\,\mathrm{P}(S_{k-1} = r)$$

- We can replace n by r in the inner sum, using that $I^+(\delta; n)$ is a.n.
- Then the inner sum becomes (1) with an extra factor k
- This helps! (Backward) induction argument allows to conclude

Thanks