GAUSSIAN LIMITS FOR SUBCRITICAL CHAOS

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ABSTRACT. We present a simple criterion, only based on second moment assumptions, for the convergence of polynomial or Wiener chaos to a Gaussian limit. We exploit this criterion to obtain new Gaussian asymptotics for the partition functions of two-dimensional directed polymers in the sub-critical regime, including a singular product between the partition function and the disorder. These results can also be applied to the KPZ and Stochastic Heat Equation. As a tool of independent interest, we derive an explicit chaos expansion which sharply approximates the *logarithm* of the partition function.

1. Introduction

In this paper we investigate the convergence to a Gaussian limit for random variables that have the structure of a *polynomial chaos*, that is a multi-linear polynomial of independent random variables, or alternatively of a *Wiener chaos*, that is a sum of multiple Wiener integrals with respect to a Gaussian random measure. Our main motivation is the study of directed polymers in random environment, whose partition function provides a discretisation of the solution of the multiplicative Stochastic Heat Equation (SHE), while its logarithm corresponds to the solution of the KPZ equation. Many convergence results to Gaussian limits have been obtained in recent years for directed polymers and for SHE and KPZ (see the discussion in Section 3) based on polynomial chaos or Wiener chaos, often exploiting the Fourth Moment Theorem and variations thereof. Our purpose is to present a general approach which makes it possible to recover these results in a simpler and unified way and, furthermore, to obtain novel results. Let us give an overview of the paper.

In Section 2 we state our first main result: a general criterion for the convergence of polynomial chaos or Wiener chaos to a Gaussian limit *only based on second moment assumptions*, see Theorems 2.1 and 2.4. Besides the fact that we do not require higher moment bounds, we can work directly with a superposition of chaos of different orders, with no need of treating them individually as in the Fourth Moment Theorem. Our criterion gives conditions that are sufficient, not necessary, but its simplicity makes it potentially suitable to many different contexts.

In Section 3 we study the partition function Z_N^β of two-dimensional directed polymers in random environment. In the limit $N \to \infty$, and for a suitable tuning of the inverse temperature $\beta = \beta_N$ (in the so-called sub-critical regime), the partition function exhibits Edwards-Wilkinson fluctuations [CSZ17b], i.e., it converges to a log-correlated Gaussian field when averaged over the starting point. An analogous result was obtained in [CSZ20] for the logarithm of the partition function. Our criterion from Section 2, besides providing alternative and more elementary proofs of Edwards-Wilkinson fluctuations, gives a natural framework to obtain new Gaussian asymptotics. We give two main illustrations.

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- We prove that a *singular product* between the partition function and the underlying disorder has a non-trivial Gaussian limit, see Theorem 3.4. This result sheds light on the mechanism which produces Edwards-Wilkinson fluctuations, explaining the source of the non-trivial factor which arises in the limiting equation.
- For the partition function Z_N^β with a fixed starting point, we obtain an *explicit chaos expansion* X_N^{dom} which sharply approximates $\log Z_N^\beta$, see Theorem 3.5; then we prove that X_N^{dom}*, hence* $\log Z_N^\beta$ *too, is asymptotically Gaussian*, see Theorem 3.6. We thus recover the main result in [CSZ17b] with a simpler and more conceptual proof.

These results can also be formulated in the continuum setting of the SHE and KPZ equation. We refer to Subsection 3.5 for a discussion and further perspectives.

The following Sections 4–7 contain the proofs of our main results, while some technical lemmas have been deferred to Appendix A.

2. Gaussian limits for polynomial and Wiener chaos

Our general convergence results can be phrased in a discrete setting (polynomial chaos) and in a continuum one (Wiener chaos). We start with the former, which is more elementary.

2.1. Polynomial chaos. Let $\mathbb{T}$ be a countable set. For each $N \in \mathbb{N}$, we consider a family $\eta^N = (\eta_t^N)_{t\in\mathbb{T}}$ of *independent random variables*, not necessarily identically distributed, with zero mean and unit variance:

$$\mathbb{E}[\eta_t^N] = 0\,, \qquad \mathbb{E}[(\eta_t^N)^2] = 1\,. \tag{2.1}$$

We further require the *uniformly integrability of the squares*:

$$\lim_{L\to\infty} \sup_{N\in\mathbb{N},\, t\in\mathbb{T}} \mathbb{E}\Big[|\eta_t^N|^2\, \mathbb{1}_{\{|\eta_t^N|>L\}}\Big] = 0\,, \tag{2.2}$$

which follows from (2.1) if the η_t^N's have the same distribution. In general, an easy sufficient condition for (2.2) is that $\sup_{N,t} \mathbb{E}[|\eta_t^N|^p] < \infty$ for some $p > 2$.

We consider a sequence of random variables $(X_N)_{N\in\mathbb{N}}$ that are polynomial chaos, i.e. multi-linear polynomials in the η_t^N's. More precisely, we assume that

$$X_N = \sum_{A\subset\mathbb{T}} q_N(A)\, \eta^N(A)\,, \qquad \text{with} \qquad \eta^N(A) := \prod_{t\in A} \eta_t^N\,, \tag{2.3}$$

where $q_N(\cdot)$ are real coefficients and the sum ranges over *finite nonempty subsets* $A \subset \mathbb{T}$ (i.e. $q_N(A) \neq 0$ only if $0 < |A| < \infty$). We can split the sum according to the cardinality of A:

$$X_N = \sum_{k=1}^\infty \sum_{\substack{\{t_1,\ldots,t_k\}\subset\mathbb{T} \\ t_i\neq t_j\ \forall i\neq j}} q_N(\{t_1,\ldots,t_k\}) \prod_{i=1}^k \eta_{t_i}^N\,. \tag{2.4}$$

We assume that $\sum_{A\subset\mathbb{T}} q_N(A)^2 < \infty$, so that X_N is a well-defined random variable with

$$\mathbb{E}[X_N] = 0, \qquad \mathbb{E}[X_N^2] = \sum_{A\subset\mathbb{T}} q_N(A)^2\,, \tag{2.5}$$

because $(\eta^N(A))_{A\subset\mathbb{T}}$ are centered and orthogonal random variables in L^2.

Our goal is to prove *convergence in distribution of* X_N *toward a Gaussian random variable.* This can be achieved via the celebrated *Fourth Moment Theorem*, formulated in our context in [NPR10] and slightly extended in [CSZ17b, Theorem 4.2]; see also the previous works

[NuaPec05, deJ90, deJ87, Rot79] and the book [NouPec12]. The Fourth Moment Theorem deals with a sequence X_N of polynomial chaos in a *fixed order chaos* (i.e. a single term k in (2.4)) and it requires to compute the *second and fourth moments of X_N*.

Our first main result gives sufficient conditions for convergence to a Gaussian limit *only based on second moment assumptions on X_N*, which can be directly applied to a superposition of chaos of different orders. Let us introduce the shorthand

$$\sigma_N^2(\mathbb{B}) := \sum_{A \subset \mathbb{B}} q_N(A)^2 \qquad \text{for} \quad \mathbb{B} \subset \mathbb{T}\,, \tag{2.6}$$

which gives the contribution to the second moment of X_N of the subsets of $\mathbb{B}$ (recall (2.5)). We can formulate our conditions as follows.

(1) *Limiting second moment*:

$$\lim_{N\to\infty} \sigma_N^2(\mathbb{T}) = \lim_{N\to\infty} \sum_{A \subset \mathbb{T}} q_N(A)^2 \;=\; \sigma^2 \in (0,\infty)\,, \tag{2.7}$$

i.e. the second moment of X_N converges to a finite limit.

(2) *Subcriticality*:

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 = 0\,, \tag{2.8}$$

i.e. the contribution of high order chaos to the second moment of X_N is negligible.

(3) *Spectral localization*: for any $M, N \in \mathbb{N}$ we can find M *disjoint subsets ("boxes")*:

$$\mathbb{B}_1, \ldots, \mathbb{B}_M \subset \mathbb{T} \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \varnothing \quad \text{for } i \neq j\,,$$

where $\mathbb{B}_i = \mathbb{B}_i^{(N,M)}$, such that the following conditions hold (recall (2.6)):

$$\lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \sigma_N^2(\mathbb{B}_i) \;=\; \sigma^2\,, \tag{2.9}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} \Big\{ \max_{i=1,\ldots,M} \sigma_N^2(\mathbb{B}_i) \Big\} \;=\; 0\,, \tag{2.10}$$

i.e. the main contribution to the second moment of X_N comes from *subsets contained in one of the boxes* $\mathbb{B}_1, \ldots, \mathbb{B}_M$, whose individual contribution is uniformly small.

Note that conditions (1), (2), (3) are *second moment assumptions.* The name "subcriticality" for condition (2) is inspired by directed polymers, that we discuss in Section 3, and more generally by marginally relevant disordered systems, see [CSZ17a], which undergo a phase transition at a critical point determined precisely by the failure of condition (2.8).

We can now state our first main result.

Theorem 2.1 (Gaussian limits for polynomial chaos). *Let X_N be polynomial chaos as in* (2.3)*, with coefficients $q_N(\cdot)$ satisfying the assumptions* (1)*,* (2)*,* (3) *(see* (2.7)*–*(2.10)*), with respect to independent random variables $\eta^N = (\eta_t^N)_{t\in\mathbb{T}}$ which satisfy* (2.1) *and* (2.2)*. Then as $N \to \infty$ we have the convergence in distribution*

$$X_N \xrightarrow{d} \mathcal{N}(0, \sigma^2)\,. \tag{2.11}$$

The proof is given in Section 4 and comes in two steps:

- first we approximate X_N in L^2 by a sum $\sum_{i=1}^M X_{N,i}$ of *independent* random variables, for a suitable $M = M_N \to \infty$;

- then we show that the random variables $(X_{N,i})_{1\leqslant i\leqslant M_N}$ satisfy the assumption of the *Central Limit Theorem for triangular arrays*, which eventually yields (2.11).

We will also replace the random variables (η_t^N) by a family of random variables with bounded p-moments for some $p>2$ (e.g. by Gaussians) to exploit the hypercontractivity of polynomial chaos, see [MOO10]. The justification of this replacement will be given at the end of the proof exploiting a suitable Lindeberg principle, see [MOO10, CSZ17a].

2.2. Wiener chaos. Theorem 2.1 has a direct translation for Wiener chaos. Let $(E,\mathcal{E},\mu)$ be a Polish (complete separable metric) space, endowed with its Borel σ-field $\mathcal{E}$ and with a non-atomic measure μ. Let $\mathcal{E}^* = \{A\in\mathcal{E}:\ \mu(A)<\infty\}$ be the class of measurable sets with finite measure. By *Gaussian random measure* on $(E,\mathcal{E},\mu)$ we mean a centered Gaussian process $W=(W(A))_{A\in\mathcal{E}^*}$ with $\mathbb{C}\mathrm{ov}[W(A),W(B)]=\mu(A\cap B)$, defined on some probability space $(\Omega,\mathcal{A},\mathbb{P})$. We often use the informal notation $W(\mathrm{d}x)$. The most important example is given by *white noise*, which corresponds to $E=\mathbb{R}^d$ with $\mu=$ Lebesgue measure.

We fix a Gaussian random measure $W(\mathrm{d}x)$ on $(E,\mathcal{E},\mu)$. For every $k\in\mathbb{N}$ and every real function $f\in L^2(E^k,\mu^{\otimes k})$, by [Ito51, NouPec12] we can define the stochastic integral

$$W^{\otimes k}(f)=\int_{E^k} f(x_1,\ldots,x_k)\,W(\mathrm{d}x_1)\cdots W(\mathrm{d}x_k)$$

which is a centered random variable in $L^2(\Omega)$ (non Gaussian as soon as $k>1$ and $f\not\equiv 0$). For *symmetric* functions $f\in L^2(E^k,\mu^{\otimes k})$ and $g\in L^2(E^{k'},\mu^{\otimes k'})$ we have the *Ito isometry*:

$$\begin{aligned}\mathbb{E}[W^{\otimes k}(f)\,W^{\otimes k'}(g)] &= \mathbb{1}_{\{k=k'\}}\,k!\,\langle f,g\rangle_{L^2(E^k,\mu^{\otimes k})}\\ &= \mathbb{1}_{\{k=k'\}}\,k!\int_{E^k} f(x_1,\ldots,x_k)\,g(x_1,\ldots,x_k)\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\,.\end{aligned}\tag{2.12}$$

In this "continuum setting", in analogy with the discrete polynomial chaos (2.4), we consider a sequence $(\tilde{X}_N)_{N\in\mathbb{N}}$ of Wiener chaos with respect to $W(\mathrm{d}x)$, that is

$$\tilde{X}_N=\sum_{k=1}^{\infty}\int_{E^k}\tilde{q}_N(x_1,\ldots,x_k)\,W(\mathrm{d}x_1)\cdots W(\mathrm{d}x_k)\,,\tag{2.13}$$

where $\tilde{q}_N$ is a *symmetric* L^2 function defined on $\bigcup_{k=1}^{\infty}(E^k,\mathcal{E}^{\otimes k},\mu^{\otimes k})$. Then, by (2.12),

$$\mathbb{E}[\tilde{X}_N]=0\,,\qquad \mathbb{E}[\tilde{X}_N^2]=\sum_{k=1}^{\infty}k!\,\|\tilde{q}_N\|^2_{L^2(E^k)}=\sum_{k=1}^{\infty}k!\int_{E^k}\tilde{q}_N(x_1,\ldots,x_k)^2\,\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_k)\,.\tag{2.14}$$

Remark 2.2. *Every centered random variable in* $L^2(\Omega)$*, which is measurable with respect to the* σ*-algebra generated by* W*, admits an expansion like* (2.13)*.*

Remark 2.3. *The factor* $k!$ *in* (2.14) *is due to the fact that* $\tilde{q}_N$ *in* (2.13) *is a symmetric function of the* ordered *variables* $x_1,\ldots,x_k$*, whereas* q_N *in* (2.4) *is a function of* unordered *variables (i.e. subsets)* $\{t_1,\ldots,t_k\}$*. To formally match* (2.4)-(2.5) *with* (2.13)-(2.14)*, we should identify* q_N *with* $k!\,\tilde{q}_N$ *and furthermore* $\sum_{\{t_1,\ldots,t_k\}\subset\mathbb{T}}$ *with* $\frac{1}{k!}\int_{E^k}W(\mathrm{d}x_1)\cdots W(\mathrm{d}x_k)$*.*

Mimicking (2.6), we set

$$\tilde{\sigma}_N^2(\mathbb{B}) := \sum_{k=1}^{\infty} k! \int_{\mathbb{B}^k} \tilde{q}_N(x_1, \ldots, x_k)^2 \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k) \qquad \text{for measurable } \mathbb{B} \subset E \,, \tag{2.15}$$

which gives the contribution to the second moment of X_N of subsets in $\mathbb{B}$, see (2.14). We can now formulate our conditions in the continuum setting.

($\tilde{1}$) *Limiting second moment*:

$$\lim_{N\to\infty} \tilde{\sigma}_N^2(\mathbb{T}) \;=\; \lim_{N\to\infty} \sum_{k=1}^{\infty} k! \, \|\tilde{q}_N\|_{L^2(E^k)}^2 \;=\; \sigma^2 \in (0,\infty) \,, \tag{2.16}$$

i.e. the second moment of $\tilde{X}_N$ converges to a finite limit.

($\tilde{2}$) *Subcriticality*:

$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{k>K} k! \, \|\tilde{q}_N\|_{L^2(E^k)}^2 = 0 \,, \tag{2.17}$$

i.e. the contribution of high order chaos to the second moment of $\tilde{X}_N$ is negligible.

($\tilde{3}$) *Spectral localization*: for any $M, N \in \mathbb{N}$ we can find M *disjoint subsets ("boxes")*:

$$\mathbb{B}_1, \ldots, \mathbb{B}_M \subset E \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \varnothing \quad \text{for } i \neq j$$

(where $\mathbb{B}_i = \mathbb{B}_i^{(N,M)}$ may depend on N, M) such that, recalling (2.15),

$$\lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \tilde{\sigma}_N^2(\mathbb{B}_i) \;=\; \sigma^2 \,, \tag{2.18}$$

$$\lim_{M\to\infty} \lim_{N\to\infty} \Big\{ \max_{i=1,\ldots,M} \tilde{\sigma}_N^2(\mathbb{B}_i) \Big\} \;=\; 0 \,, \tag{2.19}$$

i.e. the main contribution to the second moment of $\tilde{X}_N$ comes from *subsets contained in one of the M boxes* $\mathbb{B}_1, \ldots, \mathbb{B}_M$, whose individual contribution is uniformly small.

We can finally state the version of Theorem 2.1 for Wiener chaos. We omit the proof because it follows very closely that of Theorem 2.1, given in Section 4.

Theorem 2.4 (Gaussian limits for Wiener chaos). *Let $\tilde{X}_N$ be Wiener chaos as in* (2.13)*, with coefficients $\tilde{q}_N(\cdot)$ satisfying the assumptions* ($\tilde{1}$)*,* ($\tilde{2}$)*,* ($\tilde{3}$) *(see* (2.16)–(2.19)*), with respect to a Gaussian random measure $W(\mathrm{d}x)$ on a Polish measure space $(E, \mathcal{E}, \mu)$. Then as $N \to \infty$ we have the convergence in distribution*

$$\tilde{X}_N \xrightarrow{d} \mathcal{N}(0, \sigma^2) \,. \tag{2.20}$$

3. Applications to directed polymers

We now present applications of our convergence results in Section 2 to directed polymers in random environment on $\mathbb{Z}^2$.

3.1. Directed polymers and stochastic PDEs. Let $S = (S_n)_{n \geqslant 0}$ be the simple symmetric random walk on $\mathbb{Z}^2$, whose law we denote by P. Let $\omega = (\omega(n,x))_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ be a family of i.i.d. random variables, independent of S, with law $\mathbb{P}$ and such that

$$\mathbb{E}[\omega(n,x)] = 0 \,, \qquad \mathbb{E}[\omega(n,x)^2] = 1 \,, \qquad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega(n,x)}] < \infty \quad \forall \beta > 0 \,. \tag{3.1}$$

Intuitively, trajectories of the random walk S represent polymer configurations, while configurations ω describe the *disorder*, which plays the role of a *random environment*. Given a scale parameter $N \in \mathbb{N}$, a starting time-space point $(m,z) \in \{0,\ldots,N\} \times \mathbb{Z}^2$ and an interaction strength $\beta > 0$, the partition function of the directed polymer model is

$$Z_N^{\beta}(m,z) := \mathrm{E}\Big[\mathrm{e}^{\sum_{n=m+1}^{N}(\beta\omega(n,S_n)-\lambda(\beta))} \,\Big|\, S_m = z\Big] \,. \tag{3.2}$$

Directed polymers were originally introduced as an effective interface model in the framework of the Ising model with impurities, but over the years they have become an object of independent study and a prototype of a disorder system which is amenable to detailed rigorous investigation. We refer to the monograph by Comets [Com17] for a recent account.

A source of interest for directed polymers is their link with the multiplicative Stochastic Heat Equation (SHE), which is the stochastic PDE formally written as follows:

$$\partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) \;+\; \beta\, \dot W(t,x)\, u(t,x) \,, \tag{3.3}$$

where $\beta > 0$ tunes the interaction strength and $\dot W(t,x)$ denotes white noise on $(0,\infty) \times \mathbb{R}^2$. In one space dimension $d = 1$, this equation admits a rigorous integral formulation by the classical Ito-Walsh integration. In higher dimensions $d \geqslant 2$, this approach fails due to strong irregularity of white noise and no obvious meaning can be given to its solution $u(t,x)$.

By the Markov property of simple random walk, the diffusively rescaled partition function

$$U_N(t,x) := Z_N^{\beta}(\lfloor Nt \rfloor, \lfloor \sqrt{N}x \rfloor) \tag{3.4}$$

solves a discretized version of (3.3) (with ∂_t and $\frac{1}{2}\Delta_x$ replaced by $-\partial_t$ and $\frac{1}{4}\Delta_x$, see (3.24) below). This explains the interest for the convergence as $N \to \infty$ of $U_N(t,x)$, possibly for suitable $\beta = \beta_N$, since it provides an approximation of the ill-defined SHE solution $u(t,x)$.

It is also very interesting to look at the *logarithm of the partition function*

$$\log Z_N^{\beta}(\lfloor Nt \rfloor, \lfloor \sqrt{N}x \rfloor)$$

because it provides an approximation for the solution $h(t,x) = \log u(t,x)$ of the Kardar-Parisi-Zhang equation (KPZ), which is the stochastic PDE formally given by

$$\partial_t h(t,x) = \frac{1}{2}\Delta_x h(t,x) \;+\; \frac{1}{2}|\nabla_x h(t,x)|^2 \;+\; \beta\, \dot W(t,x) \text{ “} - \infty \text{”} \,, \tag{3.5}$$

where the last term “$-\infty$” indicates a form of renormalization.

Remark 3.1 (Edwards-Wilkinson equation). *The Stochastic Heat Equation* (3.3) *is singular due to the* multiplicative noise term $\dot W u$. *The additive version of this equation, known as* Edwards-Wilkinson equation, *is well-posed and reads as follows:*

$$\partial_t v(t,x) = \frac{s}{2}\Delta_x v(t,x) + c\, \dot W(t,x) \,, \tag{3.6}$$

where $s > 0$ and $c \in \mathbb{R}$ are given parameters. Starting from $v(0,\cdot) \equiv 0$, the solution $v = v^{(s,c)}$ is a random distribution *(i.e. generalized function) which is Gaussian with explicit* *covariance, see* [CSZ20, Remark 1.5]*. More precisely, if we denote by* $\langle v^{(s,c)}, \psi\rangle$ *the pairing between the distribution* $v^{(s,c)}$ *and a test function* ψ*, which formally corresponds to*

$$\langle v^{(s,c)}, \psi\rangle := \int_{\mathbb{R}^2} v^{(s,c)}(t,x)\, \psi(t,x)\, \mathrm{d}t\, \mathrm{d}x\,, \tag{3.7}$$

then $\langle v^{(s,c)}, \psi\rangle$ *for* $\psi \in C_c^\infty([0,\infty)\times\mathbb{R}^2)$ *is a centered Gaussian process with*

$$\mathbb{C}\mathrm{ov}\left[\langle v^{(s,c)}, \psi\rangle, \langle v^{(s,c)}, \psi'\rangle\right] = \int_{([0,\infty)\times\mathbb{R}^2)^2} \psi(t,x)\, K_{t,t'}^{(s,c)}(x,x')\, \psi'(t',x')\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}t'\, \mathrm{d}x'\,, \tag{3.8}$$

where the covariance kernel is given by

$$K_{t,t'}^{(s,c)}(x,x') := \frac{s\, c^2}{2}\int_{s|t-t'|}^{s(t+t')} g_u(x-x')\, \mathrm{d}u\,, \qquad \textit{where} \qquad g_u(y) := \frac{\mathrm{e}^{-\frac{|y|^2}{2u}}}{2\pi u}\,. \tag{3.9}$$

3.2. Edwards-Wilkinson fluctuations. Let us define

$$u_n := \sum_{z\in\mathbb{Z}^2} \mathrm{P}(S_n = z)^2 = \mathrm{P}(S_{2n}=0) \sim \frac{1}{\pi}\frac{1}{n}\,, \tag{3.10}$$

$$R_N := \sum_{n=1}^N \sum_{z\in\mathbb{Z}^2} \mathrm{P}(S_n = z)^2 = \sum_{n=1}^N u_n \sim \frac{1}{\pi}\log N\,, \tag{3.11}$$

where the asymptotic relations (respectively as $n\to\infty$ and as $N\to\infty$) follow by the local central limit theorem (see (A.14) below). Henceforth we are going to fix $\beta = \beta_N$ given by

$$\beta_N \sim \frac{\hat\beta}{\sqrt{R_N}} \sim \frac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{with} \qquad \hat\beta \in (0,1)\,, \tag{3.12}$$

also known as the *sub-critical regime*. This ensures that the partition function $Z_N^{\beta_N}$ has a bounded second moment as $N\to\infty$, see [CSZ17b]. It was recently shown in [LZ21+, CZ21+] that in fact *all moments of* $Z_N^{\beta_N}$ *are bounded in this regime.*

We look at the fluctuations of the diffusively rescaled partition function, encoded by

$$V_N(t,x) := \frac{1}{\beta_N}\big(Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor) - 1\big) \qquad \text{for} \quad (t,x)\in[0,1]\times\mathbb{R}^2\,. \tag{3.13}$$

It was shown in [CSZ17b, Theorem 2.13] that $Z_N^{\beta_N}$ exhibits *Edwards-Wilkinson fluctuations*, because $V_N(t,x)$ converges as $N\to\infty$ to a solution of the Edwards-Wilkinson equation (3.6):

$$V_N(t,x) \overset{\mathcal{D}}{\Longrightarrow} \tilde v(t,x) := v^{(\frac{1}{2}, c_{\hat\beta})}(1-t,x) \qquad \text{where} \quad c_{\hat\beta} := \sqrt{\frac{1}{1-\hat\beta^2}}\,, \tag{3.14}$$

where "$\overset{\mathcal{D}}{\Longrightarrow}$" denotes convergence in law *as a random distribution*:[†] for $\psi\in C_c([0,1]\times\mathbb{R}^2)$

$$\langle V_N, \psi\rangle := \int_{\mathbb{R}\times\mathbb{R}^2} V_N(t,x)\, \psi(t,x)\, \mathrm{d}t\, \mathrm{d}x \xrightarrow{\ d\ } \langle \tilde v, \psi\rangle\,. \tag{3.15}$$

The convergence (3.14) was proved in [CSZ17b] using the Fourth Moment Theorem, based on a polynomial chaos expansion of the partition function, see (3.30) below. Remarkably, our Theorem 2.1 allows for an *alternative and more elementary proof of* (3.14)*, based on second moments calculations.* The details will be presented in [Cot23].

[†]By the Cramér-Wold device [Bil95, Theorem 29.4], relation (3.15) implies convergence of all finite-dimensional distributions of the random field $(\langle V_N, \psi\rangle)_\psi$ toward $\langle \tilde v, \psi\rangle$.

Remark 3.2. *The factor* $\frac{1}{2}$ *in the parameters of* $\tilde{v}(t,x) = v^{(\frac{1}{2}, c_{\hat\beta})}(1-t,x)$*, see* (3.14)*, is due to the fact that* $\mathrm{E}[S_1^{(i)}, S_1^{(j)}] = \frac{1}{2}\mathbb{1}_{i=j}$ *for* $i,j \in \{1,2\}$*. In view of* (3.6)*, note that* $\tilde{v}$ *satisfies*

$$- \partial_t \tilde{v}(t,x) = \frac{1}{4}\Delta_x \tilde{v}(t,x) + c_{\hat\beta}\, \dot{W}(t,x)\,. \tag{3.16}$$

Edwards-Wilkinson fluctuations also hold for the logarithm of the partition function, suitably centered and rescaled as in (3.13):

$$H_N(t,x) := \frac{1}{\beta_N}\Big(\log Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor) - \mathbb{E}\big[\log Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor)\big]\Big)\,. \tag{3.17}$$

Indeed, it was shown in [CSZ20, Theorem 1.6] that a precise analogue of (3.14) holds:

$$H_N(t,x) \xrightarrow{\mathcal{D}} \tilde{v}(t,x) = v^{(\frac{1}{2}, c_{\hat\beta})}(1-t,x)\,. \tag{3.18}$$

This convergence was in fact *deduced* in [CSZ20] from (3.14) by means of a highly non trivial linearization procedure. The alternative and more elementary proof of (3.14) based on our Theorem 2.1 can then be transferred to yield a proof of (3.18) as well. We refrain from giving the details, which will be presented in [Cot23].

Remark 3.3. *A simultaneous and independent proof of* (3.18) *was given in* [G20] *for small* $\hat\beta > 0$ *in a closely related context, namely for the KPZ equation* (3.5) *where the noise* $\dot{W}(t,x)$ *is regularized by mollification (rather than by discretization, as we consider here). Previously, the existence of non-trivial subsequential limits had been shown in* [CD20]*. We refer to* [DG20+, NN21+] *for some recent extensions and generalizations.*

In this paper, we exploit Theorem 2.1 to prove two new Gaussian convergence results related to the partition function, that we now describe.

3.3. Main result I (singular product). The diffusively rescaled partition function $U_N(t,x)$ in (3.4) approximates the solution of the Stochastic Heat Equation (3.3) with *multiplicative* noise. It is not clear a priori why the fluctuations of $U_N(t,x)$, encoded by $V_N(t,x)$ in (3.13), converge to $\tilde{v}(t,x)$ which solves the Stochastic Heat Equation with *additive* noise, see (3.16), with an intensity $c_{\hat\beta}$ which *explodes* as $\hat\beta \uparrow 1$. We now present a result which sheds light on the mechanism which leads to (3.16).

Let us introduce a modified disorder $\eta_N = (\eta_N(m,z))_{m\in\mathbb{N}, z\in\mathbb{Z}^2}$, recalling (3.1):

$$\eta_N(m,z) := \frac{\mathrm{e}^{\beta_N\omega(m,z)-\lambda(\beta_N)} - 1}{\sigma_N} \qquad \text{where} \qquad \sigma_N^2 := \mathrm{e}^{\lambda(2\beta_N)-2\lambda(\beta_N)} - 1 \underset{N\to\infty}{\sim} \beta_N^2\,. \tag{3.19}$$

We denote by $\dot{W}_N(t,x)$, for $t>0$, $x \in \mathbb{R}^2$, the diffusively rescaled version of η_N:

$$\dot{W}_N(t,x) := N\, \eta_N(\lfloor Nt\rfloor, \lfloor\sqrt{N}x\rfloor)\,. \tag{3.20}$$

For any $N \in \mathbb{N}$, the modified disorder $\eta_N = (\eta_N(m,z))_{m\in\mathbb{N}, z\in\mathbb{Z}^2}$ is i.i.d. with $\mathbb{E}[\eta_N(m,z)] = 0$ and $\mathbb{E}[\eta_N(m,z)^2] = 1$, see (3.1), and higher moments of η_N are uniformly bounded (see [CSZ17a, eq. (6.7)]). It follows that $\dot{W}_N$ converges in law to the white noise:

$$\dot{W}_N(t,x) \xrightarrow{\mathcal{D}} \dot{W}(t,x)\,, \tag{3.21}$$

that is $\langle \dot{W}_N, \psi\rangle \xrightarrow{d} \langle \dot{W}, \psi\rangle \sim \mathcal{N}(0, \|\psi\|_{L^2}^2)$ as $N\to\infty$, for $\psi \in C_c^\infty([0,1]\times\mathbb{R}^2)$.

We now consider the product between $\dot W_N$ and $U_N(t,x) - 1$, i.e. the centered and diffusively rescaled partition function $Z_N^{\beta_N}(\lfloor Nt\rfloor, \lfloor \sqrt{N}x\rfloor) - 1$, see (3.4):

$$\begin{aligned} \Xi_N(t,x) &:= \dot W_N(t,x)\,\big(U_N(t,x) - 1\big) \\ &= \beta_N\,\dot W_N(t,x)\,V_N(t,x)\,, \end{aligned} \tag{3.22}$$

where we recall that $V_N(t,x) = \beta_N^{-1}(U_N(t,x) - 1)$ is defined in (3.13).

We know that $V_N \overset{\mathcal{D}}{\Longrightarrow} \tilde v$ and $\dot W_N \overset{\mathcal{D}}{\Longrightarrow} W$ as $N \to \infty$, see (3.15) and (3.21). Since $\beta_N \to 0$, one could expect that $\Xi_N \overset{\mathcal{D}}{\Longrightarrow} 0$, but *this turns out to be false.* The point is that V_N and $\dot W_N$ only converge as random distributions, and the product of distributions is not a continuous operation (it is generally not even defined). The following result shows that Ξ_N has in fact a non-trivial limit as $N \to \infty$. We prove it in Section 5 as an application of our Theorem 2.1.

Theorem 3.4 (White noise from singular product). *Let $\beta = \beta_N$ be fixed as in* (3.12), *and set $c_{\hat\beta} := (1-\hat\beta^2)^{-1/2}$. As $N \to \infty$, we have the joint convergence in law:*

$$(\dot W_N, \Xi_N) \overset{\mathcal{D}}{\Longrightarrow} \Big(\dot W,\, \sqrt{c_{\hat\beta}^2 - 1}\,\dot W'\Big)\,,$$

where $\dot W$ and $\dot W'$ denote two independent white noises on $[0,1]\times\mathbb{R}^2$. More precisely, for any $\psi \in C_c^\infty([0,1]\times\mathbb{R}^2)$, the following joint convergence in distribution holds:

$$\big(\langle \dot W_N, \psi\rangle, \langle \Xi_N, \psi\rangle\big) \xrightarrow{d} \mathcal{N}\big(0, \|\psi\|_{L^2}^2\,\Sigma_{\hat\beta}\big) \qquad \textit{where} \qquad \Sigma_{\hat\beta} = \begin{pmatrix} 1 & 0 \\ 0 & c_{\hat\beta}^2 - 1\end{pmatrix}.$$

We can finally give a heuristic explanation for equation (3.16). One can check that $Z_N^{\beta_N}(m,z)$ in (3.2) solves the following *difference equation*, for $m \leqslant N$ and $z \in \mathbb{Z}^2$:

$$Z_N^{\beta_N}(m-1,z) - Z_N^{\beta_N}(m,z) = \frac{1}{4}\Delta_{\mathbb{Z}^2} Z_N^{\beta_N}(m,z) \,+\, \sigma_N\,\frac{1}{4}\sum_{z'\sim z} \eta_N(m,z')\,Z_N^{\beta_N}(m,z')\,, \tag{3.23}$$

where $z' \sim z$ means $z' \in \{z \pm (1,0), z \pm (0,1)\}$ and $\Delta_{\mathbb{Z}^2} f(z) := \sum_{z'\sim z}\{f(z') - f(z)\}$ denotes the lattice Laplacian (we recall that σ_N and $\eta_N(m,z)$ are defined in (3.19)).

By (3.13) and (3.20), we can rewrite (3.23) as follows, for $(t,x) \in ((0,1]\cap\frac{\mathbb{Z}}{N})\times(\mathbb{R}^2\cap\frac{\mathbb{Z}^2}{\sqrt{N}})$:

$$-\,\partial_t^{(N)} U_N(t,x) = \frac{1}{4}\Delta_x^{(N)} U_N(t,x) \,+\, \sigma_N\,\frac{1}{4}\sum_{x' \overset{N}{\sim} x} \dot W_N(t,x')\,U_N(t,x')\,, \tag{3.24}$$

where $x' \overset{N}{\sim} x$ means $x' \in \{x \pm (\frac{1}{\sqrt{N}}, 0), x \pm (0, \frac{1}{\sqrt{N}})\}$ and we define the rescaled operators

$$\begin{aligned} \partial_t^{(N)} f(t,x) &:= N\big\{f(t,x) - f(t-\tfrac{1}{N}, x)\big\}\,, \\ \Delta_x^{(N)} f(t,x) &:= N \sum_{x' \overset{N}{\sim} x} \big\{f(t,x') - f(t,x)\big\}\,. \end{aligned}$$

Note that (3.24) is a discretization of the (time reversed) Stochastic Heat Equation (3.3), with the factor $\frac{1}{4}$ instead of $\frac{1}{2}$ (see Remark 3.2) and with $\sigma_N \sim \beta_N$ in place of β.

We now consider $V_N(t,x) = \beta_N^{-1}(U_N(t,x) - 1)$, see (3.14). By (3.24) we obtain

$$-\partial_t^{(N)} V_N(t,x) = \frac{1}{4}\Delta_x^{(N)} V_N(t,x) + \frac{\sigma_N}{\beta_N}\frac{1}{4}\sum_{x' \overset{N}{\sim} x}\Big\{\dot W_N(t,x') + \beta_N\,\dot W_N(t,x')\,V_N(t,x')\Big\}\,. \tag{3.25}$$

The last term $\beta_N \dot{W}_N(t,x')\, V_N(t,x')$ is nothing but $\Xi_N(t,x')$ in (3.22), which formally vanishes as $N \to \infty$ but actually *converges to an independent white noise* $\sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'(t,x)$, by Theorem 3.4 (note that $x' \overset{N}{\sim} x$ implies $|x'-x| = 1/\sqrt{N} \to 0$). If we assume that $V_N(t,x)$ converges to a limit $\tilde{v}(t,x)$, by taking the formal limit of (3.25) we finally obtain

$$-\partial_t \tilde{v}(t,x) = \frac{1}{4}\Delta_x \tilde{v}(t,x) + \dot{W}(t,x) + \sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'(t,x) \,. \tag{3.26}$$

Note that *this is equivalent to* (3.16), because $\dot{W}(t,x) + \sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'(t,x) \overset{d}{=} c_{\hat\beta}\, \dot{W}(t,x)$.

In conclusion, Theorem 3.4 provides an intuitive explanation why the random field $\tilde{v}(t,x)$ to which $V_N(t,x)$ converges should satisfy the equation (3.16), or more precisely (3.26). The factor $c_{\hat\beta}$ in (3.16) arises from the *singular product* $\Xi_N(t,x) = \beta_N \dot{W}_N(t,x)\, V_N(t,x)$ which gives rise to an *independent white noise*, by Theorem 3.4.

This result is the first step toward a *"robust analysis"* of the two-dimensional SHE (3.3), which would allow for a rigorous derivation of (3.26) from (3.25).

3.4. Main result II (log-normality). So far we have discussed the distribution of the partition function $Z_N^{\beta_N}(m,z)$, suitably rescaled, as a *random field*, i.e. averaging over the starting point (m,z). We now look at the distribution of $Z_N^{\beta_N}(m,z)$ for *fixed* starting point: we fix $(m,z) = (0,0)$ by stationarity and we set

$$Z_N^{\beta_N} := Z_N^{\beta_N}(0,0) \,. \tag{3.27}$$

It was shown in [CSZ17b, Theorem 2.8] that $Z_N^{\beta_N}$ is *asymptotically log-normal*:

$$\log Z_N^{\beta_N} \xrightarrow{d} \mathcal{N}\big(-\tfrac{1}{2}\sigma_{\hat\beta}^2,\, \sigma_{\hat\beta}^2\big) \qquad \text{where} \qquad \sigma_{\hat\beta}^2 = \log c_{\hat\beta}^2 = \log \tfrac{1}{1-\hat\beta^2} \,. \tag{3.28}$$

The original proof of this result, based on the Fourth Moment Theorem, is long and technical. Our goal is to provide a less technical and more insightful proof, based on second moment computation, exploiting our Theorem 2.1. The problem is that, unlike for $Z_N^{\beta_N}$, *we do not have a polynomial chaos expansion for* $\log Z_N^{\beta_N}$, which is essential for Theorem 2.1. We solve this problem by first proving a result of independent interest, which shows that $\log Z_N^{\beta_N}$ is sharply approximated in L^2 by an explicit polynomial chaos expansion X_N^{dom}.

We need some setup. We recall that the modified disorder $(\eta_N(n,x))_{n\in\mathbb{N}, x\in\mathbb{Z}^2}$ was defined in (3.19). We also introduce the transition kernel of the simple random walk:

$$q_n(x) := \mathrm{P}(S_n = x \,|\, S_0 = 0) \tag{3.29}$$

and we recall the polynomial chaos expansion of the partition function [CSZ17a]:

$$Z_N^{\beta_N}(m,z) := 1 + \sum_{k=1}^{\infty} (\sigma_N)^k \sum_{\substack{m = n_0 < n_1 < \ldots < n_k \leqslant N \\ x_0 := z,\, x_1, \ldots, x_k \in \mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i - n_{i-1}}(x_i - x_{i-1})\, \eta_N(n_i, x_i) \,. \tag{3.30}$$

We define a new polynomial chaos expansion X_N^{dom}, obtained from the centered partition function $Z_N^{\beta_N} - 1 = Z_N^{\beta_N}(0,0) - 1$ imposing the constraint that *all increments* $n_i - n_{i-1}$ *for*

$i \geqslant 2$ *are dominated by the first time* n_1:

$$X_N^{\mathrm{dom}} := \sum_{k=1}^{\infty} (\sigma_N)^k \sum_{\substack{0=n_0<n_1<\ldots<n_k\leqslant N: \\ \max\{n_2-n_1, n_3-n_2, \ldots, n_k-n_{k-1}\}\leqslant n_1 \\ x_0:=0,\, x_1,\ldots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})\, \eta_N(n_i, x_i)\,. \tag{3.31}$$

Our key approximation result shows that X_N^{dom} *is a sharp approximation of* $\log Z_N^{\beta_N}$.

Theorem 3.5 (Polynomial chaos for $\log Z$). *Set* $\beta = \beta_N$ *as in* (3.12). *Then*

$$\lim_{N\to\infty} \big\| \log Z_N^{\beta_N} - \big\{ X_N^{\mathrm{dom}} - \tfrac{1}{2}\,\mathbb{E}[(X_N^{\mathrm{dom}})^2] \big\} \big\|_{L^2} \;=\; 0\,. \tag{3.32}$$

We then show, by our general Theorem 2.1, that X_N^{dom} is asymptotically Gaussian.

Theorem 3.6 (Asymptotic Gaussianity of X_N^{dom}). *Set* $\beta = \beta_N$ *as in* (3.12). *Then*

$$X_N^{\mathrm{dom}} \xrightarrow{d} \mathcal{N}\big(0, \sigma_{\hat\beta}^2\big) \qquad \textit{and} \qquad \lim_{N\to\infty} \mathbb{E}\big[(X_N^{\mathrm{dom}})^2\big] = \sigma_{\hat\beta}^2 = \log \tfrac{1}{1-\hat\beta^2}\,. \tag{3.33}$$

We prove Theorems 3.5 and 3.6 in Sections 6 and 7. Note that relations (3.32) and (3.33) together provide a strengthening of the asymptotic log-normality of $\log Z_N^{\beta_N}$, see (3.28).

3.5. Conclusions and perspectives. We discussed several convergences to a Gaussian limit for directed polymers: the Edwards-Wilkinson fluctuations (3.14) and (3.18), the singular product in Theorem 3.4 and the asymptotic log-normality in Theorem 3.6. We stress that these results hold in the *sub-critical regime* (3.12) with $\hat\beta < \hat\beta_c = 1$, while they break down in the critical regime $\hat\beta = 1$ (note that $c_{\hat\beta} \to \infty$ and $\sigma_{\hat\beta} \to \infty$ as $\hat\beta \uparrow 1$).

It would be interesting to understand whether these results can be suitably extended to a "nearly critical regime", i.e. when one takes $\hat\beta = \hat\beta_N \uparrow 1$ slowly enough, strictly below the *critical window* $\hat\beta = 1 + O(\frac{1}{\log N})$ studied in [BC98, GQT21, CSZ19b, CSZ21+]. We plan to investigate this issue in future work, building on the new proofs that we presented in this paper, which are more robust and suitable for generalization.

Another direction of research is about higher dimensions $d \geqslant 3$. The Edwards-Wilkinson fluctuations (3.14) and (3.18) have been proved for $d \geqslant 3$ in the so-called "L^2 regime" in [LZ20+] and [CNN20+], sharpening previous work from [MU18, GRZ18, CCM20, DGRZ20]; see also [CCM21+] for related recent results. It would be interesting to apply the approach of our paper in this higher dimensional context, to check whether it is possible to go slightly beyond the "L^2 regime" (cf. the "nearly critical regime" mentioned above for $d = 2$).

Finally, we point out that many of the cited works focus on the "continuum setting" of the SHE (3.3) and KPZ equation (3.5) where the noise $\dot W(t,x)$ is mollified (see also Remark 3.3). Our results of this section are formulated in the discrete setting of directed polymers, which correspond to the stochastic PDEs (3.3) and (3.5) where the noise $\dot W(t,x)$ is *discretized* rather than *mollified*, but we stress that our approach can also be applied to the continuum setting with mollification, using Theorem 2.4 instead of Theorem 2.1.

4. Proofs of Theorem 2.1

As a preliminary step to prove Theorem 2.1, *we replace the random variables* $(\eta_t^N)_{t\in\mathbb{T}}$ *in the definition* (2.3) *of* X_N *by independent standard Gaussians.* We will show in Subsection 4.4 that such a replacement does not affect the asymptotic distribution of X_N as $N \to \infty$.

We therefore assume that $\eta_t^N \sim \mathcal{N}(0,1)$. We then exploit the *hypercontractivity of polynomial chaos*, which allows us to bound moments of order $p > 2$ in terms of second moments, see [MOO10, Section 3.2] and [Jan97, Theorem 5.1]:

$$\forall p > 2: \qquad \mathbb{E}\bigg[\Big|\sum_{A\subset\mathbb{T}} q_N(A)\,\eta^N(A)\Big|^p\bigg] \leqslant \bigg(\sum_{A\subset\mathbb{T}} (p-1)^{|A|}\, q_N(A)^2\bigg)^{\frac{p}{2}}. \tag{4.1}$$

Remark 4.1. *The choice of a Gaussian distribution for the η_t^N's is not fundamental here: hypercontractivity of polynomial chaos holds for* arbitrary distributions of the η_t^N's with uniformly bounded moments*: if $\sup_{N,t}\mathbb{E}[|\eta_t^N|^{\bar p}] < \infty$ for some $\bar p > p$, then*

$$\mathbb{E}\bigg[\Big|\sum_{A\subset\mathbb{T}} q_N(A)\,\eta^N(A)\Big|^p\bigg] \leqslant \bigg(\sum_{A\subset\mathbb{T}} C_p^{|A|}\, q_N(A)^2\bigg)^{\frac{p}{2}}, \tag{4.2}$$

for a suitable $C_p < \infty$ with $\lim_{p\downarrow 2} C_p = 1$: see [CSZ20, Theorem B.1].

4.1. Preparation. We consider a sequence of polynomial chaos X_N, with coefficients $q_N(\cdot)$ as in (2.3), which satisfy assumptions (1), (2), (3), see the equations (2.7)-(2.10). We now build two suitable diverging sequences of integers $M_N \to \infty$, $K_N \to \infty$.

- We fix $M_N \to \infty$ slowly enough so that assumption (3) still holds with $M = M_N$. More explicitly, for every $N \in \mathbb{N}$ we can find disjoint subsets ("boxes") $\mathbb{B}_i = \mathbb{B}_i^{(N)}$:
$$\mathbb{B}_1, \ldots, \mathbb{B}_{M_N} \subset \mathbb{T} \qquad \text{with} \qquad \mathbb{B}_i \cap \mathbb{B}_j = \varnothing \quad \text{for } i \neq j\,,$$
such that the following versions of (2.9)-(2.10) hold:
$$\lim_{N\to\infty} \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i) \;=\; \sigma^2 \qquad \text{and} \qquad \lim_{N\to\infty} \Big\{\max_{i=1,\ldots,M_N} \sigma_N^2(\mathbb{B}_i)\Big\} \;=\; 0\,. \tag{4.3}$$
- By the second relation in (4.3), we can fix $K_N \to \infty$ slowly enough so that
$$\lim_{N\to\infty} 2^{K_N}\bigg(\max_{i=1,\ldots,M_N} \sigma_N^2(\mathbb{B}_i)\bigg)^{\frac{1}{3}} \;=\; 0\,. \tag{4.4}$$
The reason for this specific choice will be clear later, see the discussion after (4.14). Note that by our assumption (2), see (2.8), for any $K_N \to \infty$ we have
$$\lim_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\ |A|>K_N}} q_N(A)^2 = 0\,. \tag{4.5}$$

Remark 4.2. *It is standard to deduce* (4.3) *from* (2.9)-(2.10)*. Indeed, given any real sequence $a_{N,M}$ which admits the limits*

$$\lim_{M\to\infty}\limsup_{N\to\infty} a_{N,M} \;=\; \lim_{M\to\infty}\liminf_{N\to\infty} a_{N,M} \;=\; \alpha\,,$$

we can always choose $M = M_N \to \infty$ slowly enough so that $\lim_{N\to\infty} a_{N,M_N} = \alpha$*, as one can check directly. Then, to obtain* (4.3) *from* (2.9)-(2.10)*, it suffices to consider*

$$a_{N,M} = \sum_{i=1}^{M} \sigma_N^2\big(\mathbb{B}_i^{(N,M)}\big)\,, \qquad \textit{resp.} \qquad a_{N,M} = \max_{i=1,\ldots,M} \sigma_N^2\big(\mathbb{B}_i^{(N,M)}\big)\,.$$

We next proceed with the actual proof of Theorem 2.1. We follow the two steps outlined after the statement of Theorem 2.1:

- first we approximate the polynomial chaos X_N in (2.3) by a sum of suitable independent random variables, see Subsection 4.2;
- then we apply the Feller-Lindeberg CLT to obtain the asymptotic Gaussianity (2.11), see Subsection 4.3.

4.2. Approximation of X_N. We recall the notation $\eta^N(A) := \prod_{t\in A} \eta_t^N$, see (2.3). We define a triangular array of random variables $(X_{N,i})_{i=1,\ldots,M_N}$ by setting

$$X_{N,i} := \sum_{\substack{A\subset \mathbb{B}_i \\ |A|\leqslant K_N}} q_N(A)\, \eta^N(A) \qquad \text{for } i = 1,\ldots,M_N\,, \tag{4.6}$$

where we recall that $M_N \to \infty$ and $K_N \to \infty$ have been fixed so that (4.3)-(4.5) hold.

We now show that the sum $\sum_{i=1}^{M_N} X_{N,i}$ is a good approximation of X_N.

Lemma 4.3. *The following holds:*

$$\lim_{N\to\infty} \left\| X_N - \sum_{i=1}^{M_N} X_{N,i} \right\|_{L^2} = 0\,. \tag{4.7}$$

Proof. Let us define a modification of the random variables $X_{N,i}$ in (4.6), where we simply remove the constraint $|A| \leqslant K_N$:

$$\tilde{X}_{N,i} := \sum_{A\subset \mathbb{B}_i} q_N(A)\, \eta^N(A) \qquad \text{for } i = 1,\ldots,M_N\,.$$

We are going to show that

$$\lim_{N\to\infty} \left\| X_N - \sum_{i=1}^{M_N} \tilde{X}_{N,i} \right\|_{L^2} = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \left\| \sum_{i=1}^{M_N} \tilde{X}_{N,i} - \sum_{i=1}^{M_N} X_{N,i} \right\|_{L^2} = 0\,. \tag{4.8}$$

The first relation is a direct consequence of our assumptions (1) and (3). Indeed, since the boxes $\mathbb{B}_i$ are disjoint, the random variable $\sum_{i=1}^{M_N} \tilde{X}_{N,i}$ is the polynomial chaos where we only sum over subsets $A \subset \bigcup_{i=1}^{M_N} \mathbb{B}_i$, hence the difference $X_N - \sum_{i=1}^{M_N} \tilde{X}_{N,i}$ is orthogonal in L^2 to $\sum_{i=1}^{M_N} \tilde{X}_{N,i}$. As a consequence, recalling also (2.6), we can write

$$\left\| X_N - \sum_{i=1}^{M_N} \tilde{X}_{N,i} \right\|_{L^2}^2 = \|X_N\|_{L^2}^2 - \left\| \sum_{i=1}^{M_N} \tilde{X}_{N,i} \right\|_{L^2}^2 = \sum_{A\subset \mathbb{T}} q_N(A)^2 - \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i)\,,$$

hence the first relation in (4.8) follows by (2.7) and the first relation in (4.3).

The second relation in (4.8) follows by our assumption (2), see (4.5), because

$$\left\| \sum_{i=1}^{M_N} \tilde{X}_{N,i} - \sum_{i=1}^{M_N} X_{N,i} \right\|_{L^2}^2 = \sum_{i=1}^{M_N} \sum_{\substack{A\subset \mathbb{B}_i \\ |A|> K_N}} q_N(A)^2 \leqslant \sum_{\substack{A\subset \mathbb{T} \\ |A|> K_N}} q_N(A)^2\,.$$

This completes the proof. □

4.3. Asymptotic Gaussianity of X_N. In view of Lemma 4.3, to prove (2.11) it remains to prove the convergence in distribution

$$\sum_{i=1}^{M_N} X_{N,i} \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\sigma^2)\,. \tag{4.9}$$

Note that $(X_{N,i})_{i=1,\ldots,M_N}$ are *independent* random variables with zero mean and finite variance, see (4.6), because the boxes $\mathbb{B}_i \subset \mathbb{T}$ are disjoint. By the *Central Limit Theorem for triangular arrays* [Bil95, Theorem 27.2], it suffices to check the convergence of the variance:

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{M_N} X_{N,i}\right)^2\right] = \sigma^2\,, \tag{4.10}$$

and the *Lindeberg condition*:

$$\forall \varepsilon > 0: \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big(X_{N,i}\big)^2\,\mathbb{1}_{\{|X_{N,i}|>\varepsilon\}}\Big] \;=\; 0\,. \tag{4.11}$$

Relation (4.10) follows by Lemma 4.3, see (4.7), and our assumption (1), see (2.7). Next we are going to prove the following *Lyapunov condition*:

$$\text{for some } p > 2: \qquad \lim_{N\to\infty} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big] \;=\; 0\,, \tag{4.12}$$

which implies Lindeberg's condition (4.11) since $\mathbb{E}\big[\big(X_{N,i}\big)^2\,\mathbb{1}_{\{|X_{N,i}|>\varepsilon\}}\big] \leqslant \varepsilon^{2-p}\,\mathbb{E}\big[\big|X_{N,i}\big|^p\big]$.

To obtain (4.12), we apply the hypercontractivity bound (4.1) to $X_{N,i}$, see (4.6), to get

$$\mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]^{\frac{2}{p}} \;\leqslant\; \sum_{\substack{A\subset\mathbb{B}_i\\|A|\leqslant K_N}} (p-1)^{|A|}\, q_N(A)^2 \;\leqslant\; (p-1)^{K_N}\,\sigma_N^2(\mathbb{B}_i)\,, \tag{4.13}$$

where we recall that $\sigma_N^2(\mathbb{B}_i) = \sum_{A\subset\mathbb{B}_i} q_N(A)^2$. Then we can write, for any $p > 2$,

$$\begin{aligned} \sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big] \;&\leqslant\; \bigg(\max_{i=1,\ldots,M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]\bigg)^{1-\frac{2}{p}} \;\sum_{i=1}^{M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]^{\frac{2}{p}} \\ &\leqslant\; \left\{(p-1)^{K_N}\bigg(\max_{i=1,\ldots,M_N} \mathbb{E}\Big[\big|X_{N,i}\big|^p\Big]\bigg)^{1-\frac{2}{p}}\right\} \sum_{i=1}^{M_N}\sum_{A\subset\mathbb{B}_i} q_N(A)^2\,. \end{aligned} \tag{4.14}$$

If we fix $p = 3$, the term in brackets vanishes as $N \to \infty$, by our choice (4.4) of K_N and the bound (4.13). The last sum is bounded by $\mathbb{E}[X_N^2] = \sum_{A\subset\mathbb{T}} q_N(A)^2$ which is uniformly bounded, thanks to our assumption (2.7). This completes the proof of (4.12).

4.4. Switching to Gaussian random variables. We finally complete the proof of Theorem 2.1 by justifying the preliminary step: we show that replacing the random variables $(\eta_t^N)_{t\in\mathbb{T}}$ in (2.3) by standard Gaussians *does not change the asymptotic distribution of X_N*. More precisely, if $(\hat\eta_t)_{t\in\mathbb{T}}$ are independent $\mathcal{N}(0,1)$ and we set

$$\hat X_N = \sum_{A\subset\mathbb{T}} q_N(A)\,\hat\eta(A)\,, \qquad \text{with} \qquad \hat\eta(A) := \prod_{t\in A} \hat\eta_t\,, \tag{4.15}$$

it suffices to show that for every bounded and smooth $f : \mathbb{R} \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \big|\mathbb{E}[f(X_N)] \,-\, \mathbb{E}[f(\hat X_N)]\big| \;=\; 0\,. \tag{4.16}$$

Indeed, since $\hat{X}_N \xrightarrow{d} \mathcal{N}(0,\sigma^2)$ by the first part of the proof, (4.16) implies $X_N \xrightarrow{d} \mathcal{N}(0,\sigma^2)$.

We exploit the Lindeberg principle [CSZ17a, Theorem 2.6], which generalizes [MOO10], to show that $\mathbb{E}[f(X_N)]$ is close to $\mathbb{E}[f(\hat{X}_N)]$. Let us fix $f : \mathbb{R} \to \mathbb{R}$ of class C^3 with

$$C_f := \max\{\|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty\} < \infty . \tag{4.17}$$

For $L > 0$, denote by $m_2^{>L}$ the second moment tail of the random variables η_t^N and $\hat{\eta}_t$:

$$m_2^{>L} := \sup_{N\in\mathbb{N},\, t\in\mathbb{T}} \ \sup_{\xi\in\{\eta_t^N,\, \hat{\eta}_t\}} \mathbb{E}\big[|\xi|^2 \mathbb{1}_{|\xi|>L}\big] , \tag{4.18}$$

Let $\mathsf{C}_{X_N^{\leqslant K}}$, $\mathsf{C}_{X_N^{>K}}$ be the second moments of X_N truncated to chaos of order $\leqslant K$ and $> K$:

$$\mathsf{C}_{X_N^{\leqslant K}} := \sum_{\substack{A\subset\mathbb{T}\\ |A|\leqslant K}} q_N(A)^2 , \qquad \mathsf{C}_{X_N^{>K}} := \sum_{\substack{A\subset\mathbb{T}\\ |A|>K}} q_N(A)^2 . \tag{4.19}$$

Finally, define the *influence* of the variable $t \in \mathbb{T}$ on X_N by†

$$\mathrm{Inf}_t[X_N] := \sum_{\substack{A\subset\mathbb{T}\\ A\ni t}} q_N(A)^2 . \tag{4.20}$$

By [CSZ17a, Theorem 2.6], for any $L > 0$ such that $m_2^{>L} \leqslant \frac{1}{4}$ and for every $K \in \mathbb{N}$ we have

$$\begin{aligned} \big|\mathbb{E}[f(X_N)] - \mathbb{E}[f(\hat{X}_N)]\big| \leqslant C_f \Bigg\{ & 2\sqrt{\mathsf{C}_{X_N^{>K}}} + 16K^2\, \mathsf{C}_{X_N^{\leqslant K}}\, m_2^{>L} \\ & + 70^{K+1}\, \mathsf{C}_{X_N^{\leqslant K}}\, L^{3K} \max_{t\in\mathbb{T}} \sqrt{\mathrm{Inf}_t[X_N]} \Bigg\} . \end{aligned} \tag{4.21}$$

It remains to show that the r.h.s. of this expression is small as $N \to \infty$, to prove (4.16). We fix any $\varepsilon > 0$ and we argue as follows:

- by assumption (2.8), we can choose $K = K_\varepsilon$ such that $\limsup_{N\to\infty} \mathsf{C}_{X_N^{>K}} \leqslant \varepsilon$;
- by assumption (2.7), for any $K \in \mathbb{N}$ we can bound $\limsup_{N\to\infty} \mathsf{C}_{X_N^{\leqslant K}} \leqslant \sigma^2$;
- by assumption (2.2), we can choose $L = L_\varepsilon$ such that $m_2^{>L_\varepsilon} \leqslant \varepsilon/(K_\varepsilon^2\, \sigma^2)$;
- finally, we show below that

$$\limsup_{N\to\infty} \max_{t\in\mathbb{T}} \sqrt{\mathrm{Inf}_t[X_N]} = 0 . \tag{4.22}$$

As a consequence, when we plug $K = K_\varepsilon$ and $L = L_\varepsilon$ in (4.21) and we let $N \to \infty$, we get

$$\limsup_{N\to\infty} \big|\mathbb{E}[f(X_N)] - \mathbb{E}[f(\hat{X}_N)]\big| \leqslant C_f \left\{ 2\sqrt{\varepsilon} + 16\,\varepsilon \right\} ,$$

from which (4.16) follows because $\varepsilon > 0$ is arbitrary.

It only remains to prove (4.22). By assumption there are disjoint boxes $\mathbb{B}_1, \ldots, \mathbb{B}_{M_N} \subset \mathbb{T}$, with $M_N \to \infty$, such that relation (4.3) holds. In particular, recalling also (2.6) and (2.7), it follows that *subsets $A \subset \mathbb{T}$ not contained in any of the boxes $\mathbb{B}_i$ give a negligible contribution*:

$$\Delta_N := \sum_{\substack{A\subset\mathbb{T}:\\ A\not\subset\mathbb{B}_i\ \forall i=1,\ldots,M_N}} q_N(A)^2 = \sigma_N^2(\mathbb{T}) - \sum_{i=1}^{M_N} \sigma_N^2(\mathbb{B}_i) \xrightarrow[N\to\infty]{} 0 . \tag{4.23}$$

†Note that we can write $\mathrm{Inf}_t[X_N] = \mathbb{E}\Big[\mathbb{V}ar\Big[X_N(\eta)|(\eta_s^N)_{s\in\mathbb{T}\setminus t}\Big]\Big]$.

Recall now the definition (4.20) of $\mathrm{Inf}_t[X_N]$. Fix $t \in \mathbb{T}$ and a subset $A \subset \mathbb{T}$ which contains t, i.e. $A \ni t$. We distinguish two cases:

- if $t \notin \mathbb{B}_i$ for all $i = 1, \ldots, M_N$, then $A \ni t$ implies $A \not\subset \mathbb{B}_i$ for all $i = 1, \ldots, M_N$, hence by (4.23) we can bound $\mathrm{Inf}_t[X_N] \leqslant \Delta_N$;
- if $t \in \mathbb{B}_j$ for some (necessarily unique) $j = 1, \ldots, M_N$, then $A \ni t$ implies that either $A \subset \mathbb{B}_j$, or $A \not\subset \mathbb{B}_i$ for all $i = 1, \ldots, M_N$ (we cannot have $A \subset \mathbb{B}_i$ for some $i \neq j$), hence by (2.6) and (4.23) we can bound $\mathrm{Inf}_t[X_N] \leqslant \sigma_N^2(\mathbb{B}_j) + \Delta_N$.

It follows that

$$\max_{t \in \mathbb{T}} \, \mathrm{Inf}_t[X_N] \;\leqslant\; \max_{j=1,\ldots,M_N} \, \sigma_N^2(\mathbb{B}_j) \,+\, \Delta_N \,,$$

hence (4.22) follows by (4.3) and (4.23). The proof of Theorem 2.1 is complete. □

5. Proof of Theorem 3.4

5.1. Preparation. We need to show that

$$(\dot{W}_N, \Xi_N) \xRightarrow{\mathcal{D}} \Big(\dot{W}, \sqrt{c_{\hat\beta}^2 - 1}\, \dot{W}'\Big),$$

that is, for any fixed $\psi \in C_c^\infty([0,1] \times \mathbb{R}^2)$ we have

$$\begin{pmatrix} \langle \dot{W}_N, \psi \rangle \\ \langle \Xi_N, \psi \rangle \end{pmatrix} \xrightarrow{d} \mathcal{N}\big(0, \|\psi\|_{L^2}^2 \, \Sigma_{\hat\beta}\big) \qquad \text{where} \qquad \Sigma_{\hat\beta} = \begin{pmatrix} 1 & 0 \\ 0 & c_{\hat\beta}^2 - 1 \end{pmatrix}, \tag{5.1}$$

By the Cramér-Wold device [Bil95, Theorem 29.4], it suffices to show that for all $\lambda, \mu \in \mathbb{R}$

$$X_N := \mu \langle \dot{W}_N, \psi \rangle + \lambda \langle \Xi_N, \psi \rangle \xrightarrow{d} \mathcal{N}\Big(0, \sigma^2 := \|\psi\|_{L^2}^2 \big(\mu^2 + \lambda^2 \,(c_{\hat\beta}^2 - 1)\big)\Big). \tag{5.2}$$

To this purpose we are going to apply Theorem 2.1.

Recall the definitions (3.20) and (3.22) of $\dot{W}_N$ and Ξ_N (see also (3.13)), we can write

$$\begin{aligned} X_N &= N \int\limits_{(0,1] \times \mathbb{R}^2} \psi(t,x)\, \eta_N\big(\lfloor Nt \rfloor, \lfloor \sqrt{N} x \rfloor\big) \Big\{ \mu + \lambda \big(Z_N^{\beta_N}(\lfloor Nt \rfloor, \lfloor \sqrt{N} x \rfloor) - 1 \big) \Big\} \, \mathrm{d}t \, \mathrm{d}x \\ &= \frac{1}{N} \int\limits_{(0,N] \times \mathbb{R}^2} \psi\big(\tfrac{t}{N}, \tfrac{x}{\sqrt{N}}\big)\, \eta_N\big(\lfloor t \rfloor, \lfloor x \rfloor\big) \Big\{ \mu + \lambda \big(Z_N^{\beta_N}(\lfloor t \rfloor, \lfloor x \rfloor) - 1 \big) \Big\} \, \mathrm{d}t \, \mathrm{d}x \,. \end{aligned} \tag{5.3}$$

Let us define $\overline{\psi}_N : \mathbb{N} \times \mathbb{Z}^2 \to \mathbb{R}$ as the average of $\psi\big(\frac{\cdot}{N}, \frac{\cdot}{\sqrt{N}}\big)$ over cubes:

$$\overline{\psi}_N(n,z) := \int\limits_{(n-1,n] \times \{(z_1-1,z_1] \times (z_2-1,z_2]\}} \psi\big(\tfrac{t}{N}, \tfrac{x}{\sqrt{N}}\big) \, \mathrm{d}t \, \mathrm{d}x \qquad \text{for} \quad (n,z) \in \mathbb{N} \times \mathbb{Z}^2 \,. \tag{5.4}$$

Recalling the polynomial chaos expansion (3.30) of $Z_N^{\beta_N}(m,z)$, we can rewrite X_N as follows:

$$\begin{aligned} X_N = \frac{1}{N} \sum_{n_0=1}^{N} \sum_{x_0 \in \mathbb{Z}^2} & \overline{\psi}_N(n_0, x_0)\, \eta_N(n_0, x_0) \\ & \Bigg\{ \mu \,+\, \lambda \sum_{k=1}^{\infty} (\sigma_N)^k \sum_{\substack{n_0 < n_1 < \ldots < n_k \leqslant N \\ x_0, x_1, \ldots, x_k \in \mathbb{Z}^2}} \prod_{j=1}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1})\, \eta_N(n_j, x_j) \Bigg\} \,. \end{aligned}$$

Renaming $(n_0, \ldots, n_k)$ as $(n_1, \ldots, n_{k+1})$ and similarly $(x_0, \ldots, x_k)$ as $(x_1, \ldots, x_{k+1})$, and subsequently renaming $k+1$ as k, we obtain the compact expression

$$X_N = \frac{1}{N} \sum_{k=1}^{\infty} (\sigma_N)^{k-1} \sum_{\substack{0<n_1<\ldots<n_k \leqslant N \\ x_1,\ldots,x_k \in \mathbb{Z}^2}} f_N(n_1, x_1, \ldots, n_k, x_k) \prod_{j=1}^{k} \eta_N(n_j, x_j)\,, \tag{5.5}$$

where we set

$$f_N(n_1, x_1, \ldots, n_k, x_k) \;:=\; \big\{\mu\, \mathbb{1}_{\{k=1\}} + \lambda\, \mathbb{1}_{\{k \geqslant 2\}}\big\}\, \overline{\psi}_N(n_1, x_1) \prod_{j=2}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1})\,. \tag{5.6}$$

In conclusion, we can write $X_N = \sum_{A \subset \mathbb{T}} q_N(A)\, \eta^N(A)$ as in (2.3)-(2.4), with the following correspondences:

- the index set is $\mathbb{T} := \mathbb{N} \times \mathbb{Z}^2$;
- the random variables $\eta_t^N = \eta_N(m,z)$, for $t = (m,z) \in \mathbb{T}$, are defined in (3.19): they satisfy (2.1) by construction, while they satisfy (2.2) because $\sup_N \mathbb{E}[|\eta_N(m,z)|^p] < \infty$ for all $p < \infty$ by (3.1) (see [CSZ17a, eq. (6.7)]);
- the kernel $q_N(A)$, for $A := \{t_1, \ldots, t_k\} = \{(n_1, x_1), \ldots, (n_k, x_k)\} \subseteq \mathbb{T}$, is
$$q_N(A) = \frac{1}{N}\, (\sigma_N)^{k-1}\, f_N(n_1, x_1, \ldots, n_k, x_k)\, \mathbb{1}_{\{0<n_1<\ldots<n_k \leqslant N\}}\,.$$

By Theorem 2.1, to prove $X_N \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ as in (5.2), we check the following conditions.

(1) *Limiting second moment:* we need to prove that $\lim_{N\to\infty} \mathbb{E}[X_N^2] = \sigma^2$.

(2) *Subcriticality*: we need to show that
$$\lim_{K\to\infty} \limsup_{N\to\infty} \sum_{\substack{A \subset \mathbb{T} \\ |A|>K}} q_N(A)^2 = 0\,. \tag{5.7}$$

(3) *Spectral localization*: for any $M, N \in \mathbb{N}$ we define the disjoint subsets
$$\mathbb{B}_j := \big(\tfrac{j-1}{M} N, \tfrac{j}{M} N\big] \qquad \text{for } j = 1, \ldots, M\,,$$
and, recalling that $\sigma_N^2(\mathbb{B}_j) := \sum_{A \subset \mathbb{B}_j} q_N(A)^2$, we need to show that
$$\lim_{M\to\infty} \sum_{j=1}^{M} \lim_{N\to\infty} \sigma_N^2(\mathbb{B}_j) \;=\; \sigma^2 \qquad \text{and} \qquad \lim_{M\to\infty} \Big\{ \max_{j=1,\ldots,M} \limsup_{N\to\infty} \sigma_N^2(\mathbb{B}_j) \Big\} \;=\; 0\,. \tag{5.8}$$

5.2. Proof of (2)**.** We need to prove (5.7). For $K \geqslant 1$ we can write, by (5.5)-(5.6),

$$\sum_{\substack{A \subset \mathbb{T} \\ |A|>K}} q_N(A)^2 = \frac{\lambda^2}{N^2} \sum_{k>K} (\sigma_N^2)^{k-1} \sum_{\substack{0<n_1<\ldots<n_k \leqslant N \\ x_1,\ldots,x_k \in \mathbb{Z}^2}} \overline{\psi}_N(n_1, x_1)^2 \prod_{j=2}^{k} q_{n_j - n_{j-1}}(x_j - x_{j-1})^2. \tag{5.9}$$

We can enlarge the sums to $0 < m_j := n_j - n_{j-1} \leqslant N$ and change variables $y_j := x_j - x_{j-1}$, for $j = 2, \ldots, k$, to get the upper bound

$$\begin{aligned}\sum_{\substack{A\subset\mathbb{T}\\|A|>K}} q_N(A)^2 &\leqslant \frac{\lambda^2}{N^2} \sum_{k>K} (\sigma_N^2)^{k-1} \sum_{\substack{0<n_1\leqslant N\\x_1\in\mathbb{Z}^2}} \overline{\psi}_N(n_1,x_1)^2 \prod_{j=2}^k \Bigg\{ \sum_{\substack{0<m_j\leqslant N\\y_j\in\mathbb{Z}^2}} q_{m_j}(y_j)^2 \Bigg\} \\ &= \lambda^2 \Bigg\{ \frac{1}{N^2} \sum_{\substack{0<n_1\leqslant N\\x_1\in\mathbb{Z}^2}} \overline{\psi}_N(n_1,x_1)^2 \Bigg\} \frac{(\sigma_N^2\, R_N)^K}{1-\sigma_N^2\, R_N}\,,\end{aligned} \tag{5.10}$$

where we used $\sum_{0<m\leqslant N}\sum_{y\in\mathbb{Z}^2} q_m(y)^2 = \sum_{0<m\leqslant N} u(m) = R_N$, see (3.10)-(3.11). Since $\sigma_N^2 \sim \hat\beta^2/R_N$, see (3.12), by Riemann sum approximation from (5.4) we get

$$\limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\|A|>K}} q_N(A)^2 \leqslant \lambda^2 \Bigg\{ \int_{[0,1]\times\mathbb{R}^2} \psi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x \Bigg\} \frac{(\hat\beta^2)^K}{1-\hat\beta^2} = \lambda^2\, \|\psi\|_{L^2}^2\, \frac{(\hat\beta^2)^K}{1-\hat\beta^2}\,, \tag{5.11}$$

from which (5.7) follows.

5.3. Proof of (1) **and** (3)**.** We are going to show that for all $M \in \mathbb{N}$ and $j \in \{1, \ldots, M\}$

$$\lim_{N\to\infty} \sigma_N^2(\mathbb{B}_j) = \big(\mu^2 + \lambda^2(c_{\hat\beta}^2 - 1)\big) \int_{(\frac{j-1}{M},\frac{j}{M}]\times\mathbb{R}^2} \psi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x\,. \tag{5.12}$$

Note that this proves (5.8) and also (for $j = M = 1$) $\lim_{N\to\infty}\mathbb{E}[X_N^2] = \sigma^2$, see (5.2).

To compute $\sigma_N^2(\mathbb{B}_j) := \sum_{A\subset\mathbb{B}_j} q_N(A)^2$ we first consider the contribution of sets $A \subset \mathbb{B}_j$ with $|A| = 1$, that is $A = \{(n_1, x_1)\}$. Since $f_N(n_1,x_1) = \mu\,\overline{\psi}_N(n_1,x_1)$, see (5.6), we get

$$\sum_{A\subset\mathbb{B}_j,\,|A|=1} q_N(A)^2 = \frac{\mu^2}{N^2} \sum_{\substack{\frac{j-1}{M}N<n_1\leqslant\frac{j}{M}N\\x_1\in\mathbb{Z}^2}} \overline{\psi}_N(n_1,x_1)^2 \xrightarrow{N\to\infty} \mu^2 \int_{(\frac{j-1}{M},\frac{j}{M}]\times\mathbb{R}^2} \psi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x\,,$$

by Riemann sum approximation. Note that this matches with the first term in (5.12).

We next focus on sets $A \subset \mathbb{B}_j$ with $|A| > 1$. Note that $\sum_{A\subset\mathbb{B}_j,\,|A|>1} q_N(A)^2$ is given by (5.9) with $K = 1$ and with the sum restricted to $\frac{j-1}{M}N < n_1 < \ldots < n_k \leqslant \frac{j}{M}N$. Then, arguing as in (5.10), we obtain an analogue of (5.11):

$$\limsup_{N\to\infty} \sum_{A\subset\mathbb{B}_j,\,|A|>1} q_N(A)^2 \leqslant \lambda^2 \Bigg\{ \int_{(\frac{j-1}{M},\frac{j}{M}]\times\mathbb{R}^2} \psi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x \Bigg\} \frac{\hat\beta^2}{1-\hat\beta^2}\,,$$

which agrees with the second term in (5.12) because $\frac{\hat\beta^2}{1-\hat\beta^2} = c_{\hat\beta}^2 - 1$, see (3.14). To complete the proof, it suffices to prove a matching lower bound, that is

$$\liminf_{N\to\infty} \sum_{A\subset\mathbb{B}_j,\,|A|>1} q_N(A)^2 \geqslant \lambda^2 \Bigg\{ \int_{(\frac{j-1}{M},\frac{j}{M}]\times\mathbb{R}^2} \psi(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x \Bigg\} \frac{\hat\beta^2}{1-\hat\beta^2}\,, \tag{5.13}$$

Let us fix $H \in \mathbb{N}$ large, such that $\frac{1}{H} < \frac{1}{M}$. Starting from the expression (5.9) for $K = 1$ and with $\frac{j-1}{M}N < n_1 < \ldots < n_k \leqslant \frac{j}{M}N$, we get a lower bound by the following restrictions:

$$1 < k \leqslant H\,, \qquad \tfrac{j-1}{M}N < n_1 \leqslant \big(\tfrac{j}{M} - \tfrac{1}{H}\big)N\,, \qquad 0 < n_j - n_{j-1} \leqslant \tfrac{1}{H^2}N \quad \forall j = 2, \ldots, k\,,$$

which ensure that $n_k \leqslant n_1 + \sum_{j=2}^k (n_j - n_{j-1}) \leqslant (\frac{j}{M} - \frac{1}{H})N + H\frac{1}{H^2}N \leqslant \frac{j}{M}N$ as required. Then, similarly to (5.10), we get the following lower bound on $\sum_{A\subset \mathbb{B}_j,\,|A|>1} q_N(A)^2$:

$$\begin{aligned} &\frac{\lambda^2}{N^2}\sum_{k=2}^{H}(\sigma_N^2)^{k-1}\sum_{\substack{\frac{j-1}{M}<n_1\leqslant(\frac{j}{M}-\frac{1}{H})N\\ x_1\in\mathbb{Z}^2}}\overline{\psi}_N(n_1,x_1)^2\prod_{j=2}^{k}\Bigg\{\sum_{\substack{0<m_j\leqslant\frac{1}{H^2}N\\ y_j\in\mathbb{Z}^2}}q_{m_j}(y_j)^2\Bigg\}\\ &\quad=\Bigg\{\frac{\lambda^2}{N^2}\sum_{\substack{\frac{j-1}{M}<n_1\leqslant(\frac{j}{M}-\frac{1}{H})N\\ x_1\in\mathbb{Z}^2}}\overline{\psi}_N(n_1,x_1)^2\Bigg\}\frac{\sigma_N^2\,R_{N/H^2}-(\sigma_N^2\,R_{N/H^2})^H}{1-\sigma_N^2\,R_{N/H^2}}\,, \end{aligned} \tag{5.14}$$

where we recall that $\sum_{k=2}^H x^{k-1} = \frac{x-x^H}{1-x}$ for $|x|<1$. Since $R_{N/H^2} \sim R_N$ for fixed $H \in \mathbb{N}$, we have shown that

$$\liminf_{N\to\infty}\sum_{A\subset\mathbb{B}_j,\,|A|>1}q_N(A)^2 \geqslant \lambda^2\Bigg\{\int_{(\frac{j-1}{M},\frac{j}{M}-\frac{1}{H}]\times\mathbb{R}^2}\psi(t,x)^2\,\mathrm{d}t\,\mathrm{d}x\Bigg\}\frac{\hat\beta^2-(\hat\beta^2)^H}{1-\hat\beta^2}\,.$$

We can finally take the limit $H \to \infty$ to see that (5.13) holds. □

6. Proof of Theorem 3.5

The proof is organised in four parts: we give different approximations of the partition function $Z_N^{\beta_N}$ and of its logarithm, which will lead us to the proof of our goal (3.32). Let us present a general overview of the strategy.

Part 1 (record times). Let us define a "constrained version" $X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z')$ of X^{dom}_N from (3.31), where we fix $(n_0,n_1;n_k)=(a,b;b')$ and $(x_0,x_1;x_k)=(x,z;z')$:

$$\begin{aligned} X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z') \;:=\; &\sum_{k=1}^{\infty}(\sigma_N)^k\,q_{b-a}(z-x)\,\eta_N(b,z)\times\\ &\times\sum_{\substack{b=:n_1<n_2<\ldots<n_{k-1}<n_k=b'\\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\}\leqslant b}}\;\sum_{\substack{x_1=z,\,x_k=z',\\ x_2,\ldots,x_{k-1}\in\mathbb{Z}^2}}\prod_{i=2}^{k}q_{n_i-n_{i-1}}(x_i-x_{i-1})\,\eta_N(n_i,x_i)\,. \end{aligned} \tag{6.1}$$

(Note that if $b=b'$ only the terms $k=1$ contributes to the sum — and we must have $z=z'$, otherwise the sum vanishes — while if $b<b'$ only the terms $k\geqslant 2$ give a contribution.)

We first show that the partition function $Z_N^{\beta_N}$ in (3.30) can be written as a concatenation of products of $X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z')$'s corresponding to suitable *record times*, see Figure 1. The next result is proved in subsection 6.1.

Lemma 6.1 (Record times). *The following equality holds, with $(b_0',z_0'):=(0,0)$:*

$$Z_N^{\beta_N} \;=\; 1+\sum_{\ell=1}^{\infty}\sum_{\substack{0<b_1\leqslant b_1'<\ldots<b_\ell\leqslant b_\ell'\leqslant N:\\ b_i-b_{i-1}'>b_{i-1}\;\forall i=2,\ldots,\ell}}\;\sum_{\underline{z},\underline{z}'\in(\mathbb{Z}^2)^\ell}\;\prod_{i=1}^{\ell}X^{\mathrm{dom}}_{N,[b_{i-1}',b_i;b_i']}(z_{i-1}',z_i;z_i')\,, \tag{6.2}$$

where we use the shortcuts $\underline{z}=(z_1,\ldots,z_\ell)$ and $\underline{z}'=(z_1',\ldots,z_\ell')$.

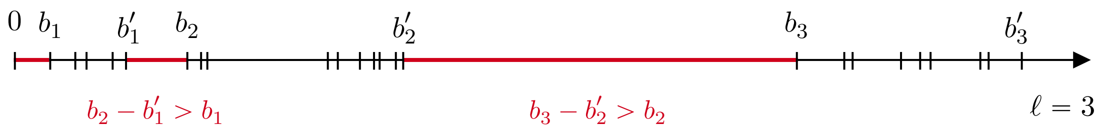

FIGURE 1. An example of the variables b_i, b_i' in (6.2). These correspond to *record times* which satisfy $b_i - b_{i-1}' > b_{i-1}$, see subsection 6.1.

Part 2 (coarse-graining and diffusive approximation). We fix a large parameter $M \in \mathbb{N}$ and we define an approximation $Z_{N,M}^{(\mathrm{diff})}$ of the partition function $Z_N^{\beta_N}$ from (6.2), as follows:[†]

(1) we set $b_{i-1}' = 0$, $z_{i-1}' = 0$ in each $X_{N,[b_{i-1}',b_i;b_i']}^{\mathrm{dom}}(z_{i-1}', z_i; z_i')$;

(2) we impose that each pair $b_i \leqslant b_i'$ belongs to the same interval $(N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]$, for some $j = 1, \ldots, M$, and we ignore the constraint $b_i - b_{i-1}' > b_{i-1}$.

This yields the following definition of $Z_{N,M}^{(\mathrm{diff})}$:

$$Z_{N,M}^{(\mathrm{diff})} := 1 + \sum_{\ell=1}^{\infty} \sum_{1 \leqslant j_1 < \ldots < j_\ell \leqslant M} \prod_{i=1}^{\ell} X_{N,M}^{\mathrm{dom}}(j_i) = \prod_{j=1}^{M} \Big(1 + X_{N,M}^{\mathrm{dom}}(j)\Big) , \tag{6.3}$$

where we set

$$X_{N,M}^{\mathrm{dom}}(j) := \sum_{b \leqslant b' \in (N^{\frac{j-1}{M}}, N^{\frac{j}{M}}]} \sum_{z,z' \in \mathbb{Z}^2} X_{N,[0,b;b']}^{\mathrm{dom}}(0, z; z') \qquad \text{for } j = 1, \ldots, M . \tag{6.4}$$

We prove that $Z_{N,M}^{(\mathrm{diff})}$ is close to $Z_N^{\beta_N}$ in L^2 for $N \gg M \gg 1$, in the following sense.

Lemma 6.2 (Coarse-graining and diffusive approximation). *The following holds:*

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \big\| Z_N^{\beta_N} - Z_{N,M}^{(\mathrm{diff})} \big\|_{L^2} = 0 . \tag{6.5}$$

The proof of this result is given in subsection 6.2 below.

Part 3 (log approximation). The product form of $Z_{N,M}^{(\mathrm{diff})}$ in (6.3) is especially suitable to take the logarithm. We thus prove a preliminary version of our goal (3.32), where we replace $\log Z_N^{\beta_N}$ by $\log Z_{N,M}^{(\mathrm{diff})}$ (and convergence in L^2 by convergence in probability).

Lemma 6.3 (log approximation). *Recall* X_N^{dom} *from* (3.31)*. For any* $\varepsilon > 0$ *we have*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big(\big| \log Z_{N,M}^{(\mathrm{diff})} - \big\{ X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] \big\} \big| > \varepsilon,\, A_{N,M} \Big) = 0 , \tag{6.6}$$

for a suitable event $A_{N,M} \subseteq \{Z_{N,M}^{(\mathrm{diff})} > 0\}$ *(so that* $\log Z_{N,M}^{(\mathrm{diff})}$ *is well-defined) which satisfies*

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\big((A_{N,M})^c\big) = 0 . \tag{6.7}$$

The proof of this result is given in subsection 6.3 below.

[†]Heuristically, these are good approximations because the main contribution to (6.2) will be shown to come from $b_{i-1}' \approx N^{\alpha_{i-1}'}$ and $b_i \approx N^{\alpha_i}$ with $\alpha_{i-1}' < \alpha_i$, hence $b_{i-1}' \ll b_i$.

Part 4 (final approximation). At last, we complete the proof of Theorem 3.5. Our final goal (3.32) is a consequence of the next lemma, where we prove convergence in probability and boundedness in L^p for some $p > 2$, which yields convergence in L^2.

Lemma 6.4 (Final approximation). *Recall* X_N^{dom} *from* (3.31). *For any* $\varepsilon > 0$ *we have*

$$\lim_{N\to\infty} \mathbb{P}\big(\big|\log Z_N^{\beta_N} - \big\{X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2]\big\}\big| > \varepsilon\big) = 0\,. \tag{6.8}$$

Moreover, for some for some $p > 2$ *we have*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\big[\big|\log Z_N^{\beta_N}\big|^p\big] < \infty\,, \qquad \sup_{N\in\mathbb{N}} \mathbb{E}\big[\big|X_N^{\mathrm{dom}}\big|^p\big] < \infty\,. \tag{6.9}$$

Intuitively, we can deduce (6.8) from (6.6) exploiting the approximation (6.5), but some care is needed to handle the logarithm.

The proof of Lemma 6.4, given in subsection 6.4, concludes the proof of Theorem 3.5. $\square$

6.1. Proof of Lemma 6.1. We rewrite the sum over $n_1, \ldots, n_k$ in (3.30) according to suitable *record times*. The first record time is n_1; the second record time is the smallest n_i for which the previous jump $n_i - n_{i-1}$ exceeds n_1; and so on. More precisely, the record times are $n_{j_1}, n_{j_2}, \ldots, n_{j_\ell}$ where we define $j_1 := 1$ and, assuming that $j_r < \infty$, we set $j_{r+1} := \min\{i \in \{j_r + 1, \ldots, k\} : \, n_i - n_{i-1} > n_{j_r}\}$, where we agree that $\min\varnothing := \infty$. The number of record times is therefore $\ell := \min\{r \geqslant 1 : \, j_{r+1} = \infty\}$.

If we rename the record times as $b_r := n_{j_r}$, and we also set $b'_{r-1} := n_{j_r - 1}$, we have by construction $b_2 - b'_1 > b_1$ and, more generally, $b_i - b'_{i-1} > b_{i-1}$ for $i = 2, \ldots, \ell$ (see Figure 1). If we name the corresponding space variables $z_r := x_{b_r}$ and $z'_{r-1} := x_{b'_{r-1}}$, then we can rewrite (3.30) equivalently as (6.2), with $X^{\mathrm{dom}}_{N,[a,b;b']}(x, z; z')$ defined in (6.1). $\square$

6.2. Proof of Lemma 6.2. The proof, which is long and structured, is based on explicit L^2 computations. A key observation is that, by the expression (6.2) for $Z_N^{\beta_N}$, we can write

$$\mathbb{E}\Big[\big(Z_N^{\beta_N}\big)^2\Big] = 1 + \sum_{\ell=1}^{\infty} \sum_{\substack{0<b_1\leqslant b'_1<\ldots<b_\ell\leqslant b'_\ell\leqslant N:\\ b_i-b'_{i-1}>b_{i-1}\ \forall i=2,\ldots,\ell}} \ \sum_{\underline{z},\underline{z}'\in(\mathbb{Z}^2)^\ell} \ \prod_{i=1}^{\ell} \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1}, z_i; z'_i)\big)^2\Big]\,. \tag{6.10}$$

To see why this holds, note that by (3.30) we can write

$$\mathbb{E}\big[\big(Z_N^{\beta_N}\big)^2\big] = 1 + \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{\substack{0=:n_0<n_1<\ldots<n_k\leqslant N\\ x_0:=0,\ x_1,\ldots,x_k\in\mathbb{Z}^2}} \ \prod_{j=1}^{k} q_{n_j-n_{j-1}}(x_j - x_{j-1})^2\,, \tag{6.11}$$

where $q_n(x) = \mathrm{P}(S_n = x)$, see (3.29), and σ_N is defined in (3.19). Similarly, by (6.1),

$$\begin{aligned}\mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b,b']}(x, z; z')\big)^2\Big] &= \sum_{k=1}^{\infty} (\sigma_N^2)^k \, q_{b-a}(z-x)^2 \times \\ &\quad\times \sum_{\substack{b=:n_1<n_2<\ldots<n_{k-1}<n_k=b'\\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\}\leqslant b}} \ \sum_{\substack{x_1=z;\, x_k=z'\\ x_2,\ldots,x_{k-1}\in\mathbb{Z}^2}} \ \prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i - x_{i-1})^2.\end{aligned} \tag{6.12}$$

When we plug (6.12) into (6.10) we obtain (6.11) by the same argument in the proof of Lemma 6.1, see subsection 6.1, because the sum over n_j, x_j in (6.11) can be rewritten in terms of record times, which lead to the variables b_r, b_r' and z_r, z_r' in (6.10).

We now turn to the proof of (6.5). We will define two "coarse-grained approximations" $Z^{(\mathrm{cg})}_{N,K,M}$ and $Z^{(\mathrm{cg}')}_{N,K,M}$, which depend on a further parameter $K \in \mathbb{N}$, and we will show that

$$Z_N^{\beta_N} \approx Z^{(\mathrm{cg})}_{N,K,M}\,, \qquad Z^{(\mathrm{cg})}_{N,K,M} \approx Z^{(\mathrm{cg}')}_{N,K,M}\,, \qquad Z^{(\mathrm{cg}')}_{N,K,M} \approx Z^{(\mathrm{diff})}_{N,M}\,,$$

where $\approx$ denotes closeness in L^2 when we let $N \to \infty$, then $K \to \infty$ and finally $M \to \infty$. More precisely, we are going to prove the following relations:

$$\limsup_{M\to\infty}\ \limsup_{K\to\infty}\ \limsup_{N\to\infty}\ \big\| Z_N^{\beta_N} - Z^{(\mathrm{cg})}_{N,K,M} \big\|_{L^2} = 0\,, \tag{6.13}$$

$$\limsup_{M\to\infty}\ \limsup_{K\to\infty}\ \limsup_{N\to\infty}\ \big\| Z^{(\mathrm{cg})}_{N,K,M} - Z^{(\mathrm{cg}')}_{N,K,M} \big\|_{L^2} = 0\,, \tag{6.14}$$

$$\limsup_{M\to\infty}\ \limsup_{K\to\infty}\ \limsup_{N\to\infty}\ \big\| Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{diff})}_{N,M} \big\|_{L^2} = 0\,, \tag{6.15}$$

which together yield (6.5). We accordingly split the proof in three steps.

6.2.1. Step 1: definition of $Z^{(\mathrm{cg})}_{N,K,M}$ and proof of (6.13). Let us fix $M, K, N \in \mathbb{N}$ with $1 \ll M \ll K \ll N$. Our first coarse-graining approximation $Z^{(\mathrm{cg})}_{N,K,M}$ of the partition function $Z_N^{\beta_N}$ in (6.2) is obtained by *suitably restricting the sums over $\underline{b}, \underline{b}'$ and $\underline{z}, \underline{z}'$*:

$$Z^{(\mathrm{cg})}_{N,K,M} := 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j} \in \{1,\ldots,M\}^\ell_{\ll}} \ \sum_{(\underline{b},\underline{b}') \in \mathcal{B}^\ell(\underline{j})} \ \sum_{(\underline{z},\underline{z}') \in \mathcal{S}^\ell(\underline{b},\underline{b}')} \ \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1}, z_i; z'_i)\ , \tag{6.16}$$

where we sum over $\underline{j} = (j_1, \ldots, j_\ell)$ in the following set:

$$\{1,\ldots,M\}^\ell_{\ll} := \Big\{ 1 \leqslant j_1 < \ldots < j_\ell \leqslant M : \quad j_i - j_{i-1} \geqslant 2 \ \ \forall i = 2,\ldots,\ell \Big\}\,, \tag{6.17}$$

then, given $\underline{j} = (j_1, \ldots, j_\ell)$, we sum over $(\underline{b}, \underline{b}')$ in the set

$$\mathcal{B}^\ell(\underline{j}) := \Big\{ (\underline{b},\underline{b}') \in \mathbb{N}^\ell \times \mathbb{N}^\ell : \quad b_i \in \big(N^{\frac{j_i-1}{M}}, \tfrac{1}{K} N^{\frac{j_i}{M}}\big]\,,\ b_i' \in [b_i, K b_i] \quad \forall i = 1,\ldots,\ell \ \Big\}\,, \tag{6.18}$$

and finally, given $(\underline{b}, \underline{b}')$, we sum over $\underline{z}, \underline{z}'$ in the "diffusive set"

$$\mathcal{S}^\ell(\underline{b},\underline{b}') := \Big\{ (\underline{z},\underline{z}') \in (\mathbb{Z}^2)^\ell \times (\mathbb{Z}^2)^\ell : \quad |z_i| \leqslant K\sqrt{b_i}\,,\ |z_i'| \leqslant K^2\sqrt{b_i} \quad \forall i = 1,\ldots,\ell \Big\}\,.$$

To see that $Z^{(\mathrm{cg})}_{N,K,M}$ *in* (6.16) *is a restriction of* $Z_N^{\beta_N}$ *in* (6.2), note that for $(\underline{b},\underline{b}') \in \mathcal{B}^\ell(\underline{j})$ we have $0 < b_1 \leqslant b_1' < \ldots < b_\ell \leqslant b_\ell' \leqslant N$, and for large N we also have $b_i - b_{i-1}' > b_{i-1}$ for $i \geqslant 2$, because $b_i > N^{\frac{j_i-1}{M}} \geqslant N^{\frac{j_{i-1}+1}{M}} \geqslant K N^{\frac{1}{M}} b_{i-1}$ (recall that $j_i - j_{i-1} \geqslant 2$) hence

$$b_i - b_{i-1}' > K N^{\frac{1}{M}} b_{i-1} - K b_{i-1} = (N^{\frac{1}{M}} - 1) K\, b_{i-1} > b_{i-1} \qquad \text{for } N > 2^M\,.$$

Thus the range of the sums in (6.16) is included in the range of the sums in (6.2). Since the terms in the polynomial chaos (3.30) are orthogonal in L^2, it follows that

$$\big\| Z_N^{\beta_N} - Z^{(\mathrm{cg})}_{N,K,M} \big\|_{L^2}^2 = \big\| Z_N^{\beta_N} \big\|_{L^2}^2 - \big\| Z^{(\mathrm{cg})}_{N,K,M} \big\|_{L^2}^2\,, \tag{6.19}$$

hence to prove (6.13) it suffices to show that

$$\limsup_{N\to\infty} \mathbb{E}\big[\big(Z_N^{\beta_N}\big)^2\big] \leqslant \frac{1}{1-\hat\beta^2}\,, \tag{6.20}$$

$$\liminf_{M\to\infty}\ \liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \mathbb{E}\big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] \geqslant \frac{1}{1-\hat\beta^2}\,. \tag{6.21}$$

Relation (6.20) can be easily deduced from the expression (6.11). Indeed, enlarging the sums to $1 \leqslant n_j - n_{j-1} \leqslant N$ and recalling the definition (3.11) of R_N, we get

$$\begin{aligned}\mathbb{E}\big[\big(Z_N^{\beta_N}\big)^2\big] &\leqslant 1 + \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{\substack{1\leqslant n_j-n_{j-1}\leqslant N\\ j=1,\ldots,k}} \ \sum_{x_0:=0,\ x_1,\ldots,x_k\in\mathbb{Z}^2} \ \prod_{j=1}^{k} q_{n_j-n_{j-1}}(x_j-x_{j-1})^2 \\ &= 1 + \sum_{k=1}^{\infty} (\sigma_N^2)^k \left(\sum_{n=1}^{N}\sum_{x\in\mathbb{Z}^2} q_n(x)^2\right)^k = 1 + \sum_{k=1}^{\infty} \big(\sigma_N^2 R_N\big)^k = \frac{1}{1-\sigma_N^2 R_N}\,.\end{aligned} \tag{6.22}$$

Since $\sigma_N \sim \beta_N \sim \hat\beta\sqrt{\pi}/\sqrt{\log N}$, see (3.19) and (3.12), and since $R_N \sim \frac{1}{\pi}\log N$, see (3.11), we see that (6.20) is proved.

We next prove (6.21). By definition (6.16) of $Z^{(\mathrm{cg})}_{N,K,M}$, in analogy with (6.10), we have

$$\mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big] = 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \ \sum_{\substack{(\underline{b},\underline{b}')\in\mathcal{B}^{\ell}(\underline{j})\\ (\underline{z},\underline{z}')\in\mathcal{S}^{\ell}(\underline{b},\underline{b}')}} \ \prod_{i=1}^{\ell} \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1},z_i;z'_i)\big)^2\Big]\,. \tag{6.23}$$

We now give a lower bound on $\mathbb{E}\big[\big(X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1},z_i;z'_i)\big)^2\big]$ when we sum over b_i, b'_i and z_i, z'_i in the sets $\mathcal{B}^{\ell}(\underline{j})$ and $\mathcal{S}^{\ell}(\underline{b},\underline{b}')$. The next result is proved in Appendix A.1.

Lemma 6.5. *For $N, M, K \in \mathbb{N}$ and $j \in \{1,\ldots,M\}$, define*

$$\Xi_{N,M,K}(j) := \inf_{\substack{0\leqslant a\leqslant N^{\frac{(j-2)^+}{M}}\\ |x|\leqslant K^2\sqrt{a}}} \ \sum_{\substack{b\in(N^{\frac{j-1}{M}},\frac{1}{K}N^{\frac{j}{M}}]\\ b'\in[b,Kb]}} \ \sum_{\substack{|z|\leqslant K\sqrt{b}\\ |z'|\leqslant K^2\sqrt{b}}} \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b,b']}(x,z;z')\big)^2\Big]\,. \tag{6.24}$$

The, for any $M \in \mathbb{N}$ and $j \in \{1,\ldots,M\}$, we have

$$\liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \Xi_{N,M,K}(j) \ = \ I_M(j) := \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s\,. \tag{6.25}$$

Coming back to (6.23), by definition (6.24) of $\Xi_{N,M,K}(j)$, we have the lower bound

$$\mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big] \ \geqslant \ 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \prod_{i=1}^{\ell} \Xi_{N,M,K}(j_i)\,, \tag{6.26}$$

which yields, by (6.25),

$$\liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big] \ \geqslant \ 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^{\ell}_{\ll}} \prod_{i=1}^{\ell} I_M(j_i)\,. \tag{6.27}$$

Recalling the definition (6.17) of $\{1,\ldots,M\}^\ell_\ll$, we can rewrite the r.h.s. of (6.27) as

$$1+\sum_{\ell=1}^\infty \frac{1}{\ell!}\left\{\left(\sum_{j=1}^M I_M(j)\right)^\ell - \sum_{\substack{j_1,\ldots,j_\ell\in\{1,\ldots,M\}\\ \exists h\neq k:\ |j_h-j_k|\leqslant 1}} I_M(j_1)\,\cdots\, I_M(j_\ell)\right\}.$$

The second term gives a vanishing contribution as $M\to\infty$, because $\max_{1\leqslant j\leqslant M} I_M(j)\leqslant \frac{C}{M}$, with $C:=\frac{\hat\beta^2}{1-\hat\beta^2}<\infty$, hence

$$\sum_{\ell=1}^\infty \frac{1}{\ell!}\sum_{\substack{j_1,\ldots,j_\ell\in\{1,\ldots,M\}\\ \exists h\neq k:\ j_h-j_k\leqslant 1}} I_M(j_1)\,\cdots\, I_M(j_\ell)\leqslant \sum_{\ell=1}^\infty \frac{1}{\ell!}\frac{C^\ell}{M^\ell}\binom{\ell+1}{2}3M^{\ell-1} = \frac{C'}{M}\xrightarrow{M\to\infty} 0\,.$$

Since $\sum_{j=1}^M I_M(j)=\int_0^1\frac{\hat\beta^2}{1-\hat\beta^2 s}\mathrm{d}s=\log\frac{1}{1-\hat\beta^2}$, we have finally shown that

$$\liminf_{M\to\infty}\ \liminf_{K\to\infty}\ \liminf_{N\to\infty}\ \mathbb{E}\Big[\big(Z^{(\mathrm{cg})}_{N,K,M}\big)^2\Big]\geqslant 1+\sum_{\ell=1}^\infty\frac{1}{\ell!}\left(\log\tfrac{1}{1-\hat\beta^2}\right)^\ell = \frac{1}{1-\hat\beta^2}\,, \tag{6.28}$$

which is (6.21). This completes the proof of (6.13). □

6.2.2. Step 2: definition of $Z^{(\mathrm{cg}')}_{N,K,M}$ and proof of (6.14). Starting from $Z^{(\mathrm{cg})}_{N,K,M}$ in (6.16), we set $b'_{i-1}=0$ and $z'_{i-1}=0$ inside each X^{dom}_N to obtain our second approximation:

$$Z^{(\mathrm{cg}')}_{N,K,M}:=1+\sum_{\ell=1}^\infty\ \sum_{\underline{j}\in\{1,\ldots,M\}^\ell_\ll}\ \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})}\ \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')}\ \prod_{i=1}^\ell X^{\mathrm{dom}}_{N,[0,b_i;b'_i]}(0,z_i;z'_i)\,. \tag{6.29}$$

Heuristically, the reason why we set $b'_{i-1}=0$ is that $b_i\gg b'_{i-1}$, hence $b_i-b'_{i-1}\approx b_i$ (indeed, note that $b_i\geqslant N^{\frac{j_i-1}{M}}\gg N^{\frac{j_{i-1}}{M}}\geqslant b'_{i-1}$ since $j_i-1>j_{i-1}$, see (6.18) and (6.17)).

We need to prove (6.14). Given $\underline{b},\underline{b}'$ and $\underline{z},\underline{z}'$, let us introduce the shortcuts

$$X_i:=X^{\mathrm{dom}}_{N,[b'_{i-1},b_i;b'_i]}(z'_{i-1},z_i;z'_i)\,,\qquad Y_i:=X^{\mathrm{dom}}_{N,[0,b_i;b'_i]}(0,z_i;z'_i)\,, \tag{6.30}$$

so that, comparing (6.16) and (6.29), we can write

$$\begin{aligned}Z^{(\mathrm{cg}')}_{N,K,M}-Z^{(\mathrm{cg})}_{N,K,M}&=\sum_{\ell=1}^\infty\ \sum_{\underline{j}\in\{1,\ldots,M\}^\ell_\ll}\ \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})}\ \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')}\left(\prod_{i=1}^\ell Y_i-\prod_{i=1}^\ell X_i\right)\\&=\sum_{\ell=1}^\infty\ \sum_{\underline{j}\in\{1,\ldots,M\}^\ell_\ll}\ \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})}\ \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')}\ \sum_{h=1}^\ell\left\{\prod_{i=1}^{h-1}Y_i\right\}(Y_h-X_h)\left\{\prod_{i=h+1}^\ell X_i\right\},\end{aligned}$$

and note that different terms in the sums are orthogonal in L^2. We justify below the following key estimate, see Lemma 6.7: for N large enough, we can bound for all $i=1,\ldots,\ell$

$$\mathbb{E}\big[(Y_i-X_i)^2\big]\leqslant\varepsilon^2\,\mathbb{E}[Y_i^2]\,. \tag{6.31}$$

This implies $\mathbb{E}[X_i^2]^{1/2} \leqslant (1+\varepsilon)\mathbb{E}[Y_i^2]^{1/2} \leqslant 2\,\mathbb{E}[Y_i^2]^{1/2}$, by the triangle inequality, hence

$$\begin{aligned}\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] &\leqslant \sum_{\ell=1}^{\infty} \sum_{\underline{j}\in\{1,\ldots,M\}^\ell_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')} \Bigg(\varepsilon^2 \sum_{h=1}^{\ell} 2^{2(\ell-h)}\Bigg) \prod_{i=1}^{\ell} \mathbb{E}[Y_i^2] \\ &\leqslant \varepsilon^2 \sum_{\ell=1}^{\infty} 4^\ell \sum_{\underline{j}\in\{1,\ldots,M\}^\ell_{\ll}} \sum_{(\underline{b},\underline{b}')\in\mathcal{B}^\ell(\underline{j})} \sum_{(\underline{z},\underline{z}')\in\mathcal{S}^\ell(\underline{b},\underline{b}')} \prod_{i=1}^{\ell} \mathbb{E}[Y_i^2]\,,\end{aligned}$$

because $\sum_{h=1}^{\ell} 2^{2(\ell-h)} = \frac{4^\ell-1}{4-1} \leqslant 4^\ell$. We now enlarge the sum ranges to obtain the factorization

$$\begin{aligned}&\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] \\ &\qquad\leqslant \varepsilon^2 \sum_{\ell=1}^{\infty} 4^\ell \sum_{1\leqslant j_1<j_2<\ldots<j_\ell\leqslant M} \prod_{i=1}^{\ell} \Bigg\{ \sum_{b_i\leqslant b_i'\in(N^{\frac{j_i-1}{M}},N^{\frac{j_i}{M}}]} \sum_{z_i,z_i'\in\mathbb{Z}^2} \mathbb{E}[Y_i^2] \Bigg\}\,.\end{aligned} \tag{6.32}$$

The following asymptotics on the term in brackets is proved in Appendix A.2.

Lemma 6.6. *For any $M\in\mathbb{N}$ and $j\in\{1,\ldots,M\}$ we have*

$$\lim_{N\to\infty} \Bigg\{ \sum_{\substack{b\leqslant b'\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}] \\ z,z'\in\mathbb{Z}^2}} \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')^2\big] \Bigg\} \;=\; I_M(j) \;=\; \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s\,. \tag{6.33}$$

We can plug (6.33) into (6.32) (where the sum is finite: it can be stopped at $\ell = M$, since for $\ell > M$ there is no choice of $1\leqslant j_1 < j_2 < \ldots < j_\ell \leqslant M$), which yields

$$\begin{aligned}\limsup_{N\to\infty}\; \mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{cg})}_{N,K,M}\big)^2\big] &\leqslant \varepsilon^2 \sum_{\ell=1}^{\infty} 4^\ell \sum_{1\leqslant j_1<j_2<\ldots<j_\ell\leqslant M} \prod_{i=1}^{\ell} I_M(j_i) \\ &\leqslant \varepsilon^2 \sum_{\ell=1}^{\infty} \frac{4^\ell}{\ell!} \Bigg(\sum_{j=1}^{M} I_M(j)\Bigg)^\ell \leqslant \varepsilon^2 \exp\Bigg(4\sum_{j=1}^{M} I_M(j)\Bigg) = \frac{\varepsilon^2}{(1-\hat\beta^2)^4}\,.\end{aligned} \tag{6.34}$$

This completes the proof of (6.14), since we can take $\varepsilon > 0$ as small as we wish.

It only remains to justify (6.31). The following result is proved in Appendix A.3.

Lemma 6.7. *Given $K, M\in\mathbb{N}$ and $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon, M, K) < \infty$ such that for all $N > N_0$ the following bound holds:*

$$\mathbb{E}\big[\big(X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z') - X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')\big)^2\big] \leqslant \varepsilon^2\, \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')^2\big]\,, \tag{6.35}$$

uniformly for $(a,x),(b,z),(b',z')\in\mathbb{Z}^3_{\mathrm{even}} = \{y\in\mathbb{Z}^3 :\; y_1+y_2+y_3 \text{ is even}\}$ such that, for some $j\in\{1,\ldots,M\}$,

$$a\in[0,N^{\frac{(j-2)^+}{M}}]\,,\quad b\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]\,,\quad |x|\leqslant K^2\sqrt{a}\,,\quad |z|\leqslant K\sqrt{b}\,. \tag{6.36}$$

6.2.3. Step 3: proof of (6.15). Recalling (6.4), we can rewrite $Z^{(\mathrm{diff})}_{N,K,M}$ in (6.3) as follows:

$$Z^{(\mathrm{diff})}_{N,M} = 1 + \sum_{\ell=1}^{\infty} \sum_{1 \leqslant j_1 < j_2 < \ldots < j_\ell \leqslant M} \sum_{\substack{\underline{b}, \underline{b}' \in \mathbb{N}^\ell : \\ b_i \leqslant b_i' \in (N^{\frac{j_i - 1}{M}}, N^{\frac{j_i}{M}}]}} \sum_{\underline{z}, \underline{z}' (\mathbb{Z}^2)^\ell} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[0,b_i;b_i']}(0, z_i; z_i') . \tag{6.37}$$

By (6.29), we see that $Z^{(\mathrm{cg}')}_{N,K,M}$ is a *restriction* of the sum which defines $Z^{(\mathrm{diff})}_{N,M}$, therefore

$$\|Z^{(\mathrm{cg}')}_{N,K,M} - Z^{(\mathrm{diff})}_{N,M}\|^2_{L^2} = \|Z^{(\mathrm{diff})}_{N,M}\|^2_{L^2} - \|Z^{(\mathrm{cg}')}_{N,K,M}\|^2_{L^2} .$$

Then, to prove (6.15), it is enough to show that

$$\liminf_{M\to\infty} \liminf_{K\to\infty} \liminf_{N\to\infty} \mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big] \geqslant \frac{1}{1-\hat\beta^2} , \tag{6.38}$$

$$\forall M \in \mathbb{N} : \qquad \limsup_{N\to\infty} \mathbb{E}\big[\big(Z^{(\mathrm{diff})}_{N,M}\big)^2\big] \leqslant \frac{1}{1-\hat\beta^2} . \tag{6.39}$$

We first consider (6.38). Recalling (6.29), in analogy with (6.10), we can write

$$\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big] = 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j} \in \{1,\ldots,M\}^\ell_{\ll}} \sum_{(\underline{b},\underline{b}') \in \mathcal{B}^\ell(\underline{j})} \sum_{(\underline{z},\underline{z}') \in \mathcal{S}^\ell(\underline{b},\underline{b}')} \prod_{i=1}^{\ell} \mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b_i;b_i']}(0,z_i;z_i')^2\big] .$$

We can now use the quantity $\Xi_{N,M,K}(j_i)$ defined in (6.24) to bound

$$\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big] \geqslant 1 + \sum_{\ell=1}^{\infty} \sum_{\underline{j} \in \{1,\ldots,M\}^\ell_{\ll}} \prod_{i=1}^{\ell} \Xi_{N,M,K}(j_i) ,$$

which coincides with the r.h.s. of (6.26). As a consequence, the bounds from (6.27) to (6.28) apply verbatim to $\mathbb{E}\big[\big(Z^{(\mathrm{cg}')}_{N,K,M}\big)^2\big]$ and show that (6.38) holds.

We finally consider (6.39), which we have essentially already proved. Indeed, note that $\mathbb{E}\big[\big(Z^{(\mathrm{diff})}_{N,K,M}\big)^2\big]$ is given by the second line of (6.32) where we replace ε^2 and 4^ℓ by 1. When we apply the bound (6.33), we obtain an analogue of (6.34), again with ε^2 and 4^ℓ replaced by 1, which yields precisely (6.39). This completes the proof of Lemma 6.6. □

6.3. Proof of Lemma 6.3. To ensure that $Z^{(\mathrm{diff})}_{N,M} > 0$, see (6.3), we define the event

$$A_{N,M} := \bigcap_{j=1}^{M} \big\{ |X^{\mathrm{dom}}_{N,M}(j)| \leqslant \tfrac{1}{2} \big\} . \tag{6.40}$$

Then, to prove (6.6), it is enough to show that the following three relations hold:

$$\lim_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\bigg(\bigg| \log Z^{(\mathrm{diff})}_{N,M} - \sum_{j=1}^{M} \big\{ X^{\mathrm{dom}}_{N,M}(j) - \tfrac{1}{2} X^{\mathrm{dom}}_{N,M}(j)^2 \big\} \bigg| > \varepsilon , \ A_{N,M} \bigg) = 0 , \tag{6.41}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} \bigg\| \sum_{j=1}^{M} X^{\mathrm{dom}}_{N,M}(j) - X^{\mathrm{dom}}_{N} \bigg\|_{L^2} = 0 , \tag{6.42}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} \bigg\| \sum_{j=1}^{M} X^{\mathrm{dom}}_{N,M}(j)^2 - \mathbb{E}[(X^{\mathrm{dom}}_{N})^2] \bigg\|_{L^1} = 0 . \tag{6.43}$$

We are going to exploit the following result.

Lemma 6.8. *Fix $\hat\beta < 1$. For every $M \in \mathbb{N}$ and $j \in \{1,\dots,M\}$ we have*

$$\lim_{N\to\infty} \mathbb{E}\big[X^{\mathrm{dom}}_{N,M}(j)^2\big] \;=\; \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s \;\leqslant\; \frac{c}{M}\,, \qquad \textit{with} \quad c = c_{\hat\beta} := \tfrac{\hat\beta^2}{1-\hat\beta^2}\,. \tag{6.44}$$

Moreover, there exist $p = p_{\hat\beta} > 2$ and $C = C_{\hat\beta} < \infty$ such that

$$\forall M \in \mathbb{N}\,,\ \forall j \in \{1,\dots,M\}: \qquad \limsup_{N\to\infty}\ \mathbb{E}\,\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big] \leqslant \frac{C}{M^{\frac{p}{2}}}\,. \tag{6.45}$$

Proof. Relation (6.44) is already proved in (6.33), by the definition (6.4) of $X^{\mathrm{dom}}_{N,M}(j)$.

Intuitively, the bound (6.45) holds because $\mathbb{E}\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big] \leqslant C\,\mathbb{E}\big[X^{\mathrm{dom}}_{N,M}(j)^2\big]^{\frac{p}{2}}$ by the *hypercontractivity of polynomial chaos*. The details are presented in Appendix A.4. ☐

It only remains to prove (6.7) and the three relations (6.41)-(6.43).

Proof of (6.7). For any $p > 2$ we can bound, by Markov's inequality,

$$\mathbb{P}\big((A_{N,M})^c\big) \leqslant \sum_{j=1}^M \mathbb{P}\big(|X^{\mathrm{dom}}_{N,M}(j)| > \tfrac12\big) \leqslant M\,2^p \max_{j\in\{1,\dots,M\}} \mathbb{E}\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big]\,,$$

and relation (6.7) follows directly by (6.45). ☐

Proof of (6.41). By (6.3) can write $\log Z^{(\mathrm{diff})}_{N,M} = \sum_{j=1}^M \log(1 + X^{\mathrm{dom}}_{N,M}(j))$. For $p > 2$ as in Lemma 6.8, we can bound $|\log(1+x) - \{x - \frac12 x^2\}| \leqslant c|x|^p$ for $|x| \leqslant \frac12$, hence

$$\mathbb{E}\Bigg[\bigg|\log Z^{(\mathrm{diff})}_{N,M} - \sum_{j=1}^M \big\{X^{\mathrm{dom}}_{N,M}(j) - \tfrac12 X^{\mathrm{dom}}_{N,M}(j)^2\big\}\bigg| \mathbb{1}_{A_{N,M}}\Bigg] \;\leqslant\; c \sum_{j=1}^M \mathbb{E}\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big] \;\leqslant\; c\,\frac{C}{M^{\frac{p}{2}-1}}\,,$$

which proves (6.41), by Markov's inequality. ☐

Proof of (6.42). The polynomial chaos $\sum_{j=1}^M X^{\mathrm{dom}}_{N,M}(j)$ contains less terms than X^{dom}_N, therefore to prove (6.42) it is enough to show that for any fixed $M \in \mathbb{N}$

$$\lim_{N\to\infty} \mathbb{E}\Bigg[\bigg(\sum_{j=1}^M X^{\mathrm{dom}}_{N,M}(j)\bigg)^2\Bigg] = \lim_{N\to\infty} \mathbb{E}\big[\big(X^{\mathrm{dom}}_N\big)^2\big] = \int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s \tag{6.46}$$

where the last equality follows by (6.44), because X^{dom}_N equals $X^{\mathrm{dom}}_{N,M}(j)$ for $M = j = 1$ (cf. (3.31) with (6.4) and (6.1)). Since the variables $X^{\mathrm{dom}}_{N,M}(j)$'s are centered and independent, a further application of (6.44) yields

$$\mathbb{E}\Bigg[\bigg(\sum_{j=1}^M X^{\mathrm{dom}}_{N,M}(j)\bigg)^2\Bigg] = \sum_{j=1}^M \mathbb{E}\big[X^{\mathrm{dom}}_{N,M}(j)^2\big] \xrightarrow{N\to\infty} \sum_{j=1}^M I_M(j) = \int_0^1 \frac{\hat\beta^2}{1-\hat\beta^2 s}\,\mathrm{d}s\,, \tag{6.47}$$

as desired. This completes the proof. ☐

Proof of (6.43). In view of the first equalities in (6.46) and (6.47), it suffices to show that

$$\lim_{M\to\infty}\ \limsup_{N\to\infty}\ \bigg\| \sum_{j=1}^M \big\{X^{\mathrm{dom}}_{N,M}(j)^2 - \mathbb{E}\big[X^{\mathrm{dom}}_{N,M}(j)^2\big]\big\}\bigg\|_{L^1} = 0\,. \tag{6.48}$$

This is a weak law of large numbers for the independent random variables $W_j := X^{\mathrm{dom}}_{N,M}(j)^2$, which satisfy the following Lyapunov condition (by (6.45) with $q := p/2$):

$$\exists q = q_{\hat\beta} > 1,\ C = C_{\hat\beta} < \infty : \qquad \forall M \in \mathbb{N} \qquad \limsup_{N\to\infty} \max_{j\in\{1,\ldots,M\}} \mathbb{E}[W_j^q] \leqslant \frac{C}{M^q} \,. \tag{6.49}$$

We prove (6.48) by truncation at level $T_M := M^{-\alpha}$, for an arbitrary $\alpha \in (\frac{1}{2}, 1)$. Note that

$$\Big\| \sum_{j=1}^M W_j \, \mathbb{1}_{\{W_j > T_M\}} \Big\|_{L^1} = \sum_{j=1}^M \mathbb{E}\big[W_j \, \mathbb{1}_{\{W_j > T_M\}}\big] \leqslant \sum_{j=1}^M \frac{\mathbb{E}[W_j^q]}{T_M^{q-1}} \leqslant M^{1+\alpha(q-1)} \max_{j\in\{1,\ldots,M\}} \mathbb{E}[W_j^q] \,,$$

which, by (6.49), vanishes as $N \to \infty$ followed by $M \to \infty$ provided $1 + \alpha(q-1) - q < 0$, that is $\alpha < 1$. To prove (6.48) it only remains to show that

$$\lim_{M\to\infty} \limsup_{N\to\infty} \Big\| \sum_{j=1}^M \Big\{ W_j \, \mathbb{1}_{\{W_j \leqslant T_M\}} - \mathbb{E}\big[W_j \, \mathbb{1}_{\{W_j \leqslant T_M\}}\big] \Big\} \Big\|_{L^1} = 0 \,.$$

It is simpler to prove convergence in L^2, because this follows by a variance computation:

$$\mathbb{V}\mathrm{ar}\bigg(\sum_{j=1}^M W_j \, \mathbb{1}_{\{W_j \leqslant T_M\}} \bigg) = \sum_{j=1}^M \mathbb{V}\mathrm{ar}\big(W_j \, \mathbb{1}_{\{W_j \leqslant T_M\}} \big) \leqslant M \, T_M^2 = M^{1-2\alpha} \,,$$

which vanishes as $M \to \infty$ provided $1 - 2\alpha < 0$, that is $\alpha > \frac{1}{2}$. □

6.4. Proof of Lemma 6.4. We first prove (6.8). In view of (6.6) and (6.7), it suffices to show that

$$\forall \varepsilon > 0 : \qquad \lim_{N\to\infty} \mathbb{P}\big(\big| \log Z_N^{\beta_N} - \log Z_{N,M}^{(\mathrm{diff})} \big| > \varepsilon \,,\ A_{N,M} \big) = 0 \,, \tag{6.50}$$

where we recall that the event $A_{N,M} \subseteq \{Z_{N,M}^{(\mathrm{diff})} > 0\}$ was defined in (6.40).

For any $a, b \in \mathbb{R}$ and $\varepsilon, \eta \in (0,1)$ we have the following inclusion:

$$\{|\log a - \log b| > \varepsilon\} \subseteq \{b < 2\eta\varepsilon\} \cup \{|a-b| > \eta\varepsilon^2\} \,.$$

Indeed, if both $b \geqslant 2\eta\varepsilon$ and $|a-b| \leqslant \eta\varepsilon^2$, then $a \geqslant b - \eta\varepsilon^2 \geqslant 2\eta\varepsilon - \eta\varepsilon^2 \geqslant \eta\varepsilon$, so that both $a, b \in [\varepsilon, \infty)$, hence $|\log a - \log b| = |\int_a^b \frac{1}{x} \mathrm{d}x| \leqslant \frac{1}{\eta\varepsilon}|b-a| \leqslant \frac{1}{\eta\varepsilon}\eta\varepsilon^2 = \varepsilon$. It follows that

$$\mathbb{P}\big(\big| \log Z_N^{\beta_N} - \log Z_{N,M}^{(\mathrm{diff})} \big| > \varepsilon,\ A_{N,M} \big) \leqslant \mathbb{P}\big(Z_{N,M}^{(\mathrm{diff})} < 2\eta\varepsilon,\ A_{N,M} \big) + \mathbb{P}\big(\big| Z_N^{\beta_N} - Z_{N,M}^{(\mathrm{diff})} \big| > \eta\varepsilon^2 \big)$$

and note that the second term in the r.h.s. vanishes as $N \to \infty$ followed by $M \to \infty$, for any fixed $\varepsilon, \eta \in (0,1)$, thanks to (6.5). It remains to show that

$$\forall \varepsilon > 0 : \qquad \lim_{\eta\downarrow 0} \limsup_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}\big(Z_{N,M}^{(\mathrm{diff})} < 2\eta\varepsilon,\ A_{N,M} \big) \ = \ 0 \,.$$

To this purpose, we can bound

$$\begin{aligned} \mathbb{P}\big(Z_{N,M}^{(\mathrm{diff})} < 2\eta\varepsilon,\ A_{N,M} \big) \leqslant \mathbb{P}\Big(\big| \log Z_{N,M}^{(\mathrm{diff})} - \big\{ X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] \big\} \big| > 1,\ A_{N,M} \Big) \\ + \mathbb{P}\Big(X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] < \log(2\eta\varepsilon) + 1 \Big) \end{aligned}$$

and note that the first term in the r.h.s. vanishes as $N \to \infty$ followed by $M \to \infty$, by (6.6). To show that the second term vanishes as $N \to \infty$ followed by $\eta \downarrow 0$, we fix $\eta > 0$ small, so that $\log(2\eta\varepsilon) + 1 < 0$, and we apply Markov's inequality to bound, for some $C < \infty$,

$$\mathbb{P}\Big(X_N^{\mathrm{dom}} - \tfrac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] < \log(2\eta\varepsilon) + 1 \Big) \leqslant \frac{\mathbb{E}\big[\big(X_N^{\mathrm{dom}} - \frac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2] \big)^2 \big]}{|\log(2\eta\varepsilon) + 1|} \leqslant \frac{C}{|\log(2\eta\varepsilon) + 1|} \,,$$

because $\mathbb{E}\big[\big(X_N^{\mathrm{dom}} - \frac{1}{2}\mathbb{E}[(X_N^{\mathrm{dom}})^2]\big)^2\big]$ converges to a finite limit as $N \to \infty$, see (6.46).

It only remains to prove (6.9). The second bound in (6.9) follows by (6.45), because we already remarked that $X_N^{\mathrm{dom}} = X_{N,M}^{\mathrm{dom}}(j)$ with $j = M = 1$, see (3.31) and (6.4), (6.1). The first bound in (6.9) was proved in [CSZ20] (see equations (3.12), (3.14) and the lines following (3.16)) exploiting *concentration of measure for the left tail of* $\log Z_N$. ☐

7. Proof of Theorem 3.6

We have already noticed in (6.46) that

$$\lim_{N\to\infty} \mathbb{E}\big[(X_N^{\mathrm{dom}})^2\big] = \sigma^2 := \log \tfrac{1}{1-\hat\beta^2}\,, \tag{7.1}$$

which follows by (6.44), because $X_N^{\mathrm{dom}} = X_{N,1}^{\mathrm{dom}}(1)$ (see (3.31) and (6.4), (6.1)). Therefore we only need to prove that

$$X_N^{\mathrm{dom}} \xrightarrow{d} \mathcal{N}\big(0,\sigma^2\big)\,. \tag{7.2}$$

We can apply Theorem 2.1 to the polynomial chaos X_N^{dom} defined in (3.31). As in the proof of Theorem 3.4, we can cast X_N^{dom} in the form (2.4) with $\mathbb{T} := \mathbb{N}_0 \times \mathbb{Z}^2$ and $\eta_t^N = \eta_N(m,z)$ defined in (3.19), while for $A := \{t_1,\ldots,t_k\} = \{(n_1,x_1),\ldots,(n_k,x_k)\} \subseteq \mathbb{T}$ we set

$$q_N(A) = (\sigma_N)^k\, \mathbb{1}_{\left\{\substack{0=:n_0<n_1<\ldots<n_k\leqslant N\\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\}\leqslant n_1-n_0}\right\}} \prod_{j=1}^k q_{n_j-n_{j-1}}(x_j-x_{j-1})\,.$$

By Theorem 2.1, to prove (7.2) we need to verify the following conditions:

(1) *Limiting second moment:* we already showed that $\lim_{N\to\infty}\mathbb{E}[(X_N^{\mathrm{dom}})^2] = \sigma^2$, see (7.1).

(2) *Subcriticality*: we need to show that

$$\lim_{K\to\infty}\ \limsup_{N\to\infty} \sum_{\substack{A\subset\mathbb{T}\\|A|\geqslant K}} q_N(A)^2 = 0\,. \tag{7.3}$$

Arguing as in (6.22), we can enlarge the sums to $1 \leqslant n_j - n_{j-1} \leqslant N$ and remove the constraint $\max\{n_2-n_1,\ldots,n_k-n_{k-1}\} \leqslant n_1 - n_0$, to get the bound

$$\begin{aligned}\sum_{\substack{A\subset\mathbb{T}\\|A|\geqslant K}} q_N(A)^2 &\leqslant \sum_{k=K}^\infty (\sigma_N^2)^k \sum_{\substack{1\leqslant n_j-n_{j-1}\leqslant N\\ j=1,\ldots,k}}\ \sum_{\substack{x_1,\ldots,x_k\in\mathbb{Z}^2\\x_0:=0}} \prod_{j=1}^k q_{n_j-n_{j-1}}(x_j-x_{j-1})^2\\ &= \sum_{k=K}^\infty (\sigma_N^2)^k \Big(\sum_{n=1}^N \sum_{x\in\mathbb{Z}^2} q_n(x)^2\Big)^k = \sum_{k=K}^\infty (\sigma_N^2\, R_N)^k \xrightarrow{N\to\infty} \frac{(\hat\beta^2)^K}{1-\hat\beta^2}\,,\end{aligned}$$

from which (7.3) follows.

(3) *Spectral localization*: given $M, N \in \mathbb{N}$, we define disjoint subsets $\mathbb{B}_j \subseteq \mathbb{T}$ by

$$\mathbb{B}_j := \big((N^{\frac{j-1}{M}}, N^{\frac{j}{M}}] \cap \mathbb{N}\big) \times \mathbb{Z}^2 \qquad \text{for } j = 1,\ldots,M\,,$$

and, recalling that $\sigma_N^2(\mathbb{B}_j) := \sum_{A\subset\mathbb{B}_j} q_N(A)^2$, see (2.6), we need to show that

$$\lim_{M\to\infty} \sum_{j=1}^M \lim_{N\to\infty} \sigma_N^2(\mathbb{B}_j) \;=\; \sigma^2 \qquad \text{and} \qquad \lim_{M\to\infty} \Big\{ \max_{j=1,\ldots,M}\ \limsup_{N\to\infty}\ \sigma_N^2(\mathbb{B}_j)\Big\} \;=\; 0\,.$$

For this it suffices to note that $\sigma_N^2(\mathbb{B}_j) = \mathbb{E}[X_{N,M}^{\mathrm{dom}}(j)^2]$ and then to apply (6.44).

The proof of Theorem 3.6 is completed. □

Appendix A. Some technical results

We collect here the proofs of some technical results.

A.1. Proof of Lemma 6.5. We are going to prove that there is a constant $C < \infty$ such that, for any given $M, K \in \mathbb{N}$ and $j \in \{1, \ldots, M\}$, we have

$$\liminf_{N\to\infty} \Xi_{N,M,K}(j) \geqslant \big(1 - (\hat\beta^2)^K\big) \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{\hat\beta^2(1-\frac{C}{K^2})}{1-\hat\beta^2(1-\frac{C}{K^2})\, s}\, \mathrm{d}s\,, \tag{A.1}$$

which clearly implies (6.25).

Given $a, b \in \mathbb{N}_0$ as in the range of the sums (6.24), we note that

$$a \leqslant \tfrac{1}{4} K^{-2} b\,. \tag{A.2}$$

This clearly holds if $a = 0$, hence for $j = 1$, because $a \leqslant N^{\frac{(j-2)^+}{M}} = 0$, while for $j \geqslant 2$ from $a \leqslant N^{\frac{j-2}{M}}$ and $b > N^{\frac{j-1}{M}}$ we get $a \leqslant N^{-\frac{1}{M}} b \leqslant \frac{1}{4}K^{-2} b$ for large N, say $N \geqslant (2K)^{2M}$. By (6.12), for fixed a, b and x, the sums over $b' \in [b, Kb]$ and $z, z' \in \mathbb{Z}^2$ in (6.24) equal

$$\begin{aligned} &\sum_{b'\in[b,Kb]} \ \sum_{\substack{|z|\leqslant K\sqrt{b} \\ |z'|\leqslant K^2\sqrt{b}}} \mathbb{E}\Big[\big(X^{\mathrm{dom}}_{N,[a,b,b']}(x,z;z')\big)^2\Big] \\ &= \sum_{k=1}^{\infty} (\sigma_N^2)^k \sum_{|x_1|\leqslant K\sqrt{b}} q_{b-a}(x_1-x)^2 \sum_{\substack{b<n_2<\ldots<n_k\leqslant Kb: \\ \max\{n_2-b,\ldots,n_k-n_{k-1}\}\leqslant b \\ x_2,\ldots,x_k\in\mathbb{Z}^2:\ |x_k|\leqslant K^2\sqrt{b}}} \prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})^2 . \end{aligned} \tag{A.3}$$

We get a lower bound by keeping just the first K terms in the sum over $k \in \mathbb{N}$. Moreover:

- we remove the constraint $n_k \leqslant Kb$ (because $\max\{n_2 - b, \ldots, n_k - n_{k-1}\} \leqslant b$ already yields $n_k = b + \sum_{i=2}^k (n_i - n_{i-1}) \leqslant Kb$) and sum freely over the increments
$$m_i := n_i - n_{i-1} \in \{1, \ldots, b\} \qquad \text{for } i = 2, \ldots, k\,; \tag{A.4}$$
- we change variables to $y_1 := x_1 - x$ and $y_i := x_i - x_{i-1}$ for $i \geqslant 2$, that we restrict to
$$|y_1| \leqslant \tfrac{1}{2} K\sqrt{b-a} \qquad \text{and} \qquad |y_i| \leqslant \tfrac{1}{2} K\sqrt{m_i} \qquad \text{for } i \geqslant 2\,,$$
which imply both $|x_1| \leqslant K\sqrt{b}$ and $|x_k| \leqslant K^2\sqrt{b}$ as required by (A.3). Indeed, recalling that $|x| \leqslant K^2\sqrt{a} \leqslant \frac{1}{2}K\sqrt{b}$ by (6.24) and (A.2), we obtain
$$|x_1| \leqslant |y_1| + |x| \leqslant \frac{1}{2}K\sqrt{b-a} + \frac{1}{2}K\sqrt{b} \leqslant K\sqrt{b}\,,$$
$$|x_k| \leqslant |x_1| + \sum_{i=2}^{k} |y_i| \leqslant K\sqrt{b} + (K-1)\frac{1}{2}K\sqrt{b} \leqslant K^2\sqrt{b}\,.$$

These restrictions yield the following lower bound on (A.3):

$$\sum_{k=1}^{K} (\sigma_N^2)^k \Bigg(\sum_{|y_1|\leqslant\frac{1}{2}K\sqrt{b-a}} q_{b-a}(y_1)^2 \Bigg) \prod_{i=2}^{k} \Bigg(\sum_{m_i=1}^{b} \sum_{|y_i|\leqslant\frac{1}{2}K\sqrt{m_i}} q_{m_i}(y_i)^2 \Bigg) . \tag{A.5}$$

Recalling that u_n and R_N are defined in (3.10) and (3.11), we define restricted versions

$$u_n^{(K)} := \sum_{|y| \leqslant \frac{1}{2}K\sqrt{n}} q_n(y)^2 , \qquad R_N^{(K)} := \sum_{m=1}^{N} u_m^{(K)} = \sum_{m=1}^{N} \sum_{|y| \leqslant \frac{1}{2}K\sqrt{m}} q_m(y)^2 , \tag{A.6}$$

so that we can rewrite (A.5) more compactly as follows:

$$\sum_{k=1}^{K} (\sigma_N^2)^k \, u_{b-a}^{(K)} \, \big(R_b^{(K)}\big)^{k-1} \;=\; \sigma_N^2 \, u_{b-a}^{(K)} \, \frac{1 - \big(\sigma_N^2 R_b^{(K)}\big)^K}{1 - \sigma_N^2 R_b^{(K)}} \, .$$

Bounding $(\sigma_N^2 R_b^{(K)})^K \leqslant (\sigma_N^2 R_N)^K$ in the numerator and recalling (6.24), we obtain

$$\Xi_{N,M,K}(j) \;\geqslant\; \big(1 - \big(\sigma_N^2 R_N\big)^K\big) \inf_{0 \leqslant a \leqslant \frac{1}{K} N^{\frac{(j-2)^+}{M}}} \; \sum_{b \in (N^{\frac{j-1}{M}} + \log N, \frac{1}{K} N^{\frac{j}{M}}]} \frac{\sigma_N^2 \, u_{b-a}^{(K)}}{1 - \sigma_N^2 \, R_b^{(K)}} \, , \tag{A.7}$$

where we restricted the sum range to $b \in (N^{\frac{j-1}{M}} + \log N, \frac{1}{K} N^{\frac{j}{M}}]$ for later convenience.

We now claim that for some $C < \infty$ we have, for n, N large enough,

$$u_n^{(K)} \geqslant (1 - \tfrac{C}{K^2}) \, \frac{1}{\pi} \frac{1}{n} \qquad \Longrightarrow \qquad R_N^{(K)} \geqslant (1 - \tfrac{C}{K^2}) \, \frac{1}{\pi} \, \log N \, . \tag{A.8}$$

This follows by (A.6) writing $u_n^{(K)} = u_n - \sum_{|y| > \frac{1}{2}K\sqrt{n}} q_n(y)^2$, recalling that $u_n \sim \frac{1}{\pi}\frac{1}{n}$ by (3.10), bounding $\sup_{y \in \mathbb{Z}^2} q_n(y) \leqslant \frac{c_1}{n}$ by the local limit theorem (see (A.14) below) and then estimating

$$\sum_{|y| > \frac{1}{2}K\sqrt{n}} q_n(y) = \mathrm{P}(|S_n| > \tfrac{1}{2}K\sqrt{n}) \leqslant 4 \, \frac{\mathrm{E}[|S_n|^2]}{K^2 \, n} = \frac{4}{K^2} \, .$$

We can plug the bounds (A.8) into (A.7) because, uniformly for a, b in the sum range, we have $b \geqslant b - a \geqslant \log N \to \infty$ as $N \to \infty$. Since $\sigma_N^2 \sim \beta_N^2 \sim \pi \hat{\beta}^2 / \log N$, see (3.12) and (3.19), for large N we have (possibly enlarging C)

$$\frac{\sigma_N^2 \, u_{b-a}^{(K)}}{1 - \sigma_N^2 R_b^{(K)}} \;\geqslant\; (1 - \tfrac{C}{K^2}) \, \frac{1}{b-a} \, \frac{\frac{\hat{\beta}^2}{\log N}}{1 - \frac{\hat{\beta}^2}{\log N} \, (1 - \frac{C}{K^2}) \, \log b} \, . \tag{A.9}$$

The r.h.s. is a decreasing function of $b - a$, hence we get a lower bound setting $a = 0$. Looking back at (A.7), a Riemann sum approximation gives

$$\Xi_{N,M,K}(j) \;\geqslant\; (1 - \tfrac{C}{K^2}) \, \big(1 - (\hat{\beta}^2)^K\big) \int_{N^{\frac{j-1}{M}} + \log N}^{\frac{1}{K} N^{\frac{j}{M}}} \frac{1}{x} \, \frac{\frac{\hat{\beta}^2}{\log N}}{1 - \frac{\hat{\beta}^2}{\log N} \, (\log x) \, (1 - \frac{C}{K^2})} \, \mathrm{d}x \, .$$

With the change of variable $x = N^s$, the integral equals

$$\int_{a_N}^{b_N} \frac{\hat{\beta}^2}{1 - \hat{\beta}^2 s \, (1 - \frac{C}{K^2})} \, \mathrm{d}s \qquad \text{with} \qquad a_N := \frac{\log(N^{\frac{j-1}{M}} + \log N)}{\log N} \, , \quad b_N := \frac{\log(\frac{1}{K^2} N^{\frac{j}{M}})}{\log N} \, .$$

Since $\lim_{N\to\infty} a_N = \frac{j-1}{M}$ and $\lim_{N\to\infty} b_N = \frac{j}{M}$, we have proved (A.1). □

A.2. Proof of Lemma 6.6. A lower bound for (6.33) is already provided by (6.25), hence it suffices to prove a matching upper bound. By (6.12) with $(a,x)=(0,0)$, we can write

$$\begin{aligned}\sum_{b\leqslant b'\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]}\ \sum_{z,z'\in\mathbb{Z}^2}\mathbb{E}\big[X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')^2\big] &\leqslant \sum_{k=1}^{\infty}(\sigma_N^2)^k \sum_{b\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]}\ \sum_{z\in\mathbb{Z}^2} q_b(z)^2 \\ &\quad\times \sum_{\substack{b=:n_1<n_2<\ldots<n_k<\infty\\ \max\{n_2-n_1,\ldots,n_k-n_{k-1}\}\leqslant b}}\ \sum_{\substack{x_1:=z\\ x_2,\ldots,x_k\in\mathbb{Z}^2}}\prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})^2\,. \end{aligned} \tag{A.10}$$

We can sum over the space variables: by (3.10) and (3.11), the r.h.s. equals

$$\sum_{k=1}^{\infty}(\sigma_N^2)^k \sum_{b\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]} u_b\,(R_b)^{k-1} \;=\; \sum_{b\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]}\frac{\sigma_N^2\,u_b}{1-\sigma_N^2\,R_b}\,. \tag{A.11}$$

Since $\sigma_N^2\,u_b\sim\frac{\hat\beta^2}{\log N}\frac{1}{b}$ and $\sigma_N^2\,R_b\sim\frac{\hat\beta^2}{\log N}\log b$, as $N\to\infty$ the r.h.s. of (A.11) is asymptotic to

$$\sum_{b\in(N^{\frac{j-1}{M}},N^{\frac{j}{M}}]}\frac{\frac{\hat\beta^2}{\log N}\,\frac{1}{b}}{1-\frac{\hat\beta^2}{\log N}\log b}\sim\int_{N^{\frac{j-1}{M}}}^{N^{\frac{j}{M}}}\frac{\frac{\hat\beta^2}{\log N}\,\frac{1}{x}}{1-\frac{\hat\beta^2}{\log N}\log x}\,\mathrm{d}x \;=\;\int_{\frac{j-1}{M}}^{\frac{j}{M}}\frac{\hat\beta^2}{1-\hat\beta^2\,s}\,\mathrm{d}s\,, \tag{A.12}$$

by the change of variable $x=N^s$. This completes the proof of (6.33). $\square$

A.3. Proof of Lemma 6.7. We can assume that $j\geqslant 2$, because if $j=1$ we have $a=0$ and $x=0$, see (6.36), hence (6.35) trivially holds.

Note that by (6.1) we can write

$$\mathbb{E}\big[X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z')^2\big] \;=\; q_{b-a}(z-x)^2\,F_{N,[b;b']}(z;z')\,,$$

where we set

$$F_{N,[b;b']}(z;z') \;:=\; \sum_{k=1}^{\infty}(\sigma_N^2)^k \sum_{\substack{b=:n_1<n_2<\ldots<n_{k-1}<n_k=b'\\ 1\leqslant n_2-n_1,\ldots,n_k-n_{k-1}\leqslant b}}\ \sum_{\substack{x_1:=z,\,x_k:=z'\\ x_2,\ldots,x_{k-1}\in\mathbb{Z}^2}}\prod_{i=2}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})^2\,.$$

The key point is that $F_{N,[b;b']}(z;z')$ does not depend on (a,x). It follows that

$$\mathbb{E}\big[\big(X^{\mathrm{dom}}_{N,[a,b;b']}(x,z;z')-X^{\mathrm{dom}}_{N,[0,b;b']}(0,z;z')\big)^2\big]=\big(q_{b-a}(z-x)-q_b(z)\big)^2\,F_{N,[b;b']}(z;z')\,,$$

therefore, to prove (6.35), it is enough to show that for $K,M\in\mathbb{N}$ and $\varepsilon>0$ there is $N_0=N_0(\varepsilon,M,K)<\infty$ such that, for $N>N_0$ and for a,b,x,z as in (6.36), we have

$$\left|1-\frac{q_b(z)}{q_{b-a}(z-x)}\right|\leqslant\varepsilon\,. \tag{A.13}$$

We recall the local limit theorem [LL10, Theorem 2.1.3]: as $n\to\infty$, uniformly for $y\in\mathbb{Z}^2$,[†]

$$q_n(y)=\frac{1}{n/2}\big(g\big(\tfrac{y}{\sqrt{n/2}}\big)+o(1)\big)\,2\,\mathbb{1}_{(n,y)\in\mathbb{Z}^3_{\mathrm{even}}}\qquad\text{with}\qquad g(x):=\frac{\mathrm{e}^{-|x|^2/2}}{2\pi}\,. \tag{A.14}$$

[†]The scaling factor in (A.14) is $n/2$ because the simple random walk on $\mathbb{Z}^2$ has covariance matrix $\frac{1}{2}I$, while the factor $2\,\mathbb{1}_{(n,y)\in\mathbb{Z}^3_{\mathrm{even}}}$ is due to periodicity.

In particular, for $(n,y) \in \mathbb{Z}^3_{\mathrm{even}}$ in the "diffusive regime" we can write

$$q_n(y) = \frac{4}{n}\, g\big(\tfrac{y}{\sqrt{n/2}}\big)\big(1+o(1)\big) \qquad \text{for } |y| = O(\sqrt{n})\,. \tag{A.15}$$

Note that a,b,x,z as in (6.36) satisfy (recall that $j \geqslant 2$)

$$0 \leqslant a \leqslant N^{\frac{j-2}{M}} \leqslant N^{-\frac{1}{M}} b\,, \qquad |z| \leqslant K\sqrt{b}\,, \qquad |x| \leqslant K^2\sqrt{a} \leqslant K^2\sqrt{N^{-\frac{1}{M}}}\,\sqrt{b}\,. \tag{A.16}$$

It follows that for any $K, M \in \mathbb{N}$, uniformly for a,b,x,z as in (6.36), we have as $N \to \infty$

$$a = o(b)\,, \qquad |z| = O(\sqrt{b})\,, \qquad |x| = o(\sqrt{b})\,,$$

which in turn imply that $|z-x| \leqslant |z| + |x| = O(\sqrt{b}) = O(\sqrt{b-a})$ and hence, by (A.15),

$$\frac{q_b(z)}{q_{b-a}(z-x)} = \frac{b-a}{b}\exp\left(\frac{|z-x|^2}{b-a} - \frac{|z|^2}{b}\right)(1+o(1)) \xrightarrow[N\to\infty]{} 1\,.$$

This completes the proof of (A.13), hence of (6.35). $\square$

A.4. Proof of (6.45)**.** The random variables η_N in (3.19) satisfy $\sup_N \mathbb{E}[|\eta_N|^{\overline{p}}] < \infty$ for all $\overline{p} < \infty$, by the assumption (3.1) (see [CSZ17a, eq. (6.7)]). We can then estimate $\mathbb{E}\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big]^{\frac{2}{p}}$ by the hypercontractive bound (4.2), which gives rise to the r.h.s. of (A.10) with σ_N^2 replaced by $C_p\,\sigma_N^2$. We can then follow the proof of Lemma 6.6 in Appendix A.2 verbatim though (A.11) and (A.12), where we note that the replacement of σ_N^2 by $C_p\,\sigma_N^2$ amounts to replace $\hat\beta^2$ by $C_p\,\hat\beta^2$, by (3.19) and (3.12). We thus obtain

$$\limsup_{N\to\infty}\ \mathbb{E}\big[|X^{\mathrm{dom}}_{N,M}(j)|^p\big]^{\frac{2}{p}} \leqslant \int_{\frac{j-1}{M}}^{\frac{j}{M}} \frac{C_p\hat\beta^2}{1-C_p\hat\beta^2 s}\,\mathrm{d}s \leqslant \frac{\tilde c}{M}\,, \qquad \text{with} \quad \tilde c := \tfrac{C_p\hat\beta^2}{1-C_p\hat\beta^2}\,. \tag{A.17}$$

Since $\hat\beta < 1$ and $\lim_{p\downarrow 2} C_p = 1$, see [CSZ20, Theorem B.1], we can fix $p > 2$ such that $C_p\hat\beta^2 < 1$, which ensures that $\tilde c < \infty$ and completes the proof. $\square$

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