TUE 8 JAN 2025

We started with the "classical" controlled ODE:

$$(*) \qquad \dot{Z}_t = \sigma(Z_t)\,\dot{X}_t \qquad t \in [0,T]$$

where $X = (X_t)_{t\in[0,T]}$ is a given path in $\mathbb{R}^d$,
$Z = (Z_t)_{t\in[0,T]}$ is the unknown solution in $\mathbb{R}^k$,
$\sigma : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^k)$ is given and sufficiently regular.

We gave a meaning to $(*)$ when X is not differentiable, more specifically $X \in \mathcal{C}^\alpha$ for some $\alpha \in (0,1]$.

YOUNG CASE $\alpha > \frac{1}{2}$: we consider the DIFFERENCE EQ.

$$(*') \qquad \underbrace{\delta Z_{st}}_{Z_t - Z_s} = \sigma(Z_s)\,\delta X_{st} + o(t-s)$$

for which we proved WELL-POSEDNESS.

ROUGH CASE $\alpha \in (\frac{1}{3}, \frac{1}{2}]$: we consider

$$(*'') \qquad \delta Z_{st} = \sigma(Z_s)\,\mathbb{X}^1_{st} + \sigma_2(Z_s)\,\mathbb{X}^2_{st} + o(t-s)$$

where $\sigma_2(z) := \nabla\sigma(z)\cdot\sigma(z)$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ is an α-ROUGH PATH over X :

$$\mathbb{X}^1_{st} = \delta X_{st} \qquad \mathbb{X}^2_{st} \text{ "}= \underbrace{\int_s^t \delta X_{su} \otimes \dot{X}_u \, du}_{\text{FORMAL}}\text{"}$$

$\mathbb{X}^2 = (\mathbb{X}^2_{st})_{0 \leq s < t \leq T}$ with values in $\mathbb{R}^{d \times d} = \mathbb{R}^d \otimes \mathbb{R}^d$ satisfies

- CHEN'S RELATION $\quad \delta \mathbb{X}^2_{sut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^2_{ut}$
- ANALYTIC BOUND $\quad \mathbb{X}^2_{st} = O\left((t-s)^{2\alpha}\right)$.

We now take a different approach to make sense of (✱) when X is not differentiable. First, we formally rewrite (✱) in integral form:

$$Z_t = Z_0 + \underbrace{\int_0^t \sigma(Z_u) \dot{X}_u \, du}_{\mathcal{I}_t(\sigma(Z), X)}$$

The new idea is to try to define a notion of integration

$$\mathcal{I}_t(Y, X) \text{ "}= \int_0^t Y_u \, dX_u\text{"}$$

which generalizes $\int_0^t Y_u \dot{X}_u \, du$ for non-differentiable X and for a wide enough class, say $\mathcal{Y}$, of integrands Y.

The hope is that $\mathcal{Y}$ has a metric structure, such that $Y \mapsto \mathcal{I}_t(Y, X)$ is continuous, and that we can solve by fixed point the following integral equation which generalises (*):

$$(*''') \qquad Z_t = Z_0 + \mathcal{I}_t(\sigma(Z), X)$$

Let us forget, for a moment, about the equation: we focus on the problem of defining a notion of integration

$$\mathcal{I}_t(Y, X) \text{ ``}=\text{'' } \int_0^t Y_u \, dX_u$$

for general Hölder paths X, Y.

For simplicity, we mainly focus on the scalar case: $X_t, Y_t \in \mathbb{R}$.

Ch 6 - THE SEWING LEMMA

Goal: define $I_t = \int_0^t Y_u \, dX_u$ for general paths

$$X, Y : [0, T] \to \mathbb{R}.$$

If $X \in C^1$, then $I_t = \int_0^t Y_u \dot{X}_u \, du$ satisfies

$$\begin{aligned} I_t - I_s &= \int_s^t Y_u \dot{X}_u \, du \\ &= Y_s (X_t - X_s) + \underbrace{\int_s^t (Y_u - Y_s) \dot{X}_u \, du}_{o(t-s)} \end{aligned}$$

Our first hope is then to characterize I by $I_0 = 0$ and

$$I_t - I_s = Y_s (X_t - X_s) + o(t-s)$$

$$(\star') \qquad \delta I_{st} = Y_s \, \delta X_{st} + o(t-s)$$

This is close in spirit to, but simpler than, the "Young" difference equation $(*')$. For instance, given ANY paths X, Y, it is immediate to show that there exists AT MOST ONE I with $I_0 = 0$ which solves $(\star')$. It is convenient to write

$$(\tilde{\star}) \qquad \delta I_{st} = A_{st} + o(t-s)$$

where the term $A = (A_{st})_{0 \le s \le t \le T}$ is given and called the GERM (for us $A_{st} = Y_s \, \delta X_{st}$).

Lemma (UNIQUENESS) Given an arbitrary germ $A = (A_{st})$, any two solutions I, I' of $(\tilde{\star})$ differ by a constant: $\exists c \in \mathbb{R}$: $I_t = I'_t + c \quad \forall t \in [0,T]$

Proof: $\delta (I - I')_{st} = o(t-s)$

$$\Rightarrow \frac{d}{dt}(I - I')_t = 0 \quad \forall t \in [0,T].$$

Since uniqueness of I holds for free, it remains to focus on conditions for EXISTENCE of I, given A.

We rewrite

$$(\tilde{\star}) \qquad \delta I_{st} = A_{st} + R_{st} \quad \text{with} \quad R_{st} = o(t-s).$$

We call A the GERM, I the INTEGRAL, R the REMAINDER.

<u>Lemma</u>. Given a germ A, a necessary condition for the existence of I which solves $(\tilde{\star})$ is

$$\underbrace{|\delta A_{sut}|}_{A_{st} - A_{su} - A_{ut}} = o(t-s) \qquad \text{for } 0 \leq s \leq u \leq t \leq T.$$

<u>Proof</u>. Assume that a solution I of $(\tilde{\star})$ exists, i.e.

defining $R_{st} := \delta I_{st} - A_{st}$ we have $R_{st} = o(t-s)$.

Then
$$|\delta R_{sut}| = |R_{st} - R_{su} - R_{ut}| = o(t-s) + o(u-s) + o(t-u) \overset{!}{=} o(t-s)$$

Moreover
$$\delta R_{sut} = \delta(\delta I - A)_{sut} \overset{!}{=} -\delta A_{sut}$$

$\square$

The key result, known as the SEWING LEMMA, shows that the necessary condition above is close to being sufficient!

Definition (COHERENCE) - Given $\eta > 1$, a germ $A = (A_{st})$ is called η-coherent if it satisfies

$$\| \delta A \|_\eta := \sup_{\substack{0 \leq s \leq u \leq t \leq T \\ s < t}} \frac{|\delta A_{sut}|}{(t-s)^\eta} < \infty$$

i.e. $\delta A_{sut} = O((t-s)^\eta)$.

Theorem (SEWING LEMMA) Let a germ $A = (A_{st})_{0 \leq s \leq t \leq T} \in C_2$ be η-coherent for some $\eta > 1$. Then there exists a (necessarily unique) function $I = (I_t)_{t \in [0,T]}$ such that $I_0 = 0$ and $\tilde{\star}$ holds, i.e.

$\tilde{\star}$ $\quad I_0 = 0, \qquad \delta I_{st} = A_{st} + R_{st}$ with $R_{st} = o(t-s)$.

(1) The remainder $R_{st} := \delta I_{st} - A_{st}$ satisfies the SEWING BOUND: $R_{st} = O((t-s)^\eta)$.

$$\| R \|_\eta \leq K_\eta \| \delta A \|_\eta = K_\eta \| \delta R \|_\eta \quad \text{with } K_\eta = \frac{1}{1-2^{1-\eta}} < \infty.$$

(2) We have $\forall t \in [0,T]$, for partitions P of $[0,t]$,

$$I_t = \lim_{|P| \to 0} \sum_{i=0}^{\#P-1} A_{t_i t_{i+1}}.$$

Proof. We already proved the sewing bound:

Thm (SEWING BOUND): If $R = (R_{st})$ satisfies $R_{st} = o(t-s)$,

then $\begin{cases} \cdot \ \|R\|_\eta \leq K_\eta \|\delta R\|_\eta \\ \cdot \ \lim\limits_{|P| \to 0} \sum\limits_{i=0}^{\#P} R_{t_i t_{i+1}} = 0 \end{cases}$

Assume that a solution I of $\tilde{\star}$ exists.

Then $R_{st} := \delta I_{st} - A_{st} = o(t-s)$ and we can apply the sewing bound that we just recalled, yielding (1).

(Recall that $\delta R = -\delta A$ since $\delta(\delta I) = 0$).

Moreover by $\delta I_{st} = A_{st} + R_{st}$ we get

$$I_t - I_0 = \sum_{i=0}^{\#P-1} \delta I_{t_i t_{i+1}} = \sum_{i=0}^{\#P-1} A_{t_i t_{i+1}} + \underbrace{\sum_{i=0}^{\#P-1} R_{t_i t_{i+1}}}_{\to 0 \text{ as } |P| \to 0}$$

which proves (2).

It remains to prove the existence of I solution of $\tilde{\star}$.

Fix a partition $P = \{t_0 < t_1 < \dots < t_K\}$

Let $P' = P \setminus \{t_i\} = \{t_0 < t_1 < \dots < t_{i-1} < t_{i+1} < \dots < t_K\}$

Set $\quad \tilde{I}_P(A) := \sum_{i=0}^{\#P-1} A_{t_i t_{i+1}}$

Then $\quad \tilde{I}_{P'}(A) - I_P(A) = A_{t_{i-1} t_{i+1}} - (A_{t_{i-1} t_i} + A_{t_i t_{i+1}})$

$$= \delta A_{t_{i-1} t_i t_{i+1}}$$

CLAIM: $\forall P$ partition of $[a,b]$ into κ intervals (i.e. $t_0 = a$, $t_\kappa = b$)

$\exists\, i$ s.t. $(t_{i+1} - t_{i-1}) \leq 2\,\frac{b-a}{\kappa-1}$

Removing from P such a point t_i we obtain

$$|\tilde{I}_{P'}(A) - I_P(A)| = |\delta A_{t_{i-1} t_i t_{i+1}}| \leq \|\delta A\|_\eta \left(2\,\frac{(b-a)}{\kappa-1}\right)^\eta$$

Iterating this argument, i.e. removing all interior points from the partition P of $[a,b]$ until we obtain the trivial partition $\{a,b\}$, we get

$$\underbrace{|\tilde{I}_{\{a,b\}}(A) - \tilde{I}_P(A)|}_{A_{ab}} \leq \sum_{n=2}^{\kappa-1} \left(\frac{2\,(b-a)}{n-1}\right)^\eta \leq C_\eta \cdot (b-a)^\eta$$

where $C_\eta = \sum_{n=2}^{\infty} \frac{2^\eta}{(n-1)^\eta} = \sum_{\ell=1}^{\infty} \frac{2^\eta}{\ell^\eta}$

MISSING $\|\delta A\|_\eta$

So far we showed the following: $\forall\ [a,b] \subseteq [0,T]$ and for any partition P of $[a,b]$ we have

$$|A_{ab} - \mathcal{I}_P(A)| \leq \|\delta A\|_\eta \cdot C_\eta \cdot (b-a)^\eta \qquad \text{𝄞}$$

We will deduce from this that, given arbitrary partitions P, P' of $[a,b]$, we have

$$|\mathcal{I}_P(A) - \mathcal{I}_{P'}(A)| \leq \|\delta A\|_\eta \cdot C_\eta \cdot T \cdot \{|P|^{\eta-1} + |P'|^{\eta-1}\} \qquad \boxdot$$

This shows that, given $[a,b] \subseteq [0,T]$, the family $(\mathcal{I}_P(A))$ as P ranges over partitions of $[a,b]$ is CAUCHY:

$$\forall \varepsilon > 0 \ \exists \delta > 0: \text{ if } |P|, |P'| < \delta: \ |\mathcal{I}_P(A) - \mathcal{I}_{P'}(A)| < \varepsilon$$

This implies that $\lim_{|P| \to 0} \mathcal{I}_P(A) =: J_{ab}$ exists!

Let us define $I_t := J_{0,t}$. We claim that

$$I_t - I_s = J_{st} \qquad \forall\ 0 \leq s < t \leq T.$$

Finally we apply 𝄞 with $[a,b] = [s,t]$ for a sequence of partitions P_n with $|P_n| \to 0$:

$$|A_{st} - \underset{\substack{\downarrow \\ J_{st} = I_t - I_s}}{\mathcal{I}_{P_n}(A)}| \leq \|\delta A\|_\eta \cdot C_\eta \cdot (t-s)^\eta$$

i.e. $|A_{st} - \delta I_{st}| = O((t-s)^\eta) = o(t-s)$

which is $\tilde{\star}$.

It remains to prove that $\oint \Rightarrow \boxed{\cdot}$.

Given P partition of $[a,b]$ and $Q \supseteq P$ we have

$$|\tilde{I}_Q(A) - \tilde{I}_P(A)| = \sum_{i=0}^{K-1} |\tilde{I}_{Q\cap[t_i,t_{i+1}]}(A) - A_{t_i t_{i+1}}|$$

$P = \{t_0 < t_1 < \dots < t_K\}$

by $\oint$

$$\leq \|\delta A\|_\eta \cdot C_\eta \sum_{i=0}^{K-1} \underbrace{(t_{i+1}-t_i)^\eta}_{\leq |P|^{\eta-1}\cdot(t_{i+1}-t_i)}$$

$$\leq \|\delta A\|_\eta \cdot C_\eta \cdot T \cdot |P|^{\eta-1}$$

Given ARBITRARY partitions P, P', we can consider $Q := P \cup P'$ and by triangle inequality we get our goal $\boxed{\cdot}$.

Last claim to check:

CLAIM: $\forall P$ partition of $[a,b]$ into K intervals (i.e. $t_0 = a$, $t_K = b$)

$$\exists\, i \text{ s.t. } (t_{i+1} - t_{i-1}) \leq 2\,\frac{b-a}{K-1}$$

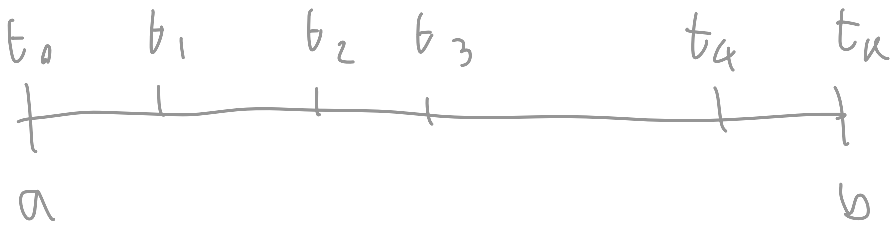
By contradiction: if no such i exists then

$$\forall i=1,\dots,k-1: \quad (t_{i+1}-t_{i-1}) > 2\,\frac{b-a}{k-1}$$

hence
$$2(b-a) \geq \sum_{i=1}^{k-1}(t_{i+1}-t_{i-1}) > 2\,\frac{(b-a)}{\cancel{k-1}}\cdot(\cancel{k-1})$$

$\downarrow$

IMPOSSIBLE!



This completes the proof of the SEWING LEMMA $\blacksquare$

Ch. 7 YOUNG INTEGRAL

Let us go back to our goal: define $I_t =$ "$\int_0^t Y_u\, dX_u$" rigorously as the solution of $I_0 = 0$ and

(★) $$\delta I_{st} = A_{st} + R_{st} \quad \text{with } R_{st} = o(t-s)$$

for $A_{st} = Y_s \cdot \delta X_{st}$

Compactly: $$\delta I_{st} = Y_s \cdot \delta X_{st} + o(t-s)$$

Theorem (YOUNG INTEGRAL) Fix any $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$. For any $X \in \mathcal{C}^\alpha$, $Y \in \mathcal{C}^\beta$ there exists a (necessarily unique) function $I : [0,T] \to \mathbb{R}$ such that

$$I_0 = 0, \qquad I_t - I_s = Y_s (X_t - X_s) + o(t-s)$$

Such a function is called YOUNG INTEGRAL and denoted

$$I_t = \int_0^t Y_u \, dX_u .$$

We have, for partitions $\mathcal{P}$ of $[0,t]$,

$$I_t = \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{\#\mathcal{P}-1} Y_{t_i} \cdot \delta X_{t_i t_{i+1}} .$$

We can define the REMAINDER $R_{st} := I_t - I_s - Y_s \cdot (X_t - X_s)$ which satisfies

$$\| R \|_{\alpha+\beta} \lesssim K_{\alpha+\beta} \| \delta X \|_\alpha \| \delta Y \|_\beta$$

This yields that $I \in \mathcal{C}^\alpha$, more precisely

$$\| \delta I \|_\alpha \lesssim \left(\| Y \|_\infty + K_{\alpha+\beta} T^\beta \| \delta Y \|_\beta \right) \| \delta X \|_\alpha .$$

Finally the map $(X, Y) \mapsto I$ is bi-linear and continuous from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^\alpha$.

Proof. Define the germ $A_{st} := Y_s \, \delta X_{st}$.

We have already computed $\delta A_{sut} = - \delta Y_{su} \, \delta X_{ut}$

$$\Rightarrow \quad \| \delta A \|_{\alpha+\beta} \leq \| \delta Y \|_{\beta} \cdot \| \delta X \|_{\alpha} < \infty$$

If $\alpha + \beta > 1$ we can apply the SEWING LEMMA since the germ A is η-coherent with $\eta = \alpha + \beta > 1$: there exists a (unique) function I which satisfies

$$I_0 = 0 \qquad \delta I_{st} = \underbrace{A_{st}}_{Y_s \cdot \delta X_{st}} + R_{st} \quad \text{with } R_{st} = o(t-s)$$

Note that, by $\delta I_{st} = Y_s \, \delta X_{st} + R_{st}$,

$$\| \delta I \|_{\alpha} \leq \| Y \|_{\infty} \cdot \| \delta X \|_{\alpha} + \underbrace{\| R \|_{\alpha}}_{\leq T^{\beta} \cdot \underbrace{\| R \|_{\alpha+\beta}}_{\leq K_{\alpha+\beta} \| \delta X \|_{\alpha} \| \delta Y \|_{\beta}}}$$

The proof is completed. ∎

We can now connect this notion with the difference equations that we studied in the first part.

Let σ be Lipschitz:

Proposition. Fix $\alpha \in (\frac{1}{2}, 1]$ and $X \in \mathcal{C}^\alpha$ $\gamma \|\nabla \sigma\|_\infty < \infty$.

A function $Z \in \mathcal{C}^\alpha$ satisfies the Young diff. eq. (*'),

$$\delta Z_{st} = \sigma(Z_s)\,\delta X_{st} + o(t-s)$$

iff it satisfies the integral eq. (*'),

$$Z_t = Z_0 + \underbrace{\underbrace{\int_0^t \sigma(Z_u)\,dX_u}_{\text{YOUNG INTEGRAL}}}_{I_t}$$

Note: $Z \in \mathcal{C}^\alpha$ + σ Lipschitz $\Rightarrow$ $\sigma(Z) \in \mathcal{C}^\alpha$

and Young integral is well-def. since $\alpha + \alpha = 2\alpha > 1$ for $\alpha > \frac{1}{2}$.