The FBSDE Approach to the Stochastic Quantisation of the sine-Gordon EQFT.

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What is a QFT?

- Not so clear, but the *Wightman-axioms* are a minimal requirement on the correlation functions
- Osterwalder-Schrader reconstruction theorem ('75):

Quantum field Theory defined on Minkowski space, $\|x\|^2 = -x_0^2 + x_1^2 + \cdots + x_d^2$ $\xleftrightarrow{x_0 \longrightarrow ix_0}$ **Euclidean Quantum Field Theory** defined on Euclidean space, $\|x\|^2 = x_0^2 + x_1^2 + \cdots + x_d^2$

- EQFTs have to satisfy the *OS*-axioms, roughly
    - Euclidean invariance
    - Reflection positivity
    - Suitable regularity

  Very restrictive, may be obtained as **probability measures** on certain distributional spaces.

Goal: Construct an interesting (e.g. non-Gaussian) EQFT.

Simplest case: Gaussian Free Field

Can check directly that the GFF that is the **Gaussian measure** on $\mathcal{S}'(\mathbb{R}^d)$ with

$$\mathrm{Cov}(\mu) = (m^2 - \Delta)^{-1},$$

satisfies the OS axioms and defines a (not terribly interesting) EQFT.

Fact: only for $d = 1$ supported on functions (in general, $\mathrm{supp}(\mu) \subset H^{\alpha}(\mathbb{R}^d)$ for any $\alpha < \frac{2-d}{2}$)

Upshot: We can try to perturb this Gaussian theory to obtain new EQFTs!

For μ the GFF, turns out that perturbations of the from

$$\text{“}\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \mathrm{e}^{-V(\varphi)} \mu(\mathrm{d}\varphi)\text{”} \quad \text{where} \quad V(\varphi) = \lambda \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$$

formally also satisfy the axioms. Some starting points:

- in $d = 2$:

$$U(x) = x^{2p} + \sum_{\ell}^{2p-1} a_\ell x^\ell \quad \text{for any } p > 0,$$
$$U(x) = \exp(\beta x),$$
$$U(x) = \cos(\beta x),$$

- in $d = 2, 3$:

$$U(x) = x^4 - b x^2.$$

Goal: Make sense of

$$\text{“}\,\nu(\mathcal{O}) = \frac{1}{\text{norm.}}\int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\mathrm{e}^{-S(\varphi)}\mathrm{d}\varphi$$
$$= \frac{1}{\text{norm.}}\int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\mathrm{e}^{-\lambda\int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x}\mu(\mathrm{d}\varphi)\text{”},$$

with μ a (massive) Gaussian free field, some $U: \mathbb{R} \longrightarrow \mathbb{R}$ and

$$S(\varphi) := \int_{\mathbb{R}^d} \frac{1}{2}(|\nabla\varphi(x)|^2 + m^2|\varphi(x)|^2) + U(\varphi(x))\mathrm{d}x.$$

Problems:

Large Scales: No decay in space: $S(\varphi) = \infty$ at best (non-sense at worst)

Small Scales: μ not supported on function spaces but only on **distributions**

$\longrightarrow$ $U(\varphi(x))$ ill-defined

With $V(\varphi) = \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$, define approximations

$$e^{-V(\varphi)}\mu(\mathrm{d}\varphi) \quad \approx \quad e^{-V^{\rho,\varepsilon}(\varphi)}\mu^{\varepsilon}(\mathrm{d}\varphi),$$

Large Scale Problem	Small Scale Problem
$\int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x = \infty?$	$\varphi(x) = ???$ for $\varphi \in H^{(2-d)/2-}(\mathbb{R}^d)$
cut-off in space $\rho \in C_c^{\infty}(\mathbb{R}^{\mathrm{d}})$:	Regularise the measure:
	$\mu^{\varepsilon} \longrightarrow \mu$,
$V^{\rho}(\varphi) = \int_{\mathbb{R}^d} \rho(x) U(\varphi(x))\mathrm{d}x$	μ^{ε} supported on a **function** space
	Additionally:
(or replace $\mathbb{R}^d$ by the torus $\mathbb{T}^d = ([0,1]/\sim)^d$)	Choose V^{ε} depending on ε

Question: Can we recover a EQFT?

$$\nu^{\rho,\varepsilon}(\mathcal{O}) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) e^{-V^{\rho,\varepsilon}(\varphi)} \mu^{\varepsilon}(d\varphi)$$

$$\xrightarrow{???}$$

$$\text{“ } \nu(\mathcal{O}) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) e^{-V(\varphi)} \mu(d\varphi) \text{ ”.}$$

Problems:

- ν is not absolutely continuous w.r.t. the Gaussian free field μ
- What does it mean to be *the* φ^4_d/SG EQFT? The construction above is always very implicit.

$\longrightarrow$ Move to different characterisations for $\nu^{\rho,\varepsilon}$ that do not rely on absolute continuity/ that make sense in the limit!

Starting point:

Given a Gaussian (here μ) and two cut-offs ρ, $\varepsilon > 0$ we can construct $\nu^{\rho,\varepsilon}$

(namely as the Gibbsian perturbation of the GFF)

Think of a map

$$\text{“ } \Phi^{\rho,\varepsilon} \colon \mu \mapsto \nu^{\rho,\varepsilon} \text{ ”} \quad \rho \in C_c^\infty(\mathbb{R}^2), \ \ \varepsilon > 0.$$

Idea: Study the maps $\Phi^{\rho,\varepsilon}$ to learn about the measures $\nu^{\rho,\varepsilon}$ and (ideally) remove both regularisations ρ, ε.

Examples: Parabolic $\mathscr{L} = \partial_t - \Delta$ or elliptic $\mathscr{L} = -\Delta$ SQ: Fomally ν^ε is invariant under

$$\mathscr{L}\varphi^\varepsilon = \xi^\varepsilon - V_\varepsilon'(\varphi^\varepsilon) \quad \text{for a discretised space time white noise } \xi^\varepsilon \longrightarrow \xi,$$

and try to obtain tightness for ν^ε from this equation.

[G. Parisi, Y. Wu · Perturbation theory without gauge fixing · Sci. Sin. · 1981]

Stochastic Quantisation and Renormalisation Group: An FBSDE for the measure

Input:

- a **scale decomposition** $(\int_0^t \dot{G}_s \mathrm{d}s)_{t\geqslant 0} =: (G_t)_{t\geqslant 0}$ of the covariance of the GFF such that

  $G_\infty = (m^2 - \Delta)^{-1}$

- a cylindrical Brownian motion $(B_t)_{t\geqslant 0}$ on $L^2(\mathbb{R}^d)$

Output:

Mod suitable UV and IR regularisations, a measure

$$\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \mathrm{e}^{-\lambda V(\varphi)} \mu(\mathrm{d}\varphi), \quad \text{where} \quad \mu := \mathrm{Law}\left(\int_0^\infty \dot{G}_t^{1/2} \mathrm{d}B_t\right) \sim \mathrm{GFF}.$$

as $\nu := \mathrm{Law}(\varphi_\infty)$, where φ is the solution to

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[-\mathrm{D}V(\varphi_\infty)] \mathrm{d}s + W_t, \qquad W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s, \qquad t \in [0, \infty]. \tag{1}$$

Intuition: $(G_t)_{t\geqslant 0}$ as a sense of **scale**

Without interaction, $V \equiv 0$,

$$\varphi_t^{\text{free}} = W_t = \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s, \qquad \mu = \mathrm{Law}(W_\infty)$$

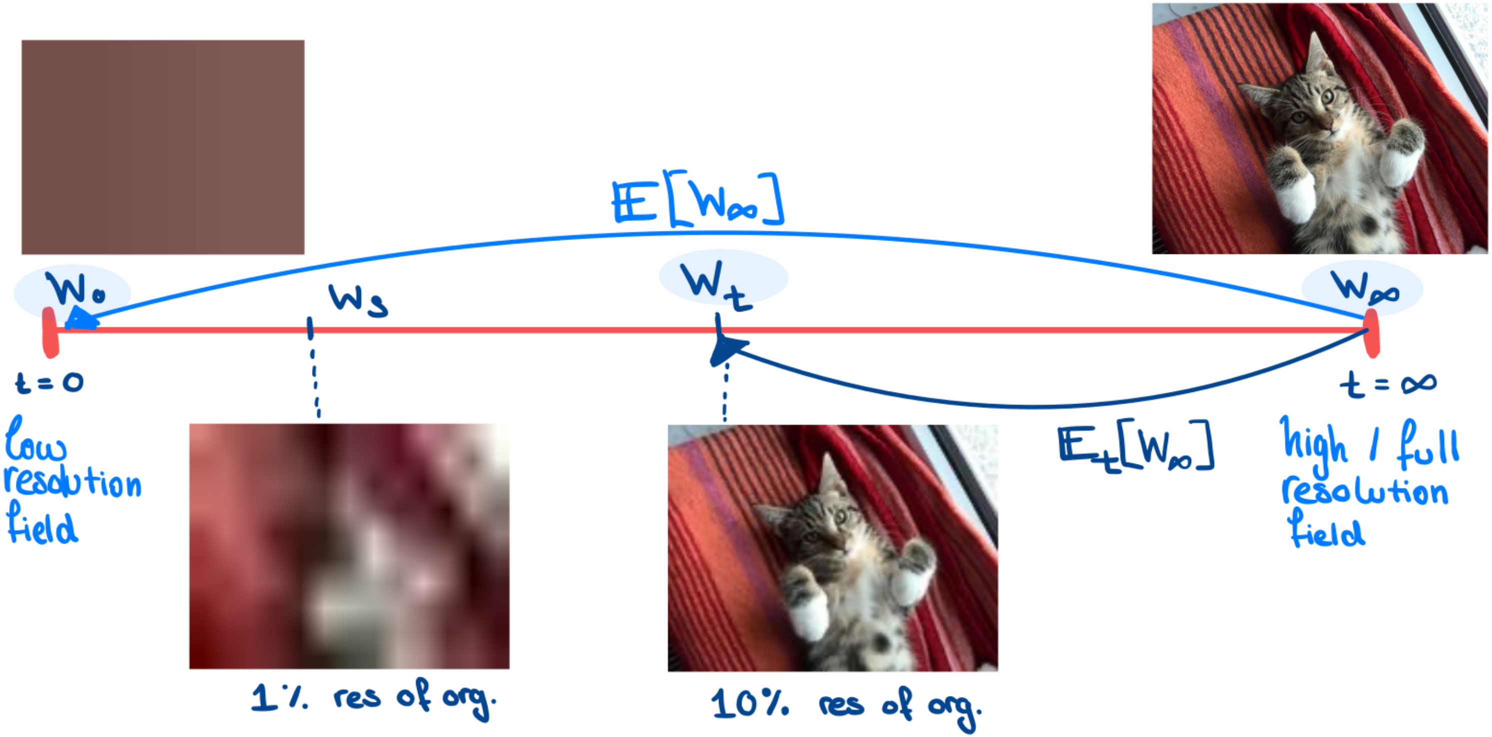
is a **martingale**: Given the **small scale** description W_∞, we can recover an **effective** description

$$W_t = \mathbb{E}_t[W_\infty] \quad \text{for any resolution } t \geqslant 0.$$

$\longrightarrow$ $(\mathbb{E}_t)_{t\geqslant 0}$ integrates out small details and **projects** the full field W_∞ down to W_t at resolution t.

$\longrightarrow$ Understand $\mathcal{F}_t$ as the information available at resolution $t \in [0, \infty]$.

Coarsening effect of $\mathbb{E}_t$



For $V \neq 0$,

$$\varphi_t^V = \int_0^t \dot{G}_s \mathbb{E}_s[-DV(\varphi_\infty)]\mathrm{d}s + \int_0^t \dot{G}_s^{1/2} \mathrm{d}B_s,$$

is not a martingale under $\mathbb{P}$, but $\varphi_t \hat{\in} \mathscr{F}_t$ for any $t \geqslant 0$.

Scale-by-scale coupling and an **effective description** of the field

$$\varphi_\infty^V = Z_\infty^V + W_\infty$$

at any scale $t \geqslant 0$.

$\longrightarrow$ The FBSDE describes **the dynamics of changing between scales**.

The UV-Problem:

Formally, for

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s + \int_0^t \dot{G}_s^{1/2}\mathrm{d}B_s, \qquad F := -\mathrm{D}V,$$

we have $\mathrm{Law}(\varphi_\infty) = \nu^V$.

Problem: The equation at resolution $t < \infty$ seems to depend on *all* small-scale information φ_∞ and we are really looking for a solution to an **FBSDE**

$$\begin{cases} \varphi_t = \int_0^t \dot{G}_s Y_s \mathrm{d}s + W_t, & \text{solved } \textit{forward} \text{ from } \varphi_0 = 0, \\ Y_t = F(\varphi_\infty) - \int_t^\infty \eta_s \mathrm{d}W_s & \text{solved } \textit{backwards} \text{ from } Y_\infty = F(\varphi_\infty). \end{cases}$$

Why is this a problem? Cannot expect φ_∞ to be better than W_∞ which is distributional $\Longrightarrow F(\varphi_\infty) = ?$

Formally, for

$$\varphi_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s + \int_0^t \dot{G}_s^{1/2}\mathrm{d}B_s, \qquad F := -\mathrm{D}V,$$

we have $\mathrm{Law}(\varphi_\infty) = \nu^V$.

Upshot: $\mathbb{E}_s$ is a *coarsening operation* $\Longrightarrow$ $\mathbb{E}_s[F(\varphi_\infty)] \hat{\in} \mathcal{F}_s$ for $s < \infty$ should be a proper *function.*

Question: Can we find/study this function $(F_s(\varphi_s))_{s\in[0,\infty)}$,

$$F_s(\varphi_s) \overset{!}{=} \mathbb{E}_s[F(\varphi_\infty)]?$$

$$Z_t = \int_0^t \dot{G}_s \mathbb{E}_s[F(\varphi_\infty)]\mathrm{d}s, \qquad \varphi_t = Z_t + W_t$$

For a generic **scale dependent** functional $(F_t)_{t\geqslant 0}$ by Ito's formula,

$$F_\infty(\varphi_\infty) = F_t(\varphi_t) + \int_t^\infty \left(\partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{Z}_s\right)(\varphi_s)\mathrm{d}s + \int_t^\infty \mathrm{D}F_s(\varphi_s)\mathrm{d}W_s$$

Therefore if $(F_t)_{t\geqslant 0}$ is a solution to the (backward) RG-flow equation,

$$H_s^F := \partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s \equiv 0, \qquad F_\infty = F$$

we find

$$\mathbb{E}_s[F(\varphi_\infty)] = \mathbb{E}_s[F_\infty(\varphi_\infty)] = F_s(\varphi_s).$$

and instead of an **FBSDE** we have the **SDE**

$$\varphi_t = \int_0^t \dot{G}_s F_s(\varphi_s)\mathrm{d}s + W_t.$$

There's a catch...

$$\partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s \equiv 0 \qquad F_\infty = F$$

is a **nonlinear** and **infinite dimensional** PDE for a function

$$F_t \colon \mathcal{S}'(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d), \qquad t \in [0,\infty].$$

Asking for **exact** solutions may be too much!

Instead: Can we find **approximate solutions** $(F_t)_{t\in[0,\infty]}$ such that $F_\infty = F$,

$$H_s^F = \partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s \approx 0,$$

is small (in some sense)?

Stable reformulation

Given **any** $(F_t)_{t\in[0,\infty]}$ define

$$H_s^F := \partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) + \mathrm{D}F_s \dot{G}_s F_s, \quad F_\infty = F$$

and

$$R_t := \mathbb{E}_t[F_\infty(\varphi_\infty) - F_t(\varphi_t)] \quad \text{with } R_\infty = 0.$$

to obtain the FBSDE

$$\begin{cases} \varphi_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s + W_t \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s. \end{cases}$$

Note: $H_s^F \equiv 0 \Longrightarrow R \equiv 0$, and we recover the SDE as before; but R allows **more freedom** in the choice of $(F_t)_{t\in[0,\infty)}$

Summary

Reformulated equation: $\varphi_t = Z_t + W_t$

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases} \qquad t \in [0,\infty] \tag{2}$$

so that $\mathrm{Law}(\varphi_\infty) = \nu^V$.

Now the problem comes down to two (related) steps

1. Find a **"good enough" approximation** $(F_t)_{t\in[0,\infty)}$ to the flow equation which allows for good estimates of $\|F_s(\varphi_s)\|$ uniformly in the (surpressed) regularisation.
2. Show **well-posedness for the FBSDE** (2) and obtain uniform a-priori estimates which allow to remove the regularisations.

Theorem 1. *Let $\beta^2 < 6\pi$. There is a choice $(F_t)_{t\in[0,\infty)}$ with*

$$F_t(W_t) = \lambda\rho\beta[\![\sin(\beta W_t)]\!] + O(\lambda^2),$$

such that there is a solution $(Z, R) \in L^\infty(\mathrm{d}\mathbb{P}; L^\infty(\mathbb{R}^2)) \times L^\infty(\mathrm{d}\mathbb{P}, L^\infty(\mathbb{R}^2))$ to the FBSDE

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases} \quad t \in [0, \infty]. \tag{3}$$

- ***Stochastic quantisation:*** *For any regularisation $\rho < 1$, $T < \infty$, it holds $\nu_{\mathrm{SG}}^{\rho,T} = \mathrm{Law}(Z_T^{\rho,T} + W_T^{\rho,T}) = \mathrm{Law}(\varphi_T^{\rho,T})$ which justifies*

$$\nu_{\mathrm{SG}} := \mathrm{Law}(Z_\infty + W_\infty) = \mathrm{Law}(\varphi_\infty).$$

- ***Regularity:*** *For any $\varepsilon > 0$, and $p \in [1, \infty]$ there is a version of the unregularised solution s.t.*

$$Z = Z_\infty \in L^\infty(\mathrm{d}\mathbb{P}; B_{p,p}^\alpha(\langle x\rangle^{-3})), \quad \alpha = 2 - \beta^2/4\pi - \varepsilon,$$

- ***Uniqueness:*** *If $|\lambda| \ll 1$ or $\rho \in C_c^\infty(\mathbb{R}^2)$, then uniqueness in Law holds for (3).*

Here $[\![\cdot]\!]$ denotes the usual Wick-ordering and $B_{p,p}^\alpha(w)$ are the usual weigthed Besov spaces.

Upshot

The FBSDE can **define** $\nu_{\mathrm{SG}} = \mathrm{Law}(Z_\infty + W_\infty)$ on the full space for $\beta^2 < 6\pi$ without reference to a limiting procedure

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s, \\ R_t = \mathbb{E}_t \int_t^\infty H_s(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases}$$

and $\boldsymbol{Z_t} + \boldsymbol{W_t}$ is an **effective description** at scale t.

$\longrightarrow$ Can try to transport properties of the GFF $W_\infty \sim \mu$ to the sine-Gordon field $Z_\infty + W_\infty \sim \nu_{\mathrm{SG}}$ along the FBSDE.

Consequences:

- **OS-Axioms:** Verify OS-Axioms from FBSDE + the limit is not Gaussian

  Caviat: In infinte volume, require the coupling to be small to get all axioms.

- **Singularity:** For $\beta^2 \geqslant 4\pi$, ν_{SG}^{ρ} is *not absolutely continuous* w.r.t. the GFF even in finite volume.

  $\longrightarrow$ so the densities from the beginning are indeed non-sense.

- **Variational description & LDP:** The Laplace-transform of the **infinite volume** measure satisfies a Boue-Dupuis type variational problem (& LDP as a result)

$$\mathcal{W}(f) := -\log \int_{\mathcal{S}'(\mathbb{R}^d)} e^{-f(\varphi)} \nu_{\mathrm{SG}}(\mathrm{d}\varphi) = \inf_{v \in \mathbb{H}^2(\langle x \rangle^n)} \mathbb{E}[f(\bar{\varphi}_\infty(\bar{r}+v)) + \mathscr{H}_s(\varphi(v+\bar{r}), \varphi(\bar{r})) + \mathcal{E}(v, \bar{r})].$$

  also requires small coupling in the infinite volume

Reminder: Started from idea that

$$\text{“}\nu_{\mathrm{SG}}^{\lambda,\beta}(\mathrm{d}\varphi) \propto \exp\Big(\int_{\mathbb{R}^2} \beta [\![\cos(\beta\varphi(x))]\!] \mathrm{d}x\Big)\mu(\mathrm{d}\varphi).\text{”}$$

The FBSDE instead allows to view an EQFT as a **diffusion** indexed by a resolution parameter $t \in [0,\infty]$,

$$\varphi_t = Z_t + W_t + \varphi_0, \quad Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s. \tag{4}$$

$\longrightarrow$ Can talk about **functionals/observables** $\mathcal{O}_t(\varphi_t)$ of the EQFT using **stochastic calculus**.

From this: Decay of correlations: For two compactly supported observables $\mathcal{O}_1, \mathcal{O}_2$,

$$\mathrm{Cov}_\nu(\mathcal{O}_1, \mathcal{O}_2) \lesssim \mathrm{e}^{-\gamma \mathfrak{l}}, \quad \mathfrak{l} := d(\mathrm{supp}(\mathcal{O}_1), \mathrm{supp}(\mathcal{O}_2)),$$

where the implicit constant depends only on regularity constants of $\mathcal{O}_1, \mathcal{O}_2$ and $\gamma = \gamma(m)$.

A Martingale Characterisation of sine-Gordon.

From this POV: We found a **semi-martingale** description for ν_{SG}.

$\longrightarrow$ We may take **the semi-martingales** $\boldsymbol{\varphi}$ as a starting point without referring to the approximate measures $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$

Natural question: Given a semi-martingale

$$\varphi_t = \varphi_0 + A_t + W_t,$$

can we tell whether it describes a sine-Gordon measure?

In other words:

Is there a martingale problem for SG?

Suppose that φ is a semi-martingale such that

$$\varphi_t = \varphi_0 + A_t + W_t$$

- $\langle W \rangle_t = G_t$ with $G_\infty = (m^2 - \Delta)^{-1}$ and $(G_t)_{t\in[0,\infty]}$ the heat kernel scale decomposition

- $A_t := \int_0^t \dot{G}_s f_s \mathrm{d}s$ for the **effective force**

$$f_t = \lim_{T\to\infty} -\mathbb{E}_t[-\lambda_T \beta \sin(\beta\varphi_T)] \qquad \lambda_T := \lambda \exp\left(\frac{\beta^2}{2}\mathbb{E}[W_T^2(0)]\right).$$

  with the additional assumptions on $\|\dot{A}_t\| \lesssim \lambda_t$ (to ensure $A_\infty \in L^\infty\left(\mathrm{d}\mathbb{P}; B^{2-\beta^2/4\pi-}_{p,p}(\langle x\rangle^{-3})\right)$; **subcriticality condition)**.

Can verify that the solution to the FBSDE has these properties.

Martingale problem:

Assume that

$$\varphi_t - \int_0^t \dot{G}_s f_s \mathrm{d}s - \varphi_0 = W_t, \qquad f_s := \lim_{T\to\infty} \mathbb{E}_s[F_T^T(\beta\varphi_T)] := \lim_{T\to\infty} \mathbb{E}_s[-\lambda_T \beta \sin(\beta\varphi_T)],$$

and that for some $\varepsilon > 0$

$$\lim_{T\to\infty} \sup_{s\geq 0} \lambda_s^{-1} s^{-\varepsilon} \|\dot{G}_s f_s - \mathbb{E}_s F_T^T(\beta\varphi_T)\| = 0, \qquad \langle W \rangle_t = G_t$$

Then, φ_t is a weak solution to the FBSDE and thus

$$\nu_{\mathrm{SG}} = \mathrm{Law}(\varphi_\infty).$$

Provided uniqueness in Law holds for the FBSDE ($\lambda \ll 1$ or finite volume) this is a **unique characterisation** of the unique SG measure.

Remark: A similar statement holds for $f_t^\varepsilon = \lim_{\varepsilon\to 0} \mathbb{E}_t[-\lambda c^\varepsilon \beta \sin(\beta(\rho^\varepsilon * \varphi_\infty))]$, $\varphi_\infty = Z_\infty + W_\infty$.

The effective force: Approximate solutions to the flow equation

Recall: The two related sub-problems

Reformulated equation: $\varphi_t = Z_t + W_t$

$$\begin{cases} Z_t = \int_0^t \dot{G}_s(F_s(\varphi_s) + R_s)\mathrm{d}s \\ R_t = \mathbb{E}_t \int_t^\infty H_s^F(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s(\varphi_s)\dot{G}_s R_s \mathrm{d}s, \end{cases} \quad t \in [0, \infty)$$

$$H_t^F := \partial_t F_t + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t) + \mathrm{D}F_t \dot{G}_t F_t, \tag{5}$$

Now the problem comes down to two steps

1. **Find a "good enough" approximation F to the flow equation (5) which allows for good estimates of $\|F_s(\varphi_s)\|_{L^\infty}$ uniformly in the UV and IR cut-offs.**
2. Show well-posedness for the FBSDE and obtain uniform a-priori estimates which allow to remove the UV and IR regularisations.

Aim: Systematic way to find "good" approximate solutions to

$$\partial_t F_t + \frac{1}{2}\mathrm{Tr}(\dot{G}_t \mathrm{D}^2 F_t) + \mathrm{D}F_t \dot{G}_t F_t \approx 0, \qquad F_\infty(\phi) = F(\phi) = \lambda\beta[\![\sin(\beta\phi)]\!].$$

Ansatz: Picard iterations starting from $F^{[0]} \equiv 0$, define inductively

$$\partial_t F_t^{[\ell]} + \frac{1}{2}\mathrm{Tr}\left(\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}\right) = \sum_{\ell'+\ell''=\ell} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']}, \qquad \ell > 0,$$

with *terminal* conditions $F_\infty^{[1]} = F$ and $F_\infty^{[\ell]} = 0$ for the levels $\ell > 1$.

Then, with $F_t := F_t^{[\leq \ell^*]} = \sum_{\ell=0}^{\ell^*} F_t^{[\ell]}$ we compute the remainder

$$H_t^F = H_t^{[\leq \ell^*]} = \sum_{\substack{\ell',\ell''<\ell^* \\ \ell'+\ell''>\ell^*}} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']}.$$

Heuristics: What bounds to expect

For concreteness, suppose that $\dot{G}_t = t^{-2} e^{-(m^2-\Delta)/t}$ so that

$$\|\dot{G}_t\|_{L^1} \lesssim \langle t \rangle^{-2}.$$

Heuristically, the bounds

$$\left\|F_t^{[\ell]}(\varphi_t)\right\|_{L^\infty} + \left\|\mathrm{D}F_t^{[\ell]}(\varphi_t)\right\|_{L^\infty} \lesssim \langle t \rangle^{-\delta\ell+1} \quad \text{where } \delta = 1 - \frac{\beta^2}{8\pi},$$

propagate: they are compatible with

$$\partial_t F_t^{[\ell]} + \frac{1}{2}\mathrm{Tr}\left(\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}\right) = \sum_{\ell'+\ell''=\ell} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']},$$

since

$$\left\|F_t^{[\ell]}(\varphi_t)\right\|_{L^\infty} \leqslant \sum_{\ell'+\ell''=\ell} \int_t^\infty \mathrm{d}s \left\|\mathrm{D}F_t^{[\ell']}(\varphi_t)\right\|_{L^\infty} \|\dot{G}_t\|_{L^1} \left\|F_t^{[\ell'']}(\varphi_t)\right\|_{L^\infty} \lesssim \int_t^\infty \mathrm{d}s \, \langle s \rangle^{-\delta\ell} \overset{!}{\lesssim} \langle t \rangle^{-\delta\ell+1}.$$

Heuristics: Why should this approximation work?

Assuming for the moment,

$$\left\|F_t^{[\ell]}(\varphi_t)\right\|_{L^\infty} + \left\|\mathrm{D}F_t^{[\ell]}(\varphi_t)\right\|_{L^\infty} \lesssim \langle t\rangle^{-\delta\ell+1} \quad \text{where } \delta = 1 - \frac{\beta^2}{8\pi}, \tag{6}$$

since $R_t = \mathbb{E}_t \int_t^\infty H_s^{[\leqslant \ell^*]}(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s^{[\leqslant \ell^*]}\dot{G}_s R_s \mathrm{d}s$ we require

$$\left\|H_t^{[\leqslant \ell^*]}(\varphi_t)\right\|_{L^\infty} \lesssim \sum_{\ell'+\ell''>\ell^*} \left\|\mathrm{D}F_t^{[\ell']}(\varphi_t)\right\|_{L^\infty} \|\dot{G}_t\|_{L^1} \left\|F_t^{[\ell'']}(\varphi_t)\right\|_{L^\infty} \lesssim \langle t\rangle^{-\delta(\ell^*+1)} \overset{!}{\in} L^1(\mathbb{R}_+) \Longleftrightarrow \beta^2 \leqslant \beta_{\ell^*}^2 := \frac{\ell^*}{\ell^*+1} 8\pi$$

Problems to overcome:

- for $\ell < \ell^*$, we cannot propagate the bounds along the flow from ∞,
  - $\longrightarrow$ this is solved by *localisation+renormalisation*
- The bounds on $F_t^{[\ell]}$ and $H_t^{[\leqslant \ell^*]}$ in general depend polynomially on $\nabla\varphi$ (degree depending on ℓ, ℓ^*).

Set-up for the SG-flow equation

Starting point: $F_t^{[1]}(\varphi) = -\lambda_t \beta \sin(\beta\varphi(x)) = \frac{\lambda_t}{2i}[\mathrm{e}^{i\beta\varphi(x)} - \mathrm{e}^{-i\beta\varphi(x)}]$ and thus

$$\partial_t F_t^{[\ell]} + \mathrm{Tr}\big(\dot{G}_t \mathrm{D}^2 F_t^{[\ell]}\big) = \sum_{\ell'+\ell''=\ell} \mathrm{D}F_t^{[\ell']} \dot{G}_t F_t^{[\ell'']}, \qquad \ell > 0, \quad F_\infty^{[1]} = F \text{ and } F_\infty^{[\ell]} = 0 \text{ for the levels } \ell > 1.$$

Naturally reproduces functionals of the form

$$F_t^{[\ell]}(\varphi)(x_1) = \sum_{\sigma_1,\dots\sigma_\ell \in \{\pm 1\}} \int \mathrm{d}x_2 \dots \int \mathrm{d}x_\ell f_t^{[\ell]}(x_1,\dots,x_\ell) \mathrm{e}^{i\sigma_1\beta\varphi(x_1)} \dots \mathrm{e}^{i\sigma_\ell\beta\varphi(x_\ell)}.$$

$\longrightarrow$ flow equation for the coefficients $f^{[\ell]}$, try to estimate these kernels in a suitable norm and recover estimates for $F^{[\ell]}$.

Fully inductive procedure: produces bounds in the full subcritical regime.

Where's the catch?

Finding approximate solutions to the flow equation is only the first step:

One still needs to show well-posedness for the resulting FBSDE,

$$\begin{cases} \varphi_t = \int_0^t \dot{G}_s\big(F_s^{[\leqslant \ell^*]}(\varphi_s) + R_s\big)\mathrm{d}s + \int_0^t \dot{G}_s^{1/2}\mathrm{d}B_s, \\ R_t = \mathbb{E}_t \int_t^\infty H_s^{[\leqslant \ell^*]}(\varphi_s)\mathrm{d}s + \mathbb{E}_t \int_t^\infty \mathrm{D}F_s^{[\leqslant \ell^*]}\dot{G}_s R_s \mathrm{d}s. \end{cases}$$

In general, the argument described yields, for $\delta = 1 - \beta^2/8\pi$

$$\|F_t(\varphi)\|_{L^\infty} \lesssim \sum_{\ell \leqslant \ell^*} \langle t \rangle^{1-\ell\delta} \big(1 + \langle t \rangle^{-1}\|\nabla\varphi\|^2 + \langle t \rangle^{-1}\|\nabla\varphi\|\big)^\ell,$$

which becomes more and more nonlinear as $\beta^2 \longrightarrow 8\pi$, and $\ell^* = \ell^*(\beta) \longrightarrow \infty$.

$\longrightarrow$ Prevents us from controlling the resulting FBSDE for $\beta^2 \geqslant 6\pi$.

Can we cover a wider parameter range $\beta \in [0, \beta^*)$ for some $(\beta^*)^2 > 6\pi$?

- For $\beta^2 \in (0, 8\pi)$: model is known to be renormalisable but with **infinitely many threshholds** requiring additional renormalisations. [G. Benfatto, G. Gallavotti, F. Nicoló, et al.· On the massive sine-Gordon equation in {the first few/ higher/ all} regions of collapse · Comm. math. phys. {1982/ 1983/ 1986}]
- Many **partial** results, valid for different ranges of λ, β or finite/infinite volume. **But:** No works covering the full subcritical regime on the full space for all λ.

Smaller questions

Can we remove some of the smallness assumptions on the coupling constant λ?

i.e. Variational problem for any λ? OS-Axioms for any λ?

Improvements for the martingale problem, e.g. relax assumptions on the scale decomposition $(G_t)_{t \in [0,\infty]}$?

More generally:

- Can we make this approach work also for other models, e.g. φ^4_3?

  $\longrightarrow$ Requires **global** solutions for the resulting FBSDE and thus strong a priori estimates for FBSDEs

- The FBSDE only uses the **force** $F = -\mathrm{D}V$; we never need to reference the **potential** V,

  $\longrightarrow$ Can we consider non-scalar models, where the potential does not make sense?

- General solution theory for these kinds of FBSDEs? Are there more general conditions about the existence/uniqueness of solutions?

Thanks!