SCHAUDER ESTIMATE FOR GERMS

FIX $\gamma > 0$, $\eta \in (0, d)$ WITH $\gamma + \eta < 1$ (FOR SIMPLICITY)

LET $F = (F_x)_{x \in \mathbb{R}^d}$ BE A γ-COHERENT GERM.

LET $K(x,y)$ BE A η-REGULARIZING KERNEL SUPPORTED IN $\{|x-y| \le 1\}$, I.E.

$$[K1] \qquad |K(x,y)| \lesssim \frac{1}{|x-y|^{d-\eta}} \mathbb{1}_{\{|x-y| \le 1\}}$$

LOCALLY INTEGRABLE IN $\mathbb{R}^d$!

WE LOOK FOR A $(\gamma+\eta)$-COHERENT GERM $H = (H_x)_{x \in \mathbb{R}^d}$ SUCH THAT

$$(\star) \qquad \mathcal{R}H = \mathcal{R}F * K = f * K$$

1. PRELIMINARIES

HOW TO MAKE SENSE OF $f * K$? FOR ANY $f \in \mathcal{D}'$ WE WRITE

$$(f * K)(\varphi) := f(\hat{K} * \varphi) := f\left(\int_{\mathbb{R}^d} \varphi(x) K(x, \cdot)\, dx \right) = f(\xi(\cdot))$$

WE NEED TO ENSURE THAT $\xi(\cdot)$ IS A TEST FUNCTION IN $\mathcal{D}$. NOTE THAT IT IS COMPACTLY SUPPORTED (BECAUSE $K(x,y)$ IS SUPPORTED IN $\{|x-y| \le 1\}$).

HOW TO ENSURE THAT IT IS SMOOTH? WE DISTINGUISH TWO CASES

a) IF $K(x,y) = k(x-y)$ ("TRUE CONVOLUTION KERNEL") THEN $\xi(\cdot)$ IS SMOOTH WITH NO EXTRA ASSUMPTION, BECAUSE DERIVATIVES CAN BE MOVED TO φ.

b) IF $K(x,y)$ IS NOT A FUNCTION OF $x-y$, WE NEED TO ASSUME THAT $K(x,\cdot)$ IS SMOOTH WITH

[K2] $\quad |\partial_2^k K(x,y)| \lesssim \frac{1}{|x-y|^{d+|k|-\eta}} \mathbb{1}_{\{|x-y|\leq 1\}} \quad \forall\, k \in \mathbb{N}_0^d.$

WE FURTHER NEED TO ASSUME THAT

[K3] $\quad y \mapsto \int_{\mathbb{R}^d} x^k K(x,y)\, dx \quad$ IS SMOOTH $\quad \forall\, k \in \mathbb{N}_0^d$

(THIS IS A MILD ASSUMPTION. MARTIN ASSUMES $\int x^k K(x,y)\, dx \equiv 0$.)

IT FOLLOWS THAT ξ IS SMOOTH BECAUSE $\forall\, r \in \mathbb{N}_0$, DENOTING BY $P_y^r(\cdot)$ THE TAYLOR POLYNOMIAL OF ψ OF ORDER r BASED AT y, WE HAVE

$$\xi(y) = \underbrace{\int_{\mathbb{R}^d} \left(\psi(x) - P_y^r(x)\right) K(x,y)\, dx}_{\substack{\text{SMOOTH BY [K2] WHICH} \\ \text{ENSURES DOMINATED CONVERGENCE}}} + \underbrace{\int_{\mathbb{R}^d} P_y^r(x) K(x,y)\, dx}_{\text{SMOOTH BY [K3]}}$$

2. CANDIDATE GERM H

ADAPTING MARTIN'S DEFINITION, WE CONJECTURE THAT THE "RIGHT" GERM $H = (H_x)_{x \in \mathbb{R}^d}$ TO SATISFY $\star$ (I.E. $\mathcal{R}H = \mathcal{R}F * K$) WHEN $\alpha + \eta < 1$ IS

$$H_x := \underbrace{F_x * K}_{A_x} + \underbrace{(\mathcal{R}F - F_x)(K(x,\cdot))}_{B_x}$$

NOTE THAT A_x IS A DISTRIBUTION (WELL-DEFINED BY PART 1. ABOVE).

ON THE OTHER HAND, B_x IS SIMPLY A CONSTANT (DEPENDING ON x). THIS IS NOT OBVIOUSLY DEFINED, SINCE $K(x,\cdot)$ IS DEFINITELY NOT A TEST FUNCTION (IT DIVERGES AT x!). HOPEFULLY, HOWEVER, THERE IS A CANONICAL DEFINITION OF B_x, THANKS TO OUR ASSUMPTIONS.

3. DECOMPOSITION OF $K(x,y)$

WE WANT TO WORK WITH THE FOLLOWING DECOMPOSITION OF $K(x,y)$:

$$[K4] \quad K(x,y) \simeq \sum_{k=0}^{\infty} 2^{-k\eta} \varphi_x^{\varepsilon_k}(y), \qquad \varepsilon_k = 2^{-k}, \qquad \text{FOR SOME } \varphi \in \mathcal{D}(B(0,1))$$

THIS IS A "TOY ASSUMPTION" WHICH MIMICS THE ASSUMPTIONS BY MARTIN.

INTUITIVELY, THIS IS CONSISTENT WITH [K1] BECAUSE FOR FIXED $y \neq x$

$$\sum_{k=0}^{\infty} 2^{-k\eta} \varphi_x^{\varepsilon_k}(y) = \sum_{k=0}^{N_{xy}} 2^{-k\eta} \cdot \varepsilon_k^{-d} \overbrace{\varphi\left(\frac{y-x}{\varepsilon_k}\right)}^{\lesssim 1} \qquad \left[N_{xy} := \log_2 \frac{1}{|x-y|}\right]$$

$$\lesssim \sum_{k=0}^{N_{xy}} 2^{k(d-\eta)} \simeq 2^{N_{xy}(d-\eta)} = \frac{1}{|x-y|^{d-\eta}}$$

TO JUSTIFY / UNDERSTAND [K4], FIX A SMOOTH $\rho : [\frac{1}{2}, 2] \to (0,\infty)$ SUCH THAT

$$\forall r \in (0,\infty) \qquad \sum_{k \in \mathbb{Z}} \rho(2^k r) \equiv 1 \qquad [\text{LITTLEWOOD-PALEY}]$$

HENCE FOR $r \in (0,1]$ WE HAVE $\sum_{k=0}^{+\infty} \rho(2^k r) \equiv 1$. IF WE DEFINE

$$\varphi^{[x,k]}(z) := 2^{k(\eta-d)} \rho(|z|)\, K(x, x+\varepsilon_k z)$$

THEN $\quad (\varphi^{[x,k]})_x^{\varepsilon_k}(y) = 2^{k\eta} \rho(2^k|x-y|)\, K(x,y)$

HENCE WE HAVE THE IDENTITY

$$K(x,y) = \sum_{k=0}^{\infty} 2^{-k\eta} (\varphi^{[x,k]})_x^{\varepsilon_k}(y)$$

THIS COINCIDES WITH [K4] EXCEPT THAT φ IS REPLACED BY $\varphi^{[x,k]}$.

HOWEVER $\varphi^{[x,\kappa]} \in \mathcal{D}(B(0,1))$ AND, FOR ANY FIXED $r \in \mathbb{N}$, WE HAVE

$$\sup_{x,\kappa} \| \varphi^{[x,\kappa]} \|_{C^r} \lesssim 1$$

THEREFORE WE CAN "MORALLY" THINK OF $\varphi^{[x,\kappa]}$ AS A SINGLE FUNCTION φ.

NOTE THAT $\forall\, x \in \mathbb{R}^d$ AND $\kappa \in \mathbb{N}_0$ THE FUNCTION $\varphi^{[x,\kappa]}(z)$ IS SUPPORTED IN THE FIXED ANNULUS $\{ \frac{1}{2} \leq |z| \leq 1 \} \subseteq B(0,1)$

THEN, IF WE FIX ANY $r \in \mathbb{N}$, THEN UNIF. FOR $\ell \in \mathbb{N}_0^d$ WITH $|\ell| \leq r$,

$$\sup_{\frac{1}{2} \leq |z| \leq 1} \left| \partial_z^\ell \varphi^{[x,\kappa]}(z) \right| \lesssim 2^{\kappa(\eta - d)} \varepsilon_\kappa^{|\ell|} \sup_{\frac{1}{2} \leq |z| \leq 1} \left| \partial_z^\ell K(x, x + \varepsilon_\kappa z) \right|$$

$$\lesssim 2^{\kappa(\eta-d)} \varepsilon_\kappa^{|\ell|} \frac{1}{\varepsilon_\kappa^{d+|\ell|-\eta}} \lesssim 1$$