# Pinning and Wetting Transition for (1+1)-Dimensional Fields with Laplacian Interaction

Francesco Caravenna

caravenna@math.unipd.it

Università degli Studi di Padova

Seminar on Stochastic Processes

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# References


- [CD1] F. Caravenna and J.-D. Deuschel
  *Pinning and wetting transition for (1+1)-dimensional fields with Laplacian interaction*, Ann. Probab. (to appear)

- [CD2] F. Caravenna and J.-D. Deuschel
  *Scaling limits of (1+1)-dimensional pinning models with Laplacian interaction*, preprint (2008).

# Outline

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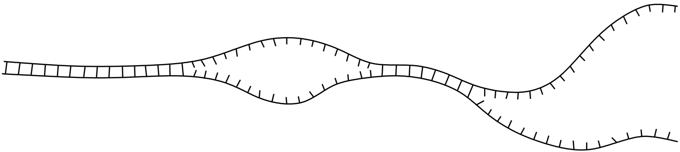
# Some motivations

DNA denaturation transition at high temperature



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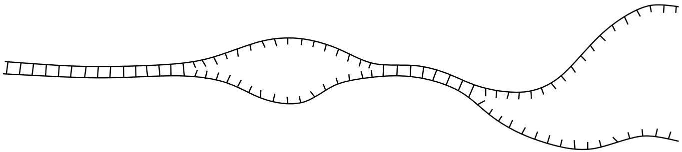
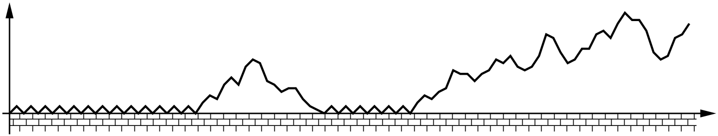


(1+1)-dimensional model: field above an impenetrable wall



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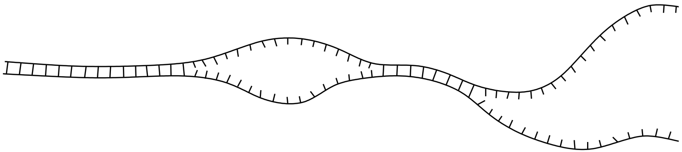

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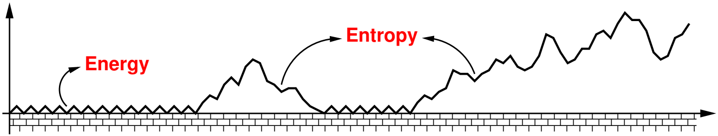

# The general wetting model

The field $\varphi = \{\varphi_i\}_{1 \le i \le N}$ in the free case:

$$\mathbb{P}^{\mathrm{w}}_{0,N}\big(\mathrm{d}\varphi_1\,,\,\ldots\,,\,\mathrm{d}\varphi_N\big) \;:=\; \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}^{\mathrm{w}}_{0,N}} \prod_{i=1}^{N} \mathrm{d}\varphi_i^+$$

- $\mathrm{d}\varphi_i^+$ is the Lebesgue measure on $[0,\infty)$
- $\mathcal{H}_N(\varphi)$ describes the structure of the chain (to be specified)
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- $\varepsilon \ge 0$ is the strength of the pinning interaction

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- $\mathrm{d}\varphi_i$ is the Lebesgue measure on $\mathbb{R}$
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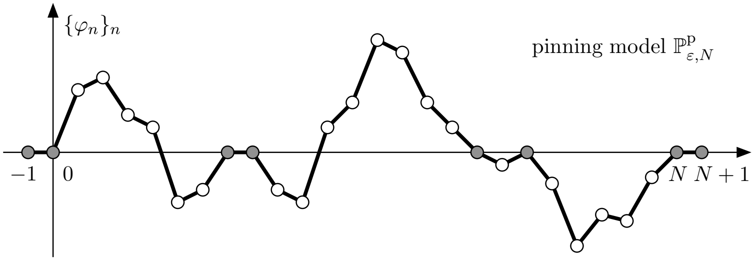

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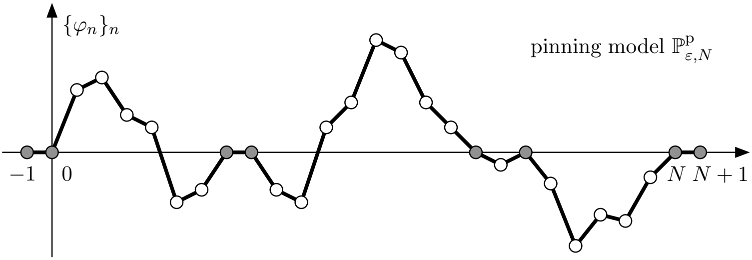


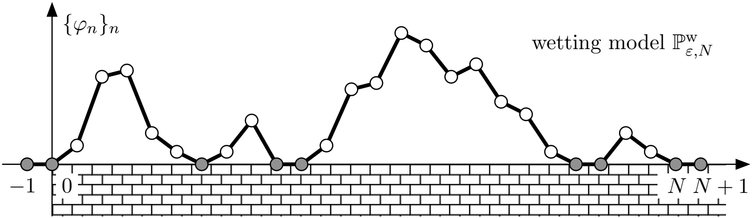

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Once $\mathcal{H}_N(\varphi)$ is chosen and $\varepsilon \geq 0$ is fixed:

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How to choose $\mathcal{H}_N(\varphi)$?

# The choice of $\mathcal{H}_N(\varphi)$

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[Isozaki, Yoshida SPA 01] [Deuschel, Giacomin, Zambotti PTRF 05] [Caravenna, Giacomin, Zambotti EJP 06]

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Interpretation of the free case $\varepsilon = 0$:

- $\mathbb{P}^{\mathrm{p}}_{0,N}$ is (the bridge of) the integral of a random walk
- $\mathbb{P}^{\mathrm{w}}_{0,N}$ is further conditioned to stay $\geq 0$

# Laplacian interaction in $(d+1)$-dimension

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Henceforth we study $\mathbb{P}^{\mathrm{p}}_{\varepsilon,N}$ and $\mathbb{P}^{\mathrm{w}}_{\varepsilon,N}$ with Laplacian interaction and boundary conditions $\varphi_{-1} = \varphi_0 = \varphi_N = \varphi_{N+1} = 0$

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Assumptions on $V$:

$$\int_{\mathbb{R}} e^{-V(x)}\, \mathrm{d}x = 1\,, \quad \int_{\mathbb{R}} x\, e^{-V(x)}\, \mathrm{d}x = 0\,, \quad \int_{\mathbb{R}} x^2\, e^{-V(x)}\, \mathrm{d}x = 1$$

$+$ regularity: $x \mapsto e^{-V(x)}$ continuous and $V(0) < +\infty$.

# Outline

# How to define localization and delocalization?

Recall the partition function: (zero boundary conditions)

$$\mathcal{Z}^{a}_{\varepsilon,N} = \int_{\Omega^a_N} e^{-\mathcal{H}_N(\varphi)} \prod_{i=1}^{N-1} \left(\mathrm{d}\varphi_i + \varepsilon\, \delta_0(\mathrm{d}\varphi_i)\right)$$

where $a \in \{\mathrm{p}, \mathrm{w}\}$ and $\Omega^{\mathrm{p}}_N = \mathbb{R}^{N-1}$ while $\Omega^{\mathrm{w}}_N = [0,\infty)^{N-1}$

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## Free Energy

$$\mathrm{F}^a(\varepsilon) := \lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}_{\varepsilon,N}^{a} \qquad \text{(super-additivity)}$$

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Basic observation: $\mathtt{F}^a(\varepsilon) \geq \mathtt{F}^a(0) = 0$ for all $\varepsilon \geq 0$ and $a \in \{\mathrm{p}, \mathrm{w}\}$

$$\mathcal{Z}^a_{\varepsilon,N} \geq \mathcal{Z}^a_{0,N} \approx N^{-c} \qquad (c > 0)$$

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  - if $\mathtt{F}^a(\varepsilon_c^a + h) \geq C\,h$ [1[st] order trans.] $\varepsilon = \varepsilon_c^a$ may be localized (phase coexistence, dependence of boundary conditions)

# The phase transition

### Theorem ([CD1])

*Both* $\mathbb{P}^{\mathrm{p}}_{\varepsilon,N}$ *and* $\mathbb{P}^{\mathrm{w}}_{\varepsilon,N}$ *undergo a non-trivial phase transition:*

$$0 < \varepsilon_c^{\mathrm{p}} < \varepsilon_c^{\mathrm{w}} < \infty$$

*and* $\mathrm{F}^a(\varepsilon)$ *is analytic on* $[0, \varepsilon_c^a) \cup (\varepsilon_c^a, \infty)$. *(variational formula)*

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- *In the wetting model the transition is of $1^{st}$ order:*

$$\mathtt{F}^{\mathrm{w}}(\varepsilon_c^{\mathrm{w}} + h) \sim C_2\, h, \qquad \ell_N \sim \mathtt{D}\, N \quad (\mathtt{D} > 0)$$

# The gradient case

## Differences in the gradient case

- the transition is non-trivial only in the wetting model:

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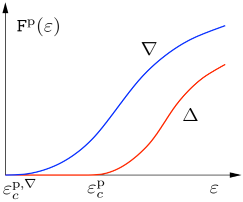


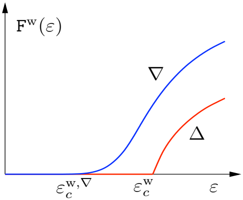

# A look at the disordered case

Disordered version of our model: $(\mathrm{d}\varphi_i^{\mathrm{p}} = \mathrm{d}\varphi_i$ and $\mathrm{d}\varphi_i^{\mathrm{w}} = \mathrm{d}\varphi_i^{+})$

$$\mathbb{P}^{a}_{\varepsilon,\beta,\omega,N}\big(\mathrm{d}\varphi_1\,,\,\ldots\,,\,\mathrm{d}\varphi_N\big) \;:=\; \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}^{a}_{\varepsilon,\beta,\omega,N}} \prod_{i=1}^{N} \big(\mathrm{d}\varphi_i^{a} + \varepsilon\, e^{\beta\omega_i}\, \delta_0(\mathrm{d}\varphi_i)\big)$$

where $\beta \geq 0$ and $\{\omega_i\}_{i\in\mathbb{N}}$ are IID $\mathcal{N}(0,1)$ (law $P$ indep. $\mathbb{P}^a$).

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## Quenched free energy

$$\mathrm{F}^a(\varepsilon,\beta) := \lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}^a_{\varepsilon,\beta,\omega,N} \geq 0$$

exists $P(\mathrm{d}\omega)$–a.s. and does not depend on $\omega$ (self-averaging)

# A look at the disordered case

Disordered version of our model: $(\mathrm{d}\varphi_i^{\mathrm{p}} = \mathrm{d}\varphi_i$ and $\mathrm{d}\varphi_i^{\mathrm{w}} = \mathrm{d}\varphi_i^+)$

$$\mathbb{P}^a_{\varepsilon,\beta,\omega,N}\big(\mathrm{d}\varphi_1\,,\,\ldots\,,\,\mathrm{d}\varphi_N\big) \;:=\; \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}^a_{\varepsilon,\beta,\omega,N}} \prod_{i=1}^N \big(\mathrm{d}\varphi_i^a \,+\, \varepsilon\, e^{\beta\omega_i}\, \delta_0(\mathrm{d}\varphi_i)\big)$$

where $\beta \geq 0$ and $\{\omega_i\}_{i\in\mathbb{N}}$ are IID $\mathcal{N}(0,1)$ (law $P$ indep. $\mathbb{P}^a$).

## Quenched free energy

$$\mathrm{F}^a(\varepsilon,\beta) \;:=\; \lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}^a_{\varepsilon,\beta,\omega,N} \;\geq\; 0$$

exists $P(\mathrm{d}\omega)$–a.s. and does not depend on $\omega$ (self-averaging)

Localization: $\mathrm{F}^a(\varepsilon,\beta) > 0 \quad\Longleftrightarrow\quad \varepsilon \;>\; \varepsilon_c^a(\beta)$ (critical line)

# Smoothing effect of disorder

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**Theorem ([Giacomin and Toninelli, CMP 06])**

*Both in the $\nabla$ and $\Delta$ case, both for $a = \mathrm{p}$ and for $a = \mathrm{w}$:*
*for every $\beta > 0$ there exists $C_\beta > 0$ such that*

$$\mathrm{F}^a\big(\varepsilon_c^a(\beta) + h\big) \;\leq\; C_\beta\, h^2$$

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Very general proof: rare-stretches in $\omega$ (Large Deviations)

# Outline

# Some deeper questions

We have established the existence of a phase transition:

$$\ell_N = o(N) \ \text{ if } \ \varepsilon < \varepsilon_c^a \qquad \text{VS} \qquad \ell_N \sim \mathtt{D} \cdot N \ \text{ if } \ \varepsilon > \varepsilon_c^a$$

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Yes in the pinning case and under additional assumptions on $V(\cdot)$:

- symmetry: $V(x) = V(-x)$ for every $x \in \mathbb{R}$
- uniform strict convexity: $\exists \gamma > 0$ s. t. $V(x) - \gamma \frac{x^2}{2}$ is convex
- regularity: $x \mapsto e^{-V(x)}$ is continuous and $V(0) < \infty$

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Let us set $M_N \;:=\; \max_{1 \le i \le N} \big| \varphi_i \big|$

# A closer look at the typical paths [CD2]

- delocalized regime $\varepsilon < \varepsilon_c^{\mathrm{p}}$: $\ell_N = O(1)$ and $M_N \approx N^{3/2}$:

$$\lim_{K\to\infty} \liminf_{N\to\infty} \mathbb{P}^{\mathrm{p}}_{\varepsilon,N}\big(\varphi_i \neq 0 \text{ for } i \in \{K, N-K\}\big) \;=\; 1$$

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# Scaling Limits

We rescale and interpolate linearly the field: for $t \in [0, 1]$

$$\widehat{\varphi}_N(t) \;:=\; \frac{\varphi_{\lfloor Nt \rfloor}}{N^{3/2}} \;+\; (Nt - \lfloor Nt \rfloor)\, \frac{\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}}{N^{3/2}}$$

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Let $\{B_t\}_{t\in[0,1]}$ standard BM, $I_t := \int_0^t B_s \, \mathrm{d}s$ integrated BM

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**Theorem (Scaling Limits [CD2])**

*The rescaled field $\{\widehat{\varphi}_N(t)\}_{t\in[0,1]}$ under $\mathbb{P}^{\mathrm{p}}_{\varepsilon,N}$ converges in distribution on $C([0,1])$ as $N \to \infty$, for every $\varepsilon \geq 0$. The limit is*

- *If $\varepsilon < \varepsilon_c^{\mathrm{p}}$, the law of $\{\widehat{I}_t\}_{t\in[0,1]}$*
- *If $\varepsilon = \varepsilon_c^{\mathrm{p}}$ or $\varepsilon > \varepsilon_c^{\mathrm{p}}$, the law concentrated on $f(t) \equiv 0$*

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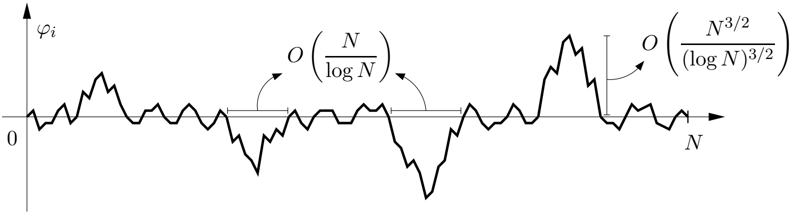

# The critical regime

Alternative idea: look at the field in a distributional sense

$$\mu_N(\mathrm{d}t) \; := \; \frac{(\log N)^{5/2}}{N^{3/2}} \, \varphi_{\lfloor Nt \rfloor} \, \mathrm{d}t \; = \; \widetilde{\varphi}_N(t) \, \mathrm{d}t$$

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Let $\{L_t\}_{t\in[0,1]}$ be the stable symmetric Lévy process of index $\frac{2}{5}$

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**Theorem ([CD2])**

*The random signed measure $\mu_N$ under $\mathbb{P}^{\mathrm{p}}_{\varepsilon_c^{\mathrm{p}},N}$ converges in distribution as $N \to \infty$ toward $\mathrm{d}L$.*

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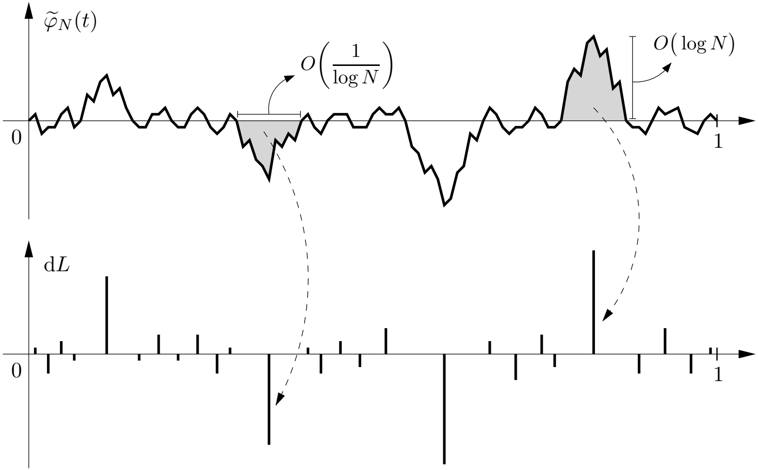

# Outline

# A random walk viewpoint ($\varepsilon = 0$)

Let $\{X_i\}_{i \in \mathbb{N}}$ be IID random variables with law

$$\mathbf{P}(X_i \in \mathrm{d}x) \; := \; e^{-V(x)}\,\mathrm{d}x \qquad \mathbf{E}(X_i) = 0 \qquad \mathbf{Var}(X_i) = 1$$

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Random walk: $Y_n := X_1 + \ldots + X_n$

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**The free case $\varepsilon = 0$**

- The field $\{\varphi_i\}_{1\le i\le N}$ under $\mathbb{P}^{\mathrm{p}}_{0,N}$ is distributed like $\{Z_i\}_{1\le i\le N}$ conditionally on $(Y_N, Z_N) = (0,0)$. $(Z_{N/2} \approx N^{3/2})$

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- Under $\mathbb{P}^{\mathrm{w}}_{0,N}$ the same, under the further conditioning $\{Z_1 \ge 0, \ldots, Z_N \ge 0\}$

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Once we know $\tau, J$, the whole field $\{\varphi_i\}_i$ is reconstructed by pasting independent excursions from $\{Z_i\}_i$ (cond. to stay $\geq 0$)

# The law of the excursions

Pinning case: good control (Donsker's inv. pr. + LLT)

$$\left\{ \frac{Z_{\langle Nt \rangle}}{N^{3/2}} \right\}_{t\in[0,1]} \text{ condit. on } (Y_N, Z_N) = (0,0) \implies \left\{ \widehat{I}_t \right\}_{t\in[0,1]}$$

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$$\mathbf{P}\big(Z_1 \geq 0\,, \,\ldots\,, \, Z_N \geq 0\big) \;\approx\; N^{-1/4} \quad \text{[Sinai (SRW)]}$$

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# Markov renewal processes

Given a (sub-)probability kernel $\mathrm{K}_{x,\mathrm{d}y}(n)$:

$$\int_{y\in\mathbb{R}} \sum_{n\in\mathbb{N}} \mathrm{K}_{x,\mathrm{d}y}(n) \;=\; c \;\leq\; 1\,, \qquad \forall x \in \mathbb{R}$$

we build the Markov renewal process $\tau$ with modulating chain $J$:

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The law of $(\tau, J)$ conditionally on $\{N, N+1\} \subseteq \tau$ is

$$\mathcal{P}\left(\tau_i = t_i\,,\; J_i \in \mathrm{d}y_i \,\middle|\, \{N, N+1\} \subseteq \tau\right) \;=\; \frac{1}{C_N} \prod_i \mathrm{K}_{y_{i-1},\mathrm{d}y_i}(t_i - t_{i-1})$$

with $C_N = \mathcal{P}(\{N, N+1\} \subseteq \tau)$.

# The law of the contact set

Consider the following kernels: for $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$

$$\mathsf{G}^{\mathrm{p}}_{x,\mathrm{d}y}(n) := \varepsilon \, \frac{\mathbf{P}_x\big(Z_{n-1} \in \mathrm{d}y \,,\, Z_n \in \mathrm{d}z\big)}{\mathrm{d}z}\bigg|_{z=0}$$

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Reminds of Markov renewal theory

# Markov renewal processes

We exploit the invariance properties: for every $\mathtt{F}$, $v(y)$

$$\mathbb{P}^a_{\varepsilon,N}\big(\tau_i = t_i\,,\; J_i \in \mathrm{d}y_i\,,\; i \le k\big)$$

$$= \frac{e^{\mathtt{F}N}}{\mathcal{Z}^a_{\varepsilon,N}} \prod_{i=1}^{k} \mathsf{G}^a_{y_{i-1},\mathrm{d}y_i}(t_i - t_{i-1})\, e^{-\mathtt{F}(t_i - t_{i-1})}\, \frac{v(y_i)}{v(y_{i-1})}$$

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If we determine $\mathtt{F}$, $v(\cdot)$ such that

$$K_{x,\mathrm{d}y}(n) \;:=\; \mathsf{G}^{a}_{x,\mathrm{d}y}(n)\, e^{-\mathtt{F}\cdot n}\, \frac{v(y)}{v(x)}$$

is a probability kernel, we have the crucial relation

$$\mathbb{P}^{a}_{\varepsilon,N}\left(\tau_i = t_i\,,\ J_i \in \mathrm{d}y_i\right) \;=\; \mathcal{P}\left(\tau_i = t_i\,,\ J_i \in \mathrm{d}y_i \,\middle|\, \{N, N+1\} \subseteq \tau\right)$$

# A Perron-Frobenius problem

It turns out that:

- $\mathtt{F}$ is the solution of the equation

$$\text{spectral radius of } \left( \sum_{n \in \mathbb{N}} \mathrm{G}^{a}_{x,\mathrm{d}y}(n)\, e^{-\mathtt{F}\cdot n} \right)_{x,y} = 1$$

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- In fact $\mathtt{F} = \mathtt{F}^a(\varepsilon)$ is the free energy
- $v(\cdot)$ is the principal eigenfunction:

$$\int_{y \in \mathbb{R}} \left( \sum_{n \in \mathbb{N}} \mathrm{G}^a_{x,\mathrm{d}y}(n)\, e^{-\mathtt{F}\cdot n} \right)_{x,y} v(y) \;=\; v(x)$$