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Large scale behavior of semiflexible heteropolymers

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Abstract. We consider a general discrete model for heterogeneous semiflexible polymer chains. Both the thermal noise and the inhomogeneous character of the chain (the *disorder*) are modeled in terms of random rotations. We focus on the *quenched* regime, i.e., the analysis is performed for a given realization of the disorder. Semiflexible models differ substantially from random walks on short scales, but on large scales a Brownian behavior emerges. By exploiting techniques from tensor analysis and non-commutative Fourier analysis, we establish the Brownian character of the model on large scales and we obtain an expression for the diffusion constant. We moreover give conditions yielding quantitative mixing properties.

Résumé. On considère un modèle discret pour un polymère semi-flexible et hétérogène. Le bruit thermique et le caractère hétérogène du polymère (le *désordre*) sont modélisés en termes de rotations aléatoires. Nous nous concentrons sur le régime de désordre *gélé*, c'est-à-dire, l'analyse est effectuée pour une réalisation fixée du désordre. Les modèles semi-flexibles diffèrent sensiblement des marches aléatoires à petite échelle, mais à grande échelle un comportement brownien apparaît. En exploitant des techniques de calcul tensoriel et d'analyse de Fourier non-commutative, nous établissons le caractère brownien du modèle à grande échelle et nous obtenons une expression pour la constante de diffusion. Nous donnons aussi des conditions qui entraînent des propriétés quantitatives de mélange.

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1. Introduction

1.1. *Homogeneous semiflexible polymer models*

In the vast polymer modeling literature an important role is played by random walks, in fact self-avoiding random walks (e.g. [2,3]). However, they are expected to model properly real polymers only on large scales. On shorter scales one observes a stiffer behavior of the chain, and other models have been proposed, notably the *semiflexible* one (see e.g. [9,16] and references therein). A *semiflexible polymer* is a natural and appealing mathematical object and, in absence of self-avoidance, it has been implicitly considered in the probability literature for a long time. Consider in fact a probability measure Q on the Lie group $SO(d)$ – the rotations in $\mathbb{R}^d$ ($d = 2, 3, \dots$) – and sample from this, in an independent fashion, a sequence of rotations $r_1, r_2, \dots$. Fixing an arbitrary rotation $R \in SO(d)$ and denoting by $e^1, \dots, e^d$ the unit coordinate vectors in $\mathbb{R}^d$, the process $\{v_n\}_{n\ge 0}$ defined by

$$v_0 := Re^d, \qquad v_n := (Rr_1r_2\cdots r_n)e^d, \quad n = 1, 2, \dots, \tag{1.1}$$

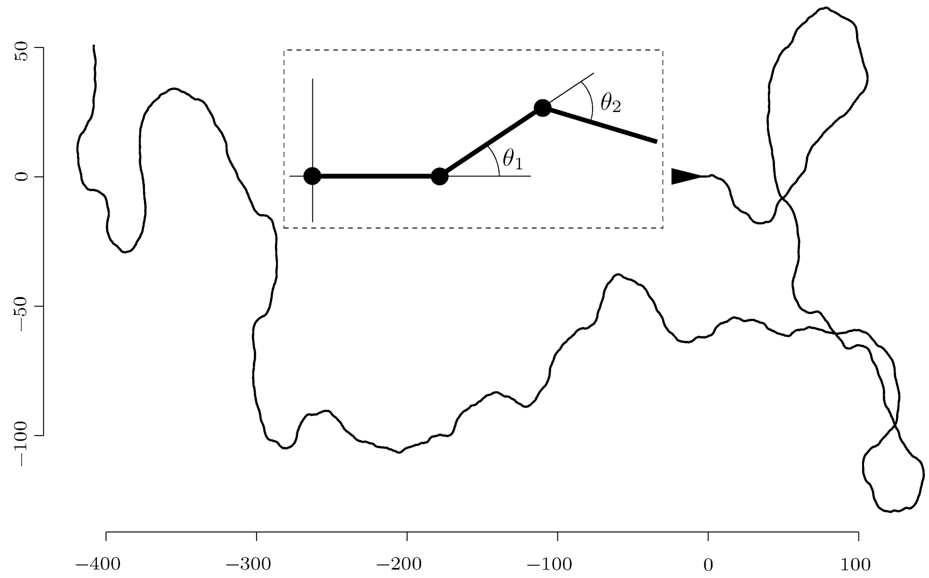

Fig. 1. A sample bidimensional trajectory, with $n = 1400$ and $\{\theta_i\}_{i=1,2,\dots}$ drawn uniformly from $(-\pi/10, \pi/10)$, while θ_0 is 0 (the notation is the one of Remark 1.1). In the inset there is a zoom of the starting portion of the polymer (the starting point is marked by the arrow). It is clear that the starting orientation $v_0 = (1, 0)$ sets up a drift that is forgotten only after a certain number of steps. Moreover, even if the starting orientation eventually fades away, in the sense that the expectation of the scalar product of v_n and v_0 vanishes as n becomes large, the local orientation is carried along for a while. A precise meaning to this is brought by the key concept of *persistence length* ℓ, that can be defined as the reciprocal of the rate of exponential decay of $\mathbf{E}\langle v_0, v_n\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$ and $\mathbf{E}$ is the average over the variables $\{r_i\}_i$. Intuitively, one expects that on a scale much larger than the persistence length, the semiflexible polymer $X_n^{v_0}$ is going to behave *like a random walk*. Note that if we view the elements of $SO(d)$ as linear operators, we can define $\overline{r} := \mathbf{E} r_1$ (not a rotation unless r_1 is trivial!) and we have $\mathbf{E}\langle v_0, v_n\rangle = \langle e^d, \overline{r}^n e^d\rangle$, which shows that the decay of $\mathbf{E}\langle v_0, v_n\rangle$ is indeed of exponential type.

is nothing but a *random walk* on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ starting at v_0, a much studied object (e.g. [11,13]). Then the process $\{X_n^{v_0}\}_{n=0,1,\dots}$ defined by

$$X_n^{v_0} := v_0 + \sum_{j=1}^{n} v_j = \sum_{j=0}^{n} v_j, \tag{1.2}$$

is a *homogeneous* semiflexible polymer model in dimension d. The reason for writing $(Rr_1r_2\cdots r_n)$ instead of $(r_nr_{n-1}\cdots r_1R)$ in (1.1) is explained in Remark 1.2 below.

Remark 1.1. *The reader can get some intuition on the process by having a look at the two-dimensional case of Fig.* 1. *This case is in reality particularly easy to analyze in detail* (*and it does not capture the full complexity of the $d > 2$ case*) *because the rotations in two dimensions commute and they are characterized by only one parameter. More precisely*, *if we identify the random rotation r_j with the angle θ_j*, *for $j \geq 1$*, *and we take θ_0 such that $v_0 = (\cos\theta_0, \sin\theta_0)$*, *by setting $\varphi_n := \theta_0 + \theta_1 + \cdots + \theta_n$ we can write*

$$X_n^{v_0} = v_0 + \left(\sum_{j=1}^{n} \cos(\varphi_j), \sum_{j=1}^{n} \sin(\varphi_j)\right). \tag{1.3}$$

This explicit expression allows an easy and complete analysis of the two-dimensional case, *cf. Appendix* A. *Of course*, *in general no such simplification is possible for $d > 2$.*

Homogeneous semiflexible chains have been used in a variety of contexts [16,17] and they do propose challenging questions that are still only partially understood (even in their continuum version, see Remark 1.3), also because it is difficult to obtain explicit expressions for very basic quantities like the *loop formation probability*, i.e. the hitting probability. As a matter of fact, a more realistic model would have to take into account a self-avoiding constraint, which is more properly called *excluded volume condition*, that imposes that the *sausage-like* trajectory does not self-intersect. This of course makes the model extremely difficult to deal with. Added to that, models need to embody the fact that often real polymers are *inhomogeneous*, i.e. they are not made up of identical monomers and that this does affect the geometry of the configurations. It is precisely on this latter direction that we are going to focus.

1.2. *Heterogeneous models*

Heterogeneous semiflexible chains have attracted a substantial amount of attention (see e.g. [1,10,15–17]), often (but not only) as a modeling frame for DNA or RNA (single or double stranded) chains. The information that we want to incorporate in the model is the fact that the monomer units may vary along the chain: for the DNA case, the four bases A, T, G and C are the origin of the inhomogeneity and couple of monomer units have an associated typical bend that depends on their bases. The model we are interested in is therefore still based on randomly sampled rotations $r_1, r_2, \dots,$ independent and identically distributed with a given marginal law Q (this represents the thermal noise in the chain), but associated to that there is a sequence of rotations $\omega_1, \omega_2, \dots$ that is fixed and does not fluctuate with the chain. If we want to stick to the DNA example, the ω-sequence is fixed once the base sequence is given. The model is then defined by giving once again the orientation $v_0 = Re^d \in S^{d-1}$ of the initial monomer and by defining for $n \ge 0$

$$X_n^{v_0,\omega} := v_0 + \sum_{j=1}^{n} v_j^{\omega} \quad \text{with } v_j^{\omega} := (R\omega_1 r_1 \cdots \omega_j r_j)e^d. \tag{1.4}$$

It should be clear that the rotation ω_i sets up the equilibrium position of the ith monomer with respect to the $(i-1)$st. In different terms, the sequence $\omega_1, \omega_2, \dots$ defines the *backbone* around which the semiflexible chain fluctuates.

The aim of this paper is to study the large scale behavior of the process $\{X_n^{v_0,\omega}\}_n$ when the sequence ω is *disordered*, i.e. it is chosen as the typical realization of a random process. The simplest example is of course the one in which the variables ω_n are independent and identically distributed, but we stress from now that we are interested in the much more general case when ω is an *ergodic process* (see Assumption 1.5 for the definition of ergodicity). This includes strongly correlated sequences of random variables and, in particular, the ones that have been proposed to mimic the base distributions along the DNA (e.g. [15] and references therein). Other aspects of this model deserve attention, notably the analysis of the *persistence length* in the heterogeneous set-up (see the caption of Fig. 1) and other kind of scaling limits, like the *Kratky–Porod* limit (see Remark 1.3): these issues are taken up in a companion paper.

Remark 1.2. *Let us comment on the order of the rotations appearing in Eqs* (1.4) *and* (1.1). *The key point is the following consideration*: *in defining the rotations r_i and ω_i, we assume that the ith monomer lies along the direction e^d. Therefore*, *before applying these rotations*, *we have to express them in the actual reference frame of the ith monomer. Let us be more precise*, *considering first the homogeneous case given by* (1.1). *The rotation r_1 describes the thermal fluctuations of the first monomer* assuming that its equilibrium position is e^d. *However, the equilibrium position of the first monomer is rather $v_0 = Re^d$, therefore we first have to express r_1 in the reference frame of v_0, obtaining Rr_1R^{-1}, and then apply it to v_0, obtaining $v_1 = (Rr_1R^{-1})v_0 = (Rr_1)e^d$. The same procedure yields $v_2 = (Rr_1)r_2(Rr_1)^{-1}v_1 = (Rr_1r_2)e^d$, and so on. The inhomogeneous case of Eq.* (1.4) *is analogous*: *we first apply ω_1 expressed in the reference frame of v_0, getting $v_0' = (R\omega_1R^{-1})v_0 = (R\omega_1)e^d$, then we apply r_1 expressed in the reference frame of v_0', obtaining $v_1^{\omega} = (R\omega_1)r_1(R\omega_1)^{-1}v_0' = (R\omega_1r_1)e^d$, and so on.*

Remark 1.3. *Most of the physical literature focuses on a continuum version of the homogeneous semiflexible model*, *often called* wormlike chain *or* Kratky–Porod *model* (*e.g.* [16] *and references therein*), *which can be obtained in a large scale/high stiffness limit of discrete models. As for the discrete semiflexible model we had a discrete length parameter n that was in fact counting the monomers along the chain, here we have a continuous parameter $t \ge 0$ and the location $\widetilde{X}_t$ of the wormlike chain at t is equal to $\int_0^t \mathtt{B}^{(d)}(s)\,\mathrm{d}s$, where $\{\mathtt{B}^{(d)}(s)\}_{s\ge 0}$ is a Brownian motion on S^{d-1}* (*e.g.* [12]). *Note that the initial orientation v_0 is here replaced by the choice of $\mathtt{B}^{(d)}(0)$. For $d = 2$, once again*, *this process becomes particularly easy to describe since $\mathtt{B}^{(2)}(t) = (\cos(B(t) + x_0), \sin(B(t) + y_0))$, where B*

is a standard Brownian motion. We point out that in the physical literature the continuum model is just used for some formal computations and, in the heterogeneous set-up, the model is often ill-defined and in fact when simulations are performed usually one goes back to a discrete model [1,10,15–17].

1.3. *The Brownian scaling*

In order to study the large scale behavior of our model, we introduce its diffusive rescaling, i.e. the continuous time process $B_N^{v_0,\omega}(t)$ defined for $N \in \mathbb{N}$ and $tN \in \mathbb{N} \cup \{0\}$ by

$$B_N^{v_0,\omega}(t) := \frac{1}{\sqrt{N}} X_{Nt}^{v_0,\omega}. \tag{1.5}$$

This definition is extended to every $t \in [0, \infty)$ by linear interpolation, so that $B_N^{v_0,\omega}(\cdot) \in C([0,\infty))$ and it is piecewise affine, where $C([0,\infty))$ denotes the space of real-valued continuous functions defined on $[0,\infty)$ and is equipped as usual with the topology of uniform converge over the compact sets and with the corresponding σ-field. The precise hypothesis we make on the thermal noise is as follows.

Assumption 1.4. *The variables $(\{r_n\}_{n\geq 1}, \mathbf{P})$ taking values in $SO(d)$ are independent and identically distributed, and the law Q of r_1 satisfies the following irreducibility condition: there do not exist linear subspaces $V, W \subseteq \mathbb{R}^d$ such that $Q(g \in SO(d)\colon\ gV = W) = 1$, except the trivial cases when $V = W = \{0\}$ or $V = W = \mathbb{R}^d$.*

We point out that this assumption on Q (actually on its support) is very mild. It is fulfilled for instance whenever the support of Q contains a non-empty open set $A \subseteq SO(d)$ (this is a direct consequence of the fact that an open subset of $SO(d)$ spans $SO(d)$), in particular when Q is absolutely continuous with respect to the Haar measure on $SO(d)$, a very reasonable assumption for thermal fluctuations (see Section 3.1 for details on the Haar measure). We stress however that absolute continuity is not necessary and in fact several interesting cases of discrete laws are allowed (e.g., for $d = 3$, when Q is supported on the symmetry group of a Platonic solid). Also notice that for $d = 2$ Assumption 1.4 can be restated more explicitly as follows: denoting by $R_\theta \in SO(2)$ the rotation by an angle θ, there does not exist $\theta \in [0, \pi)$ such that $Q(\{R_\theta, R_{\pi+\theta}\}) = 1$.

Next we state precisely our assumption on the disorder.

Assumption 1.5. *The sequence $(\{\omega_n\}_{n\geq 1}, \mathbb{P})$ is* stationary, *i.e. $\{\omega_{n+1}\}_{n\geq 1}$ and $\{\omega_n\}_{n\geq 1}$ have the same law, and ergodic, i.e. $\mathbb{P}(\{\omega_n\}_{n\geq 1} \in A) \in \{0,1\}$ for every shift-invariant measurable set $A \subseteq SO(d)^{\mathbb{N}}$. Shift-invariant means that $\{x_1, x_2, \dots\} \in A$ if and only if $\{x_2, x_3, \dots\} \in A$, while measurability is with respect to the product σ-field on $SO(d)^{\mathbb{N}}$.*

We can now state our main result.

Theorem 1.6. *If Assumptions 1.4 and 1.5 are satisfied, then $\mathbb{P}(\mathrm{d}\omega)$-almost surely and for every choice of v_0 the process $B_N^{v_0,\omega}$ converges in distribution on $C([0,\infty))$ as $N \to \infty$ toward σB, where $B = \{(B_1(t), \dots, B_d(t))\}_{t\geq 0}$ is a standard d-dimensional Brownian motion and the positive constant σ^2 is given by*

$$\sigma^2 := \frac{1}{d} + \frac{2}{d} \sum_{k=1}^{\infty} \mathbb{E}\mathbf{E}\big\langle e^d, \omega_1 r_1 \cdots \omega_k r_k e^d \big\rangle, \tag{1.6}$$

where the series in the right-hand side converges.

This result says, in particular, that the disorder affects the large scale behavior of the polymer only through the diffusion coefficient σ^2. Let us now consider some special cases in which σ^2 can be made more explicit. Notice first that, by setting $\overline{r} := \mathbf{E}(r_1)$, we can rewrite $\mathbb{E}\mathbf{E}\langle e^d, \omega_1 r_1 \cdots \omega_k r_k e^d\rangle = \mathbb{E}\langle e^d, \omega_1 \overline{r} \cdots \omega_k \overline{r} e^d\rangle$:

- When $\overline{r} = cI$, where I denotes the identity matrix and c is a constant (necessarily $|c| < 1$), the expression for σ^2 becomes

$$\sigma^2 = \frac{1}{d} + \frac{2}{d}\sum_{k=1}^{\infty} c^k \mathbb{E}\langle e^d, (\omega_1 \cdots \omega_k) e^d \rangle. \tag{1.7}$$

  Notice that the non-disordered case is recovered by setting $\omega_i \equiv I$, so that the diffusion constant becomes $1/d + 2c/(d(1-c))$. Assume now that $c > 0$ and let us switch the disorder on: if we exclude the trivial case when $\mathbb{P}(\omega_1 e^d = e^d) = 1$, we see that *the diffusion constant decreases*, whatever the disorder law is.

  We point out that by Schur's lemma the relation $\overline{r} = cI$ is fulfilled when the law of r_1 is *conjugation invariant*, i.e., $\mathbf{P}(r_1 \in \cdot) = \mathbf{P}(h r_1 h^{-1} \in \cdot)$ for every $h \in SO(d)$.

- When the variables ω_n are independent (and identically distributed), and with no extra-assumption on $\overline{r}$, by setting $\overline{\omega} := \mathbb{E}(\omega_1)$ we can write

$$\sigma^2 = \frac{1}{d} + \frac{2}{d}\sum_{k=1}^{\infty} \langle e^d, (\overline{\omega}\overline{r})^k e^d \rangle = \frac{1}{d} + \frac{2}{d}\left\langle e^d, \frac{\overline{\omega}\overline{r}}{1-\overline{\omega}\overline{r}} e^d \right\rangle. \tag{1.8}$$

  Notice in fact that Assumption 1.4 yields $\|\overline{r}\|_{\mathrm{op}} < 1$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm (see Section 2), hence the geometric series converges.

In the general case, the expression for the variance is not explicit, but of course it can be evaluated numerically.

In order to get some intuition on the model, in particular on the role of the disorder and why it leads to (1.6), we suggest to have a look at Appendix A, where we work out the computation of the asymptotic variance of $X_n^{v_0}$ in the two-dimensional case, where elementary tools are available because $SO(2)$ is Abelian. As a matter of fact, these elementary tools would allow to prove for $d = 2$ all the results we present in this paper. However, the higher dimensional setting is much more subtle and in particular the proof of Theorem 1.6 for $d > 2$ requires more sophisticated techniques: in Section 2, using tensor analysis, we prove that Theorem 1.6 follows from Assumption 1.5 plus a general condition of exponential convergence of some operator norms, cf. Hypothesis 2.1 below, and we then show that this condition is a consequence of Assumption 1.4.

Remark 1.7. *In the* homogeneous *case, i.e., when disorder is absent, our method yields a proof of the result in Theorem* 1.6 *under a generalized irreducibility condition that is weaker than Assumption* 1.4 (*see Appendix* B). *This generalized condition is fulfilled in particular whenever the support of Q generates a dense subset in $SO(d)$. We point out that this last requirement is exactly the assumption under which Theorem* 1.6 (*in the homogeneous case*) *was proven in* [7,14].

1.4. *On strong decay of correlations*

The persistence length (cf. caption of Fig. 1) does characterize the loss of the initial direction, but from a probabilistic standpoint this is not completely satisfactory, since other information could be carried on much further along the chain. For this reason, we study the mixing properties of the variables v_i^ω (see (1.4)) and this leads to a novel correlation length, that guaranties decorrelation of arbitrary local observables. As we will see, we have only a bound on this new correlation length and we can establish such a result only for a restricted (but sensible) class of models.

In order to state the result, let us introduce the σ-field $\mathcal{F}_{m,n}^\omega := \sigma(v_i^\omega : m \le i \le n)$ for $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{\infty\}$ and for fixed ω. Then the mixing index $\alpha^\omega(n)$ of the sequence $\{v_i^\omega\}_i$ is defined for $n \in \mathbb{N}$ by

$$\alpha^\omega(n) := \sup\left\{ \left|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)\right| : A \in \mathcal{F}_{1,m}^\omega, B \in \mathcal{F}_{m+n,\infty}^\omega, m \in \mathbb{N} \right\}. \tag{1.9}$$

We work under either one of the following two hypotheses:

H-1 The law Q of r_1 is conjugation invariant, i.e., $\mathbf{P}(r_1 \in \cdot) = \mathbf{P}(h r_1 h^{-1} \in \cdot)$ for every $h \in SO(d)$, and for some n_0 the law Q^{*n_0} of $(r_1 \cdots r_{n_0})$ has an L^2 density with respect to the Haar measure on $SO(d)$ (see Section 3.1).

H-2 The law Q of r_1 has an L^2 density with respect to the Haar measure on $SO(d)$.

Assumption H-1 is sensibly weaker than H-2 (of course on the conjugation invariant measure), however requiring an L^2 density is quite a reasonable assumptions for thermal fluctuations. Then we have:

Proposition 1.8. *Under assumptions* H-1 *or* H-2 *there exist two constants* $C \in (0, \infty)$ *and* $h \in (0, 1)$ *such that* $\alpha^\omega(n) \le Ch^n$ *for every* n *and every* ω.

The proof of Proposition 1.8 relies on Fourier analysis on $SO(d)$: it is given in Section 3, where one can find also an explicit characterization of the constant h (see (3.19)).

2. The invariance principle

In this section we prove the invariance principle in Theorem 1.6, including the formula (1.6) for the diffusion constant, under some abstract condition, see Hypothesis 2.1 below, which is then shown to follow from Assumption 1.4. Throughout the section we set

$$\varphi^\omega_{m,n} := \omega_m r_m \omega_{m+1} r_{m+1} \cdots \omega_n r_n, \quad m \le n, \tag{2.1}$$

so that $v^\omega_j = R\varphi^\omega_{1,j} e^d$ (see (1.4)). We recall that $v_0 = Re^d$ is an arbitrary element of S^{d-1}, with $R \in SO(d)$, and that Q denotes the law of r_1.

2.1. *Tensor products and operator norms*

Unless otherwise specified, in this section the vector spaces are assumed to be real (i.e., $\mathbb{R}$ is the underlying field) and to have finite dimension. The tensor product of two vector spaces V and W can be introduced for example by considering first the Cartesian product $V \times W$ and the (infinite-dimensional) vector space $\overline{V \times W}$ for which the elements of $V \times W$ are a basis. Then the tensor product $V \otimes W$ is defined as the quotient space of $\overline{V \times W}$ under the equivalence relations

$$\begin{aligned} &(v_1 + v_2) \times w \sim v_1 \times w + v_2 \times w, \qquad v \times (w_1 + w_2) \sim v \times w_1 + v \times w_2, \\ &c(v \times w) \sim (cv) \times w \sim v \times (cw) \end{aligned}$$

for $c \in \mathbb{R}$, $v_{(i)} \in V$ and $w_{(i)} \in W$. The equivalence class of $v \times w$ is denoted by $v \otimes w$ and we have the properties $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$, $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$ and $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$. Given a basis $\{v_i\}_{i=1,\dots,n}$ of V and a basis $\{w_i\}_{i=1,\dots,m}$ of W, $\{v_i \otimes w_j\}_{i,j}$ is a basis of $V \otimes W$, which is therefore of dimension nm. We stress that not every vector in $V \otimes W$ is of the form $v \otimes w$ for some $v \in V$, $w \in W$.

A more concrete construction of $V \otimes W$ is possible in special cases, e.g., when $V = W = \mathcal{L}(\mathbb{R}^d)$, the vector space of linear operators on $\mathbb{R}^d$ (that will be occasionally identified with the corresponding representative matrices in the canonical basis). In fact $\mathcal{L}(\mathbb{R}^d) \otimes \mathcal{L}(\mathbb{R}^d)$ is isomorphic to $\mathcal{L}(\mathcal{L}(\mathbb{R}^d))$, the space of all linear operators on $\mathcal{L}(\mathbb{R}^d)$, and this identification will be used throughout the paper. Let us be more explicit: given $g, h \in \mathcal{L}(\mathbb{R}^d)$, we can view $g \otimes h$ as the linear operator sending $m \in \mathcal{L}(\mathbb{R}^d)$ to

$$(g \otimes h)(m) := gmh^*, \quad \text{that is} \quad \big[(g \otimes h)(m)\big]_{ij} := \sum_{k,l=1}^d g_{ik} h_{jl} m_{kl}, \tag{2.2}$$

where $(h^*)_{ij} = h_{ji}$ is the adjoint of h. We are going to use this construction especially for $g, h \in SO(d)$, which of course is not a vector space, but can be viewed as a subset of $\mathcal{L}(\mathbb{R}^d)$. A useful property of this representation of $g \otimes h$ as an operator is that

$$(g_1 \otimes h_1)(g_2 \otimes h_2) = (g_1 g_2) \otimes (h_1 h_2), \tag{2.3}$$

which is readily checked from (2.2). Another crucial fact is the following one: given $s_1, s_2 \in \mathcal{L}(\mathbb{R}^d)^*$, the bilinear form $(g,h) \mapsto s_1(g)s_2(h)$ can be written as a *linear form* $s_1 \otimes s_2$ on the tensor space $\mathcal{L}(\mathbb{R}^d) \otimes \mathcal{L}(\mathbb{R}^d)$, defined on product states $g \otimes h$ by

$$(s_1 \otimes s_2)(g \otimes h) := s_1(g)s_2(h) \tag{2.4}$$

and extended to the whole space by linearity. This linearization procedure is the very reason for introducing tensor spaces, as we are going to see below.

Let us recall the definition and properties of some operator norms. Given a vector space V endowed with a scalar product $\langle \cdot, \cdot \rangle$ and an operator $A \in \mathcal{L}(V)$, we define

$$\|A\|_{\mathrm{op}} := \sup_{v,w \in V \setminus \{0\}} \frac{|\langle w, Av \rangle|}{\|v\|\|w\|} = \sup_{v \in V \setminus \{0\}} \frac{\|Av\|}{\|v\|}, \qquad \|A\|_{\mathrm{hs}} := \sqrt{\mathrm{Tr}\big(A^* A\big)}, \tag{2.5}$$

where $\mathrm{Tr}(A)$ is the trace of A and A^* is the adjoint operator of A, defined by the identity $\langle w, Av \rangle = \langle A^* w, v \rangle$ for all $v, w \in V$. If we fix an orthonormal basis $\{e^i\}_{i=1,\dots,n}$ of V and we denote by A_{ij} the matrix of A in this basis, we can write $\|A\|_{\mathrm{hs}}^2 := \sum_{i,j} |A_{ij}|^2$. It is easily checked that for all operators $A, B \in \mathcal{L}(V)$ we have

$$\|AB\|_{\mathrm{op}} \le \|A\|_{\mathrm{op}} \|B\|_{\mathrm{op}}, \qquad \|A\|_{\mathrm{op}} \le \|A\|_{\mathrm{hs}}, \qquad \|AB\|_{\mathrm{hs}} \le \|A\|_{\mathrm{op}} \|B\|_{\mathrm{hs}}. \tag{2.6}$$

In what follows, the space $\mathcal{L}(\mathbb{R}^d)$ is always equipped with the scalar product $\langle v, w \rangle_{\mathrm{hs}} := \mathrm{Tr}(v^* w) = \sum_{i,j} v_{ij} w_{ij}$. We can then give some useful bound on the operator norm of $g \otimes h$ acting on $\mathcal{L}(\mathbb{R}^d)$: by (2.2) and (2.6)

$$\|g \otimes h\|_{\mathrm{op}} = \sup_{v \in \mathcal{L}(\mathbb{R}^d) \setminus \{0\}} \frac{\|gvh^*\|_{\mathrm{hs}}}{\|v\|_{\mathrm{hs}}} \le \sup_{v \in \mathcal{L}(\mathbb{R}^d) \setminus \{0\}} \frac{\|g\|_{\mathrm{op}} \|vh^*\|_{\mathrm{hs}}}{\|v\|_{\mathrm{hs}}} \le \|g\|_{\mathrm{op}} \|h\|_{\mathrm{op}}, \tag{2.7}$$

where we have used that $\|h^*\|_{\mathrm{op}} = \|h\|_{\mathrm{op}}$.

Let us denote by Γ the orthogonal projection on the subspace of symmetric operators in $\mathcal{L}(R^d)$, defined for $v \in \mathcal{L}(\mathbb{R}^d)$ by

$$\Gamma(v) := \frac{1}{2}\big(v + v^*\big), \quad \text{i.e.} \quad \Gamma(v)_{ij} = \frac{1}{2}(v_{ij} + v_{ji}). \tag{2.8}$$

Of course $\Gamma \in \mathcal{L}(\mathcal{L}(\mathbb{R}^d))$, and for any linear operator $m \in \mathcal{L}(\mathcal{L}(\mathbb{R}^d))$ we denote by $\overline{m}$ its symmetrized version:

$$\overline{m} := \Gamma m \Gamma. \tag{2.9}$$

Note that $\overline{g \otimes g} = (g \otimes g)\Gamma = \Gamma(g \otimes g)$, for every $g \in \mathcal{L}(\mathbb{R}^d)$.

Finally, consider $s \in \mathcal{L}(\mathbb{R}^d)^*$ of the form $s(g) = \langle v, gw \rangle$, where v, w are vectors in $\mathbb{R}^d$ with $\|v\| = \|w\| = 1$. For every linear operator $m \in \mathcal{L}(\mathcal{L}(\mathbb{R}^d))$ we have

$$(s \otimes s)(m) = (s \otimes s)(\Gamma m) = (s \otimes s)(m\Gamma) = (s \otimes s)(\overline{m}), \tag{2.10}$$

as one easily checks using coordinates, since $(s \otimes s)(m) = \sum_{ijkl} v_i v_j m_{ij,kl} w_k w_l$. It is also easily seen that

$$\big|(s \otimes s)(m)\big| \le \|m\|_{\mathrm{op}}. \tag{2.11}$$

These relations are easily generalized to higher-order tensor products: in particular

$$s^{\otimes 4}(m) = s^{\otimes 4}\big(m(\Gamma \otimes \Gamma)\big) \quad \text{and} \quad \big|s^{\otimes 4}(m)\big| \le \|m\|_{\mathrm{op}} \tag{2.12}$$

for every $m \in \mathcal{L}(\mathbb{R}^d)^{\otimes 4}$.

2.2. *An abstract condition*

We are ready to state a condition on Q that will allow us to prove the invariance principle in Theorem 1.6.

Let us consider $\mathbf{E}\varphi^\omega_{m,n}$, which is an element of $\mathcal{L}(\mathbb{R}^d)$ (we recall that $\varphi^\omega_{m,n}$ is defined in (2.1)). We need to assume that, when k is large, $\mathbf{E}\varphi^\omega_{n,n+k}$ is exponentially close to the zero operator on $\mathbb{R}^d$, uniformly in n. We are also interested in the asymptotic behavior of $\mathbf{E}[\varphi^\omega_{n,n+k}\otimes\varphi^\omega_{n,n+k}]$, which by (2.2) is a linear operator on $\mathcal{L}(\mathbb{R}^d)$: we need that, when k is large and uniformly in n, the symmetrized version $\mathbf{E}[\overline{\varphi^\omega_{n,n+k}\otimes\varphi^\omega_{n,n+k}}]$ of this operator, cf. (2.9) and (2.8), is exponentially close to the linear operator Π defined as the orthogonal projection on the one-dimensional linear subspace of $\mathcal{L}(\mathbb{R}^d)$ spanned by the identity matrix $(I_d)_{i,j}=\delta_{i,j}$, $1\le i,j\le d$, that is

$$\Pi(v):=\frac{1}{d}\operatorname{Tr}(v)I_d,\quad v\in\mathcal{L}\big(\mathbb{R}^d\big). \tag{2.13}$$

The reason why the operator Π should have this form will be clear in Section 2.5. Let us now state more precisely the hypothesis we make on Q.

Hypothesis 2.1. *The law Q of r_1 is such that, for $\mathbb{P}$-almost every ω, we have*

$$C(\omega):=\sup_{n\ge 1}\left\{\sum_{k=0}^{\infty}\big\|\mathbf{E}\big[\varphi^\omega_{n,n+k}\big]\big\|_{\mathrm{op}}+\sum_{k=0}^{\infty}\big\|\mathbf{E}\big[\overline{\varphi^\omega_{n,n+k}\otimes\varphi^\omega_{n,n+k}}\big]-\Pi\big\|_{\mathrm{op}}\right\}<\infty. \tag{2.14}$$

The next paragraphs are devoted to showing that Theorem 1.6 holds if we assume Hypothesis 2.1 together with Assumption 1.5. We then show in Section 2.5 that Hypothesis 2.1 indeed follows from Assumption 1.4.

2.3. *The diffusion constant*

We start identifying the diffusion coefficient σ^2, given by Eq. (1.6). For any $\mathbb{R}^d$-valued random variable Z we denote by $\operatorname{Cov}(Z)$ its covariance matrix: $\operatorname{Cov}(Z)_{i,j}=\operatorname{cov}(Z^i,Z^j)$.

Proposition 2.2. *If Hypothesis 2.1 and Assumption 1.5 hold, then for $\mathbb{P}$-almost every ω and for every $v_0\in S^{d-1}$ we have that*

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{Cov}_{\mathbf{P}}\big(X_n^{v_0,\omega}\big)=\sigma^2 I_d, \tag{2.15}$$

where

$$\sigma^2=\frac{1}{d}\left(1+2\sum_{k=1}^{\infty}\mathbb{E}\mathbf{E}\big\langle e^d,\varphi^\omega_{1,k}e^d\big\rangle\right), \tag{2.16}$$

the series in the right-hand side being convergent.

Proof. By a standard polarization argument it is enough to prove that for any $v\in S^{d-1}$

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{var}_{\mathbf{P}}\big(\big\langle v,X_n^{v_0,\omega}\big\rangle\big)=\sigma^2,\quad \mathbb{P}(\mathrm{d}\omega)\text{-a.s.}, \tag{2.17}$$

because

$$\operatorname{cov}_{\mathbf{P}}\big(\big\langle e^i,X_n^{v_0,\omega}\big\rangle,\big\langle e^j,X_n^{v_0,\omega}\big\rangle\big)=\operatorname{var}_{\mathbf{P}}\left(\left\langle\frac{e^i+e^j}{\sqrt{2}},X_n^{v_0,\omega}\right\rangle\right)-\operatorname{var}_{\mathbf{P}}\left(\left\langle\frac{e^i-e^j}{\sqrt{2}},X_n^{v_0,\omega}\right\rangle\right). \tag{2.18}$$

We recall that $X_n^{v_0,\omega} = v_0 + \sum_{k=1}^n R\varphi_{1,k}^\omega e^d$, where we set $v_0 = Re^d$ for some $R \in SO(d)$. For notational simplicity, we redefine $X_n^{v_0,\omega} := X_n^{v_0,\omega} - v_0$ for the rest of the proof (notice that this is irrelevant for the purpose of proving (2.17)). Introducing the notation

$$s_v(g) := \langle v, Rge^d\rangle \quad \text{for } g \in \mathcal{L}(\mathbb{R}^d), \tag{2.19}$$

we have the simple estimate

$$\left|\mathbf{E}\langle v, X_n^{v_0,\omega}\rangle\right| = \left|s_v\left(\sum_{k=1}^n \mathbf{E}\varphi_{1,k}^\omega\right)\right| \le \sum_{k\in\mathbb{N}} \left\|\mathbf{E}\varphi_{1,k}^\omega\right\|_{\mathrm{op}} < \infty, \tag{2.20}$$

by Hypothesis 2.1. This shows that, in order to establish (2.17), it is sufficient to consider

$$\mathbf{E}\left[\langle v, X_n^{v_0,\omega}\rangle^2\right] = \sum_{k=1}^n \mathbf{E}\left[\left(s_v\left(\varphi_{1,k}^\omega\right)\right)^2\right] + 2\sum_{k=1}^n \sum_{l=1}^{n-k} \mathbf{E}\left[s_v\left(\varphi_{1,l}^\omega\right) s_v\left(\varphi_{1,l+k}^\omega\right)\right]. \tag{2.21}$$

By (2.4) and (2.10) we can write $(s_v(\varphi_{1,k}^\omega))^2 = (s_v \otimes s_v)(\overline{\varphi_{1,k}^\omega \otimes \varphi_{1,k}^\omega})$, and by (2.11) together with Hypothesis 2.1 we can rewrite the first sum as

$$\sum_{k=1}^n \mathbf{E}\left[\left(s_v\left(\varphi_{1,k}^\omega\right)\right)^2\right] = s_v^{\otimes 2}\left(\sum_{k=1}^n \mathbf{E}\left[\overline{\varphi_{1,k}^\omega \otimes \varphi_{1,k}^\omega}\right]\right) = n s_v^{\otimes 2}(\Pi) + \mathrm{O}(1). \tag{2.22}$$

In the same spirit the control the off-diagonal terms. We first observe that by (2.3)

$$\varphi_{1,l}^\omega \otimes \varphi_{1,l+k}^\omega = \varphi_{1,l}^\omega \otimes \left(\varphi_{1,l}^\omega \varphi_{l+1,l+k}^\omega\right) = \left(\varphi_{1,l}^\omega \otimes \varphi_{1,l}^\omega\right)\left(I_d \otimes \varphi_{l+1,l+k}^\omega\right), \tag{2.23}$$

where $I_d \in \mathcal{L}(\mathbb{R}^d)$ is the identity operator. Then by (2.4) and (2.10) we can write

$$s_v\left(\varphi_{1,l}^\omega\right) s_v\left(\varphi_{1,l+k}^\omega\right) = s_v^{\otimes 2}\left(\varGamma \varphi_{1,k}^\omega \otimes \varphi_{1,l+k}^\omega\right) = s_v^{\otimes 2}\left(\left(\overline{\varphi_{1,l}^\omega \otimes \varphi_{1,l}^\omega}\right)\left(I_d \otimes \varphi_{l+1,l+k}^\omega\right)\right). \tag{2.24}$$

By (2.11), (2.7) and (2.6) we then obtain $\mathbf{E}[s_v(\varphi_{1,l}^\omega) s_v(\varphi_{1,l+k}^\omega)] \le \|\mathbf{E}\varphi_{l+1,l+k}^\omega\|_{\mathrm{op}}$, hence by Hypothesis 2.1 it is the clear that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{k=m}^n \sum_{l=1}^{n-k} \mathbf{E}\left[s_v\left(\varphi_{1,l}^\omega\right) s_v\left(\varphi_{1,l+k}^\omega\right)\right] = 0. \tag{2.25}$$

This allows us to focus on studying the limit as $n \to \infty$ and for fixed k of

$$\frac{1}{n} \sum_{l=1}^{n-k} \mathbf{E}\left[s_v\left(\varphi_{1,l}^\omega\right) s_v\left(\varphi_{1,l+k}^\omega\right)\right] = \frac{1}{n} \sum_{l=1}^{n-k} s_v^{\otimes 2}\left(\mathbf{E}\left[\overline{\varphi_{1,l}^\omega \otimes \varphi_{1,l}^\omega}\right]\left(I_d \otimes \mathbf{E}\varphi_{l+1,l+k}^\omega\right)\right). \tag{2.26}$$

In this expression we can replace $\mathbf{E}[\overline{\varphi_{1,l}^\omega \otimes \varphi_{1,l}^\omega}]$ by its limit Π by making a negligible error (of order $1/n$), by Hypothesis 2.1. Furthermore, by the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n-k} s_v^{\otimes 2}\left(\Pi\left(I_d \otimes \mathbf{E}\varphi_{l+1,l+k}^\omega\right)\right) = s_v^{\otimes 2}\left(\Pi\left(I_d \otimes \mathbb{E}\mathbf{E}\varphi_{1,k}^\omega\right)\right), \quad \mathbb{P}(\mathrm{d}\omega)\text{-a.s.} \tag{2.27}$$

We have therefore proven that $\mathbb{P}(\mathrm{d}\omega)$-a.s.

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{var}_{\mathbf{P}}\langle v, X_n^{v_0,\omega}\rangle = s_v^{\otimes 2}(\Pi) + 2\sum_{k=1}^\infty s_v^{\otimes 2}\left(\Pi\left(I_d \otimes \mathbb{E}\mathbf{E}\varphi_{1,k}^\omega\right)\right). \tag{2.28}$$

Let us simplify this expression: by (2.13) the representative matrix of Π is $\Pi_{ij,kl} = \frac{1}{d}\delta_{ij}\delta_{kl}$ and by (2.19) we can write $s_v(g) = \sum_m (R^* v)_m g_{md}$, hence

$$s_v^{\otimes 2}(\Pi) = \sum_{ij} (R^* v)_i (R^* v)_j \Pi_{ij,dd} = \frac{1}{d} \| R^* v \|^2 = \frac{1}{d}, \tag{2.29}$$

because $R \in SO(d)$ and v is a unit vector. The second term in the right-hand side of (2.28) is analogous: setting for simplicity $m := \mathbb{E}\mathbf{E}\varphi^\omega_{l+1,l+k} \in \mathcal{L}(\mathbb{R}^d)$, the matrix of the operator $\Pi(I_d \otimes m)$ is given by

$$\big[\Pi(I_d \otimes m)\big]_{ij,kl} = \sum_{a,b=1}^d \Pi_{ij,ab}(I_d \otimes m)_{ab,kl} = \frac{1}{d} \sum_{a,b=1}^d \delta_{ij}\delta_{ab}\delta_{ak} m_{bl} = \frac{1}{d}\delta_{ij} m_{kl}, \tag{2.30}$$

hence

$$s_v^{\otimes 2}\big[\Pi(I_d \otimes m)\big] = \sum_{ij} (R^* v)_i (R^* v)_j \big[\Pi(I_d \otimes m)\big]_{ij,dd} = \frac{1}{d} m_{dd}. \tag{2.31}$$

Since $m_{dd} = \mathbb{E}\mathbf{E}\langle e^d, \varphi^\omega_{1,k} e^d\rangle$, we have shown that the right-hand side of (2.28) coincides with the formula (2.16) for σ^2 and therefore Eq. (2.17) is proven. $\square$

2.4. *The invariance principle*

Next we turn to the proof of the full invariance principle. The main tool is a projection of the increments of our process $\{X_n^{v_0,\omega}\}_n$ on martingale increments, to which the Martingale Invariance Principle can be applied.

We start setting $\hat{s}(g) := R g e^d$, so that

$$\hat{s}\big(\varphi^\omega_{1,n}\big) = v^\omega_n = X_n^{v_0,\omega} - X_{n-1}^{v_0,\omega}, \tag{2.32}$$

cf. (1.4) and (2.1). Recalling the definition (2.19) of $s_v(g)$, we have $\hat{s}(g) = \sum_{i=1}^d s_{e^i}(g) e^i$. For $n = 1, 2, \dots$ we introduce the $\mathbb{R}^d$-valued process

$$Y_n := \hat{s}\big(\varphi^\omega_{1,n}\big) - \mathbf{E}\big[\hat{s}\big(\varphi^\omega_{1,n}\big)\big]. \tag{2.33}$$

We now show that, for $\mathbb{P}$-a.e. ω,

$$\sup_{n \ge 1} \left\{ \sum_{k=0}^\infty \big\| \mathbf{E}\big[Y_{n+k} | \mathcal{F}^\omega_{1,n}\big] \big\|_{L^\infty(\mathbf{P};\mathbb{R}^d)} \right\} < \infty, \tag{2.34}$$

where we recall that $\mathcal{F}^\omega_{m,n} := \sigma(\varphi^\omega_{1,i}\colon\ m \le i \le n)$. Observe that $\mathbf{E}[Y_{n+k}|\mathcal{F}^\omega_{1,n}] = \hat{s}(\mathbf{E}[\varphi^\omega_{1,n+k}|\mathcal{F}^\omega_{1,n}] - \mathbf{E}[\varphi^\omega_{1,n+k}])$ and we can write

$$\begin{aligned} \mathbf{E}\big[\varphi^\omega_{1,n+k} | \mathcal{F}^\omega_{1,n}\big] - \mathbf{E}\big[\varphi^\omega_{1,n+k}\big] &= \varphi^\omega_{1,n} \mathbf{E}\big[\varphi^\omega_{n+1,n+k} \big| \mathcal{F}^\omega_{1,n}\big] - \mathbf{E}\big[\varphi^\omega_{1,n}\big]\mathbf{E}\big[\varphi^\omega_{n+1,n+k}\big] \\ &= \big(\varphi^\omega_{1,n} - \mathbf{E}\big[\varphi^\omega_{1,n}\big]\big)\mathbf{E}\big[\varphi^\omega_{n+1,n+k}\big]. \end{aligned} \tag{2.35}$$

Since $\|\varphi^\omega_{1,n} - \mathbf{E}[\varphi^\omega_{1,n}]\|_{\mathrm{op}} \le 2$, we have

$$\big\| \mathbf{E}\big[Y_{n+k} | \mathcal{F}^\omega_{1,n}\big] \big\|_{L^\infty(\mathbf{P};\mathbb{R}^d)} \le 2 \big\| \mathbf{E}\big[\varphi^\omega_{n+1,n+k}\big] \big\|_{\mathrm{op}}, \tag{2.36}$$

hence (2.34) follows from Hypothesis 2.1.

We are now ready to prove the invariance principle. It is actually more convenient to redefine $B_N^{v_0,\omega}(t)$, which was introduced in (1.5), as $\frac{1}{\sqrt{N}} X^{v_0,\omega}_{\lfloor Nt \rfloor}$, where $\lfloor a \rfloor \in \mathbb{N} \cup \{0\}$ denotes the integer part of a. In this way, $B_N^{v_0,\omega}(\cdot)$ is a process with trajectories in the Skorohod space $D([0,\infty))$ of càdlàg functions, which is more suitable in order to apply the Martingale Invariance Principle. However, since the limit process σB has continuous paths, it is elementary to pass from convergence in distribution on $D([0,\infty))$ to convergence on $C([0,\infty))$, thus recovering the original statement of Theorem 1.6.

Theorem 2.3. *If Hypothesis 2.1 and Assumption 1.5 hold, then $\mathbb{P}(\mathrm{d}\omega)$-a.s. and for every choice of v_0 the $\mathbb{R}^d$-valued process $B_N^{v_0,\omega}$ converges in distribution on $C([0,\infty))$ to σB, where B is a standard d-dimensional Brownian motion and σ^2 is given by (1.6).*

Proof. Let us set for $n \geq 1$

$$U_n := \sum_{k=0}^{\infty} \mathbf{E}\big[Y_{n+k} | \mathcal{F}_{1,n-1}^{\omega}\big] \quad \text{and} \quad Z_n := \sum_{k=0}^{\infty} \big(\mathbf{E}\big[Y_{n+k} | \mathcal{F}_{1,n}^{\omega}\big] - \mathbf{E}\big[Y_{n+k} | \mathcal{F}_{1,n-1}^{\omega}\big]\big), \tag{2.37}$$

where we agree that $\mathcal{F}_{1,0}^{\omega}$ is the trivial σ-field. Note that U_n and Z_n are well defined, because by Eq. (2.34) the series in (2.37) converge in $L^{\infty}(\mathbf{P}; \mathbb{R}^d)$, for $\mathbb{P}$-a.e. ω. The basic observation is that $\mathbf{E}[Z_n | \mathcal{F}_{1,n-1}^{\omega}] = 0$, hence Z_n is a *martingale difference sequence*, i.e., the process $\{T_n\}_{n\geq 0}$ defined by

$$T_0 := 0, \qquad T_n := \sum_{i=1}^{n} Z_i, \tag{2.38}$$

is a $\{\mathcal{F}_{1,n}^{\omega}\}_n$-martingale (taking values in $\mathbb{R}^d$). Moreover, we have by construction

$$Y_n = \mathbf{E}\big[Y_n | \mathcal{F}_{1,n}^{\omega}\big] = Z_n + (U_n - U_{n+1}), \tag{2.39}$$

that is Y_n is just Z_n plus a telescopic remainder. Therefore the process $\{T_n\}_n$ is very close to the original process $\{X_n^{v_0,\omega}\}_n$, because the variables Y_n are nothing but the centered increments of the process $\{X_n^{v_0,\omega}\}_n$, see (2.32) and (2.33).

For this reason, we start proving the invariance principle for the rescaled process $T^N = \{T^N(t)\}_{t\in[0,\infty)}$ defined by $T^N(t) := \frac{1}{\sqrt{N}} T_{\lfloor Nt \rfloor}$. By the Martingale Invariance Principle in the form given by [8], Corollary 3.24, Chapter VIII, the $\mathbb{R}^d$-valued process T^N converges in law to $\widetilde{\sigma} B$, where $\widetilde{\sigma} > 0$ and B denotes a standard $\mathbb{R}^d$-valued Brownian motion, provided the following conditions are satisfied:

(i) the (random) matrix $(V_n)_{i,j} = \sum_{k=1}^{n} \mathbf{E}[\langle e^i, Z_k\rangle \langle e^j, Z_k\rangle \big| \mathcal{F}_{1,k-1}^{\omega}]$, with $1 \leq i, j \leq d$, is such that

$$\frac{1}{n} V_n \overset{n\to\infty}{\longrightarrow} \widetilde{\sigma}^2 I_d \quad \text{in } \mathbf{P}\text{-probability}; \tag{2.40}$$

(ii) the following integrability condition holds:

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{E}\big[|Z_k|^2; |Z_k| > \varepsilon\sqrt{n}\big] \overset{n\to\infty}{\longrightarrow} 0. \tag{2.41}$$

The second condition is trivial because the variables Z_n are bounded, $\mathbb{P}(\mathrm{d}\omega)$-a.s. The first condition requires more work. We first show that $\mathrm{var}_{\mathbf{P}}(\frac{1}{n}(V_n)_{i,j}) \to 0$ as $n \to \infty$, for all $i, j = 1, \ldots, d$ and for $\mathbb{P}$-a.e. ω, and then we prove the convergence of $\mathbf{E}[\frac{1}{n} V_n]$.

We start controlling the variance of V_n. By definition $(V_n)_{i,j} \leq \frac{1}{2}((V_n)_{i,i} + (V_n)_{j,j})$, hence it suffices to show that $\mathrm{var}_{\mathbf{P}}(\frac{1}{n}(V_n)_{i,i}) \to 0$ for every $i = 1, \ldots, d$. We observed that Z_n has a nice explicit formula:

$$Z_n = \hat{s}\left(\varphi_{1,n-1}^{\omega}\big(\varphi_{n,n}^{\omega} - \mathbf{E}\big[\varphi_{n,n}^{\omega}\big]\big)\left(\sum_{k=0}^{\infty} \mathbf{E}\big[\varphi_{n+1,n+k}^{\omega}\big]\right)\right), \tag{2.42}$$

where we agree that $\varphi^\omega_{n+1,n}$ is the identity operator on $\mathbb{R}^d$ (this convention will be used throughout the proof). Since $\hat{s}(g) = \sum_{i=1}^d s_{e^i}(g) e^i$, where $s_v(g)$ is defined in (2.19), a simple computation then yields

$$\mathbf{E}\big[\langle e^i, Z_n\rangle^2 | \mathcal{F}^\omega_{1,n-1}\big] = s^{\otimes 2}_{e^i}\big(\mathbf{E}\big[Z_n \otimes Z_n | \mathcal{F}^\omega_{1,n-1}\big]\big) = s^{\otimes 2}_{e^i}\big(\big(\varphi^\omega_{1,n-1}\big)^{\otimes 2} \Theta^\omega_n\big), \tag{2.43}$$

where we have applied (2.4) and (2.3) and we have set

$$\Theta^\omega_n := \big(\mathbf{E}\big[\varphi^\omega_{n,n} \otimes \varphi^\omega_{n,n}\big] - \mathbf{E}\big[\varphi^\omega_{n,n}\big] \otimes \mathbf{E}\big[\varphi^\omega_{n,n}\big]\big)\left(\sum_{k=0}^\infty \mathbf{E}\big[\varphi^\omega_{n+1,n+k}\big] \otimes \sum_{l=0}^\infty \mathbf{E}\big[\varphi^\omega_{n+1,n+l}\big]\right). \tag{2.44}$$

Applying (2.43) together with (2.4) and (2.3) we obtain

$$\begin{aligned}\mathrm{var}_{\mathbf{P}}\left(\frac{(V_n)_{i,i}}{n}\right) &= \frac{1}{n^2}\mathbf{E}\left[\left(\sum_{1\le k\le n} s^{\otimes 2}_{e^i}\big(\big(\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]\big)\Theta^\omega_k\big)\right)^2\right] \\ &\le \frac{2}{n^2}\sum_{1\le k\le l\le n} s^{\otimes 4}_{e^i}\big(\mathbf{E}\big[\big(\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]\big) \\ &\qquad \otimes \big(\big(\varphi^\omega_{1,l-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,l-1}\big)^{\otimes 2}\big]\big)\big]\big(\Theta^\omega_l \otimes \Theta^\omega_k\big)\big).\end{aligned} \tag{2.45}$$

Observe that by (2.3) we can write

$$\begin{aligned}\big(\varphi^\omega_{1,l-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,l-1}\big)^{\otimes 2}\big] &= \big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big(\varphi^\omega_{k,l-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]\mathbf{E}\big[\big(\varphi^\omega_{k,l-1}\big)^{\otimes 2}\big] \\ &= \big(\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]\big)\mathbf{E}\big[\big(\varphi^\omega_{k,l-1}\big)^{\otimes 2}\big] \\ &\quad + \big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big\{\big(\varphi^\omega_{k,l-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{k,l-1}\big)^{\otimes 2}\big]\big\}\end{aligned} \tag{2.46}$$

and notice that the term inside the curly brackets is independent of $\mathcal{F}^\omega_{1,k-1}$ and vanishes when we take the expectation. Therefore we have

$$\begin{aligned}&\mathbf{E}\big[\big(\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]\big) \otimes \big(\big(\varphi^\omega_{1,l-1}\big)^{\otimes 2} - \mathbf{E}\big[\big(\varphi^\omega_{1,l-1}\big)^{\otimes 2}\big]\big)\big] \\ &\quad = \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 4} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]^{\otimes 2}\big]\big(I \otimes \mathbf{E}\big[\big(\varphi^\omega_{k,l-1}\big)^{\otimes 2}\big]\big),\end{aligned} \tag{2.47}$$

where we have applied again (2.3) and where I denotes the identity operator on $\mathcal{L}(\mathbb{R}^d)$. We can therefore rewrite the term in the sum in (2.45) as

$$\begin{aligned}&s^{\otimes 4}_{e^i}\big(\mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 4} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]^{\otimes 2}\big]\big(I \otimes \mathbf{E}\big[\varphi^\omega_{k,l-1} \otimes \varphi^\omega_{k,l-1}\big]\big)\big(\Theta^\omega_l \otimes \Theta^\omega_k\big)\big) \\ &\quad = s^{\otimes 4}_{e^i}\big(\mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 4} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]^{\otimes 2}\big]\big(I \otimes \mathbf{E}\big[\overline{\varphi^\omega_{k,l-1} \otimes \varphi^\omega_{k,l-1}}\big]\big)\big(\overline{\Theta^\omega_l} \otimes \overline{\Theta^\omega_k}\big)\big),\end{aligned} \tag{2.48}$$

where we have applied the first relation in (2.12) together with the following relations:

$$\Theta^\omega_l \Gamma = \overline{\Theta^\omega_l} \quad \text{and} \quad \mathbf{E}\big[\varphi^\omega_{k,l-1} \otimes \varphi^\omega_{k,l-1}\big]\Theta^\omega_k \Gamma = \mathbf{E}\big[\overline{\varphi^\omega_{k,l-1} \otimes \varphi^\omega_{k,l-1}}\big]\overline{\Theta^\omega_k}, \tag{2.49}$$

which follow from the fact that $(g \otimes g)\Gamma = \overline{g \otimes g}$ for every $g \in \mathcal{L}(\mathbb{R}^d)$.

We know from Hypothesis 2.1 that when $l \gg k$ the operator $\mathbf{E}[\overline{\varphi^\omega_{k,l-1} \otimes \varphi^\omega_{k,l-1}}]$ is close to Π. Furthermore, if we replace $\mathbf{E}[\overline{\varphi^\omega_{k,l-1} \otimes \varphi^\omega_{k,l-1}}]$ by Π inside (2.48) we get zero: in fact, since trivially $g^{\otimes 2}\Pi = \Pi$ for every $g \in SO(d)$, we have

$$\begin{aligned}&\mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 4} - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]^{\otimes 2}\big](I \otimes \Pi) \\ &\quad = \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2} \otimes \big(\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\Pi\big)\big] - \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big] \otimes \mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\Pi\big] = 0.\end{aligned} \tag{2.50}$$

So it remains to take into account the contribution of the error $\mathbf{E}[\overline{\varphi^\omega_{k,l-1}\otimes\varphi^\omega_{k,l-1}}]-\Pi$ inside (2.48). However, using Hypothesis 2.1, (2.7) and the triangle inequality, we have

$$\big\|\mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 4}-\mathbf{E}\big[\big(\varphi^\omega_{1,k-1}\big)^{\otimes 2}\big]^{\otimes 2}\big]\big\|_{\mathrm{op}}\le 2,\qquad \big\|\Theta^\omega_l\otimes\Theta^\omega_k\big\|_{\mathrm{op}}\le 4\big(1+C(\omega)\big)^4, \tag{2.51}$$

hence, using the second relation in (2.12), from (2.45) and (2.48) we obtain

$$\mathrm{var}_{\mathbf{P}}\bigg(\frac{(V_n)_{i,i}}{n}\bigg)\le\frac{8(1+C(\omega))^4}{n^2}\sum_{1\le k\le n,m\ge 0}\big\|\mathbf{E}\big[\overline{\varphi^\omega_{k,(k-1)+m}\otimes\varphi^\omega_{k,(k-1)+m}}\big]-\Pi\big\|_{\mathrm{op}}\le\frac{8(1+C(\omega))^5}{n}, \tag{2.52}$$

having applied Hypothesis 2.1 again. We have therefore shown that $\mathrm{var}_{\mathbf{P}}(\frac{1}{n}(V_n)_{i,j})\to 0$ as $n\to\infty$, for all $1\le i,j\le d$ and for $\mathbb{P}$-almost every ω.

It remains to prove that $\mathbf{E}[\frac{1}{n}V_n]\to\widetilde{\sigma}^2 I_d$ as $n\to\infty$ and to identify $\widetilde{\sigma}^2$. Let us first note that by (2.32) and (2.33)

$$\sum_{k=1}^n Y_k=X^{v_0,\omega}_n-\mathbf{E}\big[X^{v_0,\omega}_n\big]=:\widetilde{X}_n. \tag{2.53}$$

We also set $Z^i_n:=\langle e^i,Z_n\rangle$, $\widetilde{X}^i_n:=\langle e^i,\widetilde{X}_n\rangle$ and $U^i_n:=\langle e^i,U_n\rangle$ for short. Since $\mathbf{E}[Z_n|\mathcal{F}^\omega_{1,n-1}]=0$ and in view of (2.39), we can write

$$\mathbf{E}\big[(V_n)_{i,j}\big]=\sum_{k=1}^n\mathbf{E}\big[Z^i_kZ^j_k\big]=\sum_{k,l=1}^n\mathbf{E}\big[Z^i_kZ^j_l\big]=\mathbf{E}\big[\big(\widetilde{X}^i_n+U^i_{n+1}\big)\big(\widetilde{X}^j_n+U^j_{n+1}\big)\big] \tag{2.54}$$

(note that $U_1=0$). We recall that by Proposition 2.2 we have as $n\to\infty$, for $\mathbb{P}$-a.e. ω,

$$\mathbf{E}\big[\widetilde{X}^i_n\widetilde{X}^j_n\big]=\big(\mathrm{Cov}_{\mathbf{P}}\big(X^{v_0,\omega}_n\big)\big)_{i,j}=n\sigma^2\delta_{i,j}+\mathrm{o}(n), \tag{2.55}$$

where σ^2 is given by (2.16) (equivalently by (1.6)). Since $\sup_n\|U_n\|_{L^\infty(\mathbf{P};\mathbb{R}^d)}<\infty$ by (2.34), it follows from (2.54) that as $n\to\infty$, for $\mathbb{P}$-a.e. ω, we have

$$\mathbf{E}\big[(V_n)_{i,j}\big]=\mathbf{E}\big[\widetilde{X}^i_n\widetilde{X}^j_n\big]+\mathrm{o}(n)=n\sigma^2\delta_{i,j}+\mathrm{o}(n). \tag{2.56}$$

This completes the proof that the rescaled process $T^N=\{T^N(t)\}_{t\in[0,\infty)}$ converges in distribution as $N\to\infty$ to σ^2B, where σ^2 is given by (1.6).

It finally remains to obtain the same statement for $B^{v_0,\omega}_N(t):=\frac{1}{\sqrt{N}}X^{v_0,\omega}_{\lfloor Nt\rfloor}$. Notice that by (2.38), (2.39) and (2.53) we can write

$$\sup_{1\le k\le n}\big\|X^{v_0,\omega}_k-T_k\big\|\le\sup_{1\le k\le n}\big\|\mathbf{E}\big[X^{v_0,\omega}_k\big]\big\|+\sup_{1\le k\le n}\|U_{k+1}\|, \tag{2.57}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. However, the right-hand side is bounded in n in $L^\infty(\mathbf{P};\mathbb{R}^d)$, for $\mathbb{P}$-a.e. ω (for the first term see (2.20) while for the second term we already know that $\sup_n\|U_n\|_{L^\infty(\mathbf{P};\mathbb{R}^d)}<\infty$). Therefore, $\sup_{t\in[0,M]}\|B^{v_0,\omega}_N(t)-T^N(t)\|\le(const.)/\sqrt{N}$ for every $M>0$, and the proof is completed. □

2.5. *Proof of Theorem* 1.6

We now show that the abstract condition expressed by Hypothesis 2.1 is a consequence of Assumption 1.4. In view of Theorem 2.3, this completes the proof of Theorem 1.6.

We start by controlling $\mathbf{E}[\varphi^\omega_{n,n+k}]$, which is quite easy: the independence of the r_i yields

$$\mathbf{E}\big[\varphi^\omega_{n,n+k}\big]=\omega_n\mathbf{E}(r_1)\omega_{n+1}\mathbf{E}(r_1)\cdots\omega_{n+k}\mathbf{E}(r_1). \tag{2.58}$$

It is clear that $\|\mathbf{E}(r_1)\|_{\mathrm{op}} \le 1$. We now show that Assumption 1.4 yields $\|\mathbf{E}(r_1)\|_{\mathrm{op}} < 1$, so that for every ω we have

$$\sum_{k=1}^{\infty} \big\| \mathbf{E}\big[\varphi^{\omega}_{n,n+k}\big] \big\|_{\mathrm{op}} \le \sum_{k=1}^{\infty} \big\| \mathbf{E}(r_1)^k \big\|_{\mathrm{op}} \le \sum_{k=1}^{\infty} \big\| \mathbf{E}(r_1) \big\|_{\mathrm{op}}^k < \infty. \tag{2.59}$$

To prove that $\|\mathbf{E}(r_1)\|_{\mathrm{op}} < 1$, we argue by contradiction: if $\|\mathbf{E}(r_1)\|_{\mathrm{op}} = 1$ there would exist two vectors $x, y \in S^{d-1}$ such that

$$1 = \big\langle y, \mathbf{E}(r_1) x \big\rangle = \int_{SO(d)} \langle y, gx \rangle Q(\mathrm{d}g). \tag{2.60}$$

Since $|\langle y, gx\rangle| \le 1$, for this equality to hold it is necessary that $gx = y$ for Q-almost every $g \in SO(d)$. Setting $V := \{\lambda x \colon \lambda \in \mathbb{R}\}$ and $W := \{\lambda y \colon \lambda \in \mathbb{R}\}$, this would mean that $gV = W$ for Q-almost every $g \in SO(d)$, which is in contradiction with Assumption 1.4.

Next we turn to the analysis of $\mathbf{E}[\overline{\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}}]$, which is a linear operator on the vector space $\mathcal{L}(\mathbb{R}^d)$, equipped with the standard scalar product $\langle v, w\rangle_{\mathrm{hs}} = \mathrm{Tr}(v^* w)$. We decompose $\mathcal{L}(\mathbb{R}^d) = H_1 \oplus H_s^0 \oplus H_a$ as a sum of the orthogonal subspaces consisting respectively of the multiples of the identity, of the symmetric matrices with zero trace and of the antisymmetric matrices:

$$H_1 := \{\lambda I_d \colon \lambda \in \mathbb{R}\}, \qquad H_s^0 := \big\{ v \in \mathcal{L}\big(\mathbb{R}^d\big) \colon v^* = v \text{ and } \mathrm{Tr}(v) = 0 \big\},$$
$$H_a := \big\{ v \in \mathcal{L}\big(\mathbb{R}^d\big) \colon \ v^* = -v \big\}.$$

All of these subspaces are invariant under $g \otimes g$, for every $g \in \mathcal{L}(\mathbb{R}^d)$, hence they are invariant under $\mathbf{E}[\overline{\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}}]$. We recall that Π is the orthogonal projection on H_1, cf. (2.13), while Γ is the orthogonal projection on $H_1 \oplus H_s^0$, cf. (2.8). Since Π and $\mathbf{E}[\overline{\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}}]$ are zero on H_a and they coincide on H_1, $\|\mathbf{E}[\overline{\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}}] - \Pi\|_{\mathrm{op}}$ is nothing but the operator norm of $\mathbf{E}[\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}]$ restricted to the subspace H_s^0, therefore with obvious notation we can write for every ω

$$\sum_{k=1}^{\infty} \big\| \mathbf{E}\big[\overline{\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}}\big] - \Pi \big\|_{\mathrm{op}} = \sum_{k=1}^{\infty} \big\| \mathbf{E}\big[\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}\big] \big\|_{H_s^0, \mathrm{op}}. \tag{2.61}$$

However, from (2.3) and from the fact that the r_i are independent and identically distributed we have

$$\mathbf{E}\big[\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}\big] = (\omega_n \otimes \omega_n) \mathbf{E}[r_1 \otimes r_1] \cdots (\omega_{n+k} \otimes \omega_{n+k}) \mathbf{E}[r_1 \otimes r_1], \tag{2.62}$$

hence

$$\sum_{k=1}^{\infty} \big\| \mathbf{E}\big[\overline{\varphi^{\omega}_{n,n+k} \otimes \varphi^{\omega}_{n,n+k}}\big] - \Pi \big\|_{\mathrm{op}} \le \sum_{k=1}^{\infty} \big(\big\| \mathbf{E}[r_1 \otimes r_1] \big\|_{H_s^0, \mathrm{op}} \big)^k. \tag{2.63}$$

We are finally left with showing that $\|\mathbf{E}[r_1 \otimes r_1]\|_{H_s^0,\mathrm{op}} < 1$. Let us assume by contradiction that there exist $v, w \in H_s^0$ with $\|v\|_{\mathrm{hs}} = \|w\|_{\mathrm{hs}} = 1$ such that

$$1 = \big\langle w, \mathbf{E}[r_1 \otimes r_1] v \big\rangle_{\mathrm{hs}} = \int_{SO(d)} \big\langle w, g v g^* \big\rangle_{\mathrm{hs}} Q(\mathrm{d}g). \tag{2.64}$$

However, $\|g v g^*\|_{\mathrm{hs}} = \|v\|_{\mathrm{hs}} = 1$, hence $\langle w, g v g^*\rangle_{\mathrm{hs}} \le 1$ and we must have $w = g v g^* = g v g^{-1}$ for Q-a.e. g in $SO(d)$. In particular, the matrices v and w are similar and therefore they have the same eigenvalues $\lambda_1, \dots, \lambda_k$, with $k \le d$. Recall that by the spectral theorem v and w are diagonalizable. Denoting by K_v and K_w respectively the eigenspaces of v and w corresponding to λ_1, we have that $1 \le \dim(K_v) = \dim(K_w) \le d - 1$, where the last inequality follows from the fact that v and w, having zero-trace and not being identically zero, cannot be multiples of the identity. Let us now fix g such that $w = gvg^{-1}$ and take an arbitrary $x \in gK_v$: since $g^{-1}x \in K_v$ we have

$$wx = gvg^{-1}x = g\big(\lambda_1 g^{-1}x\big) = \lambda_1 x, \tag{2.65}$$

which yields $x \in K_w$. Therefore $gK_v \subseteq K_w$ and since the two subspaces have the same dimension we must have $gK_v = K_w$, for Q-almost every g. This being in contradiction with Assumption 1.4, we have indeed that $\|\mathbf{E}[r_1 \otimes r_1]\|_{H_s^0,\mathrm{op}} < 1$ and the proof of Theorem 1.6 is completed.

3. Decay of correlation

3.1. *General notations*

We denote by λ the normalized Haar measure on $SO(d)$. We recall that λ is the only probability measure that is left- and right-invariant, i.e., such that $\lambda(Ag) = \lambda(gA) = \lambda(A)$ for all $g \in SO(d)$ and (measurable) $A \subseteq SO(d)$. In the special case $d = 3$, λ describes a (random) rotation around the vector w of angle θ, where w is uniform on S^2 and θ is uniform on $[0, 2\pi)$. For more on the Haar measure we refer to [4].

We recall that Q denotes the law of r_1. For fixed ω, we denote by $L^\omega_{m,n}$ the law of $\varphi^\omega_{m,n}$ under $\mathbf{P}$, so that for any bounded and measurable function $F : SO(d) \to \mathbb{R}$

$$\mathbf{E}\big[F\big(\varphi^\omega_{m,n}\big)\big] = \int_{SO(d)} F(g) L^\omega_{m,n}(\mathrm{d}g). \tag{3.1}$$

We also set

$$E^\omega(k) := 2 \sup_n \big\| L^\omega_{n+1,n+k} - \lambda \big\|_{\mathrm{TV}}, \tag{3.2}$$

where the total variation (TV) distance between the probability measures μ and ν is defined as $\|\mu - \nu\|_{\mathrm{TV}} := \sup_A |\mu(A) - \nu(A)|$. We observe that $\|\mu - \nu\|_{\mathrm{TV}}$ coincides with $\frac{1}{2}\sup_{|g|\le 1} \int g\,\mathrm{d}\mu - \int g\,\mathrm{d}\nu$, in particular if μ is absolutely continuous with respect to ν, with $f := \mathrm{d}\mu/\mathrm{d}\nu$, we have $\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2}\int |f - 1|\,\mathrm{d}\nu$.

3.2. *Reminders of harmonic analysis on compact groups*

Throughout this section, we assume that G is a compact topological group, equipped with the Borel σ-field, and λ is the normalized Haar measure on G (of course we have in mind the specific case where $G = SO(d)$, $d \ge 2$). We start recalling some basic facts about harmonic analysis on G, taking inspiration from [5,6].

Given a (complex) Hilbert space H, a representation of G on H is a group homomorphism $U : G \to \mathcal{B}(H)$, i.e., $U(gh) = U(g)U(h)$ for all $g, h \in G$, where $\mathcal{B}(H)$ denotes the set of bounded linear operators from H to itself. The representation U is said to be:

- *continuous* if the map $g \mapsto \langle x, U(g)y\rangle$ from G to $\mathbb{C}$ is continuous, for all $x, y \in H$;
- *irreducible* if there is no closed subspace M of H such that $U(g)M \subseteq M$ for every $g \in G$, except the trivial case when $M = \{0\}$ or $M = H$;
- *unitary* if $U(g)$ is a unitary operator for every $g \in G$, i.e., $\langle U(g)x, U(g)y\rangle = \langle x, y\rangle$ for all $x, y \in H$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in H (that we take skew-linear in the first argument and linear in the second).

Finally, two representations U, U' of G on the Hilbert spaces H, H' are said to be *equivalent* if there exists a linear isometry $T : H \to H'$ such that $U(g) = T^{-1}U'(g)T$ for every $g \in G$. The set of equivalence classes of continuous, irreducible, unitary representations of G is denoted by Σ, which is a countable set (sometimes called the *dual object* of G).

We point out that, since G is compact, all irreducible representations are finite-dimensional, that is, they act on a finite-dimensional Hilbert space. Given $\alpha \in \Sigma$, we denote by U^α an arbitrary representation in the class α, acting on the Hilbert space H_α of finite dimension $d_\alpha \in \mathbb{N}$. In each space H_α we fix an (arbitrary) orthonormal basis

$\{\zeta_i^\alpha, i = 1, \dots, d_\alpha\}$ and we denote by $u_{ij}^\alpha(g) = \langle \zeta_i^\alpha, U^\alpha(g)\zeta_j^\alpha\rangle$ the matrix of $U^\alpha(g)$ on this basis. Notice that $u_{ij}^\alpha(\cdot)$ is a continuous function from G to $\mathbb{C}$. We have the following orthogonality relations, valid for all $\alpha, \beta \in \Sigma$, $1 \le j, k \le d_\alpha$, $1 \le l, m \le d_\beta$:

$$\int_G \overline{u_{jk}^\alpha(g)} u_{lm}^\beta(g)\lambda(\mathrm{d}g) = \frac{1}{d_\alpha}\delta_{\alpha\beta}\delta_{jl}\delta_{km}, \tag{3.3}$$

where $\overline{x}$ denotes the complex conjugate of x and δ_{ij} is the Kronecker delta. Therefore $\{\sqrt{d_\alpha} u_{i,j}^\alpha(\cdot)\}_{\alpha\in\Sigma, 1\le i,j\le d_\alpha}$ is an orthonormal set in $L^2(G, \mathrm{d}\lambda)$. A crucial result is that it is also complete, i.e., the functions $u_{i,j}^\alpha(\cdot)$ span $L^2(G, \mathrm{d}\lambda)$, by the Peter–Weil theorem.

Next we introduce the *Fourier transform* $\hat{\mu}$ of a probability measure μ on G, which is the element of the space $\mathsf{S} := \prod_{\alpha\in\Sigma} \mathcal{B}(H^\alpha)$ defined by

$$\hat{\mu}(\alpha) := \int_G U^\alpha(g)\mu(\mathrm{d}g), \quad \alpha \in \Sigma. \tag{3.4}$$

More explicitly, $\hat{\mu}(\alpha)$ is the linear operator acting on H^α whose matrix in the basis $\{\zeta_i^\alpha\}_i$ is given by $\hat{\mu}(\alpha)_{i,j} = \int_G u_{i,j}^\alpha(g)\mu(\mathrm{d}g)$, for $\alpha \in \Sigma$ and $1 \le i, j \le d_\alpha$.

It follows directly from the definition (3.4) and (2.5) that $\|\hat{\mu}(\alpha)\|_{\mathrm{op}} \le 1$ for every probability measure μ on G and for every $\alpha \in \Sigma$. As a matter of fact, when G is connected, this inequality is strict for a large class of μ, as we show in the following lemma (where we denote by $\alpha = 0$ the trivial representation, with $H_0 = \mathbb{C}$ and $U^0(g) = 1$ for every $g \in G$).

Lemma 3.1. *Let μ be a probability measure on G with support V. Assume that $V^{-1}V := \{h^{-1}g : h, g \in V\}$ generates a dense set in G, i.e., the set $\bigcup_{n=1}^\infty (V^{-1}V)^n$ is dense in G. Then $\|\hat{\mu}(\alpha)\|_{\mathrm{op}} < 1$ for every $\alpha \in \Sigma$, $\alpha \ne 0$.*

Proof. Suppose that $\|\hat{\mu}(\alpha)\|_{\mathrm{op}} = 1$. Then there must exist $x, y \in H^\alpha$ with $\|x\| = \|y\| = 1$ such that $\langle y, \hat{\mu}(\alpha)x\rangle = 1$. Now

$$1 = \Re\langle y, \hat{\mu}(\alpha)x\rangle = \int \Re\langle y, U^\alpha(g)x\rangle \nu(\mathrm{d}g). \tag{3.5}$$

The function $r(g) = \Re\langle y, U^\alpha(g)x\rangle$ is real and such that $r(g) \le 1$, and so must be constant on the support of μ and equal to 1. This implies that $U^\alpha(g)x = y$ for any $g \in V$, hence

$$U^\alpha(h^{-1}g)x = U^\alpha(h^{-1})U^\alpha(g)x = U^\alpha(h^{-1})y = x \tag{3.6}$$

for all $g, h \in V$. This means that the relation $U^\alpha(g)x = x$ holds for all $g \in V^{-1}V$ and hence for all $g \in \bigcup_{n=1}^\infty (V^{-1}V)^n$. By assumption the latter set is dense in G and the continuity of the representation U^α yields that $U^\alpha(g)x = x$ for all $g \in G$, which is impossible unless α is the trivial representation. □

We conclude this paragraph noting that the Fourier transform provides an easy tool to check whether a probability measure μ has an L^2 density with respect to the Haar measure λ. More precisely, we have the following lemma.

Lemma 3.2 (Fourier inversion theorem). *A probability measure μ on G is such that*

$$\sum_{\alpha\in\Sigma} d_\alpha \|\hat{\mu}(\alpha)\|_{\mathrm{hs}}^2 < \infty \tag{3.7}$$

if and only if it is absolutely continuous with respect to λ with density in $L^2(G, \mathrm{d}\lambda)$. In this case, the density $f = \mathrm{d}\mu/\mathrm{d}\lambda$ is given by

$$f(g) = \sum_{\alpha\in\Sigma} d_\alpha \operatorname{Tr}(\hat{\mu}(\alpha)U^\alpha(g)^*) = \sum_{\alpha\in\Sigma}\sum_{1\le i,j\le d_\alpha} d_\alpha \hat{\mu}(\alpha)_{i,j}\overline{u_{i,j}^\alpha(g)}, \tag{3.8}$$

where the series converges in $L^2(G, \mathrm{d}\lambda)$.

Proof. Since $\{\sqrt{d_\alpha} u^\alpha_{i,j}(\cdot)\}_{\alpha\in\Sigma, 1\le i,j\le d_\alpha}$ is a complete orthonormal set in $L^2(G, \mathrm{d}\lambda)$, the same is true if we replace $u^\alpha_{i,j}(\cdot)$ by $\overline{u^\alpha_{i,j}(\cdot)}$, therefore condition (3.7) guarantees that the right-hand side of (3.8) does define a function $f \in L^2(G, \mathrm{d}\lambda)$. Consider then the (a priori complex) measure $\mathrm{d}\nu := f\,\mathrm{d}\lambda$. Using (3.3) it is easy to check that

$$\hat{\nu}(\alpha)_{i,j} := \int_G u^\alpha_{i,j}(g)\nu(\mathrm{d}g) = \hat{\mu}(\alpha)_{i,j} \tag{3.9}$$

for all $\alpha \in \Sigma$ and $1 \le i, j \le d_\alpha$. By Theorem 27.42 of [6] this implies that $\mu = \nu$. Vice versa, if a function f is in $L^2(G, \mathrm{d}\lambda)$, the right-hand side of (3.8) is nothing but its Fourier series in the orthonormal set $\{\sqrt{d_\alpha}\overline{u^\alpha_{i,j}(\cdot)}\}_{\alpha\in\Sigma,1\le i,j\le d_\alpha}$, hence relation (3.7) holds true. Finally, the second equality in (3.8) is easily checked. □

3.3. *Exponential decay of the total variation norm*

In this subsection we need to assume that G is also connected (which is of course the case for $G = SO(d)$). We show that, assuming hypotheses H-1 or H-2 (cf. Section 1.4), for $\mathbb{P}$-a.e. ω, we have

$$\sum_{k\in\mathbb{N}} E^\omega(k) < \infty, \tag{3.10}$$

where we recall that $E^\omega(k)$ has been introduced in (3.2). As a matter of fact, we are going to prove the much stronger result that there exist positive constants c_1, c_2 such that

$$\sup_\omega E^\omega(k) \le c_1 \mathrm{e}^{-c_2 k} \qquad \text{for all } k \in \mathbb{N}. \tag{3.11}$$

It is convenient to introduce the convolution $\mu * \nu$ of two probability laws μ, ν on G by

$$(\mu * \nu)(A) := \int_G \mu\big(Ah^{-1}\big)\nu(\mathrm{d}h) = \int_G \nu\big(g^{-1}A\big)\mu(\mathrm{d}g), \tag{3.12}$$

so that if X, Y are two independent random elements of G with marginal laws μ, ν, then $\mu * \nu$ is the law of XY. Therefore we can express $L^\omega_{m,n}$ as

$$L^\omega_{m,n} = \delta_{\omega_m} * Q * \delta_{\omega_{m+1}} * Q * \cdots * \delta_{\omega_n} * Q, \tag{3.13}$$

where δ_g denotes the Dirac mass at $g \in G$. We stress that in general the convolution is not commutative. A basic property is that $\widehat{\mu * \nu}(\alpha) = \hat{\mu}(\alpha)\hat{\nu}(\alpha)$ for every $\alpha \in \Sigma$, or more explicitly $\widehat{\mu * \nu}(\alpha)_{i,j} = \sum_{k=1}^{d_\alpha} \hat{\mu}(\alpha)_{i,k}\hat{\nu}(\alpha)_{k,j}$, as one easily checks from (3.4).

In the next crucial lemma we give an explicit bound on $E^\omega(k)$ in terms of the Fourier transform $\widehat{Q}$ of Q. We recall that we denote by $\alpha = 0$ the trivial representation.

Lemma 3.3. *The following relation holds true for every $k \in \mathbb{N}$:*

$$\Big(\sup_\omega E^\omega(k)\Big)^2 \le \sum_{\alpha\in\Sigma,\alpha\ne 0} d_\alpha \big\|\widehat{Q}(\alpha)\big\|^2_{\mathrm{hs}} \big\|\widehat{Q}(\alpha)\big\|^{2(k-1)}_{\mathrm{op}}. \tag{3.14}$$

Proof. From (3.13) we can write

$$\widehat{L}^\omega_{n+1,n+k}(\alpha) = U^\alpha(\omega_{n+1})\widehat{Q}(\alpha)\cdots U^\alpha(\omega_{n+k})\widehat{Q}(\alpha), \tag{3.15}$$

and using the inequalities in (2.6) we get

$$\big\|\widehat{L}^\omega_{n+1,n+k}(\alpha)\big\|^2_{\mathrm{hs}} \le \big\|\widehat{Q}(\alpha)\big\|^2_{\mathrm{hs}}\big\|\widehat{Q}(\alpha)\big\|^{2(k-1)}_{\mathrm{op}} \prod_{i=1}^k \big\|U^\alpha(\omega_{n+i})\big\|^2_{\mathrm{op}} \le \big\|\widehat{Q}(\alpha)\big\|^2_{\mathrm{hs}}\big\|\widehat{Q}(\alpha)\big\|^{2(k-1)}_{\mathrm{op}}, \tag{3.16}$$

where we used that $\|U^\alpha(\omega_{n+i})\|_{\mathrm{op}}^2 = 1$ because the representation is unitary. Now assume that the right-hand side of (3.14) is finite (otherwise there is nothing to prove). By Lemma 3.2, $L^\omega_{n+1,n+k}$ has a density $f \in L^2(G, \mathrm{d}\lambda)$ with respect to λ, therefore by Jensen's inequality we can write

$$4\left\|L^\omega_{n+1,n+k} - \lambda\right\|_{\mathrm{TV}}^2 = \left(\int_G |f-1|\,\mathrm{d}\lambda\right)^2 \le \int_G f^2\,\mathrm{d}\lambda - 1 = \sum_{\alpha\in\Sigma,\alpha\neq 0} d_\alpha \left\|\widehat{L}^\omega_{n+1,n+k}(\alpha)\right\|_{\mathrm{hs}}^2, \tag{3.17}$$

where in the last equality we have used Parseval's identity, observing that $\langle f, u^\alpha_{i,j}\rangle = (\widehat{L}^\omega_{n+1,n+k}(\alpha))_{i,j}$ and that trivially $\hat{\mu}(0) = 1$ for every probability measure μ on G. Recalling the definition (3.2) of $E^\omega(k)$, relation (3.14) is proven. □

Proof of (3.11) under hypothesis H-2. Let us set $f := \mathrm{d}Q/\mathrm{d}\lambda \in L^2(G, \mathrm{d}\lambda)$. By Parseval's identity we have

$$\|f\|_2^2 := \int_G f^2\,\mathrm{d}\lambda = \sum_{\alpha\in\Sigma} d_\alpha \left\|\widehat{Q}(\alpha)\right\|_{\mathrm{hs}}^2 < \infty. \tag{3.18}$$

In particular, for every $\varepsilon > 0$, $\|\widehat{Q}(\alpha)\|_{\mathrm{hs}} \le \varepsilon$ for every $\alpha \notin \Gamma$, with Γ a finite subset of Σ. Since $\|\widehat{Q}(\alpha)\|_{\mathrm{op}} \le \|\widehat{Q}(\alpha)\|_{\mathrm{hs}}$, we have that $\|\widehat{Q}(\alpha)\|_{\mathrm{op}} \le \varepsilon$ for every $\alpha \in \Sigma$, $\alpha \notin \Gamma$. Next observe that Lemma 3.1 can be applied, because by hypothesis the support of Q contains a non-empty open set A, hence $A^{-1}A$ is open too and therefore it generates the whole G (it is easily seen that, for any non-empty open subset B, $\bigcup_{n=1}^\infty B^n$ is non-empty and both open and closed, hence it must be the whole G, which is connected). This observation yields

$$h := \sup_{\alpha\in\Sigma,\alpha\neq 0} \left\|\widehat{Q}(\alpha)\right\|_{\mathrm{op}} < 1. \tag{3.19}$$

Therefore from Lemma 3.3 we have that

$$\sup_\omega E^\omega(k) \le \|f\|_2 \cdot h^{(k-1)}, \tag{3.20}$$

which proves (3.11) under hypothesis H-2. □

Proof of (3.11) under hypothesis H-1. Since the law Q is assumed to be conjugation invariant, we have $\int_G f(g)Q(\mathrm{d}g) = \int_G f(t^{-1}gt)Q(\mathrm{d}g)$, for every $t \in G$. Then for any law ν on G and for any bounded measurable function $f : G \to \mathbb{R}$ we have

$$\int_G f\,\mathrm{d}(Q * \nu) = \int_G\int_G f(gh)Q(\mathrm{d}g)\nu(\mathrm{d}h) = \int_G\int_G f(hg)Q(\mathrm{d}g)\nu(\mathrm{d}h) = \int_G f\,\mathrm{d}(\nu * Q), \tag{3.21}$$

hence $Q * \nu = \nu * Q$. In particular, taking $\nu = \delta_g$, the operator $\widehat{Q}(\alpha)$ commutes with $U^\alpha(g)$, for every $g \in G$. Schur lemma then yields that $\widehat{Q}(\alpha)$ is a multiple of the identity I_α on H_α: $\widehat{Q}(\alpha) = c_\alpha I_\alpha$ for $c_\alpha \in \mathbb{C}$. Then from (3.15) it follows that

$$\widehat{L}^\omega_{n+1,n+k}(\alpha) = U^\alpha(\omega_{n+1}\cdots\omega_{n+k})\widehat{Q}(\alpha)^k, \tag{3.22}$$

hence $\|\widehat{L}^\omega_{n+1,n+k}(\alpha)\|_{\mathrm{hs}}^2 = \|\widehat{Q}(\alpha)^k\|_{\mathrm{hs}}^2$. Since by assumption for $k \ge n_0$ the measure Q^{*k} has a density $f_k := \mathrm{d}Q^{*k}/\mathrm{d}\lambda \in L^2(G, \mathrm{d}\lambda)$, it follows that also $L^\omega_{n+1,n+k}(\alpha)$ has a density $g^\omega_{n,k} = \mathrm{d}L^\omega_{n+1,n+k}/\mathrm{d}\lambda \in L^2(G, \mathrm{d}\lambda)$ (cf. Lemma 3.2) and by Parseval's identity we have

$$\int_G \left(g^\omega_{n,k}\right)^2\mathrm{d}\lambda = \left\|g^\omega_{n,k}\right\|_2^2 = \|f_k\|_2^2 = \sum_{\alpha\in\Sigma} d_\alpha \left\|\widehat{Q}(\alpha)^k\right\|_{\mathrm{hs}}^2 < \infty. \tag{3.23}$$

Arguing as above and recalling that $\widehat{Q}(\alpha) = c_\alpha I_\alpha$, it follows that (3.19) still holds. We therefore have for $k \ge n_0$

$$4\left\| L^{\omega}_{n+1,n+k}(\alpha) - \lambda \right\|^2_{\mathrm{TV}} \le \left(\int_G \left| g^{\omega}_{n,k} - 1 \right| \mathrm{d}\lambda \right)^2 \le \int_G \left(g^{\omega}_{n,k} \right)^2 \mathrm{d}\lambda - 1 = \sum_{\alpha \in \Sigma, \alpha \neq 0} d_\alpha \left\| \widehat{Q}(\alpha)^k \right\|^2_{\mathrm{hs}}$$
$$\le \sum_{\alpha \in \Sigma, \alpha \neq 0} d_\alpha \left\| \widehat{Q}(\alpha)^{n_0} \right\|^2_{\mathrm{hs}} \left\| \widehat{Q}(\alpha) \right\|^{2(k-n_0)}_{\mathrm{op}} = \| f_{n_0} \|^2_2 \cdot h^{2(k-n_0)}.$$

Then $\sup_\omega E^{\omega}(k) \le \| f_{n_0} \|_2 h^{k-n_0}$ and the proof of Eq. (3.11) is complete. □

Proof of Proposition 1.8. It suffices to prove that for every n and every ω we have $\alpha^{\omega}(n) \le 2E^{\omega}(n)$. Since $\{\varphi_{1,n}\}_n$ is a (inhomogeneous) Markov process we directly see that

$$\alpha^{\omega}(n) \le \sup_{u,w} \left| \mathbf{E}\left[u\left(\varphi^{\omega}_{1,m} \right) w\left(\varphi^{\omega}_{1,m+n} \right) \right] - \mathbf{E}\left[u\left(\varphi^{\omega}_{1,m} \right) \right] \mathbf{E}\left[w\left(\varphi^{\omega}_{1,m+n} \right) \right] \right|, \tag{3.24}$$

where u and w vary in the set of measurable maps from G to $[0, 1]$. Since

$$\left| \mathbf{E}\left[u\left(\varphi^{\omega}_{1,m} \right) w\left(\varphi^{\omega}_{1,m+n} \right) \right] - \mathbf{E}\left[u\left(\varphi^{\omega}_{1,m} \right) \right] \mathbf{E}\left[w\left(\varphi^{\omega}_{1,m+n} \right) \right] \right|$$
$$\le \left| \int_G u(g) \left(\int_G w\left(g g' \right) \left(L^{\omega}_{m+1,m+n}\left(\mathrm{d}g' \right) - \lambda\left(\mathrm{d}g' \right) \right) \right) L^{\omega}_{1,m}(\mathrm{d}g) \right|$$
$$+ \left| \int u(g) L^{\omega}_{1,m}(\mathrm{d}g) \int_G \left(\int_G w\left(g g' \right) \left(L^{\omega}_{m+1,m+n}\left(\mathrm{d}g' \right) - \lambda\left(\mathrm{d}g' \right) \right) \right) L^{\omega}_{1,m}(\mathrm{d}g) \right|, \tag{3.25}$$

the desired bound follows since both $|u(\cdot)|$ and $|w(\cdot)|$ are bounded by 1. □

Appendix A. The elementary approach to the two-dimensional case

We give here a partial proof of Theorem 1.6 in the 2-dimensional case. We identify in particular the variance σ^2, cf. (1.6), of the limit process. We set $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ and we denote by R_α the rotation by an angle α. With reference to (1.4), we write $\omega_j = R_{\gamma_j}$ and $r_j = R_{\theta_j}$, with γ_j and θ_j random variables taking values in $\mathbb{T}$. The Fourier coefficients of the law Q of θ_1 are $\int_{\mathbb{T}} \mathrm{e}^{imx} Q(\mathrm{d}x) =: \hat{q}_m$, for $m \in \mathbb{Z}$. Recall that we are assuming that $Q(\{\theta_0, \theta_0 + \pi\}) < 1$ for every θ_0 and this is equivalent to $|\hat{q}_n| < 1$ for $n = 1$ and $n = 2$.

We set $\Theta_n := \theta_1 + \cdots + \theta_n$ and $\Gamma_n := \gamma_1 + \cdots + \gamma_n$ for $n \in \mathbb{N}$, along with $\Phi_n := \Gamma_n + \Theta_n$. Therefore the real and complex part of the random variable $Z^{\omega}_N := \mathrm{e}^{\mathrm{i}\Phi_1} + \cdots + \mathrm{e}^{\mathrm{i}\Phi_N}$ coincide with the components of the random vector $X^{v_0,\omega}_N$, $v_0 = (1, 0)$. Our goal is to compute the asymptotic covariance matrix of $X^{v_0,\omega}_N$ as $N \to \infty$. Note that no centering is needed, since

$$\mathbf{E}\left[Z^{\omega}_N \right] = \sum_{m=1}^{N} \mathrm{e}^{\mathrm{i}\Gamma_m} \mathbf{E}\left[\mathrm{e}^{\mathrm{i}\Theta_m} \right] = \sum_{m=1}^{N} \mathrm{e}^{\mathrm{i}\Gamma_m} \hat{q}_1^m, \tag{A.1}$$

and therefore $|\mathbf{E}[Z_N]| \le |\hat{q}_1|/(1 - |\hat{q}_1|) < \infty$, because $|\hat{q}_1| < 1$.

We can therefore focus on the second moments. For simplicity, we fix an arbitrary direction $\mathrm{e}^{\mathrm{i}\xi_0}$ in $\mathbb{R}^2 \simeq \mathbb{C}$, with $\xi_0 \in \mathbb{T}$, and we look at the projection $\{Z^{\omega,\xi_0}_n\}_n$ of the process $\{Z^{\omega}_n\}_n$ in this direction, i.e.

$$Z^{\omega,\xi_0}_0 := 0, \qquad Z^{\omega,\xi_0}_n := \cos(\Phi_1 - \xi_0) + \cdots + \cos(\Phi_n - \xi_0). \tag{A.2}$$

For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ one directly computes with $x = \Theta_m + \Gamma_{m+n} - \xi_0$

$$\mathbf{E}\left[\cos(\Phi_{n+m} - \xi_0) \middle| \Theta_m \right] = \mathbf{E}\left[\cos(x + \Theta_n) \right] = \Re\left(\hat{q}_1^n \mathrm{e}^{\mathrm{i}x} \right) = |\hat{q}_1|^n \cos(\Theta_m + \bar{\theta} n + \Gamma_{m+n} - \xi_0), \tag{A.3}$$

where $\bar\theta$ is such that $\mathrm{e}^{\mathrm{i}\bar\theta} = \hat q_1/|\hat q_1|$. We observe also that for $a, b \in \mathbb{T}$

$$\mathbf{E}\big[\cos(\Theta_m + a)\cos(\Theta_m + b)\big] = \frac{1}{2}\cos(a-b) + \frac{1}{2}\Re\big(\hat q_2^m \mathrm{e}^{\mathrm{i}(a+b)}\big), \tag{A.4}$$

and from (A.3) and (A.4) we directly see that

$$\mathbf{E}\big[\cos(\Phi_m - \xi_0)\cos(\Phi_{n+m} - \xi_0)\big] = \frac{1}{2}|\hat q_1|^n\big\{\cos(\Gamma_{m+n} - \Gamma_m + \bar\theta n) + \Re\big(\hat q_2^m \mathrm{e}^{\mathrm{i}(\Gamma_{m+n}+\Gamma_m+\bar\theta n-2\xi_0)}\big)\big\}, \tag{A.5}$$

and the latter expression actually holds also for $n = 0$. We are now ready to estimate $\mathbf{E}[(Z_N^{\xi_0,\omega})^2]$. The expression contains diagonal terms and for those we have

$$\sum_{m=1}^N \mathbf{E}\big[\cos^2(\Phi_m - \xi_0)\big] = \frac{N}{2} + \mathrm{o}(N), \tag{A.6}$$

by (A.5) with $n = 0$ (recall that $|\hat q_2| < 1$). The off-diagonal terms instead give

$$2\sum_{m=1}^{N-1}\sum_{n=1}^{N-m} \mathbf{E}\big[\cos(\Phi_m - \xi_0)\cos(\Phi_{n+m} - \xi_0)\big] = \sum_{n=1}^{N-1}\hat q_1^n \sum_{m=1}^{N-n}\cos(\Gamma_{m+n} - \Gamma_m + \bar\theta n) + \mathrm{o}(N). \tag{A.7}$$

For every fixed $n \in \mathbb{N}$, by the ergodic theorem we have that $\mathbb{P}(\mathrm{d}\omega)$-a.s. as $N \to \infty$

$$\sum_{m=1}^{N-n}\cos(\Gamma_{m+n} - \Gamma_m + \bar\theta n) = \mathbb{E}\big(\cos(\Gamma_n + \bar\theta n)\big)\cdot N + \mathrm{o}(N), \tag{A.8}$$

and therefore that $\mathbb{P}$-a.s.

$$\sum_{n=1}^{N-1}\hat q_1^n \sum_{m=1}^{N-n}\cos(\Gamma_{m+n} - \Gamma_m + \bar\theta n) = \left(\sum_{n=1}^{\infty}\hat q_1^n \mathbb{E}\big(\cos(\Gamma_n + \bar\theta n)\big)\right)\cdot N + \mathrm{o}(N), \tag{A.9}$$

so that finally we have $\mathbb{P}(\mathrm{d}\omega)$-a.s.

$$\frac{1}{N}\mathbf{E}\big[\big(Z_N^{\omega,\xi_0}\big)^2\big] = \frac{1}{N}\sum_{i,j=1}^N \mathbf{E}\big(\cos(\Phi_i - \xi_0)\cos(\Phi_j - \xi_0)\big) \overset{N\to\infty}{\longrightarrow} \frac{1}{2} + \sum_{n=1}^{\infty}|\hat q_1|^n \mathbb{E}\big[\cos(\Gamma_n + \bar\theta n)\big], \tag{A.10}$$

which matches with (1.6). Note that the diffusion coefficient is independent of the direction ξ_0 and that it depends on the law of θ_1 just through the first Fourier coefficient $\hat q_1$.

Appendix B. The homogeneous case

The aim of this appendix is to argue that, if disorder is absent, Theorem 1.6 holds under the assumption that the support of Q generates a dense set in $SO(d)$.

In order to do this, let us first observe that, when disorder is absent, we can weaken Assumption 1.4 to the following *generalized condition*: there exist $m \in \mathbb{N}$ such that

$$\big\|\big(\mathbf{E}(r_1)\big)^m\big\|_{\mathrm{op}} < 1 \quad \text{and} \quad \big\|\big(\mathbf{E}(r_1 \otimes r_1)\big)^m\big\|_{H_s^0,\mathrm{op}} < 1, \tag{B.1}$$

where we recall that H_s^0 denotes the space of symmetric real matrices with zero trace. We have shown in Section 2.5 that this condition with $m = 1$ follows from Assumption 1.4. The fact that, when disorder is absent, Eq. (B.1) is sufficient to yield Hypothesis 2.1, and hence Theorem 1.6, is immediately checked: for instance, by (2.58) we can write

$$\sum_{k=1}^{\infty} \big\| \mathbf{E}\big[\varphi_{n,n+k}^{\omega}\big] \big\|_{\mathrm{op}} \le \sum_{k=1}^{\infty} \big\| \big(\mathbf{E}(r_1)\big)^m \big\|_{\mathrm{op}}^{\lfloor k/m \rfloor} < \infty \tag{B.2}$$

and analogously one shows that $\sum_{k=1}^{\infty} \|\mathbf{E}[\overline{\varphi_{n,n+k}^{\omega} \otimes \varphi_{n,n+k}^{\omega}}] - \Pi\|_{\mathrm{op}} < \infty$, cf. (2.63).

We recall that, for an arbitrary linear operator A on some vector space and for any fixed operator norm $\|\cdot\|$, the sequence $\|A^m\|^{1/m}$ converges as $m \to \infty$ toward the spectral radius of A, denoted $\mathrm{Sp}(A)$. Furthermore, by sub-additivity (since $\|A^{m+n}\|_{\mathrm{op}} \le \|A^m\|_{\mathrm{op}} \|A^n\|_{\mathrm{op}}$) we have $\mathrm{Sp}(A) = \inf_{m \in \mathbb{N}} \|A^m\|^{1/m}$, hence we can restate (B.1) as

$$\mathrm{Sp}\big(\mathbf{E}(r_1)\big) < 1, \qquad \mathrm{Sp}\big(\mathbf{E}(r_1 \otimes r_1)|_{H_s^0}\big) < 1. \tag{B.3}$$

Let us finally show that Eq. (B.3) is satisfied whenever the support V of Q generates a dense set in $SO(d)$, i.e., whenever the closure of $\bigcup_{k \in \mathbb{Z}} V^k$ is the whole $SO(d)$, where we set $V^{-1} := \{g^{-1}\colon g \in V\}$, $V^2 := \{gh\colon g, h \in V\}$, and so on. Since this fact is easily checked for $d = 2$, in the following we assume that $d \ge 3$.

We argue by contradiction: if the spectral radius of $\mathbf{E}(r_1)$ is equal to one, there exists $v \in \mathbb{C}^d$ with $\|v\| = 1$ such that $\mathbf{E}(r_1)v = \mathrm{e}^{\mathrm{i}\theta} v$, with $\theta \in [0, 2\pi)$, hence

$$1 = \Re\big\langle \mathrm{e}^{\mathrm{i}\theta} v, \mathbf{E}(r_1) v\big\rangle = \int_{SO(d)} \Re\big\langle \mathrm{e}^{\mathrm{i}\theta} v, g v\big\rangle Q(\mathrm{d}g). \tag{B.4}$$

In the preceding relations we have denoted by $\langle \cdot, \cdot \rangle$ the standard Hermitian product on $\mathbb{C}^d$, i.e., $\langle a, b\rangle := \sum_{k=1}^{d} \overline{a_k} b_k$, where $\overline{a}$ denotes the complex conjugate of a. Since $\Re\langle \mathrm{e}^{\mathrm{i}\theta} v, gv\rangle \le 1$ for every $g \in SO(d)$, we must have $gv = \mathrm{e}^{\mathrm{i}\theta} v$ for every $g \in V$, the support of Q. Writing $v_1 + \mathrm{i}v_2$ with $v_1, v_2 \in \mathbb{R}^d$ and denoting by U the linear subspace of $\mathbb{R}^d$ spanned by v_1, v_2, it follows that $gU = U$ for every $g \in V$. Since by assumption V generates a dense set in $SO(d)$, by continuity we must have $gU = U$ for every $g \in SO(d)$, which is clearly impossible because $1 \le \dim(U) \le 2$ (recall that we assume $d \ge 3$).

With analogous arguments, if the spectral radius of $\mathbf{E}(r_1 \otimes r_1)$ on the space H_s^0 equals one, there must exist $v_1, v_2 \in H_s^0$ with $\|v_1\|_{\mathrm{hs}}^2 + \|v_2\|_{\mathrm{hs}}^2 = 1$ and $\theta \in [0, 2\pi)$ such that $g(v_1 + \mathrm{i}v_2)g^{-1} = \mathrm{e}^{\theta}(v_1 + \mathrm{i}v_2)$, for every $g \in V$. Denoting by U the linear subspace of H_s^0 spanned by v_1, v_2, it follows that $gUg^{-1} = U$ for every $g \in V$. Since by assumption V generates a dense set in $SO(d)$, by continuity we must have $gUg^{-1} = U$ for every $g \in SO(d)$. However, this is not possible, because the only linear subspaces W such that $gWg^{-1} \subseteq W$ for every $g \in SO(d)$ are $W = \{0\}$ and $W = H_s^0$ (i.e., the representation $SO(d) \ni g \mapsto g \otimes g$ on the vector space H_s^0 is *irreducible*).

Let us check this fact. We take $w \in W$ not identically zero: by the spectral theorem, there exists $g \in SO(d)$ such that $v := gwg^{-1} \in W$ is diagonal: $v_{ij} = \lambda_i \delta_{ij}$. Since v is not identically zero and it has zero trace, there exist i_0, j_0 such that $\lambda_{i_0} \ne \lambda_{j_0}$. Let us now take $h \in SO(d)$ to be the matrix that permutes the coordinates i_0 and j_0, i.e., $h_{ij} := \delta_{ij}$ for $i, j \notin \{i_0, j_0\}$ while $h_{i_0 j} = h_{j i_0} := \delta_{j_0 j}$ and $h_{i j_0} = h_{j_0 i} := \delta_{i i_0}$. It is clear that $\widetilde{v} := hvh^{-1} \in W$ is such that $\widetilde{v}_{ij} = \widetilde{\lambda}_i \delta_{ij}$, where $\widetilde{\lambda}_i = \lambda_i$ for $i \notin \{i_0, j_0\}$ while $\widetilde{\lambda}_{i_0} = \lambda_{j_0}$ and $\widetilde{\lambda}_{j_0} = \lambda_{i_0}$. Therefore, $z := \frac{1}{(\lambda_{i_0} - \lambda_{j_0})}(v - \widetilde{v}) \in W$ is such that $z_{i_0 i_0} = 1$, $z_{j_0 j_0} = -1$, and $z_{ij} = 0$ for all the other values of i, j. By considering gxg^{-1}, where $g \in SO(d)$ is an arbitrary permutation matrix, we obtain all the matrices defined like z but with arbitrary i_0, j_0. These matrices span the linear subspace consisting of all the diagonal matrices with zero trace, which are therefore contained in W. However, again by the spectral theorem, for any matrix $u \in H_s^0$ we can find $g \in SO(d)$ such that gug^{-1} is diagonal with zero trace, hence we must have $W = H_s^0$ and the proof is completed.

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