THE DISCRETE-TIME PARABOLIC ANDERSON MODEL WITH HEAVY-TAILED POTENTIAL

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Abstract. We consider a discrete-time version of the parabolic Anderson model. This may be described as a model for a directed $(1+d)$-dimensional polymer interacting with a random potential, which is constant in the deterministic direction and i.i.d. in the d orthogonal directions. The potential at each site is a positive random variable with a polynomial tail at infinity. We show that, as the size of the system diverges, the polymer extremity is localized almost surely at one single point which grows ballistically. We give an explicit characterization of the localization point and of the typical paths of the model.

1. Introduction and results

The model we consider is built on two main ingredients, a random walk S and a random potential ξ. We start describing these ingredients. A word about notation: throughout the paper, we denote by $|\cdot|$ the ℓ^1 norm on $\mathbb{R}^d$, that is $|x| = |x_1| + \ldots + |x_d|$ for $x = (x_1, \ldots, x_d)$, and we set $\mathcal{B}_N := \{x \in \mathbb{Z}^d : |x| \le N\}$.

1.1. **The random walk.** Let $S = \{S_k\}_{k \ge 0}$ denote the coordinate process on the space $\Omega_S := (\mathbb{Z}^d)^{\mathbb{N}_0 := \{0,1,2,\ldots\}}$, that we equip as usual with the product topology and σ-field. We denote by P the law on Ω_S under which S is a (lazy) nearest-neighbor random walk started at zero, that is $\mathrm{P}(S_0 = 0) = 1$ and under P the variables $\{S_{k+1} - S_k\}_{k \ge 0}$ are i.i.d. with $\mathrm{P}(S_1 = y) = 0$ if $|y| > 1$. We also assume the following irreducibility conditions:

$$\mathrm{P}(S_1 = 0) =: \kappa > 0 \qquad \text{and} \qquad \mathrm{P}(S_1 = y) > 0 \quad \forall y \in \mathbb{Z}^d \text{ with } |y| = 1\,. \tag{1.1}$$

The usual assumption $\mathrm{E}(S_1) = 0$ is not necessary. For $x \in \mathbb{Z}^d$, we denote by P_x the law of the random walk started at x, that is $\mathrm{P}_x(S \in \cdot) := \mathrm{P}(S + x \in \cdot)$.

We could actually deal with random walks with finite range, i.e., for which there exists $R > 0$ such that $\mathrm{P}(S_1 = y) = 0$ if $|y| > R$, but we stick for simplicity to the case $R = 1$.

1.2. **The random potential.** We let $\xi = \{\xi(x)\}_{x \in \mathbb{Z}^d}$ denote a family of i.i.d. random variables taking values in $\mathbb{R}^+$, defined on some probability space $(\Omega_\xi, \mathcal{F}, \mathbb{P})$, which should not be confused with Ω_S. We assume that the variables $\xi(x)$ are Pareto distributed, that is

$$\mathbb{P}(\xi(0) \in \mathrm{d}x) = \frac{\alpha}{x^{1+\alpha}}\, \mathbf{1}_{[1,\infty)}(x)\, \mathrm{d}x\,, \tag{1.2}$$

for some $\alpha \in (0, \infty)$. Although the precise assumption (1.2) on the law of ξ could be relaxed to a certain extent, we prefer to keep it for the sake of simplicity.

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In the sequel we could work with the product space $\Omega_S \times \Omega_\xi$, equipped with the product probability $\mathrm{P} \otimes \mathbb{P}$, under which ξ and S are independent, but it is actually not necessary, because ξ and S will play on a separate level, as it will be clear in a moment.

1.3. **The model.** Given $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and a $\mathbb{P}$-typical realization of the variables $\xi = \{\xi(y)\}_{y \in \mathbb{Z}^d}$, our model is the probability $\mathbf{P}_{N,\xi}$ on Ω_S defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\xi}}{\mathrm{dP}}(S) := \frac{1}{U_{N,\xi}} e^{H_{N,\xi}(S)}, \qquad \text{where} \qquad H_{N,\xi}(S) := \sum_{i=1}^{N} \xi(S_i), \tag{1.3}$$

and the normalizing constant $U_{N,\xi}$ (*partition function*) is of course

$$U_{N,\xi} := \mathrm{E}\left[e^{H_{N,\xi}(S)}\right] = \mathrm{E}\left[\exp\left(\sum_{i=1}^{N} \xi(S_i)\right)\right]. \tag{1.4}$$

We stress that we are dealing with a *quenched disordered model*: we are interested in the properties of the law $\mathbf{P}_{N,\xi}$ for $\mathbb{P}$-typical but *fixed* realizations of the potential ξ.

Let us also introduce the *constrained partition function* $u_{N,\xi}(x)$, defined for $x \in \mathbb{Z}^d$ by

$$u_{N,\xi}(x) := \mathrm{E}\left[\exp\left(\sum_{i=1}^{N} \xi(S_i)\right) 1_{\{S_N = x\}}\right], \tag{1.5}$$

so that $U_{N,\xi} = \sum_{x \in \mathbb{Z}^d} u_{N,\xi}(x)$. Note that the law of S_N under $\mathbf{P}_{N,\xi}$ is given by

$$p_{N,\xi}(x) := \mathbf{P}_{N,\xi}(S_N = x) = \frac{u_{N,\xi}(x)}{U_{N,\xi}} = \frac{u_{N,\xi}(x)}{\sum_{y \in \mathbb{Z}^d} u_{N,\xi}(y)}. \tag{1.6}$$

The law $\mathbf{P}_{N,\xi}$ admits the following interpretation: the trajectories $\{(i, S_i)\}_{0 \le i \le N}$ model the configurations of a $(1+d)$-dimensional directed polymer of length N which interacts with the random potential (or environment) $\{\xi(x)\}_{x \in \mathbb{Z}^d}$. The random variable $\xi(x)$ should be viewed as a reward sitting at site $x \in \mathbb{Z}^d$, so that the energy of each polymer configuration is given by the sum of the rewards visited by the polymer. On an intuitive ground, the polymer configurations should target the sites where the potential takes very large values. The corresponding energetic gain entails of course an entropic loss, which however should not be too relevant, in view of the heavy tail assumption (1.2). As we are going to see, this is indeed what happens, in a very strong form.

Besides the directed polymer interpretation, $\mathbf{P}_{N,\xi}$ is a law on $\Omega_S = (\mathbb{Z}^d)^{\mathbb{N}_0}$ which may be viewed as a natural penalization of the random walk law P. In particular, when looking at the process $\{S_k\}_{k \ge 0}$ under the law $\mathbf{P}_{N,\xi}$, we often consider k as a time parameter.

Remark 1.1. An alternative interpretation of our model is to describe the spatial distribution of a population evolving in time. At time zero, the population consists of one individual located at the site $x = 0 \in \mathbb{Z}^d$. In each time step, every individual in the population performs one step of the random walk S, independently of all other individuals, jumping from its current site x to a site y (possibly $y = x$) and then splitting into a number of individuals (always at site y) distributed like a $\mathrm{Po}(e^{\xi(y)})$, where $\mathrm{Po}(\lambda)$ denotes the Poisson distribution of parameter $\lambda > 0$. The expected number of individuals at site $x \in \mathbb{Z}^d$ at time $N \in \mathbb{N}$ is then given by $u_{N,\xi}(x)$, as one checks easily.

Remark 1.2. Our model is somewhat close in spirit to the much studied *directed polymer in random environment* [1, 2, 8], in which the rewards $\xi(i, x)$ depend also on $i \in \mathbb{N}$ (and are usually chosen to be jointly i.i.d.). In our model, the rewards are constant in the "deterministic direction" $(1,0)$, a feature which makes the environment much more attractive from a localization viewpoint. Notice in fact that a site x with a large reward $\xi(x)$ yields a favorable straight corridor $\{0,\ldots,N\} \times \{x\}$ for the polymer $\{(i,S_i)\}_{0\le i\le N}$.

We also point out that the so-called *stretched polymer in random environment* with a fixed length, considered e.g. in [6], is a model which provides an interpolation between the directed polymer in random environment and our model.

1.4. **The main results.** The closest relative of our model is obtained considering the continuous-time analog $\hat u_{t,\xi}(x)$ of (1.5), defined for $t \in [0,\infty)$ and $x \in \mathbb{Z}^d$ by

$$\hat u_{t,\xi}(x) := \mathrm{E}\left[\exp\left(\int_0^t \xi(\hat S_u)\,\mathrm{d}u\right) 1_{\{\hat S_t = x\}}\right], \tag{1.7}$$

where $(\{\hat S_u\}_{u\in[0,\infty)}, \mathrm{P})$ denotes the continuous-time, simple symmetric random walk on $\mathbb{Z}^d$. One can check that the function $\hat u_{t,\xi}(x)$ is the solution of the following Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t}\hat u_{t,\xi}(x) = \Delta \hat u_{t,\xi}(x) + \xi(x)\,\hat u_{t,\xi}(x) \\ \hat u_{0,\xi}(x) = 1_0(x) \end{cases} \qquad \text{for } (t,x) \in (0,\infty)\times\mathbb{Z}^d,$$

known in the literature as the *parabolic Anderson problem.* We refer to [4, 3, 5] and references therein for the physical motivations behind this problem and for a survey of the main results.

When the potential ξ is i.i.d. with heavy tails like in (1.2) and $\alpha > d$, the asymptotic properties as $t \to \infty$ of the function $\hat u_{t,\xi}(\cdot)$ were investigated in [7], showing that a very strong form of localization takes place: for large t, the function $\hat u_{t,\xi}(\cdot)$ is essentially concentrated at two points almost surely and at a single point in probability. More precisely, for all $t > 0$ and $\xi \in \Omega_\xi$ there exist $\hat{\mathrm{z}}^{(1)}_{t,\xi}, \hat{\mathrm{z}}^{(2)}_{t,\xi} \in \mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{\hat u_{t,\xi}(\hat{\mathrm{z}}^{(1)}_{t,\xi}) + \hat u_{t,\xi}(\hat{\mathrm{z}}^{(2)}_{t,\xi})}{\sum_{x\in\mathbb{Z}^d} \hat u_{t,\xi}(x)} = 1, \qquad \mathbb{P}\text{-almost surely}, \tag{1.8}$$

$$\lim_{t\to\infty} \frac{\hat u_{t,\xi}(\hat{\mathrm{z}}^{(1)}_{t,\xi})}{\sum_{x\in\mathbb{Z}^d} \hat u_{t,\xi}(x)} = 1, \qquad \text{in } \mathbb{P}\text{-probability}, \tag{1.9}$$

cf. [7, Theorems 1.1 and 1.2]. The points $\hat{\mathrm{z}}^{(1)}_{t,\xi}, \hat{\mathrm{z}}^{(2)}_{t,\xi}$ are typically at superballistic distance $(t/\log t)^{1+q}$ with $q = d/(\alpha - d) > 0$, cf. [7, Remark 6]. We point out that localization at one point like in (1.9) cannot hold $\mathbb{P}$-almost surely, that is, the contribution of $\hat{\mathrm{z}}^{(2)}_{t,\xi}$ cannot be removed from (1.8): this is due to the fact that $\hat u_{t,\xi}(x)$ is a continuous function of t for every fixed $x \in \mathbb{Z}^d$, as explained in [7, Remark 1].

It is natural to ask if the discrete-time counterpart of $\hat u_{t,\xi}(\cdot)$, i.e., the constrained partition function $u_{N,\xi}(\cdot)$ defined in (1.5), exhibits similar features. Generally speaking, models built over discrete-time or continuous-time simple random walks are not expected to be very different. However, due to the heavy tail of the potential distribution, the localization points $\hat{\mathrm{z}}^{(1)}_{t,\xi}, \hat{\mathrm{z}}^{(2)}_{t,\xi}$ of the continuous-time model grow at a superballistic speed, a feature that is clearly impossible for the discrete-time model, for which $u_{N,\xi}(x) \equiv 0$ for $|x| > N$. Another interesting question is whether for the discrete model one may have localization at one single point $\mathbb{P}$-almost surely. Before answering, we need to set up some notation.

We recall that $\mathcal{B}_N := \{x \in \mathbb{Z}^d : |x| \le N\}$. It is not difficult to check that the values $\{p_{N,\xi}(x)\}_{x\in\mathcal{B}_N}$ are all distinct, for $\mathbb{P}$-a.e. $\xi \in \Omega_\xi$ and for all $N \in \mathbb{N}$, because the potential distribution is continuous, cf. (1.2). Therefore we can set

$$\mathrm{w}_{N,\xi} := \arg\max \big\{ p_{N,\xi}(x) : \; x \in \mathcal{B}_N \big\} \,, \tag{1.10}$$

and $\mathbb{P}$-almost surely $\mathrm{w}_{N,\xi}$ is a single point in $\mathbb{Z}^d$: it is the point at which $p_{N,\xi}(\cdot)$ attains its maximum. We can now state our first main result.

Theorem 1.3 (One-site localization). *We have*

$$\lim_{N\to\infty} p_{N,\xi}(\mathrm{w}_{N,\xi}) \;=\; \lim_{N\to\infty} \frac{u_{N,\xi}(\mathrm{w}_{N,\xi})}{\sum_{x\in\mathbb{Z}^d} u_{N,\xi}(x)} \;=\; 1\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}. \tag{1.11}$$

Furthermore, as $N \to \infty$ we have the following convergence in distribution:

$$\frac{\mathrm{w}_{N,\xi}}{N} \Longrightarrow \mathrm{w}\,, \qquad \text{where} \qquad \mathbb{P}(\mathrm{w} \in \mathrm{d}x) \;=\; c_\alpha\, (1-|x|)^\alpha\, \mathbf{1}_{\{|x|\le 1\}}\, \mathrm{d}x\,, \tag{1.12}$$

and $c_\alpha := (\int_{|y|\le 1} (1-|y|)^\alpha \mathrm{d}y)^{-1}$.

Recalling the definition (1.6) of $p_{N,\xi}(x)$, Theorem 1.3 shows that S_N under $\mathbf{P}_{N,\xi}$ is localized at the ballistic point $\mathrm{w}_{N,\xi} \approx \mathrm{w} \cdot N$.

Next we look more closely at the localization site $\mathrm{w}_{N,\xi}$. We introduce two points $z^{(1)}_{N,\xi}, z^{(2)}_{N,\xi} \in \mathbb{Z}^d$, defined explicitly in terms of the potential ξ, through

$$\begin{aligned} z^{(1)}_{N,\xi} &:= \arg\max \left\{ \Big(1 - \tfrac{|x|}{N+1}\Big)\, \xi(x) \;:\; x \in \mathcal{B}_N \right\}\,, \\ z^{(2)}_{N,\xi} &:= \arg\max \left\{ \Big(1 - \tfrac{|x|}{N+1}\Big)\, \xi(x) \;:\; x \in \mathcal{B}_N \setminus \big\{ z^{(1)}_{N,\xi} \big\} \right\}\,. \end{aligned} \tag{1.13}$$

Again, the values of $\{(1-\frac{|x|}{N+1})\xi(x)\}_{x\in\mathcal{B}_N}$ are $\mathbb{P}$-a.s. distinct, by the continuity of the potential distribution, therefore $z^{(1)}_{N,\xi}$ and $z^{(2)}_{N,\xi}$ are $\mathbb{P}$-a.s. single points in $\mathcal{B}_N$. We can now give the discrete-time analogues of (1.8) and (1.9).

Theorem 1.4 (Two-sites localization). *The following relations hold:*

$$\lim_{N\to\infty} \Big(p_{N,\xi}\big(z^{(1)}_{N,\xi}\big) + p_{N,\xi}\big(z^{(2)}_{N,\xi}\big) \Big) \;=\; 1 \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,, \tag{1.14}$$

$$\lim_{N\to\infty} p_{N,\xi}\big(z^{(1)}_{N,\xi}\big) \;=\; 1 \qquad \text{in } \mathbb{P}(\mathrm{d}\xi)\text{-probability}. \tag{1.15}$$

Putting together Theorems 1.3 and 1.4, we obtain the following information on $\mathrm{w}_{N,\xi}$.

Corollary 1.5. *For $\mathbb{P}$-a.e. $\xi \in \Omega_\xi$, we have $\mathrm{w}_{N,\xi} \in \{z^{(1)}_{N,\xi}, z^{(2)}_{N,\xi}\}$ for large N. Furthermore,*

$$\lim_{N\to\infty} \mathbb{P}\left(\mathrm{w}_{N,\xi} = z^{(1)}_{N,\xi} \right) \;=\; 1\,. \tag{1.16}$$

Remark 1.6. We stress that the convergence in (1.15) does not occur $\mathbb{P}(\mathrm{d}\xi)$-almost surely, i.e., $\mathrm{w}_{N,\xi}$ is not equal to $z^{(1)}_N$ for all N large enough, at least in dimension $d = 1$. In fact, in Appendix C we show explicitly that when $d = 1$

$$\mathbb{P}\left(\mathrm{w}_{N,\xi} = z^{(2)}_{N,\xi} \text{ for infinitely many } N \right) \;=\; 1\,. \tag{1.17}$$

We believe that (1.17) remains true also for $d > 1$.

The proof of two sites localization given in [7] for the continuous-time model is quite technical and exploits tools from potential theory and spectral analysis. We point out that such tools can be applied also in the discrete-time setting, but they turn out to be unnecessary. Our proof is indeed based on shorter and simpler geometric arguments. For instance, we exploit the fact that before reaching a site $x \in \mathbb{Z}^d$ a discrete-time random walk path must visit a least $|x|-1$ different sites ($\neq x$) and spend at each of them a least one time unit. Of course, this is no longer true for continuous-time random walks.

1.5. **Further path properties.** Theorem 1.3 states that $\mathbb{P}(\mathrm{d}\xi)$-a.s. the probability measure $\mathbf{P}_{N,\xi}$ concentrates on the subset of Ω_S gathering those random walk trajectories S such that $S_N = \mathrm{w}_{N,\xi}$. It turns out that this subset can be radically narrowed. In fact, we can introduce a restricted subset $\mathcal{C}_{N,\xi} \subseteq \Omega_S$ of random walk trajectories, defined as follows:

- the trajectories in $\mathcal{C}_{N,\xi}$ must reach the site $\mathrm{w}_{N,\xi}$ for the first time before time N, following an injective path, and then must remain at $\mathrm{w}_{N,\xi}$ until time N;
- the length of the injective path until $\mathrm{w}_{N,\xi}$ differs from $|\mathrm{w}_{N,\xi}|$ — which is the minimal one — at most for a small error term $h_N := (\log\log N)^{2/\alpha} N^{1-1/\alpha}$ if $\alpha > 1$ and $h_N := (\log N)^{1+2/\alpha}$ if $\alpha \le 1$ (note that in any case $h_N = o(N)$);
- all the sites x visited by the random walk before reaching $\mathrm{w}_{N,\xi}$ must have an associated field $\xi(x)$ that is strictly smaller than $\xi(\mathrm{w}_{N,\xi})$.

More formally, denoting by $\tau_x = \tau_x(S) := \inf\{n \ge 0\colon\ S_n = x\}$ the first passage time at $x \in \mathbb{Z}^d$ of a random walk trajectory S, we set

$$\begin{aligned} \mathcal{C}_{N,\xi} \;:=\; \Big\{ S \in \Omega_S\colon\ & S_i \neq S_j \ \forall i < j \le \tau_{\mathrm{w}_{N,\xi}}\,,\ S_i = \mathrm{w}_{N,\xi} \ \forall i \in \{\tau_{\mathrm{w}_{N,\xi}}, \ldots, N\}\,, \\ & \xi(S_i) < \xi(\mathrm{w}_{N,\xi}) \ \forall i < \tau_{w_{N,\xi}}\,,\ \tau_{\mathrm{w}_{N,\xi}} \le |\mathrm{w}_{N,\xi}| + h_N \Big\}\,. \end{aligned} \tag{1.18}$$

We then have the following result.

Theorem 1.7. *For $\mathbb{P}$-a.e. $\xi \in \Omega_\xi$, we have*

$$\lim_{N\to\infty} \mathbf{P}_{N,\xi}(\mathcal{C}_{N,\xi}) = 1. \tag{1.19}$$

Remark 1.8. It is worth stressing that in dimension $d = 1$ the set $\mathcal{C}_{N,\xi}$ reduces to *a single N-steps trajectory.* In fact, we have $\mathcal{C}_{N,\xi} = \mathcal{S}^{(N,\mathrm{w}_{N,\xi})}$, where we denote by $\mathcal{S}^{(N,x)}$, for $x \in \mathcal{B}_N$, the set of trajectories $S \in \Omega_S$ such that

$$S_i \;:=\; \begin{cases} i \cdot \mathrm{sign}(x) & \text{for } 0 \le i \le |x| \\ x & \text{for } |x| \le i \le N \end{cases}.$$

As stated in Corollary 1.5, for large N the site $\mathrm{w}_{N,\xi}$ is either $z^{(1)}_{N,\xi}$ or $z^{(2)}_{N,\xi}$. Note that $z^{(1)}_{N,\xi}$ and $z^{(2)}_{N,\xi}$ are easily determined, by (1.13). In order to decide whether $\mathrm{w}_{N,\xi} = z^{(1)}_{N,\xi}$ or $\mathrm{w}_{N,\xi} = z^{(2)}_{N,\xi}$, by Theorem 1.7 it is sufficient to compare the explicit contributions of just two trajectories, i.e., $\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(1)}_{N,\xi})})$ and $\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(2)}_{N,\xi})})$. More precisely, setting $\kappa(i) := \mathrm{P}(S_1 = i)$ for $i \in \{\pm 1, 0\}$ (so that $\kappa = \kappa(0)$, cf. (1.1)) and

$$b_{N,\xi}(x) \;:=\; e^{\sum_{i=1}^{|x|-1} \xi(i\,\mathrm{sign}(x)) + (N+1-|x|)\xi(x)} \kappa(\mathrm{sign}(x))^{|x|} \kappa(0)^{N-|x|}\,,$$

we have $w_{N,\xi} = z^{(1)}_{N,\xi}$ if $b_{N,\xi}(z^{(1)}_{N,\xi}) > b_{N,\xi}(z^{(2)}_{N,\xi})$ and $w_{N,\xi} = z^{(2)}_{N,\xi}$ otherwise. Therefore, in dimension $d = 1$, we have a very explicit characterization of the localization point $\mathrm{w}_{N,\xi}$.

1.6. **Organization of the paper.** The paper is organized as follows.

- In Section 2 we gather some basic estimates on the field, that will be the main tool of our analysis.
- In Section 3 we prove Theorem 1.4.
- In Section 4 we prove Theorem 1.3.
- In Section 5 we prove Theorem 1.7.
- Finally, the Appendixes contain the proofs of some technical results.

In the sequel, the dependence on ξ of various quantities, like $H_{N,\xi}$, $\mathrm{w}_{N,\xi}$, $z^{(1)}_{N,\xi}$, etc., will be frequently omitted for short.

2. Asymptotic estimates for the environment

This section is devoted to the analysis of the almost sure asymptotic properties of the random potential ξ. With the exception of Proposition 2.5, which plays a fundamental role in our analysis, the proof of the results of this section are obtained with the standard techniques of extreme values theory and are therefore deferred to the Appendices A and B.

Before starting, we set up some notation. We say that a property of the field ξ depending on $N \in \mathbb{N}$ holds *eventually* $\mathbb{P}$*-a.s.* if for $\mathbb{P}$-a.e. $\xi \in \Omega_\xi$ there exists $N_0 = N_0(\xi) < \infty$ such that the property holds for all $N \geq N_0$. We recall that $|\cdot|$ denotes the ℓ^1 norm on $\mathbb{R}^d$ and $\mathcal{B}_N = \{x \in \mathbb{Z}^d : |x| \leq N\}$. With some abuse of notation, the cardinality of $\mathcal{B}_N$ will be still denoted by $|\mathcal{B}_N|$. Note that $|\mathcal{B}_N| = c_d N^d + O(N^{d-1})$ as $N \to \infty$, where $c_d = \int_{\mathbb{R}^d} 1_{\{|x|\leq 1\}} \mathrm{d}x = 2^d/d!$.

2.1. **Order statistics for the field.** The order statistics of the field $\{\xi(x)\}_{x\in\mathcal{B}_N}$ is the set of values attained by the field rearranged in decreasing order, and is denoted by

$$X_N^{(1)} > X_N^{(2)} > \cdots > X_N^{(|\mathcal{B}_N|)} > 1\,. \tag{2.1}$$

For simplicity, when $t \in [1, |\mathcal{B}_N|]$ is not an integer we still set $X_N^{(t)} := X_N^{(\lfloor t\rfloor)}$. For later convenience, we denote by $x_N^{(k)}$ the point in $\mathcal{B}_N$ at which the value $X_N^{(k)}$ is attained, that is $X_N^{(k)} = \xi(x_N^{(k)})$. We are going to exploit heavily the following almost sure estimates.

Lemma 2.1. *For every $\varepsilon > 0$, eventually $\mathbb{P}$-a.s.*

$$\frac{N^{d/\alpha}}{(\log\log N)^{1/\alpha+\varepsilon}} \leq X_N^{(1)} \leq N^{d/\alpha}\,(\log N)^{1/\alpha+\varepsilon}\,. \tag{2.2}$$

For every $\vartheta > 1$ and $\varepsilon > 0$, eventually $\mathbb{P}$-a.s.

$$\frac{N^{d/\alpha}}{(\log N)^{\vartheta/\alpha+\varepsilon}} \leq X_N^{((\log N)^\vartheta)} \leq \frac{N^{d/\alpha}}{(\log N)^{\vartheta/\alpha-\varepsilon}}\,. \tag{2.3}$$

There exists a constant $A > 0$ such that eventually $\mathbb{P}$-a.s.

$$\sup_{(\log N)\leq k\leq |\mathcal{B}_N|} \big(k^{1/\alpha} X_N^{(k)}\big) \leq A\, N^{d/\alpha}\,. \tag{2.4}$$

The proof of Lemma 2.1 is given in Appendix A.2. For completeness, we point out that $X_N^{(1)}/(c_d N^{d/\alpha})$ converges in distribution as $N \to \infty$ toward the law μ on $(0,\infty)$ with $\mu((0,x]) = \exp(-x^{-\alpha})$, called Fréchet law of shape parameter α, as one can easily prove.

Next we give a lower bound on the gaps $X_N^{(k)} - X_N^{(k+1)}$ for moderate values of k.

Proposition 2.2. *For every $\vartheta > 0$ there exists a constant $\gamma > 0$ such that eventually $\mathbb{P}$-a.s.*

$$\inf_{1\le k\le(\log N)^{\vartheta}} \left(X_N^{(k)} - X_N^{(k+1)}\right) \ge \frac{N^{d/\alpha}}{(\log N)^{\gamma}} . \tag{2.5}$$

The proof of Proposition 2.2 is given in Appendix A.3.

2.2. **Order statistics for the modified field.** An important role is played by the modified field $\{\psi_N(x)\}_{x\in\mathcal{B}_N}$, defined by

$$\psi_N(x) := \left(1 - \frac{|x|}{N+1}\right)\xi(x) . \tag{2.6}$$

The motivation is the following: for any given point $x \in \mathcal{B}_N$, a random walk trajectory $(S_0, S_1, \dots, S_N)$ that goes to x in the minimal number of steps and sticks in x afterwards has an energetic contribution equal to $\sum_{i=1}^{|x|-1} \xi(S_i) + (N+1)\psi_N(x)$ (recall (1.3)).

The order statistics of the modified field $\{\psi_N(x)\}_{x\in\mathcal{B}_N}$ will be denoted by

$$Z_N^{(1)} > Z_N^{(2)} > \cdots > Z_N^{|\mathcal{B}_N|} ,$$

and we let $z_N^{(k)}$ be the point in $\mathcal{B}_N$ at which ψ_N attains $Z_N^{(k)}$, that is $\psi_N(z_N^{(k)}) = Z_N^{(k)}$. A simple but important observation is that $Z_N^{(k)}$ is increasing in N, for every fixed $k \in \mathbb{N}$, since $\psi_N(x)$ is increasing in N for fixed x. Also note that $Z_N^{(k)} \le X_N^{(k)}$, because $\psi_N(x) \le \xi(x)$.

Our attention will be mainly devoted to $Z_N^{(1)}$ and $Z_N^{(2)}$, whose almost sure asymptotic behaviors are analogous to that of $X_N^{(1)}$, cf. (2.2).

Lemma 2.3. *For every $\varepsilon > 0$, eventually $\mathbb{P}$-a.s.*

$$\frac{N^{d/\alpha}}{(\log\log N)^{1/\alpha+\varepsilon}} \le Z_N^{(2)} \le Z_N^{(1)} \le N^{d/\alpha}\,(\log N)^{1/\alpha+\varepsilon} . \tag{2.7}$$

The proof is given in Appendix B.2. Note that only the first inequality needs to be proved, thanks to (2.2) and to the fact that, plainly, $Z_N^{(2)} \le Z_N^{(1)} \le X_N^{(1)}$.

Next we focus on the gaps between $Z_N^{(1)}, Z_N^{(2)}$ and $Z_N^{(3)}$. The main technical tool is given by the following easy estimates, proved in Appendix B.1.

Lemma 2.4. *There is a constant c such that for all $N \in \mathbb{N}$ and $\delta \in (0,1)$*

$$\mathbb{P}(Z_N^{(2)} > (1-\delta)Z_N^{(1)}) \le c\,\delta , \qquad \mathbb{P}(Z_N^{(3)} > (1-\delta)Z_N^{(1)}) \le c\,\delta^2 . \tag{2.8}$$

As a consequence, we have the following result, which will be crucial in the sequel.

Proposition 2.5. *For every d and α, there exists $\beta \in (1,\infty)$ such that*

$$Z_N^{(1)} - Z_N^{(3)} \ge \frac{N^{d/\alpha}}{(\log N)^{\beta}} , \qquad \textit{eventually } \mathbb{P}\textit{-a.s.} . \tag{2.9}$$

Although we do not use this fact explicitly, it is worth stressing that the gap $Z_N^{(1)} - Z_N^{(2)}$ can be as small as $N^{d/\alpha-1}$ (up to logarithmic corrections), hence much smaller than the right hand side of (2.9), cf. Appendix B.3. This is the reason behind the fact that localization at the two points $\{z_{N,\xi}^{(1)}, z_{N,\xi}^{(2)}\}$ can be proved quite directly, cf. Section 3, whereas localization at a single point $\mathrm{w}_{N,\xi} \in \{z_{N,\xi}^{(1)}, z_{N,\xi}^{(2)}\}$ is harder to obtain, cf. Section 4. Furthermore, one may have $\mathrm{w}_{N,\xi} \ne z_{N,\xi}^{(1)}$ precisely when the gap $Z_N^{(1)} - Z_N^{(2)}$ is small, cf. Appendix C.

Proof of Proposition 2.5. For $r \in (0,1)$ (that will be fixed later), we set $N_k := \lfloor e^{k^r} \rfloor$, for $k \in \mathbb{N}$. By the second relation in (2.8), for $\gamma > 0$ (to be fixed later) we have

$$\sum_{k\in\mathbb{N}} \mathbb{P}\left(Z^{(1)}_{N_k} - Z^{(3)}_{N_k} \le \frac{1}{(\log N_k)^\gamma} Z^{(1)}_{N_k} \right) \le c_1 \sum_{k\in\mathbb{N}} \frac{1}{(\log N_k)^{2\gamma}} \le (const.) \sum_{k\in\mathbb{N}} \frac{1}{k^{2r\gamma}} < \infty ,$$

provided $2r\gamma > 1$. Therefore, by the Borel-Cantelli lemma and (2.7), eventually (in k) $\mathbb{P}$-a.s.

$$Z^{(1)}_{N_k} - Z^{(3)}_{N_k} \ge \frac{(N_k)^{d/\alpha}}{(\log N_k)^{\gamma+1}} . \tag{2.10}$$

Now for a generic $N \in \mathbb{N}$, let $k \in \mathbb{N}$ be such that $N_{k-1} \le N < N_k$. We can write

$$Z^{(1)}_N - Z^{(3)}_N = \big(Z^{(1)}_N - Z^{(1)}_{N_k}\big) + \big(Z^{(1)}_{N_k} - Z^{(3)}_{N_k}\big) + \big(Z^{(3)}_{N_k} - Z^{(3)}_N\big) .$$

We already observed that $Z^{(k)}_N$ is increasing in N, therefore the third term in the right hand side is non-negative and can be neglected. From (2.10) we then get for large N

$$Z^{(1)}_N - Z^{(3)}_N \ge \frac{(N_k)^{d/\alpha}}{(\log N_k)^{\gamma+1}} - \big(Z^{(1)}_{N_k} - Z^{(1)}_N\big) \ge \frac{N^{d/\alpha}}{2\,(\log N)^{\gamma+1}} - \big(Z^{(1)}_{N_k} - Z^{(1)}_N\big) , \tag{2.11}$$

because $N_k \ge N$ and $N_k \le 2N$ for large N (note that $N_k/N_{k-1} \to 1$ as $k \to \infty$).

It remains to estimate $Z^{(1)}_{N_k} - Z^{(1)}_N$. Observe that $Z^{(1)}_n = \psi_n(z^{(1)}_n) \ge \psi_n(z^{(1)}_{n+1})$, because $Z^{(1)}_n$ is the maximum of ψ_n. Therefore we obtain the estimate

$$\begin{aligned} Z^{(1)}_{n+1} - Z^{(1)}_n &= \psi_{n+1}(z^{(1)}_{n+1}) - \psi_n(z^{(1)}_n) \le \psi_{n+1}(z^{(1)}_{n+1}) - \psi_n(z^{(1)}_{n+1}) \\ &= \frac{|z^{(1)}_{n+1}|\,\xi(z^{(1)}_{n+1})}{(n+1)(n+2)} \le \frac{\xi(z^{(1)}_{n+1})}{n} , \end{aligned}$$

which yields

$$Z^{(1)}_{N_k} - Z^{(1)}_N = \sum_{n=N}^{N_k-1} \big(Z^{(1)}_{n+1} - Z^{(1)}_n\big) \le \frac{N_k - N_{k-1}}{N_{k-1}}\, \xi(z^{(1)}_{N_k}) \le \frac{N_k - N_{k-1}}{N_{k-1}}\, X^{(1)}_{N_k} . \tag{2.12}$$

Observe that as $k \to \infty$

$$\frac{e^{k^r} - e^{(k-1)^r}}{e^{(k-1)^r}} = e^{k^r-(k-1)^r} - 1 = \frac{r}{k^{1-r}}(1+o(1)) . \tag{2.13}$$

Since $N \le N_k = \lfloor e^{k^r} \rfloor$, it comes that $k \ge (\log N)^{1/r}$ and therefore (2.13) allows to write for large N

$$\frac{N_k - N_{k-1}}{N_{k-1}} \le \frac{1}{(\log N)^{1/r-1}} .$$

Looking back at (2.11) and (2.12), by (2.2) we then have eventually $\mathbb{P}$-a.s.

$$Z^{(1)}_N - Z^{(3)}_N \ge \frac{N^{d/\alpha}}{2\,(\log N)^{\gamma+1}} - \frac{N^{d/\alpha}}{(\log N)^{1/r-1/\alpha-2}} . \tag{2.14}$$

The second term in the right hand side of (2.14) can be neglected provided the parameters $r \in (0,1)$ and $\gamma \in (0,\infty)$ fulfill the condition $1/r - 1/\alpha - 2 > \gamma + 1$. We recall that we also have to obey the condition $2r\gamma > 1$. Therefore, for a fixed value of r, the set of allowed values for γ is the interval $(\frac{1}{2r}, \frac{1}{r} - \frac{1}{\alpha} - 3)$, which is non-empty if r is small enough. This shows that the two conditions on r, γ can indeed be satisfied together (a possible choice is, e.g., $r = \frac{\alpha}{6(3\alpha+1)}$ and $\gamma = \frac{4(3\alpha+1)}{\alpha}$). Setting $\beta := \gamma + 1$, it then follows from (2.14) that equation (2.9) holds true. □

3. Almost sure localization at two points

In this section we prove Theorem 1.4. We first set up some notation and give some preliminary estimates.

3.1. **Prelude.** We recall that $z_N^{(1)}$ and $z_N^{(2)}$ are the two sites in $\mathcal{B}_N$ at which the modified potential ψ_N, cf. (2.6), attains its two largest values $Z_N^{(1)} = \psi_N(z_N^{(1)})$ and $Z_N^{(2)} = \psi_N(z_N^{(2)})$.

It is convenient to define $J_1, J_2 \in \{1, \ldots, |\mathcal{B}_N|\}$ such that

$$z_N^{(1)} = x_N^{(J_1)}, \qquad z_N^{(2)} = x_N^{(J_2)}, \tag{3.1}$$

where we recall that $x_N^{(k)}$ is the point in $\mathcal{B}_N$ at which the potential ξ attains its k-th largest value, i.e., $X_N^{(k)} = \xi(x_N^{(k)})$, cf. Section 2.1. We stress that J_1 and J_2 are functions of N and ξ, although we do not indicate this explicitly. An immediate consequence of Lemma 2.3 and relation (2.3) is the following

Corollary 3.1. *For every d, α, $\varepsilon > 0$, eventually $\mathbb{P}$-a.s.*

$$\max\{J_1, J_2\} \le (\log N)^{1+\varepsilon}. \tag{3.2}$$

Next we define the local time $\ell_N(x)$ of a random walk trajectory $S \in \Omega_S$ by

$$\ell_N(x) = \ell_N(x, S) = \sum_{i=1}^{N} 1_{\{S_i = x\}}, \tag{3.3}$$

so that the Hamiltonian $H_N(S)$, cf. (1.3), can be rewritten as

$$H_N(S) = \sum_{x \in \mathcal{B}_N} \ell_N(x)\, \xi(x). \tag{3.4}$$

We also associate to every trajectory S the quantity

$$\beta_N(S) := \min\{k \ge 1 \colon \ell_N(x_N^{(k)}) > 0\}. \tag{3.5}$$

In words, $x_N^{(\beta_N(S))}$ *is the site in $\mathcal{B}_N$ which maximizes the potential ξ among those visited by the trajectory S before time N*. Finally, we introduce the basic events

$$\mathcal{A}_{1,N} := \big\{S \in \Omega_S : \beta_N(S) = J_1\big\}, \qquad \mathcal{A}_{2,N} := \big\{S \in \Omega_S : \beta_N(S) = J_2\big\}. \tag{3.6}$$

In words, the event $\mathcal{A}_{i,N}$ *consists of the random walk trajectories S that before time N visit the site $z_N^{(i)}$ (recall* (3.1)*) and do not visit any other site x with $\xi(x) > \xi(z_N^{(i)})$.*

It turns out that the localization of S_N at the point $z_N^{(i)}$ is implied by the event $\mathcal{A}_{i,N}$, i.e., for both $i = 1, 2$ we have

$$\lim_{N\to\infty} \mathbf{P}_{N,\xi}\big(\mathcal{A}_{i,N},\, S_N \ne z_N^{(i)}\big) \;=\; 0, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}. \tag{3.7}$$

The proof is simple. Denoting by $\widetilde{\tau}_{N,i}$ the *last* passage time of the random walk in $z_N^{(i)}$ before time N, that is

$$\widetilde{\tau}_{N,i} := \max\{n \le N \colon S_n = z_N^{(i)}\},$$

we can write, recalling (1.3),

$$\mathbf{P}_{N,\xi}\big(\mathcal{A}_{i,N},\, S_N \ne z_N^{(i)}\big) = \sum_{r=0}^{N-1} \frac{\mathrm{E}\big[e^{H_N(S)}\, 1_{\mathcal{A}_{i,N}}\, 1_{\{\widetilde{\tau}_{N,i} = r\}}\big]}{U_N}. \tag{3.8}$$

We stress that the sum stops at $r = N-1$, because we are on the event $S_N \neq z_N^{(i)}$. Furthermore, on the event $\mathcal{A}_{i,N} \cap \{\widetilde{\tau}_{N,i} = r\}$ we have $S_n \notin \{x_N^{(1)}, \ldots, x_N^{(J_i)}\}$ for all $n \in \{r+1, \ldots, N\}$ (we recall that $z_N^{(i)} = x_N^{(J_i)}$). By the Markov property, we can then bound the numerator in the right hand side of (3.8) by

$$\begin{aligned} \mathrm{E}\Big[e^{H_N(S)}\,\mathbf{1}_{\mathcal{A}_{i,N}}\,\mathbf{1}_{\{\widetilde{\tau}_{N,i}=r\}}\Big] \;&\le\; \mathrm{E}\Big[e^{H_r(S)}\,\mathbf{1}_{\{S_r = z_N^{(i)}\}}\Big]\,B_{N-r}^{(N,i)}\,, \\ \text{where} \qquad B_l^{(N,i)} \;&:=\; \mathrm{E}_{x_N^{(J_i)}}\Big[e^{H_l(S)}\,\mathbf{1}_{\{S_n \notin \{x_N^{(1)},\ldots,x_N^{(J_i)}\}\,\forall n=1,\ldots,l\}}\Big]\,. \end{aligned} \tag{3.9}$$

Analogously, for the denominator in the right hand side of (3.8), recalling (1.1), we have

$$U_N \ge \mathrm{E}\big[e^{H_N(S)}\,\mathbf{1}_{\{S_n = x_N^{(J_i)},\,\forall n\in\{r,\ldots,N\}\}}\big] = \mathrm{E}\Big[e^{H_r(S)}\,\mathbf{1}_{\{S_r = x_N^{(J_i)}\}}\Big]\,\kappa^{N-r}e^{(N-r)X_N^{(J_i)}}\,.$$

Plainly, $B_l^{(N,i)} \le \exp(l X_N^{(J_i+1)})$, therefore we can write

$$\begin{aligned} \mathbf{P}_{N,\xi}\big(\mathcal{A}_{i,N},\, S_N \neq z_N^{(i)}\big) &\le \sum_{r=0}^{N-1} e^{-(N-r)(X_N^{(J_i)}+\log\kappa)}\,B_{N-r}^{(N,i)} = \sum_{l=1}^{N} e^{-l(X_N^{(J_i)}+\log\kappa)}\,B_l^{(N,i)} \\ &\le \sum_{l=1}^{\infty} e^{-l(X_N^{(J_i)}-X_N^{(J_i+1)}-\log\kappa)} = \frac{e^{-(X_N^{(J_i)}-X_N^{(J_i+1)}-\log\kappa)}}{1-e^{-(X_N^{(J_i)}-X_N^{(J_i+1)}-\log\kappa)}}\,. \end{aligned} \tag{3.10}$$

From Corollary 3.1 and Proposition 2.2 it follows that $\mathbb{P}(\mathrm{d}\xi)$-almost surely $X_N^{(J_i)} - X_N^{(J_i+1)} \to +\infty$ as $N \to \infty$, therefore (3.7) is proved.

3.2. **Proof of** (1.14)**.** Let us set, for $i = 1, 2$,

$$\begin{aligned} \mathcal{W}_{i,N} :=& \big\{S \in \Omega_S :\, \beta_N(S) = J_i,\; S_N = z_N^{(i)}\big\} \\ =& \big\{S \in \Omega_S :\, S_N = z_N^{(i)},\; \ell_N(x) = 0\ \forall x \in \mathcal{B}_N \text{ such that } \xi(x) > \xi(z_N^{(i)})\big\}\,. \end{aligned} \tag{3.11}$$

In words, the event $\mathcal{W}_{i,N}$ *consists of those trajectories S such that $S_N = z_N^{(i)}$ and that before time N do not visit any site x with* $\xi(x) > \xi(z_N^{(i)})$. We are going to prove that

$$\lim_{N\to\infty}\big(\mathbf{P}_{N,\xi}(\mathcal{W}_{1,N}) + \mathbf{P}_{N,\xi}(\mathcal{W}_{2,N})\big) = 1\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,, \tag{3.12}$$

which is a stronger statement than (1.14). In view of (3.7), it is sufficient to prove that

$$\lim_{N\to\infty}\big(\mathbf{P}_{N,\xi}(\mathcal{A}_{1,N}) + \mathbf{P}_{N,\xi}(\mathcal{A}_{2,N})\big) = 1\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,. \tag{3.13}$$

We start deriving an upper bound on the Hamiltonian $H_N = H_{N,\xi}$ (recall (1.3)). For an arbitrary $k \in \{1, \ldots, |\mathcal{B}_N|\}$, to be chosen later, recalling (3.3), (3.4), (3.5) and the fact that $\sum_{x\in\mathbb{Z}^d} \ell_N(x) = N$, we can write

$$\begin{aligned} H_N(S) &= \sum_{i=\beta_N(S)}^{k} \ell_N(x_N^{(i)})\xi(x_N^{(i)}) + \sum_{i=k+1}^{|\mathcal{B}_N|} \ell_N(x_N^{(i)})\xi(x_N^{(i)}) \\ &\le \Bigg(\sum_{i=\beta_N(S)}^{k} \ell_N(x_N^{(i)})\Bigg)\,\xi\big(x_N^{(\beta_N(S))}\big) + N\,X_N^{(k+1)}\,. \end{aligned} \tag{3.14}$$

Note that $\ell_N\big(x_N^{(\beta_N(S))}\big) > 0$, that is, any trajectory S visits the site $x_N^{(\beta_N(S))}$ before time N, by the very definition (3.5) of $\beta_N(S)$. It follows that any trajectory S before time N must visit at least $|x_N^{(\beta_N(S))}|$ different sites, of which at least $|x_N^{(\beta_N(S))}|-k$ are different from $x_N^{(1)}$, $\dots$, $x_N^{(k)}$. This leads to the basic estimate

$$\sum_{i=\beta_N(S)}^{k} \ell_N(x_N^{(i)}) \le N - |x_N^{(\beta_N(S))}| + k\,. \tag{3.15}$$

By (3.14) and recalling (2.6), this yields the crucial upper bound

$$\begin{aligned} H_N(S) \;&\le\; (N+1)\psi_N\big(x_N^{(\beta_N(S))}\big) + (k-1)\,\xi\big(x_N^{(\beta_N(S))}\big) + N\,X_N^{(k+1)} \\ &\le\; (N+1)\psi_N\big(x_N^{(\beta_N(S))}\big) + (k-1)\,X_N^{(1)} + N\,X_N^{(k+1)}\,. \end{aligned} \tag{3.16}$$

We stress that this bound holds for all $k \in \{1,\dots,|\mathcal{B}_N|\}$ and for all trajectories $S \in \Omega_S$.

Next we give a lower bound on U_N (recall (1.4)). We restrict the expectation to one single N-steps random walk trajectory, denoted by $S^* = \{S_i^*\}_{0\le i\le N}$, that goes to $z_N^{(1)}$ in the minimal number of steps, i.e. $|z_N^{(1)}|$, and then stays there until epoch N. By (1.1), this trajectory has a probability larger than e^{-cN} for some positive contant c, therefore

$$U_N \ge e^{H_N(S^*)-cN} \ge e^{\xi(z_N^{(1)})(N+1-|z_N^{(1)}|)-cN} = e^{(N+1)\psi_N(z_N^{(1)})-cN} \ge e^{(N+1)(Z_N^{(1)}-c)}\,, \tag{3.17}$$

where we have used the definition of ψ_N, see (2.6).

We can finally come to the proof of (3.13). For all trajectories $S \in (\mathcal{A}_{1,N}\cup\mathcal{A}_{2,N})^c$ we have $\beta_N(S) \notin \{J_1, J_2\}$, therefore $x_N^{(\beta_N(S))} \notin \{z_N^{(1)}, z_N^{(2)}\}$ and consequently $\psi_N(x_N^{(\beta_N(S))}) \le Z_N^{(3)}$. From (3.16) and (5.9) we then obtain

$$\begin{aligned} \mathbf{P}_{N,\xi}\big((\mathcal{A}_{1,N}\cup\mathcal{A}_{2,N})^c\big) \;&=\; \frac{\mathbf{E}(e^{H_N(S)}\,\mathbb{1}_{(\mathcal{A}_{1,N}\cup\mathcal{A}_{2,N})^c})}{U_{N,\xi}} \\ &\le\; \exp\Big(-(N+1)\Big((Z_N^{(1)}-Z_N^{(3)}) - X_N^{(k+1)} - \tfrac{k-1}{N+1}X_N^{(1)} - c\Big)\Big)\,. \end{aligned} \tag{3.18}$$

By (2.9), there exists $\beta \in (1,\infty)$ such that $Z_N^{(1)} - Z_N^{(3)} \ge N^{d/\alpha}/(\log N)^\beta$ eventually $\mathbb{P}$-almost surely. We now choose $k = k_N = (\log N)^\vartheta$ with $\vartheta := 3\max\{\beta\alpha, 1\} > 1$. Applying (2.2) with $\varepsilon = 1/\alpha$ and (2.3) with $\varepsilon = \beta$, we have eventually $\mathbb{P}$-a.s.

$$\begin{aligned} &\Big((Z_N^{(1)}-Z_N^{(3)}) - X_N^{(k_N+1)} - \tfrac{k_N-1}{N+1}X_N^{(1)} - c\Big) \\ &\qquad \ge N^{d/\alpha}\bigg(\frac{1}{(\log N)^\beta} - \frac{1}{(\log N)^{2\beta}} - \frac{(\log N)^{\vartheta+2/\alpha}}{N} - \frac{c}{N^{d/\alpha}}\bigg) = \frac{N^{d/\alpha}}{(\log N)^\beta}(1+o(1))\,, \end{aligned}$$

therefore, eventually $\mathbb{P}$-almost surely,

$$\begin{aligned} \mathbf{P}_{N,\xi}(\mathcal{A}_{1,N}) + \mathbf{P}_{N,\xi}(\mathcal{A}_{2,N}) \;&=\; 1 - \mathbf{P}_{N,\xi}\big((\mathcal{A}_{1,N}\cup\mathcal{A}_{2,N})^c\big) \\ &\ge\; 1 - \exp\left(-\frac{N^{1+d/\alpha}}{(\log N)^\beta}(1+o(1))\right)\,, \end{aligned}$$

which completes the proof of (3.13).

3.3. **Proof of** (1.15)**.** Recalling (3.11), we are going to prove that

$$\lim_{N\to\infty} \mathbf{P}_{N,\xi}(\mathcal{W}_{1,N}) = 1\,, \qquad \text{in } \mathbb{P}(\mathrm{d}\xi)\text{-probability}\,, \tag{3.19}$$

which is stronger than (1.15). In view of (3.7), it suffices to show that

$$\lim_{N\to\infty} \mathbf{P}_{N,\xi}(\mathcal{A}_{1,N}) = 1\,, \qquad \text{in } \mathbb{P}(\mathrm{d}\xi)\text{-probability}\,. \tag{3.20}$$

We actually prove the following: for every $N \in \mathbb{N}$ there exists a subset $\Gamma_N \subseteq \Omega_\xi$ such that as $N \to \infty$ one has $\mathbb{P}(\Gamma_N) \to 1$ and $\inf_{\xi \in \Gamma_N} \mathbf{P}_{N,\xi}(\mathcal{A}_{1,N}) \to 1$, which implies (3.20).

For every trajectory $S \in (\mathcal{A}_{1,N})^c$ we have $\beta_N(S) \neq J_1$, therefore $x_N^{(\beta_N(S))} \neq z_N^{(1)}$ and consequently $\psi_N(x_N^{(\beta_N(S))}) \leq Z_N^{(2)}$. From (3.16) and (5.9) we then obtain

$$
\begin{aligned}
\mathbf{P}_{N,\xi}\big((\mathcal{A}_{1,N})^c\big) \;&=\; \frac{\mathbf{E}(e^{H_N(S)}\, 1_{(\mathcal{A}_{1,N})^c})}{U_{N,\xi}} \\
&\leq \exp\Big(-(N+1)\Big((Z_N^{(1)} - Z_N^{(2)}) - X_N^{(k+1)} - \tfrac{k-1}{N+1} X_N^{(1)} - c\Big)\Big) \,.
\end{aligned} \tag{3.21}
$$

We set $\Gamma_N^{(1)} := \{Z_N^{(2)} \leq (1 - \frac{1}{\log N}) Z_N^{(1)}\}$ and it follows from (2.8) that $\mathbb{P}(\Gamma_N^{(1)}) \to 1$ as $N \to \infty$. Note that for $\xi \in \Gamma_N^{(1)}$ we have

$$
\Big((Z_N^{(1)} - Z_N^{(2)}) - X_N^{(k+1)} - \tfrac{k-1}{N+1} X_N^{(1)} - c\Big) \geq \Big(\tfrac{1}{\log N} Z_N^{(1)} - X_N^{(k+1)} - \tfrac{k-1}{N+1} X_N^{(1)} - c\Big) \,.
$$

We now fix $k = k_N = (\log N)^\vartheta$ with $\vartheta := 3 \max\{2\alpha, 1\} > 1$. Applying (2.2) with $\varepsilon = 1/\alpha$, (2.3) with $\varepsilon = 2$ and (2.7), we have eventually $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\Big(\tfrac{1}{\log N} Z_N^{(1)} - X_N^{(k_N+1)} - \tfrac{k_N-1}{N+1} X_N^{(1)} - c\Big) \\
&\qquad \geq N^{d/\alpha}\bigg(\frac{1}{(\log N)^2} - \frac{1}{(\log N)^4} - \frac{(\log N)^{\vartheta+2/\alpha}}{N} - \frac{c}{N^{d/\alpha}}\bigg) = \frac{N^{d/\alpha}}{(\log N)^2}(1+o(1)) \,.
\end{aligned}
$$

In particular, defining $\Gamma_N^{(2)} := \{\frac{1}{\log N} Z_N^{(1)} - X_N^{(k+1)} - \frac{k-1}{N+1} X_N^{(1)} - c > N^{d/\alpha}/(\log N)^3\}$, we have $\mathbb{P}(\Gamma_N^{(2)}) \to 1$ as $N \to \infty$. Setting $\Gamma_N := \Gamma_N^{(1)} \cap \Gamma_N^{(2)}$, we have $\mathbb{P}(\Gamma_N) \to 1$ as $N \to \infty$; furthermore, by the preceding steps we have that, for all $\xi \in \Gamma_N$,

$$
\mathbf{P}_{N,\xi}\big(\mathcal{A}_{1,N}\big) \;=\; 1 - \mathbf{P}_{N,\xi}\big((\mathcal{A}_{1,N})^c\big) \;\geq\; 1 - \exp\left((N+1)\frac{N^{d/\alpha}}{(\log N)^3}\right) \,.
$$

This completes the proof of (3.20).

4. Almost sure localization at one point

In this section we prove Theorem 1.3. Relation (1.11) is obtained in two steps. First, we refine the results of the previous section, showing that (3.12) still holds if we replace the events $\mathcal{W}_{i,N}$, $i = 1, 2$, that were introduced in (3.11), by

$$
\begin{aligned}
\widetilde{\mathcal{W}}_{i,N} \;&:= \Big\{S \in \Omega_S \colon\; \beta_N(S) = J_i,\; S_N = z_N^{(i)},\; \ell_N(z_N^{(i)}) > \tfrac{N - |z_N^{(i)}|}{2}\Big\} \\
&= \Big\{S \in \Omega_S \colon\; S_N = z_N^{(i)},\; \ell_N(z_N^{(i)}) > \tfrac{N - |z_N^{(i)}|}{2}, \\
&\qquad\qquad \ell_N(x) = 0 \;\forall x \in \mathcal{B}_N \text{ such that } \xi(x) > \xi(z_N^{(i)})\Big\} \,,
\end{aligned} \tag{4.1}
$$

that is, if we require that the random walk trajectories spend at $z_N^{(i)}$ at least $(N - |z_N^{(i)}|)/2$ units of time (recall (3.3)). In the second step, we show that eventually $\mathbb{P}(\mathrm{d}\xi)$-almost surely

$$
\max\big\{\mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}),\, \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\big\} \;\gg\; \min\big\{\mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}),\, \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\big\} \,, \tag{4.2}
$$

which yields (1.11). Finally, we prove (1.12) in Section 4.3.

4.1. **Step 1.** In this step we refine (3.12), showing that

$$\lim_{N\to\infty} \big(\mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}) + \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\big) = 1\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,, \tag{4.3}$$

where $\widetilde{\mathcal{W}}_{i,N}$ is defined in (4.1). Consider indeed $S \in \mathcal{W}_{i,N} \setminus \widetilde{\mathcal{W}}_{i,N}$, with $i \in \{1,2\}$. Before reaching $z_N^{(i)}$, S must visit at least $|z_N^{(i)}| - 1$ different sites at which, by definition of $\mathcal{W}_{i,N}$, the field is smaller than $\xi(z_N^{(i)}) = X_N^{(J_i)}$ (recall (3.1)), hence

$$H_N(S) \le \ell_N(z_N^{(i)})\, X_N^{(J_i)} + \sum_{j=1}^{|z_N^{(i)}|-1} X_N^{(J_i+j)} + \big(N - \ell_N(z_N^{(i)}) - (|z_N^{(i)}| - 1)\big) X_N^{(J_i+1)}\,.$$

Since $\ell_N(z_N^{(i)}) \le (N - |z_N^{(i)}|)/2$ on $\mathcal{W}_{i,N} \setminus \widetilde{\mathcal{W}}_{i,N}$, we obtain

$$H_N(S) \le \tfrac{N-|z_N^{(i)}|}{2}\, X_N^{(J_i)} + \sum_{j=1}^{|z_N^{(i)}|-1} X_N^{(J_i+j)} + \Big(\tfrac{N-|z_N^{(i)}|}{2} + 1\Big) X_N^{(J_i+1)}\,.$$

Rewriting (5.9) as $U_N \ge e^{(N+1-|z_N^{(i)}|)X_N^{(J_i)} - cN}$ (recall (2.6)), we can write

$$\begin{aligned} \mathbf{P}_{N,\xi}\big(\mathcal{W}_{i,N} \setminus \widetilde{\mathcal{W}}_{i,N}\big) &= \frac{\mathrm{E}(e^{H_N(S)} \mathbb{1}_{\{S \in \mathcal{W}_{i,N} \setminus \widetilde{\mathcal{W}}_{i,N}\}})}{U_N} \\ &\le e^{cN} \exp\left(- \frac{N - |z_N^{(i)}|}{2} (X_N^{(J_i)} - X_N^{(J_i+1)}) + \sum_{j=1}^{|z_N^{(i)}|-1} X_N^{(J_i+j)} \right). \end{aligned} \tag{4.4}$$

Applying (2.7) with $\varepsilon = 1/\alpha$ and (2.2) with $\varepsilon = \varepsilon/2$, it follows that eventually $\mathbb{P}$-a.s.

$$Z_N^{(1)} \ge Z_N^{(2)} \ge \tfrac{N^{d/\alpha}}{(\log\log N)^{2/\alpha}} \quad \text{and} \quad \max\{X_N^{(J_1)}, X_N^{(J_2)}\} \le N^{d/\alpha} (\log N)^{1/\alpha + \varepsilon/2}\,.$$

Since by definition $Z_N^{(i)} = (1 - \frac{|z_N^{(i)}|}{N+1}) X_N^{(J_i)}$, it follows that for both $i \in \{1,2\}$ and for every $\varepsilon > 0$, eventually $\mathbb{P}$-a.s.

$$N - |z_N^{(i)}| \ge \frac{N}{(\log N)^{1/\alpha+\varepsilon}}\,. \tag{4.5}$$

Next we observe that, by the upper bound in (2.2) and (2.4), we have

$$\sum_{j=1}^{N} X_N^{(j)} \le (\log N) X_N^{(1)} + \sum_{j=\lceil \log N \rceil}^{N} X_N^{(j)} \le N^{d/\alpha} \left((\log N)^{1+3/2\alpha} + \sum_{j=\lceil \log N \rceil}^{N} \frac{1}{j^{1/\alpha}} \right),$$

therefore there exists a constant $c > 0$ such that, eventually $\mathbb{P}$-almost surely,

$$\sum_{j=1}^{N} X_N^{(j)} \le \begin{cases} c\, N^{d/\alpha + 1 - 1/\alpha} & \text{if } \alpha > 1 \\ (\log N)^{1+3/2\alpha} N^{d/\alpha} & \text{if } \alpha \le 1 \end{cases}. \tag{4.6}$$

Looking back at (4.4), we can apply (4.5) and (4.6) as well as Proposition 2.2 and Corollary 3.1 to conclude that $\mathbb{P}(\mathrm{d}\xi)$-a.s. the right hand side of (4.4) vanishes as $N \to \infty$. Recalling (3.12), it follows that (4.3) holds true, and the first step is completed.

4.2. **Step 2.** In this step we prove that

$$\lim_{N\to\infty} \big|\log \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}) - \log \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\big| = \infty\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,. \tag{4.7}$$

Together with (4.3), this shows that

$$\lim_{N\to\infty} \max\Big\{\mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}),\ \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\Big\} = 1\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,, \tag{4.8}$$

which yields (1.11) and, moreover, shows that

$$\mathrm{w}_{N,\xi} \;=\; \begin{cases} z_N^{(1)} & \text{if } \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}) > \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N}) \\ z_N^{(2)} & \text{if } \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N}) > \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}) \end{cases}, \quad \text{eventually } \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,.$$

It is convenient to introduce some further notation. Recalling (4.1), for $N \in \mathbb{N}$ and $x \in \mathcal{B}_N$ we define the following subsets of Ω_S:

$$\widetilde{\mathcal{W}}_N(x) \;:=\; \Big\{S \in \Omega_S : S_N = x,\ \ell_N(x) > \tfrac{N-|x|}{2},\ \ell_N(z) = 0\ \forall z \text{ s.t. } \xi(z) > \xi(x)\Big\}\,, \tag{4.9}$$

so that $\widetilde{\mathcal{W}}_{i,N} = \widetilde{\mathcal{W}}_N(z_N^{(i)})$. Next we set

$$C_N(x) \;:=\; \log \mathrm{E}\big[e^{H_N(S)}\, \mathbb{1}_{\{S\in\widetilde{\mathcal{W}}_N(x)\}}\big]\,, \tag{4.10}$$

so that we can write

$$\big|\log \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}) - \log \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\big| \;=\; \big|C_N(z_N^{(1)}) - C_N(z_N^{(2)})\big|\,. \tag{4.11}$$

Finally, given an arbitrary $\varepsilon \in (0, d/\alpha)$ and setting $N_k := \lfloor k^{2/\varepsilon}\rfloor$, we introduce the event $\mathcal{H}_k \subseteq \Omega_\xi$ defined by

$$\begin{aligned} \mathcal{H}_k \;:=\; \Big\{\xi \in \Omega_\xi :\ & \exists x, y \in \mathcal{B}_{N_{k+1}},\ x \neq y,\ \exists n \in \{\max\{|x|,|y|\},\ldots,N_{k+1}\} \text{ such that} \\ & \xi(x) > \tfrac{(N_k)^{d/\alpha}}{(\log N_{k+1})^{2/\alpha}},\ \xi(y) > \tfrac{(N_k)^{d/\alpha}}{(\log N_{k+1})^{2/\alpha}},\ |C_n(x) - C_n(y)| \le N_k^{d/\alpha-\varepsilon}\Big\}\,. \end{aligned} \tag{4.12}$$

We are going to show that

$$\sum_{k\in\mathbb{N}} \mathbb{P}(\mathcal{H}_k) < \infty\,. \tag{4.13}$$

We claim that this implies (4.7) and completes the step. Indeed, by the Borel-Cantelli lemma it follows from (4.13) that for $\mathbb{P}$-almost every $\xi \in \Omega_\xi$ there exists $\overline{k} = \overline{k}(\xi) < \infty$ such that $\xi \notin \mathcal{H}_k$ for all $k \geq \overline{k}$. For any $N \geq N_{\overline{k}}$, let $k \in \mathbb{N}, k \geq \overline{k}$ be such that $N_k < N \leq N_{k+1}$ and note that, plainly, $z_N^{(1)}, z_N^{(2)} \in \mathcal{B}_N \subseteq \mathcal{B}_{N_{k+1}}$. Recalling the lower bound in (2.7) and (4.11), since $\xi \notin \mathcal{H}_k$ for all $k \geq \overline{k}$ we conclude that eventually $\mathbb{P}(\mathrm{d}\xi)$-almost surely

$$\big|\log \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{1,N}) - \log \mathbf{P}_{N,\xi}(\widetilde{\mathcal{W}}_{2,N})\big| \geq N^{d/\alpha-\varepsilon}\,,$$

which is a stronger statement than (4.7).

We are left with proving (4.13), for which we have to estimate

$$\mathbb{P}\big(\xi(x) > t, \xi(y) > t, |C_n(x) - C_n(y)| \leq M\big)\,, \tag{4.14}$$

for suitable t and M. Recalling (4.9) and (4.10), it is useful to set

$$C_N(y;x) \;:=\; \log \mathrm{E}\big[e^{H_N(S)}\, \mathbb{1}_{S\in\widetilde{\mathcal{W}}_N(y)}\, \mathbb{1}_{\{\ell_N(x)=0\}}\big]\,. \tag{4.15}$$

Note in fact that, on the event $\xi(x) > \xi(y)$, we have $C_N(y) = C_N(y;x)$, by the definition (4.9) of $\widetilde{\mathcal{W}}_N(y)$. Therefore, splitting (4.14) on $\{\xi(x) > \xi(y)\}$ and $\{\xi(x) < \xi(y)\}$ and using the symmetry between x and y, we can easily estimate

$$\begin{aligned}
&\mathbb{P}\big(\xi(x) > t, \xi(y) > t, |C_n(x) - C_n(y)| \le M\big) \\
&\quad \le 2\,\mathbb{P}\big(\xi(x) > t, \xi(y) > t, |C_n(x) - C_n(y;x)| \le M\big) \\
&\quad \le 2\,\mathbb{E}\big[\mathbb{1}_{\{\xi(y)>t\}}\,\mathbb{P}\big(\xi(x) > t,\ |C_n(x) - C_n(y;x)| \le M \big|\mathcal{G}_x\big)\big]\,,
\end{aligned} \tag{4.16}$$

where $\mathcal{G}_x := \sigma(\{\xi(z)\}_{z\in\mathbb{Z}^d\setminus\{x\}})$. We stress that $C_n(y;x)$ is $\mathcal{G}_x$-measurable, because by definition it does not depend on $\xi(x)$ (recall (4.15)).

We now need to study the dependence of $C_n(x)$ on $\xi(x)$ conditionally on $\mathcal{G}_x$, i.e., when all the other field variables $\{\xi(z), z \neq x\}$ are fixed. Recalling (4.10), (4.9) and summing over the values of the variable $\ell_N(x)$, we can write $C_n(x) = g(\xi(x))$, where

$$g(s) := \log \sum_{k=\frac{1}{2}(n-|x|+1)}^{n-|x|} e^{ks}\, c_{n,k} \qquad \text{and} \qquad c_{n,k} := \mathrm{E}\big[e^{H_n(S)-k\xi(x)}\,\mathbb{1}_{S\in\widetilde{\mathcal{W}}_n(x)}\,\mathbb{1}_{\{\ell_N(x)=k\}}\big]\,.$$

We stress that, on the event $\{\ell_n(x) = k\}$, the term $H_n(S) - k\xi(x)$ does not depend on $\xi(x)$. Therefore the coefficients $c_{n,k}$ (and, hence, the function $g(\cdot)$) only depend on $\{\xi(z), z \neq x\}$, i.e., they are $\mathcal{G}_x$-measurable. Also note that the function $g(\cdot)$ is smooth and Lipschitz, since

$$g'(s) = \frac{\sum_{k=\frac{1}{2}(n-|x|+1)}^{n-|x|} k\, e^{ks}\, c_{n,k}}{\sum_{k=\frac{1}{2}(n-|x|+1)}^{n-|x|} e^{ks}\, c_{n,k}} \ \ge\ \frac{1}{2}(n-|x|+1)\,.$$

Therefore, by the change of variables formula, from (1.2) we obtain

$$\begin{aligned}
&\mathbb{P}\big(\xi(x) > t,\ C_n(x) \in \mathrm{d}v \,\big|\, \mathcal{G}_x\big) = \mathbb{P}\big(\xi(x) > t,\ g(\xi(x)) \in \mathrm{d}v \,\big|\, \mathcal{G}_x\big) \\
&\quad = \mathbb{1}_{\{g^{-1}(v)>\max\{1,t\}\}}\,\frac{1}{|g'(g^{-1}(v))|}\,\frac{\alpha}{(g^{-1}(v))^{1+\alpha}}\,\mathrm{d}v \le \frac{2}{(n-|x|+1)}\,\frac{\alpha}{t^{1+\alpha}}\,\mathrm{d}v\,,
\end{aligned}$$

hence

$$\begin{aligned}
&\mathbb{P}\big(\xi(x) > t,\ |C_n(x) - C_n(y;x)| \le M \big|\mathcal{G}_x\big) \\
&\quad = \mathbb{P}\big(\xi(x) > t,\ C_n(x) \in [C_n(y;x) - M, C_n(y;x) + M]\big|\mathcal{G}_x\big) \ \le\ \frac{2\alpha}{(n-|x|+1)t^{1+\alpha}} \cdot 2M\,.
\end{aligned}$$

Coming back to (4.16), since $\mathbb{P}(\xi(y) > t) \le t^{-\alpha}$, we conclude that

$$\mathbb{P}\big(\xi(x) > t, \xi(y) > t, |C_n(x) - C_n(y)| \le M\big) \ \le\ \frac{8\alpha M}{(n-|x|+1)t^{1+2\alpha}}\,. \tag{4.17}$$

We are finally ready to estimate $\mathbb{P}(\mathcal{H}_k)$. Recalling the definition (4.12) and the fact that $N_k = \lfloor k^{2/\varepsilon} \rfloor$, applying (4.17) we obtain

$$\begin{aligned}
\mathbb{P}(\mathcal{H}_k) \;&\le\; 2 \sum_{\substack{x \neq y \in \mathcal{B}_{N_{k+1}} \\ |x| \ge |y|}} \sum_{n=|x|}^{N_{k+1}} \mathbb{P}\Big(\xi(x), \xi(y) > \tfrac{(N_k)^{d/\alpha}}{(\log N_{k+1})^{2/\alpha}}, |C_n(x) - C_n(y)| \le N_k^{d/\alpha - \varepsilon}\Big) \\
&\le\; 2\,(const.)\,(N_{k+1})^{2d} \, \frac{8\alpha N_k^{d/\alpha-\varepsilon}}{\{(N_k)^{d/\alpha}/(\log N_{k+1})^{2/\alpha}\}^{(1+2\alpha)}} \sum_{n=|x|}^{N_{k+1}} \frac{1}{n - |x| + 1} \\
&\le\; (const.')\,\frac{(\log N_{k+1})^{2/\alpha+5}}{N_k^{\varepsilon}} \;\le\; (const.'')\,\frac{(\log k^{2/\varepsilon})^{2/\alpha+5}}{k^2}\,,
\end{aligned}$$

from which (4.13) follows. This completes the step.

4.3. **Proof of** (1.12)**.** In view of (1.16), it is sufficient to prove that

$$\frac{z_N^{(1)}}{N} \Longrightarrow \mathrm{w}\,, \qquad \text{where} \qquad \mathbb{P}(\mathrm{w} \in \mathrm{d}x) \;=\; c_\alpha\,(1 - |x|)^\alpha\,\mathbb{1}_{\{|x| \le 1\}}\;\mathrm{d}x\,, \tag{4.18}$$

and we recall that $c_\alpha := (\int_{|y|\le 1} (1 - |y|)^\alpha \mathrm{d}y)^{-1}$.

Setting $\varphi_N(x) := 1 - \frac{|x|}{N+1}$ and recalling (1.2), for $x \in \mathcal{B}_N$ and $t \in (1, \infty)$ we have

$$\begin{aligned}
\mathbb{P}\big(z_N^{(1)} = x\,,\ \xi(x) \in \mathrm{d}t\big) \;&=\; \mathbb{P}\big(\xi(z) < t\ \forall z \in \mathcal{B}_N \setminus \{x\}\,,\ \xi(x) \in \mathrm{d}t\big) \\
&=\; \prod_{z \in \mathcal{B}_N,\, z \neq x} \left(1 - \frac{\varphi_N(z)^\alpha}{t^\alpha \varphi_N(x)^\alpha}\right) \frac{\alpha}{t^{1+\alpha}}\,\mathrm{d}t\,,
\end{aligned}$$

therefore for all function $f : \mathbb{R}^d \to \mathbb{R}$ we can write

$$\mathbb{E}\left[f\left(\frac{z_N^{(1)}}{N}\right)\right] \;=\; \sum_{x \in \mathcal{B}_N} f\left(\frac{x}{N}\right) \int_1^\infty \mathrm{d}t \prod_{z \in \mathcal{B}_N,\, z \neq x} \left(1 - \frac{\varphi_N(z)^\alpha}{t^\alpha \varphi_N(x)^\alpha}\right) \frac{\alpha}{t^{1+\alpha}}\,. \tag{4.19}$$

Now set $t = N^{d/\alpha} s$ and note that as $N \to \infty$, uniformly in $s \in (\varepsilon, \infty)$ and $x \in \mathcal{B}_{(1-\varepsilon)N}$, where $\varepsilon > 0$ is arbitrary but fixed, by a Riemann sum approximation we have

$$\begin{aligned}
\sum_{z \in \mathcal{B}_N,\, z \neq x} \log\left(1 - \frac{\varphi_N(z)^\alpha}{t^\alpha \varphi_N(x)^\alpha}\right) \;&=\; -\frac{1}{s^\alpha (1 - \frac{|x|}{N+1})^\alpha N^d} \sum_{z \in \mathcal{B}_N,\, z \neq x} \left(1 - \frac{|z|}{N+1}\right)^\alpha (1 + o(1)) \\
&=\; -\frac{c_\alpha^{-1}}{s^\alpha\,(1 - \frac{|x|}{N+1})^\alpha}\,(1 + o(1))\,.
\end{aligned}$$

Coming back at (4.19) and noting that $\int_0^\infty \frac{\alpha}{s^{1+\alpha}}\,e^{-A/s^\alpha}\,\mathrm{d}s = \int_0^\infty e^{-Au}\,\mathrm{d}u = A^{-1}$, by a simple change of variables, it follows again by a Riemann sum argument that if f is continuous and bounded we have

$$\begin{aligned}
\lim_{N \to \infty} \mathbb{E}\left[f\left(\frac{z_N^{(1)}}{N}\right)\right] \;&=\; \lim_{N \to \infty} \frac{1}{N^d} \sum_{x \in \mathcal{B}_N} f\left(\frac{x}{N}\right) c_\alpha \left(1 - \frac{|x|}{N+1}\right)^\alpha \\
&=\; c_\alpha \int_{|y| \le 1} f(y)\,(1 - |y|)^\alpha\,\mathrm{d}y\,,
\end{aligned}$$

proving (4.18).

5. Path properties

In this section we prove Theorem 1.7, i.e., we show that $\lim_{N\to\infty} \mathbf{P}_{N,\xi}(\mathcal{C}_{N,\xi}) = 1$, $\mathbb{P}(\mathrm{d}\xi)$-almost surely, where the set $\mathcal{C}_{N,\xi}$ is defined in (1.18).

For $i = 1, 2$, we denote for simplicity by $\tau_i := \inf\{n \in \mathbb{N} : S_n = z_N^{(i)}\}$ the first time at which the random walk visits the site $z_N^{(i)}$ and we set

$$\begin{aligned} \mathcal{D}_{i,N} &:= \Big\{S \in \Omega_S \colon\ \tau_i \le N\,,\ S_m \ne S_n\ \forall m < n \le \tau_i\,,\ S_n = z_N^{(i)}\ \forall n \in \{\tau_i, \dots, N\}\Big\} \\ \mathcal{K}_{i,N} &:= \Big\{S \in \Omega_S \colon\ \tau_i \le |z_N^{(i)}| + h_N\Big\}\,, \end{aligned} \tag{5.1}$$

where we recall that $h_N := (\log\log N)^{2/\alpha} N^{1-1/\alpha}$ if $\alpha > 1$ and $h_N := (\log N)^{1+2/\alpha}$ if $\alpha \le 1$. Recalling the definition (4.1) of the set $\widetilde{\mathcal{W}}_{i,N}$, we are going show that for both $i = 1, 2$

$$\lim_{N\to\infty} \mathbf{P}_{N,\xi}\big(\widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}\big) = 0\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,, \tag{5.2}$$

$$\lim_{N\to\infty} \mathbf{P}_{N,\xi}\big((\widetilde{\mathcal{W}}_{i,N} \cap \mathcal{D}_{i,N}) \setminus \mathcal{K}_{i,N}\big) = 0\,, \qquad \mathbb{P}(\mathrm{d}\xi)\text{-almost surely}\,. \tag{5.3}$$

Recalling relation (4.8), proved in the last section, Theorem 1.7 is a consequence of (5.2) and (5.3). The rest of this section is therefore devoted to proving such relations.

5.1. **Step 1: proof of** (5.2)**.**

We fix $i \in \{1, 2\}$ throughout the section. By definition, a random walk trajectory $S \in \mathcal{W}_{i,N} \setminus \mathcal{D}_{i,N}$ makes either some *loops* before time τ_i (i.e., before reaching $z_N^{(i)}$) or some *excursions* outside $z_N^{(i)}$ between time τ_i and time N. We need to set up some notation to account for such loops and excursions.

We set $i_0 = j_0 := -1$ and, for $k \in \mathbb{N}$, we denote by $i_k = i_k(S)$, $j_k = j_k(S)$ the extremities of the k-th loop made by a trajectory $S \in \Omega_S$ before reaching $z_N^{(i)}$:

$$\begin{aligned} i_k &:= \inf\big\{n \in \{j_{k-1}+1, \dots, \tau_i - 1\} \colon\ \exists m \in \{n+1, \dots, \tau_i - 1\} \text{ s.t. } S_m = S_n\big\}\,, \\ j_k &:= \max\{n < \tau_i \colon\ S_n = S_{i_k}\}\,, \end{aligned} \tag{5.4}$$

with the usual convention $\inf \emptyset := \infty$. We also set $\mathcal{I}_k := \{i_k + 1, \dots, j_k\}$ and $|\mathcal{I}_k| := j_k - i_k$ for conciseness. Then we denote by $\mathcal{N} = \mathcal{N}(S) := \max\{k \in \mathbb{N} : i_k < \infty\}$ the total number and by $\mathcal{L} = \mathcal{L}(S) := \sum_{k=1}^{\mathcal{N}} |\mathcal{I}_k|$ the total length of the loops of the trajectory S. Note that $\mathcal{N} = \mathcal{L} = 0$ if $i_1 = \infty$, i.e., if the trajectory S has no loops. Finally, we denote by $\pi(S)$ the *injective skeleton* of S before reaching $z_N^{(i)}$, i.e., the random walk trajectory of $\tau_i - \mathcal{L}$ steps defined (with some abuse of notation) by

$$\pi(S) = \{\pi(S)_n\}_{n\in\{0,\dots,\tau_i-\mathcal{L}\}} := \{S_n\}_{n\in\{0,\dots,\tau_i\}\setminus\cup_{k=1}^{\mathcal{N}}\mathcal{I}_k}\,. \tag{5.5}$$

We let $\mathcal{V}_{i,N,r}$ denote the set of all r-steps injective paths, starting at 0 and ending at $z_N^{(i)}$, which do not visit any site $x \in \mathcal{B}_N$ with $\xi(x) > \xi(z_N^{(i)})$ (recall (3.3)):

$$\mathcal{V}_{i,N,r} := \big\{(S_n)_{n\le r} \colon\ S_r = z_N^{(i)},\ S_n \ne S_m \text{ for } m \ne n,\ \ell_r(x) = 0 \text{ when } \xi(x) > \xi(z_N^{(i)})\big\}\,. \tag{5.6}$$

Note that for $S \in \mathcal{W}_{i,N} \setminus \mathcal{D}_{i,N}$ we have $\pi(S) \in \mathcal{V}_{i,N,\tau_i - \mathcal{L}(S)}$.

Next we deal with the excursions outside $z_N^{(i)}$. Set $i'_0 = j'_0 = \tau_i - 1$ and for $k \in \mathbb{N}$ denote by $i'_k = i'_k(S)$, $j'_k = j'_k(S)$ the extremities of the k^{th} excursion outside $z_N^{(i)}$ made by the trajectory S between time τ_i and time N:

$$
\begin{aligned}
i'_k &:= \min\big\{n \in \{j_{k-1}+1, \ldots, N-1\} :\ S_n \neq z_N^{(i)}\big\}\,, \\
j'_k &:= \min\{n > i'_k :\ S_n = z_N^{(i)}\}\,.
\end{aligned} \tag{5.7}
$$

We also set $\mathcal{I}'_k := \{i'_k + 1, \ldots, j'_k\}$ and $|\mathcal{I}'_k| := j'_k - i'_k$; furthermore, we denote by $\mathcal{N}' = \mathcal{N}'(S) := \max\{k \geq 0 :\ i'_k < \infty\}$ the total number and by $\mathcal{L}' = \mathcal{L}'(S) := \sum_{k=1}^{\mathcal{N}'} |\mathcal{I}'_k|$ the total length of the excursions of the trajectory S. Note that $\mathcal{N}' = \mathcal{L}' = 0$ if $i'_1 = \infty$, i.e., if there are no excursions.

We can now start with the proof of (5.2). Recalling the definition (1.3) of our model and using the notation we have just introduced, we obtain the decomposition

$$
\mathbf{P}_{N,\xi}\big(\widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}\big) \;=\; \frac{1}{U_{N,\xi}} \sum_{r=|z_N^{(i)}|}^{N} \sum_{S^* \in \mathcal{V}_{i,N,r}} \mathrm{E}\big(e^{H_{N,\xi}(S)}\, \mathbb{1}_{\{S \in \widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}\}}\, \mathbb{1}_{\{\pi(S) = S^*\}}\big)\,. \tag{5.8}
$$

We bound the partition function $U_{N,\xi}$ from below by considering the trajectories that reach $z_N^{(i)}$ through an injective path, avoiding the sites x with $\xi(x) > \xi(z_N^{(i)})$, and stick at $z_N^{(i)}$ afterwards, getting

$$
U_{N,\xi} \;\geq\; \sum_{r=|z_N^{(i)}|}^{N} \sum_{S^* \in \mathcal{V}_{i,N,r}} e^{\sum_{n=1}^{r-1} \xi(S_n^*) + (N+1-r)\xi(z_N^{(i)})}\, \mathrm{P}(S^*)\, \kappa^{N-r}\,, \tag{5.9}
$$

where for simplicity we set $\mathrm{P}(S^*) := \mathrm{P}(S_1 = S_1^*, \ldots, S_r = S_r^*)$ and we recall (1.1).

Next we estimate the double sum in the right hand side of (5.8). Observe that for $S \in \widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}$ we have $\mathcal{L} + \mathcal{L}' \geq 1$, because S must make at least one loop before reaching $z_N^{(i)}$ or one excursion outside $z_N^{(i)}$ before time N. By definition of $\mathcal{W}_{i,N}$, cf. (4.1), any site x visited by S in the loops or excursions has an associated potential $\xi(x) < \xi(z_N^{(i)})$, hence $\xi(x) \leq X_N^{(J_i+1)} = \xi(z_N^{(i)}) - (X_N^{(J_i)} - X_N^{(J_i+1)})$, cf. (3.1). It follows that on $\{\mathcal{L} = l, \mathcal{L}' = l'\}$ we have $H_N(S) \leq \sum_{n=1}^{r-1} \xi(S_n^*) + (N+1-r)\xi(z_N^{(i)}) - (l+l')(X_N^{(J_i)} - X_N^{(J_i+1)})$, hence

$$
\begin{aligned}
&\mathrm{E}\big(e^{H_{N,\xi}(S)}\, \mathbb{1}_{\{S \in \widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}\}}\, \mathbb{1}_{\{\pi(S) = S^*\}}\big) \\
&\quad \leq \sum_{l,l' \in \mathbb{N}_0,\, l+l' \geq 1} e^{\sum_{n=1}^{r-1} \xi(S_n^*) + (N+1-r)\xi(z_N^{(i)}) - (l+l')(X_N^{(J_i)} - X_N^{(J_i+1)})}\, \mathrm{P}(\mathcal{L} = l, \mathcal{L}' = l', \pi(S) = S^*)\,.
\end{aligned}
$$

Looking back at (5.8) and (5.9), we conclude that

$$
\begin{aligned}
&\mathbf{P}_{N,\xi}\big(\widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}\big) \\
&\quad \leq \sup_{\substack{r \in \{|z_N^{(i)}|, \ldots, N\} \\ S^* \in \mathcal{V}_{i,N,r}}} \sum_{l,l' \in \mathbb{N}_0,\, l+l' \geq 1} e^{-(l+l')(X_N^{(J_i)} - X_N^{(J_i+1)})}\, \frac{\mathrm{P}(\mathcal{L} = l, \mathcal{L}' = l', \pi(S) = S^*)}{\mathrm{P}(S^*)\, \kappa^{N-r}}\,.
\end{aligned} \tag{5.10}
$$

We are left with estimating the ratio in the right hand side of (5.10). It is convenient to disintegrate the event $\{\mathcal{L} = l\}$ (resp. $\{\mathcal{L}' = l'\}$) by summing on the total number $\mathcal{N}$ and the locations $\mathcal{I} = \{\mathcal{I}_k\}_{k \leq \mathcal{N}}$ of the loops (resp. the total number $\mathcal{N}'$ and the locations $\mathcal{I}' = \{\mathcal{I}'_k\}_{k \leq \mathcal{N}'}$ of the excursions). Using the Markov property and bounding the probability of each loop and excursion (trivially) by 1, for all $n, I = \{I_k\}_{k \leq n}$, $n', I' = \{I'_k\}_{k \leq n}$ and for all injective trajectories $S^* \in \mathcal{V}_{i,N,r}$ we have

$$
\mathrm{P}(\mathcal{N} = n, \mathcal{I} = I, \mathcal{N}' = n', \mathcal{I}' = I', \pi(S) = S^*) \leq \mathrm{P}(S^*)\, \kappa^{N-r-l-l'}\,,
$$

because $|\{n \in \{\tau_i, \dots, N-1\} : S_n = S_{n+1}\}| = N - \tau_i - \mathcal{L}'$, by definition of $\mathcal{L}'$, and $\tau_i = r + \mathcal{L}$ when $\pi(S) = S^* \in \mathcal{V}_{i,N,r}$, by definition of $\mathcal{L}$. It follows that

$$\frac{\mathrm{P}(\mathcal{L} = l, \mathcal{L}' = l', \pi(S) = S^*)}{\mathrm{P}(S^*)\, \kappa^{N-r}} \le \kappa^{-l-l'} \cdot \left|\left\{(n, I, n', I') : \textstyle\sum_{k=1}^{n} |I_k| = l,\, \sum_{k=1}^{n'} |I'_k| = l'\right\}\right| .$$

It remains to bound the cardinality of the set in the right hand side. For fixed $n \in \{0, \dots, l\}$, the intervals $I = \{I_k\}_{k \le n}$ consist of $2n$ points in $\{0, \dots, \tau_i\} \subseteq \{0, \dots, N\}$, therefore the number of possible choices for I is bounded from above by $(N+1)^{2n} \le (N+1)^{2l}$. Analogously, for every $n' \in \{0, \dots, l'\}$, the number of choices for I' is bounded from above by $(N+1)^{2n'} \le (N+1)^{2l'}$. Looking back at (5.10), we can write

$$\begin{aligned}
\mathbf{P}_{N,\xi}\big(\widetilde{\mathcal{W}}_{i,N} \setminus \mathcal{D}_{i,N}\big) \;&\le \sum_{l,l' \in \mathbb{N}_0,\, l+l' \ge 1} e^{-(l+l')(X_N^{(J_i)} - X_N^{(J_i+1)} + \log\kappa - 2\log(N+1))}\,(l+1)\,(l'+1) \\
&\le\; (const.) \sum_{m=1}^{\infty} e^{-m(X_N^{(J_i)} - X_N^{(J_i+1)} + \log\kappa - 2\log(N+1))}\, m^3 \\
&\le\; (const.')\, \frac{e^{-(X_N^{(J_i)} - X_N^{(J_i+1)} + \log\kappa - 2\log(N+1))}}{\big(1 - e^{-(X_N^{(J_i)} - X_N^{(J_i+1)} + \log\kappa - 2\log(N+1))}\big)^4}\,,
\end{aligned}$$

where in the second inequality we have used that $\sum_{l,l' \in \mathbb{N}_0:\, l+l'=m} (l+1)(l'+1) \le (const.) m^3$. It then follows from Corollary 3.1 and Proposition 2.2 that relation (5.2) holds true, completing the first step.

5.2. **Step 2: proof of** (5.3)**.** Throughout the section we fix $i \in \{1, 2\}$. We recall that $\tau_i := \inf\{n \in \mathbb{N} : S_n = z_N^{(i)}\}$ denotes the first time at which the random walk visits $z_N^{(i)}$.

A random walk trajectory $S \in \widetilde{\mathcal{W}}_{i,N} \cap \mathcal{D}_{i,N}$ (cf. (4.1) and (5.1)) reaches $z_N^{(i)}$ through an injective path, avoiding sites where the potential is larger than $\xi(z_N^{(i)})$, and sticks at $z_N^{(i)}$ afterwards (from time τ_i to time N). Therefore the corresponding Hamiltonian (cf. (1.3)) is bounded from above by

$$H_{N,\xi}(S) \;\le\; \sum_{n=1}^{\tau_i - 1} \xi(S_i) \;+\; (N+1-\tau_i)\xi(z_N^{(i)}) \;\le\; \sum_{j=1}^{N} X_N^{(j)} \;+\; (N+1-\tau_i)\xi(z_N^{(i)})\,.$$

Recalling the definition (5.1) of the set $\mathcal{K}_{i,N}$, for $S \in (\widetilde{\mathcal{W}}_{i,N} \cap \mathcal{D}_{i,N}) \setminus \mathcal{K}_{i,N}$ we obtain

$$H_{N,\xi}(S) \;\le\; \sum_{j=1}^{N} X_N^{(j)} \;+\; \big(N+1-|z_N^{(i)}| - h_N\big)\xi(z_N^{(i)})\,,$$

therefore, cf. (1.3),

$$\mathbf{P}_{N,\xi}\big((\widetilde{\mathcal{W}}_{i,N} \cap \mathcal{D}_{i,N}) \setminus \mathcal{K}_{i,N}\big) \;\le\; \frac{1}{U_{N,\xi}}\, e^{\sum_{j=1}^{N} X_N^{(j)} \,+\, (N+1-|z_N^{(i)}|-h_N)\xi(z_N^{(i)})}\,.$$

As usual, we obtain a lower bound on $U_{N,\xi}$ by considering a single trajectory that reaches the site $z_N^{(i)}$ in $|z_N^{(i)}|$ steps and sticks there afterwards, getting

$$U_{N,\xi} \;\ge\; e^{(N+1-|z_N^{(i)}|)\xi(z_N^{(i)})}\, c^N\,,$$

for a suitable $c > 0$, cf. (1.1). Note that $\xi(z_N^{(i)}) \ge Z_N^{(i)} \ge N^{d/\alpha}/(\log\log N)^{3/2\alpha}$ eventually $\mathbb{P}(\mathrm{d}\xi)$-almost surely, for both $i \in \{1,2\}$, by relation (2.7). Therefore

$$\mathbf{P}_{N,\xi}\big((\widetilde{\mathcal{W}}_{i,N} \cap \mathcal{D}_{i,N}) \setminus \mathcal{K}_{i,N}\big) \;\le\; e^{\sum_{j=1}^N X_N^{(j)} - h_N\, N^{d/\alpha}/(\log\log N)^{3/2\alpha}} \,.$$

Since $h_N := (\log\log N)^{2/\alpha} N^{1-1/\alpha}$ if $\alpha > 1$ and $h_N := (\log N)^{1+2/\alpha}$ if $\alpha \le 1$, it follows from (4.6) that $\mathbf{P}_{N,\xi}\big((\widetilde{\mathcal{W}}_{i,N} \cap \mathcal{D}_{i,N}) \setminus \mathcal{K}_{i,N}\big) \to 0$ as $N \to \infty$, $\mathbb{P}(\mathrm{d}\xi)$-almost surely. This proves that (5.3) holds true and completes the second step.

Appendix A. Order statistics for the field

This section is devoted to the order statistics $X_N^{(1)}, \ldots, X_N^{(|\mathcal{B}_N|)}$ of the field $\{\xi(x)\}_{x\in\mathcal{B}_N}$. We first give some basic probability estimates, from which the proofs of Lemma 2.1 and Proposition 2.2 will be deduced.

A.1. **Basic estimates.** We start comparing the relative sizes of $X_N^{(k)}$ and $X_N^{(p)}$.

Lemma A.1. *For all $N, p, k \in \mathbb{N}$ with $1 \le p < k \le |\mathcal{B}_N|$ and for all $\delta \in (0,1)$ we have*

$$\mathbb{P}\big(X_N^{(k)} \ge (1-\delta)X_N^{(p)}\big) \le \binom{k-1}{k-p} (1-(1-\delta)^\alpha)^{k-p} \,. \tag{A.1}$$

In the special case $p = 1$ the equality holds:

$$\mathbb{P}\big(X_N^{(k)} \ge (1-\delta)X_N^{(1)}\big) = (1-(1-\delta)^\alpha)^{k-1} \,. \tag{A.2}$$

Proof. We introduce the shortcuts $M_A := \sup_{x\in A}\xi(x)$, $\{X_N^{(m)\cdots(n)}\} := \{X_N^{(m)}, \ldots, X_N^{(n)}\}$ and $A^c := \mathcal{B}_N \setminus A$ for convenience. We recall that $\mathcal{B}_N = \{z \in \mathbb{Z}^d : |z| \le N\}$. Summing over the location of the subsets $\{X_N^{(1)\cdots(k-1)}\} = A$ and $\{X_N^{(p)\cdots(k-1)}\} = B$, so that $X_N^{(k)} = M_{A^c}$ and $X_N^{(p)} = M_{(A\setminus B)^c}$, we can write

$$\begin{aligned}
&\mathbb{P}(X_N^{(k)} \ge (1-\delta)X_N^{(p)}) \\
&\quad = \sum_{\substack{A\subseteq\mathcal{B}_N,\,|A|=k-1\\ B\subseteq A,\,|B|=k-p}} \mathbb{P}\left(X_N^{(k)} \ge (1-\delta)X_N^{(p)},\ \{X_N^{(1)\cdots(k-1)}\} = A,\ \{X_N^{(p)\cdots(k-1)}\} = B\right) \\
&\quad = \sum_{\substack{A\subseteq\mathcal{B}_N,\,|A|=k-1\\ B\subseteq A,\,|B|=k-p}} \mathbb{P}\left(M_{A^c} < \xi(y) < \frac{1}{1-\delta}M_{A^c}\ \forall y \in B\,,\ \ \xi(z) > M_B\ \forall z \in A\setminus B\right) .
\end{aligned}$$

Since $M_B \ge M_{A^c}$ on the event we are considering, we can replace M_B by M_{A^c} and obtain the upper bound

$$\begin{aligned}
\mathbb{P}(X_N^{(k)} \ge (1-\delta)X_N^{(p)}) \;\le \sum_{\substack{A\subseteq\mathcal{B}_N,\,|A|=k-1\\ B\subseteq A,\,|B|=k-p}} \mathbb{P}\bigg(&\frac{(1-\delta)^\alpha}{(M_{A^c})^\alpha} < \frac{1}{\xi(y)^\alpha} < \frac{1}{(M_{A^c})^\alpha}\ \forall y \in B\,, \\
&\frac{1}{\xi(z)^\alpha} < \frac{1}{(M_{A^c})^\alpha}\ \forall z \in A\setminus B\bigg) .
\end{aligned}$$

We stress that in the special case $p = 1$ we have $A = B$, so that $A \setminus B = \emptyset$ and therefore the above inequality is an equality.

By assumption the field $\xi(\cdot)$ has a Pareto distribution with parameter $\alpha > 0$, cf. (1.2), therefore $\frac{1}{\xi^\alpha}$ is uniformly distributed on the interval $(0,1)$: $\mathbb{P}(a < \frac{1}{\xi} < b) = b - a$ for all $0 < a < b < 1$. It follows that

$$\begin{aligned}\mathbb{P}(X_N^{(k)} \geq (1-\delta) X_N^{(p)}) \;&\leq\; (1-(1-\delta)^\alpha)^{k-p} \sum_{\substack{A \subseteq \mathcal{B}_N,\, |A|=k-1 \\ B \subseteq A,\, |B|=k-p}} \mathbb{E}\left(\frac{1}{(M_{A^c})^{\alpha(k-1)}}\right) \\ &\leq\; \binom{k-1}{k-p} (1-(1-\delta)^\alpha)^{k-p} \sum_{A \subseteq \mathcal{B}_N,\, |A|=k-1} \mathbb{E}\left(\frac{1}{(M_{A^c})^{\alpha(k-1)}}\right),\end{aligned}$$

and again all these inequalities are equalities if $p = 1$. It only remains to check that the last sum equals one. To this purpose, note that for all $\ell \in \mathbb{N}$, summing on the location of the set $\{X_N^{(1)\cdots(\ell)}\}$, we can write

$$\begin{aligned}1 \;&=\; \sum_{A \subseteq \mathcal{B}_N\,,\, |A|=\ell} \mathbb{P}(\{X_N^{(1)\cdots(\ell)}\} = A) \;=\; \sum_{A \subseteq \mathcal{B}_N\,,\, |A|=\ell} \mathbb{P}(\xi(x) > M_{A^c}\ \forall x \in A) \\ &=\; \sum_{A \subseteq \mathcal{B}_N\,,\, |A|=\ell} \mathbb{P}\left(\frac{1}{\xi(x)^\alpha} < \frac{1}{(M_{A^c})^\alpha}\ \forall x \in A\right) \;=\; \sum_{A \subseteq \mathcal{B}_N\,,\, |A|=\ell} \mathbb{E}\left(\frac{1}{(M_{A^c})^{\alpha\ell}}\right). \qquad \square\end{aligned}$$

Next we give some bounds on the absolute size of $X_N^{(k)}$.

Lemma A.2. *Let $c, C > 0$ be such that $c \leq \frac{|\mathcal{B}_N|}{N^d} \leq C$. Then for all $k \in \{1, \ldots, |\mathcal{B}_N|\}$ and $t \in (0,\infty)$ the following relations hold:*

$$\mathbb{P}(X_N^{(k)} > N^{d/\alpha} t) \;\leq\; \frac{C^k}{(k-1)!} \frac{1}{t^{k\alpha}}\,, \tag{A.3}$$

$$\mathbb{P}(X_N^{(k)} \leq t N^{d/\alpha}) \;\leq\; e^{-\frac{c}{t^\alpha}} \sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{eC}{t^\alpha}\right)^m. \tag{A.4}$$

Proof. Throughout the proof we shall assume that $t \geq N^{-d/\alpha}$. In fact, for $t < N^{-d/\alpha}$ there is nothing to prove, because the left hand side of (A.4) is zero (recall that the field $\xi(\cdot)$ is bounded from below by one, cf. (1.2)) and the right hand side of (A.3) is greater than one: in fact, for $k \leq |\mathcal{B}_N|$ we have $(k-1)! \leq k^k \leq |\mathcal{B}_N|^k \leq (CN^d)^k$ and therefore for $t < N^{-d/\alpha}$

$$\frac{C^k}{k!} \frac{1}{t^{k\alpha}} \geq \frac{C^k}{(CN^d)^k} \frac{1}{t^{k\alpha}} = \frac{1}{(N^{d/\alpha} t)^\alpha} \geq 1\,.$$

We start proving (A.3). The case $k = 1$ is easy:

$$\mathbb{P}(X_N^{(1)} \leq N^{d/\alpha} t) \;=\; \mathbb{P}(\xi(x) \leq N^{d/\alpha} t\ \forall x \in \mathcal{B}_N) \;=\; \left(1 - \frac{1}{t^\alpha N^d}\right)^{|\mathcal{B}_N|},$$

and since $(1-z)^a \geq 1 - az$ for $a \geq 1$ and $z \in [0,1]$ we obtain

$$\mathbb{P}(X_N^{(1)} > N^{d/\alpha} t) \;=\; 1 - \left(1 - \frac{1}{t^\alpha N^d}\right)^{|\mathcal{B}_N|} \;\leq\; \frac{|\mathcal{B}_N|}{N^d} \frac{1}{t^\alpha} \;\leq\; \frac{C}{t^\alpha}\,. \tag{A.5}$$

For the general case, summing over the location of the set $\{X_N^{(1)\cdots(k-1)}\} := \{X_N^{(1)}, \ldots, X_N^{(k-1)}\}$ and recalling the shortcuts $M_A := \sup_{x\in A}\varphi(x)$ and $A^c := \mathcal{B}_N \setminus A$ we get

$$\begin{aligned}
\mathbb{P}(X_N^{(k)} > N^{d/\alpha}t) \;&=\; \sum_{A\subseteq\mathcal{B}_N,\, |A|=k-1} \mathbb{P}(X_N^{(k)} > N^{d/\alpha}t,\ \{X_N^{(1)\cdots(k-1)}\} = A) \\
&=\; \sum_{A\subseteq\mathcal{B}_N,\, |A|=k-1} \mathbb{P}(M_{A^c} > N^{d/\alpha}t,\ \xi(x) > M_{A^c}\ \forall x \in A) \\
&=\; \sum_{A\subseteq\mathcal{B}_N,\, |A|=k-1} \mathbb{P}\left(M_{A^c} > N^{d/\alpha}t,\ \frac{1}{\xi(x)^\alpha} < \frac{1}{M_{A^c}^\alpha}\ \forall x \in A\right).
\end{aligned}$$

We have already remarked that the random variables $1/\xi(x)^\alpha$ are uniformly distributed over the interval $(0,1)$, that is $\mathbb{P}(\frac{1}{\xi(x)^\alpha} \le s) = s$ for $s \in (0,1)$. Then with some easy bounds we obtain

$$\begin{aligned}
\mathbb{P}(X_N^{(k)} > N^{d/\alpha}t) \;&=\; \sum_{\substack{A\subseteq\mathcal{B}_N\\ |A|=k-1}} \mathbb{E}\left(\frac{1}{M_{A^c}^{\alpha(k-1)}}\,, M_{A^c} > N^{d/\alpha}t\right) \\
\le\; \frac{1}{N^{d(k-1)}t^{\alpha(k-1)}} &\sum_{\substack{A\subseteq\mathcal{B}_N\\ |A|=k-1}} \mathbb{P}(M_{A^c} > N^{d/\alpha}t) \;\le\; \frac{1}{N^{d(k-1)}t^{\alpha(k-1)}} \sum_{\substack{A\subseteq\mathcal{B}_N\\ |A|=k-1}} \mathbb{P}(X_N^{(1)} > N^{d/\alpha}t)\,.
\end{aligned}$$

where we have used that $\mathbb{P}(M_{A^c} > N^{d/\alpha}t) \le \mathbb{P}(X_N^{(1)} > N^{d/\alpha}t)$ for all $A \subseteq \mathcal{B}_N$. Since $\binom{n}{m} \le n^m/m!$ and $|\mathcal{B}_N| \le CN^d$, we obtain

$$\begin{aligned}
\mathbb{P}(X_N^{(k)} > N^{d/\alpha}t) \;&\le\; \frac{1}{N^{d(k-1)}t^{\alpha(k-1)}} \binom{|\mathcal{B}_N|}{k-1} \mathbb{P}(X_N^{(1)} > N^{d/\alpha}t) \\
&\le\; \frac{1}{N^{d(k-1)}t^{\alpha(k-1)}} \frac{|\mathcal{B}_N|^{k-1}}{(k-1)!} \frac{C}{t^\alpha} \le \frac{C^k}{(k-1)!} \frac{1}{t^{\alpha k}}\,,
\end{aligned}$$

having applied (A.5). Equation (A.3) is proved.

To prove (A.4), note that the random variable $Y := \#\{z \in \mathcal{B}_N : \xi(z) > tN^{d/\alpha}\}$ is binomial $B(n,p)$ with parameters $n = |\mathcal{B}_N|$ and $p = \mathbb{P}(\xi > tN^{d/\alpha}) = 1/(t^\alpha N^d)$, therefore

$$\begin{aligned}
\mathbb{P}(X_N^{(k)} \le tN^{d/\alpha}) \;&=\; \mathbb{P}(Y \le k-1) \;=\; \sum_{m=0}^{k-1} \binom{n}{m} p^m\,(1-p)^{n-m} \\
&=\; \sum_{m=0}^{k-1} \binom{|\mathcal{B}_N|}{m} \left(\frac{1}{t^\alpha N^d}\right)^m \left(1 - \frac{1}{t^\alpha N^d}\right)^{|\mathcal{B}_N|-m}.
\end{aligned} \tag{A.6}$$

Using the estimates $(1-x)^a \le e^{-ax}$ and $\binom{n}{m} \le n^m/m!$ we get

$$\mathbb{P}(X_N^{(k)} \le tN^{d/\alpha}) \;\le\; e^{-\frac{|\mathcal{B}_N|}{N^d}\frac{1}{t^\alpha}} \sum_{m=0}^{k-1} \frac{1}{m!} \frac{1}{t^{\alpha m}} \left(\frac{|\mathcal{B}_N|}{N^d}\, e^{\frac{1}{t^\alpha N^d}}\right)^m,$$

from which (A.4) follows, recalling that $|\mathcal{B}_N| \ge cN^d$ and $1/(t^\alpha N^d) \le 1$ by assumption. □

We are finally ready for the proof of Lemma 2.1 and Proposition 2.2, to which are devoted the next paragraphs. For convenience, the proof of Lemma 2.1 has been split in two parts, in which we consider each equation separately.

A.2. **Proof of Lemma 2.1.** We start considering equation (2.2). Let us set $N_k := 2^k$. By (A.3) we have

$$\sum_{k\in\mathbb{N}} \mathbb{P}(X_{N_k}^{(1)} > (N_k)^{d/\alpha} (\log N_k)^{1/\alpha+\varepsilon/2}) \leq \frac{C}{(\log 2)^{1+\alpha\varepsilon/2}} \sum_{k\in\mathbb{N}} \frac{1}{k^{1+\alpha\varepsilon/2}} < \infty\,,$$

and by (A.4)

$$\begin{aligned}\sum_{k\in\mathbb{N}} \mathbb{P}(X_{N_k}^{(1)} \leq (N_k)^{d/\alpha} (\log\log N_k)^{-1/\alpha-\varepsilon/2}) &\leq \sum_{k\in\mathbb{N}} \exp\Big(-c(\log\log N_k)^{1+\varepsilon\alpha/2}\Big) \\ &= \sum_{k\in\mathbb{N}} \frac{1}{(k\ \log 2)^{c(\log\log 2+\log k)^{\varepsilon\alpha/2}}} < \infty\,,\end{aligned}$$

because for large k the exponent $c(\log\log 2+\log k)^{\varepsilon\alpha/2}$ exceeds 1. By the Borel-Cantelli lemma, it follows that eventually (in k) $\mathbb{P}$-a.s.

$$\frac{(N_k)^{d/\alpha}}{(\log\log N_k)^{1/\alpha+\varepsilon/2}} \leq X_{N_k}^{(1)} \leq (N_k)^{d/\alpha} (\log N_k)^{1/\alpha+\varepsilon/2}\,. \tag{A.7}$$

Now take a generic $N \in \mathbb{N}$ and set $k := \lfloor \log_2(N)\rfloor$, so that $N_k \leq N < N_{k+1}$. Observe that $X_{N_k}^{(1)} \leq X_N^{(1)} \leq X_{N_{k+1}}^{(1)}$, because $X_N^{(1)}$ is increasing in N. Plainly, one has $N_{k+1} \leq 2N$, $N_k \geq \frac{1}{2}N$, $\log N_k \leq \log N$ and $\log N_{k+1} \leq \log 2 + \log N \leq 2\log N$ (for large N). Then it follows from (A.7) that for large N

$$2^{-d/\alpha}\frac{N^{d/\alpha}}{(\log\log N)^{\varepsilon/2}} \leq X_{N_k}^{(1)} \leq X_N^{(1)} \leq X_{N_{k+1}}^{(1)} \leq 2^{d/\alpha+1/\alpha+\varepsilon/2}\, N^{d/\alpha} (\log N)^{1/\alpha+\varepsilon/2}\,.$$

Equation (2.2) follows observing that $2^{d/\alpha} \leq (\log\log N)^{\varepsilon/2}$ and $2^{d/\alpha+1/\alpha+\varepsilon/2} \leq (\log N)^{\varepsilon/2}$ for large N.

Next we focus on the lower bound in equation (2.3). By (A.4) we can write

$$\mathbb{P}\left(X_N^{((\log N)^{\vartheta})} \leq \frac{N^{d/\alpha}}{(\log N)^{\vartheta/\alpha+\varepsilon}}\right) \;\leq\; e^{-c\,(\log N)^{\vartheta+\alpha\varepsilon}} \sum_{m=0}^{\lfloor(\log N)^{\vartheta}\rfloor-1} \frac{1}{m!}\,\Big(eC\,(\log N)^{\vartheta+\alpha\varepsilon}\Big)^m\,.$$

Observe that, for fixed $x > 0$, the sequence $m \mapsto x^m/m!$ is increasing for $m \leq x$, therefore for $k \leq x$ we have $\sum_{m=0}^{k-1} x^m/m! \leq kx^k/k! \leq k(ex/k)^k$, because $m! \geq (m/e)^m$ for all $m \in \mathbb{N}$. It follows that for some constant $C' > 0$ and for large N we can write

$$\begin{aligned}\mathbb{P}\bigg(X_N^{((\log N)^{\vartheta})} \leq \frac{N^{d/\alpha}}{(\log N)^{\vartheta/\alpha+\varepsilon}}\bigg) \;&\leq\; e^{-c\,(\log N)^{\vartheta+\alpha\varepsilon}}\,(\log N)^{\vartheta}\,\Big(C'\,(\log N)^{\alpha\varepsilon}\Big)^{(\log N)^{\vartheta}} \\ &\leq\; (\log N)^{\vartheta}\, e^{-c\,(\log N)^{\vartheta+\alpha\varepsilon} + (\log N)^{\vartheta}[\alpha\varepsilon\log\log N + \log C']} \\ &\leq\; (\log N)^{\vartheta}\, e^{-\frac{1}{2}c\,(\log N)^{\vartheta+\alpha\varepsilon}} \;\leq\; N^{-2}\,,\end{aligned} \tag{A.8}$$

because by assumption $\vartheta > 1$ and $\varepsilon > 0$ (the -2 could be replaced by any negative number). The Borel-Cantelli lemma then yields directly the lower bound in (2.3).

Finally, we prove together the upper bound in (2.3) and (2.4). By Stirling's formula we have $(k-1)! \geq (\frac{k-1}{e})^{k-1} \geq (\frac{k}{3})^k$ for large k. Applying (A.3), we can then write

$$\mathbb{P}\left(X_N^{(k)} > A\,\frac{N^{d/\alpha}}{k^{1/\alpha}}\right) \;\leq\; \frac{C^k}{(k-1)!}\left(\frac{k^{1/\alpha}}{A}\right)^{\alpha k} \;\leq\; \left(\frac{3C}{A^{\alpha}}\right)^k \;\leq\; e^{-2k}\,,$$

provided A is chosen larger than $(e^2/3C)^{1/\alpha}$. By the inclusion bound,

$$\mathbb{P}\left(\exists k \in \{(\log N), \ldots, |\mathcal{B}_N|\} : \; X_N^{(k)} > A\, \frac{N^{d/\alpha}}{k^{1/\alpha}}\right) \;\le\; \sum_{k \ge \log N} e^{-2k} \;\le\; \frac{(const.)}{N^2}\,,$$

therefore by the Borel-Cantelli lemma it follows that, eventually $\mathbb{P}$-almost surely in N, one has $X_N^{(k)} \le A\, \frac{N^{d/\alpha}}{k^{1/\alpha}}$ for all $k \ge \log N$. This yields immediately (2.4), as well as the upper bound in (2.3), because by assumption $\vartheta > 1$. ☐

A.3. **Proof of Proposition 2.2.** Since the relation (2.5) becomes stronger as β increases, we can safely assume that $\beta > 1$. Then by (2.3) we have that, eventually $\mathbb{P}$-a.s.,

$$X_N^{(k)} \ge X_N^{((\log N)^\beta)} \ge \frac{N^{d/\alpha}}{(\log N)^{2\beta/\alpha}}\,, \qquad \forall k \le (\log N)^\beta\,. \tag{A.9}$$

Since a more quantitative control will be needed later, we observe that for large N

$$\mathbb{P}(\mathcal{C}_N) \;\le\; \frac{1}{N}\,, \qquad \text{where} \qquad \mathcal{C}_N \;:=\; \bigcup_{m \ge N} \left\{ X_m^{((\log m)^\beta)} \le \frac{m^{d/\alpha}}{(\log m)^{2\beta/\alpha}} \right\}\,, \tag{A.10}$$

as it follows from (A.8).

Thanks to (A.9), in order to prove (2.5) it suffices to show that for every $\beta > 1$ there exists $\gamma > 0$ such that, eventually $\mathbb{P}$-a.s., the following event holds:

$$\mathcal{V}_N := \left\{ \forall k \le (\log N)^\beta : \; X_N^{(k)} - X_N^{(k+1)} \ge \frac{X_N^{(k)}}{(\log N)^\gamma} \right\}\,.$$

In order to apply the Borel-Cantelly lemma, it is convenient to group the events $\mathcal{V}_N$ together. More precisely, for $n \in \mathbb{N}_0$ we set $N_n := \lfloor e^{n^r} \rfloor$, where the constant $r \in (0,1)$ will be fixed later, and we define

$$\widetilde{\mathcal{V}}_n := \bigcap_{N_n < m \le N_{n+1}} \mathcal{V}_m\,.$$

The proof is then completed once we show that the event $\widetilde{\mathcal{V}}_n$ holds eventually $\mathbb{P}$-a.s. (in n).

It only remains to show that $\mathbb{P}(\widetilde{\mathcal{V}}_n^c)$ decays fast enough as $n \to \infty$. By construction, if $\widetilde{\mathcal{V}}_n$ does not hold, there must exist $m \in \{N_n + 1, \ldots, N_{n+1}\}$ and $k \le (\log m)^\beta$ such that $0 < X_m^{(k)} - X_m^{(k+1)} < (\log m)^{-\gamma} X_m^{(k)}$. Let $y, z \in \mathcal{B}_m$ be the two points at which the values $X_m^{(k)}$ and $X_m^{(k+1)}$ are attained, that is $\xi(y) = X_m^{(k)}$ and $\xi(z) = X_m^{(k+1)}$. It is convenient to distinguish three cases, according to whether y and z are in $\mathcal{B}_{N_n}$ or not.

(1) If both $y, z \in \mathcal{B}_{N_n}$, we can write $\xi(y) = X_{N_n}^{(k')}$ and $\xi(z) = X_{N_n}^{(k'')}$ for some $k' < k''$. Since by construction $\xi(z) = X_m^{(k+1)}$ and $\mathcal{B}_m \supseteq \mathcal{B}_{N_n}$, we must have $k'' \le k+1$, whence $k' \le k \le (\log m)^\beta \le (\log N_{n+1})^\beta$. Also note that

$$\begin{aligned} X_{N_n}^{(k')} - X_{N_n}^{(k'+1)} &\le X_{N_n}^{(k')} - X_{N_n}^{(k'')} = X_m^{(k)} - X_m^{(k+1)} \\ &< (\log m)^{-\gamma} X_m^{(k)} = (\log m)^{-\gamma} X_{N_n}^{(k')} \le (\log N_n)^{-\gamma} X_{N_n}^{(k')}\,. \end{aligned}$$

This shows that, if $\widetilde{\mathcal{V}}_n$ does not hold and both $y, z \in \mathcal{B}_{N_n}$, there must exist $k' \le (\log N_{n+1})^\beta$ such that $X_{N_n}^{(k')} - X_{N_n}^{(k'+1)} \le (\log N_n)^{-\gamma} X_{N_n}^{(k')}$.

(2) To handle the case when $y, z \in \mathcal{B}_m \setminus \mathcal{B}_{N_n} \subseteq \mathcal{B}_{N_{n+1}} \setminus \mathcal{B}_{N_n}$, it is sufficient to observe that $\xi(y)$ and $\xi(z)$ must take large values, because of (A.9). More precisely, on the event $\mathcal{C}_{N_n}^c$, cf. (A.10), both $\xi(y)$ and $\xi(z)$ must be larger than $m^{d/\alpha}/(\log m)^{2\beta/\alpha} \geq N_n^{d/\alpha}/(\log N_{n+1})^{2\beta/\alpha}$.

(3) Consider finally the case when exactly one of the points y, z lies in $\mathcal{B}_{N_n}$. If $y \in \mathcal{B}_{N_n}$ and $z \in \mathcal{B}_m \setminus \mathcal{B}_{N_n}$, we have $\xi(y) = X_{N_n}^{(k')}$ for some $k' \leq (\log m)^\beta$, as we have already remarked, therefore $0 < X_{N_n}^{(k')} - \xi(z) < (\log m)^{-\gamma} X_{N_n}^{(k')}$. Viceversa, if $z \in \mathcal{B}_{N_n}$ and $y \in \mathcal{B}_m \setminus \mathcal{B}_{N_n}$, we may write $0 < \xi(y) - X_{N_n}^{(k'')} < (\log m)^{-\gamma} \xi(y)$, for some $k'' \leq (\log m)^\beta$. In either case, we can state that there exists some point $x \in \mathcal{B}_{N_{n+1}} \setminus \mathcal{B}_{N_n}$ and some $\bar{k} \leq (\log N_{n+1})^\beta$ such that $(1 - (\log N_n)^{-\gamma}) < \xi(x)/X_{N_n}^{(\bar{k})} < (1 - (\log N_n)^{-\gamma})^{-1}$.

These considerations lead us directly to the following basic decomposition:

$$\widetilde{\mathcal{V}}_n^c \subseteq \mathcal{W}_n^{(1)} \cup \big(\mathcal{C}_{N_n} \cup \mathcal{W}_n^{(2)}\big) \cup \mathcal{W}_n^{(3)} ,$$

where the event $\mathcal{C}_N$ has been introduced in (A.10) and we have set

$$\begin{aligned}
\mathcal{W}_n^{(1)} &:= \bigcup_{k' \leq (\log N_{n+1})^\beta} \left\{ X_{N_n}^{(k'+1)} > \left(1 - \frac{1}{(\log N_n)^\gamma}\right) X_{N_n}^{(k')} \right\} , \\
\mathcal{W}_n^{(2)} &:= \bigcup_{y,z \in \mathcal{B}_{N_{n+1}} \setminus \mathcal{B}_{N_n},\, y \neq z} \left\{ \xi(y) \geq \frac{N_n^{d/\alpha}}{(\log N_{n+1})^{2\beta/\alpha}},\ \xi(z) \geq \frac{N_n^{d/\alpha}}{(\log N_{n+1})^{2\beta/\alpha}} \right\}, \\
\mathcal{W}_n^{(3)} &:= \bigcup_{x \in \mathcal{B}_{N_{n+1}} \setminus \mathcal{B}_{N_n},\, \bar{k} \leq (\log N_{n+1})^\beta} \left\{ 1 - \frac{1}{(\log N_n)^\gamma} < \frac{\xi(x)}{X_{N_n}^{(\bar{k})}} < \left(1 - \frac{1}{(\log N_n)^\gamma}\right)^{-1} \right\} .
\end{aligned}$$

Note that, by (A.10), $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{C}_{N_n}) \leq \sum_{n \in \mathbb{N}} \frac{1}{N_n} \leq \sum_{n \in \mathbb{N}} e^{-n^r + 1} < \infty$. By the Borel-Cantelli lemma, it suffices to show that $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{W}_n^{(i)}) < \infty$ for $i = 1, 2, 3$ and it will follow that $\widetilde{\mathcal{V}}_n$ holds eventually $\mathbb{P}$-a.s., that is what we want to prove.

Let us consider $\mathcal{W}_n^{(1)}$. By (A.1) we have $\mathbb{P}\big(X_N^{(k+1)} \geq (1-\varepsilon) X_N^{(k)}\big) \leq c\, k\, \varepsilon$ for some constant $c > 0$. Recalling that $N_n = e^{n^r}$, for large n we have

$$\begin{aligned}
\mathbb{P}(\mathcal{W}_n^{(1)}) &\leq \sum_{k=1}^{\lfloor (\log N_{n+1})^\beta \rfloor} \mathbb{P}\left(X_{N_n}^{(k+1)} > \left(1 - \frac{1}{(\log N_n)^\gamma}\right) X_{N_n}^{(k)} \right) \\
&\leq c\, \frac{1}{(\log N_n)^\gamma} \sum_{k=1}^{\lfloor (\log N_{n+1})^\beta \rfloor} k \leq c' \frac{(\log N_{n+1})^{2\beta}}{(\log N_n)^\gamma} \leq \frac{c''}{n^{r(\gamma - 2\beta)}} ,
\end{aligned} \tag{A.11}$$

for suitable $c, c'' > 0$. It follows that $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{W}_n^{(1)}) < \infty$ provided $r(\gamma - 2\beta) > 1$.

Next we consider $\mathcal{W}_n^{(2)}$. Observe that there exist constants $c, c' > 0$ such that

$$|\mathcal{B}_{N_{n+1}} \setminus \mathcal{B}_{N_n}| \leq c\,(N_{n+1} - N_n)\,(N_n)^{d-1} \leq c' \frac{(N_n)^d}{n^{1-r}} , \tag{A.12}$$

because $N_{n+1} - N_n = \lfloor e^{(n+1)^r} \rfloor - \lfloor e^{n^r} \rfloor = e^{n^r} r\, n^{r-1}(1 + o(1))$ as $n \to \infty$. Recalling that $\mathbb{P}(\xi(x) > t) \le t^{-\alpha}$ by (1.2), for a suitable $c'' > 0$ we can write

$$\begin{aligned} \mathbb{P}(\mathcal{W}_n^{(2)}) \;&\le \sum_{y,z \in \mathcal{B}_{N_{n+1}} \setminus \mathcal{B}_{N_n},\, y \ne z} \mathbb{P}\bigg(\xi(y) > \frac{N_n^{d/\alpha}}{(\log N_{n+1})^{2\beta/\alpha}}\bigg)^2 \\ &\le\; (c')^2\, \frac{(N_n)^{2d}}{n^{2(1-r)}}\, \frac{(\log N_{n+1})^{4\beta}}{(N_n)^{2d}} \;\le\; c''\, \frac{n^{4\beta r}}{n^{2(1-r)}} \;=\; \frac{c''}{n^{2-(4\beta+2)r}}\,. \end{aligned}$$

Therefore $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{W}_n^{(2)}) < \infty$ provided $2 - (4\beta + 2)r > 1$.

We finally focus on $\mathcal{W}_n^{(3)}$. Note that by (1.2) for all $t > 1$ and $\varepsilon < \frac{1}{2}$ we can write

$$\mathbb{P}\bigg((1-\varepsilon) < \frac{\xi(x)}{t} < (1-\varepsilon)^{-1}\bigg) \;=\; \int_{(1-\varepsilon)t}^{(1-\varepsilon)^{-1}t} \frac{\alpha}{s^{1+\alpha}}\, \mathrm{d}s \;\le\; c\,\alpha\, \frac{\varepsilon}{t^\alpha}\,, \tag{A.13}$$

for some universal constant $c > 0$. Note that $\xi(x)$ is independent of $X_{N_n}^{(k)}$ if $x \notin \mathcal{B}_{N_n}$. If we are on the event $\mathcal{C}_{N_n}^c$, cf. (A.10), $X_{N_n}^{(k)} \ge (N_n)^{d/\alpha}/(\log N_n)^{2\beta/\alpha}$ for $k \le (\log N_n)^\beta$, hence

$$\mathbb{P}\Bigg(1 - \frac{1}{(\log N_n)^\gamma} < \frac{\xi(x)}{X_{N_n}^{(k)}} < \bigg(1 - \frac{1}{(\log N_n)^\gamma}\bigg)^{-1},\, \mathcal{C}_{N_n}^c\Bigg) \;\le\; c\,\alpha\, \frac{1}{(\log N_n)^\gamma}\, \frac{(\log N_n)^\alpha}{(N_n)^d}\,.$$

Recalling (A.10), it follows that

$$\begin{aligned} \mathbb{P}(\mathcal{W}_n^{(3)}) \;&\le\; \mathbb{P}(\mathcal{C}_{N_n}) + \mathbb{P}(\mathcal{W}_n^{(3)},\, \mathcal{C}_{N_n}^c) \;\le\; \frac{1}{N_n} + (\log N_{n+1})^\beta\, |\mathcal{B}_{N_{n+1}}| \cdot c\,\alpha\, \frac{(\log N_n)^{\alpha-\gamma}}{(N_n)^d} \\ &\le\; c' \left(\frac{1}{e^{n^r}} + \frac{1}{n^{r(\gamma-\alpha-\beta)}}\right)\,, \end{aligned}$$

for a suitable constant $c' > 0$. If $r(\gamma - \alpha - \beta) > 1$ we then have $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{W}_n^{(3)}) < \infty$.

The proof is completed observing that the three relations we have found, namely

$$r(\gamma - 2\beta) > 1\,, \qquad 2 - (4\beta + 2)r > 1\,, \qquad r(\gamma - \alpha - \beta) > 1\,,$$

can be satisfied at the same time. In fact, for any fixed β, we can choose $r \in (0,1)$ small enough such that the second relation holds (e.g. $r := (4\beta + 3)^{-1}$) and then choose $\gamma > 0$ large enough so that the first and the third relations are satisfied (e.g. $\gamma := 6\beta + \alpha + 3$).

Appendix B. Order statistics for the modifed field

B.1. **Proof of Lemma 2.4.** We are going to prove the following stronger result.

Lemma B.1. *For all $k \ge 2$ and $\delta \in (0,1)$ one has*

$$\mathbb{P}\left(Z_N^{(k)} \ge (1-\delta) Z_N^{(1)}\right) \le (1 - (1-\delta)^\alpha)^{k-1}\,. \tag{B.1}$$

Proof. We set $L_A := \sup_{x \in A} \psi_N(x)$ (recall (2.6)) and $A^c := \mathcal{B}_N \setminus A$ for short. We also set $\varphi_N(x) := (1 - \frac{|x|}{N+1})$, so that $\psi_N(x) = \varphi_N(x)\xi(x)$. Summing over the location of the set

$A = \{Z_N^{(1)}, \ldots, Z_N^{(k-1)}\}$, so that $Z_N^{(k)} = L_{A^c}$, we can write

$$\mathbb{P}(Z_N^{(k)} \geq (1-\delta)Z_N^{(1)}) = \sum_{A \subseteq \mathcal{B}_N,\, |A|=k-1} \mathbb{P}\left(\{Z_N^{(1)}, \ldots, Z_N^{(k-1)}\} = A,\ L_{A^c} \geq (1-\delta)Z_N^{(1)}\right)$$

$$\text{(B.2)} \qquad = \sum_{A \subseteq \mathcal{B}_N,\, |A|=k-1} \mathbb{P}\left(L_{A^c} < \psi_N(x) \leq (1-\delta)^{-1} L_{A^c}\,,\ \forall x \in A\right)$$

$$= \sum_{A \subseteq \mathcal{B}_N,\, |A|=k-1} \mathbb{P}\left((1-\delta)^\alpha \left(\frac{\varphi_N(x)}{L_{A^c}}\right)^\alpha \leq \frac{1}{\xi(x)^\alpha} < \left(\frac{\varphi_N(x)}{L_{A^c}}\right)^\alpha\,,\ \forall x \in A\right).$$

It follows from (1.2) that the variable $1/\xi(x)^\alpha$ is uniformly distributed on the interval $(0,1)$, that is, its distribution function equals $J(x) := (x \wedge 1)\mathbb{1}_{(0,\infty)}(x)$, hence

$$\mathbb{P}\left((1-\delta)^\alpha t^\alpha \leq \frac{1}{\xi(x)^\alpha} < t^\alpha\right) = J(t^\alpha) - J((1-\delta)^\alpha t^\alpha)\,.$$

One checks easily that $J((1-\delta)^\alpha t^\alpha) \geq (1-\delta)^\alpha J(t^\alpha)$ for all $\delta \in (0,1)$ and $t \geq 0$ (the inequality is strict for $t > 1$), therefore

$$\text{(B.3)} \qquad \mathbb{P}(Z_N^{(k)} \geq (1-\delta)Z_N^{(1)}) \leq (1-(1-\delta)^\alpha)^{k-1} \sum_{A \subseteq \mathcal{B}_N,\, |A|=k-1} \mathbb{E}\left[\prod_{x \in A} J\left(\frac{\varphi_N(x)^\alpha}{(L_{A^c})^\alpha}\right)\right].$$

Setting $\delta = 1$ in (B.2) we see that the sum in the right hand side of the last equation equals one, and the proof is completed. □

Remark B.2. One can refine the proof of Lemma B.1 to show that

$$\mathbb{P}\left(Z_N^{(k)} \geq (1-\delta)Z_N^{(1)}\right) \geq \left(1 - C_k\, e^{-c_k N^d}\right)(1-(1-\delta)^\alpha)^{k-1}\,,$$

for suitable constants $c_k, C_k \in (0,\infty)$ and for large N. In fact, restricting the expectations in (B.2) to the event $\{Z_N^{(k)} > 1\}$, one has $\varphi_N(x)/L_{A^c} \leq 1$ and therefore (B.3) becomes

$$\mathbb{P}\big(Z_N^{(k)} \geq (1-\delta)Z_N^{(1)},\ Z_N^{(k)} > 1\big) = (1-(1-\delta)^\alpha)^{k-1}\, \mathbb{P}(Z_N^{(k)} > 1)\,.$$

It then remains to check that $\mathbb{P}(Z_N^{(k)} > 1) \leq C_k \exp(-c_k N^d)$, which can be easily done by direct computation.

B.2. **Proof of Lemma 2.3.** As already remarked, only the first inequality in (2.7) needs to be proved, because $Z_N^{(2)} \leq Z_N^{(1)} \leq X_N^{(1)}$ (recall (2.2)). We start with an auxiliary lemma.

Lemma B.3. *There exist constants c_1, c_2 such that for all $N \in \mathbb{N}$ and $t \geq 0$*

$$\text{(B.4)} \qquad \mathbb{P}(Z_N^{(2)} \leq N^{d/\alpha}\, t) \leq c_1\, e^{-\frac{c_2}{t^\alpha}}\,.$$

Proof. Setting $O_x := \sup_{x \in \mathcal{B}_N \setminus \{x\}} \psi_N(x)$ for short, we can write

$$\mathbb{P}(Z_N^{(2)} \leq N^{d/\alpha}\, t) = \sum_{x \in \mathcal{B}_N} \mathbb{P}(O_x \leq N^{d/\alpha}\, t,\ \xi(x) > O_x)$$

$$= \sum_{x \in \mathcal{B}_N} \mathbb{P}\left(\frac{1}{O_x^\alpha} \geq \frac{1}{N^d\, t^\alpha},\ \frac{1}{\xi(x)^\alpha} < \frac{1}{O_x^\alpha}\right) \leq \sum_{x \in \mathcal{B}_N} \mathbb{E}\left(\frac{1}{O_x^\alpha}\, \mathbb{1}_{\left\{\frac{1}{O_x^\alpha} \geq \frac{1}{N^d\, t^\alpha}\right\}}\right),$$

because $1/\xi(x)^\alpha$ is uniformly distributed on the interval $(0,1)$, as it follows from (1.2). We then apply the basic formula $\mathbb{E}(Z\,1_{\{Z\geq a\}}) = a\mathbb{P}(Z\geq a) + \int_a^\infty \mathbb{P}(Z\geq s)\,\mathrm{d}s$, getting

$$\mathbb{P}(Z_N^{(2)} \leq N^{d/\alpha}\,t) \leq \frac{1}{N^d}\sum_{x\in\mathcal{B}_N}\left\{\frac{1}{t^\alpha}\,\mathbb{P}\left(\frac{1}{O_x^\alpha}\geq\frac{1}{N^d\,t^\alpha}\right) + \int_{t^{-\alpha}}^\infty \mathbb{P}\left(\frac{1}{O_x^\alpha}\geq\frac{u}{N^d}\right)\mathrm{d}u\right\}.$$

We now claim that there exists $c>0$ such that for all $N\in\mathbb{N}$, $x\in\mathcal{B}_N$ and $u>0$

$$\mathbb{P}\left(\frac{1}{O_x^\alpha}\geq\frac{u}{N^d}\right)\leq e^{-c\,u}. \tag{B.5}$$

Since $|\mathcal{B}_N|\leq CN^d$ for some constants C, we get

$$\mathbb{P}(Z_N^{(2)} \leq N^{d/\alpha}\,t) \leq \frac{C}{t^\alpha}\,e^{-c\,t^{-\alpha}} + C\int_{t^{-\alpha}}^\infty e^{-c\,u}\,\mathrm{d}u \leq e^{-c\,t^{-\alpha}}\left(\frac{C}{t^\alpha}+\frac{C}{c}\right).$$

Since the function $t\mapsto t^{-\alpha}\,e^{-\frac{1}{2}c\,t^{-\alpha}}$ is bounded on $\mathbb{R}^+$, it follows that (B.4) holds true with $c_2 := \frac{1}{2}c$ and for c_1 large enough.

It remains to prove (B.5), for which we can write

$$\begin{aligned}\mathbb{P}\left(\frac{1}{O_x^\alpha}\geq\frac{u}{N^d}\right) &= \prod_{z\in\mathcal{B}_N\setminus\{x\}}\mathbb{P}\left(\frac{1}{\xi(z)^\alpha}\geq\frac{\big(1-\frac{|z|}{N+1}\big)^\alpha}{N^d}u\right)\\ &\leq \exp\left(\frac{u}{N^d}\sum_{z\in\mathcal{B}_N\setminus\{x\}}\big(1-\tfrac{|z|}{N+1}\big)^\alpha\right),\end{aligned}$$

because $\mathbb{P}(1/\xi(z)^\alpha\geq a) = 1-a\leq e^{-a}$ for $a\in[0,1]$ (recall (1.2)). By a Riemann sum approximation, as $N\to\infty$ one has

$$\frac{1}{N^d}\sum_{z\in\mathcal{B}_N\setminus\{x\}}\big(1-\tfrac{|z|}{N+1}\big)^\alpha \longrightarrow \int_{|y|\leq 1}(1-|y|)^\alpha\,\mathrm{d}y \in (0,\infty),$$

from which it follows that (B.5) holds true for some $c>0$. □

Proof of Lemma 2.3. Thanks to the inequality (B.4), the proof is identical to that of the lower bound in (2.2), cf. Appendix A.2. More precisely, one first shows, through a standard Borel-Cantelli argument, that the first inequality in (2.7) (with ε replaces by $\varepsilon/2$, say) holds along the subsequence $N_k := 2^k$; the extension to all values of N then follows easily, because $Z_N^{(2)}$ is increasing in N. We omit the details for conciseness. □

B.3. **Further results.** It may be useful to observe that if $z_{N+1}^{(1)}\neq z_N^{(1)}$ then

$$|z_{N+1}^{(1)}| > |z_N^{(1)}| \qquad\text{and}\qquad \xi(z_{N+1}^{(1)}) > \xi(z_N^{(1)}). \tag{B.6}$$

In fact, when $z_{N+1}^{(1)}\neq z_N^{(1)}$ we have by definition

$$Z_N^{(1)} = \psi_N(z_N^{(1)}) > \psi_N(z_{N+1}^{(1)}), \qquad \psi_{N+1}(z_N^{(1)}) < \psi_{N+1}(z_{N+1}^{(1)}) = Z_{N+1}^{(1)}, \tag{B.7}$$

from which we obtain, recalling the definition (2.6) of ψ_N,

$$\frac{|z_N^{(1)}|\,\xi(z_N^{(1)})}{(N+1)(N+2)} = \psi_{N+1}(z_N^{(1)}) - \psi_N(z_N^{(1)}) < \psi_{N+1}(z_{N+1}^{(1)}) - \psi_N(z_{N+1}^{(1)}) = \frac{|z_{N+1}^{(1)}|\,\xi(z_{N+1}^{(1)})}{(N+1)(N+2)},$$

hence $|z_N^{(1)}|\,\xi(z_N^{(1)}) < |z_{N+1}^{(1)}|\,\xi(z_{N+1}^{(1)})$. This shows that at least one of the two inequalities in (B.6) must hold. Two cases remain that need to be excluded:

- if $|z_{N+1}^{(1)}| \le |z_N^{(1)}|$ and $\xi(z_{N+1}^{(1)}) > \xi(z_N^{(1)})$, then
$$\psi_N(z_{N+1}^{(1)}) = \left(1 - \frac{|z_{N+1}^{(1)}|}{N+1}\right)\xi(z_{N+1}^{(1)}) > \left(1 - \frac{|z_N^{(1)}|}{N+1}\right)\xi(z_N^{(1)}) = \psi_N(z_N^{(1)}) = Z_N^{(1)}\,,$$
which is absurd, because $Z_N^{(1)}$ is by definition the maximum of ψ_N;
- analogously, if $|z_{N+1}^{(1)}| > |z_N^{(1)}|$ and $\xi(z_{N+1}^{(1)}) \le \xi(z_N^{(1)})$, then
$$Z_{N+1}^{(1)} = \psi_{N+1}(z_{N+1}^{(1)}) = \left(1 - \frac{|z_{N+1}^{(1)}|}{N+2}\right)\xi(z_{N+1}^{(1)}) < \left(1 - \frac{|z_N^{(1)}|}{N+2}\right)\xi(z_N^{(1)}) = \psi_{N+1}(z_N^{(1)})\,,$$
which is again absurd, because $Z_{N+1}^{(1)}$ is by definition the maximum of ψ_{N+1}.

Next we show that a statement analogous to (2.9) for the gap $Z_N^{(1)} - Z_N^{(2)}$ *does not hold*. Let us fix any $\bar N$ for which $z_{\bar N}^{(1)} \neq z_{\bar N+1}^{(1)}$ (note that there are almost surely infinitely many such values of $\bar N$, otherwise $Z_N^{(1)} = \psi_N(z_N^{(1)})$ would be eventually constant). We set $x := z_{\bar N}^{(1)}$ and $y := z_{\bar N+1}^{(1)}$ for short. Then $Z_{\bar N}^{(1)} = \psi_N(x)$ and $Z_{\bar N}^{(2)} \ge \psi_N(y)$, hence, recalling (2.6),
$$\begin{aligned} Z_{\bar N}^{(1)} - Z_{\bar N}^{(2)} \le \psi_{\bar N}(x) - \psi_{\bar N}(y) &= \frac{1}{\bar N+1}\Big((\bar N+1)\big(\xi(x)-\xi(y)\big) + |y|\xi(y) - |x|\xi(x)\Big) \\ &= \frac{1}{\bar N+1}\Big((\bar N+2)\big(\xi(x)-\xi(y)\big) + |y|\xi(y) - |x|\xi(x)\Big) \;+\; \frac{\xi(y)-\xi(x)}{\bar N+1} \\ &= \frac{\bar N+2}{\bar N+1}\Big(\psi_{\bar N+1}(x) - \psi_{\bar N+1}(y)\Big) \;+\; \frac{\xi(y)-\xi(x)}{\bar N+1}\,. \end{aligned}$$
By construction $y = z_{\bar N+1}^{(1)}$ and $y \neq x$, therefore $\psi_{\bar N+1}(y) = Z_{\bar N+1}^{(1)} > \psi_{\bar N+1}(x)$. Recalling (2.2), we infer that eventually $\mathbb{P}$-a.s.
$$\text{(B.8)} \qquad Z_{\bar N}^{(1)} - Z_{\bar N}^{(2)} \le \frac{\xi(z_{\bar N+1}^{(1)}) - \xi(z_{\bar N}^{(1)})}{\bar N+1} \le \frac{X_{\bar N+1}^{(1)}}{\bar N+1} \le \bar N^{d/\alpha-1}\,(\log \bar N)^{1/\alpha+\varepsilon}\,.$$
We stress that this bound differs from the one in (2.9) almost by a factor N^{-1}. It turns out that the bound (B.8) is quite sharp (up to logarithmic corrections): in fact, by the first bound in (2.8), (2.7) and a Borel-Cantelli argument, it follows that for every $\varepsilon > 0$, eventually $\mathbb{P}$-almost surely,
$$\text{(B.9)} \qquad Z_N^{(1)} - Z_N^{(2)} \ge \frac{Z_N^{(1)}}{N(\log N)^{1+\varepsilon/2}} \ge \frac{N^{d/\alpha-1}}{(\log N)^{1+\varepsilon}}\,.$$
This implies in particular that $N(Z_N^{(1)} - Z_N^{(2)}) \to +\infty$, $\mathbb{P}$-almost surely.

Appendix C. Proof of (1.17) in Remark 1.6

We want to prove, for $d = 1$, that
$$\text{(C.1)} \qquad \mathbb{P}\left(\mathrm{w}_{N,\xi} = z_{N,\xi}^{(2)} \text{ for infinitely many } N\right) \;=\; 1\,.$$
To simplify notation, we only consider the case $\alpha > 1$ and we set $m_\alpha = \mathbb{E}(\xi_1) = \alpha/(\alpha-1)$, cf. (1.2). Recalling (1.1), we set $\kappa = \mathrm{P}(S_1 = 0)$ and $\hat\kappa = \mathrm{P}(S_1 = 1)$. For the sake of simplicity,

we also assume that $\log(\hat\kappa/\kappa) > -m_\alpha$. The cases $\log(\hat\kappa/\kappa) < -m_\alpha$ and $\log(\hat\kappa/\kappa) = -m_\alpha$ are controlled with analogous arguments.

For $\alpha, \eta > 0$ and for $n \in \mathbb{N}$ and $\varepsilon > 0$ we define the event $B_{\varepsilon,\eta,n} \subseteq \Omega_\xi$ by

$$\begin{aligned} B_{\varepsilon,\eta,n} := \big\{ &\exists! x \in [n, (1+\varepsilon)n] : \xi(x) \in (1, 1+\varepsilon) n^{1/\alpha}, \\ &\exists! y \in [3n, (1+\varepsilon)3n] : \xi(y) \in (1, 1+\varepsilon)\tfrac{5}{3} n^{1/\alpha}, \\ &\forall z \in [-7n, (1+\varepsilon)7n] \setminus \{x,y\} : \xi(z) < \tfrac{1}{2} n^{1/\alpha}, \\ &\textstyle\sum_{i=x+1}^{y-1} \xi(i) > (m_\alpha - \eta)(y-x) \big\}, \end{aligned} \tag{C.2}$$

where we set $[a,b] := [a,b] \cap \mathbb{Z}$ for short. By direct computation, one checks easily that $\lim_{n\to\infty} \mathbb{P}(B_{\varepsilon,\eta,n}) > 0$ for all fixed $\varepsilon, \eta > 0$. In what follows, we denote by x, y the (random) points appearing in the definition of $B_{\varepsilon,\eta,n}$.

Recalling (2.6), for all $N \in \mathbb{N}$ we have

$$\begin{aligned} \big(1 - \tfrac{n}{N}(1+\varepsilon)\big) n^{1/\alpha} \;<\; \psi_N(x) \;<\; \big(1 - \tfrac{n}{N}\big)(1+\varepsilon) n^{1/\alpha}, \\ \big(1 - \tfrac{3n}{N}(1+\varepsilon)\big) \tfrac{5}{3} n^{1/\alpha} \;<\; \psi_N(y) \;<\; \big(1 - \tfrac{3n}{N}\big)(1+\varepsilon)\tfrac{5}{3} n^{1/\alpha}. \end{aligned} \tag{C.3}$$

It follows that for all $N \in [\frac{11}{2}n, \frac{13}{2}n]$ we have $\psi_N(x) > (1 - \frac{1+\varepsilon}{11/2}) n^{1/\alpha} = (\frac{9}{11} + O(\varepsilon)) n^{1/\alpha}$, $\psi_N(y) > (1 - \frac{3(1+\varepsilon)}{11/2}) \frac{5}{3} n^{1/\alpha} = (\frac{25}{33} + O(\varepsilon)) n^{1/\alpha}$, while $\psi_N(z) < \frac{1}{2} n^{1/\alpha}$ for all $z \in [-N, N] \setminus \{x, y\}$. Therefore, by choosing ε small enough, we can state that on the event $B_{\varepsilon,\eta,n}$ and for all $n \in \mathbb{N}$, $N \in [\frac{11}{2}n, \frac{13}{2}n]$ it comes

$$\{z_N^{(1)}, z_N^{(2)}\} = \{x, y\}. \tag{C.4}$$

For $N = \frac{11}{2}n$ we have $\psi_N(x) = (\frac{9}{11} + O(\varepsilon)) n^{1/\alpha}$ and $\psi_N(y) = (\frac{25}{33} + O(\varepsilon)) n^{1/\alpha}$, uniformly in n; on the other hand, for $N = \frac{13}{2}n$ we have $\psi_N(x) = (\frac{11}{13} + O(\varepsilon)) n^{1/\alpha}$ and $\psi_N(y) = (\frac{35}{39} + O(\varepsilon)) n^{1/\alpha}$, always uniformly in n. It follows that, if $\varepsilon > 0$ is chosen small enough we have for all $n \in \mathbb{N}$,

$$\psi_{\frac{11}{2}n}(y) - \psi_{\frac{11}{2}n}(x) < 0 \quad \text{but} \quad \psi_{\frac{13}{2}n}(y) - \psi_{\frac{13}{2}n}(x) > 0. \tag{C.5}$$

At this stage, we pick $\varepsilon_0 > 0$ such that (C.4) and (C.5) are satisfied. Next observe that

$$(N+1)(\psi_N(y) - \psi_N(x)) = x\xi(x) - y\xi(y) + (N+1)(\xi(y) - \xi(x)) \tag{C.6}$$

is increasing in N, because by construction $(\xi(y) - \xi(x)) > 0$. It follows that there is $N_n^* \in (\frac{11}{2}n, \frac{13}{2}n)$ such that:

- for $\frac{11}{2}n < N \le N_n^*$ we have $(\psi_N(y) - \psi_N(x)) < 0$, hence $x = z_N^{(1)}$ and $y = z_N^{(2)}$;
- for $N_n^* < N < \frac{13}{2}n$ we have $(\psi_N(y) - \psi_N(x)) > 0$, hence $x = z_N^{(2)}$ and $y = z_N^{(1)}$.

By (C.6) $(N+1)(\psi_N(y) - \psi_N(x))$ increases by $(\xi(y) - \xi(x))$ when N increases by 1. Since $(N_n^* + 1)(\psi_{N_n^*}(y) - \psi_{N_n^*}(x)) < 0$ and $((N_n^* + 1) + 1)(\psi_{N_n^*+1}(y) - \psi_{N_n^*+1}(x)) > 0$, it then follows that $(N_n^* + 1)(\psi_{N_n^*}(y) - \psi_{N_n^*}(x)) > -(\xi(y) - \xi(x))$, that is

$$\begin{aligned} (N_n^* + 1)(Z_{N_n^*}^{(1)} - Z_{N_n^*}^{(2)}) &= (N_n^* + 1)(\psi_{N_n^*}(x) - \psi_{N_n^*}(y)) \le (\xi(y) - \xi(x)) \\ &\le \left(\frac{2}{3} + \frac{5}{3}\varepsilon_0\right) n^{1/\alpha} =: c_0\, n^{1/\alpha}, \end{aligned} \tag{C.7}$$

by the definition of the event $B_{\varepsilon_0,\eta,n}$.

Consider now the contributions of the two N-steps random walk trajectories $\mathcal{S}^{(N,x)}$ and $\mathcal{S}^{(N,y)}$ that reach respectively x and y in the minimal number of steps and stick there until time N, i.e.,

$$\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,x)}) = \kappa^N e^{\sum_{i=1}^{x-1}\xi(i)+(N+1)\psi_N(x)+x\log(\frac{\hat\kappa}{\kappa})},$$
$$\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,y)}) = \kappa^N e^{\sum_{i=1}^{y-1}\xi(i)+(N+1)\psi_N(y)+y\log(\frac{\hat\kappa}{\kappa})},$$

so that

$$\frac{\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,y)})}{\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,x)})} = e^{\sum_{i=x}^{y-1}\xi(i)-(N+1)(\psi_N(x)-\psi_N(y))+(y-x)\log(\frac{\hat\kappa}{\kappa})}. \tag{C.8}$$

We apply this relation for $N = N_n^*$ on the event $\mathcal{B}_{\varepsilon_0,\eta_0,n}$, with $2\,\eta_0 := m_\alpha + \log(\hat\kappa/\kappa)$ (which is strictly positive, by our initial assumtion). Then $x = z^{(1)}_{N_n^*}$, $y = z^{(2)}_{N_n^*}$ and (C.8) becomes

$$\begin{aligned}\frac{\mathbf{P}_{N_n^*,\xi}\big(\mathcal{S}^{(N_n^*,z^{(2)}_{N_n^*})}\big)}{\mathbf{P}_{N_n^*,\xi}\big(\mathcal{S}^{(N_n^*,z^{(1)}_{N_n^*})}\big)} &\geq e^{\sum_{i=x+1}^{y-1}\xi(i)-(N_n^*+1)(Z^{(1)}_{N_n^*}-Z^{(2)}_{N_n^*})+(y-x)\log(\frac{\hat\kappa}{\kappa})}, \\ &\geq e^{(m_\alpha-\eta+\log(\frac{\hat\kappa}{\kappa}))(y-x)-c_0 n^{1/\alpha}} \geq e^{\eta n - c_0 n^{1/\alpha}},\end{aligned} \tag{C.9}$$

where we have used (C.7), the last condition in (C.2) and the fact that $y-x\geq n$, again by (C.2). Since $\alpha>1$ by assumption, we have shown that on the event $\mathcal{B}_{\varepsilon_0,\eta_0,n}$

$$\mathbf{P}_{N_n^*,\xi}(\mathcal{S}^{(N_n^*,z^{(2)}_{N_n^*})}) \gg \mathbf{P}_{N_n^*,\xi}(\mathcal{S}^{(N_n^*,z^{(1)}_{N_n^*})})\,.$$

If $\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(2)}_N)})+\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(1)}_N)}) > \frac{3}{4}$, this shows that $\mathrm{w}_{N,\xi} = z^{(2)}_N$. To sum up, there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$

$$\begin{aligned}B_{\varepsilon_0,\eta_0,n} \subseteq \{&\exists N\in(\tfrac{11}{2}n,\tfrac{13}{2}n):\ \mathrm{w}_{N,\xi}=z^{(2)}_N\} \\ &\cup\big\{\exists N\in(\tfrac{11}{2}n,\,\tfrac{13}{2}n)\colon \mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(2)}_N)})+\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(1)}_N)})\leq\tfrac{3}{4}\big\}\,.\end{aligned}$$

Recalling that $\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(2)}_N)})+\mathbf{P}_{N,\xi}(\mathcal{S}^{(N,z^{(1)}_N)})\to 1$ as $N\to\infty$, $\mathbb{P}(\mathrm{d}\xi)$-almost surely when $d=1$, cf. Remark 1.8, it follows that almost surely

$$\limsup_{n\to\infty} B_{\varepsilon_0,\eta_0,n} := \{B_{\varepsilon_0,\eta_0,n} \text{ for infinitely many } n\} \subseteq \{\mathrm{w}_{N,\xi}=z^{(2)}_N \text{ for infinitely many } N\}\,.$$

Finally, note that $\mathbb{P}(\limsup_{n\to\infty} B_{\varepsilon_0,\eta_0,n}) \geq \lim_{n\to\infty}\mathbb{P}(B_{\varepsilon_0,\eta_0,n})>0$, and it is not difficult to realize that indeed $\mathbb{P}(\limsup_{n\to\infty} B_{\varepsilon_0,\eta_0,n})=1$, because when $m\gg n$ the event $B_{\varepsilon_0,\eta_0,m}$ is asymptotically independent of $B_{\varepsilon_0,\eta_0,n}$. This completes the proof.

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