THE “ABRACADABRA” PROBLEM

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Abstract. We present a detailed solution of Exercise E10.6 in [Wil91]: in a random sequence of letters, drawn independently and uniformly from the English alphabet, the expected time for the first appearance of the word “ABRACADABRA” is $26^{11} + 26^4 + 26$.

We adopt the conventions $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

1. Formulation of the problem

Let $(U_i)_{i\in\mathbb{N}}$ denote random letters drawn independently and uniformly from the English alphabet. More precisely, we assume that $(U_i)_{i\in\mathbb{N}}$ are independent and identically distributed random variables, uniformly distributed in the set $\mathtt{E} := \{\text{A, B, C, D}, \dots, \text{X, Y, Z}\}$, defined on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$. For $m, n \in \mathbb{N}$ with $m \le n$, we use $U_{[m,n]}$ as a shortcut for the vector $(U_m, U_{m+1}, \dots, U_n)$.

Define τ as the random time in which the word “ABRACADABRA” first appears:

$$\tau := \min\{n \in \mathbb{N},\ n \ge 11 :\ U_{[n-10,n]} = \text{ABRACADABRA}\}\,, \tag{1.1}$$

with the convention $\min \emptyset := +\infty$. Our goal is to prove the following result.

Theorem 1. $\mathrm{E}[\tau] = 26^{11} + 26^4 + 26$.

2. Strategy

The proof is based on martingales. Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be the natural filtration of $(U_i)_{i\in\mathbb{N}}$, i.e., $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(U_1, \dots, U_n)$. We are going to prove the following results.

Proposition 2. *τ is a stopping time with $\mathrm{E}[\tau] < \infty$.*

Proposition 3. *There exists a martingale $M = (M_n)_{n\in\mathbb{N}_0}$ such that:*

(1) *$M_0 = 0$ and $M_\tau = 26^{11} + 26^4 + 26 - \tau$;*

(2) *M has bounded increments: $\exists C \in (0, \infty)$ such that $|M_{n+1} - M_n| \le C$, for all $n \in \mathbb{N}_0$.*

Let us recall (a special case of) Doob’s optional sampling theorem, cf. [Wil91, §10.10].

Theorem 4. *If $M = (M_n)_{n\in\mathbb{N}_0}$ is a martingale with bounded increments and τ is a stopping time with finite mean, then $\mathrm{E}[M_\tau] = \mathrm{E}[M_0]$.*

Combining this with Propositions 2 and 3, one obtains immediately the proof of Theorem 1.

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3. Proof of Proposition 2

Recall that τ is a stopping time if and only if $\{\tau \le n\} \in \mathcal{F}_n$ for every $n \in \mathbb{N}_0$. Note that $\{\tau \le n\} = \emptyset$ if $n \le 10$, while for $n \ge 11$

$$\{\tau \le n\} = \bigcup_{i=11}^{n} \{U_{[i-10,i]} = \text{ABRACADABRA}\},$$

which shows that the event $\{\tau \le n\}$ is in $\mathcal{F}_n$ (it is expressed as a function of $U_1, U_2, \ldots, U_n$).

To prove that $\mathrm{E}[\tau] < \infty$, we argue as in [Wil91, §10.11; Exercise E10.5].

Lemma 5. *For a positive random variable τ, in order to have $\mathrm{E}[\tau] < \infty$ it is sufficient that*

$$\exists N \in \mathbb{N},\ \varepsilon > 0 : \qquad \mathrm{P}(\tau \le n + N \,|\, \tau > n) \ge \varepsilon \qquad \forall n \in \mathbb{N}_0. \tag{3.1}$$

This result is proved in Appendix A below. In order to apply it, let A_n be the event that the word "ABRACADABRA" appears (not necessarily for the first time!) at time $n + 11$:

$$A_n := \{U_{[n+1,n+11]} = \text{ABRACADABRA}\}.$$

By assumption U_i are independent and uniformly chosen letters, hence

$$\mathrm{P}(A_n) = p^{11} > 0, \qquad \text{where} \qquad p := \frac{1}{\mathtt{E}} = \frac{1}{26}.$$

Since $A_n \subseteq \{\tau \le n + 11\}$, we have $\mathrm{P}(\tau \le n + 11 \,|\, \tau > n) \ge \mathrm{P}(A_n \,|\, \tau > n)$ for every $n \in \mathbb{N}$. However the events A_n and $\{\tau > n\}$ are *independent* (A_n is a function of $U_{[n+1,n+11]}$, while $\{\tau > n\} = \{\tau \le n\}^c \in \mathcal{F}_n$ is a function of $U_{[1,n]}$), hence $\mathrm{P}(A_n \,|\, \tau > n) = \mathrm{P}(A_n)$. Thus

$$\mathrm{P}(\tau \le n + 11 \,|\, \tau > n) \ge \mathrm{P}(A_n) = p^{11},$$

i.e. relation (3.1) holds with $N = 11$ and $\varepsilon = p^{11}$. It follows by Lemma 5 that $\mathrm{E}[\tau] < \infty$. □

4. Proof of Proposition 3

The required martingale $M = (M_n)_{n \in \mathbb{N}_0}$ will be constructed as the total net gain of a suitable family of gamblers, built as follows.

At time 0 a first gambler enters the game, with an initial capital of 1€. She bets on the event that the first letter U_1 is A (the first letter of the word "ABRACADABRA"). If she loses, her capital at time 1 drops to 0€ and she stops playing (i.e. her capital will stay 0€ at all later times). On the other hand, if she wins, her capital at time 1 becomes 26€ and she goes on playing, betting on the event that the second letter U_2 is B (the second letter of "ABRACADABRA"). If she loses, her capital at time 2 is 0€ and she stops playing, while if she wins, her capital at time 2 equals $(26)^2$€ and she goes on, betting on the event that the third letter U_3 is R (the third letter of "ABRACADABRA"), and so on, until time 11.

The gambler's capital at time 11 is either $(26)^{11}$€, if the letters $U_{[1,11]}$ have formed exactly the word "ABRACADABRA", or 0€ otherwise. In any case, the gambler stops playing after time 11, hence her capital will stay constant at all later times.

Let us denote by x_i be the i-th letter of the word "ABRACADABRA", for $1 \le i \le 11$ (so that $x_1 = \text{A}$, $x_2 = \text{B}$, $x_3 = \text{R}$, ..., $x_{11} = A$). The capital (in €) of this first gambler at time n is then given by the process $(K_n)_{n \in \mathbb{N}_0}$ defined as follows:

$$K_n := \begin{cases} 1 & \text{if } n = 0 \\ K_{n-1} \cdot 26\, \mathbb{1}_{\{U_n = x_n\}} & \text{if } 1 \le n \le 11 \\ K_{11} & \text{if } n \ge 12 \end{cases}.$$

(Note that if $K_{n-1} = 0$, then $K_n = 0$ irrespectively of U_n, as described above.)

Now a second gambler arrives, playing the same game, but with one time unit of delay. Her initial capital stays 1€ at time 0 and at time 1, then she bets on the event that $U_2 = x_1 = \mathrm{A}$: if she loses, her capital at time 2 is 0€ and she stops playing, while if she wins, her capital at time 2 is 26€ and she goes on playing, betting on the event that $U_3 = x_2 = \mathrm{B}$, etc. At time 12, the second gambler's capital will be either $(26)^{11}$€ or 0€, according to whether the letters $U_{[2,12]}$ have formed precisely the word "ABRACADABRA" or not. At this point she stops playing and her capital stays constant at all later times.

Generalizing the picture, imagine that for each $j \in \mathbb{N}$ there is a j-th gambler with an initial capital of 1€, who starts playing just before time j, betting on the event that $U_j = x_1$, then (if she wins) on $U_{j+1} = x_2$, ..., and finally (if she has won all the previous bets) on $U_{j+10} = x_{11}$. After time $j + 10$ the gambler stops playing and her capital stays constant.

Denoting by $K_n^{(j)}$ the capital (in €) of the j-th gambler at time n, for $n \in \mathbb{N}_0$, we have

$$K_n^{(j)} := \begin{cases} 1 & \text{if } n < j \\ K_{n-1}^{(j)} \cdot 26\,\mathbb{1}_{\{U_n = x_{(n-j)+1}\}} & \text{if } j \leq n \leq j+10 \\ K_{j+10}^{(j)} & \text{if } n > j+10 \end{cases} . \tag{4.1}$$

We can finally define the process we are looking for, that will be shown to be a martingale:

$$M_0 := 0, \qquad M_n := \sum_{j=1}^{n} (K_n^{(j)} - K_0^{(j)}) = \left(\sum_{j=1}^{n} K_n^{(j)} \right) - n. \tag{4.2}$$

Thus M_n is the sum of the *net gains* $K_n^{(j)} - K_0^{(j)}$ *of the first* n *gamblers* at time n.[†] For the equality in (4.2), recall that $K_0^{(j)} = 1$ for all $j \in \mathbb{N}$.

Lemma 6. *For* τ *defined as in* (1.1)*, one has* $M_\tau = (26)^{11} + (26)^4 + 26 - \tau$.

Proof. We need to evaluate

$$M_\tau = \left(\sum_{j=1}^{\tau} K_\tau^{(j)} \right) - \tau.$$

Recall that $K_\tau^{(j)}$ is the capital at time τ of the gambler who starts betting just before time j. It suffices to show that $K_\tau^{(j)} = 0$ except for $j \in \{\tau - 10, \tau - 3, \tau\}$, for which

$$K_\tau^{(\tau-10)} = (26)^{11}, \qquad K_\tau^{(\tau-3)} = (26)^4, \qquad K_\tau^{(\tau)} = 26.$$

Since the complete word "ABRACADABRA" appears at time τ, the gambler who started playing just before time $\tau - 10$ has a capital of $(26)^{11}$, i.e. $K_\tau^{(\tau-10)} = (26)^{11}$. The gambler who started playing just before time $\tau - 3$ has a capital $K_\tau^{(\tau-3)} = (26)^4$, because the *last* four letters of "ABRACADABRA" are "ABRA" and coincide with the *first* four letters of that word. Analogously, since the last letter "A" is the same as the first letter, the gambler who started playing just before time τ has won his first bet and his capital is $K_\tau^{(\tau)} = 26$.

Finally, for all $j \notin \{\tau - 10, \tau - 3, \tau\}$ all gamblers have lost at least one bet and their capital is $K_\tau^{(j)} = 0$, because τ is the *first* time the word "ABRACADABRA" appears. □

[†]We could have equivalently summed the net gains of *all* gamblers, defining $M_n := \sum_{j=1}^{\infty} (K_n^{(j)} - K_0^{(j)})$, because $K_n^{(j)} = K_0^{(j)}$ for $j > n$.

To complete the proof of Proposition 3, it remains to show that $M = (M_n)_{n\in\mathbb{N}_0}$ is a martingale with bounded increments. We start looking at the capital processes.

Lemma 7. *For every fixed $j \in \mathbb{N}$, the capital process $(K_n^{(j)})_{n\in\mathbb{N}_0}$ is a martingale.*

Proof. We argue for fixed $j \in \mathbb{N}$. Plainly, $K_0^{(j)} = 1$ is $\mathcal{F}_0$-measurable. By (4.1), $K_n^{(j)}$ is a measurable function of $K_{n-1}^{(j)}$ and U_n, assuming inductively that $K_{n-1}^{(j)}$ is $\mathcal{F}_{n-1}$-measurable, it follows that $K_n^{(j)}$ is $\mathcal{F}_n$-measurable. This shows that $(K_n^{(j)})_{n\in\mathbb{N}_0}$ is an adapted process.

Since $|K_n^{(j)}| \le 26|K_{n-1}^{(j)}|$ by (4.1), it follows inductively that $|K_n^{(j)}| \le 26^n$ for all $n \in \mathbb{N}$, hence the random variables $|K_n^{(j)}|$ are bounded (and, in particular, integrable).

Finally, the relation $\mathrm{E}[K_n^{(j)}|\mathcal{F}_{n-1}] = K_{n-1}^{(j)}$ is trivially satisfied if $n < j$ or if $n > j+10$, while for $n \in \{j, \ldots, j+10\}$, again by (4.1),

$$\mathrm{E}[K_n^{(j)}|\mathcal{F}_{n-1}] = \mathrm{E}[K_{n-1}^{(j)} \cdot 26\,\mathbb{1}_{\{U_n = x_{(n-j)+1}\}}|\mathcal{F}_{n-1}] = K_{n-1}^{(j)} \cdot 26\,\mathrm{P}(U_n = x_{(n-j)+1}) = K_{n-1}^{(j)},$$

because U_n is independent of $\mathcal{F}_{n-1}$ and $\mathrm{P}(U_n = a) = \frac{1}{26}$ for every $a \in \mathtt{E}$. □

Lemma 8. *The capital processes $(K_n^{(j)})_{n\in\mathbb{N}_0}$ have uniformly bounded increments:*

$$|K_n^{(j)} - K_{n-1}^{(j)}| \le 25^{11}, \qquad \forall j, n \in \mathbb{N}. \tag{4.3}$$

Proof. One has $|K_n^{(j)} - K_{n-1}^{(j)}| = 0$ if $n < j$ or $n > j+10$, by (4.1), while for $n \in \{j, \ldots, j+10\}$

$$|K_n^{(j)} - K_{n-1}^{(j)}| \le |26\,\mathbb{1}_{\{U_n = x_{(n-j)+1}\}} - 1||K_{n-1}^{(j)}| \le 25\,|K_{n-1}^{(j)}|\,.$$

Since $K_{j-1}^{(j)} = 1$, relation (4.3) follows. □

We can finally show that M is a martingale. Note that M_n is $\mathcal{F}_n$-measurable and in L^1, for every $n \in \mathbb{N}$, because by (4.2) M_n is a finite sum of $K_n^{(j)}$, each of which is $\mathcal{F}_n$-measurable and in L^1 by Lemma 7. Furthermore, again by (4.2), for all $n \in \mathbb{N}$ we can write

$$\mathrm{E}[M_n|\mathcal{F}_{n-1}] = \sum_{j=1}^{n} \mathrm{E}[K_n^{(j)}|\mathcal{F}_{n-1}] - n = \sum_{j=1}^{n} K_{n-1}^{(j)} - n.$$

However for $j = n$ we have $K_{n-1}^{(j)} = K_{n-1}^{(n)} = 1$ by definition, cf. (4.1), hence

$$\mathrm{E}[M_n|\mathcal{F}_{n-1}] = \sum_{j=1}^{n-1} K_{n-1}^{(j)} + 1 - n = \sum_{j=1}^{n-1} K_{n-1}^{(j)} - (n-1) = M_{n-1}.$$

This shows that M is a martingale. Finally, for all $n \in \mathbb{N}$

$$M_n - M_{n-1} = \sum_{j=1}^{n} K_n^{(j)} - n - \left(\sum_{j=1}^{n-1} K_{n-1}^{(j)} - (n-1)\right) = \sum_{j=1}^{n} \left(K_n^{(j)} - K_{n-1}^{(j)}\right),$$

again because for $j = n$ we have $K_{n-1}^{(j)} = K_{n-1}^{(n)} = 1$. Now observe that, again by (4.1), for $n \ge j + 11$ one has $K_n^{(j)} = K_{n-1}^{(j)} = K_{j+10}^{(j)}$, hence

$$|M_n - M_{n-1}| = \left| \sum_{j=n-10}^{n} \left(K_n^{(j)} - K_{n-1}^{(j)}\right) \right| \le \sum_{j=n-10}^{n} \left|K_n^{(j)} - K_{n-1}^{(j)}\right| \le 11 \cdot 25^{11},$$

having applied (4.3). This shows that M has bounded increments, completing the proof. □

Appendix A. Proof of Lemma 5

The assumptions imply that

$$\mathrm{P}(\tau > \ell N) \le (1-\varepsilon)^\ell \qquad \forall \ell \in \mathbb{N}_0 \,, \tag{A.1}$$

as we show below. We are going to use the formula

$$\mathrm{E}[\tau] = \int_0^\infty \mathrm{P}(\tau > t)\, \mathrm{d}t, \tag{A.2}$$

valid for every random variable τ taking values in $[0,\infty]$.† Breaking up the integral in the sub-intervals $[\ell N, (\ell+1)N]$, with $\ell \in \mathbb{N}_0$, since $\mathrm{P}(\tau > t) \le \mathrm{P}(\tau > \ell N)$ for $t \ge \ell N$, we get

$$\begin{aligned} \mathrm{E}[\tau] &= \sum_{\ell \in \mathbb{N}_0} \int_{\ell N}^{(\ell+1)N} \mathrm{P}(\tau > t)\, \mathrm{d}t \le \sum_{\ell \in \mathbb{N}_0} \mathrm{P}(\tau > \ell N) \int_{\ell N}^{(\ell+1)N} 1\, \mathrm{d}t \le \sum_{\ell \in \mathbb{N}_0} (1-\varepsilon)^\ell N \\ &= \frac{N}{1-(1-\varepsilon)} = \frac{N}{\varepsilon} < \infty \,, \end{aligned}$$

having applied the geometric series $\sum_{n \in \mathbb{N}_0} q^n = \frac{1}{1-q}$. This shows that $\mathrm{E}[\tau] < \infty$, as required.

It remains to prove (A.1), which we do by induction. For $\ell = 0$ there is nothing to prove. For every $\ell \in \mathbb{N}_0$, since $\{\tau > (\ell+1)N\} \subseteq \{\tau > \ell N\}$, we can write

$$\begin{aligned} \mathrm{P}(\tau > (\ell+1)N) &= \mathrm{P}(\tau > (\ell+1)N,\ \tau > \ell N) \\ &= \mathrm{P}(\tau > \ell N)\, \mathrm{P}(\tau > (\ell+1)N \,|\, \tau > \ell N) \,. \end{aligned} \tag{A.3}$$

The induction step yields $\mathrm{P}(\tau > \ell N) \le (1-\varepsilon)\ell$, while assumption (3.1) gives

$$\mathrm{P}(\tau > n + N \,|\, \tau > n) \le (1-\varepsilon) \,, \qquad \forall n \in \mathbb{N} \,.$$

Choosing $n = \ell N$ yields $\mathrm{P}(\tau > (\ell+1)N \,|\, \tau > \ell N) \le (1-\varepsilon)$, which plugged into (A.3) yields $\mathrm{P}(\tau > (\ell+1)N) \le (1-\varepsilon)^{\ell+1}$, as required. □

References

[Wil91] D. Williams (1991), *Probability with martingales*, Cambridge University Press

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† For every $T \in [0,\infty]$ one has $T = \int_0^T 1\, \mathrm{d}t = \int_0^\infty \mathbb{1}_{\{T \ge t\}}\, \mathrm{d}t$, hence $\tau(\omega) = \int_0^\infty \mathbb{1}_{\{\tau(\omega) > t\}}\, \mathrm{d}t$ for every random variable τ taking values in $[0,\infty]$. Taking expectations of both sides and exchanging the expectation with the integral (which is justified by Fubini-Tonelli, thanks to positivity) one obtains (A.2).