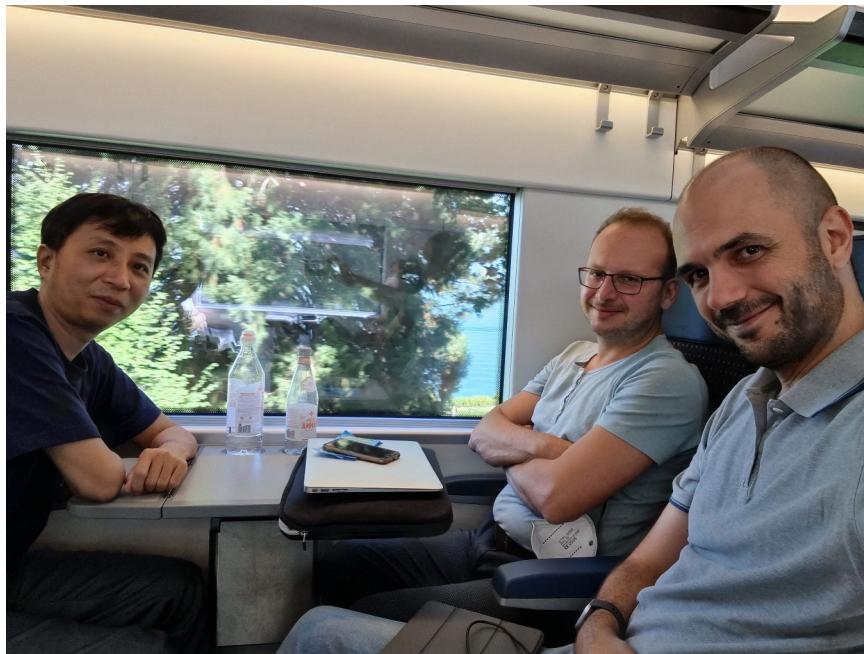


The critical 2d Stochastic Heat Flow

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Based on joint works with



Rongfeng Sun and Nikos Zygouras

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I. INTRODUCTION

THE STOCHASTIC HEAT EQUATION

$$(SHE) \quad \begin{cases} \partial_t U(t,x) = \Delta U(t,x) + \beta \xi(t,x) U(t,x) \\ U(0,x) \equiv 1 \end{cases} \quad t > 0, x \in \mathbb{R}^d$$

- $\beta > 0$ coupling constant
- $\xi(t,x)$ "space-time white noise" (δ -correlated Gaussian)

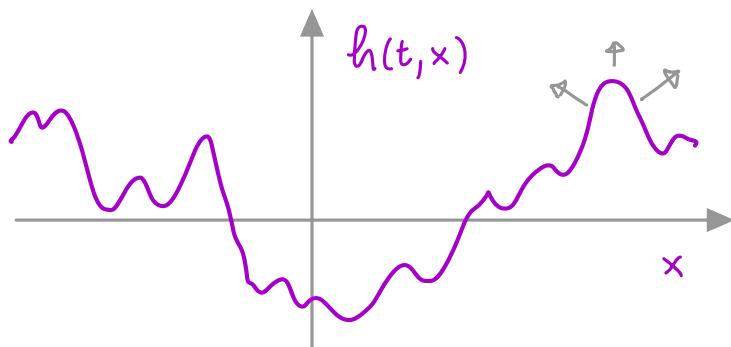
GOAL: Construct the solution $U(t,x)$ for $d=2$

THE KARDAR - PARISI - ZHANG EQUATION

[PRL 1986]

Formally $h(t,x) := \log u(t,x)$ solves

$$(KPZ) \quad \partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{GROWTH}} + \underbrace{\beta \xi(t,x)}_{\text{NOISE}}$$



SHE can help us
make sense of KPZ

SINGULARITY

(SHE) and (KPZ) are ill-defined due to singular products

$$\xi(t, x) \cup(t, x)$$

$$|\nabla h(t, x)|^2$$

$\xi(t, x)$ is a distribution and $h(t, x)$ expected to be:

$$\mathcal{C}_{\text{PAR}}^{-\frac{d}{2}-1-}$$

- non-smooth functions ($d=1$) $\mathcal{C}_{\text{PAR}}^{\frac{1}{2}-}$
- genuine distributions ($d \geq 2$) $\mathcal{C}_{\text{PAR}}^{\frac{2-d}{2}-}$

Henceforth we focus on (SHE)

THE CASE $d=1$

(1980s) $u(t,x)$ well-posed by stochastic integration (Ito-Walsh)

(2010s) Robust solution theories for "sub-critical" singular PDEs

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara]
- RENORMALIZATION [Kupiainen]

All of this breaks down in higher dimensions $d \geq 2$

THE ROLE OF DIMENSION

Space-time blow up: $\tilde{U}(t,x) := U(\delta^2 t, \delta x)$ solves

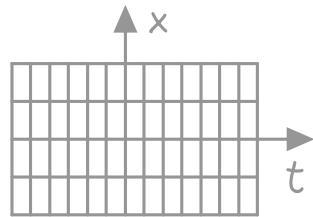
$$\partial_t \tilde{U}(t,x) = \Delta \tilde{U}(t,x) + \beta \delta^{\frac{2-d}{2}} \tilde{\xi}(t,x) \tilde{U}(t,x)$$

As $\delta \downarrow 0$ the noise formally

$\left\{ \begin{array}{ll} \text{vanishes} & (d < 2) \\ \text{stays constant} & (d = 2) \\ \text{diverges} & (d > 2) \end{array} \right.$	
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$d=2$ is CRITICAL DIMENSION for SHE

REGULARIZING SHE VIA DISCRETIZATION



We fix $d=2$. Restrict (t,x) in the lattice $\mathbb{T}_N = \frac{N}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N u_N(t,x) = \Delta^N u_N(t,x) + \beta \cdot N \cdot \xi_N(t + \frac{1}{N}, x) \langle u_N(t,x) \rangle$$

DISCR. DERIVATIVE

$$N \left\{ u(t + \frac{1}{N}, x) - u(t, x) \right\}$$

DISCR. LAPLACIAN

$$\frac{N}{4} \sum_{x' \sim x} \{ u(t, x') - u(t, x) \}$$

1.I.D. RVs

$$\mathbb{E}[\xi] = 0 \quad \mathbb{E}[\xi^2] = 1$$

$$\beta \xi \geq -1$$



$$\frac{1}{4} \sum_{x' \sim x} u(t, x')$$

Well-defined solution $u_N(t,x) \geq 0$

$$\left[u_N(0, \cdot) \equiv 1 \right]$$

CONVERGENCE ?

Can we hope that $U_N(t, x) \xrightarrow[N \rightarrow \infty]{d} U(t, x)$? (non-trivial limit)

① Convergence as (random) distributions : $\varphi \in C_c(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \varphi(x) U_N(t, x) dx \xrightarrow[N \rightarrow \infty]{d} \int_{\mathbb{R}^2} \varphi(x) U(t, dx) ?$$



RANDOM VARIABLE IN \mathbb{R}

i.e. $U_N(t, x) dx \xrightarrow{d} U(t, dx)$ as (random) measures on \mathbb{R}^2 ?

② Rescale the coupling constant $\beta = \beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}}$ $\xrightarrow[N \rightarrow \infty]{} 0$

$$\mathbb{E} \left[\int \varphi(x) U_N(t, x) dx \right] \equiv \int \varphi(x) dx$$

$$\text{VAR} \left[\int \varphi(x) U_N(t, x) dx \right] \rightarrow \begin{cases} 0 & \text{if } \hat{\beta} < \sqrt{\pi} \\ \sigma^2(\varphi) & \text{if } \hat{\beta} = \sqrt{\pi} \\ \infty & \text{if } \hat{\beta} > \sqrt{\pi} \end{cases}$$

PHASE TRANSITION

For $\hat{\beta} < \sqrt{\pi}$: $U_N(t, x) dx \xrightarrow{d} dx$ = Lebesgue measure ("trivial")

For $\hat{\beta} = \sqrt{\pi}$ do we have $U_N(t, x) dx \xrightarrow{d} \mathcal{U}(t, dx)$?

(non trivial)

II. MAIN RESULTS

THEOREM

[CSZ 23+]

Rescale $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$, more precisely

$$\star \quad \beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{g}{\log N} \right) \quad g \in \mathbb{R}$$

Then $U_N(t, x)$ converges to a non-trivial limit:

$$(U_N(t, x) dx)_{t \geq 0} \xrightarrow[N \rightarrow \infty]{\text{f.d.d.}} \mathcal{U}^g = (U_t^g(dx))_{t \geq 0}$$

We call \mathcal{U}^g critical 2d STOCHASTIC HEAT FLOW (SHF)

SHF & SHE

We have built a candidate solution of the Critical $2d$ SHE:

the SHF $\mathcal{U}^g = (\mathcal{U}_t^g(dx))_{t \geq 0}$

$$\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$$

with initial condition $\mathcal{U}_0^g(dx) \equiv dx$ $(\mathcal{U}(0, \cdot) \equiv 1)$

Remark. We actually build a two-parameter process

$$\mathcal{U}^g = (\mathcal{U}_{s,t}^g(dy, dx))_{0 \leq s \leq t < \infty}$$

"CANDIDATE SHE SOLUTION
FROM dy AT TIME s "

SOME FEATURES OF THE SHF

- $\mathbb{E} [\mathcal{U}_t^g(dx)] = dx$
- $\mathbb{E} [\mathcal{U}_t^g(dx) \mathcal{U}_t^g(dy)] = K_t^g(x, y) dx dy$ [Bertini, Cancrini 98]
 $\hookrightarrow \sim \log \frac{1}{|x-y|}$ $\Rightarrow \mathcal{U}^g$ non trivial
- $\mathcal{U}_{at}^g(d(\sqrt{a}x)) \stackrel{d}{=} a \mathcal{U}_t^{g+\log(a)}(dx)$
- Formulas for higher moments [Gu, Quastel, Tsai 21] [CSZ 19b]

CORRELATION FUNCTION

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{v}{\log N} \right)$$

$$K_N^\beta(t, x, x') := \mathbb{E} [U_N(t, x) \cdot U_N(t, x')] \xrightarrow[N \rightarrow \infty]{} K_t^g(x, x')$$

It solves the 2-body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^\beta = - \mathcal{H}_N^\beta K_N^\beta \quad \text{where} \quad \mathcal{H}_N^\beta = -\Delta^N - \beta \cdot N \cdot \mathbf{1}_{\{x=x'\}}$$

$$K_t^g = e^{-t \mathcal{H}^g} \quad \xrightarrow{\text{self-adj. ext. of}} \quad \mathcal{H}^\beta = -\Delta - \beta \delta(x-x')$$

EXPLICIT
FORMULA

[Albeverio, Gesztesy, Høegh-Krohn, Holden 87]

HIGHER ORDER CORRELATIONS

$$K_N^\beta(t, x_1, \dots, x_n) := \mathbb{E} \left[U_N(t, x_1) \cdots U_N(t, x_n) \right] \xrightarrow[N \rightarrow \infty]{} K_t^\beta(x_1, \dots, x_n)$$

It solves the n -body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{N}$:

$$\partial_t^N K_N^\beta = -\mathcal{H}_N^\beta K_N^\beta \quad \text{where} \quad \mathcal{H}_N^\beta = -\Delta^N - \beta \cdot N \cdot \sum_{1 \leq i < j \leq n} \mathbb{1}_{\{x_i = x_j\}}$$

$$K_t^\beta = e^{-t \mathcal{H}^\beta} \quad \text{self-adj. ext. of } \mathcal{H}^\beta = -\Delta - \beta \sum_{1 \leq i < j \leq n} \delta(x_i - x_j)$$

EXPLICIT FORMULAS

[dell'Antonio, Figari, Teta 94] [Dimock, Rajeev 04] [Gu, Quastel, Tsai 19]

GAUSSIAN MULTIPLICATIVE CHAOS

Generalized Gaussian field $\chi \sim \mathcal{N}(0, \kappa)$ on \mathbb{R}^d

$$\iint \varphi(x) \varphi(y) \mathbb{E} [\chi(x) \chi(y)] dx dy = \iint \kappa(x, y) \varphi(x) \varphi(y) dx dy$$

Gaussian Multiplicative Chaos (GMC) (random measure)

$$\mathcal{M}(dx) = "e^{\chi(x) - \frac{1}{2} \kappa(x, x)} dx"$$

$$\mathbb{E}[\mathcal{M}(dx)] = dx \quad \mathbb{E}[\mathcal{M}(dx) \mathcal{M}(dy)] = e^{\kappa(x, y)} dx dy$$

GMCs are "canonical": many explicit features

THEOREM

The SHF $U_t^g(dx)$ is NOT a GMC

[CSZ 22+]

Recall: Formally $h(t, x) = \log U(t, x)$ solves (KPZ)

CONJECTURE

The critical 2d KPZ equation should have a
NON GAUSSIAN SOLUTION $\mathcal{H}_t(dx)$

(KPZ) solution yet to be constructed! Cannot take $\log U_t^g(dx)$

III. IDEAS AND TECHNIQUES

FEYNMAN-KAC FORMULA

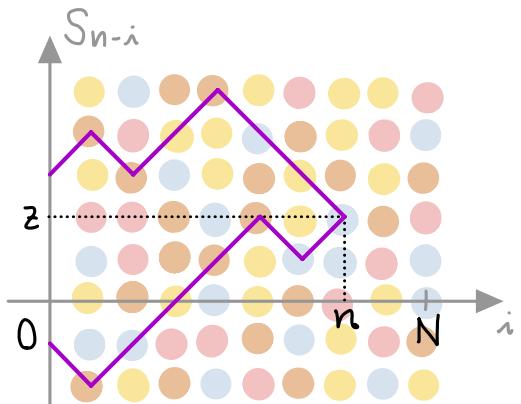
$$(t, x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}} \right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$$

$$u_N(t, x) = Z_N(u, z) = E \left[e^{\sum_{i=1}^N \beta \omega(i, S_{n-i}) - \lambda(\beta)} \mid S_0 = z \right]$$

S SIMPLE RANDOM WALK ON \mathbb{Z}^2

$$\beta \xi = e^{\beta \omega - \lambda(\beta)} - 1 \geq -1$$

Partition function of the
DIRECTED POLYMER
IN RANDOM ENVIRONMENT



SECOND MOMENT AND CRITICAL SCALING OF β

$$\mathbb{E} \left[U_N(1, x) \cdot U_N(1, x') \right] = E \left[e^{\underbrace{\beta^2 \sum_{i=0}^N \mathbb{1}_{\{S_i = S'_i\}}}_{L_N}} \mid S_0 = z, S'_0 = z' \right]$$

"REPLICA OVERLAP"

For $x = x'$:

$$\frac{\pi}{\log N} L_N \xrightarrow[N \rightarrow \infty]{d} Y \sim \text{Exp}(1) \quad [\text{Erdős-Taylor 6a}]$$

This explains the CRITICAL SCALING

$$\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$$

MAIN RESULT: STRATEGY OF THE PROOF

$$U_N(t, x) \, dx \xrightarrow{d} U_t^g(dx)$$

- Existence of subsequential limits is "easy" [Bertini-Cancrini 98]
- Non-triviality of the limit is non trivial [CSZ 19b]
- Uniqueness is **hard** [CSZ 23+]

(Moments grow too fast to determine the distribution)

How TO PROVE UNIQUENESS ?

We use a Cauchy argument:

$$U_N(t, x) dx \underset{d}{\approx} U_M(t, x) dx \quad \text{for large } N, M$$

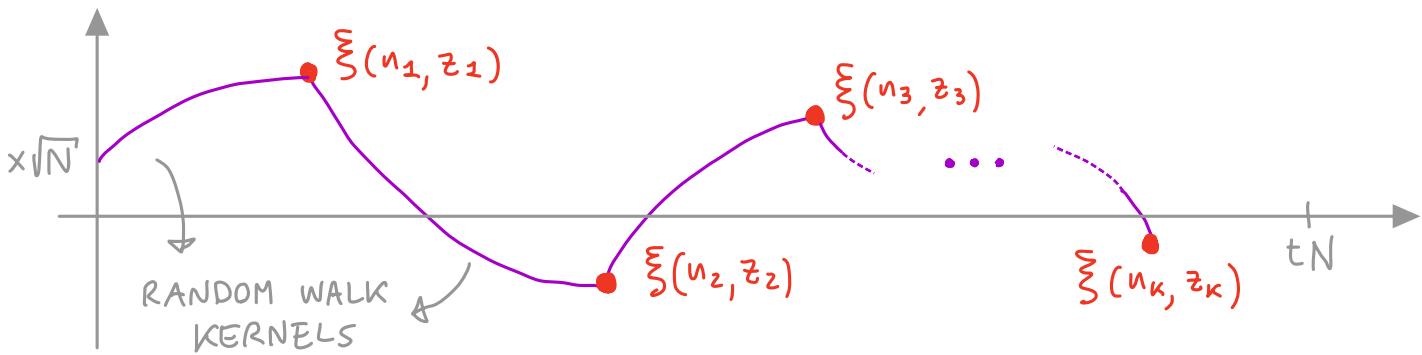
exploiting self-similarity of the model

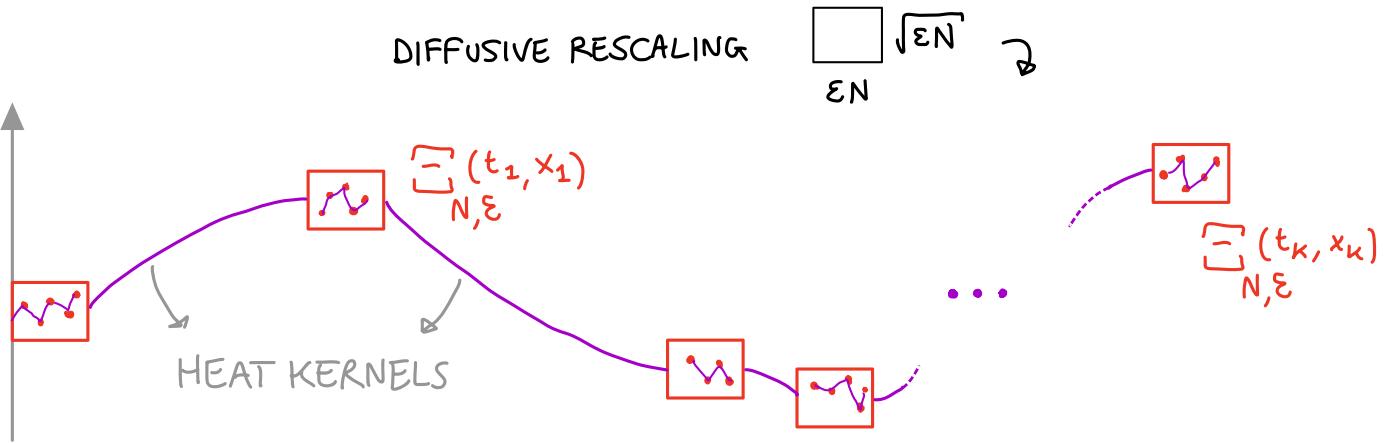
- A. COARSE-GRAINING \rightsquigarrow B. RENEWAL STRUCTURE
- C. LINDEBERG PRINCIPLE \rightsquigarrow D. FUNCTIONAL INEQUALITIES

A. COARSE - GRAINING

Polynomial chaos expansion:

$$U_N(t, x) = 1 + \sum_{k=1}^N \beta^k \sum_{(n_1, z_1), \dots, (n_k, z_k)} q((n_1, z_1), \dots, (n_k, z_k)) \cdot \prod_{i=1}^k \xi(n_i, z_i)$$





Sharp L^2 approximation via a coarse-grained model

$$U_N(t, x) dx \approx \mathcal{Z}_\epsilon^{CG}(t, dx | \Sigma_{N,\epsilon})$$

as $\epsilon \downarrow 0$

MULTI-LINEAR POLYNOMIAL \downarrow

"COARSE-GRAINED" NOISE \downarrow

B. RENEWAL STRUCTURE

Probabilistic interpretation of 2nd moment

$$\mathbb{E} \left[U_N(t, x) \cdot U_N(t, x') \right] = \sum_{k=1}^N \beta^{2k} \sum_{(u_1, z_1), \dots, (u_k, z_k)} q((u_1, z_1), \dots, (u_k, z_k))^2$$

$$\xrightarrow[N \rightarrow \infty]{} K_t^g(x, x') = 2\pi \int_0^t ds g_s(x-x') \int_s^t e^{gu} P(Y_u \leq t) du$$

[CSZ 19a]

HEAT KERNEL

"DICKMAN SUBORDINATOR"

c. LINDEBERG PRINCIPLE

The distribution of coarse-grained model $\mathcal{Z}_\varepsilon^{\text{CG}}(t, dx | \Xi)$
is insensitive to the distribution of Ξ

(as $\varepsilon \downarrow 0$, provided 1st & 2nd moments are fixed) [Röllin 2013]

~~~ We can switch  $\Xi_{N,\varepsilon}$  to  $\Xi_{M,\varepsilon}$  and get our goal

$$U_N(t, x) dx \underset{d}{\approx} U_M(t, x) dx$$

## D. FUNCTIONAL INEQUALITIES

Inequalities for Green's function of multiple random walks

"CRITICAL" HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{f(x, x') \cdot g(y, y')}{(|x-y| + |x'-y'| + |x-y'|)^{2d}} dx dx' dy dy' \leq C \|f\|_{L^p} \|g\|_{L^q}$$

Generalizes an inequality by [dell'Antonio, Figari, Teta 94]

## IV. CONCLUSIONS AND PERSPECTIVES

## CONCLUSIONS

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW  $\mathcal{U}_t^g(dx)$  as the scaling limit of solutions of discretized SHE  
 $\longleftrightarrow$  directed polymer partition functions

- Universal process of random measures on  $\mathbb{R}^2$  ( $\neq$  GMC)
- Natural candidate solution for critical 2d SHE

Many explicit features...

... and several interesting open questions:

- SINGULARITY W.R.T. LEBESGUE MEASURE ✓
- FLOW PROPERTY
- CHARACTERIZING PROPERTIES & UNIVERSALITY
- TAKING LOG  $\rightsquigarrow$  KPZ

Statistical Mechanics  $\leftrightarrow$  Singular Stochastic PDEs

## RELATED WORKS

- ANISOTROPIC KPZ:  $\Delta h = (\partial_x^2 + \partial_y^2) h \rightsquigarrow (\partial_x^2 - \partial_y^2) h$   
[Erhard, Cannizzaro, Toninelli]
- SHE WITH LÉVY NOISE:  $P(|\xi| > t) \sim \frac{C}{t^\alpha} \quad 0 < \alpha < 2$   
[Berger, Chong, Lacoin]
- DIRECTED POLYMERS, SHE & KPZ IN  $d \neq 2$   
[Magnen, Unterberger] [Chatterjee, Dunlap] [Dunlap, Gu, Ryzhik, Zeitouni]  
[Clark] [Comets, Casco, Mukherjee] [Lygkonis, Zygouras] [Junk] ...

Thanks

## NON GMC-NESS

Consider the 2d Stochastic Heat Flow  $\mathcal{U}_t^g$  for fixed  $t > 0$ ,  $g \in \mathbb{R}$

Let  $\mathcal{M}(dx)$  be the GMC with matching 1<sup>st</sup> and 2<sup>nd</sup> moments

$$\mathbb{E}[\mathcal{M}(dx) \mathcal{M}(dy)] = \mathbb{E}[\mathcal{U}_t^g(dx) \mathcal{U}_t^g(dy)] = K_t^g(x, y) dx dy$$

We prove that higher moments do not match

- 3<sup>rd</sup> MOMENT BOUND: For any  $R > 0$

$$\mathbb{E}[\mathcal{U}_t^g(B_R)^3] > \mathbb{E}[\mathcal{M}(B_R)^3]$$

- HIGHER MOMENT BOUND: there is  $\gamma > 0$  s.t. for any  $K \geq 3$

$$\liminf_{\delta \downarrow 0} \frac{\mathbb{E}[U_t^\gamma (g_\delta)^K]}{\mathbb{E}[M(g_\delta)^K]} \geq 1 + \gamma$$

HEAT KERNEL AT TIME  $\delta$

- 3<sup>rd</sup> MOMENT BOUND
 

$\left\{ \begin{array}{l} \text{explicit diagrammatic expansion} \\ \text{Gaussian calculations} \end{array} \right.$
- HIGHER MOMENT BOUND via Gaussian Correlation Inequality  
 (inspired by [Feng 16])

## MOMENT FORMULAS

$$\mathbb{E} \left[ U_t^{\vartheta}(dx) \cdot U_t^{\vartheta}(dy) \cdot U_t^{\vartheta}(dz) \right] = \underbrace{K^{(3)}(x, y, z)}_{dx dy dz}$$

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ U_N(t, x) \cdot U_N(t, y) \cdot U_N(t, z) \right]$$

$$K^{(3)}(z_1, z_2, z_3) = \sum_{m \geq 2} \int \cdots \int d\vec{a} d\vec{b} d\vec{x} d\vec{y} d\vec{z} g_{\vec{z}}^{(m)}(\vec{a}, \vec{b}, \vec{x}, \vec{y})$$

$0 < a_1 < b_1 < \dots < a_m < b_m < t$   
 $x_1, y_1, \dots, x_m, y_m \in \mathbb{R}^2$

# MOMENT FORMULAS

