THE CRITICAL 2D STOCHASTIC HEAT FLOW IS NOT A GAUSSIAN MULTIPLICATIVE CHAOS

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ABSTRACT. The critical $2d$ Stochastic Heat Flow (SHF) is a stochastic process of random measures on $\mathbb{R}^2$, recently constructed in [CSZ21]. We show that this process falls outside the class of Gaussian Multiplicative Chaos (GMC), in the sense that it cannot be realised as the exponential of a (generalised) Gaussian field. We achieve this by deriving strict lower bounds on the moments of the SHF that are of independent interest.

1. INTRODUCTION

The critical $2d$ Stochastic Heat Flow (SHF) is a stochastic process of random measures on $\mathbb{R}^2$, constructed in [CSZ21] as a universal limit of random polymer models. It is the natural candidate solution of the (ill-defined) critical $2d$ Stochastic Heat Equation:

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta\,\xi(t,x)\,u(t,x)\,, \qquad t > 0,\ x \in \mathbb{R}^2, \tag{1.1}$$

where $\xi(t,x)$ represents space-time white noise, that is a Gaussian field, delta-correlated in space and time. The term *critical* refers both to the fact that *dimension 2 is a critical dimension*, in the sense of singular stochastic PDEs [H14, GIP15] and renormalisation theory [Kup14], and that *a critical scaling in the noise strength β is needed*, see (1.21) below.

The criticality of dimension $d = 2$ for the Stochastic Heat Equation (1.1) can be seen through a scaling argument, in the spirit of renormalisation. Indeed, one can note that the rescaled function $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$ solves

$$\partial_t \tilde{u} = \frac{1}{2}\Delta\tilde{u} + \beta\,\varepsilon^{1-\frac{d}{2}}\,\tilde{\xi}\,\tilde{u}\,, \qquad \tilde{t} > 0,\ \tilde{x} \in \mathbb{R}^d,$$

where $\tilde{\xi}(t,x) := \varepsilon^{1+\frac{d}{2}}\xi(\varepsilon^2 t, \varepsilon x)$ is a new space-time white noise. One now sees that, as $\varepsilon \to 0$, when $d < 2$ the multiplicative factor $\varepsilon^{1-d/2}$ attenuates the small scale effects of the noise, while these effects are amplified when $d > 3$. On the other hand, when $d = 2$, the exponent $1 - \frac{d}{2}$ vanishes and the extent to which the noise influences the solution is not apparent.

In this paper we obtain explicit lower bounds on the moments of the SHF. Besides their own interest, these bounds imply that *the SHF is not the "exponential of a Gaussian field"* in the sense of Gaussian Multiplicative Chaos (GMC). This result provides insight on the critical $2d$ Kardar-Parisi-Zhang (KPZ) equation:

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta\,\xi(t,x)\,, \qquad t > 0,\ x \in \mathbb{R}^2\,. \tag{1.2}$$

Indeed, when the solution $u(t,x)$ of the Stochastic Heat Equation (1.1) is function valued, its logarithm $h(t,x) := \log u(t,x)$ is a solution of the KPZ equation (1.2). Since the critical $2d$ SHF is the candidate solution of (1.1), the fact that it is not a GMC suggests that *the critical $2d$ KPZ solution (yet to be constructed) is likely not a Gaussian field*.

In the rest of this introduction, we first recall the construction of the SHF from [CSZ21]; then we state our main results on the moments of the SHF and the comparison with GMC; finally, we discuss related results from the literature and outline future directions of research.

1.1. REMINDER: THE CRITICAL $2d$ SHF. A key difficulty in making sense of equation (1.1) is that its solution $u(t,x)$ is expected to be a genuine distribution on $\mathbb{R}^2$, not a function, so the product $\xi(t,x)\,u(t,x)$ is not well-defined. A natural strategy to make sense of it is to

(1) *regularise the equation*, so that a well-defined approximating solution exists;

(2) prove that the approximating solution *has a non-trivial limit* as the regularisation is removed (and the noise strength β is suitably rescaled).

This approach was recently carried out in [CSZ21], where equation (1.1) is regularised via discretisation of space and time, i.e. white noise ξ is replaced by a family of i.i.d. random variables $\omega = (\omega(n,x))_{n\in\mathbb{N}, x\in\mathbb{Z}^2}$ with law $\mathbb{P}$, called *disorder*, which satisfy

$$\mathbb{E}[\omega] = 0, \qquad \mathbb{E}[\omega^2] = 1, \qquad \exists\,\beta_0 > 0: \quad \lambda(\beta) := \log\mathbb{E}[e^{\beta\omega}] < \infty \quad \forall\beta \in [0,\beta_0]. \quad (1.3)$$

Replacing derivatives in the Stochastic Heat Equation (1.1) by suitable difference operators, the solution is the partition function of *directed polymers in random environment*:

$$Z_{M,N}^{\beta}(x,y) = \mathrm{E}\left[e^{\sum_{n=M+1}^{N-1}\{\beta\omega(n,S_n)-\lambda(\beta)\}} \, \mathbb{1}_{\{S_N=y\}} \,\Big|\, S_M = x \right], \qquad (1.4)$$

where E is the expectation with respect to $S = (S_n)$, the simple random walk on $\mathbb{Z}^2$. Note that (1.4) is a discretised Feynman-Kac formula for (1.1) on the time interval $[M,N]$, up to time-reversal and with a delta initial condition at time M. An alternative regularisation of (1.1), via mollification, is discussed in Subsection 1.4 below.

The main result of [CSZ21] is that the random field of partition functions $Z_{M,N}^{\beta}(x,y)$, under diffusive rescaling of space and time and for a suitable critical scaling of $\beta = \beta_N$, converges in law as $N \to \infty$ to a unique *measure valued random field* $\mathscr{Z}_{s,t}^{\vartheta}(\mathrm{d}x,\mathrm{d}y)$. More precisely, we define the diffusively rescaled random field of partition functions:[†]

$$\mathscr{Z}_N^{\beta} = \left(\mathscr{Z}_{N;s,t}^{\beta}(\mathrm{d}x,\mathrm{d}y) := \frac{N}{4} Z_{[\![Ns]\!],[\![Nt]\!]}^{\beta,\omega}([\![\sqrt{N}x]\!],[\![\sqrt{N}y]\!]) \,\mathrm{d}x\,\mathrm{d}y \right)_{0\leqslant s\leqslant t<\infty} \qquad (1.5)$$

where $\mathrm{d}x\,\mathrm{d}y$ is the Lebesgue measure on $\mathbb{R}^2 \times \mathbb{R}^2$ and $[\![Ns]\!]$ is the even integer closest to Ns, while $[\![\sqrt{N}x]\!]$ is the point closest to $\sqrt{N}x \in \mathbb{R}^2$ in the lattice $\mathbb{Z}_{\text{even}}^2$, where we set

$$\mathbb{Z}_{\text{even}}^d := \{(z_1,\ldots,z_d) \in \mathbb{Z}^d : z_1 + \ldots + z_d \text{ is even}\}. \qquad (1.6)$$

We next rescale $\beta = (\beta_N)_{N\in\mathbb{N}}$ in a *critical window*, defined by (A.1)-(A.2) in Appendix A, which separates the weak and strong disorder phases of $2d$ directed polymers [CSZ17b].

[†]The factor $\frac{1}{4}$ in (1.5) is due to the periodicity of the simple random walk, while the multiplication by N is due to the local limit theorem: $\mathbb{E}[Z_{M,N}^{\beta_N}(w,z)] = \mathrm{P}(S_N = z \,|\, S_M = w) = O(\frac{1}{N-M}) = O(\frac{1}{N})$ for $M/N \leqslant c < 1$.

When the disorder ω has a symmetric distribution (for simplicity), this reads as follows:

$$\beta_N^2 = \frac{\pi}{\log N}\left(1 + \frac{\varrho + o(1)}{\log N}\right) \qquad \text{for some } \varrho \in \mathbb{R}\,. \tag{1.7}$$

To have a universal parametrisation, our results will be formulated using a slightly different parameter ϑ, see (A.1), which differs from ϱ in (1.7) by a constant, see [CSZ19b, eq. (1.17)].

We can now state the main result of [CSZ21].

Theorem 1.1 (The critical $2d$ SHF [CSZ21]). *Fix β_N in the critical window* (A.1)-(A.2) *for a given $\vartheta \in \mathbb{R}$. The process of random measures $\mathcal{Z}_N^{\beta_N} = (\mathcal{Z}_{N;\,s,t}^{\beta_N}(\mathrm{d}x,\mathrm{d}y))_{0 \leqslant s \leqslant t < \infty}$ defined in (1.5) converges in finite dimensional distributions $(\text{as } N \to \infty)$ to a unique limit*

$$\mathscr{Z}^\vartheta = (\mathscr{Z}_{s,t}^\vartheta(\mathrm{d}x,\mathrm{d}y))_{0 \leqslant s \leqslant t < \infty}\,,$$

called the critical $2d$ Stochastic Heat Flow.

The convergence in distribution in Theorem 1.1 takes place in the space of locally finite measures on $\mathbb{R}^2 \times \mathbb{R}^2$, equipped with the topology of vague convergence:

$$\mu_N \to \mu \qquad \Longleftrightarrow \qquad \int \phi(x,y)\,\mu_N(\mathrm{d}x,\mathrm{d}y) \to \int \phi(x,y)\,\mu(\mathrm{d}x,\mathrm{d}y) \quad \forall \phi \in C_c(\mathbb{R}^2 \times \mathbb{R}^2)\,.$$

1.2. MAIN RESULT I: SHF vs. GMC. We focus on the SHF's one-time marginal:

$$\mathscr{Z}_t^\vartheta(\mathrm{d}x) := \int_{y \in \mathbb{R}^2} \mathscr{Z}_{0,t}^\vartheta(\mathrm{d}x,\mathrm{d}y)\,. \tag{1.8}$$

This is a stochastic process of *log-correlated random measures on* $\mathbb{R}^2$, see (1.10)-(1.11) below. Higher moments of the SHF admit explicit series expansions, see (2.15)-(2.16) below, which stem from the works [GQT21, Che21, CSZ19b, BC98]. However, as we will show below, the SHF moments grow too fast to uniquely determine the field.

In the subcritical regime $\beta_N^2 \sim \hat{\beta}\,\pi/\log N$ with $\hat{\beta} < 1$ — that is, strictly below the critical window (1.7) that we consider here — the logarithm of the directed polymer partition function displays Gaussian fluctuations [CSZ17b, Gu20, CSZ20]. This suggests that, *in the subcritical regime*, partition functions should be close to the exponential of a Gaussian field.

It is natural to wonder whether a similar picture still holds true *at criticality*: is the critical $2d$ SHF the exponential of a Gaussian field in the sense of *Gaussian Multiplicative Chaos* (GMC)? Our first main result shows that this is not the case.

Theorem 1.2. *The critical $2d$ Stochastic Heat Flow is not a Gaussian Multiplicative Chaos.*

We will recall the definition of GMC in Section 3. We point out that GMC has been studied extensively and has applications in many contexts, including Liouville quantum gravity, turbulence, zeroes of characteristic polynomials etc. A comprehensive review of its connections to various fields in probability and mathematical physics, as well as a nice introduction to its properties, is given in [RV14].

Theorem 1.2 suggests that, in the critical window (1.7), the logarithm of the partition functions has a *non-Gaussian limit*. Such a limit would then be the natural candidate solution of the critical $2d$ KPZ equation (1.2). Of course, putting this conjecture on firm ground will require further work — we cannot just take the logarithm of the SHF, which is a random measure — but our results provide an indication for the emergence of non-Gaussianity in the $2d$ KPZ equation.

1.3. MAIN RESULTS II: LOWER BOUNDS FOR THE SHF MOMENTS. Our next main results are explicit lower bounds on the moments of the critical $2d$ SHF. These bounds are the key to proving Theorem 1.2, because they show that *the moments of the SHF are strictly larger than those of a corresponding GMC*, in a sense that we now make precise.

The one-time marginal $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ of the SHF, see (1.8), is a random measure on $\mathbb{R}^2$. Let us denote by $\mathscr{M}_t^\vartheta(\mathrm{d}x)$ the GMC on $\mathbb{R}^2$ with the same first and second moments of the SHF:

$$\mathbb{E}\big[\mathscr{M}_t^\vartheta(\mathrm{d}x)\big] = \mathbb{E}\big[\mathscr{Z}_t^\vartheta(\mathrm{d}x)\big] = \tfrac{1}{2}\,\mathrm{d}x\,, \tag{1.9}$$

$$\mathbb{E}\big[\mathscr{M}_t^\vartheta(\mathrm{d}x)\,\mathscr{M}_t^\vartheta(\mathrm{d}y)\big] = \mathbb{E}\big[\mathscr{Z}_t^\vartheta(\mathrm{d}x)\,\mathscr{Z}_t^\vartheta(\mathrm{d}y)\big] = \tfrac{1}{4}\,\mathscr{K}_{t,\vartheta}^{(2)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{1.10}$$

where $\mathscr{K}_{t,\vartheta}^{(2)}(x,y)$ is known, see (2.7), and it is *log-divergent* along the diagonal (see (5.27)):

$$\mathscr{K}_{t,\vartheta}^{(2)}(x,y) \sim C_{t,\vartheta}\,\log\frac{1}{|y-x|} \qquad \text{as } |y-x| \to 0\,. \tag{1.11}$$

As will be noted after (3.7), the Gaussian field underlying such a GMC is *log-log-correlated*, i.e. its covariance kernel satisfies $k_t(x,y) \sim \log\log\frac{1}{|y-x|}$ as $|y-x| \to 0$.[†]

We first compare the third moment of the SHF $\mathscr{Z}_t^\vartheta(\varphi) := \int_{\mathbb{R}^2} \varphi(x)\,\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ with that of the GMC $\mathscr{M}_t^\vartheta(\varphi) := \int_{\mathbb{R}^2} \varphi(x)\,\mathscr{M}_t^\vartheta(\mathrm{d}x)$ averaged over integrable functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 1.3 (Third moment lower bound). *For $t > 0$ and $\vartheta \in \mathbb{R}$, let $\mathscr{M}_t^\vartheta(\mathrm{d}x)$ be the GMC with the same first and second moments as the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$, see (1.9)-(1.10). If φ is the indicator function of a ball, or the heat kernel on $\mathbb{R}^2$, see (2.1), we have*

$$\mathbb{E}\big[\mathscr{Z}_t^\vartheta(\varphi)^3\big] > \mathbb{E}\big[\mathscr{M}_t^\vartheta(\varphi)^3\big]\,, \tag{1.12}$$

hence $\mathscr{Z}_t^\vartheta(\mathrm{d}x) \neq \mathscr{M}_t^\vartheta(\mathrm{d}x)$.

Remark 1.4. *The bound (1.12) actually holds for all radially symmetric and decreasing functions φ that satisfy a basic inequality, see (4.11) below. These include, in particular, the indicator function of a ball and the heat kernel, that we single out in Theorem 1.3.*

We next turn to moments of any order $m \geqslant 3$. Since $\mathscr{M}_t^\vartheta(\mathrm{d}x)$ is a GMC with a log-divergent second moment kernel, see (1.11), one can shown that (see Proposition 5.1 below)

$$\mathbb{E}\big[\big(2\,\mathscr{M}_t^\vartheta(g_\delta)\big)^m\big] \sim \mathbb{E}\big[\big(2\,\mathscr{M}_t^\vartheta(g_\delta)\big)^2\big]^{\binom{m}{2}} \quad \text{as } \delta \downarrow 0\,, \tag{1.13}$$

where g_δ is the heat kernel on $\mathbb{R}^2$ at time δ, and the multiplicative factor 2 arises from (1.9). We show that for the SHF $\mathscr{Z}_t^\vartheta$ this asymptotic factorisation *does not hold*.

Theorem 1.5 (Higher moments lower bound). *Given any $t > 0$ and $\vartheta \in \mathbb{R}$, there exists $\eta = \eta_{t,\vartheta} > 0$ such that for any $h \in \mathbb{N}$ with $h \geqslant 3$ we have*

$$\mathbb{E}\big[\big(2\,\mathscr{Z}_t^\vartheta(g_\delta)\big)^h\big] \geqslant (1+\eta)\,\mathbb{E}\big[\big(2\,\mathscr{Z}_t^\vartheta(g_\delta)\big)^2\big]^{\binom{h}{2}} \qquad \forall \delta \in (0,1)\,. \tag{1.14}$$

As a consequence, by (1.13), for any $h \in \mathbb{N}$ with $h \geqslant 3$ we have

$$\liminf_{\delta\downarrow 0} \frac{\mathbb{E}\big[\mathscr{Z}_t^\vartheta(g_\delta)^h\big]}{\mathbb{E}\big[\mathscr{M}_t^\vartheta(g_\delta)^h\big]} \geqslant 1+\eta > 1\,, \tag{1.15}$$

[†]For a comparison, the much studied Gaussian Free Field on $\mathbb{R}^2$ is log-correlated, hence the corresponding GMC is polynomially correlated.

hence $\mathscr{Z}_t^\vartheta(\mathrm{d}x) \neq \mathscr{M}_t^\vartheta(\mathrm{d}x)$.

Remark 1.6. *For the directed polymer partition functions in the whole subcritical regime, a lower bound qualitatively similar to (1.14), but with $\eta = 0$, is also valid and matches the asymptotic behaviour of the upper bounds obtained in* [LZ21, CZ21].

Theorem 1.3 will be proved in Section 4 by exploiting a series expansion for the moments (2.15)-(2.16), which in the case of the third moment admits a renewal-type form [CSZ19b], see (2.8)-(2.11). This is quite involved and can be represented as a series of complicated diagrams. Through an explicit computation, we are able to integrate out the spatial variables in these diagrams. What remains is a multiple integral of time variables that have monotonicity properties, which we exploit in order to obtain the lower bound (1.12).

Theorem 1.5 will be proved in Section 5 via a very different approach, inspired by the work of Feng [Fen16]. A key role here is played by the Gaussian Correlation Inequality [R14, LM17], which saves us from analysing the complicated diagrammatic representation of the moments. By means of probabilistic arguments, such as bounding the variance of suitable random variables, we obtain the lower bound (1.14), which then yields (1.15).

1.4. BACKGROUND.

We recall here some results that led to the critical $2d$ SHF.

To regularise the $2d$ Stochastic Heat Equation (1.1), we used in Section 1.1 a discretisation of space and time, which led to the directed polymer partition functions. Alternatively, one can *mollify the white noise ξ in space on scale $\varepsilon > 0$* by defining $\xi^\varepsilon(t, x) := (\xi(t, \cdot) * j_\varepsilon)(x)$, where $j_\varepsilon(x) := \varepsilon^{-2} j(x/\varepsilon)$ and $j(\cdot)$ is a smooth probability kernel, say compactly supported. This leads to the mollified Stochastic Heat Equation:

$$\partial_t u^\varepsilon(t, x) = \frac{1}{2}\Delta u^\varepsilon(t, x) + \beta\, u^\varepsilon(t, x)\, \xi^\varepsilon(t, x)\,. \tag{1.16}$$

The solution admits a Feynman-Kac representation [BC95, BC98]:

$$u^\varepsilon(t, x) = \mathbf{E}_x\!\left[\mathrm{e}^{\beta \int_0^t \xi^\varepsilon(t-s,B_s)\,\mathrm{d}s - \frac{1}{2}\beta^2\|j_\varepsilon\|_2^2 t}\right] \stackrel{\mathrm{dist}}{=} \mathbf{E}_x\!\left[\mathrm{e}^{\beta \int_0^t \xi^\varepsilon(s,B_s)\,\mathrm{d}s - \frac{1}{2}\beta^2\|j_\varepsilon\|_2^2 t}\right], \tag{1.17}$$

where $\mathbf{E}_x$ denotes expectation for a standard Brownian motion B starting at x (for simplicity, we consider a flat initial condition $u^\varepsilon(0, x) \equiv 1$). The goal is then to make sense of the limit of $u^\varepsilon(\cdot, \cdot)$ as $\varepsilon \to 0$, for suitable rescaling of $\beta = \beta_\varepsilon$.

Remark 1.7. *Comparing (1.17) with (1.4), we can see $u^\varepsilon(t, x)$ as the partition function of a Brownian directed polymer in the random environment ξ^ε. Thus the two schemes of regularisation, discretisation and mollification, are conceptually (if not technically) analogous, with the correspondence $\varepsilon \longleftrightarrow 1/\sqrt{N}$. Most existing results apply to both schemes* [CSZ17b, CSZ19b, CSZ20], *so we will focus on the mollified Stochastic Heat Equation in what follows.*

Denote by $u_\varepsilon^{(\hat{\beta})}(t, x)$ the solution (1.17) with $\beta = \hat{\beta}\sqrt{4\pi}/\sqrt{\log \varepsilon^{-2}}$ for $\hat{\beta} > 0$. A phase transition on this scale with critical point $\hat{\beta}_c = 1$ was first identified in [CSZ17b], where it was shown that for any fixed (t, x), the following limit in distribution holds:

$$u_\varepsilon^{(\hat{\beta})}(t, x) \xrightarrow[\varepsilon \to 0]{d} \begin{cases} \mathrm{e}^{\sigma(\hat{\beta})\mathsf{X} - \frac{1}{2}\sigma(\hat{\beta})^2} & \text{if } \hat{\beta} < 1\,, \\ 0 & \text{if } \hat{\beta} \geqslant 1\,, \end{cases} \tag{1.18}$$

where X is a standard normal random variable and $\sigma(\hat{\beta})^2 := \log(1/(1 - \hat{\beta}^2))$.

For $\hat{\beta} < 1$, known as the *subcritical regime*, the solution $u_\varepsilon^{(\hat{\beta})}$ viewed as a random field, suitably centred and normalised, was shown in [CSZ17b] to converge in distribution to a Gaussian free field, given by the solution $v^{(\hat{\beta})}$ of the *additive* stochastic heat equation (a.k.a. Edwards-Wilkinson equation):

$$\partial_t v^{(\hat{\beta})}(t,x) = \frac{1}{2}\Delta v^{(\hat{\beta})}(t,x) + \sqrt{\tfrac{1}{1-\hat{\beta}^2}}\,\xi(t,x) \qquad \text{with} \quad v^{(\hat{\beta})}(0,x) = 0, \qquad (1.19)$$

where the noise coefficient diverges as $\hat{\beta} \uparrow 1$. More precisely, if we define

$$\mathfrak{u}_\varepsilon^{(\hat{\beta})}(t,x) := \frac{\sqrt{\log \varepsilon^{-2}}}{\sqrt{4\pi}\hat{\beta}}\big(u_\varepsilon^{(\hat{\beta})}(t,x) - 1\big), \qquad (1.20)$$

then for every test function $\phi \in C_c(\mathbb{R}^2)$ we have $\langle \mathfrak{u}_\varepsilon^{(\hat{\beta})}, \phi \rangle \xrightarrow{d} \langle v^{(\hat{\beta})}, \phi \rangle$ as $\varepsilon \to 0$.

A similar result has been established for the solution of the mollified $2d$ KPZ equation, with $u_\varepsilon^{(\hat{\beta})}(t,x) - 1$ in (1.20) replaced by $\log u_\varepsilon^{(\hat{\beta})}(t,x) - \mathbb{E}[\log u_\varepsilon^{(\hat{\beta})}(t,x)]$, see [CSZ20, Gu20]. This may be viewed as an indication that, in the subcritical regime $\hat{\beta} < 1$, the solution of the mollified $2d$ Stochastic Heat Equation is close to the exponential of a Gaussian field (as we already discussed before Theorem 1.2 in the directed polymer setting). This breaks down at criticality, as we show in Theorem 1.2.

We next review the results when $\beta = \beta_\varepsilon$ is scaled in a critical window around the critical point $\hat{\beta}_c = 1$, which for the mollified Stochastic Heat Equation reads as follows:

$$\beta_\varepsilon^2 = \frac{4\pi}{\log \varepsilon^{-2}}\left(1 + \frac{\tilde{\varrho} + o(1)}{\log \varepsilon^{-2}}\right) = \frac{2\pi}{\log \varepsilon^{-1}}\left(1 + \frac{\tilde{\varrho} + o(1)}{\log \varepsilon^{-2}}\right). \qquad (1.21)$$

Note that this is similar to (1.7) with $N = \varepsilon^{-2}$ (the different factor 4π vs. π is because the simple symmetric random walk on $\mathbb{Z}^2$ has period 2 and covariance matrix $\frac{1}{2}I$: see Subsection 5.1 and Appendix A for a more detailed comparison).

The study of the mollified Stochastic Heat Equation with $\beta = \beta_\varepsilon$ chosen in the critical window (1.21) was initiated in [BC98], where they identified the limit of the second moment of the solution $u^\varepsilon(t,\cdot)$, see (1.17). Subsequently, [CSZ19b] computed the limit of the third moment of $u^\varepsilon(t,\cdot)$ and [GQT21] identified the limit of all higher moments (see also the more recent work [Che21]). These results ensure that the mollified solutions $(u^\varepsilon(t,\cdot))_{\varepsilon>0}$ are *tight* as random measures on $\mathbb{R}^2$, hence they admit subsequential limits in distribution as $\varepsilon \downarrow 0$, and any such limit has the same moments as identified in [BC98, CSZ19b, GQT21, Che21]. However, these moments grow too fast to uniquely determine the limiting random measure.

Existence of a *unique* limit, which was named the *critical $2d$ Stochastic Heat Flow*, was finally established in [CSZ21] in the directed polymer setting, i.e. for the solution of the Stochastic Heat Equation regularised via discretisation. It is expected that the same holds for the regularisation via mollification, i.e. that $u^\varepsilon(t,\cdot)$ in (1.17) converges to the critical $2d$ Stochastic Heat Flow as $\varepsilon \downarrow 0$, although the proof of [CSZ21] needs to be adapted.

1.5. Future perspectives. We now discuss some related works and open questions.

We proved in Theorem 1.2 that the (one-time marginal of the) SHF, as a random measure on $\mathbb{R}^2$, is not a GMC. There is, however, a very different sense in which a GMC structure emerges naturally. In the Feynman-Kac formula (1.17) for the solution $u^\varepsilon(t,x)$ of the mollified Stochastic Heat Equation, the exponent $\int_0^t \xi^\varepsilon(s, B_s)\mathrm{d}s$ may be viewed as a Gaussian process (w.r.t. the randomness of the white noise ξ^ε) indexed by $(B_s)_{s\in[0,t]} \in C[0,t]$, the space of

continuous functions defined on $[0, t]$. As a consequence, on the path space $C[0, t]$, we can consider the GMC measure $\mathcal{M}_x^\varepsilon(\mathrm{d}B)$ defined by

$$\mathcal{M}_x^\varepsilon(\mathrm{d}B) := \mathrm{e}^{\beta \int_0^t \xi^\varepsilon(s, B_s)\mathrm{d}s - \frac{1}{2}\beta^2 \|j_\varepsilon\|_2^2 t} \, \mathcal{W}_x(\mathrm{d}B) \,, \tag{1.22}$$

where $\mathcal{W}_x(\cdot)$ denotes the Wiener measure on paths $B \in C[0, t]$ with $B_0 = x$. Note that $u^\varepsilon(t, x) = \mathcal{M}_x^\varepsilon(C[0, t])$ in (1.17) is the total mass of $\mathcal{M}_x^\varepsilon(\cdot)$.

This was the perspective taken in [Cla19a, Cla19b], where an analogue of the critical $2d$ directed polymer on the diamond hierarchical lattice was studied (see also [BM20] for the Euclidean setting). In [Cla19a, Cla19b], partition functions were shown to have a non-trivial limit and then used to construct *a family of critical continuum polymer measures* indexed by the analogue of $\tilde{\varrho}$ in (1.21). Interestingly, these continuum polymer measures are related to each other through a conditional GMC structure, even though they cannot be defined as a GMC w.r.t. the analogue of the Wiener measure on the continuum hierarchical lattice.

This raises the natural question whether similar results hold for the analogue of the critical $2d$ SHF in path space, namely, whether the measures $\mathcal{M}_x^\varepsilon$ on $C[0, t]$ converge as $\varepsilon \to 0$, at least when integrated over x, and whether the limits corresponding to different $\tilde{\varrho}$ in (1.21) are related to each other through a conditional GMC structure. There is ongoing work in this direction in [CM22], where the authors study the second moment measure of subsequential limits of $\mathcal{M}_x^\varepsilon \mathrm{d}x$ and found properties that are consistent with the conditional GMC structure.

Another interesting direction of research concerns the asymptotic behavior of the moments of the critical $2d$ SHF. Theorems 1.3 and 1.5 provide lower bounds and it is natural to ask whether these can be improved. The works [CSZ19b, GQT21, Che21] show that for each integer $h \geqslant 3$, there is a well-defined h-point kernel $\mathscr{K}^{(h)} : (\mathbb{R}^2)^h \to \mathbb{R} \cup \{+\infty\}$ such that for any $\varphi \in C_c(\mathbb{R}^2)$,

$$\mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)^h] = \frac{1}{2^h} \int \cdots \int_{(\mathbb{R}^2)^h} \left(\prod_{i=1}^h \varphi(x_i) \right) \mathscr{K}_t^{(h)}(x_1, \cdots, x_h) \, \mathrm{d}\vec{x} \,,$$

see Theorem 2.3 below. In light of Theorem 1.5 and (1.11), it is natural to conjecture that

$$\mathscr{K}_t^{(h)}(x_1, \cdots, x_h) \sim \mathscr{C}_{t,\vartheta;h} \prod_{1 \leqslant i < j \leqslant h} \log \frac{1}{|x_i - x_j|} \qquad \text{as} \quad \max_{1 \leqslant i \leqslant j} |x_i - x_j| \to 0 \,, \tag{1.23}$$

for some constant $\mathscr{C}_{t,\vartheta;h} > (C_{t,\vartheta})^{\binom{h}{2}}$, where $C_{t,\vartheta}$ is the constant which determines the asymptotic behavior of the second moment kernel, see (1.10)-(1.11).

1.6. Organization of the paper. The rest of the paper is structured as follows.

- In Section 2, we recall the moments formulas for the critical $2d$ SHF.
- In Section 3 we review the construction of GMC and recall its moments.
- In Sections 4 and 5 we prove our main results Theorems 1.3 and 1.5.
- In Appendix A we compare the critical windows (1.7) and (1.21).

2. Moments of the critical $2d$ SHF

In this section, we recall the moments formulas for the critical $2d$ Stochastic Heat Flow from [BC98, CSZ19a, CSZ19b, GQT21]. We denote by $g_t(x)$ the heat kernel on $\mathbb{R}^2$:

$$g_t(x) := \frac{1}{2\pi t} e^{-\frac{|x|^2}{2t}} \,. \tag{2.1}$$

An important role is played by the following special function, defined for any $\vartheta \in \mathbb{R}$:

$$G_\vartheta(t) = \int_0^\infty \frac{e^{(\vartheta - \gamma)u}\, u\, t^{u-1}}{\Gamma(u+1)}\, du \,, \tag{2.2}$$

where $\gamma = -\int_0^\infty \log u\, e^{-u}\, du \simeq 0.577$ is the Euler-Mascheroni constant.

Remark 2.1. *The function G_ϑ has a probabilistic interpretation. Denote by $Y = (Y_u)_{u\geq 0}$ the Dickman subordinator, defined as the pure jump process with Lévy measure $\mathbb{1}_{(0,1)}(x)\, x^{-1}\, dx$, see [CSZ19a]. Then G_ϑ is the* exponentially weighted renewal density *of Y:*

$$G_\vartheta(t) = \int_0^\infty e^{\vartheta u}\, \frac{\mathrm{P}(Y_u \in dt)}{dt}\, du \qquad \text{for } t \in [0,1]\,.$$

2.1. First and second moments. The first moment of the SHF is

$$\mathbb{E}[\mathscr{Z}_{s,t}^\vartheta(dx, dy)] = \tfrac{1}{2}\, g_{\frac{1}{2}(t-s)}(y-x)\, dx\, dy\,, \tag{2.3}$$

while its covariance is given by

$$\mathbb{C}\mathrm{ov}[\mathscr{Z}_{s,t}^\vartheta(dx, dy), \mathscr{Z}_{s,t}^\vartheta(dx', dy')] = \tfrac{1}{2}\, K_{t-s}^\vartheta(x, x'; y, y')\, dx\, dy\, dx'\, dy'\,, \tag{2.4}$$

where

$$K_t^\vartheta(x, x'; y, y') := \pi\, g_{\frac{t}{4}}\Big(\frac{y+y'}{2} - \frac{x+x'}{2}\Big) \iint\limits_{0<s<u<t} g_s(x'-x)\, G_\vartheta(u-s)\, g_{t-u}(y'-y)\, ds\, du\,. \tag{2.5}$$

These formulas were derived from the asymptotic results in [CSZ19a] connected to the Dickman subordinator, see [CSZ21, Proposition 3.5].

We will focus on the one-time marginal $\mathscr{Z}_t(dx)$ of the SHF, see (1.8), which we also call *the SHF with flat initial data*. The first moment of the averaged field is then

$$\mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)] = \frac{1}{2} \int\limits_{\mathbb{R}^2} \varphi(z)\, dz\,, \tag{2.6}$$

while its centered second moment can be derived from (2.4)-(2.5) and equals

$$\mathbb{E}\big[\big(\mathscr{Z}_t^\vartheta(\varphi) - \mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)]\big)^2\big] = \frac{1}{4} \int\limits_{(\mathbb{R}^2)^2} \varphi(z_1)\, \varphi(z_2)\, K_t^{(2)}(z_1, z_2)\, dz_1\, dz_2\,,$$

$$\text{with} \qquad K_t^{(2)}(z_1, z_2) := 2\pi \iint\limits_{0<s<u<t} g_s(z_1 - z_2)\, G_\vartheta(u-s)\, ds\, du\,, \tag{2.7}$$

a formula that was first derived in [BC98] in the context of the mollified Stochastic Heat Equation (see Subsection 5.1 below).

2.2. THIRD MOMENT. The centered third moment of the SHF can be written as follows:

$$\mathbb{E}\big[\big(\mathscr{Z}_t^\vartheta(\varphi) - \mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)]\big)^3\big] = \frac{1}{8} \int_{(\mathbb{R}^2)^3} \varphi(z_1)\,\varphi(z_2)\,\varphi(z_3)\,K_t^{(3)}(z_1, z_2, z_3)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,, \quad (2.8)$$

where the kernel $K_t^{(3)}(z_1, z_2, z_3)$, first obtained in [CSZ19b, Theorem 1.4], admits the following explicit but quite involved expression (see Figure 1 for a pictorial representation):[†]

$$K_t^{(3)}(z_1, z_2, z_3) := \sum_{m=2}^\infty 2^{m-1}\,(2\pi)^m\,\big\{\mathcal{I}_t^{(m)}(z_1, z_2, z_3) + \mathcal{I}_t^{(m)}(z_2, z_3, z_1) + \mathcal{I}_t^{(m)}(z_3, z_1, z_2)\big\}\,,$$
$$(2.9)$$

where the kernel $\mathcal{I}_t^{(m)}(z_1, z_2, z_3)$ is defined by

$$\mathcal{I}_t^{(m)}(z_1, z_2, z_3) := \int \cdots \int_{0<a_1<b_1<\ldots<a_m<b_m<t} \mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}^{(m)}(z_1, z_2, z_3) \left\{\prod_{\ell=1}^m G_\vartheta(b_\ell - a_\ell)\right\} \mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\,, \quad (2.10)$$

and $\mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}(z_1, z_2, z_3)$ denotes the following convolution of heat kernels:

$$\mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}^{(m)}(z_1, z_2, z_3) := \iint_{(\mathbb{R}^2)^m \times (\mathbb{R}^2)^m} \mathrm{d}\vec{x}\,\mathrm{d}\vec{y}\ g_{\frac{a_1}{2}}(x_1 - z_1)\,g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}}(y_1 - x_1)$$

$$\cdot g_{\frac{a_2}{2}}(x_2 - z_3)\,g_{\frac{a_2-b_1}{2}}(x_2 - y_1) \cdot g_{\frac{b_2-a_2}{4}}(y_2 - x_2) \quad (2.11)$$

$$\cdot \prod_{\ell=3}^m \left\{g_{\frac{a_\ell-b_{\ell-2}}{2}}(x_\ell - y_{\ell-2})\,g_{\frac{a_\ell-b_{\ell-1}}{2}}(x_\ell - y_{\ell-1}) \cdot g_{\frac{b_\ell-a_\ell}{4}}(y_\ell - x_\ell)\right\}$$

(we agree that $\prod_{\ell=3}^m\{\ldots\} := 1$ for $m = 2$). We refer again to Figure 1.

We stress that formulas (2.8)-(2.11) are the key to our proof of Theorem 1.3.

Remark 2.2. *The normalisation chosen in [CSZ21] to construct the critical 2d SHF is slightly different from the one in [CSZ19b] due to the restriction to even parity sites, see (1.5)-(1.6). As a consequence, the limiting field in [CSZ19b] corresponds to $\mathscr{Z}_t^{\vartheta,\mathrm{mix}}(\varphi) \stackrel{d}{=} \mathscr{Z}_t^\vartheta(\varphi) + \mathscr{Z}_t'^{,\vartheta}(\varphi)$, where $\mathscr{Z}_t^\vartheta(\varphi)$ and $\mathscr{Z}_t'^{,\vartheta}(\varphi)$ denote two independent copies of the SHF. It follows that*

$$\mathbb{E}[(\mathscr{Z}_t^\vartheta(\varphi) - \mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)])^3] = \frac{1}{2}\mathbb{E}[(\mathscr{Z}_t^{\vartheta,\mathrm{mix}}(\varphi) - \mathbb{E}[\mathscr{Z}_t^{\vartheta,\mathrm{mix}}(\varphi)])^3]\,,$$

that is, the third moment in (2.8) is half *of that computed in [CSZ19b, Theorem 1.4].*

[†]We remark that in [CSZ19b, eq. (1.25)] we have π^m, whereas in (2.9) we have $(2\pi)^m$. The main source of this discrepancy is a missing factor 2^{m-2} in [CSZ19b, eq. (1.25)]: indeed, a factor $2\mathbb{1}_{\{(n,x)\in\mathbb{Z}^3\}}$ due to periodicity was omitted in [CSZ19b, eq. (5.40)], which plugged in [CSZ19b, eq. (5.30)] yields a factor 2 for each $i = 3,\ldots,m$, hence the claimed factor 2^{m-2} in [CSZ19b, eq. (1.25)]. Since the third moment in (2.8) is *half* the one in [CSZ19b, Theorem 1.4], see Remark 2.2, we have a global factor $\frac{1}{2} \cdot 2^{m-2} = \frac{1}{8}2^m$: this turns π^m from [CSZ19b, eq. (1.25)] into $(2\pi)^m$ in (2.9) and accounts for the extra factor $\frac{1}{8}$ in (2.8).
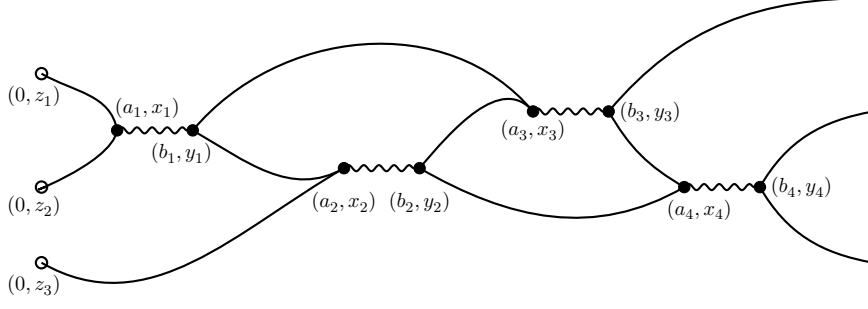
FIGURE 1. Graphical representation of the kernel $K_t^{(3)}(z_1, z_2, z_3)$ for the centered third moment, see (2.9)-(2.11). Solid-curved lines from (b, y) to (a, x) are assigned weights $g_{\frac{a-b}{2}}(x - y)$ while wiggle lines from (a, x) to (b, y) are assigned weights $G_\vartheta(b - a)g_{\frac{b-a}{4}}(y - x)$.

2.3. HIGHER MOMENTS.

A formula for higher moments of the SHF was first identified in [GQT21]. For completeness, we recall this formula in our framework.

Fix an integer $h \in \mathbb{N}$ with $h \geqslant 2$. For $t > 0$ and a pair $\{i, j\} \subset \{1, \ldots, h\}$ of *distinct* elements $i < j$, we define two measure kernels mapping from $(\mathbb{R}^2)^h$ to measures supported on the subspace

$$(\mathbb{R}^2)_{\{i,j\}}^h := \left\{ \boldsymbol{x} = (x_1, \ldots, x_h) \in (\mathbb{R}^2)^h : \; x_i = x_j \right\}. \qquad (2.12)$$

- The first measure kernel (actually a probability kernel) is called *constrained evolution*:

$$\mathsf{Q}_t^{\{i,j\}}(\boldsymbol{y}, \mathrm{d}\boldsymbol{x}) := \left(\prod_{\ell=1}^h g_{\frac{t}{2}}(x_\ell - y_\ell) \right) \cdot \left(\prod_{\ell \in \{1, \ldots, h\} \setminus \{i,j\}} \mathrm{d}x_\ell \right) \cdot \mathrm{d}x_i \cdot \delta_{x_i}(\mathrm{d}x_j), \qquad (2.13)$$

where $\delta_{x_i}(\cdot)$ denotes the Dirac mass at $x_i \in \mathbb{R}^2$ and $g_t(\cdot)$ is the heat kernel, see (2.1).

- The second measure kernel is called *replica evolution*:

$$\mathsf{G}_{\vartheta,t}^{\{i,j\}}(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) := \left(\prod_{\ell \in \{1, \ldots, h\} \setminus \{i,j\}} g_{\frac{t}{2}}(y_\ell - x_\ell) \, \mathrm{d}y_\ell \right) G_\vartheta(t) \, g_{\frac{t}{4}}(y_i - x_i) \, \mathrm{d}y_i \cdot \delta_{y_i}(\mathrm{d}y_j), \qquad (2.14)$$

where $G_\vartheta(t)$ is the function in (2.2). We will only need $\mathsf{G}_{\vartheta,t}^{\{i,j\}}(\boldsymbol{x}, \mathrm{d}\boldsymbol{y})$ with $x_i = x_j$.

We now give the higher moments formula.

Theorem 2.3. *Fix $h \in \mathbb{N}$ with $h \geqslant 2$. The h-th moment of the SHF $\mathscr{Z}_t^\vartheta$ with flat initial data, averaged over a test function $\varphi \in C_c(\mathbb{R}^2)$, admits the expression*

$$\mathbb{E}\big[\mathscr{Z}_t^\vartheta(\varphi)^h\big] = \frac{1}{2^h} \int_{(\mathbb{R}^2)^h} \varphi(z_1) \cdots \varphi(z_h) \, \mathscr{K}_t^{(h)}(z_1, \ldots, z_h) \, \mathrm{d}z_1 \cdots \mathrm{d}z_h, \qquad (2.15)$$
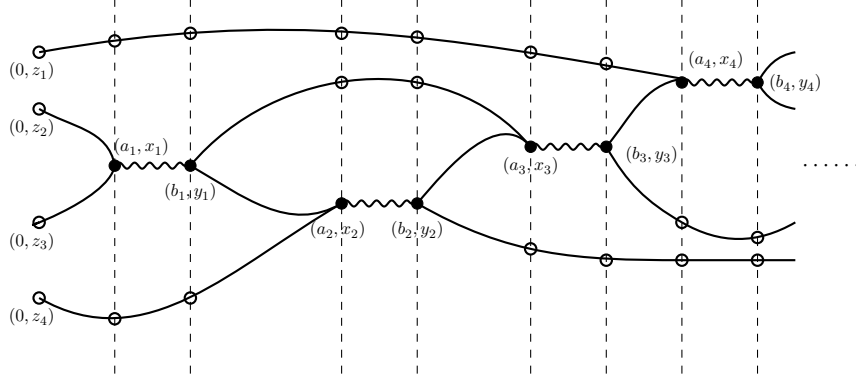
FIGURE 2. Graphical representation of the kernel $\mathscr{K}_t^{(4)}(z_1, z_2, z_3, z_4)$ for the fourth moment, see (2.16). The solid-curved and wiggle lines are assigned the same weights as in Figure 1. The hollow circles on the vertical dashed lines are where we apply the Champman-Kolmogorov decomposition (see also Remark 2.6).

with

$$
\mathscr{K}_t^{(h)}(z_1, \ldots, z_h)
$$

$$
:= 1 + \sum_{m=1}^{\infty} (2\pi)^m \sum_{\substack{\{i_1 \neq j_1\}, \ldots, \{i_m \neq j_m\} \subset \{1, \ldots, h\} \\ \text{with } \{i_\ell, j_\ell\} \neq \{i_{\ell-1}, j_{\ell-1}\} \, \forall \ell \geqslant 2}} \int \cdots \int_{0 < a_1 < b_1 < \ldots < a_m < b_m < t} \mathrm{d}\vec{a}\,\mathrm{d}\vec{b} \int \cdots \int_{(\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \in ((\mathbb{R}^2)^h)^{2m}} \tag{2.16}
$$

$$
\mathsf{Q}_{a_1}^{\{i_1, j_1\}}(\boldsymbol{z}, \mathrm{d}\boldsymbol{x}_1)\, \mathsf{G}_{\vartheta, b_1 - a_1}^{\{i_1, j_1\}}(\boldsymbol{x}_1, \mathrm{d}\boldsymbol{y}_1) \prod_{\ell=2}^{m} \mathsf{Q}_{a_\ell - b_{\ell-1}}^{\{i_\ell, j_\ell\}}(\boldsymbol{y}_{\ell-1}, \mathrm{d}\boldsymbol{x}_\ell)\, \mathsf{G}_{\vartheta, b_\ell - a_\ell}^{\{i_\ell, j_\ell\}}(\boldsymbol{x}_\ell, \mathrm{d}\boldsymbol{y}_\ell).
$$

This result can be proved by arguing as in [CSZ21, Section 6], exploiting the local limit theory for the Dickman subordinator as developed in [CSZ19a]. Formula (2.16) coincides with the one obtained in [GQT21] up to a simple scaling, see Proposition 5.3 below.

Remark 2.4. *The integral over the space variables $\vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}$ in (2.16) can be restricted to the subspace $\big((\mathbb{R}^2)_{\{i_1, j_1\}}^h\big)^2 \times \ldots \times \big((\mathbb{R}^2)_{\{i_m, j_m\}}^h\big)^2 \subseteq ((\mathbb{R}^2)^h)^{2m}$, see (2.12). This is because the kernels $\mathsf{Q}_t^{\{i,j\}}$ and $\mathsf{G}_{\vartheta, t}^{\{i,j\}}$ in (2.13)-(2.14) are measures supported on $(\mathbb{R}^2)_{\{i,j\}}^h$.*

Remark 2.5. *Centered moments $\mathbb{E}\big[\big(\mathscr{X}_t^\vartheta(\varphi) - \mathbb{E}[\mathscr{X}_t^\vartheta(\varphi)]\big)^h\big]$ admit formulas analogous to (2.15)-(2.16), with a correlation kernel $K_t^{(h)}(z_1, \ldots, z_h)$ which is obtained from (2.16) by removing the constant term "1+" and imposing the constraint $\bigcup_{\ell=1}^{m} \{i_\ell, j_\ell\} = \{1, \ldots, h\}$ in the sum over $\{i_1 \neq j_1\}, \ldots, \{i_m \neq j_m\} \subset \{1, \ldots, h\}$ (incidentally, this requires $m \geqslant \lceil \frac{h}{2} \rceil$).*

Remark 2.6. *In the special case $h = 3$, formulas (2.15)-(2.16) are consistent with formulas (2.8)-(2.11) for the centered third moment. To check this, it suffices to decompose the heat*

kernels $g_{\frac{a_\ell - b_{\ell-2}}{2}}(x_\ell - y_{\ell-2})$ *in* (2.11) *at times* $a_{\ell-1}, b_{\ell-1}$ *by Chapman-Kolmogorov:*

$$g_{\frac{a_\ell - b_{\ell-2}}{2}}(x_\ell - y_{\ell-2}) = \iint\limits_{(\mathbb{R}^2)^2} \mathrm{d}x' \, \mathrm{d}y' \, g_{\frac{a_{\ell-1} - b_{\ell-2}}{2}}(x' - y_{\ell-2}) \, g_{\frac{b_{\ell-1} - a_{\ell-1}}{2}}(y' - x') \, g_{\frac{a_\ell - b_{\ell-1}}{2}}(x_\ell - y'),$$

which gives rise to the operators $\mathsf{Q}_{a_\ell - b_{\ell-1}}^{\{i_\ell, j_\ell\}}$, $\mathsf{G}_{\vartheta, b_{\ell-1} - a_{\ell-1}}^{\{i_{\ell-1}, j_{\ell-1}\}}$ *and* $\mathsf{Q}_{a_\ell - b_{\ell-1}}^{\{i_{\ell-1}, j_{\ell-1}\}}$, *see* (2.13) *and* (2.14).
See also Figure 2 for the application of Chapman-Kolmogorov (in the case $h = 4$).

3. GMC and its moments

As already mentioned in the introduction, a nice review of the Gaussian Multiplicative Chaos (GMC) and its various connections can be found in [RV14]. Here we present its definition and the structure of its moments, which is relevant towards our goals.

3.1. Construction of GMC. Let $k : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be a kernel which is symmetric, locally integrable and positive definite, i.e. $\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x) \, k(x, y) \, \varphi(y) \, \mathrm{d}x \, \mathrm{d}y \geqslant 0$ for all $\varphi \in C_c(\mathbb{R}^2)$. Let $\mathscr{X} = (\mathscr{X}(\varphi))_{\varphi \in C_c(\mathbb{R}^2)}$ be the centered Gaussian field with covariance

$$k(\varphi, \psi) := \iint\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x) \, k(x, y) \, \psi(y) \, \mathrm{d}x \, \mathrm{d}y \qquad \text{for } \varphi, \psi \in C_c(\mathbb{R}^2).$$

Let us fix a locally finite measure μ on $\mathbb{R}^2$. The Gaussian Multiplicative Chaos (GMC) associated to $\mathscr{X}$ with respect to the measure μ, denoted by $\mathscr{M}(\mathrm{d}x)$, is formally given by

$$\mathscr{M}(\mathrm{d}x) =: \exp(\mathscr{X}(x)) \, \mu(\mathrm{d}x) : .$$

For a precise definition, for $\varepsilon > 0$ we take a continuous regularization $k_\varepsilon(x, y)$ of $k(x, y)$, still positive definite, such that $\lim_{\varepsilon \downarrow 0} k_\varepsilon(x, y) = k(x, y)$ locally uniformly in x, y. We can then consider the centered Gaussian process $\mathscr{X}_\varepsilon = (\mathscr{X}_\varepsilon(x))_{x \in \mathbb{R}^2}$ with covariance $k_\varepsilon(x, y)$, which is well-defined pointwise, and we define for $\varepsilon > 0$

$$\mathscr{M}_\varepsilon(\mathrm{d}x) := \mathrm{e}^{\mathscr{X}_\varepsilon(x) - \frac{1}{2}\mathbb{E}[\mathscr{X}_\varepsilon(x)^2]} \, \mu(\mathrm{d}x) = \mathrm{e}^{\mathscr{X}_\varepsilon(x) - \frac{1}{2}k_\varepsilon(x, x)} \, \mu(\mathrm{d}x).$$

The GMC $\mathscr{M}(\mathrm{d}x)$ is then defined as the following limit in distribution:

$$\mathscr{M}(\mathrm{d}x) := \lim_{\varepsilon \downarrow 0} \mathscr{M}_\varepsilon(\mathrm{d}x),$$

assuming that it exists in the vague sense: for $\varphi \in C_c(\mathbb{R}^2)$,

$$\mathscr{M}_\varepsilon(\varphi) := \int_{\mathbb{R}^2} \varphi(x) \, \mathscr{M}_\varepsilon(\mathrm{d}x) \xrightarrow[\varepsilon \downarrow 0]{} \mathscr{M}(\varphi) := \int_{\mathbb{R}^2} \varphi(x) \, \mathscr{M}(\mathrm{d}x).$$

3.2. Moments of GMC. By construction, for $\varepsilon > 0$ we have

$$\mathbb{E}[\mathscr{M}_\varepsilon(\varphi)] = \int_{\mathbb{R}^2} \varphi(z) \, \mu(\mathrm{d}z). \tag{3.1}$$

Since $\mathbb{E}[\mathrm{e}^{\mathscr{X}_\varepsilon(x) + \mathscr{X}_\varepsilon(y)}] = \mathbb{E}[\mathrm{e}^{\frac{1}{2}\mathrm{Var}[\mathscr{X}_\varepsilon(x) + \mathscr{X}_\varepsilon(y)]}] = \mathrm{e}^{\frac{1}{2}\{k_\varepsilon(x,x) + k_\varepsilon(y,y) + 2k_\varepsilon(x,y)\}}$, we obtain

$$\mathbb{E}[\mathscr{M}_\varepsilon(\varphi)^2] = \iint\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(z_1) \, \varphi(z_2) \, \mathrm{e}^{k_\varepsilon(z_1, z_2)} \, \mu(\mathrm{d}z_1) \, \mu(\mathrm{d}z_2). \tag{3.2}$$

Similarly, since $\mathbb{E}[\mathrm{e}^{\mathscr{X}_\varepsilon(z_1)+\cdots+\mathscr{X}_\varepsilon(z_m)}] = \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^m k_\varepsilon(z_i,z_j)}$, we have

$$\mathbb{E}[\mathscr{M}_\varepsilon(\varphi)^m] = \int_{(\mathbb{R}^2)^m} \varphi(z_1)\cdots\varphi(z_m)\, \mathrm{e}^{\sum_{1\leqslant i<j\leqslant m} k_\varepsilon(z_i,z_j)}\, \mu(\mathrm{d}z_1)\cdots\mu(\mathrm{d}z_m)\,. \qquad (3.3)$$

When we let $\varepsilon \downarrow 0$, these formulas apply to $\mathscr{M}(\varphi)$ once we replace $k_\varepsilon(z_i,z_j)$ by $k(z_i,z_j)$.

Let us now record the centered second and third moments of GMC.

- *Centered second moment*:

$$\mathbb{E}\big[\big(\mathscr{M}(\varphi)-\mathbb{E}[\mathscr{M}(\varphi)]\big)^2\big] = \int_{(\mathbb{R}^2)^2} \varphi(z_1)\,\varphi(z_2)\, K_{\mathrm{GMC}}^{(2)}(z_1,z_2)\, \mu(\mathrm{d}z_1)\,\mu(\mathrm{d}z_2) \qquad (3.4)$$

$$\text{where} \qquad K_{\mathrm{GMC}}^{(2)}(z_1,z_2) := \mathrm{e}^{k(z_1,z_2)} - 1\,.$$

- *Centered third moment*:

$$\mathbb{E}\big[\big(\mathscr{M}(\varphi)-\mathbb{E}[\mathscr{M}(\varphi)]\big)^3\big] = \int_{(\mathbb{R}^2)^3} \varphi(z_1)\varphi(z_2)\varphi(z_3)\, K_{\mathrm{GMC}}^{(3)}(z_1,z_2,z_3)\, \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\mu(\mathrm{d}z_3) \qquad (3.5)$$

$$\text{where} \qquad K_{\mathrm{GMC}}^{(3)}(z_1,z_2,z_3) := \prod_{1\leqslant i<j\leqslant 3} \mathrm{e}^{k(z_i,z_j)} - \sum_{1\leqslant i<j\leqslant 3} \mathrm{e}^{k(z_i,z_j)} + 2\,.$$

Comparing (3.5) with (3.4), we see that the following structural relation holds:

$$\begin{aligned}
K_{\mathrm{GMC}}^{(3)}(z_1,z_2,z_3) = \ & K_{\mathrm{GMC}}^{(2)}(z_1,z_2)\, K_{\mathrm{GMC}}^{(2)}(z_2,z_3)\, K_{\mathrm{GMC}}^{(2)}(z_1,z_3) \\
& + K_{\mathrm{GMC}}^{(2)}(z_1,z_2)\, K_{\mathrm{GMC}}^{(2)}(z_2,z_3) \\
& + K_{\mathrm{GMC}}^{(2)}(z_1,z_2)\, K_{\mathrm{GMC}}^{(2)}(z_1,z_3) \\
& + K_{\mathrm{GMC}}^{(2)}(z_1,z_3)\, K_{\mathrm{GMC}}^{(2)}(z_2,z_3)\,.
\end{aligned} \qquad (3.6)$$

3.3. A GMC Matching the First Two Moments of SHF.

Henceforth we denote by $\mathscr{M}_t^\vartheta(\mathrm{d}x)$ the GMC with the same first and second moments as the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$. Comparing (3.1) and (3.4) with (2.6) and (2.7), we see that this can be obtained once we fix

$$\mu(\mathrm{d}x) := \frac{1}{2}\,\mathrm{d}x\,, \qquad k_t(z_1,z_2) = \log\big(1 + K_t^{(2)}(z_1,z_2)\big)\,, \qquad (3.7)$$

where $K_t^{(2)}$ is defined in (2.7). This ensures that $K_{\mathrm{GMC}}^{(2)}(z_1,z_2) = K_t^{(2)}(z_1,z_2)$.[†] To show that $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ is not a GMC, it suffices to show that the higher moments of $\mathscr{M}_t^\vartheta(\mathrm{d}x)$ and $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ do not match.

4. Proof of Theorem 1.3: lower bounds via Gaussian integrals

In this section we prove Theorem 1.3: the third moment of the critical 2d SHF $\mathscr{Z}_t^\vartheta(\varphi)$ is strictly larger than that of a GMC $\mathscr{M}_t^\vartheta(\varphi)$ with matching first and second moments, when averaged over suitable integrable functions $\varphi : \mathbb{R}^2 \to [0,\infty)$.

Remark 4.1. *Most steps of our analysis cover any integrable function* $\varphi : \mathbb{R}^2 \to [0,\infty)$ *which is* radially symmetric and non-increasing, *that is* $\varphi(x) = \varrho(|x|)$ *for some non-increasing function* $\varrho : [0,\infty) \to [0,\infty]$, *with* $|\cdot|$ *the Euclidean norm. Only in the last step we need a*

[†]By (1.10)-(1.11) for the uncentered correlation function $\frac{1}{4}\mathscr{K}_{t,\vartheta}^{(2)}(z_1,z_2) = e^{k_t(z_1,z_2)}$, the covariance kernel of the Gaussian field underlying the GMC satisfies $k_t(z_1,z_2) \sim \log\log\frac{1}{|z_1-z_2|}$ as $|z_1 - z_2| \to 0$.

basic inequality, see Proposition 4.4, that we prove when φ is the heat kernel or the indicator function of a ball, as in Theorem 1.3. We believe that Proposition 4.4 should hold in greater generality —possibly, as soon as ϱ is log-concave— but this remains open.

Let us fix an integrable function φ, $t > 0$ and $\vartheta \in \mathbb{R}$. Our goal is to prove that

$$\mathbb{E}\big[\mathscr{L}_t^\vartheta(\varphi)^3\big] > \mathbb{E}\big[\mathscr{M}_t^\vartheta(\varphi)^3\big].$$

Since first and second moments match, it is equivalent to work with *centered* third moments:

$$\mathbb{E}\big[\big(\mathscr{L}_t^\vartheta(\varphi) - \mathbb{E}[\mathscr{L}_t^\vartheta(\varphi)]\big)^3\big] > \mathbb{E}\big[\big(\mathscr{M}_t^\vartheta(\varphi) - \mathbb{E}[\mathscr{M}_t^\vartheta(\varphi)]\big)^3\big]. \tag{4.1}$$

In view of (2.8) and (3.5), see also (3.7), we can rewrite (4.1) as

$$K_t^{(3)}(\varphi) > K_{\mathrm{GMC}}^{(3)}(\varphi), \tag{4.2}$$

where, given a kernel $H(z_1, z_2, z_3)$, we use the shorthand

$$H(\varphi) := \iiint\limits_{(\mathbb{R}^2)^3} \varphi(z_1)\,\varphi(z_2)\,\varphi(z_3)\,H(z_1, z_2, z_3)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,. \tag{4.3}$$

It remains to prove (4.2). The kernel $K_t^{(3)}$ is complicated, but we can perform an almost exact computation of the function $\mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}^{(m)}(z_1, z_2, z_3)$ in (2.11), see Proposition 4.6 below. From this we obtain a lower bound on $K_t^{(3)}(\varphi)$ (Proposition 4.2), that we complement with an upper bound on $K_{\mathrm{GMC}}^{(3)}(\varphi)$ (Proposition 4.3). At last, we will show that these bounds are compatible (Proposition 4.4), which yields our goal (4.2).

Let us introduce two key quantities $\mathscr{G}_{a_1,a_2}(\varphi)$ and $\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$ that enter our bounds:

$$\mathscr{G}_{a_1,a_2}(\varphi) := (2\pi)^2 \iiint\limits_{(\mathbb{R}^2)^3} \varphi(z_1)\,\varphi(z_2)\,\varphi(z_3)\,g_{a_1}(z_2 - z_1)\,g_{a_2}\big(z_3 - \tfrac{z_1+z_2}{2}\big)\,\mathrm{d}\vec{z}, \tag{4.4}$$

$$\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi) := (2\pi)^2 \iiint\limits_{(\mathbb{R}^2)^3} \varphi(z_1)\,\varphi(z_2)\,\varphi(z_3)\,g_{a_1}(z_2 - z_1)\,g_{a_2}(z_3 - z_2)\,\mathrm{d}\vec{z}, \tag{4.5}$$

where $g_t(z)$ denotes the heat kernel, see (2.1). We can now state our lower bound on $K_t^{(3)}(\varphi)$ which involves the quantity $\mathscr{G}_{a_1,a_2}(\varphi)$.

Proposition 4.2 (Third moment lower bound for the SHF). *Fix $\vartheta \in \mathbb{R}$ and $t > 0$. Let $K_t^{(3)}$ be the centered third moment kernel of the critical 2d SHF $\mathscr{L}_t^\vartheta$, see (2.8)-(2.9). For any integrable function $\varphi : \mathbb{R}^2 \to [0, \infty)$ which is radially symmetric and non-increasing (see Remark 4.1), we have the strict lower bound*

$$K_t^{(3)}(\varphi) > I_t^{(3)}(\varphi), \tag{4.6}$$

where we define

$$I_t^{(3)}(\varphi) := 3 \sum_{m=2}^{\infty} 2^{m-1} \int\limits_{0<a_1<b_1<\ldots<a_m<b_m<t} \cdots \int \mathscr{G}_{a_1,a_2}(\varphi)\,G_\vartheta(b_1 - a_1)\,G_\vartheta(b_2 - a_2)$$
$$\times \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}}\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{b}, \tag{4.7}$$

with $\mathscr{G}_{a_1,a_2}(\varphi)$ as in (4.4) and G_ϑ as in (2.2).

We refer to Figure 3 for a graphical representation of $I_t^{(3)}(\varphi)$ when $\varphi = g_r$ is the heat kernel, in which case $\mathscr{G}_{a_1,a_2}(\varphi)$ can be computed explicitly (see Remark 4.5).

We next state an upper bound on $K_{\mathrm{GMC}}^{(3)}(\varphi)$ which involves the quantity $\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$. Interestingly, this bound applies to any positive integrable function φ.

Proposition 4.3 (Third moment upper bound for GMC). *Fix $\vartheta \in \mathbb{R}$ and $t > 0$. Let $K_{\mathrm{GMC}}^{(3)}$ be the centered third moment kernel of the GMC $\mathscr{M}_t^\vartheta$, see (3.5) and (3.7). For any integrable function $\varphi : \mathbb{R}^2 \to [0, \infty)$ we have the strict upper bound*

$$K_{\mathrm{GMC}}^{(3)}(\varphi) \;<\; \widetilde{I}_t^{(3)}(\varphi)\,, \tag{4.8}$$

where we define

$$\widetilde{I}_t^{(3)}(\varphi) := 3 \sum_{m=2}^{\infty} 2^{m-1} \int \cdots \int_{0 < a_1 < b_1 < \ldots < a_m < b_m < t} \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)\, G_\vartheta(b_1 - a_1)\, G_\vartheta(b_2 - a_2)$$
$$\times \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}}\, \mathrm{d}\vec{a}\, \mathrm{d}\vec{b}\,, \tag{4.9}$$

with $\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$ as in (4.5) and G_ϑ as in (2.2).

Note that $\widetilde{I}_t^{(3)}(\varphi)$ in (4.9) is like $I_t^{(3)}(\varphi)$ in (4.7), just with $\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$ in place of $\mathscr{G}_{a_1,a_2}(\varphi)$. If $\mathscr{G}_{a_1,a_2}(\varphi) > \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$, then we can combine the bounds (4.6) and (4.8) to yield our goal (4.2). We finally show that this indeed holds when φ is the indicator function of a ball, or the heat kernel, which completes the proof of Theorem 1.3.

Proposition 4.4 (Comparison of bounds). *Recall $\mathscr{G}_{a_1,a_2}(\varphi)$ and $\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$ from (4.4)-(4.5). Let φ be the indicator function of a ball or the heat kernel, see (2.1):*

$$\varphi = \mathbb{1}_{\{x \in \mathbb{R}^2 : \, |x| \leqslant r\}} \qquad or \qquad \varphi = g_r\,, \qquad r > 0\,. \tag{4.10}$$

Then we have

$$\mathscr{G}_{a_1,a_2}(\varphi) > \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi) \qquad \forall a_1, a_2 > 0\,. \tag{4.11}$$

Recalling (4.7) and (4.9), it follows that for any $\vartheta \in \mathbb{R}$ and $t > 0$

$$I_t^{(3)}(\varphi) > \widetilde{I}_t^{(3)}(\varphi)\,, \tag{4.12}$$

therefore, in view of (4.6) and (4.8), one has $K_t^{(3)}(\varphi) > K_{\mathrm{GMC}}^{(3)}(\varphi)$.

Remark 4.5. *When $\varphi = g_r$ is the heat kernel, $\mathscr{G}_{a_1,a_2}(\varphi)$ and $\widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$ in (4.4)-(4.5) can be computed by an explicit Gaussian integration (see Subsection 4.3):*

$$\mathscr{G}_{a_1,a_2}(g_r) = \frac{1}{a_1 + 2r}\, \frac{1}{a_2 + \frac{3}{2}r}\,, \qquad \widetilde{\mathscr{G}}_{a_1,a_2}(g_r) = \frac{1}{a_1 a_2 + 2r(a_1 + a_2) + 3r^2}\,, \tag{4.13}$$

and in this case one sees easily that $\mathscr{G}_{a_1,a_2}(g_r) > \widetilde{\mathscr{G}}_{a_1,a_2}(g_r)$, in agreement with (4.12).

A graphical representation of $I_t^{(3)}(\varphi)$ for $\varphi = g_r$ is given in Figure 3.

It only remains to prove Propositions 4.2, 4.3 and 4.4, to which Subsections 4.1, 4.2 and 4.3 are devoted.
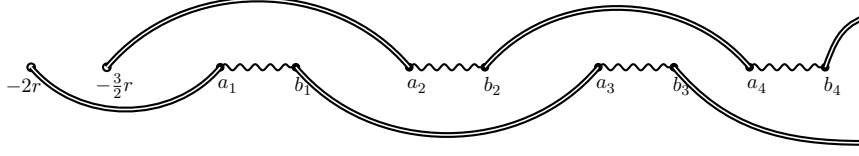
FIGURE 3. Graphical representation of $I_t^{(3)}(\varphi)$, see (4.7), when $\varphi = g_r$ is the heat kernel, see (4.13). More specifically, we represent the term $m = 4$ in the series in (4.7). Double solid lines from b_{i-2} to a_i are assigned weights $(a_i - b_{i-2})^{-1}$, while wiggle lines from a_i to b_i are assigned weights $G_\vartheta(b_i - a_i)$.

4.1. PROOF OF PROPOSITION 4.2.

The heart of the proof is the following "computation" of the function $\mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}^{(m)}(z_1, z_2, z_3)$ in (2.11), which we will prove below.

Proposition 4.6. *For $m \geqslant 2$, $0 < a_1 < b_1 < \ldots < a_m < b_m$, $z_1, z_2, z_3 \in \mathbb{R}^2$ and $\mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}^{(m)}(z_1, z_2, z_3)$ as in (2.11), we have*

$$\mathbf{g}_{a_1,b_1,\ldots,a_m,b_m}^{(m)}(z_1, z_2, z_3) = g_{a_1}(z_1 - z_2) \cdot g_{\overline{a_2}^{(m)}}\big(z_3 - \tfrac{z_1+z_2}{2}\big) \cdot \prod_{i=3}^{m} g_{\overline{a_i - b_{i-2}}^{(m)}}(0) \,, \qquad (4.14)$$

for suitable variables $\overline{a_2}^{(m)}$ and $\overline{a_i - b_{i-2}}^{(m)}$ (depending on $a_1, b_1, \ldots, a_m, b_m$) which satisfy

$$\overline{a_2}^{(m)} \leqslant a_2 - \tfrac{b_1}{4} < a_2 \,,$$
$$\overline{a_i - b_{i-2}}^{(m)} \leqslant a_i - b_{i-2} - \tfrac{b_{i-1} - a_{i-1}}{4} < a_i - b_{i-2} \,. \qquad (4.15)$$

We will also need a basic monotonicity property for the function $\mathscr{G}_{a_1,a_2}(\varphi)$ in (4.4).

Lemma 4.7. *If $\varphi : \mathbb{R}^2 \to [0, \infty)$ is integrable, radially symmetric and non-increasing (see Remark 4.1), then the function $\mathscr{G}_{a_1,a_2}(\varphi)$ in (4.4) is strictly decreasing in $a_2 > 0$.*

Proof. By the change of variables $x := z_1$, $y := z_3 - \tfrac{z_1+z_2}{2}$, $z := z_3$, we can write

$$\mathscr{G}_{a_1,a_2}(\varphi) := (2\pi)^2 \int_{\mathbb{R}^2} f(y) \, g_{a_2}(y) \, \mathrm{d}y \,, \qquad (4.16)$$

where we define

$$f(y) := \int_{\mathbb{R}^2} h(z - y) \, \varphi(z) \, \mathrm{d}z \,, \qquad h(w) := \int_{\mathbb{R}^2} \varphi(x) \, \varphi(2w - x) \, g_{a_1}(2w - 2x) \, \mathrm{d}x \,. \qquad (4.17)$$

By (4.16) we can write $\mathscr{G}_{a_1,a_2}(\varphi) = (2\pi)^2 \, \mathrm{E}[f(a_2 Z)]$, where Z is a standard Gaussian random variable on $\mathbb{R}^2$ (with density g_1). Then, to prove that $a_2 \mapsto \mathscr{G}_{a_1,a_2}(\varphi)$ is strictly decreasing, it is enough to prove that f is radially non-increasing (see Remark 4.1). We actually show that *both f and h are radially symmetric and non-increasing*, which completes the proof.

We recall the layer cake decomposition of a radially symmetric and non-increasing function:

$$\text{for a.e. } x \in \mathbb{R}^2 : \qquad \varphi(x) = \int_0^\infty \mathbb{1}_{\{|x|<r\}} \, \mu^\varphi(\mathrm{d}r) \,, \qquad (4.18)$$

where μ^φ is a positive measure on $(0, \infty)$, defined by $\mu^\varphi((r, \infty)) := \varphi((r, 0))$. Using a similar decomposition for g_{a_1}, we replace the three factors φ, φ and g_{a_1} in the definition of h by

$\mathbb{1}_{\{|\cdot|<r_1\}}$, $\mathbb{1}_{\{|\cdot|<r_2\}}$, and $\mathbb{1}_{\{|\cdot|<s\}}$ and show that for any $r_1, r_2, s > 0$ the resulting function $\hat{h}$ is radially symmetric and non-increasing:

$$
\begin{aligned}
\hat{h}(w) &:= \int_{\mathbb{R}^2} \mathbb{1}_{\{|x|<r_1\}} \, \mathbb{1}_{\{|x-2w|<r_2\}} \, \mathbb{1}_{\{|x-w|<\frac{s}{2}\}} \, \mathrm{d}x \\
&= \mathrm{Leb}(B(0,r_1) \cap B(w,\tfrac{s}{2}) \cap B(2w,r_2)) \\
&= \mathrm{Leb}(B(-w,r_1) \cap B(0,\tfrac{s}{2}) \cap B(w,r_2)) \,,
\end{aligned}
\tag{4.19}
$$

where $B(z,r) := \{x \in \mathbb{R}^2 \colon |x| < r\}$ is the ball of radius r centered at z. It is clear that $\hat{h}$, hence h, is radially symmetric and non-increasing.

Note that we can write $f = \varphi * h$ as the convolution of two radially symmetric and non-increasing functions. If we replace h and φ by $\mathbb{1}_{\{|\cdot|<t\}}$ and $\mathbb{1}_{\{|\cdot|<r\}}$, by the layer cake decomposition, we get the function

$$
\hat{f}(y) = \int_{\mathbb{R}^2} \mathbb{1}_{\{|z-y|<t\}} \, \mathbb{1}_{\{|z|<r\}} \, \mathrm{d}z = \mathrm{Leb}(B(0,r) \cap B(y,t)) \,,
$$

which is clearly radially symmetric and non-increasing, hence the same holds for f. $\qquad\square$

It is now easy to prove Proposition 4.2. When we average $\mathbf{g}^{(m)}_{a_1,b_1,\dots,a_m,b_m}(z_1, z_2, z_3)$ with respect to the function φ as in (4.3), we can apply (4.14) to write, recalling (4.4),

$$
\mathbf{g}^{(m)}_{a_1,b_1,\dots,a_m,b_m}(\varphi) \;=\; \frac{1}{(2\pi)^2} \, \mathscr{G}_{a_1,\overline{a_2}^{(m)}}(\varphi) \prod_{i=3}^{m} g_{\overline{a_i-b_{i-2}}^{(m)}}(0) \,.
$$

Since $t \mapsto g_t(0)$ and $a_2 \mapsto \mathscr{G}_{a_1,a_2}(\varphi)$ are strictly decreasing functions, we obtain the bound

$$
\mathbf{g}^{(m)}_{a_1,b_1,\dots,a_m,b_m}(\varphi) \;>\; \frac{1}{(2\pi)^2} \, \mathscr{G}_{a_1,a_2}(\varphi) \prod_{i=3}^{m} g_{a_i-b_{i-2}}(0) \;=\; \frac{1}{(2\pi)^m} \, \mathscr{G}_{a_1,a_2}(\varphi) \prod_{i=3}^{m} \frac{1}{a_i - b_{i-2}} \,.
$$

Plugging this into (2.9)-(2.10), we obtain $K_t^{(3)}(\varphi) > I_t^{(3)}(\varphi)$ with $I_t^{(3)}(\varphi)$ defined in (4.7). This completes the proof of Proposition 4.2.

We are left with proving Proposition 4.6. A key tool is the following elementary lemma.

Lemma 4.8 (Triple Gaussian integral). *Let $g_t(x)$ be the two-dimensional heat kernel, see (2.1). For all $s, t > 0$ and $x, a, b \in \mathbb{R}^2$ we have*

$$
g_s(x-a) \, g_t(x-b) = g_{s+t}(a-b) \, g_{h(s,t)}(x - m_{t,s}(a,b)) \,,
\tag{4.20}
$$

where we set

$$
h(s,t) := \left(\tfrac{1}{s} + \tfrac{1}{t}\right)^{-1} = \frac{st}{s+t} \,, \qquad m_{t,s}(x,y) := \frac{t}{s+t}x + \frac{s}{s+t}y \,.
\tag{4.21}
$$

It follows that for all $s, t, u > 0$ and $a, b, c \in \mathbb{R}^2$ we have

$$
\int_{\mathbb{R}^2} g_s(x-a) \, g_t(x-b) \, g_u(x-c) \, \mathrm{d}x = g_{s+t}(a-b) \, g_{h(s,t)+u}(c - m_{t,s}(a,b)) \,.
\tag{4.22}
$$

Proof. (4.20) follows directly from the definition (2.1) of the heat kernel and an easy algebraic manipulation. Then (4.22) follows by (4.20) and a simple Gaussian convolution. $\quad\square$

Proof of Proposition 4.6. We first prove (4.14)-(4.15) for $m = 2$. Recall that, by (2.11),

$$
\mathbf{g}^{(2)}_{a_1,b_1,a_2,b_2}(z_1, z_2, z_3) = \iiiint_{(\mathbb{R}^2)^4} g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}}(y_1 - x_1)
$$
$$
\cdot g_{\frac{a_2}{2}}(x_2 - z_3)\, g_{\frac{a_2-b_1}{2}}(x_2 - y_1) \cdot g_{\frac{b_2-a_2}{4}}(y_2 - x_2)\, \mathrm{d}x_1\, \mathrm{d}y_1\, \mathrm{d}x_2\, \mathrm{d}y_2 .
$$
(4.23)

Since $\int_{\mathbb{R}^2} g_s(x-a)\, g_t(x-b)\, \mathrm{d}x = g_{s+t}(a-b)$, we can integrate y_2, then x_2, then y_1 to get

$$
\iint_{(\mathbb{R}^2)^2} g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}}(y_1 - x_1) \cdot g_{\frac{a_2}{2}+\frac{a_2-b_1}{2}}(y_1 - z_3)\, \mathrm{d}x_1\, \mathrm{d}y_1
$$
$$
= \int_{\mathbb{R}^2} g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}+\frac{a_2}{2}+\frac{a_2-b_1}{2}}(x_1 - z_3)\, \mathrm{d}x_1 .
$$

Applying (4.22) to compute the last integral over x_1, we finally obtain

$$
\mathbf{g}^{(2)}_{a_1,b_1,a_2,b_2}(z_1, z_2, z_3) = g_{a_1}(z_1 - z_2)\, g_{\overline{a_2}^{(2)}}\left(z_3 - \tfrac{z_1+z_2}{2}\right),
$$
(4.24)

where we set

$$
\overline{a_2}^{(2)} := \frac{a_2}{2} + \frac{a_2-b_1}{2} + \frac{b_1}{4} = a_2 - \frac{b_1}{4} .
$$
(4.25)

This completes the proof of (4.14)-(4.15) for $m = 2$.

We next move to $m \geqslant 3$. In formula (2.11), the terms depending on x_m and y_m are

$$
g_{\frac{a_m-b_{m-2}}{2}}(x_m - y_{m-2})\, g_{\frac{a_m-b_{m-1}}{2}}(x_m - y_{m-1}) \cdot g_{\frac{b_m-a_m}{4}}(y_m - x_m),
$$
(4.26)

which after integration over y_m and x_m give

$$
g_{\frac{a_m-b_{m-2}}{2}+\frac{a_m-b_{m-1}}{2}}(y_{m-1} - y_{m-2}) = g_{a_m - \frac{b_{m-1}+b_{m-2}}{2}}(y_{m-1} - y_{m-2}).
$$
(4.27)

This shows that we can rewrite (2.11) for $m \geqslant 3$ as follows:

$$
\mathbf{g}^{(m)}_{a_1,b_1,\ldots,a_m}(z_1, z_2, z_3) := \iint_{(\mathbb{R}^2)^{m-1}\times(\mathbb{R}^2)^{m-1}} \mathrm{d}\vec{x}\, \mathrm{d}\vec{y}\, g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}}(y_1 - x_1)
$$
$$
\cdot g_{\frac{a_2}{2}}(x_2 - z_3)\, g_{\frac{a_2-b_1}{2}}(x_2 - y_1) \cdot g_{\frac{b_2-a_2}{4}}(y_2 - x_2)
$$
$$
\cdot \prod_{i=3}^{m-1} \left\{ g_{\frac{a_i-b_{i-2}}{2}}(x_i - y_{i-2})\, g_{\frac{a_i-b_{i-1}}{2}}(x_i - y_{i-1}) \cdot g_{\frac{b_i-a_i}{4}}(y_i - x_i) \right\}
$$
$$
\cdot g_{a_m - \frac{b_{m-1}+b_{m-2}}{2}}(y_{m-1} - y_{m-2}),
$$
(4.28)

where we agree that $\prod_{i=3}^{m-1}\{\ldots\} := 1$ for $m = 3$. We note that b_m does not appear in the r.h.s. of (4.28), hence we dropped it from the notation $\mathbf{g}^{(m)}_{a_1,b_1,\ldots,a_m}(z_1, z_2, z_3)$.

We are ready to prove (4.14)-(4.15) for $m \geqslant 3$ by induction. For $m = 3$, (4.28) becomes

$$
\mathbf{g}^{(3)}_{a_1,b_1,a_2,b_2,a_3}(z_1, z_2, z_3) = \iiiint_{(\mathbb{R}^2)^4} g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}}(y_1 - x_1)
$$
$$
\cdot g_{\frac{a_2}{2}}(x_2 - z_3)\, g_{\frac{a_2-b_1}{2}}(x_2 - y_1) \cdot g_{\frac{b_2-a_2}{4}}(y_2 - x_2)
$$
$$
\cdot g_{a_3 - \frac{b_1+b_2}{2}}(y_2 - y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1\, \mathrm{d}x_2\, \mathrm{d}y_2,
$$

and integrating over y_2 we obtain

$$\iiint\limits_{(\mathbb{R}^2)^3} g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\frac{b_1-a_1}{4}}(y_1 - x_1)$$

$$\cdot g_{\frac{a_2}{2}}(x_2 - z_3)\, g_{\frac{a_2-b_1}{2}}(x_2 - y_1) \cdot g_{a_3 - \frac{b_1}{2} - \frac{a_2+b_2}{4}}(x_2 - y_1)\, \mathrm{d}x_1\, \mathrm{d}y_1\, \mathrm{d}x_2\,. \tag{4.29}$$

When we integrate the last line over x_2, by (4.22) we get

$$g_{a_3 - \frac{b_1}{2} - \frac{a_2+b_2}{4} + \frac{a_2-b_1}{2}}(0)\, g_{\frac{a_2}{2} + h(\frac{a_2-b_1}{2}, a_3 - \frac{b_1}{2} - \frac{a_2+b_2}{4})}(y_1 - z_3) = g_{\overline{a_3 - b_1}^{(3)}}(0) \cdot g_{\overline{a_2}^{(3)} - \frac{b_1}{4}}(y_1 - z_3)$$

where we define

$$\overline{a_3 - b_1}^{(3)} := (a_3 - b_1) - \frac{b_2 - a_2}{4}\,,$$

$$\overline{a_2}^{(3)} := \frac{a_2}{2} + \frac{b_1}{4} + h\left(\frac{a_2-b_1}{2},\, a_3 - \frac{b_1}{2} - \frac{a_2+b_2}{4}\right). \tag{4.30}$$

We can then perform the integral over y_1 in (4.29) to get

$$g_{\overline{a_3-b_1}^{(3)}}(0) \int\limits_{\mathbb{R}^2} g_{\frac{a_1}{2}}(x_1 - z_1)\, g_{\frac{a_1}{2}}(x_1 - z_2) \cdot g_{\overline{a_2}^{(3)} - \frac{a_1}{4}}(x_1 - z_3)\, \mathrm{d}x_1\,,$$

and a further application of (4.22) finally yields

$$\mathbf{g}^{(3)}_{a_1,b_1,a_2,b_2,a_3}(z_1, z_2, z_3) = g_{a_1}(z_1 - z_2)\, g_{\overline{a_2}^{(3)}}\left(z_3 - \frac{z_1+z_2}{2}\right) g_{\overline{a_3-b_1}^{(3)}}(0)\,. \tag{4.31}$$

This proves (4.14) for $m = 3$. To prove (4.15), we note that $h(s,t) < s$, see (4.21), hence

$$\overline{a_2}^{(3)} < \frac{a_2}{2} + \frac{b_1}{4} + \frac{a_2-b_1}{2} = a_2 - \frac{b_1}{4}\,.$$

We finally fix $m \geqslant 3$, we assume that formulas (4.14)-(4.15) hold for $\mathbf{g}^{(m)}$ and we prove that they hold for $\mathbf{g}^{(m+1)}$. To this purpose, it is enough to show that

$$\mathbf{g}^{(m+1)}_{a_1,b_1,\ldots,a_m,b_m,a_{m+1}}(z_1, z_2, z_3) = g_{\overline{a_{m+1}-b_{m-1}}^{(m+1)}}(0) \cdot \mathbf{g}^{(m)}_{a_1,b_1,\ldots,a_{m-1},b_{m-1},\widetilde{a_m}}(z_1, z_2, z_3)$$

$$\text{for suitable} \quad \overline{a_{m+1}-b_{m-1}}^{(m+1)} \leqslant a_{m+1} - b_{m-1} - \frac{b_m-a_m}{4} \quad \text{and} \quad \widetilde{a_m} < a_m\,. \tag{4.32}$$

Indeed, by the induction step we can apply (4.14)-(4.15) to $\mathbf{g}^{(m)}$ in the r.h.s., and since $\widetilde{a_m} < a_m$ we obtain (4.14)-(4.15) for $\mathbf{g}^{(m+1)}$.

It only remains to prove (4.32). If we write formula (4.28) for $\mathbf{g}^{(m+1)}_{a_1,b_1,\ldots,a_m,b_m,a_{m+1}}(z_1, z_2, z_3)$, we see that the terms which depend on x_m and y_m are

$$g_{\frac{a_m-b_{m-2}}{2}}(x_m - y_{m-2})\, g_{\frac{a_m-b_{m-1}}{2}}(x_m - y_{m-1}) \cdot g_{\frac{b_m-a_m}{4}}(y_m - x_m)$$

$$\cdot g_{a_{m+1} - \frac{b_m+b_{m-1}}{2}}(y_m - y_{m-1})\,,$$

which after integration over y_m yield

$$g_{\frac{a_m-b_{m-2}}{2}}(x_m - y_{m-2})\, g_{\frac{a_m-b_{m-1}}{2}}(x_m - y_{m-1}) \cdot g_{a_{m+1} - \frac{b_{m-1}}{2} - \frac{a_m+b_m}{4}}(x_m - y_{m-1})\,.$$

A further integration over x_m gives, by (4.22),

$$g_{a_{m+1} - \frac{b_{m-1}}{2} - \frac{a_m+b_m}{4} + \frac{a_m-b_{m-1}}{2}}(0) \cdot g_{h(\frac{a_m-b_{m-1}}{2}, a_{m+1} - \frac{b_{m-1}}{2} - \frac{a_m+b_m}{4}) + \frac{a_m-b_{m-2}}{2}}(y_{m-1} - y_{m-2})$$

$$= g_{\overline{a_{m+1}-b_{m-1}}^{(m+1)}}(0) \cdot g_{\widetilde{a_m} - \frac{b_{m-1}+b_{m-2}}{2}}(y_{m-1} - y_{m-2})\,,$$
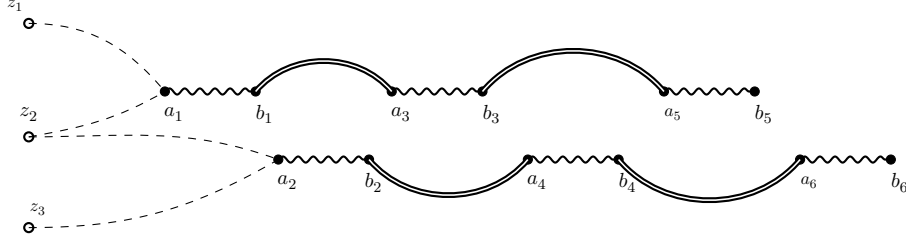
FIGURE 4. Graphical representation of the term $m = 6$ in the series (4.34) which represents $K_t^{(2)}(z_1, z_2) K_t^{(2)}(z_2, z_3)$. The total weight of the dashed lines from z_1 and z_2 to a_1 is assigned weight $g_{a_1}(z_2 - z_1)$ and the total weight of the dashed lines from z_2 and z_3 to a_2 is assigned weight $g_{a_2}(z_3 - z_2)$; a double solid line from b_{i-2} to an a_i is assigned weight $(a_i - b_{i-2})^{-1}$; a wiggle line from an a_i to b_i is assigned weight $G_\vartheta(b_i - a_i)$.

where we define

$$\overline{a_{m+1} - b_{m-1}}^{(m+1)} := (a_{m+1} - b_{m-1}) - \frac{b_m - a_m}{4} ,$$

$$\widetilde{a_m} := \frac{a_m + b_{m-1}}{2} + h\left(\frac{a_m - b_{m-1}}{2}, a_{m+1} - \frac{b_{m-1}}{2} - \frac{a_m + b_m}{4}\right).$$

Recalling (4.28), we see that (4.32) holds (note that $\widetilde{a_m} < a_m$ because $h(s, t) < s$). $\square$

4.2. PROOF OF PROPOSITION 4.3. We recall relation (3.6) satisfied by any GMC. Our choice (3.7) ensures that $K_{\text{GMC}}^{(2)}(z_1, z_2) = K_t^{(2)}(z_1, z_2)$, see (2.7), hence (3.6) becomes

$$K_{\text{GMC}}^{(3)}(z_1, z_2, z_3) = K_t^{(2)}(z_1, z_2) K_t^{(2)}(z_2, z_3) K_t^{(2)}(z_1, z_3) + K_t^{(2)}(z_1, z_2) K_t^{(2)}(z_2, z_3)$$
$$+ K_t^{(2)}(z_1, z_2) K_t^{(2)}(z_1, z_3) \quad (4.33)$$
$$+ K_t^{(2)}(z_1, z_3) K_t^{(2)}(z_2, z_3) .$$

We first give an alternative expression, that we prove below, for the product of two covariance kernels which appear in the r.h.s. of (4.33).

Lemma 4.9 (Double correlation product). *The following equality holds:*

$$K_t^{(2)}(z_1, z_2) K_t^{(2)}(z_2, z_3) = (2\pi)^2 \sum_{m=2}^{\infty} \int \cdots \int_{0 < a_1 < b_1 < \ldots < a_m < b_m < t} \mathrm{d}\vec{a} \, \mathrm{d}\vec{b}$$
$$\left\{ g_{a_1}(z_2 - z_1) g_{a_2}(z_3 - z_2) + g_{a_1}(z_3 - z_2) g_{a_2}(z_2 - z_1) \right\} \quad (4.34)$$
$$\cdot G_\vartheta(b_1 - a_1) G_\vartheta(b_2 - a_2) \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}} ,$$

see Figure 4 for a graphical representation.

When we average $K_t^{(2)}(z_1, z_2)\, K_t^{(2)}(z_2, z_3)$ with respect to a function φ as in (4.3), recalling the quantity $\widetilde{\mathscr{G}}_{a_1, a_2}(\varphi)$ from (4.5), we obtain the equality

$$\int\limits_{(\mathbb{R}^2)^3} \varphi(z_1)\, \varphi(z_2)\, \varphi(z_3)\, K_t^{(2)}(z_1, z_2)\, K_t^{(2)}(z_2, z_3)\, \mathrm{d}\vec{z}$$

$$= 2 \sum_{m=2}^{\infty} \int\limits_{0<a_1<b_1<\ldots<a_m<b_m<t} \cdots \int \mathrm{d}\vec{a}\, \mathrm{d}\vec{b}\; \widetilde{\mathscr{G}}_{a_1, a_2}(\varphi)\, G_\vartheta(b_1 - a_1)\, G_\vartheta(b_2 - a_2) \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}}\,. \tag{4.35}$$

Note that this expression resembles $\widetilde{I}_t^{(3)}(\varphi)$ in (4.9), except that $3 \cdot 2^{m-1}$ is replaced by 2.

We next consider the product of *three* covariance kernels as in (4.33). The following result is also proved below.

Lemma 4.10 (Triple correlation product). *The following equality holds:*

$$K_t^{(2)}(z_1, z_2)\, K_t^{(2)}(z_2, z_3)\, K_t^{(2)}(z_1, z_3) \;=\; \sum_{\substack{\alpha, \beta, \gamma \in \{12, 23, 13\} \\ \alpha \neq \beta,\; \beta \neq \gamma,\; \alpha \neq \gamma}} \mathcal{I}(\sigma_1, \sigma_2, \sigma_3)\,, \tag{4.36}$$

where we set

$$\mathcal{I}(12,\, 23,\, 13) := (2\pi)^3 \sum_{m=3}^{\infty} \sum_{\ell=3}^{m} \int\limits_{0<a_1<b_1<\ldots<a_m<b_m<t} \cdots \int \mathrm{d}\vec{a}\, \mathrm{d}\vec{b}$$

$$g_{a_1}(z_1 - z_2)\, g_{a_2}(z_2 - z_3)\, G_\vartheta(b_1 - a_1)\, G_\vartheta(b_2 - a_2) \prod_{i=3}^{\ell-1} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}} \tag{4.37}$$

$$\cdot\, g_{a_\ell}(z_1 - z_3)\, G_\vartheta(b_\ell - a_\ell) \sum_{\substack{\sigma_{\ell+1}, \ldots, \sigma_m \in \{12, 23, 13\} \\ \sigma_{\ell+1} \neq 13,\; \sigma_i \neq \sigma_{i-1}\, \forall i}} \prod_{i=\ell+1}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{\mathrm{prev}(i)}}\,,$$

see Figure 5 for a graphical representation, where we define

$$\mathrm{prev}(i) := \max\{j \in \{1, \ldots, i-2\} : \sigma_j = \sigma_i\}\,, \tag{4.38}$$

and we set $\sigma_j = 12$ for odd $j \leqslant \ell - 1$, $\sigma_j = 23$ for even $j \leqslant \ell - 1$, and $\sigma_\ell := 13$.

The definition of $\mathcal{I}(12, 23, 13)$ in (4.37) is complicated, but a much simpler bound will be enough for us: if we shorten the gaps $a_i - b_{\mathrm{prev}(i)} \geqslant a_i - b_{i-2}$, see (4.38), and we bound

$$g_{a_\ell}(z_1 - z_3) \leqslant g_{a_\ell}(0) = \frac{1}{2\pi\, a_\ell} < \frac{1}{2\pi\, (a_\ell - b_{\ell-2})}\,,$$

then we can estimate

$$\mathcal{I}(12, 23, 13) < (2\pi)^2 \sum_{m=3}^{\infty} \sum_{\ell=3}^{m} 2^{m-\ell} \int\limits_{0<a_1<b_1<\ldots<a_m<b_m<1} \cdots \int \mathrm{d}\vec{a}\, \mathrm{d}\vec{b}$$

$$g_{a_1}(z_1 - z_2)\, g_{a_2}(z_2 - z_3)\, G_\vartheta(b_1 - a_1)\, G_\vartheta(b_2 - a_2) \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}}\,,$$
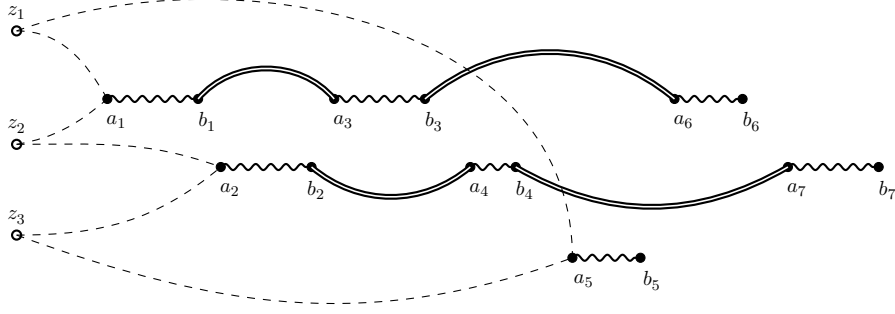
FIGURE 5. Graphical representation of the term $m = 7$ in the series (4.37), which describes $K_t^{(2)}(z_1, z_2) \, K_t^{(2)}(z_2, z_3) \, K_t^{(2)}(z_1, z_3)$, see (4.36). Pairs of dashed lines from z_i, z_j to an a are assigned *total* weight $g_a(z_i - z_j)$; double solid lines from b_{i-2} to a_i are assigned weight $(a_i - b_{i-2})^{-1}$; wiggle lines from a_i to b_i are assigned weight $G_\vartheta(b_i - a_i)$. Referring to (4.37), we have $\ell = 5$ and $\mathrm{prev}(6) = 3$, $\mathrm{prev}(7) = 4$.

where $2^{m-\ell}$ is the number of choices of $\sigma_{\ell+1}, \ldots, \sigma_m$ in (4.37). Recalling (4.5), we obtain

$$
\int\limits_{(\mathbb{R}^2)^3} \varphi(z_1) \, \varphi(z_2) \, \varphi(z_3) \, K_t^{(2)}(z_1, z_2) \, K_t^{(2)}(z_2, z_3) \, K_t^{(2)}(z_1, z_3) \, \mathrm{d}\vec{z}
$$

$$
< 6 \sum_{m=3}^{\infty} \sum_{\ell=3}^{m} 2^{m-\ell} \int \cdots \int\limits_{0<a_1<b_1<\ldots<a_m<b_m<1} \mathrm{d}\vec{a} \, \mathrm{d}\vec{b} \, \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi) \, G_\vartheta(b_1 - a_1) \, G_\vartheta(b_2 - a_2) \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}}
$$

$$
= 6 \sum_{m=2}^{\infty} (2^{m-2} - 1) \int \cdots \int\limits_{0<a_1<b_1<\ldots<a_m<b_m<1} \mathrm{d}\vec{a} \, \mathrm{d}\vec{b} \, \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi) \, G_\vartheta(b_1 - a_1) \, G_\vartheta(b_2 - a_2) \prod_{i=3}^{m} \frac{G_\vartheta(b_i - a_i)}{a_i - b_{i-2}} ,
$$

(4.39)

where in the last line we added the term $m = 2$ because the factor $(2^{m-2} - 1)$ vanishes.

We finally plug (4.39) and (three times) (4.35) into (4.33). Since $6(2^{m-2}-1)+3 \cdot 2 = 3 \cdot 2^{m-1}$, we obtain $K_{\mathrm{GMC}}^{(3)}(\varphi) < \widetilde{I}^{(3)}(\varphi)$, see (4.9). This completes the proof of Proposition 4.3.

Proof of Lemma 4.9. Our basic strategy is to approximate $K^{(2)}$ by its lattice analogue. Figure 4 provides a useful reference to the underlying structure that we will explain. In [CSZ19a], Theorem 1.4, we arrived at the Dickman renewal density G_ϑ as the limit

$$
U_N(n) = \frac{\log N}{N} G_\vartheta(\tfrac{n}{N})(1 + o(1)), \qquad \text{as } N \to \infty, \tag{4.40}
$$

where for $n \in \mathbb{N}$,

$$
U_N(n) := \sum_{k \geqslant 1} (\sigma_N^2)^k \sum_{0 = n_0 < n_1 < \cdots < n_k = n} \prod_{i=1}^{k} q_{2(n_i - n_{i-1})}(0) \tag{4.41}
$$

with $\sigma_N^2 := \frac{1}{R_N}(1 + \frac{\vartheta + o(1)}{\log N})$ as in (A.1) and $q_n(0)$ denoting the n-step transition probability from 0 to 0 for a simple symmetric random walk on $\mathbb{Z}^2$. Moreover, the following uniform

bound was established in [CSZ19a, Theorem 1.4]:

$$U_N(n) \leqslant C \frac{\log N}{N} G_\vartheta\left(\tfrac{n}{N}\right) \qquad \forall\, 0 < n \leqslant N, \tag{4.42}$$

for $C \in (0, \infty)$.

Using the local limit theorem for random walks, the asymptotic (4.40), and the bound (4.42) which allows us to apply dominated convergence, we have that (recall $\llbracket \cdot \rrbracket$ from (1.5))

$$
\begin{aligned}
&K^{(2)}(z_1, z_2) \\
&= \lim_{N\to\infty} \sigma_N^2 \sum_{1 \leqslant m_1 < m_2 \leqslant N} q_{2m_1}(\llbracket (z_1 - z_2)\sqrt{N} \rrbracket)\, U_N(m_2 - m_1) \\
&= \lim_{N\to\infty} \sum_{k \geqslant 1} \sum_{1 \leqslant m_1 < m_2 \leqslant N} (\sigma_N^2)^{k+1} \sum_{m_1 = n_0 < n_1 < \cdots < n_k = m_2} q_{2m_1}(\llbracket (z_1 - z_2)\sqrt{N} \rrbracket) \prod_{i=1}^k q_{2(n_i - n_{i-1})}(0) \\
&= \lim_{N\to\infty} \sum_{k \geqslant 1} (\sigma_N^2)^{k+1} \sum_{0 < n_0 < n_1 < \cdots < n_k \leqslant N} q_{2n_0}(\llbracket (z_1 - z_2)\sqrt{N} \rrbracket) \prod_{i=1}^k q_{2(n_i - n_{i-1})}(0).
\end{aligned}
$$

To lighten the notation below, we will drop the brackets $\llbracket \cdot \rrbracket$, i.e., when we write $z\sqrt{N}$ we refer to $\llbracket z\sqrt{N} \rrbracket$. Using this expression for the product $K^{(2)}(z_1, z_2) K^{(2)}(z_2, z_3)$ we obtain that

$$
\begin{aligned}
&K^{(2)}(z_1, z_2) K^{(2)}(z_2, z_3) \\
&= \lim_{N\to\infty} \sum_{k,k' \geqslant 1} (\sigma_N^2)^{k+k'+2} \sum_{\substack{0 < n_0 < n_1 < \cdots < n_k \leqslant N \\ 0 < n_0' < n_1' < \cdots < n_{k'}' \leqslant N}} q_{2n_0}\big((z_1 - z_2)\sqrt{N}\big)\, q_{2n_0'}\big((z_2 - z_3)\sqrt{N}\big)\cdot \\
&\hspace{6cm} \cdot \prod_{i=1}^k q_{2(n_i - n_{i-1})}(0) \prod_{i=1}^{k'} q_{2(n_i' - n_{i-1}')}(0) \tag{4.43}
\end{aligned}
$$

Let us start by assuming that the sequences $\{0 < n_0 < n_1 < \cdots < n_k \leqslant N\}$ and $\{0 < n_0' < n_1' < \cdots < n_{k'}' \leqslant N\}$ do not share common points and let us look at all possible ways they interlace, i.e.

$$0 < n_0 < \cdots < n_{\tau_1} < n_0' < \cdots < n_{\tau_1'}' < n_{\tau_1 + 1} < \cdots < n_{\tau_2} < n_{\tau_1' + 1}' < \cdots < n_{\tau_2'}' \cdots \tag{4.44}$$

for integers $\tau_1, \tau_2, \ldots \in \{1, \ldots, k\}$ and $\tau_1', \tau_2', \ldots \in \{1, \ldots, k'\}$. The case $n_0' < n_0$ is similar. We can now group together the blocks of primed or un-primed integers and sum over the possible cardinalities of the blocks as well as the values of their elements after fixing first the vector $(a_1, b_1, a_2, b_2, \ldots) = (n_0, n_{\tau_1}, n_0', n_{\tau_1'}', \ldots)$, which marks the boundaries of the blocks. Afterwards, we sum over all possible values of $(a_1, b_1, a_2, b_2, \ldots)$. Using this decomposition

in expression (4.43) we can then see that

$$K^{(2)}(z_1, z_2)K^{(2)}(z_2, z_3) = \lim_{N\to\infty} \sum_{m=2}^{\infty} \sum_{0<a_1<b_1<\ldots<a_m<b_m<N}$$

$$\left\{ q_{2a_1}\big((z_1-z_2)\sqrt{N}\big)\, q_{2a_2}\big((z_2-z_3)\sqrt{N}\big) \; + \; q_{2a_1}\big((z_2-z_3)\sqrt{N}\big)\, q_{2a_2}\big((z_1-z_2)\sqrt{N}\big) \right\} \tag{4.45}$$

$$\cdot \sigma_N^2\, U_N(b_1-a_1)\cdot \sigma_N^2 U_N(b_2-a_2)\cdot \prod_{i=3}^{m} \sigma_N^2\, U_N(b_i-a_i)\, q_{2(a_i-b_{i-2})}(0)\,.$$

After passing to the limit using the local limit theorem for random walks and the asymptotic (4.41), we arrive at expression (4.34).

It only remains to check that the interlacing blocks (4.44) are well defined, i.e. contribution to (4.43) from sequences $\{0 < n_0 < n_1 < \cdots < n_k \leqslant N\}$ and $\{0 < n_0' < n_1' < \cdots < n_{k'}' \leqslant N\}$ that share common points is negligible. So let us look at (4.43) when the sum on the right hand side is over configurations such that

$$\{0 < n_0 < n_1 < \cdots < n_k \leqslant N\} \bigcap \{0 < n_0' < n_1' < \cdots < n_{k'}' \leqslant N\} \neq \varnothing.$$

The right hand side of (4.43) would then be bounded by

$$\sum_{k,k'\geqslant 1} (\sigma_N^2)^{k+k'+2} \sum_{\ell\leqslant k\,,\,\ell'\leqslant k'} \sum_{\substack{0<n_0<n_1<\cdots<n_k\leqslant N \\ 0<n_0'<n_1'<\cdots<n_{k'}'\leqslant N}} \mathbb{1}_{\{n_\ell=n_{\ell'}'\}}\cdot$$

$$\cdot q_{2n_0}\big((z_1-z_2)\sqrt{N}\big)\, q_{2n_0'}\big((z_2-z_3)\sqrt{N}\big) \prod_{i=1}^{k} q_{2(n_i-n_{i-1})}(0) \prod_{i=1}^{k'} q_{2(n_i'-n_{i-1}')}(0),$$

accounting fo all possible places where a coincidence happens. Splitting the summation according to $n_\ell = n_{\ell'}' =: \mathfrak{n} \in \{1, \ldots, N\}$ and the corresponding indices $\ell \in \{1, \ldots, k\}$, $\ell' \in \{1, \ldots, k'\}$ and rearranging the rest of the sums, we can rewrite the above as

$$\sigma_N^4 \sum_{1\leqslant n_0, n_0'\leqslant N} q_{2n_0}\big((z_1-z_2)\sqrt{N}\big)\, q_{2n_0'}\big((z_2-z_3)\sqrt{N}\big)\cdot$$

$$\cdot \sum_{n_0\vee n_0'<\mathfrak{n}\leqslant N} U_N(\mathfrak{n}-n_0)U_N(\mathfrak{n}-n_0') \sum_{\mathfrak{n}<n,n'\leqslant N} U_N(n-\mathfrak{n})U_N(n'-\mathfrak{n}),$$

which, using (4.42), can be bounded by

$$C\sigma_N^4 \sum_{1\leqslant n_0, n_0'\leqslant N} q_{2n_0}\big((z_1-z_2)\sqrt{N}\big)\, q_{2n_0'}\big((z_2-z_3)\sqrt{N}\big)\cdot$$

$$\cdot \sum_{n_0\vee n_0'<\mathfrak{n}\leqslant N} \frac{\log N}{N}G_\vartheta\Big(\frac{\mathfrak{n}-n_0}{N}\Big)\cdot \frac{\log N}{N}G_\vartheta\Big(\frac{\mathfrak{n}-n_0'}{N}\Big) \sum_{\mathfrak{n}<n,n'\leqslant N} \frac{\log N}{N}G_\vartheta\Big(\frac{n-\mathfrak{n}}{N}\Big)\frac{\log N}{N}G_\vartheta\Big(\frac{n'-\mathfrak{n}}{N}\Big),$$

and which, by a Riemann sum approximation, is seen to converge to 0 as $N \to \infty$. $\qquad\square$

Proof of Lemma 4.10. The proof is similar to that of Lemma 4.9, so we will just give a sketch.

For the product $K^{(2)}(z_1, z_2)\, K^{(2)}(z_2, z_3)\, K^{(2)}(z_1, z_3)$ we can write a formula analogous to (4.43) and (4.45), where we now sum over three type of blocks: un-primed, primed and double-primed, to each one of which we assign a label $\sigma_i \in \{12, 23, 13\}$. Due to the interlacing

of the blocks, the assignment of labels will have the constraint that $\sigma_i \neq \sigma_{i-1}$ for all i. Thus, the only difference with the analogous formula for $K^{(2)}(z_1, z_2) K^{(2)}(z_2, z_3)$ would be that $q_{2(a_i - b_{i-2})}(0)$ would be replaced by $q_{2(a_i - b_{\mathrm{prev}(i)})}(0)$ where $\mathrm{prev}(i)$ corresponds to the previous block with the same label σ. $\qquad\square$

4.3. PROOF OF PROPOSITION 4.4. If $\varphi = g_r$ is the heat kernel, see (2.1), we can compute $\mathscr{G}_{a_1, a_2}(\varphi)$ and $\widetilde{\mathscr{G}}_{a_1, a_2}(\varphi)$, as in Remark 4.5. We start from the latter, see (4.5): integrating z_3 by Gaussian convolution, then z_2 by Lemma 4.8, and finally z_1, we get

$$
\begin{aligned}
\widetilde{\mathscr{G}}_{a_1, a_2}(g_r) &:= (2\pi)^2 \iint g_r(z_1)\, g_r(z_2)\, g_{a_1}(z_2 - z_1)\, g_{r+a_2}(z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2 \\
&= (2\pi)^2\, g_{2r+a_2}(0) \int g_r(z_1)\, g_{a_1 + h(r, r+a_2)}(z_1)\, \mathrm{d}z_1 \\
&= (2\pi)^2\, g_{2r+a_2}(0)\, g_{r+a_1+\frac{r(r+a_2)}{2r+a_2}}(0) = \frac{1}{3r^2 + 2(a_1 + a_2)r + a_1 a_2}\,,
\end{aligned}
$$

which proves the second relation in (4.13). We can compute $\mathscr{G}_{a_1, a_2}(g_r)$ from (4.4) with similar arguments, but it is easier to exploit the following basic fact: when z_1, z_2, z_3 are independent Gaussian random variables on $\mathbb{R}^2$ with density g_r, then $x := z_1 - z_2$ and $y := z_3 - \frac{z_1 + z_2}{2}$ are independent with densities g_{2r} and $g_{\frac{3}{2}r}$, therefore

$$
\begin{aligned}
\mathscr{G}_{a_1, a_2}(g_r) &= (2\pi)^2 \iint g_{a_1}(x)\, g_{2r}(x)\, g_{a_2}(y)\, g_{\frac{3}{2}r}(y)\, \mathrm{d}x\, \mathrm{d}y \\
&= (2\pi)^2\, g_{a_1 + 2r}(0)\, g_{a_2 + \frac{3}{2}r}(0) = \frac{1}{a_1 + 2r}\, \frac{1}{a_2 + \frac{3}{2}r}\,,
\end{aligned}
$$

which proves the first relation in (4.13). The fact that $\mathscr{G}_{a_1, a_2}(g_r) > \widetilde{\mathscr{G}}_{a_1, a_2}(g_r)$ then follows.

It remains to prove (4.11) when $\varphi(z) = \mathbb{1}_{\{|z| < r\}}$ is the indicator function of a ball. If we define

$$
\xi(z) := (\varphi * g_{a_2})(z) = \int \varphi(z')\, g_{a_2}(z - z')\, \mathrm{d}z'\,,
$$

then we can write, recalling (4.4) and performing a change of variables,

$$
\begin{aligned}
\mathscr{G}_{a_1, a_2}(\varphi) &:= (2\pi)^2 \iint_{(\mathbb{R}^2)^2} \varphi(z_1)\, \varphi(z_2)\, g_{a_1}(z_2 - z_1)\, \xi(\tfrac{z_1 + z_2}{2})\, \mathrm{d}z_1\, \mathrm{d}z_2\,, \\
&= (2\pi)^2 \iint_{(\mathbb{R}^2)^2} \varphi(z - \tfrac{y}{2})\, \varphi(z + \tfrac{y}{2})\, g_{a_1}(y)\, \xi(z)\, \mathrm{d}y\, \mathrm{d}z\,.
\end{aligned}
$$

Similarly, by (4.5),

$$
\widetilde{\mathscr{G}}_{a_1, a_2}(\varphi) := (2\pi)^2 \iint_{(\mathbb{R}^2)^2} \varphi(z - y)\, \varphi(z)\, g_{a_1}(y)\, \xi(z)\, \mathrm{d}y\, \mathrm{d}z\,.
$$

Note that ξ is a radially symmetric and non-increasing function since the convolution of two radially symmetric and non-increasing functions (see the proof of Lemma 4.7). Then, by a layer cake decomposition as in (4.18), it suffices to prove $\mathscr{G}_{a_1, a_2}(\varphi) > \widetilde{\mathscr{G}}_{a_1, a_2}(\varphi)$ when $\xi(z)$ is

replaced by $\mathbb{1}_{\{|z|<t\}}$, for some fixed $t > 0$. Plugging also $\varphi(x) = \mathbb{1}_{\{|x|<r\}}$, we can write

$$\mathscr{G}_{a_1,a_2}(\varphi) - \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi) = (2\pi)^2 \int_{\mathbb{R}^2} \left\{ \mathrm{Leb}\big(B(\tfrac{y}{2},r) \cap B(-\tfrac{y}{2},r) \cap B(0,t)\big) \right.$$

$$\left. - \mathrm{Leb}\big(B(y,r) \cap B(0,r) \cap B(0,t)\big) \right\} g_{a_1}(y)\,\mathrm{d}y\,,$$

where $B(z,r) := \{x \in \mathbb{R}^2 \colon |x| < r\}$ is the ball of radius r centered at z. Note that

$$A_r(y) := B(\tfrac{y}{2},r) \cap B(-\tfrac{y}{2},r)\,,$$

is a *symmetric convex set* (possibly empty), which translated by $\frac{y}{2}$ gives

$$A_r(y) + \tfrac{y}{2} = B(y,r) \cap B(0,r)\,.$$

Then it follows from Anderson's inequality [And55, Theorem 1] that we have the bound

$$\mathrm{Leb}\big(A_r(y) \cap B(0,t)\big) \geqslant \mathrm{Leb}\big((A_r(y) + \tfrac{y}{2}) \cap B(0,t)\big)\,,$$

which can also be checked directly, and is strict for suitable values of r, t, y. This shows that $\mathscr{G}_{a_1,a_2}(\varphi) > \widetilde{\mathscr{G}}_{a_1,a_2}(\varphi)$ when φ is the indicator function of a ball. $\qquad\square$

5. Proof of Theorem 1.5: lower bounds via collision local times and the Gaussian correlation inequality

In this section we prove Theorem 1.5. The key point is the lower bound (1.14) on the moments of the SHF $\mathscr{Z}_t^\vartheta$: for a suitable $\eta = \eta_{t,\vartheta} > 0$ we have, for any $m \in \mathbb{N}$ with $m \geqslant 3$,

$$\mathbb{E}\big[\big(2\,\mathscr{Z}_t^\vartheta(g_\delta)\big)^m\big] \geqslant (1+\eta)\,\mathbb{E}\big[\big(2\,\mathscr{Z}_t^\vartheta(g_\delta)\big)^2\big]^{\binom{m}{2}} \qquad \forall \delta \in (0,1)\,, \tag{5.1}$$

where g_δ is the heat kernel on $\mathbb{R}^2$, see (2.1). Then, in order to obtain (1.15) and complete the proof, it suffices to show that (1.13) holds, which follows from the next result.

Proposition 5.1 (Higher moments of GMC). *Let $\mathscr{M}_t^\vartheta(\mathrm{d}x)$ be the GMC with the same first and second moments as the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$, see Section 3.3. Then, as $\delta \downarrow 0$, we have*

$$\mathbb{E}\big[\big(2\,\mathscr{M}_t^\vartheta(g_\delta)\big)^m\big] \sim \big(C_{t,\vartheta} \log \tfrac{1}{\sqrt{\delta}}\big)^{\binom{m}{2}}\,, \tag{5.2}$$

where $C_{t,\vartheta} = \frac{1}{\pi}\int_0^t G_\vartheta(v)\,\mathrm{d}v$ is the same constant which appears in (1.11).

The rest of this section is divided in three parts.

- First we show that the moments of the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$, averaged over a test function φ, can be obtained as the limit (as $\varepsilon \downarrow 0$) of the moments of the solution $u^\varepsilon(t,x)$ of the mollified Stochastic Heat Equation (1.16), based on [GQT21].

- Then we prove the bound (5.1) by exploiting the *Gaussian Correlation Inequality* [R14, LM17], adapting the approach in Feng's Ph.D. thesis [Fen16].

- Finally, we prove Proposition 5.1, which completes the proof of Theorem 1.5.

5.1. SHF AND THE MOLLIFIED STOCHASTIC HEAT EQUATION.

We consider the mollified Stochastic Heat Equation (1.16) with spatially mollified space-time white noise

$$\xi^\varepsilon(t,x) := (\xi(t,\cdot) * j_\varepsilon)(x) = \int_{\mathbb{R}^2} j_\varepsilon(z)\,\xi(t,x-z)\,\mathrm{d}z\,,$$

where $j_\varepsilon(x) := \varepsilon^{-2} j(x/\varepsilon)$ and $j(\cdot)$ is a probability density on $\mathbb{R}^2$ (usually taken compactly supported). Assuming initial condition $u^\varepsilon(0,\cdot) = 1$, by the Feynman-Kac formula [BC95, Section 3 and eq. (3.22)], the Itô solution $u^\varepsilon(t,x) = u^\varepsilon_\beta(t,x)$ of (1.16), where we highlight the dependence on β, has the representation

$$u^\varepsilon_\beta(t,x) = \mathbf{E}_x\Big[\mathrm{e}^{\beta\int_0^t \xi^\varepsilon(t-u,B_u)\,\mathrm{d}u - \frac{1}{2}\beta^2\|j_\varepsilon\|_2^2 t}\Big] \overset{\mathrm{dist}}{=} \mathbf{E}_x\Big[\mathrm{e}^{\beta\int_0^t \xi^\varepsilon(u,B_u)\,\mathrm{d}u - \frac{1}{2}\beta^2\|j_\varepsilon\|_2^2 t}\Big], \qquad (5.3)$$

where $\mathbf{E}_x$ denotes expectation for a standard Brownian motion B starting at x. We will omit x from $\mathbf{E}_x$ if $x = 0$.

We can directly compute the moments of $u^\varepsilon_\beta(t,x)$, which do not depend on x by translation invariance, thanks to the initial condition $u(0,\cdot) \equiv 1$. Given $m \in \mathbb{N}$, let $(B^{(i)})_{1\leqslant i\leqslant m}$ denote m independent Brownian motions, and define $J_\varepsilon := \varepsilon^{-2} J(x/\varepsilon)$ with $J := j * j$. Note that

$$\mathbb{V}\mathrm{ar}\bigg[\sum_{i=1}^m \int_0^t \xi^\varepsilon(u, B_u^{(i)})\,\mathrm{d}u\bigg] = \sum_{1\leqslant i,j\leqslant m} L_{\varepsilon,t}^{i,j}, \qquad \text{where} \quad L_{\varepsilon,t}^{i,j} := \int_0^t J_\varepsilon(B_u^{(i)} - B_u^{(j)})\,\mathrm{d}u\,, \ (5.4)$$

which can be viewed as a *collision local time* at scale ε between $B^{(i)}$ and $B^{(j)}$. Note that $L_{\varepsilon,t}^{i,i} = J_\varepsilon(0)\,t = \|j_\varepsilon\|_2^2\,t$, where $\|\cdot\|_2$ denotes the L^2 norm. Given $x_1,\ldots,x_m \in \mathbb{R}^2$, if we denote by $\mathbf{P}_{\vec{x}}$ the law under which $B^{(i)}$ starts at $B_0^{(i)} = x_i$, a Gaussian computation yields

$$\mathbb{E}\bigg[\prod_{i=1}^m u^\varepsilon_\beta(t,x_i)\bigg] = \mathbb{E}\mathbf{E}_{\vec{x}}\Big[\mathrm{e}^{\beta\sum_{i=1}^m \int_0^t \xi^\varepsilon(u,B_u^{(i)})\,\mathrm{d}u - \frac{m}{2}\beta^2\|j_\varepsilon\|_2^2 t}\Big] = \mathbf{E}_{\vec{x}}\bigg[\prod_{1\leqslant i<j\leqslant m} \mathrm{e}^{\beta^2\int_0^t L_{\varepsilon,t}^{i,j}}\bigg]. \quad (5.5)$$

Remark 5.2. *In the critical window (1.21) we have $\beta_\varepsilon^2 \sim 2\pi/\log\varepsilon^{-1}$, hence $\beta_\varepsilon^2 L_{\varepsilon,t}^{i,j}$ for $i \neq j$ converges in law as $\varepsilon \downarrow 0$ to an exponential random variable Y of mean 1, by a classical result [KR53]. This explains why β_ε is critical, since $\mathrm{E}[\mathrm{e}^{\lambda Y}]$ diverges precisely at $\lambda = 1$.*

We now describe the link between the solution $u^\varepsilon_\beta(t,x)$ of the mollified Stochastic Heat Equation and the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$. We recall that the latter was obtained in [CSZ21] from the directed polymer random measure $\mathcal{Z}_{N;t}^\beta(\mathrm{d}x) = \mathcal{Z}_{N;0,t}^\beta(\mathbb{R}^2, \mathrm{d}x)$, see (1.5), based on the *simple random walk* (S_n) on $\mathbb{Z}^2$, which has covariance matrix $\mathsf{s}I$ with $\mathsf{s} = \frac{1}{2}$ and is periodic (note that S_{2n} takes values in $\mathbb{Z}^2_{\mathrm{even}}$, see (1.6)). On the other hand, the solution $u^\varepsilon_\beta(t,x)$ of the mollified Stochastic Heat Equation, see (5.3), is based on a standard Brownian motion on $\mathbb{R}^2$ with covariance matrix I and, of course, with no periodicity issues.

For these reasons, to obtain the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ from the solution $u^\varepsilon_\beta(t,x)$ of the mollified Stochastic Heat Equation, we need an *appropriate rescaling*: given $\vartheta \in \mathbb{R}$, if we scale $\beta_\varepsilon = \beta_\varepsilon(\vartheta)$ in the critical window (A.4)-(A.5) (see also (A.7)-(A.9)), we expect that

$$\frac{1}{2}\,u^\varepsilon_{\beta_\varepsilon}\big(t, x\sqrt{2}\big)\,\mathrm{d}x \overset{d}{\longrightarrow} \mathscr{Z}_t^\vartheta(\mathrm{d}x)\,, \qquad (5.6)$$

see Appendix A.3 for a heuristic derivation. We refrain from proving such a convergence, which we expect to follow from the same techniques as in the paper [CSZ21]. As a matter of fact, for our goals, it is enough to show that the two sides of (5.6) have asymptotically *the same moments*, and this follows by [GQT21] and [CSZ21], as we now describe.

Proposition 5.3 (Moments of SHF and Stochastic Heat Equation). *Fix $\vartheta \in \mathbb{R}$ and set $\beta = \beta_\varepsilon$ as in* (A.9). *Fix a mollification density $j(\cdot)$ which is radially symmetric and non-increasing. For any integrable $\varphi : \mathbb{R}^2 \to \mathbb{R}$, and for any $h \in \mathbb{N}$, we have*

$$\mathbb{E}\big[\mathscr{Z}_t^\vartheta(\varphi)^h\big] \;=\; \frac{1}{2^h} \lim_{\varepsilon \downarrow 0} \mathbb{E}\bigg[\bigg(\int_{\mathbb{R}^2} u_{\beta_\varepsilon}^\varepsilon\big(t, x\sqrt{2}\big)\,\varphi(x)\,\mathrm{d}x\bigg)^h\bigg]. \tag{5.7}$$

Proof. It is enough to compare formulas (2.15)-(2.16) with Theorem 1.1 and eq. (2.5) in [GQT21]. $\qquad\square$

Remark 5.4. *Recalling* (2.7), *we see that relation* (5.7) *for $h = 2$ reduces to*

$$\iint\limits_{(\mathbb{R}^2)^2} \varphi(x)\,\varphi(x')\,K_t^{(2)}(x, x')\,\mathrm{d}x\,\mathrm{d}x' \;=\; \lim_{\varepsilon \downarrow 0} \mathrm{Var}\bigg[\int_{\mathbb{R}^2} u_{\beta_\varepsilon}^\varepsilon(t, x\sqrt{2})\,\varphi(x)\,\mathrm{d}x\bigg]. \tag{5.8}$$

The validity of such a relation was proved in [CSZ19b, Theorem 1.7] *(note that the choice of ϑ in* (A.8)-(A.9), *which enters $K_t^{(2)} = K_{t,\vartheta}^{(2)}$ in* (2.7), *matches* [CSZ19b, eq. (1.38)]).

5.2. Proof of the lower bound (5.1). Henceforth we fix $\beta = \beta_\varepsilon$ as in (A.4)-(A.5) and omit it from notation, i.e. we set $u^\varepsilon(t, x) := u_{\beta_\varepsilon}^\varepsilon(t, x)$. It follows by (5.4)-(5.5) that

$$\mathbb{E}\bigg[\bigg(\int u^\varepsilon(t, x\sqrt{2})\,\varphi(x)\,\mathrm{d}x\bigg)^m\bigg] = \int\limits_{(\mathbb{R}^2)^m} \prod_{i=1}^m \varphi(x_i)\,\mathbf{E}_{\vec{x}\sqrt{2}}\bigg[\prod_{1 \leqslant i < j \leqslant m} \mathrm{e}^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(B_s^{(i)} - B_s^{(j)})\,\mathrm{d}s}\bigg]\mathrm{d}\vec{x}, \tag{5.9}$$

where we recall that $\mathbf{E}_{\vec{y}}$ denotes expectation w.r.t. m independent Brownian motions with $B_0^{(i)} = y_i$. We now take $\varphi = g_\delta$ to be the heat kernel, see (2.1), and note that by diffusive scaling we can write $g_\delta(x) = 2\,g_{2\delta}(x\sqrt{2})$. Then, in view of (5.7) and by a change of variables, to prove (5.1) it suffices to find $\eta = \eta_{t,\vartheta} > 0$ such that, uniformly in $m \geqslant 3$ and $\delta \in (0, 1)$,

$$\lim_{\varepsilon \downarrow 0} \int\limits_{(\mathbb{R}^2)^m} \prod_{i=1}^m g_{2\delta}(x_i)\,\mathbf{E}_{\vec{x}}\bigg[\prod_{1 \leqslant i < j \leqslant m} \mathrm{e}^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(B_s^{(i)} - B_s^{(j)})\,\mathrm{d}s}\bigg]\mathrm{d}\vec{x}$$

$$\geqslant\; (1 + \eta)\lim_{\varepsilon \downarrow 0}\bigg(\int\limits_{(\mathbb{R}^2)^2} g_{2\delta}(x_1)\,g_{2\delta}(x_2)\,\mathbf{E}_{\vec{x}}\Big[\mathrm{e}^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(B_s^{(1)} - B_s^{(2)})\,\mathrm{d}s}\Big]\mathrm{d}x_1\,\mathrm{d}x_2\bigg)^{\binom{m}{2}}. \tag{5.10}$$

We will adapt the approach in Feng's thesis [Fen16], which used the Gaussian correlation inequality [R14, LM17] to prove an analogue of (5.10) for $m = 3$ with $g_\delta(\cdot)$ replaced by $\delta_0(\cdot)$. Unfortunately, not much could be concluded in that case, because *all moments* $\mathbb{E}[u^\varepsilon(t, 0)^m]$ *of order $m > 1$ diverge as $\varepsilon \downarrow 0$*: this is due to the fact that $u^\varepsilon(t, 0) \to 0$ in distribution for $\beta = \beta_\varepsilon$ in the critical window (1.21), see [CSZ17b, Theorem 2.15], while $\mathbb{E}[u^\varepsilon(t, 0)] \equiv 1$ stays constant. We will show that the Gaussian correlation inequality can still be applied when we average $u^\varepsilon(t, x)$ w.r.t. g_δ, which will lead to the interesting bound (5.1).

Let $Z_{2\delta}^{(1)}, \ldots, Z_{2\delta}^{(m)}$ be i.i.d. normal random variables on $\mathbb{R}^2$ with probability density $g_{2\delta}$, independent of the Brownian motions $B^{(1)}, \ldots, B^{(m)}$ all starting from 0. Denoting by $\mathbf{E}$ the

joint probability, we can rewrite (5.10) as

$$
\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[\prod_{1 \leqslant i < j \leqslant m} e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(i)} + B_s^{(i)} - Z_{2\delta}^{(j)} - B_s^{(j)})\,ds} \right]
$$
$$
\geqslant (1 + \eta) \lim_{\varepsilon \downarrow 0} \mathbf{E}\left[e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(1)} + B_s^{(1)} - Z_{2\delta}^{(2)} - B_s^{(2)})\,ds} \right]^{\binom{m}{2}}. \tag{5.11}
$$

Next we Taylor expand the exponential in the l.h.s.: for each $i < j$, we write

$$
e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(i)} + B_s^{(i)} - Z_{2\delta}^{(j)} - B_s^{(j)})\,ds} = 1 + \sum_{n=1}^\infty \beta_\varepsilon^{2n} \int \cdots \int_{0 < s_1 < \cdots < s_n < t} \prod_{l=1}^n J_\varepsilon(Z_{2\delta}^{(i)} + B_{s_l}^{(i)} - Z_{2\delta}^{(j)} - B_{s_l}^{(j)})\,d\vec{s}
$$

$$
= 1 + \sum_{n=1}^\infty \beta_\varepsilon^{2n} \int \cdots \int_{\substack{0 < s_1 < \cdots < s_n < t \\ y_1, \ldots, y_n > 0}} \prod_{l=1}^n \mathbb{1}_{A_\varepsilon(y_l)}(Z_{2\delta}^{(i)} + B_{s_l}^{(i)} - Z_{2\delta}^{(j)} - B_{s_l}^{(j)})\,d\vec{s}\,d\vec{y},
$$

where we used the decomposition $J_\varepsilon(x) = \int_0^\infty \mathbb{1}_{A_\varepsilon(y)}(x)\,dy$, with

$$
A_\varepsilon(y) := \{ x \in \mathbb{R}^2 : J_\varepsilon(x) \geqslant y \}. \tag{5.12}
$$

Note that $J := j * j$ is a radially symmetric and non-increasing function, as the convolution of two radially symmetric and non-increasing functions, as we showed in the proof of Lemma 4.7. It follows that the set $A_\varepsilon(y)$ is a ball centered at the origin, for any $y > 0$.

We can substitute this Taylor expansion into the l.h.s. of (5.11) to obtain

$$
\mathbf{E}\left[\prod_{1 \leqslant i < j \leqslant m} \left(1 + \sum_{n=1}^\infty \beta_\varepsilon^{2n} \int \cdots \int_{\substack{0 < s_1 < \cdots < s_n < t \\ y_1, \ldots, y_n > 0}} \prod_{l=1}^n \mathbb{1}_{A_\varepsilon(y_l)}(Z_{2\delta}^{(i)} + B_{s_l}^{(i)} - Z_{2\delta}^{(j)} - B_{s_l}^{(j)})\,d\vec{s}\,d\vec{y} \right) \right], \tag{5.13}
$$

which, upon expansion, leads to a positive mixture of terms of the form

$$
\mathbf{E}\left[\prod_{(i,j) \in \mathcal{I}} \prod_{l=1}^{n^{(i,j)}} \mathbb{1}_{A_\varepsilon(y_l^{(i,j)})}(Z_{2\delta}^{(i)} + B_{s_l^{(i,j)}}^{(i)} - Z_{2\delta}^{(j)} - B_{s_l^{(i,j)}}^{(j)}) \right], \tag{5.14}
$$

where $\mathcal{I} \subset \{(i,j) : 1 \leqslant i < j \leqslant m\}$ and, for each $(i,j) \in \mathcal{I}$, we have $n^{(i,j)} \in \mathbb{N}$ as well as $0 < s_1^{(i,j)} < \cdots < s_{n^{(i,j)}}^{(i,j)} < t$ and $y_1^{(i,j)}, \ldots, y_{n^{(i,j)}}^{(i,j)} > 0$. Note that

$$
\mathcal{W}_{\mathcal{I}, \vec{s}, \vec{n}} := \left((Z_{2\delta}^{(i)})_{1 \leqslant i \leqslant m}, \left(B_{s_l^{(i,j)}}^{(i)}, B_{s_l^{(i,j)}}^{(j)} \right)_{(i,j) \in \mathcal{I}, 1 \leqslant l \leqslant n^{(i,j)}} \right)
$$

is a centered multi-dimensional Gaussian random vector. Since $A_\varepsilon(y)$ is a convex set symmetric about the origin (in fact, a ball), we can apply the celebrated *Gaussian correlation inequality* [R14, LM17] to lower bound (5.14) by

$$
\mathbf{E}\left[\prod_{(i,j) \in \mathcal{I} \cap \{(1,2),(1,3)\}} \prod_{l=1}^{n^{(i,j)}} \mathbb{1}_{A_\varepsilon(y_l^{(i,j)})}(Z_{2\delta}^{(i)} + B_{s_l^{(i,j)}}^{(i)} - Z_{2\delta}^{(j)} - B_{s_l^{(i,j)}}^{(j)}) \right]
$$
$$
\times \prod_{\substack{(i,j) \in \mathcal{I} \\ (i,j) \neq (1,2),(1,3)}} \mathbf{E}\left[\prod_{l=1}^{n^{(i,j)}} \mathbb{1}_{A_\varepsilon(y_l^{(i,j)})}(Z_{2\delta}^{(i)} + B_{s_l^{(i,j)}}^{(i)} - Z_{2\delta}^{(j)} - B_{s_l^{(i,j)}}^{(j)}) \right], \tag{5.15}
$$

where we have kept the factors from $(i,j) = (1,2)$ and $(1,3)$ inside the same expectation, while separating all other factors involving different $(i,j) \in \mathcal{I}$.

Substituting the bound (5.15) back into the expansion of (5.13) gives a lower bound on the l.h.s. of (5.11), namely

$$
\mathbf{E}\Bigg[\prod_{1\leqslant i<j\leqslant m} e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(i)}+B_s^{(i)}-Z_{2\delta}^{(j)}-B_s^{(j)})\,ds}\Bigg]
$$
$$
\geqslant \mathbf{E}\Bigg[\prod_{j=2,3} e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(1)}+B_s^{(1)}-Z_{2\delta}^{(j)}-B_s^{(j)})\,ds}\Bigg]\, \mathbf{E}\Big[e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(1)}+B_s^{(1)}-Z_{2\delta}^{(2)}-B_s^{(2)})\,ds}\Big]^{\binom{m}{2}-2}. \tag{5.16}
$$

Then the proof of (5.10), and hence (5.1), is complete once we prove the following Lemma.

Lemma 5.5. *There exits $\eta = \eta_{t,\vartheta} > 0$ such that, uniformly in $\delta \in (0,1)$, we have*

$$
\liminf_{\varepsilon\downarrow 0} \frac{\mathbf{E}\Big[\prod_{j=2,3} e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(1)}+B_s^{(1)}-Z_{2\delta}^{(j)}-B_s^{(j)})\,ds}\Big]}{\mathbf{E}\Big[e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(Z_{2\delta}^{(1)}+B_s^{(1)}-Z_{2\delta}^{(2)}-B_s^{(2)})\,ds}\Big]^2} \geqslant 1+\eta. \tag{5.17}
$$

Proof. Let us define $W_s^{(i)} = Z_{2\delta}^{(i)} + B_s^{(i)}$ and $W^{(i)} = (W_s^{(i)})_{0\leqslant s\leqslant t}$. We introduce the shortcuts

$$
\Phi_{\varepsilon,\delta}(W^{(1)}) := \mathbf{E}\Big[e^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(W_s^{(1)}-W_s^{(2)})\,ds}\,\Big|\,W^{(1)}\Big],
$$
$$
\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)}) := \mathbf{E}\big[\Phi_{\varepsilon,\delta}(W^{(1)})|W_0^{(1)},W_t^{(1)}\big],
$$

so that the ratio in the l.h.s. of (5.17) can be written as

$$
\frac{\mathbf{E}[\Phi_{\varepsilon,\delta}(W^{(1)})^2]}{\mathbf{E}[\Phi_{\varepsilon,\delta}(W^{(1)})]^2} = \frac{\mathbf{E}[\mathbf{E}[\Phi_{\varepsilon,\delta}(W^{(1)})^2|W_0^{(1)},W_t^{(1)}]]}{\mathbf{E}[\mathbf{E}[\Phi_{\varepsilon,\delta}(W^{(1)})|W_0^{(1)},W_t^{(1)}]]^2} \geqslant \frac{\mathbf{E}[\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)})^2]}{\mathbf{E}[\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)})]^2},
$$

by Jensen's inequality. Therefore it suffices to show that, uniformly for $\delta \in (0,1)$,

$$
\liminf_{\varepsilon\downarrow 0} \mathbf{E}\Bigg[\bigg(\frac{\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)})}{\mathbf{E}[\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)})]}\bigg)^2\Bigg] \geqslant 1+\eta. \tag{5.18}
$$

Let us show that the fraction in the l.h.s. has a limit as $\varepsilon \downarrow 0$. We treat separately numerator and denominator, starting from the latter: by (5.9) with $m=2$ and $\varphi = g_\delta$,

$$
\mathbf{E}[\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)})] = \mathbb{E}\Bigg[\bigg(\int u^\varepsilon(t,x\sqrt{2})\,g_\delta(x)dx\bigg)^2\Bigg]
$$

hence by (5.7) with $m=2$, recalling (2.7), we get

$$
\bar{\Phi}_\delta := \lim_{\varepsilon\downarrow 0} \mathbf{E}[\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)},W_t^{(1)})] = 4\,\mathbb{E}\big[\mathscr{Z}_t^\vartheta(g_\delta)^2\big]
$$
$$
= 1 + \iint_{(\mathbb{R}^2)^2} g_\delta(x_1)\,g_\delta(x_2)\,K_t^{(2)}(x_1,x_2)\,dx_1\,dx_2 \tag{5.19}
$$
$$
= 1 + 2\pi \iint_{0<s<u<t} g_{2\delta+s}(0)\,G_\vartheta(u-s)\,ds\,du \underset{\delta\downarrow 0}{\sim} \bigg(\int_0^t G_\vartheta(u)\,du\bigg)\log\frac{1}{\delta}.
$$

Next we focus on the numerator: in analogy with (5.5), we can write

$$\bar{\Phi}_{\varepsilon,\delta}(x_1, y) := \mathbf{E}\Big[\mathrm{e}^{\beta_\varepsilon^2 \int_0^t J_\varepsilon(W_s^{(1)} - W_s^{(2)})\,\mathrm{d}s} \,\Big|\, W_0^{(1)} = x_1, W_t^{(1)} = y\Big]$$
$$= \int_{\mathbb{R}^2} g_{2\delta}(x_2)\, \mathbb{E}[u^\varepsilon(t, x_1|y)\, u^\varepsilon(t, x_2)]\,\mathrm{d}x_2,$$
(5.20)

where we define $u^\varepsilon(t, x_1|y)$ as a modification of the Feynman-Kac formula (5.3):

$$u^\varepsilon(t, x_1|y) := \mathbf{E}_{x_1}\Big[\mathrm{e}^{\beta \int_0^t \xi^\varepsilon(u, B_u)\,\mathrm{d}u - \frac{1}{2}\beta^2 \|j_\varepsilon\|_2^2 t} \,\Big|\, B_t = y\Big]$$
(5.21)

(we recall that $\mathbf{E}_{x_1}$ is the expectation for a brownian motion B started at $B_0 = x_1$, so that conditioning on $B_t = y$ yields a Brownian bridge). Adapting the renewal type arguments in the proof of [CSZ19b, Theorem 1.7], see [CSZ19b, Section 8], we obtain

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[u^\varepsilon(t, x_1|y)\, u^\varepsilon(t, x_2)]$$
$$= 1 + 4\pi \int \cdots \int_{\substack{z,w\in\mathbb{R}^2 \\ 0<s<u<t}} g_s(z - x_1)\, g_s(z - x_2)\, G_\vartheta(u - s)\, g_{\frac{u-s}{2}}(w - z)\, \frac{g_{t-u}(y - w)}{g_t(y - x_1)}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}s\,\mathrm{d}u,$$

where the factor $g_{t-u}(y - w)/g_t(y - x_1)$ is originated the conditioning on $B_t = y$.[†] Therefore

$$\bar{\Phi}_\delta(x_1, y) := \lim_{\varepsilon\downarrow 0} \bar{\Phi}_{\varepsilon,\delta}(x_1, y)$$
(5.22)
$$= 1 + 4\pi \int \cdots \int_{\substack{x_2,z\in\mathbb{R}^2 \\ 0<s<u<t}} \frac{g_{2\delta}(x_2)\, g_s(z - x_1)\, g_s(z - x_2)\, g_{t-\frac{u+s}{2}}(y - z)}{g_t(y - x_1)}\, G_\vartheta(u - s)\,\mathrm{d}z\,\mathrm{d}x_2\,\mathrm{d}s\,\mathrm{d}u.$$

We can now combine (5.19) and (5.22), where $\bar{\Phi}_\delta(x_1, y)$ and $\bar{\bar{\Phi}}_\delta$ are defined: if we define

$$\Psi_\delta(x_1, y) := \frac{\bar{\Phi}_\delta(x_1, y)}{\bar{\bar{\Phi}}_\delta},$$
(5.23)

then by Fatou's lemma we can bound

$$\liminf_{\varepsilon\downarrow 0} \mathbf{E}\left[\left(\frac{\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)}, W_t^{(1)})}{\mathbf{E}[\bar{\Phi}_{\varepsilon,\delta}(W_0^{(1)}, W_t^{(1)})]}\right)^2\right] \geqslant \mathbf{E}\big[\Psi_\delta(W_0^{(1)}, W_t^{(1)})^2\big].$$

It is easy to check that $\mathbf{E}[\Psi_\delta(W_0^{(1)}, W_t^{(1)})] = 1$ (see (5.26) below). Since $\Psi_\delta(W_0^{(1)}, W_t^{(1)})$ is clearly not a constant, it follows by Jensen's inequality that for any $\delta \in (0, 1)$

$$\mathbf{E}\big[\Psi_\delta(W_0^{(1)}, W_t^{(1)})^2\big] > 1.$$

Since $\delta \mapsto \mathbf{E}\big[\Psi_\delta(W_0^{(1)}, W_t^{(1)})^2\big]$ is continuous, to prove (5.18) it only remains to show that

$$\lim_{\delta\downarrow 0} \mathbf{E}\big[\Psi_\delta(W_0^{(1)}, W_t^{(1)})^2\big] > 1.$$
(5.24)

[†]For consistency: if we remove that factor, the r.h.s. becomes $1 + 4\pi \int_{0<s<u<t} g_{2s}(x_1 - x_2)\, G_\vartheta(u - s)\,\mathrm{d}s\,\mathrm{d}u$, which is consistent with formula (5.7) once we plug in $x_1\sqrt{2}$ and $x_2\sqrt{2}$; see also (5.8).

Denote $\bar{\Psi}_\delta(W_t^{(1)}) := \mathbf{E}[\Psi_\delta(W_0^{(1)}, W_t^{(1)})|W_t^{(1)}]$. By $W_s^{(1)} = Z_{2\delta}^{(1)} + B_s^{(1)}$, we have

$$
\begin{aligned}
\bar{\Psi}_\delta(y) &= \frac{1}{\bar{\Phi}_\delta} \int \bar{\Phi}_\delta(x_1, y) \frac{g_{2\delta}(x_1) g_t(y - x_1)}{g_{t+2\delta}(y)} \mathrm{d}x_1 \\
&= \frac{1}{\bar{\Phi}_\delta} \left(1 + \frac{4\pi}{g_{t+2\delta}(y)} \int \cdots \int_{\substack{x_1, x_2, z \in \mathbb{R}^2 \\ 0 < s < u < t}} g_{2\delta}(x_1)\, g_{2\delta}(x_2)\, g_s(z - x_1)\, g_s(z - x_2) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \times\, g_{t - \frac{u+s}{2}}(y - z)\, G_\vartheta(u - s)\, \mathrm{d}z\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}s\, \mathrm{d}u \right) \\
&= \frac{1}{\bar{\Phi}_\delta} \left(1 + \frac{4\pi}{g_{t+2\delta}(y)} \iiint_{\substack{z \in \mathbb{R}^2 \\ 0 < s < u < t}} g_{2\delta+s}(z)^2\, g_{t - \frac{u+s}{2}}(y - z)\, G_\vartheta(u - s)\, \mathrm{d}z\, \mathrm{d}s\, \mathrm{d}u \right),
\end{aligned}
$$

and since $g_{2\delta+s}(z)^2 = g_{2(2\delta+s)}(0)\, g_{\delta + \frac{s}{2}}(z)$ by (4.20), we obtain

$$
\begin{aligned}
\bar{\Psi}_\delta(y) &= \frac{1}{\bar{\Phi}_\delta} \left(1 + \frac{4\pi}{g_{t+2\delta}(y)} \iint_{0 < s < u < t} g_{2(2\delta+s)}(0)\, g_{t+\delta - \frac{u}{2}}(y)\, G_\vartheta(u - s)\, \mathrm{d}s\, \mathrm{d}u \right) \\
&= \frac{1}{\bar{\Phi}_\delta} \left(1 + \iint_{0 < s < u < t} \frac{1}{2\delta + s} \frac{g_{t+\delta - \frac{u}{2}}(y)}{g_{t+2\delta}(y)}\, G_\vartheta(u - s)\, \mathrm{d}s\, \mathrm{d}u \right). \tag{5.25}
\end{aligned}
$$

Incidentally, this relation together with (5.19) shows that

$$
\mathbf{E}[\Psi_\delta(W_0^{(1)}, W_t^{(1)})] = \mathbf{E}[\bar{\Psi}_\delta(W_t^{(1)})] = \int_{\mathbb{R}^2} \bar{\Psi}_\delta(y)\, g_{t+2\delta}(y)\, \mathrm{d}y = 1. \tag{5.26}
$$

Note that as $\delta \downarrow 0$, the dominant contribution to the integral in (5.19) for $\bar{\Phi}_\delta$ comes from $s \ll 1$, since we can restrict the integral to $s < (\log \frac{1}{\delta})^{-1}$ (say) without changing the asymptotic behavior. The same is true for the integral in (5.25), hence we obtain

$$
\lim_{\delta \downarrow 0} \bar{\Psi}_\delta(y) = \bar{\Psi}_0(y) := \frac{\int_0^t g_{t - \frac{u}{2}}(y)\, G_\vartheta(u)\, \mathrm{d}u}{g_t(y) \int_0^t G_\vartheta(u)\, \mathrm{d}u},
$$

which implies that $\bar{\Psi}_\delta(W_t^{(1)}) = \bar{\Psi}_\delta(Z_{2\delta}^{(1)} + B_t^{(1)})$ converges in law to $\bar{\Psi}_0(B_t^{(1)})$ as $\delta \downarrow 0$. Therefore, by Jensen's inequality and Fatou's lemma,

$$
\lim_{\delta \downarrow 0} \mathbf{E}\big[\Psi_\delta(W_0^{(1)}, W_t^{(1)})^2\big] \geq \lim_{\delta \downarrow 0} \mathbf{E}\big[\bar{\Psi}_\delta(W_t^{(1)})^2\big] \geq \mathbf{E}\big[\bar{\Psi}_0(B_t^{(1)})^2\big] > 1,
$$

where the last inequality holds because $\mathbf{E}[\bar{\Psi}(B_t^{(1)})] = 1$ and $\bar{\Psi}(B_t^{(1)})$ is not a.s. equal to 1. This concludes the proof of (5.24), hence of Lemma 5.5. $\qquad\square$

5.3. Proof of Proposition 5.1. The log-divergence of the second moment kernel $\mathscr{K}_t^{(2)}(x, y)$ of the SHF, see (1.11), plays a crucial role. Recall from (3.7) and (2.7) that

$$
\mathscr{K}_t^{(2)}(x, y) = \mathrm{e}^{k_t(x,y)} = 1 + K_t^{(2)}(x, y) = 1 + 2\pi \iint_{0 < s < u < t} g_s(x - y)\, G_\vartheta(u - s)\, \mathrm{d}s\, \mathrm{d}u,
$$

which is a monotonically decreasing function of $|x - y|$. By a change of variable,

$$2\pi \iint\limits_{0<s<u<t} g_s(x-y)\, G_\vartheta(u-s)\, \mathrm{d}s\, \mathrm{d}u = \int_0^{t|x-y|^{-2}} \frac{\mathrm{e}^{-\frac{1}{2\tilde s}}}{\tilde s} \left(\int_0^{t-|x-y|^2 \tilde s} G_\vartheta(v)\, \mathrm{d}v \right) \mathrm{d}\tilde s,$$

and note that, as $|x - y| \downarrow 0$, the dominant contribution to the integral comes from the range of values $1 \ll \tilde s \ll |x-y|^{-2}$. Therefore, as $|x - y| \downarrow 0$,

$$\mathscr{K}_t^{(2)}(x,y) = \mathrm{e}^{k_t(x,y)} \sim \left(\int_0^t G_\vartheta(v)\mathrm{d}v \right) \log \frac{t}{|x-y|^2} \sim C_{t,\vartheta} \log \frac{1}{|x-y|}, \qquad (5.27)$$

where we set $C_{t,\vartheta} := 2 \int_0^t G_\vartheta(v)\, \mathrm{d}v$.

Applying the moment formula (3.3) and (5.27) to the l.h.s. of (5.2), we find that as $\delta \downarrow 0$,

$$\mathbb{E}\big[(2\,\mathscr{M}_t^\vartheta(g_\delta))^m\big] = \int\limits_{(\mathbb{R}^2)^m} \prod_{i=1}^m g_\delta(x_i)\, \mathrm{e}^{\sum_{1 \leqslant i<j \leqslant m} k_t(x_i,x_j)}\, \mathrm{d}\vec{x}$$

$$= (1+o(1))\, (C_{t,\vartheta})^{\binom{m}{2}} \int\limits_{(\mathbb{R}^2)^m} \prod_{i=1}^m g_\delta(x_i) \prod_{1 \leqslant i<j \leqslant m} \log \frac{1}{|x_i - x_j|}\, \mathrm{d}\vec{x}.$$

Via the change of variable $y_i = x_i/\sqrt{\delta}$, the integral in the r.h.s. can be written as

$$\int\limits_{(\mathbb{R}^2)^m} \prod_{i=1}^m g_1(y_i) \prod_{1 \leqslant i<j \leqslant m} \left(\log \frac{1}{\sqrt{\delta}} + \log \frac{1}{|y_i - y_j|} \right) \mathrm{d}\vec{y} \sim \left(\log \frac{1}{\sqrt{\delta}} \right)^{\binom{m}{2}},$$

where the asymptotic equivalence as $\delta \downarrow 0$ follows by expanding the product and noting the finiteness of the integrals. This shows that, as $\delta \downarrow 0$,

$$\mathbb{E}\big[(2\,\mathscr{M}_t^\vartheta(g_\delta))^m\big] \sim \left(C_{t,\vartheta} \log \frac{1}{\sqrt{\delta}} \right)^{\binom{m}{2}},$$

which proves (5.2) and completes the proof of Proposition 5.1. $\qquad \square$

Appendix A. On the critical windows

In this secion, we compare the critical windows for directed polymers and for the mollified Stochastic Heat Equation.

A.1. Directed polymer setting.
The critical scaling of $\beta = \beta_N$ for the directed polymer partition functions (1.4) is defined by the following asymptotic relation:

$$\sigma_N^2 := \mathrm{e}^{\lambda(2\beta_N)-2\lambda(\beta_N)} - 1 = \frac{1}{R_N}\left(1 + \frac{\vartheta + o(1)}{\log N}\right), \qquad \text{for some fixed } \vartheta \in \mathbb{R}, \qquad (A.1)$$

where $\lambda(\cdot)$ is the log-moment generating function of the disorder, see (1.3), while R_N is the expected replica overlap of two independent simple symmetric random walks S, S' on $\mathbb{Z}^2$:

$$R_N := \mathrm{E}\bigg[\sum_{n=1}^N \mathbb{1}_{\{S_n = S'_n\}} \bigg] = \sum_{n=1}^N \sum_{z \in \mathbb{Z}^2} \mathrm{P}(S_n = z)^2 = \sum_{n=1}^N \mathrm{P}(S_{2n} = 0)$$

$$= \sum_{n=1}^N \left\{ \frac{1}{2^{2n}} \binom{2n}{n} \right\}^2 = \frac{\log N}{\pi} + \frac{\alpha}{\pi} + o(1) \qquad \text{as } N \to \infty, \qquad (A.2)$$

with $\alpha = \gamma + \log 16 - \pi$ and $\gamma \approx 0.577$ the Euler constant.

Since $\lambda(\beta) \sim \frac{1}{2}\beta^2$ as $\beta \to 0$, it follows from (A.1) that $\beta_N^2 \sim \pi/\log N$ as $N \to \infty$. The parameter $\vartheta \in \mathbb{R}$ tunes the higher order asymptotic behavior of β_N, which also depends on the third and fourth cumulants κ_3, κ_4 of the disorder: see [CSZ19b, eq. (1.17)] for the exact formula, which simplifies when $\kappa_3 = 0$ (e.g. for symmetric disorder distribution) and yields

$$\beta_N^2 = \frac{\pi}{\log N}\left(1 + \frac{\vartheta - c + o(1)}{\log N}\right) \qquad \text{where} \quad \begin{aligned} c &:= \alpha + \tfrac{1}{2}\pi + \tfrac{7}{12}\pi\kappa_4 \\ &= \gamma + \log 16 - \tfrac{1}{2}\pi + \tfrac{7}{12}\pi\kappa_4 \,, \end{aligned} \tag{A.3}$$

that is (1.7) holds with $\varrho = \vartheta - c$.

A.2. STOCHASTIC HEAT EQUATION SETTING.

We next consider the Stochastic Heat Equation (1.16) with mollified noise $\xi^\varepsilon(t,x) = (\xi(t,\cdot) * j_\varepsilon)(x)$, where $j_\varepsilon(x) := \varepsilon^{-2}j(\varepsilon^{-1}x)$. The critical scaling $\beta = \beta_\varepsilon$ is (see [CSZ19b, eq. (8.28)]):

$$\beta_\varepsilon^2 = \frac{1}{\mathcal{R}_\varepsilon}\left(1 + \frac{\vartheta + o(1)}{\log \varepsilon^{-2}}\right) \tag{A.4}$$

where $\mathcal{R}_\varepsilon$ is defined as follows (see [CSZ19b, Section 8.2]):

$$\mathcal{R}_\varepsilon = \int_0^{\varepsilon^{-2}}\left(\int_{(\mathbb{R}^2)^2} J(x)\,J(y)\,g_{2t}(x-y)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}t\,. \tag{A.5}$$

Note that we can view $\mathcal{R}_\varepsilon$ as the expected replica overlap of two independent Brownian motions B, B' on $\mathbb{R}^2$ enlarged via $J := j * j$ into Wiener sausages, described by the functions $J_{B_t}(z) := J(z - B_t)$ and $J_{B'_t}(z) := J(z - B'_t)$:

$$\begin{aligned} \mathcal{R}_\varepsilon &= \int_0^{\varepsilon^{-2}}\int_{(\mathbb{R}^2)^3} J(x)\,J(y)\,g_t(z-x)\,g_t(z-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t \\ &= \int_0^{\varepsilon^{-2}}\int_{\mathbb{R}^2} \mathbb{E}[J(z-B_t)\,J(z-B'_t)]\,\mathrm{d}z\,\mathrm{d}t = \mathbb{E}\left[\int_0^{\varepsilon^{-2}} \langle J_{B_t}, J_{B'_t}\rangle_{L^2(\mathbb{R}^2)}\,\mathrm{d}t\right]. \end{aligned} \tag{A.6}$$

It was shown in [CSZ19b, end of Section 8.2] that

$$\mathcal{R}_\varepsilon = \frac{\log \varepsilon^{-2}}{4\pi} + \frac{C}{4\pi} + o(1) \qquad \text{as } \varepsilon \downarrow 0\,, \tag{A.7}$$

where

$$C = 2\int_{(\mathbb{R}^2)^2} J(x)\,\log\frac{1}{|x-y|}\,J(y)\,\mathrm{d}x\,\mathrm{d}y + \log 4 - \gamma\,. \tag{A.8}$$

Plugging this into (A.4) yields

$$\beta_\varepsilon^2 = \frac{4\pi}{\log \varepsilon^{-2}}\left(1 + \frac{\vartheta - C + o(1)}{\log \varepsilon^{-2}}\right). \tag{A.9}$$

that is (1.21) holds with $\varrho = \vartheta - C$.

A.3. MATCHING DIRECTED POLYMERS WITH THE STOCHASTIC HEAT EQUATION.
In this appendix we explain heuristically the choice of scalings in relation (5.6). Convergence
(5.6) is expected to hold by an extension of the methods in [CSZ21].

The directed polymer model is based on the simple random walk (S_n) on $\mathbb{Z}^2$, which has
covariance matrix $\mathsf{s}I$ with $\mathsf{s} = \frac{1}{2}$ and is T-periodic with $\mathsf{T} = 2$, in the sense that S_n takes
values in a sub-lattice $\mathbb{T}_n \subset \mathbb{Z}^2$ whose cells have area $\mathsf{T} = 2$:

$$\mathbb{T}_n := \begin{cases} \mathbb{Z}^2_{\text{even}} & \text{for } n \text{ even}, \\ \mathbb{Z}^2_{\text{odd}} := \mathbb{Z}^2 \backslash \mathbb{Z}^2_{\text{even}} & \text{for } n \text{ odd}, \end{cases} \tag{A.10}$$

see (1.6). This is reflected in the local limit theorem: as $n \to \infty$

$$P(S_n = z) = \big(g_{\mathsf{s}n}(z) + o(n^{-1})\big) \mathsf{T}\, \mathbb{1}_{\mathbb{T}_n}(z), \tag{A.11}$$

where $g_t(x)$ denotes the heat kernel, see (2.1). We insist on the use of general parameters
s and T, instead of the particular values $\frac{1}{2}$ and 2, as the heuristics we present should be
broad enough to encompass more general random walks.

We remark that the directed polymer partition function $Z_{0,N}^\beta(x,y)$ for $x, y \in \mathbb{Z}^2_{\text{even}}$, see
(1.4), only depend on the disorder random variables $\omega(n,z)$ for $n \in \mathbb{N}$ and $z \in \mathbb{T}_n$. When we
take the diffusive rescaling, these random variable give rise to the random field

$$\xi_N(\mathrm{d}t, \mathrm{d}x) := \sqrt{\mathsf{T}}\, \frac{1}{N} \sum_{n \in \mathbb{N}} \sum_{z \in \mathbb{T}_n} \omega(n,z)\, \delta_{(\frac{n}{N}, \frac{z}{\sqrt{N}})}(\mathrm{d}t, \mathrm{d}x), \tag{A.12}$$

which is a discretized version of the mollified space-time white noise $\xi^{\varepsilon/\sqrt{\mathsf{T}}}(t,x)\,\mathrm{d}t\,\mathrm{d}x$.[†]

As a consequence, for a proper comparison with directed polymers, we should consider
the Stochastic Heat Equation with *adjusted parameters*:

$$\begin{cases} \partial_t \tilde{u}_\beta^\varepsilon(t,x) = \dfrac{\mathsf{s}}{2} \Delta \tilde{u}_\beta^\varepsilon(t,x) + \sqrt{\mathsf{T}}\, \beta\, \tilde{u}_\beta^\varepsilon(t,x)\, \xi^{\varepsilon/\sqrt{\mathsf{T}}}(t,x) \\ \tilde{u}_\beta^\varepsilon(0, \cdot) \equiv \dfrac{1}{\mathsf{T}} \end{cases} \quad \text{with} \quad \mathsf{s} = \tfrac{1}{2},\ \mathsf{T} = 2, \quad \text{(A.13)}$$

where the initial condition $\tilde{u}_\beta^\varepsilon(0, \cdot) \equiv \frac{1}{\mathsf{T}} = \frac{1}{2}$ is due to the restriction in (1.5) to $\mathbb{Z}^2_{\text{even}}$, the
factor s in front of the Laplacian comes from the variance of the simple random walk, while
the factor $\sqrt{\mathsf{T}}$ which multiplies β arises from the factor T in the local limit theorem (A.11)
(the "other" factor $\sqrt{\mathsf{T}}$ goes into the normalization of the rescaled noise ξ_N, see (A.12)).

Summarizing, the directed polymer random measure $\mathcal{Z}_{N;0,t}^{\beta_N}(\mathbb{R}^2, \mathrm{d}x)$, see (1.5), is expected
to be close to the solution $\tilde{u}_\beta^\varepsilon(t,x)$ of (A.13), where we identify $N = \varepsilon^{-2}$:

$$\mathcal{Z}_{N;0,t}^\beta(\mathbb{R}^2, \mathrm{d}x) \stackrel{d}{\approx} \tilde{u}_\beta^\varepsilon(t,x)\,\mathrm{d}x. \tag{A.14}$$

Finally, we can relate $\tilde{u}_\beta^\varepsilon(t,x)$ to the solution $u_\beta^\varepsilon(t,x)$ of the (standard, unscaled) Stochastic
Heat Equation (1.16) by a change of variables: indeed, one can check that

$$\tilde{u}_\beta^\varepsilon(t,x) \stackrel{d}{=} \frac{1}{\mathsf{T}}\, u_{\sqrt{\frac{\mathsf{T}}{\mathsf{s}}}\beta}^\varepsilon\big(\mathsf{s}\mathsf{T}\, t, \tfrac{x}{\sqrt{\mathsf{s}}}\big) = \frac{1}{2}\, u_{2\beta}^\varepsilon\big(t, x\sqrt{2}\big). \tag{A.15}$$

Plugging this into (A.14) leads to $\mathcal{Z}_{N;0,t}^\beta(\mathbb{R}^2, \mathrm{d}x) \stackrel{d}{\approx} \frac{1}{2} u_{2\beta}^\varepsilon(t, x\sqrt{2})\,\mathrm{d}x$ which is "essentially"
(5.6) in the sense that $\beta_\varepsilon \sim 2\beta_N$, cf. (A.3) and (A.9).

[†]The spatial scale of the mollified white noise $\xi^{\varepsilon/\sqrt{\mathsf{T}}}(t,x)$ is chosen to be $\varepsilon/\sqrt{\mathsf{T}}$ because we identify
$\varepsilon = 1/\sqrt{N}$ and note that $\mathbb{T}_n$ is a rotated square lattice with lattice spacing $\sqrt{\mathsf{T}}$.

Acknowledgements

The completion of this work coincided with the passing of Francis Comets. We wish to express our deepest gratitude to Francis and dedicate this work to him for the inspiration he provided us over the years.

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