

# Pinning and Wetting Transition for (1+1)-Dimensional Fields with Laplacian Interaction

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Workshop on Gradient Models and Elasticity

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# References

- ▶ [CD1] F. Caravenna and J.-D. Deuschel  
*Pinning and wetting transition for  $(1+1)$ -dimensional fields with Laplacian interaction*, Ann. Probab. (to appear)
- ▶ [CD2] F. Caravenna and J.-D. Deuschel  
*Scaling limits of  $(1+1)$ -dimensional pinning models with Laplacian interaction*, preprint (2008).

# Outline

## 1. The Models

Introduction

Wetting and pinning models

## 2. Free Energy Results

The free energy

The phase transition

The disordered case

## 3. Path Results

Path results

Refined critical scaling limit

## 4. Sketch of the Proof

Integrated random walk

Markov renewal theory

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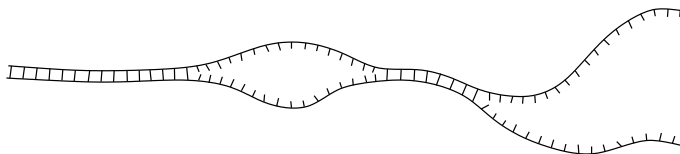
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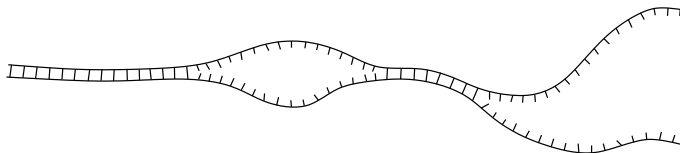
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DNA denaturation transition at high temperature

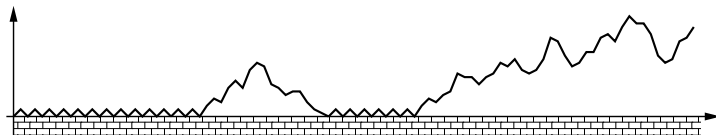


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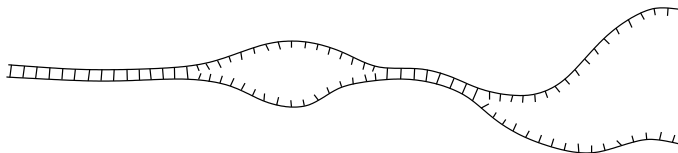


(1+1)-dimensional model: field above an impenetrable wall

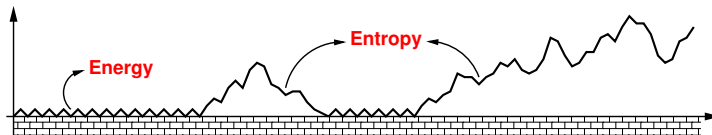


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# The general wetting model

The field  $\varphi = \{\varphi_i\}_{1 \leq i \leq N}$  in the **free case**:

$$\mathbb{P}_{0,N}^w(d\varphi_1, \dots, d\varphi_N) := \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}_{0,N}^w} \prod_{i=1}^N d\varphi_i^+$$

- ▶  $d\varphi_i^+$  is the Lebesgue measure on  $[0, \infty)$
- ▶  $\mathcal{H}_N(\varphi)$  describes the structure of the chain (**to be specified**)
- ▶  $\mathcal{Z}_{0,N}^w$  is the normalization constant (**partition function**)

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- ▶  $\delta_0(\cdot)$  is the Dirac mass at zero
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# The general pinning model

Analogous to the wetting case but **without repulsion**:  $d\varphi_i^+ \rightarrow d\varphi_i$

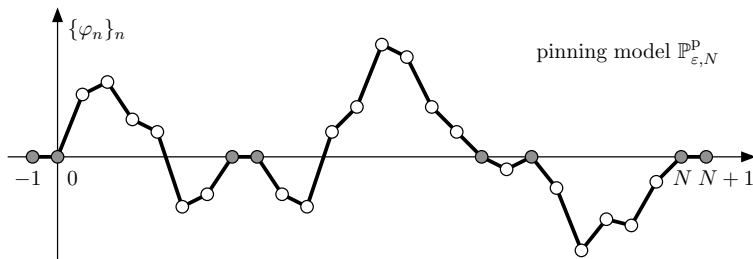
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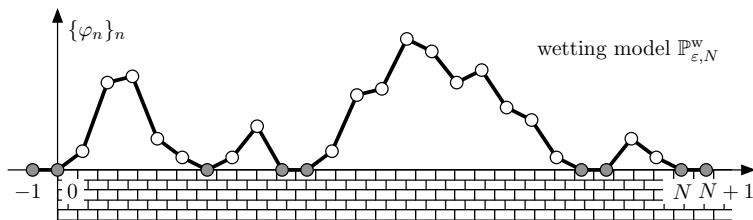
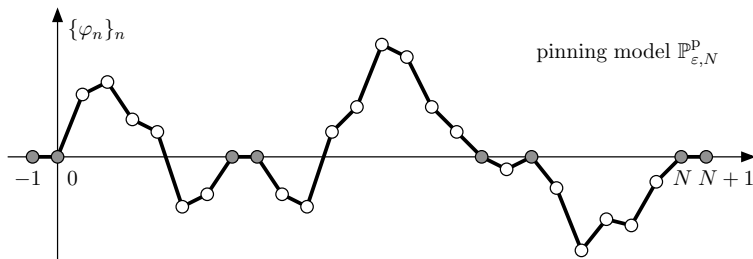
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How to choose  $\mathcal{H}_N(\varphi)$ ?

# The choice of $\mathcal{H}_N(\varphi)$

The simplest choice is the **gradient case**:

$$\mathcal{H}_N(\varphi) := \sum_{i=1}^N V(\nabla \varphi_i), \quad \nabla \varphi_i := \varphi_i - \varphi_{i-1},$$

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[Isozaki, Yoshida SPA 01] [Deuschel, Giacomin, Zambotti PTRF 05]

[Caravenna, Giacomin, Zambotti EJP 06]

# The choice of $\mathcal{H}_N(\varphi)$

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Interpretation of the free case  $\varepsilon = 0$ :

- ▶  $\mathbb{P}_{0,N}^p$  is (the bridge of) the integral of a **random walk**
- ▶  $\mathbb{P}_{0,N}^w$  is further **conditioned to stay  $\geq 0$**

# Laplacian interaction in $(d + 1)$ -dimension

Fields  $\varphi : \{1, \dots, N\}^d \rightarrow \mathbb{R}$  with Laplacian interaction for  $d \geq 2$  are models for **semiflexible membranes**

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Assumptions on  $V$ :

$$\int_{\mathbb{R}} e^{-V(x)} dx = 1, \quad \int_{\mathbb{R}} x e^{-V(x)} dx = 0, \quad \int_{\mathbb{R}} x^2 e^{-V(x)} dx = 1$$

+ regularity:  $x \mapsto e^{-V(x)}$  continuous and  $V(0) < +\infty$ .

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# How to define localization and delocalization?

Recall the **partition function**: (zero boundary conditions)

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## Free Energy

$$F^a(\varepsilon) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Z}_{\varepsilon,N}^a \quad (\text{super-additivity})$$

**Basic observation:**  $F^a(\varepsilon) \geq F^a(0) = 0$  for all  $\varepsilon \geq 0$  and  $a \in \{p, w\}$

$$\mathcal{Z}_{\varepsilon,N}^a \geq \mathcal{Z}_{0,N}^a \approx N^{-c} \quad (c > 0)$$

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## Definition

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► if  $\varepsilon > \varepsilon_c^a$  then  $\frac{\ell_N}{N} \rightarrow D^a(\varepsilon) > 0$  in  $\mathbb{P}_{\varepsilon, N}^a$ -probability

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  - ▶ if  $F^a(\varepsilon_c^a + h) \geq C h$  [**1<sup>st</sup> order trans.**]  $\varepsilon = \varepsilon_c^a$  may be **localized** (phase coexistence, dependence of boundary conditions)

# The phase transition

## Theorem ([CD1])

Both  $\mathbb{P}_{\varepsilon,N}^p$  and  $\mathbb{P}_{\varepsilon,N}^w$  undergo a non-trivial phase transition:

$$0 < \varepsilon_c^p < \varepsilon_c^w < \infty$$

and  $F^a(\varepsilon)$  is analytic on  $[0, \varepsilon_c^a) \cup (\varepsilon_c^a, \infty)$ . (*variational formula*)

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$$C_1 \frac{h}{\log \frac{1}{h}} \leq F^p(\varepsilon_c^p + h) \leq o(h)$$

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► In the *wetting model* the transition is of  $1^{st}$  order:

$$F^w(\varepsilon_c^w + h) \sim C_2 h \quad [\ell_N \sim D N, \quad D > 0]$$

# The gradient case

## Differences in the gradient case

- ▶ the transition is non-trivial only in the **wetting model**:

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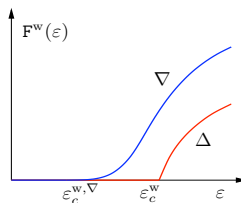
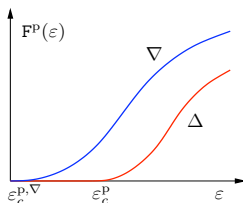
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► skip

# A look at the disordered case

Disordered version of our model: ( $d\varphi_i^p = d\varphi_i$  and  $d\varphi_i^w = d\varphi_i^+$ )

$$\mathbb{P}_{\varepsilon, \beta, \omega, N}^a(d\varphi_1, \dots, d\varphi_N) := \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}_{\varepsilon, \beta, \omega, N}^a} \prod_{i=1}^N (d\varphi_i^a + \varepsilon e^{\beta \omega_i} \delta_0(d\varphi_i))$$

where  $\beta \geq 0$  and  $\{\omega_i\}_{i \in \mathbb{N}}$  are IID  $\mathcal{N}(0, 1)$  (law  $P$  indep.  $\mathbb{P}^a$ ).

# A look at the disordered case

**Disordered version** of our model:  $(d\varphi_i^p = d\varphi_i \text{ and } d\varphi_i^w = d\varphi_i^+)$

$$\mathbb{P}_{\varepsilon, \beta, \omega, N}^a(d\varphi_1, \dots, d\varphi_N) := \frac{e^{-\mathcal{H}_N(\varphi)}}{\mathcal{Z}_{\varepsilon, \beta, \omega, N}^a} \prod_{i=1}^N (d\varphi_i^a + \varepsilon e^{\beta \omega_i} \delta_0(d\varphi_i))$$

where  $\beta \geq 0$  and  $\{\omega_i\}_{i \in \mathbb{N}}$  are IID  $\mathcal{N}(0, 1)$  (law  $P$  indep.  $\mathbb{P}^a$ ).

## Quenched free energy

$$F^a(\varepsilon, \beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Z}_{\varepsilon, \beta, \omega, N}^a \geq 0$$

exists  $P(d\omega)$ -a.s. and **does not depend on  $\omega$**  (self-averaging)

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**Localization:**  $F^a(\varepsilon, \beta) > 0 \iff \varepsilon > \varepsilon_c^a(\beta)$  (**critical line**)

# Smoothing effect of disorder

What is the behavior of  $\varepsilon_c^a(\beta)$  for small  $\beta$  ? ( $\varepsilon_c^a = \varepsilon_c^a(0)$ )

What is the **regularity of the transition** in the disordered case?

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**Theorem ([Giacomin and Toninelli, CMP 06])**

*Both in the  $\nabla$  and  $\Delta$  case, both for  $a = p$  and for  $a = w$ :  
for every  $\beta > 0$  there exists  $C_\beta > 0$  such that*

$$F^a(\varepsilon_c^a(\beta) + h) \leq C_\beta h^2$$

When disorder is present the transition is **at least of 2<sup>nd</sup> order**

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**Very general proof:** rare-stretches in  $\omega$  (Large Deviations)

► proof

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The phase transition

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# Some deeper questions

We have established the existence of a **phase transition**:

$$\ell_N = \begin{cases} o(N) & \text{if } \varepsilon < \varepsilon_c^a \\ \sim D \cdot N & \text{if } \varepsilon > \varepsilon_c^a \end{cases}$$

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Yes in the **pinning case** and under additional assumptions on  $V(\cdot)$ :

- ▶ **symmetry**:  $V(x) = V(-x)$  for every  $x \in \mathbb{R}$
- ▶ **uniform strict convexity**:  $\exists \gamma > 0$  s. t.  $V(x) - \gamma \frac{x^2}{2}$  is convex
- ▶ **regularity**:  $x \mapsto e^{-V(x)}$  is continuous and  $V(0) < \infty$

$$\int_{\mathbb{R}} e^{-V(x)} dx = 1 \quad \int_{\mathbb{R}} x^2 e^{-V(x)} dx = 1$$

# A closer look at the typical paths [CD2]

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# Scaling Limits

We rescale and interpolate linearly the field: for  $t \in [0, 1]$

$$\widehat{\varphi}_N(t) := \frac{\varphi_{\lfloor Nt \rfloor}}{N^{3/2}} + (Nt - \lfloor Nt \rfloor) \frac{\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}}{N^{3/2}}$$

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Let  $\{B_t\}_{t \in [0,1]}$  **standard BM**,  $I_t := \int_0^t B_s \, ds$  **integrated BM**

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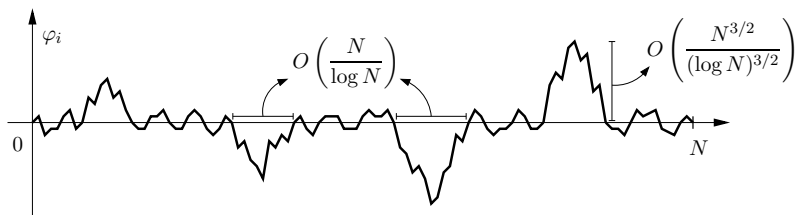
## Theorem (Scaling Limits [CD2])

*The rescaled field  $\{\widehat{\varphi}_N(t)\}_{t \in [0,1]}$  under  $\mathbb{P}_{\varepsilon, N}^p$  converges in distribution on  $C([0, 1])$  as  $N \rightarrow \infty$ , for every  $\varepsilon \geq 0$ . The limit is*

- ▶ *If  $\varepsilon < \varepsilon_c^p$ , the law of  $\{\widehat{l}_t\}_{t \in [0,1]}$*
- ▶ *If  $\varepsilon = \varepsilon_c^p$  or  $\varepsilon > \varepsilon_c^p$ , the law concentrated on  $f(t) \equiv 0$*

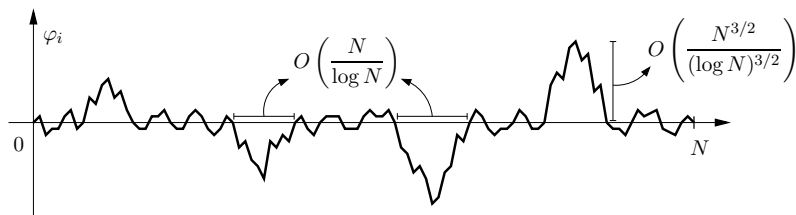
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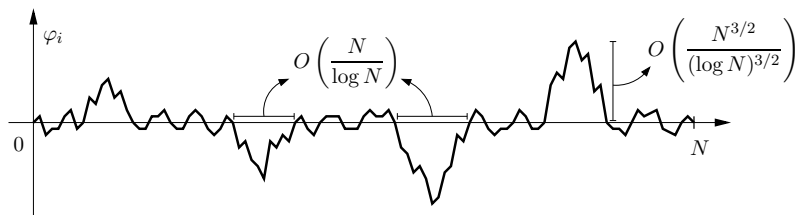
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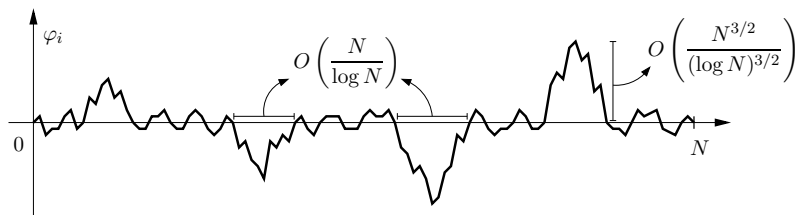
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**Not** in  $C([0, 1])$  or  $D([0, 1])$ :

the set  $\frac{1}{N} \{i \in \{1, \dots, N\} : \varphi_i = 0\}$  becomes dense in  $[0, 1]$

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**Alternative idea**: look at the field in a **distributional sense**

# The critical regime

Introduce the **random measure** (finite, signed) on  $[0, 1]$

$$\mu_N(dt) := \frac{(\log N)^{5/2}}{N^{3/2}} \varphi_{\lfloor Nt \rfloor} dt = \tilde{\varphi}_N(t) dt$$

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Let  $\{L_t\}_{t \in [0,1]}$  be the **stable symmetric Lévy process** of index  $\frac{2}{5}$

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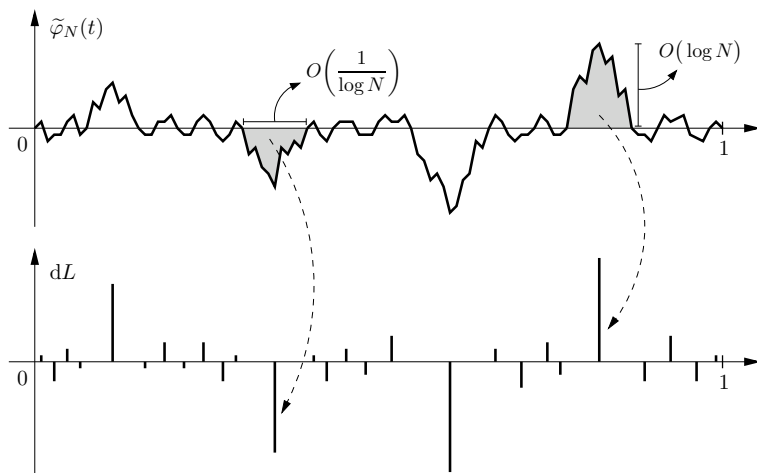
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## Theorem ([CD2])

$\mu_N$  under  $\mathbb{P}_{\varepsilon_c, N}^p$  converges in distribution as  $N \rightarrow \infty$  toward  $dL$ .

# The critical regime



► disorder

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# A random walk viewpoint ( $\varepsilon = 0$ )

Let  $\{X_i\}_{i \in \mathbb{N}}$  be IID random variables with law

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- ▶ Under  $\mathbb{P}_{0,N}^{\mathbf{W}}$  the same, under the further conditioning  $\{Z_1 \geq 0, \dots, Z_N \geq 0\}$

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Once we know  $\tau, J$ , the **whole field**  $\{\varphi_i\}_i$  is reconstructed by pasting independent excursions from  $\{Z_i\}_i$  (cond. to stay  $\geq 0$ )

# The law of the excursions

Pinning case: good control (Donsker's inv. pr. + LLT)

$$\left\{ \frac{Z_{\langle Nt \rangle}}{N^{3/2}} \right\}_{t \in [0,1]} \text{ condit. on } (Y_N, Z_N) = (0, 0) \implies \{\hat{l}_t\}_{t \in [0,1]}$$

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$$\left\{ \frac{Z_{\langle Nt \rangle}}{N^{3/2}} \right\}_{t \in [0,1]} \text{ condit. on } (Y_N, Z_N) = (0, 0) \implies \{\widehat{I}_t\}_{t \in [0,1]}$$

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## Entropic repulsion

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$$\mathbf{P}(Z_1 \geq 0, \dots, Z_N \geq 0) \approx N^{-1/4} \quad [\text{Sinai (SRW)}]$$

$$\mathbf{P}(Z_1 \geq 0, \dots, Z_N \geq 0 \mid Y_N = 0, Z_N = 0) \approx N^{-1/2} \quad [\text{conj.}]$$

# Markov renewal processes

Given a (sub-)probability kernel  $K_{x,dy}(n)$ :

$$\int_{y \in \mathbb{R}} \sum_{n \in \mathbb{N}} K_{x,dy}(n) = c \leq 1, \quad \forall x \in \mathbb{R}$$

we build the Markov renewal process  $\tau$  with modulating chain  $J$ :

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The law of  $(\tau, J)$  conditionally on  $\{N, N+1\} \subseteq \tau$  is

$$\mathcal{P}(\tau_i = t_i, J_i \in dy_i \mid \{N, N+1\} \subseteq \tau) = \frac{1}{C_N} \prod_i K_{y_{i-1}, dy_i}(t_i - t_{i-1})$$

with  $C_N = \mathcal{P}(\{N, N+1\} \subseteq \tau)$ .

# The law of the contact set

Consider the following kernels: for  $n \in \mathbb{N}$  and  $x, y \in \mathbb{R}$

$$G_{x,dy}^p(n) := \varepsilon \frac{\mathbf{P}_x(Z_{n-1} \in dy, Z_n \in dz)}{dz} \Big|_{z=0}$$

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$$\mathbb{P}_{\varepsilon, N}^a(\tau_i = t_i, J_i \in dy_i, i \leq k) = \frac{1}{Z_{\varepsilon, N}^a} \prod_{i=1}^k G_{y_{i-1}, dy_i}^a(t_i - t_{i-1})$$

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Reminds of [Markov renewal theory](#)

# Markov renewal processes

We exploit the **invariance properties**: for every  $F$ ,  $v(y)$

$$\begin{aligned} & \mathbb{P}_{\varepsilon, N}^a (\tau_i = t_i, J_i \in dy_i, i \leq k) \\ &= \frac{e^{FN}}{\mathcal{Z}_{\varepsilon, N}^a} \prod_{i=1}^k G_{y_{i-1}, dy_i}^a(t_i - t_{i-1}) e^{-F(t_i - t_{i-1})} \frac{v(y_i)}{v(y_{i-1})} \end{aligned}$$

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If we determine  $F$ ,  $v(\cdot)$  such that

$$K_{x, dy}(n) := G_{x, dy}^a(n) e^{-F \cdot n} \frac{v(y)}{v(x)}$$

is a **probability kernel**, we have the crucial relation

$$\mathbb{P}_{\varepsilon, N}^a (\tau_i = t_i, J_i \in dy_i) = \mathcal{P}(\tau_i = t_i, J_i \in dy_i \mid \{N, N+1\} \subseteq \tau)$$

# A Perron-Frobenius problem

It turns out that:

- ▶  $F$  is the solution of the equation

$$\text{spectral radius of } \left( \sum_{n \in \mathbb{N}} G_{x,dy}^a(n) e^{-F \cdot n} \right)_{x,y} = 1$$

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- In fact  $F = F^a(\varepsilon)$  is the **free energy**
- $v(\cdot)$  is the **principal eigenfunction**:

$$\int_{y \in \mathbb{R}} \left( \sum_{n \in \mathbb{N}} G_{x,dy}^a(n) e^{-F \cdot n} \right)_{x,y} v(y) = v(x)$$