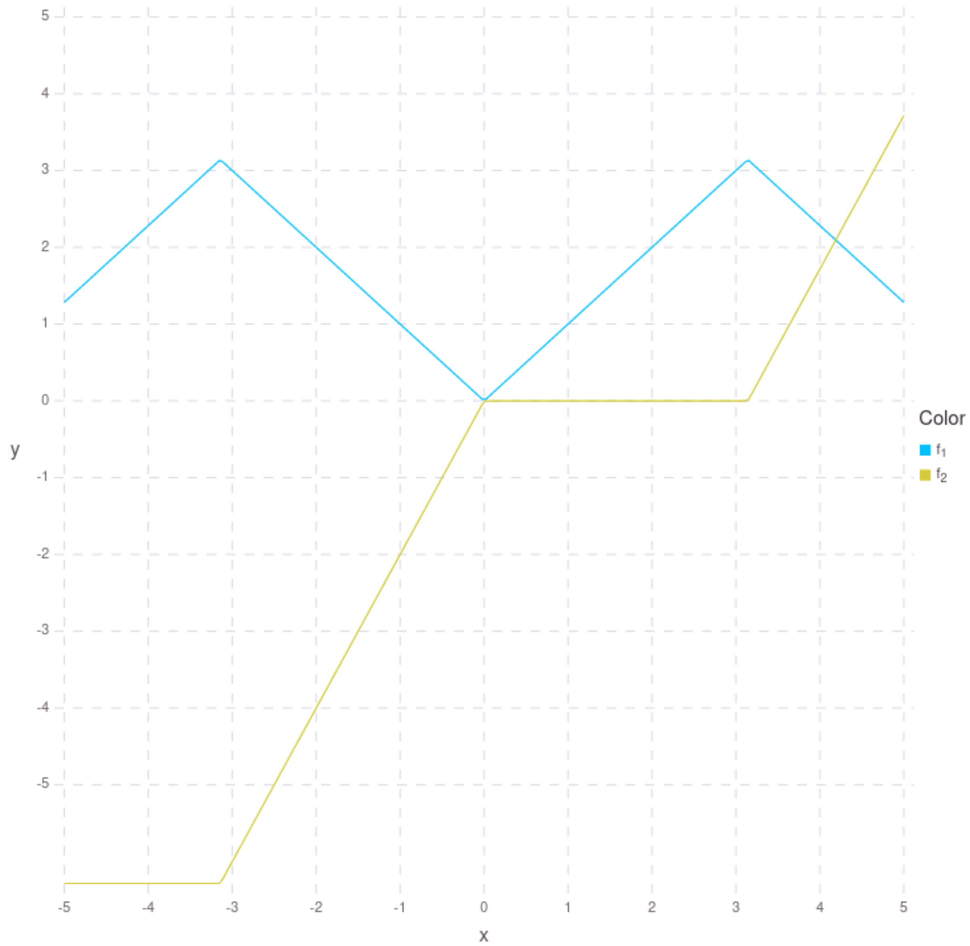
$$\arccos(cos(x))$$

$$\begin{aligned}\cos(2) &\approx -0.41 \\ \cos(2+2\pi) &\approx -0.41 \\ \arccos(-0.41) &\approx 2\end{aligned}$$

The computer can only return one result from the arccos function, it does not have the concept of a multivalued function. Normally it defines the image of arccos as between 0 and π, closed. Therefore, calling $\arccos(\cos(x))$ is the same as keeping x between 0 and π.

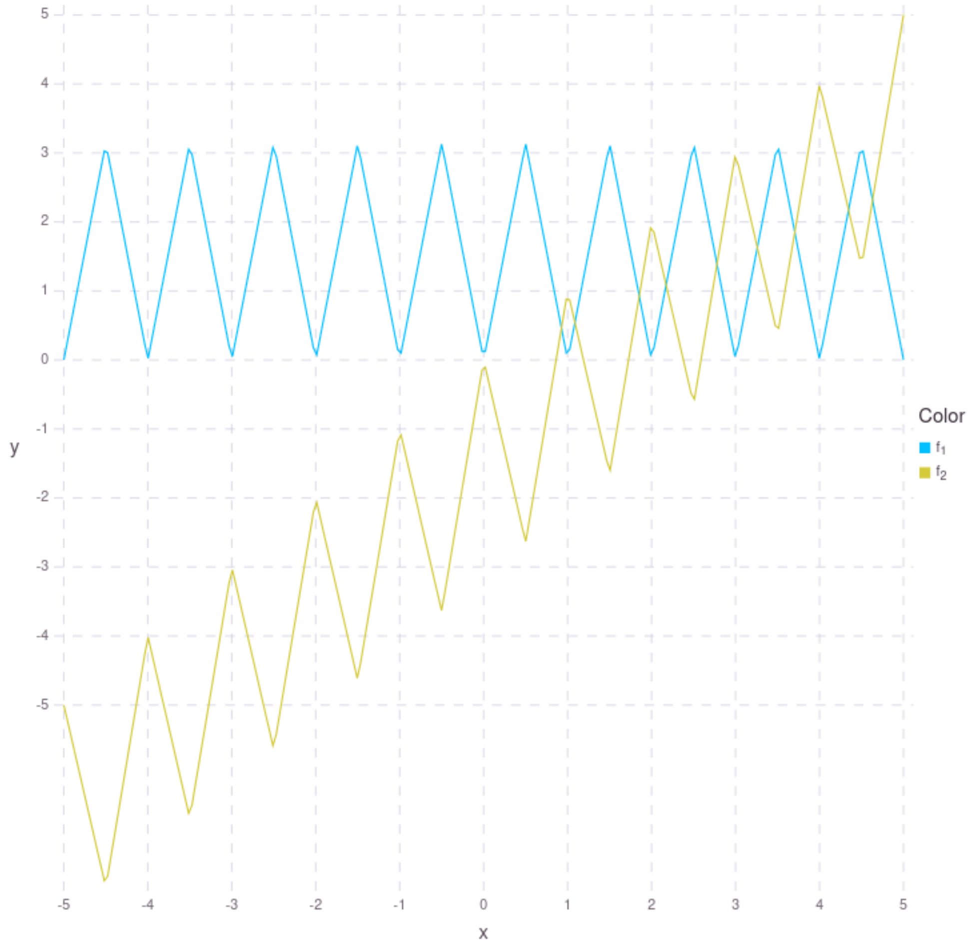
```
f1(x) = acos(cos(x))
f2(x) = x - acos(cos(x))
```



$$\arccos(cos(2\pi x))$$

By multiplying the cosine's argument by 2π, we "flatten" the function horizontally, changing its period from 2π to 1.

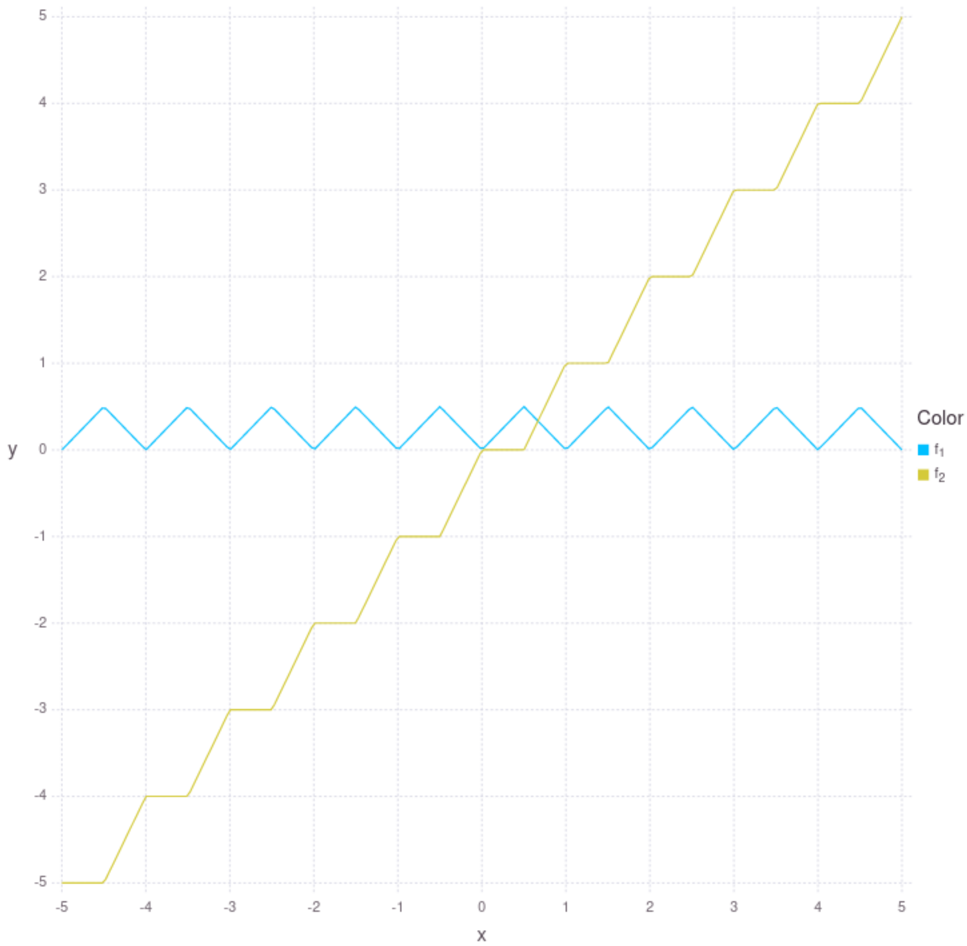
```
f1(x) = acos(cos(2pi*x))
f2(x) = x - acos(cos(2pi*x))
```



$$\frac{\arccos(cos(2\pi x))}{2\pi}$$

By dividing everything by 2π, we flatten the function vertically, making its maximum value $\frac{1}{2}$.

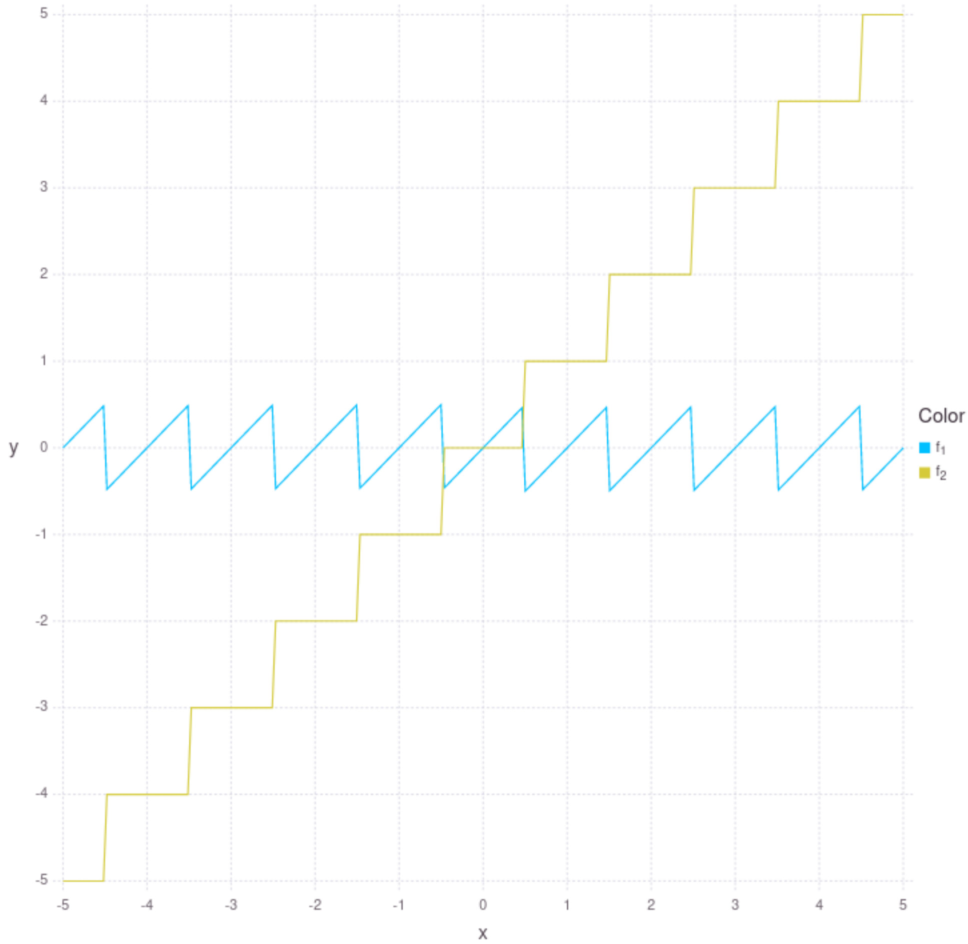
```
f1(x) = acos(cos(2pi*x))/2pi
f2(x) = x - f1(x)
```



$$\operatorname{sgn}(sin(2\pi x))\left(\frac{\arccos(cos(2\pi x))}{2\pi}\right)$$

By multiplying with the sign of the sine of $2\pi x$, we obtain a sawtooth function and a staircase function, the formats we require. We are close, but in our floor function the numbers are changing in the middle, e.g. $floor(1.4) = 1$ but $floor(1.6) = 2$.

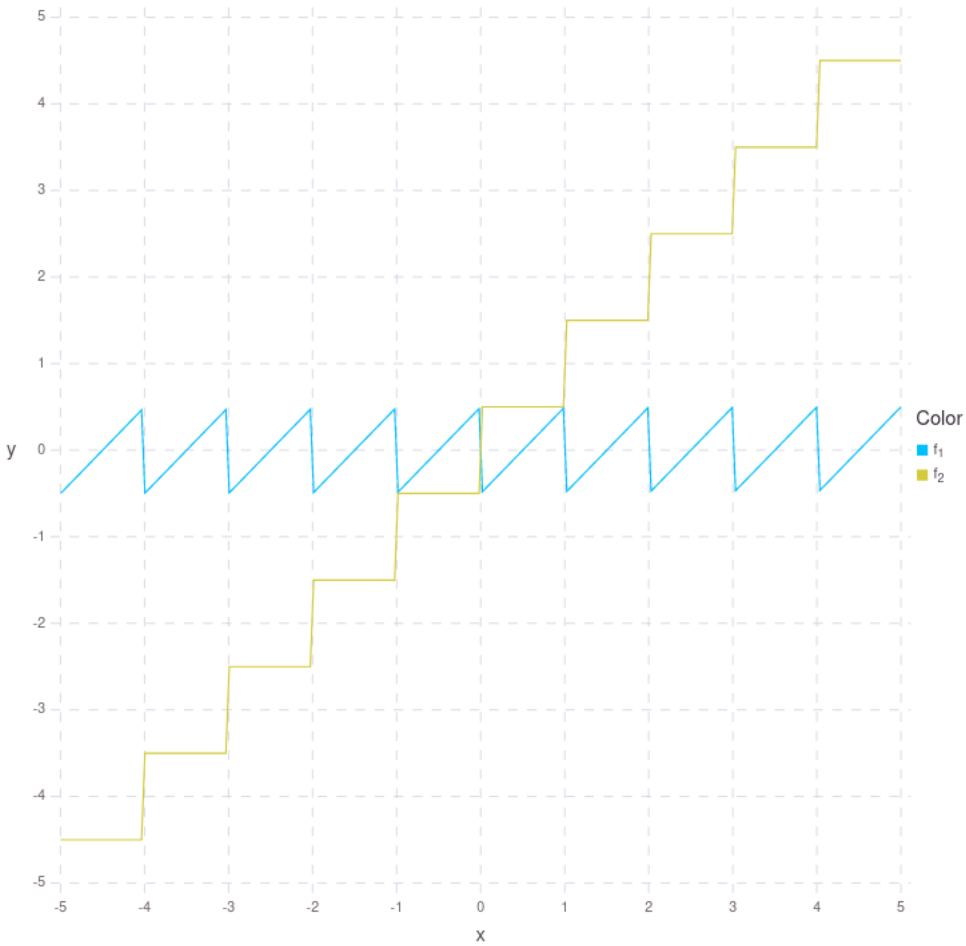
```
f1(x) = sign(sin(2pi*x)) * acos(cos(2pi*x))/2pi
f2(x) = x - f1(x)
```



$$\operatorname{sgn}(sin(2\pi x))\left(\frac{\arccos(cos(2\pi x))}{2\pi} - \frac{1}{2}\right)$$

By subtracting $\frac{1}{2}$ in the appropriate place, we translate the sawtooth function horizontally so that its minimum point coincides with the integers. This solves our last problem, but now our function is completely translated up by $\frac{1}{2}$ more than it should.

```
f1(x) = sign(sin(2pi*x)) * (acos(cos(2pi*x))/2pi - 1/2)
f2(x) = x - f1(x)
```



$$\operatorname{sgn}(sin(2\pi x))\left(\frac{\arccos(cos(2\pi x))}{2\pi} - \frac{1}{2}\right) + \frac{1}{2}$$

Adding $\frac{1}{2}$ solves our last problem, now we have our alpha and floor functions, defined entirely by an analytic method.

```
f1(x) = sign(sin(2pi*x)) * (acos(cos(2pi*x))/2pi - 1/2) + 1/2
f2(x) = x - f1(x)
```