Exercise 2.2

There are a lot of neat properties of the Fibonacci numbers that can be proved by induction. Recall that the Fibonacci numbers are defined by $f_0 = 0, f_1 = 1$ and the recursion relation $f_{n+1} = f_n + fn - 1$ for all $n \geq 1$.

Prove that $f_n \geq \frac{1}{3} \cdot 1.5^n$ for $n \geq 1$.

Solution:

The induction proof for all $n \geq 1$ begins with the base case of $n = 1$.

$$f_1 \geq \frac{1}{3} \cdot 1.5^1 \qquad 1 \geq \frac{1.5}{3} \qquad 1 \geq 0.5$$

Additionally, the case $n = 2$ will be verified to use strong induction.

$$f_2 \geq \frac{1}{3} \cdot 1.5^2 \qquad f_2 \geq 0.75 \qquad f_2 = f_1 + f_0 = 1 \geq 0.75$$

The base cases hold. Now assume that the formula to be proven is valid for all k up to an arbitrary m, with $0 \leq k \leq m$.

$$f_k \geq \frac{1}{3} \cdot 1.5^k, \; k \geq 1$$

This induction hypothesis assumes that the inequality holds for k and $k-1$. It must be proven that the assumption also holds for $k+1$. The induction step proceeds as follows.

$$f_{k+1} \geq \frac{1}{3} \cdot 1.5^{k+1}$$

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
f_{k+1} &\geq \frac{1}{3} \cdot 1.5^k + \frac{1}{3} \cdot 1.5^{k-1} && \text{(I.H.)} \\
f_{k+1} &\geq \frac{1}{3} \cdot 1.5^{k-1}(1.5 + 1) \\
f_{k+1} &\geq \frac{1}{3} \cdot 1.5^{k-1}(2.5)
\end{aligned}$$

It holds that $1.5^2 = 2.25 \leq 2.5$. Substituting 2.5 with 1.5^2 keeps the inequality valid. This allows the final steps to the solution.

$$\begin{aligned}
f_{k+1} &\geq \frac{1}{3} \cdot 1.5^{k-1}(2.5) \geq \frac{1}{3} \cdot 1.5^{k-1}(1.5^2) \\
f_{k+1} &\geq \frac{1}{3} \cdot 1.5^{k+1}
\end{aligned}$$

This demonstrates the validity of the induction step. It has thus been proven by mathematical induction that the formula holds for every integer $n \geq 1$.

Exercise 2.3

(a) For all the following functions the variable n ranges over $\mathbb{N}$. Prove or disprove the following statements. Justify your answer using Theorems 1 and/or 2.

In all items, consider the functions f and g respectively.

(1) $2n^5 + 10n^2 \leq O(\frac{1}{100}n^6)$

Solution:

$$\lim_{n\to\infty} \frac{2n^5 + 10n^2}{\frac{1}{100}n^6} = \lim_{n\to\infty} \frac{n^6}{n^6}\left(\frac{\frac{2}{n} + \frac{10}{n^4}}{\frac{1}{100}}\right) = \frac{0}{\frac{1}{100}} = 0$$

Theorem 1 proves this statement, $f \leq O(g)$.

(2) $n^{10} + 2n^2 + 7 \leq O(100n^9)$

Solution:

$$\lim_{n\to\infty} \frac{n^{10} + 2n^2 + 7}{100n^9} = \lim_{n\to\infty} \frac{n^9}{n^9}\left(\frac{n + \frac{2}{n^7} + \frac{7}{n^9}}{100}\right) = \lim_{n\to\infty} \frac{n}{100} = \infty$$

Theorem 1 disproves this statement, $f \not\leq O(g),\ g \leq O(f)$.

(3) $e^{1.2n} \leq O(e^n)$

Solution:

$$\lim_{n\to\infty} \frac{e^{1.2n}}{e^n} = \lim_{n\to\infty} \frac{e^n \cdot e^{0.2n}}{e^n} = \lim_{n\to\infty} e^{0.2n} = \infty$$

Theorem 1 disproves this statement, $f \not\leq O(g),\ g \leq O(f)$.

(4) $n^{\frac{2n+3}{n+1}} \leq O(n^2)$

Solution:

$$\lim_{n\to\infty} \frac{n^{\frac{2n+3}{n+1}}}{n^2} = \lim_{n\to\infty} n^{\frac{2n+3}{n+1}-2} = \lim_{n\to\infty} n^{\frac{2n+3-2(n+1)}{n+1}} = \lim_{n\to\infty} n^{\frac{2n+3-2n-2}{n+1}} = \lim_{n\to\infty} n^{\frac{1}{n+1}} = 1$$

Theorem 1 proves this statement, $f \leq O(g)$, and it can also be said that $g \leq O(f)$.

(b) Find f and g as in Theorem 1 such that $f \leq O(g)$, but the limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist. This proves that the first point of Theorem 1 provides a sufficient, but not a necessary condition for $f \leq O(g)$.

Solution:

As $\sin n$ is an oscillating function, the following limit does not exist.

$$\lim_{n\to\infty} \frac{n}{\sin n} = \text{DNE}$$

At the same time, it could be said that $\sin n \leq O(n)$, because as n tends to infinity, $\sin n$ continuously oscillates between 1 and -1. The existence of the limit $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is therefore not necessary for it to be true that $f \leq O(g)$.

Exercise 2.4

The goal of this exercise is to show that the sum $\sum_{i=1}^{n} \frac{1}{i}$ behaves, up to constant factors, as $\log(n)$ when n is large. Formally, we will show

$$\sum_{i=1}^{n} \frac{1}{i} \leq O(\log n) \quad \text{and} \quad \log n \leq O\left(\sum_{i=1}^{n} \frac{1}{i}\right)$$

as functions from $\mathbb{N}_{\geq 2}$ to $\mathbb{R}^+$.

For parts (a) to (c) we assume that $n = 2^k$ is a power of 2 for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will generalise the result to arbitrary $n \in \mathbb{N}$ in part (d). For $j \in \mathbb{N}$, define

$$S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i}.$$

(a) For any $j \in \mathbb{N}$, prove that $S_j \leq 1$.

Solution:

The greatest value of any term in the sum will be that which has the smallest denominator. In S_j, that would be the first term, wherein $i = 2^{j-1} + 1$. This can be shown mathematically.

$$i \geq 2^{j-1} + 1 \qquad \frac{1}{i} \leq \frac{1}{2^{j-1} + 1}$$

The sum of m terms of S_j will be lesser or equal to the sum of m identical max terms.

$$\begin{aligned} S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} &\leq \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{2^{j-1}+1} \\ &\leq \underbrace{\frac{1}{2^{j-1}+1} + ... + \frac{1}{2^{j-1}+1}}_{m \text{ times}} \\ &\leq m\left(\frac{1}{2^{j-1}+1}\right) \end{aligned}$$

Where m is the number of terms of the sum, that is, the difference between the final and starting indexes plus one (so it is inclusive).

$$m = 2^j - (2^{j-1}+1) + 1 = 2^j - 2^{j-1} - 1 + 1 = 2^{j-1}(2-1) = 2^{j-1}$$

Which means that

$$S_j \leq 2^{j-1} \cdot \frac{1}{2^{j-1}+1} \leq 1$$

because $\lim_{j\to\infty} \frac{2^{j-1}}{2^{j-1}+1} = 1$. □

(b) For any $j \in \mathbb{N}$, prove that $S_j \geq \frac{1}{2}$.

Solution:

This solution follows a similar process to the previous. This time, the lowest value of any term must be found. The lowest value will be that which has the largest denominator.

$$i \leq 2^j \qquad \frac{1}{i} \geq \frac{1}{2^j}$$

And it follows that

$$\begin{aligned} S_j = \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} &\geq \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{2^j} \\ &\geq \underbrace{\frac{1}{2^j} + ... + \frac{1}{2^j}}_{m \text{ times}} \\ &\geq m\left(\frac{1}{2^j}\right) \\ &\geq 2^{j-1} \cdot \frac{1}{2^j} \\ &\geq 2^{-1} \cdot 2^j \cdot \frac{1}{2^j} \\ &\geq \frac{1}{2} \end{aligned}$$

$\square$

(c) For any $k \in \mathbb{N}_0$, prove the following two inequalities:

$$\sum_{i=1}^{2^k} \frac{1}{i} \leq k + 1$$

and

$$\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k+1}{2}.$$

Hint: You can use that $\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j$.

Solution:

Both inequalities can be proven with the help of the hint.

$$\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j \leq k + 1 \qquad \text{(Hint)}$$

$$\sum_{j=1}^{k} S_j \leq k$$

$$\sum_{j=1}^{k} S_j \leq \sum_{j=1}^{k} 1 \leq k \qquad \text{(follows from item a)}$$

$$\underbrace{1 + ... + 1}_{k \text{ times}} \leq k$$

$$k \leq k$$

and

$$\sum_{i=1}^{2^k} \frac{1}{i} = 1 + \sum_{j=1}^{k} S_j \geq \frac{k+1}{2} \qquad \text{(Hint)}$$

$$\sum_{j=1}^{k} S_j \geq \frac{k+1}{2} - 1$$

$$\sum_{j=1}^{k} S_j \geq \sum_{j=1}^{k} \frac{1}{2} \geq \frac{k+1}{2} - 1 \qquad \text{(follows from item b)}$$

$$\frac{1}{2}k \geq \frac{k-1}{2}$$

$$\frac{1}{2}k \geq \frac{1}{2}(k-1)$$

the inequalities hold.

(d) For arbitrary $n \in \mathbb{N}$, prove that

$$\sum_{i=1}^{n} \frac{1}{i} \leq \log_2(n) + 2$$

and

$$\sum_{i=1}^{n} \frac{1}{i} \geq \frac{\log_2 n}{2}.$$

Solution:

Since $2^{\log_2 n} = n$, it will be used as a substitute.

$$\sum_{i=1}^{2^{\log_2(n)}} \frac{1}{i} \leq \log_2(n) + 2$$

$$1 + \sum_{j=1}^{\log_2(n)} S_j \leq \log_2(n) + 2 \qquad \text{(Hint)}$$

In the sum $1+\sum_{j=1}^{k} S_j$, k must be such that $k \in \mathbb{N}_0$. Since $\log_2(n) \in \mathbb{R}$, the ceiling function will be applied, as $\lceil \log_2(n) \rceil \in \mathbb{N}_0$.

$$\begin{aligned}
1 + \sum_{j=1}^{\lceil \log_2(n) \rceil} S_j &\leq \log_2(n) + 2 \\
\sum_{j=1}^{\lceil \log_2(n) \rceil} 1 &\leq \log_2(n) + 1 \qquad \text{(from item a)} \\
\lceil \log_2(n) \rceil \cdot 1 &\leq \log_2(n) + 1 \\
\log_2(n) \leq \lceil \log_2(n) \rceil &\leq \log_2(n) + 1 \\
\log_2(n) &\leq \log_2(n) + 1
\end{aligned}$$

$\square$

In the next case, the floor function will be applied so the inequality holds.

$$
\begin{aligned}
\sum_{i=1}^{2^{\log_2(n)}} \frac{1}{i} &\geq \frac{\log_2(n)}{2} \\
1 + \sum_{j=1}^{\log_2(n)} S_j &\geq \frac{\log_2(n)}{2} && \text{(Hint)} \\
1 + \sum_{j=1}^{\lfloor \log_2(n) \rfloor} S_j &\geq \frac{\log_2(n)}{2} \\
\sum_{j=1}^{\lfloor \log_2(n) \rfloor} \frac{1}{2} &\geq \frac{\log_2(n)}{2} - 1 && \text{(from item b)} \\
\lfloor \log_2(n) \rfloor \cdot \frac{1}{2} &\geq \frac{\log_2(n) - 2}{2} \\
\lfloor \log_2(n) \rfloor &\geq \log_2(n) - 2 \\
\log_2(n) \geq \lfloor \log_2(n) \rfloor &\geq \log_2(n) - 2 \\
\log_2(n) &\geq \log_2(n) - 2
\end{aligned}
$$

$\square$

The properties stated in the start of the exercise were thus shown to be true.