

Equinumerous sets — GRADED (8 points)

Please upload your solution by 30/10/2025.

Prove the following statements.

1. For all sets A , A and $\mathcal{P}(A)$ are *not* equinumerous.

Hint: You could consider the set

$$B = \{x \in A \mid x \notin f(x)\}$$

for a well-chosen function f .

Solution:

We will prove the two sets are not equinumerous by contradiction. First, assume $A \sim \mathcal{P}(A)$, this means there exists a bijection $f : A \rightarrow \mathcal{P}(A)$. This bijective function maps A to $\mathcal{P}(A)$. In other words, it takes an element of A and assigns it a subset of A (because all elements of $\mathcal{P}(A)$ are subsets of A).

This implies that there can be no elements of $\mathcal{P}(A)$ that aren't mapped by f . However, this cannot be true. Take the set $B = \{x \in A \mid x \notin f(x)\}$. B is a subset of A because all of its elements must be elements of A , but it is not mapped by our function.

Since we assumed f is a bijection, it follows that for some $b \in B$, $f(b) = B$. However, the definition of B tells us that if $b \in B$, $b \notin f(b) = B$, and this contradicts our previous claim that $b \in B$. Described in words, if an element is in B , it is not in the subset it maps to, which is B .

If $b \notin B$, we arrive at a similar contradiction, since this means that $b \in f(b) = B$. Described in words, if an element is not in b , then it is in the subset it maps to, which is B .

What these contradictions tell us is that we have defined a set B that cannot be reached by our function f , since it would otherwise lead to contradictions. This means there isn't a bijection between the sets and thus A is not equinumerous to $\mathcal{P}(A)$. This concludes the proof.

2. The sets

$$C \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\} \quad \text{and} \quad D \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid 0 \leq x < 1\}$$

are equinumerous.

Solution:

The sets C and D differ in only a single element, the number 1. Every element in D is also in C , which means $D \subseteq C$. From Lemma 3.15(iii) it follows that $D \subseteq C \implies D \preceq C$.

Now we can define an injection $f : C \rightarrow D$ as $f(x) = \frac{x}{2}$. We clearly have $f(x) \in D$, because the problematic value 1 maps to $\frac{1}{2}$ and all other $x \in C$ also get halved, approaching zero but never leaving the bounds of D .

Now to prove injectivity let $a, b \in C$ be arbitrary. We have:

$$\begin{aligned} f(a) = f(b) &\implies \frac{a}{2} = \frac{b}{2} && (\text{def. } f) \\ &\implies a = b \end{aligned}$$

Since we have defined an injection $f : C \rightarrow D$, it follows from Definition 3.42(ii) that $C \preceq D$. Now we have $D \preceq C$ (shown before) and $C \preceq D$, which together imply $C \sim D$ (Theorem 3.16). This concludes the proof.