3. Matrix multiplication and invertibility

Let $A, B, C \in \mathbb{R}^{m \times m}$ such that $BA = CA$.

a) Suppose that A is invertible. Show that $B = C$.

Solution:

We can multiply both sides by A^{-1}, making sure the inverse is on the same side of the equation, as matrix multiplication is not commutative.

$$BA = CA \implies (BA)A^{-1} = (CA)A^{-1} \implies B(AA^{-1}) = C(AA^{-1})$$

From the definition of the inverse matrix we have:

$$B(AA^{-1}) = C(AA^{-1}) \implies BI = CI \implies B = C. \qquad \square$$

b) Is it true that $AB = AC$? Give either a proof or a counterexample.

Solution:

It is not true, this will be proven by counterexample. Take three matrices $A, B, C \in \mathbb{R}^{2\times 2}$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$
We have $BA = CA$, as shown below.

$$\begin{gathered} BA = CA \\ \begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \end{gathered}$$

However, this does not imply that $AB = AC$

$$AB = AC$$
$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 5\\2 & 5\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 4\\2 & 4\end{bmatrix}$$
$$\begin{bmatrix}1 & 5\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 4\\0 & 0\end{bmatrix}$$

And this is clearly false, meaning we have found a counterexample.

c) Suppose that $B - C$ is invertible. Show that $A = 0$.

Solution:

Since we know $BA = CA$ this gives us the following equation

$$BA = CA \implies BA - CA = 0 \implies (B - C)A = 0$$

And we know $B - C$ is invertible, so multiplying both sides by the inverse from the left side gives us

$$(B - C)^{-1}(B - C)A = (B - C)^{-1} \cdot 0 \implies IA = 0 \implies A = 0 \qquad \square$$