2. Strictly diagonally dominant matrices (hand-in)

Let $A \in \mathbb{R}^{m \times m}$ be strictly diagonally dominant, that is,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}|$$

holds for all $i \in [m]$. Prove that A is invertible.

Solution:

Before we start, let's outline the thought process behind the proof. We will prove A is invertible by contradiction. A matrix is invertible if and only if its columns are linearly independent. To show that A is independent we can also show that A being linearly dependent implies it cannot be diagonally dominant. A clear contradiction should arise if we assume A fulfills both these conditions. Since this contradiction will go against the rule of Diagonally Dominant Matrices (abbreviated to DDM from now on), and the rule is described by a sum of components along a row, we will work with row-vector multiplications. This will allow us to arrive at an equation that has the same form as the rule.

To start the proof, assume A is linearly dependent. This means there is a non trivial combination of the columns of A that results in the zero vector, that is, there exists an $\mathbf{x} \in \mathbb{R}^m$ with $A\mathbf{x} = 0$ and $\mathbf{x} \neq 0$.

The fact that $\mathbf{x}$ is non-zero means that at least one of its values are non-zero. Let x_j be the largest component of $\mathbf{x}$ (absolute value). This will be helpful later when simplifying an inequality.

We can write the Matrix-Vector-Multiplication as a scalar product of every row of A with $\mathbf{x}$. The j-th row multiplied by $\mathbf{x}$ can be written as:

$$(a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m) = \sum_{i=1}^{m} a_{ji}x_i = 0$$

Remember that we are calculating the zero vector, so the result of the scalar product must be zero. We can take the term with the diagonal element out of the sum.

$$\sum_{i=1}^{m} a_{ji}x_i = a_{jj}x_j + \sum_{\substack{i=1 \\ i \neq j}}^{m} a_{ji}x_i = 0$$

$$a_{jj}x_j + \sum_{\substack{i=1 \\ i\neq j}}^{m} a_{ji}x_i = 0 \implies a_{jj}x_j = -\sum_{\substack{i=1 \\ i\neq j}}^{m} a_{ji}x_i$$

And now we take the absolute value of both sides and apply the triangle inequality

$$|a_{jj}||x_j| = |-\sum_{\substack{i=1 \\ i\neq j}}^{m} a_{ji}x_i| \implies |a_{jj}||x_j| \leq \sum_{\substack{i=1 \\ i\neq j}}^{m} |a_{ji}||x_i|$$

Note that the triangle inequality was applied in this manner:

$$|(a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m)| \leq |a_{j1}x_1| + |a_{j2}x_2| + \cdots + |a_{jm}x_m|$$

The j-th row was chosen so we can now substitute every x_i with the largest component x_j. This results in the substitution:

$$|a_{jj}||x_j| \leq \sum_{\substack{i=1 \\ i\neq j}}^{m} |a_{ji}||x_i| \leq \sum_{\substack{i=1 \\ i\neq j}}^{m} |a_{ji}||x_j|$$

And we can take x_j out of the sum and eliminate it by division since it is constant and non-zero.

$$|a_{jj}||x_j| \leq |x_j| \sum_{\substack{i=1 \\ i\neq j}}^{m} |a_{ji}|$$

$$|a_{jj}| \leq \sum_{\substack{i=1 \\ i\neq j}}^{m} |a_{ji}|$$

And this is a direct contradiction of the DDM rule. This concludes the proof.