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22 September 2025

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Algorithms & Data Structures   Exercise sheet 0   HS 25

The solutions for this sheet do not have to be submitted. The sheet will be solved in the first exercise session on 22 September 2025.

Exercises that are marked by * are challenge exercises.

You can use results from previous parts without solving those parts.

Exercise 0.1 *Induction.*

(a) Prove by mathematical induction that for any positive integer n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

(b) **(This subtask is from August 2019 exam).** Let $T : \mathbb{N} \to \mathbb{R}$ be a function that satisfies the following two conditions:

$$T(n) \geq 4 \cdot T(\tfrac{n}{2}) + 3n \qquad \text{whenever } n \text{ is divisible by } 2;$$
$$T(1) = 4.$$

Prove by mathematical induction that

$$T(n) \geq 6n^2 - 2n$$

holds whenever n is a power of 2, i.e., $n = 2^k$ with $k \in \mathbb{N}_0$. In your solution, you should address the base case, the induction hypothesis and the induction step.

Asymptotic Growth

When we estimate the number of elementary operations executed by algorithms, it is often useful to ignore smaller order terms, and instead focus on the asymptotic growth defined below. We denote by $\mathbb{R}^+$ the set of all (strictly) positive real numbers and by $\mathbb{R}_0^+$ the set of nonnegative real numbers.

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two functions. We say that f *grows asymptotically faster than* g if $\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$.

This definition is also valid for functions defined on $\mathbb{R}^+$ instead of $\mathbb{N}$. In general, $\lim_{n \to \infty} \frac{g(n)}{f(n)}$ is the same as $\lim_{x \to \infty} \frac{g(x)}{f(x)}$ if the second limit exists.

For all the following exercises, you can assume that $n \in \mathbb{N}_{\geq 10}$. We make this assumption so that all functions are well-defined and take values in $\mathbb{R}^+$.

Exercise 0.2   *Comparison of functions part 1.*

Show that

(a) $f(n) := n \log n$ grows asymptotically faster than $g(n) := n$.

(b) $f(n) := n^3$ grows asymptotically faster than $g(n) := 10n^2 + 100n + 1000$.

(c) $f(n) := 3^n$ grows asymptotically faster than $g(n) := 2^n$.

The following theorem can be useful to compute some limits.

Theorem 1 (L'Hôpital's rule). *Assume that functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable,* $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ *and for all* $x \in \mathbb{R}^+$, $g'(x) \neq 0$. *If* $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = C \in \mathbb{R}_0^+$ *or* $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty$, *then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Exercise 0.3   *Comparison of functions part 2.*

Show that

(a) $f(n) := n^{1.01}$ grows asymptotically faster than $g(n) := n \ln n$.

(b) $f(n) := e^n$ grows asymptotically faster than $g(n) := n$.

(c) $f(n) := e^n$ grows asymptotically faster than $g(n) := n^2$.

(d)* $f(n) := 1.01^n$ grows asymptotically faster than $g(n) := n^{100}$.

(e) $f(n) := \log_2 n$ grows asymptotically faster than $g(n) := \log_2 \log_2 n$.

(f) $f(n) := 2^{\sqrt{\log_2 n}}$ grows asymptotically faster than $g(n) := \log_2^{100} n$.

(g) $f(n) := n^{0.01}$ grows asymptotically faster than $g(n) := 2^{\sqrt{\log_2 n}}$.

Exercise 0.4   *Simplifying expressions.*

Simplify the following expressions as much as possible without changing their asymptotic growth rates.

Concretely, for each expression $f(n)$ in the following list, find an expression $g(n)$ that is as simple as possible and that satisfies $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$.

(a) $f(n) := 5n^3 + 40n^2 + 100$

(b) $f(n) := 5n + \ln n + 2n^3 + \frac{1}{n}$

(c) $f(n) := n \ln n - 2n + 3n^2$

(d) $f(n) := 23n + 4n \log_5 n^6 + 78\sqrt{n} - 9$

(e) $f(n) := \log_2 \sqrt{n^5} + \sqrt{\log_2 n^5}$

(f)* $f(n) := 2n^3 + (\sqrt[4]{n})^{\log_5 \log_6 n} + (\sqrt[7]{n})^{\log_8 \log_9 n}$

Exercise 0.5 *   *Finding the range of your bow.*

To celebrate your start at ETH, your parents gifted you a bow and (an infinite number of) arrows. You would like to determine the range of your bow, in other words how far you can shoot arrows with it. For simplicity we assume that all your arrow shots will cover exactly the same distance r, and we define r as the range of your bow. You also know that this range is at least $r \geq 1$ (meter).

You have at your disposition a ruler and a wall. You cannot directly measure the distance covered by an arrow shot (because the arrow slides some more distance on the ground after reaching distance r), so the only way you can get information about the range r is as follows. You can stand at a distance ℓ (of your choice) from the wall and shoot an arrow: if the arrow reaches the wall, you know that $\ell \leq r$, and otherwise you deduce that $\ell > r$. By performing such an experiment with various choices of the distance ℓ, you will be able to determine r with more and more accuracy. Your goal is to do so with as few arrow shots as possible.

(a) What is a fast strategy to find an upper bound on the range r? In other words, how can you find a distance $D \geq 1$ such that $r < D$, using few arrow shots? The required number of shots might depend on the actual range r, so we will denote it by $f(r)$. Good solutions should have $f(r) \leq 10 \log_2 r$ for large values of r.

(b) You are now interested in determining r up to some additive error. More precisely, you should find an estimate $\tilde{r}$ such that the range is contained in the interval $[\tilde{r} - 1, \tilde{r} + 1]$, i.e. $\tilde{r} - 1 \leq r \leq \tilde{r} + 1$. Denoting by $g(r)$ the number of shots required by your strategy, your goal is to find a strategy with $g(r) \leq 10 \log_2 r$ for all r sufficiently large.

(c) Coming back to part (a), is it possible to have a significantly faster strategy (for example with $f(r) \leq 10 \log_2 \log_2 r$ for large values of r)?