Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 4

This is the exercise sheet number 4. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 4.5 and your solution has to be uploaded on the Moodle page of the course **by 16/10/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 4.1 Pigeon set ($\star\star$)

Let n be an integer such that $n \geq 3$, and let S be the set $S = \{1, 2, \cdots, n-1\}$.
Prove that every subset of S of size $\lfloor \frac{n}{2} \rfloor + 1$ contains two elements whose sum is n.
The floor function $x \mapsto \lfloor x \rfloor$ is the function which takes a real number x as argument and outputs the greatest integer smaller or equal to x.
Hint 1: Try with small values of n.
Hint 2: Use case distinction and the pigeonhole principle.

Solution: Let's see what happens with a small n, let say $n = 8$. In this case, $S = \{1, 2, 3, 4, 5, 6, 7\}$, and we consider a subsets of size $\lfloor \frac{n}{2} \rfloor + 1 = 5$. Now, try to build a subset T of size 5 that **does not** satisfy the wanted property. If $1 \in T$ then $7 \notin T$ (and upside down) because $1 + 7 = 8$. Same for 2 and 6. It seems that every time we pick an element in S, another one gets forbidden (except maybe for the element $\frac{n}{2} = 3$). However, because T has size 5, you must pick a forbidden element at some point. This sounds quite similar to the pigeonhole principle, right?
Now we do the formal proof, and we distinguish the case where n is even and the case where n is odd.

Case 1: n **is odd.** For i such that $1 \leq i \leq \frac{n-1}{2}$, let $S_i = \{i, n-i\}$. Remark that $S = S_1 \cup S_2, \cdots \cup S_{\frac{n-1}{2}}$. Furthermore, it holds that $\lfloor \frac{n}{2} \rfloor + 1 = \frac{n+1}{2} > \frac{n-1}{2}$ the number of sets S_i. Therefore, using the pigeonhole principle, there exists (at least) one i such that every subset of S of size $\lfloor \frac{n}{2} \rfloor + 1$ contains two elements in S_i, thus whose sum is n.

Case 2: n **is even.** For i such that $1 \leq i \leq \frac{n}{2} - 1$, let $S_i = \{i, n-i\}$. Let also $S_{\frac{n}{2}} = \{\frac{n}{2}\}$. Remark that $S = S_1 \cup S_2, \cdots \cup S_{\frac{n}{2}}$. Furthermore, it holds that $\lfloor \frac{n}{2} \rfloor + 1 = \frac{n}{2} + 1 > \frac{n}{2}$ the number of sets S_i. Therefore, using the pigeonhole principle, there exists (at least) one i such that every subset of S of size $\lfloor \frac{n}{2} \rfloor + 1$ contains two elements in S_i, thus whose sum is n.

Exercise 4.2 Element or Subset ($\star$)

For each of the following choices of sets A and B, decide which of the statements $A \in B$ and $A \subseteq B$ are true.

1. $A = \{1, 0, \{0\}, 1\}, B = \{\{0, 1, 0, \{0, 0\}\}, 1, 10, 0\}$

> **Solution:** $A \in B$ and $A \nsubseteq B$

2. $A = \emptyset, B = \{\{\emptyset\}, \{\emptyset, \emptyset\}, \emptyset\}$

> **Solution:** $A \in B$ and $A \subseteq B$

3. $A = \{\{0\}, 0, \{0\}, \{\{0\}\}\}, B = \{0, \emptyset, \{\{0\}\}, \{0\}\}$

> **Solution:** $A \notin B$ and $A \subseteq B$

4. $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset, \emptyset, \emptyset\}\}$

> **Solution:** $A \in B$ and $A \subseteq B$

Exercise 4.3 Operations on Sets ($\star$)

In each of the following cases, give a set A such that

1. There exists an $x \in A$ such that $x \subseteq A$.

> **Solution:** $A = \{\emptyset\}$
> For $x = \emptyset$ we have $x \in A$. Also, the empty set is the subset of any other set, so $x \subseteq A$. NB: This is not the only solution. For example, $A = \{7, \{7\}\}$ also fulfills the given condition.

2. $A \nsubseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$.

> **Solution:** $A = \{\emptyset, 1\}$
> We have $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{\emptyset, 1\}\}$. Since $1 \notin \mathcal{P}(A)$, it holds that $A \nsubseteq \mathcal{P}(A)$. Also, for $x = \emptyset$ we have $x \in A$ and $x \subseteq \mathcal{P}(A)$ (since the empty set is the subset of any set).

3. $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.

> **Solution:** $A = \emptyset$
> We have $\emptyset \subseteq \mathcal{P}(A)$. The second requirement is trivially fulfilled, since A has no elements.

Exercise 4.4 Cardinality ($\star$)

Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset\}\}$ and $B = \{A, \{\emptyset\}, \{\{\emptyset\}\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality.

> **Solution:** First, notice that $A = \{\emptyset, \{\emptyset\}\}$.

1. $A \cup B$

> **Solution:** $A \cup B = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, $|A \cup B| = 4$

2. $A \cap B$

> **Solution:** $A \cap B = \{\{\emptyset\}\}$, $|A \cap B| = 1$

3. $\emptyset \times A$

> **Solution:** $\emptyset \times A = \emptyset$, $|\emptyset \times A| = 0$

4. $\{0\} \times \{3, 1\}$

> **Solution:** $\{0\} \times \{3, 1\} = \{(0, 3), (0, 1)\}$, $|\{0\} \times \{3, 1\}| = 2$

5. $\{\{1, 2\}\} \times \{3\}$

> **Solution:** $\{\{1, 2\}\} \times \{3\} = \{(\{1, 2\}, 3)\}$, $|\{\{1, 2\}\} \times \{3\}| = 1$

6. $\mathcal{P}(\{\emptyset\})$

Solution: $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$, $|\mathcal{P}(\{\emptyset\})| = 2$

Exercise 4.5 Symmetric difference ($\star\star$) — GRADED *(8 points)*
Please upload your solution by 16/10/2025

In this exercise, we introduce a new operator, the symmetric difference. The symmetric difference of sets A and B, denoted $A\Delta B$, is the set of elements belonging to A or B but not to both:

$$A\Delta B \stackrel{\text{def}}{=} \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$$

1. Rewrite the definition above using set operations. More specifically, give an expression using only "A", "B", "$\cap$", "$\cup$", and "$\setminus$" (and parentheses), which is equal to the set $A\Delta B$. **Each of the operators "$\cap$", "$\cup$", and "$\setminus$" may appear at most once**. No justification is required.

Solution:

$$A\Delta B = (A \cup B) \setminus (A \cap B)$$

2. Prove that for all sets A and B, it holds that $A\Delta B = (A \setminus B) \cup (B \setminus A)$.
Expectation: You should give a full proof for the aforementioned statement using set operations and/or the tools from previous chapters (rules of Lemma 2.1, function table, logical consequences, proof patterns). Formalism is important, but in contrast to the bonus exercise on sheet 2, you don't have to use rules of Lemma 2.1 one by one. You may use many rules at once, as soon as there is a main one (distributivity without specifying which, de Morgan's rule, absorption) used as justification and an unlimited number of minor rules (associativity, commutativity, double negation, idempotence) not necessarily mentioned. But there must be a justification for each step, even if the step uses solely a minor rule. Also, the **same** main rule can be used many times in one step. You may also use the following equivalences: $F \wedge \top \equiv F$, $F \wedge \bot \equiv \bot$, $F \vee \bot \equiv F$, $F \vee \top \equiv \top$, $F \wedge \neg F \equiv \bot$ and $F \vee \neg F \equiv \top$. **Recall that every step of your proof should be justified.**

Solution: For any x, it holds that

$$\begin{aligned}
x \in A\Delta B &\stackrel{\cdot}{\iff} (x \in A \lor x \in B) \land \neg(x \in A \land x \in B) && \text{[definition]} \\
&\stackrel{\cdot}{\iff} (x \in A \lor x \in B) \land (x \notin A \lor x \notin B) && \text{[de Morgan's rule]} \\
&\stackrel{\cdot}{\iff} (x \in A \land (x \notin A \lor x \notin B)) \lor (x \in B \land (x \notin A \lor x \notin B)) && \text{[Distributivity]} \\
&\stackrel{\cdot}{\iff} (x \in A \land x \notin A) \lor (x \in A \land x \notin B) \lor \\
&\qquad (x \in B \land x \notin A) \lor (x \in B \land x \notin B) && \text{[Distributivity (x2)]} \\
&\stackrel{\cdot}{\iff} \perp \lor (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \lor \perp && [F \land \neg F \equiv \perp] \\
&\stackrel{\cdot}{\iff} (x \in A \land x \notin B) \lor (x \in B \land x \notin A) && [F \lor \perp \equiv F] \\
&\stackrel{\cdot}{\iff} x \in (A \setminus B) \cup (B \setminus A) && \text{[definition of } \cup \text{ and } \setminus]
\end{aligned}$$

3. Prove that for all sets A, B, and C, the following holds:

$$A\Delta B = A\Delta C \implies B = C\ .$$

Expectation: Same as above.

Solution: Consider any sets A, B, C with $A\Delta B = A\Delta C$. For any x, we will show $x \in B \iff x \in C$. We do so with a case distinction:

Case 1: $x \in A$**.**

$$\begin{aligned}
x \in B &\stackrel{\cdot}{\implies} x \notin A\Delta B && [x \in A, \text{definition of } \Delta] \\
&\stackrel{\cdot}{\implies} x \notin A\Delta C && \text{[by hypothesis]} \\
&\stackrel{\cdot}{\implies} x \in C && \text{[by contradiction } (*)]
\end{aligned}$$

$(*)$: Suppose $x \notin C$: we have $x \in A$ and $x \notin C$ thus $x \in A\Delta C$, which contradicts the step just before.
By symmetry (i.e., virtually swap "B" and "C" in the proof above, this is **not** a proof pattern), $x \in C \stackrel{\cdot}{\implies} x \in B$.

Case 2: $x \notin A$**.**

$$\begin{aligned}
x \in B &\stackrel{\cdot}{\implies} x \in A\Delta B && [x \notin A, \text{definition of } \Delta] \\
&\stackrel{\cdot}{\implies} x \in A\Delta C && \text{[by hypothesis]} \\
&\stackrel{\cdot}{\implies} x \in C && \text{[by contradiction } (**)]
\end{aligned}$$

$(**)$: Suppose $x \notin C$: we have $x \notin A$ and $x \notin C$ thus $x \notin A\Delta C$, which contradicts the step just before.
By symmetry, $x \in C \stackrel{\cdot}{\implies} x \in B$.

In both cases we have shown $x \in B \iff x \in C$. Since this is true for any x, we get $B = C$.

Exercise 4.6 Relating Two Power Sets ($\star\star$)

Prove or disprove each of the following statements.

1. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ for any sets A and B.

Solution: For any C, we have

$$
\begin{aligned}
& C \in \mathcal{P}(A \cap B) \\
& \qquad \stackrel{\cdot}{\iff} C \subseteq A \cap B && \text{(definition of } \mathcal{P}) \\
& \qquad \stackrel{\cdot}{\iff} \forall c \left(c \in C \to c \in A \cap B\right) && \text{(definition of } \subseteq) \\
& \qquad \stackrel{\cdot}{\iff} \forall c \left(c \in C \to (c \in A \land c \in B)\right) && \text{(definition of } \cap) \\
& \qquad \stackrel{\cdot}{\iff} \forall c \left((c \in C \to c \in A) \land (c \in C \to c \in B)\right) && (*) \\
& \qquad \stackrel{\cdot}{\iff} \forall c\, (c \in C \to c \in A) \land \forall c\, (c \in C \to c \in B) && (**) \\
& \qquad \stackrel{\cdot}{\iff} C \subseteq A \land C \subseteq B && \text{(definition of } \subseteq) \\
& \qquad \stackrel{\cdot}{\iff} C \in \mathcal{P}(A) \land C \in \mathcal{P}(B) && \text{(definition of } \mathcal{P}) \\
& \qquad \stackrel{\cdot}{\iff} C \in \mathcal{P}(A) \cap \mathcal{P}(B) && \text{(definition of } \cap)
\end{aligned}
$$

$(*)$ We use the fact that for any formulas A_1, A_2 and A_3, we have $A_1 \to (A_2 \land A_3) \equiv \neg A_1 \lor (A_2 \land A_3) \equiv (\neg A_1 \lor A_2) \land (\neg A_1 \lor A_3) \equiv (A_1 \to A_2) \land (A_1 \to A_3)$. (This follows from Lemma 2.1.)

$(**)$ We use the fact that $\forall x P(x) \land \forall x Q(x) \equiv \forall x (P(x) \land Q(x))$ for any predicates P and Q (see Chapter 2.4.8 of the lecture notes).

2. $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ for any sets A and B.

Solution: To prove that the statement is false, we show a counterexample. Let $A = \{1\}$ and $B = \{2\}$. We have $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$. On the other hand, $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

3. $A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$ for any sets A and B.

Solution: We will prove the implication in both directions separately.

$A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$: Let B be any set and let A be any subset of B. What we have to show is that each element of $\mathcal{P}(A)$ is also an element of $\mathcal{P}(B)$. Let S be any element of $\mathcal{P}(A)$. Then, by Definition 3.7, $S \subseteq A$. By the assumption that $A \subseteq B$ and by the transitivity of $\subseteq$, it follows that $S \subseteq B$. This means that S is an element of $\mathcal{P}(B)$.

$\mathcal{P}(A) \subseteq \mathcal{P}(B) \implies A \subseteq B$: Let A, B be any sets and assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$ (which holds for any set A) and, by assumption, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have that $A \in \mathcal{P}(B)$. By Definition 3.7, this means that $A \subseteq B$.

Exercise 4.7 Special Families of Sets ($\star\star$)

Let $X \neq \emptyset$ be a set. We define the following predicate:[1]

$$Q_X(\mathcal{A}) = 1 \iff \begin{cases} \mathcal{A} \subseteq \mathcal{P}(X), \\ \mathcal{A} \neq \emptyset, \\ A \cup B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ A \cap B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ X \setminus A \in \mathcal{A} \text{ for all } A \in \mathcal{A}. \end{cases}$$

Prove or disprove the following statements.

1. $Q_X(\mathcal{P}(X)) = 1$.

Solution: We prove that the statement is true by checking that all the required properties hold for $\mathcal{A} = \mathcal{P}(X)$.

- $\mathcal{P}(X) \subseteq \mathcal{P}(X)$ trivially holds.
- Since $X \neq \emptyset$ then $\mathcal{P}(X) \neq \emptyset$.
- Let $A, B \in \mathcal{P}(X)$. We have

$$\begin{aligned}
& A \cup B \in \mathcal{P}(X) \\
\stackrel{\cdot}{\iff}\ & A \cup B \subseteq X && \text{(Definition of } \mathcal{P}\text{)} \\
\stackrel{\cdot}{\iff}\ & \forall x\, (x \in A \cup B \to x \in X) && \text{(Definition of } \subseteq\text{)} \\
\stackrel{\cdot}{\iff}\ & \forall x\, ((x \in A \lor x \in B) \to x \in X) && \text{(Definition of } \cup\text{)} \\
\stackrel{\cdot}{\iff}\ & \forall x\, ((x \in A \to x \in X) \land (x \in B \to x \in X)) && (*) \\
\stackrel{\cdot}{\iff}\ & \forall x\, (x \in A \to x \in X) \land \forall x\, (x \in B \to x \in X) && (**) \\
\stackrel{\cdot}{\iff}\ & A \subseteq X \land B \subseteq X && \text{(Definition of } \subseteq \text{ twice)} \\
\stackrel{\cdot}{\iff}\ & \top && \text{(By Assumption)}
\end{aligned}$$

$(*)$ We use the fact that $(F \lor G) \to H \equiv \neg(F \lor G) \lor H \equiv (\neg F \land \neg G) \lor H \equiv (\neg F \lor H) \land (\neg G \lor H) \equiv (F \to H) \land (G \to H)$. See Lemma 2.1.

[1] This notation stand for the logical *conjunction* of all statements on the right, meaning the predicate is true if and only if *all* statements on the right are true.

(∗∗) We use the fact that $\forall x P(x) \land \forall x Q(x) \equiv \forall x(P(x) \land Q(x))$ for any predicates P and Q (see Chapter 2.4.8 of the lecture notes).

- Let $A, B \in \mathcal{P}(X)$, that is $A, B \subseteq X$. We have

$$\begin{aligned} x \in A \cap B &\stackrel{\cdot}{\iff} x \in A \land x \in B && \text{(Definition of } \cap) \\ &\stackrel{\cdot}{\implies} x \in X \land x \in X && \text{(Definition of } \subseteq \text{ twice)} \\ &\stackrel{\cdot}{\implies} x \in X && (A \land A \equiv A) \end{aligned}$$

- Let $A \in \mathcal{P}(X)$, that is $A \subseteq X$. We have

$$x \in X \setminus A \iff x \in X \land x \notin A \stackrel{\cdot}{\implies} x \in X$$

which shows that $X \setminus A \subseteq X$, that is $X \setminus A \in \mathcal{P}(X)$.

2. $Q_X(\{X\}) = 1$.

Solution: The statement is false. Notice that $X \in \{X\}$, but $X \setminus X = \emptyset \notin \{X\}$. Therefore, the last property does not hold, and $Q_X(\{X\}) = 0$.

3. For all $\mathcal{A} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ then $X \in \mathcal{A}$.

Solution: The statement is true. Suppose that $Q_X(\mathcal{A}) = 1$. This means (by the second property) that $\mathcal{A} \neq \emptyset$. Let $A \in \mathcal{A}$. We have (by the last property) that $X \setminus A \in \mathcal{A}$. Therefore (by the third property) we have $X = (X \setminus A) \cup A \in \mathcal{A}$.

4. For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cup \mathcal{B}) = 1$.

Solution: The statement is false: we provide a counterexample. Let $X = \{1, 2, 3, 4\}$. Let $\mathcal{A} = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ and let $\mathcal{B} = \{\emptyset, \{1, 3\}, \{2, 4\}, \{1, 2, 3, 4\}\}$. It is straightforward to check that all the properties of Q_X hold for $\mathcal{A}$ and $\mathcal{B}$, so that $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$. However, consider $\mathcal{A} \cup \mathcal{B} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. While $\{1, 2\}, \{1, 3\} \in \mathcal{A} \cup \mathcal{B}$, we have $\{1, 2, 3\} = \{1, 2\} \cup \{1, 3\} \notin \mathcal{A} \cup \mathcal{B}$. This shows $Q_X(\mathcal{A} \cup \mathcal{B}) = 0$, because the third property does not hold.

5. For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cap \mathcal{B}) = 1$.

Solution: We prove that the statement is true by checking all the properties of Q_X hold for $\mathcal{A} \cap \mathcal{B}$.

- For the first property, we have

$$\begin{aligned}
& A \in \mathcal{A} \cap \mathcal{B} \\
\stackrel{\cdot}{\Longleftrightarrow}\ & A \in \mathcal{A} \wedge A \in \mathcal{B} && \text{(Definition of } \cap) \\
\stackrel{\cdot}{\Longleftrightarrow}\ & A \in \mathcal{P}(X) \wedge A \in \mathcal{P}(X) && (Q_X(\mathcal{A}) = 1 \text{ and } Q_X(\mathcal{B}) = 1,\ \text{Property 1}) \\
\stackrel{\cdot}{\Longleftrightarrow}\ & A \in \mathcal{P}(X) && (A \wedge A \equiv A)
\end{aligned}$$

- To prove the second property, we remember that from above, we know $X \in \mathcal{A}$ and $X \in \mathcal{B}$ so that $X \in \mathcal{A} \cap \mathcal{B}$. This shows the intersection is not empty.
- Let $A, B \in \mathcal{A} \cap \mathcal{B}$. Then $A, B \in \mathcal{A}$ and $A, B \in \mathcal{B}$ by definition of intersection. Since $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$, using property 3 we conclude that $A \cup B \in \mathcal{A}$ and $A \cup B \in \mathcal{B}$. By definition of intersection we get $A \cup B \in \mathcal{A} \cap \mathcal{B}$. This proves property 3.
- Let $A, B \in \mathcal{A} \cap \mathcal{B}$. Then $A, B \in \mathcal{A}$ and $A, B \in \mathcal{B}$ by definition of intersection. Since $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$, using property 4 we conclude that $A \cap B \in \mathcal{A}$ and $A \cap B \in \mathcal{B}$. By definition of intersection we get $A \cap B \in \mathcal{A} \cap \mathcal{B}$. This proves property 4.
- Let $A \in \mathcal{A} \cap \mathcal{B}$. Then $A \in \mathcal{A}$ and $A \in \mathcal{B}$ by definition of intersection. Since $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$, using property 5 we conclude that $X \setminus A \in \mathcal{A}$ and $X \setminus A \in \mathcal{B}$. By definition of intersection we get $X \setminus A \in \mathcal{A} \cap \mathcal{B}$. This proves property 5.

Exercise 4.8 Short questions (exam 2022) ($\star$)

This exercise is taken from the end of semester exam of 2022. We decided to keep the exam grading scale but this exercise is not a (graded) bonus exercise for this week. Each correct answer gives one point. No justification is required.

1. How many elements does the set $\{\{0,1\}, \{0\} \times \{1\}\} \times \{0\}$ have?

Solution: 2: $(\{0,1\}, 0)$ and $((0,1), 0)$

2. List each element of the set $\{\varnothing, (0,1)\} \times \{\{0\} \cup \{1\}, \{1,0\}\}$ exactly once.

Solution: $(\varnothing, \{0,1\})$ and $((0,1), \{0,1\})$

3. Find sets A, B, and C such that $A \setminus B \subset A \setminus C$.

> **Solution:** Taking $A = \{1,2,3\}, B = \{0,1,2\}, C = \{0,1\}$ works. Generally, any sets B, C such that $C \subset B$ works, you can see this with a quick drawing as a draft (which is very useful during exams).
> Another tip: during exams, to save time on easy questions that do not require justification, any correct answer gives the same amount of points. Therefore, if you see a really easy answer (here $A = B = C = \varnothing$), don't hesitate.

4. Find a set A such that $A \cap \mathcal{P}(A) \neq \varnothing$.

> **Solution:** Taking $A = \{1, \{1\}\}$ works since $\mathcal{P}(A) = \{\varnothing, \{1\}, \{\{1\}\}, \{1, \{1\}\}\}$.

Exercise 4.9 Short questions (exam 2021) ($\star$)

This exercise is taken from the end of semester exam of 2021. We decided to keep the exam grading scale but this exercise is not a (graded) bonus exercise for this week.
Each correct answer gives the indicated number of points. An unanswered question gives zero points. For each wrong answer, the indicated number of points are deducted. Overall, at least 0 points are given for the whole task. No justification is required.

1. (1 point) True or False: the sets $\varnothing \times \varnothing$ and $\{\varnothing\} \times \{\varnothing\}$ are equal.

> **Solution:** False. $\varnothing \times \varnothing = \varnothing$ whereas $\{\varnothing\} \times \{\varnothing\} = \{(\varnothing, \varnothing)\}$.

2. (1 point) True or False: the set $\{\varnothing, \{\varnothing\}\}$ is a subset of $\{\varnothing\}$.

> **Solution:** False, because $\{\varnothing\} \notin \{\varnothing\}$.

3. (1 point) True or False: for any two finite sets A and B we have $|A \cup B| = |A| + |B|$.

> **Solution:** False, take e.g. $A = B = \{1\}$. In fact, any finite sets A, B with $A \cap B \neq \varnothing$ yield a counterexample.

4. (1 point) True or False: for any two sets A and B there exists a set C such that

$$A = B \cap C \quad \text{or} \quad B = A \cap C.$$

Solution: False, take any two disjoint and non-empty sets, for instance $A = \{0\}$ and $B = \{1\}$. For any set C, $0 \notin B \cap C$ and $1 \notin A \cap C$.

5. (1 point) True or False: for any set A, we have $A \cap \mathcal{P}(A) = \varnothing$.

Solution: False, take $A = \{\varnothing\}$. $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}\}$ so $A \cap \mathcal{P}(A) = \{\varnothing\} \neq \varnothing$.

Due by 16/10/2025, 23:59.
Exercise 4.5 will be graded.