Orthogonality and Linear Independence

Problem 2

a) For which number $s \in \mathbb{R}$ are the two following vectors orthogonal?

$$\mathbf{v} = \begin{pmatrix} s \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ s \end{pmatrix}$$

Solution:

For the two given vectors to be orthogonal to one another, it must hold that $\mathbf{v} \cdot \mathbf{w} = \mathbf{0}$ This gives the following equation.

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} s \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ s \end{pmatrix} = s \cdot 1 + 3 \cdot 2 + 2 \cdot s = s + 6 + 2s = 3s + 6$$

Which must equal to zero, allowing us to solve for s.

$$3s + 6 = 0 \qquad 3s = -6 \qquad s = \frac{-6}{3} \qquad s = -2$$

This result can be checked by inputting $s = -2$ into $\mathbf{v}$ and $\mathbf{w}$ and seeing if their scalar product equals zero.

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -2 + 6 - 4 = 0$$

b) For which number $t \in \mathbb{R}$ are the three following vectors linearly dependent?

$$\mathbf{u} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Solution:

This item can be quickly solved by setting $t = 0$, as this makes $\mathbf{u} = \mathbf{0}$ and it is known that any set of vectors that contains the null vector is linearly dependent.

In this case, $t = 0$ is actually the only possible solution, as can be shown by attempting to write $\mathbf{u}$ as a linear combination of the two other vectors.

$$\lambda\mathbf{v} + \mu\mathbf{w} = \mathbf{u} \qquad \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$$

$$\begin{cases} \mu = t \\ \lambda = t \\ \lambda + \mu = 0 \end{cases} \qquad t + t = 0 \qquad t = 0$$

c) Show that if two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal, then they are linearly independent. Prove that the converse is not necessarily true, i.e. provide an example of two linearly independent vectors that are not orthogonal.

Solution:

To prove the first statement, it will be shown that the opposite cannot be true. That is, if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, they cannot be linearly dependent, allowing it to be concluded that orthogonal vectors must be linearly independent (considering the given constraints).

Assuming that $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, it must hold that $\mathbf{v} = \lambda\mathbf{w}, \lambda \in \mathbb{R}$. Assuming they are also orthogonal, it must hold that $\mathbf{v} \cdot \mathbf{w} = 0$.

Both of these assumptions cannot be true at the same time, as can be seen in the scalar product.

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \lambda\mathbf{v} = \lambda(\mathbf{v} \cdot \mathbf{v}) = \lambda(v_1^2 + v_2^2 + ... + v_n^2) = 0$$

This equation can only have a solution if $\lambda = 0$, making $\mathbf{w}$ a zero vector or if all the components of $\mathbf{v}$ are zero, since $v_i^2 \geq 0$. Both of these conditions clash with the constraints of this item, as both $\mathbf{v}$ and $\mathbf{w}$ must be nonzero vectors. Thus, if these two vectors are orthogonal, they must be linearly independent, as it is impossible for them to be linearly dependent.

The converse is not necessarily true. This can easily be proven by counterexample. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{w} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. These vectors are clearly linearly independent and also not orthogonal.

$$\begin{pmatrix}1\\1\end{pmatrix} = \lambda \begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{cases}1 = \lambda\\1 = 0\end{cases} \qquad \text{(the vectors are linearly independent)}$$

$$\begin{pmatrix}1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 1 \cdot 1 + 1 \cdot 0 = 1 \qquad \text{(the vectors are not orthogonal)}$$

$\square$