Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 3

This is the exercise sheet number 3. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercises are Exercise 3.2 and 3.7 and your solution has to be uploaded on the Moodle page of the course **by 09/10/2025, 23:59**. The solution to these exercises must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

1 Predicate Logic

Exercise 3.1 Expressing Relationship of Humans in Predicate Logic ($\star$)

Consider, as in the lecture, the universe of all humans (including those who died) and the following predicate:

$$\texttt{par}(x, y) = 1 \iff \text{“}x \text{ is parent of } y.\text{”}$$

Express the following statements as a formula in predicate logic, using only the above predicates (in particular, do **not** use the predicate `equals`, often also written as $=$).

1. x is great-grandparent of y.

   Solution: $\exists u \exists v\, (\texttt{par}(x, u) \wedge \texttt{par}(u, v) \wedge \texttt{par}(v, y))$

2. x and y are (first) cousins.

   Solution: $\exists u \exists v \exists w\, (\texttt{par}(u, v) \wedge \texttt{par}(u, w) \wedge \texttt{par}(v, x) \wedge \texttt{par}(w, y) \wedge \neg\texttt{par}(v, y) \wedge \neg\texttt{par}(w, x))$

Exercise 3.2 From Natural Language to a Formula ($\star$) — GRADED *(4 points)*
Please upload your solution by 09/10/2025

Consider the universe $U = \mathbb{N} \setminus \{0\}$. Express each of the following statements with a formula in predicate logic, in which the only predicates appearing are $\texttt{smallerthan}(x, y)$, $\texttt{divides}(x, y)$, $\texttt{equals}(x, y)$ and $\texttt{prime}(x)$ (instead of $\texttt{smallerthan}(x, y)$, $\texttt{divides}(x, y)$ and $\texttt{equals}(x, y)$ you can write $x < y$, $x \mid y$ and $x = y$ accordingly). You can also use the symbols $+$ and $\cdot$ to denote the addition and multiplication functions, and you can use constants (e.g., $0, 1, \ldots$). You can also use $\longrightarrow$ and $\longleftrightarrow$. No justification is required.

1. ($\star$) There does not exist a largest natural number.

   Solution: $\neg \exists x \, \forall y \, (y < x \vee x = y)$

2. ($\star$) The only divisors of a prime number are 1 and the number itself.

   Solution: $\forall x \, \forall y \, \big((\texttt{prime}(x) \, \wedge \, y \,|\, x) \, \longrightarrow \, (y = x \, \vee \, y = 1) \big)$.

3. ($\star$) 1 is the only natural number which has an inverse.

   Solution: $\forall x \, \big((\exists y \;\; (x \cdot y = 1)) \, \longleftrightarrow \, (x = 1) \big)$.

4. ($\star$) A prime number divides the product of two natural numbers if and only if it divides at least one of them.

   Solution: $\forall x \, \forall y \, \forall z \, \big(\texttt{prime}(x) \, \longrightarrow \, \big((x \,|\, (y \cdot z)) \, \longleftrightarrow \, (x \,|\, y \, \vee \, x \,|\, z) \big) \big)$.

Exercise 3.3 Winning Strategy ($\star \star$)

Alice and Bob play a game in which the stake is a chocolate bar. Rules of the game are the following: Alice chooses two integers a_1, a_2 and Bob chooses two integers b_1, b_2. Alice wins whenever $a_1 + (a_2 + b_1)^{|b_2|+1} = 1$ and Bob wins otherwise.

1. First, consider the case when Alice and Bob announce all their numbers at the same time. Give a formula that describes the statement "Alice has a winning strategy." Is this statement true?

   Solution: The numbers announced by Alice cannot depend on Bob's choice for b_1 and b_2. Therefore, the statement can be described by the following formula:

   $$\exists a_1 \exists a_2 \forall b_1 \forall b_2 \;\; \big(a_1 + (a_2 + b_1)^{|b_2|+1} = 1 \big).$$

   The above statement is false, because for each tuple (a_1, a_2), there exists a tuple $(b_1, b_2) := (2 - a_2 - a_1, 0)$ such that

   $$a_1 + (a_2 + b_1)^{|b_2|+1} = a_1 + (a_2 + 2 - a_2 - a_1) = 2.$$

   Therefore, Alice does not have a winning strategy.

2. In the second case, Alice and Bob announce their numbers one by one. That is, first Alice announces a_1, then Bob announces b_1, then Alice announces a_2, and at the end Bob replies with b_2. Once again, give a formula that describes the statement "Alice has a winning strategy." Is this statement true in this case?

> **Solution:** In this case, Alice's choice for a_2 can depend on b_1. Therefore, the statement can be described by the following formula:
>
> $$\exists a_1 \forall b_1 \exists a_2 \forall b_2 \ \left(a_1 + (a_2 + b_1)^{|b_2|+1} = 1\right).$$
>
> This statement is true. A possible winning strategy for Alice is to choose $a_1 = 1$ and $a_2 = -b_1$. For such choice, we have
>
> $$a_1 + (a_2 + b_1)^{|b_2|+1} = 1 + 0^{|b_2|+1} = 1.$$

2 Proof Patterns

Exercise 3.4 Indirect Proof of an Implication

Prove indirectly that for all natural numbers $n > 0$, we have:

1. ($\star$) If n^2 is odd, then n is also odd.

> **Solution:** Assume that n is even. Then, $n = 2k$ for some $k \in \mathbb{N}$. We have therefore $n^2 = n \cdot n = 2k \cdot 2k = 2 \cdot 2k^2$. Hence, n^2 is even.
>
> **Detailed solution:**
>
> **Statement S:** n^2 is odd.
>
> **Statement T:** n is odd.
>
> **Indirect proof:**
>
> n is not odd.
> $\stackrel{.}{\Longrightarrow}$ n is even.
> $\stackrel{.}{\Longrightarrow}$ $n = 2k$ for some $k \in \mathbb{N}$.
> $\stackrel{.}{\Longrightarrow}$ $n \cdot n = 2k \cdot 2k$ for some $k \in \mathbb{N}$.
> $\stackrel{.}{\Longrightarrow}$ $n \cdot n = 2 \cdot 2k^2$ for some $k \in \mathbb{N}$.
> $\stackrel{.}{\Longrightarrow}$ $n \cdot n = 2l$ for some $l \in \mathbb{N}$.
> $\stackrel{.}{\Longrightarrow}$ $n^2 = 2l$ for some $l \in \mathbb{N}$.
> $\stackrel{.}{\Longrightarrow}$ n^2 is even.

2. ($\star\star$) If $42^n - 1$ is a prime, then n is odd.

Solution: Assume that n is even. We show that in such case $42^n - 1$ is not a prime. To this end, notice that, since n is even, there must exist a natural number $k > 0$, such that $n = 2k$. It follows that $42^n - 1 = 42^{2k} - 1 = (42^k + 1)(42^k - 1)$. Therefore, we found two non-trivial divisors of $42^n - 1$, namely $(42^k + 1)$ and $(42^k - 1)$ (they are greater than 1, because $k > 0$). Thus, $42^n - 1$ cannot be a prime.

Detailed solution:

We consider two statements S and T. We have to show that $S \Longrightarrow T$ is true. To this end, we use an indirect direct proof, that is, we assume that T is false and show that, under this assumption S, must also be false.

Statement S: $42^n - 1$ is a prime.

Statement T: n is odd.

Indirect proof:

n is not odd.
$\stackrel{\cdot}{\Longrightarrow}$ n is even.
$\stackrel{\cdot}{\Longrightarrow}$ There exists a natural number, call it k, such that $k > 0$ and $n = 2k$.
$\stackrel{\cdot}{\Longrightarrow}$ We have $42^n - 1 = 42^{2k} - 1 = (42^k + 1)(42^k - 1)$ for $k > 0$.
$\stackrel{\cdot}{\Longrightarrow}$ There exist two non-trivial divisors of $42^n - 1$, namely $(42^k + 1)$ and $(42^k - 1)$.
$\stackrel{\cdot}{\Longrightarrow}$ $42^n - 1$ is not a prime.

Exercise 3.5 Case Distinction

Prove by case distinction that:

1. ($\star$) $n^3 + 2n + 6$ is divisible by 3 for all natural numbers $n \geq 0$.

Solution: Let n be any natural number greater or equal 0. Let $n = 3k + c$, where $0 \leq c \leq 2$ and $k \in \mathbb{N}$. We have

$$\begin{aligned} n^3 + 2n + 6 &= (3k + c)^3 + 2(3k + c) + 6 \\ &= c^3 + 9c^2k + 27ck^2 + 2c + 27k^3 + 6k + 6. \end{aligned}$$

Each summand is divisible by 3, except the term $c^3 + 2c$. Hence, we only need to show that $c^3 + 2c$ is divisible by 3 for $0 \leq c \leq 2$.

Case $c = 0$: $c^3 + 2c = 0$, which is divisible by 3.

Case $c = 1$: $c^3 + 2c = 3$, which is divisible by 3.

Case $c = 2$: $c^3 + 2c = 12$, which is divisible by 3.

Since the above cases cover all possibilities for c, we can conclude the proof.

2. ($\star\star$) If p and p^2+2 are primes, then p^3+2 is also a prime.

Solution: In the following, we let $R_3(x)$ denote the remainder of the division of x by 3 (for example, $R_3(5)=2$). For any prime number p, we can distinguish the following three cases:

$p=2$: If $p=2$, then $p^2+2=6$ is not a prime. Thus, the claim holds for $p=2$.

$p=3$: If $p=3$, then $p^2+2=11$ is a prime. However, we now have $p^3+2=29$, which is also a prime. Thus, the claim also holds for $p=3$.

$p>3$: If $p>3$ is a prime, then 3 cannot divide p. Therefore, we have $R_3(p)\in\{1,2\}$. Thus, it holds that

$$R_3(p^2)=R_3(R_3(p)\cdot R_3(p))=1.$$

It follows that

$$R_3(p^2+2)=R_3(R_3(p^2)+R_3(2))=R_3(1+2)=0$$

Therefore, p^2+2 must be divisible by 3 and so it is not a prime. Thus, the claim holds also for $p>3$.

Since the above cases cover all prime numbers, the claim holds.

Exercise 3.6 Proof by Contradiction

1. ($\star\star$) Show by contradiction that the sum of a rational number and an irrational number is irrational.

Hint: Use the fact that the difference of two rational numbers is rational.

Solution: Let x be any irrational number and let r be any rational number. Assume that $s=x+r$ is rational. To reach a contradiction, we show that in such case x must be rational. Indeed, we have $x=s-r$. Therefore, we have that x is a difference of two rational numbers and thus, by the fact from the hint, it must also be rational. This is a contradiction with the assumption that x is irrational.

Detailed solution:

Consider a statement S. To show that S is true, we will state a false statement T, and show that if S is false, then T is true.

Fix any irrational number x and any rational number r.

Statement S: The sum $x+r$ is irrational.

Statement T: x is rational.

Proof by contradiction:

We show that if S is false, then T is true:

S is false.
$\stackrel{\cdot}{\Longrightarrow}$ It is not true that the sum $x + r$ is irrational.
$\stackrel{\cdot}{\Longrightarrow}$ The sum $s = x + r$ is rational.
$\stackrel{\cdot}{\Longrightarrow}$ $x = s - r$, where s and r are some rational numbers.
$\stackrel{\cdot}{\Longrightarrow}$ x is rational. (by the fact from the hint)
$\stackrel{\cdot}{\Longrightarrow}$ T is true.

The statement T is trivially false.

2. ($\star\star\star$) Show that the number $2^{\frac{1}{n}}$ is irrational for $n > 2$, by reaching a contradiction with Fermat's Last Theorem.

Hint: Fermat's Last Theorem states that no positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for $n > 2$.

Solution: Assume for contradiction that $2^{\frac{1}{n}}$ is rational for some $n > 2$. That is, assume that there exist two positive integers, call them p and q, such that $2^{\frac{1}{n}} = \frac{p}{q}$. This implies that $2 = \frac{p^n}{q^n}$. Hence, we have $q^n + q^n = p^n$, which is a contradiction with Fermat's Last Theorem.

The contradiction with Fermat's Last Theorem follows from the counterexample $q^n + q^n = p^n$.

Detailed solution:

Fix any integer $n > 2$.

Statement S: $2^{\frac{1}{n}}$ is irrational.

Statement T: There exist positive integers p, q such that $q^n + q^n = p^n$.

Proof by contradiction:

We show that if S is false, then T is true:

S is false.
$\stackrel{\cdot}{\Longrightarrow}$ It is not true that $2^{\frac{1}{n}}$ is irrational.
$\stackrel{\cdot}{\Longrightarrow}$ $2^{\frac{1}{n}}$ is rational.
$\stackrel{\cdot}{\Longrightarrow}$ There exist positive integers p and q such that $2^{\frac{1}{n}} = \frac{p}{q}$.
$\stackrel{\cdot}{\Longrightarrow}$ There exist positive integers p and q such that $2 = \frac{p^n}{q^n}$.
$\stackrel{\cdot}{\Longrightarrow}$ There exist positive integers p and q such that $q^n + q^n = p^n$.
$\stackrel{\cdot}{\Longrightarrow}$ T is true.

The statement T is false, since it is a counterexample to Fermat's Last Theorem.

Exercise 3.7 New Proof Patterns ($\star$) — GRADED *(4 points)*

Please upload your solution by 09/10/2025

For each of the following proof patterns, **prove** or **disprove** that it is sound. Do so by first writing it as a statement involving logical consequence on formulas and then proving that the resulting statement is either true or false.

1. ($\star$) To prove a statement S, find two appropriate statements T_1 and T_2. Assume that S is false and show (from this assumption) that at least one of the statements T_1 and T_2 is true. Then show that at least one of the statements T_1 and T_2 is false.

Solution: The proof pattern described corresponds to the following statement (where statement S corresponds to propositional symbol A, T_1 to B, and T_2 to C):

$$(\neg A \to (B \vee C)) \wedge (\neg B \vee \neg C) \models A \,. \tag{1}$$

We show that the proof pattern is not sound by showing that the statement is false. To do so, we compute the function tables of the formulas involved.

A	B	C	$(\neg A \to (B \vee C)) \wedge (\neg B \vee \neg C)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

For the truth assignment in which A has truth value 0, B has truth value 0, and C has truth value 1, the formula $(\neg A \to (B \vee C)) \wedge (\neg B \vee \neg C)$ has truth value 1 despite A having truth value 0. Therefore, statement (1) is false, and the given proof pattern is **not** sound.

2. ($\star$) To prove an implication $S \Rightarrow T$, find an appropriate statement R. First, show that R is false. Then, assume that S is true and T is false, and prove that (from these assumptions) R is true.

Solution: The proof pattern described corresponds to the following statement (where statement R corresponds to propositional symbol A, S to B, and T to C):

$$\neg A \wedge ((B \wedge \neg C) \to A) \models B \to C \,. \tag{2}$$

We show that the proof pattern is sound by showing that the statement is true. To do so, we compute the function tables of the formulas involved.

B	C	A	$\neg A$	$(B \wedge \neg C) \to A$	$\neg A \wedge ((B \wedge \neg C) \to A)$	$B \to C$
0	0	0	1	1	1	1
0	0	1	0	1	0	1
0	1	0	1	1	1	1
0	1	1	0	1	0	1
1	0	0	1	0	0	0
1	0	1	0	1	0	0
1	1	0	1	1	1	1
1	1	1	0	1	0	1

We can see that for every truth assignment (on propositional symbols B, C, and A) for which the truth value of $B \to C$ is 1, the truth value of $\neg A \wedge ((B \wedge \neg C) \to A)$ is also 1. Therefore, statement (2) is true, and the proof pattern is sound.

Due by 09/10/2025, 23:59.
Exercise 3.2 and 3.7 will be graded.