Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 2

This is the exercise sheet number 2. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 2.3 and your solution has to be uploaded on the Moodle page of the course **by 02/10/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 2.1 Logical Consequence ($\star$)

Prove or disprove the following statements about formulas.

1. $A \wedge (A \to B) \models B$

Solution: We first construct the function table for the formula $A \wedge (A \to B)$.

A	B	$A \wedge (A \to B)$
0	0	0
0	1	0
1	0	0
1	1	1

The above table shows that the truth value of $A \wedge (A \to B)$ is 1 only for the truth assignment in the last row. Clearly, B is also true for that assignment. Thus, B is the logical consequence of $A \wedge (A \to B)$ and the statement holds.

2. $A \to B \models \neg A \to \neg B$

Solution: The statement is false. There exists a truth assignment, namely one in which A is false and B is true, for which $A \to B$ is true, but $\neg A \to \neg B$ is false.

Thus, $\neg A \to \neg B$ is not a logical consequence of $A \to B$.

3. $\models (A \to B) \vee (B \to A)$

Solution: The statement is true. One way to show this is to construct a function table for $F = (A \rightarrow B) \vee (B \rightarrow A)$. We present a different proof: assume, by contradiction, that some truth assignment of the propositional symbols A and B makes F false. Then, under this truth assignment, both the formula $(A \rightarrow B)$ and the formula $(B \rightarrow A)$ are false, because otherwise their conjunction would be true. The only truth assignment for which $A \rightarrow B$ is false is one where A is true but B is false. For this truth assignment, $B \rightarrow A$ is true, which is a contradiction.

4. $(A \rightarrow B) \wedge (B \rightarrow C) \models (A \rightarrow C)$

Solution: We construct the function table for both formulas: $(A \rightarrow B) \wedge (B \rightarrow C)$ and $A \rightarrow C$.

A	B	C	$A \rightarrow B$	$B \rightarrow C$	$(A \rightarrow B) \wedge (B \rightarrow C)$	$A \rightarrow C$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Analogously to question 1, we can show that the statement holds.

Exercise 2.2 Satisfiability and Tautologies (⋆)

For each of the formulas below, determine whether it is satisfiable or unsatisfiable and whether it is a tautology or not. Prove your answers.

1. $(A \vee B) \wedge \neg A$

Solution: This formula is satisfiable, since it is true for the assignment $A = 0, B = 1$. It is, however, not a tautology, since it is false for the assignment $A = 0, B = 0$.

2. $\big((A \rightarrow B) \wedge (B \rightarrow C)\big) \wedge \neg(A \rightarrow C)$

Solution: This formula is unsatisfiable (hence, it is not a tautology). In order to prove this, let $F = \big((A \to B) \land (B \to C)\big) \land \neg(A \to C)$. We notice that

$$\begin{aligned} \neg F \;&\equiv\; \neg\big((A \to B) \land (B \to C)\big) \lor (A \to C) && \text{(de Morgan, double negation)} \\ &\equiv\; (A \to B) \land (B \to C) \;\to\; (A \to C) && \text{(def. } \to\text{)} \end{aligned}$$

From question 4 of Exercise 2.1, we know that $(A \to B) \land (B \to C) \;\models\; (A \to C)$ is true. From this fact, together with Lemma 2.3, it follows that $\neg F$ is a tautology. Hence, by Lemma 2.2, F is unsatisfiable.

Exercise 2.3 Simplifying a Formula ($\star$) — GRADED *(8 points)*

Please upload your solution by 02/10/2025

Consider the propositional formula

$$F = (B \to A) \land \neg\big((\neg A \land \neg C) \land (\neg C \lor B)\big)$$

Give a formula G that is equivalent to F, but in which each atomic formula A, B, and C appears at most once. **Prove** that $F \equiv G$ by providing a sequence of equivalence transformations with **at most** 7 steps.

Expectation. Your proof should be in the form of a sequence of steps, where each step consists of applying the definition of $\to$ (that is $F \to G \equiv \neg F \lor G$), one of the rules given in Lemma 2.1 of the lecture notes[1], or one of the following rules: $F \land \neg F \equiv \bot$, $F \land \bot \equiv \bot$, $F \lor \bot \equiv F$, $F \lor \neg F \equiv \top$, $F \land \top \equiv F$, and $F \lor \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1 *3)*. Each step of your proof should apply a **single** rule **once** and state **which** rule was applied.

Solution: We choose the formula $G = A \lor (\neg B \land C)$. In the following, we prove $F \equiv G$:

$$\begin{aligned} F &= (B \to A) \land \neg\big((\neg A \land \neg C) \land (\neg C \lor B)\big) \\ &\equiv (B \to A) \land \neg\big(\neg A \land (\neg C \land (\neg C \lor B))\big) && \text{(associativity)} \\ &\equiv (B \to A) \land \neg(\neg A \land \neg C) && \text{(absorption)} \\ &\equiv (B \to A) \land \neg\neg(A \lor C) && \text{(de Morgan's rules)} \\ &\equiv (B \to A) \land (A \lor C) && \text{(double negation)} \\ &\equiv (\neg B \lor A) \land (A \lor C) && \text{(definition of } \to\text{)} \\ &\equiv (A \lor \neg B) \land (A \lor C) && \text{(commutativity of } \lor\text{)} \\ &\equiv A \lor (\neg B \land C) && \text{(second distributive law)} \\ &= G \end{aligned}$$

[1] Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).

Exercise 2.4 Knights and Knaves ($\star\star\star$)

We find ourselves on a strange island with only two types of inhabitants: knights and knaves. The knights always tell the truth, while the knaves always lie. From the outside, both groups look exactly the same and we cannot distinguish one from the other.

We have lost our way and come to a fork in the road. We know that one of the roads leads to a deadly jungle, while the other will take us to a friendly village. We see an islander standing at the fork. He is willing to answer only one question and his answer can only be "Yes" or "No". What question do we ask?

We want to use propositional logic to solve this problem. Let A be the proposition "The left road leads to the village." and let B be the proposition "The islander is a knight." We phrase our question as a formula F in A and B, asking the islander about the truth value of F. How do we choose F such that we are guaranteed to learn which road leads to the village?

Solution: Let A be the proposition "The left road leads to the village." and let B be the proposition "The islander is a knight.". We want to ask the islander about the truth value of a formula F in A and B in order to determine whether A is true. In order to be guaranteed to learn whether A is true or not, we have to receive a fixed answer (say, "Yes") from the islander in case A is true, and the opposite (say, "No") in case A is false. This has to hold **independently** of whether the islander is a knight or a knave (since we have no information about that). If the islander is a knight (B is true) the answer will be the truth value of F (since knights always tell the truth). However, if the islander is a knave (B is false) the answer will be the truth value of $\neg F$ (since knaves always lie). Hence, we derive the following partial function table:

A	B	F	$\neg F$
0	0		0
0	1	0	
1	0		1
1	1	1	

This partial function table can be completed (uniquely) to the following function table:

A	B	F	$\neg F$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

From the function table we obtain a possible formula $F = (\neg A \wedge \neg B) \vee (A \wedge B)$. Formulated as a question: "Does the left road lead to the jungle and you are a knave, or is it the case that the left road leads to the village and you are a knight?".

Exercise 2.5 Quantifiers and Predicates

In this exercise the universe is fixed to the set $\mathbb{Z}$ of integers.

1. For each of the following statements, write a formula, in which the only predicates are `less`, `equals` and `prime` (instead of $\texttt{less}(n, m)$ and $\texttt{equals}(n, m)$ you can write $n < m$ and $n = m$ accordingly).[2] You can also use the symbols $+$ and $\cdot$ to denote addition and multiplication. Which of them are true? (You don't need to justify this.)

   (a) ($\star$) If the product of two integer numbers is positive, then at least one of these numbers is positive.

   > **Solution:** $\forall m\, \forall n\, \big(0 < m \cdot n \rightarrow (0 < m \vee 0 < n)\big)$
   > This statement is false. For example, $(-2) \cdot (-2) = 4$.

   (b) ($\star$) For every natural number, one can find a strictly greater natural number that is divisible by 3.

   > **Solution:** $\forall m\, \big(-1 < m \rightarrow \exists n\big(-1 < n \wedge m < n \wedge (\exists k\, n = 3 \cdot k)\big)\big)$
   > This statement is true. For any n, one of the numbers $n+1$, $n+2$, $n+3$ must be divisible by 3.
   > NB: It is also allowed to drop the condition $-1 < n$, since it is implied by $m < n$.

   (c) ($\star\,\star$) Every even integer greater than 2 is a sum of two primes.

   > **Solution:** $\forall n\, \big(((\exists k\, n = 2{\cdot}k) \wedge 2 < n) \rightarrow \exists p\, \exists q\, (\texttt{prime}(p) \wedge \texttt{prime}(q) \wedge n = p{+}q)\big)$
   > This statement is known as the (strong) Goldbach conjecture. It is not known whether it is true.

2. Consider the following predicates $P(x)$ and $Q(x, y)$:

$$P(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise} \end{cases} \qquad Q(x, y) = \begin{cases} 1, & xy = 1 \\ 0, & \text{otherwise} \end{cases}$$

   In this context, describe the following statements in words. Also, for each statement, decide whether it is true or false.

   (a) ($\star$) $\forall x\, \exists y\, Q(x, y)$

   > **Solution:** There are many equally good ways to describe given formulas using words. We only give examples:
   > "For every integer x, there exists an integer y, such that xy is equal to 1."
   > An alternative solution would be "Each integer has a multiplicative inverse."

[2] Note that $\texttt{less}(n, m)$ is true if n is **strictly** smaller than m, so it is false for $n = m$.

This statement is false. For example, there is no integer that will give 1 when multiplied by 5.

(b) $(\star)$ $\exists x \big(\forall y \, \neg Q(x,y) \ \wedge \ \exists y \, P(y)\big)$

Solution: For instance, "There exists an integer x, such that for all integers y, the product xy is not equal to 1, and such that there exists an integer greater than 0."

This statement is true. For $x = 0$, we have that for any integer y, the product xy is not equal to 1, and that there exists a positive integer, namely 42.

Be careful, the following interpretation is **not** correct: "There exists an integer x, such that for all integers y, the product xy is not equal to 1 and y is positive."

Due by 02/10/2025, 23:59.
Exercise 2.3 will be graded.