Exercise 4.5 Symmetric difference (⋆⋆) — GRADED (8 points)

Please upload your solution by 16/10/2025

In this exercise, we introduce a new operator, the symmetric difference. The symmetric difference of sets A and B, denoted $A\Delta B$, is the set of elements belonging to A or B but not to both:

$$A\Delta B \stackrel{\text{def}}{=} \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$$

1. Rewrite the definition above using set operations. More specifically, give an expression using only "A", "B", "$\cap$", "$\cup$", and "$\setminus$" (and parentheses), which is equal to the set $A\Delta B$. **Each of the operators "$\cap$", "$\cup$", and "$\setminus$" may appear at most once.** No justification is required.

   Solution:

   $$(A \cup B) \setminus (A \cap B)$$

2. Prove that for all sets A and B, it holds that $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

   Expectation: You should give a full proof for the aforementioned statement using set operations and/or the tools from previous chapters (rules of Lemma 2.1, function table, logical consequences, proof patterns). Formalism is important, but in contrast to the bonus exercise on sheet 2, you don't have to use rules of Lemma 2.1 one by one. You may use many rules at once, as soon as there is a main one (distributivity without specifying which, de Morgan's rule, absorption) used as justification and an unlimited number of minor rules (associativity, commutativity, double negation, idempotence) not necessarily mentioned. But there must be a justification for each step, even if the step uses solely a minor rule. Also, the **same** main rule can be used many times in one step. You may also use the following equivalences: $F \wedge \top \equiv F$, $F \wedge \bot \equiv \bot$, $F \vee \bot \equiv F$ and $F \vee \top \equiv \top$. **Recall that every step of your proof should be justified.**

Solution:

Since some of the equations are too long, they are written across two rows.

$$
\begin{aligned}
A\Delta B &= \{x \mid (x \in A \lor x \in B) \land \neg(x \in A \land x \in B)\} \\
&\mathrel{\dot{\iff}} \{x \mid (x \in A \lor x \in B) \land (x \notin A \lor x \notin B)\} && \text{(De Morgan)} \\
&\mathrel{\dot{\iff}} \{x \mid ((x \in A \lor x \in B) \land x \notin A) \lor \\
&\qquad ((x \in A \lor x \in B) \land x \notin B)\} && \text{(Distributive)} \\
&\mathrel{\dot{\iff}} \{x \mid ((x \in A \land x \notin A) \lor (x \in B \land x \notin A)) \lor \\
&\qquad ((x \in A \land x \notin B) \lor (x \in B \land x \notin B)\} && \text{(Distributive)} \\
&\mathrel{\dot{\iff}} \{x \mid (\perp \lor (x \in B \land x \notin A)) \lor ((x \in A \land x \notin B) \lor \perp)\} && (F \land \neg F \equiv \perp) \\
&\mathrel{\dot{\iff}} \{x \mid (x \in B \land x \notin A) \lor (x \in A \land x \notin B)\} && (F \lor \perp \equiv F) \\
&\mathrel{\dot{\iff}} \{x \mid x \in (B \setminus A) \lor x \in (A \setminus B)\} && \text{(Def. } \setminus) \\
&\mathrel{\dot{\iff}} (B \setminus A) \cup (A \setminus B) && \text{(Def. } \cup) \\
&\mathrel{\dot{\iff}} (A \setminus B) \cup (B \setminus A) && \text{(Commutativity)}
\end{aligned}
$$

3. Prove that for all sets A, B, and C, the following holds:

$$A\Delta B = A\Delta C \implies B = C\,.$$

Expectation: Same as above.

We will prove this through case distinction. It follows from the definition of the symmetric difference that

$$x \in A\Delta B \iff (x \in A \lor x \in B) \land \neg(x \in A \land x \in B)$$

and also

$$x \in A\Delta C \iff (x \in A \lor x \in C) \land \neg(x \in A \land x \in C).$$

We will consider two cases: $x \in A$ and $x \notin A$. One of these two statements must be true at any given time and they cover all possible values of x.

Case 1: Assume $x \in A$, this gives us

$$\begin{aligned} x \in A\Delta B &\overset{\cdot}{\iff} (\top \vee x \in B) \wedge \neg(\top \wedge x \in B) && (x \in A \equiv \top) \\ &\overset{\cdot}{\iff} \top \wedge \neg(\top \wedge x \in B) && (F \ \vee \top \equiv \top) \\ &\overset{\cdot}{\iff} \neg(\top \wedge x \in B) && (F \ \wedge \top \equiv F) \\ &\overset{\cdot}{\iff} \neg(x \in B) && (F \ \wedge \top \equiv F) \end{aligned}$$

We can apply the same steps in the same order to $x \in A\Delta C$ which results in

$$x \in A\Delta C \iff \neg(x \in C)$$

And finally we have

$$(x \in A\Delta C \iff x \in A\Delta B) \implies (\neg(x \in B) \iff \neg(x \in C))$$

This can only be true if $x \in B \iff x \in C$ (achieved by negating both sides). This is the definition of equality, which shows that (under the assumption that $x \in A$)

$$(x \in A\Delta B \iff x \in A\Delta C) \implies (x \in B \iff x \in C)$$

Case 2: Assume $x \notin A$, this gives us

$$\begin{aligned} x \in A\Delta B &\overset{\cdot}{\iff} (\bot \vee x \in B) \wedge \neg(\bot \wedge x \in B) && (x \in A \equiv \bot) \\ &\overset{\cdot}{\iff} (x \in B) \wedge \neg(\bot \wedge x \in B) && (F \ \vee \bot \equiv F) \\ &\overset{\cdot}{\iff} (x \in B) \wedge \neg(\bot) && (F \ \wedge \bot \equiv \bot) \\ &\overset{\cdot}{\iff} (x \in B) && (\neg\bot \equiv \top, \ F \ \wedge \top \equiv F) \end{aligned}$$

We can apply the same steps in the same order to $x \in A\Delta C$ which results in

$$x \in A\Delta C \iff (x \in C)$$

And finally we have

$$(x \in A\Delta C \iff x \in A\Delta B) \implies (x \in B \iff x \in C)$$

This shows that, under the assumption $x \notin A$,

$$(x \in A\Delta B \iff x \in A\Delta C) \implies (x \in B \iff x \in C)$$