Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

Marian DIETZ — Milan GONZALEZ-THAUVIN — Zoé REINKE

Exercise sheet 1

This is the exercise sheet number 1. The graded exercise is Exercise 1.5 and your solution has to be uploaded to Moodle **by 25/09/2025, 23:59**. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 1.1 The Punctured Chessboard ($\star$)

In the lecture (see also Example 1.1 in the lecture notes) we considered a $k \times k$ chessboard with one of the squares punctured. We also defined the predicate $P(k)$ to be equal 1 whenever the following statement is true:

No matter which of the squares is punctured, the remaining area of the chessboard (consisting of $k^2 - 1$ squares) can be covered completely with (non-overlapping) L-shaped pieces of paper consisting of three squares.

In this exercise we consider the proof by case distinction that $P(7) = 1$.

1. What is the smallest number of cases one has to consider in the proof? (Consider symmetries of the chessboard.)

Solution: One only needs to consider 10 cases (since all other cases are symmetric). The cases are marked below.

1	2	3	4			
	5	6	7			
		8	9			
			10			

2. Carry out the proof for two of the cases.

Solution: The proof that $P(7) = 1$, including all cases, can be found on the following website:
`http://www.cut-the-knot.org/Curriculum/Games/TrominoPuzzleN.shtml`.

Exercise 1.2 A False Proof ($\star\star$)

Find the mistake in the following proof.

Claim: 1 is the largest natural number.

Proof:

$$\begin{aligned} & n \text{ is the largest natural number} \\ \overset{\cdot}{\Longrightarrow}\ & n^2 \leq n \\ \overset{\cdot}{\Longrightarrow}\ & n(n-1) = n^2 - n \leq 0 \\ \overset{\cdot}{\Longrightarrow}\ & 0 \leq n \leq 1 \\ \overset{\cdot}{\Longrightarrow}\ & n = 1 \end{aligned}$$

> **Solution:** More precisely, the claim consists of two parts: "There exists the largest natural number n" and "$n = 1$". Denote the first statement by S and the second by T. The statement to prove is S **and** T, but the proof only shows $S \implies T$, which is true, because S is false. The proof is correct, but proves the wrong statement.
>
> Note that if a statement S is false, then the statement $S \implies T$ is true for any T. In other words, it is possible to prove any statement T by starting with a false assumption.

Exercise 1.3 Interpreting Propositional Formulas in Natural Language

Let A be the proposition "Mario forgot to pay his rent" and let B be the proposition "Mario is getting evicted".

1. ($\star$) How would you interpret the following formulas?

   i) $F_1 = \neg B \rightarrow \neg A$ **ii)** $F_2 = (A \wedge B) \vee (\neg A \wedge \neg B)$

   > **Solution:** The formulas can be stated in the English language in the following way:
   >
   > **i)** F_1: "If Mario is not getting evicted, he did not forget to pay his rent."
   >
   > **ii)** F_2: "Mario forgot to pay rent and is getting evicted, or Mario did not forget to pay his rent and is not getting evicted."
   >
   > Equivalently, we could say "Mario is getting evicted if and only if he forgot to pay his rent."

2. ($\star$) Using only the propositions A, B and logical operators, write down formulas corresponding to the following sentences:

i) F_3: "Mario neither forgot to pay his rent nor is he getting evicted."

ii) F_4: "Mario either forgot to pay his rent or he is getting evicted, but not both."

Solution: The sentences can be written formally in the following way:

i) $F_3 = \neg A \wedge \neg B$

ii) $F_4 = (\neg A \wedge B) \vee (A \wedge \neg B)$

3. $(\star\,\star)$ For both formulas F_3, F_4, write down their negations both as sentences and formally as formulas.

Solution:

i) $\neg F_3$: Mario forgot to pay his rent or he is getting evicted.

$$\neg F_3 \;\equiv\; \neg(\neg A \wedge \neg B) \;\equiv A \;\vee B$$

ii) $\neg F_4$: Mario is getting evicted if and only if he forgot to pay his rent.

$$\neg F_4 \;\equiv\; (A \wedge B) \vee (\neg A \wedge \neg B) \;\equiv\; F_2$$

Exercise 1.4 Logical Equivalence via Function Tables

1. $(\star)$ Compute the function table for the following formula:

$$(B \to C) \to \big(\neg(A \to C) \wedge \neg(A \vee B)\big).$$

Solution:

A	B	C	$B \to C$	$\neg(A \to C) \wedge \neg(A \vee B)$	$(B \to C) \to \big(\neg(A \to C) \wedge \neg(A \vee B)\big)$
0	0	0	1	0	0
0	0	1	1	0	0
0	1	0	0	0	1
0	1	1	1	0	0
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	0	0	1
1	1	1	1	0	0

2. ($\star\star$) Give another formula that is equivalent to the formula from Subtask a), but in which each of the propositional symbols appears at most once.

> **Solution:** With the above function table, it becomes clear that the formula in a) is true if and only if $B \wedge \neg C$ is true. Therefore, the simple equivalent formula is $B \wedge \neg C$.

Exercise 1.5 Two New Logical Operators — GRADED *(8 points)*

Please upload your solution by 25/09/2025

We define two binary logical operators $\heartsuit$ and $\diamondsuit$ as follows:

A	B	$A \heartsuit B$
0	0	1
0	1	1
1	0	0
1	1	1

A	B	$A \diamondsuit B$
0	0	0
0	1	1
1	0	1
1	1	0

1. ($\star$) Are $\heartsuit$ and $\diamondsuit$ commutative, i.e. does it hold

$$A \heartsuit B \equiv B \heartsuit A \qquad \text{and} \qquad A \diamondsuit B \equiv B \diamondsuit A\,?$$

Argue by comparing function tables.

> **Solution:**
>
> | A | B | $B \heartsuit A$ |
> |---|---|---|
> | 0 | 0 | 1 |
> | 0 | 1 | 0 |
> | 1 | 0 | 1 |
> | 1 | 1 | 1 |
>
> | A | B | $B \diamondsuit A$ |
> |---|---|---|
> | 0 | 0 | 0 |
> | 0 | 1 | 1 |
> | 1 | 0 | 1 |
> | 1 | 1 | 0 |
>
> We can see that the function tables of $A \heartsuit B$ and $B \heartsuit A$ are **different**, therefore $\heartsuit$ is **not** commutative. On the contrary the function tables of $A \diamondsuit B$ and $B \diamondsuit A$ are **the same**, therefore $\diamondsuit$ is commutative.

2. ($\star$) Prove or disprove that

$$(\neg A \heartsuit B) \diamondsuit (B \heartsuit C) \equiv \neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$$

by computing and comparing the function tables of the left-hand-side and the right-hand-side formulas.

Solution: The function table of the left-hand-side formula $F = (\neg A \heartsuit B) \diamondsuit (B \heartsuit C)$ is

A	B	C	$\neg A \heartsuit B$	$B \heartsuit C$	F
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	0
1	0	0	1	1	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	0

and the function table of the right-hand-side formula $G = \neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$ is

A	B	C	$\neg(A \diamondsuit B)$	$\neg(A \diamondsuit C)$	G
0	0	0	1	1	1
0	0	1	1	0	0
0	1	0	0	1	1
0	1	1	0	0	1
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	1	1

The function tables of F and G are different so the formulas are not equivalent.

Let F be a formula with the following function table:

A	B	C	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

3. ($\star\,\star$) Find a formula G containing only the logical operators $\heartsuit$ and $\diamondsuit$, in which the propositional symbols A, B, and C all appear exactly once, and such that $G \equiv F$. No justification is required.

Solution: One such formula is: $B \heartsuit (A \diamondsuit C)$

Exercise 1.6 Simplifying a Formula ($\star$)

Consider the propositional formula

$$F = \Big((\neg A \vee \neg B) \wedge \neg A\Big) \wedge \Big((\neg B \wedge \neg A) \vee C\Big).$$

Give a formula G that is equivalent to F, but in which each atomic formula A, B, and C appears at most once. **Prove** that $F \equiv G$ by providing a sequence of equivalence transformations with **at most** 6 steps.

Expectation. Your proof should be in the form of a sequence of steps, where each step consists of applying the definition of $\rightarrow$ (that is $F \rightarrow G \equiv \neg F \vee G$), one of the rules given in Lemma 2.1 of the lecture notes[1], or one of the following rules: $F \wedge \neg F \equiv \bot$, $F \wedge \bot \equiv \bot$, $F \vee \bot \equiv F$, $F \vee \neg F \equiv \top$, $F \wedge \top \equiv F$, and $F \vee \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1 *3)*. Each step of your proof should apply a **single** rule **once** and state **which** rule was applied.

Solution: We choose the formula $G = \neg A \wedge (\neg B \vee C)$. In the following, we prove that $F \equiv G$:

$$\begin{aligned}
&\Big((\neg A \vee \neg B) \wedge \neg A\Big) \wedge \Big((\neg B \wedge \neg A) \vee C\Big) \\
\equiv\ &\Big((\neg A \vee \neg B) \wedge (\neg A \vee \neg A)\Big) \wedge \Big((\neg B \wedge \neg A) \vee C\Big) && \text{(idempotence (1))} \\
\equiv\ &\Big(\neg A \vee (\neg B \wedge \neg A)\Big) \wedge \Big((\neg B \wedge \neg A) \vee C\Big) && \text{(distributive law (6))} \\
\equiv\ &\Big((\neg B \wedge \neg A) \vee \neg A\Big) \wedge \Big((\neg B \wedge \neg A) \vee C\Big) && \text{(commutativity of } \vee\text{)} \\
\equiv\ &(\neg B \wedge \neg A) \vee (\neg A \wedge C) && \text{(distributive law (6))} \\
\equiv\ &(\neg A \wedge \neg B) \vee (\neg A \wedge C) && \text{(commutativity of } \wedge\text{)} \\
\equiv\ &\neg A \wedge (\neg B \vee C) && \text{(distributive law (5))}
\end{aligned}$$

Due by 25/09/2025, 23:59.
Exercise 1.5 will be graded.