2. Matrix powers (bonus, hand-in)

For a natural number $k \geq 1$, we define the k-th power of a square matrix A as the matrix multiplication

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}.$$

Moreover, we define $A^0 = I$, where I is the identity matrix.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Use induction to show that for any $k \geq 0$, the k-th power of the matrix A is

$$A^k = \begin{bmatrix} 1+k & k \\ -k & 1-k \end{bmatrix}.$$

Solution:

For the base case we have

$$A^0 = \begin{bmatrix} 1+0 & 0 \\ -0 & 1-0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

which is valid considering the definition of A^0. Now assume that the following equation is valid for an arbitrary $n \geq 1, n \in \mathbb{N}$:

$$A^n = \begin{bmatrix} 1+n & n \\ -n & 1-n \end{bmatrix}.$$

This is the induction hypothesis. The induction step will show that if the formula is valid for n, it is also valid for $n+1$.

$$
\begin{aligned}
A^{n+1} &= A^n \cdot A \\
&= \begin{bmatrix} 1+n & n \\ -n & 1-n \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} && \text{(Induction Hypothesis)} \\
&= \begin{bmatrix} (1+n)2-n & 1+n \\ -2n-(1-n) & -n \end{bmatrix} \\
&= \begin{bmatrix} 2+2n-n & 1+n \\ -2n-1+n & -n \end{bmatrix} \\
&= \begin{bmatrix} n+2 & 1+n \\ -n-1 & -n \end{bmatrix} \\
&= \begin{bmatrix} 1+(n+1) & (n+1) \\ -(n+1) & 1-(n+1) \end{bmatrix} \\
&= A^{n+1}
\end{aligned}
$$

It has thus been proven by the principle of mathematical induction that the formula is valid for any $k \geq 0$.