1 Linear combinations of vectors

a) Prove that every vector in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}-1\\1\end{pmatrix}$.

Let a general vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$. To prove that $\mathbf{u}$ can be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ it must be shown that there exist scalars $\lambda, \mu \in \mathbb{R}$ such that:

$$\lambda \begin{pmatrix}1\\1\end{pmatrix} + \mu \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}u_1\\u_2\end{pmatrix} \tag{1}$$

This linear combination results in two equations which can be solved for lambda through addition.

$$\begin{cases}\lambda - \mu = u_1\\ \lambda + \mu = u_2\end{cases} \qquad 2\lambda = u_1 + u_2 \qquad \lambda = \frac{u_1 + u_2}{2} \tag{2}$$

This value can substitute λ in the second equation, allowing it to be solved for μ.

$$\frac{u_1 + u_2}{2} + \mu = u_2 \qquad \mu = u_2 - \frac{u_1 + u_2}{2} = \frac{2u_2 - u_1 - u_2}{2} = \frac{u_2 - u_1}{2} \tag{3}$$

Proving that for any $\mathbf{u} \in \mathbb{R}^2$ there will exist $\lambda, \mu \in \mathbb{R}$ as valid coefficients for the linear combination.

b) Consider the two vectors $\mathbf{v} = \begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}0\\1\\1\end{pmatrix}$ in $\mathbb{R}^3$ Find a vector in $\mathbb{R}^3$ that cannot be written as a combination of $\mathbf{v}$ and $\mathbf{w}$. Justify your answer.

We will first find what vectors the linear combination can generate and, based on their constraints, determine an example of an impossible vector to produce. Given a vector $\mathbf{u}$ and scalars λ, μ as described in item a, the linear combination and system of equations are as follows:

$$\lambda \begin{pmatrix}1\\0\\1\end{pmatrix} + \mu \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} \qquad \begin{cases}\lambda = u_1\\ \mu = u_2\\ \lambda + \mu = u_3\end{cases} \tag{4}$$

Which provides the condition $u_3 = u_1 + u_2$. A vector that does not fulfill this will not be able to be written as a combination of $\mathbf{v}$ and $\mathbf{w}$. Take for example $\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$. Inputting its coordinates into the condition results in an error: $1 = 1 + 1$. Thus, expressing it through the linear combination discussed is not possible.