Exercise 5.4 Properties of Relations ($\star$) — GRADED (8 points)
Please upload your solution by 23/10/2025

Prove or disprove the following claims:

1. Let ρ be a relation on a set A. Then, the relation $\gamma \overset{\text{def}}{=} \rho \circ \hat{\rho}$ is symmetric.

   Solution:

   The claim is true. If γ is symmetric, this means that $(a,b) \in \gamma \iff (b,a) \in \gamma$. The proof is as follows:

$$
\begin{aligned}
(a,b) \in \gamma &\overset{\cdot}{\Longrightarrow} (a,b) \in \rho \circ \hat{\rho} && \text{(Def. } \gamma\text{)} \\
&\overset{\cdot}{\Longrightarrow} (a,x) \in \rho \text{ und } (x,b) \in \hat{\rho} \text{ für ein } x \in A && \text{(Def. } \circ\text{)} \\
&\overset{\cdot}{\Longrightarrow} (x,a) \in \hat{\rho} \text{ und } (b,x) \in \rho \text{ für ein } x \in A && \text{(Def. } \hat{\rho}\text{)} \\
&\overset{\cdot}{\Longrightarrow} (b,a) \in \rho \circ \hat{\rho} && \text{(Def. } \circ\text{)} \\
&\overset{\cdot}{\Longrightarrow} (b,a) \in \gamma && \text{(Def. } \circ\text{)}
\end{aligned}
$$

$\square$

2. Let A be a set and let σ be a symmetric relation on A and π be an antisymmetric relation on A. Then, the relation $\gamma \overset{\text{def}}{=} \sigma \circ \hat{\pi}$ is symmetric.

   Solution:

   The claim is false. A clear counterexample will be outlined.

   Let $A = \mathbb{N}, \sigma = \{(1,2),(2,1)\}, \pi = \{(3,2),(4,5)\}$. It is clear that σ and π are respectively symmetric and antisymmetric. We also have

$$
\pi = \{(3,2),(4,5)\} \Longrightarrow \hat{\pi} = \{(2,3),(5,4)\} \qquad \text{(Def. } \hat{\pi}\text{)}
$$

   and

$$
\begin{aligned}
\sigma \circ \hat{\pi} &= \{(1,3)\} && \text{(Def. } \circ\text{)} \\
\gamma &= \{(1,3)\} && \text{(Def. } \gamma\text{)}
\end{aligned}
$$

   Since $(3,1)$ is not in γ, it is not symmetric. This concludes the counterexample.

3. The intersection of two equivalence relations on the same set is an equivalence relation.

Solution:

The claim is true. It is stated in Lemma 3.10. It will be proven directly through implications.

Let A be a set and ρ, σ equivalence relations on A. This means that both relations are reflexive, symmetric and transitive. We will use these properties to prove that $\rho \cap \sigma$ is also an equivalent relation.

Reflexivity:

Let $a \in A$ be arbitrary. Since ρ and σ are reflexive, it follows that

$$a \in A \implies (a,a) \in \rho \tag{1}$$
$$a \in A \implies (a,a) \in \sigma \tag{2}$$

and these two implications allow us to demonstrate the reflexivity of the intersection

$$\begin{aligned} &\overset{(1)(2)}{\implies} (a,a) \in \rho \text{ and } (a,a) \in \sigma \\ &\overset{\cdot}{\implies} (a,a) \in \rho \cap \sigma \qquad \text{(Def. } \cap) \end{aligned}$$

Symmetry:

Let $a, b \in A$ be arbitrary with $(a,b) \in \rho \cap \sigma$

$$\begin{aligned} (a,b) \in \rho \cap \sigma &\overset{\cdot}{\implies} (a,b) \in \rho \text{ and } (a,b) \in \sigma && \text{(Def. } \cap) \\ &\implies (b,a) \in \rho \text{ and } (b,a) \in \sigma && \text{(Symm. of } \rho, \sigma) \\ &\implies (b,a) \in \rho \cap \sigma \end{aligned}$$

This proves that $\rho \cap \sigma$ is symmetrical.

Transitivity:

Let $a, b, c \in A$ arbitrary with $(a,b) \in \rho \cap \sigma$ and $(b,c) \in \rho \cap \sigma$. To start we have

$$(a,b) \in \rho \cap \sigma \tag{1}$$
$$(b,c) \in \rho \cap \sigma \tag{2}$$

$$\begin{aligned}
\overset{(1),(2)}{} \stackrel{\cdot}{\Longrightarrow} & (a,b) \in \rho \text{ and } (a,b) \in \sigma \text{ and } (b,c) \in \rho \text{ and } (b,c) \in \sigma && \text{(Def. } \cap\text{)} \\
\stackrel{\cdot}{\Longrightarrow} & (a,b) \in \rho \text{ and } (b,c) \in \rho \text{ and } (a,b) \in \sigma \text{ and } (b,c) \in \sigma && \text{(Comm.)} \\
\stackrel{\cdot}{\Longrightarrow} & (a,c) \in \rho \text{ and } (a,c) \in \sigma && \text{(Trans. of } \rho, \sigma\text{)} \\
\stackrel{\cdot}{\Longrightarrow} & (a,c) \in \rho \cap \sigma
\end{aligned}$$

This proves that $\rho \cap \sigma$ is transitive.

Since $\rho \cap \sigma$ is reflexive, symmetric and transitive, it is an equivalence relation. This concludes the proof.