2. Nullspace as a hyperplane (hand-in) $(\star\star\circ)$

Let $\mathbf{v} \in \mathbb{R}^m \setminus \{0\}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ not all zero. Consider the $m \times n$ matrix A of the form

$$A = \begin{bmatrix} | & | & & | \\ \lambda_1\mathbf{v} & \lambda_2\mathbf{v} & \cdots & \lambda_n\mathbf{v} \\ | & | & & | \end{bmatrix}.$$

(a) What is the rank of the matrix A?

Solution:

The rank is the number of independent columns. A column is independent if it is not a linear combination of the previous columns.

In the matrix A, starting from the first column, there are two possible cases. The column could be zero ($\lambda_i = 0$) or it could be a multiple of $\mathbf{v}$. Zero columns are always dependent, so they do not influence $\text{rank}(A)$. The first non-zero column will not be a linear combination of the previous vectors (and thus independent), as the set of previous vectors is either empty or composed entirely of zero vectors. This means that $\text{rank}(A) \geq 1$.

For all subsequent columns, the same property holds. If they are zero columns, they are dependent, and if they are not, they must be multiples of v and thus also dependent.

$$\mu(\lambda_i\mathbf{v}) = \lambda_j\mathbf{v} \qquad \frac{\mu\lambda_i}{\lambda_j} = \mathbf{v}$$

Since $\frac{\mu\lambda_i}{\lambda_j}$ is a scalar. This means none of the next columns will be independent, keeping the first non-zero column as the only independent one. It follows that $\text{rank}(A) = 1$.

(b) Prove that the nullspace $N(A)$ is a hyperplane through the origin.

Solution:

$N(A)$ is the set of all vectors $\mathbf{w} \in \mathbb{R}^n$ such that $A\mathbf{w} = \mathbf{0}$.

$$A\mathbf{w} = \begin{bmatrix} | & | & & | \\ \lambda_1\mathbf{v} & \lambda_2\mathbf{v} & \cdots & \lambda_n\mathbf{v} \\ | & | & & | \end{bmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{j=1}^{n} w_j(\lambda_j\mathbf{v})$$

Since $\mathbf{v}$ is a constant and $\mathbf{v} \neq 0$, we have

$$\sum_{j=1}^{n} w_j \lambda_j \mathbf{v} = \mathbf{v} \left(\sum_{j=1}^{n} w_j \lambda_j \right)$$

and

$$\mathbf{v} \left(\sum_{j=1}^{n} w_j \lambda_j \right) = \mathbf{0} \iff \sum_{j=1}^{n} w_j \lambda_j = \mathbf{0}.$$

Now let $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$. This allows us to rewrite the final sum as a scalar product.

$$\sum_{j=1}^{n} w_j \lambda_j = \mathbf{w}^T \mathbf{x} = \mathbf{0}$$

Thus, N(A) is the set of all vectors orthogonal to $\mathbf{x}$. This set can also describe a hyperplane $H_{\mathbf{x}} = \{\mathbf{w} \in \mathbb{R}^n | \mathbf{w}^T \mathbf{x} = 0\}$. This hyperplane has the vector $\mathbf{x}$ as the "table leg" and contains $\mathbf{0}$ by definition, as it is orthogonal to every vector.