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Algorithms & Data Structures Exercise sheet 1 HS 25

The solutions for this sheet are submitted on Moodle until 28 September 2025, 23:59.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

The solutions are intended to help you understand how to solve the exercises and are thus more detailed than what would be expected at the exam. All parts that contain explanation that you would not need to include in an exam are in grey.

Exercise 1.1 *Sum of Cubes* **(1 point)**.

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Solution:

- **Base Case.**
  Let $n = 1$. Then,
  $$1^3 = 1 = \frac{1^2 \cdot (1+1)^2}{4},$$
  so the property holds for $n = 1$.
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer k, that is,
  $$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$
- **Induction Step.**
  We must show that the property also holds for $k+1$. Let us add $(k+1)^3$ to both sides of the

induction hypothesis. We get

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3 \\
&= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}.
\end{aligned}$$

By the principle of mathematical induction, $1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$ is true for any positive integer n.

Guidelines for correction:

Award 1/2 point if the student correctly sets up the induction hypothesis and formulates the correct induction step. Award 1 point if the previous happens and the computations are carried out correctly.

Exercise 1.2 *Sum of reciprocals of roots* **(1 point)**.

Consider the following claim:

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\leq\sqrt{n}.$$

A student provides the following induction proof. Is it correct? If not, explain where the mistake is.

Base case: $n=1$,

$$\frac{1}{\sqrt{1}}\leq 1, \text{ which is true.}$$

Induction hypothesis: Assume the claim holds for $n=k$, i.e.

$$\frac{1}{\sqrt{1}}+\cdots+\frac{1}{\sqrt{k}}\leq\sqrt{k}.$$

Induction step: Then, starting from the claim we need to prove for $n=k+1$ and using logical equivalences:

$$\begin{aligned}
\frac{1}{\sqrt{1}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}\leq\sqrt{k+1} &\iff \frac{1}{\sqrt{1}}+\cdots+\frac{1}{\sqrt{k}}\leq\sqrt{k+1}-\frac{1}{\sqrt{k+1}} \\
&\iff \frac{1}{\sqrt{1}}+\cdots+\frac{1}{\sqrt{k}}\leq\frac{k+1}{\sqrt{k+1}}-\frac{1}{\sqrt{k+1}} \\
&\iff \frac{1}{\sqrt{1}}+\cdots+\frac{1}{\sqrt{k}}\leq\frac{k}{\sqrt{k+1}}\leq\frac{k}{\sqrt{k}}\leq\sqrt{k},
\end{aligned}$$

which is true, therefore the claim holds by the principle of mathematical induction.

Solution:

The claim is wrong, since we already even have $1/\sqrt{1}+1/\sqrt{2} > \sqrt{2}$. The mistake of the proof is hidden in the last line of the chain of equivalences, i.e. when we write:

$$\begin{aligned} &\dots \\ &\iff \frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}} \leq \frac{k}{\sqrt{k}} \leq \sqrt{k}, \end{aligned}$$

What actually happens behind the scenes is the following implication:

$$\begin{aligned} &\left(\frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}}\right) \text{ AND } \left(\frac{k}{\sqrt{k+1}} \leq \frac{k}{\sqrt{k}}\right) \\ &\qquad\qquad \implies \frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k}} = \sqrt{k}. \end{aligned}$$

In essence, we used $\frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}}$ and concluded that $\frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k}}$. But the converse implication would not be true, i.e. the latter does not suffice to conclude the former (we would need to show this implication in a proper induction proof).

The big picture is that we started from essentially assuming the induction hypothesis holds for $k+1$ and concluded that it must hold for k. But we already knew that it holds for k. No new knowledge was produced with these steps. Therefore, the principle of mathematical induction cannot be applied here.

To make this clear, think of the following toy example, where we try to "prove" $1 = 2$.

$$\begin{aligned} &1 = 2 \\ \overset{\times 0}{\implies} &0 = 0, \end{aligned}$$

which is true. The logical implication step itself is actually also valid. However, just because the conclusion is true it does not follow that the initial statement is true. This is because the step where we multiplied by 0 is not reversible. So, in the end, we *cannot* say:

$$\begin{aligned} &0 = 0 \\ \implies &1 = 2. \end{aligned}$$

A similar confusion is happening in the proof we consider. For the proof of the student to be correct, we would need to be able to start with the induction hypothesis for k and show that it holds for $k+1$. An attempt to do that could look like this. We know:

$$\left(\frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k}}\right) \text{ AND } \left(\frac{k}{\sqrt{k+1}} \leq \frac{k}{\sqrt{k}}\right).$$

We would like to conclude now that $\frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{k}} \leq \frac{k}{\sqrt{k+1}}$ (because from this statement we can further conclude the induction step). However, we are not allowed to do this. The only rule we can use for producing new inequalities is that if we know $x \leq y$ and $y \leq z$, then we can conclude $x \leq z$. But what we have here is that we know $x \leq y$ and $z \leq y$. From these it does *not* follow that $x \leq z$ (for example set $x = 2, y = 3, z = 1$). It is important to keep this common pitfall in mind, as it is easier to fall into than one might think. For more guidelines on how to write induction proofs, see also the explanation of Exercise 1.4.

Guidelines for correction:

Award 1/2 point if the student at least disproves the inequality in any way or just mentions that something is wrong in the induction step. Award 1 point if the student correctly points to the concrete mistake (we should be lenient here as this is a difficult exercise, do not punish them if they say the correct thing but with some mistakes here and there or imprecise wordings).

Exercise 1.3 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements.

(a) $f(m) = 100m^3 + 10m^2 + m$ grows asymptotically slower than $g(m) = 0.001 \cdot m^5$.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{100m^3 + 10m^2 + m}{0.001m^5} \\ &= \lim_{m\to\infty} 10^5 m^{-2} + 10^4 m^{-3} + 10^3 m^{-4} \\ &= 10^5 \lim_{m\to\infty} m^{-2} + 10^4 \lim_{m\to\infty} m^{-3} + 10^3 \lim_{m\to\infty} m^{-4} \\ &= 10^5 \cdot 0 + 10^4 \cdot 0 + 10^3 \cdot 0 = 0.\end{aligned}$$

(b) $f(m) = \log(m^3)$ grows asymptotically slower than $g(m) = (\log m)^3$.

Solution:

True, since

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{\log(m^3)}{(\log m)^3} = \lim_{m\to\infty} \frac{3\log m}{(\log m)^3} = \lim_{m\to\infty} 3 \cdot \frac{1}{(\log m)^2} = 3 \cdot 0 = 0.$$

(c) $f(m) = e^{2m}$ grows asymptotically slower than $g(m) = 2^{3m}$.

Hint: *Recall that for all $n, m \in \mathbb{N}$, we have $n^m = e^{m \ln n}$.*

Solution:

True, since

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{e^{2m}}{2^{3m}} = \lim_{m\to\infty} \frac{e^{2m}}{e^{3m\ln 2}} = \lim_{m\to\infty} e^{(2-3\ln 2)m} = \lim_{m\to\infty} e^{(-0.079\ldots)\cdot m} = 0.$$

(d)* If $f(m)$ grows asymptotically slower than $g(m)$, then $\log(f(m))$ grows asymptotically slower than $\log(g(m))$.

Solution:

False. Consider $f(m) = m$ and $g(m) = m^2$. We have $\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{m}{m^2} = \lim_{m\to\infty} \frac{1}{m} = 0$, hence f grows asymptotically slower than g. However, $\log(f(m)) = \log m$ and $\log(g(m)) = \log(m^2) = 2\log m$, therefore $\lim_{m\to\infty} \frac{\log(f(m))}{\log(g(m))} = \lim_{m\to\infty} \frac{\log m}{2\log m} = \frac{1}{2} \neq 0$ and $\log(f(m))$ does not grow asymptotically slower than $\log(g(m))$.

(e)* $f(m) = \ln(\sqrt{\ln(m)})$ grows asymptotically slower than $g(m) = \sqrt{\ln(\sqrt{m})}$.

Hint: *You can use L'Hôpital's rule from sheet 0.*

Solution:

True, since

$$
\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{\ln(\sqrt{\ln(m)})}{\sqrt{\ln(\sqrt{m})}} \\
&= \lim_{m\to\infty} \frac{\left(\ln\left(\sqrt{\ln(m)}\right)\right)'}{\left(\sqrt{\ln\left(\sqrt{m}\right)}\right)'} && \text{(L'Hôpital's rule)} \\
&= \lim_{m\to\infty} \frac{\frac{1}{2m\ln m}}{\frac{1}{4m\sqrt{\ln(\sqrt{m})}}} = \lim_{m\to\infty} \frac{2\sqrt{\ln\left(\sqrt{m}\right)}}{\ln m} \\
&= \lim_{m\to\infty} \frac{\left(2\sqrt{\ln\left(\sqrt{m}\right)}\right)'}{(\ln m)'} && \text{(L'Hôpital's rule again)} \\
&= \lim_{m\to\infty} \frac{\frac{1}{m\sqrt{\ln(\sqrt{m})}}}{\frac{1}{m}} = \lim_{m\to\infty} \frac{1}{\ln\left(\sqrt{m}\right)} = 0.
\end{aligned}
$$

Guidelines for correction:

Award 1/2 point if at least 2 of a, b, c are solved correctly. Award 1 point if all 3 of them are.

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \quad n \geq 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

Solution:

We proceed by induction. But first, we prove a lemma that will be useful for that purpose.

Lemma 1. For any $k \geq 0$, we have:

$$\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}$$

Proof. We have:

$$\begin{aligned}
\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}} &\iff \frac{2k+1}{2k+2} \leq \sqrt{\frac{3k+1}{3k+4}} \\
&\iff \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4} \\
&\iff (4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1) \\
&\iff 12k^3+28k^2+19k+4 \leq 12k^3+28k^2+20k+4 \\
&\iff 0 \leq k,
\end{aligned}$$

where all the steps above are equivalences since each time we multiplied/squared positive terms. This justification is important. If we were to square negative numbers for example, the direction of the inequality would flip (e.g. $-3 < -2$ but $9 > 4$). Always try to think if you are ignoring cases like this. □

We can now start the induction proof.

Base Case.
For $n = 1$ we have

$$\frac{1}{2} \leq \frac{1}{\sqrt{4}},$$

which is even an equality.

Induction Hypothesis.
Now we assume that it is true for $n = k$, i.e.,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}.$$

Induction Step.
We will prove that it is also true for $n = k+1$, that is we want to show that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}.$$

Using the induction hypothesis, we have:

$$\begin{aligned}
&\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}} \\
\implies &\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k}\right) \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \\
\overset{\text{Lemma 1}}{\implies} &\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}.
\end{aligned}$$

By the principle of mathematical induction, we conclude that the inequality is true for any positive integer n.

The above solution is easy to follow and verify. This is important to make sure that the reader (and not just the author) understands the solution and can tell if it is correct or wrong. We recommend that you write induction proofs (actually, proofs in general) in this manner, i.e. starting from things you know and using logical implications (the $\implies$ symbol) to reach new true statements. Or, at least, that should be your first attempt. However, when actually solving the problem for the first time on a piece of paper, steps like Lemma 1 do not magically appear. We have to discover them. A typical (and recommended!) approach would go something like this: trying to do the induction step, one almost immediately reaches (as in the first step in the solution):

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}$$

Keep in mind that we have not yet come up with Lemma 1, and are exploring ways to use the above statement to conclude the induction hypothesis for $k+1$. One way to do that is if we were somehow able to write $\frac{1}{\sqrt{3k+4}}$ instead of $\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}$ in the right hand side of the above inequality (recall the rule that $x \leq y$ and $y \leq z$ together imply $x \leq z$). We would be allowed to do that if we knew that $\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}$, and this is precisely the content of Lemma 1.

The important thing to keep in mind here is that most often the "natural" logical path we take in discovering a solution is usually not the best for actually presenting it once we have it.

(b)* Replace $3n+1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

Solution:

Sometimes it is easier to prove more than less. This simple approach does not work for the weaker inequality as we are using a weaker (and insufficiently so!) induction hypothesis in each step.

If we try to do the same proof as above, we need to show in the induction step that

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k}}{\sqrt{3k+3}}.$$

Continuing as above, we get that we want to show that

$$\begin{aligned}
\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k}{3k+3}} &\iff \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k}{3k+3} \\
&\iff (4k^2+4k+1)(3k+3) \leq (4k^2+8k+4)(3k) \\
&\iff 12k^3+24k^2+15k+3 \leq 12k^3+24k^2+12k \\
&\iff 3k+3 \leq 0,
\end{aligned}$$

which is not true.

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction. And this is because by assuming a weaker induction hypothesis, we start from a more difficult position to prove the induction step (even though the induction step might also be more relaxed). An analogy one could think of is this: a plane can definitely cross the Atlantic with two functional engines, but probably cannot even takeoff with only one.