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Assignment 1

Submission Deadline: **30 September, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. **Lines in $\mathbb{R}^m$ (in-class)** (★★☆)

   a) Let $\mathbf{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line through the origin in $\mathbb{R}^m$ if and only if there exists $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. Let now L be a line through the origin in $\mathbb{R}^m$ and let $\mathbf{u}$ be an arbitrary non-zero element of L. Prove that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.

   b) For two lines through the origin L_1 and L_2 in $\mathbb{R}^m$, prove that we have either $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.

   c) Consider an arbitrary line through the origin L in $\mathbb{R}^2$. Prove that L is a hyperplane, i.e. find a vector $\mathbf{d} \neq \mathbf{0}$ such that
   $$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}.$$

2. **Orthogonality and Linear independence (hand-in)** (★☆☆)

   a) For which number $s \in \mathbb{R}$ are the two following vectors orthogonal?
   $$\mathbf{v} = \begin{pmatrix} s \\ 3 \\ 2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ s \end{pmatrix}$$

   b) For which number $t \in \mathbb{R}$ are the three following vectors linearly dependent?
   $$\mathbf{u} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

   c) Show that if two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal, then they are linearly independent. Prove that the converse is not necessarily true, i.e. provide an example of two linearly independent vectors that are not orthogonal.

3. **Cauchy-Schwarz inequality** (★★☆)

   Consider an arbitrary vector $\mathbf{v} \in \mathbb{R}^m$.

   a) Prove the inequality
   $$\sum_{i=1}^{m} v_i \leq \sqrt{m}\,\|\,\mathbf{v}\,\|\,.$$

b) Prove the inequality

$$\sum_{i=1}^{m} \sqrt{i} v_i \leq m \, \| \, \mathbf{v} \, \| \, .$$

4. Linear independence (★★☆)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m \in \mathbb{R}^m$ be the standard unit vectors in $\mathbb{R}^m$. Consider the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^m$ with $\mathbf{v}_i \coloneqq \mathbf{e}_i + \mathbf{e}_{i+1}$ for all $i \in \{1, 2, \ldots, m-1\}$ and $\mathbf{v}_m \coloneqq \mathbf{e}_m + \mathbf{e}_1$.

For example, we get

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

in the case $m = 3$, and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

in the case $m = 4$.

a) Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent if m is even.

b) Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent if m is odd.

5. Angle between two vectors (★★★)

Consider two non-zero vectors $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$ in $\mathbb{R}^3$ with $x + y + z = 0$. Determine the value of $\cos(\alpha)$ where α denotes the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. You are not required to compute (or look up) α, but you are of course welcome to do so.

6. Challenge 1.6 (★★★)

This exercise asks you to solve Challenge 1.6 from the lecture notes (check it out for some guiding questions).

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be arbitrary vectors in $\mathbb{R}^2$ and assume that $\mathbf{v} \neq \mathbf{0}$ and that $\mathbf{w} \neq \lambda \mathbf{v}$ for all $\lambda \in \mathbb{R}$. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$ be arbitrary. Prove that $\mathbf{u}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.