Exercise 3.1 *Asymptotic growth (2 points).*

(a) Prove or disprove the following statements. Justify your answer.

(1) For all $a, b \geq 1$, suppose $n \in \mathbb{N}$ and $n > \max\{a, b\}$, then $\log_a(n) = \Theta(\log_b(n))$.

Solution:

$$\lim_{n\to\infty} \frac{\log_a n}{\log_b n} = \lim_{n\to\infty} \frac{\frac{\log n}{\log a}}{\frac{\log n}{\log b}} = \lim_{n\to\infty} \frac{\log n}{\log a} \cdot \frac{\log b}{\log n} = \lim_{n\to\infty} \frac{\log b}{\log a} = \frac{\log b}{\log a} = C$$

And $C \in \mathbb{R},\ 0 < C < \infty$, meaning $\log_a n \in O(\log_b n)$ and $\log_b n \in O(\log_a n)$, which implies $\log_a(n) = \Theta(\log_b(n))$.

(2) For all $a, b \geq 1$, suppose $n \in \mathbb{N}$, then $a^n = \Theta(b^n)$.

Solution:

$$\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \frac{e^{\ln(a^n)}}{e^{\ln(b^n)}} = \lim_{n\to\infty} \frac{e^{n \ln a}}{e^{n \ln b}} = \lim_{n\to\infty} e^{n \ln a - n \ln b} = \lim_{n\to\infty} e^{n(\ln a - \ln b)}$$

This limit has three possible values. If $\ln a > \ln b$, we have

$$\lim_{n\to\infty} n(\ln a - \ln b) = \infty \implies \lim_{n\to\infty} e^{n(\ln a - \ln b)} = \infty \implies a^n \notin O(b^n)$$

and thus $a^n \neq \Theta(b^n)$. If $\ln a < \ln b$, we have

$$\lim_{n\to\infty} n(\ln a - \ln b) = -\infty \implies \lim_{n\to\infty} e^{n(\ln a - \ln b)} = 0 \implies b^n \notin O(a^n)$$

which also means $a^n \neq \Theta(b^n)$. The final case is $\ln a = \ln b$.

$$\lim_{n\to\infty} n(\ln a - \ln b) = 0 \implies \lim_{n\to\infty} e^{n(\ln a - \ln b)} = 1 \implies a^n = \Theta(b^n)$$

This is the only case for which the statement holds. It is generally false to assume $a^n = \Theta(b^n)$ for all $a, b \geq 1, n \in \mathbb{N}$.

(b) (1) Prove that $\lim_{n\to\infty} \frac{n}{\log(n)} = \infty$.

Hint: Use L'Hôpital's rule.

Solution:

The original limit gives us something infinitely big divided by something infinitely big:

$$\lim_{n\to\infty} \frac{n}{\log(n)} = \lim_{n\to\infty} \frac{\infty}{\log(\infty)} = \frac{\infty}{\infty}$$

So L'Hôpital's rule is used[1]:

$$\lim_{n\to\infty} \frac{n}{\log(n)} = \lim_{n\to\infty} \frac{1}{\frac{1}{n\cdot\ln(10)}} = \lim_{n\to\infty} \frac{n\cdot\ln(10)}{1} = \lim_{n\to\infty} n\cdot\ln(10)$$

$\ln(10)$ is a constant factor, so it can be ignored for $n\to\infty$, so the limit yields:

$$\lim_{n\to\infty} n\cdot\ln(10) = \lim_{n\to\infty} n = \infty$$

[1] $\log(n)$ is assumed to mean $\log_{10}(n)$.

(2) Prove that $\lim_{n\to\infty} n(e^{1/n} - 1) = 1$.

You may use the following variant of L'Hôpital's rule:

Theorem 1 (L'Hôpital's rule (going to 0))**.** *Assume that functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}$ are differentiable, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ and for all $x \in \mathbb{R}^+$, $g'(x) \neq 0$. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = C \in \mathbb{R}$ or $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Solution:

The original limit is as follows:

$$\begin{aligned}\lim_{n\to\infty} n(e^{\frac{1}{n}} - 1) &= \lim_{n\to\infty} \infty \cdot (e^{\frac{1}{\infty}} - 1) = \lim_{n\to\infty} \infty \cdot (e^0 - 1)\\ &= \lim_{n\to\infty} \infty \cdot (1 - 1) = \lim_{n\to\infty} \infty \cdot 0\end{aligned}$$

This is indeterminate and hence not immediately solvable without analyzing whether the part approaching 0 or the part approaching ∞ grows faster than the other, and that to a level that cancels out the other.
Now, a little trick is used. Let $x = \frac{1}{n}$, so for $n \to \infty$, $x \to 0$. So substituting $n = \frac{1}{x}$, this yields the following limit:

$$\lim_{n\to\infty} n(e^{\frac{1}{n}} - 1) = \lim_{x\to 0} \frac{1}{x} \cdot (e^{\frac{1}{\frac{1}{x}}} - 1) = \lim_{x\to 0} \frac{1}{x} \cdot (e^x - 1) = \lim_{x\to 0} \frac{e^x - 1}{x}$$

Now, since both terms of this fraction approach 0 and since $g'(x) = 1 \neq 0$, Theorem 1 can be used:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = \lim_{x\to 0} e^x = \lim_{x\to 0} e^0 = 1$$

(3) Prove that for $n \geq 3$

$$\frac{n^{1/n} - 1}{n} = \Theta\left(\frac{\ln(n)}{n^2}\right).$$

You may use results from the earlier exercises.

You may use the following fact:

Let $f, g : \mathbb{R}^+ \to \mathbb{R}$. Suppose that $\lim_{n\to\infty} g(n) = \infty$ and there exists some constant $C \in \mathbb{R}$ such that $\lim_{n\to\infty} f(n) = C$. Then $\lim_{n\to\infty} f(g(n)) = C$.

Solution:

The following equation[2] is given:

$$\frac{n^{\frac{1}{n}} - 1}{n} = \Theta\left(\frac{\ln(n)}{n^2}\right)$$

This can be transformed:

$$\lim_{n\to\infty} \frac{n^{\frac{1}{n}} - 1}{n} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}\cdot\ln(n)} - 1}{n} = \lim_{n\to\infty} \frac{e^{\frac{\ln(n)}{n}} - 1}{n}$$

Let $x_n = \frac{\ln(n)}{n}$. Since $\lim\limits_{n\to\infty} x_n = 0$, according to exercise 3.1.b.1[3], and $\lim\limits_{n\to\infty} n(e^{\frac{1}{n}} - 1) = \lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$ (from exercise 3.1.b.2), the given fact about composition[4] above will yield the following:

$$\lim_{n\to\infty} \frac{e^{x_n} - 1}{x_n} = 1$$

This means that there exists N for which all $n \geq N$:

$$\frac{1}{2} \leq \frac{e^{x_n} - 1}{x_n} \leq 2$$

Multiplying by x_n and dividing by n yields:

$$\frac{1}{2}\frac{x_n}{n} \leq \frac{e^{x_n} - 1}{n} \leq 2\frac{x_n}{n}$$

[2]The LHS of the equation is in theory a limit of n to ∞.

[3]It's the reciprocal of what was proven there.

[4]If $\lim\limits_{n\to\infty} g(n)$ tends to ∞ and $\lim\limits_{n\to\infty} f(n)$ tends to a constant C, then $\lim\limits_{n\to\infty} f(g(n)) = C$

Re-substituting $x_n = \frac{ln(n)}{n}$ gives:

$$\frac{1}{2}\frac{\ln(n)}{n^2} \leq \frac{e^{\frac{\ln(n)}{n}} - 1}{n} = \frac{n^{\frac{1}{n}} - 1}{n} \leq 2\frac{\ln(n)}{n^2}$$

This establishes the needed two-sided bounds for Θ-notation, so the equation stands.
Inputting $n = 3$ is the last step to prove that the LHS does not exceed the limit in the Θ-notation:

$$\frac{3^{\frac{1}{3}} - 1}{3} = \frac{1.4422 - 1}{3} = \frac{0.4422}{3} = 0.1474$$
$$\frac{\ln(3)}{3^2} = \frac{1.0986}{9} = 0.1221$$

This gives us a positive ratio of $\frac{0.1221}{0.1474} = 1.2072$, which is perfectly fine, it's close enough to the ratio we expect for $n \to \infty$. Because then, the ratio approaches 1, as has been shown above. So $n \geq 3$ was simply an arbitrarily chosen threshold, any small natural number would have done the job.

Exercise 3.3 *Counting function calls in loops (1 point).*

For each of the following code snippets, compute the number of calls to f as a function of $n \in \mathbb{N}$. Provide both the exact number of calls and a maximally simplified asymptotic bound in Θ notation.

(a) **Algorithm 1**

```
i ← 0
while i ≤ n do
    f()
    f()
    i ← i + 1

j ← 0
while j ≤ 2n do
    f()
    j ← j + 1
```

For this running time bound, we let n range over natural numbers that are at least 2 so that $n \log(n) > 0$.

The number of calls is mathematically described below. Note that to find the number of iterations, one was summed to the final indexes because i and j start at zero.

$$\sum_{i=0}^{n} 2 + \sum_{j=0}^{2n} 1 = (n+1) \cdot 2 + (2n+1) \cdot 1 = 2n + 2 + 2n + 1 = 4n + 3$$

Let A be a function that describes the number of calls to f in Algorithm 1. We have $A(n) = 4n + 3$ and $A \in \Theta(n)$

(b) **Algorithm 2**

```
i ← 1
while i ≤ n do
    j ← 1
    while j ≤ i^3 do
        f()
        j ← j + 1
    i ← i + 1
```

Hint: See Exercise 1.1.

The number of calls is mathematically described below. The definition of the sum of the first n cubic numbers was used.

$$\sum_{i=1}^{n}\sum_{j=1}^{i^3} 1 = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Let B be a function that describes the number of calls to f in Algorithm 2. We have $B(n) = \frac{n^2(n+1)^2}{4}$ and $B \in \Theta(n^4)$, since $\frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$.