Assignment 4 - Solutions

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Exercise 1: Matrix multiplication with vectors and covectors

Solution. **a)** We have $v = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$.

The product $v^\top w$ is a scalar:

$$v^\top w = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix} = 1 \cdot 2 + 2 \cdot 2 + \cdots + n \cdot 2 = 2(1 + 2 + \cdots + n)$$

Using the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, we get:

$$v^\top w = 2 \cdot \frac{n(n+1)}{2} = n(n+1)$$

b) For $n = 4$, we compute $vw^\top$:

$$vw^\top = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 \\ 8 & 8 & 8 & 8 \end{pmatrix}$$

c) We compute $w^\top(vw^\top)v$. Note that this is a scalar. We can use associativity:

$$w^\top(vw^\top)v = (w^\top v)(w^\top v) = (v^\top w)^2$$

From part (a), we know $v^\top w = n(n+1)$, so:

$$w^\top(vw^\top)v = [n(n+1)]^2 = n^2(n+1)^2$$

Exercise 2: Exercise 2.47

Solution. The CR-decomposition states that $A = CR'$ where $C \in \mathbb{R}^{m \times r}$ contains r linearly independent columns of A, and $R' \in \mathbb{R}^{r \times n}$ expresses all columns of A as linear combinations of the columns of C.

a) If $r = n$, then A has full column rank. This means all n columns of A are linearly independent.

- $C = A$ (all columns of A are linearly independent)
- $R' = I_n$ (the $n \times n$ identity matrix)

b) If $r = 0$, then A has rank 0, which means $A = 0$ (the zero matrix).

- C is the $m \times 0$ empty matrix
- R' is the $0 \times n$ empty matrix
- Their product gives the $m \times n$ zero matrix

Exercise 3: Matrix multiplication and invertibility

Solution. Given: $BA = CA$ where $A, B, C \in \mathbb{R}^{m \times m}$.

a) Suppose A is invertible. Then A^{-1} exists. Multiplying both sides of $BA = CA$ on the right by A^{-1}:

$$BA \cdot A^{-1} = CA \cdot A^{-1}$$
$$B(AA^{-1}) = C(AA^{-1})$$
$$BI = CI$$
$$B = C$$

b) No, it is not necessarily true that $AB = AC$.

Counterexample: Let $A, B, C \in \mathbb{R}^{2\times 2}$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$

We have $BA = CA$, as shown below.

$$BA = CA$$
$$\begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

However, this does not imply that $AB = AC$

$$AB = AC$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

And this is clearly false, meaning we have found a counterexample.

c) Suppose $B - C$ is invertible and $BA = CA$. Then:

$$BA - CA = 0$$

$$(B - C)A = 0$$

Multiplying on the left by $(B - C)^{-1}$:

$$(B - C)^{-1}(B - C)A = (B - C)^{-1} \cdot 0$$

$$IA = 0$$
$$A = 0$$

Exercise 4: Special matrix inverses

Solution. a) For $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, this is a permutation matrix that cycles $(e_1, e_2, e_3) \to (e_2, e_3, e_1)$. To undo this, we need to cycle backwards: $(e_1, e_2, e_3) \to (e_3, e_1, e_2)$.

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Verification: $AA^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$

b) For a diagonal matrix $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$, the inverse is obtained by inverting each diagonal entry:

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$$

c) For $B = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix}$, notice that $B = PD$ where $P = A$ is the permutation matrix from part (a) and D is the diagonal matrix from part (b).

Therefore: $B^{-1} = (PD)^{-1} = D^{-1}P^{-1}$

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \\ \frac{3}{2} & 0 & 0 \end{pmatrix}$$

Exercise 5: Inverses of matrix powers

Solution. a) Yes, A^k has an inverse, and it is $(A^{-1})^k = (A^k)^{-1}$.

Proof: We show that $(A^{-1})^k$ is the inverse of A^k:

$$A^k \cdot (A^{-1})^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}} \cdot \underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ times}} = I$$

By associativity and the fact that $AA^{-1} = I$, consecutive pairs cancel out, giving I.

b) Suppose A is nilpotent, so $A^k = 0$ for some $k \in \mathbb{N}$. Assume for contradiction that A has an inverse A^{-1}. Then:

$$A^k = 0$$

Multiplying both sides on the left by $(A^{-1})^k$:

$$(A^{-1})^k A^k = (A^{-1})^k \cdot 0$$

$$I = 0$$

This is a contradiction. Therefore, a nilpotent matrix cannot have an inverse.

c) Given: $A^3 = I$ and $A^4 = I$. We need to prove $A = I$.

From $A^4 = I$, we have $A \cdot A^3 = I$, which means $A \cdot I = I$ (using $A^3 = I$), so $A = I$.

d) We need $A^k = I$ for even k and $A^k = A$ for odd k.
For $k = 2$: $A^2 = I$, so $A^{-1} = A$.
For $k = 1$: $A^1 = A$ ✓
For $k = 3$: $A^3 = A \cdot A^2 = A \cdot I = A$ ✓
So we need $A^2 = I$ with $A \neq I$. Example:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

Then $A^2 = I$, and for odd k, $A^k = (-I)^k = (-1)^k I = -I = A$ ✓
e) We need $A^k = I$ if and only if $k \equiv 0 \pmod 4$.
This means $A^4 = I$, but $A^2 \neq I$, $A^3 \neq I$, $A \neq I$.
Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (90° rotation).
Then:

$$A^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq I$$
$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \neq I$$
$$A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq I$$
$$A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

For general $k = 4m + r$ where $r \in \{0, 1, 2, 3\}$: $A^k = A^{4m} \cdot A^r = I^m \cdot A^r = A^r$. So $A^k = I$ if and only if $r = 0$, i.e., $k \equiv 0 \pmod 4$. ✓

Exercise 6: Inverse of triangular matrices

Solution. a) For $L = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, we seek $L^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $LL^{-1} = I$.

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ ax + z & ay + w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives: $x = 1$, $y = 0$, $ax + z = 0 \Rightarrow z = -a$, $ay + w = 1 \Rightarrow w = 1$.
Therefore: $L^{-1} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$

b) ($\Rightarrow$) If a lower triangular matrix L is invertible, then $\det(L) \neq 0$. For a triangular matrix, the determinant is the product of diagonal entries, so all diagonal entries must be non-zero.

($\Leftarrow$) If all diagonal entries of L are non-zero, we can perform Gaussian elimination to reduce L to the identity using only row operations that preserve the lower triangular structure. This shows L is row-equivalent to I, hence invertible.

Alternatively, we can construct the inverse explicitly using back-substitution, which works precisely when all diagonal entries are non-zero.

c) Let L be a lower triangular matrix with inverse L^{-1}. We have $LL^{-1} = I$. Let $L^{-1} = (b_{ij})$ and $L = (a_{ij})$.

For $i < j$, the (i, j)-entry of LL^{-1} is:

$$\sum_{k=1}^{n} a_{ik} b_{kj} = 0$$

Since L is lower triangular, $a_{ik} = 0$ for $k > i$. So:

$$\sum_{k=1}^{i} a_{ik} b_{kj} = 0$$

For $i < j$, we have $k \leq i < j$, so we need $b_{kj} = 0$ for all $k < j$. This means L^{-1} is lower triangular.

d) Yes, the statements are also true for upper triangular matrices. The proofs are completely analogous, with the roles of rows and columns (or upper and lower) exchanged. The determinant argument works identically, and the structure is preserved under inversion by symmetry.