Exercise 3.2 From Natural Language to a Formula ($*$) — GRADED (4 points)

Please upload your solution by 09/10/2025

Consider the universe $U = \mathbb{N} \setminus \{0\}$. Express each of the following statements with a formula in predicate logic, in which the only predicates appearing are *smallerthan*(x, y), *divides*(x, y), *equals*(x, y) and *prime*(x) (instead of *smallerthan*(x, y), *divides*(x, y) and *equals*(x, y) you can write $x < y$, $x \mid y$ and $x = y$ accordingly). You can also use the symbols $+$ and $\cdot$ to denote the addition and multiplication functions, and you can use constants (e.g., $0, 1, \ldots$). You can also use $\rightarrow$ and $\leftrightarrow$. No justification is required.

(1.) ($\star$) There does not exist a largest natural number.

Solution:

$\forall x \exists y (x < y)$

(2.) ($\star$) The only divisors of a prime number are 1 and the number itself.

Solution:

$\forall x \forall y (\text{prime}(x) \land \neg(y = x) \land \neg(y = 1) \rightarrow \neg(y|x))$

(3.) ($\star$) 1 is the only natural number which has an inverse.

Solution:

$\exists z (1 \cdot z = 1) \land (\forall x \forall y \neg(\neg(x = 1) \land (x \cdot y = 1)))$

(4.) ($\star$) A prime number divides the product of two natural numbers if and only if it divides at least one of them.

Solution:

$\forall x \forall y \forall z (\text{prime}(x) \rightarrow ((x|(y \cdot z)) \leftrightarrow (x|y \lor x|z))$

Exercise 3.7 New Proof Patterns ($*$) — GRADED (4 points)

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For each of the following proof patterns, **prove or disprove** that it is sound. Do so by first writing it as a statement involving logical consequence on formulas and then proving that the resulting statement is either true or false.

(1.) ($\star$) To prove a statement S, find two appropriate statements T_1 and T_2. Assume that S is false and show (from this assumption) that at least one of the statements T_1 and T_2 is true. Then show that at least one of the statements T_1 and T_2 is false.

Solution:

The proof pattern can be interpreted logically as:

$$F = (\neg S \rightarrow (T_1 \vee T_2)) \wedge (\neg T_1 \vee \neg T_2) \models S$$

Which can be rewritten to make writing its truth table simpler.

$$\begin{aligned}(\neg S \rightarrow (T_1 \vee T_2)) &\equiv (\neg\neg S \vee T_1 \vee T_2) && \text{(definition of implication)}\\ &\equiv (S \vee T_1 \vee T_2) && \text{(double negation)}\end{aligned}$$

To confirm the validity of this statement we will compare its truth table to that of S.

S	T_1	T_2	$S \vee T_1 \vee T_2$	$\neg T_1 \vee \neg T_2$	$(S \vee T_1 \vee T_2) \wedge (\neg T_1 \vee \neg T_2)$
0	0	0	0	1	0
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	0	0

Which shows the logical consequence does not hold, as there are instances where the formula evaluates to true while S is false (highlighted above). The proof pattern is not sound.

(2.) ($\star$) To prove an implication $S \Rightarrow T$, find an appropriate statement R. First, show that R is false. Then, assume that S is true and T is false, and prove that (from these assumptions) R is true.

Solution:

The proof pattern can be interpreted logically as:

$$\neg R \wedge ((S \wedge \neg T) \rightarrow R) \models S \rightarrow T$$

To confirm the validity of this statement, we will compare its truth table to that of $S \rightarrow T$.

R	S	T	$S \wedge \neg T$	$(S \wedge \neg T) \rightarrow R$	$\neg R$	$\neg R \wedge ((S \wedge \neg T) \rightarrow R)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	1	0	1	0
0	1	1	0	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	0
1	1	0	1	1	0	0
1	1	1	0	1	0	0

If the logical consequence holds, $\neg R \wedge ((S \wedge \neg T) \rightarrow R)$ being true will mean $S \rightarrow T$ is also true.

R	S	T	$\neg R \wedge ((S \wedge \neg T) \rightarrow R)$	$S \rightarrow T$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	0
1	1	1	0	1

It is clear to see that the logical consequence is valid because every time the formula derived from the proof pattern evaluates to true, so does the implication (highlighted above). The proof pattern is sound.