Exercise 1.5 Two New Logical Operators

We define two binary logical operators $\heartsuit$ and $\diamondsuit$ as follows:

A	B	$A \heartsuit B$
0	0	1
0	1	1
1	0	0
1	1	1

A	B	$A \diamondsuit B$
0	0	0
0	1	1
1	0	1
1	1	0

(1) Are $\heartsuit$ and $\diamondsuit$ commutative, i.e. does it hold

$$A \heartsuit B \equiv B \heartsuit A \quad \text{and} \quad A \diamondsuit B \equiv B \diamondsuit A \,?$$

Argue by comparing function tables.

Answer: To check if an operator "op" is commutative, one can compare the truth tables of A op B and B op A. Trying this out for the given operators $\heartsuit$ and $\diamondsuit$ results in:

A	B	$A \heartsuit B$	$B \heartsuit A$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

A	B	$A \heartsuit B$	$B \diamondsuit A$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	0

As the truth tables for $A \heartsuit B$ and $B \heartsuit A$ are different, one can clearly see that $A \heartsuit B \not\equiv B \heartsuit A$, as there exists cases where changing the order of A and B leads to a different result. This shows that the $\heartsuit$ operator is not commutative.

The opposite can be said when comparing $A \diamondsuit B$ and $B \diamondsuit A$. The truth tables are the same, demonstrating that the operator $\diamondsuit$ is commutative.

(2) Prove or disprove that

$$(\neg A \heartsuit B) \diamondsuit (B \heartsuit C) \;\equiv\; \neg(A \diamondsuit B) \heartsuit \neg(A \diamondsuit C)$$

by computing and comparing the function tables of the left-hand side and the right-hand side formulas.

Answer: Here are the truth tables of both equations, with additional intermediate steps to help in the process.

A	B	C	$\neg A \heartsuit B$	$B \heartsuit C$	$(\neg A \heartsuit B) \Diamond (B \heartsuit C)$
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	0
1	0	0	1	1	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	0

A	B	C	$\neg(A \Diamond B)$	$\neg(A \Diamond C)$	$\neg(A \Diamond B) \heartsuit \neg(A \Diamond C)$
0	0	0	1	1	1
0	0	1	1	0	0
0	1	0	0	1	1
0	1	1	0	0	1
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	1	1

The truth tables differ. Thus, $(\neg A \heartsuit B) \Diamond (B \heartsuit C) \not\equiv \neg(A \Diamond B) \heartsuit \neg(A \Diamond C)$

(3) Let F be a formula with the following function table:

A	B	C	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Find a formula G containing only the logical operators $\heartsuit$ and $\Diamond$, in which the propositional symbols A, B, and C all appear exactly once, and such that $G \equiv F$. No justification is required.

Answer: The formula G below induces the same function as F.

$$G \equiv B \heartsuit (A \Diamond C) \tag{1}$$