

**Equinumerous sets — GRADED (8 points)**

Please upload your solution by 30/10/2025.

Prove the following statements.

1. For all sets  $A$ ,  $A$  and  $\mathcal{P}(A)$  are *not* equinumerous.

**Hint:** You could consider the set

$$B = \{x \in A \mid x \notin f(x)\}$$

for a well-chosen function  $f$ .

**Solution:**

We will prove the two sets are not equinumerous by contradiction. First, assume  $A \sim \mathcal{P}(A)$ , this means there exists a bijection  $f : A \rightarrow \mathcal{P}(A)$ . This bijective function maps  $A$  to  $\mathcal{P}(A)$ . In other words, it takes an element of  $A$  and assigns it a subset of  $A$  (because all elements of  $\mathcal{P}(A)$  are subsets of  $A$ ).

This implies that there can be no elements of  $\mathcal{P}(A)$  that aren't mapped by  $f$ . However, this cannot be true. Take the set  $B = \{x \in A \mid x \notin f(x)\}$ .  $B$  is a subset of  $A$  because all of its elements must be elements of  $A$ , but it is not mapped by our function.

Since we assumed  $f$  is a bijection, it follows that for some  $b \in B$ ,  $f(b) = B$ . However, the definition of  $B$  tells us that if  $b \in B$ ,  $b \notin f(b) = B$ , and this contradicts our previous claim that  $b \in B$ . Described in words, if an element is in  $B$ , it is not in the subset it maps to, which is  $B$ .

If  $b \notin B$ , we arrive at a similar contradiction, since this means that  $b \in f(b) = B$ . Described in words, if an element is not in  $B$ , then it is in the subset it maps to, which is  $B$ .

What these contradictions tell us is that we have defined a set  $B$  that cannot be reached by our function  $f$ , since it would otherwise lead to contradictions. This means there isn't a bijection between the sets and thus  $A$  is not equinumerous to  $\mathcal{P}(A)$ . This concludes the proof.

2. The sets

$$C \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\} \quad \text{and} \quad D \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid 0 \leq x < 1\}$$

are equinumerous.

**Solution:**

The sets  $C$  and  $D$  differ in only a single element, the number 1. Every element in  $D$  is also in  $C$ , which means  $D \subseteq C$ . From Lemma 3.15(iii) it follows that  $D \subseteq C \implies D \preceq C$ .

Now we can define an injection  $f : C \rightarrow D$  as  $f(x) = \frac{x}{2}$ . We clearly have  $f(x) \in D$ , because the problematic value 1 maps to  $\frac{1}{2}$  and all other  $x \in C$  also get halved, approaching zero but never leaving the bounds of  $D$ .

Now to prove injectivity let  $a, b \in C$  be arbitrary. We have:

$$\begin{aligned} f(a) = f(b) &\implies \frac{a}{2} = \frac{b}{2} && (\text{def. } f) \\ &\implies a = b \end{aligned}$$

Since we have defined an injection  $f : C \rightarrow D$ , it follows from Definition 3.42(ii) that  $C \preceq D$ . Now we have  $D \preceq C$  (shown before) and  $C \preceq D$ , which together imply  $C \sim D$  (Theorem 3.16). This concludes the proof.