A Preferred-Habitat Model with a Corporate Sector[1]

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Abstract

I study a preferred-habitat model of the term structure in which the same marginal investor prices government and corporate bonds. The model endogenously generates variation in credit spreads over and above changes in credit quality. In equilibrium, credit spreads are affine functions of the aggregate risk factors, providing an equilibrium justification to credit risk valuation models. Risk premia on interest rate and credit risk are time-varying and jointly determined. Arbitrage activity strengthens the risk-neutral dependence between the aggregate risk factors beyond the observed correlation between default rates and the policy rate. Movements in credit spreads are driven by (i) variation in credit quality (ii) risk-neutral correlation of the risk factors, and (iii) portfolio rebalancing due to diversification motives. A calibrated model matches the level and the slope of the term structure of credit spreads for both investment-grade and high-yield issuers. As government bonds hedge against default risk, the strength of monetary policy transmission to corporate (Treasury) yields is weaker (stronger) when default uncertainty increases. Shocks to the short term rate move credit spreads by altering risk premia on both credit and interest rate risk. The impact of quantitative easing interventions is asymmetric and depends on the specific assets being purchased.

Keywords: Preferred-habitat demand; Credit risk valuation; Term structure; Credit spreads; Monetary policy transmission.

JEL Classification: E43, E52, G11, G12.

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1 Introduction

The Treasury yield curve reflects the borrowing costs of the U.S. government and it is a widespread indicator of financing conditions. For this reason, the term structure of interest rates is a central target of policy interventions, and it often provides a metric against which the effectiveness of policy interventions is evaluated. Yet, investment and credit supply decisions are decentralized decisions made by firms and financial intermediaries, which cannot usually borrow at the same rate as the government. The difference between Treasury and corporate yields at the same maturity is referred to as the corporate credit spread. It turns out that credit risk factors only explain a modest fraction of the variation in credit spreads (Collin-Dufresne, Goldstein, & Martin, 2001; Friewald & Nagler, 2019; He, Khorrami, & Song, 2022). This empirical puzzle is reflected in the challenges faced by standard structural models of default to match the level of credit spreads with calibrations consistent with historical default rates (Chen, Collin-Dufresne, & Goldstein, 2008; Du, Elkamhi, & Ericsson, 2019).

Motivated by the insights that (i) the variation in credit risk spread is driven by time-varying risk premia rather than default probabilities (Du et al., 2019) and that (ii) intermediary-based factors explain a substantial fraction of the common variation in credit spreads (He, Khorrami, & Song, 2022), I propose a preferred-habitat model of the term structure of Treasury and corporate yields in which corporate and Treasury bonds are jointly priced by the same intermediaries. The model extends Vayanos and Vila (2021) and Gourinchas, Ray, and Vayanos (2022) to a two asset framework with defaultable bonds, and it belongs to the class of affine term structure models (Dai & Singleton, 2000; Duffie & Kan, 1996). In the equilibrium that I fully characterize, Treasury and corporate yields are affine functions of the state variables, thereby providing an equilibrium justification to credit risk valuation approaches (Duffie & Singleton, 2003, 1999) in the context of a preferred-habitat model of the term structure with investor clienteles (Modigliani & Sutch, 1966; Vayanos & Vila, 2021).

The framework integrates elements from the literature on credit risk valuation in a preferred-habitat context where asset prices are jointly determined by the pricing kernel of arbitrageurs that are marginal in both the Treasury and the corporate bond markets. Time is continuous, and market participants trade in corporate and Treasury bonds. The corporate sector is a continuum of identical issuers of defaultable zero-coupon bonds. I model default in reduced-form as an unpredictable jump in a Poisson process with time-varying default intensity (Lando, 1998). Besides for tractability reasons, the choice is motivated by the fact that structural models of credit risk cannot usually generate credit spreads that are strictly positive even for short horizons (see for example Collin-Dufresne and Goldstein (2001). Duffie and Lando (2001) is the exception). In the cross-section, default events are assumed to be identically distributed and mutually independent. Although defaults are idiosyncratic, time-variation in the default intensity is an aggregate risk factor, which commands a risk premium.

First, I analyze a simplified environment in which arbitrageurs only trade in a risk free asset and in defaultable bonds. The decision problem of the arbitrageurs is analogous to Vayanos and Vila (2021), with the key difference that a stochastic fraction of corporate bonds defaults at any point in time. The aggregate risk factors are the short term rate, the default intensity, and shocks to habitat demand. In equilibrium, the arbitrageurs' first-order condition determines the market price of short rate risk and default intensity jointly. Corporate bond yields are affine function of the state variables.

In the same spirit as intermediary asset pricing models (Brunnermeier & Sannikov, 2014; He & Krishnamurthy, 2013), the price of corporate bonds is set such that it is optimal for the arbitrageurs to absorb habitat demand. Therefore, bond risk premia are positively related to the aggregate positions held by these arbitrageurs. However, because habitat demand is price elastic, the risk premia on each individual risk factor are interrelated. An increase in the short rate affects both interest rate and credit risk premia. Since the market price of default and short risk are state-dependent, the model endogenously generates time-varying risk premia in a setting where shocks are homoscedastic. Accordingly, my model implies economic restrictions on the joint dynamics of default intensity and the short rate under the equivalent martingale measure. Arbitrage activity strengthens the risk-neutral dependence between the risk factors over and above the observed correlation between default rates and the policy rate throughout the business cycle. Even when risk factors are assumed to be independent, the endogenous correlation arising under the risk-neutral measure implies that the variation in credit spreads is not purely determined by changes in the credit quality of corporate issuers (Duffie & Singleton, 1999).

Subsequently, I extend the segmented equilibrium so that arbitrageurs trade in both corporate and government bonds. Adding a second asset class enriches the asset pricing implications through a portfolio choice channel. While shocks to the short term rate affect corporate and government bonds in the same way, shocks to default intensity have an asymmetric effect. Government bonds hedge against default risk, as their price increases when credit risk goes up. In contrast, a deterioration in credit quality lowers the price of corporate bonds. Since corporate bonds become less attractive to arbitrageurs when default intensity increases, a portfolio rebalancing effect implies that higher default intensity widens credit spreads by both lowering Treasury yields and raising corporate yields. A key result is that, in equilibrium, credit spreads are driven by a combination of three effects, namely (i) changes in the credit quality of corporate issuers, (ii) risk-neutral dependence between credit risk and the other aggregate risk factors, and (iii) a portfolio diversification channel effect. Hence, the model endogenously generates variation in credit spreads over and above changes in credit quality without heteroscedastic shocks or jump processes (Driessen, 2005; Du et al., 2019).

Another prediction of the model is that variation in default intensity is only one of the many determinants of movements in credit spreads. Indeed, equilibrium credit spreads are affine functions of the aggregate risk factors, including the short term rate and the demand factors. The short term rate acts on credit spreads by influencing default intensity risk premia through habitat demand. I reconcile this prediction with Collin-Dufresne and Goldstein (2001) and He, Khorrami, and Song (2022) by showing that the error term in a regression of credit spreads on default intensity and the short rate includes a combination of local and global demand shocks. While these demand factors might in principle originate at any maturity or within each asset class, they propagate throughout the term structure and across markets because of the intermediation activities done by the arbitrageurs. To the extent that demand shocks are highly persistent and that they share a common factor structure, the model-implied regression provides an intermediary-based justification to the results in Collin-Dufresne and Goldstein

(2001),Friewald and Nagler (2019), and He, Khorrami, and Song (2022).

To further explore the main implications of my model, I discipline its parameters by targeting the level and the volatility of the Treasury yield curve. Any other moment of the corporate yield curve and credit spreads is untargeted. In a calibrated effort, the model provides a good fit for both the corporate and the Treasury yield curve, especially at short to intermediate maturities. Although the implied volatility of credit spreads is higher than in the data, the implied credit spreads neatly match the average credit spreads for BBB-rated issuers with a default intensity set to approximate historical default rates. Furthermore, the model captures the fact that the term structure of credit spreads is upward sloping for investment-grade issuers, but downward sloping for high-yield bonds (Sarig & Warga, 1989).

Next, I analyze the impact of monetary policy interventions on credit spreads and corporate bond yields. The first observation is that default intensity uncertainty weakens the transmission of monetary policy shocks throughout the term structure of corporate bonds. Yet, the opposite holds for the Treasury yield curve. Forward rate underreaction to a monetary policy shock decrease with the volatility of default intensity shocks. A potential explanation is that higher default uncertainty increases the value of government bonds as hedges against default risk, driving down the risk premia on government bonds. When analyzing QE interventions, the model captures spillover effects across markets. The impact of QE on bond yields is asymmetric and depends on the asset being purchased. Furthermore, the response of credit spreads to corporate-only QE is much stronger than to Treasury-only QE intervention (D'Amico & King, 2013; Krishnamurthy & Vissing-Jorgensen, 2011).

In contrast to Gertler and Karadi (2015), however, the calibration implies that a monetary tightening reduces credit spreads. The mechanism that generates this result is the same as the one that generates counterfactual responses of term premia to monetary policy shocks in Vayanos and Vila (2021), as argued by Kekre, Lenel, and Mainardi (2022). The reason is that the arbitrageurs' balance sheet shrinks when interest rates go up, so that risk premia are inversely related to the short rate and to default intensity, highlighting the challenges in introducing risky assets in a preferred-habitat framework (Costain, Nuño, & Thomas, 2022; Droste, Gorodnichenko, & Ray, 2021). In the habitat demand specification of Vayanos and Vila (2021), habitat investors do respond to prices, but not to the aggregate risk factors directly. As a result, they ignore any fundamental news about the credit quality of corporate bonds, and act as pure liquidity providers.

To address these shortcomings, I revisit the dynamics of the state variables and the preferences of habitat investors in a simplified environment where agents only trade corporate bonds. I introduce two novel elements in the habitat-demand literature while preserving the tractability of affine term structure models. First, I assume square-root dynamics for the default intensity process, such that default probabilities are guaranteed to be strictly positive under mild technical conditions on the model parameters. Second, I propose a microfoundation for habitat demand through a mean-variance objective over instantaneous changes in wealth and a restriction on the investment universe. Specifically, habitat investors are only allowed to trade a certain maturity or to invest at the risk free rate. Habitat demand responds to expected returns, economic fundamentals, and the volatility of bond returns. The key feature that allows me to preserve the affine structure is that habitat demand becomes less sensitive to expected returns precisely when the conditional volatility of the shocks is higher. The endogenous variation in habitat investors' sensitivity to expected returns exactly offsets the variation in risk premia driven by the stochastic volatility of square-root dynamics. As a result, fluctuations in arbitrageurs' position is the only source of variation on interest rate and credit risk premia. The discussion is momentarily relegated to the Appendix, as many of the main results do not rely on the functional form of habitat demand. In future work, however, I will emphasize the novel ingredients on this framework to a greater extent.

In the last section, I revisit existing empirical evidence on the term structure of credit spreads, as well as the relation between term premia and credit spreads. Rather than presenting evidence validating the model, the goal is to (i) provide an overview of the empirical evidence that I plan to collect in the next steps and (ii) think about how data can inform future improvements on the modelling side.

Related Literature First, this paper contributes to the literature on term structure models by integrating elements from credit risk valuation into a preferred-habitat framework. My model belongs to the general class of affine term structure models (Dai & Singleton, 2000; Duffie & Kan, 1996), and extends the preferred-habitat model of Vayanos and Vila (2021) by (i) introducing defaultable bonds and (ii) deriving asset pricing implications for corporate bond yields and credit spreads. Building on Duffie and Singleton (2003) and Lando (1998), I model default in reduced-form as an unpredictable jump in a Poisson process with stochastic intensity.

The preferred-habitat view of the term structure dates back to early work by Culbertson (1957) and Modigliani and Sutch (1966), but it was only recently formalized by the seminal work of Vayanos and Vila (2021). My paper is closest to Costain et al. (2022), who present a term structure model of the yield curve in a heterogeneous monetary union. In their model, the defaultable bonds are issued by periphery countries in the European union and not by the corporate sector. However, as opposed to Costain et al. (2022) (i) I model default as a flow process rather than a discrete jump (ii) the aggregate risk factor is default intensity and not the event of default (iii) in my model the market price of default risk depends on the quantity of Treasury bond outstanding and (iv) the government bonds and Treasury bonds are both affine functions of all the risk factors, including default intensity.

Greenwood and Vayanos (2014) propose a simplified version of the model to study Treasury supply effects on the term structure, whereas Gourinchas et al. (2022) discuss a two country version Vayanos and Vila (2021) to explain uncovered interest parity violations. Similarly, Greenwood, Hanson, Stein, and Sunderam (2020) develop a quantity theory of term premia and exchange rate in a two country model, whereas Droste et al. (2021) embeds habitat demand in a New Keynesian framework to explain the financial effects of QE. Beyond corporate bonds, preferred-habitat models have been applied to repos (He, Nagel, & Song, 2022; Jappelli, Pelizzon, & Subrahmanyam, 2023) and to the interest rate swaps market (Hanson, Malkhozov, & Venter, 2022).

On the other hand, the paper also relates to the literature on the determinants of credit spreads by decomposing changes in credit spreads into (i) changes in the credit quality of corporate issuers (ii) risk-neutral dependence between credit risk and the other aggregate risk factors and (iii) a portfolio rebalancing effect. Collin-Dufresne et al. (2001) show that the determinant of credit spread implied by structural models of credit risk have rather limited explanatory power. Yet, the unexplained part has a strong principal component. Friewald and Nagler (2019) and He, Khorrami, and Song (2022) link the strong principal component to OTC frictions and intermediary capital, respectively. In particular, He, Khorrami, and Song (2022) show that a large fraction of the principal component can be explained by an dealer inventory factor and a measure of intermediary distress (He, Kelly, & Manela, 2017; Hu, Pan, & Wang, 2013). Chen et al. (2008) and Du et al. (2019) document that structural credit risk models underestimate credit spreads, arguing that changes in credit spreads are driven by time-varying risk premia rather than changes in credit quality. In my paper, I generate state-dependent risk premia that depend on both the level of interest rate and default intensity in a homoscedastic environment.

Furthermore, the paper also contributes to the literature on the transmission of monetary policy to credit spreads and long term yields. I argue that the strength of the propagation of monetary diversification throughout the term structure of either government or corporate bond depends on the interaction between term premia and diversification motives of arbitrageurs. Vayanos and Vila (2021) only partially capture the evidence by Hanson and Stein (2015) in the long term forwards during FOMC announcements are due to changes in term-premia. Building on this, Kekre et al. (2022) integrate element from the intermediary asset pricing tradition into Vayanos and Vila (2021) to show that a monetary easing also revalues the wealth of the arbitrageurs, thereby flipping the sign of the relation between term premia and the short rate. Closely related to this, Gertler and Karadi (2015) show that a monetary tightening is associated to increase in various measures of credit spreads.

Finally, the model contributes to the literature on bond risk premia (e.g. Cochrane and Piazzesi (2005); Haddad and Sraer (2020)) by showing that when corporate and Treasury bonds are priced by the same marginal investors, the market price of interest rate and credit risk are interconnected. This logic borrows from the intermediary asset pricing literature (Brunnermeier & Sannikov, 2014; He & Krishnamurthy, 2013), and it implies that credit and interest rate risk are priced together. Yet, the traditional factor models of corporate bonds generally treat interest rate and credit risk separately (Acharya, Amihud, & Bharath, 2013; Kelly, Palhares, & Pruitt, 2023). An exception to this is Li (2023), who similarly claims that the pricing of credit and interest rate risk are interconnected.

Organization The rest of the paper is organized as follows. Section 2 presents a preferred-habitat model with defaultable bonds. The simplified environment is intended to analytically characterize equilibrium properties and asset pricing implications. Section 3 extends Section 2 and presents a two sector habitat model in which agents trade both corporate and government bonds. Section 4 describes the data, illustrates the model calibration, and presents the quantitative analysis. Section 5 concludes.

2 Segmented Arbitrage

Section 2 describes a simplified environment in which arbitrageurs and habitat investors only trade in a riskless asset and in corporate bonds. The goal is to analytically characterize equilibrium properties and the dependence between the market prices of risk without any complication arising from portfolio diversification motives. Because arbitrageurs are restricted to trade a strict subset of all available assets, I refer to this environment as the segmented equilibrium in the spirit of Gourinchas et al. (2022). Section 3 generalizes these insights in a two asset framework in which aggregate risk factors potentially affect asset classes asymmetrically.

2.1 Idiosyncratic Defaults

I model the corporate sector as a continuum of identical firms uniformly distributed between zero and one and indexed by $i \in [0, 1]$. Each firm issues defaultable bonds with maturity $\tau \in (0, \infty)$. For each bond i, default is modelled as an unpredictable jump in a Poisson process N_t^i with intensity λ_t. The dynamics of each individual corporate bond are given by

$$\frac{dP_t^{i,(\tau)}}{P_t^{i,(\tau)}} = \mu_{i,t}^{(\tau)} dt + \sigma_{i,t}^{(\tau)} dB_t + dN_t^i (\omega - 1)$$

where ω is the recovery rate and $\mu_{i,t}^{(\tau)}$ and $\sigma_{i,t}^{(\tau)}$ are the drift and local volatility of the bonds, which will be determined in equilibrium. The increment of the Poisson process, dN_t^i, takes the value of one if bond i defaults and zero otherwise. Given the default intensity λ_t, the probability of default within the interval $[t, t + dt]$ is $\lambda_t dt$ (Duffee, 1999). I assume that defaults are idiosyncratic and that each individual bond has the same default intensity λ_t.

Assumption 1 (Idiosyncratic Defaults)**.** *The increments dN_t^i are independent across i and all have the same intensity λ_t.*

Although the default intensity does not depend on maturity, λ_t varies over time. Since defaults are idiosyncratic, each of these bonds are ex-ante identical. Hence, it must be true that in equilibrium $\mu_{i,t}^{(\tau)} = \mu_t^{(\tau)}$ and $\sigma_{i,t}^{(\tau)} = \sigma_t^{(\tau)}$. Therefore, the instantaneous return on a well-diversified portfolio of defaultable bonds is

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} \doteq \int_0^1 \frac{dP_t^{i,(\tau)}}{P_t^{i,(\tau)}} di = \mu_t^{(\tau)} dt + \sigma_t^{(\tau)} dB_t + (\omega - 1)\lambda_t dt \tag{1}$$

where the second equality follows from the Law of Large Numbers, as detailed in Appendix A.3. For simplicity, I henceforth assume a recovery rate of zero, i.e. $\omega = 0$. Contrary to Costain et al. (2022), equation (1) interprets default as a flow process rather than a jump process.

As a result, conditional on the information at time t, the default intensity λ_t will be the same under both the physical $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$. Because the market price of risk associated with the Poisson process is zero, λ_t is equal to the physical probability of default (Duffee, 1999). In this setting, the aggregate risk factor is the default intensity and not the event of default. As default intensity varies over time and provided that arbitrageurs are risk averse, its dynamics will be different under the equivalent martingale measure $\mathbb{Q}$.

2.2 Habitat Demand and Defaultable Bonds

Timing and assets Time t is continuous and runs from zero to infinity. A zero-coupon corporate bond is a security that promises one unit of the numeraire at time $t + \tau$, where $\tau \in (0, \infty)$ denotes the maturity. Within each maturity τ, there is a continuum of firms issuing zero-coupon corporate bonds, each of which might default with intensity λ_t. With idiosyncratic defaults, a deterministic fraction $\lambda_t dt$ of bonds defaults on aggregate at any point in time. While there is no uncertainty around the fraction of defaults in the interval $[t, t+dt]$, agents are uncertain about how many corporate bonds will default in the future. Building on Costain et al. (2022), I assume that the corporate sector instantaneously issues new bonds to replace those that defaulted.

Let $P_t^{(\tau)}$ and $y_t^{(\tau)}$ be the price and the yield of the bond with maturity τ at time t, respectively. The yield is related to the price through

$$y_t^{(\tau)} = -\frac{\log P_t^{(\tau)}}{\tau}$$

and the instantaneous holding period return is $\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}$. The short rate r_t is the limit of the yield $y_t^{(\tau)}$ as τ goes to zero, and I assume it is exogenously set by an unmodeled monetary authority.

Decision problems There are two types of agents: Arbitrageurs and preferred-habitat investors. Habitat investors, indexed by $\tau \in (0, \infty)$, are uniformly distributed across maturities and only hold corporate bonds with a specific maturity τ. Investors with habitat τ at time t hold a position

$$Z_t^{(\tau)} = -\alpha(\tau) \log P_t^{(\tau)} - \beta_t^{(\tau)} \tag{2}$$

in the bond with maturity τ and hold no other bonds. The slope coefficient $\alpha(\tau) \geq 0$ only depends on maturity and regulates the sensitivity of demand to prices. The intercept coefficient $\beta_t^{(\tau)}$ is time-varying and can depend on τ. The demand intercept takes the form

$$\beta_t^{(\tau)} = \theta_0(\tau) + \sum_{k=1}^{K} \theta_k(\tau)\beta_{k,t} \tag{3}$$

where $\{\theta_k(\tau)\}_{k=0}^{K}$ are constant over time but can depend on maturity τ. Specification (2) assumes that habitat investors are price-elastic, which is analogous to Vayanos and Vila (2021) and Kekre et al. (2022). As a result, these investors hold larger (more positive) positions when the securities are cheaper. While such modeling approaches seem appropriate for default-free government bonds, it also implies that habitat investors are not responsive to changes in the fundamentals of the corporate sector, as captured by λ_t. Hence, lower prices driven by deteriorating fundamentals make habitat investors hold larger positions. This happens because habitat demand does not respond to the shocks that caused the initial price decline.

In Appendix B, I present a stylized micro-foundation of habitat demand. The stylized framework reveals that habitat demand should be specified as a function of both prices $P_t^{(\tau)}$ and a notion of fundamental cash flows that depend on either r_t and λ_t. A higher level of λ_t is associated to worse fundamentals, whereas r_t might be interpreted as the opportunity cost of locking in capital in long term bonds. I model this dependence in reduced-form by assuming that demand shocks $\beta_t^{(\tau)}$ depend on the level of λ_t and r_t. A modification of habitat demand along these lines maps into the long term investors of Kyle and Xiong (2001) and Sangvinatsos and Wachter (2005). I explicitly describe the dynamics later in equation (6) and discuss other variants in Section 3.3.

Arbitrageurs trade corporate bonds at all maturities and can also invest in the short rate r_t. W_t and τ denote arbitrageurs' wealth and dollar holdings in the bonds with maturity τ, respectively. Arbitrageurs have mean-variance preferences over instantaneous changes in wealth

$$\max_{\{X_t^{(\tau)}\}_{\tau\in\{0,\infty\}}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t)\right] \tag{4}$$

where $a \geq 0$ control arbitrageurs' risk aversion. As in Vayanos and Vila (2021), arbitrageurs can be interpreted as overlapping generations living over infinitesimal periods. The instantaneous budget constraint is given by

$$dW_t = \left(W_t - \int_0^\infty X_t^{(\tau)} d\tau\right) r_t dt + \int_0^\infty X_t^{(\tau)} \left(\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} - \lambda_t dt\right) d\tau \tag{5}$$

The first term corresponds to a position in the short rate, and the second term describes the capital gains from investing in corporate bonds. The instantaneous return on the bond index is $\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}$, which is adjusted to account for the fraction $\lambda_t dt$ of the holdings $X_t^{(\tau)}$ that is lost in the interval $[t, t+dt]$.

Risk factor dynamics In this economy, there are $K+2$ risk factors. The aggregate risk factors include the short rate r_t, the default intensity λ_t ,and the K demand factors $\beta_{k,t}$ for $k = 1, \dots K$. The $(K+2)\times 1$ vector $s_t \doteq (r_t, \lambda_t, \beta_{1,t}, \dots, \beta_{K,t})^T$ follows the process

$$ds_t = -\Gamma(s_t - \overline{s})dt + \Sigma dB_t \tag{6}$$

where $\overline{s}$ is a $(K+2)\times 1$ vector of long-term averages and $dB_t = (dB_{r,t}, dB_{\lambda,t}, dB_{\beta,1,t}, \dots, dB_{\beta,K,t})^T$ is a $(K+2)\times 1$ vector of independent Brownian motions. The matrix Γ controls the instantaneous drifts of the process. Σ describes the instantaneous covariance of the shocks, whereas the instantaneous covariance matrix of ds_t is $\Sigma\Sigma^T$. Equation (6) nests the special case in which Γ and Σ are diagonal and the risk factors are independent.

In order to simplify the quantitative analysis, Section 4 imposes the restriction that Γ and Σ are diagonal. Nevertheless, a more sensible approach would be to model the relation between habitat demand and fundamentals in reduced-form by assuming that (i) the drift of the demand factors depends on λ_t and that (ii) shocks to default intensity and the short rate are correlated to shocks to the demand factors. On the one hand, demand shocks that are correlated to default intensity allow for a more realistic dependence of habitat demand on credit quality. On the other hand, allowing demand factors and the short rate to be correlated seems enough to flip the sign of the relation between term premia and the level of r_t, as discussed in Kekre et al. (2022). While the equilibrium solution and the key results hold under general dynamics, the specification of Γ and Σ is central in determining the direction in which a shock to each of state variables affects risk premia. Since this has a significant impact on the the calibration and the quantitative exercise in Section 4, I leave this to the future.

Market clearing Bond markets clear at each maturity τ

$$Z_t^{(\tau)} + X_t^{(\tau)} = 0 \tag{7}$$

at each point in time. The equilibrium is a collection of prices and quantities $\{P_t^{(\tau)}, X_t^{(\tau)}\}_{\tau \in (0,\infty)}$ such that arbitrageurs' are optimizing and markets clear for all maturities τ.

2.3 Analytical Insights

For tractability, I present analytical results for the case without demand risk. The only aggregate risk factors are the short rate r_t and the default intensity λ_t. Hence, s_t collapses to $s_t = (r_t, \lambda_t)^T$. Because the equilibrium solution is very close to Vayanos and Vila (2021), I emphasize the novel asset pricing implications for defaultable bonds and relegate details and derivations to Appendix A.

2.3.1 Equilibrium Characterization

To analytically characterize the segmentation equilibrium, I make the simplifying assumption that the matrices Γ and Σ are diagonal, which implies that r_t and λ_t are independent. This assumption is solely for tractability purposes, and the results generalize to correlated shocks, as discussed in Section 2.4. Following Vayanos and Vila (2021), I conjecture that equilibrium yields are affine functions of the two risk factors. Under this conjecture, there exist three functions $(A_r(\tau), A_\lambda(\tau), C(\tau))$ that only depend on the maturity τ such that bond prices are exponentially-affine in the risk factors such that

$$P_t^{(\tau)} = e^{-[A_r(\tau) r_t + A_\lambda(\tau)\lambda_t + C(\tau)]} \tag{8}$$

In equilibrium, yields are linear functions of the aggregate risk factors r_t and λ_t, and the equilibrium solution pins down the unknown functions $(A_r(\tau), A_\lambda(\tau), C(\tau))$. The representation in Equation (8) is analogous to the affine term structure models discussed in Lando (1998) and Duffie and Singleton (2003). The standard arbitrage-free setting of term structure of defaultable bonds assumes the existence of an equivalent martingale measure $\mathbb{Q}$, thereby taking a stance on the $\mathbb{Q}$-dynamics of the risk factors (Duffie & Singleton, 1999). However, I do not impose any structure on the state dynamics under the risk-neutral measure, but I rather derive them as an equilibrium outcome. Lemma (1) characterises the arbitrageurs' first-order condition implied by (8) .

Lemma 1 (Arbitageurs' First-order Condition)**.** *Under conjecture (8), the arbitrageurs' first-order condition is*

$$\mu_t^{(\tau)} - r_t = \lambda_t - A_r(\tau)\sigma_r \cdot \eta_{r,t} - A_\lambda(\tau)\sigma_\lambda \cdot \eta_{\lambda,t} \tag{9}$$

where the risk prices $\eta_{s,t}$ for $s \in \{r, \lambda\}$ are given by

$$\eta_{s,t} \doteq -a\sigma_s \left(\int_0^\infty X_t^{(\tau)} A_s(\tau) d\tau \right) \tag{10}$$

The left-hand side of Equation (9) is the expected excess return on bonds with maturity τ, and it reflects the compensation that arbitrageurs require to hold risky assets in equilibrium. The first term on the right-hand side of (9) is the compensation for holding defaultable bonds. Since defaults are idiosyncratic, there is no risk correction for the event of default. As a result, λ_t enters the first-order condition with a constant coefficient of one, which clearly does not depend on risk aversion. This is due to the idiosyncratic nature of the default events, which implies that the λ_t is the same under the physical and equivalent martingale measures.

The other two terms on the right-hand side of (9) are the product of risk prices and the asset sensitivity to the corresponding aggregate risk factor. The expression (10) a reveals that both risk prices increase with the aggregate net positions in the defaultable bonds held by arbitrageurs. Lemma (9) shows that, despite r_t and λ_t being independent, the risk prices of short rate risk and default intensity risk are connected. An increase in the net positions held by the arbitrageurs causes an increase in the market prices of risk associated to both the short rate and default intensity.

The market clearing condition (7) implies that the arbitrageurs' holdings are

$$X_t^{(\tau)} = -Z_t^{(\tau)} = \theta_0(\tau) - \alpha(\tau)\left[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)\right] \tag{11}$$

Substituting equation (11) into the arbitrageurs' first-order condition (9) and matching coefficients on r_t and λ_t produces a system of two first-order linear ordinary differential equations in the two unknown functions $A_r(\tau)$ and $A_\lambda(\tau)$. The initial conditions are $A_r(0) = A_\lambda(0) = 0$.

$$A_r'(\tau) + A_r(\tau)\kappa_r - 1 = -a\sigma_r^2 A_r(\tau)\left(\int_0^\infty \alpha(\tau)A_r(\tau)^2 d\tau\right) - a\sigma_\lambda^2 A_\lambda(\tau)\left(\int_0^\infty \alpha(\tau)A_r(\tau)A_\lambda(\tau)d\tau\right)$$
$$A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda - 1 = -a\sigma_r^2 A_r(\tau)\left(\int_0^\infty \alpha(\tau)A_r(\tau)A_\lambda(\tau)d\tau\right) - a\sigma_\lambda^2 A_\lambda(\tau)\left(\int_0^\infty \alpha(\tau)A_\lambda(\tau)^2 d\tau\right)$$

Proposition (1) applies the solution approach in Vayanos and Vila (2021) to characterize $A_r(\tau)$ and $A_\lambda(\tau)$, thereby determining the equilibrium in the one sector model.

Proposition 1 (Equilibrium in the One Sector Model)**.** *Given the initial conditions $A_r(0) = A_\lambda(0) = 0$, the function $A(\tau) = (A_r(\tau), A_\lambda(\tau))^T$ is given by*

$$A(\tau) = \psi_1\left(\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right) + \psi_2\left(\frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right) \tag{12}$$

where v_k is the k-th eigenvectors of the matrix M defined by

$$M \doteq \left[\Gamma^T + a\int_0^\infty \alpha(\tau)A(\tau)A(\tau)^T d\tau\Sigma\Sigma^T\right]$$

and ψ_k are constant vectors such that $\psi_k = \mathsf{u}_k\xi_i$, where u_k is the eigenvector corresponding to v_k and ξ_i is the ith component of $\xi \doteq P^{-1}\mathbb{1}$, where $P \doteq [\mathsf{u}_1, \mathsf{u}_2]$.

Matching the terms independent of r_t and λ_t produces another ODE for the function $C(\tau)$

$$C'(\tau) = A(\tau)^T\left[\Gamma\overline{s} + a\Sigma\Sigma^T\int_0^\infty \left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right]A(\tau)d\tau\right] - \frac{1}{2}A(\tau)^T\Sigma\Sigma^T A(\tau)$$

Taking the solution (12) for $A(\tau)$ as given, the function $C(\tau)$ is

$$C(\tau) = \left(\int_0^\tau A(u)^T du\right)\chi - \frac{1}{2}\int_0^\tau A(u)^T \Sigma\Sigma^T A(u)du \tag{13}$$

where χ is a vector of constants such that

$$\chi = \begin{bmatrix}\chi_r \\ \chi_\lambda\end{bmatrix} \doteq \Gamma\overline{s} + a\Sigma\Sigma^T \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right] A(\tau)d\tau \tag{14}$$

Given Equations (12) and (13), I turn to the analytical properties of the simplified environment.

2.3.2 Pricing of Defaultable Securities

Proposition (1) implies that, in equilibrium, the price of a well-diversified portfolio of defaultable bonds is an affine function of the aggregate risk factors. Proposition (2) shows that the exponentially-affine function (8) also describes the price of an individual defaultable bond.

Proposition 2 (Equivalence with Risk-neutral Valuation)**.** *Let $\mathbb{Q}$ denote the risk-neutral measure and τ_D the (stopping) time of default of an individual corporate bond. Then*

$$P_t^{(\tau)} = e^{-[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)]} \overset{!}{=} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau}(r_u+\lambda_u)du}\right] \tag{15}$$

Proposition (1) delivers an equilibrium justification to the term structure models of defaultable bonds (see e.g. Duffie and Singleton (2003, 1999)). The conventional approach is to price defaultable bonds by directly modeling the $\mathbb{Q}$-dynamics of the risk factors, taking the existence of an equivalent martingale measure $\mathbb{Q}$ as given. In contrast, Proposition (2) takes the state dynamics under the physical measure as given, and describes the $\mathbb{Q}$-dynamics as an endogenous outcome.

2.3.3 Risk-neutral Dynamics and Risk Prices

The equilibrium in Proposition (1) and the discussion around Proposition (2) provide a complete characterization of the dynamics of the state variables under the risk-neutral measure $\mathbb{Q}$. It turns out that the $\mathbb{Q}$-dynamics maintain a stationary mean-reverting structure as Equation (6). However, the drift matrix Γ and the vector of long term averages $\overline{s}$ are different under $\mathbb{Q}$. The risk-neutral dynamics are given by

$$ds_t = -M^T\left(s_t - \overline{s}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}} \tag{16}$$

where $\overline{s}^{\mathbb{Q}}$ is implicitly defined by $M^T\overline{s}^{\mathbb{Q}} = \chi$. The matrix M is given by

$$M \doteq \left[\Gamma^T + a\int_0^\infty \alpha(\tau)A(\tau)A(\tau)^T d\tau\Sigma\Sigma^T\right]$$

If arbitrageurs are risk-neutral ($a = 0$) or habitat-investors are price-inelastic ($\alpha(\tau) = 0$), then $M^T = \Gamma$ and $\overline{s}^{\mathbb{Q}} = \bar{s}$. One the one hand, if $a = 0$, arbitrageurs will require no compensation to hold risky assets. Hence, $A(\tau)$ and $C(\tau)$ must be set such that $\mu_t^{(\tau)} - r_t = 0$. On the other hand, $\alpha(\tau) \neq 0$ generates dependence between the arbitrageurs' pricing kernel and the state variables. Equation (16) shows that the state dynamics follow a multivariate Ornstein-Uhlenbeck under both measures, although with a different drift and long-term average.

The dynamics described in (16) impose economic restrictions on the risk-neutral dynamics of the short term rate r_t and default intensity λ_t. Provided that arbitrageurs are risk-averse ($a > 0$) and that habitat investors are price elastic ($\alpha(\tau) \neq 0$), the processes for r_t and λ_t under $\mathbb{Q}$ will not generally be independent. The off-diagonal elements of M are

$$M_{12} \doteq a\sigma_\lambda^2 \int_0^\infty \alpha(\tau)A_r(\tau)A_\lambda(\tau)d\tau \qquad : \qquad M_{21} \doteq a\sigma_r^2 \int_0^\infty \alpha(\tau)A_r(\tau)A_\lambda(\tau)d\tau$$

The entries M_{12} and M_{21} are, in general, different than zero. An immediate consequence is that, under the equivalent martingale measure, the instantaneous drift of the short rate process r_t depends on λ_t, and the same holds for the instantaneous drift of λ_t, even when Γ is assumed to be diagonal. When Γ is not diagonal, and $A_r(\tau) > 0$ and $A_\lambda(\tau) > 0$ for all τ, the loading of dr_t on λ_t is higher, that is $\kappa_{r\lambda}^{\mathbb{Q}} > \kappa_{r\lambda}$, and the same applies to $d\lambda_t$. Therefore, default intensity risk and interest rate risk will become more positively correlated (unconditionally) under the equivalent martingale measure.

Standard valuation methods for defaultable bonds are flexible enough to accommodate risk-neutral dependence between r_t and λ_t. However, assumptions on the $\mathbb{Q}$-dynamics usually rely on $\mathbb{P}$-measure logic. For example, defaults are more likely to occur in bad times when interest rates tend to be lower. Hence, the dependence between r_t and λ_t is often exogenously specified, e.g. through common loadings on a latent business cycle factors (Duffie & Singleton, 2003). Equation (16) describes the implication of an additional channel that endogenously generates (or strengthens) dependence between risk factors under the equivalent martingale measure. In this framework, the dependence between r_t and λ_t endogenously arises because the arbitrageurs are holding non-zero net positions on the corporate bonds.

The dependence of instantaneous drifts on both state variables, even when Γ and Σ are diagonal, implies that risk premia are state-dependent. However, the variation in risk premia cannot come from the quantity of risk, as bond returns are homoscedastic in equilibrium. Rather, movements in risk premia must come from variation in the market prices of risk. In this model, risk prices vary because the arbitrageurs must accommodate stochastic demand from price elastic habitat investors. Because it must be optimal for the arbitrageurs to meet habitat demand, the arbitrageurs' portfolio pins down the pricing kernel. Fluctuations in arbitrageurs' positions entail that their exposure to aggregate risk factors varies over time, altering the covariance between the stochastic discount factors and bond returns. To substantiate this claim, I rewrite the arbitrageurs' first-order condition (9) as

$$\mathbb{E}_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right] - r_t = \lambda_t - \mathbb{C}\mathrm{ov}_t\left(\frac{d\pi_t}{\pi_t}, \frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right)$$

where π_t is the marginal utility process (Duffie, 2001). As in many asset pricing models, expected excess returns are proportional to the covariance between returns and the stochastic discount factor.

Applying Itô's Lemma to π_t with the dynamics of dW_t gives

$$\frac{d\pi_t}{\pi_t} = \mu_{\pi,t}dt - a\left[\int_0^\infty x_t^{(\tau)}\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}d\tau + \left(W_t - \int_0^\infty x_t^{(\tau)}d\tau\right)r_t dt\right]$$

where $\mu_{\pi,t}$ is the drift. The covariance between bond returns and the stochastic discount factor is then

$$-\mathbb{C}\text{ov}_t\left(\frac{d\pi_t}{\pi_t}, \frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right) = \sigma_r A_r(\tau)\cdot a\sigma_r\left(\int_0^\infty x_t^{(\tau)}A_r(\tau)d\tau\right) + \sigma_\lambda A_\lambda(\tau)\cdot a\sigma_\lambda\left(\int_0^\infty x_t^{(\tau)}A_\lambda(\tau)d\tau\right)$$

It follows that risk prices are proportional to the aggregate positions held by the arbitrageurs. Analogously to Vayanos and Vila (2021), the risk price on factor $s \in \{r, \lambda\}$ is

$$\eta_{s,t} = -a\sigma_s\left(\int_0^\infty x_t^{(\tau)}A_s(\tau)d\tau\right)$$

variation in risk premia come from the arbitrageurs' exposure to aggregate risk factors. Proposition (3) summarizes this discussion by describing the correction that maps the Brownian motion under the physical measure to the Brownian motion under $\mathbb{Q}$, i.e. $dB_t^{\mathbb{Q}} = dB_t + \boldsymbol{\eta_t}dt$.

Proposition 3 (Time-varying Risk Premia). *The mapping between dB_t and $dB_t^{\mathbb{Q}}$ is state-dependent and given by*

$$dB_t^{\mathbb{Q}} = dB_t - a\Sigma^T\int_0^\infty A(\tau)\left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right]d\tau + a\Sigma^T\left(\int_0^\infty \alpha(\tau)A(\tau)A(\tau)^T d\tau\right)s_t$$

Proof. The dynamics of s_t under the physical measure $\mathbb{P}$ are

$$ds_t = -\Gamma\left(s_t - \bar{s}\right)dt + \Sigma dB_t^{\mathbb{Q}}$$

Using $dB_t^{\mathbb{Q}} = dB_t + \boldsymbol{\eta_t}dt$ and substituting habitat demand into the risk prices gives

$$\begin{aligned}
ds_t &= -\Gamma\left(s_t - \bar{s}\right)dt + \Sigma\left(dB_t^{\mathbb{Q}} - \boldsymbol{\eta_t}dt\right) \\
&= -\left[\Gamma + a\Sigma\Sigma^T\left(\int_0^\infty \alpha(\tau)A(\tau)A(\tau)^T d\tau\right)\right]s_t dt \\
&\qquad + \left(\Gamma\bar{s} + a\Sigma\Sigma^T\int_0^\infty A(\tau)\left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right]d\tau dt\right)dt + \Sigma dB_t^{\mathbb{Q}} \\
&= -M^T s_t dt + \chi dt + \Sigma dB_t^{\mathbb{Q}} = -M^T\left(s_t - \bar{s}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}}
\end{aligned}$$

where $\chi = M^T\bar{s}^{\mathbb{Q}}$. The last expression describes the $\mathbb{Q}$-dynamics of s_t, completing the proof. ∎

The economic content of Proposition (3) is that the risk price of default intensity λ_t depends on the level of the short term rate r_t. The same holds for the price of short term rate risk. As a result, Proposition (3) implies that the market price of credit risk changes over time even when both the credit quality of the corporate issuers and the arbitrageurs' risk aversion are constant over time. The response of $Z_t^{j,(\tau)}$ to both λ_t and r_t through the price $P_t^{(\tau)}$ endogenously induces $\mathbb{Q}$-measure dependence between the aggregate risk factors, even in the special case that they are independent under $\mathbb{P}$.

2.4 General Case with Demand Shocks

I next present the general solution with K demand factors. The demand intercept is given in equation (3) and the dynamics follow (6). Accordingly, I conjecture that there exists $K+2$ functions $(A_r(\tau), A_\lambda(\tau), \{A_{\beta,k}(\tau)\}_{k=1}^{K}, C(\tau))$ that only depend on maturity τ such that

$$P_t^{(\tau)} = e^{-\left[A(\tau)^T s_t + C(\tau)\right]}$$

Applying Itô's Lemma, the instantaneous drift is given by

$$\mu_t^{(\tau)} \doteq A'(\tau)^T s_t + C'(\tau) + A(\tau)^T \Gamma\left(s_t - \overline{s}\right) - \frac{1}{2} A(\tau)^T \Sigma\Sigma^T A(\tau)$$

As a result, the arbitrageurs' first-order condition becomes

$$\mu_t^{(\tau)} - r_t = \lambda_t + aA(\tau)^T \Sigma\Sigma^T \int_0^\infty X_t^{(\tau)} A(\tau) d\tau \tag{17}$$

Equation (17) generalizes equation (9) by introducing K additional demand factors. The interpretation is analogous. Equation (17) balances the risk and the return required to hold risky corporate bonds. Let $\Theta(\tau) = (0, 0, \theta_1(\tau), \ldots, \theta_K(\tau))$ be a $1 \times (K+2)$ vector collecting the loadings of habitat demand on the demand factors. Imposing market clearing yields

$$x_t^{(\tau)} = -Z_t^{(\tau)} = \theta_0(\tau) + \Theta(\tau)s_t - \alpha(\tau)A(\tau)^T s_t - \alpha(\tau)C(\tau)$$

Substituting the market clearing condition into the arbitrageurs' first-order condition gives

$$\begin{aligned} A'(\tau)^T s_t + C'(\tau) + A(\tau)^T \Gamma\left(s_t - \overline{s}\right) + \frac{1}{2} A(\tau)^T \Sigma\Sigma^T A(\tau) - e_1^T s_t \\ = e_2^T s_t + aA(\tau)^T \Sigma\Sigma^T \int_0^\infty \left[\theta_0(\tau) + \Theta(\tau)s_t - \alpha(\tau)A(\tau)^T s_t - \alpha(\tau)C(\tau)\right] A(\tau) d\tau \end{aligned}$$

where e_n is the $(K+2) \times 1$ standard basis vector. Setting the linear terms in s_t on both sides to be equal gives the system of $K+2$ first-order linear ODEs

$$A'(\tau) + MA(\tau) - \boldsymbol{b} = 0 \tag{18}$$

where $\boldsymbol{b} \doteq (1, 1, 0, \ldots, 0)^T$ and M is the $(K+2) \times (K+2)$ square matrix

$$M \doteq \Gamma^T - a \int_0^\infty \left[\Theta(\tau)^T A(\tau)^T - \alpha(\tau)A(\tau)A(\tau)^T\right] d\tau \Sigma\Sigma^T \tag{19}$$

Repeating the same with the terms that are independent of s_t gives

$$C'(\tau) - A(\tau)^T \Gamma \overline{s} + \frac{1}{2} A(\tau)^T \Sigma\Sigma^T A(\tau) = aA(\tau)^T \Sigma\Sigma^T \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right] A(\tau) d\tau \tag{20}$$

Equations (18) through (20) represent the defaultable bond counterparts to the general model in Vayanos and Vila (2021). Applying the same logic as in Proposition (1) and imposing the initial conditions $A(\tau) = \mathbf{0}$ and $C(\tau) = 0$ gives

$$A(\tau) = \sum_{k=1}^{K+2} \psi_k \left(\frac{1 - e^{-\nu_k \tau}}{\nu_k} \right)$$

where ν_k denote the $K+2$ eigenvectors of M and $\psi_k = \mathsf{u}_\mathsf{k} \xi_k$ relates to the corresponding eigenvectors through $\xi = P^{-1}\boldsymbol{b}$. Similarly, the function $C(\tau)$ is

$$C(\tau) = \left[\int_0^\tau A(u)^T du \right] \chi - \frac{1}{2} \int_0^\tau A(u)^T \Sigma \Sigma^T A(u) du$$

where

$$\chi \doteq \Gamma \overline{s} + \ a \Sigma \Sigma^T \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau) C(\tau) \right] A(\tau) d\tau$$

3 Cross-market Arbitrage

I extend the segmented equilibrium by allowing arbitrageurs to trade both corporate and Treasury bonds. Allowing arbitrageurs to trade two assets enriches the asset pricing implications by introducing diversification benefits from holding corporate and government bonds, as the latter might potentially hedge against default intensity risk. While habitat investors and arbitrageurs have the same preferences as in Section 2, a set of agents is now marginal in both the government and the corporate market.

3.1 Model

Let $j \in \{G, C\}$ index government and corporate bonds, respectively. Arbitrageurs have mean-variance preferences over instantaneous changes in wealth

$$\max_{\{X_t^{j,(\tau)}\}_{\tau \in \{0,\infty\}}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2} \mathbb{V}\mathrm{ar}_t(dW_t) \right] \tag{21}$$

With the inclusion of Treasuries, the arbitrageurs' budget constraint is

$$\begin{aligned} dW_t = \left(W_t - \int_0^T \sum_j X_t^{j,(\tau)} d\tau \right) r_t dt + \int_0^T X_t^{G,(\tau)} \frac{dP_t^{G,(\tau)}}{P_t^{G,(\tau)}} d\tau \\ + \int_0^T X_t^{C,(\tau)} \left(\frac{dP_t^{C,(\tau)}}{P_t^{C,(\tau)}} - \lambda_t dt \right) d\tau \end{aligned} \tag{22}$$

The first term of the budget constraint (22) corresponds to a position in the short rate, the second term to a position in Treasury bonds, and the third term to a position in corporate bonds. The case of segmented arbitrage along the lines of Gourinchas et al. (2022) can be obtained by assuming that arbitrageurs can only trade in one market (e.g. Treasury), nesting Section 2 as a special case.

Building on Vayanos and Vila (2021), I assume that habitat investors have preferences not only for specific maturities but also for specific asset classes. For tractability, I assume that preferences take an extreme form, where investors demand only the bond closest to their preferred characteristics.

$$Z_t^{(\tau),j} = -\alpha^j(\tau) \log P_t^{j,(\tau)} - \beta_t^{j,(\tau)} \tag{23}$$

$$\beta_t^{j,(\tau)} = \theta_0^j(\tau) + \sum_{k=1}^{K} \theta_k^j(\tau)\beta_{k,t} \tag{24}$$

This way of specifying habitat demand is flexible as it can accommodate shocks to specific asset classes. Suppose that the first and the second factors are pure Treasury and corporate bond demand shocks, respectively. Then, $\theta_1^C(\tau) = 0$ and $\theta_2^G(\tau) = 0$, so that $\beta_{1,t}$ ($\beta_{2,t}$) only affects habitat demand for Treasury (corporate) bonds. These demand factors can also be interpreted as shocks to the residual supply as in Greenwood and Vayanos (2014) and He, Nagel, and Song (2022), which is helpful to understand how Quantitative Easing affect credit spreads and the term structure of defaultable bonds.

3.1.1 Equilibrium without Arbitrageurs

In a segmentation equilibrium in which only habitat investors are trading in either government or corporate bonds, yields are solely determined by local demand shocks. Yields are given by

$$y_t^{j,(\tau)} = \frac{\beta_t^{j,(\tau)}}{\alpha^j(\tau)\tau} = \frac{\theta_0^j(\tau) + \sum_{k=1}^{K} \theta_k^j(\tau)\beta_{k,t}}{\alpha^j(\tau)\tau}$$

Absent arbitrageurs, yields for each maturity and asset class are determined solely by local demand shocks of habitat investors. This represents an extreme version of the preferred-habitat hypothesis of Modigliani and Sutch (1966), extended to multiple asset classes.

3.1.2 Equilibrium with Arbitrageurs

I conjecture that, in equilibrium, yields of both government and corporate bonds are affine functions of the state variables. In particular, there exists functions $(A_j(\tau)^T, C_j(\tau))$ for $j \in \{G, C\}$ such that

$$P_t^{j,(\tau)} = e^{-\left[A_j(\tau)^T s_t + C_j(\tau)\right]} \tag{25}$$

The key component of conjecture (25) is that the Treasury yields also load on default intensity λ_t. Under conjecture (25), instantaneous returns are

$$\begin{aligned}
\frac{dP_t^{j,(\tau)}}{P_t^{j,(\tau)}} &= \mu_t^{j,(\tau)} dt - A_j(\tau)^T \Sigma dB_t \\
\mu_t^{j,(\tau)} &= A_j'(\tau)^T s_t + C_j'(\tau) + A_j(\tau)^T \Gamma(s_t - \bar{s}) + \frac{1}{2} A_j(\tau)^T \Sigma\Sigma^T A_j(\tau)
\end{aligned}$$

Substituting this into the budget constraints gives the objective

$$\max_{\{X_t^{j,(\tau)}\}_{\tau\in(0,\infty)}} \left(W_t - \int_0^\infty \sum_j X_t^{j,(\tau)}\right) r_t dt + \int_0^\infty \mu_t^{(\tau),G} X_t^{(\tau),G} d\tau dt + \int_0^\infty \left(\mu_t^{(\tau),C} - \lambda_t\right) X_t^{(\tau),C} d\tau dt$$
$$- \frac{a}{2}\left[\int_0^\infty \sum_j X_t^{j,(\tau)} A_j(\tau)^T d\tau\right] \Sigma\Sigma^T \left[\int_0^\infty \sum_j X_t^{j,(\tau)} A_j(\tau) d\tau\right] dt$$

Pointwise maximization with respect to $X_t^{j,(\tau)}$ produces the set of first-order conditions

$$\mu_t^{G,(\tau)} - r_t = aA_G(\tau)^T \Sigma\Sigma^T \left[\sum_j \int_0^\infty X_t^{j,(\tau)} A_j(\tau) d\tau\right] \tag{26}$$

$$\mu_t^{C,(\tau)} - r_t = \lambda_t + aA_C(\tau)^T \Sigma\Sigma^T \left[\sum_j \int_0^\infty X_t^{j,(\tau)} A_j(\tau) d\tau\right] \tag{27}$$

The first-order conditions pins down the arbitrageurs' pricing kernel in a no-arbitrage setting. Quoting Vayanos and Vila (2021), no-arbitrage in continuous time requires that there exist prices specific to each risk factor and common across assets, such that the expected return of any zero cost portfolio is equal to the sum across factors of the portfolio's sensitivity to each factor times the factor's price.

The first-order conditions (26) and (27) reveal that bond risk premia depends on the arbitrageurs' net positions in both the Treasury and the corporate market. As a result, a demand or a supply shock in the corporate bond propagate to Treasury yields, thereby affecting their excess return, and vice versa. The only difference between (26) and (27) is that the λ_t only shows up in the first-order condition for corporate bonds. The quantity $\left[\sum_j \int_0^\infty X_t^{j,(\tau)} A_j(\tau) d\tau\right]$ can be interpreted as a measure of intermediaries' inventories in the spirit of He, Khorrami, and Song (2022). Although, expected returns load differently on this common factor through asset-specific sensitivities $A_j(\tau)$, the first-order conditions suggest that a strong principal component is likely to capture most of the variation in credit spreads over and above λ_t and bond risk premia. This observation is consistent with Friewald and Nagler (2019) and He, Khorrami, and Song (2022).

Define the $1 \times (K+2)$ vector $\Theta_j(\tau) = (0, 0, \theta_1^j(\tau), \ldots, \theta_K^j(\tau))$ so that $\beta_t^{j,(\tau)} = \theta_0^j(\tau) + \Theta^j(\tau)s_t$. The market clearing conditions for $j \in \{G, C\}$ are

$$x_t^{j,(\tau)} = -Z_t^{(\tau),j} = \theta_0^j(\tau) + \Theta^j(\tau)s_t - \alpha^j(\tau)\left[A_j^T(\tau)s_t + C_j(\tau)\right]$$

Plugging the market clearing conditions back into the arbitrageurs' first-order conditions and setting the terms linear in s_t to be equal delivers two systems of $K+2$ linear first-order ODEs

$$A'_G(\tau) + MA_G(\tau) - e_1 = 0 \tag{28}$$

$$A'_C(\tau) + MA_C(\tau) - e_1 - e_2 = 0 \tag{29}$$

where the $(K+2)\times(K+2)$ matrix M is the same in both systems and it is given by

$$M \doteq \Gamma^T - a\sum_j \int_0^\infty \Theta_j^T(\tau)A_j(\tau)^T - \alpha^j(\tau)A_j(\tau)A_j(\tau)^T d\tau\Sigma\Sigma^T \tag{30}$$

The matrix M is the same for both $A_G(\tau)$ and $A_C(\tau)$, and it is because M characterizes the $\mathbb{Q}$-dynamics of the risk factors. The difference between this Section and Section 2.4 is that here the risk corrections depend on the combined holdings of Treasury and corporate bonds. I solve (28) and (29) under the boundary conditions that $A_G(0) = A_C(0) = 0$. The next result characterizes the equilibrium in the two asset model, which gives the two asset counterpart of Proposition (1).

Proposition 4 (Equilibrium in the Two Sector Model)**.** *Given the initial conditions $A_G(0) = A_C(0) = \mathbf{0}$, the $(K+2)$ dimensional functions $A_G(\tau) = (A_{G,r}(\tau), A_{G,\lambda}(\tau), \{A_{G,\beta_k}(\tau)\}_{k=1}^K)^T$ and $A_C(\tau) = (A_{C,r}(\tau), A_{C,\lambda}(\tau), A_{C,\beta_1}(\tau), \dots, \{A_{G,\beta_k}(\tau)\}_{k=1}^K)^T$ are given by*

$$A_G(\tau) = \sum_{k=1}^{K+2} \psi_k^G \left(\frac{1-e^{-\mathsf{v}_k\tau}}{\mathsf{v}_k}\right) \tag{31}$$

$$A_C(\tau) = \sum_{k=1}^{K+2} \psi_k^C \left(\frac{1-e^{-\mathsf{v}_k\tau}}{\mathsf{v}_k}\right) \tag{32}$$

where v_k are the eigenvectors of the matrix M defined in (30). Furthermore, ψ_k^j are constant vectors such that $\psi_k^j = \mathsf{u}_k\xi_i^j$, where u_k is the eigenvector corresponding to v_k and ξ_i^j is the asset-specific ith component of $\xi^j \doteq P^{-1}\boldsymbol{b}^j$, where $P \doteq [\mathsf{u}_1, \mathsf{u}_2]$, $\boldsymbol{b}^G = e_1$ and $\boldsymbol{b}^C = e_1 + e_2$.

What makes the system hard to solve is that the entries of M are functions of $A_j(\tau)$, which, as in Lemma (2), depends on the eigenvectors and eigenvalues of M itself. Furthermore, the matrix M has to make sure that both (31) and (32) hold simultaneously. I numerically solve (28) and (29) as in Vayanos and Vila (2021). Then, taking $A_G(\tau)$ and $A_C(\tau)$, I collect the terms independent of s_t such that

$$C_G(\tau) = \left[\int_0^\tau A_G(u)^T du\right]\chi - \frac{1}{2}\int_0^\tau A_G^T(u)\Sigma\Sigma^T A_G(u)du \tag{33}$$

$$C_C(\tau) = \left[\int_0^\tau A_C(u)^T du\right]\chi - \frac{1}{2}\int_0^\tau A_C^T(u)\Sigma\Sigma^T A_C(u)du \tag{34}$$

where χ is a $K+2$ vector of constants such that

$$\chi \doteq \Gamma\overline{s} + a\Sigma\Sigma^T\left[\sum_j \int_0^\infty A_j(\tau)\left(\theta_0^j(\tau) - \alpha^j(\tau)C_j(\tau)\right)d\tau\right] \tag{35}$$

To solve for the vector of constants χ, I substitute (33) and (34) into (35) and derive a system of $K+2$ linear equations in the $K+2$ unknown entries of χ. The functions (31) through (35) characterize the term structures of government and defaultable bonds. This system produces an equilibrium analogous to the two country framework in Gourinchas et al. (2022). Propositions (2) and (3) also apply to the two sector case, although the dynamics are harder to characterize. The matrix M and the vector χ have the same interpretation as in the segmented arbitrage case, and they fully determine the dynamics of the risk factors under the equivalent martingale measure $\mathbb{Q}$.

3.2 Credit Spreads

Proposition (4) delivers an expression that informs how monetary policy and demand shocks affect credit spreads. In equilibrium, yields for $j \in \{G, C\}$ are given by

$$y_t^{j,(\tau)} = \frac{1}{\tau}\left[A_j(\tau)^T s_t + C_j(\tau)\right]$$

The credit spread $\mathcal{S}_t^{(\tau)}$ at maturity τ is defined as the yield on corporate bonds minus the yield on Treasury bonds of the same maturity, that is

$$\mathcal{S}_t^{(\tau)} \doteq y_t^{C,(\tau)} - y_t^{G,(\tau)} = \frac{1}{\tau}\left[A_{\mathcal{S}}(\tau)^T s_t + C_{\mathcal{S}}(\tau)\right] \tag{36}$$

where $A_{\mathcal{S}} \doteq A_C(\tau) - A_G(\tau)$ and $C_{\mathcal{S}} \doteq C_C(\tau) - C_G(\tau)$. Let $\delta_t \doteq (\beta_{1,t}, \ldots, \beta_{k,t})$ be vector of demand factors only. Credit spreads can then be written as

$$\mathcal{S}_t^{(\tau)} = \frac{1}{\tau}\left[A_{\mathcal{S},r}(\tau) r_t + A_{\mathcal{S},\lambda}(\tau)\lambda_t + A_{\mathcal{S},\delta}(\tau)\delta_t + C_{\mathcal{S}}(\tau)\right] \tag{37}$$

where $A_{\mathcal{S},\delta}(\tau) \doteq (A_{C,\beta_1}, \ldots, A_{C,\beta_k})^T - (A_{G,\beta_1}, \ldots, A_{G,\beta_k})^T$. Equation (37) reveals that credit spreads are affine functions of the aggregate risk factor s_t. An immediate consequence is that credit spreads not only depend on the level of credit risk, but also on the level of the short rate r_t. To the extent that $A_{\mathcal{S},r}(\tau) \neq 0$, $A_{\mathcal{S},\lambda}(\tau) \neq 0$, and $A_{\mathcal{S},\delta}(\tau) \neq 0$, changes in credit spreads are driven by either (i) fluctuations in the credit quality of the corporate sector $d\lambda_t$, (ii) movements of the short term rate r_t, and (iii) local or global demand effects $d\delta_t$.

Changes in either the level of the short rate r_t or default intensity λ_t move credit spreads. It is natural to expect that changes in default intensity are positively related to yield spreads, such that

$$\frac{\partial \mathcal{S}_t^{(\tau)}}{\partial \lambda_t} = \frac{A_{\mathcal{S},\lambda}(\tau)}{\tau} > 0$$

where $A_{G,\lambda}(\tau) < 0$ and $A_{C,\lambda}(\tau) > 0$. The novel implication is that a deterioration in credit quality widens credit spreads by both (i) raising the yield on corporate bonds $A_{C,\lambda}(\tau) > 0$ and (ii) lowering the yield on Treasuries $A_{C,\lambda}(\tau) < 0$. This mechanism emerges as a combination of two effects. First, an increase in λ_t implies a deterioration in the economic fundamentals, which makes corporate bonds less attractive to arbitrageurs[1]. Second, an increase in λ_t alters habitat demand for corporate bonds, impacting the market prices of risk on the aggregate risk factors.

3.2.1 Credit Spreads and Short Term Rates

The sign of the relation between short rates and credit spreads is ambiguous. The theoretical insight that the level of the short rate causes credit spreads to move is not novel. Among others, Longstaff

[1] The specification of habitat demand as a function of prices obfuscates the effect of arbitrageurs' portfolio rebalancing on credit spreads. Provided that $A_{C,\lambda}(\tau) < 0$, an increase in λ_t lowers the equilibrium price of corporate bonds. As a result, habitat investors demand more, so that the net positions held by the arbitrageurs decline in equilibrium. Because of this, the market prices of risk on the aggregate risk factors also declines. I plan to address this in future iteration of the paper by specifying habitat demand as a function of fundamentals in the spirit of (Kyle & Xiong, 2001). Appendix B provides a tentative microfoundation of habitat demand curves.

and Schwartz (1995) present a structural model of default in which credit risk and interest rates turn out to be negatively correlated. The reason is that a higher r_t increase the risk-neutral drift of the firm value process. Since a higher drift reduces the probability of default, a higher short rate r_t lowers credit spreads. Furthermore, Duffee (1999), Collin-Dufresne et al. (2001), and Gertler and Karadi (2015) document a robust empirical relation between credit spreads and interest rates.

In equation (37), a unit change in r_t moves credit spreads by $A_{\mathcal{S},r}(\tau)$. The direction depends on the relative sensitivity of corporate bonds to Treasuries with respect to the policy rate, that is

$$\frac{\partial \mathcal{S}_t^{(\tau)}}{\partial r_t} = \frac{A_{\mathcal{S},r}(\tau)}{\tau} = \frac{A_{C,r}(\tau)}{\tau} - \frac{A_{G,r}(\tau)}{\tau}$$

In this model, however, the mechanism driving the negative correlation between credit spreads and the short term rate is different, and it does not rely on any structural model of credit valuation. Assuming that $A_{G,r}(\tau) > A_{C,r}(\tau) > 0$, in equilibrium, an increase in r_t raises yields throughout the term structure. Higher yields induce habitat investors to demand more bonds at all maturities. Therefore, the residual supply that arbitrageurs have to absorb shrinks, reducing their net exposure to aggregate risk factors. Since risk premia are proportional to the arbitrageurs' net positions, as in (26) and (27), the risk prices on interest rate and credit risk decline. A decline in the market price of credit risk will disproportionately affect defaultable bonds, therefore lowering credit spreads. Consistent with this reasoning, the calibrated model in Section 4 implies that $A_{\mathcal{S},r}(\tau) > 0$, although the general conditions under which $A_{G,r}(\tau) > A_{C,r}(\tau) > 0$ remains to be determined.

3.2.2 Determinants of Credit Spread Changes

The credit spread puzzle refers to the observation that only a relatively small fraction of the changes in credit spreads can be explained by standard credit risk factors (Collin-Dufresne et al., 2001; Friewald & Nagler, 2019; He, Khorrami, & Song, 2022). Including explanatory variables motivated by structural models of credit risk, Collin-Dufresne et al. (2001) run regressions of the form

$$\Delta CS_t^i = \alpha + \beta_1^i \Delta x_t^i + \beta_2^i \Delta z_t + \epsilon_t^i \tag{CGM}$$

where x_t^i is a vector of firm-specific controls (e.g. leverage), whereas z_t includes aggregate variables such as the short term rate and the slope of the yield curve. The next proposition presents the model counterpart to regression (CGM).

Proposition 5 (Determinants of Changes in Credit Spreads). *Fixing the maturity τ, instantaneous changes in credit spreads $d\mathcal{S}_t^{(\tau)}$ are given by*

$$d\mathcal{S}_t^{(\tau)} = \frac{1}{\tau} A_{\mathcal{S}}(\tau) ds_t = \frac{1}{\tau}\left[A_{\mathcal{S},r}(\tau) dr_t + A_{\mathcal{S},\lambda}(\tau) d\lambda_t + A_{\mathcal{S},\delta}(\tau) d\delta_t\right]$$

As a result, the (CGM)-regression implied by the model is

$$d\mathcal{S}_t^{(\tau)} = \beta_1^{(\tau)} dr_t + \beta_2^{(\tau)} d\lambda_t + \epsilon_t^{(\tau)}$$

where the slope coefficients and the residuals are given by

$$\beta_1^{(\tau)} \doteq \frac{1}{\tau} A_{\mathcal{S},r}(\tau) \qquad : \qquad \beta_2^{(\tau)} \doteq \frac{1}{\tau} A_{\mathcal{S},\lambda}(\tau) \qquad : \qquad \epsilon_t^{(\tau)} \doteq \frac{1}{\tau} A_{\mathcal{S},\delta}(\tau) d\delta_t$$

Furthermore, in the special case that Σ *and* Γ *are diagonal*

$$\Delta \mathcal{S}_t^{(\tau)} = \beta_0^{(\tau)} + \beta_1^{(\tau)} r_t + \beta_2^{(\tau)} \lambda_t + \epsilon_t^{(\tau)}$$

where the coefficients and the residual are given by

$$\beta_0^{(\tau)} \doteq A_{\mathcal{S},r}(\tau)\kappa_r \overline{r} + A_{\mathcal{S},\lambda}(\tau)\kappa_\lambda \overline{\lambda} \qquad : \qquad \beta_1^{(\tau)} \doteq -A_{\mathcal{S},r}(\tau)\kappa_r \qquad : \qquad \beta_2^{(\tau)} \doteq -A_{\mathcal{S},\lambda}(\tau)\kappa_\lambda$$
$$\epsilon_t^{(\tau)} \doteq A_{\mathcal{S},r}(\tau)\sigma_r \Delta B_{r,t} + A_{\mathcal{S},\lambda}(\tau)\sigma_\lambda \Delta B_{\lambda,t} - \sum_{k=1}^{K} A_{\mathcal{S},\beta_k}(\tau) \left[\kappa_{\beta_k}\beta_{k,t} + \sigma_{\beta_k}\Delta B_{\beta,k,t}\right]$$

Building on Proposition (5), I interpret the CGM R-squared as the proportion of the variation in credit spreads not explained by the short rate and the default intensity. Formally,

$$1 - R^2_{\text{CGM}} \doteq \frac{\mathbb{V}\text{ar}\left(\epsilon_t^{(\tau)}\right)}{\mathbb{V}\text{ar}\left(d\mathcal{S}_t^{(\tau)}\right)} = \frac{A_{\mathcal{S},\delta}(\tau)\Sigma_\delta \Sigma_\delta^T A_{\mathcal{S},\delta}(\tau)^T}{A_{\mathcal{S}}(\tau)\Sigma\Sigma^T A_{\mathcal{S}}(\tau)^T} \tag{38}$$

Equation (38) indicates that the proportion of variation in changes of credit spreads not explained by r_t and λ_t, i.e. CGM-Residual, is larger when demand shocks are more volatile and persistent. To the extent that demand factors have a strong principal component, equation (38) can also rationalize the strong factor structure in residuals documented by Collin-Dufresne et al. (2001) and revisited by He, Khorrami, and Song (2022). In the special case that (Γ, Σ) are diagonal

$$1 - R^2_{\text{CGM}} = \frac{\sum_{k=1}^{K} A_{\mathcal{S},\beta_k}(\tau)^2 \frac{\sigma^2_{\beta_k}}{2\kappa_{\beta_k}}}{A_{\mathcal{S},r}(\tau)^2 \frac{\sigma_r^2}{2\kappa_r} + A_{\mathcal{S},\lambda}(\tau)^2 \frac{\sigma_\lambda^2}{2\kappa_\lambda} + \sum_{k=1}^{K} A_{\mathcal{S},\beta_k}(\tau)^2 \frac{\sigma^2_{\beta_k}}{2\kappa_{\beta_k}}}$$

Hence, if demand shocks are either (i) very persistent (low κ_{β_k}) or (ii) highly volatile (high κ_{β_k}), most of the variation in credit spreads will not be explained by changes in r_t and λ_t. That local and global demand shocks potentially explain a large percentage of the variation in credit spreads is consistent with Collin-Dufresne et al. (2001) and He, Khorrami, and Song (2022). To generate a low R^2_{CGM}, the model must eventually introduce asset-specific demand shocks that have asymmetric effects on corporate and government bonds. Otherwise, changes in credit spreads are mostly driven by λ_t since it is the only aggregate factor that impact Treasuries and corporate bonds differently. However, for parsimony, the calibration in Section 4 only includes a single demand factor.

3.2.3 Risk Premia and Demand Effects, and Correlations

The rbitrageurs' first-order conditions (26)– (27) imply that the vector of factor prices $\boldsymbol{\eta}_t$ is

$$\boldsymbol{\eta}_t = a\Sigma^T \left[\sum_j \int_0^\infty X_t^{j,(\tau)} A_j(\tau) d\tau\right] \tag{39}$$

A key implication of equation (39) is that the market risk prices of the aggregate risk factors depend on the habitat demand of both corporate and Treasury bonds. As a result, even asset specific demand shocks, such as Treasury-only QE, have global effects in the sense that they affect yields across all markets. The model also predicts that an increase in either (i) government debt or (ii) corporate debt supply affects the term structure of both asset classes as well as credit spreads. This expression is analogous to the global arbitrage equilibrium of Gourinchas et al. (2022) in a two country setting.

The market prices of risk are zero if arbitrageurs are risk neutral ($a = 0$) or if habitat demand is price-inelastic ($\alpha(\tau) = 0$). The expectation hypothesis obtains in the special case that $a = 0$. On the one hand, when arbitrageurs' are risk neutral, the instantaneous expected returns on all assets, in equilibrium, must be r_t. In the one asset benchmark of Section 2, this implies that $\mu_t^{(\tau)} - \lambda_t = r_t$. In the two asset case, this means that $\mu_t^{C,(\tau)} - \lambda_t = r_t$ and $\mu_t^{G,(\tau)} = r_t$. On the other hand, $\alpha(\tau) \neq 0$ introduces dependence between the pricing kernel and the risk factors. When habitat demand is price inelastic, all the time-variation in risk premia is driven by the demand factors as in Greenwood and Vayanos (2014).

Tn equilibrium, instantaneous returns are given by

$$\frac{dP_t^{j,(\tau)}}{P_t^{j,(\tau)}} = \mu_t^{j,(\tau)} dt - A_j(\tau)^T \Sigma dB_t$$

Finally, instantaneous covariance between Treasury and corporate bond returns is

$$\mathbb{C}\mathrm{ov}\left(\frac{dP_t^{G,(\tau)}}{P_t^{G,(\tau)}}, \frac{dP_t^{C,(\tau)}}{P_t^{C,(\tau)}}\right) = A_G^T(\tau)\Sigma\Sigma^T A_C(\tau)$$

3.3 Discussion of Modelling Assumptions

I discuss some limitations and drawbacks of the model in order to assess the degree to which key results presented in Section 2 and Section 3 are driven by specific assumptions.

3.3.1 Microfoundation of Habitat Demand

In both the segmentation framework of Section 2 and in the two asset framework of Section 3, I specify habitat demand $Z_t^{j,(\tau)}$ as a function of $P_t^{j,(\tau)}$ only. Yet, specification (23) has two main drawbacks.

Substitution Patterns On the one hand, I assume an extreme form of preferences for specific maturities and asset classes. In particular, it seems reasonable that habitat investors should either (i) be responsive to the prices of bonds with very close maturities $\tau \pm d\tau$ or (ii) not be responsive at all. Vayanos and Vila (2021) provide an optimizing microfoundation based on infinitely large risk aversion and max-min preferences. Nevertheless, it seems difficult to connect this nonstandard behavior to institutional investors such as insurance companies and pension funds. A more realistic microfoundation should take into account specific mandates or constraints faced by these investors, incorporating duration matching, benchmarking, or regulations, for example. To the extent that a better microfounded demand function makes habitat investors respond to price of other bonds, habitat investors would partially behave as arbitrageurs themselves, increasing arbitrage capacity in the economy. The key propositions are likely to still hold in a more general framework, but the analysis must then consider a continuum of portfolios (Vayanos & Vila, 2021), compromising on tractability.

Prices and Fundamentals On the other hand, specification (23) is not suitable for risky assets such as corporate bonds. While the assumption is shared with Droste et al. (2021) and Costain et al. (2022), the fact that habitat demand only responds to prices and not to economic fundamentals is not innocuous. A deterioration in the credit quality of the corporate issuer should, at least initially, induce habitat investors to sell. The subsequent decline in price makes them willing to buy a bit more (or to sell less). Yet, habitat investors do not respond to fundamental news, as they only react to prices. It follows that, to the extent that $A_{j,\lambda}(\tau) > 0$, a higher default intensity makes habitat investors demand more corporate bonds. To properly accommodate risky assets, habitat demand should incorporate a notion of risk or economic fundamentals in the spirit of Kyle and Xiong (2001). Appendix B describes a different specification of habitat demand as a function of both fundamentals s_t and expected returns $\mu_t^{(\tau)}$.

What mostly matters for the main results, however, is that habitat demand respond to the state variables, which creates dependence between the arbitrageurs' pricing kernel and the risk factors. This holds if habitat investors respond to prices, to expected returns, or to fundamentals directly (e.g. the supply shock in Greenwood and Vayanos (2014)). Hence, the key results in Propositions (2) and (3) still go through even with a more realistic demand function.

3.3.2 Homoscedastic Demand Shocks

When the matrices Σ and Γ are diagonal, the dynamics of the default intensity process are

$$d\lambda_t = \kappa_\lambda(\overline{\lambda} - \lambda_t)dt + \sigma_\lambda dB_{\lambda,t}$$

These dynamics have the drawback that default intensity might become negative with non-zero probability. Although in simulations the probability that $\lambda_t < 0$ is negligible, it seems sensible to evaluate alternative specifications. A first approach is to model default intensity as a two-state Markov process as in He, Nagel, and Song (2022). A second approach, which I study in Appendix B, is to assume that $d\lambda_t$ follows Cox, Ingersoll, and Ross (1985) dynamics, that is

$$d\lambda_t = \kappa_\lambda(\overline{\lambda} - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t}$$

Unfortunately, even though these dynamics would ensure that $\lambda_t > 0$, the equilibrium yield curve will not be an affine function of the state s_t. Homoscedastic default shocks introduce an additional second source of variation in credit risk premia over and above variation in habitat demand. As a result, it turns out that covariance between the arbitrageurs' portfolio and bond returns are described by a product of two affine functions. Appendix B.1 formalizes the argument by showing that heteroscedastic default intensity shocks would lead to a violation of conjecture (25). In Appendix B.2, I revisit the segmentation framework of 2.4 by introducing stochastic volatility. I expect, however, the key results to hold under more general dynamics, and I postpone a systematic analysis to future iterations.

3.3.3 Idiosyncratic Defaults and OTC Trading

Throughout the paper, I assume that defaults are idiosyncratic. As a result, there is no risk compensation for default events, and the default probabilities are the same under both physical and risk-neutral measure. Although the introduction of some degree of correlation across defaults would bring the model closer to reality, it is unlikely that the key results would disappear. However, correlated default would imply a non-zero market price of default risk, which makes the analysis more complicated. The case of aggregate default risk is studied in Costain et al. (2022).

Further, Friewald and Nagler (2019) point out that corporate bonds are mostly traded in over-the-counter (OTC) markets. As a result, there might be additional frictions that distinguish the market for corporate and Treasury bonds other than those capture in this paper. If anything, however, the premise of a few dealers managing bond inventories to provide liquidity to customers seems to provide additional support to the mechansism of concentrated risks emphasized in Section 2 and Section 3.

4 Quantitative Analysis

4.1 Data and Descriptives

Before turning to the calibration exercise, I present some descriptive evidence on (i) the term structure of corporate and Treasury bonds, (ii) the term structure of credit spreads, (iii) the relation between term premia and credit spreads, and (iv) the determinants of credit spread changes. My goal is to preview the type of evidence that I plan to collect in order to validate and test the predictions of the general model outlined in Section 3.

Data I obtain data on Treasury bond yields from Gürkaynak, Sack, and Wright (2007). The data contains daily observations of the Treasury yield curve from June 1961 to the present. Daily data on corporate bond yields is from Bloomberg. ICE BofA publishes corporate bond indices for different maturities and credit ratings. The indices are divided between investment grade (IG) and high-yield (HY) bonds. IG indices include six maturity buckets, i.e. $\{[1,3),[3,5)[5,7),[7,10),[10,15),[15,30)\}$ and four rating categories, i.e. AAA, AA, A, and BBB. High-yield bonds only include three maturity buckets $\{[1,5),[3,5)[5,8),[8,30)\}$ and three rating categories, i.e. BB, B, and CCC. I only consider the period starting in January 1997, which coincides with the beginning of the ICE BofA sample. To be consistent with the model, I interpret the portfolio of corporate sectors as being issued by companies rated BBB. Hence, the goal is to match key moments of the BBB yield curve. To measure term premia, I obtain data from Adrian, Crump, and Moench (2013). Daily VIX data and the yield on three-month Treasury bills are from Fred. Finally, I obtain expected default frequencies (EDF) from Moody's. The monthly sample is August 1998 to December 2016.

Term Structure of Credit Spreads Table 1 and Table 2 report the historical average of corporate bond yields by maturity and credit rating. The right panel in Table 1 documents that the term structure of corporate bond is, on average, upward sloping. Effective yields at long maturities systematically exceed average short term yields at all maturities and across credit ratings. The term structure of credit spreads for investment grade issuers, measured by the option adjusted spreads in the right panel of Table 1, is also upward sloping. The difference between the 15+ year OAS and the 1–3 year OAS is virtually the same across all credit rating. The slope of the term structure of credit spreads is 0.66%, 0.60%, 0.55%, and 0.61%. for AAA, AA, A, and BBB issuers, respectively. A second observation is that credit spreads are strictly positive even at very short horizons.

	Effective Yield (%)							Option Adjusted Spread (%)						
Rating	All	1–3	3–5	5–7	7–10	10–15	15+	All	1–3	3–5	5–7	7–10	10–15	15+
IG	4.72	3.56	4.12	4.63	5.02	5.38	5.79	1.50	1.11	1.32	1.53	1.62	1.73	1.79
AAA	4.03	2.99	3.45	3.84	4.22	4.89	5.07	0.78	0.53	0.63	0.72	0.78	1.19	1.03
AA	4.07	3.14	3.66	4.07	4.48	4.83	5.31	0.96	0.69	0.84	0.96	1.06	1.11	1.29
A	4.47	3.43	3.95	4.38	4.77	5.11	5.53	1.27	0.98	1.14	1.28	1.36	1.44	1.53
BBB	5.21	4.04	4.58	5.04	5.39	5.71	6.17	1.95	1.59	1.78	1.94	1.99	2.08	2.20

Table 1: Average effective yields and option adjusted spreads (OAS) for investment grade bonds by maturity and credit rating. Data is from ICE BofA and the daily sample is from January 1997 to present.

In contrast, Table 2 shows that, on average, the term structure of credit spreads for high-yield issuers slopes down. While the 1–5 year spread and the 8+ year spread are almost identical BB rated bonds, the difference between the 8+ year and the 1–5 year OAS for CCC bonds is substantial. This observation is not new, and it has been known since the early work of Jones, Mason, and Rosenfeld (1984) and Sarig and Warga (1989). Yet, these observations are consistent with the implications of a credit downgrade in Figure 8a. Since I set all risk factors equal to their long-term average, the mechanism cannot be fully explained by mean reversion in the default intensity process.

	Effective Yield (%)				Option Adjusted Spread (%)			
Rating	All	1–5	5–8	8+	All	1–5	5–8	8+
HY	8.47	–	–	–	5.38	–	–	–
BB	6.66	6.22	6.64	7.11	3.53	3.57	3.51	3.55
B	8.48	8.66	8.40	8.39	5.39	5.95	5.28	4.94
CCC	14.17	15.83	13.24	12.78	11.20	13.18	10.19	9.44

Table 2: Average effective yields and option adjusted spreads (OAS) for high-yield bonds by maturity and credit rating. Data is from ICE BofA and the daily sample is from January 1997 to present.

Term Premia and Credit Spreads I next turn to the relation between credit spreads and term premia. Figure 1 compares the evolution of term premia, taken from Adrian et al. (2013), to the evolution of credit spreads from January 1997 to present. In the first part of the sample, the correlation between term premia and credit spreads appears to be positive at both intermediate (Figure 1a) and long (Figure 1b) maturities.

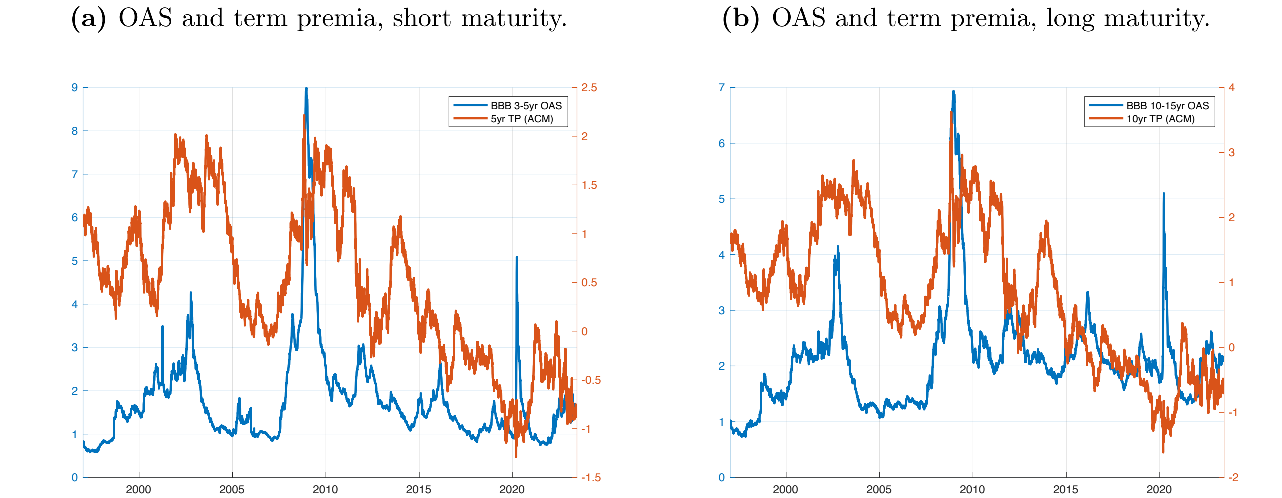


Figure 1: The figure plots term premia against credit spreads. The left panel compares the comovement of 3–5 year OAS for BBB-rated issuers to five year term premia. The right panel compares the comovement of 10–15 year OAS for BBB-rated issuers to the ten year term premia. OAS are from ICE BofA, whereas term premia are from Adrian et al. (2013). The daily sample is from January 1997 to present.

In the second part of the sample, term premia and credit spreads appear to move in opposite direction. The negative correlation is particularly striking in march 2020 at the onset of the Covid-19 pandemic. The spike in credit spreads is accompanied by a reduction in term premia at both the five year and the ten year horizon (He, Nagel, & Song, 2022). The descriptive evidence suggests that credit spreads and term premia are interconnected. The direction of the dependence mostly likely changed after 2009, and this observation is consistent with Li (2023).

Determinant of Credit Spread Changes I revisit Collin-Dufresne and Goldstein (2001) and explore the determinant of credit spread changes to provide an empirical counterpart to Proposition (5). To this purpose, I run descriptive regressions of the form

$$\Delta \mathrm{OAS}_t = \beta_0 + \beta_1 \Delta r_t + \beta_2 \Delta \mathrm{Slope}_t + \beta_3 \Delta \mathrm{VIX}_t + \beta_4 \Delta \mathrm{TP}_t + \beta_5 \Delta \mathrm{TP}_t + \beta_6 \Delta \mathrm{EDF}_t + \epsilon_t \tag{40}$$

where r_t is the yield on 3-month Treasury bills, Slope_t is the difference between the 10-year and the 1-year Treasury yield, VIX_t is the volatility index, TP_t is the 5-year term premium, and EDF_t is a 6-month ahead forecast of expected defaults. Besides r_t. the inclusion of the other controls is motivated by structural models of default (Collin-Dufresne & Goldstein, 2001; He, Khorrami, & Song, 2022). As a caveat, Equation (40) is not causal. Rather, the goal is (i) to understand correlations between credit spreads and the controls variables and (ii) to assess the explanatory power in the light of the evidence in Collin-Dufresne and Goldstein (2001). Table 3 report coefficient estimates for BBB-rated bonds of 3–5yr and 10–15yr maturities.

An increase in the short term rate is negatively correlated with changes in credit spreads across all the specifications. However, the magnitude of the coefficient increase substantially once I control for expected defaults, as shown by Column (5) in both the left and the right panel of Table 3. The magnitude of the coefficient is twice as large for long term corporate bonds. However, the negative correlation might just capture policy responses to business cycle fluctuations. Changes in credit spreads are negatively related to changes in the slope of the yield curve. In contrast, an increase in the aggregate risk, as measured by the VIX, is positively associated to credit spreads. As expected, an increase in expected defaults is strongly positively correlated to changes in corporate spreads. The magnitude of the coefficient, however, is roughly similar in both the left and the right panel.

	BBB Option Adjusted Spread – 3–5yr					BBB Option Adjusted Spread – 10–15yr				
	(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
Δr_t	-0.07^{***}	-0.07^{***}	-0.06^{**}	-0.05^{*}	-0.29^{***}	-0.06^{***}	-0.09^{***}	-0.06^{***}	-0.05^{**}	-0.65^{***}
	(0.02)	(0.03)	(0.02)	(0.02)	(0.10)	(0.02)	(0.02)	(0.02)	(0.02)	(0.09)
ΔSlope_t		-0.04^{*}	-0.03	0.05	-0.14		-0.15^{***}	-0.15^{***}	-0.03	-1.10^{***}
		(0.02)	(0.02)	(0.04)	(0.23)		(0.02)	(0.02)	(0.04)	(0.22)
ΔVIX_t			0.00^{***}	0.00^{***}	0.03^{***}			0.00^{***}	0.00^{***}	0.07^{***}
			(0.00)	(0.00)	(0.01)			(0.00)	(0.00)	(0.01)
$\Delta\text{TP}_t^{(\tau)}$				-0.00^{*}	-0.00				-0.00^{***}	0.01^{***}
				(0.00)	(0.00)				(0.00)	(0.00)
$\Delta\text{EDF}_t^{(h)}$					0.13^{**}					0.16^{***}
					(0.05)					(0.01)
Constant	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
N	6596	6596	6586	6586	194	6596	6596	6586	6586	194
R^2	0.01	0.01	0.02	0.02	0.42	0.01	0.05	0.07	0.08	0.83
Adjusted R^2	0.01	0.01	0.02	0.02	0.40	0.01	0.05	0.07	0.08	0.83

Table 3: OLS estimates of the linear regression model (40). For columns (1) through (4), the daily sample is January 1997 to present. For column (5), the monthly sample is August 1998 to December 2016. Robust standard errors are in parenthesis.

A second observation is that for Columns (1) through (4), the regressions R^2 is very small, and never exceeds 8% after controlling for short rates, slope, volatility, and term premia. However, once expected default are controlled for, the adjusted R^2 jumps to 40% for intermediate maturities and to 83% for long maturities. While the sign of the coefficients is consistent with Collin-Dufresne and Goldstein (2001) and He, Khorrami, and Song (2022), the regression R^2 are much higher.

4.2 Calibration

Following Vayanos and Vila (2021) and Kekre et al. (2022), I assume an exponential form for the price elasticity, intercept, and slope of habitat demand by maturity such that

$$\alpha^j(\tau) = \alpha^j e^{-\delta_\alpha^j} \tag{41}$$

$$\theta_1^j(\tau) = \theta_1^j \left(e^{-\delta_\alpha^j \tau} - e^{-\delta_\theta^j \tau}\right) \tag{42}$$

$$\theta_0^j(\tau) = \theta_0^j \left(e^{-\delta_\alpha^j \tau} - e^{-\delta_\theta^j \tau}\right) \tag{43}$$

for $\tau \leq 30$ and $\alpha^j(\tau) = \theta_1^j(\tau) = \theta_0^j(\tau) = 0$ otherwise. The functions (41), (42), and (42) all share an exponential structure. For parsimony, I consider a single demand factor β_t and I take the matrices Γ and Σ to be diagonal. As a result, the equilibrium term structure of government and corporate bonds is determined by 19 parameters. The first nine parameters characterize the dynamics of the aggregate risk factors, $(\kappa_r, \sigma_r, \bar{r})$ for the short rate, $(\kappa_\lambda, \sigma_\lambda, \overline{\lambda})$ for the default intensity, and $(\kappa_\beta, \sigma_\beta, \overline{\beta})$ for the demand factor. The other ten parameters control the slope $(\delta_\alpha^j, \alpha^j)$ and the intercept $(\theta_0^j, \theta_1^j, \delta_\theta^j)$ of preferred-habitat demand. The restrictions on (Γ, Σ) are akin to Gourinchas et al. (2022) and considerably simplify the estimation of the model and the interpretation of the results, but that goes at the expense of a more realistic response of habitat demand to economic fundamentals.

Given that only the product $\theta_1\sigma_\beta$ matters for the equilibrium dynamics, I normalize $\sigma_r = \sigma_\beta$. I also normalize $\overline{\beta} = 0$ without loss of generality. To further reduce the number of parameters, I assume that θ_1^j, θ_0^j, and δ_θ^j are the same for both sectors. Given that δ_α^j varies across security, however, the demand slope and the demand intercept will be different across assets. After this, there remains 14 parameters, that is seven characterizing state dynamics $(\kappa_r, \sigma_r, \overline{r}, \kappa_\lambda, \sigma_\lambda, \overline{\lambda}, \kappa_\beta)$ and the other seven describing habitat demand $(\delta_\alpha^G, \delta_\alpha^C, \alpha^G, \alpha^C, \theta_0, \theta_1, \delta_\theta)$. The arbitrageurs' risk aversion a is also a parameter to calibrate, but it is not identified since it affects equilibrium yields only through the products $(a\alpha^j, a\theta_0^j, a\theta_1^j)$. For this reason, I set a equal to the calibration in Vayanos and Vila (2021).

I denote by ϑ the vector of model parameters. I estimate ϑ to match key unconditional moments of the Treasury term structure. Although it would be likewise sensible to estimate ϑ targeting moments of credit spreads and corporate bond yields, I leave these moments untargeted. As a result, the goodness of fit for corporate yields is informative of whether the model can replicate features of the data with only minimal inputs. The average yield at maturity τ is

$$y_t^{j,(\tau)} = \frac{A_{j,r}(\tau)\overline{r} + A_{j,\lambda}(\tau)\overline{\lambda} + C_j(\tau)}{\tau} \tag{44}$$

and, since Γ and Σ are diagonal, the volatility of the yields is

$$\sigma\left(y_t^{j,(\tau)}\right) = \frac{1}{\tau}\sqrt{A_{j,r}(\tau)^2\frac{\sigma_r^2}{2\kappa_r} + A_{j,\lambda}(\tau)^2\frac{\sigma_\lambda^2}{2\kappa_\lambda} + A_{j,\beta}(\tau)^2\frac{\sigma_\beta^2}{2\kappa_\beta}} \tag{45}$$

The empirical counterparts of (44) and (45), which are the average Treasury yield and its standard deviation, respectively, are the target moments. As in Gourinchas et al. (2022), I choose ϑ to minimize the sum of the squared differences between model-implied $(\mathcal{M}_i)$ and empirical (m_i) moments. Hence

$$\widehat{\vartheta} = \arg\min L(\vartheta) \doteq \sum_i \left(\mathcal{M}_i(\vartheta) - m_i\right)^2 \tag{46}$$

To estimate ϑ, I use 40 empirical moments associated to the Treasury yield curves. These are the average yields and the volatility for maturities $\tau = 1, \ldots, 20$. To speed up computation, I take the initial guess ϑ_0 to be exact same calibration in Vayanos and Vila (2021), with the exception of $\overline{r}$ and $\overline{\lambda}$, which I pick to match the level of short term yields. The value of $\overline{r}$ is set close to the historical average of the Federal Funds Rate, whereas $\overline{\lambda}$ is approximately equal the historical percentage of BBB cumulative defaults over five years.

Description	Parameter	Value	Calibration
Risk Factor Dynamics			
Short rate mean-reversion	κ_r	0.099	Own
Short rate volatility	σ_r	0.0121	Own
Short rate average	$\overline{r}$	0.015	Average Federal Funds Rate
Demand factor mean-reversion	κ_β	0.055	Vayanos and Vila (2021)
Demand factor volatility	σ_β	0.0121	Normalized to σ_r
Demand factor average	$\overline{\beta}$	0	Vayanos and Vila (2021)
Default intensity mean-reversion	κ_λ	0.049	Own
Default intensity volatility	σ_λ	0.0101	Own
Default intensity average	$\overline{\lambda}$	0.014	S&P BBB 5yr cumulative defaults
Habitat-demand Parameters			
Government elasticity decay	δ_α^G	0.299	Vayanos and Vila (2021)
Corporate elasticity decay	δ_α^C	0.297	Vayanos and Vila (2021)
Government elasticity	$a\alpha^G$	35.3	Vayanos and Vila (2021)
Corporate elasticity	$a\alpha^C$	49.846	Own
Demand intercept	$a\theta_0$	289	Vayanos and Vila (2021)
Demand factor loading	$a\theta_1$	3155.2	Vayanos and Vila (2021)
Demand loading decay	δ_θ	0.307	Vayanos and Vila (2021)

Table 4: Calibration of model parameters for the main sample of nominal yields. The sample is January 1997 to present. The calibration only targets moments of the Treasury yield curve.

Table 4 reports the parameters used in the quantitative analysis. Although the volatility of the innovations to both r_t and λ_t is comparable, default intensity is significantly more persistent than the short rate ($\kappa_r > \kappa_\lambda$). As a result, the unconditional variance of default intensity is larger than the short term rate. At short maturities, since $\alpha^C > \alpha^G$, habitat demand for corporate bonds is more price-elastic than for Government bonds. In contrast, the exponential decay of the slope coefficient is virtually identical for both asset classes. The estimated parameter vector $\widehat{\vartheta}$ is quite close from the initial guess. A potential explanation is that the equilibrium term structure is very sensitive to the initial values of θ_1 and δ_θ, so that it becomes hard to improve on the objective. Therefore, I manually set the habitat demand parameters, except the product $a\alpha^C$ to match the calibration in Vayanos and Vila (2021). Future work is devoted to improving the estimation procedure and ensure that the algorithm converges from a broad set of initial values for ϑ_0.

4.3 Model Fit

I inspect the model fit by comparing model-implied moments and their empirical counterparts. Since I only use moments of the Treasury term structure to estimate the parameters, a good fit of corporate bond yields and credit spreads is informative about whether the model can capture, at least qualitatively, key features of the data.

4.3.1 Equilibrium Characterization and Term Structure

Figure 2a plots the equilibrium term structure of government bonds and defaultable bonds for maturities $\tau \in (0, 20)$. The model matches particularly well yields at short and intermediate maturities. The model-implied yields for $\tau > 15$ still provide a reasonable fit, although they are not as close to their empirical counterparts. Although corporate yields do not play any role in the model estimation, the implied corporate term structure provides a very good fit for maturities less than 10 years, while it deviates more at longer maturities. However, the maturity buckets of BofA ICE indices are much wider at the long end. As a result, the representative maturities of $\tau = 12.5$ and $\tau = 20$ years could be different from the actual maturity composition within each index. At shorter maturities, the brackets are narrower, and the distribution of maturities inside each bracket is more likely to be uniform.

For both corporate and government bonds, the average term structure is upward sloping. On average, the five- and ten-year Treasury yields are 2.94% (2.96% in the data) and 3.80% (3.65% in the data), respectively. The average term spread $y_t^{j,(10)} - y_t^{j,(1)}$ is 1.95% for Treasuries and 2.44% for defaultable bonds. The average term spread on Treasury yields in the data is 1.48%, which is smaller than in the model. The term structure of defaultable bonds exceeds the Treasury yield curve at all maturities, confirming that credit risk premia affect the entire corporate yield curve.

(a) Model yields, levels $y_t^{j,(\tau)}$ **(b)** Model yields, volatility $\sigma\left(y_t^{j,(\tau)}\right)$

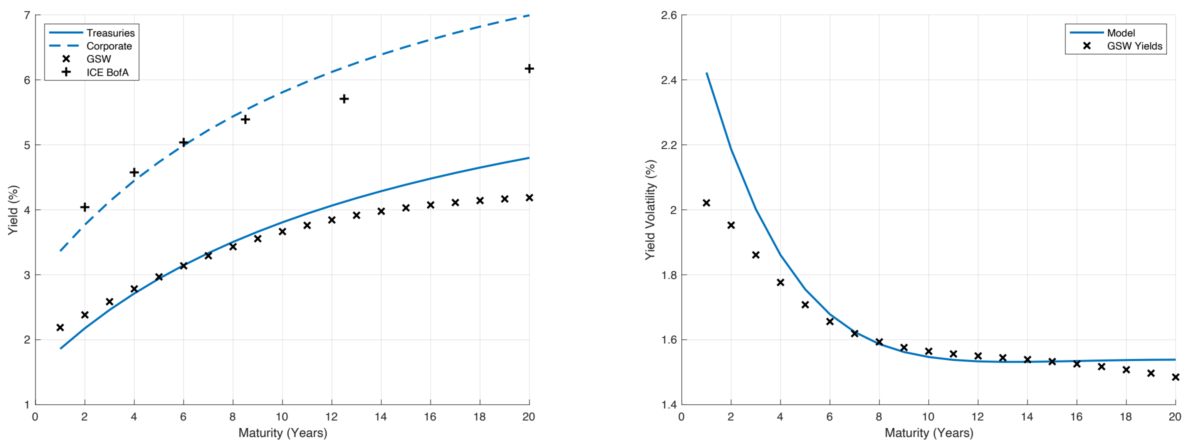


Figure 2: Panel (a) plots the model-implied yield curves for Treasury and corporate bonds against their empirical counterparts. Panel (b) plots the model-implied yield volatilities for Treasury yields against their empirical counterpart. The parameters used in the calibration are in Table 4. Treasury yields are from Gürkaynak et al. (2007), whereas bond yields are BBB effective yields from ICE BofA. The daily sample is January 1997 to present.

Figure 2b plots the model-implied Treasury yield volatilities against their empirical counterpart. The model fits the data very well at intermediate and long maturities. Although the model predicts short term yields to be slightly more volatile, the fitted yields qualitatively line up with the data. In both the model and in the data, the volatility of the yields decreases with maturity. The unconditional volatility of the 10-year yield in the model is 1.55%, against 1.56% in the data.

To analyze the propagation of short rate and default intensity throughout the Treasury yield curve, I define the instantaneous forward rate $f_t^{(\tau)}$ for maturity τ as the limit $\Delta\tau \to 0$ of the time t forward rate between $\tau - \Delta\tau$ and τ

$$f_t^{G,(\tau-\Delta\tau,\tau)} = -\frac{1}{\Delta\tau} \log\left(\frac{P_t^{G,(\tau)}}{P_t^{G,(\tau-\Delta\tau)}}\right) \tag{47}$$

As $\Delta\tau \to 0$, this gives the instantaneous forward rate

$$f_t^{G,(\tau)} \doteq \lim_{\Delta\tau \to 0} f_t^{G,(\tau-\Delta\tau,\tau)} = -\frac{\partial \log P_t^{G,(\tau)}}{\tau} = A_G'(\tau)^T s_t + C_G'(\tau) \tag{48}$$

where $A_G'(\tau) = \sum_{k=1}^{K+2} \psi_k^G e^{-\mathsf{v}_k}$. I plot the model-implied instantaneous forward rates in figure 3a and the functions $A_G'(\tau)$ in figure 3b. The model fits forward rates well at short maturities, but does not quite capture the inversion of the curve at around $\tau = 12$, where the average forward rate in the data starts declining. Forward rates respond positively to the short rate r_t and the demand factor β_t, and negatively to default intensity λ_t. A shock to the short term rate has its strongest effect at short maturities, whereas demand shocks affect long term yields more. The magnitude of the response to default intensity shocks peaks at intermediate maturities, and it weakens as τ increases.

(a) Model instantaneous forward rates $f_t^{G,(\tau)}$ **(b)** Forward rate loadings on s_t

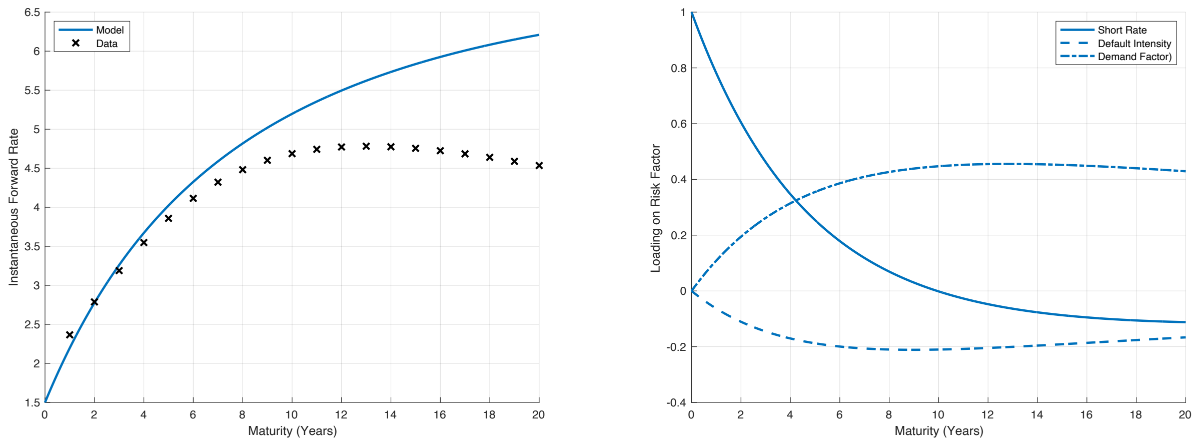


Figure 3: Panel (a) plots the term structure of instantaneous Treasury forward rate corporate bonds against the data. Panel (b) plots the loadings $A_G'(\tau)$ of instantaneous forward rates on the vector of aggregate risk factors s_t as a function of maturity. The parameters used in the calibration are in Table 4. Treasury yields are from Gürkaynak et al. (2007). The daily sample is January 1997 to present.

Overall, figure 3 validates the good model fit at short to intermediate maturities. The model also suggests that an increase in default intensity lowers forward rate, where the strongest effect is at intermediate maturities. Furthermore, it suggests that the short rate and the demand factor are relatively more important at short and long maturities, respectively. The shape of the loading on r_t, i.e. $A_{G,r}'(\tau)$, is qualitatively consistent with the monotonically decreasing responses of the nominal forward rates documented in Hanson and Stein (2015) and Kekre et al. (2022). As in Vayanos and Vila (2021) monetary policy affects long term yields through changes in the short-term rate r_t.

4.4 Model Implications and Mechanisms

4.4.1 Credit Spreads

I compute the credit spreads $\mathcal{S}_t^{(\tau)}$ implied by the model as the difference between yields on corporate bonds and Treasury bonds, as shown in equation (36). Since I interpret the corporate sector as a continuum of BBB-issuer, I compare $\mathcal{S}_t^{(\tau)}$ with the option-adjusted spreads (OAS) from ICE BofA for

BBB rated bonds at various maturities. The initial value of $\overline{\lambda}$ is chosen to approximately match the percentage of cumulative BBB defaults within a five year horizon.

Figure 4a plots $\mathcal{S}_t^{(\tau)}$ against the time-series average OAS for BBB bonds. The model matches the average level of credit spreads accurately at short and long maturities. On average, even though the default intensity process is mean-reverting, the term structure of credit spreads is upward sloping in the data and in the model. In the limit, as maturity tends to zero, i.e. $\tau \to 0$, credit spreads converge to the long-term average level of default intensity $\mathcal{S}_t^{(\tau)} \to \overline{\lambda}$. As a result, average yield spreads are strictly positive at zero maturity, consistent with the observation by Duffie and Lando (2001) in an incomplete information setting. As opposed to Duffie and Lando (2001), however, the average credit spread here is an increasing function of maturity.

(a) Credit spreads average, level

(b) Credit spread volatility, level

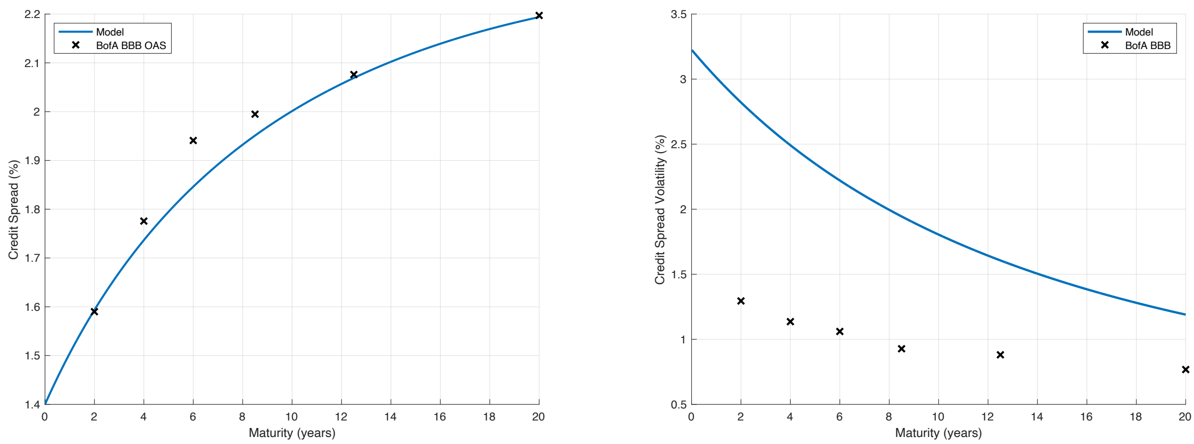


Figure 4: Panel (a) plots the term structure of credit spreads against their empirical counterpart. Panel (b) plots the volatility of credit spreads implied by the model against the data. Credit spreads are proxied by the option-adjusted spreads on BBB bonds published by ICE BofA. The parameters used in the calibration are in Table 4. Treasury yields are from Gürkaynak et al. (2007). The daily sample is January 1997 to present.

The shape of the term structure of credit spreads varies with the relative persistence of default intensity and the short rate, as well as with the volatility of default intensity shocks. Hence, the model can accommodate non-monotonic credit spreads. It should not be obvious that the model generates a realistic level of credit spread (Chen et al., 2008; Du et al., 2019), but I postpone the discussion to Section 4.4.3. Figure 4b shows the model-implied volatility of credit spreads. Overall, the model-implied volatility is qualitatively consistent with the data. Indeed, the volatility of credit spreads monotonically declines with maturity. However, credit spreads in the model are significantly more volatile than in the data.

I next study how the level of credit spreads loads on the short rate and the default intensity λ_t. Figure 5 plots the loadings of credit spreads on the state variables $A_{\mathcal{S},r}(\tau)$ and $A_{\mathcal{S},\lambda}(\tau)$. As expected, credit spreads are positively related to default rates. Short term spreads move one-to-one with λ_t, and the strength of the reaction dissipates as τ grows large.

A key implication of the model is that monetary policy affects credit spreads. The loading is negative, meaning that an increase in the short term rate r_t lowers credit spreads. The interpretation is that changes in r_t affect the market price of interest rate risk borne by corporate and Treasury bonds. A higher r_t leads to higher yields across all markets at all maturities. Because of this, habitat demand increases and arbitrageurs end up holding smaller net positions on their balance sheet. In turn, this lowers the equilibrium price of credit risk. As this disproportionately affects corporate yields, credit spreads decline at all maturities. The negative correlation between the short rate r_t and credit spreads is consistent with the evidence in Longstaff and Schwartz (1995) and Duffee (1999). Yet, the negative correlation between r_t and credit spreads in the data is confounded by policy responses to business cycle fluctuation, whereas (4) describes how credit spreads react to an exogenous change r_t.

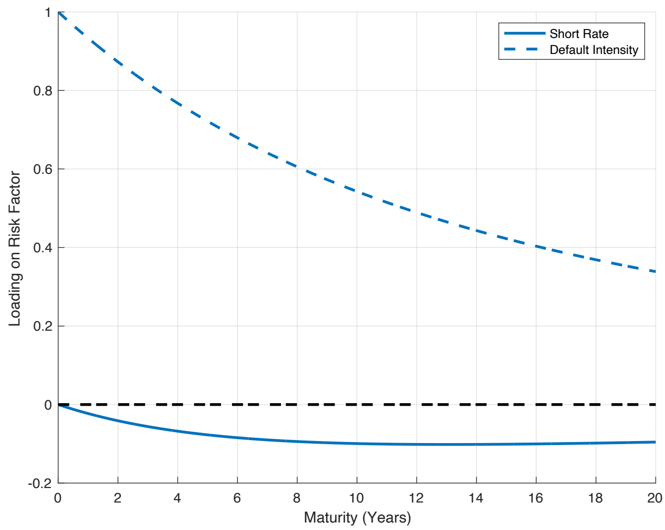


Figure 5: The figure plots the loadings $A_{\mathcal{S}}(\tau)$ of credit spreads on the aggregate risk factors s_t as a function of maturity τ. The calibration is reported in Table 4.

In this regard, Gertler and Karadi (2015) document that a contractionary monetary surprise causes an increase, rather than a decline, in various measures of credit spreads. This result is in contrast with the sign of $A_{\mathcal{S},r}(\tau)$, which implies that an exogenous shock to r_t lowers credit spreads. To reconcile this discrepancy, I explore the effects of an hypothetical monetary policy intervention. I model this intervention as a surprise increase in the level of the short rate from $r_t = \overline{r}$ to $r_t = 1.5\overline{r}$, which is a 0.75% interest rate hike. I compute risk prices as implied by the right hand side of the arbitrageurs' first-order conditions (26) and (27), and I set λ_t and β_t equal to their long-term averages $\overline{\lambda}$ and $\overline{\beta}$.

(a) Arbitrageurs' net positions $X_t^{j,(\tau)}$.

(b) Market prices of aggregate risk.

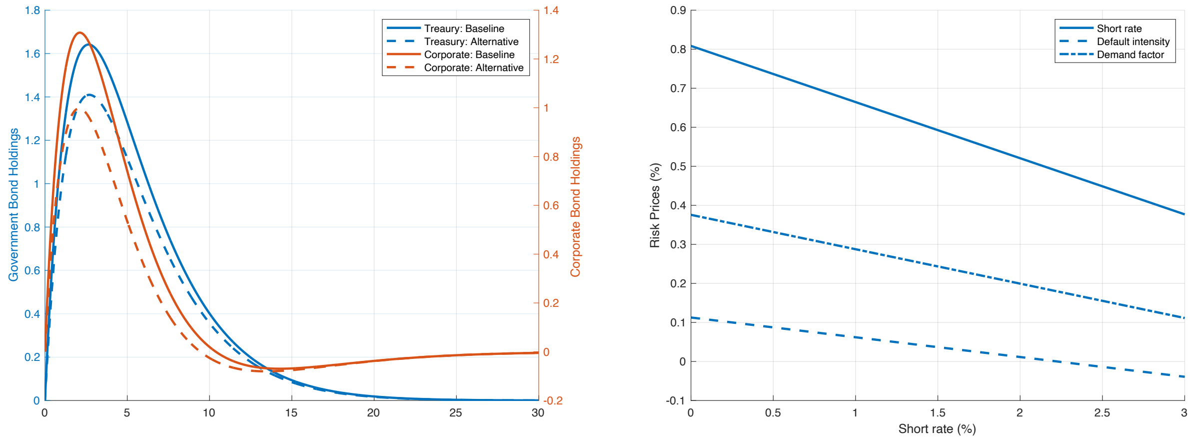


Figure 6: The left panel shows arbitrageurs' portfolio holding before and after the interest rate hike. The blue and the orange lines describe Treasury and corporate bond holdings, respectively. The right panel plots the market risk prices implied by the arbitrageurs' first-order conditions (26) and (27). The calibration is reported in Table 5. The market prices or risk are expressed as a function of r_t fixing $\lambda_t = \overline{\lambda}$ and $\beta_t = \overline{\beta} = 0$.

Figure 6a plots equilibrium portfolio holdings $X_t^{j,(\tau)}$ as a function of maturity. The solid lines represent holding at the baseline level $r_t = \overline{r}$. An increase of the short term rate from $r_t = \overline{r}$ to $r_t = 1.5\overline{r}$, holding demand and default intensity constant, raises equilibrium yields throughout the term structure, inducing habitat investors to demand more bonds. Because of market clearing, arbitrageurs' hold now smaller net positions at all maturities, as shown by the dashed lines in figure 6a. Because the arbitrageurs' exposure to the aggregate risk factors has declined, the market price of default risk is lower. Given that default intensity risk mostly affects yields of defaultable bonds, the response of the corporate bonds yields is weaker. In fact, the reduction in the price of default intensity risk acts in the opposite direction of the increase in r_t. Hence, the fact that monetary tightening r_t raises equilibrium habitat demand at all maturities generates a negative $A_{r,\mathcal{S}}(\tau)$.

Accordingly, Figure 6b plots the market prices of aggregate risk as a function of r_t fixing $\lambda_t = \overline{\lambda}$ and $\beta_t = \overline{\beta}$. There is a negative relation between r_t and the risk prices for all three state variables. An increase in the short rate induces habitat investors to save more in both bonds, reducing arbitrageurs' exposure to the aggregate risk factors, which lowers the market prices of risk. The dependence of credit risk prices on the level of the short term rate is robust to the specification of habitat demand as long as $Z_t^{j,\tau}$ responds to all three aggregate risk factors. However, the sign of the effect on credit spreads will be different if (i) an increase in the short term rate induces habitat investors to demand less bonds or (ii) if arbitrageurs are long lived and an increase in r_t causes capital losses to arbitrageurs' wealth (Kekre et al., 2022). Hence, the discrepancy between Figure 5 and the results in Gertler and Karadi (2015) is explained by the behavior of habitat investors, which implies that arbitrageurs' net exposure to aggregate risk is inversely related to r_t.

The same logic exposes the limitation of specification (23) when risky securities are introduced in a preferred-habitat framework. In the current specification, an increase in either r_t or λ_t induces habitat investors to demand more assets. Because habitat investors do not react to fundamentals, an increase in λ_t reduces the arbitrageurs' net risk exposure, lowering the market price of risk of both short rate and default intensity. In principle, a more realistic specification of habitat demand (23) or state variable dynamics (6), for example by having demand shocks negatively correlated to λ_t and r_t, is sufficient to flip the sign of this effect. Yet, arbitrageurs do respond to λ_t, and this effect is strong enough to generate a positive relation between credit spreads and default intensity.

4.4.2 State Dependent Risk Prices and Risk Neutral Dynamics

Proposition (3) states that arbitrageurs endogenously induce risk-neutral dependence between the aggregate risk factors. The top panel of Table 5 compares the drift matrix of the state variables Γ under the physical measure to the drift matrix under the equivalent martingale measure M^T. The calibration in Table 4 takes Γ to be diagonal. However, as predicted, the matrix M^T describing $\mathbb{Q}$-dynamics is neither diagonal nor symmetric. The entries of M^T reveals that under the equivalent martingale measure, changes in the short rate not only depend on the deviation $r_t - \overline{r}$, but also on λ_t and β_t. The arbitrageurs induce (or exacerbate, if Γ is not diagonal) drift dependence between r_t and λ_t. In fact, the coefficients of dr_t on λ_t is positive. The same holds for the loading of $d\lambda_t$ on r_t.

Physical Measure $\mathbb{P}$				Risk-neutral Measure $\mathbb{Q}$			
Γ	r_t	λ_t	β_t	M^T	r_t	λ_t	β_t
r_t	0.099	0	0	r_t	0.2429	0.0726	-0.1219
λ_t	0	0.049	0	λ_t	0.0506	0.1400	-0.0400
β_t	0	0	0.055	β_t	0.0881	0.0623	-0.0340
$\mathbb{V}\text{ar}(s_t)$	r_t	λ_t	β_t	$\mathbb{V}\text{ar}^{\mathbb{Q}}(s_t)$	r_t	λ_t	β_t
r_t	0.0008	0	0	r_t	0.0018	0.0002	0.0031
λ_t	0	0.0011	0	λ_t	0.0002	0.0005	0.0007
β_t	0	0	0.0014	β_t	0.0031	0.0007	0.0070

Table 5: The top left panel reports the drift matrix Γ implied by the calibration in Table 4. The top right panel reports the drift matrix M^T under the risk-neutral measure. The bottom left panel reports the unconditional covariance matrix of the state variables implied by the calibration in Table 4. The bottom right panel reports the unconditional covariance matrix of the state variables under the risk-neutral measure. x

The bottom panel of Table 5 reports the unconditional covariance matrix of the risk factors $\mathbb{V}\text{ar}(s_t)$. Under the physical measure, the non-zero entries are given by $\frac{\sigma_s^2}{2\kappa_s}$, where $s \in \{r, \lambda, \beta\}$. This should not be surprising given the restriction that Γ and Σ are diagonal. However, under the risk-neutral measure, $\mathbb{V}\text{ar}^{\mathbb{Q}}(s_t)$ is not diagonal. As a result, the state variables are positively correlated even when they are independent under $\mathbb{P}$.

On the one hand, Table 5 shows that the drift correction in the mapping between B_t and $B_t^{\mathbb{Q}}$ is state dependent. This holds provided that $a > 0$ and that $\alpha^j > 0$. On the other hand, Table 5 introduces additional economic restrictions on the joint $\mathbb{Q}$-dynamics of the short rate and the default intensity. In a setting in which risk-neutral arbitrageurs trade with habitat investors, the aggregate risk factors are in general not independent under the risk neutral measure. In fact, because of the arbitrageurs' activities, the dependence between λ_t and r_t will be higher than implied by any $\mathbb{P}$-measure logic such as defaults increase in bad times.

4.4.3 Portfolio Rebalancing Channel

The inclusion of a second asset class to the portfolio choice problem of the arbitrageurs enriches the asset pricing implications of habitat-demand models. In Vayanos and Vila (2021) and in the two country extensions of Gourinchas et al. (2022) and Greenwood et al. (2020), the aggregate risk factors enter the arbitrageurs' decision problem in a symmetric fashion. However, in my framework, corporate bonds default at a stochastic rate λ_t, whereas government bonds do not. As a result, λ_t enters directly (i.e. not through market clearing) only in the first-order condition of corporate bonds $X_t^{j,(\tau)}$. In contrast, r_t enters directly in both the first-order conditions (26) and (27). Intuitively, when λ_t is high, the arbitrageurs require a relatively higher compensation to hold corporate bonds.

Figure 7a and 7b plot the yield loadings on the state variables $\frac{1}{\tau}A_j(\tau)$ for government and corporate bonds, respectively. While the loadings on the short rate $A_{j,r}(\tau)$ and the demand shock $A_{j,\beta}(\tau)$ shares the same sign for both assets classes, the impact of default intensity λ_t on yields is asymmetric.

(a) Government bonds factor loadings **(b)** Corporate bonds factor loadings

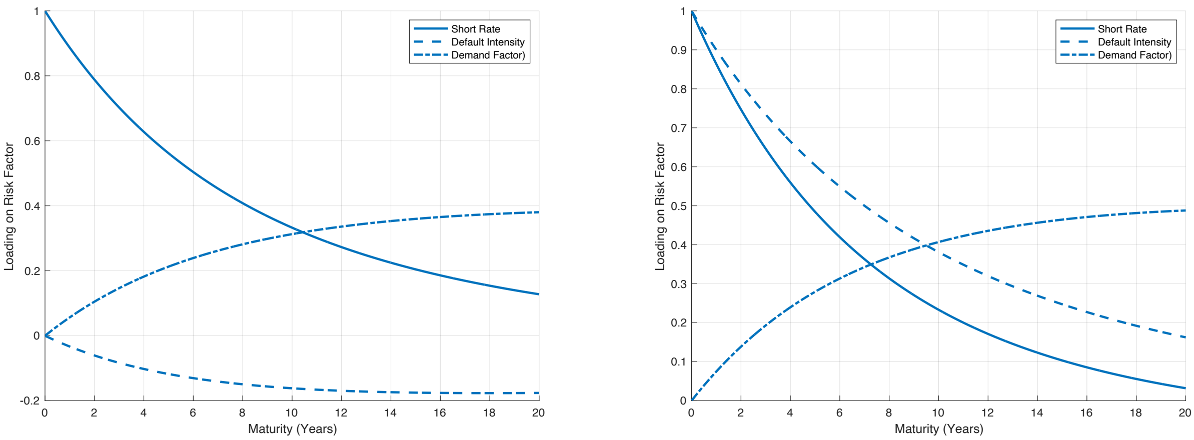


Figure 7: The figure compares the loadings of Treasury and corporate bonds on the aggregate risk factors. The loadings are the functions $A_G(\tau)$ and $A_C(\tau)$. The calibration is described in Table 4.

On the one hand, an increase in λ_t is positively related to corporate bond yields, i.e. $A_{C,\lambda}(\tau) > 0$. On the other hand, the relation between Treasury yields and λ_t is negative for all maturities, i.e. $A_{G,\lambda}(\tau) < 0$. It turns out that, in equilibrium, government bonds hedge against default intensity risk since they perform well when λ_t increases.

The model rationalizes the credit spread puzzle even without introducing stochastic volatility or jump processes in the dynamics of the aggregate risk factors (6). Du et al. (2019) argue that a major challenge of structural default models is that efforts to calibrate models to observable moments have been unable to match average credit spreads levels. Chen et al. (2008) make a similar observation, and argue that Baa–Aaa credit spreads implied by structural models of credit risk are usually significantly below historical values. Chen et al. (2008) then show that the puzzle can be resolved if the strong comovements in default rates and Sharpe ratios are properly accounted for.

In this model, the level of credit spreads is due to a combination of three effects. The first and more direct effect is fully driven by variation in the issuer credit quality λ_t. The reason is that corporate bonds default whereas government bonds do not, so that arbitrageurs require a compensation for the fraction of bonds $\lambda_t dt$ that is lost at any point in time. The second effect, which is analogous to Chen et al. (2008), is the correlation between the short rate and the default intensity. Furthermore, the dependence of the risk factors is even stronger under the risk-neutral measure, even when r_t and λ_t are assumed to be independent. The reason is that exposure to aggregate risk factors is concentrated in the arbitrageurs' portfolio, so that the equilibrium prices of credit and interest rate risk are state dependent. The third channel is a portfolio/substitution effect, and it is captured by the opposite sign of $A_{G,\lambda}(\tau)$ and $A_{C,\lambda}(\tau)$. Government bonds are hedges against default risk, and their price increase when λ_t goes up. This further contributes to widen credit spreads over and above what is implied by changes in the credit quality of the corporate issuer.

4.4.4 High-Yield Bonds and Rating Downgrades

Figure 4a reveals that, on average, the term structure of credit spreads is upward sloping. However, a long-standing observation in the corporatate bond literature is that the term structure of credit spreads is upward sloping for investment grade bonds, whereas it slopes downward for high-yield issuers (Jones et al., 1984; Sarig & Warga, 1989). The analysis so far, however, has interpreted the corporate sector as a continuum of BBB issuers. Although default risk might be of concern, BBB-rated bonds are still investment grade securities. Motivated by this, I analyze the effect of a rating downgrade on the term structure of credit spreads.

I model a rating downgrade as an unanticipated and permanent increase in the long term average level of default intensity. I consider a moderate downgrade from BBB to BB and a severe downgrade from BBB to B. I choose the average intensity after the downgrade to match the average level of the option adjusted spreads at short maturities of the corresponding rating category, The slope of the term structure of credit spreads is thus untargeted. The moderate downgrade corresponds to a change from $\overline{\lambda} = 1.4$ to $\overline{\lambda} = 3.7$, whereas the severe downgrade is a change from $\overline{\lambda} = 1.4$ to $\overline{\lambda} = 6.4$. I maintain default uncertainty constant, that is the volatility of default intensity shocks σ_λ is the same before and after each downgrade. I emphasize the case of a severe downgrade because to better capture the inversion of the term structure of credit spreads when the long term average $\overline{\lambda}$ increases.

Figure 8a, which is identical to figure 4a, and Figure 8b compare the term structure of credit spreads implied by the model before and after the downgrade. For investment grade issuers the term structure of credit spreads is upward sloping. However, the average slope of the term structure of high-yield issuers is negative. The model matches quite well the qualitative features of the data, in particular for short maturities. In unreported results, the model does a fairly good job in matching the average term structure of credit spreads after a moderate downgrade too. As opposed to 4b, the implied term structure is quite flat, and the difference between long term and short term credit spreads is minimal, as shown in the descriptives in Section 4.1.

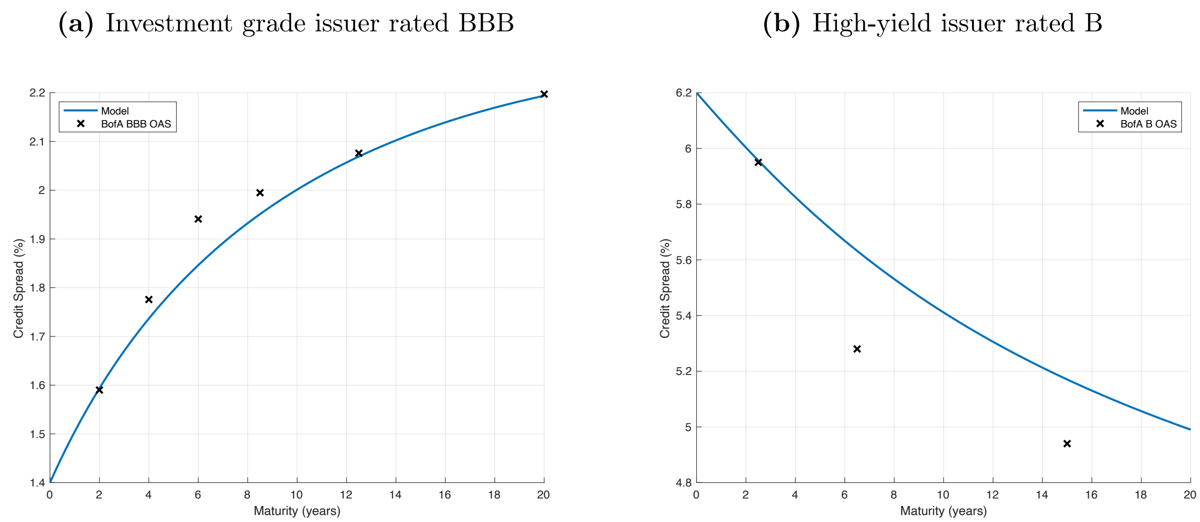


Figure 8: The figure compares the term structure of credit spreads implied by the model with its empirical counterpart. A credit downgrade is modeled as an unanticipated permanent increase in the long term average default intensity from $\overline{\lambda} = 1.4$ to $\overline{\lambda} = 6.4$. Option adjusted spreads (OAS) are obtained from ICE BofA, and the daily sample is January 1997 to present.

4.5 Monetary Policy Intervention

I now consider two alternative sets of monetary policy interventions to asses how the interactions between the corporate and the government bond market affect the propagation of monetary policy shocks throughout the yield curve. The first intervention maps into conventional monetary policy and is modelled through an unexpected increase in the level of the short rate r_t. The second intervention I analyze is Quantitative Easing (QE). I initially assume that QE purchases concern government bonds only. I model QE as an unanticipated decline $\Delta\theta_0^G(\tau)$ in the intercept of habitat demand of Treasury bonds. Proposition 4 predicts that demand shocks to the Treasury market should also affect corporate yields and credit spreads by reducing the arbitrageurs' exposure to the aggregate risk factors. Then, I consider a similar intervention where QE purchases concern corporate bonds only, which is modeled as an unanticipated decline $\Delta\theta_0^C(\tau)$ in the intercept of habitat demand of corporate bonds.

4.5.1 Conventional Monetary Policy

To analyze the propagation of short rate shocks throughout the Treasury yield curve, I compare the responses of instantaneous forward rates to the reaction of expected future short rates $\mathbb{E}_t[r_{t+\tau}]$. The expectations hypothesis implies that future expected short rates move one-to-one with forward rates. A unit shock to r_t raises instantaneous forward rates by

$$\frac{\partial f_t^{G,(\tau)}}{\partial r_t} = A'_{G,r}(\tau) = \sum_{k=1}^{K+2} \psi_k^G e^{-\nu_k \tau}$$

Conversely, the response of the expected future short rate at maturity τ is

$$\frac{\partial \mathbb{E}_t[r_{t+\tau}]}{\partial r_t} = \frac{\partial}{\partial r_t} e_1^T \mathbb{E}_t[s_{t+\tau}] = \frac{\partial}{\partial r_t} e_1^T e^{-\Gamma\tau} s_t = e^{-\kappa_r \tau}$$

where the last equality holds under the restriction that Γ is diagonal. To assess the strength of the propagation of monetary policy shocks throughout the term structure of Treasury bonds, I compare $e^{-\kappa_r \tau}$ to $\sum_{k=1}^{K+2} \psi_k^G e^{-\nu_k \tau}$ for various levels of credit risk, which I model through changes in the volatility of default intensity shocks σ_λ. For the sake of comparison, I analogously define $f_t^{C,(\tau)}$, which is the corporate bond counterpart to $f_t^{G,(\tau)}$, and I repeat the same exercise for the term structure of instantaneous forward rates of the corporate bonds.

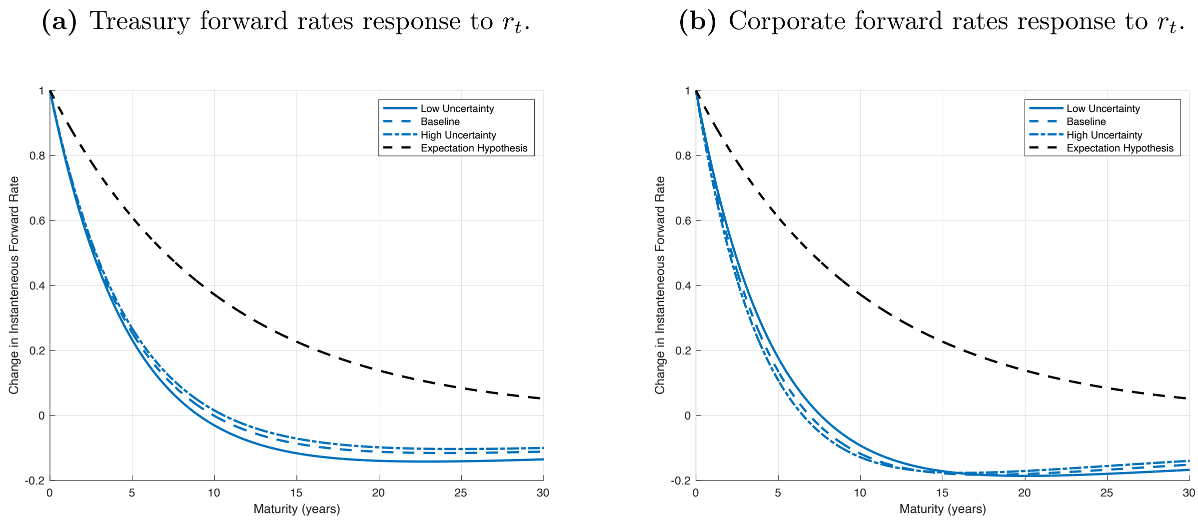


Figure 9: Underreaction of Treasury and corporate forward rates. The blue lines describe the response of forward rates to an instantaneous change in r_t for different levels of default uncertainty. The baseline uses the parameters given in Table 4. The low uncertainty case sets $\sigma_\lambda = 0.006$, whereas the high uncertainty case sets $\sigma_\lambda = 0.013$. The black dashed line plots the response of expected future short rates $\mathbb{E}_t[r_{t+\tau}]$.

Figure 9a compares how instantaneous Treasury forward rates respond to the short term rate $\frac{\partial f_t^{G,(\tau)}}{\partial r_t}$ at three different levels of default uncertainty σ_λ, namely low ($\sigma_\lambda = 0.006$), medium ($\sigma_\lambda = 0.0101$), and high ($\sigma_\lambda = 0.0130$). Similarly, Figure 9b plots the response of corporate forward rates to monetary policy shocks. In both graphs, the black dashed line represent the response of expected future short rates. As in Vayanos and Vila (2021), the model generates underreaction of forward rates to monetary policy for both asset classes. Intuitively, the extent of the overreaction is driven by arbitrageurs' risk aversion, who require a compensation to transmit monetary shocks to long term yields.

Figure 9a and 9b, however, reveal that default risk has an asymmetric impact on the strength of monetary policy transmission across asset classes. On the one hand, the underreaction in the Treasury market is inversely related to the level of default uncertainty. When σ_λ is higher, monetary policy transmission to long term Treasury yields is stronger. On the other hand, at least until intermediate maturities, the underreaction in the corporate bond market is directly proportional to the level of default uncertainty. When σ_λ is higher, monetary policy transmission to long term Treasury yields is weaker. This is in contrast to Vayanos and Vila (2021), where demand risk unambiguously weakens the transmission of short-rate shocks to bond yields by making carry trades riskier.

The effect of higher uncertainty on corporate yields is easier to interpret. Corporate bond yields load positively on default uncertainty since $A_{C,\lambda}(\tau) > 0$. As a result, an increase in σ_λ acts analogously to an increase in demand risk by making corporate bond carry-trades riskier. An immediate consequence is that, as far as the corporate yield curve is concerned, monetary policy is less effective in reducing the financing costs of firms whenever there is high default uncertainty. However, the effect is the opposite in the Treasury market. Treasury bonds hedge against default risk given that $A_{G,\lambda}(\tau) < 0$ and their price increase when λ_t goes up. A higher default uncertainty makes hedging properties even more valuable to risk averse arbitrageurs, lowering risk premia on Treasuries. A unique implication of this result is that the relative strength of monetary policy transmission across asset classes is partially determined by the interaction of (i) how risky the aggregate risk factors are (σ_λ) and (ii) how good hedges are certain assets with respect to the risk factors.

4.5.2 Quantitative Easing

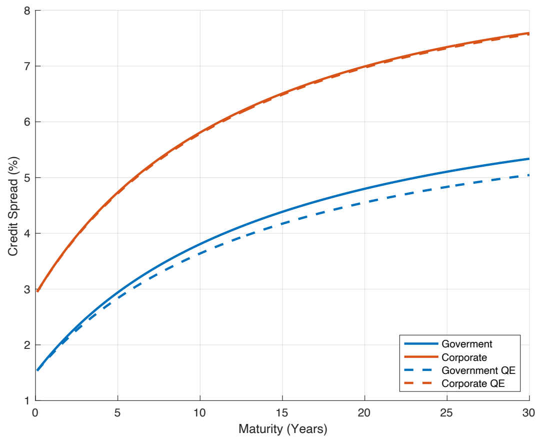
While Vayanos and Vila (2021) analyze the impact of QE on interest rates, their only policy target is the Treasury yield curve. However, QE works through different channels, and Treasury yields might not be the appropriate benchmark for evaluating the policy impact on the cost of capital for corporate issuers. Krishnamurthy and Vissing-Jorgensen (2011) evaluate the effects of QE interventions on the yields of different asset classes. A key implication is that the effects on particular assets depend critically on which assets are purchased. In particular, Treasury-only purchases had a disproportionate effect on Treasuries relative to corporate bonds. Furthermore, D'Amico and King (2013) show that QE interventions generate local supply effects, and that the effects are strongest for securities that are closer substitutes to Treasury bonds whose maturities coincide with the policy target.

Motivated by these observations, I study asymmetries in the effects of quantitative easing across markets by comparing credit spread responses to two alternative policy interventions. The first intervention captures Treasury-only QE, and the second one corporate-only QE. I model both interventions as an unanticipated and permanent decline in the demand intercept $\Delta\theta_0^j$. In this model, QE acts on yields and credit spreads by reducing the arbitrageurs' net exposure to aggregate risk factors.

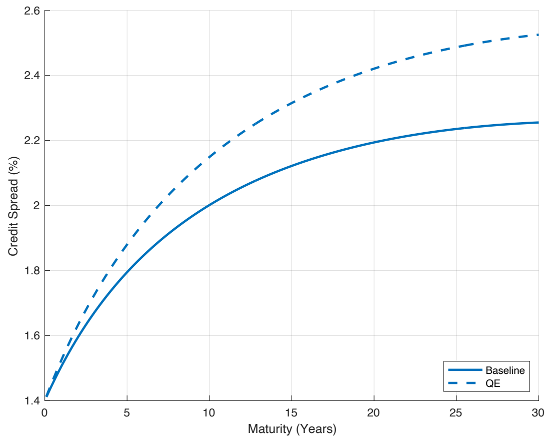
Figure 10a and Figure 10b illustrate the effect of QE purchases of government bonds only, modeled as a uniform decline in the demand intercept $\Delta\theta_0^G(\tau) < 0$ across all maturities. While the yields on Treasuries decline substantially, the impact on corporate yields is virtually zero. As a result, when QE interventions are concentrated in the Treasury market only, credit spreads increase. In contrast, Figure 10c and Figure 10d show that QE purchases of comparable magnitude but targeted to corporate bonds are much more effective in lowering corporate yields and credit spreads. Furthermore, a drop in $\Delta\theta_0^j(\tau) < 0$ also reduces the yields on government bonds, and the magnitude of the effect is comparable to the QE-only intervention.

In this model, the impact of QE on credit spreads is a combination of many effects. The direct effect is that a decline in habitat demand reduces the net positions held by the arbitrageurs. Yields fall because the market price of aggregate risk declines. However, while also government bonds fall in Figure 10c and Figure 10d, corporate bond yields are virtually unaffected by Treasury-interventions only. On the one hand, QE reduces the quantity of duration and credit risk that arbitrageurs have to absorb in equilibrium (Greenwood & Vayanos, 2014). While Treasury-only QE is mostly about extracting duration risk, corporate-only QE also reduces the quantity of credit risk the economy. On the other hand, purchases of government bonds also reduce the supply of safe assets and the supply of hedges against aggregate risk factors. Arbitrageurs value the hedging properties of Treasuries because they perform well in bad states of the world when default intensity increases. However, a reduction in Treasury supply increases the relative scarcity of hedges and safe assets, potentially raising the equilibrium price of safety (Krishnamurthy & Vissing-Jorgensen, 2011, 2012). In summary, Figure (10) is a consistent with both a portfolio rebalancing and a safety channel of QE interventions.

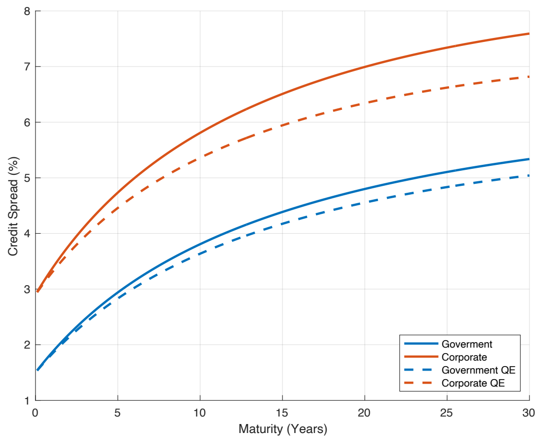
(a) Yield curves responses to $\Delta\theta_0^G(\tau) < 0$.



(b) Credit spread responses to $\Delta\theta_0^G(\tau) < 0$.



(c) Yield curves responses to $\Delta\theta_0^C(\tau) < 0$.



(d) Credit spread responses to $\Delta\theta_0^C(\tau) < 0$.

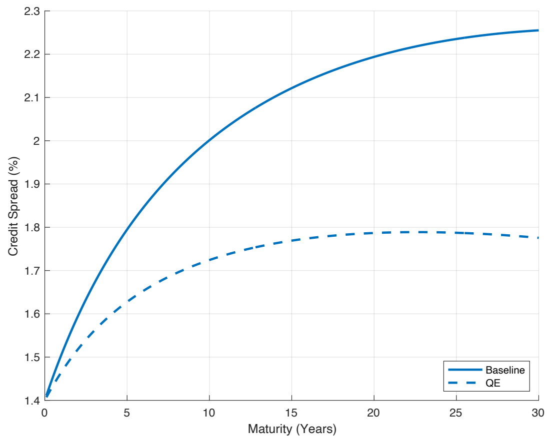


Figure 10: Impact of quantitative easing interventions (QE) across asset classes. I model Treasury-only QE as an unanticipated decline in the Treasury demand intercept from to $\theta_0^G = 289$ to $\theta_0^G = 260$. I model corporate-only QE as an unanticipated decline in the corporate demand intercept from to $\theta_0^C = 289$ to $\theta_0^C = 260$. The model parameters are described in Table 4.

5 Conclusion

Motivated by the insights that the variation in credit spreads is driven by time-varying risk premia rather than default probabilities and that intermediary-based factors explain a substantial fraction of the common variation in credit spreads, I study a model of the term structure of Treasury and corporate yields in which corporate and Treasury bonds are jointly priced by the same marginal investor. I integrate elements from the literature on credit risk valuation in a preferred-habitat context where asset prices are jointly determined by the pricing kernel of arbitrageurs that trade in both the Treasury and the corporate bond markets. I use my model to study (i) the interaction between credit and interest rate risk, (ii) the determinants of credit spreads, and (iii) how monetary policy interventions propagate throughout the term structure of credit spreads. I discipline the model to provide qualitative answers through a calibration exercise targeting empirical moments of the Treasury yield curve.

The propositions in the two sector model, as well as the calibration exercise, hint at a very strong dependence between credit risk and interest rate risk. In a context in which arbitrageurs are pricing both corporate bonds and Treasury bonds, this dependence is strengthened under the risk-neutral measure. Portfolio rebalancing effects have the potential to enrich asset pricing implications of habitat models and to shed more light on monetary policy transmission in a setting where assets are asymmetrically exposed to risk factors. The fact that risk prices of interest rate and credit risk are interconnected might explain some of the credit spread puzzles documented in the literature.

Nevertheless, the quantitative analysis reveals some limitations, which provide clear guidance onto where future efforts should directed. First, the implication that exogenous shocks to the short rate reduce credit spread is at odds with the literature. Future work is devoted to present empirical evidence of this mechanism and to understand how the model can match the data. Second, the specification of habitat demand lacks a solid microfoundation along two dimensions. On the one hand, it is unclear why habitat investors only respond to the price of a single maturity. On the other hand, fundamental news only affects habitat demand through prices, preventing these investors to react to fundamental shocks in the first place. In this regard, a better microfoundation of habitat demand is central to link habitat investors to key players in the corporate bond market as well as to generate realistic responses of risk premia to the aggregate risk factors. Third, the model suggests that intermediary inventories play a role in determining bond excess return. This results should be connected more tightly to the literature on intermediary asset pricing. Fourth, most of the asset pricing implications of the two sector model have not been tested yet. Improvements to the calibration procedure and a more thorough empirical analysis are necessary to better assess whether the model captures key features of the data.

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Appendices

A Mathematical Results

A.1 Auxiliary Lemmata and Corollaries

Lemma 2. *(Solution of System of Linear ODEs) Consider the system of linear first-order differential equations*

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$$

where A and $\boldsymbol{b}$ are constants. Suppose that A has distinct real eigenvalues and that $\boldsymbol{x}(0) = \mathbf{0}$. Let v_i denote an eigenvalue and u_i denote the corresponding eigenvector. Then

$$\boldsymbol{x} = \mathsf{u}_1\xi_1\left(\frac{e^{\mathsf{v}_1 x} - 1}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n\xi_n\left(\frac{e^{\mathsf{v}_n x} - 1}{\mathsf{v}_n}\right)$$

where $\xi = P^{-1}\boldsymbol{b}$.

Proof. Diagonalize A such that

$$A = PDP^{-1} \quad : \quad P \doteq \begin{bmatrix} \mathsf{u}_1 & \mathsf{u}_2 & \ldots & \mathsf{u}_n \end{bmatrix}$$

and consider $\boldsymbol{y} = P^{-1}\boldsymbol{x}$ (with the inverse $P\boldsymbol{y} = \boldsymbol{x}$). Then,

$$\begin{aligned} \boldsymbol{y}' &= P^{-1}\boldsymbol{x}' \\ &= P^{-1}\left(A\boldsymbol{x} + \boldsymbol{b}\right) \\ &= P^{-1}AP\boldsymbol{y} + P^{-1}\boldsymbol{b} \\ &= D\boldsymbol{y} + P^{-1}\boldsymbol{b} \end{aligned}$$

Let $\xi = P^{-1}b$, and denote ξ_i the ith element of the vector ξ. It follows that

$$y_i' = \mathsf{v}_i y + \xi_i$$

Then

$$\frac{dy_i}{dx} = \mathsf{v}_i y + \xi_i \Longrightarrow dy_i = (\mathsf{v}_i y + \xi_i)dx$$

or

$$\begin{aligned} \int \frac{1}{\mathsf{v}_i y + \xi_i} dy_i = \int dx &\Longrightarrow \frac{1}{\mathsf{v}_i}\ln(\mathsf{v}_i y + \xi_i) = x + c_i \\ &\Longrightarrow \ln(\mathsf{v}_i y + \xi_i) = \mathsf{v}_i x + \mathsf{v}_i c_i \\ &\Longrightarrow y_i = \frac{e^{\mathsf{v}_i x + \mathsf{v}_i c_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} \end{aligned}$$

Therefore

$$\boldsymbol{x} = P\boldsymbol{y} = \mathsf{u}_1\left(\frac{e^{\mathsf{v}_1 x + \mathsf{v}_1 c_1}}{\mathsf{v}_1} - \frac{\xi_1}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n\left(\frac{e^{\mathsf{v}_i x + \mathsf{v}_n c_n}}{\mathsf{v}_n} - \frac{\xi_n}{\mathsf{v}_n}\right)$$

Solving the system with the initial condition $x_1(0) = \cdots = x_n(0) = 0$ implies

$$y_i(0) = \frac{e^{\mathsf{v}_i \cdot 0 + \mathsf{v}_i c_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{e^{\mathsf{v}_i c_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = 0$$

or

$$e^{\mathsf{v}_i c_i} = \xi_i \Longrightarrow c_i = \frac{1}{\mathsf{v}_i}\ln \xi_i$$

Then

$$\frac{e^{\mathsf{v}_i x + \mathsf{v}_i c_1}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{e^{\mathsf{v}_i x + \mathsf{v}_i \frac{1}{\mathsf{v}_i}\ln \xi_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{e^{\mathsf{v}_i x} e^{\ln \xi_i}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \frac{\xi_i e^{\mathsf{v}_i x}}{\mathsf{v}_i} - \frac{\xi_i}{\mathsf{v}_i} = \xi_i\left(\frac{e^{\mathsf{v}_i x} - 1}{\mathsf{v}_i}\right)$$

Hence

$$\boldsymbol{x} = \mathsf{u}_1 \xi_1\left(\frac{e^{\mathsf{v}_1 x} - 1}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n \xi_n\left(\frac{e^{\mathsf{v}_n x} - 1}{\mathsf{v}_n}\right)$$

which is the desired result. ■

Lemma 3 (Expectation of Multivariate Ornstein-Uhlenbeck)**.** *Let s_τ be the state vector at time τ. Suppose that*

$$ds_t = -M^T\left(s_t - \overline{s}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}}$$

Under the risk-neutral measure $\mathbb{Q}$, q_τ is given by

$$s_\tau = e^{-M^T\tau}s_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{s}^{\mathbb{Q}} + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}}$$

where e^A is the matrix exponential operator. Further, since $B_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$

$$\mathbb{E}_0^{\mathbb{Q}}[s_\tau] = e^{-M^T\tau}s_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{s}^{\mathbb{Q}}$$

Proof. Define the demeaned process $\widetilde{q}_t = q_t - \overline{q}^{\mathbb{Q}}$. Because $\overline{q}^{\mathbb{Q}}$ is constant over time

$$d\widetilde{q}_t = dq_t \Longrightarrow d\widetilde{q}_t = -M^T\widetilde{q}_t + \Sigma dB_u^{\mathbb{Q}}$$

Standard arguments for the Ornstein-Uhlenbeck process (see e.g. Oksendal (1992)) give

$$\widetilde{q}_t = e^{-M^T\tau}\widetilde{q}_0 + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}}$$

Hence

$$\begin{aligned} q_\tau &= \overline{q}^{\mathbb{Q}} + e^{-M^T\tau}\left(q_0 - \overline{q}^{\mathbb{Q}}\right) + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}} \\ &= e^{-M^T\tau}q_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{q}^{\mathbb{Q}} + \int_0^\tau e^{-M^T(\tau-u)}\Sigma dB_u^{\mathbb{Q}} \end{aligned}$$

which gives the first result. Taking expectation under $\mathbb{Q}$ gives

$$\mathbb{E}_0^{\mathbb{Q}}[q_\tau] = e^{-M^T\tau}q_0 + \left(\mathbb{I} - e^{-M^T\tau}\right)\overline{q}^{\mathbb{Q}}$$

which gives the second result and completes the proof. ■

Lemma 4 (Useful Linear Operator)**.** *Let A be a (2×2) diagonal matrix and let $\boldsymbol{b}$ be a (2×1) column vector. Define the matrix function*

$$f(A,\boldsymbol{b}) = \mathbb{1}^T\left(P^T\right)^{-1}AP^Tb$$

Then

$$\mathbb{1}^T\left(P^T\right)^{-1}AP^Tb = b_1\left[a_1\psi_{rr} + a_2\psi_{r\lambda}\right] + b_2\left[a_1\psi_{\lambda r} + a_2\psi_{\lambda\lambda}\right]$$

where ψ_{rr}, $\psi_{r\lambda}$, $\psi_{\lambda r}$ and $\psi_{\lambda\lambda}$ are defined in Proposition 1.

Proof. Let

$$P = \begin{bmatrix} \mathsf{u}_{11} & \mathsf{u}_{12} \\ \mathsf{u}_{21} & \mathsf{u}_{22} \end{bmatrix} \quad : \quad A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad : \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then

$$\begin{aligned} \left(P^T\right)^{-1}AP^T &= \frac{1}{\det(P)}\begin{bmatrix} \mathsf{u}_{22} & -\mathsf{u}_{21} \\ -\mathsf{u}_{12} & \mathsf{u}_{11} \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} \mathsf{u}_{11} & \mathsf{u}_{21} \\ \mathsf{u}_{12} & \mathsf{u}_{22} \end{bmatrix} \\ &= \frac{1}{\det(P)}\begin{bmatrix} \mathsf{u}_{22}\mathsf{u}_{11}a_1 - \mathsf{u}_{21}\mathsf{u}_{12}a_2 & \mathsf{u}_{22}\mathsf{u}_{21}a_1 - \mathsf{u}_{21}\mathsf{u}_{22}a_2 \\ -\mathsf{u}_{12}\mathsf{u}_{11}a_1 + \mathsf{u}_{11}\mathsf{u}_{12}a_2 & -\mathsf{u}_{12}\mathsf{u}_{21}a_1 + \mathsf{u}_{11}\mathsf{u}_{22}a_2 \end{bmatrix} \end{aligned}$$

It follows that

$$\left(P^T\right)^{-1}AP^T\boldsymbol{b} = \frac{1}{\det(P)}\begin{bmatrix} b_1\left(\mathsf{u}_{22}\mathsf{u}_{11}a_1 - \mathsf{u}_{21}\mathsf{u}_{12}a_2\right) + b_2\left(\mathsf{u}_{22}\mathsf{u}_{21}a_1 - \mathsf{u}_{21}\mathsf{u}_{22}a_2\right) \\ b_1\left(-\mathsf{u}_{12}\mathsf{u}_{11}a_1 + \mathsf{u}_{11}\mathsf{u}_{12}a_2\right) + b_2\left(-\mathsf{u}_{12}\mathsf{u}_{21}a_1 + \mathsf{u}_{11}\mathsf{u}_{22}a_2\right) \end{bmatrix}$$

Pre-multiplication by $\mathbb{1}^T$ yields

$$\begin{aligned}
\mathbb{1}^T \left(P^T\right)^{-1} AP^T \boldsymbol{b} &= \frac{b_1}{\det(P)}\left[\mathsf{u}_{22}\mathsf{u}_{11}a_1 - \mathsf{u}_{21}\mathsf{u}_{12}a_2 - \mathsf{u}_{12}\mathsf{u}_{11}a_1 + \mathsf{u}_{11}\mathsf{u}_{12}a_2\right] \\
&\quad + \frac{b_2}{\det(P)}\left[\mathsf{u}_{22}\mathsf{u}_{21}a_1 - \mathsf{u}_{21}\mathsf{u}_{22}a_2 - \mathsf{u}_{12}\mathsf{u}_{21}a_1 + \mathsf{u}_{11}\mathsf{u}_{22}a_2\right] \\
&= b_1\left[a_1\frac{u_{11}(u_{22} - \mathsf{u}_{12})}{\det(P)} + a_2\frac{\mathsf{u}_{12}(\mathsf{u}_{11} - \mathsf{u}_{21})}{\det(P)}\right] \\
&\quad + b_2\left[a_1\frac{\mathsf{u}_{21}(\mathsf{u}_{22} - \mathsf{u}_{12})}{\det(P)} + a_2\frac{\mathsf{u}_{22}(\mathsf{u}_{11} - \mathsf{u}_{21})}{\det(P)}\right] \\
&= b_1\left[a_1\psi_{11} + a_2\psi_{12}\right] + b_2\left[a_1\psi_{21} + a_2\psi_{22}\right]
\end{aligned}$$

as desired. ■

A.2 Proofs

A.2.1 Proof of Proposition 1

Lemma 5 (Arbitageurs' First-order Condition)**.** *Under conjecture (8), the arbitrageurs' first-order condition is*

$$\mu_t^{(\tau)} - r_t = \lambda_t + A_r(\tau)\pi_{r,t} + A_\lambda(\tau)\pi_{\lambda,t}$$

where the risk prices are given by

$$\begin{aligned}
\pi_{r,t} &\doteq a\sigma_r^2\left(\int_0^T X_t^{(\tau)}A_r(\tau)d\tau\right) \\
\pi_{\lambda,t} &\doteq a\sigma_\lambda^2\left(\int_0^T X_t^{(\tau)}A_\lambda(\tau)d\tau\right)
\end{aligned}$$

Proof. Applying Itô's Lemma to equation (8) and using the fact that $dB_{r,t}$ and $dB_{\lambda,t}$ are independent gives

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \left[A_r'(\tau)r_t + A_\lambda'(\tau)\lambda_t + C'(\tau)\right]dt - A_r(\tau)dr_t - A_\lambda(\tau)d\lambda_t + \frac{1}{2}\sigma_r^2A_r(\tau)^2dt + \frac{1}{2}\sigma_\lambda^2A_\lambda(\tau)^2dt$$

This can be written as

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)}dt - \sigma_rA_r(\tau)dB_{r,t} - \sigma_\lambda A_\lambda(\tau)dB_{\lambda,t}$$

where

$$\begin{aligned}
\mu_t^{(\tau)} \doteq{}& A_r'(\tau)r_t + A_\lambda'(\tau)\lambda_t + C'(\tau) + A_r(\tau)\kappa_r(r_t - \overline{r}) + A_\lambda(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) \\
&+ \frac{1}{2}\sigma_r^2A_r(\tau)^2 + \frac{1}{2}\sigma_\lambda^2A_\lambda(\tau)^2
\end{aligned}$$

Substituting the implied dynamics of $\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}$ into the arbitrageurs' budget constraint implies that

$$\mathbb{E}_t\left[dW_t\right] = \left(W_t - \int_0^\infty X_t^{(\tau)} d\tau\right) r_t dt + \int_0^\infty X_t^{(\tau)} \left(\mu_t^{(\tau)} dt - \lambda_t dt\right) d\tau$$

and

$$\begin{aligned}\mathbb{V}\mathrm{ar}_t\left(dW_t\right) &= \mathbb{V}\mathrm{ar}_t\left(\int_0^\infty X_t^{(\tau)}\left(-\sigma_r A_r(\tau) dB_{r,t} - \sigma_\lambda A_\lambda(\tau) dB_{\lambda,t}\right) d\tau\right) \\ &= \sigma_r^2\left(\int_0^\infty X_t^{(\tau)} A_r(\tau) d\tau\right)^2 dt + \sigma_\lambda^2\left(\int_0^\infty X_t^{(\tau)} A_\lambda(\tau) d\tau\right)^2 dt\end{aligned}$$

Pointwise maximization with respect to $X_t^{(\tau)}$ gives

$$\mu_t^{(\tau)} - r_t = \lambda_t + a\sigma_r^2 A_r(\tau)\left(\int_0^\infty X_t^{(\tau)} A_r(\tau) d\tau\right) + a\sigma_\lambda^2 A_\lambda(\tau)\left(\int_0^\infty X_t^{(\tau)} A_\lambda(\tau) d\tau\right)$$

or

$$\mu_t^{(\tau)} - r_t = \lambda_t - A_r(\tau)\sigma_r \cdot \pi_{r,t} - A_\lambda(\tau)\sigma_\lambda \cdot \pi_{\lambda,t}$$

where

$$\begin{aligned}\pi_{r,t} &\doteq -a\sigma_r\left(\int_0^\infty X_t^{(\tau)} A_r(\tau) d\tau\right) \\ \pi_{\lambda,t} &\doteq -a\sigma_\lambda\left(\int_0^\infty X_t^{(\tau)} A_\lambda(\tau) d\tau\right)\end{aligned}$$

as desired. ■

Proposition (1) characterizes the solution to the system

$$\begin{aligned}A_r'(\tau) + A_r(\tau)\kappa_r - 1 &= -a\sigma_r^2 A_r(\tau)\left(\int_0^\infty \alpha(\tau) A_r(\tau)^2 d\tau\right) - a\sigma_\lambda^2 A_\lambda(\tau)\left(\int_0^\infty \alpha(\tau) A_r(\tau) A_\lambda(\tau) d\tau\right) \\ A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda - 1 &= -a\sigma_r^2 A_r(\tau)\left(\int_0^\infty \alpha(\tau) A_r(\tau) A_\lambda(\tau) d\tau\right) - a\sigma_\lambda^2 A_\lambda(\tau)\left(\int_0^\infty \alpha(\tau) A_\lambda(\tau)^2 d\tau\right)\end{aligned}$$

and, as a result, the equilibrium in the one sector model.

Proposition 6 (Equilibrium in the One Sector Model)**.** *Given the initial conditions $A_r(0) = A_\lambda(0) = 0$, the function $A(\tau) = (A_r(\tau), A_\lambda(\tau))^T$ is given by*

$$A(\tau) = \psi_1\left(\frac{1 - e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right) + \psi_2\left(\frac{1 - e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right) \tag{1}$$

where v_k is the k-th eigenvectors of the matrix M defined by

$$M \doteq \left[\Gamma^T + a\int_0^\infty \alpha(\tau) A(\tau) A(\tau)^T d\tau \Sigma\Sigma^T\right]$$

and ψ_k are constant vectors such that $\psi_k = \mathsf{u}_k\xi_i$, where u_k is the eigenvector corresponding to v_k and ξ_i is the ith component of $\xi \doteq P^{-1}\mathbb{1}$, where $P \doteq [\mathsf{u}_1, \mathsf{u}_2]$.

Proof. Using matrix notation with $s_t \doteq (r_t, \lambda_t)^T$, the arbitrageurs' first-order condition can be written as

$$\mu_t^{(\tau)} - r_t = \lambda_t + aA(\tau)^T \Sigma\Sigma^T \int_0^\infty x_t^{(\tau)} A(\tau) d\tau$$

Imposing market clearing and using the definition of $\mu_t^{(\tau)}$, I obtain

$$\begin{aligned} A'(\tau)^T s_t + C'(\tau) + A(\tau)^T \Gamma \left(s_t - \overline{q}\right) + \frac{1}{2} A(\tau)^T \Sigma\Sigma^T A(\tau) - r_t \\ = \lambda_t + aA(\tau)^T \Sigma\Sigma^T \int_0^\infty \underbrace{\left[\beta(\tau) - \alpha(\tau)A(\tau)^T s_t - \alpha(\tau)C(\tau)\right]}_{\text{scalar}} A(\tau) d\tau \end{aligned}$$

Collecting the terms multiplying s_t gives

$$A'(\tau)^T + A(\tau)^T \Gamma - e_1 = e_2 - aA(\tau)^T \Sigma\Sigma^T \int_0^\infty \alpha(\tau) A(\tau)^T A(\tau) d\tau$$

where e_1 and e_2 are the 2-dimensional basis vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$, respectively. Since $e_1 + e_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} = \mathbf{1}$, this can be written as

$$A'(\tau) + \left[\Gamma^T + a \int_0^\infty \alpha(\tau) A(\tau)^T A(\tau) d\tau \Sigma\Sigma^T\right] A(\tau) - \mathbf{1} = 0$$

or

$$A'(\tau) + MA(\tau) - \mathbf{1} = 0$$

where the matrix M is given by

$$M \doteq \left[\Gamma^T + a \int_0^\infty \alpha(\tau) A(\tau) A(\tau)^T d\tau \Sigma\Sigma^T\right]$$

Even though the entries of the matrix M depends on $A(\tau)$, the integral with respect to τ implies that these are the same for all maturities and can be treated as constants.

Given the initial conditions $A_r(0) = A_\lambda(0) = 0$, I specialize Lemma (2) such that $A = -M$ and $\boldsymbol{b} = 1$. It follows that $A(\tau)$ is given by

$$A(\tau) = \mathsf{u}_1 \xi_1 \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1}\right) + \cdots + \mathsf{u}_n \xi_n \left(\frac{1 - e^{-\mathsf{v}_n \tau}}{\mathsf{v}_n}\right)$$

Further, since $\boldsymbol{b} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Then

$$\xi = P^{-1}\boldsymbol{b} = \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} & -\mathsf{u}_{12} \\ -\mathsf{u}_{21} & \mathsf{u}_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} - \mathsf{u}_{12} \\ \mathsf{u}_{11} - \mathsf{u}_{21} \end{bmatrix}$$

Hence

$$\xi_1 = \frac{\mathsf{u}_{22} - \mathsf{u}_{12}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \quad : \quad \xi_2 = \frac{\mathsf{u}_{11} - \mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}}$$

Therefore

$$A(\tau) = \mathsf{u}_1 \frac{\mathsf{u}_{22} - \mathsf{u}_{12}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right) + \mathsf{u}_2 \frac{\mathsf{u}_{11} - \mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right)$$

or

$$\begin{aligned} A_r(\tau) &= \psi_{rr} \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right) + \psi_{r\lambda} \left(\frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right) \\ A_\lambda(\tau) &= \psi_{\lambda r} \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right) + \psi_{\lambda\lambda} \left(\frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right) \end{aligned}$$

where the constants ψ_{rr}, $\psi_{r\lambda}$, $\psi_{\lambda r}$, and $\psi_{\lambda\lambda}$ are given by

$$\psi_{rr} \doteq \frac{\mathsf{u}_{11}(\mathsf{u}_{22} - \mathsf{u}_{12})}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \quad : \quad \psi_{r\lambda} \doteq \frac{\mathsf{u}_{12}(\mathsf{u}_{11} - \mathsf{u}_{21})}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \tag{2}$$

$$\psi_{\lambda r} \doteq \frac{\mathsf{u}_{21}(\mathsf{u}_{22} - \mathsf{u}_{12})}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \quad : \quad \psi_{\lambda\lambda} \doteq \frac{\mathsf{u}_{22}(\mathsf{u}_{11} - \mathsf{u}_{21})}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \tag{3}$$

and only depend on the eigenvectors of M. This is the desired result, which concludes the proof. ■

Corollary 1 (Equilibrium in Vayanos and Vila (2021))**.** *If $\boldsymbol{b} = e_1$, then*

$$\begin{aligned} A_r(\tau) &= \frac{\mathsf{u}_{11}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{21}} \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right) - \frac{\mathsf{u}_{12}\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right) \\ A_\beta(\tau) &= \frac{\mathsf{u}_{21}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right) - \frac{-\mathsf{u}_{21}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right) \end{aligned}$$

which corresponds to the equilibrium with $K = 1$ demand shocks in Vayanos and Vila (2021).

Proof. Specialize Lemma (2) with $\boldsymbol{b} = e_1$. Then

$$\begin{aligned} \xi = P^{-1}\boldsymbol{b} &= \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} & -\mathsf{u}_{12} \\ -\mathsf{u}_{21} & \mathsf{u}_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \begin{bmatrix} \mathsf{u}_{22} \\ -\mathsf{u}_{21} \end{bmatrix} \end{aligned}$$

Hence

$$\xi_1 = \frac{\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}} \quad : \quad \xi_2 = \frac{-\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{21}\mathsf{u}_{12}}$$

Therefore

$$A(\tau) = \mathsf{u}_1 \frac{\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right) + \mathsf{u}_2 \frac{-\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}} \left(\frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right)$$

or, using the notation in Vayanos and Vila (2021),

$$A_r(\tau) = \frac{\mathsf{u}_{11}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{21}}\left(\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right) - \frac{\mathsf{u}_{12}\mathsf{u}_{21}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}}\left(\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)$$
$$A_\beta(\tau) = \frac{\mathsf{u}_{21}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}}\left(\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right) - \frac{-\mathsf{u}_{21}\mathsf{u}_{22}}{\mathsf{u}_{11}\mathsf{u}_{22} - \mathsf{u}_{12}\mathsf{u}_{12}}\left(\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)$$

which gives the desired result. ■

A.2.2 Proof of Proposition 2

Let M be the $(n \times n)$ matrix given in Proposition (1). It if useful to establish some useful properties of matrix exponentials. Provided that M is diagonalizable. Then

$$M = PDP^{-1}$$

where $D = \text{Diag}(\mathsf{v}_i)$ and $P = \begin{bmatrix} \mathsf{u}_1 & \dots & \mathsf{u}_n \end{bmatrix}$. This implies that the matrix exponential e^M is equal to

$$e^M = Pe^DP^{-1} = P \cdot \text{Diag}\left(e^{\mathsf{v}_i}\right) \cdot P^{-1}$$

Further, recall the definition of the constants ψ_{rr}, $\psi_{r\lambda}$, $\psi_{\lambda r}$, and $\psi_{\lambda\lambda}$ given by (2)–(3) from Proposition (1).

Armed with Lemmata (3) and (4), I consider the pricing of a zero-coupon defaultable bond with unitary payoff at time τ conditional on not defaulting, i.e. $\tau_D > \tau$. Let τ_D denote the default (stopping-time) and consider the indicator function $\mathbb{1}_{\{\tau^D > \tau\}}$. Then

$$P_0^{(\tau)} = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau r_u du}\mathbb{1}_{\{\tau^D>\tau\}}\right] = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau (r_u+\lambda_u)du}\right] \tag{4}$$

The proof of Proposition (2) consists in showing that the price of the defaultable bond given by (4) is the same as $P_0^{(\tau)} = e^{-[A(\tau)q_0 + C(\tau)]}$. To this purpose, I conjecture that, under the risk-neutral measure $\mathbb{Q}$, the state vector q_t evolves as

$$dq_t = -M^T\left(q_t - \overline{q}^{\mathbb{Q}}\right)dt + \Sigma dB_t^{\mathbb{Q}}$$

where M solves the ODE system, $\overline{q}^{\mathbb{Q}}$ is the long-term average under $\mathbb{Q}$, and $M^T\overline{q}^{\mathbb{Q}} = \chi$. The economic content of this proposition is that, provided that arbitrageurs are risk-averse, i.e. $a \neq 0$, M^T will not be diagonal. This induces correlation between λ_t and r_t through drift dependence under $\mathbb{Q}$.

Proposition 7 (Equivalence with Risk-neutral Valuation). *Let $\mathbb{Q}$ denote the risk-neutral measure and τ_D the (stopping) time of default of an individual corporate bond. Then*

$$P_t^{(\tau)} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau} r_u du}\mathbb{1}_{\{\tau^D>\tau\}}\right] \stackrel{!}{=} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+\tau} (r_u+\lambda_u)du}\right] \tag{5}$$

Proof. Write $\mathbb{1}^T s_t = r_t + \lambda_t$. Lemma (3) implies that, conditional on information at time 0, s_τ is multivariate Gaussian. Hence

$$\begin{aligned}
P_0^{(\tau)} &= \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau (r_u+\lambda_u)du}\right] = e^{-\mathbb{E}_0^{\mathbb{Q}}\left[-\int_0^\tau (r_u+\lambda_u)du\right]+\frac{1}{2}\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left(\int_0^\tau (r_u+\lambda_u)du\right)} \\
&= e^{-\mathbb{E}_0^{\mathbb{Q}}\left[\int_0^\tau \mathbb{1}^T s_u du\right]+\frac{1}{2}\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right)}
\end{aligned}$$

Interchanging the expectation with the integral, the first term in the exponent can be rewritten as

$$\mathbb{E}_0^{\mathbb{Q}}\left[-\int_0^\tau \mathbb{1}^T s_u du\right] = -\int_0^\tau \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}^T s_u\right] du$$

Using Lemma (3),

$$\mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T s_u] = \mathbb{1}^T e^{-M^T u} s_0 + \mathbb{1}^T\left(\mathbb{I} - e^{-M^T u}\right)\overline{s}^{\mathbb{Q}} = \mathbb{1}^T e^{-M^T u} s_0 + \mathbb{1}^T \overline{s}^{\mathbb{Q}} - \mathbb{1}^T e^{-M^T u}\overline{s}^{\mathbb{Q}}$$

Using the fact that $M = PDP^{-1}$ and $M^T\overline{s}^{\mathbb{Q}} = \chi$, it follows that

$$\overline{s}^{\mathbb{Q}} = \left(M^T\right)^{-1}\chi = \left((P^{-1})^T D P^T\right)^{-1}\chi = \left(P^T\right)^{-1} D^{-1} P^T \chi$$

and, since $e^{-M^T\tau} = \left(P^{-1}\right)^T e^{-\tau D} P^T$,

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T s_u] &= \mathbb{1}^T e^{-M^T u} s_0 + \mathbb{1}^T \overline{s}^{\mathbb{Q}} - \mathbb{1}^T e^{-M^T u}\overline{s}^{\mathbb{Q}} \\
&= \mathbb{1}^T\left(P^{-1}\right)^T e^{-\tau D} P^T s_0 + \mathbb{1}^T\overline{s}^{\mathbb{Q}} - \mathbb{1}^T\left(P^{-1}\right)^T e^{-\tau D} P^T \left(P^T\right)^{-1} D^{-1} P^T \chi \\
&= \mathbb{1}^T\left(P^{-1}\right)^T e^{-\tau D} P^T s_0 + \mathbb{1}^T\overline{s}^{\mathbb{Q}} - \mathbb{1}^T\left(P^{-1}\right)^T e^{-\tau D} D^{-1} P^T \chi \\
&\overset{\dagger}{=} r_0\left[\psi_{rr} e^{-u\mathsf{v}_1} + \psi_{r\lambda} e^{-u\mathsf{v}_2}\right] + \lambda_0\left[\psi_{\lambda r} e^{-u\mathsf{v}_1} + \psi_{\lambda\lambda} e^{-u\mathsf{v}_2}\right] \\
&\qquad + \chi_r\left[\frac{\psi_{rr}}{\mathsf{v}_1} + \frac{\psi_{r\lambda}}{\mathsf{v}_2}\right] + \chi_\lambda\left[\frac{\psi_{\lambda r}}{\mathsf{v}_1} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_2}\right] \\
&\qquad - \chi_r\left[\frac{\psi_{rr}}{\mathsf{v}_1} e^{-u\mathsf{v}_1} + \frac{\psi_{r\lambda}}{\mathsf{v}_2} e^{-u\mathsf{v}_2}\right] - \chi_\lambda\left[\frac{\psi_{\lambda r}}{\mathsf{v}_1} e^{-u\mathsf{v}_1} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_2} e^{-u\mathsf{v}_2}\right]
\end{aligned}$$

where the equality † follows from repeated application of Lemma (4) after noting that $e^{-\tau D}$ and $e^{-\tau D}D^{-1}$ are diagonal matrices. Integrating with respect to time gives

$$\begin{aligned}
\int_0^\tau \mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T s_u] du &= r_0\left[\psi_{rr}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{r\lambda}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] + \lambda_0\left[\psi_{\lambda r}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{\lambda\lambda}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] \\
&\quad + \tau\left\{\chi_r\left[\frac{\psi_{rr}}{\mathsf{v}_1} + \frac{\psi_{r\lambda}}{\mathsf{v}_2}\right] + \chi_\lambda\left[\frac{\psi_{\lambda r}}{\mathsf{v}_1} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_2}\right]\right\} \\
&\quad - \chi_r\left[\frac{\psi_{rr}}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{r\lambda}}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] - \chi_\lambda\left[\frac{\psi_{\lambda r}}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] \\
&= r_0 A_r(\tau) + \lambda_0 A_\lambda(\tau) + \tau\left\{\chi_r\left[\frac{\psi_{rr}}{\mathsf{v}_1} + \frac{\psi_{r\lambda}}{\mathsf{v}_2}\right] + \chi_\lambda\left[\frac{\psi_{\lambda r}}{\mathsf{v}_1} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_2}\right]\right\} \\
&\quad - \chi_r\left[\frac{\psi_{rr}}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{r\lambda}}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right] - \chi_\lambda\left[\frac{\psi_{\lambda r}}{\mathsf{v}_1}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_2}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right]
\end{aligned}$$

However, from Proposition (1)

$$C(\tau) = \left(\int_0^\tau A(u)^T du\right)\chi - \frac{1}{2}\int_0^\tau A(u)^T \Sigma\Sigma^T A(u) du$$

Expanding the first term gives

$$\begin{aligned}\left(\int_0^\tau A(u)du\right) &= \begin{bmatrix}\int_0^\tau \psi_{rr}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{r\lambda}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}du\\ \int_0^\tau \psi_{\lambda r}\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1} + \psi_{\lambda\lambda}\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}du\end{bmatrix}\\ &= \begin{bmatrix}\frac{\psi_{rr}}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\psi_{r\lambda}}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\}\\ \frac{\psi_{\lambda r}}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_1}\left\{\tau - \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\}\end{bmatrix}\end{aligned}$$

Hence,

$$\begin{aligned}\left(\int_0^\tau A^T(u)du\right)\begin{bmatrix}\chi_r\\ \chi_\lambda\end{bmatrix} &= \chi_r\left\{\frac{\psi_{rr}}{\mathsf{v}_1}\left(\tau + \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right) + \frac{\psi_{r\lambda}}{\mathsf{v}_1}\left(\tau + \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)\right\}\\ &\quad + \chi_\lambda\left\{\frac{\psi_{\lambda r}}{\mathsf{v}_1}\left(\tau + \frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right) + \frac{\psi_{\lambda\lambda}}{\mathsf{v}_1}\left(\tau + \frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right)\right\}\end{aligned}$$

By comparing this term with the expression for $\int_0^\tau \mathbb{E}_0^{\mathbb{Q}}[\mathbb{1}^T q_u]du$,

$$\begin{aligned}-\int_0^\tau \mathbb{1}^T\mathbb{E}_0^{\mathbb{Q}}[q_u] &= -r_0 A_r(\tau) - \lambda_0 A_\lambda(\tau) - \tau\left\{\chi_r\left[\frac{\psi_{11}}{\nu_1} + \frac{\psi_{12}}{\nu_2}\right] + \chi_\lambda\left[\frac{\psi_{21}}{\nu_1} + \frac{\psi_{22}}{\nu_2}\right]\right\}\\ &\quad + \chi_r\left[\frac{\psi_{11}}{\nu_1}\frac{1-e^{-\nu_1\tau}}{\nu_1} + \frac{\psi_{12}}{\nu_2}\frac{1-e^{-\nu_2\tau}}{\nu_2}\right] + \chi_\lambda\left[\frac{\psi_{21}}{\nu_1}\frac{1-e^{-\nu_1\tau}}{\nu_1} + \frac{\psi_{22}}{\nu_2}\frac{1-e^{-\nu_2\tau}}{\nu_2}\right]\\ &= -A^T(\tau)q_0 - \left(\int_0^\tau A^T(u)du\right)\chi\end{aligned}$$

matching the linear terms in the exponent of $e^{-[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)]}$. This gives the desired result and completes the first part of the proof. The second part of the proof matches the variance term. To compute is

$$\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T q_u du\right) = \mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T\left(\int_0^u e^{-M^T(u-v)}\Sigma dB_v^{\mathbb{Q}}\right)du\right)$$

Note that

$$\begin{aligned}\mathbb{V}\mathrm{ar}_0^{\mathbb{Q}}\left[\int_0^\tau \mathbb{1}^T q_u du\right] &= \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(\mathbb{1}^T q_u, \mathbb{1}^T q_{u'})du du'\\ &= \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(r_u + \lambda_u, r_{u'} + \lambda_{u'})du du'\\ &= \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(r_u, r_{u'})du du' + \int_0^\tau\int_0^\tau \mathbb{C}\mathrm{ov}(\lambda_u, \lambda_{u'})du du'\end{aligned}$$

The covariance functions of each of the components of q_u separately. Recall that

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(r_u, r_{u'}) &= \mathbb{C}\mathrm{ov}\left\{ e_1^T \left(\int_0^u e^{-M^T(u-v)} \Sigma dB_v^{\mathbb{Q}} \right), e_1^T \left(\int_0^{u'} e^{-M^T(u'-v)} \Sigma dB_v^{\mathbb{Q}} \right) \right\} \\
&= \mathbb{C}\mathrm{ov}\left\{ e_1^T \left(\int_0^{\min\{u,u'\}} e^{-M^T(u-v)} \Sigma dB_v^{\mathbb{Q}} \right), e_1^T \left(\int_0^{\min\{u,u'\}} e^{-M^T(u'-v)} \Sigma dB_v^{\mathbb{Q}} \right) \right\} \\
&= \mathbb{C}\mathrm{ov}\left(\int_0^{\min\{u,u'\}} e_1^T e^{-M^T(u-v)} \Sigma dB_v^{\mathbb{Q}}, \int_0^{\min\{u,u'\}} e_1^T e^{-M^T(u'-v)} \Sigma dB_v^{\mathbb{Q}} \right) \\
&= \sigma_r^2 \psi_{rr}^2 e^{-\mathsf{v}_1(u+u')} \int_0^{\min\{u,u'\}} e^{2\mathsf{v}_1 v} dv + \sigma_r^2 \psi_{rr} \psi_{r\lambda} e^{-\mathsf{v}_1 u - \mathsf{v}_2 u'} \int_0^{\min\{u,u'\}} e^{(\mathsf{v}_1+\mathsf{v}_2)v} dv \\
&\quad + \sigma_r^2 \psi_{r\lambda} \psi_{rr} e^{-\mathsf{v}_2 u - \mathsf{v}_1 u'} \int_0^{\min\{u,u'\}} e^{(\mathsf{v}_1+\mathsf{v}_2)v} dv + \sigma_r^2 \psi_{rr}^2 e^{-\mathsf{v}_2(u+u')} \int_0^{\min\{u,u'\}} e^{2\mathsf{v}_2 v} dv
\end{aligned}$$

whereas the second line uses Lemma (4) in combination with the Itô Isometry. Each of the integrals is a Riemann integral of the form

$$\int_0^{\min\{u,u'\}} e^{\mathsf{v}v} dv = \frac{1}{\mathsf{v}} e^{\mathsf{v}v} \Big|_0^{\min\{u,u'\}} = \frac{1}{\mathsf{v}} \left(e^{\mathsf{v} \min\{u,u'\}} - 1 \right)$$

Hence

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(r_u, r_{u'}) &= \sigma_r^2 \psi_{rr}^2 e^{-\mathsf{v}_1(u+u')} \frac{1}{2\mathsf{v}_1} \left(e^{2\mathsf{v}_1 \min\{u,u'\}} - 1 \right) + \sigma_r^2 \psi_{rr} \psi_{r\lambda} e^{-\mathsf{v}_1 u - \mathsf{v}_2 u'} \frac{e^{(\mathsf{v}_1+\mathsf{v}_2)\min\{u,u'\}} - 1}{\mathsf{v}_1 + \mathsf{v}_2} \\
&\quad + \sigma_r^2 \psi_{r\lambda} \psi_{rr} e^{-\mathsf{v}_2 u - \mathsf{v}_1 u'} \frac{e^{(\mathsf{v}_1+\mathsf{v}_2)\min\{u,u'\}} - 1}{\mathsf{v}_1 + \mathsf{v}_2} + \sigma_r^2 \psi_{rr}^2 e^{-\mathsf{v}_2(u+u')} \frac{1}{2\mathsf{v}_2} \left(e^{2\mathsf{v}_2 \min\{u,u'\}} - 1 \right)
\end{aligned}$$

Therefore, for the case that $u > u'$

$$\frac{1}{2} \int_t^{t+\tau} \int_t^{t+\tau} \mathbb{C}\mathrm{ov}(r_u, r_{u'}) du du' = \frac{\sigma_r^2 \psi_{rr}^2}{2\mathsf{v}_1} \left\{ \tau + \frac{1 - e^{-2\mathsf{v}_1 \tau}}{2\mathsf{v}_1} - 2 \frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right\}$$

Then

$$\begin{aligned}
\int_0^\tau \int_0^u \mathbb{C}\mathrm{ov}(r_u, r_{u'}) du' du &= \frac{\sigma_r^2 \psi_{rr}^2}{2\mathsf{v}_1{}^2} \left\{ \tau + \frac{1 - e^{-2\mathsf{v}_1 \tau}}{2\mathsf{v}_1} - 2 \frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right\} + \frac{\sigma_r^2 \psi_{r\lambda}^2}{2\mathsf{v}_2{}^2} \left\{ \tau + \frac{1 - e^{-2\mathsf{v}_2 \tau}}{2\mathsf{v}_2} - 2 \frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right\} \\
&\quad + \left[\frac{1}{\mathsf{v}_1} + \frac{1}{\mathsf{v}_2} \right] \frac{\sigma_r^2 \psi_{rr} \psi_{r\lambda}}{\mathsf{v}_1 + \mathsf{v}_2} \left\{ \tau \ + \frac{1 - e^{-(\mathsf{v}_1+\mathsf{v}_2)\tau}}{(\mathsf{v}_1 + \mathsf{v}_2)} - \frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} - \frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right\}
\end{aligned}$$

By symmetry

$$\begin{aligned}
\int_0^\tau \int_0^u \mathbb{C}\mathrm{ov}(\lambda_u, \lambda_{u'}) du' du &= \frac{\sigma_\lambda^2 \psi_{\lambda r}^2}{2\mathsf{v}_1{}^2} \left\{ \tau + \frac{1 - e^{-2\mathsf{v}_1 \tau}}{2\mathsf{v}_1} - 2 \frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} \right\} + \frac{\sigma_\lambda^2 \psi_{\lambda\lambda}^2}{2\mathsf{v}_2{}^2} \left\{ \tau + \frac{1 - e^{-2\mathsf{v}_2 \tau}}{2\mathsf{v}_2} - 2 \frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right\} \\
&\quad + \left[\frac{1}{\mathsf{v}_1} + \frac{1}{\mathsf{v}_2} \right] \frac{\sigma_r^2 \psi_{\lambda r} \psi_{\lambda\lambda}}{\mathsf{v}_1 + \mathsf{v}_2} \left\{ \tau \ + \frac{1 - e^{-(\mathsf{v}_1+\mathsf{v}_2)\tau}}{(\mathsf{v}_1 + \mathsf{v}_2)} - \frac{1 - e^{-\mathsf{v}_1 \tau}}{\mathsf{v}_1} - \frac{1 - e^{-\mathsf{v}_2 \tau}}{\mathsf{v}_2} \right\}
\end{aligned}$$

As a result, these two expressions give

$$\frac{1}{2}\mathbb{V}\text{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right) = \int_0^\tau \int_0^u \mathbb{C}\text{ov}(r_u, r_{u'})du'du + \int_0^\tau \int_0^u \mathbb{C}\text{ov}(\lambda_u, \lambda_{u'})du'du$$

The final step is to show that the second element of $C(\tau)$ is equal to these two terms. Note that

$$\frac{1}{2}\int_0^\tau A(u)^T \Sigma\Sigma^T A(u)du = \frac{1}{2}\int_0^\tau \sigma_r^2 A_r^2(u)du + \frac{1}{2}\int_0^\tau \sigma_\lambda^2 A_\lambda^2(u)du$$

Then, using the expressions for $A_r(\tau)$ and $A_\lambda(\tau)$

$$\begin{aligned}
\frac{1}{2}\sigma_r^2\int_0^\tau A_r^2(u)du &= \frac{\sigma_r^2\psi_{rr}^2}{2\mathsf{v}_1{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\sigma_r^2\psi_{r\lambda}^2}{2\mathsf{v}_2{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_2\tau}}{2\mathsf{v}_2} - 2\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\} \\
&\quad + \frac{\psi_{rr}\psi_{r\lambda}}{\mathsf{v}_1\mathsf{v}_2}\left\{\tau - \frac{1-e^{-\mathsf{v}_1 u}}{\mathsf{v}_1} - \frac{1-e^{-\mathsf{v}_2 u}}{\mathsf{v}_2} + \frac{1-e^{-(\mathsf{v}_1+\mathsf{v}_2)}}{\mathsf{v}_1+\mathsf{v}_2}\right\} \\
\frac{1}{2}\sigma_\lambda^2\int_0^\tau A_\lambda^2(u)du &= \frac{\sigma_\lambda^2\psi_{\lambda r}^2}{2\mathsf{v}_1{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_1\tau}}{2\mathsf{v}_1} - 2\frac{1-e^{-\mathsf{v}_1\tau}}{\mathsf{v}_1}\right\} + \frac{\sigma_r^2\psi_{\lambda\lambda}^2}{2\mathsf{v}_2{}^2}\left\{\tau + \frac{1-e^{-2\mathsf{v}_2\tau}}{2\mathsf{v}_2} - 2\frac{1-e^{-\mathsf{v}_2\tau}}{\mathsf{v}_2}\right\} \\
&\quad + \frac{\psi_{\lambda\lambda}\psi_{\lambda r}}{\mathsf{v}_1\mathsf{v}_2}\left\{\tau - \frac{1-e^{-\mathsf{v}_1 u}}{\mathsf{v}_1} - \frac{1-e^{-\mathsf{v}_2 u}}{\mathsf{v}_2} + \frac{1-e^{-(\mathsf{v}_1+\mathsf{v}_2)}}{\mathsf{v}_1+\mathsf{v}_2}\right\}
\end{aligned}$$

which clearly match the expressions given above. As a result

$$\frac{1}{2}\mathbb{V}\text{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right) = \frac{1}{2}\int_0^\tau A(u)^T\Sigma\Sigma^T A(u)du$$

from which it follows that

$$\begin{aligned}
P_0^{(\tau)} = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^\tau (r_u+\lambda_u)du}\right] &= e^{-\mathbb{E}_0^{\mathbb{Q}}\left[\int_0^\tau \mathbb{1}^T s_u du\right] + \frac{1}{2}\mathbb{V}\text{ar}_0^{\mathbb{Q}}\left(\int_0^\tau \mathbb{1}^T s_u du\right)} \\
&= e^{-A^T(\tau)s_0 - \left(\int_0^\tau A^T(u)du\right)\chi + \frac{1}{2}\int_0^\tau A(u)^T\Sigma\Sigma^T A(u)du} \\
&= e^{-[A_r(\tau)r_0 + A_\lambda(\tau)\lambda_0 + C(\tau)]}
\end{aligned}$$

This is the desired result and it concludes the proof. ■

A.3 Poisson Processes and Idiosyncratic Defaults

Let the increment of the Poisson process N_t be

$$dN_t = \begin{cases} 0 & : \text{ wp } \quad 1-\lambda dt \\ 1 & : \text{ wp } \quad \lambda dt \\ \geq 2 & : \text{ wp } \quad 0 \end{cases}$$

Again, the intuition is that in an interval dt, the probability of two or more jumps goes to zero because $(dt)^k \approx 0$ for $k \geq 2$. Consider a continuum of bonds $i \in [0,1]$. Each of these bonds follows the dynamics

$$\frac{dP_t^i}{P_t^i} = \mu dt + \sigma dW_t + dN_t^i(\omega - 1)$$

where α is the recovery rate. The dN_t^i describes whether bond i defaults or not. I assume that dN_t^i are independent across i, but they have the same intensity λ. The dynamics of the continuum of bonds is

$$\frac{dP_t}{P_t} \doteq \int_0^1 \frac{dP_t^i}{P_t^i} di = \int_0^1 (\mu dt) di + \int_0^1 (\sigma dW_t) di + \int_0^1 dN_t^i (\omega - 1) = \mu dt + \sigma dW_t + (\omega - 1) \int_0^1 dN_t^i$$

The quantity $\int_0^1 dN_t^i$ can be thought of as the cross-sectional average defaults across all bonds. Consider an equally spaced partition of the unit interval $\Pi \doteq \left\{0, \frac{1}{N}, \frac{2}{N}, \dots \frac{N-1}{N}, 1\right\}$. The norm of the partition is $\sup_n \Pi = \frac{1}{N}$. Hence, $\Pi \to 0$ can be written as $N \to \infty$. As a result

$$\int_0^1 dN_t^i = \lim_{\Pi \to 0} \sum_{k=1}^N dN_t^i \cdot (i_{k+1} - i_k) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N dN_t$$

Since $\mathbb{E}[dN_t] = \lambda dt < \infty$ and dN_t^i are i.i.d across bonds, the Law of Large numbers gives

$$\int_0^1 dN_t^i = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N dN_t \stackrel{\text{LLN}}{=} \mathbb{E}[dN_t] = \lambda dt$$

Therefore

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t + (\omega - 1)\lambda dt$$

and the same argument goes through if λ_t is time-varying but known at t. In the special case that the recovery rate is $\omega = 0$, the arbitrageurs' problem can finally be written as

$$\begin{aligned} dW_t = \left(W_t - \int_0^T x_t^{G,(\tau)} d\tau - \int_0^T x_t^{C,(\tau)} d\tau \right) r_t dt + \int_0^T x_t^{G,(\tau)} \frac{dP_t^{G,(\tau)}}{P_t^{G,(\tau)}} d\tau \\ + \int_0^T x_t^{C,(\tau)} \left(\frac{dP_t^{C,(\tau)}}{P_t^{C,(\tau)}} - \lambda_t dt \right) d\tau \end{aligned}$$

where λ_t is the fraction of bonds that defaults in each interval dt. This can be interpreted as the depreciation rate of capital in standard macro models.

B Model Extensions

B.1 Square Root Dynamics

When shocks to default intensity are heteroscedastic, yields are no longer affine in the risk factors s_t. As a result, the exponentially-affine conjecture as in Vayanos and Vila (2021) breaks, hinting at a pricing function with a different functional form. To show why, I consider the segmented version of the model in which arbitrageurs only invest in the short term rate and in corporate bonds at all maturities. I omit the security index j and I abstract for demand shocks.

The decision problem of the arbitrageurs and the specification of habitat demand is the same as in

Section 2. However, I assume that default intensity has square root dynamics of the form

$$d\lambda_t = \kappa_\lambda(\overline{\lambda} - \lambda_t) + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} \tag{6}$$

Square-root dynamics ensure that $\lambda_t > 0$ and introduce heteroscedasticity. I conjecture that

$$P_t^{(\tau)} = e^{-[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)]}$$

Following the same steps implies that the arbitrageurs' first-order condition is

$$\mu_t^{(\tau)} - r_t = \lambda_t - A_r(\tau)\sigma_r \cdot \eta_{r,t} - A_\lambda(\tau)\sigma_\lambda \cdot \eta_{\lambda,t}$$

where the market price of default risk is

$$\pi_{\lambda,t} \doteq -\sigma_\lambda\lambda_t\left(\int_0^\infty X_t^{(\tau)}A_\lambda(\tau)d\tau\right)$$

Market clearing requires $X_t^{(\tau)} + Z_t^{(\tau)}$, so that

$$X_t^{(\tau)} = \theta(\tau) - \alpha(\tau)\left[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)\right]$$

After substituting the market clearing condition into the market prices of risk, I obtain

$$\eta_{\lambda,t} = -\sigma_\lambda\lambda_t\left(\int_0^\infty \left\{\theta(\tau) - \alpha(\tau)\left[A_r(\tau)r_t + A_\lambda(\tau)\lambda_t + C(\tau)\right]\right\}A_\lambda(\tau)d\tau\right) \tag{7}$$

Equation (7) shows that heteroscedastic shocks introduce a second source of variation in market risk premia on top of stochastic habitat demand. As a result, the right-hand side of the arbitrageurs' first-order condition includes a product of two affine functions, whereas the Itô term on the left-hand side is linear in the state variables. In turns out that there are higher powers of r_t and λ_t, and matching coefficients on λ_t^2 implies

$$0 = -A_\lambda(\tau)a\sigma_\lambda^2\int_0^\infty \alpha(\tau)A_r(\tau)A_\lambda(\tau)d\tau$$

Clearly, this only holds provided that $A_\lambda(\tau)$, which leads to contradiction.

B.2 Microfoundation of Habitat Demand with Square-root Dynamics

The specification of habitat demand in Section 3 has the convenient properties that (i) the $\mathbb{Q}$-dynamics of the state vector can be characteriyed analytically and (ii) that, in equilibrium, yields are affine functions of the state variables. Although this particular specification of habit demand is common in the literature (Costain et al., 2022; Gourinchas et al., 2022; Vayanos & Vila, 2021), it suffers from three main shortcomings. First, there is no guarantee that default intensity is strictly positive. Second, habitat agents only respond to prices, but not to the economic fundamentals s_t. As a result, habitat investors act as liquidity providers and trade favorably to the arbitrageurs, meaning that they want to buy when intermediaries wish to sell. Third, the specification of habitat demand lacks a clear microfoundation.

Building on these insights, I revisit the segmentation model without demand shocks in Section 2 to incorporate two additional elements. First, I assume that the dynamics of the risk factors r_t and λ_t are given by the square-root process

$$
\begin{aligned}
dr_t &= \kappa_r(\overline{r} - r_t)dt + \sigma_r\sqrt{\lambda_t}dB_{r,t} \\
d\lambda_t &= \kappa_\lambda(\overline{\lambda} - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t}
\end{aligned}
$$

where the shocks to the short rate and to default intensity $dB_{r,t}$ and $dB_{\lambda,t}$ are independent. These dynamics ensure that $\lambda_t > 0$ almost surely and imply that both r_t and λ_t are heteroscedastic. However, I make the simplifying assumption that the heteroscedasticity is entirely driven by λ_t. Technical conditions on the existence of strong solutions to the SDE are given in Duffie and Kan (1996). The only restriction required for the existence of a unique strong solution is $\kappa_\lambda\overline{\lambda} > \frac{\sigma_\lambda^2}{2}$, which I assume to be true.

The main challenge with square-root dynamics is that the stochastic volatility introduces a second source of variation in risk premia. While in Section 2.4 risk premia only vary with the quantities absorbed by the arbitrageurs, square-root dynamics entail that the quantity of risk also varies over time. Without any further change, this alone would imply that the covariance between the pricing kernel and bond returns is a product of two affine functions. Unfortunately, such a result would rule out an equilibrium yield curve that is still affine in the aggregate risk factors,

Second, I specify habitat demand such that habitat investors respond to economic fundamentals and consider the risk profile of the securities they are allowed to invest in. As before, habitat investors, indexed by $\tau \in (0, \infty)$, are uniformly distributed across maturities and only hold corporate bonds with a specific maturity τ. Investors with habitat τ at time t hold a position

$$
Z_t^{(\tau)} = \frac{\alpha(\tau)}{\lambda_t}\left[\mu_t^{(\tau)} - r_t - \lambda_t\right] + \frac{\alpha(\tau)}{\lambda_t}\beta_t^{(\tau)} + \theta(\tau) \tag{8}
$$

in the bond with maturity τ and hold no other bonds. Specification (8) has three components. The first term is the standard mean-variance demand, adjusted to account for the fact that a fraction of bonds $\lambda_t dt$ is defaulting at any instant. The second term collects demand shocks that affect expected returns, such as investment managers' skills, regulatory costs, or information frictions. The third term captures the inelastic component of demand, which can be driven by compensation schemes linear in a benchmark (Pavlova & Sikorskaya, 2022) or by duration matching of liabilities.

Henceforth, I consider the case with $K = 0$ demand factors. Accordingly, $\alpha(\tau)$ is given by

$$
\alpha(\tau) = \frac{1}{a^H\left[A_r(\tau)^2\sigma_r^2 + A_\lambda(\tau)^2\sigma_\lambda^2\right]} \tag{9}
$$

where a^H denotes the risk-aversion of the habitat investors. As in the standard mean-variance framework, the sensitivity of demand to expected return is inversely proportional to fundamental risk, i.e. σ_r and σ_λ, and to risk-aversion a^H. The function (9) can be exactly microfounded by assuming that habitat investors maximize an instantaneous mean-variance objective, but are only allowed to trade bonds with maturity τ or invest at the risk-free rate. Specification (8) is very similar to the habitat demand in Vayanos and Vila (2021). It features a price-elastic term, demand shocks, and a demand intercept. However, there are important differences that make equation (8) more suitable for risky assets such as corporate bonds.

First, habitat demand depends on the risk-return profile of the asset. Not only does $Z_t^{(\tau)}$ respond to expected returns, but it also accounts for the volatility of bond returns, as captured by the the denominator of $\alpha(\tau)$. Furthermore, habitat investors become less price elastic when default intensity is higher. Second, $Z_t^{(\tau)}$ directly responds to the economic fundamentals r_t and λ_t, and not only through their effect on prices. Third, the sensitivity of demand to expected returns $\frac{\alpha(\tau)}{\lambda_t}$ is endogenous and varies over time. The fact that λ_t appears in the denominator is a convenient property that allows me to solve for an equilibrium affine yield curve even in presence of stochastic volatility. I next describe two different ways to justify Equations (8) and (9), giving a concrete identity to habitat investors.

B.2.1 Habitat Investors: Mutual Funds & ETFs

I interpret habitat investors as delegated portfolio managers, whose compensation is linked to a bond benchmark. The benchmark varies across maturity and it includes bonds that the funds also trade. In the spirit of Pavlova and Sikorskaya (2022), I assume that the compensation of the fund manager has three components.

$$W = R_{t+1}^x + b(R_{t+1}^x - R_{t+1}^b) + c = (a+b)R_{t+1}^x - bR_{t+1}^b + c$$

First, the manager gets a fraction of the return he generates on the portfolio. Second, the manager gets paid depending on the fund's performance relative to a benchmark R_{t+1}^b. In Vayanos and Vila (2021), the benchmark can be assumed to be a government bond index. Here, I assume that the benchmark is a corporate bond with maturity τ. Given the assumption of a continuum of firms subject to idiosyncratic defaults, the benchmark can be roughly thought as a general bond index for a given rating category and maturity. Third, there is a fixed fee c. The return on the manager's portfolio is

$$R_{t+1}^x = \left(W_t - Z_t^{(\tau)}\right) r_t dt + Z_t^{(\tau)} \left(\frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}} - \lambda_t dt\right) + \Delta_t Z_t^{(\tau)} dt$$

where, Δ_t captures the alpha that managers can generate net of a private costs of monitoring (Kashyap, Kovrijnykh, Li, & Pavlova, 2023). When habitat investors have CARA utility over next period compensation $-e^{-a^H W}$, the problem is equivalent to the standard mean-variance formulation. Substituting the compensation function, and assuming that the return on the benchmark is linear in the return on the defaultable bond, i.e. $R_{t+1}^b = \omega_b R_{t+1}^{(\tau)}$, I obtain

$$\begin{aligned}\max_{X_t^{(\tau)}}(a+b)\left[\left(W_t - X_t^{(\tau)}\right) r_t + X_t^{(\tau)}\mu_t^{(\tau)} + \Delta_t X_t^{(\tau)}\right] - b\mu_t^b + c \\ - \frac{a^H}{2}\left\{ (a+b)^2 (X_t^{(\tau)})^2 (\sigma_t^{(\tau)})^2 + b\left(\sigma_t^b\right)^2 - 2(a+b)bX_t^{(\tau)}\mathbb{C}\text{ov}_t\left(R_{t+1}^{(\tau)}, \omega_b R_{t+1}^{(\tau)}\right)\right\}\end{aligned}$$

The fund manager's demand for $Z_t^{(\tau)}$ is

$$Z_t^{(\tau)} = \frac{1}{a+b} \cdot \frac{\mu_t^{(\tau)} - r_t - \lambda_t}{a^H \sigma_t^{2,(\tau)}} + \frac{\Delta_t}{a^H (a+b) \sigma_t^{2,(\tau)}} + \frac{b}{a+b} \cdot \omega_b$$

The third term is just a constant and does not depend neither on prices nor on risk aversion and the return variance. The quantities $\mu_t^{(\tau)}$ and $\sigma_t^{2,(\tau)}$ are determined in equilibrium. Habitat demand is an affine function of expected returns, demand shocks, here captured by the managers' skill Δ_t, and a demand intercept $\frac{b}{a+b} \cdot \omega_b$. The demand function generalizes to multiple assets.

When the risk factors follow square-root dynamics, the sensitivity to expected return takes the form

$$\frac{1}{(a+b)a^H \sigma_t^{2,(\tau)}} = \frac{1}{(a+b)a^H \lambda_t [\sigma_r^2 A_r(\tau)^2 + \sigma_\lambda^2 A_\lambda(\tau)]} = \frac{\alpha(\tau)}{\lambda_t}$$

which is the same expression as Equation (9) but with effective risk aversion $(a+b)a^H$.

B.2.2 Habitat Investors: Pension Funds & Insurance Companies

I interpret habitat investors as pension funds and insurance companies. I assume that each of these agents (i) seek to maximize expected returns but try to match the duration of their liabilities and (ii) are subject to time-varying regulatory costs that are proportional to the quantity of risky assets. There are two key ingredients. An **exogenous** liability $L^{(\tau)}$ evolves as $dL_t = L^{(\tau)} \frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}}$, where $\frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}}$ is the return on the bond with maturity τ. I interpret the liability as a "portfolio" of bonds with the same returns as the Treasury bonds in the market. Time-varying regulatory costs, linear in the positions they held in the risky asset $\psi_t X_t^{(\tau)}$. I assume that P&Is seek to maximize the objective

$$\max_{X_t^{(\tau)}} \mathbb{E}[dA_t] - \frac{a^H}{2} \mathbb{V}\mathrm{ar}_t \left[dA_t - dL_t \right] - \psi_t X_t^{(\tau)}$$

where assets A_t evolve as

$$dA_t = \left(A_t - X_t^{(\tau)} \right) r_t dt + X_t^{(\tau)} \frac{dP_{t+1}^{(\tau)}}{P_t^{(\tau)}}$$

Let $\mu_t^{(\tau)}$ and $\sigma_t^{(\tau)}$ denote the expected return and the volatiltiy of the bond at maturity τ. The objective can be written as

$$\max_{X_t^{(\tau)}} \left(A_t - X_t^{(\tau)} \right) r_t dt + X_t^{(\tau)} \mu_t dt - \frac{a^H}{2} \left(X_t^{(\tau)} - L^{(\tau)} \right)^2 \left(\sigma_t^{(\tau)} \right)^2 - \psi_t X_t^{(\tau)}$$

The first-order condition with respect to $X_t^{(\tau)}$ is

$$\mu_t^{(\tau)} - r_t - \psi_t - a^H \left(\sigma_t^{(\tau)} \right)^2 \left(X_t^{(\tau)} - L^{(\tau)} \right) = 0$$

Solving for $X_t^{(\tau)}$ gives

$$X_t^{(\tau)} = \frac{\mu_t^{(\tau)} - r_t}{a^H \sigma_t^{2,(\tau)}} - \frac{\psi_t}{a^H \sigma_t^{2,(\tau)}} + L^{(\tau)}$$

where $\sigma_t^{2,(\tau)}$ is the same as in the previous section.

B.3 Equilibrium

I study a simplified environemnt similar to segmentation model in Section 2 to show that specification (8) together with square-root dynamics preserve a term structure affine in s_t. For simplicity, I consider the case without demand shocks, so that the only two risk factors are r_t and λ_t. I also set the demand intercept $\theta(\tau)$ equal to zero.

I conjecture that prices are exponentially-affine in the risk factors. That is

$$P_t^{(\tau)} = e^{-[A_r(\tau) r_t + A_\lambda(\tau)\lambda_t + C(\tau)]}$$

Itô's Lemma gives

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)} dt - A_r(\tau)\sigma_r \sqrt{\lambda_t} dB_{r,t} - A_\lambda(\tau)\sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}$$

where expected returns at maturity τ are given by

$$\begin{aligned}\mu_t^{(\tau)} = A_r'(\tau) r_t + A_\lambda'(\tau)\lambda_t + C'(\tau) + A_r(\tau)\kappa_r(r_t - \overline{r}) + A_\lambda(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) \\ + \frac{1}{2} A_r(\tau)^2 \sigma_r^2 \lambda_t + \frac{1}{2} A_\lambda(\tau)^2 \sigma_\lambda^2 \lambda_t\end{aligned}$$

Taking the bond dynamics as given, the variance of arbitrageurs' wealth changes dW_t is given by

$$\mathbb{V}\text{ar}(dW_t) = \sigma_r^2 \lambda_t \left(\int_0^\infty X_t^{(\tau)} A_r(\tau) d\tau \right)^2 dt + \sigma_\lambda^2 \lambda_t \left(\int_0^\infty X_t^{(\tau)} A_\lambda(\tau) d\tau \right)^2 dt$$

It follows that the arbitrageurs' first-order condition is

$$\mu_t^{(\tau)} - r_t = \lambda_t + a\sigma_r^2 \lambda_t A_r(\tau) \left(\int_0^\infty X_t^{(\tau)} A_r(\tau) d\tau \right) + a\sigma_\lambda^2 A_\lambda(\tau)\lambda_t \left(\int_0^\infty X_t^{(\tau)} A_\lambda(\tau) d\tau \right)$$

Habitat Investors Motivated by the discussion on delegated fund managers and P&Is, I assume that habitat investors have mean-variance preferences over instantaneous changes in wealth. To illustrate the main mechanism, I abstract from hedging demand and demand shifters. The objective is

$$\max_{Z_t^{(\tau)}} \mathbb{E}\left[dW_t^h\right] - \frac{a^h}{2} \mathbb{V}\text{ar}\left(dW_t^h\right)$$

where a^h denotes the risk aversion of the habitat investors. Wealth W_t^H evolves as

$$dW_t^h = \left(W_t^h - Z_t^{(\tau)}\right) r_t dt + Z_t^{(\tau)} \left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} - \lambda_t\right]$$

The main difference between habitat investors and arbitrageurs is that the former only trade bonds with maturity τ or invest in the short rate, whereas the latter trade bonds with all maturities $\tau \in [0, \infty]$. Under the exponentially-affine conjecture, I obtain

$$\mathbb{E}[dW_t^h] = \left(W_t^h - Z_t^{(\tau)}\right) r_t dt - Z_t^{(\tau)} \left[\mu_t^{(\tau)} - \lambda_t\right] dt$$
$$\mathbb{V}\text{ar}\left(dW_t^h\right) = \mathbb{V}\text{ar}\left(Z_t^{(\tau)} \frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right) = \left(Z_t^{(\tau)}\right)^2 \left[A_r(\tau)^2\sigma_r^2 + A_\lambda(\tau)^2\sigma_\lambda^2\right] \lambda_t dt$$

As a result, the optimization problem of the habitat investors becomes

$$\max_{Z_t^{(\tau)}} \left(W_t^h - Z_t^{(\tau)}\right) r_t - Z_t^{(\tau)} \left[\mu_t^{(\tau)} - \lambda_t\right] - \frac{a^h}{2} \left(Z_t^{(\tau)}\right)^2 \left[A_r(\tau)^2\sigma_r^2 + A_\lambda(\tau)^2\sigma_\lambda^2\right] \lambda_t$$

Habitat demand at maturity τ is

$$Z_t^{(\tau)} = \frac{\mu_t^{(\tau)} - r_t - \lambda_t}{a^h \left[A_r(\tau)^2\sigma_r^2 + A_\lambda(\tau)^2\sigma_\lambda^2\right] \lambda_t} = \frac{\alpha(\tau)}{\lambda_t} \left[\mu_t^{(\tau)} - r_t - \lambda_t\right]$$

where

$$\alpha(\tau) \doteq \frac{1}{a^H \left[A_r(\tau)^2\sigma_r^2 + A_\lambda(\tau)^2\sigma_\lambda^2\right]}$$

only varies with τ but is constant over time.

Market Clearing The market clearing condition is

$$X_t^{(\tau)} + Z_t^{(\tau)} = 0$$

Hence

$$X_t^{(\tau)} = -Z_t^{(\tau)} = -\frac{\alpha(\tau)}{\lambda_t} \left[\mu_t^{(\tau)} - r_t - \lambda_t\right]$$

Substituting back into the arbitrageurs' first-order condition gives

$$\begin{aligned}\mu_t^{(\tau)} - r_t = \lambda_t - a\sigma_r^2 A_r(\tau) &\left(\int_0^\infty \left(\alpha(\tau) \left[\mu_t^{(\tau)} - r_t - \lambda_t\right]\right) A_r(\tau) d\tau\right) \\ &- a\sigma_\lambda^2 A_\lambda(\tau)\lambda_t \left(\int_0^\infty \alpha(\tau) \left[\mu_t^{(\tau)} - r_t - \lambda_t\right] A_\lambda(\tau) d\tau\right)\end{aligned}$$

where again

$$\mu_t^{(\tau)} = A_r'(\tau) r_t + A_\lambda'(\tau)\lambda_t + C'(\tau) + A_r(\tau)\kappa_r(r_t - \overline{r}) + A_\lambda(\tau)\kappa_\lambda(\lambda_t - \overline{\lambda}) + \frac{1}{2}A_r(\tau)^2\sigma_r^2\lambda_t + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\lambda_t$$

Ordinary Differential Equations I obtain ODEs for the unknown functions by matching coefficients. Matching coefficients on r_t gives

$$\begin{aligned} A_r'(\tau) + A_r(\tau)\kappa_r - 1 = &- a\sigma_r^2 A_r(\tau) \int_0^\infty \alpha(\tau) A_r(\tau) \left[A_r'(\tau) + A_r(\tau)\kappa_r - 1\right] d\tau \\ &- a\sigma_\lambda^2 A_\lambda(\tau) \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[A_r'(\tau) + A_r(\tau)\kappa_r - 1\right] d\tau \end{aligned}$$

Matching coefficients on λ_t gives

$$\begin{aligned} A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2 = 1 \\ - a\sigma_r^2 A_r(\tau) \int_0^\infty \alpha(\tau) A_r(\tau) \left[A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\right] d\tau \\ - a\sigma_\lambda^2 A_\lambda(\tau) \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\right] d\tau \end{aligned}$$

Finally, setting the constant terms to be equal gives

$$\begin{aligned} C'(\tau) - A_r(\tau)\kappa_r\overline{r} - A_\lambda(\tau)\kappa_\lambda\overline{\lambda} = &-a\sigma_r^2 A_r(\tau) \int_0^\infty \alpha(\tau) A_r(\tau) \left[C'(\tau) - A_r(\tau)\kappa_r\overline{r} - A_\lambda(\tau)\kappa_\lambda\overline{\lambda}\right] d\tau \\ &- a\sigma_\lambda^2 A_\lambda(\tau) \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[C'(\tau) - A_r(\tau)\kappa_r\overline{r} - A_\lambda(\tau)\kappa_\lambda\overline{\lambda}\right] d\tau \end{aligned}$$

Solving the ODE system The ODEs for $A_r(\tau)$ and $A_\lambda(\tau)$ are

$$\begin{aligned} A_r'(\tau) + A_r(\tau)\kappa_r - 1 = &- a\sigma_r^2 A_r(\tau) \int_0^\infty \alpha(\tau) A_r(\tau) \left[A_r'(\tau) + A_r(\tau)\kappa_r - 1\right] d\tau \\ &- a\sigma_\lambda^2 A_\lambda(\tau) \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[A_r'(\tau) + A_r(\tau)\kappa_r - 1\right] d\tau \end{aligned}$$

$$\begin{aligned} A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2 = 1 \\ - a\sigma_r^2 A_r(\tau) \int_0^\infty \alpha(\tau) A_r(\tau) \left[A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\right] d\tau \\ - a\sigma_\lambda^2 A_\lambda(\tau) \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\right] d\tau \end{aligned}$$

Let

$$\begin{aligned} \kappa_{rr}^* &= a\sigma_r^2 \int_0^\infty \alpha(\tau) A_r(\tau) \left[A_r'(\tau) + A_r(\tau)\kappa_r - 1\right] d\tau \\ \kappa_{rl}^* &= a\sigma_\lambda^2 \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[A_r'(\tau) + A_r(\tau)\kappa_r - 1\right] d\tau \end{aligned}$$

and

$$\begin{aligned} \kappa_{lr}^* &= a\sigma_r^2 \int_0^\infty \alpha(\tau) A_r(\tau) \left[A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\right] \\ \kappa_{ll}^* &= a\sigma_\lambda^2 \int_0^\infty \alpha(\tau) A_\lambda(\tau) \left[A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2\right] d\tau \end{aligned}$$

Then

$$\begin{aligned} A_r'(\tau) + A_r(\tau)\kappa_r - 1 &= -\kappa_{rr}^* A_r(\tau) - \kappa_{rl}^* A_\lambda(\tau) \\ A_\lambda'(\tau) + A_\lambda(\tau)\kappa_\lambda + \frac{1}{2}A_r(\tau)^2\sigma_r^2 + \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2 &= 1 - \kappa_{lr}^* A_r(\tau) - \kappa_{ll}^* A_\lambda(\tau) \end{aligned}$$

Solving for the first order terms gives

$$\begin{aligned} A_r'(\tau) &= 1 - A_r(\tau)\kappa_r - \kappa_{rr}^* A_r(\tau) - \kappa_{rl}^* A_\lambda(\tau) \\ A_\lambda'(\tau) &= 1 - A_\lambda(\tau)\kappa_\lambda - \kappa_{lr}^* A_r(\tau) - \kappa_{ll}^* A_\lambda(\tau) - \frac{1}{2}A_r(\tau)^2\sigma_r^2 - \frac{1}{2}A_\lambda(\tau)^2\sigma_\lambda^2 \end{aligned}$$

In terms of coefficients

$$\begin{aligned} A_r'(\tau) &= 1 - c_1 A_r(\tau) - c_2 A_\lambda(\tau) \\ A_\lambda'(\tau) &= 1 - c_3 A_r(\tau) - c_4 A_\lambda(\tau) - c_5 A_r(\tau)^2 - c_6 A_\lambda(\tau)^2 \end{aligned}$$

where

$$\begin{aligned} c_1 \doteq \kappa_r + \kappa_{rr}^* \quad &: \quad c_2 \doteq \kappa_{rl}^* \\ c_3 = \kappa_{lr}^* \quad &: \quad c_4 = \kappa_\lambda + \kappa_{ll}^* \\ c_5 = \frac{1}{2}\sigma_r^2 \quad &: \quad c_6 = \frac{1}{2}\sigma_\lambda^2 \end{aligned}$$

The system describes a coupled Riccati equation in the two unknown functions $A_r(\tau)$ and $A_\lambda(\tau)$. The initial conditions are $A_r(\tau) = A_\lambda(\tau) = 0$. What makes the ODE system complicated to solve is that the coefficients c_k, $k = 1, \ldots, 4$ are functions of κ_{rr}^*, κ_{rl}^*, κ_{lr}^*, and κ_{ll}^*, which are integrals involving the functions $A_r(\tau)$ and $A_\lambda(\tau)$. I next describe the numerical algorithm that I use to solve for the unknown coefficients as a function of the model parameters κ_r, κ_λ, σ_r, σ_λ, a and a^h. Note that what matters is the ratio of the arbitrageurs' risk aversion to the habitat investor $\frac{a}{a^h}$.

Algorithm I solve the system

$$A_r'(\tau) = 1 - c_1 A_r(\tau) - c_2 A_\lambda(\tau) \tag{10}$$

$$A_\lambda'(\tau) = 1 - c_3 A_r(\tau) - c_4 A_\lambda(\tau) - c_5 A_r(\tau)^2 - c_6 A_\lambda(\tau)^2 \tag{11}$$

where

$$\begin{aligned} c_1 \doteq \kappa_r + \kappa_{rr}^* \quad &: \quad c_2 \doteq \kappa_{rl}^* \\ c_3 = \kappa_{lr}^* \quad &: \quad c_4 = \kappa_\lambda + \kappa_{ll}^* \\ c_5 = \frac{1}{2}\sigma_r^2 \quad &: \quad c_6 = \frac{1}{2}\sigma_\lambda^2 \end{aligned}$$

Let $\mathcal{C} \doteq [c_1, \ldots, c_6]^T$ denote the vector of all coefficients. I propose the following numerical procedure so solve for $\mathcal{C}$. The procedure is initialized for an initial guess $\mathcal{C}^{(0)}$.

- **Step 1**: Start with an initial guess $\mathcal{C}^{(k)}$.

- **Step 2**: Solve the coupled Riccati equations (10)–(11), where the coefficients $c_1^{(k)}, \ldots, c_n^{(k)}$ are given by the initial guess. I use a pre-programmed routine to solve the system of first-order non-linear ordinary differential equations. Let $A_r^{(k)}(\tau)$ and $A_\lambda^{(k)}(\tau)$ denote the solution to the k-step system. I impose the boundary condition $A_r^{(k)}(\tau) = 0$ and $A_\lambda^{(k)}(\tau) = 0$.

- **Step 3**: Compute the integrals κ^*_{rr}, κ^*_{rl}, κ^*_{lr}, and κ^*_{ll} by plugging in $A_r^{(k)}(\tau)$ and $A_\lambda^{(k)}(\tau)$. Then, compute the updated vector of coefficients $\mathcal{C}^{(k+1)}$. Note that $A_r^{(k)}(\tau)$ and $A_\lambda^{(k)}(\tau)$ also control habitat investors' sensitivity to expected returns.

- **Step 4**: If $\|\mathcal{C}^{(k)} - \mathcal{C}^{(k+1)}\| < \varepsilon$, then terminate. Otherwise, set $k = k+1$ and go to step 1.

Figure 11 plots $A_r(\tau)$ and $A_\lambda(\tau)$ for the following parametrization

$$\kappa_r = 0.12 \quad : \quad \kappa_\lambda = 0.05 \quad : \quad \sigma_r = 0.023 \quad : \quad \sigma_\lambda = 0.054$$

The model generalizes to accommodate demand shocks. The full characterization of the equilibrium with demand shocks is still work in progress.

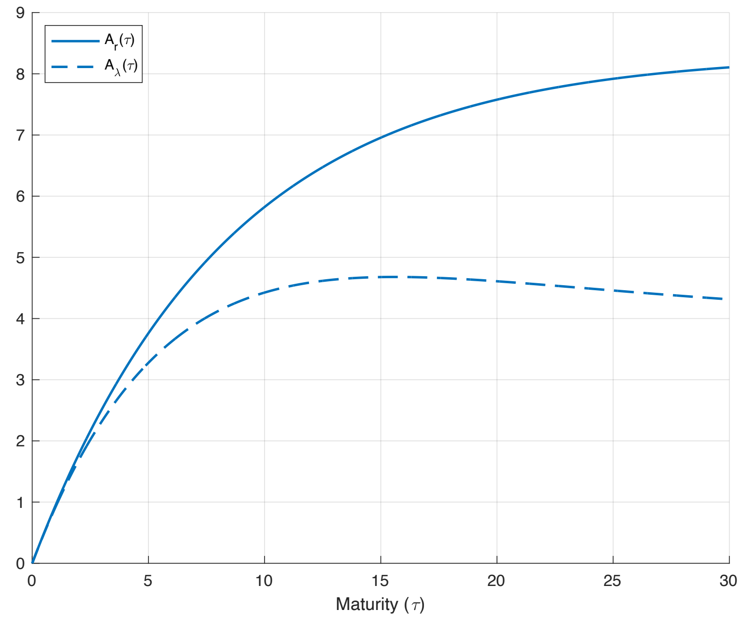

Figure 11: Coefficient $A_r(\tau)$ and $A_\lambda(\tau)$.