1 Markov Chain [2 points]

Given the matrix

$$M = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

and a vector $u_0 = [1, 0]^T$

Calculate $u_1 = Au_0$, $u_2 = Au_1$, and $u_3 = Au_2$. What property do you notice for vectors u_0, u_1, u_2, u_3?

2 Roots of a matrix [2 points]

Given the matrix

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$$

1. find a matrix M, such that $A = M^2$.
2. find a matrix C, such that $A = C^T C$.

3 Critical Points [2 points]

Let

$$f(x, y) = xy - x^3 y - xy^3$$

1. Find critical points of f.
2. Determine the nature of critical points.

4 Probability Density [2 points]

Let

$$f(x) = \begin{cases} \frac{x^3}{5000}(10 - x) & \text{if } 0 \leq x \leq 10 \\ 0 & \text{if } else \end{cases}$$

1. Show that $f(x)$ is a probability density

   Hint: $f(x)$ needs to integrate to 1, and be non-negative.
2. Find $P(1 \leq x \leq 4)$

5 Gradient and Hessian [3 points]

Given an $f : \mathcal{R}^n \to \mathcal{R}$ of class $\mathcal{C}^2$, recall that the *Hessian Matrix* $H = (h_{ij})$ is the n-by-n matrix given by

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Suppose $M = (m_{ij})$ is a symmetric 2×2 matrix (i.e. $m_{ij} = m_{ji}$), suppose $\vec{a} \in \mathcal{R}^2$, and let $f : \mathcal{R}^2 \to \mathcal{R}$ be given by

$$f(\vec{x}) = \vec{a} \cdot \vec{x} + \frac{1}{2}\vec{x} \cdot (M\vec{x}) + c\sin^3(\hat{\imath} \cdot \vec{x})$$

This can also be written as

$$f(x_1, x_2) = a_1x_1 + a_2x_2 + \frac{1}{2}\left(x_1m_{11}x_1 + x_1m_{12}x_2 + x_2m_{21}x_1 + x_2m_{22}x_2\right) + c\sin^3(x_1)$$

1. calculate the gradient of f at $\vec{x} \in \mathcal{R}$
2. calculate f's Hessian.
3. Assume now that $\det M \neq 0$, so that M is invertible and $c = 0$. Solve the equation $\nabla f(p) = \vec{0}$

6 Romeo and Juliet [2 points]

Romeo and Juliet have agreed to meet for a date somewhere between 9pm and 10pm. To avoid being seen, each of them picks a random time in the above interval and waits for 15 minutes for the partner. What is the probability that they will actually meet?

7 Coin Toss [3 points]

You have a jar containing 99 fair coins and one double headed coin. You randomly pick a coin and toss it 10 times getting 10 heads. What is the probability you will toss head one more time with the same coin?

8 Matrix Rotation [2 points]

Solve the following problem, preferably in Python, but otherwise in your favorite programming language.

Given an image represented by an NxN matrix, where each pixel in the image is a number, write a function to rotate the image by 90 degrees clockwise. Can you do this in place?

9 Ternary Tree Symmetry [3 points]

Solve the following problem, preferably in Python, but otherwise in your favorite programming language.

In this problem we are testing your ability to understand and solve a programming problem. We have intentionally left the problem a little ambiguous. Do your best to understand what the problem wants and to give a solution.

You are given a Ternary tree, where each node has a left child, a middle child, and a right child (there are no weights or labels). You need to determine if the given tree is symmetric with respect to the vertical line passing through the root.

First write a class "Tree" that has a left tree, a middle tree, and a right tree.

Then, write a function that takes an argument root(of type Tree), and determines whether the tree is symmetric.

10 Bonus (If you are bored): Points on a Circle [¿3 points]

n points are drawn uniform randomly on a circle.

1. What's the probability that all n points belong to some half circle when $n = 3$?
2. What's the probability that all n points belong to some half circle for generic n? (I.e. write a formula containing n)

11 Bonus (If you are still bored): Volume [¿3 points]

Find the volume of the region enclosed by the plane $z = 9$ and the surface

$$z = (2x - y)^2 + (x + y - 1)^2$$