Some remarks on the uncertainty analysis of $\mathcal{R}_0$ in the SIR model

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Abstract

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1 Background

$\mathcal{R}_0$ is important, a key quantity in epidemic modelling.

Acknowledging uncertainty on parameter values is important.

logarithmic pooling is a nice and robust way of combining multiple sources of info. [4] discuss the issue of propagating uncertainty through a deterministic model.

This begs the question, however, of in which order the pooling and propagation (inducing) operations should be performed.

1.1 SIR model

$$\begin{aligned}
\frac{dS}{dt} &= -\beta S I \\
\frac{dI}{dt} &= \beta S I - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}$$

where $S(t) + I(t) + R(t) = N \quad \forall t$, β is the transmission (infection) rate and γ is the recovery rate.

$$\mathcal{R}_0 = \frac{\beta N}{\gamma}. \tag{1}$$

1.2 Uncertainty analysis

$p(\beta, \gamma)$

$M(\cdot)$

$M(p(\beta, \gamma)) = p(\mathcal{R}_0)$

For simplicity we will assume that $p(\beta, \gamma) = p(\beta)p(\gamma)$.

uncertainty about parameters can be represented by Gamma distributions.

$$\begin{aligned}
f_\beta(b) &= \frac{1}{\Gamma(k_1)\theta_1^{k_1}} b^{k_1} exp(-\frac{b}{\theta_1}) \\
f_\gamma(g) &= \frac{1}{\Gamma(k_2)\theta_2^{k_2}} g^{k_2} exp(-\frac{g}{\theta_2})
\end{aligned}$$

1.3 Logarithmic pooling

Logarithmic pooling is a popular method for combining opinions on an agreed quantity, specially when these opinions can be framed as probability distributions. Let $\mathbf{F}_\theta := \{f_1(\theta), f_2(\theta), \ldots, f_K(\theta)\}$ be a set of distributions representing the opinions of K experts and let $\mathbf{w} := \{w_1, w_2, \ldots, w_K\}$ be the vector of weights, such that $w_i > 0\, \forall i$ and $\sum_{i=0}^K w_i = 1$. The **logarithmic pooling operator** $\mathcal{LP}(\mathbf{F}_\theta, \mathbf{w})$ is defined as

$$\mathcal{LP}(\mathbf{F}_\theta, \mathbf{w}) := \pi(\theta|\mathbf{w}) = t(\mathbf{w}) \prod_{i=0}^{K} f_i(\theta)^{w_i}, \tag{2}$$

where $t(\mathbf{w}) = \int_{\mathbf{\Theta}} \prod_{i=0}^K f_i(\theta)^{w_i} d\theta$. This pooling method enjoys several desirable properties and yields tractable distributions for a large class of distribution families [3, 1].

2 The Gamma ratio distribution

To derive the distribution, we begin by noting that for $N > 1$, the distribution of $\beta^* = \beta N$ is a Gamma distribution with parameters k_1 and $N\theta_1$. Under the assumption of independence $p(\beta^*, \gamma) = p(\beta^*)p(\gamma)$, thus

$$\begin{aligned}
\mathcal{R}_0 &= \beta^*/\gamma & (3)\\
f_{\mathcal{R}_0}(r) &= A \int_0^\infty \gamma(\gamma r)^{k_1-1} e^{-\frac{\gamma r}{N\theta_1}} \gamma^{k_2-1} e^{-\frac{\gamma}{\theta_2}} d\gamma & (4)\\
A &= \frac{1}{\Gamma(k_1)(N\theta_1)^{k_1}\Gamma(k_2)\theta_2^{k_2}} & (5)
\end{aligned}$$

Rearranging, yields

$$\begin{aligned}
f_{\mathcal{R}_0}(r) &= A \int_0^\infty r^{k_1-1} \gamma^{k_1+k_2-1} e^{-B\gamma} d\gamma & (6)\\
B &= \frac{\theta_2 r + N\theta_1}{N\theta_1\theta_2} & (7)
\end{aligned}$$

$$\begin{aligned}
f_{\mathcal{R}_0}(r) &= \phi \times r^{k_1-1}(\theta_2 r + N\theta_1)^{-(k_1+k_2)} & (8)\\
\phi &= \frac{(N\theta_1\theta_2)^{k1+k2}}{\mathcal{B}(k_1, k_2)(N\theta_1)^{k_1}\theta_2^{k_2}} & (9)
\end{aligned}$$

where $\mathcal{B}(a, b) = \Gamma(a + b)/\Gamma(a)\Gamma(b)$ is the Beta function and ϕ is the normalisation constant. The probability distribution in (8) will be called Gamma ratio distribution henceforth. The expectation of the Gamma ratio distribution is then

$$\begin{aligned}
E(\mathcal{R}_0) &= \int_0^\infty r f_{R0}(r) dr & (10)\\
&= \frac{N\theta_1}{\theta_2} \frac{k_1}{(k_2 - 1)} & (11)
\end{aligned}$$

and its variance can be computed as

$$\begin{aligned}
Var(\mathcal{R}_0) &= E(\mathcal{R}_0^2) - E(\mathcal{R}_0)^2 & (12)\\
&= \left(\frac{N\theta_1}{\theta_2}\right)^2 \frac{(k_1 + k_2 - 1)k_1}{(k_2 - 2)(k_2 - 1)^2} & (13)
\end{aligned}$$

which only exists for $k_2 > 2$.

The mode is

$$\frac{N\theta_1}{\theta_2}\frac{k_1 - 1}{(k_2 + 1)} \tag{14}$$

For a slightly different derivation, based on generalised Gamma distributions, see [2].

Now suppose we have two sets of prior probability distributions – elicited by the same K experts – on β and γ, $\mathbf{F}_\beta$ and $\mathbf{G}_\gamma$, respectively. We will consider that all distributions for β and γ are Gamma distributions, parametrised as above. Thus, for instance, the parameter θ_{2i} is the the scale parameter of the prior for the recovery rate (γ) given by the $i-$th expert. Analogous to the above, assume the experts have a vector $\mathbf{w}$ of weights associated with them. Suppose further that the components of these sets are independent, i.e., $p_i(\beta, \gamma) = f_i(\beta)g_i(\gamma)\ \forall i$. One can either:

(a) construct $\pi(\beta|\mathbf{w}) = \mathcal{LP}(\mathbf{F}_\beta, \mathbf{w})$ and $\pi(\gamma|\mathbf{w}) = \mathcal{LP}(\mathbf{G}_\gamma, \mathbf{w})$ and then apply the transform in (1) to obtain $\pi(\mathcal{R}_0|\mathbf{w})$ or;

(b) apply the transform to each component i of $\mathbf{F}_\beta$ and $\mathbf{G}_\gamma$ to build

$$\mathbf{R}_{\mathcal{R}_0} := \{r_i(\mathcal{R}_0), r_2(\mathcal{R}_0), \ldots, r_K(\mathcal{R}_0)\}$$

and obtain $\pi'(\mathcal{R}_0|\mathbf{w}) = \mathcal{LP}(\mathbf{R}_{\mathcal{R}_0}, \mathbf{w})$.

Notice that the transform in (1) is not invertible, and thus does not enjoy the property discussed in Remark **??**. This means that in general, pooling-then-inducing (a) will yield a different distribution than inducing-then-pooling (b). In fact, procedure (a) will lead to

$$\pi_{\mathcal{R}_0}(r|\mathbf{w}) \propto r^{k_1^*-1}(\theta_2^* r + N\theta_1^*)^{-(k_1^*+k_2^*)} \tag{15}$$

where $k_1^* = \sum_{i=0}^K w_i k_{1i}$, $k_2^* = \sum_{i=0}^K w_i k_{2i}$, $\theta_1^* = \sum_{i=0}^K w_i \theta_{1i}$ and $\theta_2^* = \sum_{i=0}^K w_i \theta_{2i}$. The distribution resulting from procedure (b) will be

$$\pi'_{\mathcal{R}_0}(r|\mathbf{w}) \propto \prod_{i=0}^K \left[r^{k_{1i}-1}(\theta_{2i} r + N\theta_{1i})^{-(k_{1i}+k_{2i})}\right]^{w_i} \tag{16}$$

$$\propto r^{k_1^*-1}\prod_{i=0}^K (\theta_{2i} r + N\theta_{1i})^{-w_i(k_{1i}+k_{2i})} \tag{17}$$

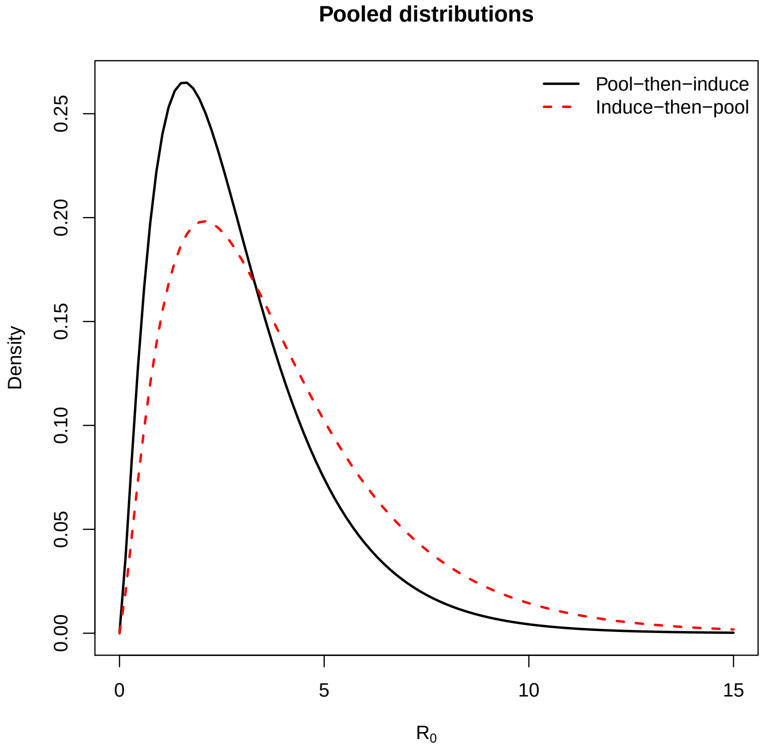
where k_1^* is defined as before.

[LM: WE SHOULD TRY TO REPRESENT THE P.D.F. ABOVE IN A DIFFERENT WAY. WHY, YOU ASK? BECAUSE WHILST FOR THE POOL-THEN-INDUCE ONE WE KNOW THE FIRST AND SECOND MOMENTS, FOR THIS ONE WE DO NOT. WILL INDUCE-THEN-POOL *ALWAYS* GIVE HIGHER VARIANCE? MAYBE IT CAN BE ARGUED THAT IT WILL ALWAYS HAVE HEAVIER TAILS, BUT I DON'T KNOW AT THIS POINT.]

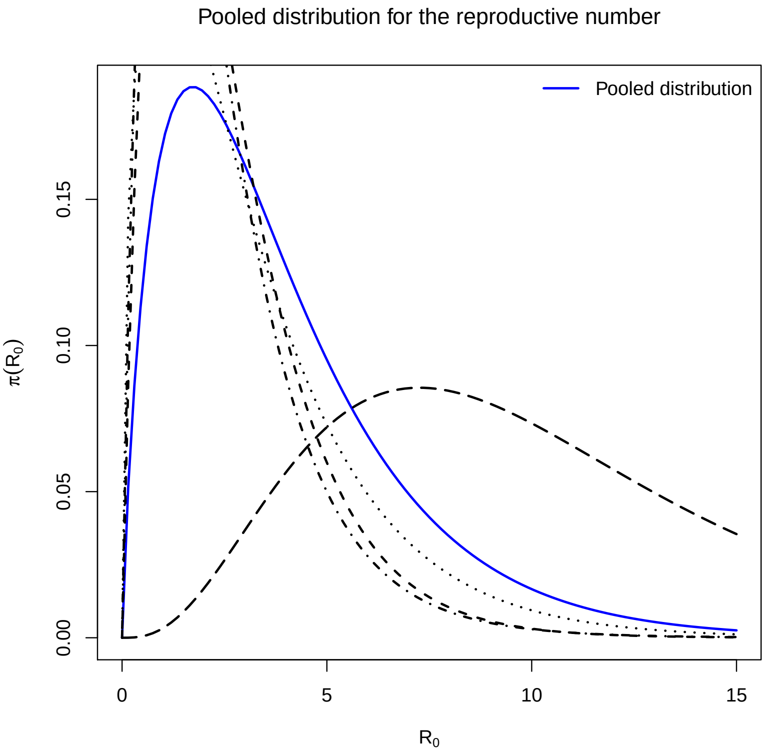
An example plot of the resulting densities is shown in Figure 1A. We note that $\pi'(\mathcal{R}_0|\mathbf{w})$ – procedure (b) – has thicker tails and therefore allows for more extreme values with higher probability. This makes sense intuitively, because this distribution propagates uncertainty resulting from the model for each distribution.

3 Application: $\mathcal{R}_0$ for Ebola in West Africa

[SCAN THE LITERATURE FOR BAYESIAN ESTIMATES]



(a)



(b)

Figure 1: **Distributions for $\mathcal{R}_0$.** In panel A we present the 'induce-then-pool" and "pool-then-induce" distributions using equal weights ($\alpha_i = 1/K \ ,\forall i$). Panel B shows the pooled distribution for $\mathcal{R}_0$ that minimises KL divergence with the "induce-then-pool" distribution, i.e., that minimises discrepancy in transformed space.

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