SIR Models

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Alternative formulations for SIR Models

Assuming the typical SIR formulation as 3 ordinary differential equations on a closed population: $S(t) + I(t) + R(t) = N$, $\forall t \geq 0$. We have that for this formulation the basic reproduction number $\mathcal{R}_0 = \frac{\beta}{\gamma}$.

If we take the ratio:

$$\frac{dS}{dR} = \frac{-\beta SI}{\gamma I} = -\mathcal{R}_0 S \tag{1}$$

It can be integrated to $S(t) = S_0 e^{-\mathcal{R}_0 R}$. We can then substitute this definition of $S(t)$ into the standard $\frac{dR}{dt} = \gamma I$, and obtain

$$\frac{dR}{dt} = \gamma \left(N - R - S_0 e^{-\mathcal{R}_0 R}\right) \tag{2}$$

With this we have reduce a 3-dimensional system to a single equation embodying the full SIR dynamics.

1 Approximate Solution

Equation 2 does not have a direct solution. but an approximate solution can be obtained by assuming $\mathcal{R}_0 R$ remains small for then $e^{-\mathcal{R}_0 R} \approxeq 1 - (\mathcal{R}_0 R) + (\mathcal{R}_0 R)^2$ and (2) reduces to the first order quadratic ODE

$$\frac{dZ}{dt} \approxeq \gamma \left(N - S_0 + [S_0 \mathcal{R}_0 - 1]R - (S_0 \mathcal{R}_0^2/2)R^2\right) \tag{3}$$

which takes the standard solution

$$R(t) = \frac{1}{\mathcal{R}_0^2 S_0} \{(S_0 \mathcal{R}_0) - 1 + \alpha \tanh[(\alpha\gamma t/2) - \phi] \tag{4}$$

where the amplitude $\alpha = \sqrt{[S_0 \mathcal{R}_0 - 1]^2 + 2 S_0 I_0 \mathcal{R}_0^2}$, and the phase $\phi = \tanh^{-1}\{(S_0 \mathcal{R}_0 - 1)/\alpha\}$