# Propiedades Test del Signo



## Eficiencia asintótica relativa (ARE)

$ARE = \frac{c_1^2}{c_2^2}$ , $c_i$ = eficacia de $\phi_i$

De alguna manera necesito $c_i$ p/alcanzar mismo potencia c/el test $\phi_i$

$H_0: \theta = 0$ vs $H_1: \theta > 0$

Concepto local, en alternativas contiguas ($\to 0$), no son $\theta$ fijos.

$X_1, \dots, X_m \sim F(x-\theta)$ , $F \in \Omega_0 = \{F$ abs continua, $\underbrace{\text{est. creciente}, \ F(0)=1/2}_{\text{Único mediana en } 0}\}$

bajo $H_0$: $\theta = 0$ , $X_1, \dots, X_m \sim F(x)$

bajo $H_1$: $\theta > 0$ , $X_1, \dots, X_m \sim F(x-\theta)$

Vamos a probar que:

1) La eficacia del Test del signo es $2f(0) = c \to c^2 = 4f^2(0)$.

2) Asumir con algunos sugerencias del test $t$, $T = \underbrace{\sqrt{m}\,\bar{X}_m}_{S}$ bajo $H_0: \theta = 0$.
es $c = \frac{1}{\sigma_f}$ donde $\sigma_f^2$ es la varianza de $x_i$ : $\sigma_f^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$ $\Big(\underbrace{E_F(x) = 0}_{\theta=0 \text{ por sup.}}\Big)$.

3) $ARE = \frac{C_{TS}^2}{C_{TT}^2} = 4f^2(0) \cdot \sigma_f^2$

4) $N(0,1)$ : $ARE = 0{,}63$

5) D. Exp $f(x) = \frac{1}{2}e^{-|x|}$ : $ARE = 2$

Dem: 1) Miro las condiciones de Pittman (1.1, 1.2, 1.3, 1.4, 1.5).

1.1) Necesito ver consistencia del test del test del signo (clase pasada) ■

4) $N(0,1)$ : $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \to f(0) = \frac{1}{\sqrt{2\pi}}$ , $\sigma_f^2 = 1$ : $ARE = \frac{2}{\pi} \simeq 0.63$ ■

5) Doble Exponencial $f(x) = \frac{1}{2}e^{-|x|} \to f(0) = \frac{1}{2}$

$$\sigma_f^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{2} e^{-|x|}\,dx = 2\int_0^{+\infty} x^2 \frac{1}{2} e^{-x}\,dx = \int_0^{+\infty} x^2 e^{-x}\,dx = 2 \cdot \int_0^{+\infty} \frac{x^2 e^{-x}}{2}\,dx = 2$$

Nota: $\Gamma(m, \lambda) = \frac{x^{m-1}\lambda^m e^{-\lambda x}}{\Gamma(m)} \to \Gamma(3,1) = \frac{x^2 \cdot e^{-x}}{\Gamma(3)}$

$ARE = 2 \cdot 4 \cdot \left(\frac{1}{2}\right)^2 = 2$ ■

1.2), 1.3), 1.4) y 1.5):

$$\phi(x) = \begin{cases} 1 & \text{si } S_n \geq k \\ 0 & \text{si } S_n < k \end{cases} = \begin{cases} 1 & \text{si } \frac{S_n}{n} \geq \frac{k}{n} \\ 0 & \text{si } \frac{S_n}{n} < \frac{k}{n} \end{cases}$$

Entonces: $T_n = \frac{S_n}{n}$.

Luego, las sucesiones son:

$$\mu_n(\theta) = 1 - F(-\theta)$$

$$\sigma_n(\theta) = \frac{\sqrt{F(-\theta)\cdot[1-F(-\theta)]}}{\sqrt{n}}$$

1.3): $\mu_n'(\theta) = -f(-\theta)\,(-1) = f(-\theta)$

$\mu_n'(0) = f(0)$

Si f es continua en 0

1.4) Sea $\theta_n \to 0$: $\frac{\mu_n'(\theta_n)}{\mu_n'(0)} = \frac{f(-\theta_n)}{f(0)} \longrightarrow \frac{f(0)}{f(0)} = 1$

ojo, no está en $\Omega_0$.

Nota: g es continua en a $\leftrightarrow \forall (a_n)_{n\in\mathbb{N}}$ : $a_n \to a$, $f(a_n) \to f(a)$.

$$\sigma_n(\theta_n) \; \frac{\sqrt{F(-\theta_n)(1-F(-\theta_n))}}{\sqrt{n}} = \frac{\frac{\sqrt{F(-\theta_n)(1-F(-\theta_n))}}{\sqrt{n}}}{\frac{\sqrt{1/2\; 1/2}}{\sqrt{n}}} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}} = 1$$

$\theta_n \to 0$, $-\theta_n \to 0$

$F(-\theta_n) \to F(0) = \frac{1}{2}$ pues F cont.

1.5) $$\lim_{n\to+\infty} \frac{\mu_n'(0)}{\sqrt{n}\cdot\sigma_n(0)} = \lim_{n\to+\infty} \frac{f(0)}{\sqrt{n}\,\frac{\sqrt{\frac{1}{2}\cdot\frac{1}{2}}}{\sqrt{n}}} = 2f(0) = c_1$$

Nota: $c_1 > 0$ pues como F es est. crec en 0 hace que $F'(0) > 0$
$F'(0) = f(0)$

1.2) Falta ver que: $$\frac{T_n - \mu_n(\theta)}{\sigma_n(\theta)} = \frac{\bar{S}_n - (1-F(-\theta))}{\sqrt{F(-\theta)(1-F(-\theta))}}\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1)$$

uniformemente en $\mathcal{E}(\theta)$.

La convergencia "normal" la tenemos por TCL pero la otra parte no.

Esto es:

$$\underbrace{\left| P\left(\sqrt{n}\ \frac{\bar{S}_n-(1-F(-\theta))}{\sqrt{F(-\theta)(1-F(-\theta))}} \le t\right) - \Phi(t)\right| \underset{|\theta|\le\delta}{\le} \text{constante}(n,\delta)}_{\text{convergencia uniformemente en un entorno de } \theta=0.} \xrightarrow[\delta \text{ fijo}]{n\to+\infty} 0$$

TEOREMA DE BARRY-ESSEN:

$X_1,\dots,X_n$ iid con $E(X_i)=\mu$, $V(X_i)=\sigma^2$, $0<\sigma^2<\infty$, $E(|X_i-\mu|^3)=\rho^3$
$0<\rho^3<\infty$. Entonces:

$$\left| P\left[\sqrt{n}\ \frac{(\bar{X}-\mu)}{\sigma} \le t\right] - \Phi(t)\right| \le \frac{d\rho^3}{\sqrt{n}\,\sigma^3}$$

$d$, constante indep. de $n$.

/ Por Barry Essen: cota $= d\rho^3/\sqrt{n}\sigma^3$

$$\left| P\left(\sqrt{n}\ \frac{\bar{S}_n-(1-F(-\theta))}{\sqrt{F(-\theta)(1-F(-\theta))}} \le t\right) - \Phi(t)\right| \le \frac{d\,E(|\overset{S(x_i)}{x_i}-\mu|^3)}{\sqrt{n}\,\sigma^3}$$

$$\frac{S_n}{n}=\bar{S}_n=\sum_{i=1}^{n} S(x_i)/n \quad , \quad S(x_i)=\begin{cases}1 & \text{si } x_i>0\\ 0 & \text{si no}\end{cases}$$

$$\mu = E(s(x_i)) = P(s(x_i)=1) = P(x_i>0) = 1-F(-\theta) = p(\theta)$$

$$\sigma^2 = V(s(x_i)) = F(-\theta)(1-F(-\theta))$$

$$y_i = s(x_i)-p(\theta) = \begin{cases} 1-(1-F(-\theta)) & \text{si } x_i>0\\ 0-(1-F(-\theta)) & \text{si no}\end{cases}$$

$$y_i = \begin{cases} F(-\theta) & \text{si } x_i>0\\ F(-\theta)-1 & \text{si no}\end{cases}$$

Entonces:

$$E\left(|s(x_i)-u|^3\right) = E\left(|y_i|^3\right)$$

$$V(y_i) = V(s(x_i)) = F(-\theta)(1-F(-\theta)) = \sigma^2$$

$$E\left(|y_i|^3\right) = |F(-\theta)|^3 \underbrace{P(x_i>0)}_{1-F(-\theta)} + (1-F(-\theta))^3 \underbrace{P(x_i<0)}_{F(-\theta)}$$

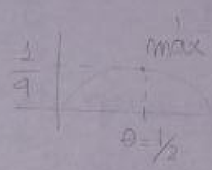
$$= \underbrace{F(-\theta)(1-F(-\theta))}_{\sigma^2}\left[F(-\theta)^2 + (1-F(-\theta))^2\right]$$

Entonces la cota:

$$\text{cota} = \frac{d \cdot \cancel{\sigma^2}\overbrace{\left[(F(-\theta))^2 + (1-F(-\theta))^2\right]}^{\leq 1 \text{ pues:}}}{\sqrt{n}\,\cancel{\sigma^3}}$$

$$1 = (p+(1-p))^2 = p^2 + \underbrace{2p(1-p)}_{\geq 0} + (1-p)^2 \geq p^2 + (1-p)^2.$$

Luego resta acotar $\sigma = \sqrt{F(-\theta)(1-F(-\theta))}$




$$\frac{1}{16} \leq \underbrace{F(-\theta)(1-F(-\theta))}_{g(\theta)} \leq \frac{1}{4} \longrightarrow \frac{1}{16} \leq \sigma \leq \frac{1}{4}$$

pues g es continua

$$\text{cota} = \frac{d}{\sqrt{n}\,\sigma} \leq \frac{d}{\sqrt{n}\sqrt{\frac{1}{16}}} = \frac{4d}{\sqrt{n}} \xrightarrow{n\to\infty} 0$$

vale $\forall\theta$: $|\theta| \leq \delta$

Demostrada la convergencia uniforme $\varepsilon(\theta)=0$.

EFICACIA DEL TEST T:

- Lehman. (p.537)
- Hettmansperger. (p.70)

$T = \frac{\sqrt{n}\,\overline{X}_n}{S}$, $\sigma_f^2 = \int x^2 f(x)\,dx$ var bajo $F \in \Omega_0$

$\mu_n(\theta) = \frac{\sqrt{n}\,\theta}{\sigma_f}$, $\sigma_n(0) = 1$

# TEST DE WILCOXON



Ordeno la muestra:

$$X^{(1)} \leq X^{(2)} \leq \dots \leq X^{(n)}$$

$$\underbrace{R(X_j) = R_j}_{\downarrow} = \# \{ X_i : X_i < X_j \}$$

$$X^{(R_j)} = X_j$$

Antirango: $D_j : X_{D_j} = X^{(j)}$

Test:

$$H_0 : \theta = 0 \quad \text{vs} \quad H_1 : \theta > 0$$

$$\Omega_s = \{ F / F \text{ abs continua, simétrica, único mediana} \}$$

$$W_j = \begin{cases} 1 & \text{si } |X|^{(j)} \text{ corresponde a una obs} > 0 \\ 0 & \text{caso contrario} \end{cases}$$

$$T^+ = \sum_{j=1}^{n} R_j \, s(X_j) = \sum_{j=1}^{n} j \cdot \underbrace{s(X_{D_j})}_{W_j} = \sum_{j=1}^{n} j \cdot W_j$$

Se rechazará $H_0 : \theta = 0$ vs $H_1 : \theta$ si $T^+ > W_{1-\alpha}$

$$T^- = \sum_{j=1}^{n} R_j \tilde{s}(X_j) \quad , \quad \tilde{s}(X_j) = \begin{cases} 1 & \text{si } X_j < 0 \end{cases}$$

$$T^+ + T^- = \sum_{j=1}^{n} R_j \, s(X_j) + \sum_{j=1}^{n} R_j \tilde{s}(X_j)$$

si no hay ceros

$$= \sum_{j=1}^{n} R_j = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

VER FOTO, Ejemplo con n°s y caso de ceros.

En general: (si hay empates, si $n$ es grande, si NO hay 0's)

$$T = \frac{T_0^+}{\left( \sum_{i=1}^{n} R_i^2 \right)^{1/2}} \quad , \quad T^+ + T^- = \frac{n(n+1)}{2}$$

$$T_0 = T^+ - T^- = T^+ - \left( \frac{n(n+1)}{2} - T^+ \right)$$

$$= 2T^+ - \frac{n(n+1)}{2}.$$

Bastaría probar $s(x_i)$ es indep de $|x_i|$:

$$P(s(x_i)=1, |x_i|\le y_i) = P(x_i>0, -y_i \le x_i \le y_i)$$

$$= P(0\le x_i \le y_i)$$

$$= F(y_i) - \underbrace{F(0)}_{1/2}$$

$$= \frac{1}{2}\left(2F(y_i)-1\right)$$

$$= P(s(x_i)=1)\cdot P(|x_i|\le y_i)$$

$$s(x_i)=\begin{cases} 1 & x_i>0 \\ 0 & \text{si no}\end{cases}$$

$$P(x_i>0) = 1-F(0) = 1-\frac{1}{2} = \frac{1}{2}$$

$$P(|x_i|\le y_i) = P(-y_i\le x_i\le y_i) = F(y_i)-F(-y_i) = F(y_i)-(1-F(y_i)) = 2F(y_i)-1$$

Vale la independencia.

Análogamente: $(s(x_1),\dots,s(x_n))$ y $\underbrace{(D_1,\dots,D_n)}_{h(|x_1|,\dots,|x_n|)}$ son indep.

b) Ver notas Elena.

$$\sum_{i=1}^{m} R_i^2 = \sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$$

$\downarrow$ si no hay empates

Luego:

$$T = \frac{2T^+ - \frac{m(m+1)}{2}}{\sqrt{\frac{m(m+1)(2m+1)}{6}}} = \frac{T^+ - \overbrace{\left[\frac{m(m+1)}{4}\right]}^{E(T^+)}}{\underbrace{\sqrt{\frac{m(m+1)(2m+1)}{24}}}_{V(T^+)}} \xrightarrow{\mathcal{D}} N(0,1)$$

(ojo acá no uso TCL clásico sino lindeberg.)

Ojo con p-valor.

DISTRIBUCIÓN DE $T^+$

$Rg(T^+) = \{0, 1, \ldots, \frac{m(m+1)}{2}\}$

$$S(x_i) = \begin{cases} 1 & \text{si } x_i > 0 \\ 0 & \end{cases}$$

Quiero ver que $(s(x_1), \ldots s(x_m))$ es indep. de $(R_1, \ldots, R_m)$.

$(R_1, \ldots, R_m) = g(|x_1|, \ldots, |x_m|)$

Equivale a probar que:

$(s(x_1), \ldots, s(x_m))$ es indep. de $(|x_1|, \ldots, |x_m|)$

Esto equivale a probar que $P(S = s, |x| = y) = P(S = s) \cdot P(|x| = y)$

$$P\Big( (s(x_1), \ldots, s(x_m)) = (s_1, \ldots, s_m) \ ; \ (|x_1|, \ldots, |x_m|) \overset{\text{"}\le\text{"}}{=} (y_1, \ldots, y_m) \Big) =$$

$\hookrightarrow$ son continuas!

$$= P\big( s(x_1) = s_1, \ldots, s(x_m) = s_m \ ; \ |x_1| \le y_1, \ldots, |x_m| \le y_m \big) =$$

$$= P\Big( \bigcap_{i=1}^{m} (s(x_i) = s_i, |x_i| \le y_i) \Big)$$

$$= \prod_{i=1}^{m} P(s(x_i) = s_i, |x_i| \le y_i)$$