Computing distance-regular graph and association scheme parameters in SageMath with `sage-drg`

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Live slides on Binder

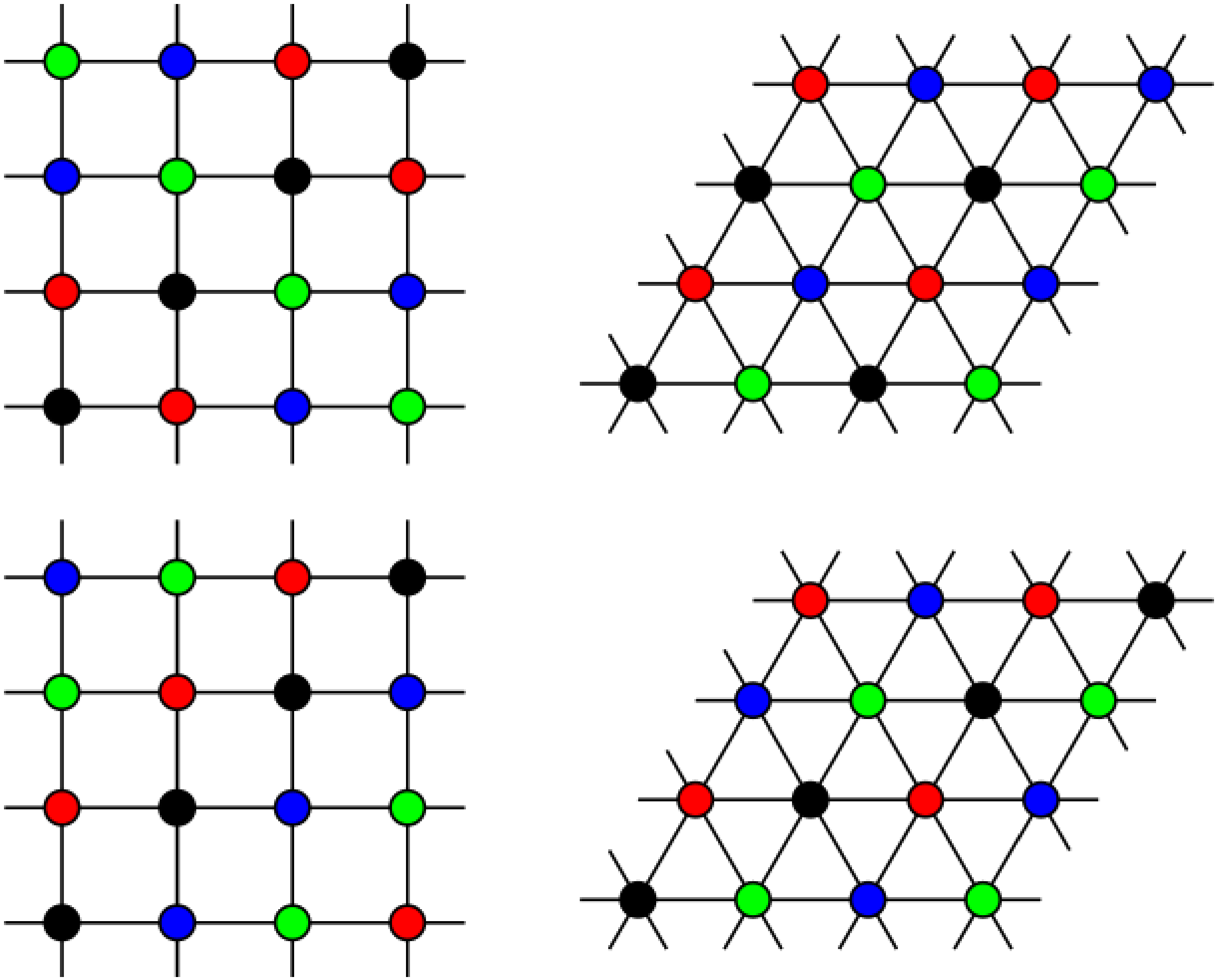
https://github.com/jaanos/sage-drg

Association schemes

- Association schemes were defined by Bose and Shimamoto in 1952 as a theory underlying experimental design.
- They provide a unified approach to many topics, such as
  - combinatorial designs,
  - coding theory,
  - generalizing groups, and
  - strongly regular and distance-regular graphs.

Examples

- Hamming schemes: $X = \mathbb{Z}_n^d, x\ R_i\ y \Leftrightarrow \mathrm{weight}(x-y) = i$
- Johnson schemes: $X = \{S \subseteq \mathbb{Z}_n \mid |S| = d\}$ $(2d \leq n)$, $x\ R_i\ y \Leftrightarrow |x \cap y| = d - i$



Definition

- Let X be a set of vertices and $\mathcal{R} = \{R_0 = \mathrm{id}_X, R_1, \ldots, R_D\}$ a set of symmetric relations partitioning X^2.
- $(X, \mathcal{R})$ is said to be a D-class association scheme if there exist numbers p_{ij}^h ($0 \le h, i, j \le D$) such that, for any $x, y \in X$,
$$x \; R_h \; y \Rightarrow |\{z \in X \mid x \; R_i \; z \; R_j \; y\}| = p_{ij}^h$$
- We call the numbers p_{ij}^h ($0 \le h, i, j \le D$) intersection numbers.

Main problem

- Does an association scheme with given parameters exist?
  - If so, is it unique?
  - Can we determine all such schemes?
- Lists of feasible parameter sets have been compiled for strongly regular and distance-regular graphs.
- Recently, lists have also been compiled for some Q-polynomial association schemes.
- Computer software allows us to efficiently compute parameters and check for existence conditions, and also to obtain new information which would be helpful in the construction of new examples.

Bose-Mesner algebra

- Let A_i be the binary matrix corresponding to the relation R_i ($0 \le i \le D$).
- The vector space $\mathcal{M}$ over $\mathbb{R}$ spanned by A_i ($0 \le i \le D$) is called the Bose-Mesner algebra.
- $\mathcal{M}$ has a second basis $\{E_0, E_1, \ldots, E_D\}$ consisting of projectors to the common eigenspaces of A_i ($0 \le i \le D$).
- There are nonnegative constants q_{ij}^h, called Krein parameters, such that

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{d} q_{ij}^h E_h,$$

where $\circ$ is the entrywise matrix product.

Parameter computation: general association schemes

In [2]:

```
import drg
p = [[[1, 0, 0, 0], [0, 6, 0, 0], [0, 0, 3, 0], [0, 0, 0, 6]],
     [[0, 1, 0, 0], [1, 2, 1, 2], [0, 1, 0, 2], [0, 2, 2, 2]],
     [[0, 0, 1, 0], [0, 2, 0, 4], [1, 0, 2, 0], [0, 4, 0, 2]],
     [[0, 0, 0, 1], [0, 2, 2, 2], [0, 2, 0, 1], [1, 2, 1, 2]]]
scheme = drg.ASParameters(p)
scheme.kreinParameters()
```

Out[2]:

```
0: [1 0 0 0]
   [0 6 0 0]
   [0 0 3 0]
   [0 0 0 6]

1: [0 1 0 0]
   [1 2 1 2]
   [0 1 0 2]
   [0 2 2 2]

2: [0 0 1 0]
   [0 2 0 4]
   [1 0 2 0]
   [0 4 0 2]

3: [0 0 0 1]
   [0 2 2 2]
   [0 2 0 1]
   [1 2 1 2]
```

Metric and cometric schemes

- If $p^h_{ij} \neq 0$ (resp. $q^h_{ij} \neq 0$) implies $|i-j| \leq h \leq i+j$, then the association scheme is said to be metric (resp. cometric).

- The parameters of a metric association scheme can be determined from the intersection array
$$\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\} \quad (b_i = p^i_{1,i+1}, c_i = p^i_{1,i-1}).$$

- The parameters of a cometric association scheme can be determined from the Krein array
$$\{b^*_0, b^*_1, \ldots, b^*_{D-1}; c^*_1, c^*_2, \ldots, c^*_D\} \quad (b^*_i = q^i_{1,i+1}, c^*_i = q^i_{1,i-1}).$$

- Metric association schemes correspond to distance-regular graphs.

Parameter computation: metric and cometric schemes

In [3]:
```
from drg import DRGParameters
syl = DRGParameters([5, 4, 2], [1, 1, 4])
syl
```

Out[3]:
```
Parameters of a distance-regular graph with intersection array {5, 4, 2; 1, 1,
4}
```

In [4]:
```
syl.order()
```

Out[4]:
```
36
```

In [5]:
```
from drg import QPolyParameters
q225 = QPolyParameters([24, 20, 36/11], [1, 30/11, 24])
q225
```

Out[5]:
```
Parameters of a Q-polynomial association scheme with Krein array {24, 20, 36/1
1; 1, 30/11, 24}
```

In [6]:
```
q225.order()
```

Out[6]:
```
225
```

In [7]:
```
syl.pTable()
```

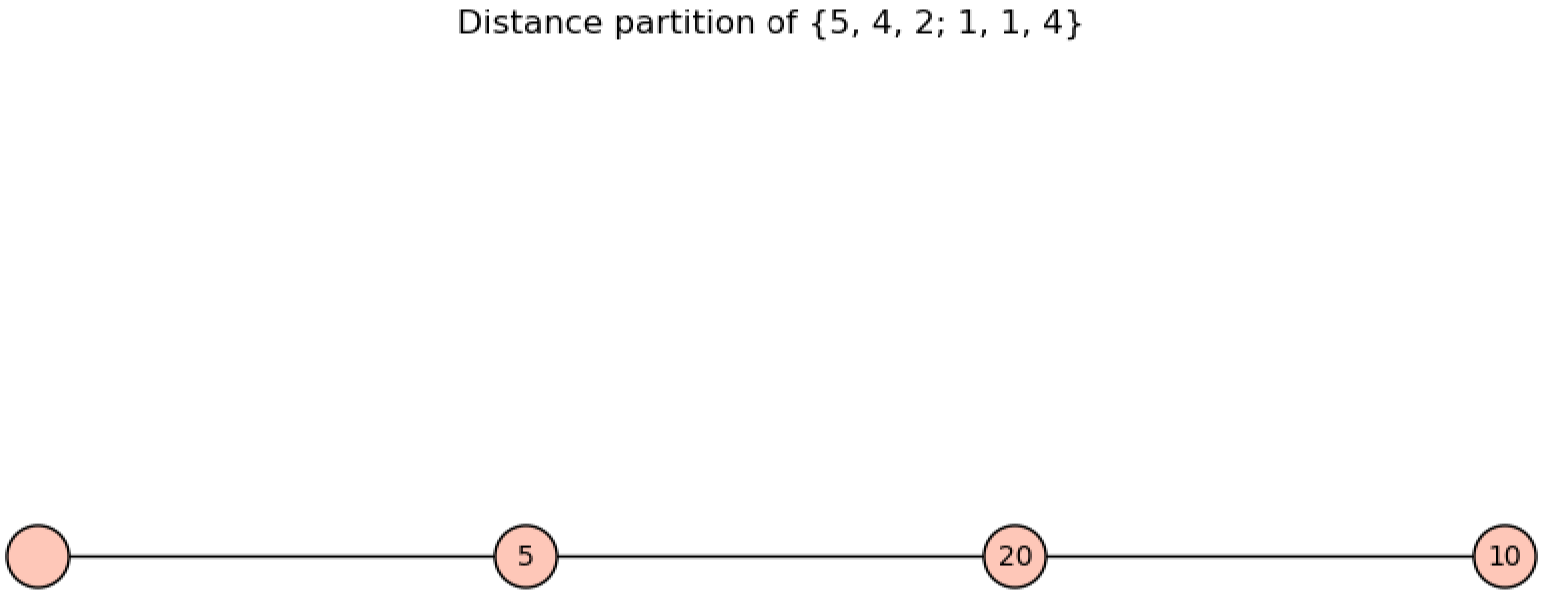
Out[7]:
```
0: [ 1  0  0  0]
   [ 0  5  0  0]
   [ 0  0 20  0]
   [ 0  0  0 10]

1: [0 1 0 0]
   [1 0 4 0]
   [0 4 8 8]
   [0 0 8 2]

2: [ 0  0  1  0]
   [ 0  1  2  2]
   [ 1  2 11  6]
   [ 0  2  6  2]

3: [ 0  0  0  1]
   [ 0  0  4  1]
   [ 0  4 12  4]
   [ 1  1  4  4]
```

In [8]:
```
syl.kreinParameters()
```

Out[8]:
```
0: [ 1  0  0  0]
   [ 0 16  0  0]
   [ 0  0 10  0]
   [ 0  0  0  9]

1: [    0     1     0     0]
   [    1  44/5  22/5   9/5]
   [    0  22/5     2  18/5]
   [    0   9/5  18/5  18/5]

2: [      0       0       1       0]
   [      0  176/25   16/5  144/25]
   [      1    16/5       4     9/5]
   [      0  144/25     9/5   36/25]

3: [    0     0     0     1]
   [    0  16/5  32/5  32/5]
   [    0  32/5     2   8/5]
   [    1  32/5   8/5     0]
```

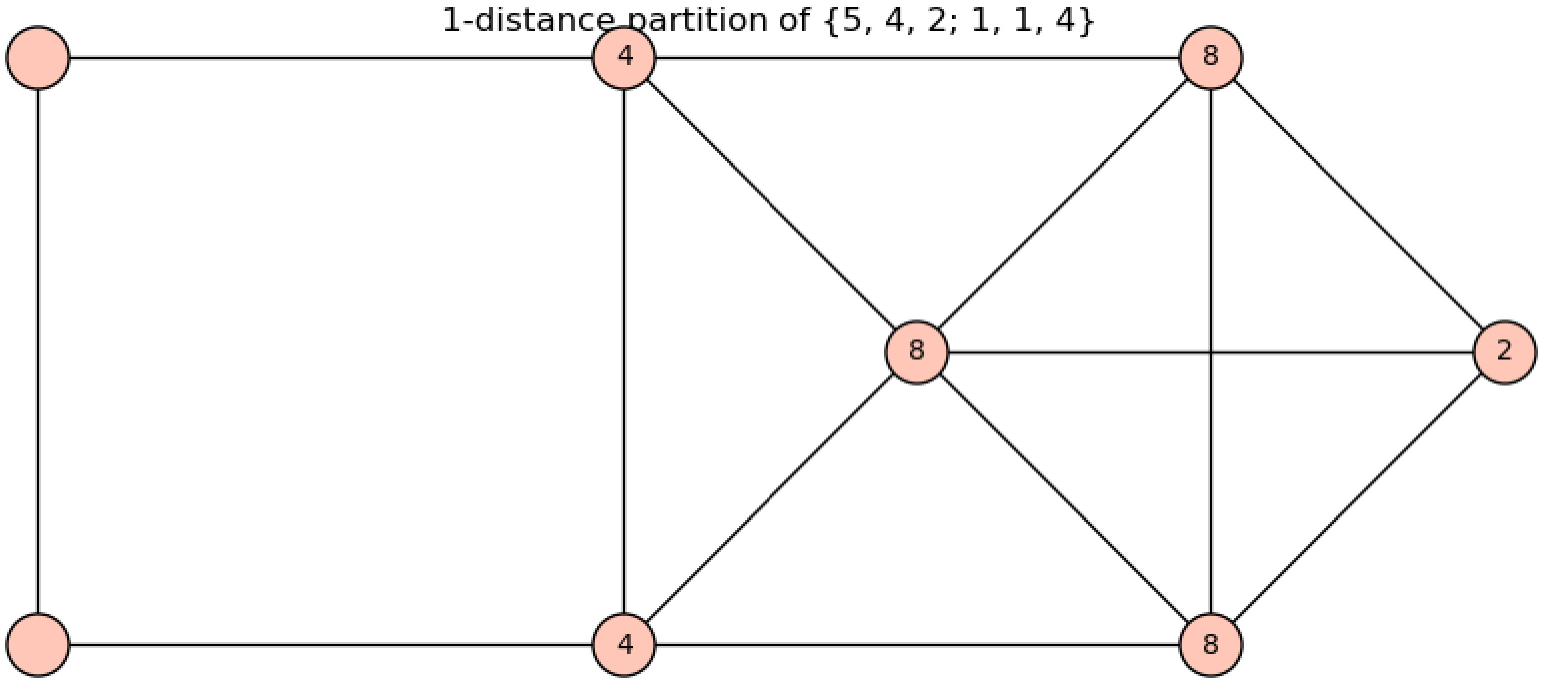
In [9]:
```
syl.distancePartition()
```

Out[9]:



In [10]:
```
syl.distancePartition(1)
```

Out[10]:



Parameter computation: parameters with variables

Let us define a one-parametric family of intersection arrays.

In [11]:
```
r = var("r")
f = DRGParameters([2*r^2*(2*r+1), (2*r-1)*(2*r^2+r+1), 2*r^2], [1, 2*r^2, r*(4*r^2-1)])
f
```

Out[11]:
```
Parameters of a distance-regular graph with intersection array {4*r^3 + 2*r^2, 4*r^3 + r - 1, 2*r^2; 1, 2*r^2, 4*r^3 - r}
```

In [12]:
```
f1 = f.subs(r == 1)
f1
```

Out[12]:
```
Parameters of a distance-regular graph with intersection array {6, 4, 2; 1, 2, 3}
```

The parameters of `f1` are known to uniquely determine the Hamming scheme $H(3, 3)$.

In [13]:
```
f2 = f.subs(r == 2)
f2
```

Out[13]:
```
Parameters of a distance-regular graph with intersection array {40, 33, 8; 1, 8, 30}
```

Feasibility checking

A parameter set is called feasible if it passes all known existence conditions.

Let us verify that $H(3,3)$ is feasible.

In [14]:
```
f1.check_feasible()
```

No error has occured, since all existence conditions are met.

Let us now check whether the second member of the family is feasible.

In [15]:
```
f2.check_feasible()
```

```
---------------------------------------------------------------------------
InfeasibleError                           Traceback (most recent call last)
<ipython-input-15-83a4aafdb73c> in <module>()
----> 1 f2.check_feasible()

/home/janos/repos/git/sage-drg/jupyter/2019-07-04-fpsac/drg/drg.pyc in check_f
easible(self, checked, skip, derived)
    682         for name, check in checks:
    683             if name not in skip:
--> 684                 check()
    685         if not derived:
    686             return

/home/janos/repos/git/sage-drg/jupyter/2019-07-04-fpsac/drg/drg.pyc in check_f
amily(self)
    643                                 in zip(self.b[:-1] + self.c[1:], b + c)], v
ars)
    644             if any(checkConditions(cond, sol) for sol in sols):
--> 645                 raise InfeasibleError(refs = ref)
    646
    647     def check_feasible(self, checked = None, skip = None, derived = Tr
ue):

InfeasibleError: nonexistence by JurišićVidali12
```

In this case, nonexistence has been shown by matching the parameters against a list of nonexistent families.

Triple intersection numbers

- In some cases, triple intersection numbers can be computed.
- Nonexistence of some Q-polynomial association schemes has been proven by obtaining a contradiction in double counting with triple intersection numbers.

In [16]:
```
q225.check_quadruples()
```

```
---------------------------------------------------------------------------
InfeasibleError                           Traceback (most recent call last)
<ipython-input-16-40f750f5d8a3> in <module>()
----> 1 q225.check_quadruples()

/home/janos/repos/git/sage-drg/jupyter/2019-07-04-fpsac/drg/assoc_scheme.py in
check_quadruples(self, solver)
    685                                         "d(w, y) = %d, d(w, z) = %d, "
    686                                         "d(x, y) = %d, d(x, z) = %d, "
--> 687                                         "d(y, z) = %d" % (sd + st))
    688                                 if len(r[st]) < l:
    689                                     zero[st] = {(sh, si, sj)

InfeasibleError: found forbidden quadruple wxyz with d(w, x) = 1, d(w, y) = 1,
d(w, z) = 1, d(x, y) = 3, d(x, z) = 3, d(y, z) = 3
```

Integer linear programming has been used to find solutions to multiple systems of linear Diophantine equations, eliminating inconsistent solutions.