DampedOscillation

May 28, 2024

0.0.1 Appendix: Fourier Analysis of a damped oscillation

Consider a generalised description of the oscillation by

$$f(t) = a(t)\cos(\omega_S t) \tag{1}$$

Where $a(t)$ is the time varying amplitude. This product of two functions can be treated with the help of the so-called convolution theorem. A convolution is represented by

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \tag{2}$$

which means, that one sums up all contributions of a function $g(t)$ centered at a time value of τ with amplitude $f(\tau)$. Convolutions play an important role for example in optics, where the microscope resolution function convolutes the structural images of all objects.

This convolution integral can be transformed into a product of the Fourier transforms ($\mathscr{F}$) of both functions

$$H(\omega)G(\omega) = \mathscr{F}(f * g) \tag{3}$$

So the product of the Fourier transform of the individual functions in the product is the same as the Fourier transform of the convolution integral Eq. 2.

What is relevant in the discussed case of the oscillating guitar string is the inverse relation of the convolution theorem

$$H(\omega) * G(\omega) = \mathscr{F}(f(t)g(t)) \tag{4}$$

This means that the Fourier transform of a product of two functions is equivalent to a convolution of the Fourier transforms of the individual functions.

The convolution integral of the Fourier transformed function then corresponds to

$$H(\omega) * G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\Omega)G(\omega-\Omega)d\Omega \tag{5}$$

Using this relation and

$$H(\omega) = \mathscr{F}(\cos(\omega_S t)) \tag{6}$$

and

$$G(\omega) = \mathscr{F}(\Theta(t)e^{-t/\tau}) \tag{7}$$

I can compute all types of amplitude modulated harmonic oscillations in Fourier space. In particular, the Fourier transform $\mathscr{F}$ of a harmonic function $\cos(\omega_S t)$ results in a so-called delta function ($\delta(t)$)

$$H(\omega) = \mathscr{F}(\cos(\omega_S t)) = \sqrt{\frac{\pi}{2}}\delta(\omega + \omega_S) - \sqrt{\frac{\pi}{2}}\delta(\omega + \omega_S) \tag{8}$$

The δ-function has the properties as described in section **??**.

$$G(\omega) = \mathscr{F}(\Theta(t)e^{-t/\tau}) = \frac{1}{\sqrt{2\pi}}\frac{i\tau}{(i - \tau\omega)} \tag{9}$$

The squared magnitude of Eq. 9 yields a Lorentzian lineshape

$$|G(\omega)|^2 = \frac{1}{2\pi}\frac{\tau^2}{1 + \tau^2\omega^2} \tag{10}$$

for the frequency spectrum, which is very common in physics. This Lorentzian function has a maximum at $\omega = 0$ with $|G(\omega)|^2 = \tau^2/2\pi$.

With the help of Eq. 8 and Eq. 9 I can write the convolution integral 5 as

$$H(\omega) * G(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{2}}[\delta(\Omega - \omega_S) + \delta(\Omega + \omega_S)]\frac{1}{\sqrt{2\pi}}\frac{i\tau}{(i - \tau(\omega - \Omega))}d\Omega \tag{11}$$

The integration can be carried out using the integration rule in Eq. **??** for the delta function which leaves me with the integral of the function displayed in Eq. 11 as equal to

$$F(\omega) = H(\omega) * G(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}\sqrt{2\pi}}\left[\frac{i\tau}{(i - \tau(\omega - \omega_S))} + \frac{i\tau}{(i - \tau(\omega + \omega_S))}\right] \tag{12}$$

This yields

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1/\tau + i\omega}{(1/\tau + i\omega)^2 + \omega_S^2} \tag{13}$$