Reflection and Refraction of Electromagnetic Waves

Effect of the Refractive Index

The effect of the material on the wave propagation can be understood when considering a plane wave E_s incident on a thin material slab of thickness Δz.

$$\begin{aligned} E &= E_0 \cdot e^{-i(\omega t - kz - (n-1)k \cdot \Delta z)} \\ &= E_0 e^{-i(n-1)k\Delta z} e^{i(\omega t - kz)} \\ &= e^{-i\theta} E_s \quad \text{ with } \theta = k(n-1)\Delta z \end{aligned} \tag{1}$$

For small values of θ the exponential function can be approximated by

$$e^x \sim 1 + x + \frac{x^2}{2} \tag{2}$$

such that we obtain

$$e^{-ik(n-1)\Delta z} \approx 1 - ik(n-1)\Delta z - \frac{k^2(n-1)^2 \Delta z^2}{2} \tag{3}$$

The total field behind the thin slab therefore is

$$E(z) = \underbrace{E_0 e^{i(\omega t - kz)}}_{E_e} - \underbrace{ik(n-1)\Delta z E_0 e^{i(\omega t - kz)}}_{E_{\text{medium}}} \tag{4}$$

As shown in Figure 1, the resulting wave is delayed by a phase factor $k(n-1)\Delta z$ (see Equation 1) and the amplitude is reduced by a factor $k(n-1)\Delta z$.

```
# Parameters
E0 = 1.0  # Initial amplitude
k = 2*np.pi  # Wave number
n = 1.5  # Refractive index
dz = 0.1  # Thickness of slab
omega = 2*np.pi  # Angular frequency
t = 0  # Fixed time point

# Spatial grid
z = np.linspace(0, 2, 1000)

# Calculate components
E_vacuum = E0 * np.exp(1j*(omega*t - k*z))
E_medium = -1j * k * (n-1) * dz * E0 * np.exp(1j*(omega*t - k*z))
E_total = E_vacuum + E_medium

# Create figure
fig, ax1 = plt.subplots(1, 1, figsize=get_size(12, 8))

# Plot real parts
ax1.plot(z, E_vacuum.real, 'b-', label='Incident', alpha=0.7)
ax1.plot(z, E_medium.real, 'r-', label='Medium', alpha=0.7)
ax1.plot(z, E_total.real, 'g-.', label='Total', alpha=0.7)
ax1.set_xlabel('position (z)')
ax1.set_ylabel('Re[E(z)]')
ax1.legend()

plt.show()
```

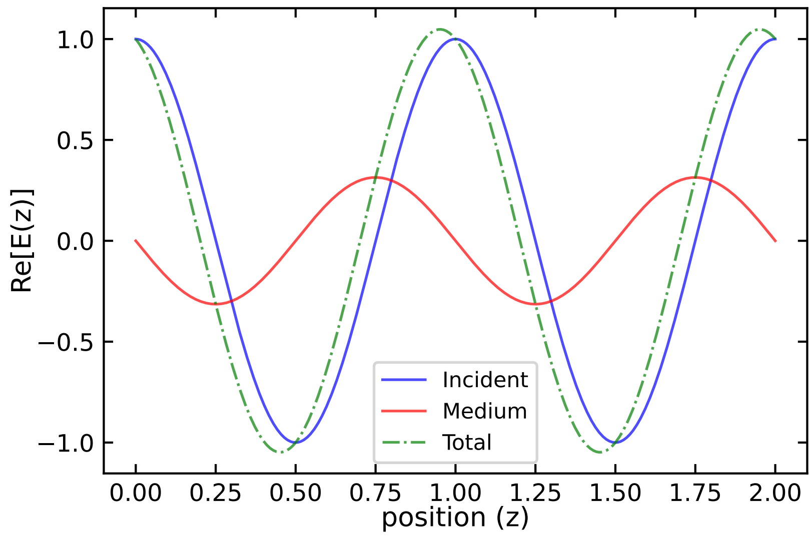

Figure 1: Components of the electric field in medium

General Description of Reflection and Refraction

Before we examine the behavior of electromagnetic waves at boundaries, we need to define the geometry and examine the electric and magnetic fields present. Figure 2 shows the wavevectors of the incident $\vec{k}_I$, reflected $\vec{k}_R$ and transmitted $\vec{k}_T$ waves in the two materials with the refractive indices n_1 and n_2.

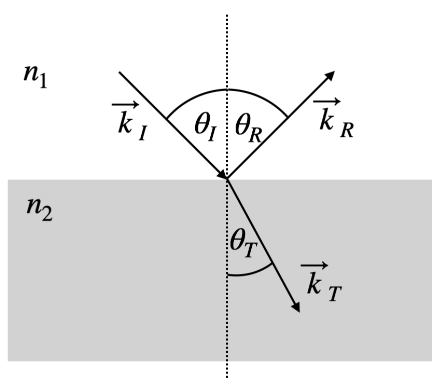

Figure 2: Reflection and refraction of an electromagnetic wave at a boundary.

These wavevectors are connected to the following plane waves:

$$\vec{E}_{inc} = \vec{E}_I e^{i(\omega_I t - \vec{k}_I \cdot \vec{r})} \tag{5}$$

$$\vec{E}_{ref} = \vec{E}_R e^{i(\omega_R t - \vec{k}_R \cdot \vec{r})} \tag{6}$$

$$\vec{E}_{tra} = \vec{E}_T e^{i(\omega_T t - \vec{k}_T \cdot \vec{r})} \tag{7}$$

As discussed in Equation 5 through Equation 7, we must consider the direction of polarization of their electric or magnetic fields. We differentiate between:

- **p-polarized light:** electric field is in the plane of incidence (given by the k-vector and the surface normal), also called transverse magnetic (TM)
- **s-polarized light:** electric field is perpendicular to the plane of incidence, also called transverse electric (TE)

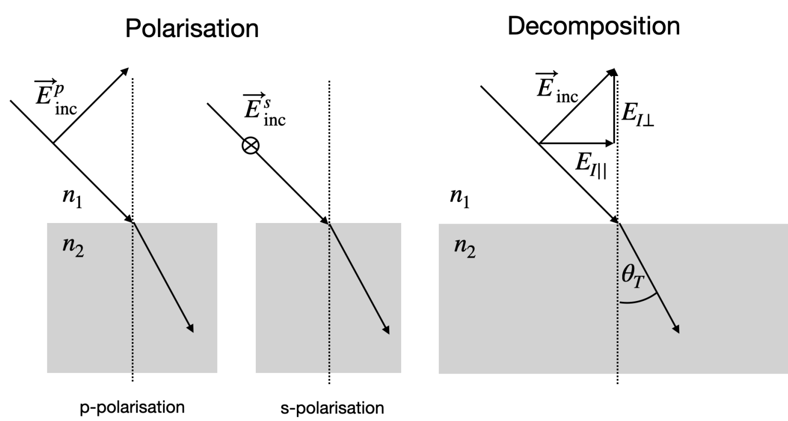

Figure 3: Decomposition of the incident electric field into components normal and tangential to the dielectric boundary.

As shown in Figure 3 , we need to split these polarisation vectors into components that are **parallel** (||) or **perpendicular** ($\perp$) to the interface. This is required for applying boundary conditions for the electric and magnetic fields.

Boundary Conditions

The boundary conditions for the electric or magnetic field passing an interface are derived from the Maxwell equations.

Let us take the divergence of the displacement field $\nabla \cdot \vec{D} = \rho_f$ which is equal to the density of free charges. When integrating both sides over the volume:

$$\int \nabla \vec{D} dV = \int \rho_f dV \tag{8}$$

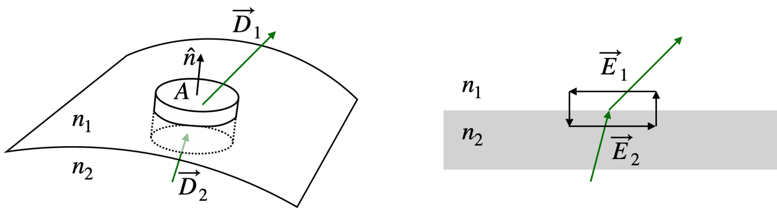

Figure 4: Integration over a closed surface (left) and close path (right) to obtain the boundary conditions for the electric field components.

we can apply Gauss' theorem and replace the volume integral over the divergence with an integral over closed surface of that volume:

$$\oint \vec{D} d\vec{A} = q_f \tag{9}$$

where $d\vec{A}$ is a vector standing perpendicular on the surface element dA and q_f are the free charges in the volume. When we consider a pillbox-shaped volume straddling the interface between two materials, the surface integral can be broken down into three parts:

1. Top surface (in material 1)
2. Bottom surface (in material 2)
3. Side surface (cylindrical part)

The total surface integral becomes:

$$\oint \vec{D} \cdot d\vec{A} = \int_{\text{top}} \vec{D}_1 \cdot d\vec{A} + \int_{\text{bottom}} \vec{D}_2 \cdot d\vec{A} + \int_{\text{side}} \vec{D} \cdot d\vec{A}$$

For a pillbox of height h and radius r:

- The top surface contributes: $\vec{D}_1 \cdot \hat{n} A$ (negative because $d\vec{A}$ points along $\hat{n}$)
- The bottom surface contributes: $-\vec{D}_2 \cdot \hat{n} A$
- The side surface contribution goes to zero as $h \to 0$ (as area $\sim 2\pi r h$)

Therefore:

$$\oint \vec{D} \cdot d\vec{A} = A(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = q_f$$

Key Points

1. The side surface contribution vanishes as $h \to 0$ because:

- Its area scales with h ($2\pi rh$)
- The field components remain finite

2. Only the normal components of $\vec{D}$ contribute because:

   - $d\vec{A}$ is parallel to $\hat{n}$ for top and bottom surfaces
   - The tangential components don't contribute to the dot product

3. The ratio q_f/A becomes the surface charge density σ_f as $h \to 0$

Following that, we obtain the boundary condition for the normal component of the displacement field:

$$D_{1\perp} = D_{1\perp} \tag{10}$$

This implies a jump in the normal electric field component:

$$\frac{E_{1\perp}}{E_{2\perp}} = \frac{\epsilon_2}{\epsilon_1} \tag{11}$$

as $D = \epsilon E$. Similar calculations with the magnetic flux density $\nabla \cdot \vec{B} = 0$ show that the normal component of the flux density is conserved.

Another boundary condition arises from the curl of the electric field. Using the Maxwell equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

Integrating both sides over an area and applying Stokes theorem:

$$\oint \vec{E} d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{A} \tag{13}$$

Consider a rectangular loop straddling the interface between two media, with a height h a width w and a normal to the interface $\hat{n}$ the line integral can be broken down into four parts:

1. Top segment (in medium 1)
2. Bottom segment (in medium 2)
3. Two vertical segments connecting them

The line integral becomes:

$$\oint \vec{E} \cdot d\vec{l} = w\vec{E}_1 \cdot \hat{t} - w\vec{E}_2 \cdot \hat{t} + \text{(vertical segments)}$$

where $\hat{t}$ is the unit vector tangent to the interface.

As $h \to 0$:

1. The contribution from vertical segments vanishes (as $h \to 0$)
2. The area of the loop approaches zero, making the right-hand side zero:

$$-\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{A} \to 0$$

Therefore:

$$w(\vec{E}_1 - \vec{E}_2) \cdot \hat{t} = 0$$

Since $w \neq 0$, this implies:

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{t} = 0$$

This can be rewritten in terms of the cross product with the normal vector:

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \tag{14}$$

Therefore:

$$E_{2||} = E_{1||} \tag{15}$$

indicating that the tangential component of the electric field is conserved.

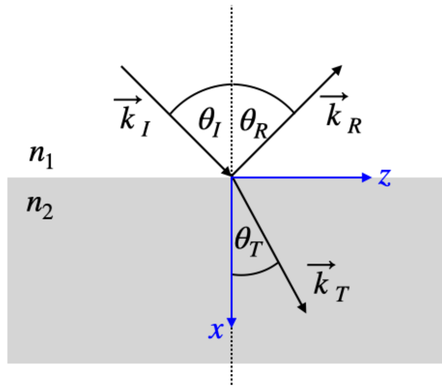

Figure 5: Coordinate system for wave at interfaces.

Reflection/Refraction

Frequency and Wavevector Matching

Referring to Figure 5, we can explicitly write down the components of the wavevectors of the three waves in our coordinate system:

According to the coordinate system shown in Figure 5, we have:

$$\begin{aligned} \vec{k}_I &= k_I \cos(\theta_I)\hat{e}_x + k_I \sin(\theta_I)\hat{e}_z \\ \vec{k}_R &= -k_R \cos(\theta_R)\hat{e}_x + k_R \sin(\theta_R)\hat{e}_z \\ \vec{k}_T &= k_T \cos(\theta_T)\hat{e}_x + k_T \sin(\theta_T)\hat{e}_z \end{aligned} \tag{16}$$

Note that the component $\hat{e}_x$ always provides the wavevector component perpendicular to the interface, while $\hat{e}_z$ is the tangential (parallel) component. The total field on both sides is given by:

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{ref} \quad (\text{for } x < 0) \tag{17}$$

and

$$\vec{E} = \vec{E}_{tra} \quad (\text{for } x > 0) \tag{18}$$

For these fields to match according to our previously described boundary conditions, we require:

$$\omega_I = \omega_R = \omega_T \tag{19}$$

This frequency matching confirms our initial intuition. Along the interface, we also need phase matching of the waves as discussed in the wave optics chapter:

$$\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r} \tag{20}$$

at all positions $\vec{r}$ that belong to the interface (i.e., $x = 0$). Therefore $\vec{r} = \{0, y, z\}$ and the equalities yield:

$$k_I \sin(\theta_I) = k_R \sin(\theta_R) = k_T \sin(\theta_T) \tag{21}$$

Since the magnitude of the wavevector of the incident and the reflected light is the same (both waves travel in the same material), we find:

$$\theta_I = \theta_R \tag{22}$$

For the incident and transmitted waves, we must account for the change in wavenumber:

$$\begin{gathered} n_1 k_0 \sin(\theta_I) = n_2 k_0 \sin(\theta_T) \\ n_1 \sin(\theta_I) = n_2 \sin(\theta_T) \end{gathered} \tag{23}$$

Equation 23 represents Snell's law, which results from the conservation of the parallel component of the wavevector across an interface, while the normal component must have a jump according to the refractive indices.