Mathematical Derivation

Your Name

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Derivation of Tangential Components

Starting with the equations for the tangential components:

$$\vec{k}_{1p} = \left(\vec{k}_1 \cdot \hat{N}\right) \hat{N}$$

$$\vec{k}_{1t} = \vec{k}_1 - \vec{k}_{1p} = \vec{k}_1 - \left(\vec{k}_1 \cdot \hat{N}\right) \hat{N}$$

$$\vec{k}_{2p} = \left(\vec{k}_2 \cdot \hat{N}\right) \hat{N}$$

$$\vec{k}_{2t} = \vec{k}_2 - \left(\vec{k}_2 \cdot \hat{N}\right) \hat{N}$$

Now, setting $\vec{k}_{1t} = \vec{k}_{2t}$.

Relating Incident and Refracted Directions

Since $|\vec{k}_1| = (n_1 k_0)\,\hat{k}_1$ and $|\vec{k}_2| = (n_2 k_0)\,\hat{k}_2$, we can write:

$$n_1 k_0 \hat{k}_1 - \left[\left(\vec{k}_1 \cdot \hat{N}\right) \hat{N}\right] = n_2 k_0 \left(\hat{k}_2 - \left(\vec{k}_2 \cdot \hat{N}\right) \hat{N}\right)$$

which simplifies to:

$$n_1 k_0 \left(\hat{k}_1 \times \hat{N}\right) = n_2 k_0 \left(\hat{k}_2 \times \hat{N}\right)$$

Therefore,

$$n_1 \left(\hat{k}_1 \times \hat{N}\right) = n_2 \left(\hat{k}_2 \times \hat{N}\right)$$

Angle Relations

Expanding the expression using trigonometric functions:

$$n_1 \sin(\alpha + \theta_1) = n_2 \sin(\alpha - (\pi - \theta_2))$$

$$= n_2 \sin(\alpha + \theta_2')$$

Thus,

$$n_1 \alpha + n_1 \theta_1 = n_2 \alpha + n_2 \theta_2'$$

leading to the result:

$$\theta_2' = \frac{n_1}{n_2}\theta_1 + \frac{n_1 - n_2}{n_2}\frac{y}{R}$$

Unit Vectors in Cartesian Coordinates

For the components:

$$\hat{k}_1 = \frac{1}{\sqrt{x_1^2 + y^2}} \begin{pmatrix} x_1 \\ y \end{pmatrix}$$

$$\hat{k}_2 = \frac{1}{\sqrt{x_2^2 + y^2}} \begin{pmatrix} x_2 \\ -y \end{pmatrix}$$

$$\hat{N} = \frac{1}{R} \begin{pmatrix} x_R \\ -y \end{pmatrix}$$

Cross Products of Unit Vectors

Calculating the cross products:

$$\hat{k}_1 \times \hat{N} = \frac{1}{R\sqrt{x_1^2 + y^2}} \begin{pmatrix} 0 \\ 0 \\ -x_1 y - y x_R \end{pmatrix}$$

$$\hat{k}_2 \times \hat{N} = \frac{1}{R\sqrt{x_2^2 + y^2}} \begin{pmatrix} 0 \\ 0 \\ -x_2 y + y x_R \end{pmatrix}$$

Equating the Cross Product Terms

From the relations, we get:

$$\frac{n_1}{\sqrt{x_1^2 + y^2}} \begin{pmatrix} 0 \\ 0 \\ -x_1 y - y x_R \end{pmatrix} = \frac{n_2}{\sqrt{x_2^2 + y^2}} \begin{pmatrix} 0 \\ 0 \\ -x_2 y + y x_R \end{pmatrix}$$

After simplification:

$$-n_1 \left(\frac{x_1}{\sqrt{x_1^2 + y^2}} \right) + \frac{x_R}{y} = -n_2 \left(\frac{x_2}{\sqrt{x_2^2 + y^2}} + \frac{x_R}{y} \right)$$

Final Form

Expanding and simplifying the trigonometric expressions, we obtain:

$$n_1 \sin(\alpha + \theta_1) = n_2 \sin(\alpha + \theta_2')$$

which leads to:

$$\theta_2' = \frac{n_1 - n_2}{n} \alpha + \frac{n_1}{n_2} \theta_1$$

This transcription retains the structure and layout from the handwritten derivations. Make sure to adjust or expand as necessary, depending on the level of detail you want to include.

Let me know if you would like further refinements or additional sections for explanations and commentary.