Proof that a Mirror Must Have a Parabolic Shape

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Introduction

In this proof, we will show that a mirror must have a parabolic shape for all the rays to meet at the focus. We will use **Fermat's principle** to derive the function that represents the parabolic mirror.

Fermat's Principle

Fermat's principle states that a ray of light will always take the path that requires the least time. For the mirror to reflect parallel rays (rays coming from infinity) to the focus, the time taken by light to travel from the dashed line to the focus must be the same for all rays.

Setup

We consider a parallel ray coming from infinity, parallel to the principal axis (along the y-axis), which is reflected by the mirror and converges at the focus.

The time taken for light to travel consists of two parts:

$$t = t_A + t_B$$

where:

- t_A is the time taken to travel from the dashed line to the mirror.
- t_B is the time taken to travel from the mirror to the focus.

Let's define:

- The dashed line is at $y = D$,
- The speed of light is v,

- The point on the mirror is (x, y),
- The focus is at $(0, C)$.

Derivation of the Parabolic Shape

Time for Path A

The length of path A is $D - y$ (since y is the coordinate of the point on the mirror). Thus, the time for path A is:

$$t_A = \frac{D - y}{v}$$

Time for Path B

The light travels from the point (x, y) on the mirror to the focus at $(0, C)$. The distance for this part of the path is:

$$\sqrt{x^2 + (y - C)^2}$$

Thus, the time for path B is:

$$t_B = \frac{\sqrt{x^2 + (y - C)^2}}{v}$$

Total Time

The total time for the ray is:

$$t = \frac{D - y}{v} + \frac{\sqrt{x^2 + (y - C)^2}}{v}$$

Since Fermat's principle requires that the time is the same for all rays, we set the total time equal to a constant k:

$$\frac{D - y}{v} + \frac{\sqrt{x^2 + (y - C)^2}}{v} = k$$

Special Case for a Ray Along the Y-Axis

Consider a ray travelling along the y-axis. This ray reflects off the mirror at $(0, 0)$, and the total distance traveled by this ray is $D + C$. The time taken by this ray is:

$$\frac{D + C}{v}$$

Thus, $k = \frac{D+C}{v}$, and we substitute this into the equation for the general case:

$$\frac{D - y}{v} + \frac{\sqrt{x^2 + (y - C)^2}}{v} = \frac{D + C}{v}$$

Simplification

Multiplying through by v and simplifying:

$$D - y + \sqrt{x^2 + (y - C)^2} = D + C$$

$$\sqrt{x^2 + (y - C)^2} = y + C$$

Squaring both sides:

$$x^2 + (y - C)^2 = (y + C)^2$$

$$x^2 + y^2 - 2Cy + C^2 = y^2 + 2Cy + C^2$$

$$x^2 = 4Cy$$

This is the equation of a parabola with a focus at $(0, C)$.

Conclusion

Thus, we have proven that a mirror must have a parabolic shape for all the rays to meet at the focus, as per Fermat's principle.