Light Propagation in Anisotropic Materials

Principles of Photonics - Lecture Notes

Based on Saleh & Teich

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1 Introduction to Anisotropic Materials

Anisotropic materials exhibit direction-dependent optical properties, unlike isotropic materials where optical properties are the same in all directions. This lecture explores how light propagates through such materials, following the treatment in Saleh and Teich's "Principles of Photonics."

Key Learning Objectives: - Understand the concept of optical anisotropy - Derive the wave equation in anisotropic media - Explore birefringence and its applications - Analyze ordinary and extraordinary ray propagation

2 Mathematical Framework

2.1 Permittivity Tensor

In anisotropic materials, the electric displacement field $\mathbf{D}$ and electric field $\mathbf{E}$ are related through the permittivity tensor:

$$\mathbf{D} = \varepsilon_0 \overleftrightarrow{\varepsilon}_r \mathbf{E}$$

Where $\overleftrightarrow{\varepsilon}_r$ is the relative permittivity tensor:

$$\overleftrightarrow{\varepsilon}_r = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

2.2 Principal Axes and Dielectric Constants

For a lossless, non-magnetic anisotropic medium, the permittivity tensor can be diagonalized:

$$\overleftrightarrow{\varepsilon}_r = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}$$

Where n_x, n_y, and n_z are the principal refractive indices.

2.3 Wave Equation in Anisotropic Media

Starting from Maxwell's equations, the wave equation in an anisotropic medium becomes:

$$\nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mu_0\varepsilon_0\frac{\partial^2\mathbf{D}}{\partial t^2}$$

For a plane wave propagating in the z-direction with wave vector $\mathbf{k}$, this leads to the Fresnel equation.

3 Birefringence

3.1 Uniaxial Crystals

Uniaxial crystals have two distinct principal refractive indices: - **Ordinary index** (n_o): $n_x = n_y = n_o$ - **Extraordinary index** (n_e): $n_z = n_e$

The optical axis is along the z-direction.

3.1.1 Positive and Negative Uniaxial Crystals

- **Positive uniaxial**: $n_e > n_o$ (e.g., quartz)
- **Negative uniaxial**: $n_e < n_o$ (e.g., calcite)

3.2 Biaxial Crystals

Biaxial crystals have three distinct principal indices: $n_x \neq n_y \neq n_z$.

4 Wave Propagation Analysis

4.1 Ordinary and Extraordinary Rays

When unpolarized light enters a uniaxial crystal, it splits into two linearly polarized rays:

1. **Ordinary ray (o-ray)**:
   - Follows Snell's law
   - Refractive index = n_o (independent of direction)
   - Electric field perpendicular to optical axis

2. **Extraordinary ray (e-ray)**:
    - Does not follow ordinary Snell's law
    - Refractive index depends on propagation direction
    - Electric field has component along optical axis

4.2 Index Ellipsoid

The index ellipsoid provides a geometric representation of the refractive index variation:

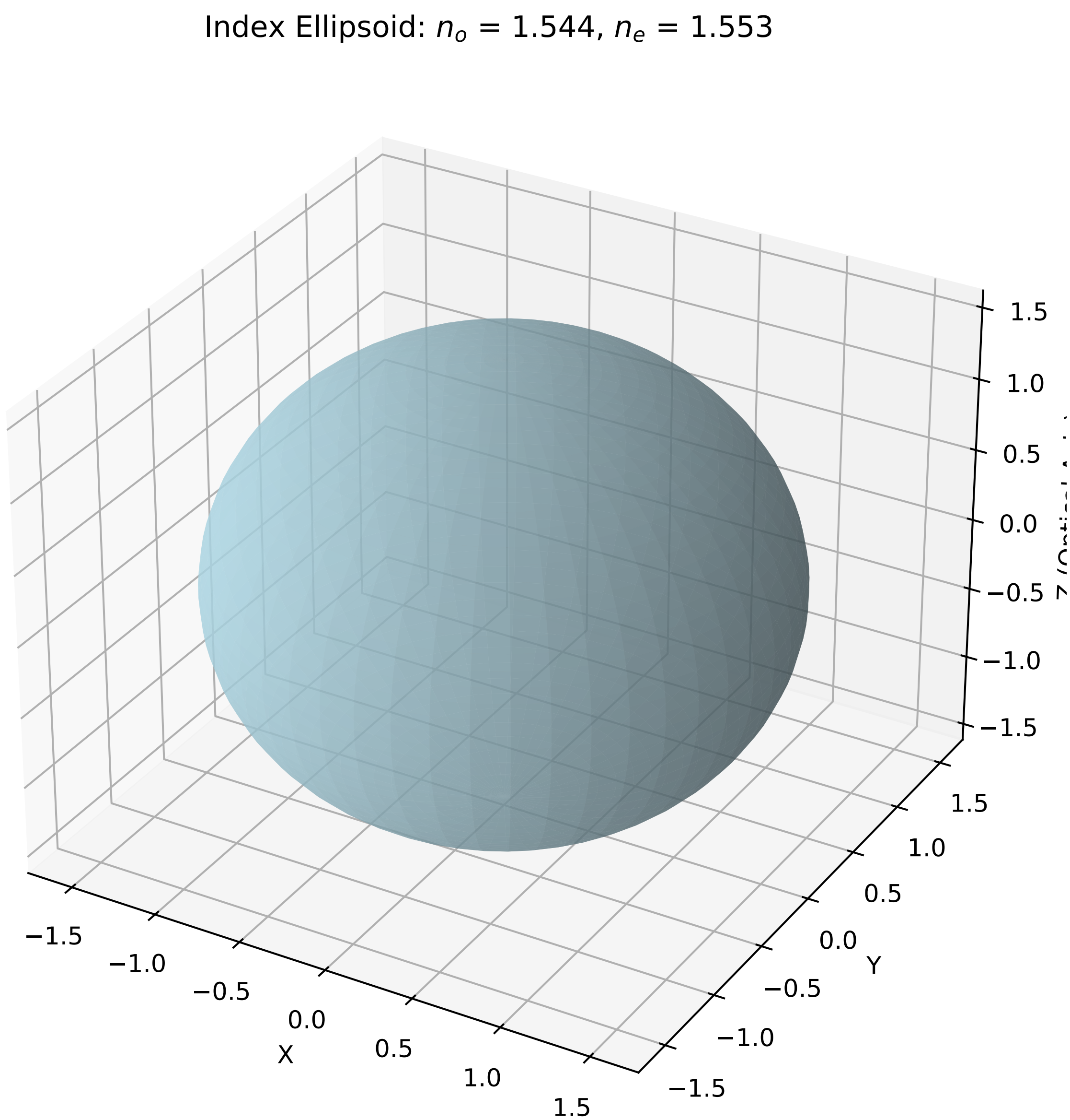
$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

For uniaxial crystals ($n_x = n_y = n_o$, $n_z = n_e$):

$$\frac{x^2 + y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create index ellipsoid for uniaxial crystal
def plot_index_ellipsoid(n_o, n_e):
    fig = plt.figure(figsize=(10, 8))
    ax = fig.add_subplot(111, projection='3d')

    # Create sphere coordinates
    u = np.linspace(0, 2 * np.pi, 50)
    v = np.linspace(0, np.pi, 50)

    # Ellipsoid coordinates
    x = n_o * np.outer(np.cos(u), np.sin(v))
    y = n_o * np.outer(np.sin(u), np.sin(v))
    z = n_e * np.outer(np.ones(np.size(u)), np.cos(v))

    # Plot ellipsoid
    ax.plot_surface(x, y, z, alpha=0.6, color='lightblue')

    # Add axes labels
    ax.set_xlabel('X')
    ax.set_ylabel('Y')
```

```
    ax.set_zlabel('Z (Optical Axis)')
    ax.set_title(f'Index Ellipsoid: $n_o$ = {n_o}, $n_e$ = {n_e}')

    plt.show()

# Example: Quartz (positive uniaxial)
plot_index_ellipsoid(1.544, 1.553)
```

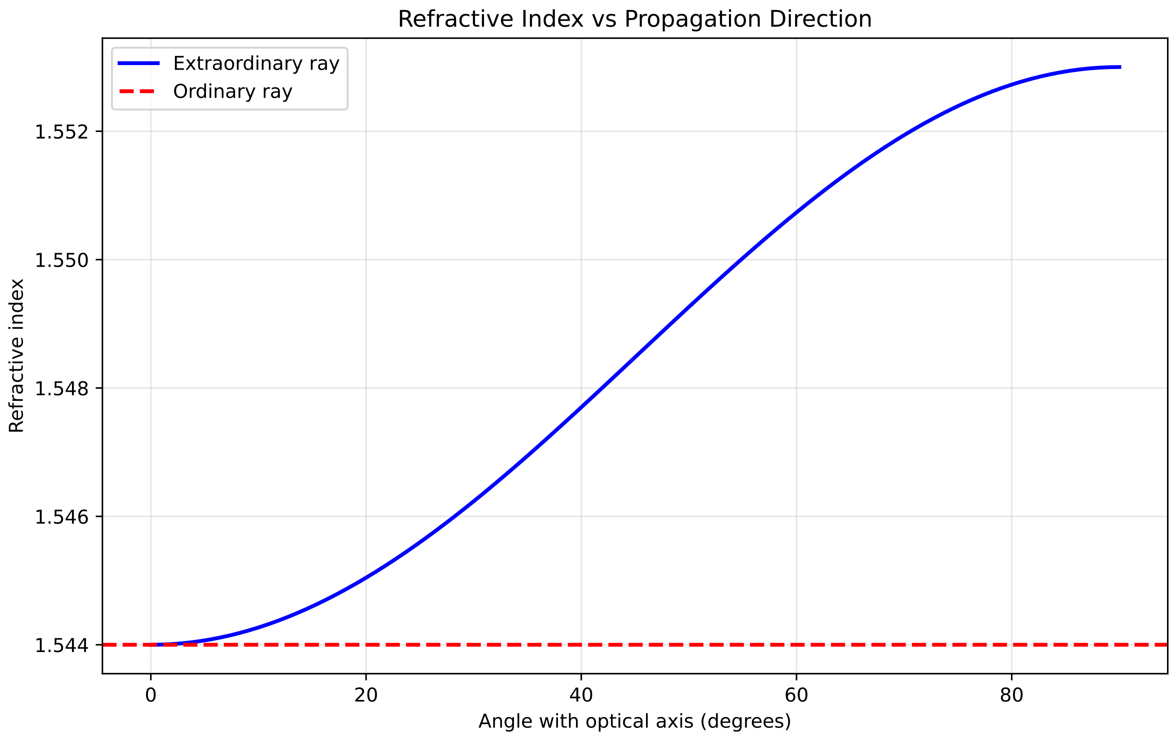


4.3 Extraordinary Ray Refractive Index

For a uniaxial crystal, the extraordinary ray refractive index depends on the angle between the wave vector and optical axis:

$$\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}$$

```python
# Plot extraordinary ray index vs angle
def plot_extraordinary_index(n_o, n_e):
    theta = np.linspace(0, np.pi/2, 100)
    n_e_theta = 1/np.sqrt(np.sin(theta)**2/n_e**2 + np.cos(theta)**2/n_o**2)

    plt.figure(figsize=(10, 6))
    plt.plot(theta*180/np.pi, n_e_theta, 'b-', linewidth=2, label='Extraordinary ray')
    plt.axhline(y=n_o, color='r', linestyle='--', linewidth=2, label='Ordinary ray')
    plt.xlabel('Angle with optical axis (degrees)')
    plt.ylabel('Refractive index')
    plt.title('Refractive Index vs Propagation Direction')
    plt.legend()
    plt.grid(True, alpha=0.3)
    plt.show()

# Example for quartz
plot_extraordinary_index(1.544, 1.553)
```



5 Applications

5.1 Wave Plates

Wave plates introduce controlled phase differences between ordinary and extraordinary rays:

5.1.1 Quarter-Wave Plate

- Phase difference: $\delta = \pi/2$
- Thickness: $d = \frac{\lambda}{4|n_e - n_o|}$
- Converts linearly polarized light to circularly polarized light

5.1.2 Half-Wave Plate

- Phase difference: $\delta = \pi$
- Thickness: $d = \frac{\lambda}{2|n_e - n_o|}$
- Rotates linear polarization by 2θ

5.2 Polarization Beam Splitters

Utilize the different propagation characteristics of ordinary and extraordinary rays to separate orthogonal polarization components.

5.3 Optical Isolators

Combine birefringent materials with Faraday rotators to create non-reciprocal optical devices.

6 Numerical Examples

6.1 Example 1: Calcite Crystal

Given: - $n_o = 1.658$ (ordinary index) - $n_e = 1.486$ (extraordinary index) - Wavelength: $\lambda = 633$ nm

Calculate the thickness of a quarter-wave plate:

```
# Calcite quarter-wave plate calculation
n_o_calcite = 1.658
n_e_calcite = 1.486
wavelength = 633e-9  # meters

delta_n = abs(n_e_calcite - n_o_calcite)
thickness_qwp = wavelength / (4 * delta_n)

print(f"Birefringence (Δn): {delta_n:.4f}")
print(f"Quarter-wave plate thickness: {thickness_qwp*1e6:.2f}  m")
```

```
Birefringence (Δn): 0.1720
Quarter-wave plate thickness: 0.92  m
```

6.2 Example 2: Beam Deviation

Calculate the angular separation between ordinary and extraordinary rays in calcite for light incident at 45°:

```python
import numpy as np

# Beam deviation calculation
def beam_deviation_angle(n_o, n_e, incident_angle_deg):
    theta_i = np.radians(incident_angle_deg)

    # Ordinary ray (follows Snell's law)
    theta_o = np.arcsin(np.sin(theta_i) / n_o)

    # Extraordinary ray (approximate for small angles)
    n_e_eff = 1/np.sqrt(np.sin(theta_i)**2/n_e**2 + np.cos(theta_i)**2/n_o**2)
    theta_e = np.arcsin(np.sin(theta_i) / n_e_eff)

    deviation = np.degrees(abs(theta_o - theta_e))
    return deviation, np.degrees(theta_o), np.degrees(theta_e)

deviation, theta_o_deg, theta_e_deg = beam_deviation_angle(n_o_calcite, n_e_calcite, 45)
print(f"Ordinary ray refraction angle: {theta_o_deg:.2f}°")
print(f"Extraordinary ray refraction angle: {theta_e_deg:.2f}°")
print(f"Angular separation: {deviation:.2f}°")
```

```
Ordinary ray refraction angle: 25.24°
Extraordinary ray refraction angle: 26.86°
Angular separation: 1.62°
```

7 Summary

This lecture covered the fundamental principles of light propagation in anisotropic materials:

1. **Anisotropic materials** exhibit direction-dependent optical properties described by the permittivity tensor
2. **Birefringence** causes light to split into ordinary and extraordinary rays with different refractive indices
3. **Wave plates** and other optical devices exploit these properties for polarization control
4. **Applications** include polarization optics, beam splitters, and optical isolators

7.1 Key Equations

- Permittivity relation: $\mathbf{D} = \varepsilon_0 \overleftrightarrow{\varepsilon}_r \mathbf{E}$
- Index ellipsoid: $\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$

- Extraordinary index: $\frac{1}{n_e^2(\theta)} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}$
- Quarter-wave plate thickness: $d = \frac{\lambda}{4|n_e - n_o|}$

7.2 Further Reading

- Saleh, B.E.A. and Teich, M.C., "Fundamentals of Photonics," Chapters 5 and 6
- Born, M. and Wolf, E., "Principles of Optics," Chapter 15
- Yariv, A. and Yeh, P., "Optical Waves in Crystals"