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Procedia Engineering 114 (2015) 330 – 337

Procedia Engineering

www.elsevier.com/locate/procedia

1st International Conference on Structural Integrity

Similitude and Scaling Laws - Static and dynamic behaviour beams and plates

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Abstract

This investigation concerns the determination of the scaling laws based on the similitude theory for beams and plates cases when loaded statically and with low velocity impact, respectively. Understanding the relationship between model and prototype behaviour is essential in scaled down models designs. The similarity conditions are established directly from the system field equations. Theoretical predictions of the model were projected with each scaling law and compared with the prototype theoretical predictions. Both finite element analysis and experimental tests were compiled for studied cases. The results obtained, using the similitude theory, were in good agreement with the experimental and numerical results. The scaling laws, which produced good predictions, were identified. The prediction depended on the scaling variables range used to stay within acceptable accuracy.

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Keywords: similitude; scaling; low velocity impact

1. Introduction

In order to develop similitude laws for simple plates, it is important to test whether the numerical, experimental and theoretical results are consistent. In this experiment, two different plates, with different thicknesses, were used. In addition to the plate thickness, all the other variables such as the drop height (which concerns the energy at which the plate is impacted with the indenter), the total mass of both the drop test and the indenter and also the type of material used (in our case Aluminium plates) were used.

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Peer-review under responsibility of INEGI - Institute of Science and Innovation in Mechanical and Industrial Engineering
doi:10.1016/j.proeng.2015.08.076

Nomenclature

d	Striker diameter
G	Striker mass
H	Plate thickness
M_0	Fully plastic bending moment per unit length
Q_0	Fully plastic transverse shear force per unit length
R	Outer radius of plate
V_i	Impact velocity
w	Transverse displacement
μ	Mass per unit area

1.1. Similarity

Similarity of systems requires identical systems parameters. A system is governed by a unique set of characteristic equations. Therefore, if a relation or an equation of variables is written for a system, it is valid for all systems similar to it. A scale factor was introduced as the ratio of each variable in a model to corresponding variable in prototype. Understanding the relationship between the model and prototype behaviour is essential in designing scaled down models. These scaled down models are not widely used due to the associated uncertainty with scaling operations. Before scaled down models can be used, the following technical barriers must be overcome:

- Determination of proper scale factors of structural geometry parameters
- Establishment of the necessary conditions, which relate the response behaviour of the scaled down model to the original system
- Understanding the effect of scaling on the accuracy of the models in predicting the prototype behaviour

Similarity conditions can be established either directly from the field equations of the system or through dimensional analysis. For dimensional analysis all the variables and parameters affecting the system behaviour must be known. Using dimensional analysis produces an incomplete form of the system characteristic equation. Similarity conditions can be established on the basis of this characteristic equation. The differential equations describing the dynamic system can be used to develop model scaling-laws. Solving these differential equations is not required to develop the appropriate scaling laws.

1.2. Similarity Conditions

The necessary and sufficient conditions of similitude between the prototype and its scaled model requires that the mathematical model of the scaled model could be transformed to that of the prototype by Bi-unique mapping or vice versa [1].

In many cases it is impossible to satisfy a complete similarity between model and prototype due to its size, shape, material properties, boundary and environmental conditions. When at least one scaling law cannot be satisfied, distorted models with partial similarity are obtained. It is extremely useful to find a scaling law, which can predict the behaviour of a distorted prototype with desired accuracy.

1.3. Literature review

Rayleigh [2] first discussed a small-scale modelling, based on dimensional analysis. The principal has been reviewed and modified by Riabounchinsky [3], Bridgman [4], and Langhaar [5], (more details are presented in Macango [6]). Kline [7] gives prospective of the method based on both dimensional analysis and the direct use of governing equations.

Experimental investigations of the scaling laws for metal plates struck by large masses were introduced by Wen et. al [8, 9]. Jones et. al. [10] and Jones [11] developed a solution for ductile circular plates struck by a mass. Tabiei et. al. [12] implemented the similitude analysis to predict scaling laws of cylindrical shells under lateral pressure. Tabiei and Balawi [13] developed scaling laws for dynamically loaded plates.

2. Theoretical and Finite Element

2.1. Theoretical similarity conditions

For the first case we study the case of a cantilever beam. Following the Euler-Bernoulli beam theory, the governing equation is as follows:

$$\frac{d^2w}{dx^2} = -\frac{1}{EI}M(x) \tag{1}$$

For the cantilever beam show in Fig. 1, the moment function is as follows

Fig. 1. Cantilever beam boundaries and force application.

$$M(x) = -Px \tag{2}$$

where I is the cross sectional moment of inertia for the rectangular section of height H and width B. The beam equation; with typical boundary conditions of no rotation and displacement at the root of the beam, is:

$$\frac{d^2w}{dx^2} = \frac{1}{EI}Px \tag{3}$$

where $I = \frac{1}{12}BH^3$ (4)

Similitude between the prototype (p) and model (m) can be achieved as shown in the following:

$$\frac{d^2w_p}{dx_p^{\ 2}} = \frac{1}{E_pI_p}P_px_p \tag{5}$$

$$\frac{d^2w_m}{dx_m^{\ 2}} = \frac{1}{E_mI_m}P_mx_m \tag{6}$$

$$\lambda_w = \frac{w_p}{w_m},\ \lambda_x = \frac{x_p}{x_m},\ \lambda_E = \frac{E_p}{E_m},\ \lambda_I = \frac{I_p}{I_m},\ \lambda_P = \frac{P_p}{P_m} \tag{7}$$

$$\frac{\lambda_w}{\lambda_x^2}\frac{d^2w}{dx^2} = \frac{\lambda_P\lambda_x}{\lambda_E\lambda_I}\frac{1}{EI}Px \tag{8}$$

$$\lambda_I I = \frac{1}{12}\lambda_B\lambda_H^3 BH^3 \tag{9}$$

$$\frac{\lambda_w}{\lambda_x^2} = \frac{\lambda_P\lambda_x}{\lambda_E\lambda_I} \tag{10}$$

$$\lambda_I = \lambda_B\lambda_H^3 \tag{11}$$

Thus, combining equation (10) and (11) produces the similarity conditions needed for the beam.

$$\lambda_w = \frac{\lambda_P\lambda_x^3}{\lambda_E\lambda_B\lambda_H^3} \tag{12}$$

The same result could be obtained from the beam equation solution with the boundary conditions, once this equation is specialized for the free end of the beam, where:

$$w = 4\frac{Px^3}{EBH^3} \tag{13}$$

$$\lambda_w w = 4\frac{\lambda_P\lambda_x^3}{\lambda_E\lambda_B\lambda_H^3}\frac{Px^3}{EBH^3} \tag{14}$$

In cases of similarity where the beam is made of the same material for both prototype and model the value for λ_E will be unity. For the free end deflection, this reduces the similarity conditions to four independent similarity conditions, i.e λ_E, λ_B, λ_H and λ_L, and one dependent condition for the deflection.

2.2. FEA Simulation

Using ANSYS FEA software, a cantilever beam case was studied at various similarity conditions. Comparisons between the exact solution and the FEA prediction for the free tip deflection were made on a range of these similarity ratios. Not all the length ratios were the same, which was expected to produce a distorted shape for the model with respect to the prototype. Fig. 2(a) depicts the change of the prediction between the theoretical and FEA with the change in the length ratio. It shows excellent correlation between the two. A similar behaviour is shown when fixing all of the similarity conditions and changing the load ratio, but in this case the error is between 1.4 and 1.8 %.

Table 1 shows the change of the FEA prediction values based on the change of the beam height, H. The table shows that after a certain ratio the error starts accumulating faster and the higher the ratio, the greater the prediction error. This was due to the fact that the higher the value for the height ratio, with a fixed length ratio, the shorter the beam becomes, where the Euler-Bernoulli beam fails and a more involved Timoshenko beam is needed to describe it.

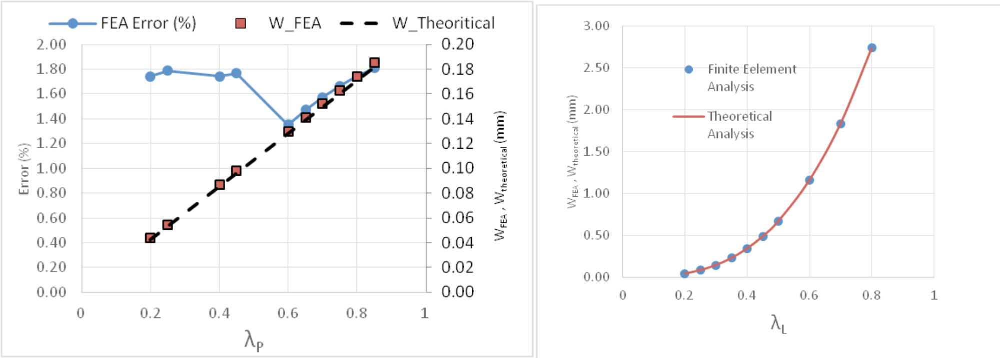


Fig. 2. Comparisons of predicted and theoretical deflections with errors where (a) $\lambda_H = 0.3$, $\lambda_L = 0.2$, $\lambda_B = 0.2$; (b) $\lambda_H = 0.3$, $\lambda_P = 0.2$, $\lambda_B = 0.2$.

Table 1. Comparisons for the cantilever beam based on . λ_P, λ_L and λ_B having a value of 0.2

λ_H	λ_E	λ_w	w_{Th} (m)	w_{FEA} (m)	Error %
0.4	0.66	0.19	1.80E-05	1.86E-05	3.12
0.45	0.66	0.13	1.27E-05	1.32E-05	4.20
0.5	0.66	0.10	9.24E-06	9.68E-06	4.82
0.55	0.66	0.07	6.94E-06	7.34E-06	5.79
0.6	0.66	0.06	5.34E-06	5.72E-06	7.03
0.65	0.66	0.04	4.20E-06	4.55E-06	8.24
0.7	0.66	0.04	3.37E-06	3.68E-06	9.34
0.75	0.66	0.03	2.74E-06	3.03E-06	10.73

3. Circular fixed plate struck by a mass at low velocity

3.1. Similitude analysis

The case of a ductile plate being struck of a mass had already been studied by Jones [10] and [11]. Fig. 3 shows a schematic representation of the problem.

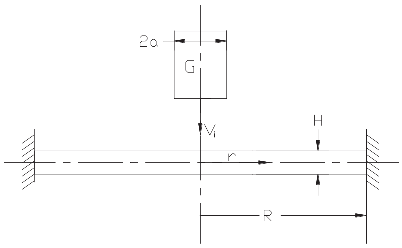


Fig. 3. Rigid cylindrical mass G striking a fully clamped circular plate with an impact velocity v_i .

The governing equations for the system are presented in equations (15) and (16) [10].

$$\left(G+\mu\pi a^2\right)\ddot{W}_o = -2\pi a Q_0 \tag{15}$$

$$\ddot{W}_1 - \frac{12\left(aQ_0\left(R-a\right)-2RM_0\right)}{\mu\left(R+3a\right)\left(R-a\right)^2} = 0 \tag{16}$$

To apply the similitude theory to the governing equations, the following scaling factors were introduced:

$$\lambda_{Q_0} = \frac{(Q_0)_m}{(Q_0)_p}, \lambda_\mu = \frac{\mu_m}{\mu_p}, \lambda_R = \frac{R_m}{R_p}, \lambda_a = \frac{a_m}{a_p}, \lambda_{W_1} = \frac{(W_1)_m}{(W_1)_p},$$
$$\lambda_{M_0} = \frac{(M_0)_m}{(M_0)_p}, \lambda_t = \frac{t_m}{t_p}, \lambda_{V_0} = \frac{(V_0)_m}{(V_0)_p} \tag{17}$$

When applying the similitude theory to the governing equations of the system the following scaling laws are obtained:

$$\lambda_{W_1} = \frac{\lambda_a \lambda_{Q_0} \lambda_t^2}{\lambda_R^2 \lambda_\mu} \tag{18}$$

$$\lambda_{W_1} = \frac{\lambda_{M_0} \lambda_t^2}{\lambda_R^2 \lambda_\mu} \tag{19}$$

Similar to the cantilever beam case, the same similarity conditions may be obtained from the solution provided for the above governing equations. Jones [11] presents the solution for the case in hand as:

$$\frac{w_f}{H} = \sqrt{1 + \frac{\gamma\lambda}{4}} - 1 \tag{20}$$

where the values for γ and λ meet the criteria of large striking mass with respect to the thin plate mass. It can be shown based on the similitude analysis that

$$\lambda_{w_f} = \sqrt{\frac{\lambda_\gamma \lambda_H}{\lambda_{M_0}}} \tag{21}$$

$$\lambda_{w_f} = \sqrt{\frac{\lambda_\gamma}{\lambda_H}} \tag{22}$$

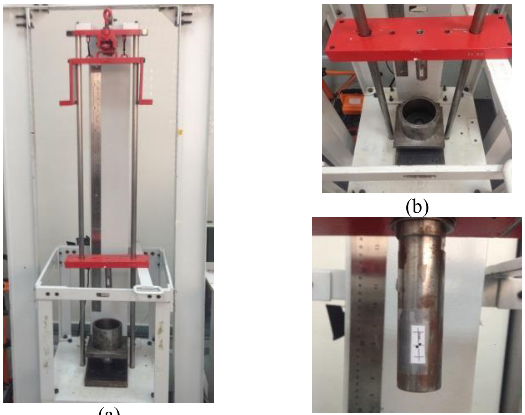
assuming that the model and the prototype have similar yield strength and same impact velocity. Another case considered is the case where the same material is used but changes are made to the thickness and impact velocity. The reduced similarity conditions can be represented as:

$$\lambda_{w_f} = \sqrt{\frac{\lambda_\gamma \lambda_{V_0}^2}{\lambda_H}} \tag{23}$$

Results for these two cases are presented in *3.3*.

3.2. Experimental test setup

The experimental part of the study consisted in using an in-house built, low-velocity drop tower, capable of up to 300J as depicted in Fig. 4(a). The drop tower was fitted with a 120kN Kistler© load cell. The sample was securely fit on the bottom loading cylinder as shown in Fig.4(b). Once the plate was in place, a top ring with eight screws, was used to fix it to the bottom cylinder. A torque wrench was used to tighten these screws to maintain the same boundary conditions on the plate. Multiple tests were done to reach to the optimum loading force in these screws. For direct measurement of the plate deflection a Photron© high-speed camera was used and a marker was attached to the indenter as in Fig. 4(c). Photron FPA software was used to trace the marker and calculate the instant deflections and velocities with time. Table 2 presents some of the loading conditions and other parameters used in the experiments.



(a) (b) (c)

Fig. 4. Low-velocity drop tower(a) overall system; (b) plate loading cylinder; (c) indenter showing displacement marker.

Table 2: Some properties of the experimental parts and conditions

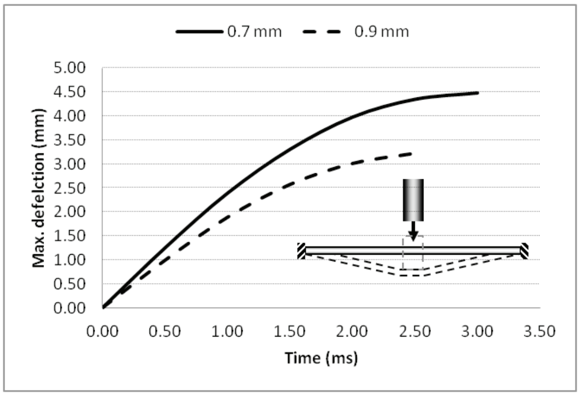
Experimental Part	Mass [*kg*]	Thickness [*mm*]	Impact velocity 1 [*m/s*]	Impact velocity 2 [*m/s*]
Prototype	0.0429	0.90	1.40	1.17
Model	0.0334	0.70	1.40	1.17
Impact mass	6.0290	-		

3.3. Comparisons between theoretical similarity conditions and experimental results

As introduced in Table 2, the two experimentally studied cases had distorted models based on the thickness change only and a second case with changes in both thickness and impact velocity. An example of the maximum deflection under the impact load is presented in Fig. 5. In this figure both model and prototype have the same impact velocity of 1.40 m/s. The second case is for the prototype with an impact velocity of 1.17 m/s and the model having an impact velocity of 1.4 m/s. The percentage error in Table 3 was calculated according to the following expression:

$$Error = \frac{\left(\lambda_{w_f}\right)_{th} - \left(\lambda_{w_f}\right)_{\exp}}{\left(\lambda_{w_f}\right)_{th}} \times 100\% \tag{24}$$

Table 3 shows acceptable errors when comparing the theoretical values of the final deflection according to Jones [10] and [11] when compared with the experimental data provided. This shows a case of where the similitude analysis correctly predicted the prototype behaviour from a distorted model. The ability to even change the velocity gives more flexibility in terms of future impact testing.



(a)

(b)

Fig. 5. (a) Maximum deflection at centre of plate with impact velocity of 1.40 m/s; (b) pictures of tested samples.

Table 3. Comparisons between model and prototype in cases 1 and 2

Experimental	λ_γ	λ_H	λ_{v_0}	Error (%)
Case 1 (thickness change only)	0.7784	0.7778	1.0000	6.33
Case 2 (thickness and velocity change)	0.7784	0.7778	1.1955	2.45

4. Conclusions

This investigation considered the determination of the scaling laws for beams and plates loaded statically and with low-velocity impact, based on similitude theory. The similarity conditions were established directly from the field equations of the system for both described cases. Similarity FEA results were compared with theoretical values for the case of the beam. Experimental tests were compared to theoretical results for the case of the plates impacted by a mass. The results obtained using the similitude theory were in good agreement with the experimental and numerical results and highly depend on the selection of the scaling variables. In case of the plate impact, it was possible to scale the velocity and still achieve results within the acceptable error range Further investigations will be needed to fully describe the ranges at which the distorted models will accurately predict the prototype behaviour.

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