Week 2: Analysis of Algorithms

Analysis of Algorithms

Running Time

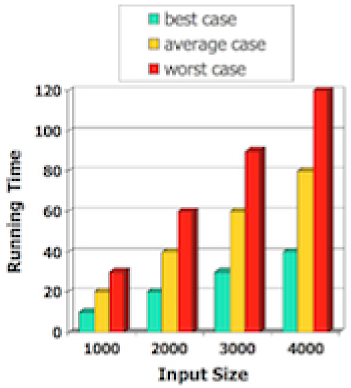
2/87

An algorithm is a step-by-step procedure

- for solving a problem
- in a finite amount of time

Most algorithms map input to output

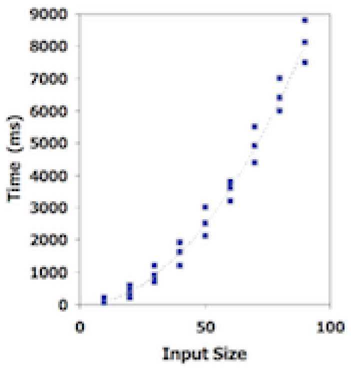
- running time typically grows with input size
- *average time* often difficult to determine
- Focus on *worst case* running time
  - easier to analyse
  - crucial to many applications: finance, robotics, games, …



Empirical Analysis

3/87

1. Write program that implements an algorithm
2. Run program with inputs of varying size and composition
3. Measure the actual running time
4. Plot the results



Limitations:

- requires to implement the algorithm, which may be difficult
- results may not be indicative of running time on other inputs

- same hardware and operating system must be used in order to compare two algorithms

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Theoretical Analysis

4/87

- Uses high-level description of the algorithm instead of implementation ("pseudocode")
- Characterises running time as a function of the input size, *n*
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

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Pseudocode

5/87

worst case:5n-2
best case:4n-1，if中无操作

Example: Find maximal element in an array

```
arrayMax(A):
|  Input  array A of n integers
|  Output maximum element of A
|
|  currentMax=A[0]              1
|  for all i=1..n-1 do  n+(n-1),n表示检查每一次i是否<n-1,结束时i=n，共n次比较操作。n-1表示n-1次赋值操作，i=i+1
|  |  if A[i]>currentMax then   2(n-1), 每次2个操作：获取A[i]的值和一次比较操作，在for循环进行n-1次
|  |     currentMax=A[i]   n-1,1次赋值操作，n-1次循环
|  |  end if
|  end for
|  return currentMax      1
```

... Pseudocode

6/87

Control flow

- **if** … **then** … [**else**] … **end if**
- **while .. do** … **end while**
  repeat … **until**
  for [all][each] .. do … **end for**

Function declaration

- f(arguments):
  Input …
  Output …
  …

Expressions

- = assignment
- = equality testing
- n^2 superscripts and other mathematical formatting allowed
- swap A[i] and A[j] verbal descriptions of *simple* operations allowed

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... Pseudocode

7/87

- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms

- Hides program design issues

Exercise #1: Pseudocode

8/87

Formulate the following verbal description in pseudocode:

To reverse the order of the elements on a stack S with the help of a queue:

1. *In the first phase, pop one element after the other from S and enqueue it in queue Q until the stack is empty.*
2. *In the second phase, iteratively dequeue all the elements from Q and push them onto the stack.*

As a result, all the elements are now in reversed order on S.

Sample solution:

```
while S is not empty do
   pop e from S, enqueue e into Q
end while
while Q is not empty do
   dequeue e from Q, push e onto S
end while
```

Exercise #2: Pseudocode

10/87

Implement the following pseudocode instructions in C

1. `A` is an array of `int`s

   ```
   ...
   swap A[i] and A[j]
   ...
   ```

2. `S` is a stack

   ```
   ...
   swap the top two elements on S
   ...
   ```

1. ```
   int temp = A[i];
   A[i] = A[j];
   A[j] = temp;
   ```

2. ```
   x = StackPop(S);
   y = StackPop(S);
   StackPush(S, x);
   StackPush(S, y);
   ```

The following pseudocode instruction is problematic. Why?

```
...
swap the two elements at the front of queue Q
```

...

The Abstract RAM Model

12/87

RAM = Random Access Machine

- A CPU (central processing unit)
- A potentially unbounded bank of memory cells
  - each of which can hold an arbitrary number, or character
- Memory cells are numbered, and accessing any one of them takes CPU time

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Primitive Operations

13/87

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent of the programming language
- Exact definition not important (we will shortly see why)
- Assumed to take a constant amount of time in the RAM model

Examples:

- evaluating an expression
- indexing into an array
- calling/returning from a function

Counting Primitive Operations

14/87

By inspecting the pseudocode …

- we can determine the maximum number of primitive operations executed by an algorithm
- as a function of the input size

Example:

```
arrayMax(A):
|   Input  array A of n integers
|   Output maximum element of A
|
|   currentMax=A[0]                  1
|   for all i=1..n-1 do              n+(n-1)
|   |  if A[i]>currentMax then       2(n-1)
|   |     currentMax=A[i]            n-1
|   |  end if
|   end for
|   return currentMax                1
                                     -------
                             Total   5n-2
```

Estimating Running Times

15/87

Algorithm `arrayMax` requires $5n - 2$ primitive operations in the *worst* case

- *best* case requires $4n - 1$ operations (why?)

Define:

- *a* … time taken by the fastest primitive operation
- *b* … time taken by the slowest primitive operation

Let *T(n)* be worst-case time of `arrayMax`. Then

$a \cdot (5n - 2) \le T(n) \le b \cdot (5n - 2)$

Hence, the running time *T(n)* is bound by two linear functions

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... Estimating Running Times

16/87

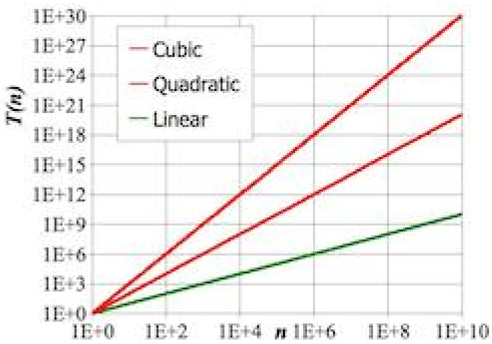
Seven commonly encountered functions for algorithm analysis

- Constant ≅ 1
- Logarithmic ≅ $\log n$ 对数的
- Linear ≅ n
- N-Log-N ≅ $n \log n$
- Quadratic ≅ n^2
- Cubic ≅ n^3
- Exponential ≅ 2^n

... Estimating Running Times

17/87

In a log-log chart, the slope of the line corresponds to the growth rate of the function

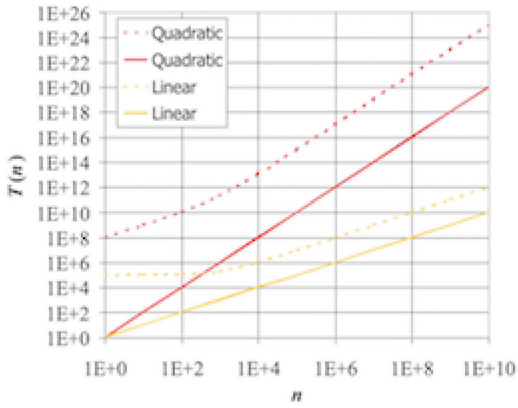


... Estimating Running Times

18/87

The growth rate is not affected by constant factors or lower-order terms

- Examples:
  - $10^2n + 10^5$ is a linear function
  - $10^5n^2 + 10^8n$ is a quadratic function



... Estimating Running Times

19/87

Changing the hardware/software environment

- affects *T(n)* by a constant factor
- but does not alter the growth rate of *T(n)*

固有的、本身的

⇒ *Linear* growth rate of the running time *T(n)* is an intrinsic property of algorithm `arrayMax`

Exercise #3: Estimating running times

20/87

Determine the number of primitive operations

```
matrixProduct(A,B):
|  Input  n×n matrices A, B
|  Output n×n matrix A·B
|
|  for all i=1..n do
|  |  for all j=1..n do
|  |  |  C[i,j]=0
|  |  |  for all k=1..n do
|  |  |     C[i,j]=C[i,j]+A[i,k]·B[k,j]
|  |  |  end for
|  |  end for
|  end for
|  return C
```

```
matrixProduct(A,B):
|  Input  n×n matrices A, B
|  Output n×n matrix A·B
|
|  for all i=1..n do                    2n+1
|  |  for all j=1..n do                 n(2n+1)
|  |  |  C[i,j]=0                       n^2
|  |  |  for all k=1..n do              n^2(2n+1)
|  |  |     C[i,j]=C[i,j]+A[i,k]·B[k,j] n^3·4   三次取值（3个不同的值），一次计算
|  |  |  end for
|  |  end for
|  end for
|  return C                             1
                                        ------------
                              Total     6n^3+4n^2+3n+2
```

Big-Oh

Big-Oh Notation

23/87

Given functions f(n) and g(n), we say that

$$f(n) \in O(g(n))$$

if there are positive constants c and n_0 such that

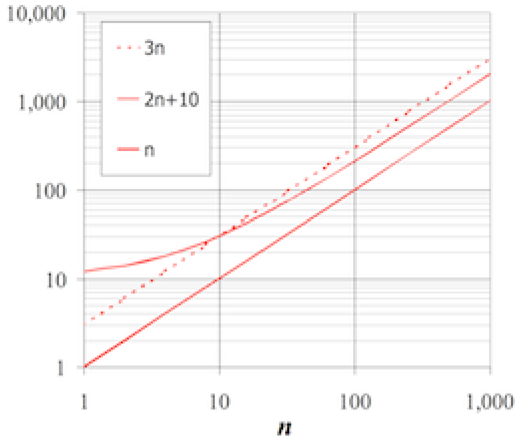
$$f(n) \leq c \cdot g(n) \quad \forall n \geq n_0$$

Hence: *O(g(n)) is the set of all functions that do not grow faster than g(n)*

... Big-Oh Notation

24/87

Example: function $2n + 10$ is in O(n)

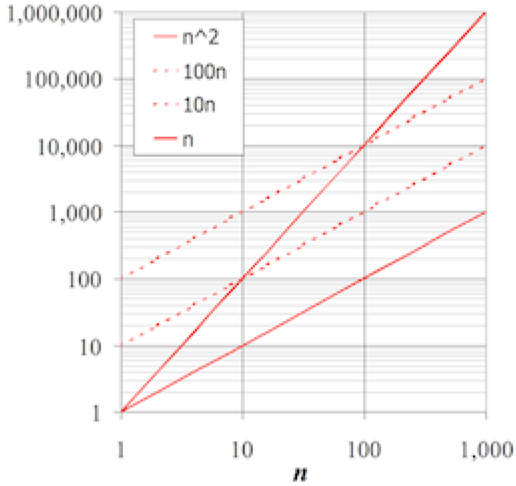


- $2n+10 \leq c \cdot n$
  $\Rightarrow$ $(c-2)n \geq 10$
  $\Rightarrow$ $n \geq 10/(c-2)$
- pick c=3 and n_0=10

... Big-Oh Notation

25/87

Example: function n^2 is not in O(n)



- $n^2 \le c \cdot n$
  $\Rightarrow \quad n \le c$
- inequality cannot be satisfied since c must be a constant

Exercise #4: Big-Oh

26/87

Show that

1. 7n-2 is in O(n)
2. $3n^3 + 20n^2 + 5$ is in $O(n^3)$
3. 3·log n + 5 is in O(log n)

---

1. 7n-2 ∈ O(n)
   need c>0 and $n_0 \ge 1$ such that $7n-2 \le c \cdot n$ for $n \ge n_0$
   ⇒ true for c=7 and $n_0=1$
2. $3n^3 + 20n^2 + 5 \in O(n^3)$
   need c>0 and $n_0 \ge 1$ such that $3n^3+20n^2+5 \le c \cdot n^3$ for $n \ge n_0$
   ⇒ true for c=4 and $n_0=21$
3. 3·log n + 5 ∈ O(log n)
   need c>0 and $n_0 \ge 1$ such that $3 \cdot \log n+5 \le c \cdot \log n$ for $n \ge n_0$
   ⇒ true for c=8 and $n_0=2$

Big-Oh and Rate of Growth

28/87

- Big-Oh notation gives an upper bound on the growth rate of a function
  - "f(n) ∈ O(g(n))" means growth rate of f(n) no more than growth rate of g(n)
- use big-Oh to rank functions according to their rate of growth

	f(n) ∈ O(g(n))	g(n) ∈ O(f(n))
g(n) grows faster	yes	no
f(n) grows faster	no	yes
same order of growth	yes	yes

Big-Oh Rules

29/87

- If f(n) is a polynomial of degree d ⇒ f(n) is $O(n^d)$
  - lower-order terms are ignored
  - constant factors are ignored
- Use the smallest possible class of functions
  - say "2n is O(n)" instead of "2n is $O(n^2)$"
    - but keep in mind that, 2n is in $O(n^2)$, $O(n^3)$, …
- Use the simplest expression of the class
  - say "3n + 5 is O(n)" instead of "3n + 5 is O(3n)"

Exercise #5: Big-Oh

30/87

Show that $\sum_{i=1}^{n} i$ is $O(n^2)$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

which is $O(n^2)$

Asymptotic Analysis of Algorithms

32/87

Asymptotic analysis of algorithms determines running time in big-Oh notation:

- find worst-case number of primitive operations as a function of input size
- express this function using big-Oh notation

Example:

- algorithm `arrayMax` executes at most 5n – 2 primitive operations
  ⇒ algorithm `arrayMax` "*runs in O(n) time*"

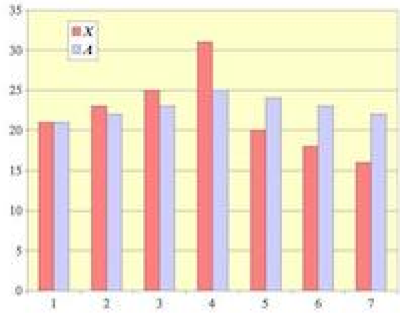
Constant factors and lower-order terms eventually dropped
⇒ can disregard them when counting primitive operations

Example: Computing Prefix Averages 前缀平均值

33/87

- The *i-th prefix average* of an array X is the average of the first i elements:

A[i] = (X[0] + X[1] + … + X[i]) / (i+1)



NB. computing the array A of prefix averages of another array X has applications in financial analysis

... Example: Computing Prefix Averages

34/87

A *quadratic* algorithm to compute prefix averages:

```
prefixAverages1(X):
|  Input   array X of n integers
|  Output  array A of prefix averages of X
|
|  for all i=0..n-1 do          O(n)
|  |  s=X[0]                    O(n)
|  |  for all j=1..i do         O(n^2)
|  |     s=s+X[j]               O(n^2)
|  |  end for
|  |  A[i]=s/(i+1)              O(n)
|  end for
|  return A                     O(1)
```

$$2 \cdot O(n^2) + 3 \cdot O(n) + O(1) = O(n^2)$$

⇒ *Time complexity* of algorithm `prefixAverages1` is $O(n^2)$

... Example: Computing Prefix Averages

35/87

The following algorithm computes prefix averages by keeping a running sum:

```
prefixAverages2(X):
|  Input   array X of n integers
|  Output  array A of prefix averages of X
|
|  s=0
|  for all i=0..n-1 do          O(n)
|     s=s+X[i]                  O(n)
|     A[i]=s/(i+1)              O(n)
|  end for
|  return A                     O(1)
```

Thus, `prefixAverages2` is O(n)

Example: Binary Search

36/87

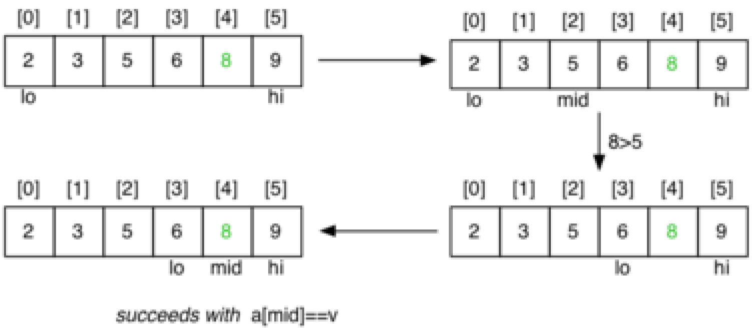
The following recursive algorithm searches for a value in a *sorted* array:

```
search(v,a,lo,hi):
|  Input  value v
|         array a[lo..hi] of values
|  Output true if v in a[lo..hi]
|         false otherwise
|
|  mid=(lo+hi)/2
|  if lo>hi then return false
|  if a[mid]=v then
|     return true
|  else if a[mid]<v then
|     return search(v,a,mid+1,hi)
|  else
|     return search(v,a,lo,mid-1)
|  end if
```

... Example: Binary Search

37/87

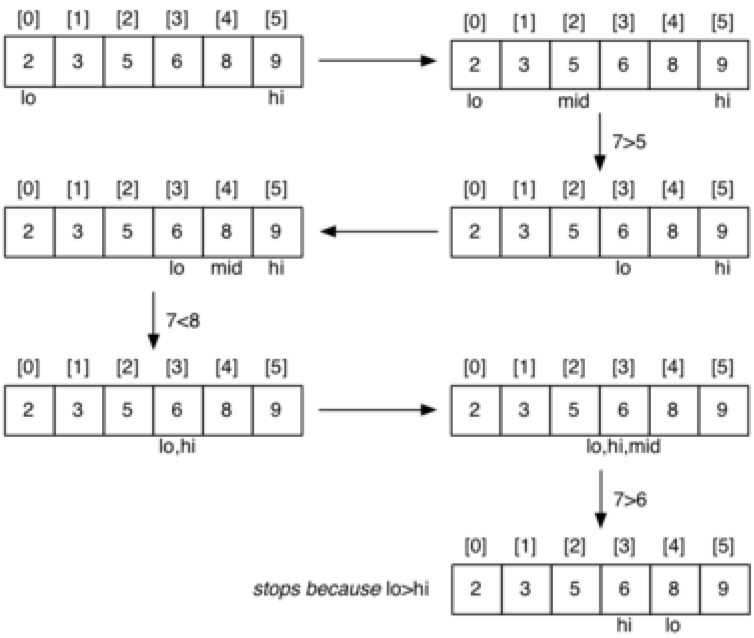
Successful search for a value of 8:



... Example: Binary Search

38/87

Unsuccessful search for a value of 7:



39/87

... Example: Binary Search

Cost analysis:

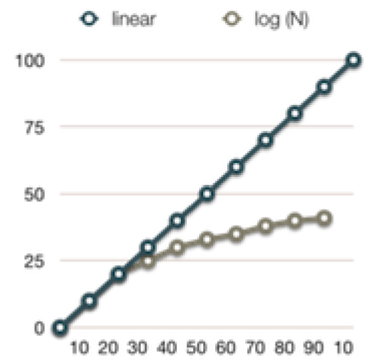
- C_i = #calls to `search()` for array of length i
- for best case, $C_n = 1$
- for `a[i..j]`, `j<i` (length=0)
  - $C_0 = 0$
- for `a[i..j]`, `i≤j` (length=n)
  - $C_n = 1 + C_{n/2} \;\; \Rightarrow C_n = \log_2 n$

Thus, binary search is $O(\log_2 n)$ or simply $O(\log n)$ (why?)

... Example: Binary Search

40/87

Why logarithmic complexity is good:



Math Needed for Complexity Analysis

41/87

- Logarithms
  - $\log_b (xy) = \log_b x + \log_b y$
  - $\log_b (x/y) = \log_b x - \log_b y$
  - $\log_b x^a = a \log_b x$
  - $\log_a x = \log_b x \cdot (\log_c a / \log_c b)$
- Exponentials
  - $a^{(b+c)} = a^b a^c$
  - $a^{bc} = (a^b)^c$
  - $a^b / a^c = a^{(b-c)}$
  - $b = a^{\log_a b}$
  - $b^c = a^{c \cdot \log_a b}$
- Proof techniques
- Summation (addition of sequences of numbers)
- Basic probability (for average case analysis, randomised algorithms)

Exercise #6: Analysis of Algorithms

42/87

What is the complexity of the following algorithm?

```
enqueue(Q,Elem):
|  Input  queue Q, element Elem
```

```
|   Output Q with Elem added at the end
|
|   Q.top=Q.top+1
|   for all i=Q.top down to 1 do
|      Q[i]=Q[i-1]
|   end for
|   Q[0]=Elem
|   return Q
```

Answer: O(|Q|)

Exercise #7: Analysis of Algorithms

44/87

What is the complexity of the following algorithm?

```
binaryConversion(n):
|   Input  positive integer n
|   Output binary representation of n on a stack
|
|   create empty stack S
|   while n>0 do
|   |   push (n mod 2) onto S
|   |   n=⌊n/2⌋
|   end while
|   return S
```

Assume that creating a stack and pushing an element both are O(1) operations ("constant")

Answer: O(log n)

Relatives of Big-Oh

46/87

big-Omega

- $f(n) \in \Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that

$$f(n) \geq c \cdot g(n) \quad \forall n \geq n_0$$

big-Theta

- $f(n) \in \Theta(g(n))$ if there are constants $c',c'' > 0$ and an integer constant $n_0 \geq 1$ such that

$$c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n) \quad \forall n \geq n_0$$

... Relatives of Big-Oh

47/87

- f(n) belongs to O(g(n)) if f(n) is asymptotically *less than or equal* to g(n)
- f(n) belongs to Ω(g(n)) if f(n) is asymptotically *greater than or equal* to g(n)
- f(n) belongs to Θ(g(n)) if f(n) is asymptotically *equal* to g(n)

... Relatives of Big-Oh

Examples:

- *$\frac{1}{4}n^2 \in \Omega(n^2)$*
  - need c > 0 and $n_0 \geq 1$ such that $\frac{1}{4}n^2 \geq c \cdot n^2$ for $n \geq n_0$
  - let c=¼ and n_0=1
- *$\frac{1}{4}n^2 \in \Omega(n)$*
  - need c > 0 and $n_0 \geq 1$ such that $\frac{1}{4}n^2 \geq c \cdot n$ for $n \geq n_0$
  - let c=1 and n_0=2
- *$\frac{1}{4}n^2 \in \Theta(n^2)$*
  - since $\frac{1}{4}n^2$ belongs to $\Omega(n^2)$ and $O(n^2)$

Complexity Analysis: Arrays vs. Linked Lists

Static/Dynamic Sequences

Previously we have used an *array* to implement a stack

- fixed size collection of homogeneous elements
- can be accessed via index or via "moving" pointer

The "fixed size" aspect is a potential problem:

- how big to make the (dynamic) array? (big … just in case)
- what to do if it fills up?

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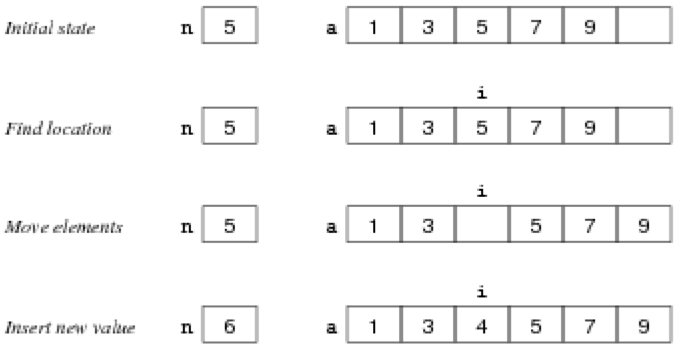
The rigid sequence is another problems:

- inserting/deleting an item in middle of array

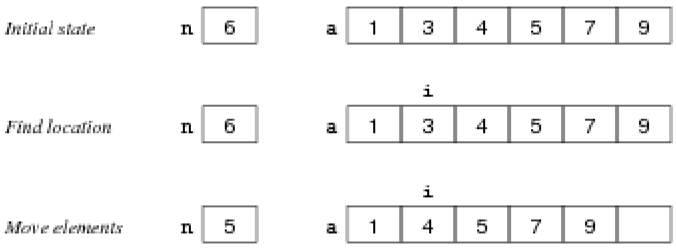
... Static/Dynamic Sequences

Inserting a value (*4*) into a sorted array `a` with `n` elements:



... Static/Dynamic Sequences

Deleting a value (*3*) from a sorted array `a` with `n` elements:

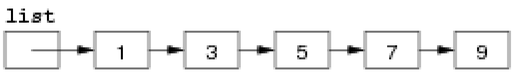


... Static/Dynamic Sequences

53/87

The problems with using arrays can be solved by

- allocating elements individually
- linking them together as a "chain"



Benefits:

- insertion/deletion have minimal effect on list overall
- only use as much space as needed for values

Self-referential Structures

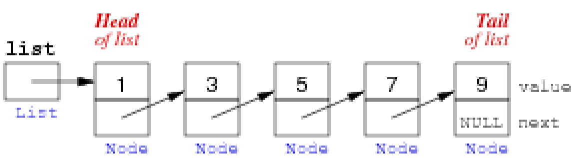
54/87

To realise a "chain of elements", need a *node* containing

- a value
- a link to the next node

To represent a chained (linked) *list* of nodes:

- we need a *pointer* to the first node
- each node contains a pointer to the next node
- the `next` pointer in the last node is `NULL`



... Self-referential Structures

55/87

Linked lists are more flexible than arrays:

- values do not have to be adjacent in memory
- values can be rearranged simply by altering pointers
- the number of values can change dynamically
- values can be added or removed in any order

Disadvantages:

- it is not difficult to get pointer manipulations wrong
- each value also requires storage for `next` pointer 指针变量在16位系统中占4字节，在32位中占6字节，64中占8字节

... Self-referential Structures

56/87

Create a new list node:

```
makeNode(v)
|  Input  value v
|  Output new linked list node with value v
|
|  new.value=v       // initialise data
|  new.next=NULL     // initialise link to next node
|  return new        // return pointer to new node
```

Exercise #8: Creating a Linked List

57/87

Write pseudocode to create a linked list of three nodes with values 1, 42 and 9024.

```
mylist=makeNode(1)
mylist.next=makeNode(42)
(mylist.next).next=makeNode(9024)
```

Iteration over Linked Lists

59/87

When manipulating list elements

- typically have pointer `p` to current node
- to access the data in current node: `p.value`
- to get pointer to next node: `p.next`

To iterate over a linked list:

- set `p` to point at first node (head)
- examine node pointed to by `p`
- change `p` to point to next node
- stop when `p` reaches end of list (`NULL`)

... Iteration over Linked Lists

60/87

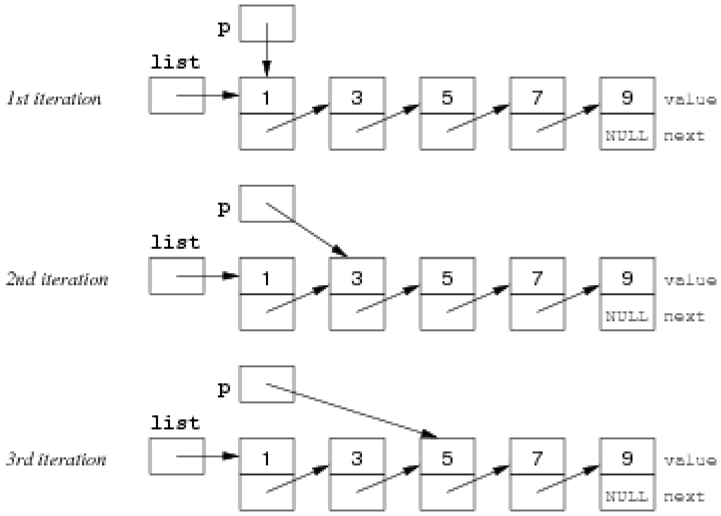
Standard method for scanning all elements in a linked list:

```
list  // pointer to first Node in list
p     // pointer to "current" Node in list

p=list
while p≠NULL do
|  … do something with p.value …
|  p=p.next
end while
```

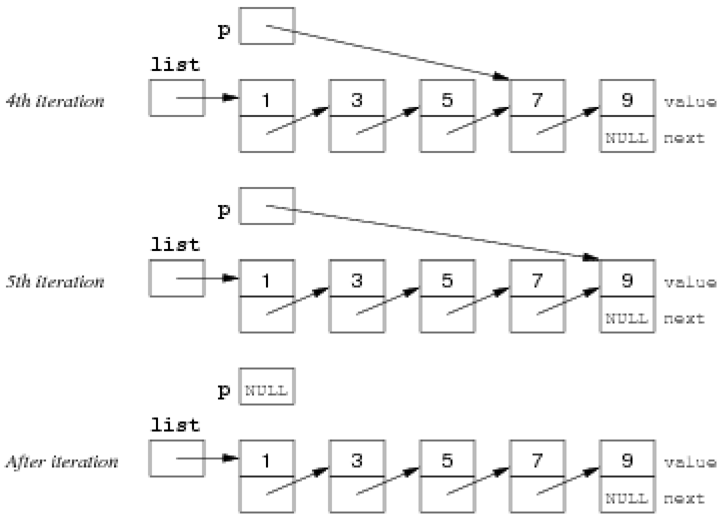
... Iteration over Linked Lists

61/87



... Iteration over Linked Lists

62/87



... Iteration over Linked Lists

63/87

Check if list contains an element:

```
inLL(L,d):
|  Input  linked list L, value d
|  Output true if d in list, false otherwise
|
|  p=L
|  while p≠NULL do
|  |  if p.value=d then    // element found
|  |     return true
|  |  end if
|  |  p=p.next
|  end while
|  return false            // element not in list
```

Time complexity: O(|L|)

... Iteration over Linked Lists

64/87

Print all elements:

```
showLL(L):
|  Input linked list L
|
|  p=L
|  while p≠NULL do
|     print p.value
|     p=p.next
|  end while
```

Time complexity: O(|L|)

Exercise #9: Traversing a linked list

65/87

What does this code do?

```
1  p=list
2  while p≠NULL do
3  |  print p.value
4  |  if p.next≠NULL then
5  |     p=p.next.next
6  |  else
7  |     p=NULL
8  |  end if
9  end while
```

What is the purpose of the conditional statement in line 4?

Every second list element is printed.

If `p` happens to be the last element in the list, then `p.next.next` does not exist.
The if-statement ensures that we do not attempt to assign an undefined value to pointer `p` in line 5.

Exercise #10: Traversing a linked list

67/87

Rewrite **`showLL()`** as a recursive function.

```
showLL(L):
|  Input linked list L
|
|  if L≠NULL do
|     print p.value
|     showLL(L.next)
|  end if
```

Modifying a Linked List

69/87

Insert a new element at the beginning:

```
insertLL(L,d):
|  Input  linked list L, value d
|  Output L with d prepended to the list
|
```

```
|   new=makeNode(d)  // create new list element
|   new.next=L       // link to beginning of list
|   return new       // new element is new head
```

Time complexity: O(1)

... Modifying a Linked List

Delete the *first* element:

```
deleteHead(L):
|   Input  non-empty linked list L, value d
|   Output L with head deleted
|
|   return L.next    // move to second element
```

Time complexity: O(1)

Delete a *specific* element (recursive version):

```
deleteLL(L,d):
|   Input  linked list L
|   Output L with element d deleted
|
|   if L=NULL then              // element not in list
|      return L
|   else if L.value=d then      // d found at front
|      return deleteHead(L)     // delete first element
|   else                        // delete element in tail list
|      L.next=deleteLL(L.next,d)
|   end if
|   return L
```

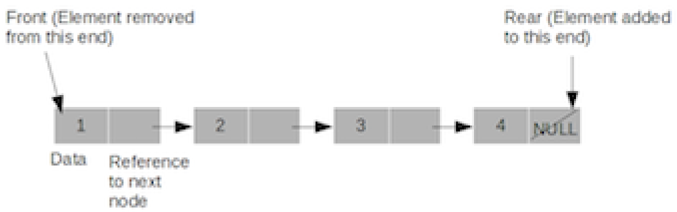
Time complexity: O(|L|)

Exercise #11: Implementing a Queue as a Linked List

Develop a datastructure for a queue based on linked lists such that …

- enqueuing an element takes constant time
- dequeuing an element takes constant time

Use pointers to both ends



Dequeue from the front …

```
dequeue(Q):
|   Input  non-empty queue Q
|   Output front element d, dequeued from Q
|
|   d=Q.front.value          // first element in the list
|   Q.front=Q.front.next     // move to second element
|   return d
```

Enqueue at the rear …

```
enqueue(Q,d):
|   Input queue Q
|
|   new=makeNode(d)       // create new list element
|   Q.rear.next=new       // add to end of list
|   Q.rear=new            // link to new end of list
```

Comparison Array vs. Linked List

Complexity of operations, *n* elements

	array	linked list
insert/delete at beginning	$O(n)$	$O(1)$
insert/delete at end	$O(1)$	$O(1)$ ("doubly-linked" list, with pointer to rear)
insert/delete at middle	$O(n)$	$O(n)$
find an element	$O(n)$ ($O(\log n)$, if array is sorted)	$O(n)$
index a specific element	$O(1)$	$O(n)$

Complexity Classes

Complexity Classes

Problems in Computer Science …

- some have *polynomial* worst-case performance (e.g. n^2)
- some have *exponential* worst-case performance (e.g. 2^n)

Classes of problems:

- P = problems for which an algorithm can compute answer in polynomial time
- NP = includes problems for which no P algorithm is known

Beware: NP stands for "nondeterministic, polynomial time (on a theoretical *Turing Machine*)"

... Complexity Classes

Computer Science jargon for difficulty:

- tractable … have a polynomial-time algorithm (useful in practice)
- intractable … no tractable algorithm is known (feasible only for small n)
- non-computable … no algorithm can exist

Computational complexity theory deals with different degrees of intractability

---

Generate and Test

77/87

In scenarios where

- it is simple to test whether a given state is a solution
- it is easy to generate new states (preferably likely solutions)

then a *generate and test* strategy can be used.

It is necessary that states are generated systematically

- so that we are guaranteed to find a solution, or know that none exists
  - some randomised algorithms do not require this, however
    (more on this later in this course)

... Generate and Test

78/87

Simple example: checking whether an integer n is prime

- generate/test all possible factors of n
- if none of them pass the test $\Rightarrow$ n is prime

Generation is straightforward:

- produce a sequence of all numbers from 2 to $n-1$

Testing is also straightforward:

- check whether next number divides n exactly

... Generate and Test

79/87

Function for primality checking:

```
isPrime(n):
|  Input  natural number n
|  Output true if n prime, false otherwise
|
|  for all i=2..n-1 do       // generate
|  |  if n mod i = 0 then    // test
|  |     return false        // i is a divisor => n is not prime
|  |  end if
|  end for
|  return true               // no divisor => n is prime
```

Complexity of `isPrime` is O(n)

Can be optimised: check only numbers between 2 and $\lfloor\sqrt{n}\rfloor$ $\Rightarrow$ $O(\sqrt{n})$

Example: Subset Sum

80/87

Problem to solve ...

Is there a subset S of these numbers with $\Sigma_{x\in S}\, x = 1000$?

```
 34,  38,  39,  43,  55,  66,  67,  84,  85,  91,
101, 117, 128, 138, 165, 168, 169, 182, 184, 186,
234, 238, 241, 276, 279, 288, 386, 387, 388, 389
```

General problem:

- given n arbitrary integers and a target sum k
- is there a subset that adds up to exactly k?

... Example: Subset Sum

81/87

Generate and test approach:

```
subsetsum(A,k):
|  Input  set A of n integers, target sum k
|  Output true if Σx∈Sx=k for some S⊆A
|         false otherwise
|
|  for each subset B⊆A do
|  |  if Σb∈Bb=k then
|  |     return true
|  |  end if
|  end for
|  return false
```

- How many subsets are there of n elements?
- How could we generate them?

... Example: Subset Sum

82/87

Given: a set of `n` distinct integers in an array `A` ...

- produce all subsets of these integers

A method to generate subsets:

- represent sets as n bits (e.g. $n=4$, `0000`, `0011`, `1111` etc.)
- bit i represents the i^{th} input number
- if bit i is set to 1, then `A[i]` is in the subset
- if bit i is set to 0, then `A[i]` is not in the subset
- e.g. if `A[]=={1,2,3,5}` then `0011` represents `{1,2}`

... Example: Subset Sum

83/87

Algorithm:

```
subsetsum1(A,k):
|  Input  set A of n integers, target sum k
```

```
|  Output true if Σx∈Sx=k for some S⊆A
|         false otherwise
|
|  for s=0..2^n-1 do
|  |  if k = Σ(ith bit of s is 1) A[i] then
|  |     return true
|  |  end if
|  end for
|  return false
```

Obviously, `subsetsum1` is $O(2^n)$

... Example: Subset Sum

84/87

Alternative approach …

`subsetsum2(A,n,k)`
(returns true if any subset of A[0..n-1] sums to k; returns false otherwise)

- if the n^{th} value A[n-1] is part of a solution …
  - then the first n-1 values must sum to k – A[n-1]
- if the n^{th} value is not part of a solution …
  - then the first n-1 values must sum to k
- base cases: k=0 (solved by {}); n=0 (unsolvable if k>0)

```
subsetsum2(A,n,k):
|  Input  array A, index n, target sum k
|  Output true if some subset of A[0..n-1] sums up to k
|         false otherwise
|
|  if k=0 then
|     return true   // empty set solves this
|  else if n=0 then
|     return false  // no elements => no sums
|  else
|     return subsetsum(A,n-1,k-A[n-1]) or subsetsum(A,n-1,k)
|  end if
```

... Example: Subset Sum

85/87

Cost analysis:

- C_i = #calls to `subsetsum2()` for array of length i
- for worst case,
  - $C_1 = 2$
  - $C_n = 2 + 2 \cdot C_{n-1} \quad \Rightarrow C_n \cong 2^n$

Thus, `subsetsum2` also is $O(2^n)$

... Example: Subset Sum

86/87

Subset Sum is typical member of the class of *NP-complete problems*

- intractable … only algorithms with exponential performance are known
  - increase input size by 1, double the execution time
  - increase input size by 100, it takes 2^{100} = 1,267,650,600,228,229,401,496,703,205,376 times as long to execute
- but if you can find a polynomial algorithm for Subset Sum, then any other *NP*-complete problem becomes *P* …

Summary

87/87

- Big-Oh notation
- Asymptotic analysis of algorithms
- Examples of algorithms with logarithmic, linear, polynomial, exponential complexity
- Linked lists vs. arrays

- Suggested reading:
  - Sedgewick, Ch. 2.1-2.4, 2.6

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