COMP9414: Artificial Intelligence

Lecture 4b: Automated Reasoning

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This Lecture

- Proof systems
  - Soundness, completeness, decidability
- Resolution and Refutation 决议和反驳
- Horn clauses and SLD resolution
- Prolog 霍恩子句（Horn Clause）是带有最多一个肯定文字的子句(文字的析取)
- Tableau method

Summary So Far

- Propositional Logic
  - Syntax: Formal language built from $\wedge$, $\vee$, $\neg$, $\rightarrow$
  - Semantics: Definition of truth table for every formula
  - $S \models P$ if whenever all formulae in S are True, P is True
- Proof System
  - System of axioms and rules for deduction
  - Enables computation of proofs of P from S
- **Basic Questions**
  - Are the proofs that are computed always correct? (soundness)
  - If $S \models P$, is there always a proof of P from S (completeness)

Mechanizing Proof

- A proof of a formula P from a set of premises S is a sequence of lines in which any line in the proof is
  1. An axiom of logic or premise from S, or
  2. A formula deduced from previous lines of the proof using a rule of inference

  and the last line of the proof is the formula P
- Formally captures the notion of mathematical proof
- S proves P ($S \vdash P$) if there is a proof of P from S; alternatively, P follows from S
- Example: Natural Deduction proof

Soundness and Completeness

- A proof system is sound if (intuitively) it preserves truth
  - 1、有效性定理是指逻缉系统的证明规则绝不会允许从真前提推导出假结论
  - Whenever $S \vdash P$, if every formula in S is True, P is also True
  - Whenever $S \vdash P$, $S \models P$ 2、可靠性定理是指逻辑系统的证明规则绝不会允许从真前提推导出假结论，而且前提已被证明为真
  - If you start with true assumptions, any conclusions must be true
- A proof system is complete if it is capable of proving all consequences of any set of premises (including infinite sets)
  - Whenever P is entailed by S, there is a proof of P from S
  - Whenever $S \models P$, $S \vdash P$
- 可判定的
  A proof system is decidable if there is a mechanical procedure (computer program) which when asked whether $S \vdash P$, can always answer 'yes' – **or 'no'** – correctly

Resolution

- Another type of proof system based on refutation
- Better suited to computer implementation than systems of axioms and rules (can give correct 'no' answers)
- Decidable in the case of Propositional Logic
- Generalizes to First-Order Logic (see later in term)
- Needs all formulae to be converted to clausal form

小句形式

Normal Forms

- 文字
  A literal ℓ is a propositional variable or the negation of a propositional variable (P or $\neg P$)
- 分句，从句
  A clause is a disjunction of literals $\ell_1 \vee \ell_2 \vee \cdots \vee \ell_n$
- 合取范式：并的交
  Conjunctive Normal Form (CNF) — a conjunction of clauses, e.g. $(P \vee Q \vee \neg R) \wedge (\neg S \vee \neg R)$ – or just one clause, e.g. $P \vee Q$
- Disjunctive Normal Form (DNF) — a disjunction of conjunctions of literals, e.g. $(P \wedge Q \wedge \neg R) \vee (\neg S \wedge \neg R)$ – or just one conjunction, e.g. $P \wedge Q$ 析取范式：交的并
- Every Propositional Logic formula can be converted to CNF and DNF
- Every Propositional Logic formula is equivalent to its CNF and DNF

Conversion to Conjunctive Normal Form

- Eliminate $\leftrightarrow$ rewriting $P \leftrightarrow Q$ as $(P \rightarrow Q) \wedge (Q \rightarrow P)$
- Eliminate $\rightarrow$ rewriting $P \rightarrow Q$ as $\neg P \vee Q$
- Use De Morgan's laws to push $\neg$ inwards (repeatedly)
  - Rewrite $\neg(P \wedge Q)$ as $\neg P \vee \neg Q$
  - Rewrite $\neg(P \vee Q)$ as $\neg P \wedge \neg Q$
- Eliminate double negations: rewrite $\neg\neg P$ as P
- Use the distributive laws to get CNF [or DNF] – if necessary
  - Rewrite $(P \wedge Q) \vee R$ as $(P \vee R) \wedge (Q \vee R)$ [for CNF]
  - Rewrite $(P \vee Q) \wedge R$ as $(P \wedge R) \vee (Q \wedge R)$ [for DNF]

Example Clausal Form

Clausal Form = set of clauses in the CNF

- $\neg(P \rightarrow (Q \wedge R))$
- $\neg(\neg P \vee (Q \wedge R))$
- $\neg\neg P \wedge \neg(Q \wedge R)$
- $\neg\neg P \wedge (\neg Q \vee \neg R)$
- $P \wedge (\neg Q \vee \neg R)$
- Clausal Form: $\{P, \neg Q \vee \neg R\}$

Resolution Rule of Inference

$$\frac{A_1 \vee \cdots \vee A_m \vee B \qquad \neg B \vee C_1 \vee \cdots \vee C_n}{A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n}$$

命题变量

where B is a propositional variable and A_i and C_j are literals

- B and $\neg B$ are complementary literals互补文字
- 归结式
  $A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$ is the resolvent of the two clauses
- Special case: If no A_i and C_j, resolvent is empty clause, denoted $\Box$

决议

Resolution Rule: Key Idea

- Consider $A_1 \vee \cdots \vee A_m \vee B$ and $\neg B \vee C_1 \vee \cdots \vee C_n$
  - Suppose both are True
  - If B is True, $\neg B$ is False and $C_1 \vee \cdots \vee C_n$ is True
  - If B is False, $A_1 \vee \cdots \vee A_m$ is True
  - Hence $A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$ is True

Hence the resolution rule is sound

- Starting with true premises, any conclusion made using resolution must be true

Applying Resolution: Naive Method

- Convert knowledge base into clausal form
- Repeatedly apply resolution rule to the resulting clauses
- 从句
  P follows from the knowledge base if and only if each clause in the CNF of P can be derived using resolution from the clauses of the knowledge base (or subsumption)
- Example
  - $\{P \rightarrow Q, Q \rightarrow R\} \vdash P \rightarrow R$
  - Clauses $\neg P \vee Q$, $\neg Q \vee R$, show $\neg P \vee R$
  - Follows from one resolution step

Refutation Systems

- To show that P follows from S (i.e. $S \vdash P$) using refutation, start with S and $\neg P$ in clausal form and derive a contradiction using resolution
- A contradiction is the "empty clause" (a clause with no literals)
- The empty clause $\square$ is unsatisfiable (always False)
- So if the empty clause $\square$ is derived using resolution, the original set of clauses is unsatisfiable (never all True together)
- That is, if we can derive $\square$ from the clausal forms of S and $\neg P$, these clauses can never be all True together
- Hence whenever the clauses of S are all True, at least one clause from $\neg P$ must be False, i.e. $\neg P$ must be False and P must be True
- By definition, $S \models P$ (so P can correctly be concluded from S)

Applying Resolution Refutation

- Negate query to be proven (resolution is a refutation system)
- Convert knowledge base and negated query into CNF
- Repeatedly apply resolution until either the empty clause (contradiction) is derived or no more clauses can be derived
- If the empty clause is derived, answer 'yes' (query follows from knowledge base), otherwise answer 'no' (query does not follow from knowledge base)

Resolution: Example 1

$(G \lor H) \rightarrow (\neg J \land \neg K),\ G \vdash \neg J$

Clausal form of $(G \lor H) \rightarrow (\neg J \land \neg K)$ is
$\{\neg G \lor \neg J,\ \neg H \lor \neg J,\ \neg G \lor \neg K,\ \neg H \lor \neg K\}$

1. $\neg G \lor \neg J$ [Premise]
2. $\neg H \lor \neg J$ [Premise]
3. $\neg G \lor \neg K$ [Premise]
4. $\neg H \lor \neg K$ [Premise]
5. G [Premise]
6. J [$\neg$ Query]
7. $\neg G$ [1, 6 Resolution]
8. $\square$ [5, 7 Resolution]

Resolution: Example 2

$P \rightarrow \neg Q,\ \neg Q \rightarrow R \vdash P \rightarrow R$

Recall $P \rightarrow R \Leftrightarrow \neg P \lor R$

Clausal form of $\neg(\neg P \lor R)$ is $\{P,\ \neg R\}$

1. $\neg P \lor \neg Q$ [Premise]
2. $Q \lor R$ [Premise]
3. P [$\neg$ Query]
4. $\neg R$ [$\neg$ Query]
5. $\neg Q$ [1, 3 Resolution]
6. R [2, 5 Resolution]
7. $\square$ [4, 6 Resolution]

Resolution: Example 3

$\vdash ((P \lor Q) \land \neg P) \to Q$

Clausal form of $\neg(((P \lor Q) \land \neg P) \to Q)$ is $\{P \lor Q,\ \neg P,\ \neg Q\}$

1. $P \lor Q$ [¬ Query]
2. $\neg P$ [¬ Query]
3. $\neg Q$ [¬ Query]
4. Q [1, 2 Resolution]
5. $\square$ [3, 4 Resolution]

Soundness and Completeness Again

For Propositional Logic

- Resolution refutation is sound, i.e. it preserves truth (if a set of premises are all true, any conclusion drawn from those premises must also be true)
- Resolution refutation is complete, i.e. it is capable of proving all consequences of any knowledge base (not shown here!)
- Resolution refutation is decidable, i.e. there is an algorithm implementing resolution which when asked whether $S \vdash P$, can always answer 'yes' or 'no' (correctly)

Heuristics in Applying Resolution

从句消除

- Clause elimination — can disregard certain types of clauses
  - Pure clauses: contain literal L where $\neg L$ doesn't appear elsewhere
  - Tautologies: clauses containing both L and $\neg L$
  - Subsumed clauses: another clause is a subset of the literals
- Ordering strategies
  - Resolve unit clauses (only one literal) first
  - Start with query clauses
  - Aim to shorten clauses

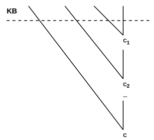
Horn Clauses

Idea: Use less expressive language

- Review
  - literal — proposition variable or negation of proposition variable
  - clause — disjunction of literals
- Definite Clause – exactly one positive literal
  - e.g. $B \lor \neg A_1 \lor \ldots \lor \neg A_n$, i.e. $B \leftarrow A_1 \land \ldots \land A_n$
- Negative Clause – no positive literals
  - e.g. $\neg Q_1 \lor \neg Q_2$ (negation of a query)
- Horn Clause – clause with at most one positive literal

SLD Resolution – $\vdash_{SLD}$

- Selected literals Linear form Definite clauses resolution 明确的
- SLD refutation of a clause C from a set of clauses KB is a sequence
  1. First clause of sequence is C
  2. Each intermediate 中间的 clause C_i is derived by resolving the previous clause C_{i-1} and a clause from KB
  3. The last clause in the sequence is $\square$



- **Theorem.** For a definite KB and negative clause query Q: $KB \cup Q \vdash \square$ if and only if $KB \cup Q \vdash_{SLD} \square$

Prolog

- Horn clauses in First-Order Logic (see later in term)
- SLD resolution
- Depth-first search strategy with backtracking
- User control
  - Ordering of clauses in Prolog database (facts and rules)
  - Ordering of subgoals in body of a rule
- Prolog is a programming language based on resolution refutation relying on the programmer to exploit search control rules

Prolog Example

```
r.                 # facts
u.
v.

q :- r, u.         # rules
s :- v.
p :- q, r, s.

?- p.              # query
yes
```


Prolog Interpreter

Input: A query Q and a logic program KB
Output: 'yes' if Q follows from KB, 'no' otherwise

Initialize current goal set to $\{Q\}$
while the current goal set is not empty do
  Choose G from the current goal set; (first in goal set)
  Choose a copy G' :- $B_1, \dots, B_n$ of a clause from KB (try all in KB)
  (if no such rule, try alternative rules)
  Replace G by $B_1, \dots, B_n$ in current goal set
if current goal set is empty
  output 'yes'
else output 'no'

- Depth-first, left-right with backtracking

Tableau Method

Alpha Rules:

$$\frac{A \wedge B}{\begin{array}{c}A\\B\end{array}} \qquad \frac{\neg(A \vee B)}{\begin{array}{c}\neg A\\\neg B\end{array}} \qquad \frac{\neg(A \rightarrow B)}{\begin{array}{c}A\\\neg B\end{array}}$$

¬¬-Elimination:

$$\frac{\neg\neg A}{A}$$

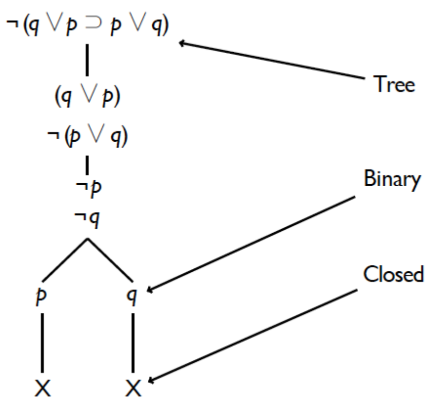
Beta Rules:

$$\frac{A \vee B}{A \mid B} \qquad \frac{A \rightarrow B}{\neg A \mid B} \qquad \frac{\neg(A \wedge B)}{\neg A \mid \neg B}$$

Branch Closure:

$$\frac{\begin{array}{c}A\\\neg A\end{array}}{\times}$$

Tableau Method Example



Conclusion: Propositional Logic

- Propositions built from $\wedge$, $\vee$, $\neg$, $\rightarrow$
- Sound, complete and decidable proof systems (inference procedures)
  - Natural deduction
  - Resolution refutation
  - Prolog for special case of definite clauses
  - Tableau method
- Limited expressive power
  - Cannot express ontologies, e.g. AfPak Ontology
- First-Order Logic can express knowledge about objects, properties and relationships between objects