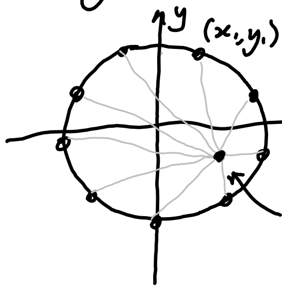
Q2. (1.22 BRML) Example 2 explosion:

This is extension of explosion example on p20.

N sensors evenly spread on Earth surface with locations (x_i, y_i) $i \in \{1, 2, \dots, N\}$



Signal at each location $\left(\dfrac{1}{d_i^2 + 0.1}\right)$ $d_i >> 0$ $\Rightarrow s \in (0, 1)$

$$d_i^2 = (x_i - e_i)^2 + (y_i - e_y)^2$$

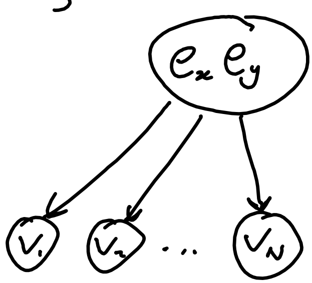
Sensors have noise $p(v_i | d_i) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(v_i - \frac{1}{d_i^2 + 0.1}\right)^2}$

v_i → observed signal at (x_i, y_i)

Signals are independent so:

$$p(v_1, v_2, \dots, v_N, e_x, e_y) = \left[\prod_{i=1}^{N} p(v_i | d_i)\right] p(e_x, e_y)$$

Belief Network:



$$p(a|b) = \frac{p(a,b)}{p(b)} \propto p(a,b)$$

if $p(e_x, e_y) = c$

$\Rightarrow p(e_x, e_y) \propto \prod_{i=1}^{N} p(v_i | d_i)$

Q1.1.a)

$$P(V_1, \dots, V_N, S_1, S_2) = P(S_1)P(S_2)\prod_{i=1}^{N} P(V_i \mid d_i(1), d_i(2))$$

Note: $P(V_i \mid d_i(1), d_i(2)) = \dfrac{1}{d_i^2(1)+0.1} + \dfrac{1}{d_i^2(2)+0.1}$

and $d_i(1)$ is $\|S_1 - V_i\|$, $d_i(2) = \|S_2 - V_i\|$ ← L_2 norm

Now marginalize out S_2:

$$P(V_1, \dots, V_N, S_1) = \sum_{S_2} P(S_1)P(S_2)\prod_{i=1}^{N} P(V_i \mid d_i(1), d_i(2))$$

$$P(S_1 \mid \underline{V}) = \frac{\sum_{S_2} P(S_1)P(S_2)\prod_{i=1}^{N} P(V_i \mid d_i(1), d_i(2))}{P(\underline{V})}$$

note: let priors be constant, and $P(\underline{V})$ doesn't depend on S_1

$$= \frac{P(S_1)P(S_2)}{P(\underline{V})}\sum_{S_2}\prod_{i=1}^{N} P(V_i \mid d_i(1), d_i(2))$$

$$P(S_1 \mid \underline{V}) = k\sum_{S_2}\prod_{i=1}^{N} P(V_i \mid d_i(1), d_i(2))$$

$$= k\sum_{S_2}\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}\left(V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right)^2\right\}$$

$$\boxed{P(S_1 \mid \underline{V}) = \frac{k}{\left(\sqrt{2\pi\sigma^2}\right)^N}\sum_{S_2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left[V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right]^2\right\}}$$

Q2.1.b) Find $\log\left[p(\underline{v}|H_2)\right] - \log\left[p(\underline{v}|H_1)\right]$

where H_2: There a 2 explosions, H_1: 1 explosion

$$p(s_1|\underline{v}, H_2) = k \sum_{s_2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left[v_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right]^2\right\}$$

$$p(\underline{v}, s_1, s_2|H_2) = p(s_1)p(s_2)\prod_{i=1}^{N} p(v_i|s_1, s_2, H_2)$$

$$p(\underline{v}, s_1|H_1) = p(s_1)\prod_{i=1}^{N} p(v_i|s_1, H_1)$$

Note: $p(v_i|s_1, H_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}\left(v_i - \frac{1}{d_i^2+0.1}\right)^2\right\}$

$$p(v_i|s_1, s_2, H_2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}\left(v_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right)^2\right\}$$

marginalise out s_1:

$$p(\underline{v}|H_1) = \sum_{s_1} p(\underline{v}, s_1|H_1)$$ Assuming we have a constant prior

$$= \sum_{s_1} p(s_1)\prod_{i=1}^{N} p(v_i|s_1, H_1)$$

$$= p(s_1)\sum_{s_1}\prod_{i=1}^{N} p(v_i|s_1, H_1)$$

$$P(\underline{V}, S_1, S_2 | H_2) = p(S_1)p(S_2)\prod_{i=1}^{N} p(V_i | S_1, S_2, H_2)$$

$$P(\underline{V} | H_2) = \sum_{S_1 S_2} p(S_1)p(S_2)\prod_{i=1}^{N} p(V_i | S_1, S_2, H_2)$$

$$= P(S_1)P(S_2)\sum_{S_1 S_2}\prod_{i=1}^{N} p(V_i | S_1, S_2, H_2)$$

note:

$$\log(P(\underline{V}|H_2)) - \log(P(\underline{V}|H_1)) = \log\left[\frac{P(\underline{V}|H_2)}{P(\underline{V}|H_1)}\right]$$

$$\frac{P(\underline{V}|H_2)}{P(\underline{V}|H_1)} = \frac{\cancel{P(S_1)}P(S_2)\sum_{S_1 S_2}\exp\left\{\sum_{i=1}^{N}\frac{-1}{2\sigma^2}\left(V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right)^2\right\}}{\cancel{P(S_1)}\sum_{S_1}\exp\left\{\sum_{i=1}^{N}\frac{-1}{2\sigma^2}\left(V_i - \frac{1}{d_i^2(1)+0.1}\right)^2\right\}}$$

Note: $p(S_i)$ is independent and uniform $\Rightarrow p(S_i) = \frac{1}{S}$

(where S is number of discritized points in world space)

$$\frac{P(\underline{V}|H_2)}{P(\underline{V}|H_1)} = \frac{\sum_{S_1 S_2}\exp\left\{\sum_{i=1}^{N}\frac{-1}{2\sigma^2}\left(V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right)^2\right\}}{S\sum_{S_1}\exp\left\{\sum_{i=1}^{N}\frac{-1}{2\sigma^2}\left(V_i - \frac{1}{d_i^2(1)+0.1}\right)^2\right\}}$$

H_2: 2 explosions: Q 2.22.2

$$p(S_1|\underline{V}) = \sum_{S_2} p(S_1, S_2|\underline{V}) = \sum_{S_2} \frac{p(S_1, S_2, \underline{V})}{p(\underline{V})}$$

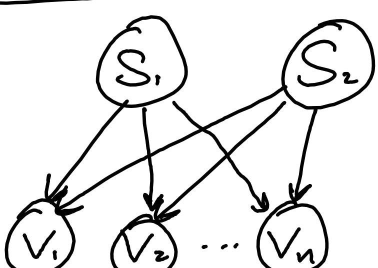
$$= \sum_{S_2} \left[\frac{p(S_1)p(S_2)\prod_{i=1}^{n} p(V_i|d_i(1), d_i(2))}{p(\underline{V})} \right]$$

$$= \frac{p(S_1)p(S_2)}{p(\underline{V})} \sum_{S_2} p(V_i|d_i(1), d_i(2))$$

$$= \frac{p(S_1)p(S_2)}{p(\underline{V})} \sum_{S_2} \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}\left[V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right]^2 \right\} \right]$$

$$= \frac{p(S_1)p(S_2)}{p(\underline{V})} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \sum_{S_2} \prod_{i=1}^{n} \exp\left\{ -\frac{1}{2\sigma^2}\left[V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right]^2 \right\}$$

$$p(S_1|\underline{V}) = k_1 \sum_{S_2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[V_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right]^2 \right\}$$



$$p(S_1, S_2, \underline{V}) = p(S_1)p(S_2)\prod_{i=1}^{n} p(V_i|S_1, S_2)$$

Let $p(S_1) = P_1$, $p(S_2) = P_2$.

Given H_1: 1 explosion only:

$$p(\underline{v}|H_1) = \sum_{S_1} p(\underline{v}, S_1) = \sum_{S_1} p(S_1) \prod_{i=1}^{n} p(v_i|S_1)$$

$$= P_1 \sum_{S_1} \prod_{i=1}^{n} p(v_i|S_1) = P_1 \sum_{S_1} \prod_{i=1}^{n} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left(v_i - \frac{1}{d_i^2(1)+0.1}\right)^2\right\}$$

$$= P_1 \left(\frac{1}{2\pi\sigma^2}\right)^n \sum_{S_1} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(v_i - \frac{1}{d_i^2(1)+0.1}\right)^2\right\}$$

under H_2: 2 explosions:

$$p(\underline{v}|H_2) = P_1 P_2 \left(\frac{1}{2\pi\sigma^2}\right)^n \sum_{S_1 S_2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(v_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right)^2\right\}$$

$$\log\left[\frac{P(\underline{v}|H_2)}{P(\underline{v}|H_1)}\right] = \log\left[\frac{P_2 \sum_{S_1 S_2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(v_i - \frac{1}{d_i^2(1)+0.1} - \frac{1}{d_i^2(2)+0.1}\right)^2\right\}}{\sum_{S_1} \exp\left\{\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(v_i - \frac{1}{d_i^2(1)+0.1}\right)^2\right\}}\right]$$

?

Q1.22.3

if $p(H_1) = p(H_2) = k \in \mathbb{R}^+$ why does $\log p(\underline{v}|H_1) - \log p(\underline{v}|H_2)$ relate to prob of 2 explosions compared to 1?

$$p(\underline{v}|H_1) = \frac{p(\underline{v}, H_1)}{p(H_1)} \quad \text{and} \quad p(\underline{v}|H_2) = \frac{p(\underline{v}, H_2)}{p(H_2)}$$

then $\log p(\underline{v}|H_2) - \log p(\underline{v}|H_1) = \log\left[\frac{p(\underline{v}|H_2)}{p(\underline{v}|H_1)}\right]$

Let E be random variable for number of explosions.

$\Rightarrow p(\underline{v}|H_1) = p(\underline{v}|E=1), \quad p(\underline{v}|H_2) = p(\underline{v}|E=2).$

$$\Rightarrow \log\left[\frac{p(\underline{v}|H_2)}{p(\underline{v}|H_1)}\right] = \log\left[\frac{p(H_1)\ p(\underline{v}, E=1)}{p(H_2)\ p(\underline{v}, E=2)}\right] = \log\left[\frac{k\ \ p(\underline{v}, E=1)}{k\ \ p(\underline{v}, E=2)}\right]$$

$$= \log\left(\frac{p(\underline{v}, E=1)}{p(\underline{v}, E=2)}\right) = \log\left(\frac{p(E=1|\underline{v})\, p(\underline{v})}{p(E=2|\underline{v})\, p(\underline{v})}\right) = \underline{\underline{\log\left(\frac{p(E=1|\underline{v})}{p(E=2|\underline{v})}\right)}}$$

1.22.4/

Assume k explosions, what is computational complexity of $p(\underline{v}|H_k)$?

- Using same methods as before we get:

$$p(\underline{v}|H_k) = \prod_{i=1}^{k}\left[p(s_i)\right]\left(\frac{1}{2\pi\sigma^2}\right)^n \sum_{s_1 \cdots s_k} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[v_i - \sum_{j=1}^{k}\frac{1}{d_i^2(j)+0.1}\right]^2\right\}$$

Therefore we have to marginalise over all^k possible explosion locations. This means calculations grow $O(S^k)$ where S is the number of discretized world points we are iterating over for each S_i.