How to go from partial to full retroactivity in detail

Cristina Gomes Fernandes, Felipe Castro de Noronha

IME-USP – Brazil

LAGOS 25 – November 10-14, 2025

What is a spanning tree?

- Let $G = (V, E)$ be a connected graph
- **Spanning tree:** A tree with all vertices of G

What is a spanning tree?

- Let $G = (V, E)$ be a connected graph
- **Spanning tree:** A tree with all vertices of G
- **Properties:**
  - Connected (path between any two vertices)
  - Acyclic (no cycles)
  - Contains exactly $n - 1$ edges for n vertices

What is a spanning tree?

- Let $G = (V, E)$ be a connected graph
- **Spanning tree:** A tree with all vertices of G
- **Properties:**
  - Connected (path between any two vertices)
  - Acyclic (no cycles)
  - Contains exactly $n - 1$ edges for n vertices

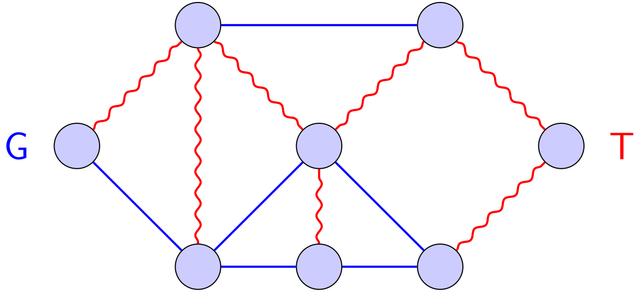


Figure: Graph G (blue edges) and spanning tree T (red wavy edges)

Minimum Spanning Tree and Forest

- **Minimum Spanning Tree (MST):** spanning tree with minimum total cost

Minimum Spanning Tree and Forest

- **Minimum Spanning Tree (MST):** spanning tree with minimum total cost
- **Minimum Spanning Forest (MSF):** generalization for disconnected graphs

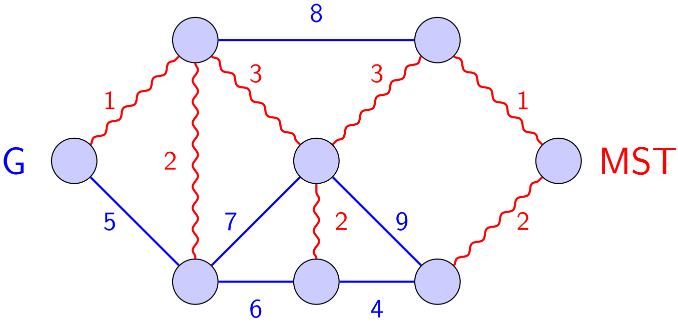


Figure: Graph G (blue edges) and Minimum Spanning Tree (red wavy edges)

Incremental MSF problem

- **Problem:** Keep track of an MSF in a graph that grows over time
- Graph starts empty, edges are added one by one

Incremental MSF problem

- **Problem:** Keep track of an MSF in a graph that grows over time
- Graph starts empty, edges are added one by one
- **Operations:**
  - `add_edge`(u, v, w): add edge with cost w between vertices u and v
  - `get_msf`$()$: return current MSF

Incremental MSF problem

- **Problem:** Keep track of an MSF in a graph that grows over time
- Graph starts empty, edges are added one by one
- **Operations:**
  - `add_edge`(u, v, w): add edge with cost w between vertices u and v
  - `get_msf`$()$: return current MSF
- **Solution:** Frederickson (1983) using link-cut trees

Incremental MSF example - Step 1

- **add_edge(G, H, 4):** Add edge with cost 4

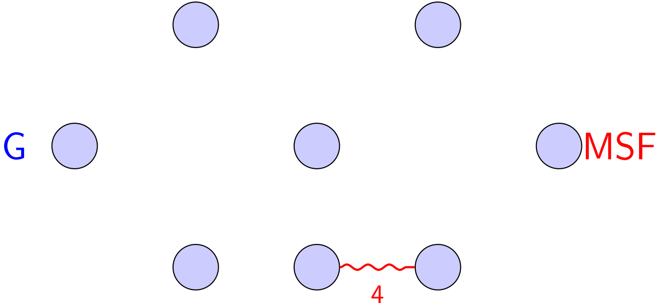


Figure: Step 1: Added edge (G,H) with cost 4

- **MSF:** {G-H}

Incremental MSF example - Step 2

- **add_edge(C, A, 1):** Add edge with cost 1

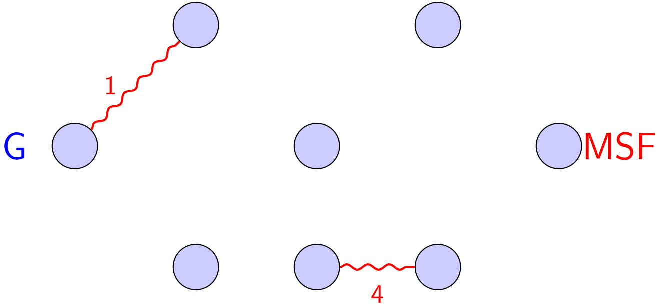


Figure: Step 2: Added edge (C,A) with cost 1

- **MSF:** {G-H, C-A}

Incremental MSF example - Step 3

- **add_edge(F, G, 6):** Add edge with cost 6

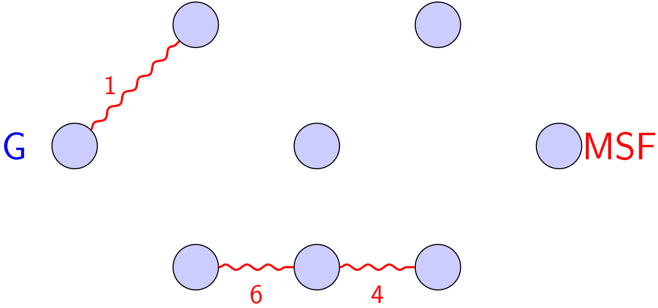


Figure: Step 3: Added edge (F,G) with cost 6

- **MSF:** {G-H, C-A, F-G}

Incremental MSF example - Step 4

- **add_edge(A, F, 2):** Add edge with cost 2

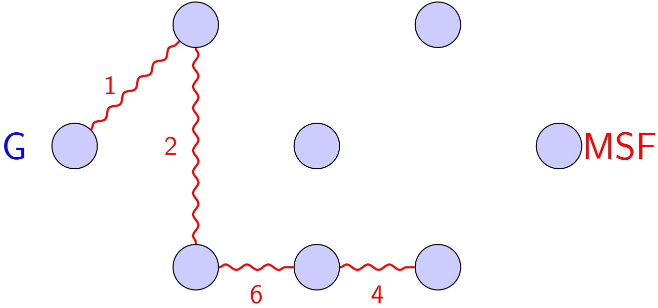


Figure: Step 4: Added edge (A,F) with cost 2

- **MSF:** {G-H, C-A, F-G, A-F}

Incremental MSF example - Step 5

- **add_edge(C, F, 5):** Add edge with cost 5

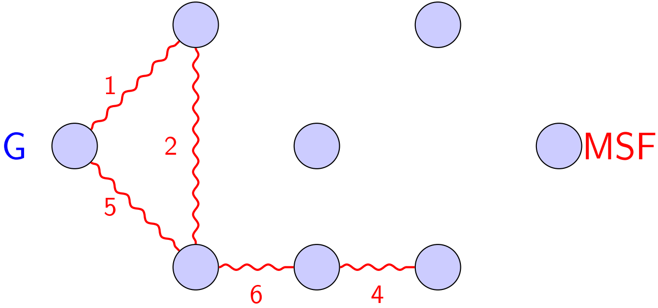


Figure: Step 5: Added edge (C,F) with cost 5

- **MSF:** {G-H, C-A, F-G, A-F, C-F}

Incremental MSF example - Step 6

- **add_edge(F, D, 7):** Add edge with cost 7

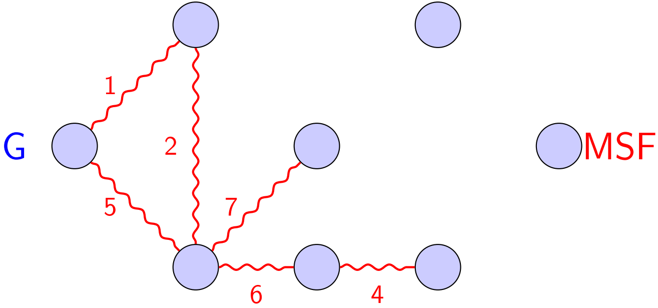


Figure: Step 6: Added edge (F,D) with cost 7

- **MSF:** {G-H, C-A, F-G, A-F, C-F, F-D}

Incremental MSF example - Step 7

- **add_edge(A, B, 8):** Add edge with cost 8

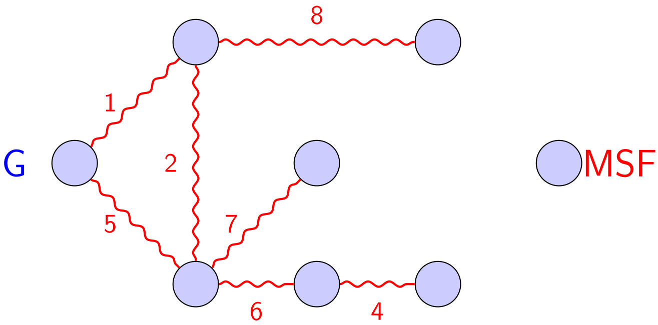


Figure: Step 7: Added edge (A,B) with cost 8

- **MSF:** {G-H, C-A, F-G, A-F, C-F, F-D, A-B}

Incremental MSF example - Step 8

- **add_edge(G, D, 2):** Add edge with cost 2
- **Cycle detected!** Replace expensive edge

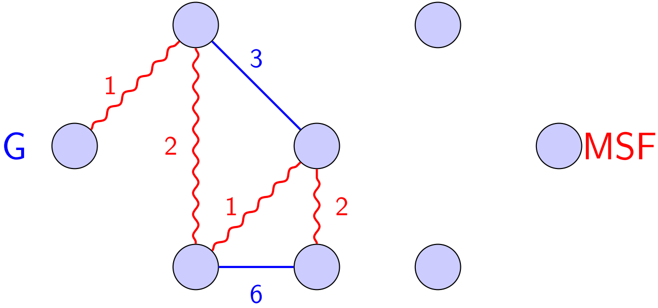


Figure: Step 8: Added edge (G,D) with cost 2 - replaced edge (F,G)

- **MSF:** {C-A, A-F, F-D, G-D} (cost improved)

Incremental MSF example - Step 9

- **add_edge(B, D, 3):** Add edge with cost 3

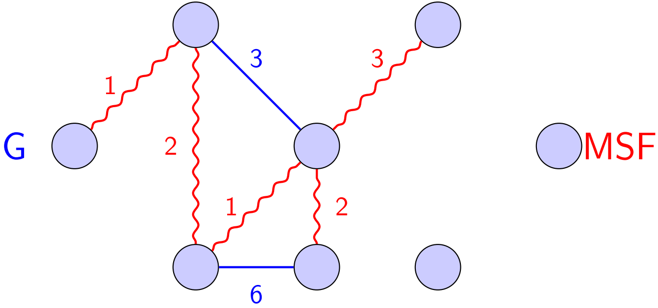


Figure: Step 9: Added edge (B,D) with cost 3

- **MSF:** {C-A, A-F, F-D, G-D, B-D}

Incremental MSF example - Step 10

- **add_edge(B, E, 1):** Add edge with cost 1

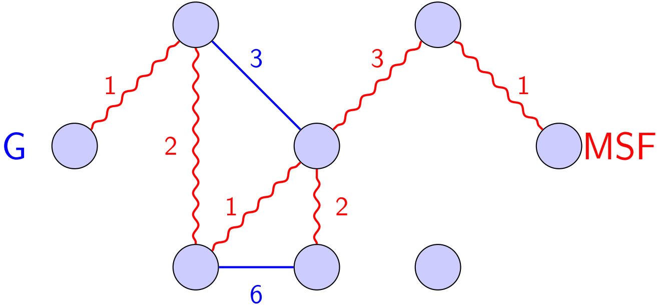


Figure: Step 10: Added edge (B,E) with cost 1

- **MSF:** {C-A, A-F, F-D, G-D, B-D, B-E}

Incremental MSF example - Step 11

- **add_edge(H, E, 2):** Add edge with cost 2

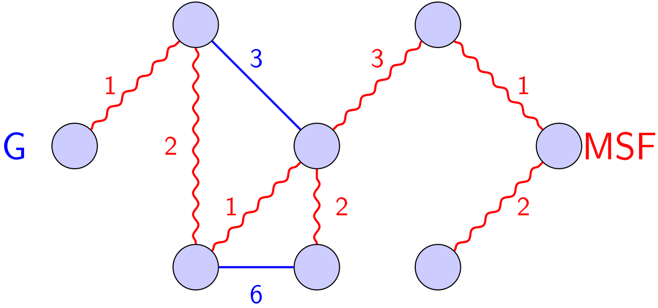


Figure: Step 11: Added edge (H,E) with cost 2

- **MSF:** {C-A, A-F, F-D, G-D, B-D, B-E, H-E}

Incremental MSF example - Final Result

- **Continue adding edges...**

Incremental MSF example - Final Result

- **Continue adding edges...**
- **Final MSF:** Minimum spanning forest with optimal cost

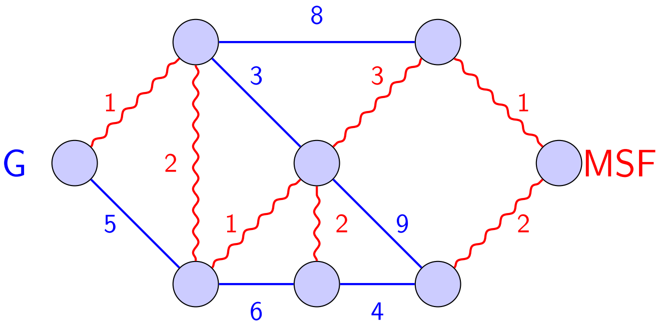


Figure: Final MSF with optimal cost = 12

- **Solution:** Frederickson (1983) using link-cut trees

Frederickson's link-cut tree solution

- **Key insight:** Use link-cut trees to maintain MSF dynamically

Frederickson's link-cut tree solution

- **Key insight:** Use link-cut trees to maintain MSF dynamically
- **Algorithm for adding edge (u, v, w):**
  1. Check if u and v are in same component
  2. If not: add edge to forest
  3. If yes: find max cost edge on u-v path
  4. If $w <$ max cost: replace max edge with new edge

Frederickson's link-cut tree solution

- **Key insight:** Use link-cut trees to maintain MSF dynamically
- **Algorithm for adding edge (u, v, w):**
  1. Check if u and v are in same component
  2. If not: add edge to forest
  3. If yes: find max cost edge on u-v path
  4. If $w <$ max cost: replace max edge with new edge
- **Link-cut tree operations:**
  - $\texttt{find\_max}(u, v)$: $\mathcal{O}(\log n)$ amortized
  - $\texttt{link}(u, v)$: $\mathcal{O}(\log n)$ amortized
  - $\texttt{cut}(u, v)$: $\mathcal{O}(\log n)$ amortized

Frederickson's link-cut tree solution

- **Key insight:** Use link-cut trees to maintain MSF dynamically
- **Algorithm for adding edge (u, v, w):**
  1. Check if u and v are in same component
  2. If not: add edge to forest
  3. If yes: find max cost edge on u-v path
  4. If $w <$ max cost: replace max edge with new edge
- **Link-cut tree operations:**
  - `find_max`(u, v): $\mathcal{O}(\log n)$ amortized
  - `link`(u, v): $\mathcal{O}(\log n)$ amortized
  - `cut`(u, v): $\mathcal{O}(\log n)$ amortized
- **Total cost:** Amortized $\mathcal{O}(\log n)$ per edge addition

What is retroactivity?

- **Problem:** Data structures usually support updates and queries
- Order of updates affects the state of the data structure

What is retroactivity?

- **Problem:** Data structures usually support updates and queries
- Order of updates affects the state of the data structure
- **Retroactivity:** Manipulate the sequence of updates

What is retroactivity?

- **Problem:** Data structures usually support updates and queries
- Order of updates affects the state of the data structure
- **Retroactivity:** Manipulate the sequence of updates
- **Operations:**
  - Insert update at time t (possibly in the past)
  - Remove update at time t
  - Query at time t (not just present)

Partial vs Full retroactivity

Partially Retroactive

- Queries only on **current** state
- Insert/remove updates at any time
- Example: Dynamic MSF $\rightarrow$ Partially retroactive MSF

Partial vs Full retroactivity

Partially Retroactive

- Queries only on **current** state
- Insert/remove updates at any time
- Example: Dynamic MSF $\rightarrow$ Partially retroactive MSF

Fully Retroactive

- Queries at **any** time t
- Insert/remove updates at any time
- Complete retroactive functionality

Partial vs Full retroactivity

Partially Retroactive

- Queries only on **current** state
- Insert/remove updates at any time
- Example: Dynamic MSF $\rightarrow$ Partially retroactive MSF

Fully Retroactive

- Queries at **any** time t
- Insert/remove updates at any time
- Complete retroactive functionality

Semi-Retroactive

- Queries at **any** time t
- Insert updates at any time
- **No removal** of updates

The challenge

Challenge

How to transform partial $\rightarrow$ full retroactivity?

The challenge

Challenge

How to transform partial $\rightarrow$ full retroactivity?

- **Problem:** Need to support queries at any time t
- **Solution approach:** Square-root decomposition

The challenge

Challenge

How to transform partial $\rightarrow$ full retroactivity?

- **Problem:** Need to support queries at any time t
- **Solution approach:** Square-root decomposition
- **Key insight:** Keep checkpoints with data structure states
- **Implementation:** Demaine, Iacono & Langerman (2007)

Demaine, Iacono & Langerman's solution

Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

- $\mathcal{O}(\sqrt{m})$ *slowdown per operation*
- $\mathcal{O}(m)$ *space usage*
- **Requirement:** *Need persistent version of the data structure*

Demaine, Iacono & Langerman's solution

Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

- $\mathcal{O}(\sqrt{m})$ *slowdown per operation*
- $\mathcal{O}(m)$ *space usage*
- **Requirement:** *Need persistent version of the data structure*

- **Key idea:** Square-root decomposition
- Keep $\sqrt{m}$ checkpoints with data structure states

Demaine, Iacono & Langerman's solution

Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

- $\mathcal{O}(\sqrt{m})$ *slowdown per operation*
- $\mathcal{O}(m)$ *space usage*
- **Requirement:** *Need persistent version of the data structure*

- **Key idea:** Square-root decomposition
- Keep $\sqrt{m}$ checkpoints with data structure states
- **Query at time t:**
  1. Find closest checkpoint before t
  2. Apply updates from checkpoint to t
  3. Answer query, then rollback

The space problem

- **Naive approach:** Keep $\sqrt{m}$ independent copies
- Space usage: $\Theta(m\sqrt{m})$

The space problem

- **Naive approach:** Keep $\sqrt{m}$ independent copies
- Space usage: $\Theta(m\sqrt{m})$
- **Demaine et al. solution:** Use persistent data structures
- Space usage: $\mathcal{O}(m)$

The space problem

- **Naive approach:** Keep $\sqrt{m}$ independent copies
- Space usage: $\Theta(m\sqrt{m})$
- **Demaine et al. solution:** Use persistent data structures
- Space usage: $\mathcal{O}(m)$

Problem

What if we don't have or don't want to use persistent data structures?

The space problem

- **Naive approach:** Keep $\sqrt{m}$ independent copies
- Space usage: $\Theta(m\sqrt{m})$
- **Demaine et al. solution:** Use persistent data structures
- Space usage: $\mathcal{O}(m)$

Problem

What if we don't have or don't want to use persistent data structures?

Our contribution

Simple rebuilding strategy without persistent data structures

- Same time complexity: $\mathcal{O}(\sqrt{m})$ per operation
- Space usage: $\Theta(m\sqrt{m})$

Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - `add_edge`(u, v, w, t): add edge at time t
  - `get_msf`(t): get MSF at time t

Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - `add_edge`(u, v, w, t): add edge at time t
  - `get_msf`(t): get MSF at time t
- **Implementation:** Square-root decomposition
- **Checkpoints:** $t_i = i\sqrt{m}$ for $i = 1, \ldots, \sqrt{m}$

Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - `add_edge`(u, v, w, t): add edge at time t
  - `get_msf`(t): get MSF at time t
- **Implementation:** Square-root decomposition
- **Checkpoints:** $t_i = i\sqrt{m}$ for $i = 1, \ldots, \sqrt{m}$
- **Data structures:** D_i contains edges before time t_i
- **Time:** $\mathcal{O}(\sqrt{m}\log n)$ per operation

Limitations

Problems with their approach

- **Fixed m:** Must know sequence length beforehand
- **Fixed time range:** Operations must have timestamps 1 to m
- **No rebuilding:** Cannot handle arbitrary growth

Limitations

Problems with their approach

- **Fixed m:** Must know sequence length beforehand
- **Fixed time range:** Operations must have timestamps 1 to m
- **No rebuilding:** Cannot handle arbitrary growth

Our goal

Remove these limitations while maintaining efficiency

Limitations

Problems with their approach

- **Fixed m:** Must know sequence length beforehand
- **Fixed time range:** Operations must have timestamps 1 to m
- **No rebuilding:** Cannot handle arbitrary growth

Our goal

Remove these limitations while maintaining efficiency

- **Key insight:** Implement rebuilding process
- **Challenge:** How to rebuild without persistent data structures?
- **Solution:** Reuse existing data structures during rebuilding

Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)

Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)
- **Strategy:**
  1. Create new empty structures D'_0, D'_1
  2. Reuse $D_i \rightarrow D'_{i+2}$ for $i = 0, \ldots, k-1$
  3. Apply missing updates to each D'_i

Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)
- **Strategy:**
  1. Create new empty structures D'_0, D'_1
  2. Reuse $D_i \to D'_{i+2}$ for $i = 0, \ldots, k-1$
  3. Apply missing updates to each D'_i

Key Lemma

Every update in D_i is within the first $(i+2)(k+1)$ updates in the new sequence.

Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)
- **Strategy:**
  1. Create new empty structures D'_0, D'_1
  2. Reuse $D_i \to D'_{i+2}$ for $i = 0, \ldots, k-1$
  3. Apply missing updates to each D'_i

Key Lemma

Every update in D_i is within the first $(i+2)(k+1)$ updates in the new sequence.

- **Time per rebuilding:** $\mathcal{O}(m \log n)$
- **Amortized cost:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

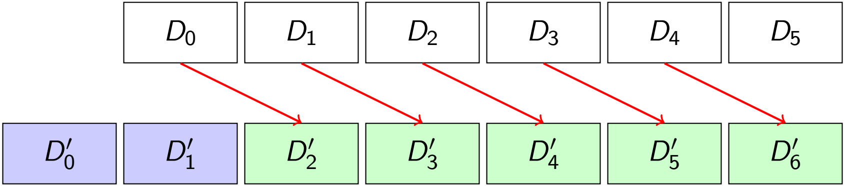
Rebuilding algorithm

1. $D'_0 \leftarrow \textsc{newIncrementalMSF}()$
2. $D'_1 \leftarrow \textsc{newIncrementalMSF}()$
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ $\quad\triangleright$ reuse existing
4. For $i = 1$ to $k+1$:
   - $p \leftarrow \textsc{kth}(S, i(k+1))$ $\quad\triangleright$ $i(k+1)$th edge
   - $t'_i \leftarrow p.\text{time}$
   - $\textsc{addEdges}(S, t_{i-2}, t'_i, D'_i)$
5. Return $k+1, D', t'$

Rebuilding algorithm

1. $D'_0 \leftarrow \textsc{newIncrementalMSF}()$
2. $D'_1 \leftarrow \textsc{newIncrementalMSF}()$
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ $\qquad\triangleright$ reuse existing
4. For $i = 1$ to $k+1$:
   - $p \leftarrow \textsc{kth}(S, i(k+1))$ $\qquad\triangleright$ $i(k+1)$th edge
   - $t'_i \leftarrow p.\text{time}$
   - $\textsc{addEdges}(S, t_{i-2}, t'_i, D'_i)$
5. Return $k+1, D', t'$

Original



New

$$D_i \to D'_{i+2}$$

Results

Our contribution

- **General transformation:** Partial $\rightarrow$ Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
- **Space trade-off:** $\Theta(m\sqrt{m})$ vs $\mathcal{O}(m)$

Results

Our contribution

- **General transformation:** Partial $\to$ Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
- **Space trade-off:** $\Theta(m\sqrt{m})$ vs $\mathcal{O}(m)$

Semi-retroactive MSF implementation

- **Operations:** $\texttt{add\_edge}(u, v, w, t)$, $\texttt{get\_msf}(t)$
- **Time:** $\mathcal{O}(\sqrt{m}\log n)$ per operation
- **Space:** $\Theta(m\sqrt{m})$
- **No fixed m or time range restrictions**

Extending for full retroactivity

- **General applicability:** Works for any partially retroactive data structure

Extending for full retroactivity

- **General applicability:** Works for any partially retroactive data structure

- **Supporting removals:** To achieve full retroactivity
  - Adapt rebuilding trigger: when $|\lfloor\sqrt{m'}\rfloor - \lfloor\sqrt{m}\rfloor| \leq 1$
  - Handle both insertions and removals in update sequence
  - Rebuilding frequency: every $2\lfloor\sqrt{m}\rfloor - 1$ operations

Extending for full retroactivity

- **General applicability:** Works for any partially retroactive data structure
- **Supporting removals:** To achieve full retroactivity
  - Adapt rebuilding trigger: when $|\lfloor\sqrt{m'}\rfloor - \lfloor\sqrt{m}\rfloor| \leq 1$
  - Handle both insertions and removals in update sequence
  - Rebuilding frequency: every $2\lfloor\sqrt{m}\rfloor - 1$ operations
- **Requirements:**
  - Partially retroactive data structure
  - Rollback capability
  - No persistent version needed

Thank you!

Questions?