# How to go from partial to full retroactivity in detail

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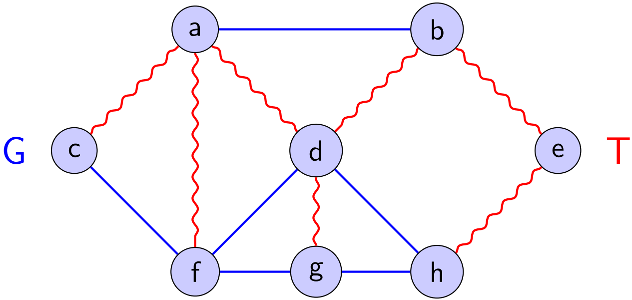

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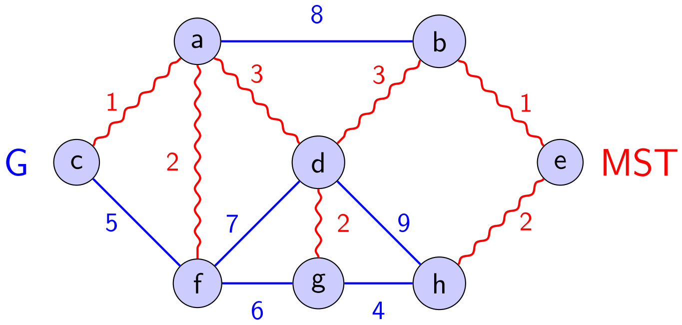

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- **Solution:** Frederickson (1983) using link-cut trees

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  - `is_connected`$(u, v)$: $\mathcal{O}(\log n)$ amortized

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  1. Check if $u$ and $v$ are in connected in the same component
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  3. If yes: find the edge with maximum cost on the $u$-$v$ path
  4. If $w <$ maximum cost: replace maximum cost edge with $(u, v, w)$

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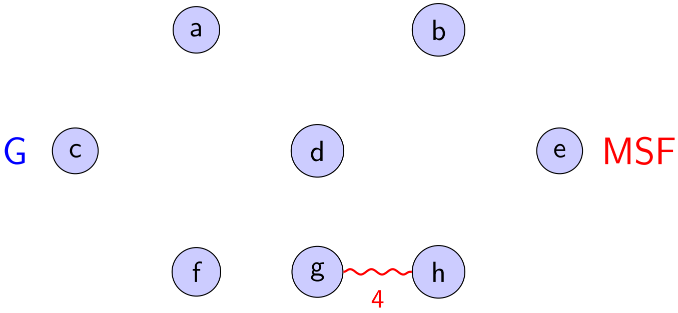
- **Total cost:** Amortized $\mathcal{O}(\log n)$ per edge addition

# Incremental MSF example - Step 1

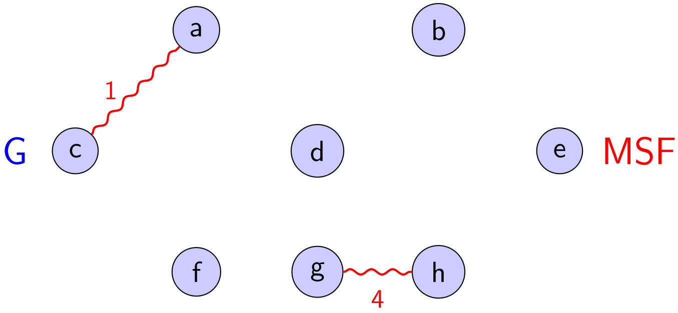
- **add_edge(g, h, 4):** Add edge $(g, h)$ with cost 4




- **MSF:** {g-h}

# Incremental MSF example - Step 2

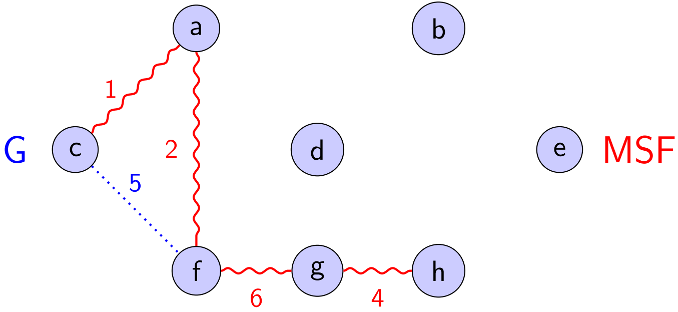
- **add_edge(c, a, 1):** Add edge with cost 1




- **MSF:** {g-h, c-a}

# Incremental MSF example - Step 3 (Cycle Check)

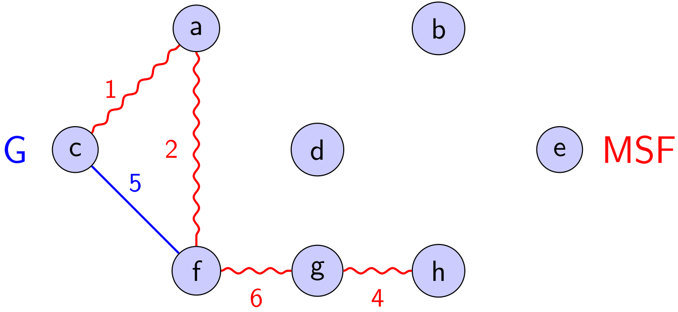
- **add_edge(c, f, 5):** Add edge with cost 5




- **MSF:** {g-h, c-a, f-g, a-f}

# Incremental MSF example - Step 3 (Result)

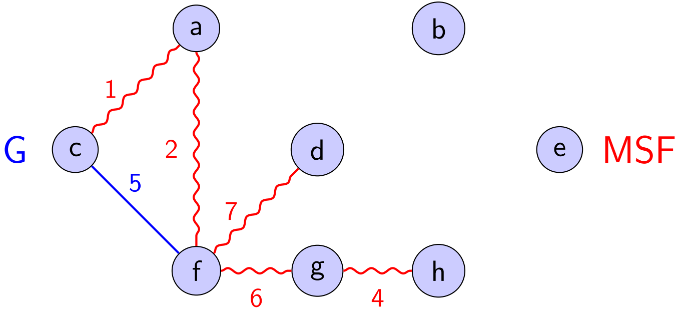
- **add_edge(c, f, 5):** Edge cost is greater than max cost (2), not added




- **MSF:** {g-h, c-a, f-g, a-f}

# Incremental MSF example - Step 4

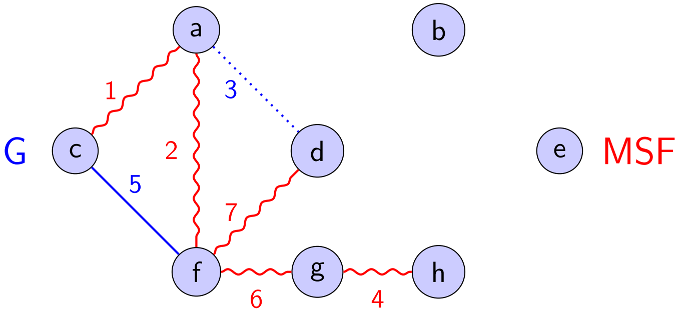
- **add_edge(f, d, 7):** Add edge with cost 7




- **MSF:** {g-h, c-a, f-g, a-f, f-d}

# Incremental MSF example - Step 5 (Cycle Check)

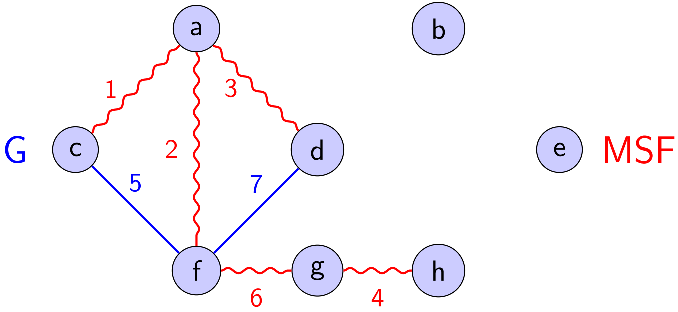
- **add_edge(a, d, 3):** Add edge with cost 3




- **MSF:** {g-h, c-a, f-g, a-f, f-d}

# Incremental MSF example - Step 5 (Result)

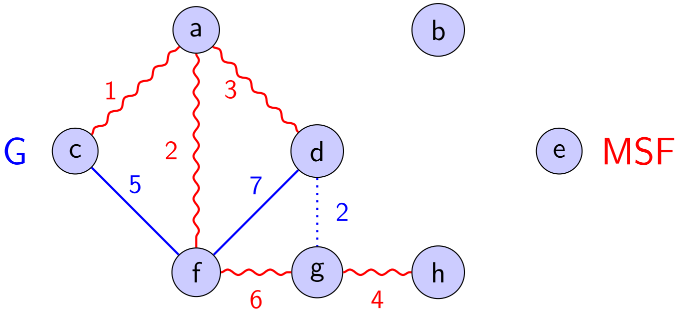
- **add_edge(a, d, 3):** Cost 3 ¡ max cost 7, swap edges




- **MSF:** {g-h, c-a, f-g, a-f, a-d}

# Incremental MSF example - Step 6 (Cycle Check)

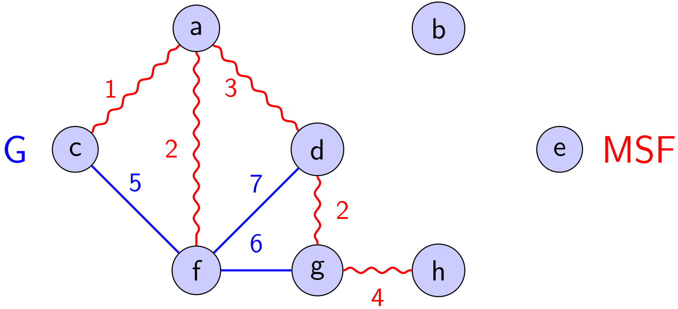
- **add_edge(d, g, 2):** Add edge with cost 2




- **MSF:** {g-h, c-a, f-g, a-f, a-d}

# Incremental MSF example - Step 6 (Result)

- **add_edge(d, g, 2):** Cost 2 ¡ max cost 6, swap edges




- **MSF:** {g-h, c-a, a-f, a-d, d-g}

# Incremental MSF example - Step 7 (Final Result)

- **Continue adding edges...**

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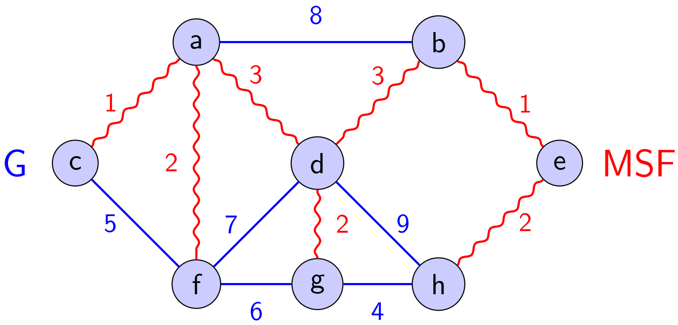
- **Final MSF:** Minimum spanning forest with optimal cost

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- **Retroactivity:** Manipulate the sequence of updates
- **Operations:**
  - Insert update at time $t$ (possibly in the past)
  - Remove update at time $t$
  - Query at time $t$ (not just present)

# Partial vs Full retroactivity

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### Semi-Retroactive

- Queries at **any** time $t$
- Insert updates at any time
- **No removal** of updates

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- **Problem:** Need to support queries at any time $t$
- **Solution approach:** Square-root decomposition
- **Key insight:** Keep checkpoints with data structure states
- **Implementation:** Demaine, Iacono & Langerman (2007)

# Demaine, Iacono & Langerman's solution

**Theorem (Theorem 05)**

*Any partially retroactive data structure can be transformed into a fully retroactive one with:*

- $\mathcal{O}(\sqrt{m})$ *slowdown per operation*
- $\mathcal{O}(m)$ *space usage*
- **Requirement:** *Need persistent version of the data structure*

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- **Key idea:** Square-root decomposition
- Keep $\sqrt{m}$ checkpoints with data structure states
- **Query at time $t$:**
  1. Find closest checkpoint before $t$
  2. Apply updates from checkpoint to $t$
  3. Answer query, then rollback

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### Problem

What if we don't have or don't want to use persistent data structures?

### Our contribution

Simple rebuilding strategy without persistent data structures

- Same time complexity: $\mathcal{O}(\sqrt{m})$ per operation
- Space usage: $\Theta(m\sqrt{m})$

# Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - `add_edge`$(u, v, w, t)$: add edge at time $t$
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- **Implementation:** Square-root decomposition
- **Checkpoints:** $t_i = i\sqrt{m}$ for $i = 1, \ldots, \sqrt{m}$
- **Data structures:** $D_i$ contains edges before time $t_i$
- **Time:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

# Limitations → Key Insight

**Problems with the existing static approach**

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## Our dynamic goal and solution

**Goal:** Remove the **Fixed m** dependency while preserving time complexity.

- **Key Insight:** Introduce a dynamic rebuilding process to handle arbitrary growth.
- **Challenge:** Rebuilding $\sqrt{\mathbf{m}}$ checkpoints must be fast, avoiding complex persistent data structures.
- **Solution:** Reuse the existing data structures to efficiently reconstruct new checkpoints.

# Our solution - Rebuilding strategy

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**Key Lemma**

Every update in $D_i$ is within the first $(i+2)(k+1)$ updates in the new sequence.

- **Time per rebuilding:** $\mathcal{O}(m \log n)$
- **Amortized cost:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

# Rebuilding algorithm

1. $D'_0 \leftarrow \text{NEWINCREMENTALMSF}()$
2. $D'_1 \leftarrow \text{NEWINCREMENTALMSF}()$
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ $\quad \triangleright$ reuse existing
4. For $i = 1$ to $k+1$:
   - $p \leftarrow \text{KTH}(S, i(k+1))$ $\quad \triangleright$ $i(k+1)$th item in the sequence of updates
   - $t'_i \leftarrow p.\text{time}$
   - $\text{ADDEDGES}(S, t_{i-2}, t'_i, D'_i)$
5. Return $k+1, D', t'$

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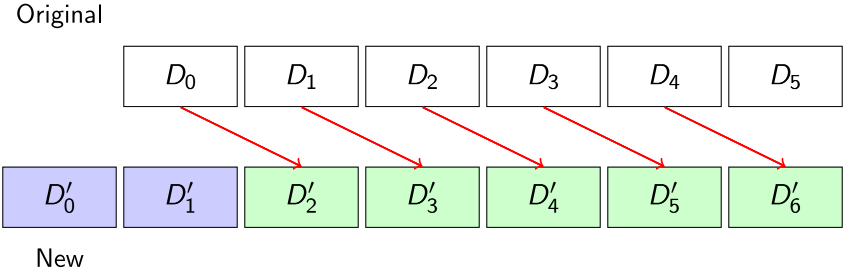


$$D_i \to D'_{i+2}$$

# Results

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- **General transformation:** Partial $\rightarrow$ Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
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## Semi-retroactive MSF implementation

- **Operations:** $\texttt{add\_edge}(u, v, w, t)$, $\texttt{get\_msf}(t)$
- **Time:** $\mathcal{O}(\sqrt{m}\log n)$ per operation
- **Space:** $\Theta(m\sqrt{m})$
- **No fixed** $m$ **or time range restrictions**

Thank you!

Questions?