How to go from partial to full retroactivity in detail

Cristina Gomes Fernandes, Felipe Castro de Noronha

IME-USP – Brazil

LAGOS 25 – November 10-14, 2025

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1. Hello there everybody, my name is Felipe Noronha and today I'm gonna gonna be doing a quick presentation about the paper professor Cristina and I did at the IME of USP
2. with the topic being: going from partial to full retroactivity in detail
3. This work addresses a practical limitation in Demaine, Iacono & Langerman's 2007 transformation
4. And also iterates over Junior & Seabra's solution from 2022
5. Our contribution shows how to go from partial to full retroactivity with same time complexity without requiring persistent data structures
6. As our main focus of study we had the minimum spanning fores problem, and that's what we are going to start with today

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3. Explain key properties: connected (path between any two vertices), acyclic (no cycles), contains exactly n-1 edges for n vertices
4. ———- SKIP SLIDE ———-
5. Show visual example with graph G (blue edges) and spanning tree T (red wavy edges)
6. In the example: 8 vertices, so spanning tree has exactly 7 edges
7. This builds up the concepts step by step for the incremental MSF problem
8. Emphasize that spanning trees are not unique - there can be many valid spanning trees

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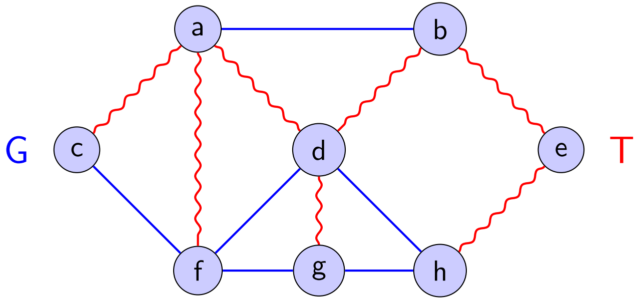

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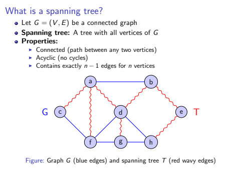

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1. Define MST as spanning tree in a graph with weighted edges with minimum total cost - optimization problem
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3. Generalize to MSF for disconnected graphs - collection of MSTs for each component
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5. Show visual example with weighted edges: blue edges show graph G, red wavy edges show MST
6. Demonstrate that red edges form MST with cost 14 $(1+2+3+2+3+1+2=14)$
7. Explain that any other spanning tree would have higher cost - this is the optimal solution
8. This prepares for the incremental MSF problem where we maintain optimality dynamically
9. Key insight: we need to maintain optimality as edges are added one by one

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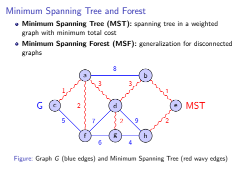

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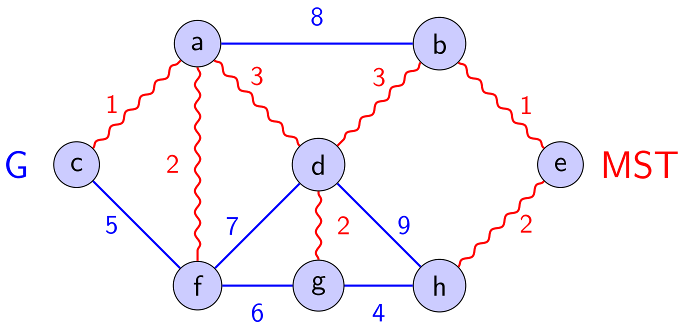

Figure: Graph G (blue edges) and Minimum Spanning Tree (red wavy edges)

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3. ———- SKIP SLIDE ———-
4. Show the two key operations: add_edge(u,v,w) and get_msf()
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6. Mention Frederickson's breakthrough solution from 1983 using link-cut trees
7. Note the cost is $O(logn)$ amortized per edge addition using link-cut trees
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3. List the specific link-cut tree operations: find_max, link, cut - all $O(logn)$ amortized
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6. 2. If not connected: add edge directly $linkoperation$
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10. Emphasize the logarithmic time complexity: $O(logn)$ per edge addition
11. Key insight: link-cut trees support efficient rollback, which we'll need for retroactivity

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- **Key insight:** Use link-cut trees to maintain MSF dynamically
- **Link-cut tree operations:**
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Incremental MSF example - Step 1

- **add_edge(g, h, 4):** Add edge with cost 4

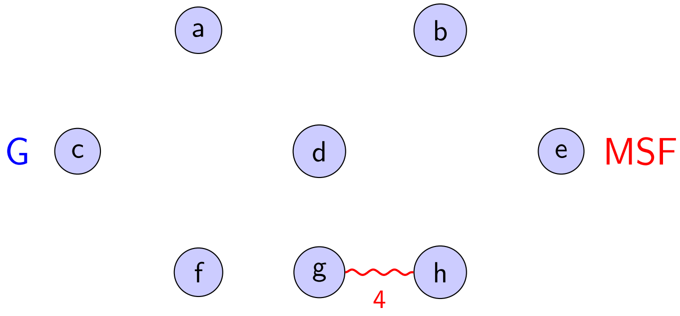
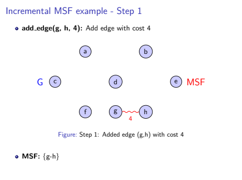


Figure: Step 1: Added edge (g,h) with cost 4

- **MSF:** {g-h}



1. Show first edge being added: (g,h) with cost 4
2. Explain it's automatically added to MSF since no cycle exists yet
3. Current MSF: g-h with total cost 4
4. This demonstrates the incremental nature: we start with empty graph
5. Each step shows how MSF evolves as edges are added
6. Link-cut tree operations: link(g,h) - O(log n) time

Incremental MSF example - Step 2

- **add_edge(c, a, 1):** Add edge with cost 1

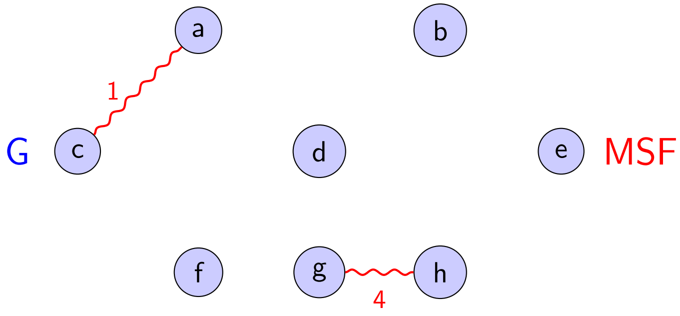
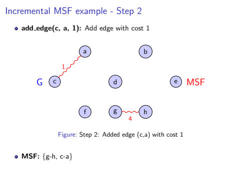


Figure: Step 2: Added edge (c,a) with cost 1

- **MSF:** {g-h, c-a}



1. Show second edge being added: (c,a) with cost 1
2. Still no cycle, so added to MSF directly
3. Current MSF: g-h, c-a with total cost 5
4. Link-cut tree operations: link(c,a) - O(log n) time
5. We now have two separate components: g,h and c,a
6. This shows how MSF grows incrementally without cycles

Incremental MSF example - Step 3

- **add_edge(f, g, 6):** Add edge with cost 6

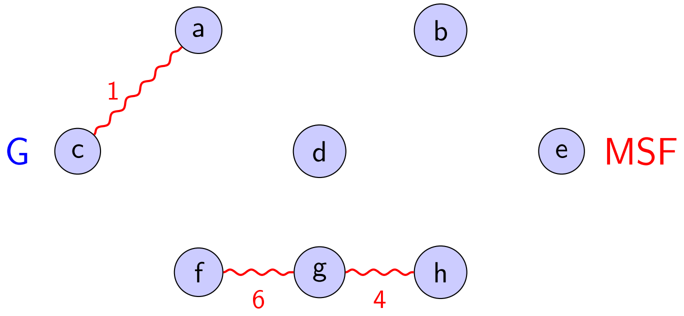
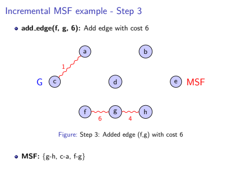


Figure: Step 3: Added edge (f,g) with cost 6

- **MSF:** {g-h, c-a, f-g}



1. Show third edge being added: (f,g) with cost 6
2. Still no cycle, so added to MSF directly
3. Current MSF: g-h, c-a, f-g with total cost 11
4. Link-cut tree operations: link(f,g) - O(log n) time
5. Now we have components: g,h,f and c,a
6. This continues the incremental growth pattern

Incremental MSF example - Step 4

- **add_edge(a, f, 2):** Add edge with cost 2

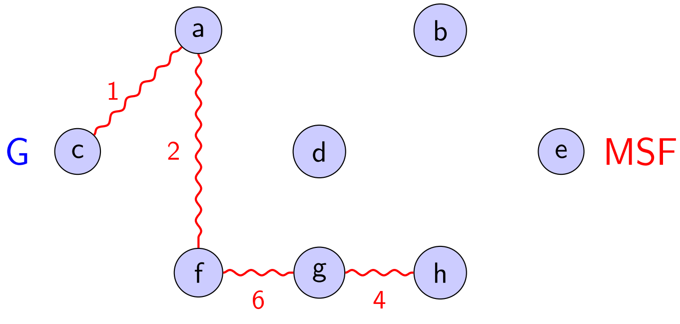
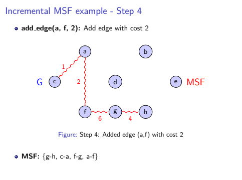


Figure: Step 4: Added edge (a,f) with cost 2

- **MSF:** {g-h, c-a, f-g, a-f}



1. Show fourth edge being added: (a,f) with cost 2
2. Still no cycle, so added to MSF directly
3. Current MSF: g-h, c-a, f-g, a-f with total cost 13
4. Link-cut tree operations: link(a,f) - $O(\log n)$ time
5. Now we have component: g,h,f,a,c - all vertices connected!
6. This shows how components merge as edges are added

Incremental MSF example - Step 5

- **add_edge(c, f, 5):** Add edge with cost 5

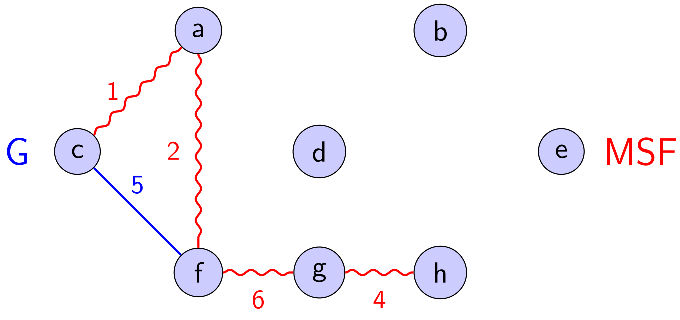
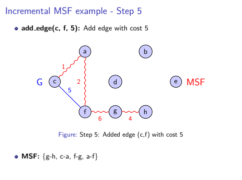


Figure: Step 5: Added edge (c,f) with cost 5

- **MSF:** {g-h, c-a, f-g, a-f}



1. Show fifth edge being added: c, f with cost 5
2. This creates a cycle! c-a-f-g-h-c forms a cycle
3. Link-cut tree operations: find_max(c,f) finds edge a, f with cost 2
4. Since new edge cost 5, which is grater than 2, it's not added to the MSF
5. Current MSF: g-h, c-a, f-g, a-f with total cost 13
6. This demonstrates the cycle-breaking optimization in Frederickson's algorithm
7. Key insight: we maintain optimality by replacing expensive edges with cheaper ones

Incremental MSF example - Step 6

- **add_edge(f, d, 7):** Add edge with cost 7

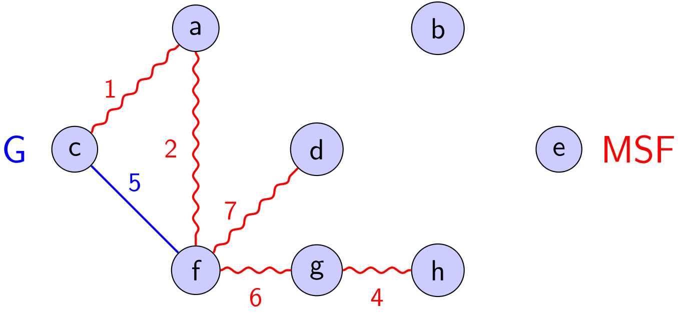
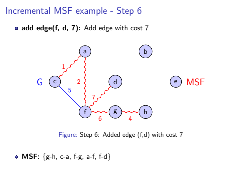


Figure: Step 6: Added edge (f,d) with cost 7

- **MSF:** {g-h, c-a, f-g, a-f, f-d}



1. Show sixth edge being added: f, d with cost 7
2. This creates a cycle! f-d-g-h-f forms a cycle
3. Link-cut tree operations: find_max(f,d) no cycle returned
4. Edge is added to the MSF
5. Current MSF: g-h, c-a, f-g, a-f, f-d with total cost 20

Incremental MSF example - Step 7

- **add_edge(a, d, 3):** Add edge with cost 3

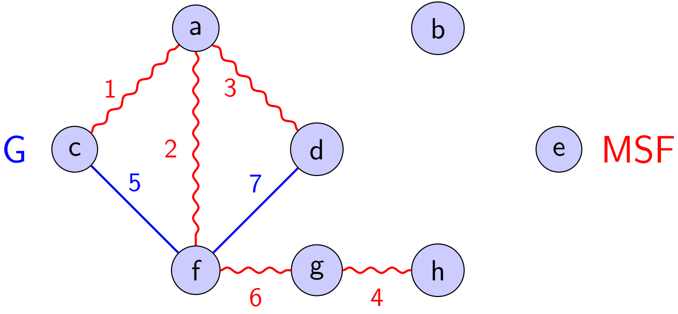
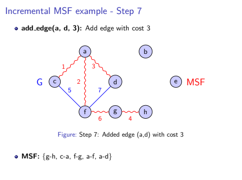


Figure: Step 7: Added edge (a,d) with cost 3

- **MSF:** {g-h, c-a, f-g, a-f, a-d}



1. Show seventh edge being added: a, d with cost 3
2. This creates a cycle! a-d-f-a forms a cycle
3. Link-cut tree operations: find_max(a,d finds edge f, d with cost 7
4. Since new edge cost 3 ¡ max cost 7, we replace (f,d) with (a,d)
5. Current MSF: g-h, c-a, f-g, a-f, a-d with total cost 16 *improved*!
6. This shows continued optimization as better edges are found
7. Key insight: the algorithm continuously improves the MSF as new edges arrive

Incremental MSF example - Step 8

- **add_edge(d, g, 2):** Add edge with cost 2

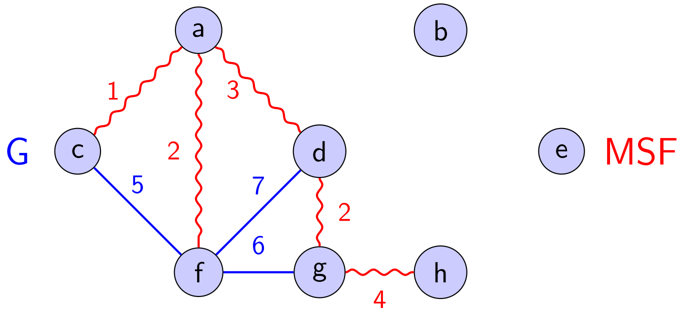
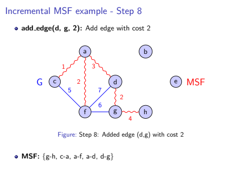


Figure: Step 8: Added edge (d,g) with cost 2

- **MSF:** {g-h, c-a, a-f, a-d, d-g}



1. Show eighth edge being added: d, g with cost 2
2. This creates a cycle! d-g-f-a-d forms a cycle
3. Link-cut tree operations: find_max(d,g) finds edge (f,g) with cost 6
4. Since new edge cost 2 ¡ max cost 6, we replace (f,g) with (d,g)
5. Current MSF: {g-h, c-a, a-f, a-d, d-g} with total cost 12 *improved*!
6. This shows the final optimization step
7. Key insight: the algorithm finds the optimal MSF through incremental improvements
8. Total cost reduced from 14 to 8 through smart edge replacements

Incremental MSF example - Final Result

- **Continue adding edges...**

Incremental MSF example - Final Result
- Continue adding edges...

1. If we continue adding edges and applying this algorithm
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3. Show final complete MSF with optimal cost = 12
4. Summarize the incremental process: started empty, added edges one by one
5. Transition to Frederickson's solution: $O(logn)$ amortized per edge addition
6. Key insight: link-cut trees enable efficient cycle detection and edge replacement
7. This sets up the retroactive version: what if we want to query MSF at any time t?
8. The challenge: maintain MSF not just for current state, but for any historical time
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Incremental MSF example - Final Result

- **Continue adding edges...**
- **Final MSF:** Minimum spanning forest with optimal cost

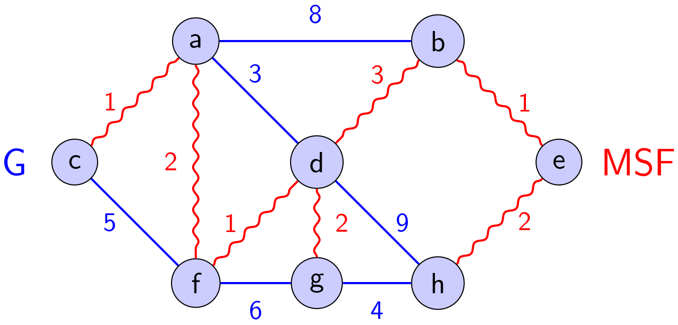
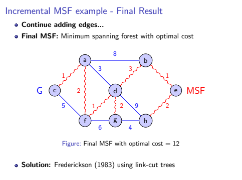


Figure: Final MSF with optimal cost = 12

- **Solution:** Frederickson (1983) using link-cut trees



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6. Show the three key operations: insert, remove, query at any time
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Any partially retroactive data structure can be transformed into a fully retroactive one with:

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- **Fixed m:** Must know sequence length beforehand
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6. Present our key insight: implement rebuilding process
7. Explain the challenge: how to rebuild without persistent structures
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- **Solution:** Reuse existing data structures during rebuilding

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Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
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1. Explain our key insight: reuse existing data structures
2. Show rebuilding moments: when m is a perfect square ($m = k^2$)
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4. Walk through the three-step strategy:
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9. Present the key lemma: every update in D_i is within first $(i+2)(k+1)$ updates
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11. Analyze time complexity: $O(mlogn)$ total, $O(\sqrt{m}logn)$ amortized
12. This sets up the detailed algorithm in the next slide
13. Key insight: we can reuse most of the work from previous structures
14. The offset (i+2) is crucial for correctness

Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)
- **Strategy:**
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- **Time per rebuilding:** $\mathcal{O}(m \log n)$
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Rebuilding algorithm

1. $D'_0 \leftarrow \textsc{newIncrementalMSF}()$
2. $D'_1 \leftarrow \textsc{newIncrementalMSF}()$
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ $\qquad \triangleright$ reuse existing
4. For $i = 1$ to $k+1$:
   - $p \leftarrow \textsc{kth}(S, i(k+1))$ $\qquad \triangleright$ $i(k+1)$th edge
   - $t'_i \leftarrow p.\text{time}$
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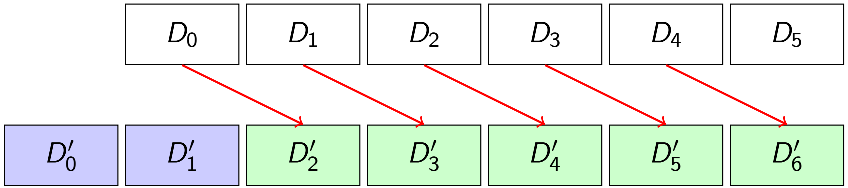
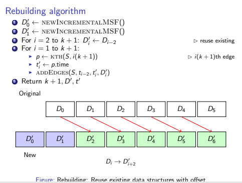


New

$D_i \rightarrow D'_{i+2}$



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Our contribution

- **General transformation:** Partial $\rightarrow$ Full retroactivity
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Semi-retroactive MSF implementation

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1. Invite questions from the audience
2. Be prepared to answer questions about:
3. * The rebuilding algorithm details
4. * Space vs time trade-offs
5. * Implementation challenges
6. * Comparison with persistent data structures
7. * Applications beyond MSF
8. Key points to emphasize if asked:
9. * Our approach is simpler to implement
10. * Same time complexity as Demaine et al.
11. * No persistent data structure requirement
12. * General applicability to any partially retroactive structure
13. Thank the audience for their attention