How to go from partial to full retroactivity in detail

Cristina Gomes Fernandes, Felipe Castro de Noronha

IME-USP – Brazil

LAGOS 25 – November 10-14, 2025

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1. Hello everyone. My name is Felipe Noronha, and today I'll be presenting our work with Professor Cristina Fernandes from IME-USP.
2. Our paper details a method for transforming partially retroactive data structures into fully retroactive ones.
3. This work is motivated by a practical limitation in the well-known 2007 transformation by Demaine, Iacono, and Langerman...
4. ...and it also builds upon a 2022 solution by Junior and Seabra.
5. Our key contribution is a method to achieve this transformation with the same time complexity, but *without* the need for complex persistent data structures.
6. To illustrate our approach, we'll use the minimum spanning forest problem as our main example. So, let's start by defining what that is.

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4. It will have 3 main properties: connected (path between any two vertices), acyclic (no cycles), contains exactly n-1 edges for n vertices
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6. Show visual example with graph G (blue edges) and spanning tree T (red wavy edges)
7. In the example: 8 vertices, so spanning tree has exactly 7 edges
8. Emphasize that spanning trees are not unique - there can be many valid spanning trees

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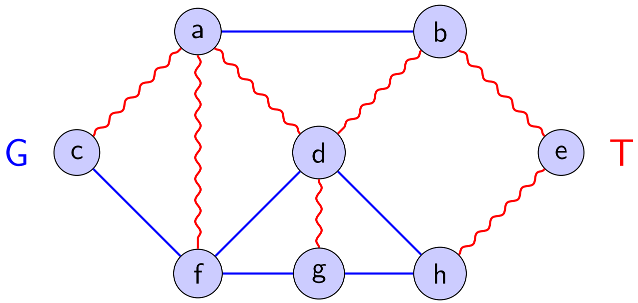
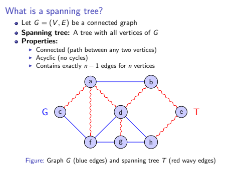


Figure: Graph G (blue edges) and spanning tree T (red wavy edges)



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1. Now, let's add weights or costs to the edges. In a weighted graph, a Minimum Spanning Tree, or MST, is a spanning tree that has the minimum possible total cost. It's an optimization problem.
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3. This concept generalizes to disconnected graphs as well. We call this a Minimum Spanning Forest, or MSF, which is simply the collection of MSTs for each connected component.
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5. In the visual example, you can see the same graph as before, but now with costs on the edges. The red wavy edges again show the tree, but this time, they've been chosen to be the MST.
6. If we sum the costs of the red edges, we get a total of 14. Any other spanning tree you could build for this graph would have a total cost greater than or equal to 14.
7. This idea of maintaining an optimal-cost forest is central to our problem. Specifically, how to maintain this optimality as the graph changes.

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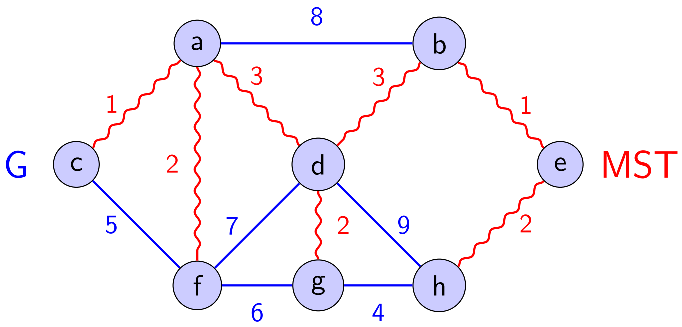
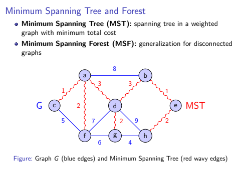


Figure: Graph G (blue edges) and Minimum Spanning Tree (red wavy edges)



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4. This problem is defined by two operations: *add_edge*, which inserts a new weighted edge, and *get_msf*, which returns the current minimum spanning forest.
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6. The solution to this was given by Frederickson in 1983. He used a dynamic data structure called link-cut trees which achieves a $O(\log n)$ amortized time per edge addition.
7. This incremental solution is the foundation for the *retroactive* version, which is what we're interested in.

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5. With this, we can construct a straightforward algorithm that supports adding a new edge (u, v, w):
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Incremental MSF example - Step 1

- **add_edge(g, h, 4):** Add edge with cost 4

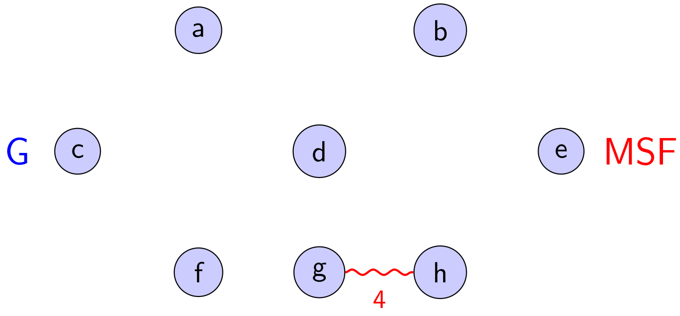
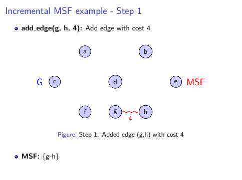


Figure: Step 1: Added edge (g,h) with cost 4

- **MSF:** {g-h}



1. Let's walk through a quick example. We start with an empty graph.
2. First, we add edge (g, h) with cost 4.
3. Are 'g' and 'h' connected? No. So, by step 2 of the algorithm, we simply add the edge to our MSF.
4. The MSF is now just {g-h}.

Incremental MSF example - Step 2

- **add_edge(c, a, 1):** Add edge with cost 1

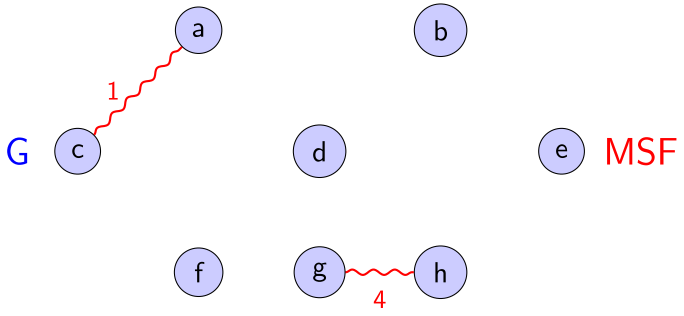
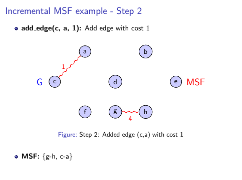


Figure: Step 2: Added edge (c,a) with cost 1

- **MSF:** {g-h, c-a}



1. Next, we add (c, a) with cost 1.
2. Again, are 'c' and 'a' connected? No. They are in a different component from 'g' and 'h'.
3. So, we add it directly. The MSF now has two components: {g-h} and {c-a}.

Incremental MSF example - Step 3

- **add_edge(f, g, 6):** Add edge with cost 6

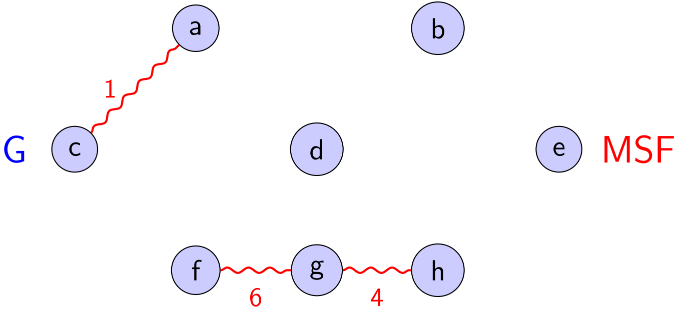
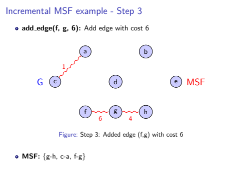


Figure: Step 3: Added edge (f,g) with cost 6

- **MSF:** {g-h, c-a, f-g}



1. Now, add (f, g) with cost 6.
2. Are 'f' and 'g' connected? No. 'f' is isolated, and 'g' is in the {g-h} component.
3. We link them. The MSF now contains {c-a} and {f-g-h}.

Incremental MSF example - Step 4

- **add_edge(a, f, 2):** Add edge with cost 2

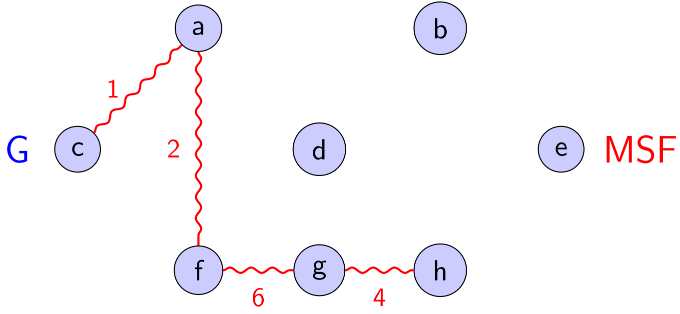
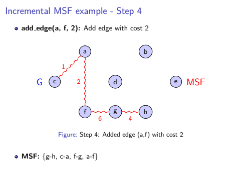


Figure: Step 4: Added edge (a,f) with cost 2

- **MSF:** {g-h, c-a, f-g, a-f}



1. Add (a, f) with cost 2.
2. Are 'a' and 'f' connected? No. 'a' is in the {c-a} component and 'f' is in the {f-g-h} component.
3. We link these two components. Our forest now becomes a single tree, and all vertices shown so far are connected.

Incremental MSF example - Step 5

- **add_edge(c, f, 5):** Add edge with cost 5

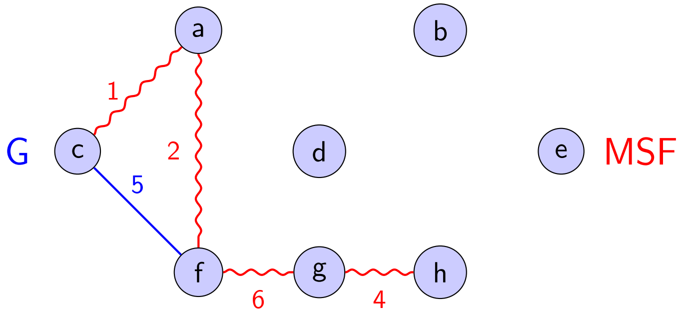
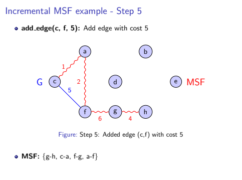


Figure: Step 5: Added edge (c,f) with cost 5

- **MSF:** {g-h, c-a, f-g, a-f}



1. Okay, now we add (c, f) with cost 5. This is our first interesting case.
2. Are 'c' and 'f' connected? Yes, they are. Adding this edge will create a cycle: c-a-f-c.
3. So, we go to step 3. We find the max-cost edge on the path c-a-f. The edges are (c,a) with cost 1 and (a,f) with cost 2. The max cost is 2.
4. Our new edge costs 5. Since 5 is *not* less than the max cost of 2, we *do not* add this edge. It's discarded.
5. The MSF remains unchanged.

Incremental MSF example - Step 6

- **add_edge(f, d, 7):** Add edge with cost 7

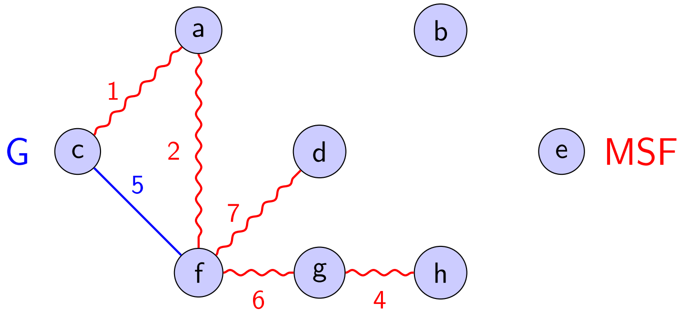
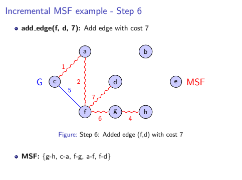


Figure: Step 6: Added edge (f,d) with cost 7

- **MSF:** {g-h, c-a, f-g, a-f, f-d}



1. Next, add (f, d) with cost 7.
2. Are 'f' and 'd' connected? No. 'f' is in the main tree, but 'd' is a new, isolated vertex.
3. Therefore, we simply add the edge. The MSF is updated.

Incremental MSF example - Step 7

- **add_edge(a, d, 3):** Add edge with cost 3

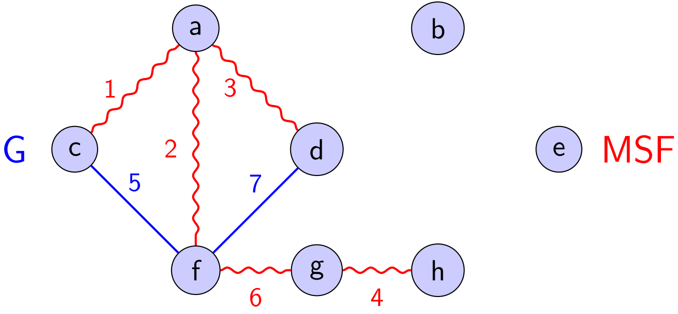
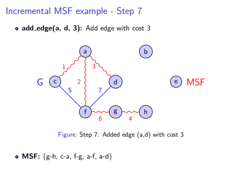


Figure: Step 7: Added edge (a,d) with cost 3

- **MSF:** {g-h, c-a, f-g, a-f, a-d}



1. Now, add (a, d) with cost 3.
2. Are 'a' and 'd' connected? Yes. This creates the cycle a-f-d-a.
3. We find the max-cost edge on the path a-f-d. The edges are (a,f) with cost 2 and (f,d) with cost 7. The max cost is 7.
4. Our new edge costs 3. Since 3 *is* less than 7, we swap them.
5. We ‘cut‘ the expensive edge (f,d) and ‘link‘ our new, cheaper edge (a,d).
6. The MSF is now {g-h, c-a, f-g, a-f, a-d} and its total cost has improved.

Incremental MSF example - Step 8

- **add_edge(d, g, 2):** Add edge with cost 2

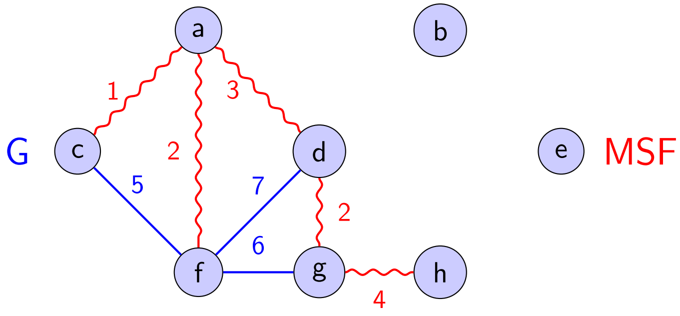
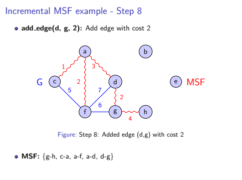


Figure: Step 8: Added edge (d,g) with cost 2

- **MSF:** {g-h, c-a, a-f, a-d, d-g}



1. Finally, let's add (d, g) with cost 2.
2. Are 'd' and 'g' connected? Yes. This creates the cycle d-a-f-g-d.
3. We find the max-cost edge on the path d-a-f-g. The edges are (d,a) cost 3, (a,f) cost 2, and (f,g) cost 6. The max cost is 6, from edge (f,g).
4. Our new edge costs 2. Since 2 *is* less than 6, we swap them.
5. We ‘cut‘ edge (f,g) and ‘link‘ our new edge (d,g).
6. The MSF is updated again, and the total cost is now 12.

Incremental MSF example - Final Result

- **Continue adding edges...**

Incremental MSF example - Final Result
- Continue adding edges...

1. If we continue this process, adding all the remaining edges from our original graph...
2. ———- SKIP SLIDE ———-
3. ...we would eventually arrive at the final, optimal Minimum Spanning Tree. The one shown here, for example, has a total cost of 12.
4. So, to summarize, Frederickson's solution gives us an efficient $O(\log n)$ amortized time per *incremental* update.
5. It perfectly handles cycle detection and edge replacement to maintain optimality.
6. But this only answers queries about the *present*. What if we want to ask: "What did the MSF look like 10 updates ago?"
7. This is the core question of retroactivity. How do we efficiently query the past?

Incremental MSF example - Final Result

- **Continue adding edges...**
- **Final MSF:** Minimum spanning forest with optimal cost

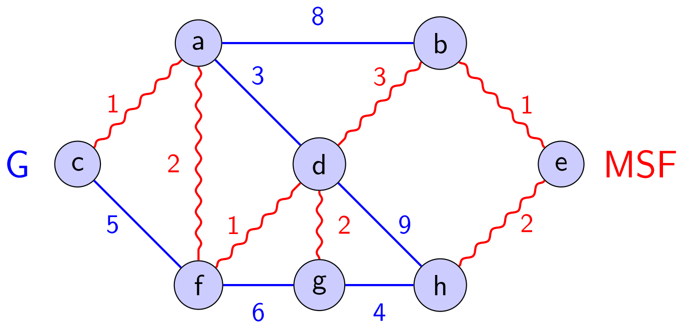
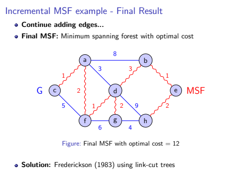


Figure: Final MSF with optimal cost = 12

- **Solution:** Frederickson (1983) using link-cut trees



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1. There are a few different "flavors" of retroactivity. The most powerful is Fully Retroactive, which supports all the operations we just saw: insert, remove, and query, all at any time t.
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3. Partially Retroactive is more limited. You can still insert or remove updates anywhere in the timeline, but you can only query the state of the structure at the *current* time, "now". This is a key limitation.
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5. And finally, there's Semi-Retroactive, which is a bit of a mix. You can query at any time t and insert updates at any time, but you are *not allowed* to remove updates.
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Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

- $\mathcal{O}(\sqrt{m})$ *slowdown per operation*
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2. The cost is an $O(\sqrt{m})$ slowdown per operation and $O(m)$ space, where m is the number of updates.
3. But there's a catch: this transformation *requires* a persistent version of the data structure. This is the key limitation we want to address.
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5. So, how does it work? The idea is to break the m updates into $\sqrt{m}$ blocks, each of size $\sqrt{m}$.
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- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - add_edge(u, v, w, t): add edge at time t
  - get_msf(t): get MSF at time t

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1. Our starting point was the 2022 work by Junior and Seabra on a semi-retroactive MSF.
2. Remember, "semi-retroactive" means they can add edges at any time t in the past, and query the MSF at any time t, but they cannot *remove* edges.
3. ——— SKIP SLIDE ———
4. They also use a square-root decomposition. They maintain $\sqrt{m}$ checkpoints, t_i, spaced $\sqrt{m}$ updates apart.
5. ——— SKIP SLIDE ———
6. They use a set of data structures, D_i, where each D_i stores the incremental MSF containing all edges added *before* its checkpoint time t_i.
7. This approach gives them a final time complexity of $O(\sqrt{m}\log n)$ per operation.
8. However, their solution has some significant practical limitations...

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Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - add_edge(u, v, w, t): add edge at time t
  - get_msf(t): get MSF at time t
- **Implementation:** Square-root decomposition
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Problems with their approach

- **Fixed m:** Must know sequence length beforehand
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6. Our goal was to remove these limitations, especially the need to know m in advance, while keeping the same efficiency.
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Rebuilding algorithm

1. $D'_0 \leftarrow \textsc{newIncrementalMSF}()$
2. $D'_1 \leftarrow \textsc{newIncrementalMSF}()$
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ $\quad\triangleright$ reuse existing
4. For $i = 1$ to $k+1$:
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5. Return $k+1, D', t'$

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1. This slide shows the algorithm in more detail.
2. Lines 1 and 2 create the two new empty structures, D'_0 and D'_1.
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6. The diagram at the bottom visualizes this reuse. The new D'_0 and D'_1 are built from scratch, but all the others, D'_2 through D'_{k+1}, are just the old D_0 through D_{k-1}, shifted over and updated.
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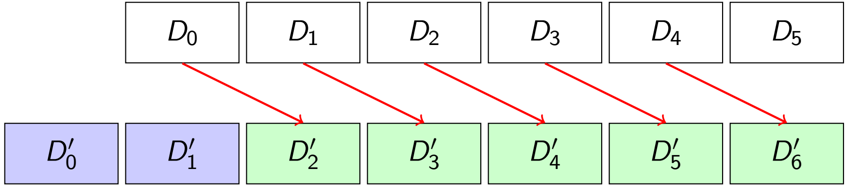
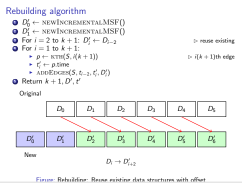


New

$D_i \rightarrow D'_{i+2}$



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Our contribution

- **General transformation:** Partial $\rightarrow$ Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
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1. So, to summarize our contributions:
2. We've developed a general transformation to take a partially retroactive data structure and make it fully retroactive.
3. Crucially, our method *does not require persistent data structures*.
4. We match the $O(\sqrt{m})$ slowdown per operation from the Demaine et al. paper...
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7. Applying this to our test case, we get a semi-retroactive MSF implementation.
8. It supports adding edges and querying the MSF at any time t in $O(\sqrt{m}\log n)$ amortized time.
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Semi-retroactive MSF implementation

- **Operations:** $\texttt{add\_edge}(u, v, w, t)$, $\texttt{get\_msf}(t)$
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1. Finally, while we focused on the semi-retroactive MSF, our transformation is general.
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4. To do this, we just adapt the rebuilding trigger. Instead of rebuilding only when m grows, we rebuild whenever the *number of blocks* (the floor of $\sqrt{m}$) changes, whether from insertions or removals.
5. This just means rebuilding happens a bit more frequently, but the amortized cost remains the same.
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  - Adapt rebuilding trigger: when $|\lfloor\sqrt{m'}\rfloor - \lfloor\sqrt{m}\rfloor| \leq 1$
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1. Invite questions from the audience
2. Be prepared to answer questions about:
3. * The rebuilding algorithm details
4. * Space vs time trade-offs
5. * Implementation challenges
6. * Comparison with persistent data structures
7. * Applications beyond MSF
8. Key points to emphasize if asked:
9. * Our approach is simpler to implement
10. * Same time complexity as Demaine et al.
11. * No persistent data structure requirement
12. * General applicability to any partially retroactive structure
13. Thank the audience for their attention