How to go from partial to full retroactivity in detail

Cristina Gomes Fernandes, Felipe Castro de Noronha

IME-USP – Brazil

LAGOS 25 – November 10-14, 2025

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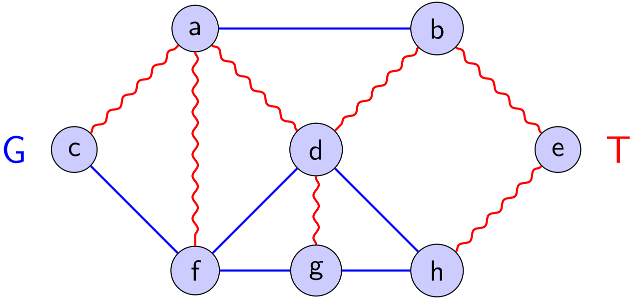

Figure: Graph G (blue edges) and spanning tree T (red wavy edges)

Minimum Spanning Tree and Forest

- **Minimum Spanning Tree (MST):** spanning tree in a weighted graph with minimum total cost

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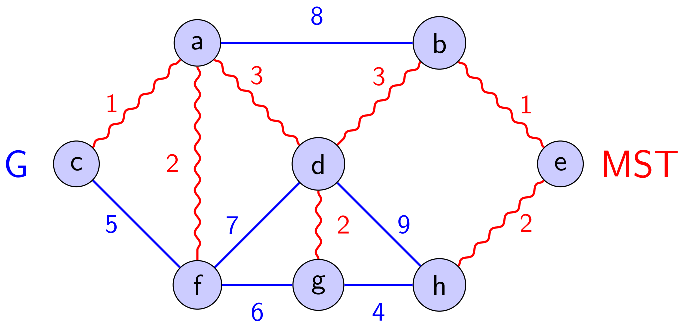

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  - `find_max`(u, v): $\mathcal{O}(\log n)$ amortized
  - `link`(u, v, w): $\mathcal{O}(\log n)$ amortized
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- **Algorithm for adding edge (u, v, w):**
  1. Check if u and v are in same component
  2. If not: add edge to forest
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- **Total cost:** Amortized $\mathcal{O}(\log n)$ per edge addition

Incremental MSF example - Step 1

- **add_edge(g, h, 4):** Add edge with cost 4

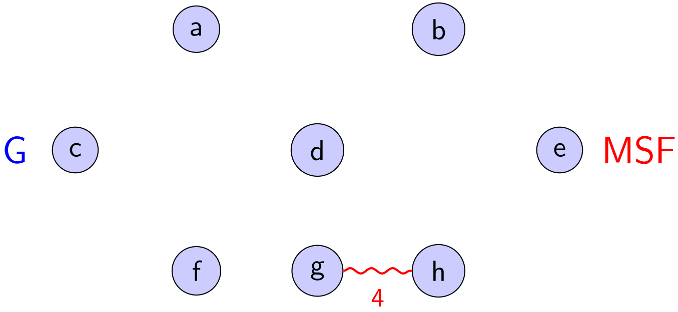

Figure: Step 1: Added edge (g,h) with cost 4

- **MSF:** {g-h}

Incremental MSF example - Step 2

- **add_edge(c, a, 1):** Add edge with cost 1

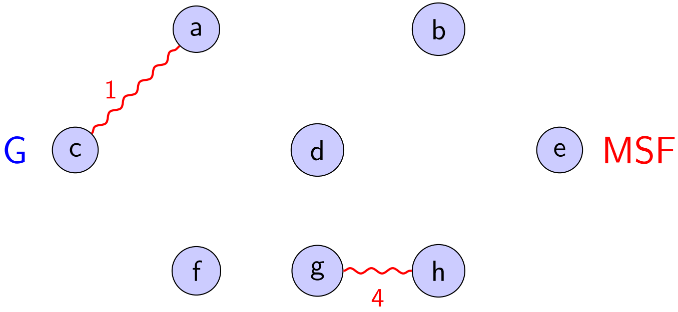

Figure: Step 2: Added edge (c,a) with cost 1

- **MSF:** {g-h, c-a}

Incremental MSF example - Step 3

- **add_edge(f, g, 6):** Add edge with cost 6

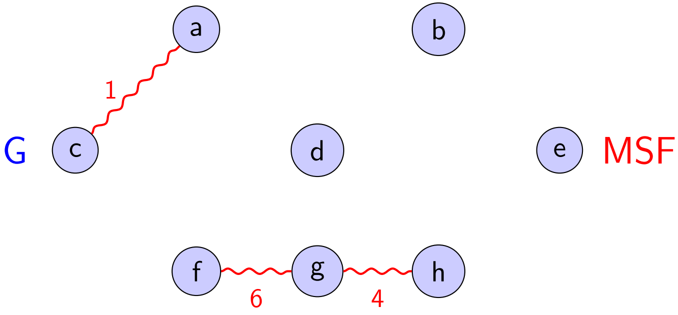

Figure: Step 3: Added edge (f,g) with cost 6

- **MSF:** {g-h, c-a, f-g}

Incremental MSF example - Step 4

- **add_edge(a, f, 2):** Add edge with cost 2

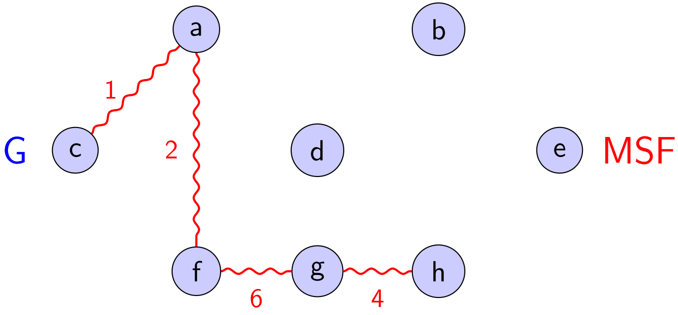

Figure: Step 4: Added edge (a,f) with cost 2

- **MSF:** {g-h, c-a, f-g, a-f}

Incremental MSF example - Step 5

- **add_edge(c, f, 5):** Add edge with cost 5

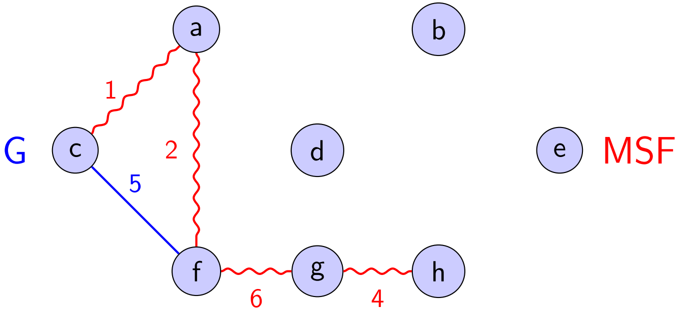

Figure: Step 5: Added edge (c,f) with cost 5

- **MSF:** {g-h, c-a, f-g, a-f}

Incremental MSF example - Step 6

- **add_edge(f, d, 7):** Add edge with cost 7

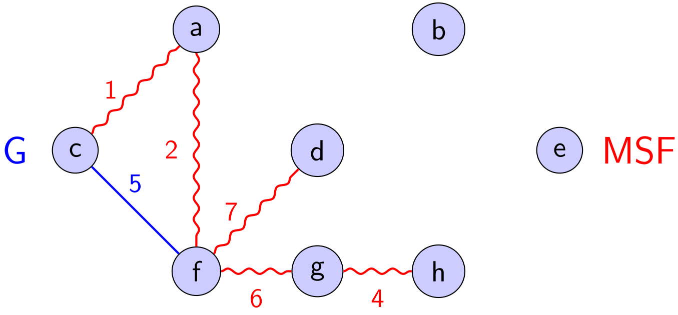

Figure: Step 6: Added edge (f,d) with cost 7

- **MSF:** {g-h, c-a, f-g, a-f, f-d}

Incremental MSF example - Step 7

- **add_edge(a, d, 3):** Add edge with cost 3

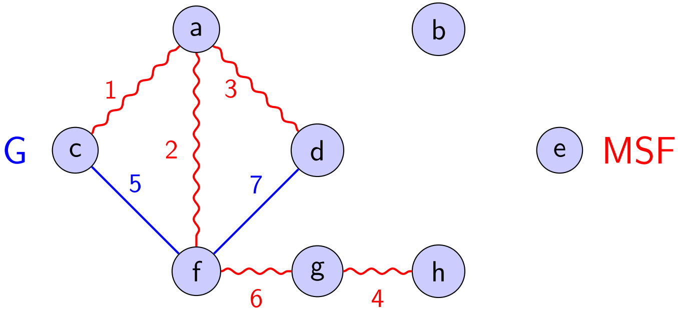

Figure: Step 7: Added edge (a,d) with cost 3

- **MSF:** {g-h, c-a, f-g, a-f, a-d}

Incremental MSF example - Step 8

- **add_edge(d, g, 2):** Add edge with cost 2

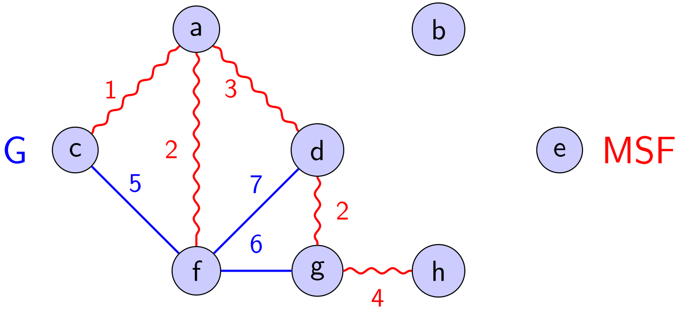

Figure: Step 8: Added edge (d,g) with cost 2

- **MSF:** {g-h, c-a, a-f, a-d, d-g}

Incremental MSF example - Final Result

- **Continue adding edges...**

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- **Final MSF:** Minimum spanning forest with optimal cost

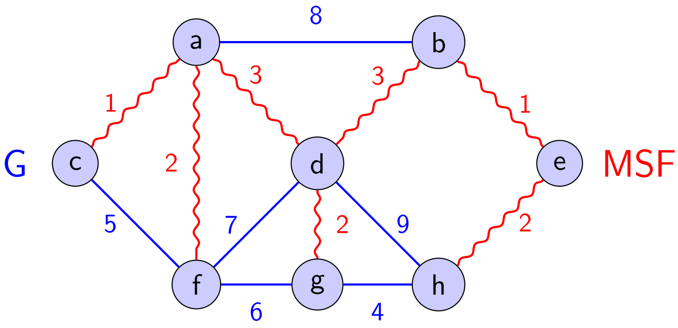

Figure: Final MSF with optimal cost $= 14$

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- The order of updates affects the state of the data structure
- **Retroactivity:** Manipulate the sequence of updates
- **Operations:**
  - Insert update at time t (possibly in the past)
  - Remove update at time t
  - Query at time t (not just present)

Partial vs Full retroactivity

Fully Retroactive

- Queries at **any** time t
- Insert/remove updates at any time

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Semi-Retroactive

- Queries at **any** time t
- Insert updates at any time
- **No removal** of updates

The challenge

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- **Solution approach:** Square-root decomposition
- **Key insight:** Keep checkpoints with data structure states
- **Implementation:** Demaine, Iacono & Langerman (2007)

Demaine, Iacono & Langerman's solution

Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

- $\mathcal{O}(\sqrt{m})$ *slowdown per operation*
- $\mathcal{O}(m)$ *space usage*
- **Requirement:** *Need persistent version of the data structure*

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- **Key idea:** Square-root decomposition
- Keep $\sqrt{m}$ checkpoints with data structure states
- **Query at time t:**
  1. Find closest checkpoint before t
  2. Apply updates from checkpoint to t
  3. Answer query, then rollback

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Our contribution

Simple rebuilding strategy without persistent data structures

- Same time complexity: $\mathcal{O}(\sqrt{m})$ per operation
- Space usage: $\Theta(m\sqrt{m})$

Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - `add_edge`(u, v, w, t): add edge at time t
  - `get_msf`(t): get MSF at time t

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- **Checkpoints:** $t_i = i\sqrt{m}$ for $i = 1, \dots, \sqrt{m}$
- **Data structures:** D_i contains edges before time t_i
- **Time:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

Limitations → Key Insight

Problems with the Existing Static Approach

- **Fixed** m**:** Requires knowing the maximum sequence length ($\mathbf{m}$) beforehand. Meaning that it cannot handle arbitrary growth or dynamic operation counts.

Our Dynamic Goal and Solution

Goal: Remove the **Fixed m** dependency while preserving time complexity.

- **Key Insight:** Introduce a **dynamic rebuilding process** to handle arbitrary growth.
- **Challenge:** Rebuilding $\sqrt{\mathbf{m}}$ checkpoints must be fast, avoiding complex persistent data structures.
- **Solution:** **Reuse** the existing data structures to efficiently reconstruct new checkpoints.

Our solution - Rebuilding strategy

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- **Time per rebuilding:** $\mathcal{O}(m \log n)$
- **Amortized cost:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

Rebuilding algorithm

1. $D'_0 \leftarrow$ NEWINCREMENTALMSF()
2. $D'_1 \leftarrow$ NEWINCREMENTALMSF()
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ ▷ reuse existing
4. For $i = 1$ to $k+1$:
   - $p \leftarrow$ KTH$(S, i(k+1))$ ▷ $i(k+1)$th edge
   - $t'_i \leftarrow p$.time
   - ADDEDGES(S, t_{i-2}, t'_i, D'_i)
5. Return $k+1, D', t'$

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4. For $i = 1$ to $k+1$:
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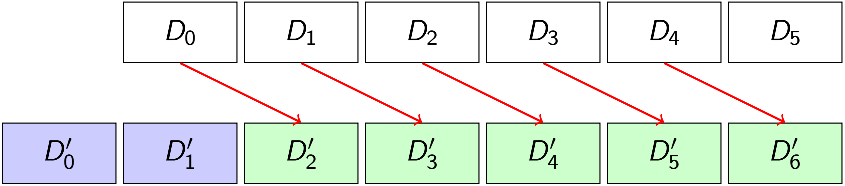
Original



New

$$D_i \rightarrow D'_{i+2}$$

Results

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- **General transformation:** Partial $\rightarrow$ Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
- **Space trade-off:** $\Theta(m\sqrt{m})$ vs $\mathcal{O}(m)$

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Semi-retroactive MSF implementation

- **Operations:** `add_edge`(u, v, w, t), `get_msf`(t)
- **Time:** $\mathcal{O}(\sqrt{m}\log n)$ per operation
- **Space:** $\Theta(m\sqrt{m})$
- **No fixed** m **or time range restrictions**

Thank you!

Questions?