

## How to go from partial to full retroactivity in detail

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IME-USP – Brazil

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1. Hello everyone. My name is Felipe Noronha, and today I'll be presenting the work done by Professor Cristina Fernandes and I at IME-USP.
2. Our paper details a method for transforming partially retroactive data structures into fully retroactive ones.
3. This work is motivated by a practical limitation in the well-known 2007 transformation by Demaine, Iacono, and Langerman and it also builds upon a 2022 solution by Junior and Seabra.
4. Our key contribution is a method to achieve this transformation with the same time complexity, but *\*without\** the need for complex persistent data structures.
5. To illustrate our approach, we'll use the minimum spanning forest problem as our main example. So, let's start by defining what that is.

└─What is a spanning tree?

# What is a spanning tree?

- Let  $G = (V, E)$  be a connected graph
- **Spanning tree:** A tree with all vertices of  $G$

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2. A spanning tree will be a tree will all the vertices of  $G$
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4. It will have 3 main properties: it is connected (path between any two vertices), acyclic (no cycles), contains exactly  $n-1$  edges for  $n$  vertices
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6. Show visual example with graph  $G$  (blue edges) and spanning tree  $T$  (red wavy edges)
7. In the example: 8 vertices, so spanning tree has exactly 7 edges
8. Emphasize that spanning trees are not unique - there can be many valid spanning trees

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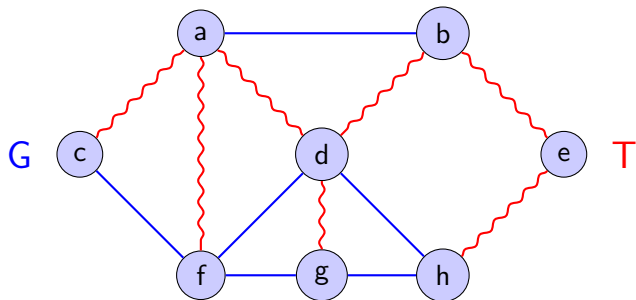
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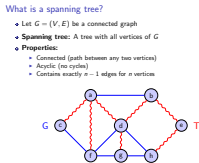
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## Minimum Spanning Tree and Forest

- **Minimum Spanning Tree (MST)**: spanning tree in a weighted graph with minimum total cost

1. Now, let's add weights or costs to the edges. In a weighted graph, a Minimum Spanning Tree, or MST, is a spanning tree that has the minimum possible total cost. It's an optimization problem.
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3. This concept generalizes to disconnected graphs as well. We call this a Minimum Spanning Forest, or MSF, which is simply the collection of MSTs for each connected component.
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5. In the visual example, you can see the same graph as before, but now with costs on the edges. The red edges again show the tree, but this time, they've been chosen to be the MST.
6. If we sum the costs of the red edges, we get a total of 14. Any other spanning tree you could build for this graph would have a total cost greater than or equal to 14.
7. This idea of maintaining an optimal-cost forest is central to our problem. Specifically, how to maintain this optimality as the graph changes.

# Minimum Spanning Tree and Forest

- **Minimum Spanning Tree (MST):** spanning tree in a weighted graph with minimum total cost
- **Minimum Spanning Forest (MSF):** generalization for disconnected graphs

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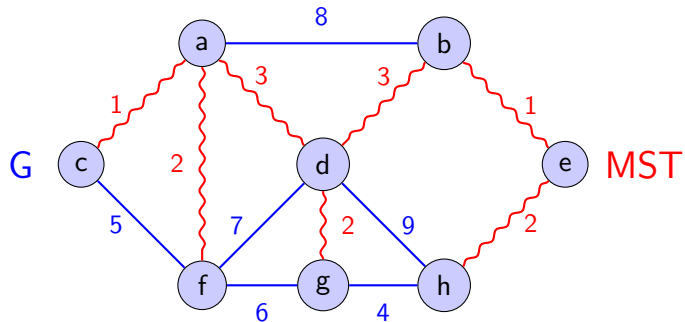
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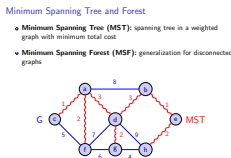
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4. This problem is defined by two operations: *add\_edge*, which inserts a new weighted edge, and *get\_msf*, which returns the current minimum spanning forest.
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6. The solution to this was given by Frederickson in 1983. He used a dynamic data structure called link-cut trees which achieves a  $O(\log n)$  amortized time per edge addition, where  $n$  is the number of vertices.



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- **Key insight:** Use link-cut trees to maintain MSF dynamically

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4. With this, we can construct a straightforward algorithm that supports adding a new edge  $(u, v, w)$ :
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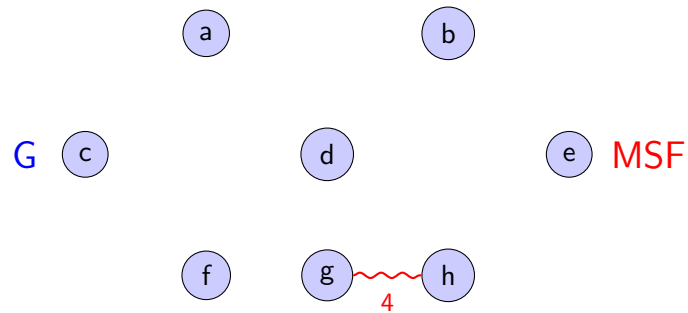
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# Incremental MSF example - Step 1

- **add\_edge(g, h, 4):** Add edge (g, h) with cost 4

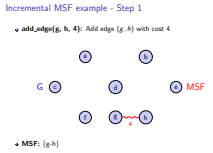


- **MSF:** {g-h}

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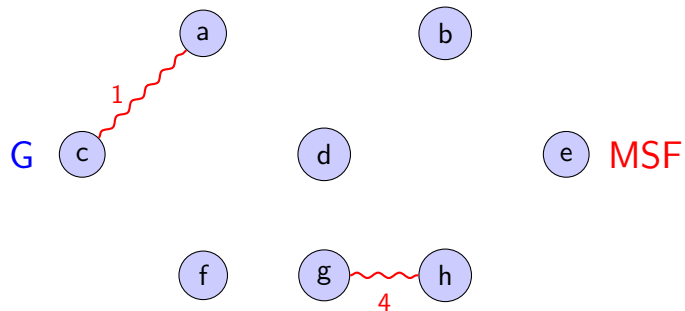
### Incremental MSF example - Step 1



1. Let's walk through a quick example. We start with an empty graph.
2. First, we add edge (g, h) with cost 4.
3. Are 'g' and 'h' connected? No. So, by step 2 of the algorithm, we simply add the edge to our MSF.
4. The MSF is now just {g-h}.

## Incremental MSF example - Step 2

- **add\_edge(c, a, 1):** Add edge with cost 1

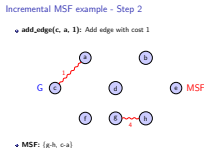


- **MSF:**  $\{g-h, c-a\}$

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└ Incremental MSF example - Step 2

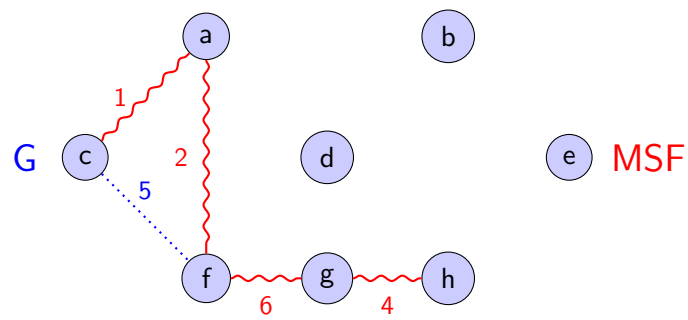


1. Next, we add  $(c, a)$  with cost 1.
2. Again, are 'c' and 'a' connected? No. They are in a different component from 'g' and 'h'.
3. So, we add it directly. The MSF now has two components:  $\{g-h\}$  and  $\{c-a\}$ .



# Incremental MSF example - Step 3 (Cycle Check)

- **add\_edge(c, f, 5):** Add edge with cost 5

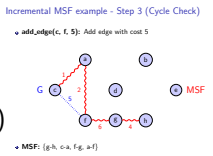


- **MSF:** {g-h, c-a, f-g, a-f}

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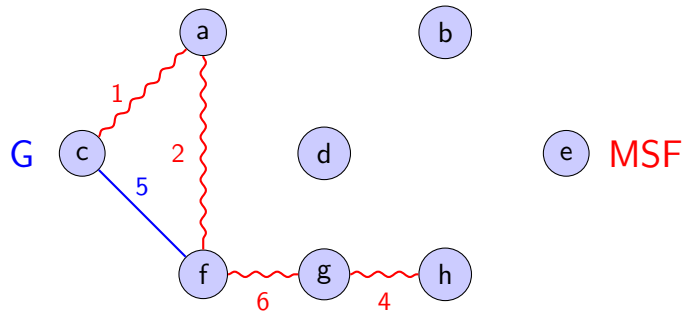
## Incremental MSF example - Step 3 (Cycle Check)



1. After fast-forwarding through a few steps (adding f-g and a-f), our forest is more connected.
2. Now, we add (c, f) with cost 5. This is our first interesting case.
3. Are 'c' and 'f' connected? Yes, they are. Adding this edge will create a cycle: c-a-f-c.
4. So, we go to step 3. We find the max-cost edge on the path c-a-f. The edges are (c,a) with cost 1 and (a,f) with cost 2. The max cost is 2.

## Incremental MSF example - Step 3 (Result)

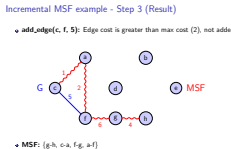
- **add\_edge(c, f, 5):** Edge cost is greater than max cost (2), not added



- **MSF:**  $\{g-h, c-a, f-g, a-f\}$

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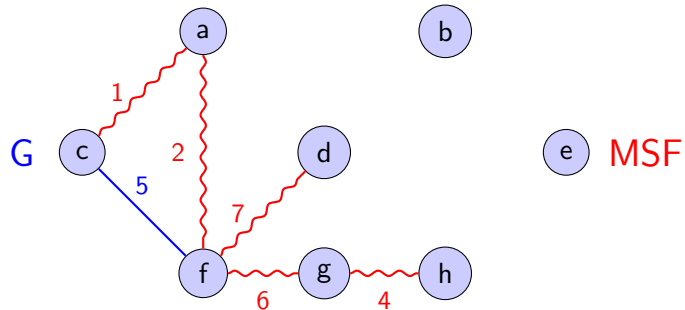
└ Incremental MSF example - Step 3 (Result)



1. Our new edge costs 5. Since 5 is *\*not\** less than the max cost of 2, we *\*do not\** add this edge. It's discarded.
2. The MSF remains unchanged.

## Incremental MSF example - Step 4

- **add\_edge(f, d, 7):** Add edge with cost 7

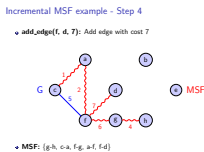


- **MSF:**  $\{g-h, c-a, f-g, a-f, f-d\}$

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Partial to full retroactivity

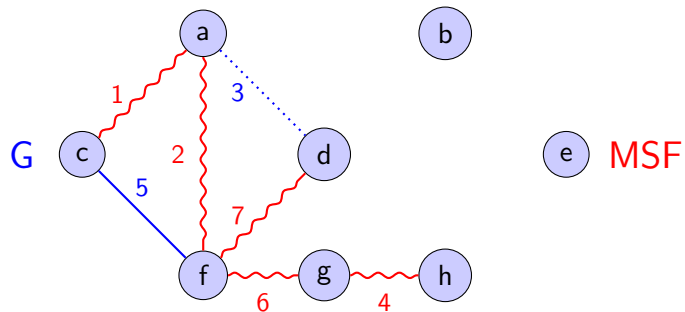
└ Incremental MSF example - Step 4



1. Next, add  $(f, d)$  with cost 7.
2. Are 'f' and 'd' connected? No. 'f' is in the main tree, but 'd' is a new, isolated vertex.
3. Therefore, we simply add the edge. The MSF is updated.

## Incremental MSF example - Step 5 (Cycle Check)

- **add\_edge(a, d, 3):** Add edge with cost 3

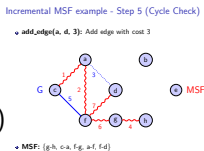


- **MSF:** {g-h, c-a, f-g, a-f, f-d}

2025-10-30

Partial to full retroactivity

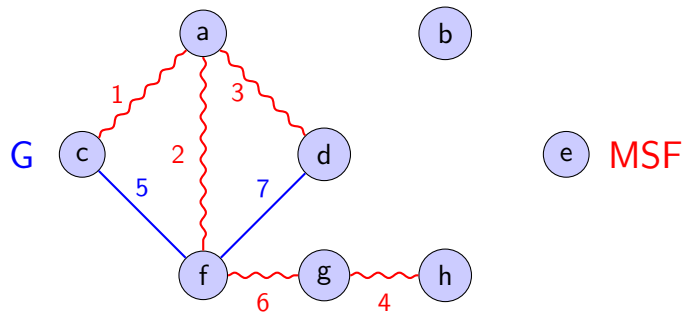
└ Incremental MSF example - Step 5 (Cycle Check)



1. Now, add (a, d) with cost 3.
2. Are 'a' and 'd' connected? Yes. This creates the cycle a-f-d-a.
3. We find the max-cost edge on the path a-f-d. The edges are (a,f) with cost 2 and (f,d) with cost 7. The max cost is 7.

## Incremental MSF example - Step 5 (Result)

- **add\_edge(a, d, 3):** Cost 3 ; max cost 7, swap edges

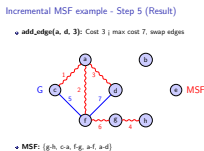


- **MSF:**  $\{g-h, c-a, f-g, a-f, a-d\}$

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Partial to full retroactivity

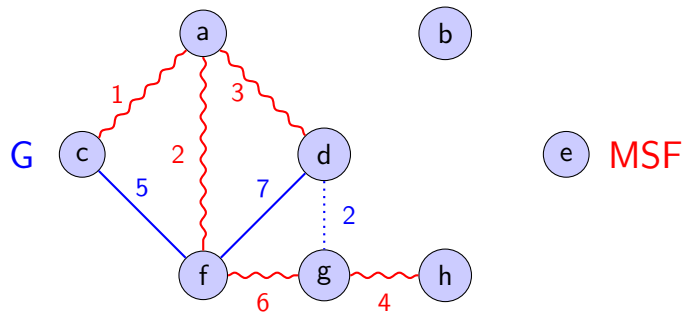
└ Incremental MSF example - Step 5 (Result)



1. Our new edge costs 3. Since 3 \*is\* less than 7, we swap them.
2. We 'cut' the expensive edge  $(f,d)$  and 'link' our new, cheaper edge  $(a,d)$ .
3. The MSF is now  $\{g-h, c-a, f-g, a-f, a-d\}$  and its total cost has improved.

## Incremental MSF example - Step 6 (Cycle Check)

- **add\_edge(d, g, 2):** Add edge with cost 2

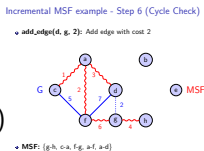


- **MSF:** {g-h, c-a, f-g, a-f, a-d}

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Partial to full retroactivity

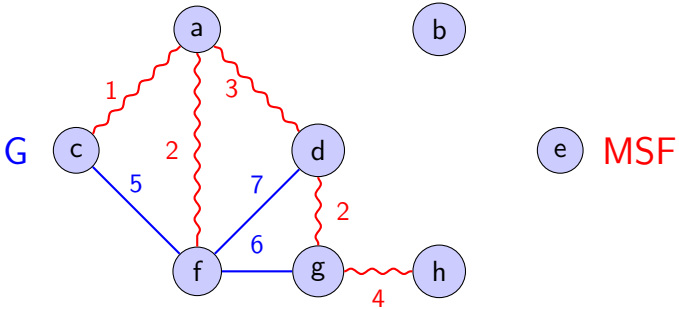
└ Incremental MSF example - Step 6 (Cycle Check)



1. Finally, let's add (d, g) with cost 2.
2. Are 'd' and 'g' connected? Yes. This creates the cycle d-a-f-g-d.
3. We find the max-cost edge on the path d-a-f-g. The edges are (d,a) cost 3, (a,f) cost 2, and (f,g) cost 6. The max cost is 6, from edge (f,g).

# Incremental MSF example - Step 6 (Result)

- **add\_edge(d, g, 2):** Cost 2 ; max cost 6, swap edges

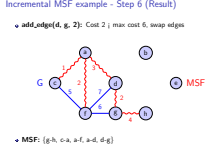


- **MSF:** {g-h, c-a, a-f, a-d, d-g}

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## Partial to full retroactivity

### Incremental MSF example - Step 6 (Result)



1. Our new edge costs 2. Since 2 \*is\* less than 6, we swap them.
2. We 'cut' edge (f,g) and 'link' our new edge (d,g).
3. The MSF is updated again, and the total cost is now 12.

# Incremental MSF example - Step 7 (Final Result)

- **Continue adding edges...**

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Partial to full retroactivity

Incremental MSF example - Step 7 (Final Result)  
• Continue adding edges...

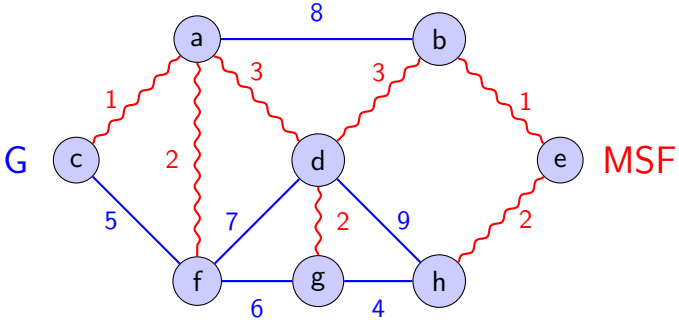
└ Incremental MSF example - Step 7 (Final Result)

1. If we continue this process, adding all the remaining edges from our original graph...
2. ——— SKIP SLIDE ———
3. ...we would eventually arrive at the final, optimal Minimum Spanning Tree. The one shown here, for example, has a total cost of 14.
4. But this only answers queries about the \*present\*. What if we want to ask: "What did the MSF look like 10 updates ago?"
5. This is the core question of retroactivity. How do we efficiently query the past?



# Incremental MSF example - Step 7 (Final Result)

- Continue adding edges...
- Final MSF:** Minimum spanning forest with optimal cost

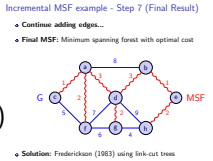


- Solution:** Frederickson (1983) using link-cut trees

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## Partial to full retroactivity

Incremental MSF example - Step 7 (Final Result)



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- SKIP SLIDE ——
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- But this only answers queries about the \*present\*. What if we want to ask: "What did the MSF look like 10 updates ago?"
- This is the core question of retroactivity. How do we efficiently query the past?

# What is retroactivity?

- **Problem:** Data structures usually support updates and queries
- The order of updates affects the state of the data structure

## Partial to full retroactivity

└─What is retroactivity?

1. In a normal data structure, the order of updates is important. Most of the time, the state of the structure, and thus the answers to queries, depends on this sequence.
2. This means we usually don't have a good way to go back and correct mistakes or insert operations we forgot.
3. ——— SKIP SLIDE ———
4. That's where retroactivity comes in. A retroactive data structure allows us to manipulate this sequence of updates.
5. ——— SKIP SLIDE ———
6. Specifically, it adds operations to: Insert a new update at some time  $t^*$  in the past\*...
7. ...Remove an update that \*already happened\* at time  $t$ ....
8. ...and, most importantly, Query the state of the structure at \*any\* time  $t$ , not just the present.
9. The key challenge is how to do this efficiently, maintaining the state for every possible time.

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- **Retroactivity:** Manipulate the sequence of updates
- **Operations:**
  - ▶ Insert update at time  $t$  (possibly in the past)
  - ▶ Remove update at time  $t$
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# Partial vs Full retroactivity

## Fully Retroactive

- Queries at **any** time  $t$
- Insert/remove updates at any time

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## Partial to full retroactivity

### └ Partial vs Full retroactivity

Partial vs Full retroactivity

Fully Retroactive

- Queries at any time  $t$
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1. There are a few different "flavors" of retroactivity. The most powerful is Fully Retroactivity, which supports all the operations we just saw: insert, remove, and query, all at any time  $t$ .
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3. Partially Retroactive is more limited. You can still insert or remove updates anywhere in the timeline, but you can only query the state of the structure at the *\*current\** time, "now". This is a key limitation.
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5. And finally, there's Semi-Retroactive, which is a bit of a mix. You can query at any time  $t$  and insert updates at any time, but you are *\*not allowed\** to remove updates.
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# Partial vs Full retroactivity

## Fully Retroactive

- Queries at **any** time  $t$
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- Queries only on **current** state
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# Partial vs Full retroactivity

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- Queries at **any** time  $t$
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## Partial to full retroactivity

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# The challenge

## Challenge

How to transform partial  $\rightarrow$  full retroactivity?

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## Partial to full retroactivity

### └ The challenge

1. ...which is this: How can we transform a simple partially retroactive structure into a fully retroactive one?
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5. A general solution for this was proposed by Demaine, Iacono, and Langerman in 2007.
6. Their approach uses a classic technique called square-root decomposition.
7. Let's see how that works.



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How to transform partial  $\rightarrow$  full retroactivity?

- **Problem:** Need to support queries at any time  $t$
- **Solution approach:** Square-root decomposition

2025-10-30

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- **Implementation:** Demaine, Iacono & Langerman (2007)

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- $O(\sqrt{m})$  slowdown per operation
  - $O(m)$  space usage
  - **Requirement:** Need persistent version of the data structure
- Where  $m$  is the number of updates.

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## Partial to full retroactivity

└ Demaine, Iacono &amp; Langerman's solution

## Demaine, Iacono &amp; Langerman's solution

## Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

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Where  $m$  is the number of updates.

1. Their paper presented this theorem: any partially retroactive data structure can be made fully retroactive.
2. The cost is an  $O(\sqrt{m})$  slowdown per operation and  $O(m)$  space, where  $m$  is the number of updates.
3. But there's a catch: this transformation *\*requires\** a persistent version of the data structure.
4. ——— SKIP SLIDE ———
5. So, how does it work? The idea is to break the  $m$  updates into  $\sqrt{m}$  blocks, each of size  $\sqrt{m}$ .
6. At the beginning of each block, we store a "checkpoint" of the data structure's state.
7. ——— SKIP SLIDE ———
8. Now, to query at some time  $t$ :
9. First, we find the closest checkpoint *\*before\**  $t$ . We load this saved state.
10. Then, we "roll forward" by applying all the updates between that checkpoint and time  $t$ . There are at most  $\sqrt{m}$  of them.
11. We answer the query, and then we "roll back" the changes to restore the checkpoint, which is where persistence comes in handy.

- $O(\sqrt{m})$  slowdown per operation
- $O(m)$  space usage
- **Requirement:** Need persistent version of the data structure

- **Key idea:** Square-root decomposition
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## Partial to full retroactivity

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- **Query at time  $t$ :**
  - 1 Find closest checkpoint before  $t$
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## Partial to full retroactivity

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# The space problem

- **Naive approach:** Keep  $\sqrt{m}$  independent copies
- Space usage:  $\Theta(m\sqrt{m})$

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## Partial to full retroactivity

└ The space problem

The space problem

- Naive approach: Keep  $\sqrt{m}$  independent copies
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1. ...and that leads to the space problem. If we're naive and just store  $\sqrt{m}$  \*independent copies\* of the data structure, one for each checkpoint...
2. ...and each copy can have up to  $m$  updates, our space usage explodes to  $\Theta(m\sqrt{m})$ .
3. ——— SKIP SLIDE ———
4. This is why Demaine et al. use persistent data structures. A persistent structure cleverly shares memory between versions, so all  $\sqrt{m}$  checkpoints can be stored efficiently in just  $O(m)$  total space.
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6. But this raises a practical problem: What if we don't have a persistent version of our data structure? Or what if it's just too complex to implement?
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9. We achieve the \*same  $O(\sqrt{m})$  time\* per operation.
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## Problem

What if we don't have or don't want to use persistent data structures?

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## Partial to full retroactivity

### └ The space problem

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## Our contribution

Simple rebuilding strategy without persistent data structures

- Same time complexity:  $\mathcal{O}(\sqrt{m})$  per operation
- Space usage:  $\Theta(m\sqrt{m})$

## Partial to full retroactivity

2025-10-30

### └ The space problem

1. ...and that leads to the space problem. If we're naive and just store  $\sqrt{m}$  \*independent copies\* of the data structure, one for each checkpoint...
2. ...and each copy can have up to  $m$  updates, our space usage explodes to  $\Theta(m\sqrt{m})$ .
3. ——— SKIP SLIDE ———
4. This is why Demaine et al. use persistent data structures. A persistent structure cleverly shares memory between versions, so all  $\sqrt{m}$  checkpoints can be stored efficiently in just  $\mathcal{O}(m)$  total space.
5. ——— SKIP SLIDE ———
6. But this raises a practical problem: What if we don't have a persistent version of our data structure? Or what if it's just too complex to implement?
7. ——— SKIP SLIDE ———
8. This is our contribution. We propose a simple rebuilding strategy that \*doesn't\* require persistence.
9. We achieve the \*same  $\mathcal{O}(\sqrt{m})$  time\* per operation.
10. The trade-off is that we go back to using  $\Theta(m\sqrt{m})$  space, but we argue this is a practical trade-off for a much, much simpler implementation.

The space problem

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# Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF

- **Operations:**

- ▶ `add_edge(u, v, w, t)`: add edge at time  $t$
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2025-10-30

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6. They use a set of data structures,  $D_i$ , where each  $D_i$  stores the incremental MSF containing all edges added \*before\* its checkpoint time  $t_i$ .
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2025-10-30

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### Problems with the existing static approach

- ♦ **Fixed  $m$ :** Requires knowing the maximum sequence length ( $m$ ) beforehand. Meaning that it cannot handle arbitrary growth or dynamic operation counts.

1. The existing approach is based on a static constraint: we must assume a fixed  $m$ .
2. This means you have to know the total number of operations in advance.
3. Crucially, they lack a mechanism for **rebuilding**, making them unable to handle a growing or unknown number of operations.
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5. Our goal is simple: remove the dependence on a fixed  $m$  while keeping the time efficiency.
6. Our key insight is to introduce a **dynamic rebuilding process** to handle growth.
7. The challenge is doing this efficiently. Rebuilding  $\sqrt{m}$  checkpoints non-persistently usually takes too long.
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**Goal:** Remove the **Fixed  $m$**  dependency while preserving time complexity.

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## Partial to full retroactivity

2025-10-30

## └ Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When  $m = k^2$  (perfect square)

1. Here's our strategy. The key idea is to reuse the existing structures.
2. We trigger a rebuild whenever the total number of operations,  $m$ , becomes a perfect square, say  $k^2$ .
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4. When we rebuild, we're going from  $k$  checkpoints to  $k + 1$  new ones. Our strategy is:
  1. We create two new, \*empty\* structures,  $D'_0$  and  $D'_1$ .
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  3. Finally, we just apply the "missing" updates to each of these reused structures to get them up to date for their new checkpoint times.
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9. The reason this is efficient is based on a key lemma we prove: The updates needed for the new  $D'_{i+2}$  are just a continuation of the updates from the old  $D_i$ . We don't have to restart from scratch.
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# Rebuilding algorithm

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- 4 For  $i = 1$  to  $k + 1$ :
  - ▷  $p \leftarrow \text{KTH}(S, i(k + 1))$  ▷  $i(k + 1)$ th item in the sequence of updates
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  - ▷  $\text{ADDEDGES}(S, t_{i-2}, t'_i, D'_i)$
- 5 Return  $k + 1, D', t'$

## Partial to full retroactivity

2025-10-30

### └ Rebuilding algorithm

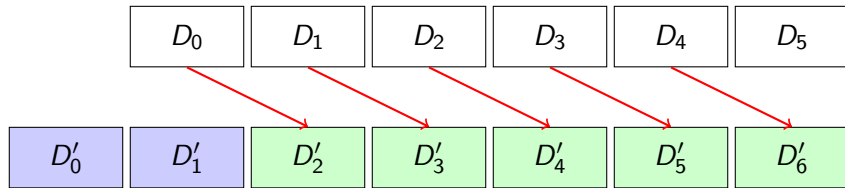
```
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2. Lines 1 and 2 create the two new empty structures,  $D'_0$  and  $D'_1$ .
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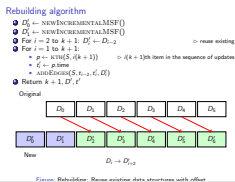
New

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2025-10-30

## Partial to full retroactivity

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## Our contribution

- **General transformation:** Partial  $\rightarrow$  Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:**  $\mathcal{O}(\sqrt{m})$  per operation
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## Partial to full retroactivity

### Results

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2. We've developed a general transformation to take a partially retroactive data structure and make it fully retroactive.
3. Crucially, our method *\*does not require persistent data structures\**.
4. We match the  $\mathcal{O}(\sqrt{m})$  slowdown per operation from the Demaine et al. paper...
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7. Applying this to our test case, we get a semi-retroactive MSF implementation.
8. It supports adding edges and querying the MSF at any time  $t$  in  $\mathcal{O}(\sqrt{m} \log n)$  amortized time.
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## Semi-retroactive MSF implementation

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- **Time:**  $\mathcal{O}(\sqrt{m} \log n)$  per operation
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# Thank you!

# Questions?

2025-10-30

Partial to full retroactivity

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Questions?

1. Invite questions from the audience
2. Be prepared to answer questions about:
  3. \* The rebuilding algorithm details
  4. \* Space vs time trade-offs
  5. \* Implementation challenges
  6. \* Comparison with persistent data structures
  7. \* Applications beyond MSF
8. Key points to emphasize if asked:
  9. \* Our approach is simpler to implement
  10. \* Same time complexity as Demaine et al.
  11. \* No persistent data structure requirement
  12. \* General applicability to any partially retroactive structure
13. Thank the audience for their attention