Partial to full retroactivity

How to go from partial to full retroactivity in detail

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1. Hello everyone. My name is Felipe Noronha, and today I'll be presenting the work done by Professor Cristina Fernandes and I at IME-USP.
2. Our paper details a method for transforming partially retroactive data structures into fully retroactive ones.
3. This work is motivated by a practical limitation in the well-known 2007 transformation by Demaine, Iacono, and Langerman and it also builds upon a 2022 solution by Junior and Seabra.
4. Our key contribution is a method to achieve this transformation with the same time complexity, but *without* the need for complex persistent data structures.
5. To illustrate our approach, we'll use the minimum spanning forest problem as our main example. So, let's start by defining what that is.

What is a spanning tree?

- Let $G = (V, E)$ be a connected graph
- **Spanning tree:** A tree with all vertices of G

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2. A spanning tree will be a tree will all the vertices of G
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4. It will have 3 main properties: it is connected (path between any two vertices), acyclic (no cycles), contains exactly n-1 edges for n vertices
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6. Show visual example with graph G (blue edges) and spanning tree T (red wavy edges)
7. In the example: 8 vertices, so spanning tree has exactly 7 edges
8. Emphasize that spanning trees are not unique - there can be many valid spanning trees

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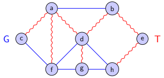

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Minimum Spanning Tree and Forest

- **Minimum Spanning Tree (MST):** spanning tree in a weighted graph with minimum total cost

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3. This concept generalizes to disconnected graphs as well. We call this a Minimum Spanning Forest, or MSF, which is simply the collection of MSTs for each connected component.
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5. In the visual example, you can see the same graph as before, but now with costs on the edges. The red edges again show the tree, but this time, they've been chosen to be the MST.
6. If we sum the costs of the red edges, we get a total of 14. Any other spanning tree you could build for this graph would have a total cost greater than or equal to 14.
7. This idea of maintaining an optimal-cost forest is central to our problem. Specifically, how to maintain this optimality as the graph changes.

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- **Minimum Spanning Forest (MSF):** generalization for disconnected graphs

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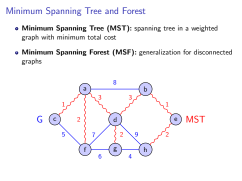

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Incremental MSF problem

- **Problem:** Keep track of an MSF in a graph that grows over time
- Graph starts empty, edges are added one by one

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Frederickson's link-cut tree solution

- **Key insight:** Use link-cut trees to maintain MSF dynamically

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2. Link-cut trees provide all the operations we need, all in $O(\log n)$ amortized time: *find_max* to find the most expensive edge on a path, *link* to add a weighted edge, *cut* to remove one, and *is_connected* to check if u and v are in the same component
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4. With this, we can construct a straightforward algorithm that supports adding a new edge (u, v, w):
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6. If they *are* connected, adding this new edge creates a cycle. We find the most expensive edge on the path in that cycle using *find_max*.
7. If our new edge's cost is cheaper than that maximum cost, we swap them: we *cut* the old, expensive edge and *link* our new, cheaper edge .
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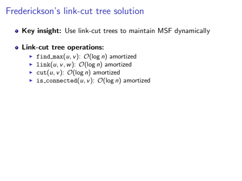
9. With these steps using LCT operations, the time per edge addition is $O(\log n)$ amortized.



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- **Algorithm for adding edge** (u, v, w)**:**
  1. Check if u and v are in connected in the same component
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  3. If yes: find the edge with maximum cost on the u-v path
  4. If $w <$ maximum cost: replace maximum cost edge with (u, v, w)

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- **Total cost:** Amortized $\mathcal{O}(\log n)$ per edge addition

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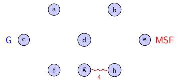
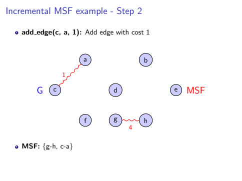
Incremental MSF example - Step 1

- **add_edge(g, h, 4):** Add edge (g, h) with cost 4

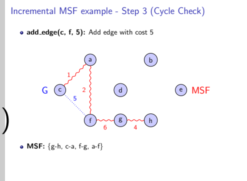


- **MSF:** {g-h}

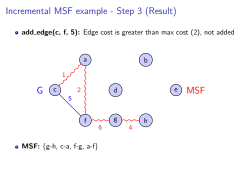
1. Let's walk through a quick example. We start with an empty graph.
2. First, we add edge (g, h) with cost 4.
3. Are 'g' and 'h' connected? No. So, by step 2 of the algorithm, we simply add the edge to our MSF.
4. The MSF is now just {g-h}.



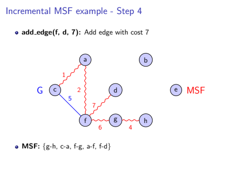
1. Next, we add (c, a) with cost 1.
2. Again, are 'c' and 'a' connected? No. They are in a different component from 'g' and 'h'.
3. So, we add it directly. The MSF now has two components: {g-h} and {c-a}.



1. After fast-forwarding through a few steps (adding f-g and a-f), our forest is more connected.
2. Now, we add (c, f) with cost 5. This is our first interesting case.
3. Are 'c' and 'f' connected? Yes, they are. Adding this edge will create a cycle: c-a-f-c.
4. So, we go to step 3. We find the max-cost edge on the path c-a-f. The edges are (c,a) with cost 1 and (a,f) with cost 2. The max cost is 2.



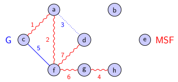
1. Our new edge costs 5. Since 5 is *not* less than the max cost of 2, we *do not* add this edge. It's discarded.
2. The MSF remains unchanged.



1. Next, add (f, d) with cost 7.
2. Are ’f’ and ’d’ connected? No. ’f’ is in the main tree, but ’d’ is a new, isolated vertex.
3. Therefore, we simply add the edge. The MSF is updated.

Incremental MSF example - Step 5 (Cycle Check)

- **add_edge(a, d, 3):** Add edge with cost 3



- **MSF:** {g-h, c-a, f-g, a-f, f-d}

1. Now, add (a, d) with cost 3.
2. Are 'a' and 'd' connected? Yes. This creates the cycle a-f-d-a.
3. We find the max-cost edge on the path a-f-d. The edges are (a,f) with cost 2 and (f,d) with cost 7. The max cost is 7.

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Partial to full retroactivity

└Incremental MSF example - Step 5 (Result)

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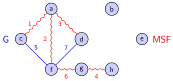
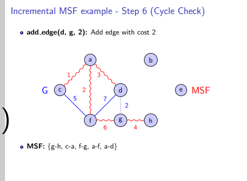
- **add_edge(a, d, 3):** Cost 3 ¡ max cost 7, swap edges

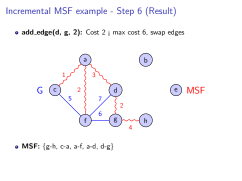


- **MSF:** {g-h, c-a, f-g, a-f, a-d}

1. Our new edge costs 3. Since 3 *is* less than 7, we swap them.
2. We ‘cut‘ the expensive edge (f,d) and ‘link‘ our new, cheaper edge (a,d).
3. The MSF is now {g-h, c-a, f-g, a-f, a-d} and its total cost has improved.



1. Finally, let's add (d, g) with cost 2.
2. Are 'd' and 'g' connected? Yes. This creates the cycle d-a-f-g-d.
3. We find the max-cost edge on the path d-a-f-g. The edges are (d,a) cost 3, (a,f) cost 2, and (f,g) cost 6. The max cost is 6, from edge (f,g).



1. Our new edge costs 2. Since 2 *is* less than 6, we swap them.
2. We ‘cut‘ edge (f,g) and ‘link‘ our new edge (d,g).
3. The MSF is updated again, and the total cost is now 12.

└Incremental MSF example - Step 7 (Final Result)

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- **Continue adding edges...**

1. If we continue this process, adding all the remaining edges from our original graph...
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3. ...we would eventually arrive at the final, optimal Minimum Spanning Tree. The one shown here, for example, has a total cost of 14.
4. But this only answers queries about the *present*. What if we want to ask: "What did the MSF look like 10 updates ago?"
5. This is the core question of retroactivity. How do we efficiently query the past?

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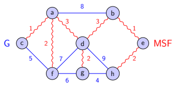
- **Continue adding edges...**
- **Final MSF:** Minimum spanning forest with optimal cost



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- **Problem:** Data structures usually support updates and queries
- The order of updates affects the state of the data structure

1. In a normal data structure, the order of updates is important. Most of the time, the state of the structure, and thus the answers to queries, depends on this sequence.
2. This means we usually don't have a good way to go back and correct mistakes or insert operations we forgot.
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4. That's where retroactivity comes in. A retroactive data structure allows us to manipulate this sequence of updates.
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7. ...Remove an update that *already happened* at time t....
8. ...and, most importantly, Query the state of the structure at *any* time t, not just the present.
9. The key challenge is how to do this efficiently, maintaining the state for every possible time.

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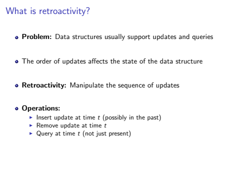

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Fully Retroactive

- Queries at **any** time t
- Insert/remove updates at any time

1. There are a few different "flavors" of retroactivity. The most powerful is Fully Retroactivity, which supports all the operations we just saw: insert, remove, and query, all at any time t.
2. ———- SKIP SLIDE ———-
3. Partially Retroactive is more limited. You can still insert or remove updates anywhere in the timeline, but you can only query the state of the structure at the *current* time, "now". This is a key limitation.
4. ———- SKIP SLIDE ———-
5. And finally, there's Semi-Retroactive, which is a bit of a mix. You can query at any time t and insert updates at any time, but you are *not allowed* to remove updates.
6. Generally, partially retroactive structures are much simpler to design. And this leads to an interesting challenge...

Partial vs Full retroactivity

Fully Retroactive

- Queries at **any** time t
- Insert/remove updates at any time

Partially Retroactive

- Queries only on **current** state
- Insert/remove updates at any time

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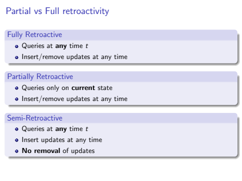
Partial to full retroactivity

└Partial vs Full retroactivity



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How to transform partial → full retroactivity?

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Any partially retroactive data structure can be transformed into a fully retroactive one with:

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- **Naive approach:** Keep $\sqrt{m}$ independent copies
- Space usage: $\Theta(m\sqrt{m})$

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Simple rebuilding strategy without persistent data structures

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Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
  - add_edge(u, v, w, t): add edge at time t
  - get_msf(t): get MSF at time t

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4. They also use a square-root decomposition. They maintain $\sqrt{m}$ checkpoints, t_i, spaced $\sqrt{m}$ updates apart.
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6. They use a set of data structures, D_i, where each D_i stores the incremental MSF containing all edges added *before* its checkpoint time t_i.
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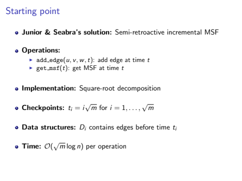

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Limitations → Key Insight

Problems with the existing static approach

- **Fixed** m: Requires knowing the maximum sequence length (**m**) beforehand. Meaning that it cannot handle arbitrary growth or dynamic operation counts.

1. The existing approach is based on a static constraint: we must assume a fixed m.
2. This means you have to know the total number of operations in advance.
3. Crucially, they lack a mechanism for **rebuilding**, making them unable to handle a growing or unknown number of operations.
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5. Our goal is simple: remove the dependence on a fixed m while keeping the time efficiency.
6. Our key insight is to introduce a **dynamic rebuilding process** to handle growth.
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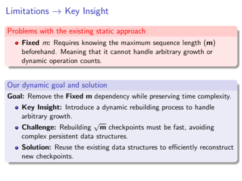

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- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)

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2. We trigger a rebuild whenever the total number of operations, m, becomes a perfect square, say k^2.
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4. When we rebuild, we're going from k checkpoints to $k+1$ new ones. Our strategy is:
5. 1. We create two new, *empty* structures, D'_0 and D'_1.
6. 2. Then, we *reuse* our old structures: the old D_0 becomes the new D'_2, the old D_1 becomes the new D'_3, and so on. We shift them over by two spots.
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Key Lemma
Every update in D_i is within the first $(i+2)(k+1)$ updates in the new sequence.

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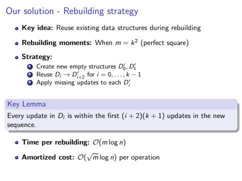

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Rebuilding algorithm

1. $D'_0 \leftarrow \textsc{newIncrementalMSF}()$
2. $D'_1 \leftarrow \textsc{newIncrementalMSF}()$
3. For $i = 2$ to $k+1$: $D'_i \leftarrow D_{i-2}$ ▷ reuse existing
4. For $i = 1$ to $k+1$:
   - $p \leftarrow \textsc{kth}(S, i(k+1))$ ▷ $i(k+1)$th item in the sequence of updates
   - $t'_i \leftarrow p.\text{time}$
   - $\textsc{addEdges}(S, t_{i-2}, t'_i, D'_i)$
5. Return $k+1, D', t'$

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2. Lines 1 and 2 create the two new empty structures, D'_0 and D'_1.
3. Line 3 is the reuse: we loop from $i = 2$ up to $k+1$, and simply assign the old D_{i-2} to be the new D'_i. This is just a pointer swap; it's instant.
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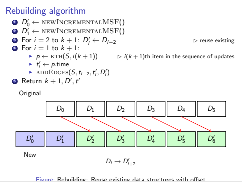

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Our contribution

- **General transformation:** Partial $\rightarrow$ Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
- **Space trade-off:** $\Theta(m\sqrt{m})$ vs $\mathcal{O}(m)$

1. So, to summarize our contributions:
2. We've developed a general transformation to take a partially retroactive data structure and make it fully retroactive.
3. Crucially, our method *does not require persistent data structures*.
4. We match the $O(\sqrt{m})$ slowdown per operation from the Demaine et al. paper...
5. ...at the cost of $\Theta(m\sqrt{m})$ space, which we argue is a very practical trade-off for simplicity.
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7. Applying this to our test case, we get a semi-retroactive MSF implementation.
8. It supports adding edges and querying the MSF at any time t in $O(\sqrt{m}\log n)$ amortized time.
9. And, we have successfully removed the limitations from the previous work: our structure works *without* a fixed m or a fixed time range.

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Semi-retroactive MSF implementation

- **Operations:** `add_edge`(u, v, w, t), `get_msf`(t)
- **Time:** $\mathcal{O}(\sqrt{m}\log n)$ per operation
- **Space:** $\Theta(m\sqrt{m})$
- **No fixed m or time range restrictions**

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2. We've developed a general transformation to take a partially retroactive data structure and make it fully retroactive.
3. Crucially, our method *does not require persistent data structures*.
4. We match the $O(\sqrt{m})$ slowdown per operation from the Demaine et al. paper...
5. ...at the cost of $\Theta(m\sqrt{m})$ space, which we argue is a very practical trade-off for simplicity.
6. ———- SKIP SLIDE ———-
7. Applying this to our test case, we get a semi-retroactive MSF implementation.
8. It supports adding edges and querying the MSF at any time t in $O(\sqrt{m}\log n)$ amortized time.
9. And, we have successfully removed the limitations from the previous work: our structure works *without* a fixed m or a fixed time range.

Partial to full retroactivity

2025-10-30

Thank you!

Questions?

1. Invite questions from the audience
2. Be prepared to answer questions about:
3. * The rebuilding algorithm details
4. * Space vs time trade-offs
5. * Implementation challenges
6. * Comparison with persistent data structures
7. * Applications beyond MSF
8. Key points to emphasize if asked:
9. * Our approach is simpler to implement
10. * Same time complexity as Demaine et al.
11. * No persistent data structure requirement
12. * General applicability to any partially retroactive structure
13. Thank the audience for their attention