

How to go from partial to full retroactivity in detail

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IME-USP – Brazil

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1. Hello everyone. My name is Felipe Noronha, and today I'll be presenting the work done by Professor Cristina Fernandes and I at IME-USP.
2. Our paper details a method for transforming partially retroactive data structures into fully retroactive ones.
3. This work is motivated by a practical limitation in the well-known 2007 transformation by Demaine, Iacono, and Langerman and it also builds upon a 2022 solution by Junior and Seabra.
4. Our key contribution is a method to achieve this transformation with the same time complexity, but **without** the need for complex persistent data structures.
5. To illustrate our approach, we'll use the minimum spanning forest problem as our main example. So, let's start by defining what that is.

What is a spanning tree?

- Let $G = (V, E)$ be a connected graph
- **Spanning tree:** A tree with all vertices of G

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Partial to full retroactivity

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3. ——— SKIP SLIDE ———
4. It will have 3 main properties: it is connected (path between any two vertices), acyclic (no cycles), contains exactly $n-1$ edges for n vertices
5. ——— SKIP SLIDE ———
6. Show visual example with graph G (blue edges) and spanning tree T (red wavy edges)
7. In the example: 8 vertices, so spanning tree has exactly 7 edges
8. Emphasize that spanning trees are not unique - there can be many valid spanning trees

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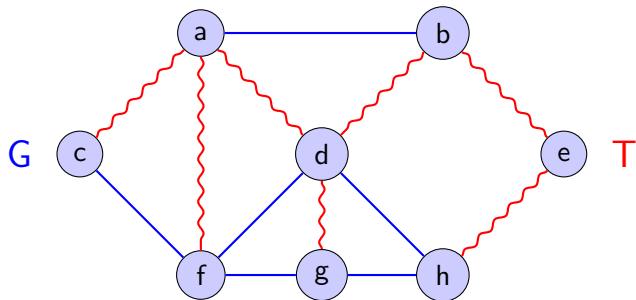
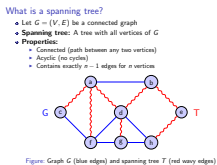


Figure: Graph G (blue edges) and spanning tree T (red wavy edges)

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Minimum Spanning Tree and Forest

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- **Minimum Spanning Tree (MST):** spanning tree in a weighted graph with minimum total cost

1. Now, let's add weights or costs to the edges. In a weighted graph, a Minimum Spanning Tree, or MST, is a spanning tree that has the minimum possible total cost. It's an optimization problem.
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3. This concept generalizes to disconnected graphs as well. We call this a Minimum Spanning Forest, or MSF, which is simply the collection of MSTs for each connected component.
4. ——— SKIP SLIDE ———
5. In the visual example, you can see the same graph as before, but now with costs on the edges. The red edges again show the tree, but this time, they've been chosen to be the MST.
6. If we sum the costs of the red edges, we get a total of 14. Any other spanning tree you could build for this graph would have a total cost greater than or equal to 14.
7. This idea of maintaining an optimal-cost forest is central to our problem. Specifically, how to maintain this optimality as the graph changes.

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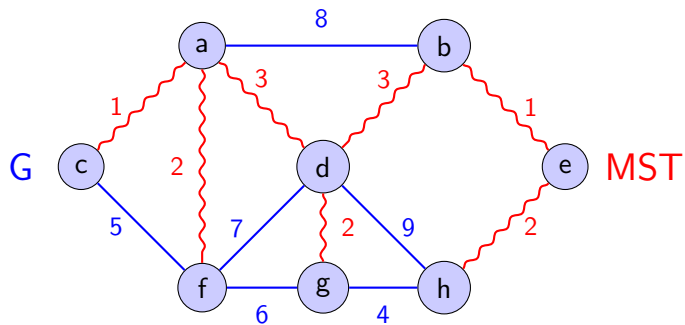
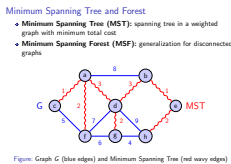


Figure: Graph G (blue edges) and Minimum Spanning Tree (red wavy edges)

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Incremental MSF problem

- **Problem:** Keep track of an MSF in a graph that grows over time
- Graph starts empty, edges are added one by one

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2. Crucially, the graph starts empty, and edges are only added one by one.
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4. This problem is defined by two operations: *add_edge*, which inserts a new weighted edge, and *get_msf*, which returns the current minimum spanning forest.
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6. The solution to this was given by Frederickson in 1983. He used a dynamic data structure called link-cut trees which achieves a $O(\log n)$ amortized time per edge addition, where n is the number of vertices.

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 - ▶ `add_edge(u, v, w)`: add edge with cost w between vertices u and v
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- **Key insight:** Use link-cut trees to maintain MSF dynamically

1. So, what was Frederickson's solution? He showed that link-cut trees can efficiently maintain this incrementing forest.
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3. Link-cut trees provide all the operations we need, all in $O(\log n)$ amortized time: *find_max* to find the most expensive edge on a path, *link* to add a weighted edge, and *cut* to remove one.
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5. With this, we can construct a straightforward algorithm that supports adding a new edge (u, v, w) :
6. First, we check if u and v are already connected. If they're not, the new edge can't create a cycle, so we just add it to the forest using *link*.
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10. With these steps using LCT operations, the time per edge addition is $O(\log n)$ amortized.

- ♦ **Key insight:** Use link-cut trees to maintain MSF dynamically
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- **Algorithm for adding edge (u, v, w) :**

- 1 Check if u and v are in same component
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Incremental MSF example - Step 1

- **add_edge(g, h, 4):** Add edge with cost 4

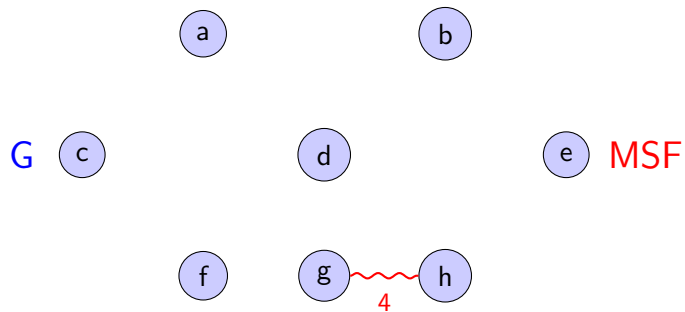


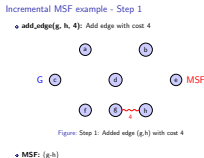
Figure: Step 1: Added edge (g,h) with cost 4

- **MSF:** $\{g-h\}$

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└ Incremental MSF example - Step 1



1. Let's walk through a quick example. We start with an empty graph.
2. First, we add edge (g, h) with cost 4.
3. Are 'g' and 'h' connected? No. So, by step 2 of the algorithm, we simply add the edge to our MSF.
4. The MSF is now just $\{g-h\}$.

Incremental MSF example - Step 2

- **add_edge(c, a, 1):** Add edge with cost 1

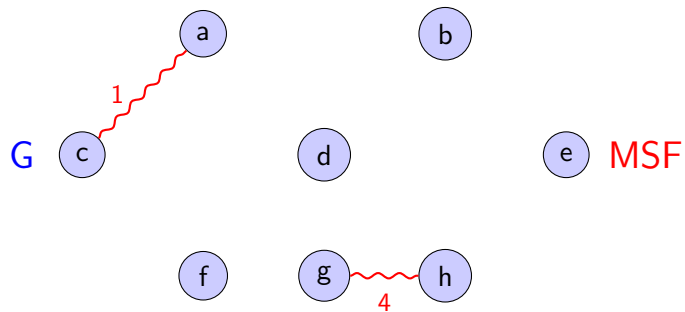


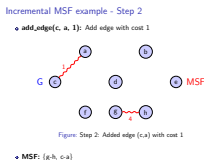
Figure: Step 2: Added edge (c,a) with cost 1

- **MSF:** $\{g-h, c-a\}$

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└ Incremental MSF example - Step 2



1. Next, we add (c, a) with cost 1.
2. Again, are 'c' and 'a' connected? No. They are in a different component from 'g' and 'h'.
3. So, we add it directly. The MSF now has two components: $\{g-h\}$ and $\{c-a\}$.

Incremental MSF example - Step 3

- **add_edge(f, g, 6):** Add edge with cost 6

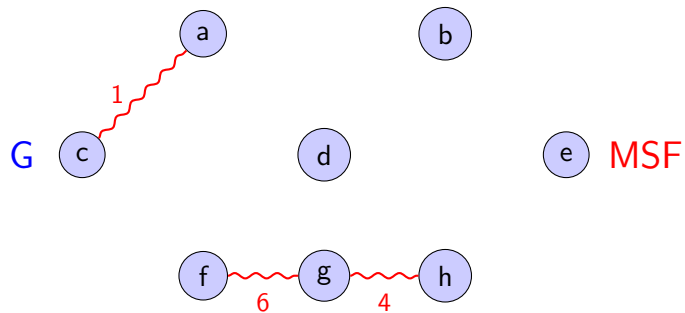


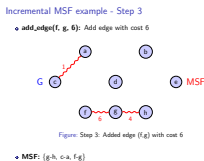
Figure: Step 3: Added edge (f,g) with cost 6

- **MSF:** {g-h, c-a, f-g}

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└ Incremental MSF example - Step 3



1. Now, add (f, g) with cost 6.
2. Are 'f' and 'g' connected? No. 'f' is isolated, and 'g' is in the {g-h} component.
3. We link them. The MSF now contains {c-a} and {f-g-h}.

Incremental MSF example - Step 4

- **add_edge(a, f, 2):** Add edge with cost 2

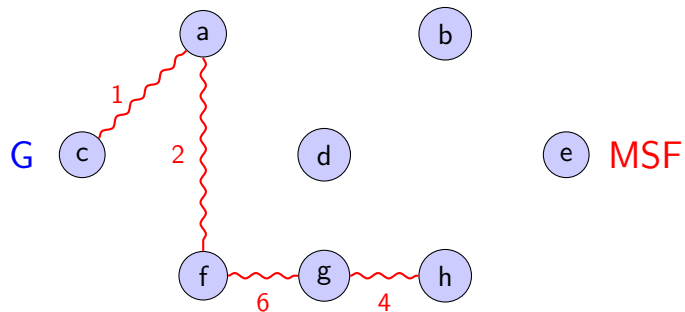


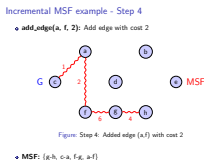
Figure: Step 4: Added edge (a,f) with cost 2

- **MSF:** $\{g-h, c-a, f-g, a-f\}$

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└ Incremental MSF example - Step 4



1. Add (a, f) with cost 2.
2. Are 'a' and 'f' connected? No. 'a' is in the $\{c-a\}$ component and 'f' is in the $\{f-g-h\}$ component.
3. We link these two components. Our forest now becomes a single tree, and all vertices shown so far are connected.

Incremental MSF example - Step 5

- **add_edge(c, f, 5):** Add edge with cost 5

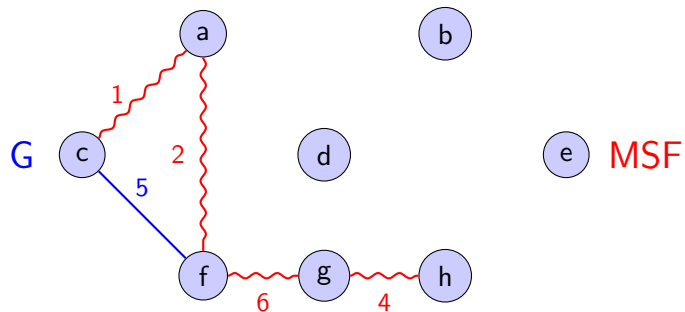


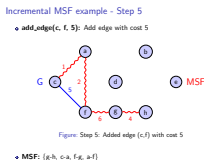
Figure: Step 5: Added edge (c,f) with cost 5

- **MSF:** {g-h, c-a, f-g, a-f}

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└ Incremental MSF example - Step 5



1. Okay, now we add (c, f) with cost 5. This is our first interesting case.
2. Are 'c' and 'f' connected? Yes, they are. Adding this edge will create a cycle: c-a-f-c.
3. So, we go to step 3. We find the max-cost edge on the path c-a-f. The edges are (c,a) with cost 1 and (a,f) with cost 2. The max cost is 2.
4. Our new edge costs 5. Since 5 is **not** less than the max cost of 2, we **do not** add this edge. It's discarded.
5. The MSF remains unchanged.

Incremental MSF example - Step 6

- **add_edge(f, d, 7):** Add edge with cost 7

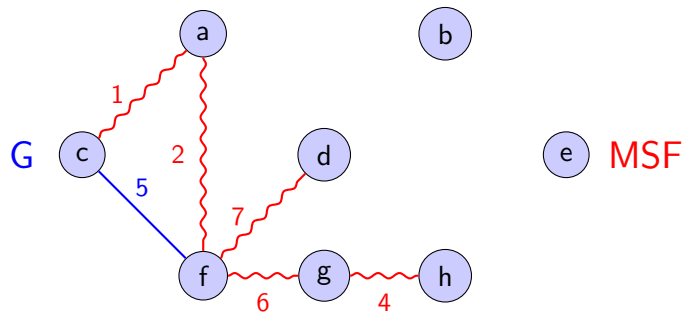


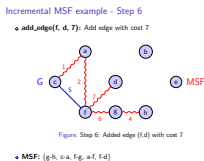
Figure: Step 6: Added edge (f,d) with cost 7

- **MSF:** {g-h, c-a, f-g, a-f, f-d}

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└ Incremental MSF example - Step 6



1. Next, add (f, d) with cost 7.
2. Are 'f' and 'd' connected? No. 'f' is in the main tree, but 'd' is a new, isolated vertex.
3. Therefore, we simply add the edge. The MSF is updated.

Incremental MSF example - Step 7

- **add_edge(a, d, 3):** Add edge with cost 3

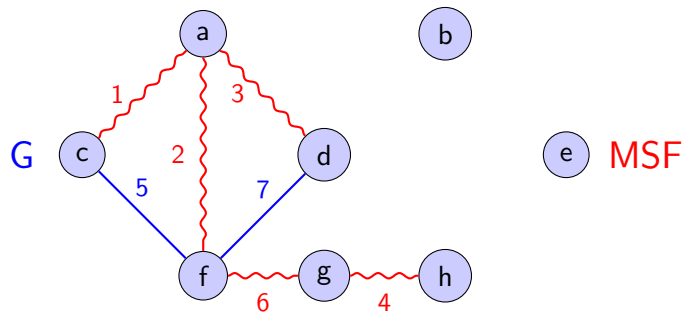


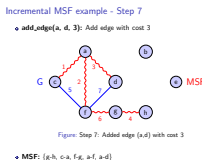
Figure: Step 7: Added edge (a,d) with cost 3

- **MSF:** {g-h, c-a, f-g, a-f, a-d}

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Incremental MSF example - Step 7

1. Now, add (a, d) with cost 3.
2. Are 'a' and 'd' connected? Yes. This creates the cycle a-f-d-a.
3. We find the max-cost edge on the path a-f-d. The edges are (a,f) with cost 2 and (f,d) with cost 7. The max cost is 7.
4. Our new edge costs 3. Since 3 *is* less than 7, we swap them.
5. We 'cut' the expensive edge (f,d) and 'link' our new, cheaper edge (a,d).
6. The MSF is now {g-h, c-a, f-g, a-f, a-d} and its total cost has improved.



Incremental MSF example - Step 8

- **add_edge(d, g, 2):** Add edge with cost 2

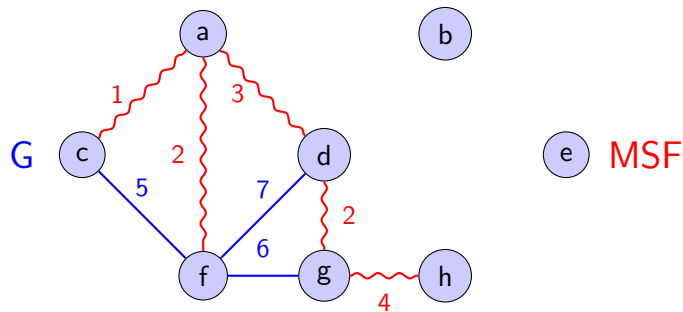


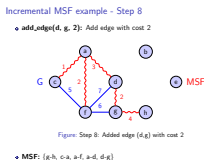
Figure: Step 8: Added edge (d,g) with cost 2

- **MSF:** {g-h, c-a, a-f, a-d, d-g}

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Partial to full retroactivity

└ Incremental MSF example - Step 8



1. Finally, let's add (d, g) with cost 2.
2. Are 'd' and 'g' connected? Yes. This creates the cycle d-a-f-g-d.
3. We find the max-cost edge on the path d-a-f-g. The edges are (d,a) cost 3, (a,f) cost 2, and (f,g) cost 6. The max cost is 6, from edge (f,g).
4. Our new edge costs 2. Since 2 *is* less than 6, we swap them.
5. We 'cut' edge (f,g) and 'link' our new edge (d,g).
6. The MSF is updated again, and the total cost is now 12.

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Partial to full retroactivity

└ Incremental MSF example - Final Result

Incremental MSF example - Final Result

- Continue adding edges...

1. If we continue this process, adding all the remaining edges from our original graph...
2. ——— SKIP SLIDE ———
3. ...we would eventually arrive at the final, optimal Minimum Spanning Tree. The one shown here, for example, has a total cost of 12.
4. But this only answers queries about the *present*. What if we want to ask: "What did the MSF look like 10 updates ago?"
5. This is the core question of retroactivity. How do we efficiently query the past?

Incremental MSF example - Final Result

- Continue adding edges...
- **Final MSF:** Minimum spanning forest with optimal cost

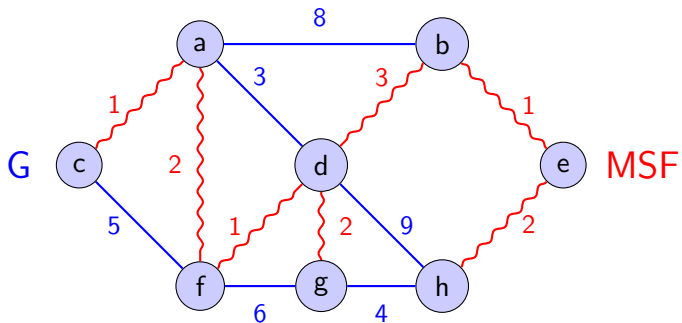
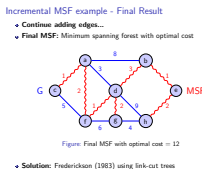


Figure: Final MSF with optimal cost = 12

- **Solution:** Frederickson (1983) using link-cut trees

Partial to full retroactivity

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What is retroactivity?

- **Problem:** Data structures usually support updates and queries
- The order of updates affects the state of the data structure

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Partial to full retroactivity

└─What is retroactivity?

1. In a normal data structure, the order of updates is important. Most of the time, the state of the structure, and thus the answers to queries, depends on this sequence.
2. This means we usually don't have a good way to go back and correct mistakes or insert operations we forgot.
3. ——— SKIP SLIDE ———
4. That's where retroactivity comes in. A retroactive data structure allows us to manipulate this sequence of updates.
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6. Specifically, it adds operations to: Insert a new update at some time t *in the past*...
7. ...Remove an update that *already happened* at time t
8. ...and, most importantly, Query the state of the structure at *any* time t , not just the present.
9. The key challenge is how to do this efficiently, maintaining the state for every possible time.

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Partial vs Full retroactivity

Fully Retroactive

- Queries at **any** time t
- Insert/remove updates at any time

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Partial to full retroactivity

└ Partial vs Full retroactivity

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3. Partially Retroactive is more limited. You can still insert or remove updates anywhere in the timeline, but you can only query the state of the structure at the **current** time, "now". This is a key limitation.
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5. And finally, there's Semi-Retroactive, which is a bit of a mix. You can query at any time t and insert updates at any time, but you are **not allowed** to remove updates.
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How to transform partial \rightarrow full retroactivity?

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Partial to full retroactivity

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Partial to full retroactivity

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Theorem (Theorem 05)

Any partially retroactive data structure can be transformed into a fully retroactive one with:

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1. Their paper presented this theorem: any partially retroactive data structure can be made fully retroactive.
2. The cost is an $O(\sqrt{m})$ slowdown per operation and $O(m)$ space, where m is the number of updates.
3. But there's a catch: this transformation **requires** a persistent version of the data structure.
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5. So, how does it work? The idea is to break the m updates into \sqrt{m} blocks, each of size \sqrt{m} .
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Simple rebuilding strategy without persistent data structures

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2025-10-26

Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF

- **Operations:**

- ▶ `add_edge(u, v, w, t)`: add edge at time t
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- **Time:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

2025-10-26

Partial to full retroactivity

└ Starting point

1. Our starting point was the 2022 work by Junior and Seabra on a semi-retroactive MSF.
2. Remember, "semi-retroactive" means they can add edges at any time t in the past, and query the MSF at any time t , but they cannot *remove* edges.
3. _____ SKIP SLIDE _____
4. They also use a square-root decomposition. They maintain \sqrt{m} checkpoints, t_i , spaced \sqrt{m} updates apart.
5. _____ SKIP SLIDE _____
6. They use a set of data structures, D_i , where each D_i stores the incremental MSF containing all edges added *before* its checkpoint time t_i .
7. This approach gives them a final time complexity of $\mathcal{O}(\sqrt{m} \log n)$ per operation.
8. However, their solution has some significant practical limitations...

Starting point

- **Junior & Seabra's solution:** Semi-retroactive incremental MSF
- **Operations:**
 - ▶ `add_edge(u, v, w, t)`: add edge at time t
 - ▶ `get_msf(t)`: get MSF at time t
- **Implementation:** Square-root decomposition
- **Checkpoints:** $t_i = i\sqrt{m}$ for $i = 1, \dots, \sqrt{m}$
- **Data structures:** D_i contains edges before time t_i
- **Time:** $\mathcal{O}(\sqrt{m} \log n)$ per operation

Limitations → Key Insight

Problems with the Existing Static Approach

- **Fixed m :** Requires knowing the maximum sequence length (m) beforehand. Meaning that it cannot handle arbitrary growth or dynamic operation counts.

Our Dynamic Goal and Solution

Goal: Remove the **Fixed m** dependency while preserving time complexity.

- **Key Insight:** Introduce a ****dynamic rebuilding process**** to handle arbitrary growth.
- **Challenge:** Rebuilding \sqrt{m} checkpoints must be fast, avoiding complex persistent data structures.
- **Solution:** ****Reuse**** the existing data structures to efficiently reconstruct new checkpoints.

Partial to full retroactivity

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- **Solution:** ****Reuse**** the existing data structures to efficiently reconstruct new checkpoints.

1. The existing approach is based on a static constraint: we must assume a fixed m .
2. This means you have to know the total number of operations in advance.
3. Crucially, they lack a mechanism for ****rebuilding****, making them unable to handle a growing or unknown number of operations.
4. Our goal is simple: remove the dependence on a fixed m while keeping the time efficiency.
5. Our key insight is to introduce a ****dynamic rebuilding process**** to handle growth.
6. The challenge is doing this efficiently. Rebuilding \sqrt{m} checkpoints non-persistently usually takes too long.
7. Our solution is a clever trick: we ****reuse**** the data structures already present in our system to reconstruct new checkpoints quickly.

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Partial to full retroactivity

└ Our solution - Rebuilding strategy

- **Key idea:** Reuse existing data structures during rebuilding
- **Rebuilding moments:** When $m = k^2$ (perfect square)

1. Here's our strategy. The key idea is to reuse the existing structures.
2. We trigger a rebuild whenever the total number of operations, m , becomes a perfect square, say k^2 .
3. ——— SKIP SLIDE ———
4. When we rebuild, we're going from k checkpoints to $k + 1$ new ones. Our strategy is:
 1. We create two new, *empty* structures, D'_0 and D'_1 .
 2. Then, we *reuse* our old structures: the old D_0 becomes the new D'_2 , the old D_1 becomes the new D'_3 , and so on. We shift them over by two spots.
 3. Finally, we just apply the "missing" updates to each of these reused structures to get them up to date for their new checkpoint times.
8. ——— SKIP SLIDE ———
9. The reason this is efficient is based on a key lemma we prove: The updates needed for the new D'_{i+2} are just a continuation of the updates from the old D_i . We don't have to restart from scratch.
10. ——— SKIP SLIDE ———
11. This rebuilding process takes $O(m \log n)$ time in total.

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Partial to full retroactivity

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Every update in D_i is within the first $(i+2)(k+1)$ updates in the new sequence.

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Rebuilding algorithm

- 1 $D'_0 \leftarrow \text{NEWINCREMENTALMSF}()$
- 2 $D'_1 \leftarrow \text{NEWINCREMENTALMSF}()$
- 3 For $i = 2$ to $k + 1$: $D'_i \leftarrow D_{i-2}$ ▷ reuse existing
- 4 For $i = 1$ to $k + 1$:
 - ▷ $p \leftarrow \text{KTH}(S, i(k + 1))$ ▷ $i(k + 1)$ th edge
 - ▷ $t'_i \leftarrow p.\text{time}$
 - ▷ $\text{ADDEDGES}(S, t_{i-2}, t'_i, D'_i)$
- 5 Return $k + 1, D', t'$

Partial to full retroactivity

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└ Rebuilding algorithm

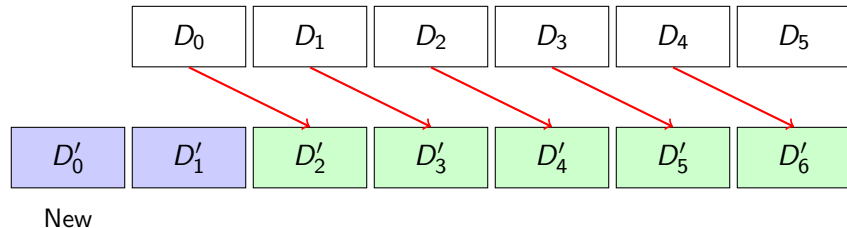
```
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1. This slide shows the algorithm in more detail.
2. Lines 1 and 2 create the two new empty structures, D'_0 and D'_1 .
3. Line 3 is the reuse: we loop from $i = 2$ up to $k + 1$, and simply assign the old D_{i-2} to be the new D'_i . This is just a pointer swap; it's instant.
4. Line 4 is where the work happens. We loop through our new structures and apply the missing updates to each one, from its old checkpoint time t_{i-2} to its new checkpoint time t'_i .
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6. The diagram at the bottom visualizes this reuse. The new D'_0 and D'_1 are built from scratch, but all the others, D'_2 through D'_{k+1} , are just the old D_0 through D_{k-1} , shifted over and updated.
7. Again, this gives us the $O(\sqrt{m} \log n)$ amortized time...
8. ...but it requires $\Theta(m\sqrt{m})$ space, because we are storing these \sqrt{m} independent copies.

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Original

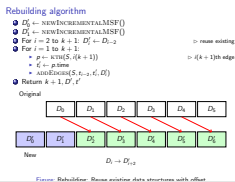


$$D_i \rightarrow D'_{i+2}$$

Partial to full retroactivity

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Our contribution

- **General transformation:** Partial \rightarrow Full retroactivity
- **No persistent data structures needed**
- **Same time complexity:** $\mathcal{O}(\sqrt{m})$ per operation
- **Space trade-off:** $\Theta(m\sqrt{m})$ vs $\mathcal{O}(m)$

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Partial to full retroactivity

└ Results

Results

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1. So, to summarize our contributions:
2. We've developed a general transformation to take a partially retroactive data structure and make it fully retroactive.
3. Crucially, our method **does not require persistent data structures**.
4. We match the $\mathcal{O}(\sqrt{m})$ slowdown per operation from the Demaine et al. paper...
5. ...at the cost of $\Theta(m\sqrt{m})$ space, which we argue is a very practical trade-off for simplicity.
6. ——— SKIP SLIDE ———
7. Applying this to our test case, we get a semi-retroactive MSF implementation.
8. It supports adding edges and querying the MSF at any time t in $\mathcal{O}(\sqrt{m} \log n)$ amortized time.
9. And, we have successfully removed the limitations from the previous work: our structure works **without** a fixed m or a fixed time range.

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Semi-retroactive MSF implementation

- **Operations:** `add_edge(u, v, w, t)`, `get_msf(t)`
- **Time:** $\mathcal{O}(\sqrt{m} \log n)$ per operation
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Partial to full retroactivity

Results

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Thank you!

Questions?

2025-10-26

Partial to full retroactivity

Thank you!

Questions?

1. Invite questions from the audience
2. Be prepared to answer questions about:
 3. * The rebuilding algorithm details
 4. * Space vs time trade-offs
 5. * Implementation challenges
 6. * Comparison with persistent data structures
 7. * Applications beyond MSF
8. Key points to emphasize if asked:
 9. * Our approach is simpler to implement
10. * Same time complexity as Demaine et al.
11. * No persistent data structure requirement
12. * General applicability to any partially retroactive structure
13. Thank the audience for their attention