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Optimal Timing of Non-Renewable Resource Process Execution

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Abstract This article considers the problem of how to optimally time the execution of a project consisting of N jobs that ought to be processed sequentially according to a known sequence. This setting arises in various industries, including real estate and mining. While the time and cost of processing each job is deterministic, their return depend on a market price (or economic index) that evolves stochastically over time. Upon completion of a job, the decision maker can opt to either start the next job in the sequence, suspend the project or abandon it altogether. We use a dynamic programming formulation of the problem to find upper and lower bounds on the optimal expected net return and develop a family of approximations for which the optimal policy is shown to be of the threshold-type. We also derive approximating policies whose performance is tested numerically. Finally, the methodology is applied to a real case in the context of underground metal mine project to derive a long-term extraction plan.

Keywords Real options · optimal stopping time · approximate dynamic programming - mining industry

1 Introduction

Consider the problem faced by a decision maker who must decide the timing at which to execute a project consisting of N jobs that must be processed according to a known, endogenously determined, sequence. The processing of each job can start only if the previous job in the sequence has been completed. Upon completion, each job generates a net profit that depends on the value of a stochastically evolving market indicator or price. Specifically, we assume that profit generated by a job is equal to the product of the market price at the time of completion times a deterministic job-dependent weight. (Depending on the application, we can interpret this weight as the amount of a marketable asset that each job contains.) The cost of processing a job is deterministic and depends on the aforementioned weight. Upon completion of a job the decision maker can opt to process the next job, suspend the project and wait for the price to evolve, or shut down and abandon the project altogether. Suspension and waiting incurs a cost per unit of time associated with maintaining the productive capacity necessary for processing the jobs. Shutdown and abandonment of the project typically involves a fixed cost penalty for non-completion of commitments, although in some cases it may actually bring a residual benefit. The problem thus consists

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of computing the expected maximum economic value of the project and identifying the optimal policy for obtaining that value.

There are numerous real-world situations in which a problem of this type may arise. Take, for example, the case of a large-scale real-estate development project that must follow a given geographically ordered sequence of lots. The project manager develops one of the lots and then, depending on the state of the real-estate market, either continues developing a new lot or waits until conditions improve. If prices are very low and not expected to rise in the near future, the project manager may abandon further development and sell off the undeveloped lots for a residual benefit.

Another example is the working of an ore deposit at a mine. Blocks of ore are extracted from the deposit in a known sequence and the benefit derived from each one depends on the market price of the metal it contains, which evolves randomly over time. Assuming that both the extraction cost and the metal content of ore blocks are known, the mine operator may suspend operations if the market price of the metal is too low and wait until it improves, while in the meantime incurring a cost for maintaining the mine's production capacity. If the operator decides to shut down the mine permanently, it will incur environmental mitigation costs but will also have the option of attaining a residual benefit by selling the property.

In terms of related literature, the class of problems illustrated by these two specific cases can be viewed as an application of real options theory (see [6, 21]) when the underlying market price is correlated to some tradable security (e.g., a futures market for the output of a mining project). Previous work in this area analyze situations similar to those we have just presented. Titman [20], for instance, develops a model that provides a model for pricing vacant lots and provides intuition regarding the conditions under which it is rational to defer construction to a later date. The author adopts a real options approach, first introduced by Black and Scholes [2] and Merton [12], to determine the explicit value of vacant lots in an urban area. Another example is Guthrie [7], who uses a real options approach and a binomial tree to evaluate a real-estate project consisting of five separate stages with the options of continuing, suspending or abandoning development.

In yet another case, Dixit [5] examines a firm's entry and exit decisions when the output price is a geometric Brownian motion, shown that the optimal policy is characterized by a pair of trigger prices for entry and exit. McDonald and Siegel [10] develop and study a methodology for valuing risky investment projects where there is an option of temporarily and costlessly shutting down production whenever variable costs exceed operating revenues. In [11] the same authors consider the optimal timing of investment in a project whose benefits and investment cost behave stochastically over time. They obtain the optimal investment policy and the value of the investment option in closed-form, they also analyze the scrapping decision. Schwartz et al [18] develop a real options model for valuing natural resources exploration investments when there is joint price and geological-technical uncertainty. The authors allow for a multiple-stage exploration phase, a development investment phase and an extraction phase. They consider a timing option for the development investment phase and closure, opening and abandonment options for the extraction phase. See [3, 9, 14, 15] for further applications of the methodology.

For many assets such as company shares or gold, price is commonly assumed to follow geometric Brownian motion (GBM), both in academia and industry. For others assets, such as copper or petroleum, while is widely accepted that price trajectories exhibit regression to a mean, the GBM assumption either can not be statistically discarded, or remains true after a change of measure arguments (usually involving assumptions on the market price of risk and/or convenience yields; see [19, Chapter 5]). In the present analysis we assume that the price of the underlying asset follows a GBM, which seems to be aligned with practice. Our model is formulated as a dynamic program where the state variable is the market price of the underlying asset and solving each stage is equivalent to solving an optimal stopping problem. We construct upper and lower bounds to solve the problem approximately, develop a family of approximations based on these bounds that can be solved by a threshold-type policy, and specify a procedure for calculating such thresholds.

The principal contribution of this paper is the identification and formalization of a type of decision-making problem under uncertainty that is frequently encountered in many industries. The problem is a complex one and we have not found an analytic solution, but by exploiting the structure and properties of our formulation we are able to devise an algorithm that can find high quality approximate solutions.

The remainder of this paper is organized into five sections. In Section 2 we describe our model and formulate the problem to be solved as a sequential optimal stopping problem. In Section 3 we use optimal stopping theory to propose a solution. We also derive some properties for the value function and compute upper and lower bounds for the optimal value. In Section 4 we generalize these bounds to a family of approximations that share the same problem structure. For these approximations we provide a result ensuring that the optimal policies are of the threshold type. We also discuss how to compute the thresholds and develop a procedure to efficiently compute the optimal policies. In section 5 we use a test instance to demonstrate the quality of our proposed approximations and then apply our methodology to a very large-scale real-world mining project. Our conclusions and suggestions for future research are presented in Section 6. Proofs are relegated to an appendix.

2 The Model

Consider a stochastic process $\{S_t\}_{t\geq 0}$ defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration generated by the $\{S_t\}$ process. Here S_t represents the price of an underlying asset at time t. We assume that $\{S_t\}_{t\geq 0}$ evolves as a geometric Brownian motion and solves the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = S, \tag{1}$$

where $\mu > 0$ is the drift coefficient, $\sigma > 0$ is the process volatility and B_t is a one-dimensional Brownian motion.

Now consider a project consisting of a fixed sequence of jobs $\pi := \{\pi_1, \ldots, \pi_N\}$, where π_N is the first job in the sequence and π_1 the last. Each job is characterized by two parameters; Q_i, which denotes job π_i's weight (or alternatively, its size); and L_i, which denotes the fraction of job π_i's weight that corresponds to the underlying asset. Job π_i is processed in time T_i, which is equal to Q_i/K where K is a processing capacity. Processing job π_i incurs an execution cost of A_i per unit of weight. Decision variable τ_i indicates the time at which π_i starts its processing. Thus, τ_i is an $\mathcal{F}_t$-stopping time satisfying

$$\tau_i \geq \tau_{i+1} + T_{i+1} \quad \text{and} \quad \tau_N \geq 0. \tag{2}$$

While being processed, job π_i generates revenue due to the sales of the asset contained in the job. (We assume the asset contained in a job is extracted at a constant rate during processing). If the decision maker decides to abandon all further jobs a shutdown cost C_0 is incurred, which if negative represents a residual benefit. If it is decided to suspend further processing temporarily, a waiting cost M is incurred for each time unit of waiting (which reflects the expense of maintaining production capacity).

Let r be the time discount rate for the project. Also let $\widetilde{W}_i$ be the net present value of processing job π_i at start time τ_i and $W_i(S_{\tau_i})$ be the expected value of $\widetilde{W}_i$. Then

$$\widetilde{W}_i = \int_0^{T_i} e^{-rt}(L_i K S_{\tau_i+t} - A_i K)dt \quad \text{and,} \quad W_i(S_{\tau_i}) = \mathbb{E}[\widetilde{W}_i | \mathcal{F}_{\tau_i}]. \tag{3}$$

Note that $W_i(S_{\tau_i}) = R_i S_{\tau_i} - C_i$, where

$$R_i \triangleq \mathbb{E}\Big[\int_0^{T_i} e^{-rt} L_i K S_{\tau_i+t} dt | \mathcal{F}_{\tau_i}\Big] = L_i K \left\{\frac{1 - e^{-(r-\mu)T_i}}{r-\mu}\right\}$$

and

$$C_i \triangleq \int_0^{T_i} e^{-rt} A_i K dt = A_i K \left\{\frac{1 - e^{-rT_i}}{r}\right\}.$$

We assume that the decision-maker aims at maximizing the expected net present value of the costs and revenues associated with the job execution, suspension (waiting) and shutdown (abandonment) decisions.

Consider the following dynamic programming formulation of the problem where the state variable is the asset's price and the stages are the various jobs. The value function $V_i(S)$ denotes the expected maximum net present value at the moment job π_{i+1} has just been completed and the asset's price is currently S. The optimal value of the objective function is therefore $V_N(S_0)$. We can write the Bellman equation for the value function as follows

$$(\mathcal{P}) \qquad V_i(S) = \sup_{\tau>0} \mathbb{E}\left[e^{-r\tau} \max\left\{ W_i(S_\tau) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\tau+T_i})|\mathcal{F}_\tau], -C_0 \right\} - \int_0^\tau e^{-rt} M dt \Big| S_0 = S \right]$$

subject to $\quad dS_t = \mu S_t dt + \sigma S_t dB_t$

τ is a valid $\mathcal{F}_t -$ stopping time

and $V_0(S) = -C_0$ for all S.

The optimal strategy takes into account the options to wait, continue processing (i.e., continuing to the next job) or abandon the project. If the option to continue is chosen, profit is given by

$$W_i(S_\tau) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\tau+T_i})|\mathcal{F}_\tau],$$

which includes the profit obtained from the execution of π_i plus those associated with the decisions relating to the next job T_i time units later. If the decision maker opts for abandonment, shutdown cost C_0 is incurred. The integral represents the cost of maintaining production capacity until τ, at which point either job π_{i-1} is started or the project is abandoned.

3 Model Analysis and Properties

In this section we study a solution for problem $(\mathcal{P})$ using optimal stopping theory, We also develop lower and upper bounds for $V_i(S)$ that will allow us to solve the problem approximately.

3.1 Model Analysis

To find the optimal value of our problem we proceed iteratively, solving for $V_1(S)$ first, then for $V_2(S)$ and each successive term until we find $V_N(S)$. As demonstrated in [13], solving each instance of $V_i(S)$ is equivalent to finding the time of an optimal stopping problem. Furthermore, it turns out that a solution to this class of problems is of the threshold type, with optimal stopping time given by

$$\tau^* = \inf\{t \geq 0 : S_t \notin (S_i^a, S_i^b)\}.$$

On this policy, if the price is below S_i^a the project should be immediately shut down and if it is above S_i^b the processing of job π_i should start immediately. If, on the other hand, the price is at an intermediate level in the interval (S_i^a, S_i^b), processing of job π_i should wait until either the price drops below S_i^a or rises above S_i^b, at which point the appropriate action should be taken. The above-mentioned interval is called the continuation region.

For $(\mathcal{P})$ to have a solution, according to the verification theorem in [13], it must be satisfied that

$$-rV_i(S) + \mu S \frac{\partial}{\partial S} V_i(S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} V_i(S) = M, \quad S_i^a < S < S_i^b, \tag{3}$$

where S_i^a and S_i^b satisfy the following boundary conditions:

$$V_i(S_i^a) = -C_0, \tag{4a}$$

$$V_i(S_i^b) = W_i(S_i^b) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{T_i})|S_0 = S_i^b], \tag{4b}$$

$$\frac{\partial}{\partial S} V_i(S_i^a) = 0, \tag{4c}$$

$$\frac{\partial}{\partial S} V_i(S_i^b) = R_i + e^{-rT_i}\frac{\partial}{\partial S}\mathbb{E}[V_{i-1}(S_{T_i})|S_0 = S]\Big|_{S=S_i^b}. \tag{4d}$$

Conditions (4a)-(4d), known as value matching and smooth pasting, are smoothness and continuity conditions. The solution of partial differential equation (3) is known and given by

$$V_i(S) = B_i S^{\lambda_1} + D_i S^{\lambda_2} - M/r, \quad S_i^a < S < S_i^b, \tag{5}$$

where

$$\lambda_{1,2} \triangleq \frac{1}{\sigma^2}\Big[\frac{1}{2}\sigma^2 - \mu \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}\Big], \quad \text{and} \quad \lambda_2 < 0 < 1 < \lambda_1. \tag{6}$$

Finally, $V_i(S)$ satisfies

$$V_i(S) = \begin{cases} -C_0 & \text{if } S \le S_i^a \\ B_i S^{\lambda_1} + D_i S^{\lambda_2} - M/r & \text{if } S_i^a < S < S_i^b \\ W_i(S) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{T_i})|S_0 = S] & \text{if } S_i^b \le S. \end{cases} \tag{7}$$

In the above, we have that when market price is less than S_i^a, the project must be abandoned and the shutdown cost paid. If such a price is greater than S_i^b on the other hand, job π_i should be processed, thus obtaining the expected profit, $W_i(S)$, for the job plus the expected profit on revenues from future jobs, all discounted to the start time of job π_i. The terms $B_i S^{\lambda_1}$ and $D_i S^{\lambda_2}$ on the interval (S_i^a, S_i^b) are related to the expected discount for the time taken by the process to exit the interval, see [16, Proposition 8.4.1]. Finally, the $-M/r$ term represents the cost of maintaining production capacity for an indefinite period without either abandoning the project or processing any further jobs, discounted at rate r.

Constants S_i^a, S_i^b, B_i and D_i are the solution to a system of non-linear equations consisting of (7) and (4). Unfortunately, there is no closed formula for these constants, even for the simplest case of $i = 1$.

Next, we approximate $V_i(S)$ by constructing two functions $V_i^U(S)$ and $V_i^L(S)$, which we show are upper- and lower-bounds for the value function. Interestingly, associated with these new functions, in Section 4.1 we show that they satisfy a similar set of optimality conditions as those in (4) but without the conditional expectation, which is a property that significantly simplify their analysis. In what follows, we present these bounds and other useful properties of $V_i(S)$.

3.2 Properties of $V_i(S)$

In order to simplify the exposition, we consider the following notation. For two vectors X and α define

$$X_{k,i}^+ \triangleq \sum_{h=k+1}^{i} X_h, \quad \alpha_{k,i}^\times \triangleq \prod_{h=k+1}^{i} \alpha_h \quad \text{and} \quad (\alpha^\times X)_{k,i}^+ \triangleq \sum_{h=k+1}^{i} \alpha_{h,i}^\times X_h.$$

We begin with the following proposition regarding the asymptotic behaviour of $V_i(S)$:

Proposition 1 *For all $i \ge 1$, $V_i(S)$ has the following linear asymptotic behaviour:*

$$(\mathcal{R}_i S - \mathcal{C}_i) \le V_i(S) \le G_i(S) + (\mathcal{R}_i S - \mathcal{C}_i), \tag{8}$$

where

$$G_i(S) \triangleq \frac{i(i+1)}{2} \cdot S_i^m \cdot R_i^m \cdot \exp\left(-\frac{1}{\sigma^2} \min_{1\le k\le i} \frac{1}{T_{k,i}^+}\Big[\ln\Big(\frac{S}{S_i^m}\Big) + \{r - \rho + \frac{\sigma^2}{2}\}T_{k,i}^+\Big]^2\right),$$

$$\mathcal{R}_i \triangleq \sum_{h=1}^{i} e^{-(r-\mu)T_{h,i}^+} R_h, \quad \mathcal{C}_i \triangleq \sum_{h=0}^{i} e^{-rT_{h,i}^+} C_h, \quad S_i^m \triangleq \max_{1\le k\le i} S_k^b \quad \text{and} \quad R_i^m \triangleq \max_{1\le k\le i} R_k.$$

Clearly, $G_i(S) > 0$ and $G_i(S) \to 0$ when $S \to \infty$.

This proposition implies that, roughly speaking, when S is sufficiently high the decision-maker executes the entire set of jobs without suspensions or abandonment. The next proposition provides further insight on the value function.

Proposition 2 *For all $i \geq 1$, the function $V_i(S)$ is convex and increasing.*

The growth of $V_i(S)$ reflects the intuitive notion that the higher the asset's market price, the higher the project's expected value. Given the convexity of $V_i(S)$ and recalling Jensen's inequality, we can construct a lower bound denoted $V_i^L(S)$ that is a proxy of the value function. For this, we simply assume that the market price at the time job π_i finishes being processed equals its expected value. More precisely, if at the start of the π_i process the price is S_τ, then its expected value at completion is $e^{\mu T_i} S_\tau$. The lower-bound function $V_i^L(S)$ is then defined as

$$V_i^L(S) \triangleq \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-r\tau} \max\big\{W_i(S_\tau) + e^{-rT_i} V_{i-1}^L(e^{\mu T_i} S_\tau), -C_0\big\} - \int_0^\tau e^{-rt} M dt \Big| S_0 = S\Big]. \tag{9}$$

In order to construct an upper-bound function, we use the following lemma.

Lemma 1 *For all $i \geq 1$ and $t \geq 0$, $V_i(\cdot)$ satisfies the following inequality:*

$$e^{-rT_i} V_i(S_{t+T_i}(\omega)) - \int_0^{T_i} e^{-rs} M ds \leq V_i(S_t(\omega)), \quad \textit{for all } \omega \in \Omega. \tag{10}$$

The value of $V_i(S_t)$ is the optimum among all possible strategies if there are i jobs yet to be processed and the price is S_t. One such strategy may be to suspend processing and wait for a period of time T_i, thus incurring the corresponding production capacity maintenance cost. The lemma merely establishes that this particular strategy cannot exceed the value of the optimal strategy.

To define an upper bound function consider a relaxation of the problem where the decision-maker does not need to wait for a job to complete its processing before considering starting processing the next one. With this result we define the following upper bound function, denoted $V_i^U(\cdot)$:

$$V_i^U(S) \triangleq \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-r\tau} \max\{W_i(S_\tau) + V_{i-1}^U(S_\tau) + \int_0^{T_i} e^{-rt} M dt, -C_0\} - \int_0^\tau e^{-rt} M dt \Big| S_0 = S\Big]. \tag{11}$$

The following corollary establishes that the above are valid lower- and upper- bounds to the value function.

Corollary 1 *Assume that $C_0 < M/r$. Then, for all $i \geq 1$,*

$$V_i^L(S) \leq V_i(S) \leq V_i^U(S). \tag{12}$$

4 Approximations and Algorithm

In this section we present a new family of functions $\mathcal{V}_i(S)$ that generalizes the lower and upper bounds $V_i^L(S)$ and $V_i^U(S)$. We also set out an algorithm for computing this family of approximations and analyze their asymptotic behaviour when the price tends to infinity.

4.1 (α, η, γ)-Aproximaciones

Consider the function

$$\mathcal{V}_i(S) \triangleq \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-r\tau} \max\{W_i(S_\tau) + \gamma_i + \alpha_i \mathcal{V}_{i-1}(\eta_i S_\tau), -C_0\} - \int_0^\tau e^{-rt} M dt \Big| S_0 = S\Big], \tag{13}$$

where $\mathcal{V}_0(S) = -C_0$. This function generalizes V_i^L and V_i^U: if α_i and η_i are equal to 1 and γ_i is equal to $\int_0^{T_i} e^{-rt} M dt$, then we obtain $V_i^U(S)$; similarly, if α_i is equal to e^{-rT_i}, η_i is equal to $e^{\mu T_i}$ and γ_i is equal to 0, then we get $V_i^L(S)$.

As with $(\mathcal{P})$, the problem associated with $\mathcal{V}_i(S)$ is also a sequential optimal stopping problem. Making use once again of the verification theorem, we propose the following solution for (13)

$$\mathcal{V}_i(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_i^a \\ \mathcal{B}_i S^{\lambda_1} + \mathcal{D}_i S^{\lambda_2} - M/r & \text{if } \mathcal{S}_i^a < S < \mathcal{S}_i^b \\ R_i S - (C_i - \gamma_i) + \alpha_i \mathcal{V}_{i-1}(\eta_i S) & \text{if } \mathcal{S}_i^b \leq S, \end{cases} \tag{14}$$

where the thresholds $\mathcal{S}_i^a, \mathcal{S}_i^b$ and the constants $\mathcal{B}_i, \mathcal{D}_i$ are solutions of the following non-linear system

$$\mathcal{B}_i \mathcal{S}_i^{a^{\lambda_1}} + \mathcal{D}_i \mathcal{S}_i^{a^{\lambda_2}} - M/r = -C_0 \tag{15a}$$

$$\mathcal{B}_i \mathcal{S}_i^{b^{\lambda_1}} + \mathcal{D}_i \mathcal{S}_i^{b^{\lambda_2}} - M/r = (R_i \mathcal{S}_i^b - (C_i - \gamma_i)) + \alpha_i \mathcal{V}_{i-1}(\eta_i \mathcal{S}_i^b) \tag{15b}$$

$$\lambda_1 \mathcal{B}_i \mathcal{S}_i^{a^{\lambda_1 - 1}} + \lambda_2 \mathcal{D}_i \mathcal{S}_i^{a^{\lambda_2 - 1}} = 0 \tag{15c}$$

$$\lambda_1 \mathcal{B}_i \mathcal{S}_i^{b^{\lambda_1 - 1}} + \lambda_2 \mathcal{D}_i \mathcal{S}_i^{b^{\lambda_2 - 1}} = R_i + \alpha_i \eta_i \mathcal{V}'_{i-1}(\eta_i \mathcal{S}_i^b). \tag{15d}$$

To prove the validity of (14) we must find the conditions under which (15) has a solution. The following theorem establishes two conditions for solving $\mathcal{V}_1$, the simplest case of just one job.

Theorem 1 *Consider a problem with one job. If i) $C_0 < M/r$ and ii) $C_0(1-\alpha_1) < C_1 - \gamma_1$, then* (15) *has a solution where $0 < \mathcal{S}_1^a < \mathcal{S}_1^b$ and $\mathcal{B}_1, \mathcal{D}_1 > 0$. Furthermore, $\tau^* = \inf\{t \geq 0 : S_t \notin (\mathcal{S}_1^a, \mathcal{S}_1^b)\}$ is an optimal stopping time for $\mathcal{V}_1(S)$ and its solution is given by* (14).

The first condition in Theorem 1 imposes that the present value of the cost of maintaining production capacity indefinitely in a state of waiting be greater than the shutdown cost. This gives a positive value to the abandonment option as otherwise shutdown would never be optimal. When $\gamma_1 = 0$ and $\alpha_1 = e^{-rT_1}$ the second condition reduces to $C_0 < A_1 K/r$, which is the equivalent of requiring that shutdown cost be less than the present value of continuing job processing to infinity. When $\gamma_1 = \int_0^{T_1} e^{-rt} M dt$ and $\alpha_1 = 1$, this condition implies that the present cost of processing must be greater than that of simply maintaining capacity during the same period. Both conditions are reasonable in practice and make economic sense.

In order to compute $\mathcal{V}_i$ for $i \geq 1$ we have to know $\mathcal{V}_{i-1}$ evaluated at $\eta_i \mathcal{S}_i^b$. Since S_i^b is unknown, we must iterate (14) using $\mathcal{V}_{i-1}(\eta_i \mathcal{S}_i^b)$ to identify what interval $\eta_i \mathcal{S}_i^b$ belongs to. Next, we show how to solve the case for two jobs, which can be directly generalized to the case of i jobs. Let

$$\mathcal{V}_1(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_1^a \\ \mathcal{B}_1 S^{\lambda_1} + \mathcal{D}_1 S^{\lambda_2} - M/r & \text{if } \mathcal{S}_1^a < S < \mathcal{S}_1^b \\ R_1 S - (C_1 - \gamma_1) - \alpha_1 C_0 & \text{if } \mathcal{S}_1^b \leq S, \end{cases}$$

For $\mathcal{V}_2(S)$ we propose the following function:

$$\mathcal{V}_2(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_2^a \\ \mathcal{B}_2 S^{\lambda_1} + \mathcal{D}_2 S^{\lambda_2} - M/r & \text{if } \mathcal{S}_2^a < S < \mathcal{S}_2^b \\ R_2 S - (C_2 - \gamma_2) + \alpha_2 \mathcal{V}_1(\eta_2 S) & \text{if } \mathcal{S}_2^b \leq S. \end{cases}$$

To calculate S_2^a, S_2^b, B_2, D_2 it is sufficient to know $\mathcal{V}_1(\eta_2 S_2^b)$. We therefore propose a method that iterates assuming $\eta_2 \mathcal{S}_2^b$ belongs to one of the three intervals defined for $\mathcal{V}_1(S)$, after which we solve the system defined by (15) and then check whether the assumption is satisfied. If it is not, we repeat the procedure and check for a second interval and, if need be, for the third. For example, if we assume that $\eta_2 \mathcal{S}_2^b \leq \mathcal{S}_1^a$, we then solve the problem of the second job using the fact that $\mathcal{V}_1(\eta_2 S_2^b)$ is equal to $-C_0$. Here we can apply Theorem 1 using (15) evaluated for this situation and calculate $\mathcal{B}_2, \mathcal{D}_2, \mathcal{S}_2^a$ and $\mathcal{S}_2^b$. The value function $\mathcal{V}_2(S)$ is then given by

$$\mathcal{V}_2(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_2^a \\ \mathcal{B}_2 S^{\lambda_1} + \mathcal{D}_2 S^{\lambda_2} - \frac{M}{r} & \text{if } \mathcal{S}_2^a < S < \mathcal{S}_2^b \\ (R_2 S - (C_2 - \gamma_2) - \alpha_2 C_0 & \text{if } \mathcal{S}_2^b \leq S \leq \mathcal{S}_1^a/\eta_2 \\ (R_2 S - (C_2 - \gamma_2) + \alpha_2(\mathcal{B}_1(\eta_2 S)^{\lambda_1} + \mathcal{D}_1(\eta_2 S)^{\lambda_2} - \frac{M}{r}) & \text{if } \mathcal{S}_1^a/\eta_2 < S < \mathcal{S}_1^b/\eta_2 \\ (R_2 S - (C_2 - \gamma_2)) + \alpha_2(R_1 \eta_2 S - (C_1 - \gamma_1) - \alpha_1 C_0) & \text{if } \mathcal{S}_1^b/\eta_2 \leq S. \end{cases} \tag{16}$$

If solution (16) contradicts the initial assumption that $\eta_2 \mathcal{S}_2^b \leq \mathcal{S}_1^a$, we assume that $\mathcal{S}_1^a < \eta_2 \mathcal{S}_2^b < \mathcal{S}_1^b$. In this case, $\mathcal{V}_1(\eta_2 S_2^b) = \mathcal{B}_1(\eta_2 S_2^b)^{\lambda_1} + \mathcal{D}_1(\eta_2 S_2^b)^{\lambda_2} - M/r$ and the solution obtained is the same as (16) except that $\mathcal{S}_2^b > \mathcal{S}_1^a/\eta_2$, meaning that the third interval disappears. If this solution also contradicts the initial assumption that $\mathcal{S}_1^a < \eta_2 \mathcal{S}_2^b < \mathcal{S}_1^b$, we assume that $\eta_2 \mathcal{S}_2^b \geq \mathcal{S}_1^b$. In this case, $\mathcal{V}_1(\eta_2 S_2^b) = R_1(\eta_2 S_2^b) - (C_1 - \gamma_1) - \alpha_1 C_0$ and the solution is the same as (16) with the exception that $\mathcal{S}_1^b/\eta_2 < \mathcal{S}_2^b$ and thus the third and fourth intervals disappear.

4.2 Asymptotic Approximations

It can be shown that if S is sufficiently large, then $\mathcal{V}_i(S)$ is a linear function. More precisely, when the market price is above $\bar{\mathcal{S}}_i \triangleq \max_{1\le k\le i}\{\mathcal{S}^b_k/\eta^\times_{k,i}\}$, we have that

$$\mathcal{V}_i(S) = ((\alpha\eta)^\times R)^+_i S - (\alpha^\times(C-\gamma))^+_i - \alpha^\times_{0,i}C_0, \quad \text{for all } S \ge \bar{\mathcal{S}}_i.$$

Specializing this result for the case $\alpha_i = \eta_i = 1$, we have that

$$V^U_i(S) = R^+_i S - (C-\gamma)^+_i - C_0, \quad \text{for all } S \ge \bar{\mathcal{S}}_i.$$

In similar fashion, if we specialize for the case $\alpha_i = e^{-rT_i}$, $\eta_i = e^{\mu T_i}$ and $\gamma_i = 0$, we have that

$$V^L_i(S) = \mathcal{R}_i S - \mathcal{C}_i, \quad \text{for all } S \ge \bar{\mathcal{S}}_i.$$

This asymptotic behaviour of V^L_i is the same as that presented by V_i described in Proposition 1. With this result we can now calculate a new function for the asymptotic behaviour of $V_i(S)$. First, we know that for a sufficiently large S, $V_{i-1}(S) \approx V^L_{i-1}(S)$ and therefore $\mathbb{E}[V_{i-1}(S_{T_i})|S_0 = S] \approx \mathbb{E}[V^L_{i-1}(S)|S_0 = S]$.

Using the observation above and (7) we find a new asymptotic approximation for V_i, which we refer to as $\widehat{V^L_i}$, defined by

$$\widehat{V^L_i}(S) \triangleq \begin{cases} -C_0 & \text{if } S \le \widehat{S^a_i} \\ \widehat{B_i}S^{\lambda_1} + \widehat{D_i}S^{\lambda_2} - M/r & \text{if } \widehat{S^a_i} < S < \widehat{S^b_i} \\ (R_i + e^{-(r-\mu)T_i}\mathcal{R}_{i-1})S - (C_i + e^{-rT_i}\mathcal{C}_{i-1}) & \text{if } \widehat{S^b_i} \le S, \end{cases}$$

where $\widehat{B_i}$, $\widehat{D_i}$ and $\widehat{S^a_i}$ and $\widehat{S^b_i}$ are calculated by imposing the smooth pasting and value matching conditions as in (15). The convergence of V_i and V^L_i, as S becomes large, suggests that $\widehat{V^L_i}$ is a good quality approximation. Similarly, we can construct an approximation $\widehat{V^U_i}$ based on the asymptotic behaviour of V^U_i. In such a case, however, there is no result suggesting that this approximation is of good quality. As we will see later, numerical examples confirm these hypotheses.

4.3 V-approx Algorithm

The following proposition characterizes the number of intervals in the function V_i.

Proposition 3 *Let m_i be the number of intervals in the function V_i and l_i be the index of the corresponding interval if the price equals $\eta_{i+1}\mathcal{S}^b_{i+1}$. Then*

$$m_i = 2 + (m_{i-1} - l_{i-1} + 1).$$

Note that m_i is at least 3 and at most $2i+1$.

To facilitate the notation we introduce a family of matrices Y^i, each one associated with the function $\mathcal{V}_i(S)$. The rows of Y^i represent the pieces of $\mathcal{V}_i$ while the columns are the 6 constants that define each function: R_i, C_i, $\mathcal{B}_i$, $\mathcal{D}_i$, M/r and C_0. As an example, for the function $\mathcal{V}_2$ given by $\mathcal{V}_2$, the matrix is

$$Y^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & C_0 \\ 0 & 0 & \mathcal{B}_2 & \mathcal{D}_2 & M/r & 0 \\ R_2 & (C_2-\gamma_2) & 0 & 0 & 0 & \alpha_2 C_0 \\ R_2 & (C_2-\gamma_2) & \alpha_2\eta_1^{\lambda_1}\mathcal{B}_1 & \alpha_2\eta_1^{\lambda_2}\mathcal{D}_1 & \alpha_2 M/r & 0 \\ (R_2+\alpha_2\eta_2R_1) & ((C_2-\gamma_2)+\alpha_2(C_1-\gamma_1+\alpha_1C_0)) & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We also introduce the family of matrices P^i and the family of vectors $v^i(S)$ and $I^i(S)$. The P^i rows represent the $\mathcal{V}_i$ thresholds, the $v^i(S)$ rows represent the values at which $\mathcal{V}_i$ is evaluated and each $I^i(S)$ column indicates the piece of $\mathcal{V}_i$ that S belongs to. For the case of (16),

$$P^2 = \begin{pmatrix} -\infty & \mathcal{S}_2^a \\ \mathcal{S}_2^a & \mathcal{S}_2^b \\ \mathcal{S}_2^b & \mathcal{S}_1^a/\eta_2 \\ \mathcal{S}_1^a/\eta_2 & \mathcal{S}_1^b/\eta_2 \\ \mathcal{S}_1^b/\eta_2 & \infty \end{pmatrix}, \quad v^2(S) = \begin{pmatrix} S \\ -1 \\ S^{\lambda_1} \\ S^{\lambda_2} \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad I^2(S) = \begin{pmatrix} \mathbf{1}_{\{(P^2_{1,1}, P^2_{1,2})\}} \\ \mathbf{1}_{\{(P^2_{2,1}, P^2_{2,2})\}} \\ \mathbf{1}_{\{(P^2_{3,1}, P^2_{3,2})\}} \\ \mathbf{1}_{\{(P^2_{4,1}, P^2_{4,2})\}} \\ \mathbf{1}_{\{(P^2_{5,1}, P^2_{5,2})\}} \end{pmatrix},$$

so that

$$\mathcal{V}_2(S) = I^2(S)^\top Y^2 v^2(S).$$

Using this notation we can write (15) in compact form. Generalizing the results of Theorem 1 we propose the following V-approx algorithm for computing the values of $\mathcal{B}_i, \mathcal{D}_i, \mathcal{S}_i^a$ and $\mathcal{S}_i^b$, thus giving the value of the approximation function $\mathcal{V}_i$.

Algorithm 1 V-approx

Require: $\{(R_k), (C_k), (\alpha_k, \eta_k, \gamma_k)\}_{k=1}^i, M/r, C_0$
1: Create Y^1, P^1 and solve the System (15) for $k = 1$. Set $k = 2$.
2: Compute $m =$ number of rows of Y^{k-1}.
3: **for** $l = 1 \dots m$ **do**
4: Create $\tilde{Y}^l, \tilde{P}^l$, both of which are zero matrices with $2 + (m - l + 1)$ rows, and 6 and 2 columns, respectively.
5: Set $\tilde{Y}^l_{1,\cdot} = (0\ 0\ 0\ 0\ 0\ C_0)$, $\tilde{Y}^l_{2,\cdot} = (0\ 0\ B_k\ D_k\ M/r\ 0)$, $\tilde{P}^l_{1,\cdot} = \left(-\infty\ \mathcal{S}_k^a\right)$ and $\tilde{P}^l_{2,\cdot} = \left(\mathcal{S}_k^a\ \mathcal{S}_k^b\right)$.
6: **for** $j = 1 \dots (m - l + 1)$ **do**
7: Set

$$\begin{aligned} \tilde{Y}^l_{(2+j),1} &= R_k + \alpha_k \cdot \eta_k \cdot Y^{k-1}_{(l-1+j),1} & \tilde{Y}^l_{(2+j),2} &= C_k + \alpha_k \cdot Y^{k-1}_{(l-1+j),2} \\ \tilde{Y}^l_{(2+j),3} &= \alpha_k \cdot \eta_k^{\lambda_1} \cdot Y^{k-1}_{(l-1+j),3} & \tilde{Y}^l_{(2+j),4} &= \alpha_k \cdot \eta_k^{\lambda_2} \cdot Y^{k-1}_{(l-1+j),4} \\ \tilde{Y}^l_{(2+j),5} &= \alpha_k \cdot Y^{k-1}_{(l-1+j),5} & \tilde{Y}^l_{(2+j),6} &= \alpha_k \cdot Y^{k-1}_{(l-1+j),6} \\ \tilde{P}^l_{(2+j),2} &= P^{k-1}_{(l-1+j),2}/\eta_k. \end{aligned}$$

8: If $j = 1$ then $\tilde{P}^l_{(2+j),1} = \mathcal{S}_k^b$, Else $\tilde{P}^l_{(2+j),1} = P^{k-1}_{(l-1+j),1}/\eta_k$.
9: **end for**
10: Solve System (15) using Y^l. If $\eta_k \mathcal{S}_k^b \in (P^{k-1}_{l,1}, P^{k-1}_{l,2})$ then Break and define $l^* = l$.
11: **end for**
12: Set $Y^k = \tilde{Y}^{l^*}$ and $P^k = \tilde{P}^{l^*}$. If $k = i$ Stop, Else $k = k + 1$ and go to step 2.
Ensure: $\{Y^k, P^k\}_{k=1}^i$.

The following corollary bounds the number of calculations required by the algorithm.

Corollary 2 *Consider a project consisting of i jobs. The V-approx algorithm terminates in no more than $(3i - 2)$ iterations each of which solves exactly one non-linear system of the same type as* (15).

5 Numerical Experiments

We first illustrate the approximations and algorithms in the previous sections by means of numerical examples. Then, we apply our methods to a case study.

5.1 Synthetic examples.

Let us now consider an example with 5 jobs, where $C_0 = 5$, $M = 3.8$, $\mu = 0.057$, $\sigma = 0.233$ and $r = 0.12$. The characteristics of each job are summarized in Table 1.

Using the V-approx algorithm, we compute $V_5^U(S)$ and $V_5^L(S)$. In Figure 1 the resulting approximations are compared with $V_5(S)$, which is computed numerically. We observe that all three functions are asymptotically linear and that $V_5^L(S)$ converges to $V_5(S)$.

Job	R_k	C_k	T_k
1	0.25	14	1.2
2	0.3	9	1.6
3	0.4	16	1
4	0.32	10	2
5	0.35	12.25	0.7

Table 1: 5-Job Example.

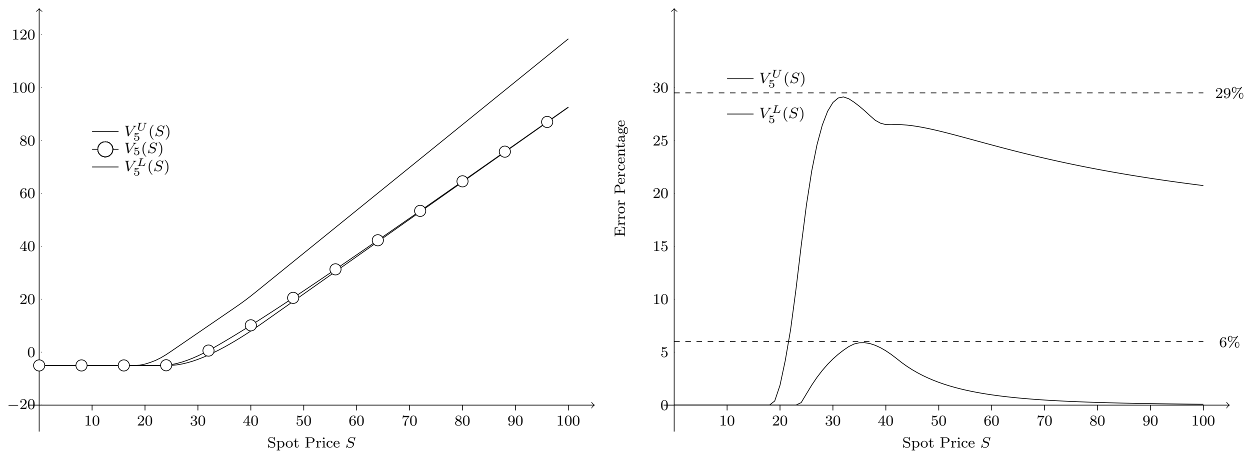

Fig. 1: *Left panel:* Value of the functions V_5 (circle),V_5^U (triangle) and V_5^L (square) as the price varies. *Right panel:* Relative error of V_5^U (triangle) and V_5^L (square) with respect to V_5. The dashed lines mark the maximum relative error of the two approximations.

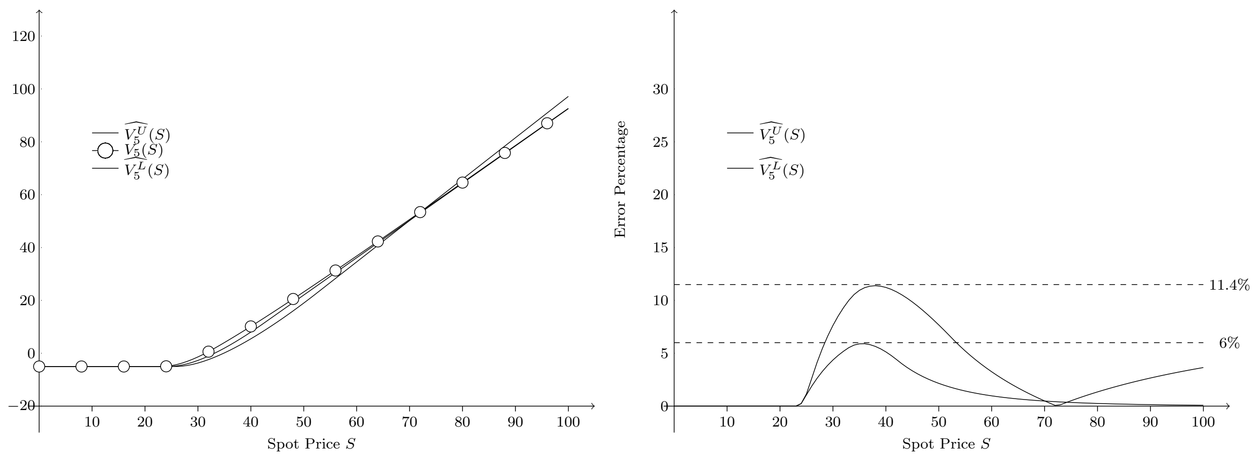

Fig. 2: *Left panel:* Value of the functions V_5 (circle),$\widehat{V_5^U}$ (triangle) and $\widehat{V_5^L}$ (square) as the price varies. *Right panel:* Relative error of $\widehat{V_5^U}$ (triangle) and $\widehat{V_5^L}$ (square) with respect to V_5. The dashed lines mark the maximum relative error of the two approximations.

Figure 2 compares $\widehat{V_5^U}(S)$ and $\widehat{V_5^L}(S)$ with $V_5(S)$. As can be seen, $\widehat{V_5^L}(S)$ provides a reasonable approximation of $V_5(S)$ at most spot prices S and converges to it. $\widehat{V_5^U}$ also provides a reasonable approximation for low prices but degrades as price increases.

Let $\mathcal{E}(\mathcal{V}_i)$ denote the average relative error of the approximation function $\mathcal{V}_i$ with respect to function V_i over the range $[S_{\min}, S_{\max}]$. That is,

$$\mathcal{E}(\mathcal{V}_i) \triangleq \frac{100}{S_{\max} - S_{\min}} \int_{S_{\min}}^{S_{\max}} \frac{|\mathcal{V}_i(S) - V_i(S)|}{V_i(S)} dS.$$

The average relative error of our approximations for the interval $[0, 100]$ is shown in Table 2. The expression V_5^{EL} is the value of V_5 under the optimal policy generated by V_5^L. These results suggest that $\widehat{V_i^L}$ and V_i^L are the best approximations of V_i. Note also that V_5^{EL} has an error close to 0.

Approximation $\mathcal{V}$	$\mathcal{E}(\mathcal{V})$
V_5^L	1.33%
V_5^U	18.66%
$\widehat{V_5^L}$	1.34%
$\widehat{V_5^U}$	3.54%
V_5^{EL}	0.32%

Table 2: Mean relative error.

5.2 Case Study

We apply the methodogy developed so far to evaluate a real project at Chuquicamata, the iconic copper mine in northern Chile owned by Codelco, the world's largest copper producer. While still the biggest and longest-running open pit operation in the world, declining economic viability will force its closure towards the end of the present decade, by which time a replacement underground mine will begin operating. According to plans for the massive new operation, the underground mine is to be divided into four sectors which are projected to yield up to 1,700 million tonnes of ore averaging 0.71% Cu. The undertaking represents an investment totalling about US$5 billion.

The instance we present is one of the four sectors of the new Chuquicamata mine, which we will call S1. It consists of 20 blocks containing a total of 440 million tonnes of ore with an average grade of 0.68%. Its spatial distribution is shown in Figure 3 and the specific characteristics of each block in the sector (tonnage, ore content and processing time) are detailed in Table 3.

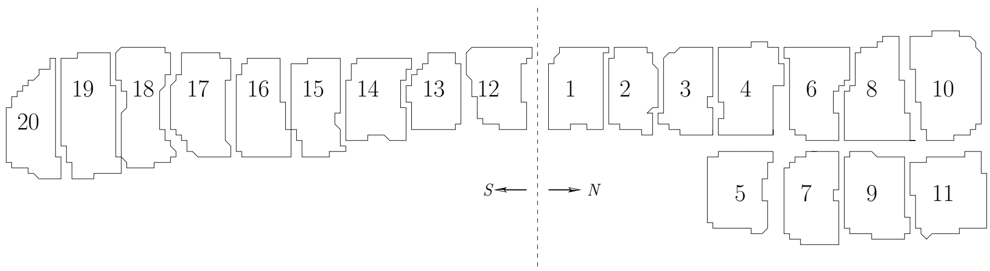

Fig. 3: S1 spatial distribution.

Block	Q_k[Ton]	L_k[%]	T_k[years]
1	4333285	0.742	0.59
2	7646668	0.674	1.04
3	10451116	0.844	1.43
4	11801936	0.766	1.61
5	14495435	0.761	1.98
6	29220025	0.753	4
7	16695793	0.706	2.28
8	33268812	0.707	4.55
9	15297264	0.736	2.09
10	43069997	0.696	5.89

Block	Q_k[Ton]	L_k[%]	T_k[years]
11	22640604	0.699	3.1
12	3708814	0.805	0.5
13	8882645	0.803	1.21
14	12883028	0.796	1.76
15	21390450	0.753	2.93
16	26742924	0.676	3.66
17	35356325	0.589	4.84
18	44094300	0.586	6.04
19	40080765	0.611	5.49
20	34275177	0.562	4.69

Table 3: S1 tonnage, ore content and processing time, by block.

The blocks in the north section of S1 can be processed, that is, their ore extracted, in any of the 6 different sequences set out in Table 4. The south section, on the other hand, can only be processed in one block sequence: 12, 13, 14, 15, 16, 17, 18, 19 and 20.

Sequence	Block order
N1	1-2-3-4-5-6-7-8-9-10-11
N2	1-2-3-5-4-6-7-9-8-10-11
N3	1-2-5-3-4-6-7-9-8-10-11
N4	1-2-5-3-4-6-7-8-9-10-11
N5	1-2-3-4-5-7-6-8-9-11-10
N6	1-2-5-3-4-6-7-9-8-11-10

Table 4: Extraction sequences, North section

The marginal cost of processing a block depends on which other blocks have already been extracted, implying that each sequence has its own marginal cost. For a given sequence π we assume this cost increases linearly with the distance from the first block in the sequence (π_N) to the block currently being extracted (π_j) according to

$$A_j^\pi = 9.514 + 0.0008 \cdot d_{\pi_N \pi_j},$$

where $d_{\pi_N \pi_j}$ is the distance between π_N and π_j. The parameters of the formula represent the projected production costs of the underground mine and the distances between blocks are displayed in Table 5. Production capacity K is assumed to be 7.3 [million Tonnes/year], shutdown cost C_0 is 10 [million \$] and production capacity maintenance cost M is 30 [million \$/year].

We model the spot price of copper as a GBM with $\mu = r - \rho$, where $r = 12\%$ is the discount factor and $\rho = 6.3\%$ is the convenience yield [1] and $\sigma = 0.233$ is the price volatility. [2]

d_{ij}	1	2	3	4	5	6	7	8	9	10	11
1	0	159	318	476	595	633	731	791	870	949	1014
2	159	0	159	317	465	474	588	632	721	790	861
3	318	159	0	158	358	315	457	474	576	632	710
4	476	317	158	0	312	158	358	316	449	474	569
5	595	465	358	312	0	350	157	443	314	564	472
6	633	474	315	158	350	0	322	158	356	316	447
7	731	588	457	358	157	322	0	357	158	448	316
8	791	632	474	316	443	158	357	0	318	158	352
9	870	721	576	449	314	356	158	318	0	353	158
10	949	790	632	474	564	316	448	158	353	0	312
11	1014	861	710	569	472	447	316	352	158	312	0

d_{ij}	12	13	14	15	16	17	18	19	20	-	-
12	0	158	323	485	647	813	970	1129	1290	-	-

Table 5: Distance between S1 blocks, in metres.

The north and south sections of S1 were extracted simultaneously following the sequences indicated above. The value functions $V_9^S(S)$ and $V_{11}^N(S)$ were used to represent the project value of the north and south sections, respectively. Total project value is given by $V_{20} = V_9^S(S) + V_{11}^N(S)$. The results of the project evaluation for the six sequences considered (recall that the south section has only one possible sequence) when the price is varied from 50[¢/lb] to 600[¢/lb] are summarized in Tables 6 and 7.

[1] The *convenience yield* is the benefit or premium associated with holding an underlying product or physical good.
[2] Values reported in [4, 17].

Precio	$N1$			$N2$			$N3$		
S	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)
50	207.38	189.45	8.65	207.17	189.32	8.62	207.37	189.57	8.58
100	1401.65	1393.39	0.59	1401.34	1393.15	0.58	1401.82	1393.79	0.57
150	2679.38	2688.49	0.34	2679.15	2688.24	0.34	2679.84	2689.18	0.35
200	3960.11	3983.58	0.59	3959.96	3983.34	0.59	3960.86	3984.57	0.60
250	5240.84	5278.68	0.72	5240.77	5278.43	0.72	5241.87	5279.96	0.73
300	6521.57	6573.77	0.80	6521.58	6573.53	0.80	6522.88	6575.35	0.80
350	7802.29	7868.87	0.85	7802.39	7868.63	0.85	7803.9	7870.74	0.86
400	9083.02	9163.97	0.89	9083.19	9163.72	0.89	9084.91	9166.13	0.89
450	10363.75	10459.06	0.92	10364	10458.82	0.91	10365.92	10461.52	0.92
500	11644.48	11754.16	0.94	11644.81	11753.91	0.94	11646.94	11756.91	0.94
550	12925.2	13049.26	0.96	12925.62	13049.01	0.95	12927.95	13052.3	0.96
600	14205.93	14344.35	0.97	14206.43	14344.11	0.97	14208.96	14347.69	0.98

Table 6: Exact value of value function V_{20} in millions of US$, approximate value of $\widehat{V_{20}^L}$ and relative error Err(%) for North section sequences $N1$, $N2$ and $N3$, for the price range 50 to 600 U.S. cents.

Precio	$N4$			$N5$			$N6$		
S	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)
50	207.57	189.7	8.61	204.85	189.07	7.70	204.89	189.24	7.64
100	1402.12	1394.05	0.58	1400.88	1392.66	0.59	1401.14	1393.17	0.57
0	2680.05	2689.44	0.35	2678.61	2687.76	0.34	2679.11	2688.56	0.35
200	3960.98	3984.83	0.60	3959.43	3982.86	0.59	3960.18	3983.95	0.60
250	5241.9	5280.22	0.73	5240.26	5277.95	0.72	5241.26	5279.34	0.73
0	6522.82	6575.61	0.81	6521.09	6573.05	0.80	6522.33	6574.73	0.80
350	7803.74	7871	0.86	7801.92	7868.15	0.85	7803.4	7870.12	0.85
400	9084.66	9166.39	0.90	9082.75	9163.24	0.89	9084.47	9165.51	0.89
450	10365.58	10461.78	0.93	10363.58	10458.34	0.91	10365.55	10460.89	0.92
500	11646.5	11757.17	0.95	11644.41	11753.43	0.94	11646.62	11756.28	0.94
550	12927.42	13052.56	0.97	12925.24	13048.53	0.95	12927.69	13051.67	0.96
600	14208.35	14347.94	0.98	14206.06	14343.63	0.97	14208.77	14347.06	0.97

Table 7: Continuation of 6 for north section sequences $N4$, $N5$ and $N6$.

These results show that as price S increases, value $V_i(S)$ converges to a linear function similar to the linear segment of $\widehat{V_i^L}$ and the relative error remains low.

6 Conclusions and Future Research

We present a solution approach to the problem of processing N jobs in a known and exogenously ordered sequence with the objective of maximizing expected return where the value of each job depends on the market price of an underlying asset which we assume follows a GBM. The proposed solution involves a dynamic program whose various stages are the jobs, for each of which the decision maker has the options of processing, suspension (waiting) or abandonment. Although the proposed approach does not solve the problem analytically, it was shown to behave in a linear manner for large values of the asset price. Other properties of the value function such as its convexity and growth enabled us to construct upper- or lower-bound functions of the value function, which can be computed using an algorithm that performs no more than $(3N-2)$ iterations.

Our results demonstrate that some of the proposed approximations produce very good results (close to optimality) and had threshold-type structures. They also show that the best approximations are given by the lower bounds, a conclusion corroborated by the empirical evidence drawn from our application of the proposed approach to a real-world case of a major mining investment project.

As for extensions of our work, there are a number of interesting variants that could be investigated both for their application to real problems and the theoretical challenges they pose. One example is the option of dynamically adjusting the execution sequence of n jobs as new information on the asset price is obtained. This adds a combinatorial aspect to the basic problem. For a given state $(S, \mathfrak{N})$ where $\mathfrak{N}$ is the set of jobs already processed with $||\mathfrak{N}|| = M$, the expected optimal profit would have to be computed $\binom{N}{M}$ times, which in practice would be impossible for realistic values of M and N. Another possible line of research is considering the processing capacity K as a decision variable. A first approach might be to use the asymptotic behaviour of the upper and lower bounds in K to develop a method that approximates the optimal value. Yet another extension is to simultaneous job execution, i.e. processing multiple jobs simultaneously.

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A Proof of main results

Proof ***(Proposition 1)*** We must prove both inequalities in (8). First, for the lower bound we demonstrate that

$$\sum_{k=1}^{i} e^{-rT^+_{k,i}}\mathbb{E}[W_k(S_{T^+_{k,i}})|S_0 = S] - e^{-rT^+_{0,i}}C_0 \leq V_i(S), \tag{A-1}$$

and then, noting that

$$\sum_{k=1}^{i} e^{-rT^+_{k,i}}\mathbb{E}[W_k(S_{T^+_{k,i}})|S_0 = S] - e^{-rT^+_{0,i}}C_0 = \mathcal{R}_i S - \mathcal{C}_i,$$

we have the desired inequality. Proceeding by induction on i, for i equal to 1 the result is immediate. If (A-1) is satisfied for $i-1$, we have

$$\begin{aligned}
V_i(S) &= \sup_{\tau\geq 0}\mathbb{E}[e^{-r\tau}\max\Big\{W_i(S_\tau) + e^{-rT_i}\mathbb{E}^{S_\tau}[V_{i-1}(S_{T_i})], -C_0\Big\} - \int_0^\tau e^{-rt}M dt|S_0 = S]\\
&\geq \max\Big\{W_i(S) + e^{-rT_i}\mathbb{E}^{S}[V_{i-1}(S_{T_i})], -C_0\Big\}\\
&\geq \max\left\{W_i(S) + e^{-rT_i}\mathbb{E}^{S}\Big[\sum_{k=1}^{i-1} e^{-rT^+_{k,i-1}}\mathbb{E}[W_k(S_{T^+_{k,i-1}})|S_0 = S_{T_i}]\Big] - e^{-rT^+_{0,i-1}}C_0, -C_0\right\}\\
&= \max\left\{\sum_{k=1}^{i} e^{-rT^+_{k,i}}\mathbb{E}^{S}[W_k(S_{T^+_{k,i}})] - e^{-rT^+_{0,i}}C_0, -C_0\right\}\\
&\geq \sum_{k=1}^{i} e^{-rT^+_{k,i}}\mathbb{E}^{S}[W_k(S_{T^+_{k,i}})] - e^{-rT^+_{0,i}}C_0,
\end{aligned}$$

where in the first equality we use the strong Markov property [8, Theorem 5.4.20], in the first inequality we define $\tau \equiv 0$ and in the second inequality we apply the inductive hypothesis.

For the upper bound we consider a modified price process $\mathcal{Z}_t$ defined by

$$\mathcal{Z}_t \triangleq S_t + \sum_{k:T^+_{k,i}\leq t}(S^b_k - \mathcal{Z}_{T^+_{k,i}-})^+, \qquad \mathcal{Z}_{0-} = S_0.$$

The idea behind the use of this process is that each time the execution of a job π_i is completed, process S_t moves to the price S^b_{i-1} that allows execution of the next job to start. In this way, the entire sequence of jobs can be processed consecutively with no suspensions.

Let $\mathcal{H}_i(S)$ be the expected value of a project with i jobs under process $\mathcal{Z}_t$. Since $S_t \leq \mathcal{Z}_t$ for every realization of these processes, it is clear that $V_i(S) \leq \mathcal{H}_i(S)$. Therefore,

$$\begin{aligned}
V_i(S) \leq \mathcal{H}_i(S) &= \sum_{k=1}^{i} e^{-rT^+_{k,i}}\mathbb{E}[W_k(\mathcal{Z}_{T^+_{k,i}})|S_0 = S] - e^{-rT^+_{0,i}}C_0\\
&= \sum_{k=1}^{i} e^{-rT^+_{k,i}}(R_k\mathbb{E}[\mathcal{Z}_{T^+_{k,i}}|S_0 = S] - C_k) - e^{-rT^+_{0,i}}C_0\\
&= \sum_{k=1}^{i} e^{-rT^+_{k,i}}R_k\mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S] + \sum_{k=1}^{i} e^{-rT^+_{k,i}}(R_k\mathbb{E}[S_{T^+_{k,i}}|S_0 = S] - C_k) - e^{-rT^+_{0,i}}C_0\\
&= \sum_{k=1}^{i} e^{-rT^+_{k,i}}R_k\mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S] + (\mathcal{R}_i S - \mathcal{C}_i). \qquad \text{(A-2)}
\end{aligned}$$

We now bound the term $\mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S]$ on the right-hand side of the last expression. Since $S_t = S_0\exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$, we have

$$\mathbb{P}(\{S^b_k \geq S_{T^+_{k,i}}\}|S_0 = S) = \mathbb{P}(\frac{1}{\sigma}\Big[\ln\Big(\frac{S}{S^*_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big] \leq B_{T^+_{k,i}}). \tag{A-3}$$

By a theorem of [1, Theorem XIII-2.1], for $S > S^m_i \cdot \exp(-\min_{1\leq k\leq i}\{(\mu + \frac{\sigma^2}{2})T^+_{k,i}\})$ we can bound the left-hand side of (A-3) by an exponential term as follows:

$$\mathbb{P}(\frac{1}{\sigma}\Big[\ln\Big(\frac{S}{S^*_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big] \leq \tilde{B}_{T^+_{k,i}}) \leq \exp\left(-\frac{1}{T^+_{k,i}\sigma^2}\Big[\ln\Big(\frac{S}{S^*_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big]^2\right).$$

Putting it all together, we get

$$\begin{aligned}
\mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S] &= \sum_{n=k}^{i} \mathbb{E}[(S^b_k - \mathcal{Z}_{T^+_{k,i}-})\mathbf{1}_{\{\{S^b_k \geq \mathcal{Z}_{T^+_{k,i}-}\}\}}|S_0 = S] \\
&\leq \sum_{n=k}^{i} S^b_k \mathbb{P}(\{S^b_k \geq \mathcal{Z}_{T^+_{k,i}-}\}|S_0 = S) \\
&\leq \sum_{n=k}^{i} S^b_k \mathbb{P}(\{S^b_k \geq S_{T^+_{k,i}}\}|S_0 = S) \\
&= \sum_{n=k}^{i} S^b_k \mathbb{P}(\frac{1}{\sigma}\Big[\ln\Big(\frac{S}{S^b_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big] \leq B_{T^+_{k,i}}) \\
&\leq (i+1-k)\cdot S^m_i \cdot \exp\Bigg(-\frac{1}{\sigma^2}\min_{1\leq k\leq i}\frac{1}{T^+_{k,i}}\Big[\ln\Big(\frac{S}{S^m_i}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big]^2\Bigg).
\end{aligned}$$

Finally, using the bound in (A-2), we deduce that

$$V_i(S) \leq \frac{i(i+1)}{2}\cdot S^m_i \cdot R^m_i \cdot \exp\Bigg(-\frac{1}{\sigma^2}\min_{1\leq k\leq i}\frac{1}{T^+_{k,i}}\Big[\ln\Big(\frac{S}{S^m_i}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big]^2\Bigg) + (\mathcal{R}_i S - \mathcal{C}_i).$$

Proof ***(Proposition 2)*** We star with the convexity. Define $X_t \triangleq \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$ so S_t is equal to $S_0 X_t$ and, therefore, we can write $V_1(S)$ as

$$V_1(S) = \sup_{\tau\geq 0}\mathbb{E}\Bigg[e^{-r\tau}\max\{W_1(SX_\tau) - e^{-rT_1}C_0, -C_0\} - \int_0^\tau e^{-rt}Mdt\Bigg].$$

Since the maximum of convex functions is itself a convex function, so is $V_1(S)$. The result follows immediately for $i > 1$ by induction.

The monotonicity of $V_i(S)$ simply follows from the monotonicity of $W_1(S)$ and induction.

Proof ***(Lemma 1)*** Let τ be an $\mathcal{F}_t$ - stopping time. Then $V_i(S_{\tau+T_{i+1}})$ is equal to

$$\sup_{\xi\geq 0}\mathbb{E}\Big[e^{-r\xi}\max\Big\{W_i(S_\xi) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\xi+T_i})|\mathcal{F}_\xi], -C_0\Big\} - \int_0^\xi e^{-rt}Mdt|S_0 = S_{\tau+T_{i+1}}\Big]. \tag{A-4}$$

Using the change of variable $\xi = \lambda - T_{i+1}$, the fact that Brownian motion is a Lévy process and recalling that S_t is equal to $S_0\exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$, it is the case that (A-4) equals

$$\sup_{\lambda\geq T_{i+1}}\mathbb{E}\Big[e^{-r(\lambda - T_{i+1})}\max\Big\{W_i(S_\lambda) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\lambda+T_i})|\mathcal{F}_\lambda], -C_0\Big\} - \int_0^{\lambda - T_{i+1}} e^{-rt}Mdt|S_0 = S_\tau\Big].$$

Rearranging this last expression yields

$$\begin{aligned}
&\sup_{\lambda\geq T_{i+1}}\mathbb{E}\Big[e^{-r(\lambda - T_{i+1})}\max\Big\{W_i(S_\lambda) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\lambda+T_i})|\mathcal{F}_\lambda], -C_0\Big\} - \int_0^\lambda e^{-r(u-T_{i+1})}Mdu|S_0 = S_\tau\Big] \\
&+ \int_0^{T_{i+1}} e^{-r(u-T_{i+1})}Mdu,
\end{aligned}$$

which is less than or equal to

$$e^{rT_{i+1}}\sup_{\lambda>0}\mathbb{E}\Big[e^{-r\lambda}\max\Big\{W_i(S_\lambda) + e^{-rT_i}V_{i-1}(S_{\lambda+T_i}), -C_0\Big\} - \int_0^\lambda e^{-ru}Mdu|S_0 = S_\tau\Big] + e^{rT_{i+1}}\int_0^{T_{i+1}} e^{-ru}Mdu.$$

We therefore conclude that

$$V_i(S_{\tau+T_{i+1}}) \leq e^{rT_{i+1}}V_i(S_\tau) + e^{rT_{i+1}}\int_0^{T_{i+1}} e^{-ru}Mdu,$$

as required.

Proof ***(Corollary 1)***

We prove the two bounds by induction. For i equal to 1, the lower bound is satisfied because $V_1^L(S)$ is equal to $V_1(S)$ by definition. If the lower bound is valid for $i-1$, then

$$
\begin{aligned}
V_i^L(S) &= \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\Big\{W_i(S_\tau)+e^{-rT_i}V_{i-1}^L(e^{\mu T_i}S_\tau),-C_0\Big\}-\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&\overset{\text{Ind. Hyp}}{\leq} \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\{W_i(S_\tau)+e^{-rT_i}V_{i-1}(e^{\mu T_i}S_\tau),-C_0\}-\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&\overset{(a)}{=} \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\{W_i(S_\tau)+e^{-rT_i}V_{i-1}(\mathbb{E}[S_{\tau+T_i}|\mathcal{F}_\tau]),-C_0\}-\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&\overset{\text{Jensen}}{\leq} \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\Big\{W_i(S_\tau)+e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\tau+T_i})|\mathcal{F}_\tau],-C_0\Big\}-\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&= V_i(S).
\end{aligned}
$$

In (a) we have made use of the fact that $e^{-\mu t}S_t$ is a $\mathbb{P}$-martingale. Notice that we can make use of Jensen's inequality since by Proposition 2, $V_i(S)$ is a convex function.

The validity of the upper bound for i equal to 1 immediately follows from the condition that $C_0<\frac{M}{r}$. To prove this bound is valid for $i>1$ we assume that it is valid for $i-1$. Then,

$$
\begin{aligned}
V_i(S) &= \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\Big\{W_i(S_\tau)+e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\tau+T_i})|\mathcal{F}_\tau],-C_0\Big\}-\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&\overset{(a)}{\leq} \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\{W_i(S_\tau)+e^{-rT_i}(e^{rT_i}V_{i-1}(S_\tau)+e^{rT_i}\int_0^{T_i}e^{-rt}Mdt),-C_0\} \\
&\quad -\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&\overset{\text{Ind. Hyp}}{\leq} \sup_{\tau\geq 0}\mathbb{E}\Big[e^{-r\tau}\max\{W_i(S_\tau)+V_{i-1}^U(S_\tau)+\int_0^{T_i}e^{-rt}Mdt,-C_0\}-\int_0^\tau e^{-rt}Mdt\Big|S_0=S\Big] \\
&= V_i^U(S),
\end{aligned}
$$

where the inequality proved in Lemma 1 is used in (a).

*Proof (**Theorem 1**)*

This proof is divided into two parts. In the first part we use condition $I)$ to find a solution for the system of equations (15) when i equals 1. Then, in the second part, we use this solution and condition $II)$ to check the hypotheses in [13, Theorem 10.4.1], from which we can conclude the desired result. Throughout the proof we will assume $(C_1-\gamma_1+\alpha_1C_0)\leq M/r$, the proof of the case where $(C_1-\gamma_1+\alpha_1C_0)>M/r$ is analogous and we omit it.

To simplify notation we let

$$x_1\triangleq\mathcal{S}_1^a \quad\text{and}\quad x_2\triangleq\mathcal{S}_1^b.$$

System (15) with i equals to 1 then becomes

$$\mathcal{B}_1x_1^{\lambda_1}+\mathcal{D}_1x_1^{\lambda_2}-M/r=-C_0, \tag{A-5}$$

$$\mathcal{B}_1x_2^{\lambda_1}+\mathcal{D}_1x_2^{\lambda_2}-M/r=R_1x_2-(C_1-\gamma_1)-\alpha_1C_0, \tag{A-6}$$

$$\lambda_1\mathcal{B}_1x_1^{\lambda_1-1}+\lambda_2\mathcal{D}_1x_1^{\lambda_2-1}=0, \tag{A-7}$$

$$\lambda_1\mathcal{B}_1x_2^{\lambda_1-1}+\lambda_2\mathcal{D}_1x_2^{\lambda_2-1}=R_1. \tag{A-8}$$

Since $M/r>C_0$, it cannot be the case that $\mathcal{D}_1$ equals 0. Also, since $\lambda_2<0$ it must be the case that $x_1>0$. By a similar argument we conclude that $\mathcal{B}_1$ is not equal to 0. If we combine equations (A-5) and (A-7), we can solve for $\mathcal{B}_1$ and $\mathcal{D}_1$ as functions of x_1, thereby obtaining

$$\mathcal{B}_1=-\frac{(M/r-C_0)\lambda_2}{(\lambda_1-\lambda_2)x_1^{\lambda_1}},\qquad \mathcal{D}_1=\frac{(M/r-C_0)\lambda_1}{(\lambda_1-\lambda_2)x_1^{\lambda_2}}.$$

Then, exploiting the fact that $M/r-C_0>0$ and $\lambda_2<0<1<\lambda_1$, we may conclude that $\mathcal{B}_1>0$ and $\mathcal{D}_1>0$. All that remains is to show that $x_1<x_2$ and that a solution of the system (A-5) -(A-8) exists. From equations (A-6) and (A-8), we can also solve for $\mathcal{B}_1,\mathcal{D}_1$ but this time as functions of x_2. Therefore,

$$\mathcal{B}_1=\frac{(C_1-\gamma_1+\alpha_1C_0-M/r)\lambda_2-R_1x_2(\lambda_2-1)}{(\lambda_1-\lambda_2)x_2^{\lambda_1}},\qquad \mathcal{D}_1=-\frac{(C_1-\gamma_1+\alpha_1C_0-M/r)\lambda_1-R_1x_2(\lambda_1-1)}{(\lambda_1-\lambda_2)x_2^{\lambda_2}}.$$

At this point it is convenient to define the following functions for $x>0$:

$$
\begin{aligned}
I_1(x) &\triangleq \frac{\lambda_2(C_0-M/r)}{x^{\lambda_1}}, \\
I_2(x) &\triangleq \frac{\lambda_1(C_0-M/r)}{x^{\lambda_2}}, \\
J_1(x) &\triangleq \frac{(C_1-\gamma_1+\alpha_1C_0-M/r)\lambda_2-R_1x(\lambda_2-1)}{x^{\lambda_1}}, \\
J_2(x) &\triangleq \frac{(C_1-\gamma_1+\alpha_1C_0-M/r)\lambda_1-R_1x(\lambda_1-1)}{x^{\lambda_2}}.
\end{aligned}
$$

The derivatives of these functions are:

$$
\begin{aligned}
I_1'(x) &= \frac{-\lambda_1\lambda_2(C_0 - M/r)}{x^{\lambda_1+1}},\\
I_2'(x) &= -\frac{\lambda_1\lambda_2(C_0 - M/r)}{x^{\lambda_2+1}},\\
J_1'(x) &= \frac{R_1x(\lambda_1-1)(\lambda_2-1) - (C_1-\gamma_1+\alpha_1C_0-M/r)\lambda_1\lambda_2}{x^{\lambda_1+1}},\\
J_2'(x) &= \frac{R_1x(\lambda_1-1)(\lambda_2-1) - (C_1-\gamma_1+\alpha_1C_0-M/r)\lambda_1\lambda_2}{x^{\lambda_2+1}}.
\end{aligned}
$$

With these definitions, what we have to prove is the existence of a pair (x_1, x_2), with $x_1 < x_2$ such that

$$
I_1(x_1) = J_1(x_2) \quad \text{y} \quad I_2(x_1) = J_2(x_2). \tag{A-9}
$$

Note that $I_1(x)$ is positive and decreasing over its domain, with $I_1(x) \overset{x\to\infty}{\to} 0$ and $I_1(x) \overset{x\to 0^+}{\to} \infty$, while $I_2(x)$ is negative and decreasing over its domain with $I_2(x) \overset{x\to\infty}{\to} -\infty$ and $I_2(x) \overset{x\to 0^+}{\to} 0$. Also, $J_1(x)$ takes positive values and is decreasing over its entire domain, with $J_1(x) \overset{x\to\infty}{\to} 0$ and $J_1(x) \overset{x\to 0^+}{\to} \infty$, while $J_2(x)$ is negative and decreasing over its domain with $J_2(x) \overset{x\to\infty}{\to} -\infty$ and $J_2(x) \overset{x\to 0^+}{\to} 0$. Notice that at the points

$$
\underline{x} = \frac{(C_1-\gamma_1+\alpha_1C_0-C_0)\lambda_2}{R_1(\lambda_2-1)}, \qquad \overline{x} = \frac{(C_1-\gamma_1+\alpha_1C_0-C_0)\lambda_1}{R_1(\lambda_1-1)},
$$

it is the case that $I_1(\underline{x}) = J_1(\underline{x})$ and $I_2(\overline{x}) = J_2(\overline{x})$, and $\underline{x} < \overline{x}$.
From this analysis we can conclude as follows regarding the solution of (A-9). Firstly, $x_1, x_2 \in (\underline{x}, \overline{x})$ and $x_1 < x_2$. This is so because for $x \geq \overline{x}$, it is the case that $I_2(x) \geq J_2(x)$ and therefore all pairs of points (x_1, x_2) greater than $\overline{x}$ such that $I_2(x_1) = J_2(x_2)$ satisfy $x_1 \geq x_2$. At the same time, since $J_1(x) > I_1(x)$ for $x \geq \overline{x}$, all pairs of points (x_1, x_2) greater than $\overline{x}$ such that $I_1(x_1) = J_1(x_2)$ satisfy $x_1 < x_2$. Therefore, it must also be the case that $x_1, x_2 < \overline{x}$. Similarly, for $x \leq \underline{x}$ it is the case that $I_1(x) \geq J_1(x)$ and therefore the pairs of points (x_1, x_2) less than $\underline{x}$ such that $I_1(x_1) = J_1(x_2)$ satisfy $x_1 \geq x_2$. At the same time, since $J_2(x) > I_2(x)$ for $x \leq \underline{x}$, all pairs of points (x_1, x_2) less than $\underline{x}$ such that $I_2(x_1) = J_2(x_2)$ satisfy $x_1 < x_2$. From the foregoing we may conclude that $x_1, x_2 > \underline{x}$ and therefore that $x_1, x_2 \in (\underline{x}, \overline{x})$. Also, on this interval all possible solutions of (A-9) satisfy $x_1 < x_2$.

Secondly, consider $x_2 \in (\underline{x}, \overline{x})$ and let $x_1^1(x_2)$ be the only solution of $I_1(x_1^1) = J_1(x_2)$ with $x_1^1 < x_2$, and $x_1^2(x_2)$ the only solution of $I_2(x_1^2) = J_2(x_2)$ with $x_1^2 < x_2$. Clearly, $x_1^1(x_2)$ and $x_1^2(x_2)$ are continuous functions defined in $(\underline{x}, \overline{x})$. We must prove that there exists an $x_2 \in (\underline{x}, \overline{x})$ such that $x_1^1(x_2) = x_1^2(x_2)$. For x_2 close to $\underline{x}$, $x_2 - x_1^2(x_2) > 0$, but since $x_2 - x_1^1(x_2) \approx 0$ there must exist an x_2 such that $x_2 - x_1^1(x_2) = x_2 - x_1^2(x_2)$, that is, $x_1^1(x_2) = x_1^2(x_2)$.
A graphical description of the above points is given in Figure 4. Briefly, it was demonstrated that assuming $M/r > (C_1 - \gamma_1 + \alpha_1 C_0) > C_0$, system (A-5)-(A-8) has a solution in the variables $x_1, x_2, \mathcal{B}_1, \mathcal{D}_1$.

To conclude our proof we continue with the verification of the conditions required by [13, Theorem 10.4.1], which allow us to establish that our optimal stopping problem has a solution. We propose the following continuation region

$$
D = \{S : \mathcal{S}_k^a < S < \mathcal{S}_k^b\},
$$

and the following candidate solution

$$
\phi(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_1^a \\ \mathcal{B}_1 S^{\lambda_1} + \mathcal{D}_1 S^{\lambda_2} - M/r & \text{if } \mathcal{S}_1^a < S < \mathcal{S}_1^b \\ R_1 S - (C_1 - \gamma_1 + \alpha_1 C_0) & \text{if } \mathcal{S}_1^b \leq S, \end{cases}
$$

where $\mathcal{S}_1^a, \mathcal{S}_1^b$ are found by imposing the value matching and smooth pasting conditions. Conditions $iii), iv), v), viii)$ and $ix)$ of [13, Theorem 10.4.1] are easily verified. And since we have already demonstrated that (15) has a solution, condition $i)$ of the theorem is proved and condition $vii)$ is verified by construction of $\phi(S)$ (see Equation (3) and the related discussion). This leaves only conditions $ii)$ and $vi)$ to be verified. In the case of $ii)$, it must be proved that $\phi(S) \geq \max\{R_1 S - (C_1 - \gamma_1 + \alpha_1 C_0), -C_0\}$ for all $S \geq 0$. This can be done using $II)$. As for $vi)$, we must show that

$$
-r\phi(S) + \mu S\phi'(S) + \frac{1}{2}\sigma^2 S^2 \phi''(S) \leq M, \quad \text{en } D^c.
$$

For $S \leq \mathcal{S}_1^a$, since $\phi(S)$ is equal to $-C_0$ the inequality is equivalent to $C_0 \leq M/r$ and since we are considering that $M/r > C_0$, the condition is satisfied. For $S \geq \mathcal{S}_1^b$, $\phi(S)$ is equal to $R_1 S - (C_1 - \gamma_1 + \alpha_1 C_0)$ and the inequality is equivalent to

$$
\frac{r(C_1 - \gamma_1 + \alpha_1 C_0) - M}{(r-\mu)R_1} \leq S.
$$

Given that $S \geq \mathcal{S}_1^b$ and that the numerator on the left-hand side of the above inequality is negative, the inequality is true (for present purposes we have assumed that $r > \mu$). The desired result is therefore shown.

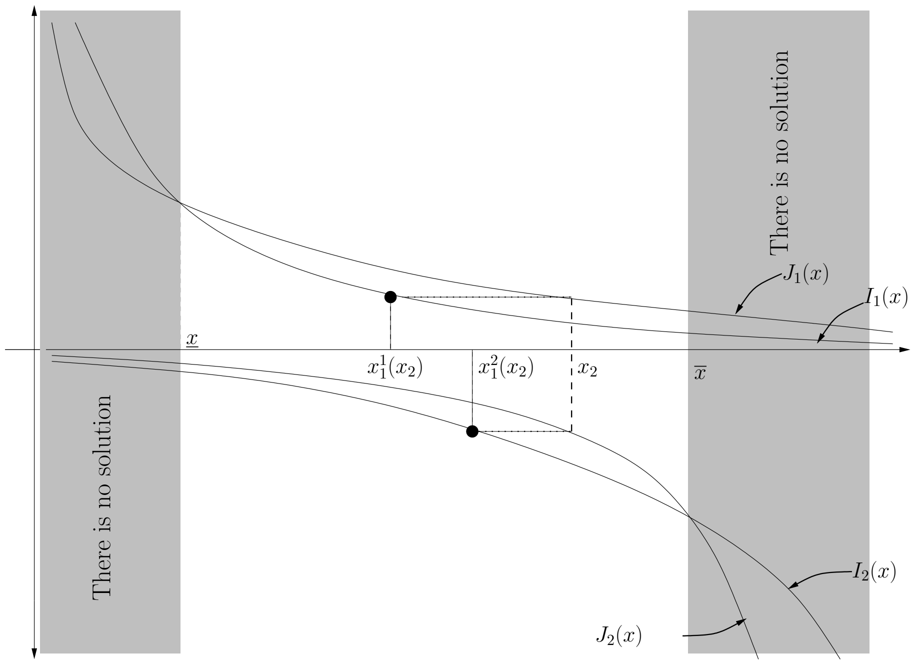

Fig. 4: The curves are the functions I_1, I_2, J_1 and J_2 defined in Theorem 1. The last point in the demonstration of this proposition is that close to $\underline{x}$, the value of $x_2 - x_1^1(x_2)$ is close to 0 while the value of $x_2 - x_1^2(x_2)$ is greater than 0. Therefore, as x_2 grows towards $\overline{x}$, the value of $x_2 - x_1^1(x_2)$ rises while the value of $x_2 - x_1^2(x_2)$ falls to 0. This implies that there must exist some $x_2 \in (\underline{x}, \overline{x})$ such that $x_2 - x_1^1(x_2) = x_2 - x_1^2(x_2)$, that is, $x_1^1(x_2) = x_1^2(x_2)$.

Proof ***(Proposition 3)***

Let I_j^i be the *jth* piece defining $\mathcal{V}_i$. By (14) we know that $\mathcal{V}_i$ always has two invariable pieces, $I_1^i = (-\infty, \mathcal{S}_i^a]$ and $I_2^i = (\mathcal{S}_i^a, \mathcal{S}_i^b)$. The third and last piece $I_3^i = [\mathcal{S}_i^b, \infty)$ divides depending on the quantity m_{i-1} of pieces in $\mathcal{V}_{i-1}$. Since we only need to know $\mathcal{V}_{i-1}$ for the values of S such that $\eta_i S > \eta_i \mathcal{S}_i^b$, and $\eta_i \mathcal{S}_i^b \in (a^*, b^*) \triangleq I_{l_{i-1}}^{i-1}$, then for $S > \mathcal{S}_i^b$, $\mathcal{V}_i(S)$ will have a different definition for each interval: $(\mathcal{S}_i^b, b^*/\eta_i), I_{l_{i-1}+1}^{i-1}/\eta_i, \ldots, I_{m_{i-1}}^{i-1}/\eta_i$. Therefore,

$$m_i = 2 + (m_{i-1} - l_{i-1} + 1). \tag{A-10}$$

If m_{i-1} equals l_{i-1} then m_i is at least 3, and by summing both sides of (A-10) from 2 to i and setting l_{i-1} equal to 1 we get that m_i is at most $2i + 1$.

Proof ***(Corollary 2)***

When i equals 1, only one system needs to be solved and therefore only one iteration is required. We now show by induction that the result is also true for $i \geq 2$. First, we define ν_j, where $j \in \{1, \ldots, i-1\}$, as the number of iterations executed by the algorithm to calculate Y^j, not counting the iterations used to calculate Y^k for $k \in \{1, \ldots, j-1\}$. Also, let m^j be the number of rows in matrix Y^j.

Calculating Y^{i-1} will involve a total of $\sum_{j=1}^{i-1} \nu_i$ iterations, while calculating Y^i may entail from 1 to m^{i-1} iterations, then it follows that the number of iterations for the algorithm to finish in the worst case is given by $\sum_{j=1}^{i-1} \nu_j + m^{i-1}$. For example, in the case illustrated by the dotted line in Figure 5, Y^3 is calculated in $\nu_1 = 1, \nu_2 = 1$ iterations. ν_3 can therefore take values between 1 and 5, the worst case thus being $\nu_1 + \nu_2 + m^2 = 7$.
Continuing now with the induction on i, note that by Proposition 3 we have

$$m^{i-1} = 2 + (m^{i-2} - \nu_{i-1} + 1). \tag{A-11}$$

Assume that the result is true for $i - 1$. Then,

$$\begin{aligned}
\sum_{j=1}^{i-1} \nu_j + m^{i-1} &= \sum_{j=1}^{i-2} \nu_j + \nu_{i-1} + m^{i-1} \\
&\leq (3(i-1) - 2 - m^{i-2}) + \nu_{i-1} + m^{i-1} \\
&= 3i - 2 + \left(m^{i-1} - (2 + (m^{i-2} - \nu_{i-1} + 1))\right) \\
&= 3i - 2,
\end{aligned}$$

Were the inequality follows from the induction hypothesis and the las equality follows from equation (A-11). Since the ν_j were chosen arbitrarily, the result is proved.

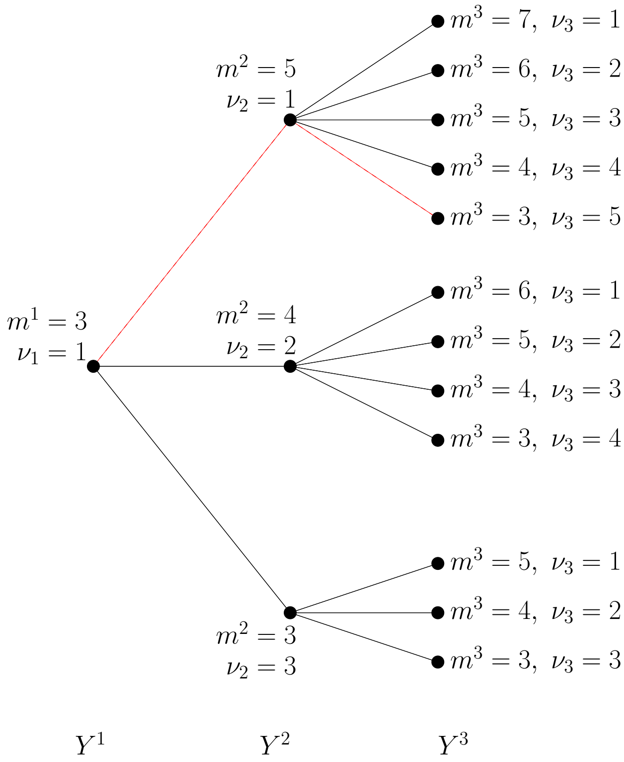

Fig. 5: Possible number of rows and iterations for calculating Y^3.