Proof Strategies for Backward Reasoning

1. If your goal is $\neg A$ then
   (a) Do $[\neg I]$; now it is enough to prove $\bot$ using A as an assumption.
   This looks like : $\dfrac{\bot}{\neg A}\,[\neg I]$
2. If your goal is $A \to B$ then
   (a) Do $[\to I]$; now it is enough to prove B using A as an assumption.
   This looks like : $\dfrac{B}{A \to B}\,[\to I]$
3. If your goal is $A \wedge B$ then
   (a) Do $[\wedge I]$; now it is enough to prove both A and B separately.
   This looks like : $\dfrac{A \qquad B}{A \wedge B}\,[\wedge I]$
4. If your goal is $A \vee B$ then
   (a) Do $[\vee I]$ choosing one of A or B to prove — which one to choose is not always clear! Looking at the given hypotheses or the current context might give a clue as to which one to prove.
   Warning: Choosing the wrong option to prove can lead to a dead-end!
   This looks like : $\dfrac{A}{A \vee B}\,[\vee I]$ or $\dfrac{B}{A \vee B}\,[\vee I]$
5. If your goal is A then
   (a) Try $[\wedge E]$ if there is a hypothesis/context of the form $A \wedge B$
   This looks like : $\dfrac{A \wedge B}{A}\,[\wedge E]$
   (b) Try $[\to E]$ if there are hypotheses/contexts of the form $X \to A$ and X
   This looks like : $\dfrac{X \to A \qquad X}{A}\,[\to E]$
   (c) Try $[\vee E]$ if there are hypotheses/contexts of the form $X \vee Y,\ X \to A$ and $Y \to A$
   This looks like : $\dfrac{X \vee Y \qquad X \to A \qquad Y \to A}{A}\,[\vee E]$
6. If your goal is $\bot$ then
   (a) Try $[\neg E]$ if there are hypotheses/contexts of the form Z and $\neg Z$
   This looks like : $\dfrac{Z \qquad \neg Z}{\bot}\,[\neg E]$ Note that this is a special case of 5b above.

In addition to the aforementioned strategies, for predicate logic we have the following additional strategies:

7. If your goal is $\neg\forall x, P(x)$ then
   (a) Do $[\neg I]$; now it is enough to prove $\perp$ using $\forall x, P(x)$ as an assumption.
   This looks like : $\dfrac{\perp}{\neg\forall x, P(x)}\,[\neg I]$
8. If your goal is $\neg\exists x, P(x)$ then
   (a) Do $[\neg I]$; now it is enough to prove $\perp$ using $\exists x, P(x)$ as an assumption.
   This looks like : $\dfrac{\perp}{\neg\exists x, P(x)}\,[\neg I]$
9. If your goal is $\forall x, P(x)$ then
   (a) Try $[\forall I]$; now it is enough to prove $P(t)$ for a **general** variable t in the domain
   This looks like : $\dfrac{P(t)}{\forall x, P(x)}\,[\forall I]$
10. If your goal is $\exists x, P(x)$ then you need to supply an element $t : U$ of the domain and a proof of $P(t)$, i.e., show that t satisfies P.
    (a) The $t : U$ to be supplied could be a constant (e.g., $2 : \mathbb{N}$) or a variable already in the context (e.g., some $n : \mathbb{N}$). Use $[\exists I]$ supplying a suitable element $a : U$. This looks like : $\dfrac{P(a)}{\exists x, P(x)}\,[\exists I]$
11. If your goal is $P(t)$ for a **general** variable t in the domain then check your hypotheses to see what might allow to prove $P(t)$. Eliminate that hypothesis/these hypotheses, using the corresponding elimination rule.

The two hardest (most non-intuitive) rules to apply are $[\vee E]$ and $[\exists E]$. In general, if you are stuck and don't see any way to proceed with your proof, then maybe consider the following three things:

1. Try $[\vee E]$
2. Try $[\exists E]$
3. Try either DNE or EM from classical logic. **Warning**: Do not use rules from classical logic if the question explicitly says to only use rules from intuitionistic logic only.