

Week 10: Uniformly integrable martingales, stopping times

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 27/11/2023 17:00 (online) following the instructions on the course website

<https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/>

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(M_n)_{n \geq 0}$ be a $(\mathcal{F}_n)_{n \geq 0}$ martingale and let T be a $(\mathcal{F}_n)_{n \geq 0}$ stopping time.

- (1) Assume that T is bounded. Show that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.
- (2) Assume that $\mathbb{E}[T] < \infty$ and there exists $K > 0$ such that a.s. we have $\mathbb{E}[|M_{n+1} - M_n| | \mathcal{F}_n] \leq K$ for every $n \geq 0$. Show that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Hint. Justify that $|M_{T \wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| \mathbb{1}_{T > i}$ and use dominated convergence.

- (3) Let $(X_n)_{n \geq 1}$ be i.i.d. integrable real-valued random variables. Set $M_0 = 0$, $M_n = X_1 + \dots + X_n$ for $n \geq 1$ and $\mathcal{F}_n = \sigma(M_i : 0 \leq i \leq n)$ for $n \geq 0$. Finally, let T be a (\mathcal{F}_n) -stopping time with $\mathbb{E}[T] < \infty$. Show that

$$\mathbb{E}[S_T] = \mathbb{E}[X_1] \mathbb{E}[T].$$

Solution:

- (1) Let $N > 0$ be such that $T \leq N$. We know that $(M_{n \wedge T})_{n \geq 0}$ is a martingale, so that $\mathbb{E}[M_{n \wedge T}] = \mathbb{E}[M_0]$ for every $n \geq 0$. It suffices to take $n = N$, since $N \wedge T = T$.

- (2) Let us prove

$$|M_{T \wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| \mathbb{1}_{T > i}. \quad (1)$$

Write

$$M_{T \wedge n} = M_0 + \sum_{i=0}^{n \wedge T - 1} (M_{i+1} - M_i) = M_0 + \sum_{i=0}^{n \wedge T - 1} (M_{i+1} - M_i) \mathbb{1}_{T > i},$$

and we get (1) by triangular inequality.

Since $\mathbb{E}[T] < \infty$, it follows that $T < \infty$ a.s. As a consequence, $M_{T \wedge n}$ converges almost surely to M_T . In addition, by (1), we are in position to use dominated convergence to get $\mathbb{E}[M_{T \wedge n}] \rightarrow$

$\mathbb{E}[M_T]$ as $n \rightarrow \infty$. Since $\mathbb{E}[M_{T \wedge n}] = \mathbb{E}[M_0]$ for every $n \geq 0$, we get the desired result.

- (3) We use (2) with the martingale $M_n = S_n - \mathbb{E}[X_1]n$. We just have to check that there exists $K > 0$ such that a.s. we have $\mathbb{E}[|M_{n+1} - M_n| | \mathcal{F}_n] \leq K$ for every $n \geq 0$. To this end write

$$\mathbb{E}[|M_{n+1} - M_n| | \mathcal{F}_n] = \mathbb{E}[|X_{n+1} - \mathbb{E}[X_1]| | \mathcal{F}_n] \leq 2\mathbb{E}[|X_1|].$$

□

2 Training exercises

Exercise 2. Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration and let S, T be two stopping times with respect to $(\mathcal{F}_n)_{n \geq 0}$. Let $S, T : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ be (\mathcal{F}_n) stopping times. Prove or disprove with a counter-example the following statements:

- (1) $S \vee T$ is a stopping time.
- (2) $S \wedge T$ is a stopping time.
- (3) $S + T$ is a stopping time.
- (4) $S + 1$ is a stopping time.
- (5) $S - 1$ is a stopping time.

Solution:

- (1) This is true. Indeed for $n \geq 0$ we have $\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$ for $n \geq 0$ since $\{S \leq n\}, \{T \leq n\} \in \mathcal{F}_n$.
- (2) This is true. For $n \geq 0$ we have $\{S \wedge T > n\} = \{S > n\} \cap \{T > n\}$ which is an element of \mathcal{F}_n since \mathcal{F}_n is stable under intersections and $\{S > n\} = \{S \leq n\}^c, \{T > n\} = \{T \leq n\}^c \in \mathcal{F}_n$. Therefore also $\{S \wedge T \leq n\} = \{S \wedge T > n\}^c \in \mathcal{F}_n$ as required.
- (3) This is also true. Indeed, we have

$$\{S + T \leq n\} = \bigcup_{k+\ell \leq n} \{S \leq k\} \cap \{T \leq \ell\}.$$

Also $\{S \leq k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ and $\{T \leq \ell\} \in \mathcal{F}_\ell \subset \mathcal{F}_n$ for $k, \ell \leq n$. Thus $\{S + T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$ as required.

- (4) This is true. Indeed, for $n \geq 1$ we have $\{S + 1 \leq n\} = \{S \leq n - 1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ and $\{S + 1 = 0\} = \emptyset$.

- (5) This is not true. For instance, consider a Bernoulli random variable B with parameter $1/2$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(B)$ for $n \geq 1$. Then $T := B + 1$ is an (\mathcal{F}_n) stopping time but $\{T - 1 = 0\} = \{B = 0\} \notin \mathcal{F}_0$.

□

Exercise 3. Let (\mathcal{F}_n) be a filtration and $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n)$. Let (X_n) be a sequence of integrable random variables such that $X_n \rightarrow X$ as $n \rightarrow \infty$ both a.s. and in L^1 , where X is an integrable random variable. Assume that for all $n \geq 0$, $|X_n| \leq Y$ a.s., where Y is a non-negative integrable random variable.

- (1) Define $Z_n = \sup_{m \geq n} |X_m - X|$. Show that $Z_n \rightarrow 0$ as $n \rightarrow \infty$ a.s. and in L^1 .
- (2) Show that $\mathbb{E}[X_n | \mathcal{F}_n] \rightarrow \mathbb{E}[X | \mathcal{F}_\infty]$ as $n \rightarrow \infty$ a.s.
- (3) Let (Y_n) and (Z_n) be two independent sequences of independent random variables such that $\mathbb{P}(Y_n = n) = n^{-2} = 1 - \mathbb{P}(Y_n = 0)$ and $\mathbb{P}(Z_n = n) = n^{-1} = 1 - \mathbb{P}(Z_n = 0)$. Set $X_n = Y_n Z_n$ and $\mathcal{A} = \sigma(Z_n : n \geq 0)$.

Show that $X_n \rightarrow 0$ as $n \rightarrow \infty$ both a.s. and in L^1 , but $\mathbb{E}[X_n | \mathcal{A}]$ does not converge to 0 almost surely.

Solution:

- (1) Since $X_n \rightarrow X$ a.s. it is clear that $Z_n \rightarrow 0$ a.s. as $n \rightarrow \infty$. To see that this convergence is also in L^1 it suffices to prove that (Z_n) is uniformly integrable. This follows from the fact that $Z_n \leq 2Y$ for all $n \geq 1$ a.s. with Y is integrable.
- (2) For $m \leq n$ a.s.

$$\begin{aligned} |\mathbb{E}[X_n | \mathcal{F}_n] - \mathbb{E}[X | \mathcal{F}_\infty]| &\leq \mathbb{E}[|X - X_n| | \mathcal{F}_n] + |\mathbb{E}[X | \mathcal{F}_n] - \mathbb{E}[X | \mathcal{F}_\infty]| \\ &\leq \mathbb{E}[Z_m | \mathcal{F}_n] + |\mathbb{E}[X | \mathcal{F}_n] - \mathbb{E}[X | \mathcal{F}_\infty]|. \end{aligned}$$

As seen in the lecture, the right hand side converges to $\mathbb{E}[Z_m | \mathcal{F}_\infty]$ a.s. and in L^1 as $n \rightarrow \infty$ for fixed m . Therefore

$$\limsup_{n \rightarrow \infty} |\mathbb{E}[X_n | \mathcal{F}_n] - \mathbb{E}[X | \mathcal{F}_\infty]| \leq \inf_{m \geq 1} \mathbb{E}[Z_m | \mathcal{F}_\infty] \quad \text{a.s.}$$

The claim follows since $\mathbb{E}[Z_m | \mathcal{F}_\infty] \rightarrow 0$ as $m \rightarrow \infty$ a.s. by dominated convergence for conditional expectations.

- (3) We have

$$\mathbb{E}(X_n) = \frac{1}{n} \quad \text{and} \quad \mathbb{E}[X_n | \mathcal{A}] = \mathbb{E}[Y_n] Z_n = \frac{1}{n} Z_n \quad \text{a.s.}$$

As a consequence $X_n \rightarrow 0$ in L^1 . Also, since $\mathbb{P}(X_n > 0) \leq \mathbb{P}(Y_n, Z_n > 0) = \frac{1}{n^2}$ and since $\sum_{n \geq 1} \frac{1}{n^2} < \infty$, it follows that a.s. $X_n = 0$ for n sufficiently large, so that $X_n \rightarrow 0$ a.s.

Finally

$$\sum_{n \geq 1} \mathbb{P}(Z_n n^{-1} \geq 1) \geq \sum_{n \geq 1} \mathbb{P}(Z_n = n) = \sum_{n \geq 1} \frac{1}{n} = \infty$$

and so by Borel-Cantelli we have $\mathbb{E}[X_n | \mathcal{A}] \not\rightarrow 0$ almost surely.

□

Exercise 4. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n \geq 0}$. Assume that there exist $\varepsilon > 0$ and $n_0 \geq 1$ such that for every $n \geq 0$, almost surely

$$\mathbb{P}(T \leq n + n_0 | \mathcal{F}_n) > \varepsilon.$$

- (1) Show that for every $k \geq 0$ we have $\mathbb{P}(T \geq kn_0) \leq (1 - \varepsilon)^k$.
- (2) Show that T is almost surely finite and that $\mathbb{E}[T] < \infty$.

Solution:

- (1) We argue by induction on k . For $k = 0$, the result is true. Assume that $\mathbb{P}(T \geq kn_0) \leq (1 - \varepsilon)^k$. Then write, using the fact that $\mathbb{1}_{T \geq kn_0}$ is measurable with respect to \mathcal{F}_{kn_0} ,

$$\mathbb{P}(T \geq (k+1)n_0) = \mathbb{E}[\mathbb{1}_{T \geq kn_0, T \geq (k+1)n_0}] = \mathbb{E}[\mathbb{1}_{T \geq kn_0} \mathbb{P}(T \geq kn_0 + n_0 | \mathcal{F}_{kn_0})] \leq \mathbb{E}[\mathbb{1}_{T \geq kn_0} (1 - \varepsilon)]$$

which is at most $(1 - \varepsilon)^{k+1}$ by induction hypothesis.

- (2) Write

$$\mathbb{E}[T] = \sum_{k=0}^{\infty} \mathbb{E}[T \mathbb{1}_{kn_0 \leq T < (k+1)n_0}] \leq \sum_{k=0}^{\infty} (k+1)n_0 \mathbb{P}(kn_0 \leq T) \leq \sum_{k=0}^{\infty} (k+1)n_0 (1 - \varepsilon)^k < \infty.$$

A fortiori, this shows that $T < \infty$ a.s.

□

Exercise 5. Let $(M_n)_{n \geq 0}$ be a uniformly integrable martingale with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$.

- (1) Is it true that the collection $\{M_T : T \text{ stopping time with respect to } (\mathcal{F}_n)_{n \geq 0}\}$ is uniformly integrable?
- (2) Let T be a stopping time. Is it true that $(M_{n \wedge T})_{n \geq 0}$ is a uniformly integrable martingale? Justify your answer.

Solution:

- (1) Yes it is true. It follows from the following two facts seen in the lecture : $M_T = \mathbb{E}[Z|\mathcal{F}_T]$ and for any collection of σ -fields $(\mathcal{A}_i)_{i \in I}$ the collection $(\mathbb{E}[Z|\mathcal{A}_i])_{i \in I}$ is uniformly integrable.
- (2) Yes it is true, as a consequence of (1) since $n \wedge T$ is a stopping time for every $n \geq 0$.

□

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Let T be a stopping time with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$ with $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$. Recall that the σ -field \mathcal{F}_T is defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T = n\} \in \mathcal{F}_n \forall n \geq 0\}.$$

- (1) Let S be a stopping time with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ such that $S \leq T$. Show that $\mathcal{F}_S \subset \mathcal{F}_T$.
- (2) Show that T is \mathcal{F}_T measurable.
- (3) Here we assume that $(X_n)_{n \geq 0}$ is a sequence of random variables and that $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Show that $\mathcal{F}_T = \sigma(X_{T \wedge n} : n \geq 0)$.

Solution:

- (1) Take $A \in \mathcal{F}_S$ and $n \geq 0$. We show that $A \cap \{T = n\} \in \mathcal{F}_n$. To this end, using $S \leq T$ write

$$A \cap \{T = n\} = \left(\bigcup_{s=0}^n A \cap \{T = s\} \right) \cap \{T = n\} \in \mathcal{F}_n.$$

- (2) It is enough to show that for every $k \geq 0$, $\{T = k\} \in \mathcal{F}_T$. To this end, take $n \geq 0$ and write $\{T = k\} \cap \{T = n\} = \emptyset \in \mathcal{F}_n$ for $n \neq k$ and $\{T = k\} \cap \{T = k\} \in \mathcal{F}_n$ for $n = k$ by definition of a stopping time.
- (3) We argue by double inclusion.

* First, we have seen in the lecture that $X_{n \wedge T}$ is $\mathcal{F}_{n \wedge T}$ measurable. Since $T \wedge n \leq T$, we have $\mathcal{F}_{n \wedge T} \subset \mathcal{F}_T$ by (1), so $X_{n \wedge T}$ is \mathcal{F}_T measurable. Thus

$$\sigma(X_{n \wedge T} : n \geq 0) \subset \mathcal{F}_T.$$

* We now show that

$$\mathcal{F}_T \subset \sigma(X_{n \wedge T} : n \geq 0).$$

We show by strong induction that

$$\forall k \geq 0, \forall A \in \mathcal{F}_T, \quad A \cap \{T = k\} \in \sigma(X_{n \wedge T} : n \geq 0).$$

– $k = 0$. For $A \in \mathcal{F}_T$,

$$A \cap \{T = 0\} \in \mathcal{F}_0 = \sigma(X_0) = \sigma(X_{0 \wedge T}) \subset \sigma(X_{n \wedge T} : n \geq 0)$$

– Assume that for every $0 \leq i \leq k-1$ and for every $A \in \mathcal{F}_T$ we have $A \cap \{T = i\} \in \sigma(X_{n \wedge T} : n \geq 0)$. Take $A \in \mathcal{F}_T$. We have

$$A \cap \{T = k\} \in \mathcal{F}_k = \sigma(X_0, \dots, X_k),$$

so by the Doob-Dynkin lemma applied with the \mathbb{R}^{k+1} -valued random variable (X_0, \dots, X_k) , there exists a measurable function $f : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ such that

$$\mathbb{1}_{A \cap \{T=k\}} = f(X_0, X_1, \dots, X_k).$$

As $\mathbb{1}_{\{T=k\}} = \mathbb{1}_{\{T=k\}} \mathbb{1}_{\{T \geq k\}}$ we get

$$\mathbb{1}_{A \cap \{T=k\}} = f(X_1, \dots, X_k) \mathbb{1}_{\{T \geq k\}} = f(X_{0 \wedge T}, X_{1 \wedge T}, \dots, X_{k \wedge T}) \mathbb{1}_{\{T \geq k\}}.$$

Because of the induction hypothesis (applied with $A = \Omega$), we have

$$\{T \geq k\} = \{T \leq k-1\}^c \in \sigma(X_{n \wedge T} : n \geq 0),$$

and therefore it follows that $\mathbb{1}_{A \cap \{T=k\}}$ is $\sigma(X_{n \wedge T} : n \geq 0)$ -measurable; this is equivalent to $A \cap \{T = k\} \in \sigma(X_{n \wedge T} : n \geq 0)$.

□

Exercise 7. Let (M_n) be a martingale with respect to a filtration (\mathcal{F}_n) and let S and T be stopping times. Show that for every $n \geq 0$ we almost surely have

$$\mathbb{E}(M_{n \wedge S} \mid \mathcal{F}_T) = M_{n \wedge S \wedge T}$$

Solution:

First of all, for $n \geq 0$, $M_{S \wedge T \wedge n}$ is measurable with respect to $\mathcal{F}_{S \wedge T \wedge n}$, so is is measurable with respect to \mathcal{F}_T (Exercise 6 (1))

We now fix $A \in \mathcal{F}_T$ and show that

$$\mathbb{E}[M_{n \wedge S} \mathbb{1}_A] = \mathbb{E}[M_{n \wedge S \wedge T} \mathbb{1}_A]$$

which will imply the result.

To this end start with writing

$$\mathbb{E}[M_{n \wedge S} \mathbb{1}_A] = \sum_{k=0}^n \mathbb{E}[M_{n \wedge S} \mathbb{1}_{A \cap \{T=k\}}] + \mathbb{E}[M_{n \wedge S} \mathbb{1}_{A \cap \{T>n\}}]$$

We know that $(M_{n \wedge S})_{n \geq 0}$ is a martingale and since $A \cap \{T = k\} \in \mathcal{F}_k$ (by definition) we have for $k \leq n$,

$$\mathbb{E}[M_{n \wedge S} \mathbb{1}_{A \cap \{T=k\}}] = \mathbb{E}[\mathbb{E}[M_{n \wedge S} \mid \mathcal{F}_k] \mathbb{1}_{A \cap \{T=k\}}] = \mathbb{E}[M_{k \wedge S} \mathbb{1}_{A \cap \{T=k\}}].$$

Therefore

$$\mathbb{E}[M_{n \wedge S} \mathbb{1}_A] = \sum_{k=0}^n \mathbb{E}[M_{k \wedge S} \mathbb{1}_{A \cap \{T=k\}}] + \mathbb{E}[M_{n \wedge S} \mathbb{1}_{A \cap \{T>n\}}] = \mathbb{E}[M_{n \wedge S \wedge T} \mathbb{1}_A]$$

which completes the argument. □

Exercise 8. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a Lipschitz function, i.e. there exists $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [0, 1]$. Let $f'_n: [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f'_n(x) = \begin{cases} 2^n \left(f\left(\frac{i+1}{2^n}\right) - f\left(\frac{i}{2^n}\right) \right) & : x \in [i/2^n, (i+1)/2^n), \\ 0 & : x = 1. \end{cases}$$

Note that f'_n is the derivative of the piecewise linear extension of $f|_{(2^{-n}\mathbb{Z}) \cap [0, 1]}$ to $[0, 1]$.

- (1) Show that $f'_n \rightarrow f'$ almost everywhere and in L^1 (with respect to the Lebesgue measure) as $n \rightarrow \infty$ for some integrable function $f': [0, 1] \rightarrow \mathbb{R}$.

Hint. Use the martingale convergence theorem after defining a suitable probability space together with a martingale on it.

- (2) Deduce that

$$f(x) - f(0) = \int_0^x f'(y) dy \quad \text{for all } x \in [0, 1].$$

Solution:

- (1) Let $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$ and let \mathbb{P} be the Lebesgue measure on $[0, 1]$. We also define the

filtration

$$\mathcal{F}_n = \sigma([i2^{-n}, (i+1)2^{-n}]: i = 0, \dots, 2^n - 1).$$

Clearly the sequence (f'_n) is adapted with respect to (\mathcal{F}_n) . Moreover, $|f'_n| \leq K$ by the Lipschitz property, so (f'_n) is bounded in L^∞ and in particular uniformly integrable. Lastly, let us verify the martingale property for (f'_n) . We have

$$\mathbb{E}[f'_{n+1} | \mathcal{F}_n] = \sum_{i=0}^{2^{n+1}-1} 2^{n+1} \left(f\left(\frac{i+1}{2^{n+1}}\right) - f\left(\frac{i}{2^{n+1}}\right) \right) \mathbb{P}([i/2^{n+1}, (i+1)/2^{n+1}) | \mathcal{F}_n)$$

almost surely. The key is now that if $a = j/2^n$ and $b = (j+1)/2^n$ then

$$\mathbb{P}([a, a + (b-a)/2) | \mathcal{F}_n) = \mathbb{P}([a + (b-a)/2, b) | \mathcal{F}_n) = \frac{1}{2} 1_{[a,b]} \quad \text{a.s.}$$

This follows since \mathcal{F}_n is a σ -algebra which is generated by a collection of disjoint sets and hence the conditional expectation is explicit. Using this observation, the terms in the expression $\mathbb{E}[f'_{n+1} | \mathcal{F}_n]$ with $i = 2j$ and with $i = 2j+1$ combine and we see that

$$\mathbb{E}[f'_{n+1} | \mathcal{F}_n] = f'_n \quad \text{a.s.}$$

Hence (f'_n) is a uniformly integrable martingale and hence converges to an integrable limit f' almost surely and in L^1 .

(2) For $n \geq m \geq 0$ and $i \in \{0, \dots, 2^m - 1\}$ we have by definition that

$$f(i/2^m) - f(0) = \int_0^{i/2^m} f'_n(y) dy \rightarrow \int_0^{i/2^m} f'(y) dy \quad \text{as } n \rightarrow \infty.$$

Therefore the result holds for all x of the form $x = i/2^m$ and the full result follows by continuity of f and integrability of f' (using dominated convergence).

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. You throw a fair six-sided die until you get 6. What is the expected number of throws (including the throw giving 6) conditioned on the event that all throws gave even numbers?

Solution:

See <https://www.yichijin.com/files/elchanan.pdf>

□