Week 4: Kolmogorov 0 – 1 law, integration

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 16/10/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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All random variables are defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let μ and ν be measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

1) Assume that $\int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f(x)\nu(\mathrm{d}x)$ when f is any nonnegative measurable function. Show that $\mu = \nu$.
(2) Assume that $\mu(\mathbb{R}) = 1$. Assume that $\int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f(x)\nu(\mathrm{d}x)$ when f is any function of the form $f(x) = \mathbb{1}_{x\in(a,b)}$ with $a < b$ where $a, b \in \mathbb{R}$. Show that $\mu = \nu$.
(3) Assume that $\mu(\mathbb{R}) = 1$. Assume that $\int_{\mathbb{R}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} f(x)\nu(\mathrm{d}x)$ when f is any continuous function with compact support. Show that $\mu = \nu$.

2 Training exercises

Exercise 2.

(1) Let $X \geq 0$ be a non-negative real-valued random variable. Show that $\mathbb{E}[\min(X, n)] \to \mathbb{E}[X]$ as $n \to \infty$.
(2) Let $X \geq 0$ be a non-negative real-valued random variable.
  (a) Assume that $\mathbb{E}[X] < \infty$. Show that $n\mathbb{E}\left[\ln\left(1 + \frac{X}{n}\right)\right] \to \mathbb{E}[X]$ as $n \to \infty$.
  (b) Assume that $\mathbb{E}[X] = \infty$. Show that $n\mathbb{E}\left[\ln\left(1 + \frac{X}{n}\right)\right] \to \infty$ as $n \to \infty$.
(3) Let X be a real-valued integrable random variable. Show that $\mathbb{E}\left[X\mathbb{1}_{|X|\geq n}\right] \to 0$ as $n \to \infty$.

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Exercise 3. Let $X \geq 0$ be a non-negative real-valued random variable.

(1) Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing continuously differentiable function such that $g(0) = 0$. Show that

$$\mathbb{E}[g(X)] = \int_0^{+\infty} g'(t)\mathbb{P}(X \geq t)\,\mathrm{d}t.$$

Hint. Write $g(X)$ as an integral and use Fubini-Tonnelli's theorem.

(2) (Application) Let X be a nonnegative random variable. Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t)\mathrm{d}t$.

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Exercise 4. Let $X : (\Omega, \mathcal{A}) \to \mathbb{R} \cup \{\pm\infty\}$ a random variable.

(1) Let $\mathcal{B} \subset \mathcal{A}$ be a σ-field such that for every $B \in \mathcal{B}$ we have $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$. Assume that $X : (\Omega, \mathcal{B}) \to \mathbb{R} \cup \{\pm\infty\}$ is measurable. Show that there exists a (deterministic) constant $c \in \mathbb{R} \cup \{\pm\infty\}$ such that $\mathbb{P}(X = c) = 1$.

(2) (Application) Let (X_n) be a sequence of independent random variables and let (a_n) be a deterministic sequence with $a_n \to 0$ as $n \to \infty$. Show that there exists (deterministic) constants $C_\pm \in [-\infty, \infty]$ such that

$$\liminf_{n\to\infty} a_n \cdot (X_1 + \cdots + X_n) = C_- \quad \text{and} \quad \limsup_{n\to\infty} a_n \cdot (X_1 + \cdots + X_n) = C_+$$

almost surely.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let X be a random variable taking values in $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let (X_n) be a sequence of i.i.d. random variables with the same law as X.

(1) Show that $\mathbb{E}[X] = \sum_{n\geq 1} \mathbb{P}(X \geq n)$.

Now assume that $\mathbb{E}[X] = \infty$

(2) Show that $\limsup_{n\to\infty} X_n/n \geq k$ almost surely for all $k \in \mathbb{N}$.

Hint. First show that $\sum_{n\geq 1} \mathbb{P}(X \geq nk) = \infty$.

(3) Deduce that $\limsup_{n\to\infty} X_n/n = \infty$ almost surely.

Now consider any real-valued random variable Y satisfying $\mathbb{E}[|Y|] = \infty$ and let (Y_n) be i.i.d. random variables, each of which has the same law as Y.

(4) Using the previous questions, show that $\limsup_{n\to\infty} |Y_n|/n = \infty$ almost surely.

(5) Deduce that $\limsup_{n\to\infty} |Y_1 + \cdots + Y_n|/n = \infty$ almost surely.

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Exercise 6. (The Doob–Dynkin lemma does not hold for general σ-algebras) Set $E = F = \{1, 2\}$ and equip F with the σ-algebra $\{\emptyset, F\}$. Let $f : E \to F$ be defined by $f(1) = f(2) = 1$. Find a measurable function $g = (E, \sigma(f)) \to (\mathbb{R}, \{\emptyset, \mathbb{R}\})$ which cannot be written in the form $g = h \circ f$ with $h : (F, \mathcal{F}) \to (\mathbb{R}, \{\emptyset, \mathbb{R}\})$ measurable.

Remark. Here $\mathbb{R}$ is equipped with the trivial σ-algebra $\{\emptyset, \mathbb{R}\}$ and not with the Borel σ-algebra.

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Exercise 7.

(1) Let f be a differentiable function on $[0,1]$, with bounded derivative f'. Show that

$$\int_0^1 f'(x)\mathrm{d}x = f(1) - f(0).$$

(2) Find a continuous function, almost everywhere differentiable on $[0,1]$ such that $f(0) = 0, f(1) = 1$ and $\int_0^1 f'(x)\mathrm{d}x = 0$.

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Exercise 8. Let $\varphi : ([0,1], \mathcal{B}([0,1])) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be an integrable function for the Lebesgue measure. Define $G : \mathbb{R} \to \mathbb{R}_+$ by

$$G(t) = \int_{[0,1]} |\varphi(x) - t| \mathrm{d}x.$$

(1) Show that G is continuous.
(2) Show that G is differentiable at $t \in \mathbb{R}$ if and only if $\lambda(\{\varphi = t\}) = 0$, where λ denotes the Lebesgue measure.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9.

(1) In the logicians' prison, the following game is proposed to 100 prisoners. The 100 prisoners are lined up in a single line, so that each logician can see the logicians in front of her but not those behind her. A black or white hat is placed on the head of each of them, and then the logicians are asked one by one to guess the color of their hat. They start by asking the one at the back (the one who can see 99 hats), and then they work their way up to the one at the front (the one who can't see any). Each logician hears all the previous answers. If a logician correctly guesses the color of his hat, she is released. If she does not, she is sentenced to death. The logicians have the right to confer before the game and establish a strategy. What strategy can minimize the number of deaths?

(2) In Hilbert's prison, there is an infinite (countable) number of logicians. It is decided to make them play the same hat game. Again, each logician can see all the hats in front of her (now an infinite number), and must determine the color of her own hat. What strategy can save the maximum number of logicians?