Tensor Spherical Harmonics and Tensor Spherical Splines

by

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Abstract. In this paper, we deal with the problem of spherical interpolation of discretely given data of tensorial type. To this end, spherical tensor fields are investigated and a decomposition formula is described. It is pointed out that the decomposition formula is of importance for the spectral analysis of the gravitational tensor in (spaceborne) gradiometry. Tensor spherical harmonics are introduced as eigenfunctions of a tensorial analogue to the Beltrami operator and discussed in detail. Based on these preliminaries, a spline interpolation process is described and error estimates are presented. Furthermore, some relations between the spline basis functions and the theory of radial basis functions are developed.

Introduction

In the last decade, the interpolation problem of discretely given data by spherical splines has been solved in numerous papers. Both theoretical and numerical aspects of spherical splines have been analyzed in depth, for scalar–valued as well as for vector–valued data (cf. Freeden (1981), Freeden (1982), Freeden (1990), Freeden and Gervens (1993), Freeden and Hermann (1986), Wahba (1981), Wahba (1984), and the references therein). It turns out that the spherical splines are natural generalizations of the spherical harmonics respectively the vector spherical harmonics, having quite similar properties as the well–known one–dimensional polynomial splines.

Spherical splines are well–suited for various interpolation and best approximation problems preferably in the geosciences. In particular, they have been used for the macro– and micro–modelling of the earth's gravitational field from discretely given data (cf. e.g. Euler et al. (1985), Euler et al. (1986), Freeden (1990), v. Gysen and Merry (1990), Kling et al. (1987)). Moreover, based on the ideas of the scalar theory, a successfull application of spherical splines has been given to the analysis of meteorological data (Wahba 1982).

In this paper, we develop the general theory of tensor spherical splines for the interpolation of spherical tensor fields, its components given in discrete points on the sphere. The main idea hereby is the decomposition of the tensor fields in nine scalar functions and to use the basic results from the theory of scalar spherical splines. This decomposition is motivated by the theory of tensor spherical harmonics, as described in Backus (1966) and Backus (1967). (Other approaches for the definition of tensor spherical harmonics are described e.g. in Zerilli (1970), James (1976) and Jones (1980). A review article about tensor spherical harmonics used in the general relativity literature is due to Thorne (1980).) We extend this theory and show that the tensor spherical harmonics are the eigenfunctions of the tensorial analogue of the well–known Beltrami operator. This theory is the main tool for the establishment of tensor spherical splines which is in fact a generalization of these harmonics. As in the scalar theory, the interpolation problem is a well–posed problem of minimizing a semi–norm of a suitable Sobolev space under interpolatory constraints.

Tensor spherical splines are useful, for example, in multipole expansions of gravitational radiation (cf. e.g. Regge and Wheeler (1957), Zerilli (1970)), the deformation analysis of the earth's surface (cf. e.g. Grafarend (1986), Georgiadou and Grafarend (1986), Mochizuki (1988), McClung (1989)), in the investigation of the influence of rotation on the free elastic–gravitational oscillations of the earth (cf. e.g. Backus and Gilbert (1961), Dahlen and Smith (1975), etc.). Furthermore, they offer possibilities to solve geodetic problems in gradiometry, namely the determination of the earth's gravitational field from the gravitational tensor of second derivatives (Hesse matrix).

Terrestrial gradiometer measurements are very much affected by density variation and topographic features in the immediate vicinity of the observation point. Thus, in spite of the very laborious work, they are very suitable for exploration geophysics (cf. Jung (1961)). Gravity gradiometry measurements, taken at certain altitude above the earth's surface, are highly sensitive to the fine structure of the earth's gravity field by observing in principle the relative motion of neighbouring, free–falling test masses. Spaceborne gradiometry, as planned for the next decade, will improve the "microscopic structure" of the gravity field. This will lead us to a better understanding of the composition and dynamics of the solid earth (cf. e.g. Rummel (1986), Rummel and Colombo (1984), Sacerdote and Sansó (1989), Schwarz and Krynski (1977)). The mathematical characteristics of gradiometry are commonly explained employing spectral analysis as applied to spherical harmonic expansions (cf. Rummel (1975), Rummel and van Gelderen (1992)). To discuss the gravity tensor of second derivatives tensor spherical harmonics are needed similar to the application of vector spherical harmonics to the expansion of the deflections of the vertical (cf. Meissl (1971a), Groten and Moritz (1964)). To obtain gravity information from discrete gradiometer data tensor spherical splines as natural generalizations of "tensor spherical polynomials" i.e. tensor spherical harmonics, seem to be adequate structures thereby avoiding specific difficulties (e.g. oscillation phenomena, convergence problems, determination of higher order terms, etc.) in interpolation and approximation. Moreover, spline techniques become available for smoothing noisy data (cf. e.g. Freeden and Witte (1982), Wahba (1984), Girard (1991), Utreras (1990)) thereby using control parameters for the trade off between "approximation to the data" and "smoothness" of the solution.

Some emphasis is drawn on the choice of the norm. Here, we develop tools to combine the spherical spline theory with generalizations of the concept of radial basis functions (cf. Micchelli (1986), Madych and Nelson (1988) and many others).

The paper is organized as follows: In Section 1 some preliminaries are given. The theory of tensor spherical harmonics is established in Section 2. In detail, we consider spherical tensor fields, their decomposition in normal and tangential parts to get nine scalar functions. Tensor spherical harmonics are recognized as eigenfunctions of a tensorial analogue of the Beltrami operator. To define the tensor splines in the third section, we first have a look at adequate Sobolev spaces. After that we consider the interpolation algorithm and prove the convergence of our method. Appendix A illustrates the explicit decomposition of the gravitational tensor (Hesse matrix) e.g. for single poles and multipoles (i.e. solid spherical harmonics). Finally, Appendix B gives a mathematical overview about possible choices of norms and relations to the theory of radial basis functions.

1 Preliminaries

We begin by introducing some notations that will be used throughout the paper.

1.1 Definitions and Notations

Let us use $x, y, \ldots$ to represent the elements of Euclidean space $\mathbb{R}^3$. For all $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, different from the origin, we have

$$x = r\xi \ , \qquad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \ ,$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of $x \in \mathbb{R}^3$. The unit sphere in $\mathbb{R}^3$ will be denoted by Ω. If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$, we may represent the points $\xi \in \Omega$ in polar coordinates by

$$\begin{aligned} \xi &= t\ \varepsilon^3 + \sqrt{1-t^2}\ (\cos\varphi\ \varepsilon^1 + \sin\varphi\ \varepsilon^2), \\ -1 &\le t \le 1\ , \quad 0 \le \varphi < 2\pi\ , \ t = \cos\theta\ . \end{aligned} \tag{1.1}$$

As usual, inner–, and vector–, and dyadic (tensor) product of two vectors $x, y \in \mathbb{R}^3$, respectively, are defined by

$$\begin{aligned} x \cdot y &= \sum_{i=1}^{3} x_i\ y_i\ , \\ x \wedge y &= (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)^T\ , \\ x \otimes y &= \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}\ . \end{aligned} \tag{1.2}$$

In terms of the polar coordinates (1.1) the gradient ∇ in $\mathbb{R}^3$ reads

$$\nabla_x = \xi \frac{\partial}{\partial r} + \frac{1}{r}\ \nabla^*_\xi\ ,$$

where ∇^* is the surface gradient of the unit sphere Ω. Moreover, the Laplace operator $\Delta = \nabla \cdot \nabla$ in $\mathbb{R}^3$ has the representation

$$\Delta_x = (\frac{\partial}{\partial r})^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi\ ,$$

where $\Delta^* = \nabla^* \cdot \nabla^*$ is the Beltrami operator of the unit sphere Ω (for explicit representations in terms of polar coordinates see e.g. Meissl (1971a)).

A function $F : \Omega \to \mathbb{R}$ (resp. $f : \Omega \to \mathbb{R}^3$) [1] possessing k continuous derivatives on the unit sphere Ω is said to be of class $\mathcal{C}^{(k)}(\Omega)$ (resp. $c^{(k)}(\Omega)$). $\mathcal{C}^{(0)}(\Omega)$ (resp. $c^{(0)}(\Omega)$) is the class of real continuous scalar valued (resp. vector valued) functions on Ω .

[1] Scalar valued (resp. vector valued, tensor valued) functions are denoted by capital (resp. small, small bold) letters throughout this paper.

$\mathcal{C}^{(0)}(\Omega)$ is a complete normed space endowed with

$$\|F\|_{\mathcal{C}^{(0)}} = \sup_{\xi\in\Omega} \mid F(\xi) \mid .$$

In $\mathcal{C}^{(0)}(\Omega)$ we have the inner product $(\cdot,\cdot)_{\mathcal{L}^2}$ corresponding to the norm

$$\|F\|_{\mathcal{L}^2} = \sqrt{\int_\Omega \mid F(\xi) \mid^2 d\omega(\xi)},$$

($d\omega$ denotes the surface element). In connection with $(\cdot,\cdot)_{\mathcal{L}^2}$, $\mathcal{C}^{(0)}(\Omega)$ is a pre–Hilbert space. For each $F \in \mathcal{C}^{(0)}(\Omega)$ we have the inequality

$$\|F\|_{\mathcal{L}^2} \le \sqrt{4\pi}\|F\|_{\mathcal{C}^{(0)}}.$$

By $\mathcal{L}^2(\Omega)$ we denote the space of (Lebesgue) square–integrable scalar functions on Ω. $\mathcal{L}^2(\Omega)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathcal{L}^2}$. $\mathcal{L}^2(\Omega)$ is the completion of $\mathcal{C}^{(0)}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{L}^2}$.

Since the operators ∇^* and Δ^* are of particular interest throughout this paper, we list some of their properties. Especially, for $\xi\in\Omega$, we have

$$\begin{aligned}
\nabla^*_\xi\cdot(\nabla^*_\xi F(\xi)) &= \Delta^*_\xi F(\xi),\\
\nabla^*_\xi\cdot(\xi\wedge\nabla^*_\xi F(\xi)) &= 0,\\
\nabla^*_\xi\cdot(\xi\wedge(\xi\wedge\nabla^*_\xi F(\xi))) &= -\Delta^*_\xi F(\xi),\\
\nabla^*_\xi\cdot(F(\xi)f(\xi)) &= (\nabla^*_\xi F(\xi))\cdot f(\xi)\\
&\quad +F(\xi)(\nabla^*_\xi\cdot f(\xi)).
\end{aligned}$$

By the theorems of Gauß and Stokes we obtain

$$\begin{aligned}
\int_\Omega \nabla^*_\xi\cdot f(\xi)d\omega(\xi) &= 0,\\
\int_\Omega \nabla^*_\xi\cdot(f(\xi)\wedge\xi)d\omega(\xi) &= 0,\\
\int_\Omega f(\xi)\cdot\nabla^*_\xi F(\xi)d\omega(\xi) &= \int_\Omega F(\xi)\nabla^*_\xi\cdot f(\xi)d\omega(\xi),\\
\int_\Omega \nabla^*_\xi F(\xi)\cdot\nabla^*_\xi G(\xi)d\omega(\xi) &= -\int_\Omega F(\xi)\Delta^*_\xi G(\xi)d\omega(\xi)\\
&= -\int_\Omega G(\xi)\Delta^*_\xi F(\xi)d\omega(\xi),
\end{aligned}$$

where $\nabla^*_\xi\cdot f(\xi)$ (resp. $\nabla^*_\xi\cdot(f(\xi)\wedge\xi)$) is the *surface divergence* (resp. *surface curl*) of the vector field f at $\xi\in\Omega$. Note that the surface curl $\xi\mapsto\nabla^*_\xi\cdot(f(\xi)\wedge\xi)$, $\xi\in\Omega$, defines a scalar–valued function on the unit sphere $\Omega\subset\mathbb{R}^3$ (that is the reason why we do not use the notation $\nabla^*_\xi\wedge f(\xi)$ instead of $\nabla^*_\xi\cdot(f(\xi)\wedge\xi)$). Furthermore, Green's identities read

$$\begin{aligned}
F(\xi) &= \frac{1}{4\pi}\int_\Omega F(\eta)d\omega(\eta)\\
&\quad+\frac{1}{4\pi}\int_\Omega\left(\nabla^*_\eta\left(2\ln\frac{2}{\mid\xi-\eta\mid}\right)\right)\cdot(\nabla^*_\eta F(\eta))d\omega(\eta),
\end{aligned}$$

$$\begin{aligned}
F(\xi) &= \frac{1}{4\pi}\int_\Omega F(\eta)d\omega(\eta)\\
&\quad-\frac{1}{4\pi}\int_\Omega\left(2\ln\frac{2}{\mid\xi-\eta\mid}\right)(\Delta^*_\eta F(\eta))d\omega(\eta),
\end{aligned}$$

provided that $F:\Omega\to\mathbb{R}$ resp. $f:\Omega\to\mathbb{R}^3$ are sufficiently often continuously differentiable. Moreover, for $H\in\mathcal{C}^{(2)}[-1,1]$, we have

$$\begin{aligned}
\nabla^*_\xi H(\xi\cdot\eta) &= H'(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi),\\
\Delta^*_\xi H(\xi\cdot\eta) &= -2(\xi\cdot\eta)H'(\xi\cdot\eta)\\
&\quad+(1-(\xi\cdot\eta)^2)H''(\xi\cdot\eta).
\end{aligned}$$

In particular,

$$\begin{aligned}
\nabla^*_\xi(\xi\cdot\eta) &= \eta-(\xi\cdot\eta)\xi,\\
\Delta^*_\xi(\xi\cdot\eta) &= -2(\xi\cdot\eta).
\end{aligned}$$

1.2 Scalar Spherical Harmonics

As usual, the spherical harmonics Y_n of order n are defined as the everywhere on Ω infinitely differentiable eigenfunctions of the Beltrami operator Δ^* corresponding to the eigenvalues $-n(n+1)$, $n=0,1,2,\ldots$, i.e.,

$$(-\Delta^*_\xi-\lambda_n)\,Y_n(\xi)=0,\ \lambda_n=n(n+1),\ \xi\in\Omega.$$

The functions $H_n:\mathbb{R}^3\to\mathbb{R}$ defined by $H_n(x)=r^nY_n(\xi), r=|x|$, are homogeneous polynomials of degree n in rectangular coordinates which satisfy the Laplace equation $\Delta_x H_n(x)=0,\ x\in\mathbb{R}^3$. Conversely, every homogeneous harmonic polynomial of degree n when restricted to the unit sphere Ω is a spherical harmonic of order n. (An efficient algorithm for the *exact* generation of homogeneous harmonic polynomials in $\mathbb{R}^3$ can be found in Freeden and Reuter (1990).)

The Legendre polynomials $P_n:[-1,1]\to[-1,1], n=0,1,\ldots$ given by

$$P_n(t)=\sum_{s=0}^{[\frac{n}{2}]}(-1)^s\frac{(2n-2s)!}{2^n(n-2s)!(n-s)!s!}t^{n-2s},$$

are the only everywhere on the interval $[-1,1]$ infinitely differentiable eigenfunctions of the Legendre operator, i.e.,

$$\left(-(1-t^2)\left(\frac{d}{dt}\right)^2+2t\frac{d}{dt}-\lambda_n\right)P_n(t)=0$$

for $t\in[-1,1]$, which satisfy $P_n(1)=1$. Apart from a constant factor, the functions $P_n(\varepsilon^3\cdot):\Omega\to\mathbb{R},\ \xi\mapsto P_n(\varepsilon^3\cdot\xi)$, $\xi\in\Omega$, are the only spherical harmonics which are invariant under orthogonal transformations which leave ε^3 fixed. As is well-known, $|P_n(t)|\le P_n(1)=1$ for all $t\in[-1,+1]$ and all n.

The linear space Harm_n of all spherical harmonics of order n is of dimension $2n+1$. Thus, there exist $2n+1$ linearly independent spherical harmonics $Y_{n,1},...,Y_{n,2n+1}$. We assume this system to be orthonormalized in the sense of the $\mathcal{L}^2(\Omega)$–inner product

$$(Y_{n,j}, Y_{m,k})_{\mathcal{L}_2} = \int_\Omega Y_{n,j}(\eta)\, Y_{m,k}(\eta)\, d\omega(\eta) = \delta_{j,k}\, \delta_{n,m}.$$

Throughout the paper, for convenience, the capital letter Y followed by double indices (for example $Y_{n,j}$) denotes a member of a $\mathcal{L}^2(\Omega)$–orthonormal system $\{Y_{n,1}, \ldots, Y_{n,2n+1}\}$.

For any pair $\xi, \eta \in \Omega$, the sum

$$F_n(\xi,\eta) = \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta)$$

is invariant under all orthogonal transformations $\mathbf{t}$, i.e., $F_n(\mathbf{t}\xi, \mathbf{t}\eta) = F_n(\xi,\eta)$. For a fixed vector $\xi \in \Omega$, $F_n(\xi,\cdot) : \Omega \to \mathbb{R}$ is a spherical harmonic of order n. $F_n(\xi,\eta)$ is symmetric in ξ and η and depends only on the inner product of ξ and η . Hence, we have apart from a multiplicative constant a_n

$$F_n(\xi,\eta) = a_n\ P_n(\xi\cdot\eta)\ .$$

For the evaluation of a_n we set $\xi = \eta$. This gives

$$F_n(\xi,\xi) = a_n\ P_n(1) = a_n\ .$$

Integration over Ω yields $2n+1 = 4\pi\ a_n$. Therefore we finally obtain the *addition theorem of scalar spherical harmonics* (cf. Müller (1966), Freeden (1978))

$$\frac{2n+1}{4\pi}\ P_n(\xi\cdot\eta) = \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta)\ .$$

For $\xi = \eta$ we get the formula

$$\sum_{j=1}^{2n+1} (Y_{n,j}(\xi))^2 = \frac{2n+1}{4\pi},$$

and

$$\sup_{\xi\in\Omega} |Y_{n,j}(\xi)|\ \le\ \left(\frac{2n+1}{4\pi}\right)^{\frac{1}{2}},\quad j = 1,\ldots,2n+1\ .$$

Let H be a function of class $\mathcal{L}^1[-1,1]$ and Y_n be a spherical harmonic of order n. Then *Hecke's formula* states

$$\int_\Omega H(\xi\cdot\eta)\ Y_n(\eta)\ d\omega(\eta) = \hat{H}(n) Y_n(\xi)\ , \qquad (1.3)$$

where

$$\hat{H}(n) = 2\pi \int_{-1}^{1} H(t)\ P_n(t)\ dt. \qquad (1.4)$$

This formula establishes the close connection between the orthogonal invariance of the sphere and the addition theorem. The principle of the Hecke formula is that of isotropic operators, e.g. developed by P. Meissl (1971b). Since the kernel H depends only on the inner product $\xi\cdot\eta$, or equivalently on the distance between ξ and η, the spherical harmonics Y_n are the eigenfunctions of the integral operator defined by the left hand side of (1.3) corresponding to the eigenvalues $\hat{H}(n)$. Therefore, Hecke's formula greatly simplifies most manipulations with spherical harmonics, in particular when spectral analysis is pursued.

Frequently, we will make use of the *Green function* $G(\Delta^* + \lambda_n;\cdot,\cdot)$ with respect to the operator $\Delta^* + \lambda_n$. The *defining properties of* $G(\Delta^* + \lambda_n;\cdot,\cdot)$ read as follows:

(i) for every $\xi \in \Omega$, $G(\Delta^* + \lambda_n;\xi,\eta)$ is infinitely differentiable for all $\eta \in \Omega\backslash\{\xi\}$, and we have the *differential equation*

$$(\Delta^*_\eta + \lambda_n) G(\Delta^* + \lambda_n;\xi,\eta) = -\frac{2n+1}{4\pi} P_n(\xi\cdot\eta)$$

for all $\eta \in \Omega\backslash\{\xi\}$.

(ii) for every $\xi \in \Omega$, the function

$$G(\Delta^* + \lambda_n;\xi,\eta) - 1/(4\pi)\ln(1-\xi\cdot\eta)$$

is continuously differentiable for all $\eta \in \Omega$ (*characteristic singularity*).

(iii) $G(\Delta^*+\lambda_n;\xi,\eta)$ depends only on the inner product of ξ and η (*spherical symmetry*), i.e.

$$G(\Delta^* + \lambda_n;\mathbf{t}\xi,\mathbf{t}\eta) = G(\Delta^* + \lambda_n;\xi,\eta)$$

for all orthogonal transformations $\mathbf{t}$.

(iv) the *normalizing condition*

$$\int_\Omega G(\Delta^* + \lambda_n;\xi,\eta) P_n(\xi\cdot\eta) d\omega(\eta) = 0$$

holds.

$G(\Delta^* + \lambda_n;\cdot,\cdot)$ is uniquely determined by the conditions (i)–(iv). Especially, for $\xi,\eta \in \Omega, \xi \neq \eta$, the Green functions are given by

$$G(\Delta^*;\xi,\eta) = \frac{1}{4\pi}\ln(1-\xi\cdot\eta) + \frac{1}{4\pi} - \frac{1}{4\pi}\ln 2 \quad , \qquad (1.5)$$

and for $n = 1, 2, \ldots$ by

$$\begin{aligned} G(\Delta^* + \lambda_n;\xi,\eta) &= \frac{1}{4\pi} P_n(\xi\cdot\eta)\ln(1-\xi\cdot\eta) \\ &-\frac{1}{2\pi}\sum_{k=0}^{n-1} \frac{2k+1}{\lambda_k - \lambda_n} P_k(\xi\cdot\eta) \qquad (1.6) \\ &-\frac{1}{4\pi}\left(\frac{2n+1}{2}\int_{-1}^{+1} P_n^2(t)\ln(1-t)dt\right) P_n(\xi\cdot\eta). \end{aligned}$$

The spherical harmonics of order n, i.e. the eigenfunctions of the Beltrami operator Δ^* with respect to the eigenvalue $-\lambda_n$, are eigenfunctions of Green's function with respect to the operator $\Delta^* + \lambda_k$ in the sense

$$Y_n(\xi) = \frac{\lambda_n - \lambda_k}{4\pi} \int_\Omega G(\Delta^* + \lambda_k; \xi, \eta) Y_n(\eta) d\omega(\eta), \; k \neq n.$$

Let $G((\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_m); \cdot, \cdot)$, for $m = 1, 2, \ldots$, be defined by convolution as follows

$$G((\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_m); \xi, \eta) = \int_\Omega G((\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_{m-1}); \xi, \zeta) \times G(\Delta^* + \lambda_m; \zeta, \eta) d\omega(\zeta).$$

Then $G((\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_m); \cdot, \cdot)$ is the Green function with respect to the operator $(\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_m)$. For $m > 0$ and all $\xi, \eta \in \Omega$, $G((\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_m); \cdot, \cdot)$ admits a representation in terms of the bilinear expansion

$$\sum_{k=m+1}^{\infty} \frac{2k+1}{4\pi} \frac{1}{(\lambda_k - \lambda_0) \cdot \ldots \cdot (\lambda_k - \lambda_m)} P_k(\xi \cdot \eta)$$

which converges absolutely and uniformly both in ξ and η.

The kernel (null space) of the operator $(\Delta^* + \lambda_0) \cdot \ldots \cdot (\Delta^* + \lambda_m)$ is the space $\mathrm{Harm}_{0,\ldots,m}$ of all spherical harmonics of order m or less. $\mathrm{Harm}_{0,\ldots,m}$ has the dimension

$$M = \sum_{n=0}^{m} (2n+1) = (m+1)^2 .$$

With respect to the inner product $(\cdot, \cdot)_{\mathcal{L}^2}$ we have the orthogonal decomposition

$$\mathrm{Harm}_{0,\ldots,m} = \mathrm{Harm}_0 \oplus \ldots \oplus \mathrm{Harm}_m .$$

Furthermore, the space $\mathcal{L}^2(\Omega)$ resp. $\mathcal{C}^{(0)}(\Omega)$ is the completion of $\mathrm{Harm}_{0,\ldots,\infty} = \mathrm{span}(Y_{n,j})$ with respect to the $\mathcal{L}^2(\Omega)$–topology resp. $\mathcal{C}^{(0)}(\Omega)$–topology.

A set $X_M = \{\eta_1, \ldots, \eta_M\}$ of M points $\eta_1, \ldots, \eta_M$ on Ω is called $\mathrm{Harm}_{0,\ldots,m}$–*unisolvent* if the rank of the (M, M)-matrix

$$\left(\sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\eta_j \cdot \eta_k) \right)_{\substack{j=1,\ldots,M \\ k=1,\ldots,M}} \tag{1.7}$$

is equal to M. The following statements are equivalent:

(i) $X_M = \{\eta_1, \ldots, \eta_M\}$ is $\mathrm{Harm}_{0,\ldots,m}$–unisolvent,

(ii) rank $\left(Y_{n,j}(\eta_k)\right)_{n=0,\ldots,m; j=1,\ldots,2n+1; k=1,\ldots,M} = M$,

(iii) the functions

$$\sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\cdot \eta_1), \ldots, \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\cdot \eta_M)$$

are linearly independent.

The equivalence "(i) $\Leftrightarrow$ (ii)" follows from the addition theorem, while "(ii) $\Leftrightarrow$ (iii)" is obtained using the fact that

$$P_n(\eta_j \cdot \eta_k) = \frac{2n+1}{4\pi} \int_\Omega P_n(\eta_j \cdot \beta) P_n(\eta_k \cdot \beta) d\omega(\beta) .$$

Hence, (1.7) can be regarded as a matrix of Gram type. If X_M is a $\mathrm{Harm}_{0,\ldots,m}$–unisolvent set, then we are able to interpolate given real numbers $w_1, \ldots, w_M$ by a unique $Y \in \mathrm{Harm}_{0,\ldots,m}$, i.e. $Y(\eta_k) = w_k, \; k = 1, \ldots, M$.

1.3 Vector Spherical Harmonics

A function $f : \Omega \to \mathbb{R}^3$ is called *spherical vector field.* By $l^2(\Omega)$ we denote the space of (Lebesgue) square integrable vector fields on Ω, i.e.

$$l^2(\Omega) = \left\{ f : \Omega \to \mathbb{R}^3 \,\middle|\, \int_\Omega f(\xi) \cdot f(\xi) d\omega(\xi) < \infty \right\}.$$

$l^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(f, g)_{l^2} = \left(\int_\Omega f(\xi) \cdot g(\xi) d\omega(\xi) \right)^{1/2} .$$

$c^{(k)}(\Omega)$ denotes the space of vector fields with k–times continously differentiable components on Ω.

For a given vector field $f : \Omega \to \mathbb{R}^3$ the field

$$f_{norm} : \xi \mapsto f_{norm}(\xi) = (\xi \cdot f(\xi))\xi, \; \xi \in \Omega$$

is called the *normal part* of f, while

$$f_{tan} : \xi \mapsto f_{tan}(\xi) = f(\xi) - f_{norm}(\xi), \; \xi \in \Omega$$

is called the *tangential part* of f. A vector field f is called *tangential* (resp. *normal*), if $f(\xi) = f_{tan}(\xi)$ (resp. $f(\xi) = f_{norm}(\xi)$) for all $\xi \in \Omega$.

The study of vector fields on the sphere can be greatly simplified by the following decomposition theorem for continously differentiable vector fields $f : \Omega \to \mathbb{R}^3$

$$f(\xi) = f_{norm}(\xi) + f_{tan}(\xi), \; \xi \in \Omega,$$

where

$$f_{tan}(\xi) = f^{(1)}_{tan}(\xi) + f^{(2)}_{tan}(\xi), \; \xi \in \Omega,$$

and

$$\begin{aligned} \nabla^*_\xi \cdot f^{(1)}_{tan}(\xi) &= \nabla^*_\xi \cdot f_{tan}(\xi), \\ \nabla^*_\xi \cdot (f^{(1)}_{tan}(\xi) \wedge \xi) &= 0, \\ \nabla^*_\xi \cdot f^{(2)}_{tan}(\xi) &= 0, \\ \nabla^*_\xi \cdot (f^{(2)}_{tan}(\xi) \wedge \xi) &= \nabla^*_\xi \cdot (f(\xi) \wedge \xi). \end{aligned}$$

Due to Backus (1966) (see also Meissl (1971a)) there exist uniquely determined functions $F_i \in C^{(2)}(\Omega), i = 1, 2$ satisfying

$$\int_\Omega F_i(\xi) d\omega(\xi) = 0, i = 1, 2,$$

such that

$$\begin{aligned} f_{tan}^{(1)}(\xi) &= \nabla_\xi^* F_1(\xi),\ \xi \in \Omega, \\ f_{tan}^{(2)}(\xi) &= \xi \wedge \nabla_\xi^* F_2(\xi),\ \xi \in \Omega. \end{aligned}$$

Freeden and Gervens (1993) give the explicit representations of $F_i, i = 1,2$, in terms of Green's function $G(\Delta^*;\cdot,\cdot)$, namely

$$\begin{aligned} F_1(\xi) &= \int_\Omega G(\Delta;\xi,\eta)\nabla_\eta^* \cdot (f(\eta) - (\eta\cdot f(\eta))\eta) d\omega(\eta) \\ F_2(\xi) &= -\int_\Omega G(\Delta;\xi,\eta)\nabla_\eta^* \cdot (\eta \wedge f(\eta)) d\omega(\eta) \end{aligned}$$

Using the operators $O^{(i)} : c^{(1)}(\Omega) \to C^{(0)}(\Omega), i = 1,2,3$, defined by

$$\begin{aligned} O_\xi^{(1)} f(\xi) &= \xi \cdot f(\xi) \\ O_\xi^{(2)} f(\xi) &= -\nabla_\xi^* \cdot (f(\xi) - (\xi\cdot f(\xi))\xi) \\ O_\xi^{(3)} f(\xi) &= \nabla_\xi^* \cdot (\xi \wedge f(\xi)) \end{aligned}$$

we are able to rewrite the decomposition formula as follows

$$\begin{aligned} f_{norm}(\xi) &= \left(O_\xi^{(1)} f(\xi)\right)\xi \\ f_{tan}^{(1)}(\xi) &= -\nabla_\xi^* \int_\Omega G(\Delta^*;\xi,\eta) O_\eta^{(2)} f(\eta) d\omega(\eta) \\ f_{tan}^{(2)}(\xi) &= -\xi \wedge \nabla_\xi^* \int_\Omega G(\Delta^*;\xi,\eta) O_\eta^{(3)} f(\eta) d\omega(\eta). \end{aligned}$$

Corresponding to the operators $O^{(i)}$ we introduce operators $o^{(i)} : C^{(1)}(\Omega) \to c^{(0)}(\Omega), i = 1,2,3$, by setting

$$\begin{aligned} o_\xi^{(1)} F(\xi) &= F(\xi)\xi \\ o_\xi^{(2)} F(\xi) &= \nabla_\xi^* F(\xi) \\ o_\xi^{(3)} F(\xi) &= \xi \wedge \nabla_\xi^* F(\xi). \end{aligned}$$

Then it is not difficult to show that for $F \in C^{(2)}(\Omega)$

$$O^{(j)}(o^{(i)} F(\xi)) = 0,\ j \neq i.$$

Moreover, for $i = j$, we get

$$\begin{aligned} O^{(1)}(o^{(1)} F(\xi)) &= F(\xi) \\ O^{(2)}(o^{(2)} F(\xi)) &= -\Delta_\xi^* F(\xi) \\ O^{(3)}(o^{(3)} F(\xi)) &= -\Delta_\xi^* F(\xi). \end{aligned}$$

By definition, let

$$o_i = \begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 2,3. \end{cases}$$

Now, let Y_n be a scalar spherical harmonic of order n. Then it follows that

$$O^{(i)}(o^{(i)} Y_n(\xi)) = \mu_n^{(i)} Y_n(\xi)$$

for all $\xi \in \Omega$ and $i = 1,2,3$, and for $n = o_i, o_i + 1, \ldots$, where

$$\mu_n^{(i)} = \begin{cases} 1 & \text{for } i = 1 \\ \lambda_n & \text{for } i = 2,3. \end{cases}$$

The vector fields

$$y_n^{(i)}(\xi) = o_\xi^{(i)} Y_n(\xi),\ \xi \in \Omega,\ n = o_i, o_i + 1, \ldots$$

are called *vector spherical harmonics of order n and kind i*. $y_n^{(1)}$ describes a normal field, while $y_n^{(2)}, y_n^{(3)}$ are tangential fields of order n. Obviously, according to our construction (cf. Freeden and Gervens (1993)),

$$\begin{aligned} \xi \wedge y_n^{(1)}(\xi) &= 0, & & \\ \xi \cdot y_n^{(2)}(\xi) &= 0, & \nabla_\xi^* \cdot (\xi \wedge y_n^{(2)}(\xi)) &= 0, \\ \xi \cdot y_n^{(3)}(\xi) &= 0, & \nabla_\xi^* \cdot y_n^{(3)}(\xi) &= 0, \\ \xi \wedge y_n^{(2)}(\xi) &= y_n^{(3)}(\xi) & & \end{aligned}$$

and

$$y_n^{(i)}(\xi) \cdot y_n^{(j)}(\xi) = 0, i \neq j.$$

The vector fields $y_{n,j}^{(i)} = (\mu_n^{(i)})^{-1/2} o_\xi^{(i)} Y_{n,j}(\xi)$, $n = o_i, o_i + 1, \ldots$, form a l^2–orthonormal system ($\{Y_{n,j}\}$ being always assumed to be $\mathcal{L}^2$–orthonormal); more explicitly

$$\int_\Omega y_{n,k}^{(i)}(\xi) \cdot y_{m,l}^{(j)}(\xi) d\omega(\xi) = \delta_{i,j}\delta_{n,m}\delta_{k,l}. \tag{1.8}$$

The vector spherical harmonics of order n are eigenfunctions of the vectorial analogue $\mathbf{\Delta}^*$ to the Beltrami operator Δ^* corresponding to the eigenvalues $-\lambda_n = -n(n+1)$, that is

$$(-\mathbf{\Delta}_\xi^* - \lambda_n) y_n^{(i)}(\xi) = 0, \xi \in \Omega, \tag{1.9}$$

where $\mathbf{\Delta}^*$ is defined by

$$\mathbf{\Delta}_\xi^* f(\xi) = \Delta_\xi^* f(\xi) - 2(\xi \wedge \nabla_\xi^*) \wedge f(\xi) - 2f(\xi) \tag{1.10}$$

for $f \in c^{(2)}(\Omega)$.

Consider the kernel

$$\frac{2n+1}{4\pi} \mathbf{p}_n^{(i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(i)}(\eta), \xi,\eta \in \Omega.$$

In can be deduced that for every vector spherical harmonic $y_n^{(i)}$ of order n and kind i the reproducing property

$$\frac{2n+1}{4\pi} \int_\Omega \mathbf{p}_n^{(i,i)}(\xi,\eta) y_n^{(i)}(\eta) d\omega(\eta) = y_n^{(i)}(\xi), \xi \in \Omega, \tag{1.11}$$

is valid. Let $\mathbf{t}$ be an orthogonal transformation. Then it follows that $\mathbf{p}_n^{(i,k)}(\mathbf{t}\xi, \mathbf{t}\eta) = \mathbf{t}\mathbf{p}_n^{(i,k)}(\xi,\eta)\mathbf{t}^T$ for any pair of unit vectors ξ, η and $i = 1,2,3$. Therefore, $\mathbf{p}_n^{(i,k)}(\xi,\eta)$ is invariant under orthogonal transformations. By straightforward calculations and observing the structure of the tensor product (1.3) we obtain the following *vectorial analogue of the addition theorem*:

$$\begin{aligned}
\mathbf{p}_n^{(1,1)}(\xi,\eta) &= P_n(\xi\cdot\eta)\xi\otimes\eta,\\
\mathbf{p}_n^{(1,2)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)\xi\otimes[\xi-(\xi\cdot\eta)\eta],\\
\mathbf{p}_n^{(1,3)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)\xi\otimes\eta\wedge\xi,\\
\mathbf{p}_n^{(2,1)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)[\eta-(\xi\cdot\eta)\xi]\otimes\eta,\\
\mathbf{p}_n^{(3,1)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)\xi\wedge\eta\otimes\eta,\\
\mathbf{p}_n^{(2,2)}(\xi,\eta) &= \frac{1}{n(n+1)}\times\\
&\quad \{P_n''(\xi\cdot\eta)[\eta-(\xi\cdot\eta)\xi]\otimes[\xi-(\xi\cdot\eta)\eta]\\
&\quad +P_n'(\xi\cdot\eta)[\mathbf{i}-\xi\otimes\xi-[\eta-(\xi\cdot\eta)\xi]\otimes\eta]\},\\
\mathbf{p}_n^{(3,3)}(\xi,\eta) &= \frac{1}{n(n+1)}\{P_n''(\xi\cdot\eta)\xi\wedge\eta\otimes\eta\wedge\xi\\
&\quad +P_n'(\xi\cdot\eta)((\xi\cdot\eta)\mathbf{i}-\eta\otimes\xi)\},\\
\mathbf{p}_n^{(3,2)}(\xi,\eta) &= \frac{1}{n(n+1)}\{P_n''(\xi\cdot\eta)\xi\wedge\eta\otimes[\xi-(\xi\cdot\eta)\eta]\\
&\quad +P_n'(\xi\cdot\eta)[(\xi\cdot\eta)\mathbf{i}-\xi\otimes\eta]\},\\
\mathbf{p}_n^{(2,3)}(\xi,\eta) &= \frac{1}{n(n+1)}\{P_n''(\xi\cdot\eta)[\eta-(\xi\cdot\eta)\xi]\otimes\eta\wedge\xi\\
&\quad +P_n'(\xi\cdot\eta)(-\mathbf{j}(\eta)-\xi\otimes\eta\wedge\xi)\},
\end{aligned}$$

for all $\xi,\eta\in\Omega$, where the identity matrix $\mathbf{i}$ and the surface rotation matrix $\mathbf{j}$ are given by

$$\mathbf{i} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{1.12}$$

$$\mathbf{j}(\xi) = \begin{pmatrix} 0 & -\xi\cdot\varepsilon^3 & \xi\cdot\varepsilon^2\\ \xi\cdot\varepsilon^3 & 0 & -\xi\cdot\varepsilon^1\\ -\xi\cdot\varepsilon^2 & \xi\cdot\varepsilon^1 & 0\end{pmatrix}. \tag{1.13}$$

Several simple results can be derived directly from the addition theorem of vector spherical harmonics. The identities

$$\begin{aligned}
(\xi\wedge\eta)\cdot(\eta\wedge\xi) &= -(1-(\xi\cdot\eta)^2),\\
(\eta-(\xi\cdot\eta)\xi)\cdot(\xi-(\xi\cdot\eta)\eta) &= -(\xi\cdot\eta)(1-(\xi\cdot\eta)^2)
\end{aligned}$$

enable us to calculate the trace of $\mathbf{p}_n^{(i,k)}(\xi,\eta)$, $\xi,\eta\in\Omega$:

$$\begin{aligned}
\mathrm{tr}(\mathbf{p}_n^{(1,1)}(\xi,\eta)) &= P_n(\xi\cdot\eta)\\
\mathrm{tr}(\mathbf{p}_n^{(1,2)}(\xi,\eta)) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)(1-(\xi\cdot\eta)^2)\\
\mathrm{tr}(\mathbf{p}_n^{(2,1)}(\xi,\eta)) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)(1-(\xi\cdot\eta)^2)\\
\mathrm{tr}(\mathbf{p}_n^{(2,2)}(\xi,\eta)) &= \frac{1}{\sqrt{n(n+1)}}((1-(\xi\cdot\eta)^2)P_n'(\xi\cdot\eta)\\
&\quad +n(n+1)(\xi\cdot\eta)P_n(\xi\cdot\eta))\\
\mathrm{tr}(\mathbf{p}_n^{(3,3)}(\xi,\eta)) &= P_n(\xi\cdot\eta)\\
\mathrm{tr}(\mathbf{p}_n^{(1,3)}(\xi,\eta)) &= \mathrm{tr}(\mathbf{p}_n^{(3,1)}(\xi,\eta))=0\\
\mathrm{tr}(\mathbf{p}_n^{(3,2)}(\xi,\eta)) &= \mathrm{tr}(\mathbf{p}_n^{(2,3)}(\xi,\eta))=0.
\end{aligned}$$

For all $\xi,\eta\in\Omega$ and $i,k,j\in\{1,2,3\}$,

$$|\,\mathbf{p}_n^{(i,k)}(\xi,\eta)\varepsilon^j\,|\le 1.$$

In particular,

$$\sum_{j=1}^{2n+1}\left|y_{n,j}^{(i)}(\xi)\right|^2=\frac{2n+1}{4\pi},\ \xi\in\Omega$$

so that

$$\sup_{\xi\in\Omega}\left|y_{n,j}^{(i)}(\xi)\right|\le\left(\frac{2n+1}{4\pi}\right)^{1/2},\ j=1,\dots 2n+1.$$

A vectorial analogue of the Hecke formula reads as follows:

$$\begin{aligned}
&\int_\Omega H(\xi\cdot\eta)\mathbf{p}_n^{(1,1)}(\xi,\zeta)d\omega(\zeta)=\\
&\qquad \hat H_{(1,1)}(n)\mathbf{p}_n^{(1,1)}(\xi,\eta)+\hat H_{(1,2)}(n)\mathbf{p}_n^{(1,2)}(\xi,\eta),\\
&\int_\Omega H(\xi\cdot\eta)\mathbf{p}_n^{(2,2)}(\xi,\zeta)d\omega(\zeta)=\\
&\qquad \hat H_{(2,1)}(n)\mathbf{p}_n^{(2,1)}(\xi,\eta)+\hat H_{(2,2)}(n)\mathbf{p}_n^{(2,2)}(\xi,\eta),\\
&\int_\Omega H(\xi\cdot\eta)\mathbf{p}_n^{(3,3)}(\xi,\zeta)d\omega(\zeta)=\\
&\qquad \hat H_{(3,3)}(n)\mathbf{p}_n^{(3,3)}(\xi,\eta),
\end{aligned}$$

where the numbers $\hat H_{(i,j)}$ are given as follows

$$\begin{aligned}
\hat H_{(1,1)}(n) &= \frac{n+1}{2n+1}\hat H(n+1)+\frac{n}{2n+1}\hat H(n-1),\\
\hat H_{(1,2)}(n) &= -\frac{\sqrt{n(n+1)}}{2n+1}(\hat H(n+1)-\hat H(n-1)),\\
\hat H_{(2,1)}(n) &= -\frac{\sqrt{n(n+1)}}{2n+1}(\hat H(n+1)-\hat H(n-1)),\\
\hat H_{(2,2)}(n) &= \frac{3n+2}{2n+1}\hat H(n+1)+\frac{n+1}{2n+1}\hat H(n-1)\\
\hat H_{(3,3)}(n) &= \hat H(n),
\end{aligned}$$

and $\hat H(n)$ is defined by (1.4).

The cartesian components of vector spherical harmonics of order n and kind 1 and 2 can be shown to be linear combinations of scalar spherical harmonics of order $n-1$ and $n+1$, while the cartesian components of a vector spherical harmonic of order n and kind 3 is a linear combination of scalar spherical harmonics of order n. We obtain two consequences, namely

$$\begin{aligned}
y_{n,j}^{(1)}(-\xi) &= (-1)^{n+1}y_{n,j}^{(1)}(\xi),\ \xi\in\Omega,\\
y_{n,j}^{(2)}(-\xi) &= (-1)^{n+1}y_{n,j}^{(2)}(\xi),\ \xi\in\Omega,\\
y_{n,j}^{(3)}(-\xi) &= (-1)^{n}y_{n,j}^{(3)}(\xi),\ \xi\in\Omega,
\end{aligned}$$

and

$$\begin{aligned}\int_\Omega Y_l(\xi) y^{(1)}_{n,j}(\xi)d\omega(\xi) &= 0 \text{ if } |l-n| \neq 1\\ \int_\Omega Y_l(\xi) y^{(2)}_{n,j}(\xi)d\omega(\xi) &= 0 \text{ if } |l-n| \neq 1\\ \int_\Omega Y_l(\xi) y^{(3)}_{n,j}(\xi)d\omega(\xi) &= 0 \text{ if } l \neq n\end{aligned}$$

provided that Y_l is a scalar spherical harmonic of order l.

We conclude our excursion into vector spherical harmonic theory with a third significant consequence. Suppose that f is a continuous vector field on Ω. Then, of course, for any given $\varepsilon > 0$, there exists a linear combination

$$s_N = \sum_{l=1}^{3}\sum_{n=0}^{N}\sum_{j=1}^{2n+1} d^{(l)}_{n,j} Y_{n,j}\varepsilon^l \tag{1.14}$$

such that $||f - s_N||_c \leq \varepsilon$, i.e. the system $\{Y_{n,j}\varepsilon^l \mid n = 0,1,\ldots, j = 1,\ldots,2n+1, l = 1,2,3\}$ is closed in $c(\Omega)$ (with respect to $||\cdot||_c$). But we know already that for $l = 1,2,3$

$$Y_{n,j}\varepsilon^l = \sum_{s=1}^{3}\sum_{r=n-1}^{n+1}\sum_{t=1}^{2r+1} c^{(s)}_{r,t} y^{(s)}_{r,t}. \tag{1.15}$$

This shows that the system $\{y^{(i)}_{n,j}\}$ is closed in $c(\Omega)$ with respect to $||\cdot||_c$. Because of the norm estimate $||f||_{l^2} \leq \sqrt{4\pi}||f||_c$ for $f \in c(\Omega)$, the closure of $\{y^{(i)}_{n,j}\}$ in $c(\Omega)$ with respect to $||\cdot||_c$ implies the closure of $\{y^{(i)}_{n,j}\}$ in $c(\Omega)$ with respect to $||\cdot||_{l^2}$. Since the space $c(\Omega)$ is dense in $l^2(\Omega)$ (with respect to $||\cdot||_{l^2}$), we finally obtain the closure of $\{y^{(i)}_{n,j}\}$ in $l^2(\Omega)$ with respect to $||\cdot||_{l^2}$, which, by a functional analytic argument (cf. e.g. Davis (1963)), is equivalent to the completeness of $\{y^{(i)}_{n,j}\}$ in $l^2(\Omega)$, i.e. for a given function $f \in l^2(\Omega)$, $(f, y^{(i)}_{n,j}) = 0$ for $n = o_i, o_i+1,\ldots, j = 1,\ldots,2n+1$, $i = 1,2,3$, implies $f = 0$. But this means, in particular, that

$$\lim_{N\to\infty}\left|\left| f - \sum_{i=1}^{3}\sum_{n=o_i}^{N}\sum_{j=1}^{2n+1}(f, y^{(i)}_{n,j})_{l^2} y^{(i)}_{n,j}\right|\right|_{l^2} = 0 \tag{1.16}$$

holds for all fields $f \in l^2(\Omega)$.

For more details about vector spherical harmonics and splines the interested reader is referred to Freeden and Gervens (1993) and the references therein.

2 Tensor Spherical Harmonics

Before we turn to the introduction of tensor spherical harmonics we deal with some basic settings about tensor fields.

2.1 Tensor Fields

As usual (cf. e.g. Gurtin (1972)), a *second order tensor* $\mathbf{f}$ is understood to be a linear mapping that assigns each $x \in \mathbb{R}^3$ a vector $y \in \mathbb{R}^3$. The (cartesian) components F_{ij} of $\mathbf{f}$ are defined by

$$F_{ij} = \varepsilon^i \cdot (\mathbf{f}\varepsilon^j) = (\varepsilon^i)^T(\mathbf{f}\varepsilon^j), \tag{2.1}$$

so that $y = \mathbf{f}x$ is equivalent to

$$y \cdot \varepsilon^i = \sum_{j=1}^{3} F_{ij}(x \cdot \varepsilon^j).$$

We write $\mathbf{f}^T$ for the transpose of $\mathbf{f}$; it is the unique tensor satisfying $(\mathbf{f}y)\cdot x = y \cdot (\mathbf{f}^T x)$ for all $x, y \in \mathbb{R}^3$. Moreover, we write $\mathrm{tr}(\mathbf{f})$ for the trace and $\det(\mathbf{f})$ for the determinant of $\mathbf{f}$.

The dyadic (tensor) product $x \otimes y$ of two elements $x, y \in \mathbb{R}^3$ (cf. (1.2)) is the tensor that assigns to each $u \in \mathbb{R}^3$ the vector $(y \cdot u)x$:

$$(x \otimes y)u = (y \cdot u)x \tag{2.2}$$

for every $u \in \mathbb{R}^3$.

The inner product $\mathbf{f}\cdot\mathbf{g}$ of two second order tensors $\mathbf{f}, \mathbf{g}$ is defined by

$$\mathbf{f}\cdot\mathbf{g} = \mathrm{tr}(\mathbf{f}^T\mathbf{g}) = \sum_{i,j=1}^{3} F_{ij}G_{ij}, \tag{2.3}$$

while

$$|\mathbf{f}| = (\mathbf{f}\cdot\mathbf{f})^{1/2} \tag{2.4}$$

is called the norm of $\mathbf{f}$.

Given any tensor $\mathbf{f}$ and any pair x and y,

$$x \cdot (\mathbf{f}y) = \mathbf{f}\cdot(x \otimes y). \tag{2.5}$$

This identity implies that

$$(\varepsilon^i \otimes \varepsilon^j)\cdot(\varepsilon^k \otimes \varepsilon^l) = \delta_{ik}\delta_{jl}, \tag{2.6}$$

so that the nine tensors $\varepsilon^i \otimes \varepsilon^j$ are orthonormal. Moreover

$$\sum_{i,j=1}^{3}(F_{ij}\varepsilon^i \otimes \varepsilon^j)x = \sum_{i,j=1}^{3} F_{ij}(x\cdot\varepsilon^j)\varepsilon^i = \mathbf{f}x \tag{2.7}$$

and thus $\mathbf{f} \in \mathbb{R}^{3\times 3}$ with

$$\mathbf{f} = \sum_{i,j=1}^{3} F_{ij}\varepsilon^i \otimes \varepsilon^j. \tag{2.8}$$

In particular, the identity tensor $\mathbf{i}$ (cf. (1.12)) is given by $\mathbf{i} = \sum_{i=1}^{3}\varepsilon^i \otimes \varepsilon^i$.

By a *spherical second order tensor field* we mean a function $\mathbf{f}$ that assigns to each point $\xi \in \Omega$ a second order tensor $\mathbf{f}(\xi) \in \mathbb{R}^{3\times 3}$.

By $\mathbf{l}^2(\Omega)$ we denote the space of all quadratic integrable tensor fields, i.e.,

$$\mathbf{l}^2(\Omega) = \left\{ \mathbf{f} : \Omega \to \mathbb{R}^{3\times 3} \middle| \int_\Omega \mathbf{f}(\eta)\cdot\mathbf{f}(\eta)d\omega(\eta) < \infty \right\}.$$

With respect to the scalar product

$$\begin{aligned}(\mathbf{f},\mathbf{g})_{\mathbf{l}^2} &= \int_\Omega \mathbf{f}(\xi)\cdot\mathbf{g}(\xi)d\omega(\xi)\\ &= \int_\Omega \left(\sum_{i,j=1}^3 F_{i,j}(\xi)G_{i,j}(\xi)\right) d\omega(\xi)\end{aligned}$$

the space $\mathbf{l}^2(\Omega)$ forms a Hilbert space. With $\mathbf{c}^{(k)}(\Omega)$ we denote the space of all tensor fields $\mathbf{f}$ with k continuously differentiable components $F_{i,j}$. $\mathbf{c}^{(0)}(\Omega)$ is the class of continuous tensor fields (i.e. continuous components). Each tensor field $\mathbf{f} : \Omega \to \mathbb{R}^{3\times 3}$ can be written as sum of elementary dyadic fields

$$\mathbf{f} : \xi \mapsto \mathbf{f}(\xi) = \sum_{i,j=1}^3 F_{i,j}(\xi)\varepsilon^i \otimes \varepsilon^j, \quad \xi\in\Omega. \tag{2.9}$$

For later use we introduce some abbreviations. Let $\mathbf{f}\in\mathbf{l}^2(\Omega)$, $\mathbf{g}\in\mathbf{c}^{(1)}(\Omega)$, and $F\in C^{(2)}(\Omega)$, then, by definition, we let

$$\begin{aligned}\xi\wedge\mathbf{f}(\xi) &= \xi\wedge\left(\sum_{i=1}^3 \left(\mathbf{f}(\xi)\varepsilon^i\right)\otimes\varepsilon^i\right)\\ &= \sum_{i=1}^3 \left(\xi\wedge\left(\mathbf{f}(\xi)\varepsilon^i\right)\right)\otimes\varepsilon^i,\\ \mathbf{f}(\xi)\wedge\xi &= \left(\sum_{i=1}^3 \varepsilon^i\otimes\left(\mathbf{f}^T(\xi)\varepsilon^i\right)\right)\\ &= \sum_{i=1}^3 \varepsilon^i\otimes\left(\xi\wedge\left(\mathbf{f}^T(\xi)\varepsilon^i\right)\right),\\ \nabla^*_\xi\cdot\mathbf{g}(\xi) &= \nabla^*_\xi\cdot\left(\sum_{i=1}^3 \mathbf{g}(\xi)(\varepsilon^i\otimes\varepsilon^i)\right)\\ &= \sum_{i=1}^3 \left(\nabla^*_\xi\cdot(\mathbf{g}(\xi)\varepsilon^i)\right)\varepsilon^i,\end{aligned}$$

and

$$\begin{aligned}&\nabla^*_\xi\otimes\nabla^*_\xi F(\xi) =\\ &\qquad\sum_{i=1}^3 \nabla^*_\xi(\varepsilon^i\cdot\nabla^*_\xi F(\xi))\otimes\varepsilon^i,\\ &\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi F(\xi) =\\ &\qquad\sum_{i=1}^3 \nabla^*_\xi(\varepsilon^i\cdot(\xi\wedge\nabla^*_\xi F(\xi)))\otimes\varepsilon^i,\\ &\xi\wedge\nabla^*_\xi\otimes\nabla^*_\xi F(\xi) =\\ &\qquad\sum_{i=1}^3 \xi\wedge\nabla^*_\xi(\varepsilon^i\cdot\nabla^*_\xi F(\xi))\otimes\varepsilon^i,\\ &\xi\wedge\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi F(\xi) =\\ &\qquad\sum_{i=1}^3 \xi\wedge\nabla^*_\xi(\varepsilon^i\cdot(\xi\wedge\nabla^*_\xi F(\xi)))\otimes\varepsilon^i.\end{aligned}$$

$\nabla^*_\xi\cdot\mathbf{g}(\xi)$ is the *surface divergence* of the tensor field $\mathbf{g}$ at $\xi\in\Omega$, while $\nabla^*_\xi\cdot(\mathbf{g}(\xi)\wedge\xi)$ is the *surface curl* of the tensor field $\mathbf{g}$ at $\xi\in\Omega$.

For a given spherical tensor field $\mathbf{f}$, the field

$$\xi\mapsto\mathbf{f}_{nor}(\xi) = (\xi^T\mathbf{f}(\xi)\xi)\xi\otimes\xi, \xi\in\Omega$$

is called the *normal part* of $\mathbf{f}$, while for $\xi\in\Omega$

$$\begin{aligned}\xi\mapsto\mathbf{f}_{nor,tan}(\xi) &= \xi\otimes(\mathbf{f}^T(\xi)\xi - (\xi^T\mathbf{f}(\xi)\xi)\xi),\\ \xi\mapsto\mathbf{f}_{tan,nor}(\xi) &= (\mathbf{f}(\xi)\xi - (\xi^T\mathbf{f}(\xi)\xi)\xi)\otimes\xi,\end{aligned}$$

respectively, is called the *left normal/right tangential* and *left tangential/right normal part* of $\mathbf{f}$. For a given tensor field $\mathbf{f}$, the field

$$\begin{aligned}&\xi\mapsto\mathbf{f}_{tan}(\xi) =\\ &\qquad\mathbf{f}(\xi) - \mathbf{f}_{nor}(\xi) - \mathbf{f}_{nor,tan}(\xi) - \mathbf{f}_{tan,nor}(\xi),\ \xi\in\Omega,\end{aligned}$$

is said to be the *tangential part* of $\mathbf{f}$ (cf. Backus (1966)). A tensor field is called *tangential* (resp. *normal*) if it coincides with its tangential (resp. normal) part. Obviously,

$$\mathbf{f}_{tan}(\xi)\xi = 0,\ \xi^T\mathbf{f}_{tan}(\xi) = 0,\ \xi\in\Omega. \tag{2.10}$$

Due to Backus (1966) a general tensor field can be represented in a fashion analogous to the aforementioned vector decomposition, where nine scalar functions are involved in the representation. From these nine scalar functions, four functions are used for the representation of the tangential part.

The following investigations present, in addition, the explicit structure of these scalar fields by using the concept of (scalar) Green's functions on Ω with respect to (iterated) Beltrami derivatives.

We identify fourth–order tensors with linear transformations on the space of all (second–order) tensors. Thus, a fourth–order tensor $\mathbf{C} : \mathbb{R}^{3\times3}\to\mathbb{R}^{3\times3}$ is a linear mapping of type

$$\mathbf{C} : \mathbf{g}\mapsto\mathbf{f} = \mathbf{C}\mathbf{g},\quad \mathbf{f},\mathbf{g}\in\mathbb{R}^{3\times3},$$

where the components are defined by

$$C_{i,j,k,l} = \varepsilon^i\cdot\mathbf{C}(\varepsilon^k\otimes\varepsilon^l)\varepsilon^j = (\varepsilon^i\otimes\varepsilon^j)\cdot\mathbf{C}(\varepsilon^k\otimes\varepsilon^l).$$

The mapping relation can be reformulated in its components as follows

$$F_{i,j} = \sum_{k,l=1}^3 C_{i,j,k,l}G_{k,l}.$$

If $\hat{C}_{i,j,k,l}$ are the components of $\mathbf{C}$ with respect to a second orthonormal basis $\hat{\varepsilon}^1, \hat{\varepsilon}^2, \hat{\varepsilon}^3$ in $\mathbb{R}^3$ given by $\hat{\varepsilon}^i = \mathbf{t}\varepsilon^i$ ($\mathbf{t}$ orthogonal), then it is not difficult to see that

$$\hat{C}_{i,j,k,l} = \sum_{m,n,p,q=1}^{3} t_{m,i} t_{n,j} t_{p,k} t_{q,l} C_{m,n,p,q}.$$

The transpose $\mathbf{C}^T = (C^T_{i,j,k,l})$ of $\mathbf{C}$ is defined by

$$C^T_{i,j,k,l} = C_{k,l,i,j},$$

and we get the following property

$$\mathrm{tr}\big(\mathbf{f}(\mathbf{C}\mathbf{g})^T\big) = \mathrm{tr}\big((\mathbf{C}^T\mathbf{f})\mathbf{g}^T\big).$$

As usual (cf. e.g. Gurtin (1972)), we define the trace of a fourth-order tensor $\mathbf{C}$ by

$$\mathrm{tr}(\mathbf{C}) = \mathrm{tr}(C_{i,j,k,l}) = \sum_{i,j=1}^{3} C_{i,j,i,j}.$$

Obviously,

$$\mathrm{tr}(\mathbf{C}^T) = \mathrm{tr}(\mathbf{C}).$$

It is clear that the trace is invariant with respect to orthonormal basis transformations, i.e. $\mathrm{tr}(\mathbf{C}) = \mathrm{tr}(\hat{\mathbf{C}})$, where $\hat{\mathbf{C}}$ is defined as before

If $\mathbf{f}, \mathbf{g}$ are second–order tensors we understand by the *dyadic fourth-order tensor* $\mathbf{B}$ of $\mathbf{f}$ and $\mathbf{g}$

$$\mathbf{B} = \mathbf{f} \odot \mathbf{g}, \quad B_{i,j,k,l} = F_{i,j} G_{k,l},$$

the mapping assigning to each second–order tensor $\mathbf{h}$ the second–order tensor

$$\mathbf{B}\mathbf{h} = \mathrm{tr}(\mathbf{g}\mathbf{h}^T)\mathbf{f} = (\mathbf{g}\cdot\mathbf{h})\mathbf{f}.$$

The transpose and trace of $\mathbf{B}$ are then given by

$$\mathbf{B}^T = \mathbf{g} \odot \mathbf{f},$$
$$\mathrm{tr}(\mathbf{B}) = \mathrm{tr}(\mathbf{f}\mathbf{g}^T) = \mathbf{f}\cdot\mathbf{g}.$$

In analogy to (2.9), each fourth–order tensor possesses the following representation

$$\mathbf{B} = \sum_{i,j,k,l=1}^{3} B_{i,j,k,l} \left(\varepsilon^i \otimes \varepsilon^j\right) \odot \left(\varepsilon^k \otimes \varepsilon^l\right). \qquad (2.11)$$

2.2 Decomposition Formula

Let us consider a tensor field $\mathbf{f} \in \mathbf{c}^{(1)}(\Omega)$. Of course, $\mathbf{f}$ can be decomposed by using the nine elementary dyadic tensors $\varepsilon^i \otimes \varepsilon^j,\ i,j = 1,2,3$:

$$\mathbf{f}(\xi) = \sum_{i,j=1}^{3} F_{i,j}(\xi)\varepsilon^i \otimes \varepsilon^j = \sum_{i=1}^{3} f_i(\xi) \otimes \varepsilon^i, \qquad (2.12)$$

where $F_{i,j} : \Omega \to \mathbb{R}$ are differentiable functions with $F_{i,j}(\xi) = \varepsilon^i \cdot (\mathbf{f}(\xi)\varepsilon^j)$ and $f_i(\xi) = \sum_{j=1}^{3} F_{i,j}(\xi)\varepsilon^j$. Formula (2.12) can be used to reduce tensorial differential or integral equations to scalar ones, but it has the drawback that essential properties (e.g. spherical symmetry) of the tensor fields are ignored. This difficulty can be overcome by the already mentioned decomposition formula which will be established now.

Starting point is the sum

$$\mathbf{f} = \mathbf{f}_{nor} + \mathbf{f}_{nor,tan} + \mathbf{f}_{tan,nor} + \mathbf{f}_{tan} \qquad (2.13)$$

where, according to our construction,

$$\begin{array}{rlrl} \xi \wedge \mathbf{f}_{nor}(\xi) = & 0, & \mathbf{f}_{nor}(\xi) \wedge \xi = & 0, \\ \xi \wedge \mathbf{f}_{nor,tan}(\xi) = & 0, & \mathbf{f}_{nor,tan}(\xi)\xi = & 0, \\ \mathbf{f}_{tan,nor}(\xi) \wedge \xi = & 0, & \xi^T\mathbf{f}_{tan,nor}(\xi) = & 0, \\ \mathbf{f}^T_{tan}(\xi)\xi = & 0, & \mathbf{f}_{tan}(\xi)\xi = & 0 \end{array} \qquad (2.14)$$

for all $\xi \in \Omega$. It is easily seen that the decomposition is unique. The orthogonal projection operator P mapping a spherical tensor field to its tangential part is introduced by

$$P : \mathbf{f} \mapsto P(\mathbf{f})(\xi) = \mathbf{f}_{tan}(\xi), \quad \xi \in \Omega.$$

Explicitly written out we have

Lemma 2.1 *Let $\mathbf{f} \in \mathbf{c}^{(1)}(\Omega)$ be a tensor field. There exist uniquely determined tangent vector functions $v_1, v_2 \in c^{(1)}(\Omega)$ and a function $F \in C^{(1)}(\Omega)$ such that*

$$\mathbf{f}(\xi) - P(\mathbf{f})(\xi) = F(\xi)\xi\otimes\xi + \xi\otimes v_1(\xi) + v_2(\xi)\otimes\xi,\ \xi \in \Omega, \quad (2.15)$$

where

$$\begin{aligned} F(\xi) &= \xi^T\mathbf{f}(\xi)\xi, \\ v_1(\xi) &= \mathbf{f}^T(\xi)\xi - F(\xi)\xi, \\ v_2(\xi) &= \mathbf{f}(\xi)\xi - F(\xi)\xi. \end{aligned}$$

For each $\mathbf{f} \in \mathbf{c}^{(1)}(\Omega)$, Lemma 2.1 enables us to express the projection operator P as follows:

$$\begin{aligned} P(\mathbf{f})(\xi) &= (\mathbf{i} - \xi\otimes\xi)^T\mathbf{f}(\xi)(\mathbf{i} - \xi\otimes\xi) \\ &= \mathbf{f}(\xi) - \xi\otimes(\mathbf{f}^T(\xi)\xi) \\ &\quad -(\mathbf{f}(\xi)\xi)\otimes\xi + (\xi^T\mathbf{f}(\xi)\xi)\xi\otimes\xi. \end{aligned}$$

The *spherical tangential identity operator* $\mathbf{i}_{tan}$ is given by

$$P(\mathbf{i})(\xi) = \mathbf{i}_{tan}(\xi), \quad \xi \in \Omega.$$

For any tangent vector field f the relations

$$\begin{array}{rclrcl} f^T(\xi)\mathbf{i}_{tan}(\xi) & = & f^T(\xi), & \mathbf{i}_{tan}(\xi)f(\xi) & = & f(\xi) \\ \xi^T\mathbf{i}_{tan}(\xi) & = & 0, & \mathbf{i}_{tan}(\xi)\xi & = & 0 \end{array}$$

hold. Setting $\mathbf{f} = \mathbf{i}$ we obtain

$$P(\mathbf{i})(\xi) = \mathbf{i}_{tan}(\xi) = \mathbf{i} - \xi\otimes\xi,\ \xi \in \Omega.$$

Furthermore, it is easy to see that (2.13) and (2.15) represent orthogonal decompositions with respect to the scalar product $(\cdot,\cdot)_{\mathbf{l}^2}$.

The calculation of the surface divergences

$$\begin{aligned}\nabla^*_\xi \cdot (F(\xi)\mathbf{i}_{tan}(\xi)) &= \nabla^*_\xi F(\xi) - 2F(\xi)\xi,\\ \nabla^*_\xi \cdot (F(\xi)\mathbf{i}_{tan}(\xi)\wedge\xi) &= \xi\wedge\nabla^*_\xi F(\xi),\end{aligned}$$

resp. the surface curls

$$\begin{aligned}\nabla^*_\xi \cdot (F(\xi)\mathbf{i}_{tan}(\xi)\wedge\xi) &= \xi\wedge\nabla^*_\xi F(\xi),\\ \nabla^*_\xi \cdot (F(\xi)(\mathbf{i}_{tan}(\xi)\wedge\xi)\wedge\xi) &= 2F(\xi)\xi - \nabla^*_\xi F(\xi)\end{aligned}$$

can be performed by elementary operations provided that F is continuously differentiable on Ω.

According to our construction we have

Lemma 2.2

$$\begin{aligned}(\mathbf{f}_{nor},\mathbf{f}_{tan})_{\mathbf{l}^2} &= 0, & (\mathbf{f}_{nor},\mathbf{f}_{nor,tan})_{\mathbf{l}^2} &= 0,\\ (\mathbf{f}_{nor,tan},\mathbf{f}_{tan})_{\mathbf{l}^2} &= 0, & (\mathbf{f}_{nor},\mathbf{f}_{tan,nor})_{\mathbf{l}^2} &= 0,\\ (\mathbf{f}_{tan,nor},\mathbf{f}_{tan})_{\mathbf{l}^2} &= 0, & (\mathbf{f}_{nor,tan},\mathbf{f}_{tan,nor})_{\mathbf{l}^2} &= 0.\end{aligned}$$

From the vector theory we know that the vector fields v_1 and v_2 occuring in (2.15) can be decomposed in their surface divergence and surface curl parts, i.e., there exist functions $F_i\in C^{(2)}(\Omega)$, $i=1,\ldots,4$, such that

$$\begin{aligned}v_1(\xi) &= \nabla^*_\xi F_1(\xi) + \xi\wedge\nabla^*_\xi F_2(\xi),\\ v_2(\xi) &= \nabla^*_\xi F_3(\xi) + \xi\wedge\nabla^*_\xi F_4(\xi),\end{aligned} \tag{2.16}$$

where F_i, $i=1,...,4$ can be represented as described in the vector theory as follows

$$\begin{aligned}&F_1(\xi) =\\ &\quad \int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(v_1(\eta)-(v_1(\eta)\cdot\eta)\eta)d\omega(\eta),\\ &F_2(\xi) =\\ &\quad -\int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(\eta\wedge v_1(\eta)))d\omega(\eta),\\ &F_3(\xi) =\\ &\quad \int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(v_2(\eta)-(v_2(\eta)\cdot\eta)\eta)d\omega(\eta),\\ &F_4(\xi) =\\ &\quad -\int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(\eta\wedge v_2(\eta)))d\omega(\eta),\end{aligned}$$

and $G(\Delta^*;\cdot,\cdot)$ denotes the Green function defined in (1.5). Combining (2.16) with (2.15) we find

Theorem 2.1 *Suppose that $\mathbf{f}\in\mathbf{c}^{(1)}(\Omega)$. Then there exist uniquely determined functions $F_i\in C^{(2)}(\Omega)$, $i=1,\ldots,5$, with*

$$\int_\Omega F_i(\xi)d\omega(\xi) = 0,\ i=2,...,5$$

such that for all $\xi\in\Omega$

$$\begin{aligned}&\mathbf{f}(\xi) - P(\mathbf{f})(\xi) =\\ &\quad F_1(\xi)\xi\otimes\xi + \xi\otimes\nabla^*_\xi F_2(\xi) + \xi\otimes\xi\wedge\nabla^*_\xi F_3(\xi)\\ &\quad +\nabla^*_\xi F_4(\xi)\otimes\xi + \xi\wedge\nabla^*_\xi F_5(\xi)\otimes\xi,\end{aligned}$$

where

$$\begin{aligned}&F_1(\xi) = \xi^T\mathbf{f}(\xi)\xi,\\ &F_2(\xi) =\\ &\quad \int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(\mathbf{f}^T(\eta)\eta - \eta F_1(\eta))d\omega(\eta),\\ &F_3(\xi) =\\ &\quad -\int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(\eta\wedge(\mathbf{f}^T(\eta)\eta))d\omega(\eta),\\ &F_4(\xi) =\\ &\quad \int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(\mathbf{f}(\eta)\eta - \eta F_1(\eta))d\omega(\eta),\\ &F_5(\xi) =\\ &\quad -\int_\Omega G(\Delta^*;\xi,\eta)\nabla^*_\eta\cdot(\eta\wedge(\mathbf{f}(\eta)\eta))d\omega(\eta).\end{aligned}$$

Next we want to decompose the tangential part $P(\mathbf{f})(\xi)$. It is not difficult to show (cf. Backus (1966)) that there exist scalar functions $G_1,...,G_4\in C^2(\Omega)$ such that

$$P(\mathbf{f})(\xi) = \mathbf{f}^{(1)}_{tan}(\xi) + \mathbf{f}^{(2)}_{tan}(\xi) + \mathbf{f}^{(3)}_{tan}(\xi) + \mathbf{f}^{(4)}_{tan}(\xi),$$

where

$$\begin{aligned}\mathbf{f}^{(1)}_{tan}(\xi) &= G_1(\xi)\mathbf{i}_{tan}(\xi),\\ \mathbf{f}^{(2)}_{tan}(\xi) &= G_2(\xi)\mathbf{i}_{tan}(\xi)\wedge\xi,\\ \mathbf{f}^{(3)}_{tan}(\xi) &= 2P(\nabla^*_\xi\otimes\nabla^*_\xi G_3)(\xi)\\ &\quad -\Delta^*_\xi G_3(\xi)\mathbf{i}_{tan}(\xi),\\ \mathbf{f}^{(4)}_{tan}(\xi) &= P((\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi G_4)(\xi) +\\ &\quad \xi\wedge\nabla^*_\xi\otimes\nabla^*_\xi G_4(\xi)),\end{aligned}$$

and

$$(\mathbf{f}^{(i)}_{tan},\mathbf{f}^{(j)}_{tan})_{\mathbf{l}^2} = 0 \text{ for } i\neq j.$$

Thereby, the scalar functions G_3, G_4 are orthogonal to the spherical hamonics of order 0 and 1. With this condition $G_1,\ldots,G_4$ are uniquely determined by $\mathbf{f}_{tan}$. In order to determine the tangential projections of $\mathbf{f}^{(3)}_{tan}$ and $\mathbf{f}^{(4)}_{tan}$ we use

Lemma 2.3 *Assume that $F\in C^{(2)}(\Omega)$. Then*

$$\begin{aligned}&\nabla^*_\xi\otimes\nabla^*_\xi F(\xi) =\\ &\quad (\nabla^*_\xi\otimes\nabla^*_\xi F(\xi))^T - \nabla^*_\xi F(\xi)\otimes\xi + \xi\otimes\nabla^*_\xi F(\xi),\\ &\xi\wedge\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi F(\xi) =\\ &\quad (\xi\wedge\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi F(\xi))^T - \mathbf{i}\wedge(\xi\wedge\nabla^*_\xi F(\xi)),\\ &\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi F(\xi) =\\ &\quad (\xi\wedge\nabla^*_\xi\otimes\nabla^*_\xi F(\xi))^T - \mathbf{i}\wedge\nabla^*_\xi F(\xi),\\ &\xi\wedge\nabla^*_\xi\otimes\nabla^*_\xi F(\xi) =\\ &\quad (\nabla^*_\xi\otimes\xi\wedge\nabla^*_\xi F(\xi))^T - \mathbf{i}\wedge\nabla^*_\xi F(\xi).\end{aligned}$$

Proof: Simple calculations in cartesian components show

$$
\begin{aligned}
&\nabla_x \otimes \nabla_x F(x) = (\nabla_x \otimes \nabla_x F(x))^T, \\
&(x \wedge \nabla_x) \otimes (x \wedge \nabla_x) F(x) = \\
&\qquad (x \wedge \nabla_x \otimes x \wedge \nabla_x F(x))^T - \mathbf{i} \wedge (x \wedge \nabla_x F(x)), \\
&\nabla_x \otimes x \wedge \nabla_x F(x) = \\
&\qquad (x \wedge \nabla_x \otimes \nabla_x F(x))^T - \mathbf{i} \wedge \nabla_x F(x)
\end{aligned}
$$

for every $F \in C^{(2)}(\Omega)$. Thus, the usual process of separating radial and angular parts provides us with the required result. Observe that the last equation can be obtained by forming the transpose of the matrices occuring in the third equation. ■

By right hand side multiplication with ξ of the formulae written down in Lemma 2.3 we obtain

$$
\begin{aligned}
(\nabla^*_\xi \otimes \nabla^*_\xi F(\xi))\xi &= -\nabla^*_\xi F(\xi), \\
(\xi \wedge \nabla^*_\xi \otimes \xi \wedge \nabla^*_\xi F(\xi))\xi &= \nabla^*_\xi F(\xi), \\
(\nabla^*_\xi \otimes \xi \wedge \nabla^*_\xi F(\xi))\xi &= -\xi \wedge \nabla^*_\xi F(\xi), \\
(\xi \wedge \nabla^*_\xi \otimes \nabla^*_\xi F(\xi))\xi &= -\xi \wedge \nabla^*_\xi F(\xi).
\end{aligned}
$$

Therefore we get from (2.16)

$$
P(\nabla^*_\xi \otimes \nabla^*_\xi F)(\xi) = \nabla^*_\xi \otimes \nabla^*_\xi F(\xi) + \nabla^*_\xi F(\xi) \otimes \xi
$$

and

$$
\begin{aligned}
&P(\nabla^*_\xi \otimes \xi \wedge \nabla^*_\xi F + \xi \wedge \nabla^*_\xi \otimes \nabla^*_\xi F)(\xi) = \\
&\qquad \nabla^*_\xi \otimes \xi \wedge \nabla^*_\xi F(\xi) + \xi \wedge \nabla^*_\xi \otimes \nabla^*_\xi F(\xi) \\
&\qquad +2\xi \wedge \nabla^*_\xi F(\xi) \otimes \xi.
\end{aligned}
$$

Summarizing our results we obtain

Theorem 2.2 *Suppose that $\mathbf{f} \in \mathbf{c}^{(1)}(\Omega)$. Then there exist uniquely determined functions $F_i \in C^{(2)}(\Omega)$, $i = 1, \ldots, 4$, with*

$$
\begin{aligned}
\int_\Omega F_i(\xi) d\omega(\xi) &= 0, \ i = 3, 4, \\
\int_\Omega F_i(\xi) \xi d\omega(\xi) &= 0, \ i = 3, 4,
\end{aligned}
$$

such that

$$
P(\mathbf{f})(\xi) = \mathbf{f}^{(1)}_{tan}(\xi) + \mathbf{f}^{(2)}_{tan}(\xi) + \mathbf{f}^{(3)}_{tan}(\xi) + \mathbf{f}^{(4)}_{tan}(\xi),
$$

where

$$
\begin{aligned}
\mathbf{f}^{(1)}_{tan}(\xi) &= F_1(\xi)\mathbf{i}_{tan}(\xi), \\
\mathbf{f}^{(2)}_{tan}(\xi) &= F_2(\xi)\mathbf{i}_{tan}(\xi) \wedge \xi, \\
\mathbf{f}^{(3)}_{tan}(\xi) &= 2\nabla^*_\xi \otimes \nabla^*_\xi F_3(\xi) \\
&\quad +2\nabla^*_\xi F_3 \otimes \xi - \Delta^*_\xi F_3(\xi)\mathbf{i}_{tan}(\xi), \\
\mathbf{f}^{(4)}_{tan}(\xi) &= \nabla^*_\xi \otimes \xi \wedge \nabla^*_\xi F_4(\xi) \\
&\quad +\xi \wedge \nabla^*_\xi \otimes \nabla^*_\xi F_4(\xi) + 2\xi \wedge \nabla^*_\xi F_4(\xi) \otimes \xi
\end{aligned}
$$

satisfy the orthogonal relations

$$
\begin{aligned}
0 &= \int_\Omega \mathbf{f}^{(i)}_{tan}(\xi) \cdot \mathbf{f}^{(j)}_{tan}(\xi) d\omega(\xi), \ i \neq j, \\
0 &= \int_\Omega \mathbf{f}_{nor}(\xi) \cdot \mathbf{f}^{(j)}_{tan}(\xi) d\omega(\xi), \ j = 1, \ldots, 4, \\
0 &= \int_\Omega \mathbf{f}_{nor,tan}(\xi) \cdot \mathbf{f}^{(j)}_{tan}(\xi) d\omega(\xi), \ j = 1, \ldots, 4, \\
0 &= \int_\Omega \mathbf{f}_{tan,nor}(\xi) \cdot \mathbf{f}^{(j)}_{tan}(\xi) d\omega(\xi), \ j = 1, \ldots, 4.
\end{aligned}
$$

Theorem 2.1 and Theorem 2.2 allow the following reformulation by using the operators $\mathbf{o}^{(i,k)} : C^{(2)}(\Omega) \rightarrow \mathbf{c}^{(0)}(\Omega)$, introduced as follows

$$
\begin{aligned}
\mathbf{o}^{(1,1)}_\xi F(\xi) &= F(\xi)\xi \otimes \xi, \\
\mathbf{o}^{(1,2)}_\xi F(\xi) &= \xi \otimes \nabla^*_\xi F(\xi), \\
\mathbf{o}^{(1,3)}_\xi F(\xi) &= \xi \otimes \xi \wedge \nabla^*_\xi F(\xi), \\
\mathbf{o}^{(2,1)}_\xi F(\xi) &= \nabla^*_\xi F(\xi) \otimes \xi, \\
\mathbf{o}^{(3,1)}_\xi F(\xi) &= \xi \wedge \nabla^*_\xi F(\xi) \otimes \xi, \qquad (2.17) \\
\mathbf{o}^{(2,2)}_\xi F(\xi) &= F(\xi)(\mathbf{i} - \xi \otimes \xi), \\
\mathbf{o}^{(3,3)}_\xi F(\xi) &= F(\xi)(\mathbf{i} \wedge \xi), \\
\mathbf{o}^{(2,3)}_\xi F(\xi) &= 2\nabla^*_\xi \otimes \nabla^*_\xi F(\xi) \\
&\quad +2\nabla^*_\xi F(\xi) \otimes \xi - \Delta^*_\xi F(\xi)(\mathbf{i} - \xi \otimes \xi), \\
\mathbf{o}^{(3,2)}_\xi F(\xi) &= \nabla^*_\xi \otimes \xi \wedge \nabla^*_\xi F(\xi) \\
&\quad +\xi \wedge \nabla^*_\xi \otimes \nabla^*_\xi F(\xi) + 2\xi \wedge \nabla^*_\xi F(\xi) \otimes \xi.
\end{aligned}
$$

Theorem 2.3 *Let $\mathbf{f} \in \mathbf{c}^{(1)}(\Omega)$. Then there exist uniquely determined functions $F_{i,j} \in C^{(2)}(\Omega), i, j = 1, 2, 3,$ with $\int_\Omega F_{i,j}(\xi) d\omega(\xi) = 0$ if $(i, j) \in \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, resp. $\int_\Omega F_{i,j}(\xi)\xi d\omega(\xi) = 0$ if $(i, j) \in \{(2, 3), (3, 2)\}$, such that*

$$
\mathbf{f} = \sum_{i,j=1}^{3} \mathbf{o}^{(i,j)} F_{i,j}
$$

and

$$
\left(\mathbf{o}^{(i,j)} F_{i,j}, \mathbf{o}^{(k,l)} F_{k,l}\right)_{\mathbf{l}^2} = 0, \quad (i, j) \neq (k, l).
$$

By elementary calculations we find

Lemma 2.4 *The surface divergence of the tangential tensor fields $\mathbf{o}^{(i,j)} F$, $(i, j) \in \{(2, 2), (3, 3), (2, 3), (3, 2)\}$, respectively, reads as follows*

$$
\begin{aligned}
\nabla^*_\xi \cdot (\mathbf{o}^{(2,2)}_\xi F(\xi)) &= \nabla_\xi F(\xi) - 2F(\xi)\xi, \\
\nabla^*_\xi \cdot (\mathbf{o}^{(3,3)}_\xi F(\xi)) &= \xi \wedge \nabla_\xi F(\xi), \\
\nabla^*_\xi \cdot (\mathbf{o}^{(2,3)}_\xi F(\xi)) &= \nabla_\xi (\Delta^*_\xi F(\xi) + 2F(\xi)), \\
\nabla^*_\xi \cdot (\mathbf{o}^{(3,2)}_\xi F(\xi)) &= \xi \wedge \nabla_\xi (\Delta^*_\xi F(\xi) + 2F(\xi)),
\end{aligned}
$$

whereas the surface curl, respectively, is given by

$$\begin{aligned}
\nabla^*_\xi \cdot (\xi \wedge \mathbf{o}^{(2,2)}_\xi F(\xi)) &= -\xi \wedge \nabla_\xi F(\xi),\\
\nabla^*_\xi \cdot (\xi \wedge \mathbf{o}^{(3,3)}_\xi F(\xi)) &= \nabla_\xi F(\xi) - 2F(\xi)\xi,\\
\nabla^*_\xi \cdot (\xi \wedge \mathbf{o}^{(2,3)}_\xi F(\xi)) &= \xi \wedge \nabla_\xi(\Delta^*_\xi F(\xi) + 2F(\xi)),\\
\nabla^*_\xi \cdot (\xi \wedge \mathbf{o}^{(3,2)}_\xi F(\xi)) &= -\nabla_\xi(\Delta^*_\xi F(\xi) + 2F(\xi)),
\end{aligned}$$

provided that F is sufficiently often differentiable.

In particular, we obtain as immediate consequence

$$\begin{aligned}
\xi \cdot (\nabla^*_\xi \cdot (\mathbf{o}^{(2,2)}_\xi F(\xi))) &= -2F(\xi),\\
\xi \cdot (\nabla^*_\xi \cdot (\xi \wedge \mathbf{o}^{(3,3)}_\xi F(\xi))) &= -2F(\xi),\\
\nabla_\xi \cdot \nabla^*_\xi \cdot (\mathbf{o}^{(2,3)}_\xi F(\xi)) &= \Delta^*_\xi(\Delta^*_\xi + 2)F(\xi),\\
\nabla_\xi \cdot \nabla^*_\xi \cdot (\xi \wedge \mathbf{o}^{(3,2)}_\xi F(\xi)) &= -\Delta^*_\xi(\Delta^*_\xi + 2)F(\xi),
\end{aligned}$$

for all $\xi \in \Omega$. These identities give rise to introduce the operators $O^{(i,j)} : \mathbf{c}^2(\Omega) \to C^{(0)}(\Omega)$ by

$$\begin{aligned}
O^{(1,1)}_\xi \mathbf{f}(\xi) &= \xi^T \mathbf{f}(\xi)\xi,\\
O^{(1,2)}_\xi \mathbf{f}(\xi) &= -\nabla^*_\xi \cdot (\mathbf{f}^T(\xi)\xi - \xi O^{(1,1)}_\xi \mathbf{f}(\xi)),\\
O^{(1,3)}_\xi \mathbf{f}(\xi) &= \nabla^*_\xi \cdot (\xi \wedge (\mathbf{f}^T(\xi)\xi)),\\
O^{(2,1)}_\xi \mathbf{f}(\xi) &= -\nabla^*_\xi \cdot (\mathbf{f}(\xi)\xi - \xi O^{(1,1)}_\xi \mathbf{f}(\xi)),\\
O^{(3,1)}_\xi \mathbf{f}(\xi) &= \nabla^*_\xi \cdot (\xi \wedge (\mathbf{f}(\xi)\xi)), \qquad (2.18)\\
O^{(2,2)}_\xi \mathbf{f}(\xi) &= -\xi \cdot \left(\nabla^*_\xi \cdot (P(\mathbf{f})(\xi))\right),\\
O^{(3,3)}_\xi \mathbf{f}(\xi) &= -\xi \cdot \left(\nabla^*_\xi \cdot (\xi \wedge P(\mathbf{f})(\xi))\right),\\
O^{(2,3)}_\xi \mathbf{f}(\xi) &= \\
&\hspace{-4em} 2\nabla^*_\xi \cdot \left(\nabla^*_\xi \cdot (P(\mathbf{f}) - \frac{1}{2}\mathbf{i}_{tan} O^{(2,2)}_\xi \mathbf{f}(\xi))\right),\\
O^{(3,2)}_\xi \mathbf{f}(\xi) &= \\
&\hspace{-4em} -2\nabla^*_\xi \cdot \Big(\left(\nabla^*_\xi \cdot (\xi \wedge P(\mathbf{f}) - \frac{1}{2}\mathbf{i}_{tan} O^{(2,2)}_\xi (\xi \wedge P(\mathbf{f})\right)\Big).
\end{aligned}$$

The operators $O^{(i,k)}$ assign to each tensor $\mathbf{f}$ nine functions $F_{i,k}$ determined by Theorem 2.3. It is easy to see that for $F \in C^{(2)}(\Omega)$

$$O^{(i,j)}_\xi (\mathbf{o}^{(r,s)}_\xi F(\xi)) = 0,$$

if $(i,j) \neq (r,s)$. Moreover, for $(i,j) = (r,s)$, we get

$$\begin{aligned}
O^{(1,1)}_\xi (\mathbf{o}^{(1,1)}_\xi F(\xi)) &= F(\xi),\\
O^{(1,2)}_\xi (\mathbf{o}^{(1,2)}_\xi F(\xi)) &= -\Delta^*_\xi F(\xi),\\
O^{(1,3)}_\xi (\mathbf{o}^{(1,3)}_\xi F(\xi)) &= -\Delta^*_\xi F(\xi),\\
O^{(2,1)}_\xi (\mathbf{o}^{(2,1)}_\xi F(\xi)) &= -\Delta^*_\xi F(\xi),\\
O^{(3,1)}_\xi (\mathbf{o}^{(3,1)}_\xi F(\xi)) &= -\Delta^*_\xi F(\xi),\\
O^{(2,2)}_\xi (\mathbf{o}^{(2,2)}_\xi F(\xi)) &= 2F(\xi),\\
O^{(3,3)}_\xi (\mathbf{o}^{(3,3)}_\xi F(\xi)) &= 2F(\xi),\\
O^{(2,3)}_\xi (\mathbf{o}^{(2,3)}_\xi F(\xi)) &= 2(\Delta^*_\xi(\Delta^*_\xi + \lambda_1))F(\xi),\\
O^{(3,2)}_\xi (\mathbf{o}^{(3,2)}_\xi F(\xi)) &= 2(\Delta^*_\xi(\Delta^*_\xi + \lambda_1))F(\xi).
\end{aligned}$$

In particular, we have for every scalar spherical harmonic Y_n of order n

$$O^{(i,j)}(\mathbf{o}^{(i,j)} Y_n) = \mu^{(i,j)}_n Y_n, \; n \geq o_{i,j}, \qquad (2.19)$$

where we have used the abbreviations

$$o_{i,j} = \begin{cases} 0 & \text{for } (i,j) \in \{(1,1),(2,2),(3,3)\} \\ 1 & \text{for } (i,j) \in \{(1,2),(2,1),(1,3),(3,1)\} \\ 2 & \text{for } (i,j) \in \{(2,3),(3,2)\} \end{cases},$$

and

$$\mu^{(i,j)}_n = \begin{cases} 1 & \text{for } (i,j) = (1,1) \\ 2 & \text{for } (i,j) \in \{(2,2),(3,3)\} \\ \lambda_n & \text{for } (i,j) \in \{(1,2),(2,1), \\ & \qquad (1,3),(3,1)\} \\ 2\lambda_n(\lambda_n - \lambda_1) & \text{for } (i,j) \in \{(2,3),(3,2)\} \end{cases}.$$

We are able to generalize the definition of $O^{(i,j)}$ to fourth–order tensors in the following way. Let $\mathbf{C}$ be a fourth–order tensor. Then we set for $\xi \in \Omega$

$$O^{(i,j)}_\xi \mathbf{C}(\xi) = \left(O^{(i,j)}_\xi (C_{r,s,k,l}(\xi))_{r,s}\right)_{k,l}. \qquad (2.20)$$

If $\mathbf{C}$ is a fourth–order tensor of the dyadic form $\mathbf{C}(\xi) = \mathbf{s}(\xi) \odot \mathbf{t}(\xi)$ we have

$$O^{(i,j)}_\xi \mathbf{C}(\xi) = (O^{(i,j)}_\xi \mathbf{s}(\xi))\mathbf{t}(\xi).$$

Using (2.18) in Theorem 2.3 we see that every twice continuously differentiable tensor field defined on Ω can be expressed by its $O^{(i,j)}$–components in the following way:

Theorem 2.4 *Let $\mathbf{f} \in \mathbf{c}(\Omega)$. Then there exist uniquely determined scalar functions $F_{i,j} \in C^2(\Omega)$, $i,j = 1,2,3$, with $\int_\Omega F_{i,j}(\xi) d\omega(\xi) = 0$ if $(i,j) \in \{(1,2),(2,1),(1,3),(3,1)\}$, resp. $\int_\Omega F_{i,j}(\xi)\xi d\omega(\xi) = 0$ if $(i,j) \in \{(2,3),(3,2)\}$, such that*

$$\mathbf{f} = \sum_{i,j=1}^{3} \mathbf{o}^{(i,j)} F_{i,j}$$

and

$$\begin{aligned}
F_{1,1}(\xi) &= O^{(1,1)}_\xi \mathbf{f}(\xi),\\
F_{2,2}(\xi) &= \frac{1}{2} O^{(2,2)}_\xi \mathbf{f}(\xi),\\
F_{3,3}(\xi) &= \frac{1}{2} O^{(3,3)}_\xi \mathbf{f}(\xi),
\end{aligned}$$

$$
\begin{aligned}
F_{1,2}(\xi) &= -\int_\Omega G(\Delta^*;\xi,\eta)O_\eta^{(1,2)}\mathbf{f}(\eta)d\omega(\eta),\\
F_{2,1}(\xi) &= -\int_\Omega G(\Delta^*;\xi,\eta)O_\eta^{(2,1)}\mathbf{f}(\eta)d\omega(\eta),\\
F_{1,3}(\xi) &= -\int_\Omega G(\Delta^*;\xi,\eta)O_\eta^{(1,3)}\mathbf{f}(\eta)d\omega(\eta),\\
F_{3,1}(\xi) &= -\int_\Omega G(\Delta^*;\xi,\eta)O_\eta^{(3,1)}\mathbf{f}(\eta)d\omega(\eta),\\
F_{3,2}(\xi) &= \frac{1}{2}\int_\Omega G(\Delta^*(\Delta^*+\lambda_1);\xi,\eta)O_\eta^{(3,2)}\mathbf{f}(\eta)d\omega(\eta),\\
F_{2,3}(\xi) &= \frac{1}{2}\int_\Omega G(\Delta^*(\Delta^*+\lambda_1);\xi,\eta)O_\eta^{(2,3)}\mathbf{f}(\eta)d\omega(\eta).
\end{aligned}
$$

A geodetically relevant example of this decomposition formula is given in Appendix A.

2.3 Definition of Tensor Spherical Harmonics

Let Y_n be a scalar spherical harmonic of order n. For $i,k=1,2,3$ and $n=o_{i,k},o_{i,k}+1,\ldots$, the tensor fields

$$\mathbf{y}_n^{(i,k)}(\xi)=\mathbf{o}_\xi^{(i,k)}Y_n(\xi),\ \xi\in\Omega$$

are called *tensor spherical harmonics of order n and kind (i,k)*. $\mathbf{y}_n^{(1,1)}$ describes a normal tensor field on Ω, while $\mathbf{y}_n^{(1,2)},\mathbf{y}_n^{(2,1)},\mathbf{y}_n^{(1,3)}$ and $\mathbf{y}_n^{(3,1)}$ are tensor fields consisting of combined normal and tangential components. Finally, according to Theorem 2.2, the fields $\mathbf{y}_n^{(2,2)},\mathbf{y}_n^{(2,3)},\mathbf{y}_n^{(3,2)}$ and $\mathbf{y}_n^{(3,3)}$ are tangential fields. Obviously,

$$
\begin{aligned}
\xi\wedge\mathbf{y}_n^{(1,k)}(\xi) &= 0, && \text{for } k=1,2,3,\\
\xi\wedge\mathbf{y}_n^{(2,1)}(\xi) &= \mathbf{y}_n^{(3,1)}(\xi),\\
\mathbf{y}_n^{(2,2)}(\xi)\wedge\xi &= \mathbf{y}_n^{(3,3)}(\xi),\\
\left(\mathbf{y}_n^{(i,k)}(\xi)\right)^T\xi &= \mathbf{y}_n^{(i,k)}(\xi)\xi=0, && \text{if } i,k=2,3.
\end{aligned}
$$

We denote by $\mathbf{harm}_n^{(i,k)}$ the space of all tensor spherical harmonics of order n and kind (i,k). $\mathbf{harm}_n$ is defined by

$$
\begin{aligned}
\mathbf{harm}_0 &= \mathbf{harm}_0^{(1,1)}\oplus\mathbf{harm}_0^{(2,2)}\oplus\mathbf{harm}_0^{(3,3)},\\
\mathbf{harm}_1 &= \bigoplus_{\substack{i,k=1\\(i,k)\neq(2,3),(3,2)}}^{3}\mathbf{harm}_1^{(i,k)}, \qquad (2.21)\\
\mathbf{harm}_n &= \bigoplus_{i,k=1}^{3}\mathbf{harm}_n^{(i,k)},\ \text{if } n\geq 2.
\end{aligned}
$$

Equipped with the scalar product $(\cdot,\cdot)_{\mathbf{l}^2}$ the space $\mathbf{harm}_n^{(i,k)}$ is a $(2n+1)$–dimensional Hilbert space.

The system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$ of tensor fields $\mathbf{y}_{n,j}^{(i,k)}:\Omega\to\mathbb{R}^{3\times3}$, defined by

$$\mathbf{y}_{n,j}^{(i,k)}(\xi)=(\mu_n^{(i,k)})^{-1/2}\mathbf{o}_\xi^{(i,k)}Y_{n,j}(\xi)$$

for $n=o_{i,k},o_{i,k}+1,\ldots$, forms a $\mathbf{l}^2(\Omega)$–orthonormal system, i.e.,

$$\int_\Omega\mathbf{y}_{n,j}^{(r,s)}(\xi)\cdot\mathbf{y}_{m,k}^{(u,t)}(\xi)d\omega(\xi)=\delta_{r,u}\delta_{s,t}\delta_{n,m}\delta_{j,k}.$$

In analogy to the scalar and vectorial theory it can be proved that the space $\mathbf{harm}_n^{(i,k)}$ is invariant in the sense that $\mathbf{y}\in\mathbf{harm}_n^{(i,k)}$ implies $\mathbf{t}^T\mathbf{y}(\mathbf{t}\cdot)\mathbf{t}\in\mathbf{harm}_n^{(i,k)}$ for all orthogonal transformations $\mathbf{t}$. Furthermore, known from the theory of scalar and vector spherical harmonics, $\mathbf{harm}_n^{(i,k)}$ is irreducible, i.e. there exists no real invariant subspace of $\mathbf{harm}_n^{(i,k)}$. From this fact it follows that all results known from the theory of scalar or vector spherical harmonics which are based on these algebraic facts, carry over to the tensorial case.

Given an element $\mathbf{x}_n^{(i,k)}\in\mathbf{harm}_n^{(i,k)}$, we consider the space $\mathbf{w}^{(i,k)}$ of all linear combinations of elements $\mathbf{t}^T\mathbf{x}_n^{(i,k)}(\mathbf{t}\cdot)\mathbf{t}$. Since $\mathbf{w}^{(i,k)}$ is an invariant subspace of $\mathbf{harm}_n^{(i,k)}$, it follows that $\mathbf{w}^{(i,k)}=\mathbf{harm}_n^{(i,k)}$. Therefore we obtain

Lemma 2.5 *Let $\mathbf{x}_n^{(i,k)}$ be an element of $\mathbf{harm}_n^{(i,k)}$. Then there exist $2n+1$ orthogonal transformations $\mathbf{t}_1,\ldots,\mathbf{t}_{2n+1}$ such that every tensor spherical harmonic $\mathbf{y}_n^{(i,k)}$ of order n and kind (i,k) can be expressed in the form*

$$\mathbf{y}_n^{(i,k)}=\sum_{j=1}^{2n+1}c_j\mathbf{t}_j^T\mathbf{x}_n^{(i,k)}(\mathbf{t}_j\cdot)\mathbf{t}_j,$$

where $c_1,\ldots,c_{2n+1}$ are real numbers.

The rôle of the Legendre function will be played by the tensor field

$$\mathbf{p}_{Leg,n}^{(i,k)}(\cdot,\eta)=\mathbf{o}^{(i,k)}P_n(\cdot\eta),\ \eta\in\Omega, \qquad (2.22)$$

which represents uniquely (apart from a multiplicative factor) that orthogonal invariant tensor field satisfying

$$\mathbf{p}_{Leg,n}^{(i,k)}(\mathbf{t}\xi,\eta)=\mathbf{t}^T\mathbf{p}_{Leg,n}^{(i,k)}(\xi,\eta)\mathbf{t},$$

for all orthogonal transformations $\mathbf{t}$ having $\eta\in\Omega$ unchanged.

In connection with Lemma 2.5 each tensor spherical harmonic can be expressed by means of Legendre functions. We obtain

Lemma 2.6 *There exist $2n+1$ points $\eta_1,\ldots,\eta_{2n+1}\in\Omega$ such that every tensor spherical harmonic $\mathbf{y}_n^{(i,k)}\in\mathbf{harm}_n^{(i,k)}$ can be written as*

$$\mathbf{y}_n^{(i,k)}=\sum_{j=1}^{2n+1}c_j\mathbf{p}_{Leg,n}^{(i,k)}(\cdot,\eta_j)$$

with real numbers $c_1,\ldots,c_{2n+1}$.

Next, we consider the fourth–order tensor

$$\frac{2n+1}{4\pi}\mathbf{P}_n^{(i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} (\mathbf{y}_{n,j}^{(i,k)}(\xi)) \odot (\mathbf{y}_{n,j}^{(i,k)}(\eta)) \tag{2.23}$$

for $\xi, \eta \in \Omega$. Then, for every fixed $\mathbf{a} \in \mathbb{R}^{3\times 3}$ and $\eta \in \Omega$, the tensor field $\mathbf{P}_n^{(i,k)}(\cdot,\eta)\mathbf{a}$ is an element of $\mathbf{harm}_n^{(i,k)}$, and for every element $\mathbf{y}_n^{(i,k)} \in \mathbf{t}_n^{(i,k)}$ we have using (2.23) the *reproducing property*

$$\begin{aligned} &\frac{2n+1}{4\pi}\int_\Omega \mathbf{P}_n^{(i,k)}(\xi,\eta)(\mathbf{y}_n^{(i,k)}(\eta))^T\, d\omega(\eta) \\ &= \sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi)\int_\Omega \mathrm{tr}\left(\mathbf{y}_{n,j}^{(i,k)}(\eta)(\mathbf{y}_n^{(i,k)}(\eta))^T\right) d\omega(\eta) \\ &= \sum_{j=1}^{2n+1} (\mathbf{y}_{n,j}^{(i,k)}, \mathbf{y}_n^{(i,k)})_{\mathbf{l}^2}\mathbf{y}_{n,j}^{(i,k)}(\xi) \\ &= \mathbf{y}_n^{(i,k)}(\xi), \quad \xi \in \Omega. \end{aligned}$$

More explicitly,

$$\mathbf{y}_n^{(i,k)}(\xi) = \frac{2n+1}{4\pi}\int_\Omega \mathbf{P}_n^{(i,k)}(\xi,\eta)(\mathbf{y}_n^{(i,k)}(\eta))^T\, d\omega(\eta).$$

The determination of (2.23) in terms of scalar Legendre functions leads us to the *addition theorem for tensor spherical harmonics*. As an example, we just state the following formula. We have for $\xi, \eta \in \Omega$, and $n \geq 0$

$$\sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(2,2)}(\xi) \odot \mathbf{y}_{n,j}^{(2,2)}(\eta) = \frac{2n+1}{4\pi}\mathbf{P}_n^{(2,2)}(\xi,\eta), \tag{2.24}$$

where the fourth order tensor $\mathbf{P}_n^{(2,2)}(\xi,\eta)$ is defined by

$$\mathbf{P}_n^{(2,2)}(\xi,\eta) = \frac{1}{2}P_n(\xi,\eta)(\mathbf{i} - \xi\otimes\xi)\odot(\mathbf{i} - \eta\otimes\eta).$$

Of special interest is the trace of $\mathbf{P}_n^{(i,k)}(\xi,\eta)$. For that purpose we first show the following invariance property:

Lemma 2.7 *Let $\xi, \eta \in \Omega$. Then, for each orthogonal transformation $\mathbf{t}$, we have*

$$\mathrm{tr}(\mathbf{P}_n^{(i,k)}(\xi,\eta)) = \mathrm{tr}(\mathbf{P}_n^{(i,k)}(\mathbf{t}\xi,\mathbf{t}\eta)).$$

Proof: From the orthogonal invariance of $\mathbf{harm}_n^{(i,k)}$ we know that $\mathbf{t}^T\mathbf{y}_{n,j}^{(i,k)}(\mathbf{t}\cdot)\mathbf{t}$ can be written as

$$\mathbf{t}^T\mathbf{y}_{n,j}^{(i,k)}(\mathbf{t}\cdot)\mathbf{t} = \sum_{r=1}^{2n+1} \mathbf{c}_{j,r}\mathbf{y}_{n,r}^{(i,k)}.$$

Now, we have

$$\begin{aligned} &\int_\Omega \mathrm{tr}\left[(\mathbf{t}^T\mathbf{y}_{n,r}^{(i,k)}(\xi)\mathbf{t})(\mathbf{t}^T\mathbf{y}_{n,s}^{(i,k)}(\xi)\mathbf{t})^T\right] d\omega(\xi) \\ &= \sum_{l=1}^{2n+1}\sum_{l'=1}^{2n+1}\int_\Omega \mathrm{tr}\left[c_{r,l}c_{s,l'}\mathbf{y}_{n,l}^{(i,k)}(\xi)\mathbf{y}_{n,l'}^{(i,k)}(\xi)\right] d\omega(\xi) \\ &= \sum_{l=1}^{2n+1} c_{r,l}c_{s,l}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_\Omega \mathrm{tr}\left[(\mathbf{t}^T\mathbf{y}_{n,r}^{(i,k)}(\xi)\mathbf{t})(\mathbf{t}^T\mathbf{y}_{n,s}^{(i,k)}(\xi)\mathbf{t})^T\right] d\omega(\xi) = \\ &\qquad \int_\Omega \mathrm{tr}\left[\mathbf{y}_{n,r}^{(i,k)}(\xi)(\mathbf{y}_{n,s}^{(i,k)}(\xi))^T\right] d\omega(\xi) = \delta_{r,s}. \end{aligned}$$

Combining these relations, we get that the coefficients $c_{r,l}$ are the components of an orthogonal matrix

$$\sum_{l=1}^{2n+1} c_{r,l}c_{s,l} = \delta_{r,s}.$$

This fact can be used to complete the proof:

$$\begin{aligned} &\mathrm{tr}\left(\frac{2n+1}{4\pi}\mathbf{P}_n^{(i,k)}(\mathbf{t}\xi,\mathbf{t}\eta)\right) \\ &= \sum_{j=1}^{2n+1} \mathrm{tr}\left[(\mathbf{y}_{n,j}^{(i,k)}(\mathbf{t}\xi))(\mathbf{y}_{n,j}^{(i,k)}(\mathbf{t}\eta))^T\right] \\ &= \sum_{j=1}^{2n+1} \mathrm{tr}\left[(\mathbf{t}^T\mathbf{y}_{n,j}^{(i,k)}(\mathbf{t}\xi)\mathbf{t})(\mathbf{t}^T\mathbf{y}_{n,j}^{(i,k)}(\mathbf{t}\eta)\mathbf{t})^T\right] \\ &= \sum_{j=1}^{2n+1}\sum_{l=1}^{2n+1}\sum_{l'=1}^{2n+1} c_{j,l}c_{j,l'}\mathrm{tr}\left[(\mathbf{y}_{n,l}^{(i,k)}(\xi))(\mathbf{y}_{n,l'}^{(i,k)}(\eta))^T\right] \\ &= \mathrm{tr}\left(\frac{2n+1}{4\pi}\mathbf{P}_n^{(i,k)}(\xi,\eta)\right). \end{aligned}$$

■

Lemma 2.7 means that the scalar function $\mathrm{tr}(\mathbf{P}_n^{(i,k)}(\xi,\eta))$ depends only on the scalar product $\xi\cdot\eta$. Therefore, the scalar function $\xi \mapsto \mathrm{tr}(\mathbf{P}_n^{(i,k)}(\xi,\xi))$ is constant. Hence, there exists a number $\alpha_n \in \mathbb{R}$ such that

$$\mathrm{tr}(\mathbf{P}_n^{(i,k)}(\xi,\xi)) = \alpha_n$$

for all $\xi \in \Omega$, i.e.,

$$\sum_{j=1}^{2n+1} \mathrm{tr}((\mathbf{y}_{n,j}^{(i,k)}(\xi))(\mathbf{y}_{n,j}^{(i,k)}(\xi))^T) = \alpha_n, \quad \xi \in \Omega.$$

Integrating both sides over the unit sphere Ω we get

Lemma 2.8

$$\sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi)\cdot\mathbf{y}_{n,j}^{(i,k)}(\xi) = \frac{2n+1}{4\pi}.$$

Now, let

$$\mathbf{y}_n^{(i,k)}(\xi) = \sum_{j=1}^{2n+1} a_j\mathbf{y}_{n,j}^{(i,k)}(\xi), \quad a_j = (\mathbf{y}_n^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2}.$$

The application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
|\, \mathbf{y}_n^{(i,k)}(\xi)\,|^2 &\leq \sum_{j=1}^{2n+1} a_j^2 \sum_{j=1}^{2n+1} |\, \mathbf{y}_{n,j}^{(i,k)}(\xi)\,|^2 \\
&\leq \frac{2n+1}{4\pi} \sum_{j=1}^{2n+1} a_j^2 .
\end{aligned}
$$

Note that $\sum_{j=1}^{2n+1} a_j^2 = (\mathbf{y}_n^{(i,k)}, \mathbf{y}_n^{(i,k)})_{\mathbf{l}^2}$. Thus we are able to derive the following estimate for tensor spherical harmonics:

Lemma 2.9

$$
\sup_{\xi\in\Omega} |\, \mathbf{y}_n^{(i,k)}(\xi)\,| \leq \sqrt{\frac{2n+1}{4\pi}} \left(\mathbf{y}_n^{(i,k)}, \mathbf{y}_n^{(i,k)}\right)_{\mathbf{l}^2}^{1/2}.
$$

We know already (cf. Section 1.3) that each vector spherical harmonic of order n can be characterized componentwisely by certain linear combinations of scalar spherical harmonics. The operator ξ (resp. $\nabla_\xi^*, \xi\wedge\nabla_\xi^*$) transforms a spherical harmonic of order n in those of order $n-1$ and $n+1$ (resp. $n-1$ and $n+1$, n). If we apply one of the operators $\xi, \nabla_\xi^*, \xi\wedge\nabla_\xi^*$ once again to construct tensor spherical harmonics we see that for $(i,k) \in \{(1,1),(1,2),(2,1),(2,2),(2,3)\}$ the tensor field $\mathbf{y}_n^{(i,k)}$ is a linear combination of scalar spherical harmonics of order $n-2, n$ and $n+2$ in each component while for $(i,k)\in\{(1,3),(3,1),(3,3),(3,2)\}$ the tensor field $\mathbf{y}_n^{(i,k)}$ componentwisely forms a linear combination of scalar spherical harmonics of order $n-1$ and $n+1$. From this fact we are able to obtain

Lemma 2.10 *Let $\xi\in\Omega$. Then*

$$
\mathbf{y}_n^{(i,k)}(-\xi) = (-1)^n \mathbf{y}_n^{(i,k)}(\xi)
$$

if $(i,k) \in \{(1,1),(1,2),(2,1),(2,2),(2,3)\}$, and

$$
\mathbf{y}_n^{(i,k)}(-\xi) = (-1)^{n+1} \mathbf{y}_n^{(i,k)}(\xi)
$$

if $(i,k)\in\{(1,3),(3,1),(3,3),(3,2)\}$.

Furthermore, we have

Lemma 2.11 *Let Y_m be a scalar spherical harmonic of order m. Then*

$$
\int_\Omega Y_m(\xi) \mathbf{y}_n^{(i,k)}(\xi) d\omega(\xi) = 0,
$$

if $(i,k) \in \{(1,1),(1,2),(2,1),(2,2),(2,3)\}$ and $m\neq n-2, n, n+2$, or if $(i,k)\in\{(1,3),(3,1),(3,3),(3,2)\}$ and $m \neq n-1, n+1$.

Applying the operator $O^{(i,k)}$ to tensor spherical harmonics we get using (2.19)

$$
O_\xi^{(k,l)} \mathbf{y}_n^{(k,l)}(\xi) = \mu_n^{(k,l)} Y_n,\ n\geq o_{i,k}. \tag{2.25}
$$

Finally, if we apply $O_\eta^{(i,k)} O_\xi^{(i,k)}$ to the fourth–order tensor $\mathbf{P}_n^{(i,k)}(\xi,\eta)$, we get for all $\xi,\eta\in\Omega$

$$
O_\eta^{(i,k)}(O_\xi^{(i,k)} \mathbf{P}_n^{(i,k)}(\xi,\eta)) = \mu_n^{(i,k)} P_n(\xi\cdot\eta)
$$

for $n = o_{i,k}, o_{i,k}+1, \ldots$.

A tensorial analogue to the well known Hecke formula can be derived, if both sides of the usual Hecke formula (1.3) are multiplied with the operator $O^{(i,k)}$. We obtain

Lemma 2.12 *Let H be a function of class $\mathcal{L}^1[-1,+1]$. Then, for all $\xi,\eta\in\Omega$,*

$$
\int_\Omega H(\eta\cdot\zeta) \mathbf{p}_{Leg,n}^{(i,k)}(\xi,\zeta) d\omega(\zeta) = \hat{H}(n) \mathbf{p}_{Leg,n}^{(i,k)}(\xi,\eta).
$$

In what follows we study closure and completeness of the system of tensor spherical harmonics in the topology of the spaces $\mathbf{c}(\Omega)$ and $\mathbf{l}^2(\Omega)$, respectively. First we mention

Lemma 2.13 *The system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$ is closed in $\mathbf{c}(\Omega)$.*

Proof: Let $\mathbf{f}$ be of class $\mathbf{c}(\Omega)$. Then from the closure of scalar spherical harmonics it is clear that the tensorial system $\left\{Y_{n,j}\varepsilon^i\otimes\varepsilon^k\right\}$ forms a closed system in $\mathbf{c}(\Omega)$ in the sense that for every $\varepsilon>0$ there exist $N\in\mathbb{N}$ and $d_{n,j}^{(i,k)}\in\mathbb{R}$ such that

$$
\left\| \mathbf{f} - \sum_{i,k=1}^{3} \sum_{n=o_{i,k}}^{N} \sum_{j=1}^{2n+1} d_{n,j}^{(i,k)} Y_{n,j} \varepsilon^i\otimes\varepsilon^k \right\|_{\mathbf{c}} \leq \varepsilon. \tag{2.26}
$$

From Lemma 2.11 it can be deduced that each $Y_{n,j}\varepsilon^i\otimes\varepsilon^k$ can be represented by finite linear combinations of tensor spherical harmonics of order $n-2,\ldots,n+2$, i.e.,

$$
Y_{n,j}\varepsilon^i\otimes\varepsilon^k = \sum_{o,p=1}^{3} \sum_{r=n-2}^{n+2} \sum_{m=1}^{r} c_{r,m}^{(o,p)} \mathbf{y}_{r,m}^{(o,p)}. \tag{2.27}
$$

Combining (2.26) and (2.27), we have the required result. ■

As an immediate consequence we obtain

Lemma 2.14 *The system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$ is closed and consequently complete in $\mathbf{l}^2(\Omega)$ with respect to $\|\cdot\|_{\mathbf{l}^2}$.*

Proof: According to standard arguments in Functional Analysis (cf. e.g. Davis (1963)), Lemma 2.14 is equivalent to the completeness of the system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$, i.e., $\mathbf{f}\in\mathbf{l}^2(\Omega)$ with $(\mathbf{f},\mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2} = 0$ for all $n = o_{i,k}, o_{i,k}+1,\ldots$; $j=1,\ldots,2n+1$; $i,k=1,2,3$, implies that $\mathbf{f}=0$. ■

Lemma 2.14 is equivalent to the fact that every tensor field $\mathbf{f} \in \mathbf{l}^2(\Omega)$ can be written as orthogonal expansion in terms of tensor spherical harmonics. More explicitly,

$$\lim_{N\to\infty} \left\| \mathbf{f} - \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=o_{i,k}}^{N}\sum_{j=1}^{2n+1} (\mathbf{f}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2}\mathbf{y}_{n,j}^{(i,k)} \right\|_{\mathbf{l}^2} = 0.$$

At the end of this section, we want to define the tensorial analogue of the Beltrami operator Δ^*. So we look for a differential operator for spherical tensor fields such that the system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$ represents the eigenfunctions of this operator corresponding to the spectrum $\{-\lambda_n | n = o_{i,k}, o_{i,k}+1, \ldots\}$. For the definition of this new operator, denoted by $\hat{\mathbf{\Delta}}^*$, we use the (usual) Beltrami operator Δ^* and its vectorial counterpart $\mathbf{\Delta}^*$.

We start by defining $\hat{\mathbf{\Delta}}^*$ for the normal part of a tensor field $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$, decomposed as in (2.13). We set

$$\begin{aligned}
\left(\hat{\mathbf{\Delta}}^*_{nor}\right)_\xi \mathbf{f}(\xi) &= \Delta^*_\xi \left(\xi^T \mathbf{f}^{(1)}_{nor}(\xi)\xi\right) \xi \otimes \xi, \\
\left(\hat{\mathbf{\Delta}}^*_{nor,tan}\right)_\xi \mathbf{f}(\xi) &= \xi \otimes \mathbf{\Delta}^*_\xi \left(\xi^T \mathbf{f}^{(2)}_{nor}(\xi)\right)^T, \\
\left(\hat{\mathbf{\Delta}}^*_{tan,nor}\right)_\xi \mathbf{f}(\xi) &= \mathbf{\Delta}^*_\xi \left(\mathbf{f}^{(3)}_{nor}(\xi)\xi\right) \otimes \xi.
\end{aligned}$$

Then it is clear, that the sum of these three operators $\hat{\mathbf{\Delta}}^*_{nor}$, $\hat{\mathbf{\Delta}}^*_{nor,tan}$, and $\hat{\mathbf{\Delta}}^*_{tan,nor}$, mapping $\mathbf{c}^{(2)}(\Omega)$ into $\mathbf{c}^{(0)}(\Omega)$, has the tensor spherical harmonics $\mathbf{y}_{n,j}^{(i,k)}$, $(i,k) \in \{(1,1),(1,2),(1,3),(2,1),(3,1)\}$, as its eigenfunctions to the eigenvalue $-\lambda_n = -n(n+1)$, $n = o_{i,k}, o_{i,k}+1, \ldots$.

Since the tangential tensor field $\mathbf{f}_{tan}$ can be written as a finite sum of tensor fields in dyadic form, it is enough to define the tangential part $\hat{\mathbf{\Delta}}^*_{tan}$ of $\hat{\mathbf{\Delta}}^*$ for tangential tensor fields of dyadic form. Hence, assume that $\mathbf{f}_{tan}$ is given by $\mathbf{f}_{tan}(\xi) = u(\xi) \otimes v(\xi)$, where u and v are tangential vector fields. Then we define

$$\begin{aligned}
\left(\hat{\mathbf{\Delta}}^*_{tan}\right)_\xi \mathbf{f}(\xi) &= \left(\hat{\mathbf{\Delta}}^*_{tan}\right)_\xi \mathbf{f}_{tan}(\xi) \\
&= \left(\hat{\mathbf{\Delta}}^*_{tan}\right)_\xi \left(u(\xi) \otimes v(\xi)\right),
\end{aligned}$$

where

$$\begin{aligned}
&\left(\hat{\mathbf{\Delta}}^*_{tan}\right)_\xi (u(\xi) \otimes v(\xi)) = \\
&\quad \mathbf{\Delta}^*_\xi u(\xi) \otimes v(\xi) + u(\xi) \otimes \mathbf{\Delta}^*_\xi v(\xi) + \\
&\quad 2\left(\xi \wedge u(\xi) \otimes \xi \wedge v(\xi)\right) + 2\left(\nabla^*_\xi \otimes u(\xi) + u(\xi) \otimes \xi\right)^T \\
&\quad \left(\nabla^*_\xi \otimes v(\xi) + v(\xi) \otimes \xi\right),
\end{aligned}$$

and where $\nabla^*_\xi \otimes u(\xi)$ is defined analogously to (2.10) by

$$\nabla^*_\xi \otimes u(\xi) = \sum_{i=1}^{3} \nabla^*_\xi \left(\varepsilon^i \cdot u(\xi)\right) \otimes \varepsilon^i.$$

It can be proved that the tensor spherical harmonics $\mathbf{y}_{n,j}^{(i,k)}$, $i,k = 2,3$ are the eigenfunctions of $\hat{\mathbf{\Delta}}^*_{tan}$ to the eigenvalue $-\lambda_n$ and that the definition of $\hat{\mathbf{\Delta}}^*_{tan}$ is independent on the decomposition of the tangential tensor field in dyadic form. Hence, the definition $\hat{\mathbf{\Delta}}^* := \hat{\mathbf{\Delta}}^*_{nor} + \hat{\mathbf{\Delta}}^*_{nor,tan} + \hat{\mathbf{\Delta}}^*_{tan,nor} + \hat{\mathbf{\Delta}}^*_{tan}$ provides the tensorial analogue of the Beltrami operator. Since the system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$ is closed in $\mathbf{l}^2(\Omega)$, it is clear that all eigenfunctions are found.

3 Tensor Spherical Splines

The theory of tensor spherical harmonics will be used now to develop the tensor spherical spline theory.

3.1 Sobolev Spaces

Let $\mathcal{A}$ denote the linear space of all sequences $\{A_n\}$ of all real numbers A_n, $n = 0, 1, \ldots$.

Definition 3.1 Let $\{A_n\}, \{B_n\} \in \mathcal{A}$. The sequence $\{A_n\}$ is called $\{B_n\}$*–summable*, if $A_n \neq 0$ for $n = 0, 1, \ldots$ and

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{B_n}{{A_n}^2} < \infty.$$

Of special interest are sequences $\{A_n\}$ that are $\{(n+1/2)^\tau\}$–summable for a real number τ.

For $i,k = 1,2,3$ we consider the series expansion

$$\mathbf{f}^{(i,k)}(\xi) = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} A_n^{-1} (\hat{\mathbf{f}}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2} \mathbf{y}_{n,j}^{(i,k)}$$

for $\xi \in \Omega$, corresponding to a function $\hat{\mathbf{f}}^{(i,k)} \in \mathbf{l}^2(\Omega)$. Then, by virtue of the Cauchy-Schwarz inequality, we get for all $\xi \in \Omega$,

$$\begin{aligned}
\mid \mathbf{f}^{(i,k)}(\xi) \mid &\leq \Big(\sum_{n=o_{i,k}}^{\infty} \frac{2n+1}{4\pi} \mid A_n \mid^{-2} \Big)^{1/2} \\
&\quad \Big(\sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} (\hat{\mathbf{f}}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})^2_{\mathbf{l}^2} \Big)^{1/2} < \infty,
\end{aligned}$$

provided that $\{A_n\}$ is $\{(n+1/2)^\tau\}$–summable ($\tau \geq 0$). In other words, $\mathbf{h}$ and $\mathbf{h}^{(i,k)}$, $i,k = 1,2,3$, defined by

$$\mathbf{h}^{(i,k)} = \Big\{ \mathbf{f}^{(i,k)} \mid \mathbf{f}^{(i,k)} = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} A_n^{-1} (\hat{\mathbf{f}}^{(i)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2} \mathbf{y}_{n,j}^{(i,k)},\ \hat{\mathbf{f}}^{(i,k)} \in \mathbf{l}^2(\Omega) \Big\},$$

and

$$\mathbf{h} = \Big\{ \mathbf{f} \mid \mathbf{f} = \sum_{i,k=1}^{3} \mathbf{f}^{(i,k)},\ \mathbf{f}^{(i,k)} \in \mathbf{h}^{(i,k)},\ i,k = 1,2,3 \Big\},$$

are linear subspaces of the space $\mathbf{c}^{(0)}(\Omega)$ of all continuous tensor fields, on which we are able to define the structure of a separable Hilbert space by introducing the inner product for elements $\mathbf{f}^{(i,k)}$ and $\mathbf{g}^{(i,k)}$ defined by $\hat{\mathbf{f}}^{(i,k)} \in \mathbf{l}^2(\Omega)$ resp. $\hat{\mathbf{g}}^{(i,k)} \in \mathbf{l}^2(\Omega)$ as

$$\mathbf{f}^{(i,k)} = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} A_n^{-1} (\hat{\mathbf{f}}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2} \mathbf{y}_{n,j}^{(i,k)},$$

$$\mathbf{g}^{(i,k)} = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} A_n^{-1} (\hat{\mathbf{g}}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2} \mathbf{y}_{n,j}^{(i,k)},$$

by setting

$$(\mathbf{f}^{(i,k)}, \mathbf{g}^{(i,k)})_{\mathbf{h}^{(i,k)}} = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} (\hat{\mathbf{f}}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2} (\hat{\mathbf{g}}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2}$$

$$(\mathbf{f}, \mathbf{g})_{\mathbf{h}} = \sum_{i,k=1}^{3} (\mathbf{f}^{(i,k)}, \mathbf{g}^{(i,k)})_{\mathbf{h}^{(i,k)}} = (\hat{\mathbf{f}}, \hat{\mathbf{g}})_{\mathbf{l}^2}.$$

Furthermore, consider the kernel $\mathbf{K}_{\mathbf{h}^{(i,k)}} : \Omega \times \Omega \to \mathbb{R}^3$ defined by

$$\mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \eta) = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} (A_n^{-1} \mathbf{y}_{n,j}^{(i,k)}(\xi)) \odot (A_n^{-1} \mathbf{y}_{n,j}^{(i,k)}(\eta)).$$

The addition theorem (2.23) yields

$$\mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \eta) = \sum_{n=o_{i,k}}^{\infty} A_n^{-2} \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k)}(\xi, \eta).$$

Suppose now, in addition, that for $i, k \in \{1,2,3\}$, $\{A_n\}$ is $\{(n+1/2)^{2o_{i,k}}\}$–summable. Then the tensor functions $O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \cdot)$ are uniformly bounded. More explicitly,

$$| O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \cdot) | \leq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \mu_n^{(i,k)} A_n^{-2}. \tag{3.1}$$

Thus, a neccessary and sufficient condition that $\mathbf{h}^{(i,k)}$ have a reproducing kernel is fulfilled (cf. Davis (1963)).

Theorem 3.1 *Let $\{A_n\}$ be $\left\{(n+1/2)^{2o_{i,k}}\right\}$–summable. Then the space $\mathbf{h}^{(i,k)}$ is a Hilbert subspace of $\mathbf{c}^{(o_{i,k})}(\Omega)$. Each space $\mathbf{h}^{(i,k)}$, $i,k = 1,2,3$ possesses a uniquely determined reproducing kernel*

$$\begin{aligned}\mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \eta) &= \sum_{n=o_{i,k}}^{\infty} | A_n |^{-2} \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k)}(\xi, \eta) \\ &= \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} | A_n |^{-2} \mathbf{y}_{n,j}^{(i,k)}(\xi) \odot \mathbf{y}_{n,j}^{(i,k)}(\eta),\end{aligned}$$

for $\xi, \eta \in \Omega$ in the following sense:

(i) For any fixed $\xi \in \Omega$, $O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \cdot) \in \mathbf{h}^{(i,k)}$, where

$$O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \cdot) = \sum_{n=o_{i,k}}^{\infty} \sum_{j=1}^{2n+1} | A_n |^{-2} \left(O_\xi^{(i,k)} \mathbf{y}_{n,j}^{(i,k)}(\xi)\right) \mathbf{y}_{n,j}^{(i,k)}.$$

(ii) For every $\mathbf{f}^{(i,k)} \in \mathbf{h}^{(i,k)}$ and all $\xi \in \Omega$ the reproducing property holds

$$O_\xi^{(i,k)} \mathbf{f}^{(i,k)}(\xi) = \left(O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \cdot), \mathbf{f}^{(i,k)}\right)_{\mathbf{h}^{(i,k)}} \tag{3.2}$$

for $i, k = 1,2,3$.

Corollary 3.1 *Let $\{A_n\}$ be $\{(n+1/2)^\tau\}$–summable with $\tau \geq 4$. Then the space $\mathbf{h}$ is a Hilbert subspace of $\mathbf{c}^{(0)}(\Omega)$. $\mathbf{h}$ possesses the uniquely determined reproducing kernel*

$$\mathbf{K}_{\mathbf{h}}(\xi, \eta) = \sum_{i,k=1}^{3} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \eta), \ \xi, \eta \in \Omega,$$

in the following sense:

(i) For all $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(r,s)}}(\xi, \cdot) = 0, \ (i,k) \neq (r,s).$$

(ii) For every $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}}(\xi, \cdot) = O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}^{(i,k)}}(\xi, \cdot) \in \mathbf{h}$$

(iii) For every $\mathbf{f} \in \mathbf{h}$ and all $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{f}(\xi) = \left(O_\xi^{(i,k)} \mathbf{K}_{\mathbf{h}}(\xi, \cdot), \mathbf{f}\right)_{\mathbf{h}}, \ i,k = 1,2,3.$$

The space $\mathbf{h}_m^{(i,k)} = \mathbf{harm}_{o_{i,k},\ldots,m}^{(i,k)}$, $m \geq o_{i,k}$, $i,k = 1,2,3$, of all tensor spherical harmonics of order $\leq m$ and kind (i,k) is a finite–dimensional Hilbert space with the semi–inner product $(\cdot,\cdot)_{\mathbf{h}_m^{(i,k)}}$ corresponding to the norm

$$\|\mathbf{f}^{(i,k)}\|_{\mathbf{h}_m^{(i,k)}} = \left(\sum_{n=o_{i,k}}^{m} \sum_{j=1}^{2n+1} | A_n |^2 (\mathbf{f}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2}^2 \right)^{\frac{1}{2}}.$$

$\mathbf{h}_m^{(i,k)}$ has the reproducing kernel

$$\mathbf{K}_{\mathbf{h}_m^{(i,k)}}(\xi, \eta) = \sum_{n=o_{i,k}}^{m} \sum_{j=1}^{2n+1} | A_n |^{-2} \mathbf{y}_{n,j}^{(i,k)}(\xi) \odot \mathbf{y}_{n,j}^{(i,k)}(\eta).$$

Let us denote by $\mathbf{h}_m^{(i,k)-}$ the orthogonal complement of $\mathbf{h}_m^{(i,k)}$ in $\mathbf{h}^{(i,k)}$. Then the linear space $\mathbf{h}_m^{(i,k)-}$

is a Hilbert space with the inner product $(\cdot,\cdot)_{\mathbf{h}_m^{(i,k)\perp}}$ corresponding to the norm

$$\|\mathbf{f}^{(i,k)}\|_{\mathbf{h}_m^{(i,k)\perp}} = \Big(\sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} | A_n |^2 \, (\mathbf{f}^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})^2_{\mathbf{l}^2} \Big)^{\frac{1}{2}} .$$

$\mathbf{h}_m^{(i,k)-}$ has the reproducing kernel

$$\mathbf{K}_{\mathbf{h}_m^{(i,k)\perp}}(\xi,\eta) = \sum_{n=m+1}^{\infty} \sum_{j=1}^{2n+1} | A_n |^{-2} \, \mathbf{y}_{n,j}^{(i,k)}(\xi) \odot \mathbf{y}_{n,j}^{(i,k)}(\eta).$$

Hence, $\mathbf{h}^{(i,k)} = \mathbf{h}_m^{(i,k)} \oplus \mathbf{h}_m^{(i,k)-}$ and $(\cdot,\cdot)_{\mathbf{h}^{(i,k)}} = (\cdot,\cdot)_{\mathbf{h}_m^{(i,k)}} + (\cdot,\cdot)_{\mathbf{h}_m^{(i,k)\perp}}$.

For $i,k \in \{1,2,3\}$, we denote by $proj_{\mathbf{h}_m^{(i,k)}}$, resp. $proj_{\mathbf{h}_m^{(i,k)\perp}}$, the projection operators in $\mathbf{h}^{(i,k)}$ onto $\mathbf{h}_m^{(i,k)}$, resp. $\mathbf{h}_m^{(i,k)-}$,

$$proj_{\mathbf{h}_m^{(i,k)}} : \mathbf{h}^{(i,k)} \to \mathbf{h}_m^{(i,k)}, \qquad \mathbf{f} \mapsto proj_{\mathbf{h}_m^{(i,k)}}(\mathbf{f}),$$
$$proj_{\mathbf{h}_m^{(i,k)\perp}} : \mathbf{h}^{(i,k)} \to \mathbf{h}_m^{(i,k)-}, \qquad \mathbf{f} \mapsto proj_{\mathbf{h}_m^{(i,k)\perp}}(\mathbf{f}).$$

The operators are linear and bounded. The kernels $proj^{-1}_{\mathbf{h}_m^{(i,k)}}(0)$ and $proj^{-1}_{\mathbf{h}_m^{(i,k)\perp}}(0)$ are as $\mathbf{h}^{(i,k)}$–subspaces orthogonal to $\mathbf{h}_m^{(i,k)}$ and $\mathbf{h}_m^{(i,k)-}$, respectively. $\mathbf{h}_m^{(i,k)}$ is a closed linear subspace of $\mathbf{h}^{(i,k)}$, and it is clear that $\mathbf{h}_m^{(i,k)} = (\mathbf{h}_m^{(i,k)-})^-$. Using the reproducing property of the kernels we find for $\mathbf{f}^{(i,k)} \in \mathbf{h}^{(i,k)}$, $\xi \in \Omega$:

$$proj_{\mathbf{h}_m^{(i,k)}}(\mathbf{f}^{(i,k)})(\xi) = (\mathbf{K}_{\mathbf{h}_m^{(i,k)}}(\xi,\cdot),\ \mathbf{f}^{(i,k)})_{\mathbf{h}_m^{(i,k)}},$$
$$proj_{\mathbf{h}_m^{(i,k)\perp}}(\mathbf{f}^{(i,k)})(\xi) = (\mathbf{K}_{\mathbf{h}_m^{(i,k)\perp}}(\xi,\cdot),\ \mathbf{f}^{(i,k)})_{\mathbf{h}_m^{(i,k)\perp}}.$$

We let

$$\mathbf{h}_m^- = \left\{ \mathbf{f} \mid \mathbf{f} = \sum_{i,k=1}^{3} \mathbf{f}^{(i,k)},\ \mathbf{f}^{(i,k)} \in \mathbf{h}_m^{(i,k)-},\ i,k = 1,2,3 \right\}$$

with

$$(\mathbf{f},\mathbf{g})_{\mathbf{h}_m^\perp} = \sum_{i,k=1}^{3} (\mathbf{f}^{(i,k)}, \mathbf{g}^{(i,k)})_{\mathbf{h}_m^{(i,k)\perp}}$$

for $\mathbf{f}^{(i,k)}, \mathbf{g}^{(i,k)} \in \mathbf{h}_m^{(i,k)-}$, $i,k = 1,2,3$, and

$$\mathbf{K}_{\mathbf{h}_m^\perp}(\xi,\eta) = \sum_{i,k=1}^{3} \mathbf{K}_{\mathbf{h}_m^{(i,k)\perp}}(\xi,\eta).$$

3.2 Definition of Tensor Spherical Splines

The linear space $\mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m}$ of all tensor spherical harmonics of order $\le m$ and kind (i,j) has the dimension $M - o_{i,j}$. Consequently, the linear space

$$\mathbf{harm}_{0,\ldots,m} = \bigoplus_{n=0}^{m} \mathbf{harm}_n \ , \ m \ge 2,$$

of all tensor spherical harmonics of order m or less (cf. (2.21) is of dimension

$$3 + 7 \cdot 3 + \sum_{n=2}^{m} 9(2n+1) = 9M - 12, \ m \ge 2.$$

Now, for $i,j \in \{1,2,3\}$ consider the subset $X_M^{(i,j)}$ of $M - o_{i,j}$ points on the unit sphere

$$X_M^{(i,j)} = \left\{ \eta^{(i,j)}_{o_{i,j}}, \ldots, \eta^{(i,j)}_{M-1} \right\}, \quad i,j = 1,2,3.$$

We set

$$X_M = \bigcup_{i,j=1}^{3} X_M^{(i,j)}.$$

Therefore, X_M is a subset of Ω consisting of at least M and of at most $9M-12$ points.

For $i,j \in \{1,2,3\}$, the system $X_M^{(i,j)}$ is called $\mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m}$*–unisolvent*, if the matrix

$$\Big(\sum_{n=o_{i,j}}^{m} \frac{2n+1}{4\pi} P_n(\eta_k^{(i,j)} \cdot \eta_l^{(i,j)}) \Big)_{l,k}$$

is regular. The following statements are equivalent:

(i) $X_M^{(i,j)}$ is $\mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m}$–unisolvent,

(ii) the matrix

$$\Big(\sum_{n=o_{i,j}}^{m} O^{(i,j)}_{\eta_l^{i,j}} \big(\frac{2n+1}{4\pi} \mathbf{p}^{(i,j)}_{Leg,n}(\eta_k^{(i,j)}, \eta_l^{(i,j)}) \big) \Big)_{l,k}$$

is regular,

(iii) the matrix

$$\Big(\sum_{n=o_{i,j}}^{m} O^{(i,j)}_{\eta_k^{(i,j)}} \big(O^{(i,j)}_{\eta_l^{(i,j)}} \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,j)}(\eta_k^{(i,j)}, \eta_l^{(i,j)}) \big) \Big)_{l,k}$$

is regular.

The system X_M is called $\mathbf{harm}_{0,\ldots,m}$*-unisolvent*, if the nine matrices

$$\Big(\sum_{n=o_{i,j}}^{m} \frac{2n+1}{4\pi} P_n(\eta_k^{(i,j)} \cdot \eta_l^{(i,j)}) \Big)_{l,k}, \ i,j = 1,2,3$$

are regular. Assuming that X_M is $\mathbf{harm}_{0,\ldots,m}$–unisolvent, any tensor spherical harmonic $\mathbf{y}$ of order $\le m$ can be written as

$$\mathbf{y}(\xi) = \sum_{i,j=1}^{3} \sum_{k=o_{i,j}}^{M-1} c_k^{(i,j)} \sum_{n=o_{i,j}}^{m} \frac{2n+1}{4\pi} \mathbf{p}^{(i,j)}_{Leg,n}(\xi, \eta_k^{(i,j)})$$

for $\xi \in \Omega$.

Now, we start with the precise treatment of *tensor spline interpolation*. For $i,j \in \{1,2,3\}$, a set $X_N^{(i,j)} \subset \Omega$, $N \geq M$, given by

$$X_N^{(i,j)} = \{\eta_{o_{i,j}}^{(i,j)}, \ldots, \eta_{N-1}^{(i,j)}\}$$

is called $\mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)}$–*admissible*, if it contains a $\mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)}$–unisolvent subset. In the following, we assume that $X_M^{(i,j)} \subset X_N^{(i,j)}$ is $\mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)}$–unisolvent. This is always achievable by reordering. Therefore it makes sense to call a set

$$X_N = \bigcup_{i,j=1}^{3} X_N^{(i,j)},\ X_N^{(i,j)} \subset \Omega,\ N \geq M,$$

$\mathbf{harm}_{0,\ldots,m}$–*admissible*, if it contains X_M as $\mathbf{harm}_{0,\ldots,m}$–unisolvent subset.

Definition 3.2 Given a $\mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)}$–admissible system $X_N^{(i,j)}$, then any function $\mathbf{s}^{(i,j)} \in \mathbf{h}^{(i,j)}$, $i,j = 1,2,3$, of the form

$$\mathbf{s}^{(i,j)} = \mathbf{p}^{(i,j)} + \sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_k^{(i,j)}, \cdot)$$

with $\mathbf{p}^{(i,j)} \in \mathbf{h}_m^{(i,j)}$ is called *tensor spherical spline* in $\mathbf{h}^{(i,j)}$ relative to $X_N^{(i,j)}$, if the coefficients $a_k^{(i,j)}$ satisfy the linear system

$$\sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{y}_{n,l}^{(i,j)}(\eta_k^{(i,j)}) = 0,$$
$$n = o_{i,j}, \ldots, m;\, l = 1, ..., 2n+1.$$

If X_N is a $\mathbf{harm}_{0,\ldots,m}$–admissible system, then, $\mathbf{s}(\xi) = \sum_{i,j=1}^{3} \mathbf{s}^{(i,j)}(\xi)$ is called *tensor spherical spline* in $\mathbf{h}$ relative to X_N. The space of all tensor spherical splines in $\mathbf{h}^{(i,j)}$ relative to $X_N^{(i,j)}$ is denoted by $\mathcal{S}_m(X_N^{(i,j)})$, whereas $\mathcal{S}_m(X_N)$ denotes the set of all tensor spherical splines in $\mathbf{h}$ relative to X_N.

Now let there be given the data points $(\eta_k^{(i,j)}, w_k^{(i,j)}) \in X_N^{(i,j)} \times \mathbb{R}$, $i,j = 1,2,3; k = o_{i,j}, \ldots, N-1$, corresponding to a $\mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)}$–admissible system $X_N^{(i,j)}$. For fixed $i,j \in \{1,2,3\}$, we consider the spline problem of finding the "$\mathbf{h}_m^{(i,j)-}$–smallest interpolant"

$$\|\mathbf{s}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} = \inf_{\mathbf{f}^{(i,j)} \in \mathcal{I}_N^{(i,j)}(w^{(i,j)})} \|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}},$$

where

$$\mathcal{I}_N^{(i,j)}(w^{(i,j)}) = \Big\{\mathbf{f} \in \mathbf{h}^{(i,j)} \mid O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{f}(\eta_k^{(i,j)}) = w_k^{(i,j)}, \quad k = o_{i,j}, \ldots, N-1\Big\}.$$

First we prove some lemmata. We start with

Lemma 3.1 *If* $\mathbf{f}^{(i,j)} \in \mathcal{I}_N^{(i,j)}(w^{(i,j)})$, $\mathbf{s}^{(i,j)} \in \mathcal{S}_m(X_N^{(i,j)})$, *then*

$$(\mathbf{s}^{(i,j)}, \mathbf{f}^{(i,j)})_{\mathbf{h}_m^{(i,j)\perp}} = \sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} w_k^{(i,j)}.$$

Proof: It follows from (3.2) that

$$\sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{f}^{(i,j)}(\eta_k^{(i,j)}) = \sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} (O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{K}_{\mathbf{h}_m^{(i,j)}}(\eta_k^{(i,j)}, \cdot), \mathbf{f}^{(i,j)})_{\mathbf{h}^{(i,j)}} + \sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} (O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_k^{(i,j)}, \cdot), \mathbf{f}^{(i,j)})_{\mathbf{h}^{(i,j)}}.$$

Now the coefficients $a_k^{(i,j)}$ satisfy the linear system, where

$$\sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{y}_{n,l}^{(i,j)}(\eta_k^{(i,j)}) = 0,$$
$$n = o_{i,j}, \ldots, m;\, l = 1, \ldots, 2n+1.$$

Hence,

$$(\mathbf{s}^{(i,j)}, \mathbf{f}^{(i,j)})_{\mathbf{h}_m^{(i,j)\perp}} = \sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} (O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_k^{(i,j)}, \cdot), \mathbf{f}^{(i,j)})_{\mathbf{h}_m^{(i,j)\perp}}.$$

This is the desired result. ■

Lemma 3.2 *Let the data points* $(\eta_k^{(i,j)}, w_k^{(i,j)}) \in X_N^{(i,j)} \times \mathbb{R}$, $i,j = 1,2,3$ *corresponding to a* $\mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)}$–*admissible system* $X_N^{(i,j)}$ *be given. Then* $\mathcal{S}_m(X_N^{(i,j)}) \cap \mathcal{I}_N^{(i,j)}(w^{(i,j)})$ *consists of only one element, denoted briefly by* $\mathbf{s}_N^{(i,j)}$. *If* $N = M$, *then* $\mathcal{S}_m(X_N^{(i,j)}) \cap \mathcal{I}_N^{(i,j)}(w^{(i,j)}) = \mathbf{harm}_{o_{i,j},\ldots,m}^{(i,j)} \cap \mathcal{I}_N^{(i,j)}(w^{(i,j)})$.

Proof: The conditons $O^{(i,j)} \mathbf{s}_N^{(i,j)}(\eta_k^{(i,j)}) = w_k^{(i,j)}$, $k = o_{i,j}, \ldots, N-1$ give us the linear equations

$$w_k^{(i,j)} = \sum_{n=o_{(i,j)}}^{m} \sum_{l=1}^{2n+1} c_{n,l}^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{y}_{n,l}^{(i,j)}(\eta_k^{(i,j)}) + \sum_{l=o_{i,j}}^{N-1} a_k^{(i,j)} \Big(O_{\eta_k^{(i,j)}}^{(i,j)} \big(O_{\eta_l^{(i,j)}}^{(i,j)} \mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_l^{(i,j)}, \eta_k^{(i,j)})\big)\Big). \tag{3.3}$$

The linear equations

$$\sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{y}_{n,l}^{(i,j)}(\eta_k^{(i,j)}) = 0, \tag{3.4}$$
$$n = o_{i,j}, \ldots, m;\, l = 1, \ldots, 2n+1, \tag{3.5}$$

provide $M - o_{i,j}$ further equations. We have to show that the whole linear system is non–singular. This will be established, if it can be shown that the corresponding homogeneous system possesses only the trivial solution in which all coefficients vanish. To prove this, let $\hat{\mathbf{s}}^{(i,j)} \in \mathcal{S}_m(X_N^{(i,j)}) \cap \mathcal{I}_N^{(i,j)}(0)$. From Lemma 3.1 we get $\|\hat{\mathbf{s}}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)}} = 0$, i.e. $\hat{\mathbf{s}}^{(i,j)} \in \mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m} \cap \mathcal{I}_N^{(i,j)}(0)$. With our assumptions imposed on $X_N^{(i,j)}$ we get $\hat{\mathbf{s}}^{(i,j)} = 0,\ i,j = 1,2,3$, as required. ∎

As immediate consequences we find

Lemma 3.3 (First minimum property) *If* $\mathbf{f}^{(i,j)} \in \mathcal{I}_N^{(i,j)}(w^{(i,j)})$, *then*

$$\|\mathbf{f}^{(i,j)}\|^2_{\mathbf{h}_m^{(i,j)\perp}} = \|\mathbf{s}_N^{(i,j)}\|^2_{\mathbf{h}_m^{(i,j)\perp}} + \|\mathbf{s}_N^{(i,j)} - \mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}.$$

Lemma 3.4 (Second minimum property) *If* $\mathbf{f}^{(i,j)} \in \mathcal{I}_N^{(i,j)}(w^{(i,j)})$, $\mathbf{s}^{(i,j)} \in \mathcal{S}_m(X_N^{(i,j)})$, *then*

$$\|\mathbf{s}^{(i,j)} - \mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} = \|\mathbf{s}_N^{(i,j)} - \mathbf{f}^{(i,j)}\|^2_{\mathbf{h}_m^{(i,j)\perp}} + \|\mathbf{s}^{(i,j)} - \mathbf{s}_N^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}.$$

Lemma 3.3 and Lemma 3.4 follow by straightforward calculation. Summarizing our results we therefore obtain

Theorem 3.2 *The spline interpolation problem*

$$\|\mathbf{s}_N^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} = \inf_{\mathbf{f}^{(i,j)} \in \mathcal{I}_N^{(i,j)}(w^{(i,j)})} \|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}$$

is well–posed in the sense that its solution exists, is unique and depends continuously on the data $w_k^{(i,j)}$, $k = o_{i,j}, \ldots, N-1$. *The uniquely determined solution* $\mathbf{s}_N^{(i,j)}$ *is given in the explicit form*

$$\begin{aligned}\mathbf{s}_N^{(i,j)} &= \sum_{n=o_{i,j}}^{m} \sum_{l=1}^{3} c_{n,l}^{(i,j)} \mathbf{y}_{n,l}^{(i,j)} \\ &\quad + \sum_{k=o_{i,j}}^{N-1} a_k^{(i,j)} O^{(i,j)}_{\eta_k^{(i,j)}} \mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_k^{(i,j)}, \cdot),\end{aligned}$$

where the coefficients $c_{n,l}^{(i,j)}, a_k^{(i,j)}$ *satisfy the linear equations (3.3) and (3.4).*

Corollary 3.2 $\mathbf{s}_N \in \mathbf{h}$ *given by*

$$\mathbf{s}_N = \sum_{i,j=1}^{3} \mathbf{s}_N^{(i,j)}$$

is the unique solution of the interpolation problem

$$\|\mathbf{s}_N\|_{\mathbf{h}_m^\perp} = \inf_{\mathbf{f}} \|\mathbf{f}\|_{\mathbf{h}_m^\perp},$$

where $\mathbf{f} = \sum_{i,j=1}^{3} \mathbf{f}^{(i,j)}$ *and* $\mathbf{f}^{(i,j)} \in \mathcal{I}_N^{(i,j)}(w^{(i,j)})$.

Numerical experiences have shown that the linear systems stated above tend to be ill–conditioned unless m and N are both small, and this may cause difficulties if the attempt is made to solve these systems directly. Therefore we are now interested in giving an alternate formulation of the solution $\mathbf{s}_N^{(i,j)}$ that is more suitable for computational purposes. To this end we remember that, given (i,j), there exists in $\mathbf{h}_m^{(i,j)}$ a unique basis $\mathbf{l}_{o_{i,j}}^{(i,j)}, \ldots, \mathbf{l}_{M-1}^{(i,j)}$ of the form

$$\mathbf{l}_k^{(i,j)} = \sum_{r=o_{i,j}}^{M-1} c_{k,r}^{(i,j)} \sum_{n=o_{i,j}}^{m} \frac{2n+1}{4\pi} \mathbf{p}^{(i,j)}_{Leg,n}(\cdot, \eta_r^{(i,j)}) \tag{3.6}$$

for $k = o_{i,j}, \ldots, M-1$, satisfying the conditions

$$\delta_{r,k} = O^{(i,j)}_{\eta_k^{(i,j)}} \mathbf{l}_r^{(i,j)}(\eta_k^{(i,j)}).$$

Then every $\mathbf{y}^{(i,j)} \in \mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m}$ can be represented as follows ($\xi \in \Omega$)

$$\mathbf{y}^{(i,j)}(\xi) = \sum_{k=o_{i,j}}^{M-1} (O^{(i,j)}_{\eta_k^{(i,j)}} \mathbf{y}^{(i,j)}(\eta_k^{(i,j)})) \mathbf{l}_k^{(i,j)}(\xi).$$

Therefore, for every $\mathbf{f}^{(i,j)} \in \mathbf{h}^{(i,j)}$, the unique $\mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m}$–interpolant $p_{\mathbf{h}_m^{(i,j)}} \mathbf{f}^{(i,j)}$ of $\mathbf{f}^{(i,j)}$ for the $\mathbf{harm}^{(i,j)}_{o_{i,j},\ldots,m}$–unisolvent set $X_M^{(i,j)}$ is given by the "Lagrange form"

$$p_{\mathbf{h}_m^{(i,j)}} \mathbf{f}^{(i,j)} = \sum_{k=o_{i,j}}^{M} (O^{(i,j)}_{\eta_k^{(i,j)}} \mathbf{f}^{(i,j)}(\eta_k^{(i,j)})) \mathbf{l}_k^{(i,j)}.$$

The mapping $p_{\mathbf{h}_m^{(i,j)}} : \mathbf{h}^{(i,j)} \to \mathbf{h}_m^{(i,j)}$ is a continuous linear projector of $\mathbf{h}^{(i,j)} \subset \mathbf{c}^{(0)}(\Omega)$ onto $\mathbf{h}_m^{(i,j)}$. Hence, $p_{\mathbf{h}_m^{(i,j)}}$ determines the decomposition of $\mathbf{h}^{(i,j)}$ as direct sum $\mathbf{h}^{(i,j)} = \mathbf{h}_m^{(i,j)} \oplus \dot{\mathbf{h}}_m^{(i,j)-}$, where

$$\begin{aligned}\dot{\mathbf{h}}_m^{(i,j)-} = \Big\{ &\dot{\mathbf{f}}^{(i,j)} \in \mathbf{h}^{(i,j)} \mid \\ &O^{(i,j)}_{\eta_k^{(i,j)}} \dot{\mathbf{f}}^{(i,j)}(\eta_k^{(i,j)}) = 0,\ k = o_{i,j}, \ldots, M-1 \Big\}.\end{aligned} \tag{3.7}$$

Consequently, any $\mathbf{f}^{(i,j)} \in \mathbf{h}^{(i,j)}$ admits the unique representation

$$\mathbf{f}^{(i,j)} = p_{\mathbf{h}_m^{(i,j)}} \mathbf{f}^{(i,j)} + \dot{\mathbf{f}}^{(i,j)},\ \dot{\mathbf{f}}^{(i,j)} \in \dot{\mathbf{h}}_m^{(i,j)-}. \tag{3.8}$$

$\dot{\mathbf{h}}_m^{(i,j)-}$ defined by (3.7) and equipped with the scalar product $(\cdot,\cdot)_{\mathbf{h}_m^{(i,j)\perp}}$ is a Hilbert function subspace of $\mathbf{c}^{(0)}(\Omega)$. Furthermore it can be readily seen that for $\xi \in \Omega$ (cf. Freeden and Hermann (1986))

$$\big(\dot{\mathbf{f}}^{(i,j)}, O_\xi^{(i,j)} \mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\xi, \cdot)\big)_{\mathbf{h}_m^{(i,j)\perp}} = O_\xi^{(i,j)} \dot{\mathbf{f}}^{(i,j)}(\xi), \tag{3.9}$$

where the tensor $\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\cdot,\cdot)$, is given by

$$\begin{aligned}\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\xi,\eta) &= \mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\xi,\eta)\\ &- \sum_{r=o_{i,j}}^{M-1} \mathbf{l}_r^{(i,j)}(\xi)\odot \mathbf{g}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_r^{(i,j)},\eta)\\ &- \sum_{r=o_{i,j}}^{M-1} \mathbf{g}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_r^{(i,j)},\xi)\odot \mathbf{l}_r^{(i,j)}(\eta)\\ &+ \sum_{r=o_{i,j}}^{M-1}\sum_{k=o_{i,j}}^{M-1} G_{\mathbf{h}_m^{(i,j)\perp}}(\eta_r^{(i,j)},\eta_k^{(i,j)})\mathbf{l}_k^{(i,j)}(\xi)\odot \mathbf{l}_r^{(i,j)}(\eta)\end{aligned}$$

with

$$\mathbf{g}_{\mathbf{h}_m^{(i,j)\perp}}(\xi,\eta) = O_\xi^{(i,j)}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\xi,\eta) = \sum_{n=m+1}^{\infty}\sum_{r=1}^{2n+1} | A_n |^{-2} \left(O_\xi^{(i,j)}\mathbf{y}_{n,r}^{(i,j)}(\xi)\right)\mathbf{y}_{n,r}^{(i,j)}(\eta),$$

$$G_{\mathbf{h}_m^{(i,j)\perp}}(\xi,\eta) = \sum_{n=m+1}^{\infty}\sum_{r=1}^{2n+1} | A_n |^{-2} \left(O_\xi^{(i,j)}\mathbf{y}_{n,r}^{(i,j)}(\xi)\right)\left(O_\eta^{(i,j)}\mathbf{y}_{n,r}^{(i,j)}(\eta)\right),$$

for $i,j=1,2,3$. By standard arguments of the theory of Hilbert spaces (cf. Freeden (1990)) we therefore find

Theorem 3.3 *The uniquely determined solution $\mathbf{s}_N^{(i,j)}$ of the spline interpolation problem*

$$\|\mathbf{s}_N^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} = \inf_{\mathbf{f}^{(i,j)}\in\mathcal{I}_N^{(i,j)}(w^{(i,j)})} \|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}$$

is given in the explicit form

$$\begin{aligned}\mathbf{s}_N^{(i,j)}(\xi) &= \sum_{k=o_{i,j}}^{M-1} w_k^{(i,j)}\mathbf{l}_k^{(i,j)}(\xi) +\\ &\quad \sum_{k=M}^{N-1} d_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\eta_k^{(i,j)},\xi),\end{aligned}$$

where the coefficients $d_k^{(i,j)}$, $k=M,\dots,N-1$ are the solutions of the linear equations

$$\sum_{k=M}^{N-1} d_k^{(i,j)} O_{\eta_k^{(i,j)}}^{(i,j)}\left(O_{\eta_l^{(i,j)}}^{(i,j)}\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\eta_k^{(i,j)},\eta_l^{(i,j)})\right) = w_l^{(i,j)} - \sum_{k=o_{i,j}}^{M-1} w_k^{(i,j)} O_{\eta_l^{(i,j)}}^{(i,j)}\mathbf{l}_k^{(i,j)}(\eta_l^{(i,j)})$$

with $l=M,\dots,N-1$.

The coefficient matrix in the unknowns $d_k^{(i,j)}$ is a Gram matrix, thus, it is symmetric and positive definite. Therefore, the linear system has a unique solution and the solution process can be carried out by standard algorithms, for which powerful routines are available.

3.3 Convergence

For any $\mathbf{harm}_{0,\dots,m}$–admissible system $X_N = \bigcup_{i,j=1}^3 X_N^{(i,j)}$ we define the X_N–*width* by

$$\Theta_N = \max_{i,j=1,2,3}(\Theta_N^{(i,j)}),$$

where

$$\Theta_N^{(i,j)} = \max_{\xi\in\Omega}(\min_{\eta\in X_N^{(i,j)}} | \xi-\eta |).$$

Theorem 3.4
Let $\mathbf{f}^{(i,j)}$ be of class $\mathbf{h}^{(i,j)}$, $i,j\in\{1,2,3\}$. Assume that the system $X_N^{(i,j)}$ is $\mathbf{harm}_{o_{i,j},\dots,m}^{(i,j)}$–admissible. Denote by $\mathbf{s}_N^{(i,j)}\in\mathbf{h}^{(i,j)}$ the uniquely determined solution of the interpolation problem

$$\|\mathbf{s}_N^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} = \inf_{\mathbf{f}^{(i,j)}\in\mathcal{I}_N^{(i,j)}(w^{(i,j)})} \|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}},$$

where

$$\mathcal{I}_N^{(i,j)}(w^{(i,j)}) = \Big\{\mathbf{f}\in\mathbf{h}^{(i,j)} \mid O_{\eta_k^{(i,j)}}^{(i,j)}\mathbf{f}(\eta_k^{(i,j)}) = w_k^{(i,j)},\ k=o_{i,j},\dots,N-1\Big\}.$$

Furthermore, suppose that $\sigma\in[0,1]$, and let $\{A_n\}$ be $\{(n+1/2)^\tau\}$–summable with $\tau\ge 1+2(2\sigma+o_{i,j})$. Then

$$\sup_{\xi\in\Omega} | O_\xi^{(i,j)}\mathbf{f}(\xi) - O_\xi^{(i,j)}\mathbf{s}_N^{(i,j)}(\xi) | \le 4F_{m,\sigma,\{A_n\}}^{(i,j)}(\Theta_N^{(i,j)})^\sigma \|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}},$$

where

$$F_{m,\sigma,\{A_n\}}^{(i,j)} = \Big(\sum_{n=m+1}^\infty \tfrac{2n+1}{4\pi}A_n^{-2}\Big((2n+1)\big(\tfrac{\lambda_n}{2}\big)^{2\sigma+o_{i,j}} +2\big(\tfrac{\lambda_n}{2}\big)^{\sigma+1}C_{M,\sigma}^{(i,j)} + (C_{M,\sigma}^{(i,j)})^2\tfrac{\lambda_n}{2}\Big)\Big)^{\frac12}$$

and

$$C_{M,\sigma}^{(i,j)} = \Big(\sum_{k=o_{i,j}}^{M-1}\sum_{l=o_{i,j}}^{M-1} | c_{l,k}^{(i,j)} |\Big)\Big(\sum_{n=0}^m \frac{2n+1}{4\pi}\Big(\frac{\lambda_n}{2}\Big)^\sigma\Big).$$

Here, as usual, $\lambda_n=n(n+1)$ and $c_{l,k}^{(i,j)}$ are the constituting coefficients of $\mathbf{l}_k^{(i,j)}$ (cf. (3.6)).

Proof: It is clear that for $\xi\in\Omega$ there exists a point $\eta_k^{(i,j)}\in X_N^{(i,j)}$ $(i=1,2,3)$ with $| \xi-\eta_k^{(i,j)} | \le \Theta_N^{(i,j)}$. Observing the fact that for $k=o_{i,j},\dots,N-1$

$$O_{\eta_k^{(i,j)}}^{(i,j)}\mathbf{s}_N^{(i,j)}(\eta_k^{(i,j)}) = O_{\eta_k^{(i,j)}}^{(i,j)}\mathbf{f}^{(i,j)}(\eta_k^{(i,j)}) = w_k^{(i,j)},$$

we obtain using the triangle inequality and the representation (3.8)

$$\begin{aligned}| O_\xi^{(i,j)}\mathbf{s}_N^{(i,j)}(\xi) - O_\xi^{(i,j)}\mathbf{f}^{(i,j)}(\xi) | &\le\\ | O_\xi^{(i,j)}\dot{\mathbf{s}}_N^{(i,j)}(\xi) - O_{\eta_k^{(i,j)}}^{(i,j)}\dot{\mathbf{s}}_N^{(i,j)}(\eta_k^{(i,j)}) |&\\ + | O_\xi^{(i,j)}\dot{\mathbf{f}}^{(i,j)}(\xi) - O_{\eta_k^{(i,j)}}^{(i,j)}\dot{\mathbf{f}}^{(i,j)}(\eta_k^{(i,j)}) | .&\end{aligned}$$

From (3.9) it can be deduced that

$$\begin{aligned} &O_\xi^{(i,j)}\dot{\mathbf{s}}_N(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\dot{\mathbf{s}}_N^{(i,j)}(\eta_k^{(i,j)}) = \\ &\quad (O_\xi^{(i,j)}\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\xi,\cdot) - \\ &\quad O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\eta_k^{(i,j)},\cdot), \dot{\mathbf{s}}_N^{(i,j)})_{\mathbf{h}_m^{(i,j)\perp}} \end{aligned}$$

and

$$\begin{aligned} &O_\xi^{(i,j)}\dot{\mathbf{f}}^{(i,j)}(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\dot{\mathbf{f}}^{(i,j)}(\eta_k^{(i,j)}) = \\ &\quad (O_\xi^{(i,j)}\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\xi,\cdot) - \\ &\quad O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{K}_{\dot{\mathbf{h}}_m^{(i,j)\perp}}(\eta_k^{(i,j)},\cdot), \dot{\mathbf{f}}^{(i,j)})_{\mathbf{h}_m^{(i,j)\perp}}. \end{aligned}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned} &| O_\xi^{(i,j)}\dot{\mathbf{s}}_N^{(i,j)}(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\dot{\mathbf{s}}_N^{(i,j)}(\eta_k^{(i,j)}) | \leq \\ &\quad (\kappa^{(i,j)}(\xi,\eta_k^{(i,j)}))^{\frac{1}{2}}\|\dot{\mathbf{s}}_N^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}, \\ &| O_\xi^{(i,j)}\dot{\mathbf{f}}^{(i,j)}(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\dot{\mathbf{f}}^{(i,j)}(\eta_k^{(i,j)}) | \leq \\ &\quad (\kappa^{(i,j)}(\xi,\eta_k^{(i,j)}))^{\frac{1}{2}}\|\dot{\mathbf{f}}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}\ , \end{aligned}$$

where

$$\begin{aligned} \kappa^{(i,j)}(\xi,\eta_k^{(i,j)}) &= O_\xi^{(i,j)}O_\xi^{(i,j)}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\xi,\xi) \\ &-2O^{(i,j)}_{\eta_k^{(i,j)}}O_\xi^{(i,j)}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\xi,\eta_k^{(i,j)}) \\ &+O^{(i,j)}_{\eta_k^{(i,j)}}O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_k^{(i,j)},\eta_k^{(i,j)}) \\ &-2\sum_{r=o_{i,j}}^{M-1}\left(O_\xi^{(i,j)}\mathbf{l}_r^{(i,j)}(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{l}_r^{(i,j)}(\eta_k^{(i,j)})\right)\times \\ &\quad \left(O_\xi^{(i,j)}O^{(i,j)}_{\eta_r^{(i,j)}}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_r^{(i,j)},\xi)\right. \\ &\quad \left.-O^{(i,j)}_{\eta_k^{(i,j)}}O^{(i,j)}_{\eta_r^{(i,j)}}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_r^{(i,j)},\eta_k^{(i,j)})\right) \\ &+\sum_{s=o_{i,j}}^{M-1}\sum_{r=o_{i,j}}^{M-1}\left(O_\xi^{(i,j)}\mathbf{l}_r^{(i,j)}(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{l}_r^{(i,j)}(\eta_k^{(i,j)})\right)\times \\ &\quad \left(O_\xi^{(i,j)}\mathbf{l}_s^{(i,j)}(\xi) - O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{l}_s^{(i,j)}(\eta_k^{(i,j)})\right)\times \\ &\quad O^{(i,j)}_{\eta_s^{(i,j)}}O^{(i,j)}_{\eta_r^{(i,j)}}\mathbf{K}_{\mathbf{h}_m^{(i,j)\perp}}(\eta_r^{(i,j)},\eta_s^{(i,j)})\ . \end{aligned}$$

Using the inequalities (Freeden and Hermann 1986)

$$| P_n(\xi\cdot\zeta) - P_n(\eta\cdot\zeta) | \leq 2\left(\frac{\lambda_n}{2}\mid \xi-\eta\mid\right)^\sigma$$

and

$$\begin{aligned} &| P_n(\xi\cdot\xi) - 2P_n(\xi\cdot\eta) + P_n(\eta\cdot\eta) | \leq \\ &\quad 4(2n+1)\left(\frac{\lambda_n}{2}\mid \xi-\eta\mid\right)^{2\sigma} \end{aligned}$$

for any $\sigma\in[0,1]$ and all $\xi,\eta,\zeta\in\Omega$ we obtain after elementary calculations

$$| \kappa^{(i,j)}(\xi,\eta_k^{(i,j)}) | \leq 4(F^{(i,j)}_{m,\sigma,\{A_n\}})^2 \mid \xi-\eta_k^{(i,j)}\mid^{2\sigma}.$$

As an interpolating spline, $\mathbf{s}_N^{(i,j)}$ is the smoothest interpolant, i.e.,

$$\|\dot{\mathbf{s}}_N^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} \leq \|\dot{\mathbf{f}}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}.$$

Summarizing our results we find

$$\begin{aligned} &| O^{(i,j)}\mathbf{f}^{(i,j)}(\xi) - O^{(i,j)}\mathbf{s}_N^{(i,j)}(\xi) | \\ &\quad\leq 2\left(\kappa^{(i,j)}(\xi,\eta_k^{(i,j)})\right)^{\frac{1}{2}}\|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} \\ &\quad\leq 4F^{(i,j)}_{m,\sigma,\{A_n\}}\mid\xi-\eta_k^{(i,j)}\mid^\sigma\|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}} \\ &\quad\leq 4F^{(i,j)}_{m,\sigma,\{A_n\}}(\Theta_N^{(i,j)})^\sigma\|\mathbf{f}^{(i,j)}\|_{\mathbf{h}_m^{(i,j)\perp}}. \end{aligned}$$

This proves Theorem 3.4. ■

Theorem 3.4 allows the following reformulation.

Corollary 3.3 *The uniquely determined solution* $\mathbf{s}_N\in\mathbf{h}$ *of the spline interpolation problem*

$$\|\mathbf{s}_N\|_{\mathbf{h}_m^\perp} = \inf_{\mathbf{f}\in\mathcal{I}_N(w)}\|\mathbf{f}\|_{\mathbf{h}_m^\perp}, \tag{3.10}$$

where

$$\begin{aligned} \mathcal{I}_N(w) = \{\mathbf{f}\in\mathbf{h}\mid & O^{(i,j)}_{\eta_k^{(i,j)}}\mathbf{f}(\eta_k^{(i,j)}) = w_k^{(i,j)}, \\ & k=o_{i,j},\ldots,N-1, i,j=1,2,3\} \end{aligned}$$

satisfies the inequality

$$\begin{aligned} &\sup_{\xi\in\Omega}| O_\xi^{(i,j)}\mathbf{f}(\xi) - O_\xi^{(i,j)}\mathbf{s}_N(\xi) | \leq \\ &\quad 4F_{m,\sigma,\{A_n\}}(\Theta_N)^\sigma\|\mathbf{f}\|_{\mathbf{h}_m^\perp},\ i,j=1,2,3, \end{aligned}$$

with

$$F_{m,\sigma,\{A_n\}} = \max_{i,j=1,..,3}\left(F^{(i,j)}_{m,\sigma,\{A_n\}}\right).$$

Now we want to show that $\mathbf{f}-\mathbf{s}_N$ does not only tend to zero in its $O^{(i,j)}$–components, if $\Theta_N\to 0$ for $N\to\infty$, but also in the components and, in addition, in the Euclidean norm. For that purpose we use the decomposition Theorem 2.4 and we are able to obtain

Corollary 3.4 *The uniquely determined solution* $\mathbf{s}_N\in\mathbf{h}$ *of the interpolation problem (3.10) satisfies*

$$\begin{aligned} &\sup_{\xi\in\Omega}| \mathbf{f}(\xi) - \mathbf{s}_N(\xi) | \leq \\ &\quad 4(13+2C)F_{m,\sigma,\{A_n\}}(\Theta_N)^\sigma\|\mathbf{f}\|_{\mathbf{h}_m^\perp}, \end{aligned}$$

where $C>0$ *denotes a real number satisfying*

$$C \geq \left| \int_\Omega \mathbf{o}_\xi^{(i,j)}G(\Delta^*(\Delta^*+\lambda_1);\xi,\eta)d\omega(\eta)\right|. \tag{3.11}$$

Proof: From Theorem 2.4 we know that for $\xi \in \Omega$

$$\begin{aligned} \mathbf{f}(\xi) - \mathbf{s}_N(\xi) = & \\ & \mathbf{o}_\xi^{(1,1)}(O_\xi^{(1,1)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi))) \\ + & \sum_{(i,j)\in\{(2,2),(3,3)\}} \frac{1}{2}\mathbf{o}_\xi^{(i,j)}(O_\xi^{(i,j)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi))) \\ - & \sum_{(i,j)\in\{(1,2),(1,3),(2,1),(3,1)\}} \\ & \mathbf{o}_\xi^{(i,j)} \int_\Omega G(\Delta^*;\xi,\eta)(O_\eta^{(i,j)}(\mathbf{f}(\eta) - \mathbf{s}_N(\eta)))d\omega(\eta) \\ + & \frac{1}{2} \sum_{(i,j)\in\{(2,3),(3,2)\}} \mathbf{o}_\xi^{(i,j)} \int_\Omega G(\Delta^*(\Delta^*+\lambda_1);\xi,\eta) \times \\ & (O_\eta^{(i,j)}(\mathbf{f}(\eta) - \mathbf{s}_N(\eta)))d\omega(\eta). \end{aligned}$$

In what follows we want to give an estimate for each component. We have

$$\begin{aligned} & \mid \mathbf{o}_\xi^{(1,1)}(O_\xi^{(1,1)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi))) \mid \\ \leq & \mid \xi \otimes \xi \mid\mid O_\xi^{(1,1)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid \\ \leq & \mid O_\xi^{(1,1)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid, \end{aligned}$$

and for $i = 2,3$

$$\begin{aligned} & \mid \frac{1}{2}\mathbf{o}_\xi^{(i,i)}(O_\xi^{(i,i)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi))) \mid \\ \leq & \frac{1}{2} 2 \mid O_\xi^{(i,i)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid \\ \leq & \mid O_\xi^{(i,i)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid . \end{aligned}$$

Since $\mathbf{o}_\xi^{(i,j)} \int_\Omega \ldots d\omega(\eta) = \int_\Omega \mathbf{o}^{(i,j)} \ldots d\omega(\eta)$, we find for $(i,j) \in \{(1,2),(1,3),(2,1),(3,1)\}$

$$\begin{aligned} & \mid \mathbf{o}_\xi^{(i,j)} \int_\Omega G(\Delta^*;\xi,\eta)(O_\eta^{(i,j)}(\mathbf{f}(\eta) - \mathbf{s}_N(\eta)))d\omega(\eta) \mid \\ \leq & \sup_{\xi\in\Omega} \mid O_\xi^{(i,j)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid \times \\ & \sup_{\xi\in\Omega} \int_\Omega \mid \mathbf{o}_\xi^{(i,j)} G(\Delta^*;\xi,\eta) \mid d\omega(\eta). \end{aligned}$$

Using

$$\begin{aligned} & \int_\Omega \mid \mathbf{o}_\xi^{(i,j)} G(\Delta^*;\xi,\eta) d\omega(\eta) \mid \\ = & \int_\Omega \mid \nabla_\xi^* G(\Delta^*;\xi,\eta) \mid d\omega(\eta) \\ = & \frac{1}{4\pi} \int_\Omega \mid \nabla_\xi^* \ln(1-\xi\cdot\eta) \mid d\omega(\eta) \\ = & \frac{1}{4\pi} \int_\Omega \mid \frac{\eta - (\xi\cdot\eta)\xi}{1-\xi\cdot\eta} \mid d\omega(\eta) \\ = & \frac{1}{4\pi} \int_\Omega \mid \frac{\sqrt{1-(\xi\cdot\eta)^2}}{1-\xi\cdot\eta} \mid d\omega(\eta) \\ \leq & \frac{1}{2} \int_{-1}^{1} \frac{\sqrt{1+t}}{\sqrt{1-t}} dt \leq 2, \end{aligned}$$

we get for $(i,j) \in \{(1,2),(1,3),(2,1),(3,1)\}$

$$\begin{aligned} & \mid \mathbf{o}_\xi^{(i,j)} \int_\Omega G(\Delta^*;\xi,\eta)(O_\eta^{(i,j)}(\mathbf{f}(\eta) - \mathbf{s}_N(\eta)))d\omega(\eta) \mid \\ & \leq 2 \sup_{\xi\in\Omega} \mid O^{(i,j)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid . \end{aligned}$$

The same process for $(i,j) \in \{(2,3),(3,2)\}$ leads us to

$$\begin{aligned} & \mid \mathbf{o}_\xi^{(i,j)} \int_\Omega G(\Delta^*(\Delta^*+\lambda_1);\xi,\eta) \times \\ & (O_\eta^{(i,j)}(\mathbf{f}(\eta) - \mathbf{s}_N(\eta)))d\omega(\eta) \mid \\ \leq & \sup_{\xi\in\Omega} \mid O_\xi^{(i,j)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid \times \\ & \sup_{\xi\in\Omega} \mid \int_\Omega \mathbf{o}_\xi^{(i,j)} G(\Delta^*(\Delta^*+\lambda_1);\xi,\eta) d\omega(\eta) \mid \\ \leq & C \sup_{\xi\in\Omega} \mid O_\xi^{(i,j)}(\mathbf{f}(\xi) - \mathbf{s}_N(\xi)) \mid, \end{aligned}$$

for all $\xi \in \Omega$ and $(i,j) \in \{(2,3),(3,2)\}$, where $C > 0$ satisfies (3.11).

Summarizing our results we get by virtue of Corollary 3.3

$$\begin{aligned} & \mid \mathbf{f}(\xi) - \mathbf{s}_N(\xi) \mid \leq \\ & \quad (1+4+8+2C) F_{m,\sigma,\{A_n\}}(\Theta_N)^\sigma \|\mathbf{f}\|_{\mathbf{h}_m^\perp} \end{aligned}$$

for every $\xi \in \Omega$. This proves Corollary 3.4. ■

Corollary 3.4 means that we are able to approximate any $\mathbf{f} \in \mathbf{h}$ together with its $O^{(i,j)}$-components uniformly on Ω in a constructive way using spline interpolation provided that the X_N–width Θ_N tends to zero as $N \to \infty$. As is well–known, the set of all finite linear combinations of tensor spherical harmonics is dense in the space $\mathbf{c}^{(2)}(\Omega)$. Hence, $\mathbf{h}$ is a dense subset of $\mathbf{c}^{(2)}(\Omega)$, too. An extended version of Helly's theorem due to Yamabe (1959) shows us that to any $\mathbf{g} \in \mathbf{c}^{(2)}(\Omega)$ and any (admissible) system X_T, there exists an element $\mathbf{f} \in \mathbf{h}$ in an ε-neighbourhood of $\mathbf{g}$ with

$$O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{f}(\eta_k^{(i,j)}) = O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{g}(\eta_k^{(i,j)})$$

for $k = o_{i,j}, \ldots, T-1$, $i,j = 1,2,3$. Thus we finally obtain

Theorem 3.5 *Let X_T be a $\mathbf{harm}_{0,\ldots,m}$–admissible system. Suppose, that $\{X_N\}$, $N \to \infty$ is a sequence of $\mathbf{harm}_{0,\ldots,m}$–admissible systems such that $X_T \subset X_N$ for all N and $\Theta_N \to 0$ as $N \to \infty$. Then, to any $\mathbf{g} \in \mathbf{c}^{(2)}(\Omega)$ and to every $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and a spherical spline $\mathbf{s}_N \in h$ such that*

$$O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{s}_N(\eta_i^{(i,j)}) = O_{\eta_k^{(i,j)}}^{(i,j)} \mathbf{g}(\eta_i^{(i,j)})$$

for $k = o_{i,j}, \ldots, T-1, i,j = 1,2,3,$ and

$$\sup_{\xi\in\Omega} \mid O_\xi^{(i,j)} \mathbf{s}_N(\xi) - O_\xi^{(i,j)} \mathbf{g}(\xi) \mid \leq \varepsilon,$$

$$\sup_{\xi\in\Omega} \mid \mathbf{s}_N(\xi) - \mathbf{g}(\xi) \mid \leq \varepsilon.$$

Theorem 3.5 gives the theoretical justification of the use of tensor spherical spline functions as basis system in problems of uniform approximation on the unit sphere.

Appendix A — The Decomposition of the Hesse Matrix

In order to clarify the decomposition of spherical tensor fields as described in Theorem 2.4, we discuss the decomposition of the Hesse matrix of a twice continuously differentiable function. The Hesse matrix plays an important rôle in physical geodesy, in particular for the determination of the earth's gravity field via satellite gradiometry (cf. e.g. Rummel (1986)). In this context, the Hesse matrix denotes the gravitational tensor.

We assume that $x \in \mathbb{R}^3$ has the representation $x = r\xi$, $\xi \in \Omega$, in polar coordinates. Then the Hesse matrix $\mathbf{h}$ of a twice continuously differentiable function H is defined on Ω by

$$\begin{aligned} \mathbf{h}(\xi) &= \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} H(x)|_{r=1}\right)_{i,j=1,2,3} \\ &= \nabla_x \otimes \nabla_x H(x)|_{r=1}. \end{aligned}$$

The separation process of the differential operators in radial and angular parts yields

$$\begin{aligned} \nabla_x \otimes \nabla_x &= (\xi\frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_\xi) \otimes (\xi\frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_\xi) \\ &= \xi \otimes \xi(\frac{\partial}{\partial r})^2 - \frac{1}{r^2}\xi \otimes \nabla^*_\xi \\ &\quad +\frac{1}{r}\xi \otimes \nabla^*_\xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_\xi \otimes \xi\frac{\partial}{\partial r} \\ &\quad +\frac{1}{r^2}\nabla^*_\xi \otimes \nabla^*_\xi, \end{aligned}$$

such that

$$\begin{aligned} \mathbf{h}(\xi) &= \xi \otimes \xi \; ((\frac{\partial}{\partial r})^2 H(x)\Big|_{r=1}) \\ &\quad +\xi \otimes \nabla^*_\xi \left(-H(\xi) + \left(\frac{\partial}{\partial r}H(x)\Big|_{r=1}\right)\right) \\ &\quad +\nabla^*_\xi \otimes \xi(\frac{\partial}{\partial r}H(x)|_{r=1}) \\ &\quad +\nabla^*_\xi \otimes \nabla^*_\xi H(\xi). \end{aligned}$$

Furthermore,

$$\begin{aligned} \nabla^*_\xi \otimes \nabla^*_\xi H(\xi) &= 2\nabla^*_\xi \otimes \nabla^*_\xi(\frac{1}{2}H(\xi)) \\ &= 2\nabla^*_\xi \otimes \nabla^*_\xi(\frac{1}{2}H(\xi)) \\ &\quad +\nabla^*_\xi(\frac{1}{2}H(\xi)) \otimes \xi \\ &\quad -(\mathbf{i} - \xi \otimes \xi)\Delta^*_\xi(\frac{1}{2}H(\xi)) \\ &\quad -\nabla^*_\xi(\frac{1}{2}H(\xi)) \otimes \xi \\ &\quad -(\mathbf{i} - \xi \otimes \xi)\Delta^*_\xi(\frac{1}{2}H(\xi)) \\ &= \mathbf{o}^{(2,3)}_\xi(\frac{1}{2}H(\xi)) \\ &\quad -2\nabla^*_\xi(\frac{1}{2}H(\xi)) \otimes \xi \\ &\quad -(\mathbf{i} - \xi \otimes \xi)\Delta^*_\xi(\frac{1}{2}H(\xi)), \end{aligned}$$

and

$$\begin{aligned} \nabla^*_\xi \otimes \xi(\frac{\partial}{\partial r}(H(x))|_{r=1}) &= \nabla^*_\xi(\frac{\partial}{\partial r}H(x)|_{r=1}) \otimes \xi \\ &\quad +(\mathbf{i} - \xi \otimes \xi)(\frac{\partial}{\partial r}H(x)|_{r=1}). \end{aligned}$$

Summarizing our results we finally arrive at the following decomposition of the Hesse matrix:

$$\begin{aligned} \mathbf{h}(\xi) &= \mathbf{o}^{(1,1)}_\xi\left(\left(\frac{\partial}{\partial r}\right)^2 H(x)\Big|_{r=1}\right) \\ &\quad +\mathbf{o}^{(1,2)}_\xi\left(-H(\xi) + \frac{\partial}{\partial r}H(x)\Big|_{r=1}\right) \\ &\quad +\mathbf{o}^{(2,1)}_\xi\left(-H(\xi) + \frac{\partial}{\partial r}H(x)\Big|_{r=1}\right) \\ &\quad +\mathbf{o}^{(2,2)}_\xi\left(-\frac{1}{2}\Delta^*_\xi H(\xi) + \frac{\partial}{\partial r}H(x)\Big|_{r=1}\right) \\ &\quad +\mathbf{o}^{(2,3)}_\xi\left(\frac{1}{2}H(\xi)\right). \end{aligned}$$

Finally, we are interested in expanding the gravitational tensor (at "altidude" $r_0 > R$)

$$\mathbf{v}(r_0\xi) = \nabla_x \otimes \nabla_x V(x)|_{r=r_o}$$

into a series in terms of tensor spherical harmonics, where V is the uniquely determined solution of the classical Dirichlet problem of potential theory corresponding to continuous boundary values on the sphere around the origin with radius R and r_0 is assumed to be strictly greater than R (cf. Rummel and van Gelderen (1992) using Zerilli (1970) nomenclature):

$$V(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}\left(\frac{R}{|x|}\right)^{n+1} Y_{n,j}(\xi),$$

with $x = r\xi$, $r > R$, and

$$V_{n,j} = \int_\Omega V(R\eta)Y_{n,j}(\eta)d\omega(\eta).$$

It is clear that $\Delta_x V(x)|_{|x|=r_0} = 0$ is equivalent to

$$\mathrm{tr}\;(\nabla_x \otimes \nabla_x V(x))|_{|x|=r_0} = 0.$$

Therefore an easy calculation shows that

$$\begin{aligned}
\mathbf{v}(r_0\xi) = \\
&\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}(n+1)(n+2)\frac{R^{n+1}}{{r_0}^{n+3}}\mathbf{y}_{n,j}^{(1,1)}(\xi) \\
&-\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}(n+2)\sqrt{\lambda_n}\frac{R^{n+1}}{{r_0}^{n+3}}\mathbf{y}_{n,j}^{(1,2)}(\xi) \\
&\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}(n+2)\sqrt{\lambda_n}\frac{R^{n+1}}{{r_0}^{n+3}}\mathbf{y}_{n,j}^{(2,1)}(\xi) \\
&-\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}(n+1)(n+2)\frac{R^{n+1}}{{r_0}^{n+3}}\mathbf{y}_{n,j}^{(2,2)}(\xi) \\
&+\frac{1}{\sqrt{2}}\sum_{n=2}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}\sqrt{\lambda_n(\lambda_n-\lambda_1)}\frac{R^{n+1}}{{r_0}^{n+3}}\mathbf{y}_{n,j}^{(2,3)}(\xi)
\end{aligned}$$

holds for all $\xi \in \Omega$.

As an example we mention the gravitational tensor $\mathbf{v}$ of a fixed mass point y inside the sphere (i.e. $y = |y|\eta,\ |y| < R$):

$$V(x) = \frac{1}{|x-y|},\ |x| > R.$$

On the one hand, $\mathbf{v}(r_0\xi)$ allows a series expansion in terms of tensor spherical harmonics with coefficients $V_{n,j}$ given by

$$V_{n,j} = \frac{4\pi}{2n+1}\frac{1}{R}\left(\frac{|y|}{R}\right)^n Y_{n,j}(\eta).$$

Written out in terms of Legendre polynomials we find

$$\begin{aligned}
\mathbf{v}(r_0\xi) &= V_1(\xi)\xi\otimes\xi \\
&+ \quad V_2(\xi)\left(\xi\otimes(\eta-(\xi\cdot\eta)\xi)+(\eta-(\xi\cdot\eta)\xi)\otimes\xi\right) \\
&+ \quad V_3(\xi)\mathbf{i}_{tan} + V_4(\xi)(\eta-(\xi\cdot\eta)\xi)\otimes(\eta-(\xi\cdot\eta)\xi),
\end{aligned}$$

where via the addition theorem the functions V_i are expressible as series in terms of the Legendre polynomials P_n and their derivatives. We omit the explicit expansions.

On the other hand simple calculations yield

$$\begin{aligned}
\mathbf{v}(r_0\xi) &= \nabla_x\otimes\nabla_x\left(|x-y|^{-1}\right)\Big|_{|x|=r_0} \\
&= \frac{3}{|r_0\xi-y|^5}(r_0\xi-y)\otimes(r_0\xi-y) - \frac{1}{|r_0\xi-y|^3}\mathbf{i}.
\end{aligned}$$

Appendix B — The Choice of the Norm

In the described approach for the tensor spherical spline interpolation all conclusions were drawn under a predefined Hilbert space structure. Now we fix our attention on this Hilbert space structure which is given by the summable sequence $\{A_n\}$ or, equivalently, by the choice of the norm. Of course it is desired to choose a norm which is adopted to the specific problem and to the specific data. However, for numerical reasons, there are some restrictions for this choice. The condition number of the resulting linear equation is influenced by this norm, as well as the possibility of a representation of the reproducing kernel in closed form. For this reason, the choice of the norm must be a compromise between numerical restrictions and theoretical intentions. In this context, we carry over some ideas of positive definite functions (Micchelli 1986, Madych and Nelson 1988) to the spherical case, to have an appropriate tool for the decision, whether a given kernel is applicable or not.

To simplify our investigations, we first restrict ourselves to the scalar case, i. e. we consider only scalar spherical functions and data. This theory is immediately applicable to tensor functions of type $(1,1)$. To transfer these results to tensors with tangential parts, some straightforward modifications are necessary. Since some emphasis lies on the choice of the sequence $\{A_n\}$, we change our notations in an obvious way.

We start with an extension of Definition 3.1.

Definition B.1 Let $\{A_n\},\{B_n\} \in \mathcal{A}$. $\{A_n\}$ is called $\{B_n\}$*–summable of order* m, if $A_n \neq 0$ for $n \geq m+1$ and

$$\sum_{n=m+1}^{\infty}\frac{2n+1}{4\pi}\frac{B_n}{{A_n}^2} < \infty.$$

Furthermore, $\{A_n\}$ is called *summable of order* m, if $A_n \neq 0$ for $n \geq m+1$ and

$$\sum_{n=m+1}^{\infty}\frac{2n+1}{4\pi}\frac{1}{{A_n}^2} < \infty.$$

To stress the dependence on the sequence $\{A_n\}$, we denote by $\mathcal{H}(\{A_n\};\Omega)$ the completion of the space of all infinitely differentiable functions satisfying

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \mid A_n \mid^2 (F,Y_{n,j})^2_{\mathcal{L}^2} < \infty$$

with respect to the norm corresponding to the inner product

$$\begin{aligned}
&(F,G)_{\mathcal{H}(\{A_n\};\Omega)} = \\
&\quad\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \mid A_n \mid^2 (F,Y_{n,j})_{\mathcal{L}^2}(G,Y_{n,j})_{\mathcal{L}^2}.
\end{aligned}$$

If no confusion is likely to arise we write $\mathcal{H}$ instead of $\mathcal{H}(\{A_n\};\Omega)$.

We know (cf. Theorem 3.1) that, if $\{A_n\}$ is summable, each function of $\mathcal{H} = \mathcal{H}(\{A_n\};\Omega)$ corresponds to a continuous function in $C^{(0)}(\Omega)$ and that there exists a reproducing kernel in $\mathcal{H}(\{A_n\};\Omega)$. It is given by

$$K_{\mathcal{H}}(\xi,\eta) = K_{\mathcal{H}(\{A_n\};\Omega)}(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \mid A_n \mid^{-2} Y_{n,j}(\xi)Y_{n,j}(\eta);\ \xi,\eta\in\Omega.$$

The space $\mathcal{H}_m = \mathrm{Harm}_{0,\ldots,m}$, spanned by the set of all spherical harmonics of order $\leq m$, is a finite dimensional Hilbert space with inner product corresponding to the norm

$$\|F\|_{\mathcal{H}_m} = \left(\sum_{n=0}^{m}\sum_{j=1}^{2n+1} \mid A_n \mid^{2} (F,Y_{n,j})^2\right)^{\frac{1}{2}}$$

and the reproducing kernel

$$K_{\mathcal{H}_m}(\xi,\eta) = \sum_{n=0}^{m}\sum_{j=1}^{2n+1} \mid A_n \mid^{-2} Y_{n,j}(\xi)Y_{n,j}(\eta).$$

Let us denote by $\mathcal{H}_m^- = \mathcal{H}_m^-(\{A_n\};\Omega)$ the orthogonal complement of $\mathcal{H}_m^-$ in $\mathcal{H}(\{A_n\},\Omega)$. The linear space $\mathcal{H}_m^-(\{A_n\};\Omega)$ is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}_m^\perp}$ corresponding to the norm

$$\|F\|_{\mathcal{H}_m^\perp} = \left(\sum_{n=m+1}^{\infty}\sum_{j=1}^{2n+1} \mid A_n \mid^{2} (F,Y_{n,j})^2_{\mathcal{L}^2}\right)^{\frac{1}{2}}$$

and the reproducing kernel

$$K_{\mathcal{H}_m^\perp}(\xi,\eta) = \sum_{n=m+1}^{\infty}\sum_{j=1}^{2n+1} \mid A_n \mid^{-2} Y_{n,j}(\xi)Y_{n,j}(\eta).$$

Hence, $\mathcal{H}$ is the orthogonal direct sum of $\mathcal{H}_m$ and $\mathcal{H}_m^-$ with the inner product

$$(\cdot,\cdot)_{\mathcal{H}} = (\cdot,\cdot)_{\mathcal{H}_m} + (\cdot,\cdot)_{\mathcal{H}_m^\perp}$$

and the reproducing kernel

$$K_{\mathcal{H}}(\cdot,\cdot) = K_{\mathcal{H}_m}(\cdot,\cdot) + K_{\mathcal{H}_m^\perp}(\cdot,\cdot).$$

Before we can draw our attention on different kernels, we want to give conditions for the kernels to be suitable for spline interpolation. For this we introduce the concept of positive definite functions as mentioned before.

Definition B.2 Let $K:[-1,+1]\to\mathbb{R}$ be continuous. K is said to be *conditionally positive definite of order m*, if for any distinct (admissible) system $X_N = \{\eta_1,\ldots,\eta_N\}$ on Ω and scalars $a_1,\ldots,a_N$ satisfying

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \ldots & Y_{0,1}(\eta_N) \\ \vdots & & \vdots \\ Y_{m,2m+1}(\eta_1) & \ldots & Y_{m,2m+1}(\eta_N) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = 0\,, \tag{B.1}$$

the quadratic form

$$\sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l K(\eta_k\cdot\eta_l)$$

is nonnegative. K is said to be *conditionally strictly positive definite of order m* if the quadratic form is positive. Conditionally positive definiteness of order -1 is simply called conditionally positive definiteness. In this case, the condition (B.1) has to be omitted.

As an immediate consequence of Definition B.2 we obtain the following lemma.

Lemma B.1 *If $K \in C^{(0)}[-1,+1]$ is conditionally strictly positive definite, then K_m^- defined by*

$$K_m^-(t) = K(t) - \sum_{n=0}^{m}\frac{2n+1}{2}\left(\int_{-1}^{+1} K(u)P_n(u)du\right)P_n(t),$$

for $t\in[-1,+1]$ is conditionally strictly positive definite of order m.

Often it is not easy to determine whether a given continuous function K is conditionally strictly positive definite. A direct method is to guarantee one of the following equivalent conditions.

Theorem B.1 *Let $K:[-1,+1]\to\mathbb{R}$ be continuous. Then the following statements are equivalent:*

(i)

$$\int_\Omega\int_\Omega K_m^-(\xi\cdot\eta)\Phi(\xi)\Phi(\eta)d\omega(\xi)d\omega(\eta) > 0$$

for all $\Phi\in C^{(\infty)}(\Omega)$ with $(\Phi,Y_{n,j})_{\mathcal{L}^2}\neq 0$ for $n = m+1,m+2,\ldots; j=1,\ldots,2n+1$.

(ii)

$$\int_\Omega\int_\Omega K_m^-(\xi\cdot\eta)Y_{n,j}(\xi)Y_{n,j}(\eta)d\omega(\xi)d\omega(\eta) > 0$$

for all $Y_{n,j}$ with $n = m+1,m+2,\ldots; j = 1,\ldots,2n+1$.

(iii) K is conditionally strictly positive definite of order m.

(iv) The sequence $\{A_n\}$ with

$$A_n = \left(\frac{2n+1}{2}\int_{-1}^{+1} K_m^-(u)P_n(u)du\right)^{-1/2} \tag{B.2}$$

is summable of order m.

Proof: The statement (i) $\Rightarrow$ (ii) is clear. Therefore we turn to (ii) $\Rightarrow$ (iii). Assume that (ii) is true. Then it is obvious that

$$0 \leq r^{2n}\left(\sum_{k=1}^{N} a_k Y_{n,j}(\eta_k)\right)^2 \times \int_\Omega\int_\Omega K_m^-(\xi\cdot\eta)Y_{n,j}(\xi)Y_{n,j}(\eta)d\omega(\xi)d\omega(\eta)$$

for positive values r and coefficients $a_1,\ldots,a_N$ satisfying (B.1). But this shows that

$$0 \leq \int_\Omega\int_\Omega K_m^-(\xi\cdot\eta)\Phi_r(\xi)\Phi_r(\eta)d\omega(\xi)d\omega(\eta),$$

where

$$\Phi_r(\xi) = \sum_{l=1}^{N} a_l Q_r(\eta_j\cdot\xi), \xi\in\Omega$$

and Q_r is the Poisson kernel defined by

$$Q_r(t) = \frac{1}{4\pi}\frac{1-r^2}{(1+r^2-2rt)^{3/2}}.$$

Letting $r\to 1$ we see that K_m^- is conditionally strictly positive definite of order m. Next we prove (iii) $\Rightarrow$ (iv). Suppose that

$$0 \leq \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l\ K_m^-(\eta_k\cdot\eta_l)$$

for all $a_1,\ldots,a_N$ satisfying (B.1). Then, due to Aronszjain (1950), there corresponds one and only one class of functions forming a real Hilbert space (denoted by) $H_m^-(\{A_n\};\Omega)$ and admitting $K_m^-(\xi\cdot\eta), \xi,\eta\in\Omega$ as reproducing kernel. Obviously, the system $\{A_n^{-1}Y_{n,j}\}$, $n=m+1,m+2,\ldots;\ j=1,\ldots,2n+1$, with coefficients A_n given by (B.2) is complete in $H_m^-(\{A_n\};\Omega)$, i.e.

$$K_m^-(\xi\cdot\eta) = \sum_{n=m+1}^{\infty}\sum_{j=1}^{2n+1} \mid A_n\mid^{-2} Y_{n,j}(\xi)Y_{n,j}(\eta) \quad (B.3)$$

in the sense of uniform convergence for all $\xi,\eta\in\Omega$ with $-1\leq\xi\cdot\eta\leq +1$. Moreover, for all $\xi,\eta\in\Omega$,

$$|K_m^-(\xi\cdot\eta)| \leq (K_m^-(\xi\cdot\xi))^{1/2}(K_m^-(\eta\cdot\eta))^{1/2},$$

so that

$$\sup_{(\xi,\eta)\in\Omega^2} \mid K_m^-(\xi\cdot\eta)\mid \;\leq\; \sum_{n=m+1}^{\infty}\frac{2n+1}{4\pi}\frac{1}{A_n^2}P_n(1) = \sum_{n=m+1}^{\infty}\frac{2n+1}{4\pi}\frac{1}{A_n^2}.$$

But this means that $\{A_n\}$ is summable of order m. Finally we prove (iv) $\Rightarrow$ (i). Since $\{A_n\}$ is assumed to be summable of order m, the representation (B.3) is valid. Consequently,

$$\int_\Omega\int_\Omega K_m^-(\xi\cdot\eta)\Phi(\xi)\Phi(\eta)d\omega(\xi)d\omega(\eta) = \sum_{n=m+1}^{\infty}\sum_{n=1}^{2n+1}\frac{1}{A_n^2}(\Phi,Y_{n,j})^2_{\mathcal{L}^2} > 0$$

for all $\Phi\in C^{(\infty)}(\Omega)$ with $(\Phi,Y_{n,j})_{\mathcal{L}^2}\neq 0, n=m+1,m+2,\ldots; j=1,\ldots,2n+1$. This states Theorem B.1. ■

The last theorem gives conditions for the decision whether a given kernel is suitable for the spline interpolation method. As examples, we want to define some possible kernels and hence the corresponding Hilbert spaces which are defined by the underlying sequences. We will show examples of kernels of different types:

- classical Green functions with respect to Δ^* and its iterations
- kernels related to pseudodifferential operators
- multiquadric kernels.
- locally supported kernels ("axisymmetric finite elements")

Let us construct the Hilbert space $\mathcal{H}=\mathcal{H}(\{A_n\};\Omega)$ by choosing the inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\Omega)}$ corresponding to the norm $\|\cdot\|_{\mathcal{H}(\{A_n\};\Omega)}$, given by

$$\|F\|^2_{\mathcal{H}(\{A_n\};\Omega)} = \frac{1}{4\pi}\left|\int_\Omega F(\xi)Y_{0,1}(\xi)d\omega\right|^2 + \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}(\lambda_n)^2 \quad \left|\int_\Omega F(\xi)Y_{n,j}(\xi)d\omega\right|^2.$$

That means the sequence $\{A_n\}$ is given by

$$A_n = \begin{cases}\sqrt{\frac{1}{4\pi}} & n=0\\ -\lambda_n & n=1,2,\ldots\end{cases}$$

Obviously, $\{A_n\}$ is $\{(n+\frac{1}{2})^\tau\}$–summable for all $\tau<2$. For the reproducing kernel in $\mathcal{H}(\{A_n\};\Omega)$ we find

$$K_{\mathcal{H}}(\xi,\eta) = 1+\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{2n+1}{(\lambda_n)^2}P_n(\xi\cdot\eta).$$

Apart from an additive constant the series on the right hand side coincides with Green's function on the unit sphere corresponding to the differential operator $(\Delta^*)^2$. It can be seen (cf. Freeden and Gervens (1993)) that in this case the semi–norm which is minimized by the spline interpolant is related to the linearized bending energy of a thin beam.

As an example of kernels related to pseudodifferential operators on Ω, we consider $\{A_n\} = \{r^{-n/2}\Lambda_{(n)}\}$, $0 < r < 1$, where Λ is a pseudodifferential operator on Ω with symbol $(\Lambda_{(n)})$ such that

(i) $n \mapsto \Lambda_{(n)}$ is a real rational function

(ii) there exists a positive constant C' with

$$0 < \mid \Lambda_{(n)} \mid \leq C' \, (n + (\frac{1}{2}))^{\tau}$$

for some τ.

The norm is then given by

$$\|F\|^2_{\mathcal{H}} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{r^n} \Big| \int_\Omega \Lambda^2_{(n)} F(\xi) Y_{n,j}(\xi) d\omega \Big|^2$$

and the reproducing kernel in $\mathcal{H}(\{A_n\};\Omega)$ is

$$K_{\mathcal{H}}(\xi,\eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \Lambda^{-2}_{(n)} r^n P_n(\xi\cdot\eta).$$

The sequence $\{A_n\}$ is $\{(n+\frac{1}{2})^\tau\}$–summable for all τ. Moreover, different types of kernel representations are available in closed form (Magnus et al. 1966). We only mention here

(i) *Poisson kernel:* $\{A_n\} = \{r^{-n/2}\Lambda_{(n)}\}$ with $\Lambda_{(n)} = 1$, $n = 0, 1, \ldots$

$$K_{\mathcal{H}}(\xi,\eta) = \frac{1}{4\pi} \frac{1-r^2}{(L_r(\xi\cdot\eta))^{3/2}}$$

(ii) *Singularity kernel:* $\{A_n\} = \{r^{-n/2}\Lambda_{(n)}\}$ with $\Lambda_{(n)} = (n+\frac{1}{2})^{\frac{1}{2}}$, $n = 0, 1, \ldots$

$$K_{\mathcal{H}}(\xi,\eta) = \frac{1}{2\pi} \frac{1}{(L_r(\xi\cdot\eta))^{1/2}}$$

(iii) *Logarithmic kernel:* $\{A_n\} = \{r^{-n/2}\Lambda_{(n)}\}$ with $\Lambda_{(n)} = ((2n+1)(n+1))^{\frac{1}{2}}$, $n = 0, 1, \ldots$

$$K_{\mathcal{H}}(\xi,\eta) = \frac{1}{2\pi r} \ln\Big(1 + \frac{2r}{(L_r(\xi\cdot\eta))^{1/2} + 1 - r}\Big),$$

where we have used the abbreviation

$$L_r(\xi\cdot\eta) = 1 + r^2 - 2r(\xi\cdot\eta).$$

According to (2.18) and (3.9) the tensor spline interpolation method demands the knowledge of the series

$$G_{\mathbf{h}_m^{(k,l)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta) = \sum_{n=m+1}^{\infty} \frac{2n+1}{4\pi} \mu_n^{(i,k)} \Lambda^{-2}_{(n)} r^n P_n(\xi\cdot\eta),$$

where m is a positive integer $\geq o_{k,l}$. Observing the fact that $D_r(r^n) = n(n+1)r^n = \lambda_n r^n$, $D_r = \frac{d}{dr} r^2 \frac{d}{dr}$, we find

$$G_{\mathbf{h}_m^{(k,l)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta) = \begin{cases} G_{\mathbf{h}_m^{(1,1)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta), \\ \quad \text{if } (k,l) = (1,1) \\ D_r G_{\mathbf{h}_m^{(1,1)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta), \\ \quad \text{if } (k,l) \in \{(1,2),(2,1),(1,3),(3,1)\} \\ 2G_{\mathbf{h}_m^{(1,1)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta), \\ \quad \text{if } (k,l) \in \{(2,2),(3,3)\} \\ 2D_r(D_r - \lambda_1) G_{\mathbf{h}_m^{(1,1)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta), \\ \quad \text{if } (k,l) \in \{(2,3),(3,2)\}. \end{cases}$$

But this means that all $G_{\mathbf{h}_m^{(k,l)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta)$ are known
as elementary functions, if $G_{\mathbf{h}_m^{(1,1)}(\{r^{\perp n/2}\Lambda_{(n)}\};\Omega)}(\xi,\eta)$ is available as elementary function, which is of basic interest in numerical computations.

To define kernels of multiquadric type, suppose that H is in $C^{(0)}[0,\infty) \cap C^{(\infty)}(0,\infty)$ with

$$\begin{array}{rcl} (-1)^j H^{(j)}(\sqrt{t}) & \geq & 0,\ t > 0, j \geq m+1, \\ H^{(m+1)}(\sqrt{t}) & \neq & \text{const.} \end{array}$$

Then it can be readily seen from results due to Micchelli (1986) that

$$K : t \mapsto K(t) = H(\sqrt{2-2t}),\ t \in [-1,+1],$$

is a conditionally strictly positive function (of order m). Examples of functions H satisfying the aforementioned conditions include (cf. Dyn (1989)):

$$\begin{array}{rcl} H(t) & = & (-1)^{m+1} t^\beta,\ 2m < \beta < 2m+2 \\ H(t) & = & (-1)^{m+1} t^m \ln t \\ H(t) & = & (-1)^{m+1} (t^2+c^2)^{\beta/2}, \\ & & 2m < \beta < 2m+2, c > 0 \\ H(t) & = & (-1)^{m+1} (t^2+c^2)^m \ln (t^2+c^2)^{1/2}. \end{array}$$

These functions include the multiquadric for $m = 0$. For geophysical applications, kernels of multiquadric type have been used by Hardy (1983).

Finally, we turn to the "finite element" type of kernel functions mentioned before. Consider e.g. the auxiliary function $B_h^{(k)}$ defined by

$$B_h^{(k)}(t) = \begin{cases} 0 & \text{for } -1 \leq t \leq h \\ \frac{(t-h)^k}{(1-h)^k} & \text{for } h < t \leq 1 \end{cases}$$

for $k = 0, 1, \ldots$, $h \in [0,1)$ (other types of $B_h^{(k)}$ can be found in Cui et al. (1992)). Obviously, the system $\Big\{B_h^{(k)}(\eta_1\,\cdot\,\cdot), \ldots, B_h^{(k)}(\eta_N\,\cdot\,\cdot)\Big\}$, $\eta_i \neq \eta_k$ for $i \neq k$, forms a linearly independent system. Now let

$$L_h^{(k)}(\eta_i,\eta_k) = L_h^{(k)}(\eta_i\cdot\eta_k) = \int_\Omega B_h^{(k)}(\eta_i\cdot\xi) B_h^{(k)}(\eta_k\cdot\xi) d\omega(\xi)$$

for $i, k = 1, \ldots, N$. Hence, according to our construction, $t \mapsto L_h^{(k)}(t)$, $t \in [-1, 1]$, is conditionally strictly positive. Note that, for every $\xi \in \Omega$, $L_h^{(k)}(\xi \cdot .)$ is an axisymmetric locally supported function showing as support

$$\mathrm{supp} L_h^{(k)}(\xi \cdot .) = \{\eta \in \Omega | -1 + 2h^2 \leq \xi \cdot \eta \leq 1\}.$$

Thus band matrix solvers can be applied in spline interpolation (of order -1, i.e. when no polynomial exactness is required) which is of particular importance for numerical purposes.

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