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THE RICCI FLOW ON SURFACES

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In this paper we will discuss the evolution of a Riemannian metric g_{ij} on a compact surface M by its curvature R under the equation

$$* \qquad \frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij}$$

where r is the average value of R. We shall prove the following results.

1.1 **Theorem.** *For any initial data, the solution exists for all time. (See also Cao [1]).*

1.2 **Theorem.** *If* $r \le 0$, *the metric converges to one of constant curvature.*

1.3. **Theorem.** *If* $R > 0$, *the metric converges to one of constant curvature.*

Of course we conjecture that any metric on a compact surface converges to one of constant curvature, but the case of a metric in S^2 with curvature of varying sign is still open.

As consequences we obtain new proofs of many classical results, such as the uniformization theorem for Riemann surfaces, the topological classification of surfaces, and the topological type of the diffeomorphism group of surfaces. The proofs depend on

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some new a *priori* estimates on higher derivatives which are interesting in their own right. We hope these results will generalize to higher dimensional Kähler manifolds, and maybe throw light on the problem of S^2 necks pinching off on three manifolds with positive scalar curvature under the Ricci flow.

2. We have previously studied the flow of a metric by its Ricci curvature

$$\frac{\partial}{\partial t} g_{ij} = 2\left[\frac{r}{n} g_{ij} - R_{ij}\right]$$

on three-manifolds [3] and on four-manifolds [4]. In some repects the higher dimensional cases are easier, due to the information contained in the second Bianchi identity. For surfaces with positive curvature, the gradient estimate on the scalar curvature fails for this reason. Therefore a new approach is needed.

On a surface, the Ricci flow equation simplifies, because all of the information about curvature is contained in the scalar function R. In our notation, $R = 2K$ where K is the Gauss curvature, with $K = 1$ on the sphere of radius 1. Thus the Ricci curvature is given by

$$R_{ij} = \frac{1}{2} R\, g_{ij}$$

and the Ricci flow equation simplifies the following to the equation for the metric:

$$\frac{\partial}{\partial t} g_{ij} = (r-R) g_{ij}. \tag{2.1}$$

Notice that the change in the metric is pointwise a multiple of the metric, so the conformal structure is preserved. The term r in the equation is added to keep the area of the surface constant; if $\mu = \sqrt{\det g_{ij}}$ is the area element then

$$\frac{\partial}{\partial t} \mu = (r-R)\mu$$

and as a result, if A is the total area

$$\frac{d}{dt} A = \frac{d}{dt} \int 1 d\mu = \int (r-R) d\mu = 0$$

since r is the mean scalar curvature

$$r = \int R d\mu / \int 1 d\mu .$$

The integral of R over the surface M gives the Euler class $\chi(M)$ by the Gauss-Bonnet formula

$$\int R d\mu = 4\pi \chi(M)$$

and as a consequence, on a surface we see that r is constant; indeed

$$r = 4\pi \chi(M)/A.$$

We could choose to normalize with $A = 4\pi$ and $r = 2$ on the sphere, but we prefer not.

The equation 2.1 makes perfectly good sense in higher dimensions, but differs from the Ricci flow. It is in fact the gradient flow for the Yamabe problem, where we fix the conformal structure and the volume and try to minimize the mean scalar curvature r. Thus in higher dimensions r will decrease. We can prove that the solution exists for all time. When $R < 0$, the solution converges exponentially to a metric with constant scalar curvature. When $R > 0$, we can show the solution exists for all time, and the curvature approaches a constant; however there is some problem with the convergence of the metric. Presumably this problem could be overcome using the positive mass estimate.

3. When the metric g_{ij} evolves, so will its scalar curvature R. The equation for the evolution of the curvature on a surface is particularly simple and elegant. 3.1 The equation for the curvature:

$$\frac{\partial R}{\partial t} = \Delta R + R^2 - rR.$$

It can be found by a straightforward calculation. It has the following interpretation. Let B be any region on the surface with a smooth boundary curve ∂B, and let N be the unit normal to the boundary and λ the arc length measure along the boundary. Then

$$\frac{d}{dt} \int\int_B R d\mu = \int_{\partial B} \nabla R \cdot N \, d\lambda.$$

This says that the curvature R flows across any boundary curve with a speed equal to the negative of its gradient. This produces the usual Laplacian term for diffusion in the equation 3.1, while the quadratic self-interaction term $R(R-r)$ is due entirely to the fact that the curvature density R increases or decreases with the change in the area element μ, which changes by the factor $r-R$.

Applying the maximum principle to equation 3.1 gives the following result.

3.2 **Theorem.** *If $R \geq 0$ at the start, it remains so for all time. Likewise if $R \leq 0$ at the start, it remains so for all time. Thus both positive and negative curvature are preserved for surfaces. The same is true in the higher dimensional Yamabe flow, but not in the higher dimensional Ricci flow, where only positive curvature is preserved.*

For negative curvature, the preceeding result can be improved considerably. As an immediate consequence of the maximum principle we get the following result.

3.3 **Theorem.** *If $-C \leq R \leq -\varepsilon < 0$ at the start, then it remains so, and*

$$re^{-\varepsilon t} \leq r - R \leq Ce^{rt}$$

so R approaches r exponentially.

Proof. The maximum of R satisfies the differential inequality

$$\frac{d}{dt} R_{\max} \leq R_{\max}(R_{\max} - r) \leq -\varepsilon(R_{\max} - r)$$

while the minimum of R satisfies

$$\frac{d}{dt} R_{\min} \geq R_{\min}(R_{\min} - r) \geq r(R_{\min} - r).$$

3.4 **Corollary.** *On a compact surface, if $R < 0$ then the solution exists for all time and converges exponentially to a metric of constant negative curvature.*

For positive curvature, the situation is much worse, because $R = r$ is now a repulsive fixed point for the ordinary differential equation

$$\frac{dR}{dt} = R^2 - rR$$

and hence the reaction term in equation 3.1 is fighting the diffusion term. The best we can do is the following.

3.5 **Theorem.** *If* $r > 0$ *and* $R/r \geq c > 0$ *at the start, then* $c < 1$ *and for all time*

$$\frac{R}{r} \geq \frac{1}{1 + (\frac{1}{c} - 1)e^{rt}}.$$

This gives a positive lower bound which deteriorates to zero as $t \to \infty$. Notice that the corresponding upper bound goes to infinity in a finite time.

3.6 **Theorem.** *If* $r > 0$ *and* $R/r \leq C$ *at the start, then* $C > 1$ *and*

$$\frac{R}{r} \leq \frac{1}{1 - (1 - \frac{1}{C})e^{rt}}$$

at least for time

$$t < \frac{1}{r} \log \frac{c}{c - 1}.$$

4. To get results when $R > 0$ somewhere, we need better methods. The first step is to introduce the potential function f.

4.1 **Definition.** The potential f is the solution of the equation

$$\Delta f = R - r$$

with mean value zero.

Note we can always solve the equation since $R - r$ has mean value zero, and the solution is unique up to a constant, so we can make f have mean value zero. The potential function f satisfies a particularly simple equation.

4.2 The equation for the potential:

$$\frac{\partial f}{\partial t} = \Delta f + rf - b$$

where

$$b = \int |Df|^2 d\mu / \int 1\, d\mu$$

is a constant over space and a function only of time.

Proof. Since $\Delta f = R - r$, differentiating in time we compute

$$\Delta \frac{\partial f}{\partial t} = \Delta(\Delta f + rf)$$

which shows that

$$\frac{\partial f}{\partial t} = \Delta f + rf - b$$

for some number b which is a constant over space and a function only of time. It is easy to compute b from the relation

$$\int f\, d\mu = 0.$$

To continue the argument, we introduce a new function h and a tensor M_{ij}.

4.3 **Definition.** We let

$$h = \Delta f + |Df|^2$$

and

$$M_{ij} = D_i D_j f - \frac{1}{2} \Delta f \cdot g_{ij}.$$

Note that M_{ij} is the trace-free part of the second covariant derivative of f.

4.4 The equation for h:

$$\frac{\partial h}{\partial t} = \Delta h - 2|M_{ij}|^2 + rh.$$

4.5 **Corollary.** *If $h \le C$ at the start, then $h \le C\, e^{rt}$ for all time.*

The significance of this estimate is that

$$R = h - |Df|^2 + r$$

so $R \le C\, e^{rt} + r$. This gives a bound on R from above for all t, which unfortunately deteriorates as $t \to \infty$ if $r > 0$. We also have a lower bound from the maximum principle on equation 3.1. Not using the best possible result, we have the following. If $r \ge 0$ and the minimum of R is negative, it increases. If $r \le 0$ and the minimum of R is less than r, it increases. This proves the following estimate.

4.6 **Theorem.** For any initial metric on a compact surface, there is a constant C with

$$-C \le R \le C\, e^{rt} + r.$$

4.7 **Corollary.** *For any initial metric on a compact surface, the Ricci flow equation has a solution for all time.*

4.8 **Corollary.** *If in addition* $r \le 0$, *then the scalar curvature* R *remains bounded both above and below.*

When $r < 0$, this bound actually shows that $R < 0$ for large time. We can then apply Corollary 3.4. This proves the following.

4.9 **Theorem.** *On a compact surface with* $r < 0$, *for any initial metric the solution exists for all time and converges to a metric with constant negative curvature.*

5. The case $r = 0$ merits separate attention. We already know the solution exists for all time, and the curvature remains bounded above and below. It remains to see why the solution converges to a flat metric.

Let us write $g_{ij} = e^u \bar{g}_{ij}$ for a conformal change of metric. Then it is easy to compute

$$R = e^{-u}(\bar{R} - \bar{\Delta} u)$$

where $\bar{R}$ is the curvature of $\bar{g}_{ij}$ and $\bar{\Delta}$ is the Laplacian in the metric $\bar{g}_{ij}$. Given $\bar{g}_{ij}$, if we solve $\bar{\Delta}u = \bar{R}$ (which is possible since $\bar{R}$ has mean value zero) then $R = 0$. Hence we can produce a flat metric.

Let us assume now that $\bar{g}_{ij}$ is the flat metric, and study the evolution of g_{ij} by studying its conformal factor u. We easily derive the following equation.

5.1 The equation for the conformal factor

$$\frac{\partial u}{\partial t} = e^{-u}\,\bar{\Delta}u.$$

Applying the maximum principle, we have the following.

5.2 **Corollary.** *There exists a constant* C *with* $-C \le u \le C$. *As a result, the metrics* $g_{ij}(t)$ *are uniformly equivalent for all* t. *This gives us control of the diameter, the injectivity radius, and the constant in the Sobolev inequality.*

To produce some exponential convergence, we calculate

$$\frac{d}{dt}\int |\bar{D}u|^2\, d\bar{\mu} + 2\int e^{-u}(\bar{\Delta}u)^2\, d\bar{\mu} = 0$$

and use

$$\int (\bar{\Delta}u)^2 d\bar{\mu} \ge c \int |\bar{D}u|^2 d\bar{\mu}$$

and the bounds above and below on u to conclude that for some constant $c > 0$

$$\frac{d}{dt}\int |\bar{D}u|^2\, d\bar{\mu} + c\int |\bar{D}u|^2 d\bar{\mu} \le 0$$

so the integral goes to zero exponentially. Thus

$$\int |\bar{D}u|^2\, d\bar{\mu} \le C\, e^{-ct}$$

for some $c > 0$.

Now if we integrate the previous equation over time, we get

$$2\int_T^{\infty}\int e^{-u}(\bar{\Delta}u)^2\, d\bar{\mu}\, dt \le \int |\bar{D}u|^2\, d\bar{\mu}(T)$$

which shows that

$$\int_T^\infty \int R^2 d\mu dt \leq C\, e^{-cT}.$$

Hence at some point in each interval $T \leq t \leq T+1$ we will have

$$\int R^2 d\mu \leq C\, e^{-ct}.$$

Moreover

$$\frac{d}{dt}\int R^2 d\mu + 2\int |DR|^2 d\mu = \int R^3 d\mu.$$

Since R is bounded, we have

$$\frac{d}{dt}\int R^2 d\mu \leq C\int R^2 d\mu.$$

Now since the integral is frequently small, and its growth is controlled, it follows that

$$\int R^2 d\mu \leq C\, e^{-ct}$$

for all t. Since R is bounded, any L_p norm of R will go to zero exponentially. Then integrating the previous equation gives

$$\int_0^T \int |DR|^2 d\mu dt \leq C\, e^{-cT}$$

for some $c > 0$. This shows

$$\int |DR|^2 d\mu \leq C\, e^{-ct}$$

at least once in each interval $T \leq t \leq T+1$. But we can also bound the growth of this integral.

$$\frac{d}{dt}\int |DR|^2 d\mu + 2\int (\Delta R)^2 d\mu \leq -2\int R^2 \Delta R\, d\mu$$

and

$$-2\int R^2 \Delta R\, d\mu \leq \int (\Delta R)^2 d\mu + \int R^4 d\mu$$

so

$$\frac{d}{dt}\int |DR|^2 d\mu + \int (\Delta R)^2 d\mu \leq \int R^4 d\mu.$$

Since the last term is itself exponentially small, we get

$$\int |DR|^2 d\mu \le C\, e^{-ct}$$

for some $c > 0$ and all t. Then, again integrating over time

$$\int_0^T \int (\Delta R)^2 d\mu dt \le C\, e^{-cT}$$

so

$$\int (\Delta R)^2 d\mu \le C\, e^{-ct}$$

at least once in each interval $T \le t \le T+1$. Now we use the uniform Sobolev inequality to bound the maximum of R by the L^2 norm of R, DR, and D^2R; the bound on D^2R follows from that on ΔR and DR by integrating by parts

$$\int |D^2R|^2 d\mu = \int (\Delta R)^2 d\mu - \frac{1}{2} \int R\, |DR|^2 d\mu$$

and using the fact that R is bounded. Thus the maximum of R goes to zero exponentially. It now follows by a general result in [3] that the metric converges exponentially to the flat metric.

6. Now we turn our attention to the case where $R > 0$. First we prove a Harnack inequality on R by deriving a maximum principle estimate on the space and time derivatives of $\log R$. This was inspired by a similar proof of the Harnack inequality for the ordinary linear heat equation shown to us by S. T. Yau (see [5]). A similar Harnack estimate can be shown for the curvature of a plane curve moving by its mean curvature vector. We believe that similar Harnack inequalities will play an important role in many geometric problems.

The classical Harnack inequality says the following.

6.1 **Theorem.** *Let M be a compact manifold of dimension n with a fixed metric of non-negative Ricci curvature. Let f be a solution of the ordinary heat equation*

$$\frac{\partial f}{\partial t} = \Delta f$$

for $0 < t < T$ *with* $f > 0$ *everywhere. Then for any two points* (ξ,τ) *and* (X,T) *in space-time with* $0 < \tau < T$ *we have*

$$\tau^{n/2} f(\xi,\tau) \le e^{\Delta/4} T^{n/2} f(X,T)$$

where $\Delta = d(\xi,X)^2/(T-\tau)$ *and* $d(\xi,X)$ *is the distance along the shortest geodesic.*

Proof. Let $L = \log f$. Then

$$\frac{\partial L}{\partial t} = \Delta L + |DL|^2.$$

Next let $Q = \frac{\partial L}{\partial t} - |DL|^2 = \Delta L$ and compute

$$\frac{\partial Q}{\partial t} \ge \Delta Q + 2DL \cdot DQ + \frac{2}{n} Q^2$$

using

$$|D^2 L|^2 \ge \frac{1}{n}(\Delta L)^2 \text{ and } Rc(DL, DL) \ge 0.$$

It follows from the maximum principle that

$$Q \ge -2/nt$$

regardless of how negative Q might be to start.

Choose a geodesic path from ξ to X parametrized by time t proportionally to the arc length. Along the path

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial s}\frac{ds}{dt}.$$

Integrating along the path and using

$$\frac{\partial L}{\partial t} \ge |DL|^2 - \frac{2}{nt} \text{ and } |DL|^2 + \frac{\partial L}{\partial s}\frac{ds}{dt} \ge -\frac{1}{4}\left[\frac{ds}{dt}\right]^2,$$

by Cauchy-Schwartz we get

$$L(X,T) - L(\xi,\tau) = \int_\tau^T \frac{dL}{dt}\,dt$$

$$\geq \int_\tau^T \left\{ |DL|^2 - \frac{2}{nt} + \frac{\partial L}{\partial s}\frac{ds}{dt} \right\} dt$$

$$\geq -\frac{2}{n}\log(T/\tau) - \frac{1}{4}\int_\tau^T \left(\frac{ds}{dt}\right)^2 dt.$$

Now

$$\Delta = d(X,\xi)^2/(T-\tau) = \int_\tau^T \left(\frac{ds}{dt}\right)^2 dt$$

and the result follows by exponentiation.

We imitate the theorem and its proof for the Ricci flow on a surface. Since the metric is changing, we get a more complicated version of Δ.

6.2 **Definition.** On a manifold with a Riemannian metric $g_{ij}(x,t)$ which changes over time t, we define

$$\Delta(\xi,\tau,X,T) = \inf_\gamma \int_\tau^T \left(\frac{ds}{dt}\right)^2 dt$$

taking the infimum over all paths from (ξ,τ) to (X,T) parametrized by time t for $\tau \leq t \leq T$, where ds/dt is the velocity in space at time t.

This agrees with $\Delta(\xi,\tau,X,T) = d(\xi,X)^2/(T-\tau)$ when the metric is fixed. For a varying metric it gives a reasonable notion of distance between points at different times. If we have two fixed metrics $\gamma_{ij}(x)$ and $G_{ij}(x)$ independent of t, with distances $\delta(\xi,X)$ and $D(\xi,X)$ along geodesics, then clearly

$$\delta(\xi,X)^2/(T-\tau) \leq \Delta(\xi,\tau,X,T) \leq D(\xi,X)^2/(T-\tau)$$

whenever $\gamma_{ij}(x) \leq g_{ij}(x,t) \leq G_{ij}(x)$.

We can now state our new Harnack inequality.

6.3 **Theorem.** *Suppose we have a solution of the Ricci flow equation on a compact surface with $R > 0$ for $0 < t \le T$. Then for any two points (ξ, τ) and (X, T) in space-time with $0 < \tau < T$ we have*

$$(e^{r\tau} - 1) R(\xi, \tau) \le e^{\Delta/4}(e^{rT} - 1) R(X, T)$$

where $\Delta = \Delta(\xi, \tau, X, T)$ is defined as before.

Proof. Let $L = \log R$ and let

$$Q = \frac{\partial L}{\partial t} - |DL|^2 = \Delta L + R - r.$$

Then using

$$2\,|D_i D_j L - \frac{1}{2}(R - r) g_{ij}|^2 \ge Q^2$$

we compute

$$\frac{\partial Q}{\partial t} \ge \Delta Q + 2\nabla L \cdot \nabla Q + Q^2 + rQ.$$

It follows from the maximum principle that

$$Q \ge -r\, e^{rt} / (e^{rt} - 1)$$

no matter how negative Q is to start, by comparing to the solution of the ordinary differential equation we would get if Q were constant in space. Now take any path γ from (ξ, τ) to (X, T) parameterized by time t for $\tau \le t \le T$, and compute as before

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial s}\frac{ds}{dt}$$

$$L(X, T) - L(\xi, \tau) = \int_\tau^T \frac{dL}{dt}\, dt$$

$$\ge \int_\tau^T \left\{ |DL|^2 - \frac{re^{rt}}{e^{rt} - 1} + \frac{\partial L}{\partial s}\frac{ds}{dt} \right\} dt$$

$$\ge -\log\, (e^{rT} - 1)/(e^{r\tau} - 1) - \frac{1}{4}\int_\tau^T \frac{ds}{dt}^2 dt\,.$$

The infimum of the last integral over all such paths is the definition of Δ. The result follows by exponentiation.

7. The next step is rather unusual. We let

$$Z = \int Q\, R\, d\mu / \int R\, d\mu$$

and compute using the equation for Q that

$$\frac{dZ}{dt} \geq Z^2 + rZ \,.$$

Now if Z were ever to become positive, then it would blow up to infinity in a finite time. But we already know that the solution exists for all time. The only possible conclusion is that $Z \leq 0$! Using

$$Q = \frac{\Delta R}{R} - \frac{|DR|^2}{R^2} + (R - r)$$

this gives us the following result.

7.1 **Lemma.** *For any solution with* $R > 0$ *we have*

$$\int (R-r)^2 d\mu \leq \int \frac{|DR|^2}{R} d\mu.$$

Since

$$\frac{d}{dt}\int R \log R\, d\mu = \int (R-r)^2 d\mu - \int \frac{|DR|^2}{R} d\mu$$

we get the following suprising result.

7.2 **Theorem.** *For the Ricci flow on a compact surface with* $R > 0$ *the integral*

$$\int R \log R\, d\mu$$

is decreasing.

Note that $x \log x \geq -1/e$ is bounded below. Therefore R cannot be large except on a set where $\int R\, d\mu$ is small, which is a set whose Gauss image is small. Therefore this estimate by itself precludes the formation of a cone-like singularity. We think of this estimate as a statement about entropy. Since the integral of R is constant, it represents a probability measure. Then the integral of $R \log R$ is the negative of the entropy. The estimate says that entropy is increasing.

8. We now combine the Harnack inequality and the entropy estimate to conclude that R is bounded. Pick a point ξ at time τ where the curvature R is largest. Then wait for a time $T - \tau = 1/2\, R_{\max}(\tau)$. During that time

$$\frac{d}{dt} R_{\max} \leq R_{\max}^2$$

and so $R_{\max}(T) \leq 2 R_{\max}(\tau)$. On the other hand

$$\frac{\partial}{\partial t} g_{ij} = (r - R)\, g_{ij}$$

so distances will grow at most by a constant factor (since $R_{\max} \geq r$ the time interval is bounded). Hence if $d(\xi, X)$ is the geodesic distance at time T we will have

$$\Delta(\xi, \tau, X, T) \leq C\, d(\xi, X)^2 / (T - \tau).$$

Then (again using the bound on the time period, and assuming $\tau \geq 1$) the Harnack inequality gives

$$R(\xi, \tau) \leq C\, R(X, T)$$

for all X in a ball around ξ of radius

$$\rho = \pi / \sqrt{R_{\max}(T)/2}\,.$$

On the other hand, if our surface is oriented then Theorem 5.9 in Cheeger and Ebin [2] tells us that the injectivity radius of M is at least ρ at time T. (If M is not oriented, pass to the double cover.) In the ball of radius ρ around ξ we have R comparable to $R(\xi, \tau) = R_{\max}(\tau)$, which is at least half of $R_{\max}(T)$ by the choice of T. Therefore if we integrate over the ball B of radius ρ around at time T

$$\int_B R \log R \, d\mu \geq c \log R_{\max}(T)$$

at time T for some $c > 0$. Then the entropy estimate shows $R_{\max}(T)$ is bounded, and hence $R_{\max}(\tau)$ is bounded. This is true for all $\tau \geq 1$, so R is bounded.

Once R is bounded, we get as before a lower bound on the injectivity radius, and since the volume is bounded this gives an upper bound on the diameter. Using the diameter bound, plus the fact that the growth of distances is bounded, we easily get for

$T - \tau \leq 1$ the estimate

$$\Delta(\xi, \tau, X, T) \leq C/(T-\tau)$$

and hence the Harnack inequality tells us that for $t \geq 1$ and any two points x and y we have

$$R(x,t) \leq C\,R(y,t+1).$$

Therefore we also get a lower bound on R.

8.1 **Theorem.** *If we have a solution to the Ricci flow equation with $R > 0$ on a compact surface, then there exist constants $c > 0$ and $C < \infty$ with $0 < c \leq R \leq C$ for all time.*

8.2 **Corollary.** *All of the derivatives of the curvature remain bounded for all time also.*

Proof. Having control of the diameter, the volume, and the injectivity radius, we can control the Sobolev constant. From [3], Theorem 13.4, we have

$$\frac{d}{dt}\int |D^n R|^2 d\mu + 2\int |D^{n+1} R|^2 d\mu \leq C_n \int |D^n R|^2 d\mu$$

and using

$$\int |D^n R|^2 d\mu \leq \varepsilon \int |D^{n+1} R|^2 d\mu + C_n(\varepsilon)\int R^2 d\mu$$

we deduce

$$\frac{d}{dt}\int |D^n R|^2 d\mu + \int |D^n R|^2 d\mu \leq C_n$$

which shows that

$$\int |D^n R|^2 d\mu \leq C_n$$

for all n and all time. Then the Sobolev inequality gives supremum bounds for all derivatives.

9. We apply the lower bound on R to the evolution equation for M_{ij}. Recall from section 4 that we chose the potential function f to solve $\Delta f = R - r$, and let

$$M_{ij} = D_i D_j f - \frac{1}{2}\Delta f \cdot g_{ij}.$$

9.1 The equation for $|M_{ij}|^2$:

$$\frac{\partial}{\partial t}|M_{ij}|^2 = \Delta |M_{ij}|^2 - 2|D_k M_{ij}|^2 - 2R|M_{ij}|^2.$$

This follows from a straightforward calculation.

9.2 **Corollary.** *If* $R \geq c > 0$ *then*

$$|M_{ij}| \leq C\, e^{-ct}$$

for some constant C. *Hence* $M_{ij} \to 0$ *exponentially. This follows from the maximum principle.*

Next we consider a modification of the Ricci flow. Consider the equation

$$\frac{\partial}{\partial t} g_{ij} = 2M_{ij} = (r-R)g_{ij} - 2D_iD_jf.$$

This equation differs from the Ricci flow only by transport along a one parameter family of diffeomorphisms generated by the gradient vector field of the potential f. Since M_{ij} converges to zero exponentially, the modified metrics will converge as $t \to \infty$. We shall show that their derivatives also converge, and the limiting metric is smooth.

First note that the bound $0 < c \leq R \leq C$ on R still holds for the modified flow, since it differs only by a diffeomorphism. Next note that the metrics $g_{ij}(x,t)$ are all equivalent, since they converge. Then to prove convergence of the $g_{ij}(x,t)$ as $t \to \infty$, it suffices to show that all the covariant derivatives of M_{ij} go to zero exponentially.

To obtain the higher derivatives bounds on M_{ij} it is convenient to switch to complex notation. This happens by viewing the surface as a Kähler manifold of complex dimension one. On a Kähler manifold each real tensor, say T_{ij}, has complex components $T_{\alpha\beta}, T_{\alpha\bar\beta}, T_{\bar\alpha\beta}, T_{\alpha\bar\beta}$ chosen so that if $T = T_{ij}\,dx^i\,dx^j$ in real coordinates then

$$T = T_{\alpha\beta}\,dz^\alpha\,dz^\beta + T_{\alpha\bar\beta}\,dz^\alpha\,d\bar z^\beta + T_{\bar\alpha\beta}\,d\bar z^\alpha\,dz^\beta + T_{\alpha\bar\beta}\,d\bar z^\alpha\,d\bar z^\beta$$

in complex coordinates. Since we have only one dimension, all unbarred indices may be interchanged, as may all bared ones. Moreover any pair of a bared and unbarred index may be contracted. This allows us to represent any tensor as equivalent to several fully symmetric tensors with all indices unbarred or barred. These correspond to the irreducible representations of the Lie group S^1. We say that a tensor with k unbarred indices has weight k, whereas one with k barred indices has weight $-k$. We can then drop the indices, and regard the tensor of weight k as a complex function on the principal tangent bundle. The exterior derivatives of a tensor T are given by DT and $\bar{D}T$, where if T has weight k then DT has weight $k+1$ and $\bar{D}T$ has weight $k-1$. For example, if $T = T_{\alpha\beta}$ then $DT = D_\alpha T_{\beta\gamma}$ and $\bar{D}T = g^{\bar{\alpha}\beta} D_{\bar{\alpha}} T_{\beta\gamma}$

It is now easy to derive the following formulas for a tensor T of weight k.

9.3 Formulas on a surface

$$D\bar{D}T - \bar{D}DT = -\frac{1}{2}kRT$$

$$\Delta T = D\bar{D}T + \bar{D}DT$$

$$D\Delta T = \Delta DT - \frac{1}{2}k\,DR\cdot T - \left(k+\frac{1}{2}\right)R\,DT$$

$$\frac{\partial}{\partial t}DT = D\frac{\partial}{\partial t}T + \frac{1}{2}k\,DR\cdot T$$

$$\frac{\partial}{\partial t}|T|^2 = \bar{T}\frac{\partial}{\partial t}T + T\frac{\partial}{\partial t}\bar{T} + k(R-r)|T|^2.$$

Using these, we compute

$$\frac{\partial f}{\partial t} = \Delta f + rf - b$$

$$\frac{\partial}{\partial t}Df = \Delta Df - \frac{1}{2}R\,Df + r\,Df$$

$$\frac{\partial}{\partial t}D^2f = \Delta D^2f - 2R\,D^2f + r\,D^2f$$

$$\frac{\partial}{\partial t}D^3f = \Delta D^3f - 2DR\cdot D^2f - \frac{9}{2}R\,D^3f + r\,D^3f$$

$$\frac{\partial}{\partial t}D^4f = \Delta D^4f - 2D^2R\cdot D^2f - \frac{13}{2}DR\cdot D^3f - 8R\,D^4f + r\,D^4f$$

and consequently

$$\frac{\partial}{\partial t}|D^2f|^2 \le \Delta |D^2f|^2 - 2R\ |D^2f|^2$$

$$\frac{\partial}{\partial t}|D^3f|^2 \le \Delta |D^3f|^2 - (6R+r)|D^3f|^2 + 4\,|DR|\,|D^2f|\,|D^3f|$$

$$\frac{\partial}{\partial t}|D^4f|^2 \le \Delta |D^4f|^2 - (12R+2r)|D^4f|^2$$

$$+4\,|D^2R|\,|D^2f|\,|D^4f| + 13\,|DR|\,|D^3f|\,|D^4f|$$

and so on.

The importance here for us is just that D^2f is the complex form of

$$M_{ij} = D_iD_jf - \frac{1}{2}\Delta f \cdot g_{ij}$$

the trace-free part of the second covariant derivative. We already know that since $R \ge C > 0$ we have $|D^2f|$ going to zero exponentially. It now follows easily since $|DR|$ is bounded that $|D^3f|$ goes to zero exponentially; and since $|D^2R|$ is bounded we also have $|D^4f|$ going to zero exponentially. In fact all $|D^kf|$ go to zero exponentially.

However, if we integrate by parts we can bound the L^2 norms of $|D^k\,\overline{D}^l\,D^2f|$ in terms of the L^2 norms of $|D^{m+2}f|$. Hence all the derivatives of D^2f go to zero in L^2. Since we have uniform control of the Sobolev constant independent of time, we therefore have all the derivatives of $M_{ij} = D^2f$ going to zero exponentially in the supremum norm as $t \to \infty$. This proves that the solution of the modified equation converges exponentially to a limit metric with $M_{ij} = 0$.

10. A metric g_{ij} with $M_{ij} = 0$ is a soliton solution for the Ricci flow. It moves only by a diffeomorphism, so its shape remains unchanged. We shall show that on a compact surface there are no soliton solutions with $R > 0$, other than constant curvature where there is no motion. On a non-compact surface solitons with $R > 0$ do exist. For example, on the plane the metric

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

is a soliton with $R > 0$ which flows by conformal dilation. It is asymptotic to a flat cylinder at infinity, with maximum curvature at the origin.

10.1 **Theorem.** *On a compact surface there are no soliton solutions other than constant curvature.*

Proof. A soliton solution for the Ricci flow consists of a metric g_{ij} and a vector field v_i such that g_{ij} flows along v_i under the Ricci deformation. This happens when

$$2\left[R_{ij} - \frac{r}{n} g_{ij}\right] = D_i v_j + D_j v_i$$

and on a surface this simplifies to

$$(R - r) g_{ij} = D_i v_j + D_j v_i .$$

A soliton of the Ricci flow has

$$\frac{\partial R}{\partial t} = \Delta R + R(R - r)$$

and for a soliton the minimum value of R is constant. Then at that point $\partial R / \partial t = 0$. This shows that $R \geq 0$, and by the strong maximum principle we even have $R > 0$. From the evolution of the tensor $|M_{ij}|^2$ in 9.1 we see from the maximum principle that if a soliton has $R > 0$ then it must have $M_{ij} = 0$. For at the point where $|M_{ij}|^2$ is largest the equation says it should decrease, while if it flows by a diffeomorphism the maximum is unchanged. Therefore the vector field v_i along which the soliton flows must be $D_i f$, the gradient of the potential function f.

The vector field v_i must be conformal, since flowing along v_i changes the metric g_{ij} conformally. Now very few conformal vector fields can be gradients of a function. In complex coordinates the conformal vector field is holomorphic, and hence is locally given by $v(z)\partial/\partial z$ for a holomorphic function $v(z)$. At a zero of v there will be a power series expansion

$$v(z) = a\,z^p + \ldots \qquad (a \neq 0)$$

and if $p > 1$ the vector field will have closed orbits in any neighborhood of zero. Now a gradient flow cannot have a closed orbit. Hence v has only simple zeros, and f is a Morse function on the surface with only maxima and minima, and no saddles. Then there is one maximum and one minimum, and the surface is a union of two discs, thus a sphere.

Consider a soliton solution on the sphere S^2. The gradient of f must be a holomorphic vector field. Then it has exactly two zeros, which we can take to be at 0 and ∞. If we take $z = u + iv$ as complex coordinate, the holomorphic vector field must be $c\,z\,\partial/\partial z$ for some complex number c.

10.2 **Lemma.** *If $c\,z\,\partial/\partial z$ is a gradient vector field then c is real.*

Proof. Write the metric as

$$ds^2 = g(u,v)\,(du^2 + dv^2).$$

Then $\nabla f = c\,z\,\partial/\partial z$ means that if $c = a + bi$ then

$$\frac{\partial f}{\partial u} = (au - bv)g \qquad \frac{\partial f}{\partial v} = (bv + au)g.$$

Taking the mixed partials $\partial^2 f/\partial u\,\partial v$ and equating them at the origin $u = v = 0$ gives $b = 0$, so c is real.

Consequently our soliton is defined on the cylinder and moves by translation down the cylinder. Let x and y be coordinates on the cylinder, with translation in the x direction being the flow and identifying $y \to y + 2\pi$. Since the gradient of f is just $a\,\partial/\partial x$ for a real constant a, if the metric is given by

$$ds^2 = g(x,y)\,(dx^2 + dy^2)$$

we get the equations

$$\frac{\partial f}{\partial x} = ag \qquad \frac{\partial f}{\partial y} = 0 .$$

The second shows that $f = f(x)$ is a function of x only, and then the first shows that $g = g(x)$ is also a function of x only, if $a \neq 0$. (If $a = 0$ then f is constant and R is constant.)

Consider a metric on the cylinder regarded as a quotient of the xy plane by $y \to y + 2\pi$, and independent of y. Then we can write the metric as

$$ds^2 = g(x)(dx^2 + dy^2).$$

The condition that the metric extend as $x \to -\infty$ to a metric on the plane given by

$$u = e^x \cos y$$

$$v = e^x \sin y$$

is that $g(x)$ be a smooth function of e^{2x}, with no constant term. For

$$ds^2 = g(x) e^{-2x} (du^2 + dv^2)$$

and so $e^{-2x} g(x)$ must be a smooth function of $u^2 + v^2 = e^{2x}$. Likewise the condition that the metric extend as $x \to +\infty$ is that $g(x)$ be a smooth function of e^{-2x} with no constant term.

The curvature of the metric is given by

$$R = -\frac{1}{g}\left[\frac{g'}{g}\right]'$$

as can be easily computed, where prime denotes differentiation in x. If g is a soliton with velocity c moving by translation in x, then $g = g(x + ct)$ satisfies

$$\frac{\partial g}{\partial t} = (r - R)g$$

which becomes

(10.3) $$c\, g' = rg + \left[\frac{g'}{g}\right]' .$$

To solve this, we substitute $g = v'$, where v is determined up to a constant. By including the constant in v, this integrates to

$$v'' - C\, v'^2 + r\, v\, v' = 0 \tag{10.4}$$

which can be integrated again to

$$v' = \frac{r}{c} v + \frac{r}{c^2}\left[1 - ke^{cv}\right] \tag{10.5}$$

with an arbitrary constant k. Now if we substitute $y = cv + 1$ and $u = rx/c$ we get the equation

$$\frac{dy}{du} = y - ke^{y-1}$$

whose solutions are given by

$$u = \int \frac{dy}{y - ke^{y-1}}\,. \tag{10.6}$$

Now suppose we have a solution $g(x)$ where as $x \to -\infty$ we have an expansion

$$g(x) = b\, e^{\lambda x} + c\, e^{2\lambda x} + \ldots$$

in powers of $e^{\lambda x}$ with no constant term. Then $v(x)$ will have an expansion

$$v(x) = a + \lambda b\, e^{\lambda x} + 2\lambda c\, e^{\lambda x} + \ldots$$

and hence be bounded as $x \to -\infty$. Likewise if $g(x)$ has an expansion

$$g(x) = b\, e^{-\mu x} + c\, e^{-2\mu x} + \ldots$$

as $x \to +\infty$ then $v(x)$ will be

$$v(x) = a - b\mu e^{-\mu x} - 2c\mu e^{-2\mu x} - \ldots$$

and hence will also be bounded as $x \to +\infty$. If v is bounded so is y, while $u \to \pm\infty$ when x does. Therefore the denominator in the previous integral must have two zeros. This happens precisely when $0 < k < 1$.

Suppose then that $0 < k < 1$ and that the equation

$$y = k\, e^{y-1}$$

has two solutions $y = k - p < 1$ and $y = 1 + q > 1$.

10.8 **Lemma.** *We always have* $p < q$, *and* $p/q \to 1$ *as* $k \to 1$, *while* $p/q \to 0$ *as* $k \to 0$.

Proof. As $k \to 0$, $p \to 1$ and $q \to \infty$, so $p/q \to 0$. In general

$$e^{p+q} = \frac{1+q}{1-p}$$

and since

$$\frac{e^x - 1 - x}{x^2} \to \frac{1}{2} \text{ as } x \to 0$$

letting $x = p + q$, using the previous expression, and using $p \to 0$ and $q \to 0$ as $k \to 1$, we get

$$\frac{e^{(p+q)} - 1 - (p+q)}{(p+q)^2} = \frac{p}{(1-p)(p+q)} \to \frac{1}{2} \text{ as } k \to 1$$

which shows $p/q \to 1$. Finally since

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \ldots$$

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \ldots$$

we can see that

$$e^{2x} < \frac{1+x}{1-x} \quad \text{for} \quad x > 0.$$

This means that $p \neq q$ for $0 < k < 1$. Since $p/q \to 0$ as $k \to 0$, we must always have $p/q < 1$ by continuity.

To return to our discussion, for each value of R between 0 and 1 we get a soliton solution on the cylinder with bounded diameter. We now examine the asymptotics as $x \to \pm\infty$. Near $y = 1 - p$ write $y = 1 - p + z$. Then we have a power series expansion

$$y - ke^{y-1} = pz - \frac{1}{2}(1-p)z^2 + \ldots$$

which integrates to an expansion of

$$u = \int \frac{dy}{y - ke^{y-1}} = \int \frac{dz}{pz - \frac{1}{2}(1-p)z^2 + \ldots}$$

starting out as

$$u = \frac{1}{p} \log z + \ldots$$

which gives in turn an expansion of z in powers of e^{pu}. This in turn gives an expansion of $g(x)$ in powers of $e^{\lambda x}$ with $\lambda = rp/c$ as $x \to -\infty$. Likewise we get an expansion of $g(x)$ in powers of $e^{-\mu x}$ where $\mu = rq/c$ as $x \to +\infty$. Then $\lambda/\mu = p/q$.

By an appropriate choice of k we can make the ratio $\lambda/\mu = p/q$ any number with $0 < \lambda/\mu < 1$. Then by an appropriate choice of the velocity c we can make λ and μ any numbers we want with $0 < \lambda < \mu$. Notice that we do not attain $\lambda = \mu$ except in the limiting case where $c = 0$. Thus to get a solution on S^2 we need $\lambda = \mu = 2$, which makes the velocity $c = 0$, so we just have the constant curvature solutions. The other solutions we have found exist on orbifolds.

10.9 **Corollary.** *On a compact surface with $R > 0$ the heat flow*

$$\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij}$$

converges exponentially to a constant curvature metric.

Proof. For the modified flow we have seen that the curvature R converges to its limiting value exponentially. But since there are no soliton solutions on S^2, we must have R converging to the constant r exponentially. This then implies that the unmodified flow will also converge exponentially.

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