Appendix A
Spin-Weighted Spherical Harmonic Function

Here, we review the properties of the spin-weighted spherical harmonic function. In the past, this was mainly applied to the analysis of the gravitational wave (see e.g. Ref. [1]). This discussion is based on Refs. [2–4].

The spin-weighted spherical harmonic function on 2D sphere, ${}_sY_{lm}(\theta,\phi)$, is more general expression than the ordinary spherical harmonic function, $Y_{lm}(\theta,\phi)$, and has additional $U(1)$ symmetry characterized by a spin weight s. The spin-s function such as ${}_sY_{lm}(\theta,\phi)$ obeys the spin raising and lowering rule as $(\eth\, {}_sf)' = e^{-i(s+1)\psi}\eth\, {}_sf$ and $(\bar{\eth}\, {}_sf)' = e^{-i(s-1)\psi}\bar{\eth}\, {}_sf$. Here, the spin raising and lowering operators are given by

$$\begin{aligned} \eth\, {}_sf(\theta,\phi) &= -\sin^s\theta\left[\partial_\theta + i\csc\theta\,\partial_\phi\right]\sin^{-s}\theta\, {}_sf(\theta,\phi)\ , \\ \bar{\eth}\, {}_sf(\theta,\phi) &= -\sin^{-s}\theta\left[\partial_\theta - i\csc\theta\,\partial_\phi\right]\sin^{s}\theta\, {}_sf(\theta,\phi)\ , \end{aligned} \tag{A.1}$$

Specifically, the spin raising and lowering operators acting twice on the spin-±2 function ${}_{\pm2}f(\mu,\phi)$ such as the CMB polarization fields can be expressed as

$$\begin{aligned} \bar{\eth}^{\,2}\, {}_2f(\theta,\phi) &= \left(-\partial_\mu + \frac{m}{1-\mu^2}\right)^2\left[(1-\mu^2)\, {}_2f(\mu,\phi)\right]\ , \\ \eth^{\,2}\, {}_{-2}f(\theta,\phi) &= \left(-\partial_\mu - \frac{m}{1-\mu^2}\right)^2\left[(1-\mu^2)\, {}_{-2}f(\mu,\phi)\right]\ , \end{aligned} \tag{A.2}$$

where $\mu \equiv \cos\theta$ and ${}_{\pm2}f(\theta,\phi) = {}_{\pm2}\tilde{f}(\mu)e^{im\phi}$. Utilizing these properties, we can express ${}_sY_{lm}(\theta,\phi)$ in terms of ${}_0Y_{lm}(\theta,\phi) = Y_{lm}(\theta,\phi)$ as

$$\begin{aligned} {}_sY_{lm}(\theta,\phi) &= \left[\frac{(l-s)!}{(l+s)!}\right]^{\frac{1}{2}}\eth^{\,s}Y_{lm}(\theta,\phi)\ \ (0\le s\le l)\ , \\ {}_sY_{lm}(\theta,\phi) &= \left[\frac{(l+s)!}{(l-s)!}\right]^{\frac{1}{2}}(-1)^s\bar{\eth}^{\,-s}Y_{lm}(\theta,\phi)\ \ (-l\le s\le 0)\ , \end{aligned} \tag{A.3}$$

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Table A.1 Dipole ($l = 1$) harmonics for spin-0 and 1

m	Y_{1m}	${}_1Y_{1m}$
± 1	$-m\sqrt{\frac{3}{8\pi}}\sin\theta e^{mi\phi}$	$-\frac{1}{2}\sqrt{\frac{3}{4\pi}}(1 - m\cos\theta)e^{mi\phi}$
0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\sqrt{\frac{3}{8\pi}}\sin\theta$

Table A.2 Quadrupole ($l = 2$) harmonics for spin-0 and 2

m	Y_{2m}	${}_2Y_{2m}$
± 2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{mi\phi}$	$\frac{1}{8}\sqrt{\frac{5}{\pi}}\left(1 - \frac{m}{2}\cos\theta\right)^2 e^{mi\phi}$
± 1	$-m\sqrt{\frac{15}{8\pi}}\sin\theta\,\cos\theta\, e^{mi\phi}$	$-\frac{1}{4}\sqrt{\frac{5}{\pi}}\sin\theta\,(1 - m\cos\theta)\, e^{mi\phi}$
0	$\frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1)$	$\frac{3}{4}\sqrt{\frac{5}{6\pi}}\sin^2\theta$

where these equations contain

$$\begin{aligned}
\eth\, {}_sY_{lm}(\theta,\phi) &= [(l-s)(l+s+1)]^{\frac{1}{2}}\, {}_{s+1}Y_{lm}(\theta,\phi)\ , \\
\bar{\eth}\, {}_sY_{lm}(\theta,\phi) &= -\,[(l+s)(l-s+1)]^{\frac{1}{2}}\, {}_{s-1}Y_{lm}(\theta,\phi)\ , \\
\bar{\eth}\,\eth\, {}_sY_{lm}(\theta,\phi) &= -(l-s)(l+s+1)\,{}_sY_{lm}(\theta,\phi)\ .
\end{aligned} \tag{A.4}$$

These properties reduce to an explicit expression:

$$\begin{aligned}
{}_sY_{lm}(\theta,\phi) = e^{im\phi}\left[\frac{(l+m)!(l-m)!}{(l+s)!(l-s)!}\frac{2l+1}{4\pi}\right]^{1/2}\sin^{2l}(\theta/2) \\
\times \sum_r \binom{l-s}{r}\binom{l+s}{r+s-m}(-1)^{l-r-s+m}\cot^{2r+s-m}(\theta/2)\ .
\end{aligned} \tag{A.5}$$

This holds the orthogonality and completeness conditions as

$$\begin{aligned}
\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, {}_sY^*_{l'm'}(\theta,\phi)\, {}_sY_{lm}(\theta,\phi) = \delta_{l',l}\delta_{m',m}\ , \\
\sum_{lm} {}_sY^*_{lm}(\theta,\phi)\, {}_sY_{lm}(\theta',\phi') = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta').
\end{aligned} \tag{A.6}$$

The reactions to complex conjugate and parity transformation are given by

$$\begin{aligned}
{}_sY^*_{lm}(\theta,\phi) &= (-1)^{s+m}\, {}_{-s}Y_{l-m}(\theta,\phi)\ , \\
{}_sY_{lm}(\pi-\theta,\phi+\pi) &= (-1)^l\, {}_{-s}Y_{lm}(\theta,\phi)\ .
\end{aligned} \tag{A.7}$$

Finally, we give the specific expressions for some simple cases in Tables A.1 and A.2.

Appendix B
Wigner D-matrix

Here, on the basis of Refs. [3, 5, 6], we introduce the properties of the Wigner D-matrix $D^{(l)}_{mm'}$, which is the unitary irreducible matrix of rank $2l+1$ that forms a representation of the rotational group as $SU(2)$ and $SO(3)$. With this matrix, the change of the spin wighted spherical harmonic function under the rotational transformation as $\hat{\mathbf{n}} \to R\hat{\mathbf{n}}$ is expressed as

$$ {}_sY^*_{lm}(R\hat{\mathbf{n}}) = \sum_{m'} D^{(l)}_{mm'}(R) {}_sY^*_{lm'}(\hat{\mathbf{n}}) \ . \tag{B.1}$$

This satisfies the relation as

$$D^{(l)*}_{mm'}(R) = (-1)^{m-m'} D^{(l)}_{-m,m'}(R) = D^{(l)}_{m'm}(R^{-1}) \ . \tag{B.2}$$

When we express the rotational matrix with three Euler angles (α, β, γ) under the $z-y-z$ convention as

$$R = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\beta\sin\gamma\cos\alpha - \cos\gamma\sin\alpha & \cos\alpha\sin\beta \\ \cos\alpha\sin\gamma + \cos\gamma\cos\beta\sin\alpha & \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & \sin\beta\sin\alpha \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}, \tag{B.3}$$

we can write a general relationship between the Wigner D-matrix and the spin weighted spherical harmonics as

$$D^{(l)}_{ms}(\alpha, \beta, \gamma) = (-1)^s \sqrt{\frac{4\pi}{2l+1}} {}_{-s}Y^*_{lm}(\beta, \alpha) e^{-is\gamma} \ . \tag{B.4}$$

Like the spin weighted spherical harmonics, there also exists the orthogonality of the Wigner D-matrix as

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$$\int_0^{2\pi} d\alpha \int_{-1}^{1} d\cos\beta \int_0^{2\pi} d\gamma\, D^{(l')*}_{m's'}(\alpha,\beta,\gamma) D^{(l)}_{ms}(\alpha,\beta,\gamma) = \frac{8\pi^2}{2l+1}\delta_{l',l}\delta_{m',m}\delta_{s',s}\ . \tag{B.5}$$

Appendix C
Wigner Symbols

Here, we briefly review the useful properties of the Wigner-$3j$, $6j$ and $9j$ symbols. The following discussions are based on Refs. [5, 7–12].

C.1 Wigner-$3j$ symbol

In quantum mechanics, considering the coupling of two angular momenta as

$$\mathbf{l_3} = \mathbf{l_1} + \mathbf{l_2} \ , \tag{C.1}$$

the scalar product of eigenstates between the right-handed term and the left-handed one, namely, a Clebsch-Gordan coefficient, is related to the Wigner-$3j$ symbol:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \equiv \frac{(-1)^{l_1 - l_2 + m_3} \langle l_1 m_1 l_2 m_2 | (l_1 l_2) l_3 m_3 \rangle}{\sqrt{2l_3 + 1}} \ . \tag{C.2}$$

This symbol vanishes unless the selection rules are satisfied as follows:

$$\begin{aligned} &|m_1| \leq l_1 \ , \quad |m_2| \leq l_2 \ , \quad |m_3| \leq l_3 \ , \quad m_1 + m_2 = m_3 \ , \\ &|l_1 - l_2| \leq l_3 \leq l_1 + l_2 \ \text{(the triangle condition)} \ , \quad l_1 + l_2 + l_3 \in \mathbb{Z} \ . \end{aligned} \tag{C.3}$$

Symmetries of the Wigner-$3j$ symbol are given by

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$$\begin{aligned}
\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= (-1)^{\sum_{i=1}^3 l_i} \begin{pmatrix} l_2 & l_1 & l_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{\sum_{i=1}^3 l_i} \begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix} \\
&\quad \text{(odd permutation of columns)} \\
&= \begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \\
&\quad \text{(even permutation of columns)} \\
&= (-1)^{\sum_{i=1}^3 l_i} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \\
&\quad \text{(sign inversion of } m_1, m_2, m_3) \, . \qquad \text{(C.4)}
\end{aligned}$$

The Wigner-$3j$ symbols satisfy the orthogonality as

$$\begin{aligned}
\sum_{l_3 m_3} (2l_3 + 1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3 \end{pmatrix} &= \delta_{m_1, m_1'} \delta_{m_2, m_2'} \, , \\
(2l_3 + 1) \sum_{m_1 m_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3' \\ m_1 & m_2 & m_3' \end{pmatrix} &= \delta_{l_3, l_3'} \delta_{m_3, m_3'} \, . \qquad \text{(C.5)}
\end{aligned}$$

For a special case that $\sum_{i=1}^3 l_i = \text{even}$ and $m_1 = m_2 = m_3 = 0$, there is an analytical expression as

$$\begin{aligned}
\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^{\sum_{i=1}^3 \frac{-l_i}{2}} \\
&\times \frac{\left(\sum_{i=1}^3 \frac{l_i}{2}\right)! \sqrt{(-l_1 + l_2 + l_3)!} \sqrt{(l_1 - l_2 + l_3)!} \sqrt{(l_1 + l_2 - l_3)!}}{\left(\frac{-l_1+l_2+l_3}{2}\right)! \left(\frac{l_1-l_2+l_3}{2}\right)! \left(\frac{l_1+l_2-l_3}{2}\right)! \sqrt{\left(\sum_{i=1}^3 l_i + 1\right)!}} \, .
\end{aligned} \qquad \text{(C.6)}$$

This vanishes for $\sum_{i=1}^3 l_i = \text{odd}$. The Wigner-$3j$ symbol is related to the spin-weighted spherical harmonics as

$$\prod_{i=1}^2 {}_{s_i}Y_{l_i m_i}(\hat{\mathbf{n}}) = \sum_{l_3 m_3 s_3} {}_{s_3}Y^*_{l_3 m_3}(\hat{\mathbf{n}}) I^{-s_1 -s_2 -s_3}_{l_1 \, l_2 \, l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \, , \qquad \text{(C.7)}$$

which leads to the "extended" Gaunt integral including spin dependence:

$$\int d^2\hat{\mathbf{n}} {}_{s_1}Y_{l_1 m_1}(\hat{\mathbf{n}}) {}_{s_2}Y_{l_2 m_2}(\hat{\mathbf{n}}) {}_{s_3}Y_{l_3 m_3}(\hat{\mathbf{n}}) = I^{-s_1 -s_2 -s_3}_{l_1 \, l_2 \, l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \, . \qquad \text{(C.8)}$$

Here $I^{s_1 s_2 s_3}_{l_1 l_2 l_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$.

C.2 Wigner-6j symbol

Considering two other ways in the coupling of three angular momenta as

$$\mathbf{l_5} = \mathbf{l_1} + \mathbf{l_2} + \mathbf{l_4} \tag{C.9}$$

$$= \mathbf{l_3} + \mathbf{l_4} \tag{C.10}$$

$$= \mathbf{l_1} + \mathbf{l_6} \,, \tag{C.11}$$

the Wigner-6j symbol is defined using a Clebsch-Gordan coefficient between each eigenstate of $\mathbf{l_5}$ corresponding to Eqs. (C.10) and (C.11) as

$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} \equiv \frac{(-1)^{l_1+l_2+l_4+l_5} \langle (l_1 l_2) l_3; l_4; l_5 m_5 | l_1; (l_2 l_4) l_6; l_5 m_5 \rangle}{\sqrt{(2l_3+1)(2l_6+1)}} \,. \tag{C.12}$$

This is expressed with the summation of three Wigner-3j symbols:

$$\sum_{m_4 m_5 m_6} (-1)^{\sum_{i=4}^{6} l_i - m_i} \begin{pmatrix} l_5 & l_1 & l_6 \\ m_5 & -m_1 & -m_6 \end{pmatrix} \begin{pmatrix} l_6 & l_2 & l_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix} \begin{pmatrix} l_4 & l_3 & l_5 \\ m_4 & -m_3 & -m_5 \end{pmatrix}$$
$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} \,; \tag{C.13}$$

hence, the triangle conditions are given by

$$|l_1 - l_2| \le l_3 \le l_1 + l_2, \; |l_4 - l_5| \le l_3 \le l_4 + l_5 \,,$$
$$|l_1 - l_5| \le l_6 \le l_1 + l_5, \; |l_4 - l_2| \le l_6 \le l_4 + l_2 \,. \tag{C.14}$$

The Wigner-6j symbol obeys 24 symmetries such as

$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} = \begin{Bmatrix} l_2 & l_1 & l_3 \\ l_5 & l_4 & l_6 \end{Bmatrix} = \begin{Bmatrix} l_2 & l_3 & l_1 \\ l_5 & l_6 & l_4 \end{Bmatrix} \text{ (permutation of columns)}$$
$$= \begin{Bmatrix} l_4 & l_5 & l_3 \\ l_1 & l_2 & l_6 \end{Bmatrix} = \begin{Bmatrix} l_1 & l_5 & l_6 \\ l_4 & l_2 & l_3 \end{Bmatrix}$$
$$\text{(exchange of two corresponding elements between rows)} \,. \tag{C.15}$$

Geometrically, the Wigner-6j symbol is expressed using the tetrahedron composed of four triangles obeying Eq. (C.14). It is known that the Wigner-6j symbol is suppressed by the square root of the volume of the tetrahedron at high multipoles.

C.3 Wigner-9j symbol

Considering two other ways in the coupling of four angular momenta as

$$\mathbf{l_9} = \mathbf{l_1} + \mathbf{l_2} + \mathbf{l_4} + \mathbf{l_5} \tag{C.16}$$

$$= \mathbf{l_3} + \mathbf{l_6} \tag{C.17}$$

$$= \mathbf{l_7} + \mathbf{l_8}\ , \tag{C.18}$$

where $\mathbf{l_3} \equiv \mathbf{l_1} + \mathbf{l_2}$, $\mathbf{l_6} \equiv \mathbf{l_4} + \mathbf{l_5}$, $\mathbf{l_7} \equiv \mathbf{l_1} + \mathbf{l_4}$, $\mathbf{l_8} \equiv \mathbf{l_2} + \mathbf{l_5}$, the Wigner 9$j$ symbol expresses a Clebsch-Gordan coefficient between each eigenstate of $\mathbf{l_9}$ corresponding to Eqs. (C.17) and (C.18) as

$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{Bmatrix} \equiv \frac{\langle (l_1 l_2) l_3;\ (l_4 l_5) l_6;\ l_9 m_9 | (l_1 l_4) l_7;\ (l_2 l_5) l_8;\ l_9 m_9 \rangle}{\sqrt{(2l_3+1)(2l_6+1)(2l_7+1)(2l_8+1)}}\ . \tag{C.19}$$

This is expressed with the summation of five Wigner-3j symbols:

$$\sum_{\substack{m_4 m_5 m_6 \\ m_7 m_8 m_9}} \begin{pmatrix} l_4 & l_5 & l_6 \\ m_4 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} l_7 & l_8 & l_9 \\ m_7 & m_8 & m_9 \end{pmatrix} \times \begin{pmatrix} l_4 & l_7 & l_1 \\ m_4 & m_7 & m_1 \end{pmatrix} \begin{pmatrix} l_5 & l_8 & l_2 \\ m_5 & m_8 & m_2 \end{pmatrix} \begin{pmatrix} l_6 & l_9 & l_3 \\ m_6 & m_9 & m_3 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{Bmatrix}\ , \tag{C.20}$$

and that of three Wigner-6j symbols:

$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{Bmatrix} = \sum_x (-1)^{2x} (2x+1) \begin{Bmatrix} l_1 & l_4 & l_7 \\ l_8 & l_9 & x \end{Bmatrix} \begin{Bmatrix} l_2 & l_5 & l_8 \\ l_4 & x & l_6 \end{Bmatrix} \begin{Bmatrix} l_3 & l_6 & l_9 \\ x & l_1 & l_2 \end{Bmatrix}; \tag{C.21}$$

hence, the triangle conditions are given by

$$|l_1 - l_2| \le l_3 \le l_1 + l_2\ ,\ |l_4 - l_5| \le l_6 \le l_4 + l_5\ ,\ |l_7 - l_8| \le l_9 \le l_7 + l_8\ ,$$
$$|l_1 - l_4| \le l_7 \le l_1 + l_4\ ,\ |l_2 - l_5| \le l_8 \le l_2 + l_5\ ,\ |l_3 - l_6| \le l_9 \le l_3 + l_6\ . \tag{C.22}$$

The Wigner-$9j$ symbol obeys 72 symmetries:

$$\begin{aligned}
\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{Bmatrix} &= (-1)^{\sum_{i=1}^{9} l_i} \begin{Bmatrix} l_2 & l_1 & l_3 \\ l_5 & l_4 & l_6 \\ l_8 & l_7 & l_9 \end{Bmatrix} = (-1)^{\sum_{i=1}^{9} l_i} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_7 & l_8 & l_9 \\ l_4 & l_5 & l_6 \end{Bmatrix} \\
&\quad \text{(odd permutation of rows or columns)} \\
&= \begin{Bmatrix} l_2 & l_3 & l_1 \\ l_5 & l_6 & l_4 \\ l_8 & l_9 & l_7 \end{Bmatrix} = \begin{Bmatrix} l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \\ l_1 & l_2 & l_3 \end{Bmatrix} \\
&\quad \text{(even permutation of rows or columns)} \\
&= \begin{Bmatrix} l_1 & l_4 & l_7 \\ l_2 & l_5 & l_8 \\ l_3 & l_6 & l_9 \end{Bmatrix} = \begin{Bmatrix} l_9 & l_6 & l_3 \\ l_8 & l_5 & l_2 \\ l_7 & l_4 & l_1 \end{Bmatrix} \\
&\quad \text{(reflection of the symbols)} \ .
\end{aligned} \tag{C.23}$$

C.4 Analytic expressions of the Wigner symbols

Here, we show some analytical formulas of the Wigner symbols.

The I symbols, which are defined as

$$I_{l_1 l_2 l_3}^{s_1 s_2 s_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{pmatrix} , \tag{C.24}$$

are expressed as

$$\begin{aligned}
I_{l_1 l_2 l_3}^{0\,0\,0} &= \sqrt{\frac{\prod_{i=1}^{3}(2l_i+1)}{4\pi}} (-1)^{\sum_{i=1}^{3} \frac{-l_i}{2}} \\
&\quad \times \frac{\left(\sum_{i=1}^{3} \frac{l_i}{2}\right)! \sqrt{(-l_1+l_2+l_3)!} \sqrt{(l_1-l_2+l_3)!} \sqrt{(l_1+l_2-l_3)!}}{\left(\frac{-l_1+l_2+l_3}{2}\right)! \left(\frac{l_1-l_2+l_3}{2}\right)! \left(\frac{l_1+l_2-l_3}{2}\right)! \sqrt{(\sum_{i=1}^{3} l_i + 1)!}} \\
&\quad \text{(for } l_1+l_2+l_3 = \text{even)} \\
&= 0 \quad \text{(for } l_1+l_2+l_3 = \text{odd)} \ ,
\end{aligned} \tag{C.25}$$

$$\begin{aligned}
I_{l_1\,l_2\,l_3}^{0\,1\,-1} &= \sqrt{\frac{5}{8\pi}} (-1)^{l_2+1} \sqrt{\frac{(l_2-1)(l_2+1)}{l_2-1/2}} \quad \text{(for } l_1 = l_2-2, l_3 = 2) \\
&= \sqrt{\frac{15}{16\pi}} (-1)^{l_2} \sqrt{\frac{l_2+1/2}{(l_2-1/2)(l_2+3/2)}} \quad \text{(for } l_1 = l_2, l_3 = 2) \\
&= \sqrt{\frac{5}{8\pi}} (-1)^{l_2} \sqrt{\frac{l_2(l_2+2)}{l_2+3/2}} \quad \text{(for } l_1 = l_2+2, l_3 = 2)
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{3}{8\pi}}(-1)^{l_3+1}\sqrt{l_3+1} \ \ (\text{for } l_1 = l_3-1, l_2 = 1) \\
&= \sqrt{\frac{3}{4\pi}}(-1)^{l_3+1}\sqrt{l_3+1/2} \ \ (\text{for } l_1 = l_3, l_2 = 1) \\
&= \sqrt{\frac{3}{8\pi}}(-1)^{l_3+1}\sqrt{l_3} \ \ (\text{for } l_1 = l_3+1, l_2 = 1) \ . \qquad \text{(C.26)}
\end{aligned}
$$

The Wigner-9j symbols are calculated as

$$
\begin{aligned}
\begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ 1 & 1 & 2 \end{Bmatrix} &= \sqrt{\frac{2(l_3 \pm 1)+1}{5}} \begin{Bmatrix} l_1 & l_4 & 1 \\ l_3 \pm 2 & l_3 \pm 1 & l_5 \end{Bmatrix} \begin{Bmatrix} l_2 & l_5 & 1 \\ l_3 \pm 1 & l_3 & l_1 \end{Bmatrix} \ \ (\text{for } l_6 = l_3 \pm 2) \\
&= \sqrt{\frac{(2l_3-1)(2l_3+2)(2l_3+3)}{30(2l_3)(2l_3+1)}} \begin{Bmatrix} l_1 & l_4 & 1 \\ l_3 & l_3-1 & l_5 \end{Bmatrix} \begin{Bmatrix} l_2 & l_5 & 1 \\ l_3-1 & l_3 & l_1 \end{Bmatrix} \\
&\quad + \sqrt{\frac{2(2l_3-1)(2l_3+1)(2l_3+3)}{15(2l_3)(2l_3+2)}} \begin{Bmatrix} l_1 & l_4 & 1 \\ l_3 & l_3 & l_5 \end{Bmatrix} \begin{Bmatrix} l_2 & l_5 & 1 \\ l_3 & l_3 & l_1 \end{Bmatrix} \\
&\quad + \sqrt{\frac{(2l_3-1)(2l_3)(2l_3+3)}{30(2l_3+1)(2l_3+2)}} \begin{Bmatrix} l_1 & l_4 & 1 \\ l_3 & l_3+1 & l_5 \end{Bmatrix} \begin{Bmatrix} l_2 & l_5 & 1 \\ l_3+1 & l_3 & l_1 \end{Bmatrix} \\
&\qquad (\text{for } l_6 = l_3) \ , \qquad \text{(C.27)}
\end{aligned}
$$

where these Wigner-6j symbols are analytically given by

$$
\begin{aligned}
\begin{Bmatrix} l_1 & l_2 & 1 \\ l_4 & l_5 & l_6 \end{Bmatrix} &= (-1)^{l_1+l_4+l_6+1} \sqrt{\frac{{}_{l_1+l_4+l_6+2}P_2 \ {}_{l_1+l_4-l_6+1}P_2}{{}_{2l_4+3}P_3 \ {}_{2l_1+1}P_3}} \\
&\qquad (\text{for } l_2 = l_1-1, l_5 = l_4+1) \\
&= (-1)^{l_1+l_4+l_6+1} \sqrt{\frac{2(l_1+l_4+l_6+2)(l_1+l_4-l_6+1)}{{}_{2l_4+3}P_3}} \\
&\quad \times \sqrt{\frac{(-l_1+l_4+l_6+1)(l_1-l_4+l_6)}{{}_{2l_1+2}P_3}} \ \ (\text{for } l_2 = l_1, l_5 = l_4+1) \\
&= (-1)^{l_1+l_4+l_6+1} \sqrt{\frac{{}_{-l_1+l_4+l_6+1}P_2 \ {}_{l_1-l_4+l_6+1}P_2}{{}_{2l_4+3}P_3 \ {}_{2l_1+3}P_3}} \\
&\qquad (\text{for } l_2 = l_1+1, l_5 = l_4+1) \\
&= (-1)^{l_1+l_4+l_6+1} \\
&\quad \times [l_4(l_4+1) + l_1(l_1-1)(l_4+1) - l_6(l_6+1) - l_1(l_1+1)l_4] \\
&\quad \times \sqrt{\frac{2(l_1+l_4+l_6+1)(l_1+l_4-l_6)}{(-l_1+l_4+l_6+1)(l_1-l_4+l_6)_{2l_4+2}P_3 \ {}_{2l_1+1}P_3}}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{for } l_2 = l_1 - 1, l_5 = l_4) \\
&= 2(-1)^{l_1+l_4+l_6+1} \\
&\quad \times \frac{l_4(l_4+1) + l_1(l_1+1)(l_4+1) - l_6(l_6+1) - l_1(l_1+1)l_4}{\sqrt{{}_{2l_4+2}P_3 \; {}_{2l_1+2}P_3}} \\
&\qquad\qquad\qquad (\text{for } l_2 = l_1, l_5 = l_4) \\
&= (-1)^{l_1+l_4+l_6+1} \\
&\quad \times [l_4(l_4+1) + (l_1+1)(l_1+2)(l_4+1) - l_6(l_6+1) - l_1(l_1+1)l_4] \\
&\quad \times \sqrt{\frac{2(-l_1+l_4+l_6)(l_1-l_4+l_6+1)}{(l_1+l_4+l_6+2)(l_1+l_4-l_6+1)\,{}_{2l_4+2}P_3 \; {}_{2l_1+3}P_3}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{for } l_2 = l_1 + 1, l_5 = l_4) \; . \qquad (\text{C.28})
\end{aligned}
$$

Using these analytical formulas, one can reduce the time cost involved with calculating the CMB bispectrum from PMFs.

Appendix D
Polarization Vector and Tensor

We summarize the relations and properties of a divergenceless polarization vector $\epsilon_a^{(\pm 1)}$ and a transverse and traceless polarization tensor $e_{ab}^{(\pm 2)}$ [6, 11].

The polarization vector with respect to a unit vector $\hat{\mathbf{n}}$ is expressed using two unit vectors $\hat{\theta}$ and $\hat{\phi}$ perpendicular to $\hat{\mathbf{n}}$ as

$$\epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) = \frac{1}{\sqrt{2}}\left[\hat{\theta}_a(\hat{\mathbf{n}}) \pm i\,\hat{\phi}_a(\hat{\mathbf{n}})\right] . \tag{D.1}$$

This satisfies the relations:

$$\begin{aligned} \hat{n}^a \epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) &= 0 , \\ \epsilon_a^{(\pm 1)*}(\hat{\mathbf{n}}) &= \epsilon_a^{(\mp 1)}(\hat{\mathbf{n}}) = \epsilon_a^{(\pm 1)}(-\hat{\mathbf{n}}) , \\ \epsilon_a^{(\lambda)}(\hat{\mathbf{n}})\epsilon_a^{(\lambda')}(\hat{\mathbf{n}}) &= \delta_{\lambda,-\lambda'} \quad (\text{for } \lambda, \lambda' = \pm 1) . \end{aligned} \tag{D.2}$$

By defining a rotational matrix, which transforms a unit vector parallel to the z axis, namely $\hat{\mathbf{z}}$, to $\hat{\mathbf{n}}$, as

$$S(\hat{\mathbf{n}}) \equiv \begin{pmatrix} \cos\theta_n \cos\phi_n & -\sin\phi_n & \sin\theta_n \cos\phi_n \\ \cos\theta_n \sin\phi_n & \cos\phi_n & \sin\theta_n \sin\phi_n \\ -\sin\theta_n & 0 & \cos\theta_n \end{pmatrix} , \tag{D.3}$$

we specify $\hat{\theta}$ and $\hat{\phi}$ as

$$\hat{\theta}(\hat{\mathbf{n}}) = S(\hat{\mathbf{n}})\hat{\mathbf{x}} , \quad \hat{\phi}(\hat{\mathbf{n}}) = S(\hat{\mathbf{n}})\hat{\mathbf{y}} , \tag{D.4}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors parallel to x- and y-axes. By using Eq. (D.1), the polarization tensor is constructed as

$$e_{ab}^{(\pm 2)}(\hat{\mathbf{n}}) = \sqrt{2}\epsilon_a^{(\pm 1)}(\hat{\mathbf{n}})\epsilon_b^{(\pm 1)}(\hat{\mathbf{n}}) . \tag{D.5}$$

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To utilize the polarization vector and tensor in the calculation of this thesis, we need to expand Eqs. (D.1) and (D.5) with spin spherical harmonics. An arbitrary unit vector is expanded with the spin-0 spherical harmonics as

$$\hat{r}_a = \sum_m \alpha_a^m Y_{1m}(\hat{\mathbf{r}}) ,$$
$$\alpha_a^m \equiv \sqrt{\frac{2\pi}{3}} \begin{pmatrix} -m(\delta_{m,1} + \delta_{m,-1}) \\ i\ (\delta_{m,1} + \delta_{m,-1}) \\ \sqrt{2}\delta_{m,0} \end{pmatrix} . \tag{D.6}$$

Here, note that the repeat of the index implies the summation. The scalar product of α_a^m is calculated as

$$\alpha_a^m \alpha_a^{m'} = \frac{4\pi}{3}(-1)^m \delta_{m,-m'} , \quad \alpha_a^m \alpha_a^{m'*} = \frac{4\pi}{3}\delta_{m,m'} . \tag{D.7}$$

Through the substitution of Eq. (D.4) into Eq. (D.6), $\hat{\theta}$ is expanded as

$$\hat{\theta}_a(\hat{\mathbf{n}}) = \sum_m \alpha_a^m Y_{1m}(\hat{\theta}(\hat{\mathbf{n}})) = \sum_m \alpha_a^m \sum_{m'} D_{mm'}^{(1)*}(S(\hat{\mathbf{n}}))Y_{1m'}(\hat{\mathbf{x}})$$
$$= -\frac{s}{\sqrt{2}}(\delta_{s,1} + \delta_{s,-1}) \sum_m \alpha_a^m {}_sY_{1m}(\hat{\mathbf{n}}) . \tag{D.8}$$

Here, we use the properties of the Wigner D-matrix as described in Appendix B [3, 5, 6, 13]

$$Y_{\ell m}(S(\hat{\mathbf{n}})\hat{\mathbf{x}}) = \sum_{m'} D_{mm'}^{(\ell)*}(S(\hat{\mathbf{n}}))Y_{\ell m'}(\hat{\mathbf{x}}) ,$$
$$D_{ms}^{(\ell)}(S(\hat{\mathbf{n}})) = \left[\frac{4\pi}{2\ell+1}\right]^{1/2} (-1)^s {}_{-s}Y_{\ell m}^*(\hat{\mathbf{n}}) . \tag{D.9}$$

In the same manner, $\hat{\phi}$ is also calculated as

$$\hat{\phi}_a(\hat{\mathbf{n}}) = \frac{i}{\sqrt{2}}(\delta_{s,1} + \delta_{s,-1}) \sum_m \alpha_a^m {}_sY_{1m}(\hat{\mathbf{n}}) ; \tag{D.10}$$

hence, the explicit form of Eq. (D.1) is calculated as

$$\epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) = \mp \sum_m \alpha_a^m {}_{\pm 1}Y_{1m}(\hat{\mathbf{n}}) . \tag{D.11}$$

Substituting this into Eq. (D.5) and using the relations of Appendix C and $I_{2\,1\,1}^{\mp 2 \pm 1 \pm 1} = \frac{3}{2\sqrt{\pi}}$, the polarization tensor can also be expressed as

$$e_{ab}^{(\pm 2)}(\hat{\mathbf{n}}) = \frac{3}{\sqrt{2\pi}} \sum_{M m_a m_b} {}_{\mp 2}Y_{2M}^*(\hat{\mathbf{n}}) \alpha_a^{m_a} \alpha_b^{m_b} \begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix} . \tag{D.12}$$

This obeys the relations:

$$\begin{aligned} e_{aa}^{(\pm 2)}(\hat{\mathbf{n}}) &= \hat{n}_a e_{ab}^{(\pm 2)}(\hat{\mathbf{n}}) = 0 , \\ e_{ab}^{(\pm 2)*}(\hat{\mathbf{n}}) &= e_{ab}^{(\mp 2)}(\hat{\mathbf{n}}) = e_{ab}^{(\pm 2)}(-\hat{\mathbf{n}}) , \\ e_{ab}^{(\lambda)}(\hat{\mathbf{n}}) e_{ab}^{(\lambda')}(\hat{\mathbf{n}}) &= 2\delta_{\lambda,-\lambda'} \quad (\text{for } \lambda, \lambda' = \pm 2) . \end{aligned} \tag{D.13}$$

Using the projection operators as

$$\begin{aligned} O_a^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}} &\equiv k^{-1} \nabla_a e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{k}_a e^{i\mathbf{k}\cdot\mathbf{x}} , \\ O_{ab}^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}} &\equiv \left(k^{-2} \nabla_a \nabla_b + \frac{\delta_{a,b}}{3} \right) e^{i\mathbf{k}\cdot\mathbf{x}} = \left(-\hat{k}_a \hat{k}_b + \frac{\delta_{a,b}}{3} \right) e^{i\mathbf{k}\cdot\mathbf{x}} , \\ O_a^{(\pm 1)} e^{i\mathbf{k}\cdot\mathbf{x}} &\equiv -i\epsilon_a^{(\pm 1)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} , \\ O_{ab}^{(\pm 1)} e^{i\mathbf{k}\cdot\mathbf{x}} &\equiv k^{-1} \left(\nabla_a O_b^{(\pm 1)} + \nabla_b O_a^{(\pm 1)} \right) e^{i\mathbf{k}\cdot\mathbf{x}} = \left(\hat{k}_a \epsilon_b^{(\pm 1)}(\hat{\mathbf{k}}) + \hat{k}_b \epsilon_a^{(\pm 1)}(\hat{\mathbf{k}}) \right) e^{i\mathbf{k}\cdot\mathbf{x}} , \\ O_{ab}^{(\pm 2)} e^{i\mathbf{k}\cdot\mathbf{x}} &\equiv e_{ab}^{(\pm 2)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} , \end{aligned} \tag{D.14}$$

the arbitrary scalar, vector and tensor are decomposed into the helicity states as

$$\eta(\mathbf{k}) = \eta^{(0)}(\mathbf{k}), \tag{D.15}$$

$$\omega_a(\mathbf{k}) = \omega^{(0)}(\mathbf{k}) O_a^{(0)} + \sum_{\lambda=\pm 1} \omega^{(\lambda)}(\mathbf{k}) O_a^{(\lambda)} , \tag{D.16}$$

$$\begin{aligned} \chi_{ab}(\mathbf{k}) = &-\frac{1}{3} \chi_{\rm iso}(\mathbf{k}) \delta_{a,b} + \chi^{(0)}(\mathbf{k}) O_{ab}^{(0)} \\ &+ \sum_{\lambda=\pm 1} \chi^{(\lambda)}(\mathbf{k}) O_{ab}^{(\lambda)} + \sum_{\lambda=\pm 2} \chi^{(\lambda)}(\mathbf{k}) O_{ab}^{(\lambda)}. \end{aligned} \tag{D.17}$$

Then, using Eq. (D.9) and (D.13), we can find the inverse formulae as

$$\omega^{(0)}(\mathbf{k}) = -O_a^{(0)} \omega_a(\mathbf{k}) , \tag{D.18}$$

$$\omega^{(\pm 1)}(\mathbf{k}) = -O_a^{(\mp 1)}(\hat{\mathbf{k}}) \omega_a(\mathbf{k}) , \tag{D.19}$$

$$\chi^{(0)}(\mathbf{k}) = \frac{3}{2} O_{ab}^{(0)}(\hat{\mathbf{k}}) \chi_{ab}(\mathbf{k}) , \tag{D.20}$$

$$\chi^{(\pm 1)}(\mathbf{k}) = \frac{1}{2} O_{ab}^{(\mp 1)}(\hat{\mathbf{k}}) \chi_{ab}(\mathbf{k}) , \tag{D.21}$$

$$\chi^{(\pm 2)}(\mathbf{k}) = \frac{1}{2} O_{ab}^{(\mp 2)}(\hat{\mathbf{k}}) \chi_{ab}(\mathbf{k}) . \tag{D.22}$$

From these, we can derive the relations of several projection operators as

$$\begin{aligned}
O^{(0)}_{ab}(\hat{\mathbf{k}}) &= -\hat{k}_a\hat{k}_b + \frac{1}{3}\delta_{ab} \\
&= -\sqrt{\frac{3}{2\pi}}\sum_{Mm_am_b} Y^*_{2M}(\hat{\mathbf{k}})\alpha_a^{m_a}\alpha_b^{m_b}\begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix}\,, \\
O^{(\pm1)}_{ab}(\hat{\mathbf{k}}) &= \hat{k}_a\epsilon^{(\pm1)}_b(\hat{\mathbf{k}}) + \hat{k}_b\epsilon^{(\pm1)}_a(\hat{\mathbf{k}}) \\
&= \pm\frac{3}{\sqrt{2\pi}}\sum_{Mm_am_b} {}_{\mp1}Y^*_{2M}(\hat{\mathbf{k}})\alpha_a^{m_a}\alpha_b^{m_b}\begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix}\,, \\
O^{(\pm2)}_{ab}(\hat{\mathbf{k}}) &= e^{(\pm2)}_{ab}(\hat{\mathbf{k}}) \\
&= \frac{3}{\sqrt{2\pi}}\sum_{Mm_am_b} {}_{\mp2}Y^*_{2M}(\hat{\mathbf{k}})\alpha_a^{m_a}\alpha_b^{m_b}\begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix}\,, \qquad \text{(D.23)} \\
P_{ab}(\hat{\mathbf{k}}) &\equiv \delta_{ab} - \hat{k}_a\hat{k}_b \\
&= -2\sum_{L=0,2} I^{01-1}_{L11}\sum_{Mm_am_b} Y^*_{LM}(\hat{\mathbf{k}})\alpha_a^{m_a}\alpha_b^{m_b}\begin{pmatrix} L & 1 & 1 \\ M & m_a & m_b \end{pmatrix}\,, \\
O^{(0)}_{ab}(\hat{\mathbf{k}})P_{bc}(\hat{\mathbf{k}}) &= \frac{1}{3}P_{ac}(\hat{\mathbf{k}})\,, \\
O^{(\pm1)}_{ab}(\hat{\mathbf{k}})P_{bc}(\hat{\mathbf{k}}) &= \hat{k}_a\epsilon^{(\pm1)}_c(\hat{\mathbf{k}})\,, \\
O^{(\pm2)}_{ab}(\hat{\mathbf{k}})P_{bc}(\hat{\mathbf{k}}) &= e^{(\pm2)}_{ac}(\hat{\mathbf{k}})\,, \\
\hat{k}_c &= i\eta^{abc}\epsilon^{(+1)}_a(\hat{\mathbf{k}})\epsilon^{(-1)}_b(\hat{\mathbf{k}})\,, \\
\eta^{abc}\hat{k}_a\epsilon^{(\pm1)}_b(\hat{\mathbf{k}}) &= \mp i\epsilon^{(\pm1)}_c(\hat{\mathbf{k}})\,.
\end{aligned}$$

Appendix E
Calculation of $f^{(a)}_{W^3}$ and $f^{(a)}_{WW^2}$

Here, we calculate each product between the wave number vectors and the polarization tensors of $f^{(a)}_{W^3}$ and $f^{(a)}_{\widetilde{W}W^2}$ mentioned in Chap. 8 [14].

Using the relations discussed in Appendix D, the all terms of $f^{(a)}_{W^3}$ are written as

$$\begin{aligned}
e^{(-\lambda_1)}_{ij} e^{(-\lambda_2)}_{jk} e^{(-\lambda_3)}_{ki} &= -(8\pi)^{3/2} \sum_{M,M',M''} {}_{\lambda_1}Y^*_{2M}(\hat{\mathbf{k_1}})_{\lambda_2}Y^*_{2M'}(\hat{\mathbf{k_2}})_{\lambda_3}Y^*_{2M''}(\hat{\mathbf{k_3}}) \\
&\quad \times \frac{1}{10}\sqrt{\frac{7}{3}} \begin{pmatrix} 2 & 2 & 2 \\ M & M' & M'' \end{pmatrix} , \\
e^{(-\lambda_1)}_{ij} e^{(-\lambda_2)}_{kl} e^{(-\lambda_3)}_{kl} \hat{k}_{2i} \hat{k}_{3j} &= -(8\pi)^{3/2} \sum_{L',L''=2,3} \frac{4\pi}{15} (-1)^{L'} I^{\lambda_2 0 -\lambda_2}_{L'12} I^{\lambda_3 0 -\lambda_3}_{L''12} \\
&\quad \times \sum_{M,M',M''} {}_{\lambda_1}Y^*_{2M}(\hat{\mathbf{k_1}})_{\lambda_2}Y^*_{L'M'}(\hat{\mathbf{k_2}})_{\lambda_3}Y^*_{L''M''}(\hat{\mathbf{k_3}}) \\
&\quad \times \begin{pmatrix} 2 & L' & L'' \\ M & M' & M'' \end{pmatrix} \begin{Bmatrix} 2 & L' & L'' \\ 2 & 1 & 1 \end{Bmatrix} , \\
e^{(-\lambda_1)}_{ij} e^{(-\lambda_2)}_{ki} e^{(-\lambda_3)}_{jl} \hat{k}_{2l} \hat{k}_{3k} &= -(8\pi)^{3/2} \sum_{L',L''=2,3} \frac{4\pi}{3} (-1)^{L'} I^{\lambda_2 0 -\lambda_2}_{L'12} I^{\lambda_3 0 -\lambda_3}_{L''12} \\
&\quad \times \sum_{M,M',M''} {}_{\lambda_1}Y^*_{2M}(\hat{\mathbf{k_1}})_{\lambda_2}Y^*_{L'M'}(\hat{\mathbf{k_2}})_{\lambda_3}Y^*_{L''M''}(\hat{\mathbf{k_3}}) \\
&\quad \times \begin{pmatrix} 2 & L' & L'' \\ M & M' & M'' \end{pmatrix} \begin{Bmatrix} 2 & L' & L'' \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{Bmatrix} , \qquad \text{(E.1)} \\
e^{(-\lambda_1)}_{ij} e^{(-\lambda_2)}_{ik} e^{(-\lambda_3)}_{kl} \hat{k}_{2l} \hat{k}_{3j} &= -(8\pi)^{3/2} \sum_{L',L''=2,3} \frac{4\pi}{3} (-1)^{L'} I^{\lambda_2 0 -\lambda_2}_{L'12} I^{\lambda_3 0 -\lambda_3}_{L''12} \\
&\quad \times \sum_{M,M',M''} {}_{\lambda_1}Y^*_{2M}(\hat{\mathbf{k_1}})_{\lambda_2}Y^*_{L'M'}(\hat{\mathbf{k_2}})_{\lambda_3}Y^*_{L''M''}(\hat{\mathbf{k_3}})
\end{aligned}$$

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$$\times \begin{pmatrix} 2 & L' & L'' \\ M & M' & M'' \end{pmatrix} \begin{Bmatrix} 2 & 1 & L' \\ 2 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 2 & L' & L'' \\ 2 & 1 & 1 \end{Bmatrix} .$$

In the calculation of $f^{(a)}_{\widetilde{W}W^2}$, we also need to consider the dependence of the tensor contractions on η^{ijk}. Making use of the relation:

$$\eta^{abc}\alpha_a^{m_a}\alpha_b^{m_b}\alpha_c^{m_c} = -i\left(\frac{4\pi}{3}\right)^{3/2}\sqrt{6}\begin{pmatrix} 1 & 1 & 1 \\ m_a & m_b & m_c \end{pmatrix} , \tag{E.2}$$

the first two terms of $f^{(a)}_{\widetilde{W}W^2}$ reduce to

$$\begin{aligned} i\eta^{ijk}e_{kq}^{(-\lambda_1)}e_{jm}^{(-\lambda_2)}e_{iq}^{(-\lambda_3)}\hat{k}_{3m} &= -(8\pi)^{3/2}\sum_{L''=2,3}\sqrt{\frac{2\pi}{5}}(-1)^{L''}I^{\lambda_3 0 -\lambda_3}_{L''12} \\ &\times \sum_{M,M',M''} {}_{\lambda_1}Y^*_{2M}(\hat{\mathbf{k}}_1)_{\lambda_2}Y^*_{2M'}(\hat{\mathbf{k}}_2)_{\lambda_3}Y^*_{L''M''}(\hat{\mathbf{k}}_3) \\ &\times \begin{pmatrix} 2 & 2 & L'' \\ M & M' & M'' \end{pmatrix}\begin{Bmatrix} 2 & 2 & L'' \\ 1 & 2 & 1 \end{Bmatrix} , \\ i\eta^{ijk}e_{kq}^{(-\lambda_1)}e_{mi}^{(-\lambda_2)}e_{mq}^{(-\lambda_3)}\hat{k}_{3j} &= -(8\pi)^{3/2}\sum_{L''=2,3}2\sqrt{2\pi}(-1)^{L''}I^{\lambda_3 0 -\lambda_3}_{L''12} \\ &\times \sum_{M,M',M''} {}_{\lambda_1}Y^*_{2M}(\hat{\mathbf{k}}_1)_{\lambda_2}Y^*_{2M'}(\hat{\mathbf{k}}_2)_{\lambda_3}Y^*_{L''M''}(\hat{\mathbf{k}}_3) \\ &\times \begin{pmatrix} 2 & 2 & L'' \\ M & M' & M'' \end{pmatrix}\begin{Bmatrix} 2 & 2 & L'' \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{Bmatrix} . \end{aligned} \tag{E.3}$$

For the other terms, by using the relation

$$\eta^{abc}\hat{k}_a e_{bd}^{(\lambda)}(\hat{\mathbf{k}}) = -\frac{\lambda}{2}ie_{cd}^{(\lambda)}(\hat{\mathbf{k}}) , \tag{E.4}$$

we have

$$
\begin{aligned}
i\eta^{ijk} e_{pj}^{(-\lambda_1)} e_{pm}^{(-\lambda_2)} \hat{k}_{1k} \hat{k}_{2l} e_{il}^{(-\lambda_3)} \hat{k}_{3m} = & -\frac{\lambda_1}{2}(8\pi)^{3/2} \sum_{L', L''=2,3} \sum_{M, M', M''} \\
& \times \frac{4\pi}{3}(-1)^{L''} I_{L'12}^{\lambda_2 0 -\lambda_2} I_{L''12}^{\lambda_3 0 -\lambda_3} \\
& \times {}_{\lambda_1}Y_{2M}^*(\hat{\mathbf{k_1}}) {}_{\lambda_2}Y_{L'M'}^*(\hat{\mathbf{k_2}}) {}_{\lambda_3}Y_{L''M''}^*(\hat{\mathbf{k_3}}) \\
& \times \begin{pmatrix} 2 & L' & L'' \\ M & M' & M'' \end{pmatrix} \begin{Bmatrix} 2 & L' & L'' \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{Bmatrix} ,
\end{aligned} \tag{E.5}
$$

$$
\begin{aligned}
i\eta^{ijk} e_{pj}^{(-\lambda_1)} e_{pm}^{(-\lambda_2)} \hat{k}_{1k} \hat{k}_{2l} e_{im}^{(-\lambda_3)} \hat{k}_{3l} = & -\frac{\lambda_1}{2}(8\pi)^{3/2} \sum_{L', L''=2,3} \sum_{M, M', M''} \\
& \times \frac{2\pi}{15}\sqrt{\frac{7}{3}}(-1)^{L''} I_{L'12}^{\lambda_2 0 -\lambda_2} I_{L''12}^{\lambda_3 0 -\lambda_3} \\
& \times {}_{\lambda_1}Y_{2M}^*(\hat{\mathbf{k_1}}) {}_{\lambda_2}Y_{L'M'}^*(\hat{\mathbf{k_2}}) {}_{\lambda_3}Y_{L''M''}^*(\hat{\mathbf{k_3}}) \\
& \times \begin{pmatrix} 2 & L' & L'' \\ M & M' & M'' \end{pmatrix} \begin{Bmatrix} 2 & L' & L'' \\ 1 & 2 & 2 \end{Bmatrix} .
\end{aligned}
$$

Appendix F
Graviton Non-Gaussianity from the Weyl Cubic Terms

Here, let us derive the bispectra of gravitons coming from the parity-even and parity-odd Weyl cubic terms, namely, Eqs. (8.16) and (8.17) [14]. For convenience, we decompose the interaction Hamiltonians of W^3 and $\widetilde{W}W^2$ (8.15) into

$$H_{int} = \sum_{n=1}^{4} H_{int}^{(n)}. \tag{F.1}$$

Depending on this, we also split the graviton non-Gaussianity as

$$\left\langle \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}) \right\rangle_{int} = \sum_{m=1}^{4} \left\langle \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}) \right\rangle_{int}^{(m)} . \tag{F.2}$$

In what follows, we shall show the computation of each fraction.

F.1 W^3

The bracket part of Eq. (8.12) in terms of $H_{W^3}^{(1)}$ is expanded as

$$\begin{aligned}
&\left\langle 0 \left| \left[: H_{W^3}^{(1)}(\tau') :, \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \right] \right| 0 \right\rangle \\
&= \left\langle 0 \right| : H_{W^3}^{(1)}(\tau') : \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \left| 0 \right\rangle \\
&\quad - \left\langle 0 \left| \left[\prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \right] : H_{W^3}^{(1)}(\tau') : \right| 0 \right\rangle
\end{aligned}$$

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$$
\begin{aligned}
&= -\Lambda^{-2}(H\tau')^2 \left(\frac{\tau'}{\tau_*}\right)^A \frac{1}{4}(2\pi)^3 \delta\left(\sum_{n=1}^{3} \mathbf{k_n}\right) \\
&\quad \times e_{ij}^{(\lambda_1)}(-\hat{\mathbf{k}}_1) e_{jk}^{(\lambda_2)}(-\hat{\mathbf{k}}_2) e_{ki}^{(\lambda_3)}(-\hat{\mathbf{k}}_3) \\
&\quad \times 6 \left[\left\{ \prod_{n=1}^{3} (\ddot{\gamma}_{dS} - k_n^2 \gamma_{dS})(k_n, \tau') \gamma_{dS}^*(k_n, \tau) \right\} \right. \\
&\quad \left. - \left\{ \prod_{n=1}^{3} \gamma_{dS}(k_n, \tau)(\ddot{\gamma}_{dS}^* - k_n^2 \gamma_{dS}^*)(k_n, \tau') \right\} \right] \\
&= -\frac{3}{2}\Lambda^{-2}(H\tau')^2 \left(\frac{\tau'}{\tau_*}\right)^A (2\pi)^3 \delta\left(\sum_{n=1}^{3} \mathbf{k_n}\right) \\
&\quad \times e_{ij}^{(-\lambda_1)}(\hat{\mathbf{k}}_1) e_{jk}^{(-\lambda_2)}(\hat{\mathbf{k}}_2) e_{ki}^{(-\lambda_3)}(\hat{\mathbf{k}}_3) \\
&\quad \times 2i \mathrm{Im} \left[\prod_{n=1}^{3} (\ddot{\gamma}_{dS} - k_n^2 \gamma_{dS})(k_n, \tau') \gamma_{dS}^*(k_n, \tau) \right] . \qquad \text{(F.3)}
\end{aligned}
$$

Here, we use

$$
\begin{aligned}
\left\langle 0 \right| : \prod_{n=1}^{3} a_{k_n'}^{(\lambda_n')} a_{-k_n}^{(\lambda_n)\dagger} : \left| 0 \right\rangle &= (2\pi)^9 \delta(\mathbf{k_1} + \mathbf{k_3'}) \delta_{\lambda_1, \lambda_3'} \delta(\mathbf{k_1'} + \mathbf{k_3}) \delta_{\lambda_1', \lambda_3} \\
&\quad \times \delta(\mathbf{k_2} + \mathbf{k_2'}) \delta_{\lambda_2, \lambda_2'} + 5 \text{ perms.} \qquad \text{(F.4)} \\
&= \left\langle 0 \right| : \prod_{n=1}^{3} a_{k_n}^{(\lambda_n)} a_{-k_n'}^{(\lambda_n')\dagger} : \left| 0 \right\rangle \\
e_{ij}^{(-\lambda)}(\hat{\mathbf{k}}) &= e_{ij}^{(\lambda)}(-\hat{\mathbf{k}}) .
\end{aligned}
$$

Furthermore, since

$$
\begin{aligned}
\ddot{\gamma}_{dS} - k^2 \gamma_{dS} &= \frac{2H\tau'}{M_{\rm pl}} k^{3/2} e^{-ik\tau'} , \\
\prod_{n=1}^{3} \gamma_{dS}^*(k_n, \tau) &\xrightarrow{\tau \to 0} i \frac{H^3}{M_{\rm pl}^3} (k_1 k_2 k_3)^{-3/2} , \qquad \text{(F.5)}
\end{aligned}
$$

the time integral at $\tau \to 0$ is performed as

$$
\begin{aligned}
&\mathrm{Im}\left[\int_{-\infty}^{\tau} d\tau'(H\tau')^2\left(\frac{\tau'}{\tau_*}\right)^A \prod_{n=1}^{3}(\ddot{\gamma}_{dS}-k_n^2\gamma_{dS})(k_n,\tau')\gamma_{dS}^*(k_n,\tau)\right] \\
&\quad = \frac{8H^5}{M_{\rm pl}^3}\sqrt{k_1^3k_2^3k_3^3}\,\mathrm{Im}\left[\left(\prod_{n=1}^{3}\gamma_{dS}^*(k_n,\tau)\right)\tau_*^{-A}\int_{-\infty}^{\tau} d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right] \\
&\quad \xrightarrow{\tau\to 0} \frac{8H^8}{M_{\rm pl}^6}\mathrm{Re}\left[\tau_*^{-A}\int_{-\infty}^{0} d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right], \qquad \text{(F.6)}
\end{aligned}
$$

where $k_t \equiv \sum_{n=1}^{3} k_n$. Thus, the graviton non-Gaussianity in the late time limit arising from $H_{W^3}^{(1)}$ is

$$
\begin{aligned}
\left\langle \prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n})\right\rangle_{W^3}^{(1)} &= (2\pi)^3\delta\left(\sum_{n=1}^{3}\mathbf{k_n}\right)8\left(\frac{H}{M_{\rm pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2 \\
&\quad \times \mathrm{Re}\left[\tau_*^{-A}\int_{-\infty}^{0} d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right] \\
&\quad \times 3e_{ij}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{jk}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{ki}^{(-\lambda_3)}(\hat{\mathbf{k}}_3)\,. \qquad \text{(F.7)}
\end{aligned}
$$

The bracket part in terms of $H_{W^3}^{(2)}$ is given by

$$
\begin{aligned}
&\left\langle 0\left|\left[:H_{W^3}^{(2)}(\tau'):, \prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n},\tau)\right]\right|0\right\rangle \\
&= \left\langle 0\left|:H_{W^3}^{(2)}(\tau'):\prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n},\tau)\right|0\right\rangle \\
&\quad - \left\langle 0\left|\left[\prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n},\tau)\right]:H_{W^3}^{(2)}(\tau'):\right|0\right\rangle \\
&= \frac{3}{2}\Lambda^{-2}(H\tau')^2\left(\frac{\tau'}{\tau_*}\right)^A k_2k_3(2\pi)^3\delta\left(\sum_{n=1}^{3}\mathbf{k_n}\right) \\
&\quad \times \hat{k}_{2i}\hat{k}_{3j}e_{ij}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{kl}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{kl}^{(-\lambda_3)}(\hat{\mathbf{k}}_3) \\
&\quad \times 2i\,\mathrm{Im}\left[(\ddot{\gamma}_{dS}-k_1^2\gamma_{dS})(k_1,\tau')\dot{\gamma}_{dS}(k_2,\tau')\dot{\gamma}_{dS}(k_3,\tau')\prod_{n=1}^{3}\gamma_{dS}^*(k_n,\tau)\right] \\
&\quad + 5\text{ perms.} \qquad \text{(F.8)}
\end{aligned}
$$

Using

$$\dot{\gamma}_{dS} = i\frac{H\tau}{M_{\rm pl}}\sqrt{k}e^{-ik\tau'} \ , \tag{F.9}$$

we can reduce the time integral to

$$\begin{aligned}
&\mathrm{Im}\left[\int_{-\infty}^{\tau} d\tau'(H\tau')^2\left(\frac{\tau'}{\tau_*}\right)^A k_2k_3(\ddot{\gamma}_{dS} - k_1^2\gamma_{dS})(k_1,\tau')\right.\\
&\quad\left.\times\ \dot{\gamma}_{dS}(k_2,\tau')\dot{\gamma}_{dS}(k_3,\tau')\prod_{n=1}^{3}\gamma_{dS}^*(k_n,\tau)\right]\\
&\quad\xrightarrow{\tau\to 0} -\frac{2H^8}{M_{\rm pl}^6}\mathrm{Re}\left[\tau_*^{-A}\int_{-\infty}^{0} d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right] ,
\end{aligned} \tag{F.10}$$

and obtain

$$\begin{aligned}
\left\langle\prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n})\right\rangle_{W^3}^{(2)} &= (2\pi)^3\delta\left(\sum_{n=1}^{3}\mathbf{k_n}\right)8\left(\frac{H}{M_{\rm pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2\\
&\quad\times\mathrm{Re}\left[\tau_*^{-A}\int_{-\infty}^{0} d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right]\\
&\quad\times\frac{3}{4}\hat{k}_{2i}e_{ij}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)\hat{k}_{3j}e_{kl}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{kl}^{(-\lambda_3)}(\hat{\mathbf{k}}_3)\\
&\quad+5\text{ perms.}\ .
\end{aligned} \tag{F.11}$$

The graviton non-Gaussianities from $H_{W^3}^{(3)}$ and $H_{W^3}^{(4)}$ are derived in the same manner as that from $H_{W^3}^{(2)}$:

$$\begin{aligned}
\sum_{m=3}^{4}\left\langle\prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n})\right\rangle_{W^3}^{(m)} &= (2\pi)^3\delta\left(\sum_{n=1}^{3}\mathbf{k_n}\right)8\left(\frac{H}{M_{\rm pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2\\
&\quad\times\mathrm{Re}\left[\tau_*^{-A}\int_{-\infty}^{0} d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right]\\
&\quad\times\left[\frac{3}{4}\hat{k}_{3k}e_{ki}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{ij}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{jl}^{(-\lambda_3)}(\hat{\mathbf{k}}_3)\hat{k}_{2l}\right.\\
&\quad\left.-\frac{3}{2}\hat{k}_{3j}e_{ji}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{ik}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{kl}^{(-\lambda_3)}(\hat{\mathbf{k}}_3)\hat{k}_{2l}\right]\\
&\quad+5\text{ perms.}
\end{aligned} \tag{F.12}$$

F.2 WW^2

At first, we shall focus on the contribution of $H^{(1)}_{\widetilde{W}W^2}$. The bracket part is computed as

$$\begin{aligned}
&\left\langle 0\right| \left[: H^{(1)}_{\widetilde{W}W^2}(\tau') :, \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \right] \left|0\right\rangle \\
&= \left\langle 0\right| : H^{(1)}_{\widetilde{W}W^2}(\tau') : \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \left|0\right\rangle \\
&\quad - \left\langle 0\right| \left[\prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \right] : H^{(1)}_{\widetilde{W}W^2}(\tau') : \left|0\right\rangle \\
&= -\int d^3x' \Lambda^{-2}(H\tau')^2 \left(\frac{\tau'}{\tau_*}\right)^A (-3) \left[\prod_{n=1}^{3} \int \frac{d^3\mathbf{k}'_\mathbf{n}}{(2\pi)^3} e^{i\mathbf{k}'_\mathbf{n}\cdot\mathbf{x}'} \sum_{\lambda'_n=\pm 2} \right] \\
&\quad \times \eta^{ijk} e^{(\lambda'_1)}_{kq}(\hat{\mathbf{k}}'_1) e^{(\lambda'_2)}_{jm}(\hat{\mathbf{k}}'_2) e^{(\lambda'_3)}_{iq}(\hat{\mathbf{k}}'_3)(ik'_{3m}) \\
&\quad \times \Big[\left(\ddot{\gamma}_{dS} - k'^2_1 \gamma_{dS}\right)(k'_1, \tau') \left(\ddot{\gamma}_{dS} - k'^2_2 \gamma_{dS}\right)(k'_2, \tau') \dot{\gamma}_{dS}(k'_3, \tau') \\
&\qquad\quad \times : \left\langle 0\right| \left\{ \prod_{m=1}^{3} a^{(\lambda'_m)}_{k'_m} \right\} \left\{ \prod_{n=1}^{3} \gamma^*_{dS}(k_n, \tau) a^{(\lambda_n)\dagger}_{-k_n} \right\} \left|0\right\rangle : \\
&\qquad - \left(\ddot{\gamma}^*_{dS} - k'^2_1 \gamma^*_{dS}\right)(k'_1, \tau') \left(\ddot{\gamma}^*_{dS} - k'^2_2 \gamma^*_{dS}\right)(k'_2, \tau') \dot{\gamma}^*_{dS}(k'_3, \tau') \\
&\qquad\quad \times : \left\langle 0\right| \left\{ \prod_{n=1}^{3} \gamma_{dS}(k_n, \tau) a^{(\lambda_n)}_{k_n} \right\} \left\{ \prod_{m=1}^{3} a^{(\lambda'_m)\dagger}_{-k'_m} \right\} \left|0\right\rangle : \Big] \\
&= \Lambda^{-2}(H\tau')^2 \left(\frac{\tau'}{\tau_*}\right)^A (-3i) k_3 (2\pi)^3 \delta\left(\sum_{n=1}^{3} \mathbf{k_n}\right) \\
&\quad \times \eta^{ijk} e^{(-\lambda_1)}_{kq}(\hat{\mathbf{k}}_1) e^{(-\lambda_2)}_{jm}(\hat{\mathbf{k}}_2) e^{(-\lambda_3)}_{iq}(\hat{\mathbf{k}}_3) \hat{k}_{3m} \\
&\quad \times 2i\,\mathrm{Im}\Big[\left(\ddot{\gamma}_{dS} - k^2_1 \gamma_{dS}\right)(k_1, \tau') \left(\ddot{\gamma}_{dS} - k^2_2 \gamma_{dS}\right)(k_2, \tau') \\
&\qquad\qquad \times \dot{\gamma}_{dS}(k_3, \tau') \left\{ \prod_{n=1}^{3} \gamma^*_{dS}(k_n, \tau) \right\} \Big] + 5 \text{ perms.}
\end{aligned} \tag{F.13}$$

Via the time integral:

$$
\begin{aligned}
&\mathrm{Im}\left[\int_{-\infty}^{\tau} d\tau'(H\tau')^2\left(\frac{\tau'}{\tau_*}\right)^A k_3\left(\ddot{\gamma}_{dS}-k_1^2\gamma_{dS}\right)(k_1,\tau')\left(\ddot{\gamma}_{dS}-k_2^2\gamma_{dS}\right)(k_2,\tau')\right.\\
&\quad\left.\times\dot{\gamma}_{dS}(k_3,\tau')\left\{\prod_{n=1}^{3}\gamma_{dS}^*(k_n,\tau)\right\}\right]\\
&\qquad\xrightarrow{\tau\to 0}-\frac{4H^8}{M_{\rm pl}^6}\mathrm{Im}\left[\tau_*^{-A}\int_{-\infty}^{0}d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right],
\end{aligned}
\tag{F.14}
$$

we have

$$
\begin{aligned}
\left\langle\prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n})\right\rangle_{\widetilde{W}W^2}^{(1)} &= (2\pi)^3\delta\left(\sum_{n=1}^{3}\mathbf{k_n}\right)8\left(\frac{H}{M_{\rm pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2\\
&\times\mathrm{Im}\left[\tau_*^{-A}\int_{-\infty}^{0}d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right]\\
&\times(-3i)\eta^{ijk}e_{kq}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{jm}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{iq}^{(-\lambda_3)}(\hat{\mathbf{k}}_3)\hat{k}_{3m}\\
&+5\text{ perms.}
\end{aligned}
\tag{F.15}
$$

Like this, we can gain the second counterpart:

$$
\begin{aligned}
\left\langle\prod_{n=1}^{3}\gamma^{(\lambda_n)}(\mathbf{k_n})\right\rangle_{\widetilde{W}W^2}^{(2)} &= (2\pi)^3\delta\left(\sum_{n=1}^{3}\mathbf{k_n}\right)8\left(\frac{H}{M_{\rm pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2\\
&\times\mathrm{Im}\left[\tau_*^{-A}\int_{-\infty}^{0}d\tau'\tau'^{5+A}e^{-ik_t\tau'}\right]\\
&\times i\,\eta^{ijk}e_{kq}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{mi}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{mq}^{(-\lambda_3)}(\hat{\mathbf{k}}_3)\hat{k}_{3j}\\
&+5\text{ perms.}
\end{aligned}
\tag{F.16}
$$

The bracket part with respect to $H_{\widetilde{W}W^2}^{(3)}$ is

$$
\begin{aligned}
&\left\langle 0 \left| \left[: H^{(3)}_{\widetilde{W}W^2}(\tau') :, \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \right] \right| 0 \right\rangle \\
&= \left\langle 0 \right| : H^{(3)}_{\widetilde{W}W^2}(\tau') : \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \left| 0 \right\rangle \\
&\quad - \left\langle 0 \left| \left[\prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}, \tau) \right] : H^{(3)}_{\widetilde{W}W^2}(\tau') : \right| 0 \right\rangle \\
&= - \int d^3x' \Lambda^{-2} (H\tau')^2 \left(\frac{\tau'}{\tau_*} \right)^A 4 \left[\prod_{n=1}^{3} \int \frac{d^3\mathbf{k}'_\mathbf{n}}{(2\pi)^3} e^{i\mathbf{k}'_\mathbf{n} \cdot \mathbf{x}'} \sum_{\lambda'_n = \pm 2} \right] \\
&\quad \times \eta^{ijk} e^{(\lambda'_1)}_{pj}(\hat{\mathbf{k}}'_1) e^{(\lambda'_2)}_{pm}(\hat{\mathbf{k}}'_2) e^{(\lambda'_3)}_{il}(\hat{\mathbf{k}}'_3)(ik'_{1k})(ik'_{2l})(ik'_{3m}) \\
&\quad \times \left[: \left\langle 0 \left| \left\{ \prod_{n=1}^{3} \dot{\gamma}_{dS}(k'_n, \tau') a^{(\lambda'_n)}_{k'_n} \right\} \left\{ \prod_{m=1}^{3} \gamma^*_{dS}(k_m, \tau) a^{(\lambda_m)\dagger}_{-k_m} \right\} \right| 0 \right\rangle : \right. \\
&\qquad \left. - : \left\langle 0 \left| \left\{ \prod_{m=1}^{3} \gamma_{dS}(k_m, \tau) a^{(\lambda_m)}_{k_m} \right\} \left\{ \prod_{n=1}^{3} \dot{\gamma}^*_{dS}(k'_n, \tau') a^{(\lambda'_n)\dagger}_{-k'_n} \right\} \right| 0 \right\rangle : \right] \\
&= \Lambda^{-2} (H\tau')^2 \left(\frac{\tau'}{\tau_*} \right)^A (-4)(-i)^3 k_1 k_2 k_3 (2\pi)^3 \delta \left(\sum_{n=1}^{3} \mathbf{k_n} \right) \\
&\quad \times \eta^{ijk} e^{(-\lambda_1)}_{pj}(\hat{\mathbf{k}}_1) e^{(-\lambda_2)}_{pm}(\hat{\mathbf{k}}_2) e^{(-\lambda_3)}_{il}(\hat{\mathbf{k}}_3) \hat{k}_{1k} \hat{k}_{2l} \hat{k}_{3m} \\
&\quad \times 2i \,\mathrm{Im} \left[\prod_{n=1}^{3} \dot{\gamma}_{dS}(k_n, \tau') \gamma^*_{dS}(k_n, \tau) \right] + 5 \text{ perms.}
\end{aligned} \tag{F.17}
$$

The time integral is

$$
\begin{aligned}
&\mathrm{Im} \left[\int_{-\infty}^{\tau} d\tau' (H\tau')^2 \left(\frac{\tau'}{\tau_*} \right)^A \prod_{n=1}^{3} k_n \dot{\gamma}_{dS}(k_n, \tau') \gamma^*_{dS}(k_n, \tau) \right] \\
&\quad \xrightarrow{\tau \to 0} \frac{H^8}{M_{\rm pl}^6} \mathrm{Im} \left[\tau_*^{-A} \int_{-\infty}^{0} d\tau' \tau'^{5+A} e^{-ik_t \tau'} \right],
\end{aligned} \tag{F.18}
$$

so that the bispectrum of gravitons becomes

$$\left\langle \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}) \right\rangle_{\widetilde{W}W^2}^{(3)} = (2\pi)^3 \delta\left(\sum_{n=1}^{3} \mathbf{k_n}\right) 8 \left(\frac{H}{M_{\rm pl}}\right)^6 \left(\frac{H}{\Lambda}\right)^2 \times {\rm Im}\left[\tau_*^{-A} \int_{-\infty}^{0} d\tau' \tau'^{5+A} e^{-ik_t\tau'}\right] \times i\, \eta^{ijk} e_{pj}^{(-\lambda_1)}(\hat{\mathbf{k}}_1) e_{pm}^{(-\lambda_2)}(\hat{\mathbf{k}}_2) e_{il}^{(-\lambda_3)}(\hat{\mathbf{k}}_3) \hat{k}_{1k} \hat{k}_{2l} \hat{k}_{3m} + 5 \text{ perms.} \tag{F.19}$$

Through the same procedure, the bispectrum from $H^{(4)}_{\widetilde{W}W^2}$ is estimated as

$$\left\langle \prod_{n=1}^{3} \gamma^{(\lambda_n)}(\mathbf{k_n}) \right\rangle_{\widetilde{W}W^2}^{(4)} = (2\pi)^3 \delta\left(\sum_{n=1}^{3} \mathbf{k_n}\right) 8 \left(\frac{H}{M_{\rm pl}}\right)^6 \left(\frac{H}{\Lambda}\right)^2 \times {\rm Im}\left[\tau_*^{-A} \int_{-\infty}^{0} d\tau' \tau'^{5+A} e^{-ik_t\tau'}\right] \times (-i) \eta^{ijk} e_{pj}^{(-\lambda_1)}(\hat{\mathbf{k}}_1) e_{pm}^{(-\lambda_2)}(\hat{\mathbf{k}}_2) e_{im}^{(-\lambda_3)}(\hat{\mathbf{k}}_3) \hat{k}_{1k} \hat{k}_{2l} \hat{k}_{3l} + 5 \text{ perms.} \tag{F.20}$$

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