Representing Vector Fields on a 2-Sphere Using Vector Spherical Harmonics

Frank Corapi and Erik Schnetter
Perimeter Institute for Theoretical Physics, Waterloo, ON, N2L 2Y5 Canada

The initial work required to determine the properties of simulated black hole horizons, using spherical harmonic series representations, is shown here. The different properties of scalar and vector spherical harmonics, and their numerical implementation for a round 2-sphere is also explored throughout this paper. A numerical technique for solving Poisson's equation using spherical harmonics series is demonstrated, and different properties for black hole horizons are calculated.

I. INTRODUCTION

It is often difficult to have a definition of angular momentum for black holes that do not have any symmetries, however, there are have been a few proposed definitions for such a property. This work explores the definition proposed by Korzynski [1], and introduces the preliminary work to calculate the angular momentum of black hole horizons using this definition, which includes representing scalar and vector fields on a round 2-sphere. Representing functions, on a sphere as scalar or vector spherical harmonics series can prove to be very useful, since they allow for different techniques for solving differential equations, such as Poisson's equation.

This paper will define scalar spherical harmonics, as well as polar and axial vector spherical harmonics, and discuss some of their properties and their usefulness. These properties include orthogonality, as well as how to determine the coefficients for their series representation of scalar and vector fields. It also discusses the definition and properties of conformally spherical surfaces, and the properties that can be calculated for them, including the angular momentum proposed by Korzynski [1].

The numerical implementation of the spherical harmonics series representations, using Python, is also discussed, touching upon the errors achievable for different integration techniques, as well as the limits on the length of the series that can be determined. A spherical harmonic series technique for solving Poisson's equation is also discussed. Different properties of a spherical surface, defined in Section III C, are also calculated. All results presented in this paper are for a round 2-sphere.

Future implementations as well as the next steps for this project are discussed in Section V. Eventually, the representation of scalar and vector fields on a conformally spherical surface will be implemented, and the properties of black hole horizons from numerical simulations will be calculated, using Ricci flow techniques (described in [1, 2]).

II. SPHERICAL HARMONICS DEFINITIONS

Before the spherical harmonic functions can be used to represent scalar and vector functions on spherical surface, their definitions and properties must be made known. There are different definitions for scalar, vector and tensor spherical harmonics, however, for the purposes of this paper, the definitions of only the scalar and vector spherical harmonics will be stated.

A. Scalar Spherical Harmonics

The scalar spherical harmonics, Y^{lm}, are functions of θ and ϕ, using the convention where θ is the polar angle from the positive z-axis, and ϕ is the azimuthal angle from the positive x-axis. They are defined as the following:

$$Y^{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P^{lm}(\cos(\theta))e^{im\phi} \qquad (1)$$

where P^{lm} are the Associated Legendre Polynomials, with $l = 0, 1, 2, ...$ and $m = -l, -(l-1), ..., 0, ..., (l-1), l$ [3]. This definition for the scalar spherical harmonics was determined through determining the angular solution to Laplace's equation, since the scalar spherical harmonics are said solutions. They are eigenfunctions of the Laplace operator in spherical coordinates, with the following eigenvalue equation: [3]

$$\nabla^2 Y^{lm} = -l(l+1)Y^{lm} \qquad (2)$$

The square root factor is included for normalization purposes. The scalar spherical harmonics have very useful properties, which allow them to be used as a way of representing scalar functions on the surface of a sphere. These properties will now be discussed.

1. *Orthogonality*

One of the more useful properties of the scalar spherical harmonics is that they form an orthonormal basis, with an orthogonality given by the scalar product, defined as:

$$\langle Y^{lm}, Y^{l'm'}\rangle = \int Y^{*lm}Y^{l'm'}d\Omega = \delta^{ll'}\delta^{mm'} \qquad (3)$$

where $d\Omega$ is the area form of the surface, with a metric given by (8), and the * denotes a complex conjugation [4]. With this property in mind, it can now be discussed how a scalar function can be represented using these spherical harmonics.

2. *Series Representation*

Given that the scalar spherical harmonics form a set of orthonormal basis functions, one can represent a function, f, as a series of these basis functions

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y^{lm}(\theta,\phi) \tag{4}$$

where c_{lm} are scalar coefficients, which can be determined using the orthogonality relation given by (3) [5]. One can see that performing a scalar product with $Y^{l'm'}$ on both sides of (4) gives the values of each of the c_{lm} coefficients,

$$c_{l'm'} = \langle Y^{l'm'}, f\rangle. \tag{5}$$

Representing a scalar function using a list of these spherical harmonic series coefficients can be very convenient for different types of numerical analysis, and from this representation it is very easy to go back to representing a function in terms of θ and ϕ again.

B. Vector Spherical Harmonics

The vector spherical harmonics are defined in terms of the derivatives of the scalar spherical harmonics, and depending on the parity of the field there are different definitions for them. For a polar vector field, the polar vector spherical harmonics are defined as

$$Y_A^{lm} = \nabla_A Y^{lm} = (Y^{lm}_{,\theta}, Y^{lm}_{,\phi}) \tag{6}$$

where Y^{lm} are the scalar spherical harmonics, and the $,\theta$ and $,\phi$ denote partial derivatives with respect to θ and ϕ, respectively [4]. The last equality is its representation in the spherical coordinate basis $[\theta,\phi]$. One can see that this definition is independent of the metric, and only depends on the coordinates used.

For an axial vector field, the axial vector spherical harmonics are defined as

$$\begin{aligned} S_A^{lm} &= \epsilon_{AB} q^{BC} \nabla_C Y^{lm} \\ &= \left(\frac{1}{\sin(\theta)} Y^{lm}_{,\phi}, -\sin(\theta) Y^{lm}_{,\theta}\right) \end{aligned} \tag{7}$$

where ϵ_{AB} is the Levi-Civita tensor, q^{BC} is the inverse metric and the last equality is its representation in the spherical coordinate basis $[\theta,\phi]$ [4]. The spherical metric in the $[\theta,\phi]$ basis is defined as [4]

$$q_{AB} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{bmatrix} \tag{8}$$

and the Levi-Civita tensor in the same basis is defined as [4]

$$\epsilon_{AB} = \sqrt{\det q}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{9}$$

The axial vector spherical harmonics clearly depend upon the metric, however, it can also be seen that for a conformal spherical metric, the representation in the $[\theta,\phi]$ basis is unchanged, since the conformal terms from the Levi-Civita tensor and the inverse metric will eliminate one another.

1. *Orthogonality*

Similar to the scalar spherical harmonics, both the polar and axial vector spherical harmonics also have an orthogonality relationship, however they are not orthonormal, like the scalar spherical harmonics are. For vectors, the scalar product is defined as the following [4]

$$\langle h_A, g_A\rangle = \int h_A^* g^A d\Omega. \tag{10}$$

From (10), the orthogonality relationship between the polar vector spherical harmonics and the axial vector spherical harmonics is given by

$$\langle Y_A^{lm}, Y_A^{l'm'}\rangle = \langle S_A^{lm}, S_A^{l'm'}\rangle = l(l+1)\delta^{l'l}\delta^{m'm} \tag{11}$$

which is an important relationship, since it allows for vector fields to be represented as a series of vector spherical harmonics [4]. It is important to note that this orthogonality is only defined for the round spherical metric, given by (8).

2. *Series Representation*

Given that each of the polar vector spherical harmonics are orthogonal to each other, and that each of the axial vector spherical harmonics are also orthogonal to each other, a vector field can be represented as a series of either polar or axial vector spherical harmonics, depending on its parity. If a given vector field f_A is polar, it can be represented as

$$f_A(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} p_{lm}Y_A^{lm}(\theta,\phi) \tag{12}$$

where p_{lm} are the coefficients in the series [5]. These coefficients can be determined by implementing the orthogonality relation given by (11), resulting in

$$p_{l'm'}=\frac{\langle Y_A^{l'm'},f_A\rangle}{l'(l'+1)}. \tag{13}$$

Similarly, if a vector field g_A is axial, it can be represented as

$$g_A(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}S_A^{lm}(\theta,\phi) \tag{14}$$

where a_{lm} are the coefficients in the series [5]. Once again, by implementing (11), the coefficients can be determined by calculating

$$a_{l'm'}=\frac{\langle S_A^{l'm'},g_A\rangle}{l'(l'+1)}. \tag{15}$$

III. PROPERTIES OF THE 2-SPHERE

There are many properties of a 2-sphere that are fairly interesting to calculate and utilize for the purposes of this paper, as well as for other future applications, both physical and mathematical.

A. Conformally Spherical Metric

A conformally spherical metric is given by the following,

$$q'_{AB}=\Psi^4 q_{AB} \tag{16}$$

where $\Psi^4(\theta,\phi)$ is a positive conformal factor that preserves orientation [1, 6]. Ψ^4 can be determined in many ways, including Ricci flow [2] techniques, and is not unique [1]. The choice of Ψ^4 will define a specific comformally spherical coordinate system, and each of these coordinate systems are related by a conformal transformation of the spherical metric q_{AB} [1].

These transformations are a part of a group that contains the SO(3) subgroup, whose action on a surface is generated by six vector fields [1]. These six vector fields consist of the three fields that rotate about these three orthogonal axes, [1]

$$\begin{aligned}\varphi_1&=-\sin\phi\partial_\theta-\cot\theta\cos\phi\partial_\phi\\ \varphi_2&=\cos\phi\partial_\theta-\cot\theta\sin\phi\partial_\phi\\ \varphi_3&=\partial_\phi\end{aligned} \tag{17}$$

as well as three that generate conformal transformations along these, also orthogonal, axes [1]

$$\begin{aligned}\xi_1&=-\cos\theta\cos\phi\partial_\theta+\frac{\sin\phi}{\sin\theta}\partial_\phi\\ \xi_2&=-\cos\theta\sin\phi\partial_\theta-\frac{\cos\phi}{\sin\theta}\partial_\phi\\ \xi_3&=\sin\theta\partial_\theta.\end{aligned} \tag{18}$$

These six axes can be analytically represented in terms of vector spherical harmonics. The axes given by (17) are all axial and are represented in terms of the axial vector spherical harmonics as follows:

$$\begin{aligned}\varphi_{1A}&=-\sqrt{\frac{2\pi}{3}}\left(S_A^{11}-S_A^{1-1}\right)\\ \varphi_{2A}&=i\sqrt{\frac{2\pi}{3}}\left(S_A^{11}+S_A^{1-1}\right)\\ \varphi_{3A}&=2\sqrt{\frac{\pi}{3}}S_A^{10}.\end{aligned} \tag{19}$$

The axes given by (18) are all polar and are represented in terms of the polar vector spherical harmonics as follows.

$$\begin{aligned}\xi_{1A}&=\sqrt{\frac{2\pi}{3}}\left(Y_A^{11}-Y_A^{1-1}\right)\\ \xi_{2A}&=-i\sqrt{\frac{2\pi}{3}}\left(Y_A^{11}+Y_A^{1-1}\right)\\ \xi_{3A}&=-2\sqrt{\frac{\pi}{3}}Y_A^{10}.\end{aligned} \tag{20}$$

The above representations shown in (19) and (20) are for a 2-sphere with a metric given by q_{AB}. For a conformally spherical metric, the representations in (19) and (20) are then multiplied by the conformal factor Ψ^4, since they are represented in covariant form. Given that Ψ^4 is a function of θ and ϕ, it is evident that the inclusion of a conformal factor can change the parity of each of the axes given above.

B. Angular Momentum

The angular momentum, J, of a spherical surface can be determined through the evaluation of the following six integrals,

$$J_i = \frac{-1}{8\pi G}\int_\Omega \omega(\varphi_i)d\Omega \tag{21}$$

$$K_i = \frac{-1}{8\pi G}\int_\Omega \omega(\xi_i)d\Omega \tag{22}$$

where

$$\omega_A = -(\nabla_A l^\mu)k_\mu = (\nabla_A k^\mu)l_\mu \tag{23}$$

is the rotation one form of the surface Ω, $d\Omega$ is the area form of the surface and φ_i and ξ_i, where $i = 1, 2, 3$, are given by (17) and (18), respectively [1].

The J_i and K_i shown in (21) and (22) can be considered as components of 3-dimensional vectors $\vec{J}$ and $\vec{K}$ in Euclidean space, and using these vectors two invariants, which do not depend on the choice of the conformally spherical coordinate system, can be calculated [1]. These invariants are of second degree and are given by

$$A = |\vec{J}|^2 - |\vec{K}|^2 \tag{24}$$

$$B = \vec{K}\cdot\vec{J} \tag{25}$$

which can then be used to calculate the angular momentum of the surface, using the following relationship [1]

$$J = \sqrt{\frac{A + \sqrt{A^2 + 4B^2}}{2}}. \tag{26}$$

This definition of angular momentum is invariant and can apply to any conformally spherical coordinate system [1]. With the angular momentum, J, known, other properties of the surface can be calculated.

C. Other Properties

There are few other properties of a 2-sphere that are interesting to calculate, and they will be defined in this section. One of the more basic properties of the 2-sphere that can be calculated is the area of the surface

$$A_\Omega = \int_\Omega d\Omega \tag{27}$$

where $d\Omega$ is the area form of the surface Ω [7]. From this the areal radius can be defined in terms of the area of the surface,

$$R_{areal} = \sqrt{\frac{A_\Omega}{4\pi}}. \tag{28}$$

If one considers the surface to be the horizon of a black hole, we can also calculate different properties of the horizon, such as the irreducible mass, the mass energy and the spin. The irreducible mass can be defined as [7]

$$M_{irr} = \frac{R_{areal}}{2}. \tag{29}$$

Using the definition for angular momentum given in (26), the mass energy of the system, M, can be calculated as follows,

$$M^2 = M_{irr}^2 + \frac{J^2}{4M_{irr}^2} \tag{30}$$

and from this the spin of the black hole

$$a = \frac{J}{M^2} \tag{31}$$

can also be calculated [7, 8].

IV. NUMERICAL IMPLEMENTATION

A python script was written and used to represent different functions on a 2-sphere, as well as perform other calculations, such as determining derivatives of spherical harmonic series and solving Poisson's equation, using scalar and vector spherical harmonics. Both built in python functions and manually created functions were used in this numerical implementation. A few of the built in python functions used were "legendre" and "sph_harm" from the scipy.special library which return the legendre polynomials of order n as a function of x, and the value of the spherical harmonic functions at a given value of l, m, θ, ϕ. Another built in function that was used primarily for the integration method was the "legroots" function from the numpy.polynomial.legendre library, which returns a list of all the zeros of a given Legendre polynomial of order n.

The main method used for integrating over the 2-sphere incorporates both a Gauss-Legendre quadrature integration method, as well as a trapezoidal integration method. The Guass-Legendre quadrature method is used to integrate along the z-axis from $z = -1$ to $z = 1$, and the trapezoidal integration method is used to integrate along ϕ, using evenly spaced points in ϕ [9]. All integrations over the 2-sphere are performed using this method, whether it be for determining the coefficients for the spherical harmonic series, or for calculating the angular momentum of the 2-sphere using (26).

Scalar functions were represented using scalar spherical harmonics numerically by calculating the coefficients shown in (5), and storing them in a list structured as follows:

$$c_{lm} = \begin{bmatrix} c_{00} \\ c_{1-1} \\ c_{10} \\ c_{11} \\ \cdots \\ c_{l-m} \\ \cdots \\ c_{l0} \\ \cdots \\ c_{lm} \end{bmatrix} \tag{32}$$

Using the coefficients in this list, and the relationship shown in (4), functions were then represented in terms of θ and ϕ and compared to their corresponding analytic versions. Functions that are explicitly a linear combination of the scalar spherical harmonics converged once the coefficients for each spherical harmonic in the function were calculated, as is shown in Figure 1. The value of N, in Figures 1-3 equals up to which l-value is calculated, with every corresponding m-value for each l term. The coefficients were calculated using the zeros from the 80th order Legendre polynomial as the integration points along the z-axis, and 60 evenly spaced integration points along the ϕ-direction, for both Figures 1 and 2. It was determined that the number of points along the ϕ-direction must be greater than the l-value of the coefficient that is being calculated, otherwise errors are introduced into the solution. An example of this is in Figure 3, where only 30 points were used for integrating along ϕ.

For functions that are not explicitly one of the scalar spherical harmonics, or a linear combination of them, many coefficients are required for the error to converge to a low enough value. In Figure 4, where the zeros of the 86th Legendre polynomial were used as the integration points along the z-axis, and 100 evenly spaced points were used to integrate along the ϕ-direction, one can see that the error does not converge to a low value. Many more coefficients are required for representing such a function in terms of a scalar spherical harmonic series, however, due to how the built in Legendre polynomial python functions are written, runtime errors begin to occur when Legendre polynomials of order 87 or higher are used. This limits this python algorithm to calculating coefficients for values of $l \leq 86$. For the parameters used in 4, the runtime of the program was approximately 6 hours, where as calculating up to $l = 20$ takes around 5 minutes to run.

The structure of the list in (32) matters since the matrix method used for solving Poisson's equation has the Laplace matrix operator defined in such a way to act on a list of the spherical harmonic coefficients that is structured as in (32). The Poisson equation in spherical coordinates

$$\nabla^2 \Phi = \rho \tag{33}$$

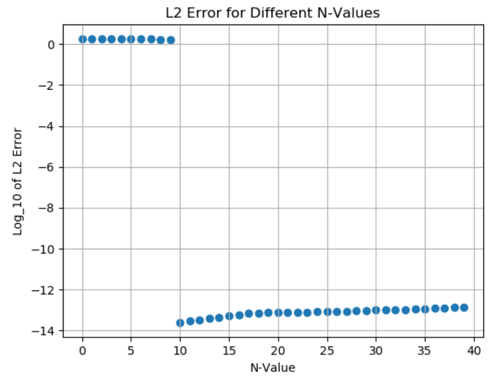

FIG. 1: L2 Errors between the numerical series solution and the analytic solution for the function $\sin(10\theta)\cos(\phi)$ calculated for varying amounts of calculated coefficients.

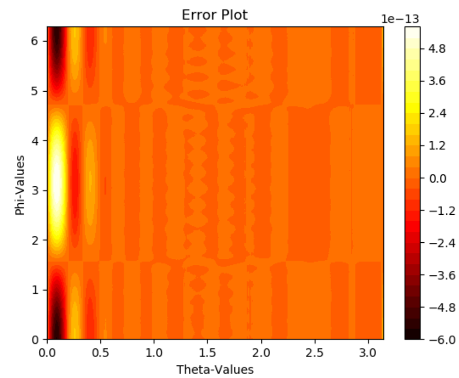

FIG. 2: Error between the numerical series solution and the analytic solution for the function $\sin(10\theta)\cos(\phi)$ calculated for N = 24. Same integration parameters as Figure 1.

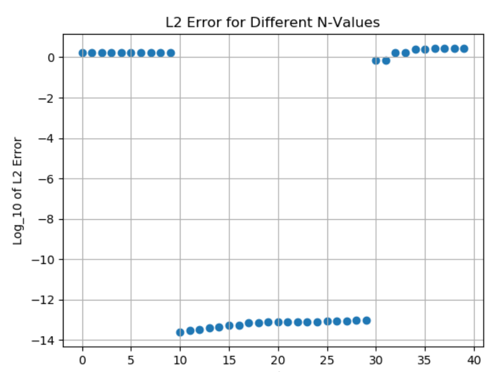

FIG. 3: L2 Errors between the numerical series solution and the analytic solution for the function $sin(10\theta)cos(\phi)$ calculated for varying amounts of calculated coefficients. Error introduced at N=30 is due to only using 30 points when integrating along ϕ.

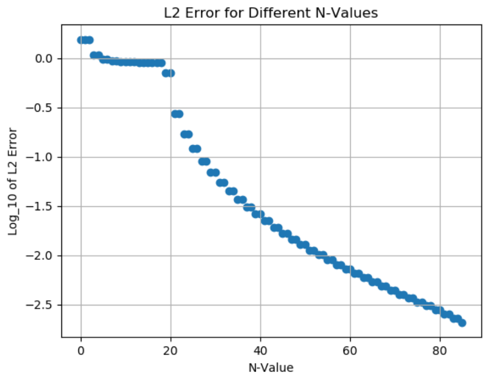

FIG. 4: L2 Errors between the numerical series solution and the analytic solution for the function $sin^2(10\theta)cos(3\phi)$ calculated for varying amounts of calculated coefficients.

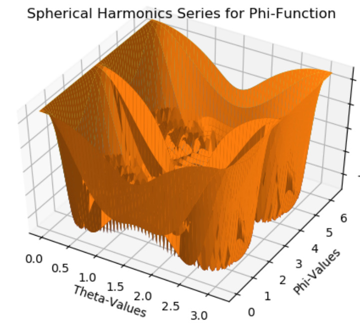

FIG. 5: The series solution and analytic solution for a test Φ function plotted together as a function of θ and ϕ.

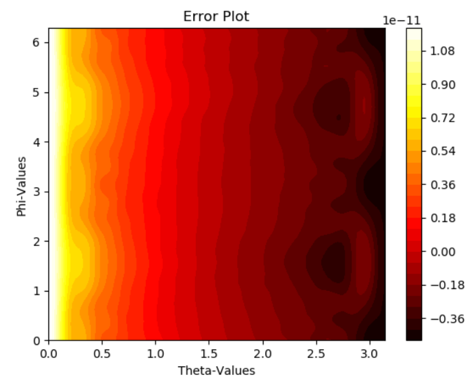

FIG. 6: Error between the numerical series solution and the analytic solution for a test Φ function (N=24).

can be solved by solving a system of linear equations

$$L\phi_{lm} = \rho_{lm} \tag{34}$$

where ϕ_{lm} is the list of the spherical harmonic coefficients for the function Φ in (33) and ρ_{lm} is the list of the spherical harmonic coefficients for the function ρ in (33). Making use of (2), the matrix operator L from (34) that will act on the list of coefficients given by (5) is defined as:

$$L = \begin{bmatrix} 0 & 0 & 0 & \ldots\ldots\ldots\ldots & & & & 0 \\ 0 & -2 & 0 & 0 & \ldots\ldots\ldots & & & 0 \\ 0 & 0 & -2 & 0 & 0 & \ldots\ldots & & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l(l+1) \end{bmatrix}. \tag{35}$$

The matrix is diagonal since the spherical harmonics are eigenvectors of the Laplace operator, and each non-zero element in the matrix is the eigenvalue corresponding to the value of l for each coefficient in the list that this matrix acts upon. The first row of L can be modified depending on the conditions of the problem. The first value in the list for ρ_{lm} will be zero, since the derivative of a constant (ϕ_{00}) is zero. By setting the L_{00} coefficient equal to 1 and modifying the ρ_{00} value to some value, p, we can set a condition on what the ϕ_{00} value can be. Performing these modifications would impose the condition $\phi_{00} = \rho_{00} = p$ when solving the system of linear equations.

For a given function ρ from (33) of θ and ϕ, the spherical harmonic coefficients, ρ_{lm}, can be determined by numerically solving (5). Using L and ρ_{lm}, in this Python implementation, the list of coefficients ϕ_{lm} is determined using the linear equation solver numpy.linalg.solve, and is expanded into a function of θ and ϕ as in (4). The analytic solution for Φ was determined and compared to the numerical solution to determine the numerical error. Figures 5 and 6, are plots for a Φ function given by a linear combination of all the scalar spherical harmonics ranging from $l = 0$ to $l = 24$, with every other m-value for each l term, ranging from $-l$ to l.

The representation of vector fields numerically as a series of vector spherical harmonics was also implemented. However, this was done for vector fields that were either polar or axial, exclusively, rather than for vector fields that were a combination of a polar and axial vector field. This numerical implementation was tested on the 6 vector fields given by (19) and (20), on a surface with (8) as the metric, and L2 errors of the order of 10^{-11} were obtained for each of the different fields. The derivatives of the scalar spherical harmonics were calculated using a finite difference method, with a difference value of $\epsilon = 10^{-5}$. The series only went up to N=1, and the integration method used the zeros of the 6th order Legendre polynomials for the z-axis points, and 100 evenly spaced points were used for the phi integration. The field given by the vector spherical harmonics series, as well as for the analytic vector field, for the two vector fields ξ_{1A}

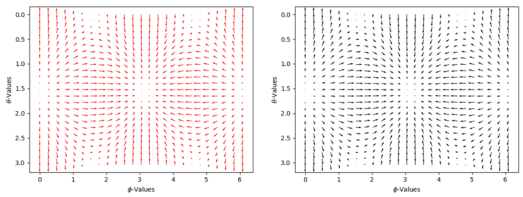

FIG. 7: The field ξ_{1A}, from (20), given by the analytic solution (red), as well as vector spherical harmonic series solution (black). (L2 Error = 5.45×10^{-11})

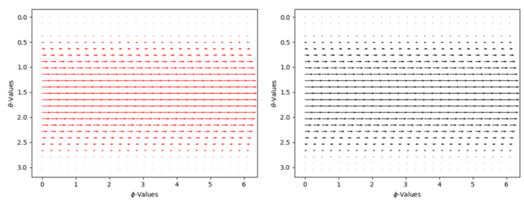

FIG. 8: The field φ_{3A}, from (19), given by the analytic solution (red), as well as vector spherical harmonic series solution (black). (L2 Error = 3.09×10^{-11})

and φ_{3A} are shown in Figures 7 and 8, respectively.

In addition to representing scalar and vector fields on a round 2-sphere using scalar and vector spherical harmonics, the properties defined in Section III C, were also calculated for different ω_A test cases. The different test cases used for ω_A were each of the different co-vectors given in (19) and (20). The results for each of the different properties of the surface are given in Table I.

It turns out that when ω_A was equal to any of the fields from (19), the angular momentum was always equal to 0.33 and when ω_A was equal to any of the fields from (20), the angular momentum was always zero.

V. DISCUSSION AND FUTURE STEPS

This paper discusses the initial steps required for creating a numerical algorithm in python that will be able to determine the properties of the horizon of numerically simulated black hole. It describes the methods used to represent scalar and/or vector fields on a round 2-sphere as a list of coefficients for a scalar and/or vector spherical harmonics series. This representation is quite powerful since it is an elegant way to represent scalar and vector fields on a 2-sphere, and allows for different techniques to be implemented, such as the method used for solving Poisson's equation, which was outlined in Section IV.

ω_A	$\varphi_{1A}, \varphi_{2A}, \varphi_{3A}$	$\xi_{1A}, \xi_{2A}, \xi_{3A}$
J	0.3	0
A_Ω	12.6	12.6
R_{areal}	1	1
M_{irr}	0.5	0.5
M	0.6	0.5
a	0.9	0

TABLE I: Numerically calculated surface (round 2-sphere) properties for six different test cases for ω_A.

When representing the scalar function on the round 2-sphere as a series of scalar spherical harmonics, L2 errors on the order of 10^{-13} were able to be obtained for functions that were explicitly a linear combination of the scalar spherical harmonics. For functions that were not, it seems that many coefficients are required in order for the error to converge to a low enough value. Given that the numerical implementation described in this paper is limited to a series with $N \leq 86$, more complex functions cannot be represented with acceptable errors.

Similarly, when representing the vector fields given by (19) and (20), L2 errors on the order of 10^{-11} were obtained, demonstrating that the numerical method used to represent vector fields as a series of vector spherical harmonics is viable, at least for fields that are explicitly a linear combination of either polar vector spherical harmonics, or axial vector spherical harmonics. A modification that should be implemented in the future is to not assume whether a field is axial or polar before finding the series coefficients for the given field. In practice, it will not be known whether a more complicated field is axial or polar, and thus the coefficients for the axial, as well as the polar, vector spherical harmonics series must be found for any given field. Adding the two series together should reproduce the analytic form of the field, but this needs to be numerically tested once implemented.

All of the results shown in Section IV were for a round 2-sphere, with a metric given by, (8), however, implementing the ability to perform these calculations for a conformally spherical metric (16) will be required for future work. The definitions of the vectors given by (19) and (20) will be changed since a conformal factor must be introduced in front of each definition. Finding the series coefficients for these functions will require finding both the axial and polar series coefficients, since the conformal factor can depend on both θ and ϕ.

Once the above is implemented, an interesting and useful test for the program would be to determine ω_A for a known black hole spacetime metric, such as the Kerr-Schild Metric [10], and calculate the various properties outlined in Section III C. Once the properties, such as the mass and spin, are determined numerically, they can be compared to the expected results from the spacetime metric, in order to determine whether the program is working correctly.

After these additions and tests are performed, the next stage can begin, where an algorithm for finding the horizon properties of a numerically simulated black hole can be made. In order to do this, first a 2-metric from a numerical relativity simulation must be obtained and then writing this metric as a conformal metric, with a conformal factor equal to 1. Using Ricci flow, which is discussed in more detail in [2], a conformally round metric will be found through the modification of the conformal term introduced in from of the original spacetime metric. Then, this newly found metric must undergo a coordinate transformation to the standard θ and ϕ coordinates, in order to obtain a conformally spherical metric. With this newly obtained conformally spherical metric, the properties of the black hole horizon can be calculated using the definitons and methods outlined in Sections III C and IV.

[1] M. Korzynski, "Quasi-local angular momentum of non-symmetric isolated and dynamical horizons from the conformal decomposition of the metric," *Classical and Quantum Gravity*, vol. 24, no. 23, pp. 5935–5943, 2007.

[2] R. S. Hamilton, "The ricci flow on surfaces," *Mathematics and General Relativity Contemporary Mathematics*, pp. 237–262, 1988.

[3] "Wolfram mathworld: Spherical harmonic." `http://mathworld.wolfram.com/SphericalHarmonic.html`. Accessed: 2018-08-15.

[4] E. Berti, *Black Hole Perturbation Theory.* International Center for Theoretical Sciences Summer School on Gravitational-Wave Astronomy, 2016.

[5] M. J. Mohlenkamp, *A Users Guide to Spherical Harmonics.* 2016.

[6] G. B. Cook, "Initial data for numerical relativity," *Living Reviews in Relativity*, vol. 3, no. 1, 2000.

[7] R. M. Wald, *General relativity.* The University of Chicago Press, 1984.

[8] D. Christodoulou, "Reversible and irreversible transformations in black-hole physics," *Physical Review Letters*, vol. 25, no. 22, pp. 1596–1597, 1970.

[9] C. H. L. Beentjes, *Quadrature on a Spherical Surface.* Mathematical Institute, University of Oxford, Oxford, UK, 2015.

[10] S. A. Teukolsky, "The kerr metric," *Classical and Quantum Gravity*, vol. 32, pp. 124006–124038, Jan 2015.