

Isolated and dynamical horizons from a common perspective

Mikołaj Korzyński

*Institute of Theoretical Physics, Warsaw University**

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Abstract

A framework is developed in which one can write down the constraint equations on a three-dimensional hypersurface of arbitrary signature. It is then applied to isolated and dynamical horizons. The derived equations can be used to extract physical relevant quantities describing the horizon irrespective to whether it is isolated (null) or dynamical at a given instant of time. Furthermore, small perturbation of isolated horizons are considered, and finally a family of axially-symmetric exact solution of the constraint equations on a dynamical horizon is presented.

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*Electronic address: mkorz@fuw.edu.pl

I. INTRODUCTION

Since their introduction, isolated and dynamical horizons have been used to describe black holes in general, dynamical situations, in both analytic [1, 2, 3, 4, 5] and numerical [6, 7, 8] context. Both are defined as three-dimensional hypersurfaces foliated by marginally trapped surfaces (so called marginally trapped tubes, MTT's). They are supposed to represent the boundary of a black hole in a non-stationary spacetime. Isolated horizons (IH) are null hypersurfaces and describe black holes which are in equilibrium with the environment at the moment. Dynamical horizons (DH) on the other hand are spacelike and describe a black hole absorbing matter or gravitational radiation.

In physical situations we expect an isolated horizon to become dynamical if matter or gravitational radiation falls on it. On the other hand, it is believed that a perturbed black hole will return to a stationary state after the matter inflow ends and all gravitational radiation is radiated away. In this case a DH should asymptotically become null. In both processes we are led to investigate a hypersurface which can change its signature (from null to spacelike) depending on circumstances.

However, geometrically null and spacelike surfaces are very different. The latter are equipped with a well-defined, positive definite metric, which makes them Riemannian manifolds. Such hypersurfaces are completely described by their metrics and extrinsic curvature. On the other hand, on null surfaces one usually uses a different set of geometrical objects [1].

It is tempting to use perturbative techniques to investigate small perturbations of an IH [9, 10, 11, 12]. This is not easy, as in presence of smallest perturbations the IH is not strictly null anymore. On the other hand, the orthogonal vector cannot be normalized if a DH becomes null and therefore the standard definitions of extrinsic curvature and covariant derivative do not make sense any more [8].

This demonstrates the need to introduce a way of description of a MTT (or any hypersurface) which is applicable to any signature. It should involve “signature-blind” variables which do not refer to any normalization of its orthogonal vector or to the three-dimensional metric on the horizon. It should however contain all information needed to recover the standard geometrical objects if the surface happens to be spacelike.

Although variables of that kind have been used in calculations by several authors [9, 13,

14], there hasn't been a complete discussion of the Einstein constraint equations in this context, neither on MTT's, nor in general. The aim of this paper is to introduce such variables and discuss the Einstein constraint equations on any hypersurface – spacelike, null or timelike. We then focus on MTT's. We also analyse a linearly perturbed IH and finally present a family of axially symmetric, exact solutions of the Einstein constraint equations.

II. “SIGNATURE-BLIND” VARIABLES

We consider a 3-dimensional hypersurface H foliated by a family of 2-dimensional space-like surfaces of spherical topology, called sections. The induced metric on H may have arbitrary signature, or even be degenerate.

By convention, the greek indices will denote spacetime objects, while the latin indices objects defined on sections. In particular, if X_α is a one-form defined on spacetime, X_A will denote its pullback to a section, whereas if X^α is a spacetime vector, X^A will be the orthogonal projection to the tangent space to sections.

On each section we have a positive definite metric q_{AB} . At each point we can pick up two null vectors, k^μ and l^μ , orthogonal to the leaves of the foliation and normalized $l_\mu k^\mu = -1$. We also choose a vector field n^α tangent to H , preserving the foliation and orthogonal to sections.

We may now decompose n according to $n^\mu = n_l l^\mu + n_k k^\mu$. The vector field $\tau^\mu = n_l l^\mu - n_k k^\mu$ is orthogonal to both leaves of the foliation and n , and therefore orthogonal to H .

The choice of l and k is not unique, as we may rescale them simultaneously by any non-vanishing function A : $l'^\mu = A l^\mu$, $k'^\mu = A^{-1} k^\mu$. This gauge freedom can be partially fixed by first fixing l and k at one section and then imposing the conditions

$$\begin{aligned} \nabla_n l^A &= -\nabla^A l_\mu k^\mu \\ \nabla_n l^\alpha l_\alpha &= \nabla_n l^\alpha k_\alpha = 0 \\ \nabla_n k^A &= -\nabla^A k_\mu l^\mu \\ \nabla_n k^\alpha l_\alpha &= \nabla_n k^\alpha k_\alpha = 0. \end{aligned} \tag{1}$$

They can be checked to preserve the normalization of l and k , their null character and their orthogonality to the foliation (see Appendix A).

We have therefore specified the n component of the covariant derivative of l and k . Its pullback to sections constitutes so called Weingarten tensors or deformation tensors

$$\nabla_A l_B = L_{AB} = \sigma_{(AB)}^l + \frac{1}{2} \theta^l q_{AB} \quad (2)$$

$$\nabla_A k_B = K_{AB} = \sigma_{(AB)}^k + \frac{1}{2} \theta^k q_{AB}. \quad (3)$$

They have been decomposed to traceless, symmetric part, called shear, and the trace called expansion. The antisymmetric part vanished since k and l are surface-forming.

We also have the rotation one-form $\omega_A = -\nabla_A l^\mu k_\mu = \nabla_A k^\mu l_\mu$.

If H is spacelike, the data specified here are equivalent to the standard objects in constraint equations, *i.e.* the three-dimensional metric and the extrinsic curvature. The precise transformation is given in Appendix B.

We still have the gauge freedom to multiply l^μ and k^μ by one function *at a given section*:

$$\begin{aligned} l'^\mu &= F l^\mu \\ k'^\mu &= F^{-1} k^\mu \\ \omega'_A &= \omega_A - D_A \ln |F| \end{aligned} \quad (4)$$

III. THE EINSTEIN CONSTRAINT EQUATIONS

The constraint equations on any three-dimensional hypersurface are the Einstein equations involving the $G_{\alpha\beta}\tau^\alpha$ component of the Einstein tensor, where τ is ortogonal to the surface.

The $G_{\alpha\beta}\tau^\alpha$ one-form can be decomposed to the pullback to sections $G_{\alpha B}\tau^\alpha$ and two sections-ortogonal components $G_{\alpha\beta}\tau^\alpha l^\beta$ and $G_{\alpha\beta}\tau^\alpha k^\beta$.

We shall denote the covariant derivative on the section 2-spheres, compatible with q_{AB} , by D , its Laplacian $q^{AB} D_A D_B$ by Δ and its Ricci scalar by \mathcal{R} . The constraint equations read

$$\begin{aligned} \mathcal{L}_n \omega_A &= G_{\mu A} \tau^\mu - n_l (D_B \bar{L}^B_A + \omega_A \theta^l) - D_B n_l L^B_A + \\ &\quad + n_k (D_B \bar{K}^B_A - \omega_A \theta^k) + D_B n_k K^B_A \end{aligned} \quad (5)$$

$$\begin{aligned} \mathcal{L}_n \theta^l &= -G_{\mu\nu} \tau^\mu l^\nu + \Delta n_k - 2\omega^A D_A n_k - n_k \left(\frac{\mathcal{R}}{2} + D_A \omega^A - \omega_A \omega^A + \theta^l \theta^k \right) + \\ &\quad - n_l L_{AB} L^{AB} \end{aligned} \quad (6)$$

$$\begin{aligned}\mathcal{L}_n \theta^k &= G_{\mu\nu} \tau^\mu k^\nu + \Delta n_l + 2\omega^A D_A n_l - n_l \left(\frac{\mathcal{R}}{2} - D_A \omega^A - \omega_A \omega^A + \theta^l \theta^k \right) + \\ &\quad - n_k K_{AB} K^{AB}.\end{aligned}\tag{7}$$

We have introduced the notation $\bar{K}_{AB} = \sigma_{AB}^k - \frac{1}{2}\theta^k q_{AB}$ and analogically for \bar{L}_{AB} . These equations, together with the evolution equation for the metric q_{AB}

$$\mathcal{L}_n q_{AB} = 2n_l L_{AB} + 2n_k K_{AB}\tag{8}$$

constitute a system, which yields initial data for Cauchy problem in the spacelike case. It can be analysed and solved, as we shall see, at least in some special cases.

If we introduce a “time” coordinate on H compatible with n and the foliation, *i.e.* t is constant on sections and $n^\mu t_{,\mu} = 1$, we can replace the Lie derivatives in (5)-(8) by ordinary partial derivatives with respect to t .

Equation (5) was discussed in [8] in the null case and in [14] in a general situation. Equations (6) and (7) appeared in [15] and [10].

IV. CONSTRAINT EQUATIONS ON A MTT

We now set $\theta_l = 0$. The equations (5)–(8) become now

$$\mathcal{L}_n q_{AB} = 2n_l \sigma_{AB} + 2n_k K_{AB}\tag{9}$$

$$\mathcal{L}_n \omega_A = G_{\mu A} \tau^\mu - D_B (n_l \sigma^{l B}_A) + n_k (D_B \bar{K}^B_A - \omega_A \theta^k) + D_B n_k K^B_A\tag{10}$$

$$0 = -G_{\mu\nu} \tau^\mu l^\nu + P n_k - n_l \sigma^{l AB} \sigma^{l AB}\tag{11}$$

$$\mathcal{L}_n \theta^k = G_{\mu\nu} \tau^\mu k^\nu + P^* n_l - n_k \left(\sigma_{AB}^k \sigma^{k AB} + \frac{1}{2} \theta^k{}^2 \right).\tag{12}$$

We have introduced two elliptic differential operators

$$P = \Delta - 2\omega^A D_A - \left(\frac{\mathcal{R}}{2} + \text{div } \omega - \omega^2 \right)\tag{13}$$

$$P^* = \Delta + 2\omega^A D_A - \left(\frac{\mathcal{R}}{2} - \text{div } \omega - \omega^2 \right).\tag{14}$$

They are adjoint to each other [15]: for any section S and complex functions f, g on S we have

$$\int_S \bar{f} P g \sqrt{\det q} d^2 x = \int_S g P^* \bar{f} \sqrt{\det q} d^2 x.\tag{15}$$

Given the matter sources (represented by the Einstein tensor), we have a system of 7 differential equations for 12 quantities. All but one are evolution equations. Equations (11) and (12) appeared in [15] and their combination was used in [10]

Equation (11) doesn't involve time derivatives. It is a differential equation for n_k and algebraic for n_l . Given a spacetime and a marginally trapped surface therein, we can use (11) to construct a family of marginally trapped surfaces constituting a horizon. We may solve it by specifying n_k and then deriving n_l or, if P is invertible, the other way round. In both cases we have a freedom of specifying one function.

The character of H is determined by $\tau^\mu \tau_\mu = 2n_l n_k$. It is null if $n_k = 0$ or $n_l = 0$, spacelike if the signs of n_l and n_k are opposite and timelike if the signs are the same.

If $\sigma^l = 0$ and $G_{\mu\nu} = 0$, we can take $n_k = 0$ and any n_l . In this way we obtain a non-expanding horizon. Although we have a complete freedom of specifying one, positive function n_l , various choices do not lead to different submanifolds of the spacetime, but rather change the slicing of a given H . With proper choice of slicing we can make H a weakly isolated horizon.

If any of those quantities does not vanish and P turns out to be strictly negative, one can prove that H is strictly spacelike provided that $n_l > 0$. This can be done by applying the strong maximum principle [16]. In simple terms, one proves that $P^{-1}f$ is negative if $f > 0$. Note that although P itself is not invariant with respect to gauge transformations (4), the property stated above holds even after rescaling of l and k , provided that we choose $F > 0$ (we don't flip the direction the vectors point).

In spacelike case different solutions of (11) lead to truly different horizons, *i.e.* different submanifolds of the spacetime [17]. Furthermore, if $\theta^k < 0$, the hypersurface is a DH. In [17] it is also proved that, under some reasonable technical assumptions, there are strong restriction concerning the location of DH's with respect to each other. In particular, they cannot foliate any neighbourhood of DH.

A. Angular momentum

Following [1] and [18], we define the angular momentum of H with respect to a specified vector field X^A , tangent to a section, as

$$J_X = -\frac{1}{8\pi G} \int_S X^A \omega_A \sqrt{\det q} d^2x. \quad (16)$$

In the null case this definition agrees completely with the standard one. In the DH case it is slightly different from the definition in [1], which in our terms can be written as

$$J'_X = -\frac{1}{8\pi G} \int_S X^A \left(\omega_A + \frac{1}{2} D_A \ln \left| \frac{n_l}{n_k} \right| \right) \sqrt{\det q} d^2x. \quad (17)$$

However, as was pointed out in [1], the definition is not reliable if X^A is not a symmetry, or at least divergence-free. If $D_A X^A = 0$, both definitions actually coincide. Moreover, (16) is gauge invariant with respect to (4).

B. Small perturbations of an IH

The framework introduced above is particularly well-suited to discuss the perturbation of an IH. As we mentioned, an IH can be perturbed by matter inflow or ($T_{\alpha\beta}\tau^\alpha \neq 0$) or by external gravitational field ($\sigma_l \neq 0$). In this paper we will analyse the perturbation by a gravitational field in the linear order.

External gravitational field, for example a gravitational wave, causes $\sigma^l \neq 0$. This affects the equations (10) and (9), but not (11), as it involves $\sigma^{l\,2}$. Therefore in the linear order n_k remains 0.

This has important consequences: first, the second term in (9) vanishes and therefore the metric is perturbed only by $n_l \sigma_l$. This means that even if the geometry of sections is deformed (by tidal forces), the deformation does not affect their total area.

Moreover, one can integrate equations (9) and (10) with the n_k terms absent. They yield

$$q_{AB}(t) = q_{AB}(t_0) + \int_{t_0}^t 2n_l(t') \sigma_{AB}^l(t') dt' + o(\sigma^l) \quad (18)$$

$$\omega_A(t) = \omega_A(t_0) + D^B(q_{AB}(t) - q_{AB}(t_0)) + o(\sigma^l) \quad (19)$$

(D^B in the second equation is take with respect to $q_{AB}(t_0)$). This means that while ω (and therefore the angular momentum of the horizon) may change in the result of influence of external gravitational field, its change is rigidly connected with the change of geometry. If the horizon returns to its original shape after the perturbation, the angular momentum returns to its original value.

The conclusions above are valid for, among other things, Booth and Fairhurst's slowly evolving horizons [9, 10] without matter accretion. Under their assumptions, σ^l is of the order of $\epsilon a_H^{-1/2}$ [9], ϵ being the Booth-Fairhurst "small parameter" and a_H denoting the horizon

area. However, we do not make any use of the slow change of θ^k

$$|\mathcal{L}_n \theta^k| \sim \epsilon a_H^{-1}$$

(condition 3 in their papers).

V. A FAMILY OF EXACT SOLUTIONS

Bartnik and Isenberg have investigated the constraint equations on a DH in the case of full spherical symmetry [19]. We can generalize their result to a special class of axially symmetric horizons.

The set of equations (9)–(11) seems very difficult to treat at first glance. However, we may achieve a great deal of simplification if we assume $n_l, n_k, \theta_l, \theta_k, T_{\alpha\beta} \tau^\alpha l^\beta$ and $T_{\alpha\beta} \tau^\alpha k^\beta$ to be constant on sections. In this manner we hope to reduce the equations (9)–(11) to a system of ODE's. For simplicity we also put $\sigma_k = \sigma_l = 0$. This implies, due to (9), that the horizon geometry simply scales by a constant conformal factor with time $q(t) = e^{F(t)} q_0$.

The assumption of scalar quantities to be constant on sections imposes further conditions on the geometry of H . In order to satisfy both (11) and (12) we must assume $\frac{1}{2}\mathcal{R} - \text{div } \omega - \omega^2$ and $\frac{1}{2}\mathcal{R} + \text{div } \omega - \omega^2$ to be constant on sections as well. This cannot be achieved unless

$$\text{div } \omega = 0 \tag{20}$$

(on a sphere there are no vector fields of constant and non-vanishing divergence). We therefore have to assume

$$\frac{1}{2}\mathcal{R} - \omega^2 = \text{const} \tag{21}$$

on sections.

We may construct a metric q_0 and a smooth vector field ω_0 satisfying (20) and (21) in the following way: assume q_0 to be axially symmetric

$$q_0 = d\theta^2 + f(\theta)^2 \sin^2 \theta d\varphi^2. \tag{22}$$

$$f(0) = f(\pi) = 1$$

$$f'(0) = f'(\pi) = 0$$

and ω_0 to be proportional to the axial Killing vector $\omega_0 = A(\theta) \partial_\varphi$. ω_0 is now automatically divergence-free and it vanishes at the two poles of the metric $\theta = 0$ and $\theta = \pi$.

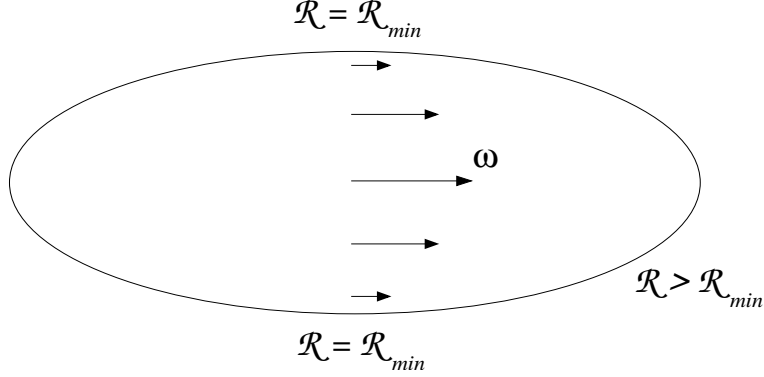


FIG. 1: Geometry of the horizon section

In order to solve (21) assume that the curvature \mathcal{R} attains its global minimum \mathcal{R}_{min} at the poles, *i.e.* $\mathcal{R}(\theta) \geq \mathcal{R}_{min}$ with equality at $\theta = 0$ and $\theta = \pi$. Note that \mathcal{R} is axially symmetric and therefore always has a local maximum, minimum or is constant near the poles. Moreover, all odd terms of its Taylor expansion at $\theta = 0$ and $\theta = \pi$ vanish. Therefore

$$A(\theta) = \sqrt{\frac{\mathcal{R}(\theta) - \mathcal{R}_{min}}{2f(\theta)^2 \sin^2 \theta}}, \quad (23)$$

is well-defined and smooth everywhere. It also solves (21), because $\frac{1}{2}\mathcal{R}(\theta) - \omega_0^2 = \frac{1}{2}\mathcal{R}_{min}$.

The constructed horizon will have, in general, non-vanishing angular momentum with respect to ∂_φ (one can construct examples with strictly vanishing J_φ as well – see Appendix C). It's interesting to note the geometry of sections is necessarily flattened on the poles (*i.e.* \mathcal{R} is smaller) and rounded in the middle due to its rotation, just like in the case of a rotating drop of fluid, and in good agreement with intuition (see fig. 1).

We have to deal with the last equation now, (10). We need ω to rescale together with q in such a way that (21) is preserved. It's straightforward to see that we need the vector $\omega^A(t)$ to be $e^{-F(t)}\omega_0^A$ (or $\omega_A(t) = \omega_0{}_A$ as a one-form). (10) yields now an equation for $G_{\alpha A}\tau^\alpha$: it must be proportional to ω

$$G_{\alpha A}\tau^\alpha = \alpha(t)\omega_A. \quad (24)$$

Let's summarize the considerations now. We take the section metric to be $q = e^{2F(t)}q_0$, the one-form $\omega_A = \omega_0{}_A$, where q_0 and ω_0 are constant in time and constructed according to the prescription stated above. \mathcal{R}_{min} is the minimum of the curvature of q_0 , also constant in time. We introduce the short-hand notation $G_{\mu\nu}\tau^\mu l^\nu = \rho_1(t)$ and $G_{\mu\nu}\tau^\mu k^\nu = \rho_2(t)$.

(9)–(12) yield the following set of equations:

$$\dot{F} = \frac{1}{2} n_k \theta^k \quad (25)$$

$$\alpha = n_k \theta^k \quad (26)$$

$$0 = -\frac{\mathcal{R}_{min}}{2} n_k - \rho_1 e^{2F} \quad (27)$$

$$\dot{\theta}^k = \rho_2 - \frac{\mathcal{R}_{min}}{2} n_l e^{-2F} - \frac{1}{2} n_k \theta^k{}^2. \quad (28)$$

All quantities except \mathcal{R}_{min} are functions of time. For a spacelike surface and $\mathcal{R}_{min} > 0$ this system of ODE's can be reduced to the one in [19], which can be seen by substituting $e^F = \frac{\mathcal{R}_{min}}{2} y^{2/3}$, $\rho_1 = -n_k(\rho + \xi)$, $\rho_2 = n_l(\rho - \xi)$, $n_k n_l = -1/2$ and eliminating θ^k .

We may obtain a Cauchy surface of an Einstein metric if we assume \mathcal{R}_{min} to be equal to 0 (sections flat at the poles). We take $n_k = \rho_1 = \rho_2 = \theta_k = 0$, $F = \text{const}$, $n_l(t)$ and $n_k(t)$ arbitrary. We can choose them in such a way that the horizon is spacelike. By comparing with (B4) and (B3) we conclude that in general the resulting initial data surface is neither time-symmetric, nor extremal. It is not a DH, as expansions of both null vector fields vanish.

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APPENDIX A: GAUGE CONDITION FOR THE NULL ORTOGONAL VECTORS

We assume that k^μ and l^μ are ortogonal to sections of H , which are preserved by the flow of n^α . This is equivalent to

$$\mathcal{L}_n l_\alpha = A l_\alpha + B k_\alpha \quad (A1)$$

$$\mathcal{L}_n k_\alpha = C l_\alpha + D k_\alpha \quad (A2)$$

for some functions A , B , C and D . Furthermore, the normalization conditions, differentiated, yield

$$\nabla_n k_\alpha k^\alpha = \nabla_n l_\alpha l^\alpha = 0 \quad (A3)$$

$$\nabla_n k_\alpha l^\alpha = -\nabla_n l_\alpha k^\alpha. \quad (\text{A4})$$

It follows that $A = D = 0$ and $B = -C$. Without loosing generality and in order to simplify the equations we can put $B = 0$. This, substituted back to (A2), leads directly to (1).

APPENDIX B: EXTRINSIC CURVATURE OF A SPACELIKE SURFACE

For a spacelike surface ($n_l > 0$ and $n_k < 0$) we can normalize τ and n

$$\hat{\tau}^\alpha = (-2n_l n_k)^{-1/2} \tau^\alpha \quad \hat{n}^\alpha = (-2n_l n_k)^{-1/2} n^\alpha. \quad (\text{B1})$$

The extrinsic curvature, *i.e.* the pullback of the covariant derivative of $\hat{\tau}$, expressed in terms of introduced variables, reads:

$$\mathcal{K}_{AB} = n_l L_{AB} + n_k K_{AB} \quad (\text{B2})$$

$$\mathcal{K}_{\alpha\beta} \hat{n}^\alpha \hat{n}^\beta = |8n_l n_k|^{-1/2} \mathcal{L}_n \ln \left| \frac{n_l}{n_k} \right| \quad (\text{B3})$$

$$\mathcal{K}_{A\beta} \hat{n}^\beta = \omega_A + \frac{1}{2} D_A \ln \left| \frac{n_l}{n_k} \right|. \quad (\text{B4})$$

APPENDIX C: NON-SPHERICALLY SYMMETRIC DH'S WITH VANISHING ANGULAR MOMENTUM

We aim to construct a non-spherically symmetric dynamical horizon, of the family presented in Section V, with strictly vanishing angular momentum. This can be achieved if we take $f(\theta)$ such that

1. $\mathcal{R}(\pi/2 + \theta) = \mathcal{R}(\pi/2 - \theta)$,
2. $\mathcal{R}(\theta) \geq \mathcal{R}_{min}$ and $\mathcal{R}(0) = \mathcal{R}(\pi/2) = \mathcal{R}(\pi) = \mathcal{R}_{min}$,
3. all derivatives of \mathcal{R} vanish at $\theta = \pi/2$,

and take $\omega = A(\theta) \partial_\varphi$, with

$$A(\theta) = \pm \sqrt{\frac{\mathcal{R}(\theta) - \mathcal{R}_{min}}{2f(\theta)^2 \sin^2 \theta}}. \quad (\text{C1})$$

The positive sign is taken for $\theta < \pi/2$ and negative for $\theta \geq \pi/2$. $A(\theta)$ is a smooth function despite the sign change, due to condition 3. J_φ is 0 due to conditions 1 and 2 and (C1).

Clearly, in this example the matter falling in does have angular momentum but its inflow is balanced to 0 thanks to different directions of rotation above and below $\theta = \pi/2$.

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- [1] A. Ashtekar and B. Krishnan, *LIVING REV.REL.* **7**, 10 (2004), gr-qc/0407042.
 - [2] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski, and J. Wisniewski, *Physical Review Letters* **85**, 3564 (2000), gr-qc/0006006.
 - [3] A. Ashtekar, C. Beetle, and J. Lewandowski, *Physical Review D* **64**, 044016 (2001), gr-qc/0103026.
 - [4] A. Ashtekar, J. Engle, T. Pawłowski, and C. V. D. Broeck, *Classical and Quantum Gravity* **21**, 2549 (2004), gr-qc/0401114.
 - [5] M. Korzynski, J. Lewandowski, and T. Pawłowski, *Classical and Quantum Gravity* **22**, 2001 (2005), gr-qc/0412108.
 - [6] J. L. Jaramillo, E. Gourgoulhon, and G. A. M. Marugan, *Physical Review D* **70**, 124036 (2004), gr-qc/0407063.
 - [7] E. Schnetter, B. Krishnan, and F. Beyer, *Introduction to dynamical horizons in numerical relativity* (2006), gr-qc/0604015.
 - [8] E. Gourgoulhon and J. L. Jaramillo, *Physics Reports* **423**, 159 (2006), gr-qc/0503113.
 - [9] I. Booth and S. Fairhurst, *Physical Review Letters* **92**, 011102 (2004), gr-qc/0307087.
 - [10] W. Kavanagh and I. Booth, *Spacetimes containing slowly evolving horizons* (2006), gr-qc/0603074.
 - [11] E. Poisson, *Physical Review D* **70**, 084044 (2004), gr-qc/0407050.
 - [12] E. Poisson, *Physical Review Letters* **94**, 161103 (2005), gr-qc/0501032.
 - [13] I. Booth and S. Fairhurst, *Classical and Quantum Gravity* **22**, 4515 (2005), gr-qc/0505049.
 - [14] E. Gourgoulhon, *Physical Review D* **72**, 104007 (2005), gr-qc/0508003.
 - [15] D. M. Eardley, *Physical Review D* **57**, 2299 (1998), gr-qc/9703027.
 - [16] L. Andersson, M. Mars, and W. Simon, *Physical Review Letters* **95**, 111102 (2005), gr-qc/0506013.
 - [17] A. Ashtekar and G. J. Galloway, *Some uniqueness results for dynamical horizons* (2005), gr-qc/0503109.
 - [18] S. A. Hayward, *Angular momentum and conservation laws for dynamical black holes* (2006),

gr-qc/0601058.

- [19] R. Bartnik and J. Isenberg, Classical and Quantum Gravity **23**, 2559 (2006), gr-qc/0512091.