

## ch1: Markov Decision Process

Static Concepts:

state(S) - policy(pi) - action(A) - model(p) - state(S')

reward(R') from transition

return(G) from trajectory

value(V+Q)

策略与价值:

①reward、return、value，用来评估一个策略的好坏。

策略与价值一一对应。

②价值比较，策略比较，策略改进定理。

③强化学习的终极目标，求取最优策略，

最优策略不唯一，最优价值唯一。

最优动作价值，意味着选取这个动作，未来回报的期望最大。

④r线性变换，V+Q线性变换，改变最优价值，不改变greedy最优策略。

⑤迭代时，最优策略可能已经稳定了，但是对应的最优价值还没稳定。

⑥从终止状态反向迭代更新价值，速度更快。但是哪里是终止状态？上帝视角。

$$p(s', r|s, a) = \Pr \{S_t = s', R_t = r | S_{t-1} = s, A_{t-1} = a\}$$

$$\sum_{s' \in S} \sum_{r \in R} p(s', r|s, a) = 1$$

$$p(s'|s, a) = \sum_{r \in R} p(s', r|s, a)$$

$$r(s, a) = \sum_{s' \in S} \sum_{r \in R} (p(s', r|s, a) * r)$$

## ch2: Bellman Equations

Static Relationship

实质：描述状态值之间的静态关系（单项形式、矩阵形式）

求解：（矩阵求逆、数值迭代） — （policy-evaluation）

$$\begin{aligned}
 v_{\pi}(s) &= E_{\pi} [G_t | S_t = s] && (Definition) \\
 &= E_{\pi} [R_{t+1} + \gamma G_{t+1} | S_t = s] && (TD - 0) \\
 &= E_{\pi} [R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s] && (TD - n)(TD - \infty = MC) \\
 &= \sum_{a \in A} \pi(a|s) \sum_{s', r} p(s', r | s, a) * [r + \gamma E_{\pi} [G_{t+1} | S_{t+1} = s']] \\
 &= \sum_{a \in A} \pi(a|s) \sum_{s', r} p(s', r | s, a) * [r + \gamma v_{\pi}(s')] && (BEs) \\
 &= \sum_{a \in A} \pi(a|s) * q_{\pi}(s, a)
 \end{aligned}$$

$$\begin{aligned}
 q_{\pi}(s, a) &= E_{\pi} [G_t | S_t = s, A_t = a] && (Definition) \\
 &= E_{\pi} [R_{t+1} + \gamma G_{t+1} | S_t = s, A_t = a] && (TD - 0) \\
 &= E_{\pi} [R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s, A_t = a] && (TD - n)(TD - \infty = MC) \\
 &= \sum_{s', r} p(s', r | s, a) * [r + \gamma E_{\pi} [G_{t+1} | S_{t+1} = s']] \\
 &= \sum_{s', r} p(s', r | s, a) * [r + \gamma v_{\pi}(s')] \\
 &= \sum_{s', r} p(s', r | s, a) * \left[ r + \gamma \sum_{a' \in A} \pi(a' | s') * q_{\pi}(s', a') \right] && (BEs)
 \end{aligned}$$

policy-comparison:

$$\pi' \geq \pi \quad \longleftrightarrow \quad v_{\pi'}(s) \geq v_{\pi}(s) \quad \forall s \in S$$

policy-improvement:

$$E_{\pi'} [q_{\pi}(s, \pi'(s))] \geq v_{\pi}(s) = E_{\pi} [q_{\pi}(s, \pi(s))] \quad \forall s \in S$$

Bellman Optimal Equations:

$$\begin{aligned} v_*(s) &= \max_{\pi} v_{\pi}(s) && (Definition) \\ &= \max_{\pi} (r_{\pi} + \gamma P_{\pi} v) && (BOEs) \\ &= \max_{a \in A} q_{\pi^*}(s, a) && \forall s \in S \end{aligned}$$

Contraction Mapping Theorem (迭代收敛至唯一不动点)  
 贝尔曼最优方程的收缩迭代过程，即是value iteration算法

## ch3: Dynamic Programming

理解:

Model-based. Dynamics with Model  $p$ .

①已知模型 $p$ , 给定策略 $\pi$ , 解BEs, 得到价值 $V$ 。

两种解法: 矩阵求逆、数值迭代。

(1) Policy Iteration:

Policy Evaluation:

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

Policy Improvement:

$$\pi_{k+1} = \arg \max_{\pi} (r_{\pi} + \gamma P_{\pi} v_k)$$

(2) Value Iteration:

$$v_{k+1} = \max_{\pi} (r_{\pi} + \gamma P_{\pi} v_k)$$

Policy Update:

$$\pi_{k+1} = \arg \max_{\pi} (r_{\pi} + \gamma P_{\pi} v_k)$$

Value Update:

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

(3) Turncated Iteration:

值迭代有限次数（介于1次与无穷次之间）；值也未稳定，就进行策略改进

## ch4: Monte Carlo

Sample:

$$\begin{aligned}v_{\pi}(s) &= E_{\pi} [G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s] & (TD - \bowtie = MC) \\q_{\pi}(s, a) &= E_{\pi} [G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s, A_t = a] & (TD - \bowtie = MC)\end{aligned}$$

理解:

Model-free. Dynamics with Trajectory.

- ①采样进行估计，依据概率论的大数定理。
- ②episode长度（探索半径是否覆盖终点？）对估值影响，最优价值是否反向传播。
- ③估计的更新方式，非增长式（等着一起算）和增长式（来一个算一个）。
- ④epsilon关乎采样策略的探索性和最优性，  
大则探索性强、最优性弱，小则探索性弱、最优性强，
- ⑤如果epsilon大到一定程度，可能会导致epsilon-greedy与最优greedy不一致。

### (1) MC-Basic

二次循环，遍历所有(s, a)；某个策略下，每对采足够样，非增长式估计相应Q。  
策略相应的，一套稳定Q值下，策略改进。  
迭代。

### (2) MC-Exploring-Starts

起始分布，覆盖(s, a)全集。

Pi下，充分利用每一个trajectory里的所有(s, a)对，访问，增长式估计相应Q。  
每一个trajectory结束后，Q值未必稳定，都进行策略改进。  
迭代。

### (3) MC-epsilon-greedy

过程分布，覆盖(s, a)全集。

e-Pi下，充分利用每一个trajectory里的所有(s, a)对，访问，增长式估计相应Q。  
每一个trajectory结束后，Q值未必稳定，都进行策略改进，生成e-Pi。  
迭代。

## \*stochastic approximation

理解:

以某形式的公式，为理论依据，进行实际采样与近似估计。

(1) Incremental-Estimation:

$$\begin{aligned}w_k &= \frac{1}{k} \sum_{i=1}^k x_i & w_{k-1} &= \frac{1}{k-1} \sum_{i=1}^{k-1} x_i \\w_k &= \frac{1}{k} [(k-1)w_{k-1} + x_k] & &= w_{k-1} + \frac{1}{k} [x_k - w_{k-1}]\end{aligned}$$

(2) Robbins-Monro:

$$\begin{aligned}g(w) &= 0 && (\text{g is unknown, w is input, 0 is output}) \\g(w) &= \nabla_w L(w) = 0 && (\text{w is parameters}) \\g(w) &= L(w) - C = 0 \\w^* &\text{ is the solution (Convergence Condition)}\end{aligned}$$

$$\begin{aligned}w_{k+1} &= w_k + a_k [\tilde{g}(w_k, \eta_k) - 0] \\&= w_k + a_k (g(w_k) + \eta_k) \\iteration : & \quad \{w_k\} + \{\tilde{g}_k\} + \{a_k\}\end{aligned}$$

(3) Optimazation:

$$\begin{aligned}\min_w J(w) &= E[f(w, X)] \\&\Rightarrow \Rightarrow \quad \nabla_w E[f(w, X)] = 0 \\&\Rightarrow \Rightarrow \quad \text{Gradient : } \quad \text{InputSpace, } \quad \text{direction} + \text{magnitude}\end{aligned}$$

GD + (Mini)Batch GD + Stochastic GD:

$$\begin{aligned}w_{k+1} &= w_k - \alpha_k \nabla_w E[f(w_k, X)] = w_k - \alpha_k E[\nabla_w f(w_k, X)] \\w_{k+1} &= w_k - \alpha_k \frac{1}{n} \sum_{i=1}^n \nabla_w f(w_k, x_i) \\w_{k+1} &= w_k - \alpha_k \nabla_w f(w_k, x_k)\end{aligned}$$

## ch5: Temporal Difference

理解:

Model-free. Dynamics with Transition.

①TD时序差分: 在不同时刻, 对同一个量估计, 有差, 利用差改进估计。

②没有模型p、只有数据t, 进行估计:

MC利用整条trajectory, 估计V、Q, 离线/无偏/大方差;

TD利用片段transition, 估计V、Q, 在线/有偏/小方差。

③SARSA, 用TD估计某个Pi的Q。

④Q-Learning, 用TD直接估计Q\*。(异轨, 从数据中提取共性MDP-Q\*信息)

⑤行为策略采集数据, 利用数据计算TD-target (数据组织形式),

目标策略的估计值趋向TD-target, 即TD-target代表着目标策略。

⑥on-policy: 用一个策略和环境交互, 得到experience, 估计这个策略, 改进这个策略。

再用改进的策略循环, 交互、估计、改进, 直至最优 (这个系列的最优)。

off-policy: 一个策略和环境交互, 得到experience,

另一个策略被估计、被改进。(目标策略不易采集数据, 所以需要行为策略, 异轨)

experience应当具有一些性质, 能够架通两个策略,

行为策略能够采到, 目标策略能够使用。

(1) TD-V:      s-a-r-s

$$v_{t+1}(s_t) = v_t(s_t) + \alpha_t [[r_{t+1} + \gamma v_t(s_{t+1})] - v_t(s_t)]$$

$$v_{t+1}(s) = v_t(s) \quad \forall s \neq s_t$$

(2) TD-Q: SA-R-SA

Policy Evaluation:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) + \alpha_t(s_t, a_t)[r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1}) - q_t(s_t, a_t)]$$

$$q_{t+1}(s, a) = q_t(s, a) \quad \forall (s, a) \neq (s_t, a_t)$$

Policy Improvement:

某个策略下的Q值可能还未估计稳定,  
就可以进行策略提升。最终目标提升至Q\*。

(3) TD-Q\*: Q\*-Learning (s-a-r-s)

Value Improvement:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) + \alpha_t(s_t, a_t)[r_{t+1} + \gamma \max_a q_t(s_{t+1}, a) - q_t(s_t, a_t)]$$

$$q_{t+1}(s, a) = q_t(s, a) \quad \forall (s, a) \neq (s_t, a_t)$$

\*(DP+MC+TD)-Summary:

value	DP-model(p)	TD-data(t)
v	Policy-Evaluation(vBEs)	TD(0/n/...)
v*	Value-iteration(vBOEs)	-
q	Policy-Evaluation(qBEs)	TD(0)=SARSA
q*	Value-iteration(qBOEs)	Q*-Learning



## ch6: Value Function Approximation

理解:

①函数拟合的优势: 用更少的参数量 (存储), 估计, 更多的状态量 (泛化);  
原因在于, 函数这种表达形式, 存储了更本质的信息。

函数拟合的劣势: 存在精度损失。

函数拟合的关键: 最优的结构 + 最优的参数, 提升拟合准确程度。

②状态分布(.): 平均分布U、稳态分布D ( $D + (\pi + p) = D$ )。

优化时, SGD将不同分布吸收拉平了。

③某个真实分布中的一系列真值, 形成一个超面;

经过各种采样, 得到超面的部分观测真值;

建立一个模型并优化参数, 用来拟合部分观测真值得到的超面。

④自举, 让不均匀的高低估扩展, 可能导致最优性发生变化。

(1) Objective Function:

$$J(w) = E_{S \sim (\cdot)} [(v_\pi(S) - \tilde{v}(S, w))^2]$$

(2) Optimazation:

$$\begin{aligned} w_{k+1} &= w_k - \alpha_k \nabla_w J(w_k) \\ w_{t+1} &= w_t + \alpha_t (v_\pi(s_t) - \tilde{v}(s_t; w_t)) \nabla_w \tilde{v}(s_t; w_t) \end{aligned}$$

(3) Algorithm:

$$\begin{aligned} w_{t+1} &= w_t + \alpha_t (g_t - \tilde{v}(s_t; w_t)) \nabla_w \tilde{v}(s_t; w_t) & MC + VA \\ w_{t+1} &= w_t + \alpha_t [r_{t+1} + \gamma \tilde{v}(s_{t+1}; w_t) - \tilde{v}(s_t; w_t)] \nabla_w \tilde{v}(s_t; w_t) & TD + VA \end{aligned}$$

(4) Approximator:

$$\begin{aligned} linear &= kx + b \\ nonlinear &= neural \ network \end{aligned}$$

(5) SARSA + VA: (SA-R-SA + q- + w)

Value Update(w):

$$w_{t+1} = w_t + \alpha_t [r_{t+1} + \gamma \tilde{q}(s_{t+1}, a_{t+1}; w_t) - \tilde{q}(s_t, a_t; w_t)] \nabla_w \tilde{q}(s_t, a_t; w_t)$$

Policy Improvement: epsilon-greedy

(6) Q\*-Learning + VA: (S-A-R-S + q- + w)

Value Update(w):

$$w_{t+1} = w_t + \alpha_t [r_{t+1} + \gamma \max_a \tilde{q}(s_{t+1}, a; w_t) - \tilde{q}(s_t, a_t; w_t)] \nabla_w \tilde{q}(s_t, a_t; w_t)$$

Policy Improvement: (epsilon-)greedy

(7) Deep Q\*-Learning: (Buffer(S-A-R-S) + NN(w)\*2)  
(continuous states + discrete actions)

Value Update(w):

$$w_{t+1} = w_t + \alpha_t [r_{t+1} + \gamma \max_a \tilde{q}(s_{t+1}, a; w_t) - \tilde{q}(s_t, a_t; w_t)] \nabla_w \tilde{q}(s_t, a_t; w_t)$$

Policy Improvement: (epsilon-)greedy

(8) Double Q\*-Net: (Buffer(S-A-R-S) + NN(w)\*2)

## ch7: Policy Gradient

\*Policy-Summary:

Policy		
Form	Distribution	Network Model
Evaluation	Values (V + Q)	Performance Metrics
Optimization	(epsilon)-greedy	Gradient-Descent

(1) Policy Network: (Storage + Generalization)

$$\pi(a|s; \theta) \quad \text{from Distribution to Network Model}$$

(2) Optimal Policy: (!= Optimal Values)

$$\max_{\theta} \text{metrics}[\pi(a|s; \theta)] \quad \text{from Values to Metrics}$$

(3) Distribution(d):

$$\begin{aligned} d - \pi : e.g. \quad & \text{Uniform Distribution :} & d(s) = 1/|S| & \quad \forall s \in S \\ d + \pi : e.g. \quad & \text{Stationary Distribution :} & d_{\pi}^T \cdot P_{\pi} = d_{\pi}^T & \\ d_0 : e.g. \quad & \text{Specific Distribution :} & d_0(s_0) = 1 & \quad d_0(s) = 0 \quad \forall s \neq s_0 \end{aligned}$$

(4) Metric-1: average state value (?) = average average return

$$\begin{aligned} J_1(\theta) &= \overline{v_{\pi}} = E_{s \sim d}[v_{\pi}(s)] = \sum_{s \in d} d(s) v_{\pi}(s) = d^T \cdot v_{\pi} \\ &= E \left[ \sum_{t=0}^{\infty} \gamma^t R_{t+1} \right] \end{aligned}$$

(5) Metric-2: average one-step reward = average infinite reward

$$\begin{aligned} J_2(\theta) &= \overline{r_{\pi}} = E_{s \sim d}[r_{\pi}(s)] = \sum_{s \in d} d(s) r_{\pi}(s) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} E \left[ \sum_{k=1}^n R_{t+k} \right] \end{aligned}$$

(6)Metrics-Relationship:

$$\bar{r}_\pi = (1 - \gamma)\bar{v}_\pi$$

(7)Policy Gradient Theorem:

$$\nabla_\theta J(\theta) = \sum_{s \in S} \eta(s) \sum_{a \in A} \nabla_\theta \pi(a|s; \theta) q_\pi(s, a)$$

proof. Zhang Weinan

$$= E [\nabla_\theta \ln \pi(A|S; \theta) q_\pi(S, A)]$$

$$\theta_{t+1} = \theta_t + \alpha \nabla_\theta \ln \pi(a_t|s_t; \theta_t) q_\pi(s_t, a_t)$$

stochastic gradient ascent

$$= \theta_t + \alpha \frac{q_\pi(s_t, a_t)}{\pi(a_t|s_t; \theta_t)} \nabla_\theta \pi(a_t|s_t; \theta_t)$$

exploitation + exploration

(8)REINFORCE:

$$\theta_{t+1} = \theta_t + \alpha \nabla_\theta \ln \pi(a_t|s_t; \theta_t) q_t(s_t, a_t)$$

$$q_\pi(s_t, a_t) : \quad q_t(s_t, a_t)(MC)$$

## ch8: Actor Critic

(1) QAC: (SARSA-VA) + PG

$$\begin{aligned}w_{t+1} &= w_t + \alpha_w [r_{t+1} + \gamma \tilde{q}(s_{t+1}, a_{t+1}; w_t) - \tilde{q}(s_t, a_t; w_t)] \nabla_w \tilde{q}(s_t, a_t; w_t) \\ \theta_{t+1} &= \theta_t + \alpha_\theta \nabla_\theta \ln \pi(a_t | s_t; \theta_t) \tilde{q}(s_t, a_t; w_{t+1})\end{aligned}$$

(2) A2C: (TD-VA) + PG

$$\begin{aligned}q_\pi(s_t, a_t) - BS &= \tilde{q}(s_t, a_t; w'_t) - \tilde{v}(s_t; w_t) \\ = \delta_t &= r_{t+1} + \gamma \tilde{v}(s_{t+1}; w_t) - \tilde{v}(s_t; w_t) \\ w_{t+1} &= w_t + \alpha_w \delta_t \nabla_w \tilde{v}(s_t; w_t) \\ \theta_{t+1} &= \theta_t + \alpha_\theta \nabla_\theta \ln \pi(a_t | s_t; \theta_t) \delta_t\end{aligned}$$