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An expression for the radiation force exerted by an acoustic beam with arbitrary wavefront (L)

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Most studies investigating the acoustic radiation force upon a target are based on symmetry considerations between the object and the incident beam. Even so, this symmetry condition is not always fulfilled in several cases. An expression for the radiation force is obtained as a function of the beam-shape and the scattering coefficients of an incident wave and the object, respectively. The expression for the radiation force caused by a plane wave on a rigid sphere is used to validate the formula. This method represents a theoretical advance permitting different interpretations and predictions concerned to the acoustic radiation force phenomenon. © *2011 Acoustical Society of America*. [DOI: 10.1121/1.3652894]

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I. INTRODUCTION

Static radiation force caused by single-frequency acoustic waves has been intensively investigated over the last century.[1–9] More recently, investigations of radiation force have been performed considering focused[10,11] and limited-diffracting beams[12–15] acting upon a sphere in the on-axis configuration. Many useful applications have emerged based on acoustic radiation force (static or dynamic): measuring ultrasound power,[16] particle manipulation,[17] acoustic tweezer,[18] ultrasound stimulated spectrography,[19] biomedical imaging,[20] lab-on-a-chip tweezers,[21] and acoustic levitator,[22] to name a few.

Most theoretical analysis carried out for acoustic radiation force have considered the incident beam as either a plane wave or an axisymmetric beam with respect to the target object. Few exceptions to these assumptions have been done so far. Gor'kov[23] proposed a radiation force model considering an incident beam of arbitrary shape, but his method is limited to compressional spheres smaller than the incident wavelength. Computational methods based on ray acoustics,[24] Boltzmann lattice,[25] and finite-differences[26] have been proposed to calculate radiation force on objects. The ray acoustics approach is only valid for objects that are much larger than the incident wavelength. The other computational methods mentioned here might be both time and memory consuming for three-dimensional (3D) radiation force problems. Therefore a method to calculate the radiation force due to an arbitrary shaped beam over an object in 3D without restrictions would be certainly useful in the study of radiation force and its applications.

We present a method to compute the acoustic radiation force upon an object suspended in a nonviscous fluid due to an arbitrary shaped wave. The nonviscous approximation is valid when the acoustic boundary layer $\delta = \sqrt{\nu/\omega}$ (with ν and ω being the dynamic viscosity and the angular frequency, respectively) is much smaller than the dimensions of the target object.[27] A series expression for the radiation force components in the Cartesian coordinates is obtained. This expression is given in terms of the beam-shape and the scattering coefficients for the object. In turn, the beam-shape and the scattering coefficients are those present in the partial wave expansion of the incident beam and the scattered wave. When the incident wave is asymmetric with respect to the target object, the beam-shape coefficients can be obtained by numerical quadrature.[28,29] The scattering coefficients are obtained through appropriate boundary conditions on the surface of the target. To illustrate the method, we compute the radiation force on a rigid sphere due to a plane wave which propagates in arbitrary direction. A discussion on the applications of the proposed method as a new tool to inquire the physical aspects of the radiation force phenomenon is also outlined.

II. ACOUSTIC RADIATION FORCE MODEL

Consider a homogeneous and lossless fluid with ambient density ρ_0 and speed of sound c_0. Let $\mathbf{v} = -\nabla\phi$ be the particle velocity, where ϕ is the velocity potential function. The excess of pressure due to the waves in the fluid is $p = \rho_0(\partial\phi/\partial t)$. These acoustic fields are function of the position vector $\mathbf{r}$, with respect to a fixed coordinate system, and the time t.

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To calculate the acoustic radiation force upon an target object due to an incident beam, we have to obtain the scattered fields by the target. The total acoustic fields is the sum of the incident and the scattered components. Accordingly, we have $\phi = \phi_i + \phi_s, p = p_i + p_s, \mathbf{v} = \mathbf{v}_i + \mathbf{v}_s$, where the indexes "$i$" and "$s$" stand for the incident and scattered fields. We also introduce the complex amplitude of these fields $\hat{\phi}$, $\hat{p}$, and $\hat{\mathbf{v}}$ normalized to their respective perturbation magnitudes ϕ_0, p_0, and v_0.

Assume that the incident beam has angular frequency ω and insonifies a suspended object in the fluid. The amplitude of the incident and scattered potential functions are, respectively,[30]

$$\hat{\phi}_i = \sum_{l,m} a_l^m j_l(kr) Y_l^m(\Omega), \tag{1}$$

$$\hat{\phi}_s = \sum_{l,m} s_l^m h_l^{(1)}(kr) Y_l^m(\Omega), \tag{2}$$

where $\sum_{l,m} \to \sum_{l=0}^{\infty} \sum_{m=-l}^{l}$, $k = \omega/c_0$, $\{a_l^m\}$ and $\{s_l^m\}$ are the beam-shape and scattering coefficients, $\{j_l\}$ are the spherical Bessel functions, $\{h_l^{(1)}\}$ are the first-kind spherical Hankel functions, $\{Y_l^m\}$ are the spherical harmonics, and $\Omega = (\theta, \varphi)$ is the solid angle in spherical coordinates. The scattering coefficient can be determined for targets made out of fluid, elastic, or viscoelastic materials through appropriate boundary conditions. The beam-shape coefficient is given by

$$a_l^m = \frac{1}{j_l(kR)} \int_{4\pi} \hat{\phi}_i(kR, \Omega) Y_l^{m*}(\Omega) d\Omega, \tag{3}$$

where $l \geq 0$, $|m| \leq l$, and R is the radius of the spherical region in which the incident beam propagates. This region contains the target object. To numerically evaluate Eq. (3), one should choose the size factor kR different than the zeros of the spherical Bessel function $j_l(kR)$. This is always possible because, in principle, the propagation spherical region can have arbitrary size as long as it encloses the object.

In the farfield $kr \gg 1$, the velocity potential amplitudes are

$$\hat{\phi}_i = \frac{1}{kr} \sum_{l,m} a_l^m \sin\left(kr - \frac{l\pi}{2}\right) Y_l^m(\Omega), \tag{4}$$

$$\hat{\phi}_s = \frac{e^{ikr}}{kr} \sum_{l,m} i^{-l-1} s_l^m Y_l^m(\Omega). \tag{5}$$

The radiation force acting upon the target object can be found by integrating the radiation stress tensor produced by the incident and scattered waves over the object surface. The radiation stress tensor is given, in the second-order approximation, by[5]

$$\mathbf{S} = \rho_0 \overline{\mathbf{v}\mathbf{v}} - \overline{\mathcal{L}}\mathbf{I}, \tag{6}$$

where the over bar denotes time-average, $\mathbf{I}$ is the second-rank unit-tensor in $\mathbb{R}^3$, and $\rho_0 \mathbf{v}\mathbf{v}$ is the Reynolds stress tensor. The acoustic Lagrangian density is

$$\mathcal{L} = \frac{\rho_0 v^2}{2} - \frac{p^2}{2\rho_0 c_0^2},$$

where $v = \|\mathbf{v}\|$.

Owing to the fact that $\nabla \cdot \mathbf{S} = 0$ for lossless fluids, the radiation force can be obtained in the farfield, which significantly simplifies the calculations. It is also acknowledged that nearfield calculations in a lossless fluid provide equivalent results. This has been verified in the context of the radiation force perpendicular to the axis of a cylinder.[31,32] The radiation force might be obtained by integrating $\mathbf{S}$ over the surface of a control sphere whose radius $r \to \infty$. Thus, using Eq. (6) one can show that the radiation force on the object is given by[5]

$$\mathbf{f} = \lim_{r\to\infty} r^2 \int_{4\pi} \Big\{ \overline{\mathcal{L}_{is}} \mathbf{e}_r - \rho_0 \Big[\overline{(\mathbf{v}_i \cdot \mathbf{e}_r)\mathbf{v}_s} + \overline{(\mathbf{v}_s \cdot \mathbf{e}_r)\mathbf{v}_i} + \overline{(\mathbf{v}_s \cdot \mathbf{e}_r)\mathbf{v}_s} \Big] \Big\} d\Omega,$$

where $\mathbf{e}_r = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ is the unit-vector in the outward radial direction, and

$$\mathcal{L}_{is} = \rho_0 (\mathbf{v}_i \cdot \mathbf{v}_s) - \frac{p_i p_s}{\rho_0 c_0^2}.$$

is the interacting Lagrangian density. In terms of the incident and scattered potential functions, we have

$$\mathbf{f} = E_0 \lim_{r\to\infty} \int_{4\pi} \mathrm{Re}\{\Phi_{is}\} \mathbf{e}_r d\Omega, \tag{7}$$

where $E_0 = p_0^2 / 2\rho_0 c_0^2$ is the potential energy density averaged in time of the incident wave, "Re" means the real-part, and

$$\Phi_{is} = -r^2 \left[\left(\hat{\phi}_i - \frac{i}{k} \frac{\partial \hat{\phi}_i}{\partial r} \right) \hat{\phi}_s^* + |\hat{\phi}_s|^2 \right]. \tag{8}$$

Using the farfield expressions Eqs. (4), (5), and (8) into Eq. (7), we find

$$\Phi_{is} = -\frac{1}{k^2} \sum_{l,m} \sum_{l',m'} i^{l'-l} (a_l^m + s_l^m) s_{l'}^{m'*} Y_l^m(\Omega) Y_{l'}^{m'*}(\Omega). \tag{9}$$

We can write the radiation force in the Cartesian coordinates as follows

$$\mathbf{f} = \pi a^2 E_0 (Y_x \mathbf{e}_x + Y_y \mathbf{e}_y + Y_z \mathbf{e}_z), \tag{10}$$

where $\{\mathbf{e}_i, i = x, y, z\}$ are the unit-vectors along the Cartesian axes, and Y_x, Y_y, and Y_z are the radiation force functions. Now, using the integral relations of the spherical harmonics given in Ref. 33, one obtains

$$Y_x = \frac{1}{2\pi (ka)^2} \mathrm{Im} \sum_{l,m} (a_l^m + s_l^m) \big(s_{l+1}^{m-1*} b_{l+1}^{-m} + s_{l-1}^{m-1*} b_l^{m-1} - s_{l+1}^{m+1*} b_{l+1}^{m} - s_{l-1}^{m+1*} b_l^{-m-1} \big), \tag{11}$$

$$Y_y = \frac{1}{2\pi(ka)^2}\mathrm{Re}\sum_{l,m}(a_l^m + s_l^m)\big(s_{l+1}^{m+1*}b_{l+1}^m + s_{l-1}^{m+1*}b_l^{-m-1} + s_{l+1}^{m-1*}b_{l+1}^{-m} + s_{l-1}^{m-1*}b_l^{m-1}\big), \tag{12}$$

$$Y_z = \frac{1}{\pi(ka)^2}\mathrm{Im}\sum_{l,m}(a_l^m + s_l^m)\big(s_{l+1}^{m*}c_{l+1}^m - s_{l-1}^{m*}c_l^m\big), \tag{13}$$

where

$$b_l^m = \sqrt{\frac{(l+m)(l+m+1)}{(2l-1)(2l+1)}}, \quad c_l^m = \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}.$$

According to Eqs. (11)–(13), the radiation force exerted on an object due to an arbitrary shaped beam can be obtained by computing the beam-shape and the scattering coefficients. Furthermore, these equations are the generalization of the one-dimensional radiation force function for a plane wave introduced in Ref. 6.

Consider an axisymmetric beam with a spherical target placed on the beam axis, which coincides to the z-axis. Hence the beam is invariant under an azimuthal rotation with respect to the coordinate system. Thus, $\hat{\phi}_i(r,\theta,\varphi-\varphi_0) = \hat{\phi}_i(r,\theta,\varphi)$, where $0<\varphi_0<2\pi$. Using Eq. (3), one can show that the beam-shape coefficient of the rotated beam is $a_l^m e^{-im\varphi_0}$. Because φ_0 is arbitrary, we have that $m=0$. Thus $a_l^m \propto \delta_{m,0}$, where δ_{mn} is the Kronecker delta function. Substituting this coefficient into Eqs. (11) and (12) and noting that $\delta_{m,0}\delta_{m\pm1,0}=0$, one finds $Y_x = Y_y = 0$, as expected.

III. NUMERICAL EVALUATION

We assume that the target object is a rigid sphere of radius a. The origin of the coordinate system is set to the center of the sphere. The scattering coefficient can be obtained from the Neumann boundary for the velocity potential function. Thus $s_l^m = -[j_l'(ka)/h_l^{(1)'}(ka)]a_l^m$, with the prime symbol denoting differentiation.

The radiation force upon a sphere caused by a plane wave propagating along the z-axis in the positive direction is given by[34]

$$Y = -\frac{4}{(ka)^2}\sum_{l=0}^{\infty}(l+1)[\alpha_l + \alpha_{l+1} + 2(\alpha_l\alpha_{l+1} + \beta_l\beta_{l+1})], \tag{14}$$

where α_l and β_l are the real and the imaginary part of $-j_l'(ka)/h_l^{(1)'}(ka)$, respectively. The symmetry of the problem implies that $Y_x = Y_y = 0$. The magnitude of the radiation force exerted by a plane wave over the sphere is invariant under a rotation of the coordinate system. We will use this invariance to verify Eqs. (11)–(13).

Consider now an arbitrary plane wave whose the wave-vector is represented, in spherical coordinates, as $\mathbf{k} = (k,\alpha,\beta)$, where α and β are the polar and azimuthal angles, respectively. The beam-shape coefficient for this plane wave is given by[35]

$$a_l^m = 4\pi i^l Y_l^{m*}(\alpha,\beta).$$

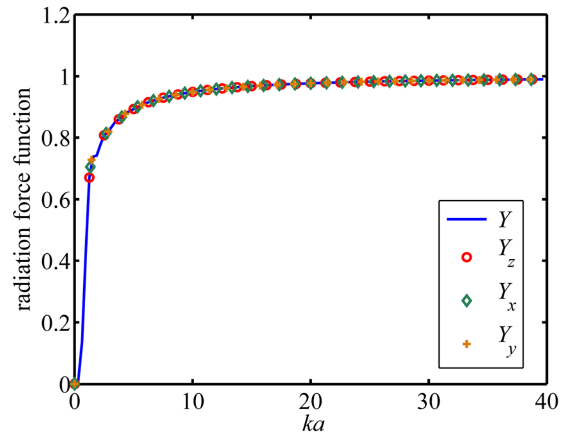

FIG. 1. (Color online) The radiation force functions on a rigid sphere caused by plane waves traveling along the x, y, and $z$axis varying with the sphere size factor ka. The solid line is the classical result for a plane wave.[34]

To numerically evaluate Eqs. (11)–(14), we have to truncate the series at $l=L$. The truncation order L depends of the magnitude of the function s_l^m, which in turn is related to the sphere size factor ka. Thus to determine the truncation order, we have used the condition $|s_0^m/s_{l+L}^m| < 0.01$ for $l = 1,2,\ldots,$ which yielded the rule $L = ka + 6$. This rule may need to be re-evaluated for each type of the sphere's materials.

In Fig. 1, we show the radiation force functions Y_x, Y_y, and Y_z with varying sphere size factor ka, produced by plane waves propagating along the x, y, and z axis, respectively. These functions were computed through Eqs. (11)–(13), and they were compared to the result obtained from Eq. (14). The relative error between the computed radiation force functions and Eq. (14) is about 10^{-8}.

We show the radiation force functions for a plane wave with varying angle α and fixed $\beta = \pi/4$ in Fig. 2. The z component of the radiation force is proportional to $\cos\alpha$; while the x and y components have the same amplitude and they vary as $\sin\alpha$. Both angular dependences of the radiation force are expected due to the geometry of the problem.

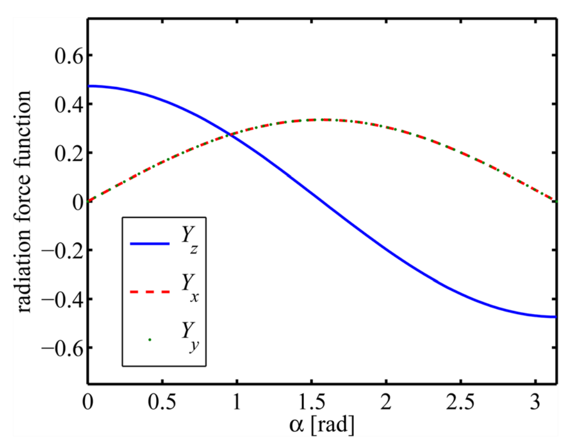

FIG. 2. (Color online) The radiation force functions for a rigid sphere as a function of the incident angle α of the plane wave. The azimuthal angle of the incident plane wave is $\beta = \pi/4$ and the sphere size factor is $ka = 1$.

IV. SUMMARY AND CONCLUSION

We have derived the acoustic radiation force in Cartesian coordinates due to an arbitrary shaped wave acting upon a suspended object. The problem was reduced to compute the beam-shape coefficients of the incident beam and the scattering coefficients that are related to the object's material properties (fluid, elastic, viscoelastic, etc.). The obtained result was validated by recovering the classical radiation force result due to a plane progressive wave on a rigid sphere. The proposed method can be applied to at least two important problems in applied acoustics: the axial and the transverse radiation forces due to actual beams upon an off-axis sphere and the radiation force in a cluster of particles. The method allows a full description of the radiation force upon objects when the fluid viscosity can be neglected.

In conclusion, we believe that our method provides a better understanding of radiation force problems found in acoustic tweezers, particle manipulation and levitation, and some biomedical imaging techniques.

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