# Chapter 1.2: Heuristic Diversification: Equal Weighting and Risk Parity

## Contents



## Heuristic Diversification: Why and How?

To diversify a portfolio, we should "spread eggs across many baskets". There are, however, several possible definitions for eggs and baskets: eggs can be dollar contributions or risk contributions, and baskets can be physical constituents or underlying risk factors.

### Mean-Variance Optimization and Parameter Uncertainty

As explained in Chapter 1.1, the definition of efficient portfolios as maximum Sharpe ratio (MSR) portfolios raises difficulties when it comes to practical implementation because the MSR portfolio involves a large amount of estimation risk. Estimation errors plague mainly expected returns, but also covariances. As a result, the out-of-sample Sharpe ratio of an estimated MSR portfolio often ends up being lower than that of a global minimum variance (GMV) portfolio – which ignores expected returns – and even than that of a naively diversified equally-weighted portfolio – which ignores any difference between assets. DeMiguel et al. (2009) even show that attempts to improve expected return estimates with Bayesian methods produce disappointing results in that the improved estimate for the MSR portfolio fails to dominate the equally-weighted benchmark out-of-sample. Therefore, a consensus has emerged among practitioners and researchers that the shortcomings of mean-variance theory in practice are not simply due to implementation frictions such as transaction costs, but that they are inherent to its use with imperfect parameter estimates.

## A Proverbial Definition of Diversification

In view of the disappointing performance of estimated mean-variance efficient portfolios in practice, one may be tempted to go back to a more heuristic definition of diversification, and seek to hold a well-balanced portfolio in an attempt to try and achieve a better Sharpe ratio. While a well-balanced portfolio is often taken to be an equally-weighted portfolio, common wisdom suggests instead a more general formulation, namely having "eggs spread across many baskets". At this stage, the true meaning of the very words "eggs", "many" and "baskets" should be questioned in order to translate the principle into a workable portfolio construction rule. The most straightforward (but not necessarily deepest) interpretation is to define eggs as dollars and baskets as assets, and assume that the best spreading mechanism is achieved via the allocation of an equal number of eggs/dollars to each basket/asset. In these lecture notes, we first introduce the notion of "effective number of constituents" in order to provide a more meaningful measure of the number of eggs in each basket. We then examine another interpretation for the proverbial eggs and baskets analogy where baskets are still assets but eggs are re-interpreted as contributions to risk (volatility). In Chapter 1.3, we will explore another definition for the baskets, as underlying risk factors.

# 1. Counting Eggs per Baskets: The Effective Number of Constituents

> The effective number of constituents measures the concentration of a portfolio in terms of dollars across assets. It is defined as the inverse of the sum of squared weights, a number equal to 1 for a fully concentrated portfolio and to the nominal number of constituents for an equally-weighted portfolio.

The nominal number of constituents in a portfolio is a misleading measure of diversification because it does not account for the fact that some constituents may be overweighted with respect to some others. To understand the nuance, let us consider the extreme case of a fictitious portfolio invested in 100 constituents, and allocating 99% of the assets to one constituent while spreading the remaining 1% of the wealth to the 99 remaining constituents. Although 100 constituents have a non-zero weight, this portfolio is clearly heavily concentrated in the first asset.

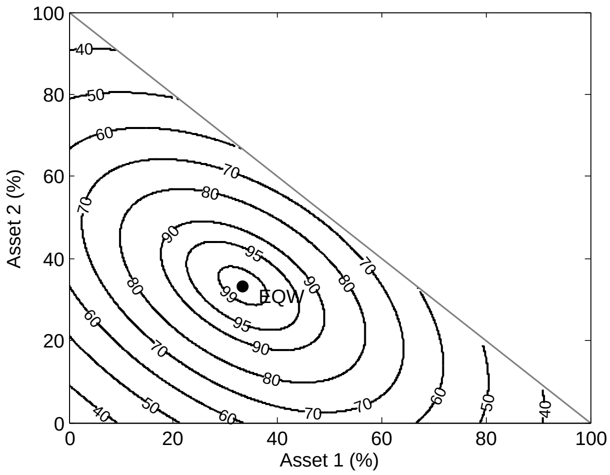
## The Effective Number of Constituents

We now introduce a more meaningful measure of the number of constituents in a portfolio known as the *effective number of constituents* (ENC), defined as the reciprocal of the sum of the squared weights:

$$ENC = \frac{1}{\sum_{i=1}^{N} w_i^2}.$$

It can be shown (see Section 1 of the Technical Supplement) that the ENC is always less than or equal to $N$, and the maximum is attained only when the portfolio is equally weighted. On the other hand, a portfolio invested in a single asset has an ENC of 1, which reflects its severe concentration. In the previous example, the ENC is (assuming that the

**Figure 1:** Effective number of constituents expressed as a percentage of the nominal number of constituents (here, 3) as a function of portfolio composition.




*Note: The investment universe contains three assets, and the weight of the third constituent is one minus the sum of the first two asset weights. Each contour represents a set of portfolio compositions that give the same ENC, and is labeled with the value of the constant ENC, expressed as a percentage of the nominal number of assets. The point labeled EQW is the equally-weighted portfolio. The area plotted on the figure is restricted to long-only allocations.*

remaining 1% of wealth is equally split across the 99 constituents)

$$ENC = \frac{1}{0.99^2 + 99 \times \left[\frac{0.01}{99}\right]^2} \approx 1.02,$$

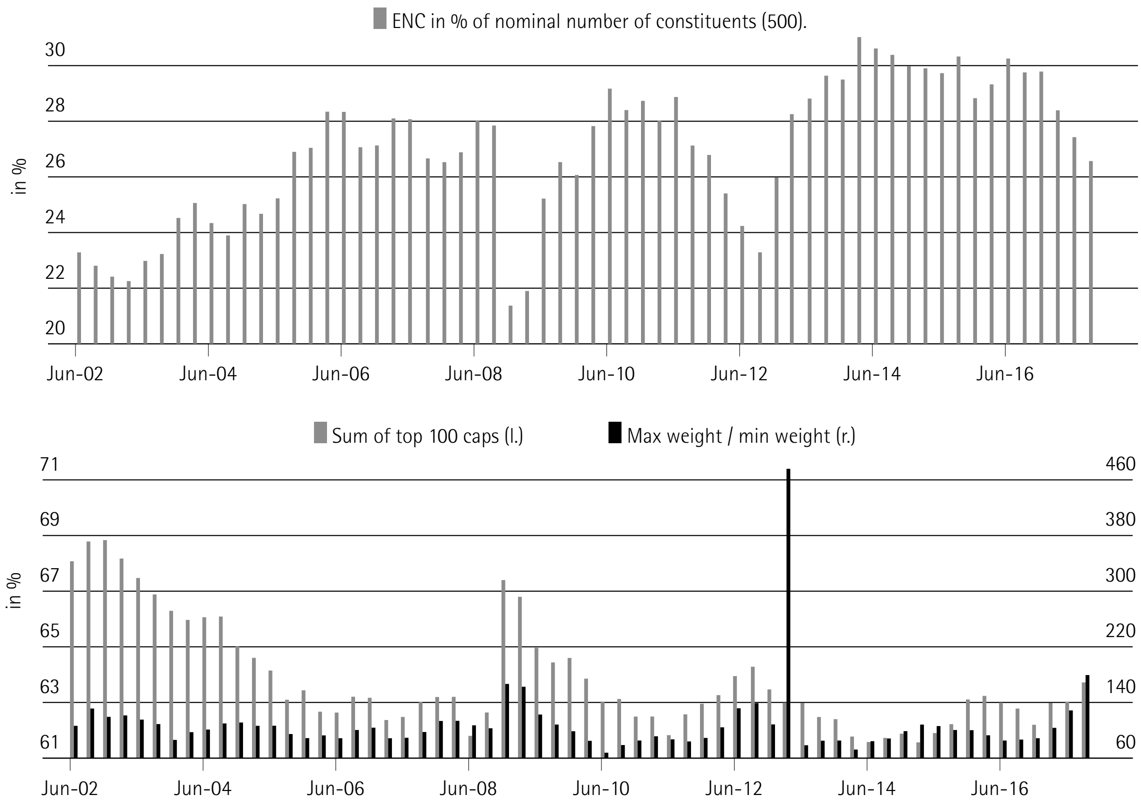
which is hardly greater than 1. Hence, the ENC appears to be a formal measure of diversification consistent with the intuition.

Cap-weighted stock indices tend to be concentrated in a limited number of stocks. Figure 2 shows that for a broad cap-weighted index made of the 500 largest US stocks, the ENC ranges approximately from 21% to 31% of the nominal number of constituents, with an average at 26.9%. This corresponds to an ENC comprised between 105 and 155, while the nominal number of constituents is 500. For an equally-weighted portfolio, the ENC would be 500.

The portfolio concentration could also be measured through the ratio of the largest to the smallest weight, or as the cumulative weight of top holdings, e.g. the 100 largest stocks. For the equally-weighted portfolio, the ratio would be 1 and the cumulative sum would be 20%, while for the cap-weighted index, the ratio is often greater than 100 and the sum is always greater than 60%. Both measures appear to negatively correlated with the ENC, which confirms that the ENC captures the intuitive notion of diversification.

The ENC is a function of the $N$ portfolio weights, hence of $N-1$ independent parameters since the weights sum up to 1. Thus, a graphical representation can be provided up to two assets. Figure 1 shows a "contour plot" of the ENC as a function of the weights of the first two assets. The ENC is expressed as a percentage of the maximum total value (here, 3) and each contour represents a set of portfolios that have the same ENC. Note that there is no need to specify any risk or return parameter for the assets since the ENC is independent from these parameters. The black point at the center of the trian-

**Figure 2:** Diversification metrics for US broad cap-weighted index.




*Note: Data is from the ERI Scientific Beta database and spans the period from June 2002 to September 2017. Each bar corresponds to a quarterly rebalancing date.*

gle corresponds to the equally-weighted portfolio and it is the only portfolio that has an ENC equal to 100% of the nominal number of assets. As indicated before, it can be shown (see Section 1 of the Technical Supplement) that for long-only portfolios, the minimum possible ENC is 1 and that it is attained by portfolios fully invested in one asset. This can be verified from the plot, where the ENC is minimal at the vertices of the triangle: it is then equal to 33.3% of the maximum which is $N = 3$.

## Maximizing the ENC with Constraints

Maximizing the ENC in the absence of weight constraints other than the budget constraint (weights should add up to 1) leads to weighting constituents equally, so that the equally-weighted portfolio can be seen as the output of a formal optimization exercise. The procedure can be used in a number of practically relevant applications such as to find the closest approximation to an equally-weighted portfolio subject to weight constraints like, for instance, minimum or maximum weights in some constituents, tracking error constraints for an equity portfolio, or duration constraints for a bond portfolio. Maximizing the effective number of constituents subject to these constraints is a method for obtaining the most balanced portfolio compatible with the constraints. Introducing weight constraints amounts to changing the region of feasible weights in Figure 1. The new region may not include the equally-weighted portfolio, in which case the maximum ENC is strictly lower than $N$.

## 2. A Better Definition for Eggs: Risk as Opposed to Dollar Contributions

> The ENC is based on dollar weights and does not recognize that some constituents can have much more impact than others on the risk of a portfolio. The effective number of correlated bets (ENCB) measures the deconcentration of the portfolio in terms of contributions of constituents to volatility.

### Limits of ENC as a Measure of Diversification

While insightful, the ENC measure can be deceiving when applied to assets with non homogenous risks. The classical example is that of a simple equally-weighted stock-bond portfolio: since equities are more volatile than bonds (at least for relatively short maturity sovereign bonds), investing equally in both asset classes leads to a portfolio with a risk largely explained by that of the more volatile class. Let us assume for instance that a stock index and a very short duration bond index have a volatility respectively given by $\sigma_S = 30\%$ and $\sigma_B = 1\%$ per year, and let us also assume for simplicity that the two assets are uncorrelated. Then, portfolio variance can be written as

$$\sigma_p^2 = w_S^2\sigma_S^2 + w_B^2\sigma_B^2, \tag{2.1}$$

and the numerical values of the two terms in the right-hand side are

$$w_S^2\sigma_S^2 = 0.5^2 \times 0.3^2 \approx 0.0225,$$
$$w_B^2\sigma_B^2 = 0.5^2 \times 0.01^2 \approx 0.000025,$$

so that the portfolio variance is 0.0225, and the portfolio volatility is $\sqrt{0.0225} \approx 15.0\%$ per year. Observe that the two terms in the right-hand side of Equation (2.1) are of very different sizes: the equity contribution to portfolio risk is 900 times higher than the bond contribution, so that the proportion of portfolio variance that comes from equity risk is

$$\frac{1}{1 + \frac{1}{900}} \approx 99.9\%.$$

Hence, this apparently well-balanced portfolio ($ENC/N = 100\%$) happens to be extremely concentrated when it comes to the distribution of risk contributions.

### The Two-Asset Case

This example suggests an alternative definition for eggs as risk contributions, as opposed to dollar contributions. The formal definition for risk contributions is based on Equation (2.1): the portfolio variance is the sum of a stock term and a bond term, so that the relative contribution of each asset is defined as the ratio of its specific term in the right-hand side of Equation (2.1) to the portfolio variance. In detail, the relative contributions are

$$c_{rel,S} = \frac{w_S^2\sigma_S^2}{\sigma_p^2}, \quad c_{rel,B} = \frac{w_B^2\sigma_B^2}{\sigma_p^2},$$

and they add up to 1.

Clearly, to make the relative contributions equal, it suffices to weight constituents by the inverse of their volatility. With the assumed volatilities, the weighting scheme that would lead to equal contributions to volatility (the *risk parity* allocation, which we define in the general sense below) is

$$w_S = \frac{\sigma_S^{-1}}{\sigma_S^{-1} + \sigma_B^{-1}} \approx 3.2\%,$$
$$w_B = \frac{\sigma_B^{-1}}{\sigma_S^{-1} + \sigma_B^{-1}} \approx 96.8\%.$$

The intuition is that it is only if the most volatile constituents are underweighted that their contribution to the risk of the portfolio does not dominate the contribution from lower volatility components. Such examples are given in the early papers on risk parity, e.g. Qian (2005) and Maillard, Roncalli, and Teiletche (2010).

## Risk Contributions of Constituents

In general, we may have more than two assets, and they may be correlated, which introduces covariance terms in the expression of a portfolio variance. To provide a general definition for the *risk contribution* of an asset to a portfolio volatility, start from the general expression for the portfolio variance:

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}.$$

Here, $\sigma_{ij}$ is the covariance between assets $i$ and $j$. Next, divide both sides by $\sigma_p$ to arrive at portfolio volatility:

$$\sigma_p = \sum_{i=1}^{N} \frac{w_i}{\sigma_p}\left[\sum_{j=1}^{N} w_j \sigma_{ij}\right]. \qquad (2.2)$$

This equation expresses portfolio volatility as the sum of contributions from the various assets. The contribution of asset $i$ to volatility is formally defined as

$$c_i = \frac{w_i}{\sigma_p}\left[\sum_{j=1}^{N} w_j \sigma_{ij}\right]. \qquad (2.3)$$

The biggest contributors are assets with large volatilities and/or large covariances with the others. Indeed, a highly volatile asset is obviously a big contributor to risk, and an asset that covaries positively with the other constituents also contributes to risk because shocks to the various assets add up.

A second expression for the contribution of an asset is in terms of the covariance between the asset return ($R_i$) and the portfolio return ($R_p$):

$$c_i = \frac{w_i}{\sigma_p}\mathbb{C}\mathrm{ov}\left[R_i, R_p\right].$$

This formula makes it clear that the asset contribution to portfolio risk has the same sign as the covariance between the asset return and the portfolio return.

A third expression for the contribution involves the sensitivity of portfolio volatility with respect to the allocation to asset $i$. Mathematically, the bracketed term in Equation (2.3) is one half the partial derivative of portfolio variance with respect to $w_i$, that is the change in portfolio variance for an infinitesimal change of $w_i$, other weights remaining fixed. The contribution of asset $i$ can be rewritten as

$$c_i = w_i \frac{\partial \sigma_p}{\partial w_i}. \qquad (2.4)$$

The biggest contributors are the assets for which the allocation has a large impact on portfolio volatility.

Asset contributions add up to the portfolio volatility. By using the expression given in Equation (2.4), portfolio volatility can be rewritten as

$$\sigma_p = \sum_{i=1}^{N} w_i \frac{\partial \sigma_p}{\partial w_i}.$$

This equation is known as *Euler identity*. In fact, it holds not only for volatility, but also for any risk measure that is "homogenous of degree 1 in the weights".[1] This allows us to define contributions for other risk measures than volatility, as long as they satisfy the homogeneity condition (which is the case for instance for semi-volatility or Value-at-Risk; see Roncalli (2013a) and Martellini, Milhau, and Tarelli (2015) for examples). In the special case of volatility, Qian (2005) shows that under some assumptions, the contributions $c_i$ can also be interpreted as contributions to portfolio losses.

### The Effective Number of Correlated Bets

Since the portfolio volatility is the sum of contributions, we may define the *relative contribution* of asset $i$ to portfolio volatility as $c_{rel,i} = c_i/\sigma_p$. These quantities add up to 1, a property that is shared with dollar weights. Adapting the definition of the ENC, we can measure the dispersion of risk contributions as the reciprocal of the sum of squared relative contributions. This new metrics is called the *effective number of correlated bets* (ENCB):

$$ENCB = \frac{1}{\sum_{i=1}^{N} c_{rel,i}^2}.$$

Just like the ENC measures the "deconcentration" in terms of dollar weights, the ENCB measures the extent to which risk contributions differ from one asset to the other across the portfolio. The ENCB has the same upper bound as the ENC: it is always less than or equal to $N$, the nominal number of constituents, and this maximum is attained if, and only if, all risk contributions are equal. This property defines the *risk parity* or *equal risk contribution* portfolio, which we study in more detail in the next section. On the other hand, a portfolio with risk concentrated in a few constituents will have low ENCB. In the previous two-asset example, the ENCB is

$$ENCB = \frac{\sigma_p^4}{\left[w_S^2\sigma_S^2\right]^2 + \left[w_B^2\sigma_B^2\right]^2} \approx 1.002,$$

which is hardly greater than 1 because most of the risk comes from a single constituent.

Figure 3 shows a contour plot of the ENCB as a function of asset weights in a three-asset universe. Unlike for the ENC, which is a function of the sole weights, we now have to specify volatilities and correlations. For the sake of the illustration, we set the three volatilities to 10%, 20% and 30%, and the three correlations to moderate values: $\rho_{12} = 10\%$, $\rho_{13} = 0$ and $\rho_{23} = -10\%$. With the assumed parameters, the volatility of an equally-weighted portfolio of the three assets is $\sigma_p = 12.1\%$, and the relative contribution of the first asset is

$$\begin{aligned} c_{rel,1} &= \frac{w_1\sigma_1}{\sigma_p^2} \times \left[w_1\sigma_1 + w_2\sigma_2\rho_{12} + w_3\sigma_3\rho_{13}\right] \\ &= 9.1\%. \end{aligned}$$

The contributions of the other two assets are obtained in a similar manner and are 27.3% and 63.6%. Finally, the ENCB of the equally-weighted portfolio is

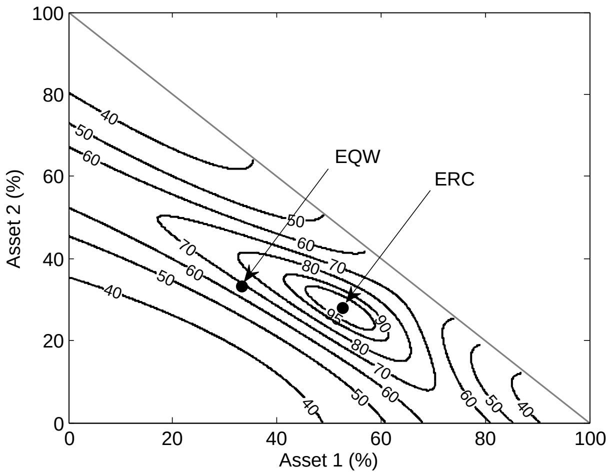
$$\begin{aligned} ENCB &= \frac{1}{c_{rel,1}^2 + c_{rel,2}^2 + c_{rel,3}^2} \\ &\approx 2.05, \end{aligned}$$

which is 68.4% of the maximum of 3.

It is visually clear from Figure 3 that the risk parity portfolio, which equates all risk contributions, is

1 – Homogeneity of degree 1 means that if all weights are multiplied by a positive constant (which amounts to diluting the portfolio with cash or to adding leverage), the function is multiplied by the same constant.

**Figure 3:** Effective number of correlated bets expressed as a percentage of the nominal number of constituents (here, 3) as a function of portfolio composition.




*Note: The investment universe contains three assets, and the weight of the third constituent is one minus the sum of the first two asset weights. Each contour represents a set of portfolio compositions that have the same ENCB, and is labeled with the value of the constant ENCB, expressed as a percentage of the nominal number of assets. Asset volatilities are* $\sigma_1 = 10\%$, $\sigma_2 = 20\%$ *and* $\sigma_3 = 30\%$, *and correlations are* $\rho_{12} = 10\%$, $\rho_{13} = 0$ *and* $\rho_{23} = -10\%$. *The points labeled EQW and ERC are respectively the equally-weighted and the equal risk contribution portfolios. The area plotted on the figure is restricted to long-only allocations.*

distinct from the equally-weighted one. Its composition can be computed by an adapted numerical routine (see next section) and is 52.7% in asset 1, 27.9% in asset 2 and 19.4% in asset 3.

## 3. Risk Parity Portfolio

A risk parity portfolio is defined by the condition that all assets contribute equally to volatility. Its effective number of correlated bets equals the nominal number of constituents.

The ENCB attains its maximum when all assets contribute equally to portfolio volatility. This condition defines a *risk parity* (in short, RP) portfolio, also known as *equal risk contribution* portfolio. This weighting scheme was introduced by Qian (2005) and Maillard, Roncalli, and Teiletche (2010) as a response to the problem of excessive risk concentration in equally-weighted portfolios. It is the exact counterpart of the equally-weighted allocation with a different definition for eggs and baskets: baskets are still assets, but eggs are risk contributions as opposed to dollars, so that the highest extent of diversification is achieved by spreading risk contributions evenly across assets.

### Computing the RP Portfolio

It is neither clear ex-ante that a risk parity portfolio exists for any choice of volatilities and correlations, nor that it is unique, but existence and uniqueness (subject to long-only constraints) are proved by Spinu (2013). In some cases, it can be directly computed. In particular, if all assets are uncorrelated from each other, the portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2,$$

so the risk contribution of asset $i$ is

$$c_{rel,i} = \frac{[w_i \sigma_i]^2}{\sigma_p^2},$$

and risk contributions are identical across constituents if weights are inversely proportional to volatilities, leading to a portfolio known as the *inverse volatility portfolio*. Remarkably, this property still holds in the presence of non-zero correlations across assets provided all correlations are equal (see Section 3 of the Technical Supplement for a proof). With a more general correlation structure across assets, and possibly different correlations, the inverse volatility portfolio does not achieve risk parity, and no analytical expression is available for the weights. In this situation, numerical routines have to be employed (see Section 4 of the Technical Supplement).

The RP portfolio is not an optimal portfolio according to mean-variance theory (in general it does not lie on the ex-ante efficient frontier), but it may prove ex-post better in terms of performance or risk-adjusted performance (Sharpe ratio) than proxies for efficient portfolios because it is not plagued by the errors in expected return estimates that dramatically affect these proxies. However, the RP portfolio can be related to mean-variance theory under some restrictive conditions on parameter values: it can be shown (see Section 5 of the Technical Supplement) that risk parity weights maximizes the Sharpe ratio when all assets have the same Sharpe ratio and the same pairwise correlations.

## Some Properties of RP Portfolios

A fundamental property of the RP strategy, which follows from the equality of risk contributions, is that it tends to overweight low volatility constituents with respect to high volatility ones. It also tends to overweight constituents that have low correlations with the others. It shares these properties with the global minimum variance portfolio, but one key difference is that it allocates non-zero weights to all constituents, thus avoiding the corner solution problem typical in portfolio optimization procedures with nonnegativity constraints (a corner solution occurs when the optimizer returns a vector where all weights are nonnegative, but some -and often many- of them are zero). As a result, it tends to be less concentrated and to have a higher ENC.[2]

It should be noted that the marginal contribution of an asset to the risk of the portfolio does not only depends on the characteristics of asset $i$ but also on the characteristics of all assets in the portfolio. As a consequence, the computation of the ENCB involves some choice as to the attribution of the correlated components across constituents, and this choice is necessarily arbitrary. This feature leads to the undesirable result that the risk parity portfolio suffers from a so-called *lack of invariance by replication*. Suppose for example that one adds one additional asset to a $N$ asset universe, where the additional $[N+1]^{th}$ asset is chosen to be identical to the first one. In this context, it would obviously be desirable to obtain that the aggregate allocation to the first and $[N+1]^{th}$ assets would coincide with what was the allocation to the first asset before the artificial completion of the universe with the redundant asset. This invariance by replication property, however, is not satisfied by the RP portfolio, as shown in Section 6 of the Technical Supplement. In next chapter, we discuss a possible solution to this conceptual problem by introducing a framework where

2 – Note that the concentration of the GMV can be reduced by imposing norm constraints on weights (more on this in the lecture notes of Chapter 1.3).

risk contributions are not computed for correlated assets but for uncorrelated underlying risk factors.

### Extensions of Risk Parity Approach

The RP portfolio is a special case of a more general allocation method known as *risk budgeting*, where the contributions of assets are set to target values that are not necessarily equal. In this approach, the decision is how much risk, as opposed to how many dollars, an investor decides to allocate to each constituent, even if this decision must obviously eventually be translated into dollar weights in order to be implemented (see Chapter 2 in Roncalli (2013b) for details). As for the RP portfolio, numerical techniques are required to obtain the weights of a risk budgeting portfolio. Another extension to the risk parity method consists in keeping the idea of balanced risk contributions but adding weight constraints, in the spirit of maximum ENC portfolios. Examples of practically relevant constraints are upper and lower bounds on portfolio weights, tracking error constraints or factor exposure constraints (such as duration constraints in bond portfolios). The idea is to find the best approximation to a RP portfolio while satisfying the constraints, and this is done by performing a constrained maximization of the ENCB.

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