EDHEC-Risk Institute

Chapter 2.2 - Technical Supplement: Liability-Driven Investing

Contents

This Technical Supplement provides details on the mathematics of liability-driven investing (LDI). While Section 1 relies on basic notions of stochastic calculus in continuous time, Sections 2 and 3 use the more advanced tools developed in the literature on portfolio optimization in continuous time (vector Brownian motions, stochastic discount factors, characterization of incomplete markets).

1. Fixed-Mix LDI Strategies

Continuous-Time Dynamics of Assets and Liabilities

The discrete-time dynamics of the performance-seeking and the liability-hedging portfolios (PSP and LHP) read

$$\log \frac{S_{t+h}}{S_t} = m_S h + \sigma_S \varepsilon_S \sqrt{h},$$
$$\log \frac{L_{t+h}}{L_t} = m_L h + \sigma_L \varepsilon_L \sqrt{h},$$

where ε_S and ε_L are two normally distributed random variables with zero mean, unit variance and correlation ρ, and h is the time step, expressed as a number of years. For instance, if the sampling frequency is monthly, we have $h = 1/12$. If it is quarterly, then $h = 1/4$ and so on.

By letting h shrink to zero, we obtain the continuous-time dynamics:

$$\mathrm{d}\log S_t = m_S\,\mathrm{d}t + \sigma_S\,\mathrm{d}z_{St},$$
$$\mathrm{d}\log L_t = m_L\,\mathrm{d}t + \sigma_L\,\mathrm{d}z_{Lt},$$

where z_S and z_L are two Brownian motions representing PSP risk and liability risk. The time step $\mathrm{d}t$ is infinitesimal and the quantities $\mathrm{d}z_{St}$ and $\mathrm{d}z_{Lt}$ are the increments to the Brownian motions over the infinitesimal period $[t, t + \mathrm{d}t]$.

Consider a fixed-mix strategy that allocates a proportion x of assets to the PSP and the remainder, $1 - x$, to the LHP. If the rebalancing period is h, the return to the asset portfolio between two rebalancing dates t and $t+h$ is

$$\frac{A_{t+h}}{A_t} - 1 = x\left[\frac{S_{t+h}}{S_t} - 1\right] + [1-x]\left[\frac{L_{t+h}}{L_t} - 1\right].$$

By letting h converge to zero, we obtain the continuous-time budget equation:

$$\frac{\mathrm{d}A_t}{A_t} = x\frac{\mathrm{d}S_t}{S_t} + [1-x]\frac{\mathrm{d}L_t}{L_t}, \tag{1.1}$$

The instantaneous volatility of assets (i.e., the annualized volatility of asset returns over the infinitesimal time period $[t, t+\mathrm{d}t]$) is

$$\sigma_A = \sqrt{x^2\sigma_S^2 + [1-x]^2\sigma_L^2 + 2x[1-x]\sigma_S\sigma_L\rho}.$$

Asset Returns

Ito's lemma allows us to find the dynamics of logarithmic returns from the dynamics of the arithmetic returns (see Chapter I.3 of Cvitanić and Zapatero (2004)). Apply it to Equation (1.1) to get

$$\begin{aligned}\mathrm{d}\log A_t + \frac{1}{2}\sigma_A^2\,\mathrm{d}t &= x\,\mathrm{d}\log S_t + [1-x]\,\mathrm{d}\log L_t \\ &\quad + \frac{1}{2}x\sigma_S^2\,\mathrm{d}t + \frac{1}{2}[1-x]\sigma_L^2\,\mathrm{d}t.\end{aligned}$$

By re-arranging terms, this can be rewritten as

$$\begin{aligned}\mathrm{d}\log A_t &= x\,\mathrm{d}\log S_t + [1-x]\,\mathrm{d}\log L_t \\ &\quad - \frac{1}{2}x[1-x]TE_S^2\,\mathrm{d}t, \end{aligned}\tag{1.2}$$

where TE_S is the annualized tracking error of the PSP with respect to the LHP, that is

$$TE_S = \sqrt{\sigma_S^2 + \sigma_L^2 - 2\sigma_S\sigma_L\rho}.$$

It is the annualized volatility of the excess logarithmic return of the PSP over the LHP, $\log[S_t/L_t]$.

By integrating Equation (1.2) over the entire time period, $[0, T]$, we obtain

$$\begin{aligned}\log\frac{A_T}{A_0} &= x\log\frac{S_T}{S_0} + [1-x]\log\frac{L_T}{L_0} \\ &\quad + \frac{1}{2}x[1-x]TE_S^2 T. \end{aligned}\tag{1.3}$$

This proves Equation (2.1) in the Lecture Notes, with a rebalancing premium

$$\Lambda = \frac{x[1-x]}{2}TE_S^2 T.$$

Expectation and Variance of Funding Ratio

The funding ratio is $F = A/L$. By subtracting $\log[L_T/L_0]$ from both sides of Equation (1.3), we obtain

$$\log\frac{F_T}{F_0} = x\left[\log\frac{S_T}{S_0} - \log\frac{L_T}{L_0}\right] + \Lambda. \tag{1.4}$$

Hence, the variance of the logarithmic change in the funding ratio is

$$\mathbb{V}\left[\log\frac{F_T}{F_0}\right] = x^2 TE_S^2 T.$$

By Equation (1.4), the expected logarithmic return on the funding ratio is

$$\mathbb{E}\left[\log\frac{F_T}{F_0}\right] = x\mathbb{E}\left[\log\frac{S_T}{S_0} - \log\frac{L_T}{L_0}\right] + \Lambda,$$

that is

$$\mathbb{E}\left[\log\frac{F_T}{F_0}\right] = x m_{S/L}T - \frac{x^2}{2}TE_S^2 T,$$

where $m_{S/L}$ is given by

$$m_{S/L} = m_S - m_L + \frac{TE_S^2}{2}.$$

This proves Equations (2.2) and (2.3) of the Lecture Notes.

Optimal Fixed-Mix Allocation

To find the optimal fixed-mix allocation to the PSP, we maximize the quadratic utility from the funding ratio with respect to x. Quadratic utility is given by

$$\begin{aligned} U &= \mathbb{E}\left[\log F_T\right] - \frac{\gamma - 1}{2}\mathbb{V}\left[\log F_T\right] \\ &= \log F_0 + x m_{S/L} T - \frac{\gamma}{2} x^2 T E_S^2 T. \end{aligned}$$

It is a quadratic concave function of x, so it reaches its maximum at

$$\begin{aligned} x^* &= \frac{m_{S/L} T}{\gamma T E_S^2 T} \\ &= \frac{m_{S/L}}{\gamma T E_S^2}. \end{aligned}$$

2. Optimal LDI Strategy

In this section, we derive the optimal portfolio strategy for an investor who faces liabilities and securities as described in Section 3 of the Lecture Notes. A general presentation of the theory of optimal portfolio choice in multi-period models is given in Chapter I.4 of Cvitanić and Zapatero (2004).

Some Notation

The framework is a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with $N+1$ correlated Brownian motions z_1, ..., z_N, z_L representing the sources of uncertainty associated with the N securities and with liability value. The time span is the range $[0, T]$, where 0 is the initial date and T is the final date.

For notational ease, it is convenient to add vector notation for the sources of risk. Each unexpected asset return is rewritten as

$$\sigma_i \, \mathrm{d}z_{it} = \boldsymbol{\sigma}_i' \, \mathrm{d}\mathbf{z}_t,$$

where $\mathbf{z}$ is a $[N+1]$-dimensional Brownian motion and $\boldsymbol{\sigma}_i$ is a $[N+1] \times 1$ volatility vector. The vector Brownian motion $\mathbf{z}$ contains N independent sources of risk which summarize all correlated sources of risk associated with asset returns: the passage from the correlated Brownian motions to the uncorrelated ones is a procedure similar to the "orthogonalization" of risk factors in Chapter 1.3. The vector $\boldsymbol{\sigma}_i$ contains the loadings of security i on the various sources of risk. The probability space is equipped with the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the Brownian motion $\mathbf{z}$: $\mathcal{F}_t$ is the sigma-field representing the information set available at date t.

Similarly, unexpected liability return is rewritten as

$$\sigma_L \, \mathrm{d}z_{Lt} = \boldsymbol{\sigma}_L' \, \mathrm{d}\mathbf{z}_t,$$

The N volatility vectors of the locally risky assets are collected in the $[N+1] \times N$ volatility matrix $\boldsymbol{\sigma}$. This matrix is constant over time and across states of the world because volatilities and correlations are assumed to be constant. The instantaneous covariance matrix of assets and the vector of covariances between assets and the liability process are given by

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{\sigma}' \boldsymbol{\sigma}, \\ \boldsymbol{\Sigma}_L &= \boldsymbol{\sigma}' \boldsymbol{\sigma}_L. \end{aligned}$$

The *spanned market price of risk vector* is defined as

$$\boldsymbol{\lambda} = \boldsymbol{\sigma} \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N. \end{bmatrix}.$$

It satisfies

$$\boldsymbol{\sigma}' \boldsymbol{\lambda} = \tilde{\boldsymbol{\mu}},$$

where $\tilde{\boldsymbol{\mu}}$ is the vector of expected excess returns.

The Optimization Problem

The dynamics of the asset portfolio reads:

$$\frac{dA_t}{A_t} = \sum_{i=1}^{N} w_{it} \frac{dS_{it}}{S_{it}} + \left[1 - \sum_{i=1}^{N} w_{it}\right] r_t \, dt.$$

It can be rewritten in the following form, which disentangles the expected return (the dt term in the right-hand side) and the unexpected return (the $d\mathbf{z}_t$ term):

$$\frac{dA_t}{A_t} = \left[r_t + \mathbf{w}_t' \tilde{\boldsymbol{\mu}}\right] dt + \mathbf{w}_t' \boldsymbol{\sigma}' \, d\mathbf{z}_t. \tag{2.1}$$

The optimization problem consists in maximizing the expected utility of the terminal funding ratio subject to the budget constraint summarized in the previous equation:

$$\max_{(w_{1,t}, w_{2,t}, \ldots, w_{N,t})} \mathbb{E}\left[U(F_T)\right]. \tag{2.2}$$

There are two main techniques to solve Program (2.2): dynamic programming, which was the technique originally used by Merton, and the duality technique, which was introduced by Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989). Of course, both methods eventually lead to the same optimal strategy, but they have a different focus. In the dynamic programming approach, one searches directly for the optimal control $\mathbf{w}^*$, while the duality technique focuses on the computation of the optimal terminal wealth. The optimal portfolio policy is then obtained as the strategy that replicates the optimal wealth.

As pointed by Cochrane (2014), the separation between the search for the optimal payoff and the derivation of the corresponding strategy clarifies the optimization process and is relevant in many contexts: the first step is the computation of an investor-specific optimal payoff, and the second step is the engineering of a strategy that replicates it. In theory, the absence of frictions and arbitrage opportunities implies that financial securities with identical payoffs are perfect substitutes for each other, so the replicating strategy within a given set of assets is uniquely defined. In practice, the existence of frictions such as transaction costs and taxes implies that a given payoff can be synthesized in different, more or less costly, ways. The design of the strategy is thus a really applied financial engineering problem, where one has to take into account trading costs and the tax treatment of the various securities, in order to deliver the cheapest replicating portfolio.

Stochastic Discount Factors

A stochastic discount factor (SDF) is a stochastic process $(M_t)_{0 \le t \le T}$ such that all asset prices multiplied by M follow martingales under the probability measure $\mathbb{P}$. When the market is complete, the SDF is unique, but when it is incomplete, there exist infinitely many of them (He and Pearson 1991). Here, the market is potentially incomplete due to the possible presence of unspanned liability risk. He and Pearson (1991) show that the generic expression for the SDFs is

$$M_t = \exp\left[-\int_0^t \left[r_s + \frac{\|\boldsymbol{\lambda} + \boldsymbol{\nu}\|^2}{2}\right] ds - \int_0^t [\boldsymbol{\lambda} + \boldsymbol{\nu}]' \, d\mathbf{z}_s\right], \tag{2.3}$$

where $\| \cdot \|$ denotes the Euclidian norm of a vector and $\boldsymbol{\nu}$ is a vector such that $\boldsymbol{\sigma}'\boldsymbol{\nu} = \mathbf{0}$.

The SDF depends on the vector $\boldsymbol{\nu}$, and the vector $\boldsymbol{\lambda}+\boldsymbol{\nu}$ contains the prices of the uncorrelated sources of risk. When liability risk is completely spanned, the matrix $\boldsymbol{\sigma}$ is non-singular, and only the null vector satisfies $\boldsymbol{\sigma}'\boldsymbol{\nu} = \mathbf{0}$, so that $\boldsymbol{\lambda}$ is the only possible price of risk vector, and the SDF is unique.

Optimal Payoff

It follows from He and Pearson (1991) (see their Theorems 7 and 8) that the optimal terminal asset value has the form

$$A_T^* = L_T U'^{-1}(\eta M_T L_T),$$

where M_T is a suitably chosen SDF and η is a scalar. If the market is complete, the SDF is unique, and there is no degree of freedom to specify, but if it is incomplete, one has to find the SDF such that A_T^* is a replicable payoff, and this SDF depends on preferences.

The optimal terminal asset value can be rewritten as

$$A_T^* = \eta^{-\frac{1}{\gamma}} \frac{[M_T L_T]^{1-\frac{1}{\gamma}}}{M_T}.$$

The constant coefficient $\eta^{-1/\gamma}$ can be explicitly computed by using the budget equality $\mathbb{E}[M_T A_T^*] = A_0$. We obtain

$$A_T^* = \frac{A_0}{\mathbb{E}\left[[M_T L_T]^{1-\frac{1}{\gamma}}\right]} \frac{[M_T L_T]^{1-\frac{1}{\gamma}}}{M_T}. \tag{2.4}$$

The next step in the solution of Program (2.2) is the derivation of a strategy that replicates A_T^*. Because A_T^* is a function of $A_T^{*,0}$ and L_T, the strategy will involve some mixture of the portfolio that would be optimal with constant liabilities, and the LHP. We just have to find the relative weights of these constituents, which is the goal of the next section.

Optimal Portfolio Strategy

To obtain the optimal asset value at any date t, it suffices to take the expected discounted value of A_T^*, as implied by the definition of the SDF:

$$A_t^* = \mathbb{E}_t\left[\frac{M_T A_T^*}{M_t}\right]$$

$$= \eta^{-\frac{1}{\gamma}} M_t^{-\frac{1}{\gamma}} L_t^{1-\frac{1}{\gamma}} \mathbb{E}_t\left[\left[\frac{M_T L_T}{M_t L_t}\right]^{1-\frac{1}{\gamma}}\right]. \tag{2.5}$$

The next step is to apply Ito's lemma to the right-hand side of Equation (2.5) to identify the optimal exposures to the Brownian risk factors, to deduce the optimal weights.

The dynamics of M and L read

$$\begin{aligned} \mathrm{d}M_t &= M_t\left[-r_t\,\mathrm{d}t - [\boldsymbol{\lambda} + \boldsymbol{\nu}]'\,\mathrm{d}\mathbf{z}_t\right], \\ \mathrm{d}L_t &= L_t\left[[r_t + \tilde{\mu}_L]\,\mathrm{d}t + \boldsymbol{\sigma}_L'\,\mathrm{d}\mathbf{z}_t\right], \end{aligned}$$

where $\boldsymbol{\nu}$ is a vector such that $\boldsymbol{\sigma}'\boldsymbol{\nu} = \mathbf{0}$. At this stage, it is unknown, and we have to compute it, in such a way that A_T^* is a replicable payoff.

By integrating the equations for M and L, it is seen that

$$\begin{aligned} \frac{M_T L_T}{M_t L_t} = \exp\Bigg[&\left[\tilde{\mu}_L - \frac{\|\boldsymbol{\lambda} + \boldsymbol{\nu}\|^2 + \sigma_L^2}{2}\right][T-t] \\ &+ [\boldsymbol{\sigma}_L - \boldsymbol{\lambda} - \boldsymbol{\nu}]'[\mathbf{z}_T - \mathbf{z}_t]\Bigg]. \end{aligned}$$

Because the coefficients are constant, this quantity is log-normally distributed, so the conditional expectation of $[M_T L_T/[M_t L_t]]^{1-1/\gamma}$, which appears in the right-hand side of Equation (2.5), is a function of time only, and it does not contribute to the volatility of A_t^*. As a result, applying Ito's lemma to Equation (2.5) leads to

$$\begin{aligned} \mathrm{d}A_t^* &= (\cdots)\,\mathrm{d}t \\ &+ A_t^*\left[\frac{1}{\gamma}[\boldsymbol{\lambda} + \boldsymbol{\nu}] + \left[1 - \frac{1}{\gamma}\right]\boldsymbol{\sigma}_L\right]'\mathrm{d}\mathbf{z}_t. \end{aligned} \tag{2.6}$$

We have not written the details of the $\mathrm{d}t$ term, which are irrelevant for the computation of the optimal strategy. To find the optimal weights, we equate the Brownian terms in Equations (2.1) and

(2.6), which leads to

$$\boldsymbol{\sigma}\mathbf{w}^* = \frac{1}{\gamma}[\boldsymbol{\lambda} + \boldsymbol{\nu}] + \left[1 - \frac{1}{\gamma}\right]\boldsymbol{\sigma}_L. \qquad (2.7)$$

By multiplying both sides by $\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}'$, we arrive at

$$\mathbf{w}^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}'\boldsymbol{\lambda} + \left[1 - \frac{1}{\gamma}\right]\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L. \qquad (2.8)$$

Fund Separation Result

Equation (2.8) expresses the optimal portfolio as a combination of three funds. The first one is the *maximum Sharpe ratio portfolio*, which maximizes the instantaneous Sharpe ratio of a portfolio of the available assets:

$$\mathbf{w}_{MSR} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}},$$

where $\mathbf{1}$ is a column vector of N ones.

The second building block is the *liability-hedging portfolio*, defined here as the portfolio fully invested in risky assets that maximizes the instantaneous squared correlation with liabilities. The instantaneous correlation is equal to

$$\frac{\mathbf{w}'\boldsymbol{\Sigma}_L}{\sigma_L\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}},$$

and the portfolio that maximizes its square is

$$\mathbf{w}_{LHP} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L.$$

LHP weights are proportional to the elements of the vector $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L$, which are the multivariate betas of liabilities with respect to the assets. The third and last fund is the cash account.

The instantaneous Sharpe ratio of a portfolio, and the instantaneous beta of liabilities with respect to the portfolio return are given by

$$\lambda_p = \frac{\mathbf{w}'\tilde{\boldsymbol{\mu}}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}},$$
$$\beta_{L/p} = \frac{\mathbf{w}'\boldsymbol{\Sigma}_L}{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}.$$

A little algebra shows that we have, for the two building blocks,

$$\frac{\lambda_{MSR}}{\sigma_{MSR}} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}},$$
$$\beta_{L/LHP} = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L.$$

Substituting these expressions back into Equation (2.8) gives the formal expression for the fund separation theorem written in the Lecture Notes.

Tracking Error Minimization

Consider the objective of minimizing the conditional tracking error with respect to liabilities at each point in time. If $\mathbb{V}_t$ denotes the variance conditional on the information available at date t and TE denotes the annualized conditional tracking error, then

$$\begin{aligned} TE_t\,\mathrm{d}t &= \sqrt{\mathbb{V}_t[\,\mathrm{d}A_t - \mathrm{d}L_t]} \\ &= \sqrt{\mathbf{w}_t'\boldsymbol{\Sigma}\mathbf{w}_t + \sigma_L^2 - 2\mathbf{w}_t'\boldsymbol{\Sigma}_L}. \end{aligned}$$

Maximizing TE_t is equivalent to maximizing

$$\mathbf{w}_t'\boldsymbol{\Sigma}_t\mathbf{w}_t - 2\mathbf{w}_t'\boldsymbol{\Sigma}_L.$$

No budget constraint is enforced here, so cash is allowed to enter the portfolio. The solution to this quadratic optimization program is straightforward, and the optimal allocation to the N risky assets is

$$\begin{aligned} \mathbf{w}_{TE} &= \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L \\ &= \beta_{L/LHP}\mathbf{w}_{LHP}. \end{aligned}$$

3. Fund Interaction Result with the Optimal LDI Strategy

In this section, we calculate the indirect utility, that is the expected utility for an investor who follows the optimal strategy, and we use this expression to compute the certainty equivalent, defined as the constant funding ratio that leads to the same expected utility as the optimal strategy. This leads to a mathematical statement of the fund interaction result in this setting.

Expression of Indirect Utility

Indirect utility is defined as the maximum possible expected utility. It is traditionally denoted with the symbol J:

$$J = \mathbb{E}\left[U\left(\frac{A_T^*}{L_T}\right)\right].$$

Given the expression (2.4) for A_T^*, this can be rewritten as:

$$\begin{aligned} J &= \frac{1}{1-\gamma} \times \frac{A_0^{1-\gamma}}{\mathbb{E}\left[(M_T L_T)^{1-\frac{1}{\gamma}}\right]^{1-\gamma}} \times \mathbb{E}\left[[M_T L_T]^{-\frac{1}{\gamma}}\right] \\ &= \frac{A_0^{1-\gamma}}{1-\gamma}\mathbb{E}\left[[M_T L_T]^{-\frac{1}{\gamma}}\right]^{\gamma}, \end{aligned}$$

where M is the preference-dependent stochastic discount factor that makes A_T^* a replicable payoff.

The stochastic discount factor has the form given in Equation (2.3), with a $\boldsymbol{\nu}$ that must be computed explicitly before we proceed with utility calculation. To do this, start from Equation (2.7), and multiply both sides by the matrix $\mathbf{P} = \mathbf{I}_N - \boldsymbol{\sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}'$, where $\mathbf{I}_N$ is the identity matrix of size N. For any vector $\mathbf{u}$, the vector $\mathbf{Pu}$ is the residual of the orthogonal projection of $\mathbf{u}$ on the columns of $\boldsymbol{\sigma}$. Since $\boldsymbol{\lambda}$ lies in the span of $\boldsymbol{\sigma}$, we have $\mathbf{P}\boldsymbol{\lambda} = \mathbf{0}$, hence

$$\boldsymbol{\nu} = [1-\gamma]\mathbf{P}\boldsymbol{\sigma}_L.$$

We recover the fact that if liabilities are redundant, i.e. if $\boldsymbol{\sigma}_L$ is spanned by the columns of $\boldsymbol{\sigma}$, we have $\boldsymbol{\nu} = \mathbf{0}$.

After some algebra, we obtain an expression for J, which can be converted into a *certainty equivalent*, defined as the constant funding ratio that yields the same expected utility as the optimal strategy:

$$J = U\left(FR_{eq}\right).$$

The certainty equivalent is given by

$$\begin{aligned} FR_{eq} = &-\left[\tilde{\mu}_L - \boldsymbol{\sigma}_L'\boldsymbol{\lambda}\right]T + \frac{1}{2\gamma}\|\boldsymbol{\sigma}_L - \boldsymbol{\lambda}\|^2 T \\ &- \frac{[1-\gamma]^2}{2\gamma}\boldsymbol{\sigma}_L'\mathbf{P}\boldsymbol{\sigma}_L T. \end{aligned} \quad (3.1)$$

Expression of Certainty Equivalent

A more interesting expression of the certainty equivalent in Equation (3.1) is obtained by highlighting the dependence with respect to the properties of the building blocks. The first two important characteristics are the Sharpe ratio of the MSR portfolio and the correlation between the LHP and the value of liabilities. The maximum squared Sharpe ratio is

$$\lambda_{MSR}^2 = \tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}} = \|\boldsymbol{\lambda}\|^2.$$

The correlation of a portfolio with liabilities is

$$\rho_{p,L} = \frac{\mathbf{w}'\boldsymbol{\Sigma}_L}{\sigma_L\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}}.$$

Hence, we have, for the LHP

$$\rho_{LHP,L} = \frac{1}{\sigma_L} \times \frac{\boldsymbol{\Sigma}_L'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L} \times \frac{\left|\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L\right|}{\sqrt{\boldsymbol{\Sigma}_L'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L}}, \quad (3.2)$$

so the maximum squared correlation is

$$\rho^2_{LHP,L} = \frac{\boldsymbol{\Sigma}_L'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L}{\sigma_L^2}.$$

The squared residual of the projection of liability risk on the risks of assets is thus

$$\begin{aligned}\boldsymbol{\sigma}_L'\mathbf{P}\boldsymbol{\sigma}_L &= \boldsymbol{\sigma}_L'\boldsymbol{\sigma}_L - \boldsymbol{\sigma}_L'\boldsymbol{\sigma}[\boldsymbol{\sigma}'\boldsymbol{\sigma}]^{-1}\boldsymbol{\sigma}'\boldsymbol{\sigma}_L \\ &= \sigma_L^2 - \boldsymbol{\Sigma}_L'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L \\ &= \sigma_L^2\left[1-\rho^2_{LHP,L}\right].\end{aligned}$$

The certainty equivalent depends also on the cross term $\boldsymbol{\sigma}_L'\boldsymbol{\lambda}$, for which we need an alternative expression. It can be rewritten as

$$\begin{aligned}\boldsymbol{\sigma}_L'\boldsymbol{\lambda} &= \boldsymbol{\sigma}_L'\boldsymbol{\sigma}\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}} \\ &= \boldsymbol{\Sigma}_L'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}.\end{aligned}$$

Next, observe that the correlation between the MSR portfolio and liability value is

$$\rho_{MSR,L} = \frac{1}{\sigma_L} \times \frac{\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}} \times \frac{\left|\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\right|}{\sqrt{\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}},$$

while the Sharpe ratio of the MSR portfolio is

$$\lambda_{MSR} = \frac{\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}} \times \frac{\left|\mathbf{1}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\right|}{\sqrt{\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}}.$$

By multiplying the two quantities, it follows that

$$\lambda_{MSR}\rho_{MSR,L} = \frac{\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_L}{\sigma_L}. \tag{3.3}$$

By substituting the expressions for $\|\boldsymbol{\lambda}\|^2$, $\boldsymbol{\sigma}_L'\mathbf{P}\boldsymbol{\sigma}_L$ and $\boldsymbol{\sigma}_L'\boldsymbol{\lambda}$ and rearranging terms, we obtain the expression for the certainty equivalent given in the text, that is

$$\begin{aligned}FR_{eq} = \frac{A_0}{L_0}\exp\Bigg[&-\left[\tilde{\mu}_L - \frac{2-\gamma}{2}\sigma_L^2\right]T + \frac{\lambda^2_{MSR}}{2\gamma}T \\ &+ \left[1-\frac{1}{\gamma}\right]\sigma_L\lambda_{MSR}\rho_{MSR,L}T + \frac{[1-\gamma]^2}{2\gamma}\sigma_L^2\rho^2_{LHP,L}T\Bigg].\end{aligned} \tag{3.4}$$

Fund Interaction Result

The terms on the first line of the decomposition in Equation (3.4) do not depend on the characteristics of the two funds, but those on the second line do. There are three of them. The first one is a pure performance contribution from the MSR portfolio, proportional to its squared Sharpe ratio and independent from its hedging properties, as expected. The last term is a pure hedging contribution from the LHP, and it is proportional to the squared correlation between this fund and liability value, as also expected. The central term, on the other hand, is a cross term involving both the Sharpe ratio of the MSR portfolio and its correlation with liabilities. It is remarkable that while the MSR portfolio has a strict focus on short-term Sharpe ratio maximization, regardless of the hedging properties, and while the optimal allocation to the three funds does not depend at all on the hedging ability of this fund, investor welfare eventually depends on the correlation of the MSR portfolio with liabilities. For $\gamma > 1$, the sign of the cross term is identical to that of the product $\lambda_{MSR}\rho_{MSR,L}$. In the usual case where λ_{MSR} is positive, welfare is increasing in the correlation of the MSR portfolio with liabilities. This can be interpreted as follows: investors have an implicit short exposure to liability risk, so they benefit from holding assets that covary positively with liabilities. Since they have a long position in the MSR portfolio when λ_{MSR} is positive, they are better off if this fund is positively, as opposed to negatively, correlated with liabilities.

This result is a *fund interaction result* in the sense that it highlights the importance of the interaction between the performance and hedging motives. One can equivalently say that investor welfare is increasing in the correlation of the PSP with the liabilities, or increasing in the Sharpe ratio of the LHP, as can be seen from the following result. Define the LHP as the portfolio that maximizes the instantaneous squared correlation with liabilities. Then, the product of Sharpe ratio times correlation is the same for the MSR portfolio and the LHP:

$$\lambda_{LHP}\rho_{LHP,L} = \lambda_{MSR}\rho_{MSR,L}.$$

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