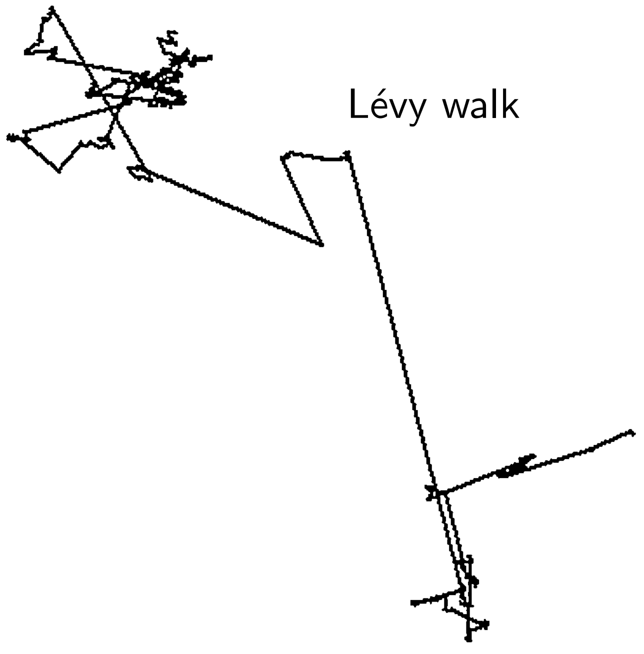
On the Search Efficiency of Parallel Lévy Walks on $\mathbb{Z}^2$

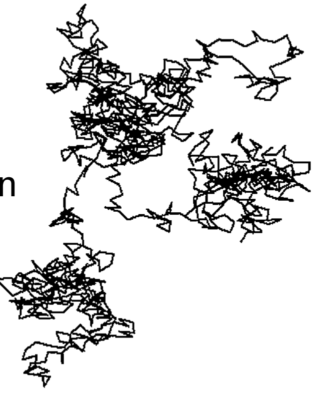
Francesco d'Amore

Joint work with Andrea Clementi, George Giakkoupis, and Emanuele Natale

Irif seminar, 9 June 2020

What are Lévy Walks?





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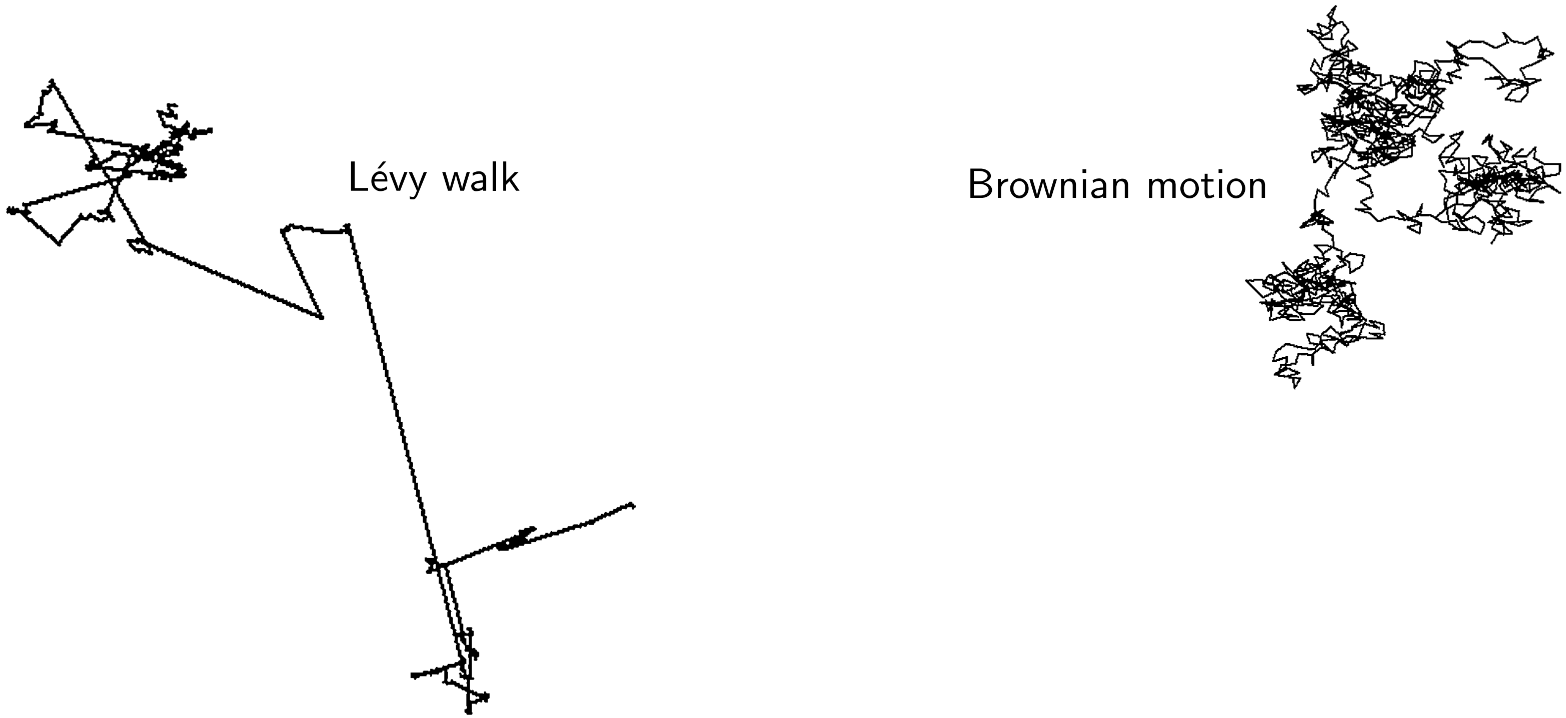

Lévy walk (informal):
A Lévy walk is a random walk whose step-length density distribution is proportional to a power-law, namely, for each $d \in \mathbb{R}$, $f(d) \sim 1/d^{\alpha}$, *for some* $\alpha > 1$

Note: the speed of the walk is constant

Why are Lévy walks interesting?

Lévy walks are used to model **movement patterns** [Biology Open, '18]

Examples:

- T cells within the brain
- swarming bacteria
- midge swarms
- termite broods
- fishes
- Australian desert ants
- a variety of molluscs

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Some fun: mussels Lévy walk video [Science, '11]

The Theory of Foraging

Scenario:
- a density distribution ρ in $\mathbb{R}^n$ describing food locations
- an uninformed walker searching for food in $\mathbb{R}^n$

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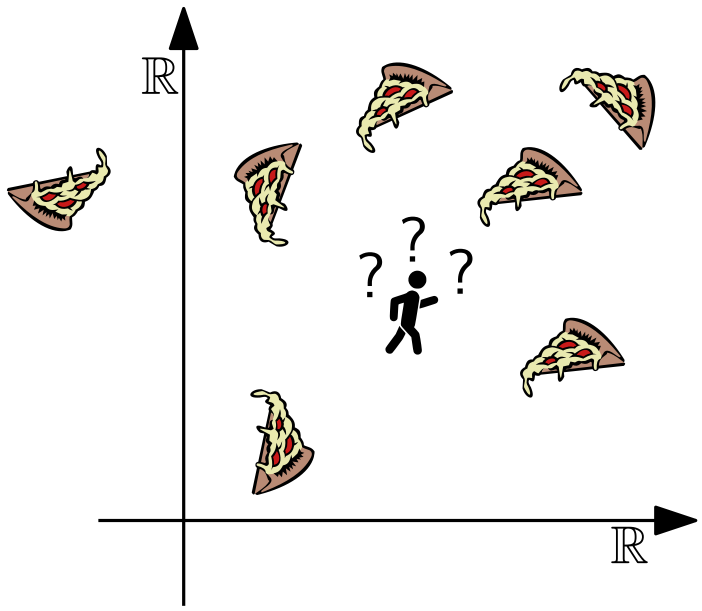

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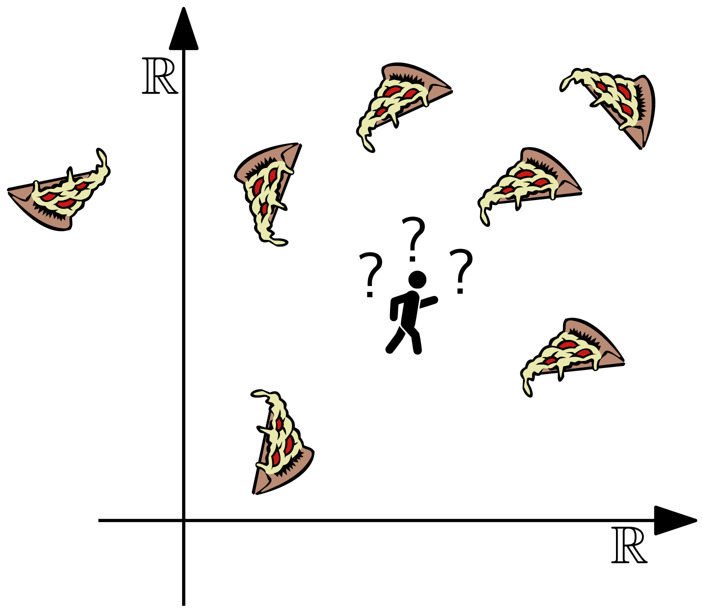

Question: which strategy maximizes the expected food discovery rate?

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[Nature, '99] analyzes three random search strategies:

(a) normal diffusion **(b)** ballistic diffusion **(c)** super diffusion

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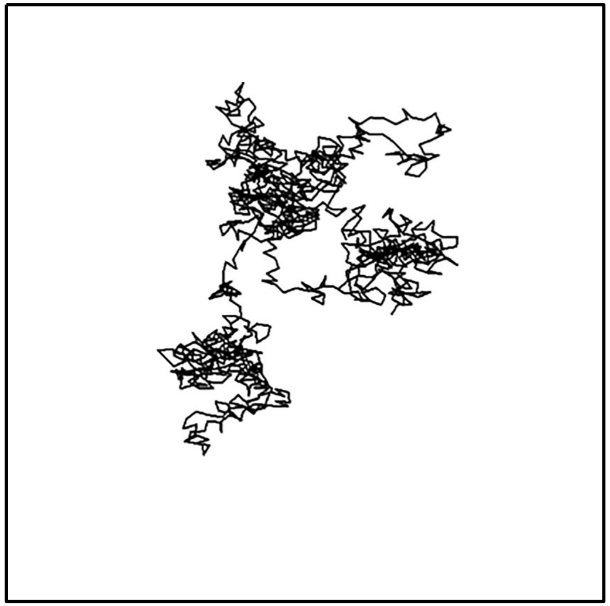

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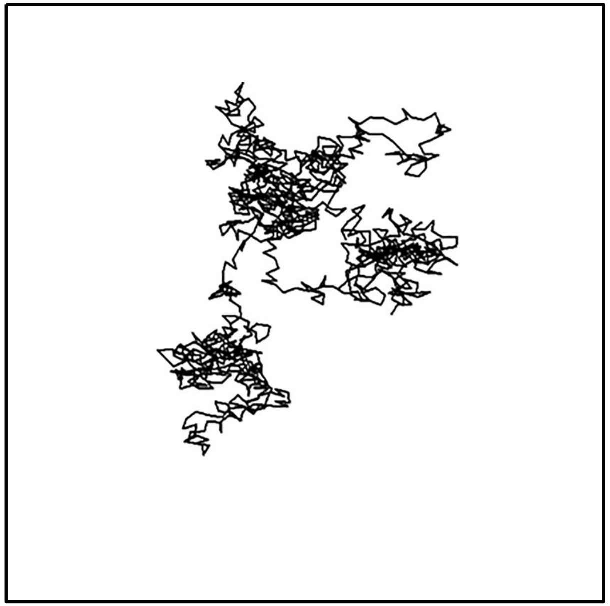

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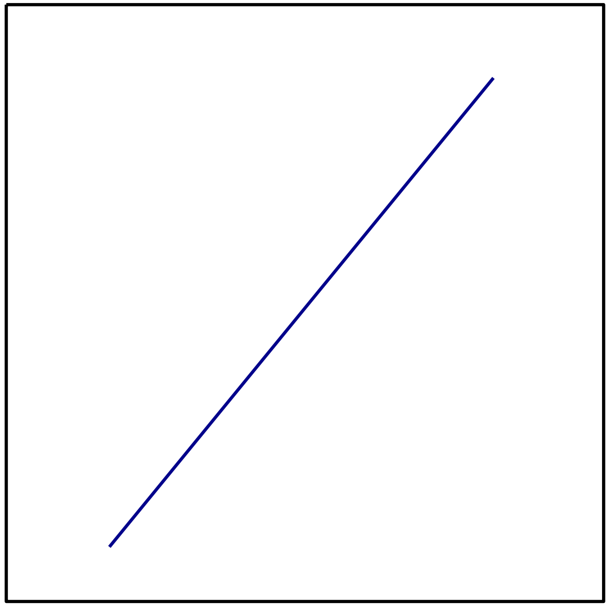

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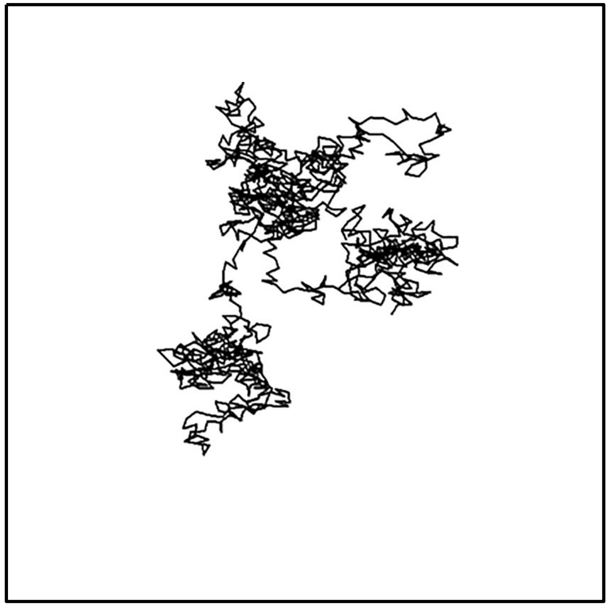

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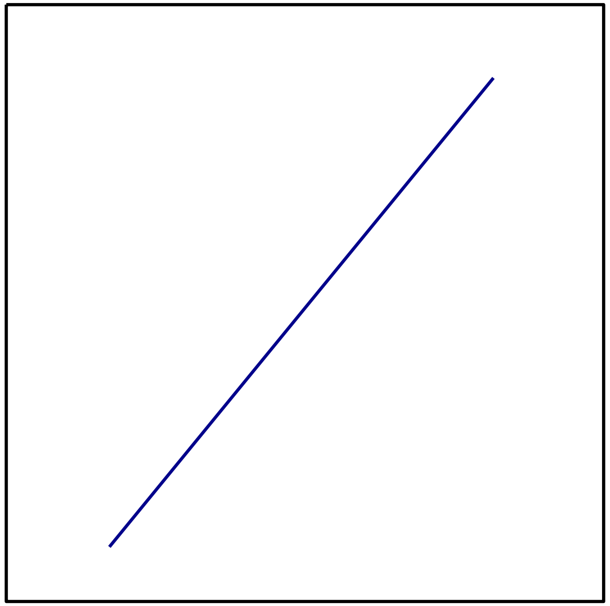
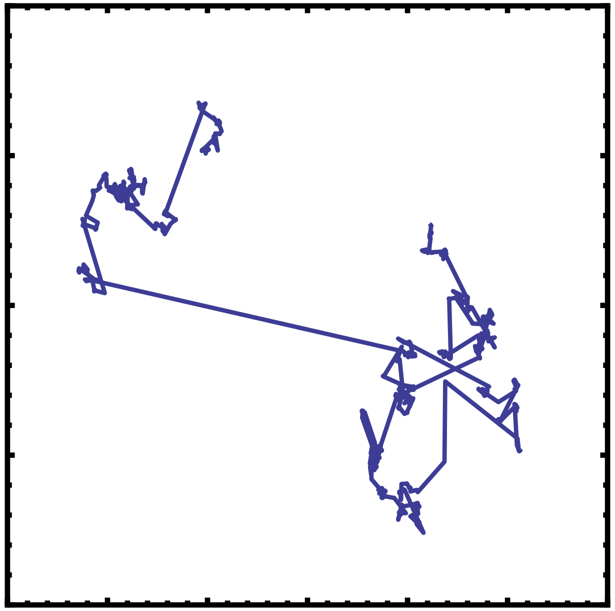


(c) super diffusion

(between (a) and (b))



The Lévy Walk Regimes

Reminder: the density distribution of the step-length is $f(d) \sim 1/d^{\alpha}$

Case $\alpha \geq 3$: the Lévy walk has normal diffusion

(Idea) In one dimension, and for $\alpha > 3$.

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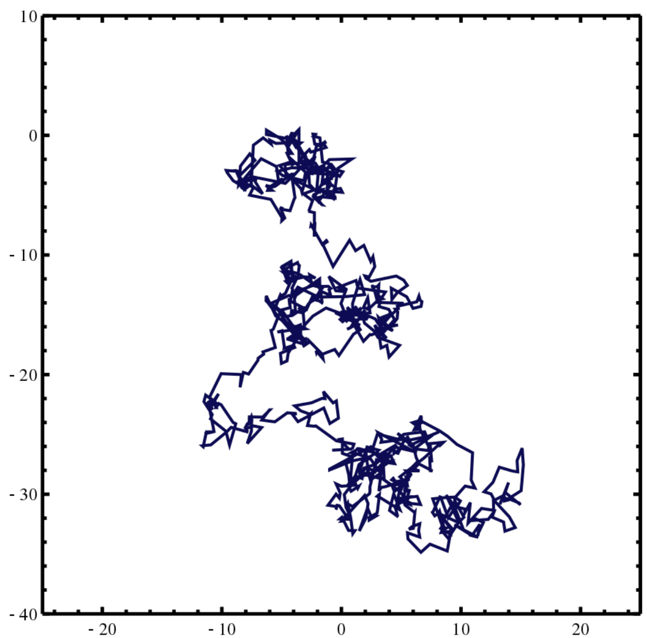

A Lévy walk with parameter $\alpha = 3$ approximates a brownian motion

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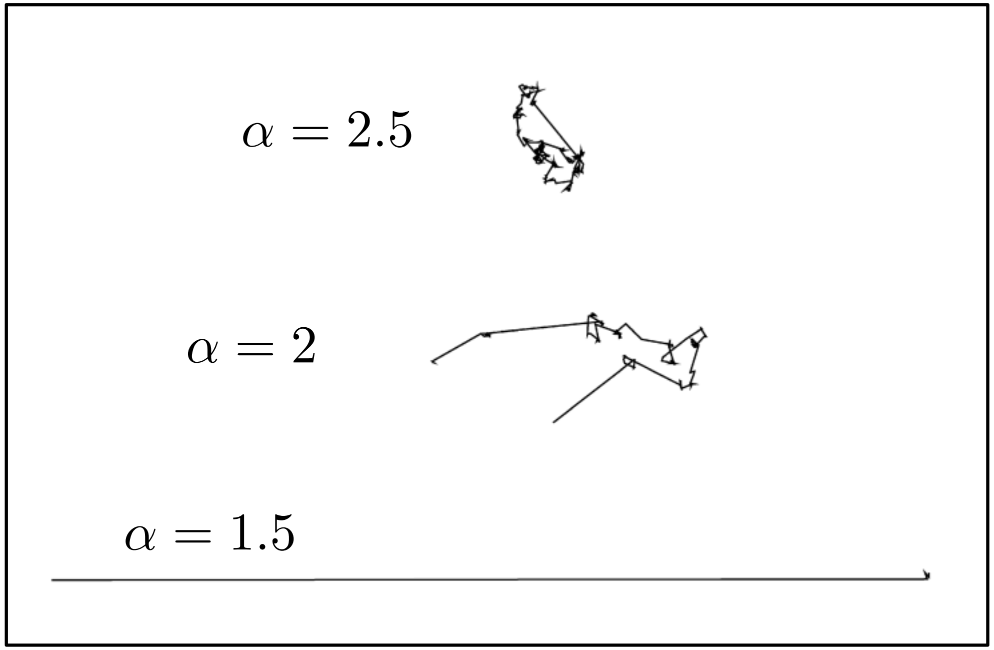

Examples of Lévy walks for different values of α

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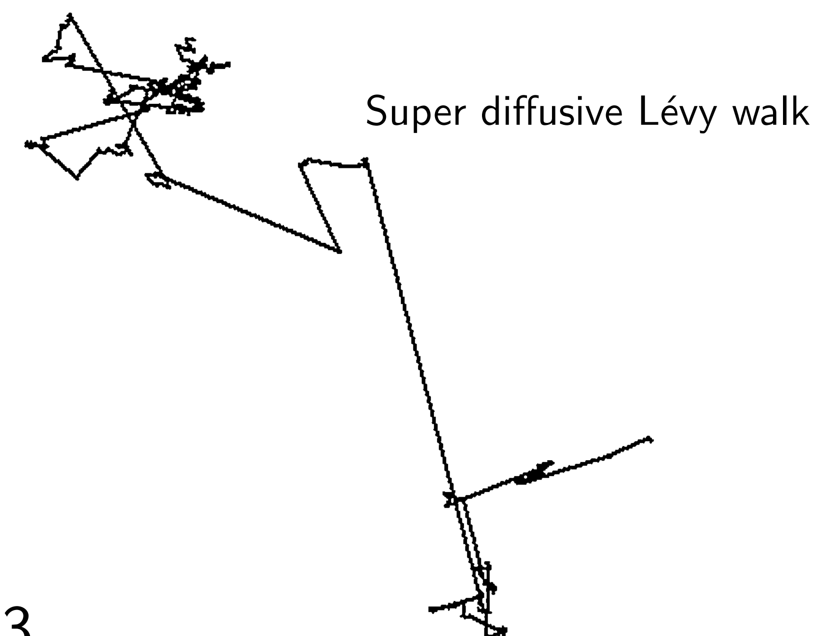
Note: in between normal and ballistic diffusion



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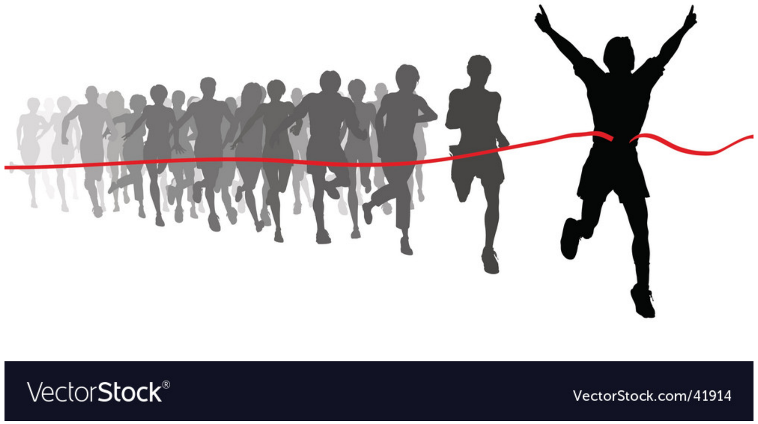
- the Lévy walk with exponent $\alpha = 2$, for non-destructive foraging
- the ballistic walk, for destructive foraging



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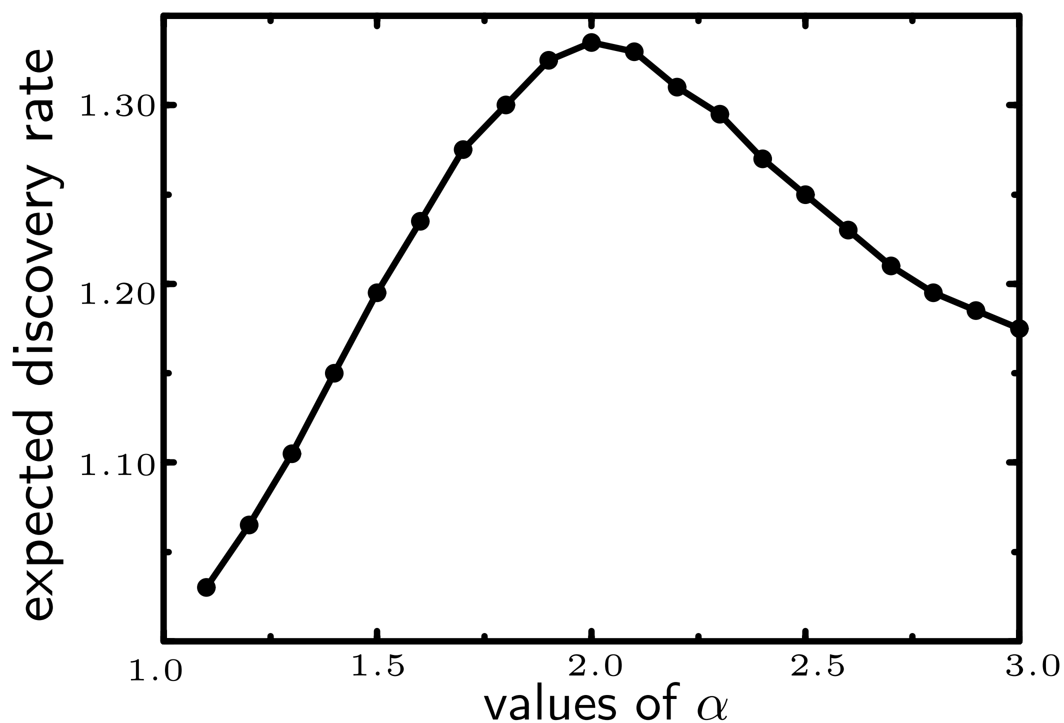

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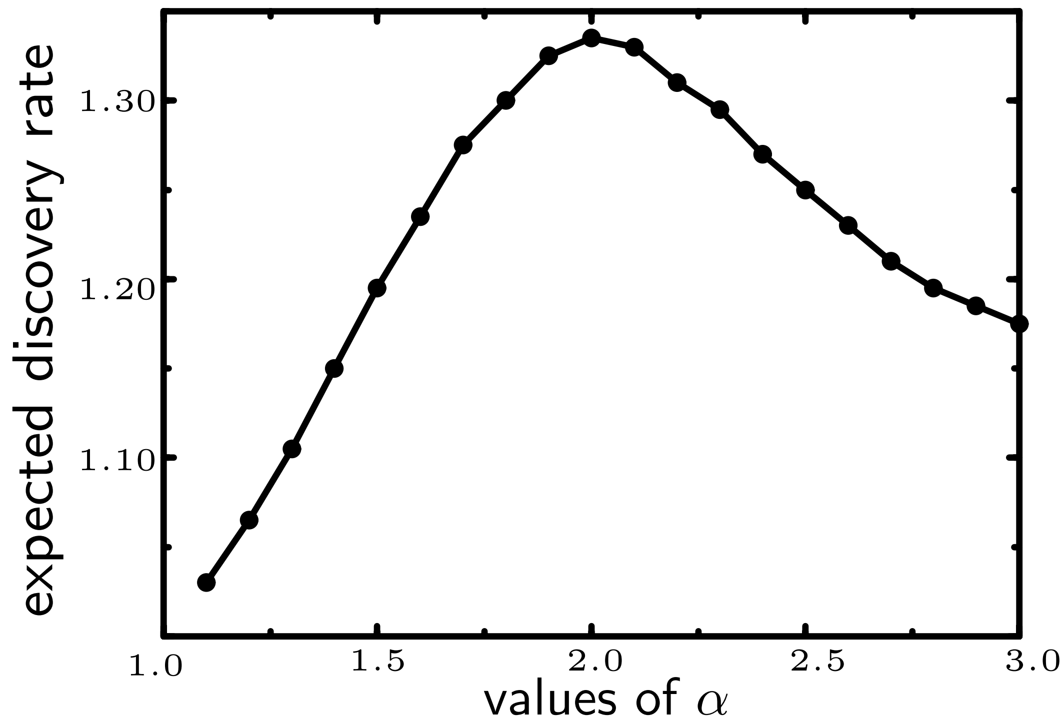

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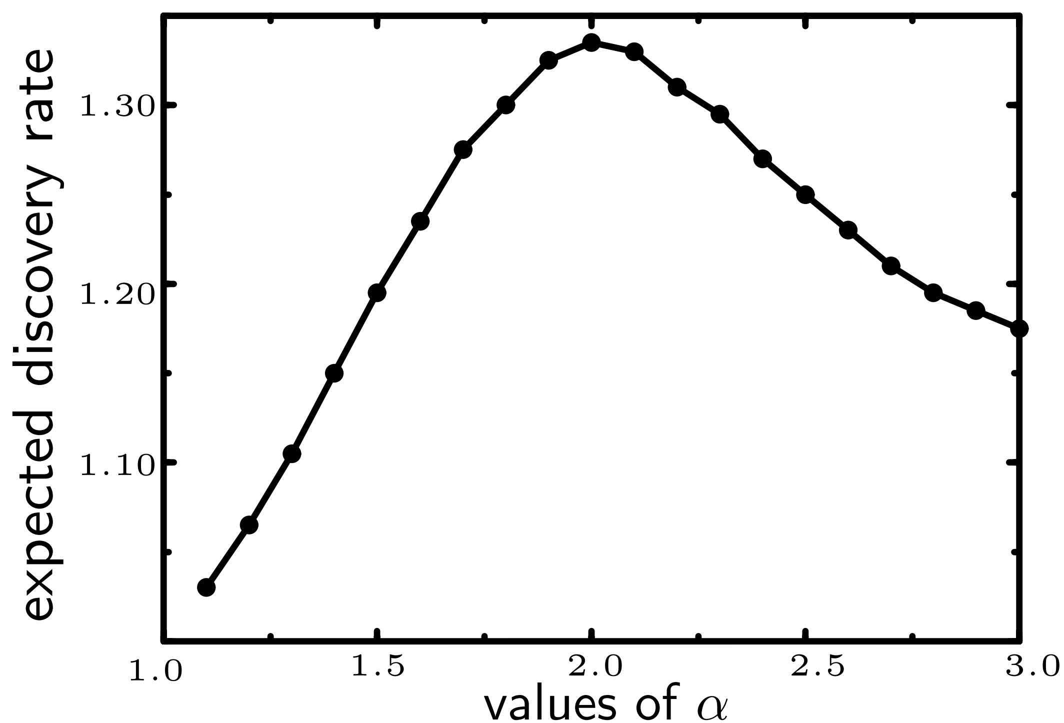

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HOWEVER...

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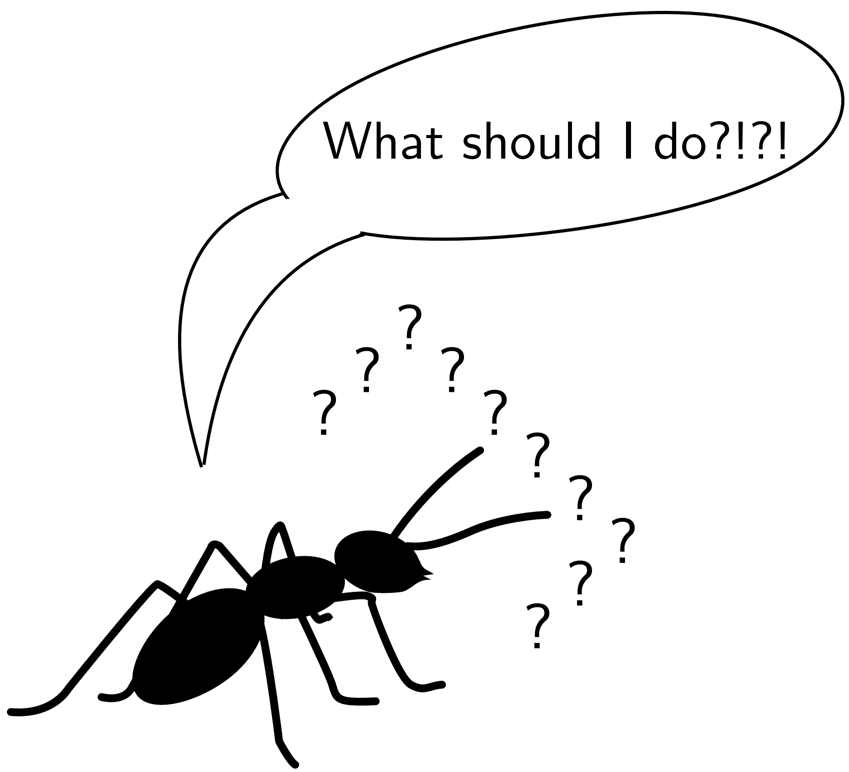

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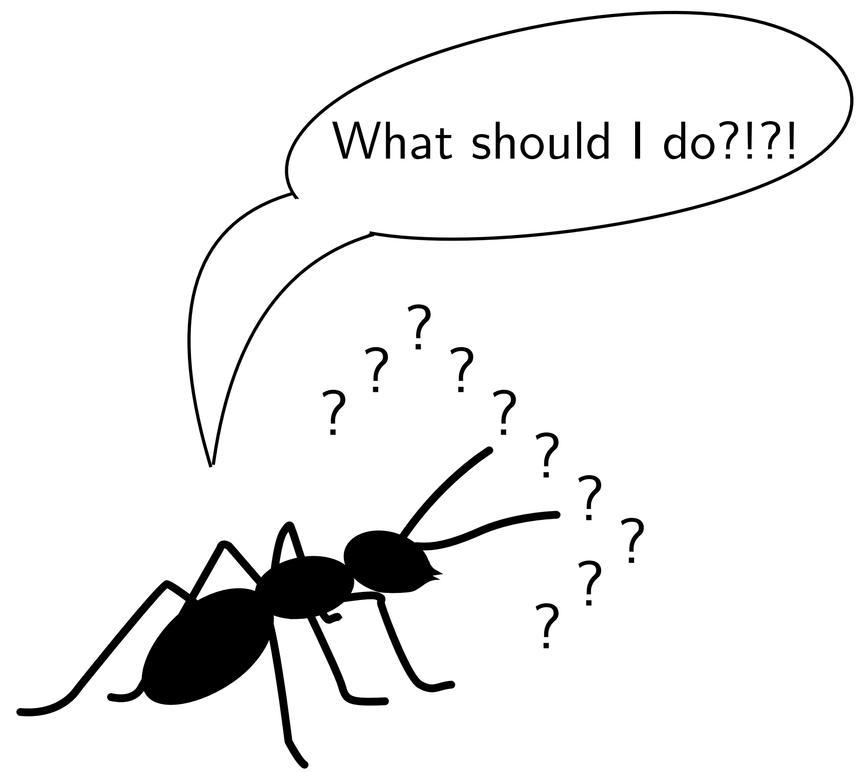
The Lévy walk has never been studied in the discrete setting



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Setting:
- k (mutually) independent walkers (agents) start moving on $\mathbb{Z}^2$ from the origin
- time is synchronous and marked by a global clock
- one special node $\mathcal{P} \in \mathbb{Z}^2$, the *treasure*, at (Manhattan) distance ℓ from the origin

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Question: which strategy is the best one to find the treasure?

Some Preliminaries

We denote by

- *step* a move that takes one time unit
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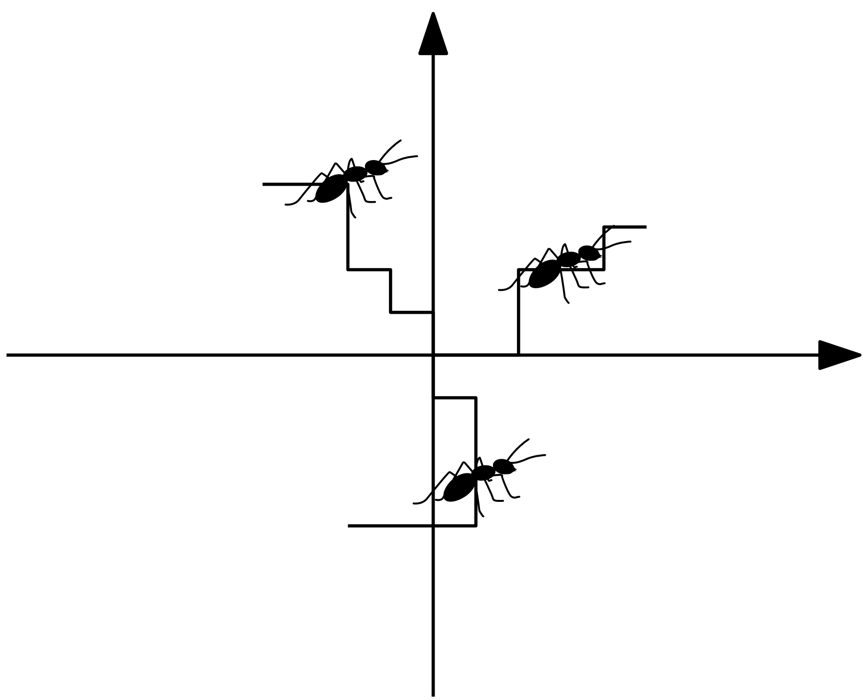
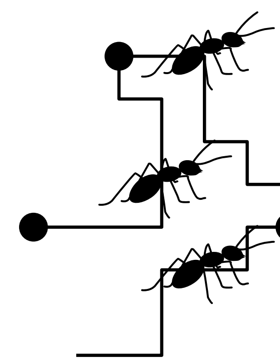
Definition (*work*): k agents moving for t steps make a work equal to $k \cdot t$





work $=$ total path covered

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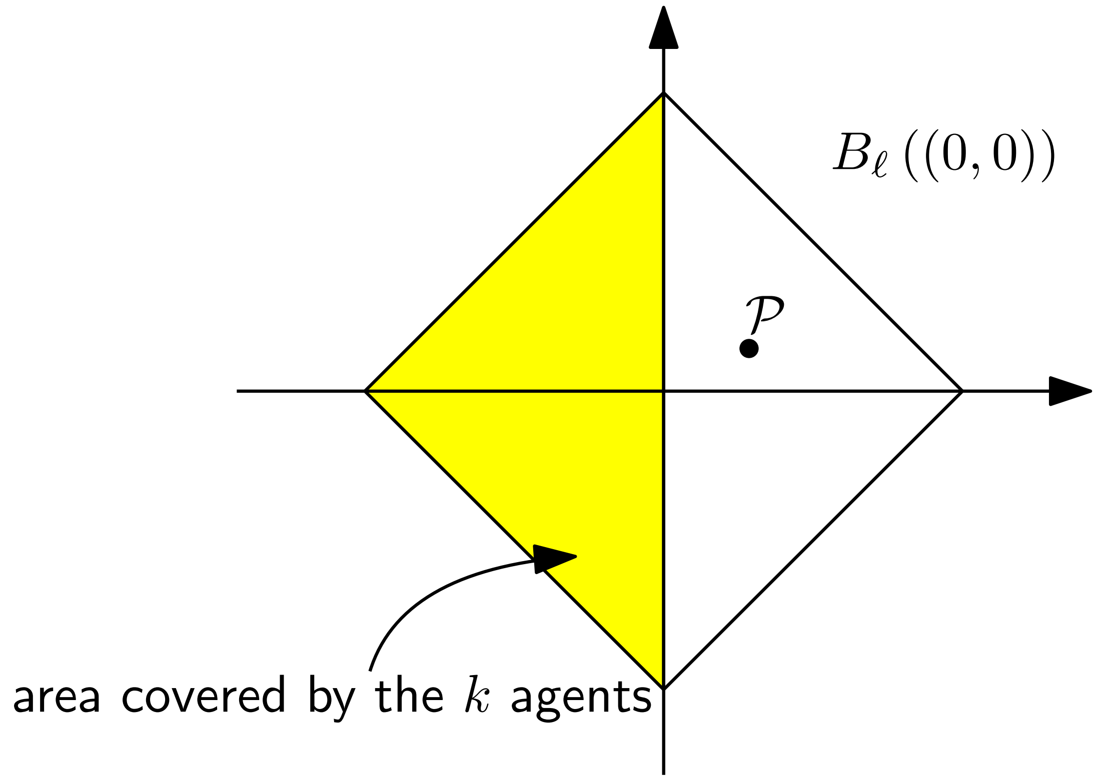
Lemma: locate $\mathcal{P}$ u.a.r. in one node in $B_\ell\left((0,0)\right)$. For any $k \geq 1$, and for any search algorithm $\mathcal{A}$, the work required to find $\mathcal{P}$ is $\Omega\left(\ell^2\right)$ both with constant probability and in expectation

Proof:

- $|B_\ell\left((0,0)\right)| = \ell^2$
- set $t = \ell^2/(4k)$



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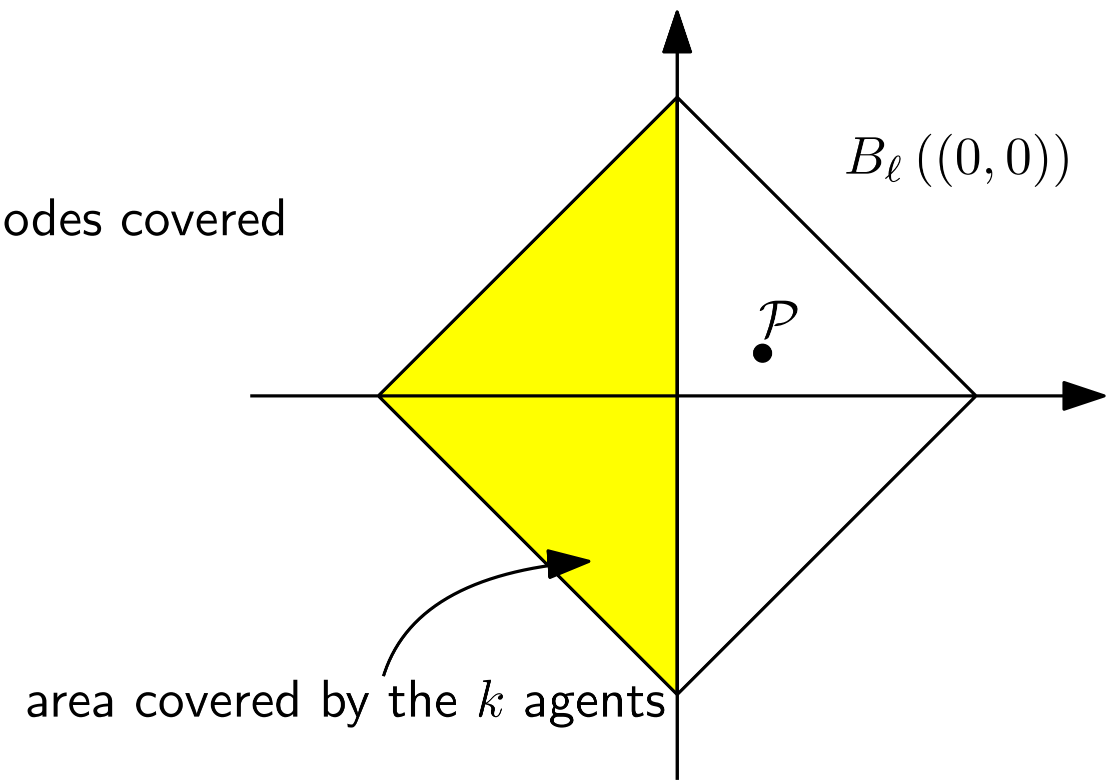

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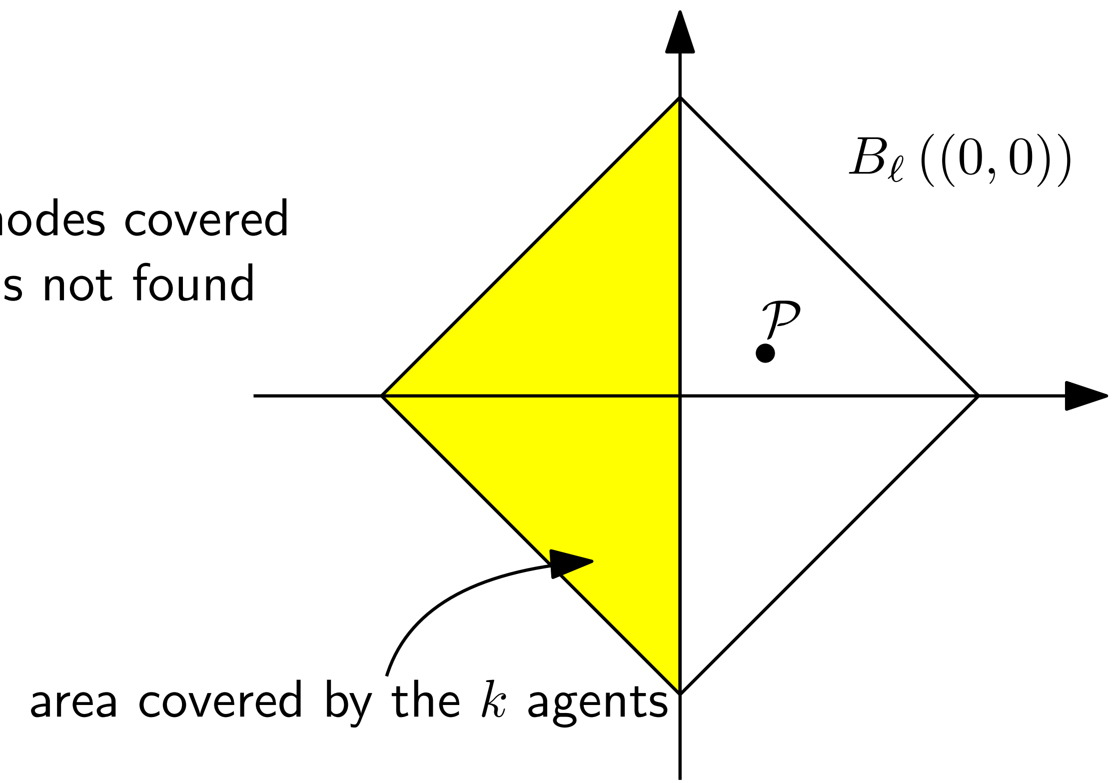

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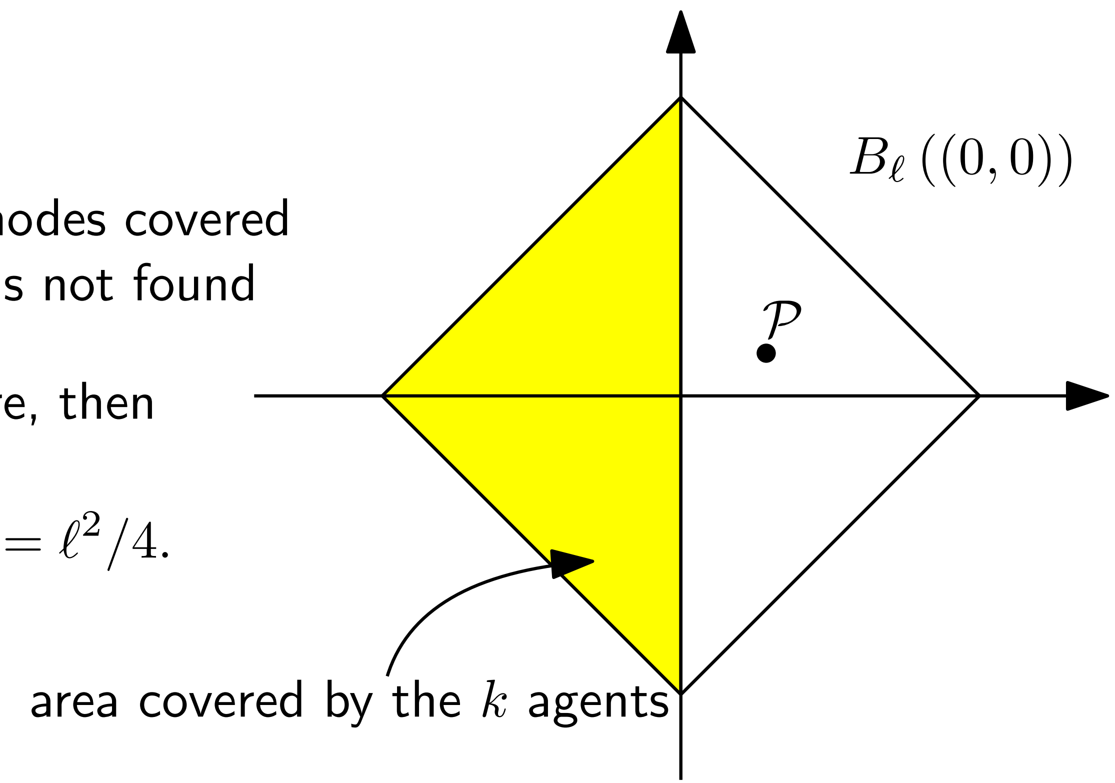
- $H =$ first hitting time for the treasure, then

$$\mathbb{E}\left[\text{work}\right] = \mathbb{E}\left[kH\right] \geq 2kt \cdot \frac{1}{2} = \ell^2/4.$$



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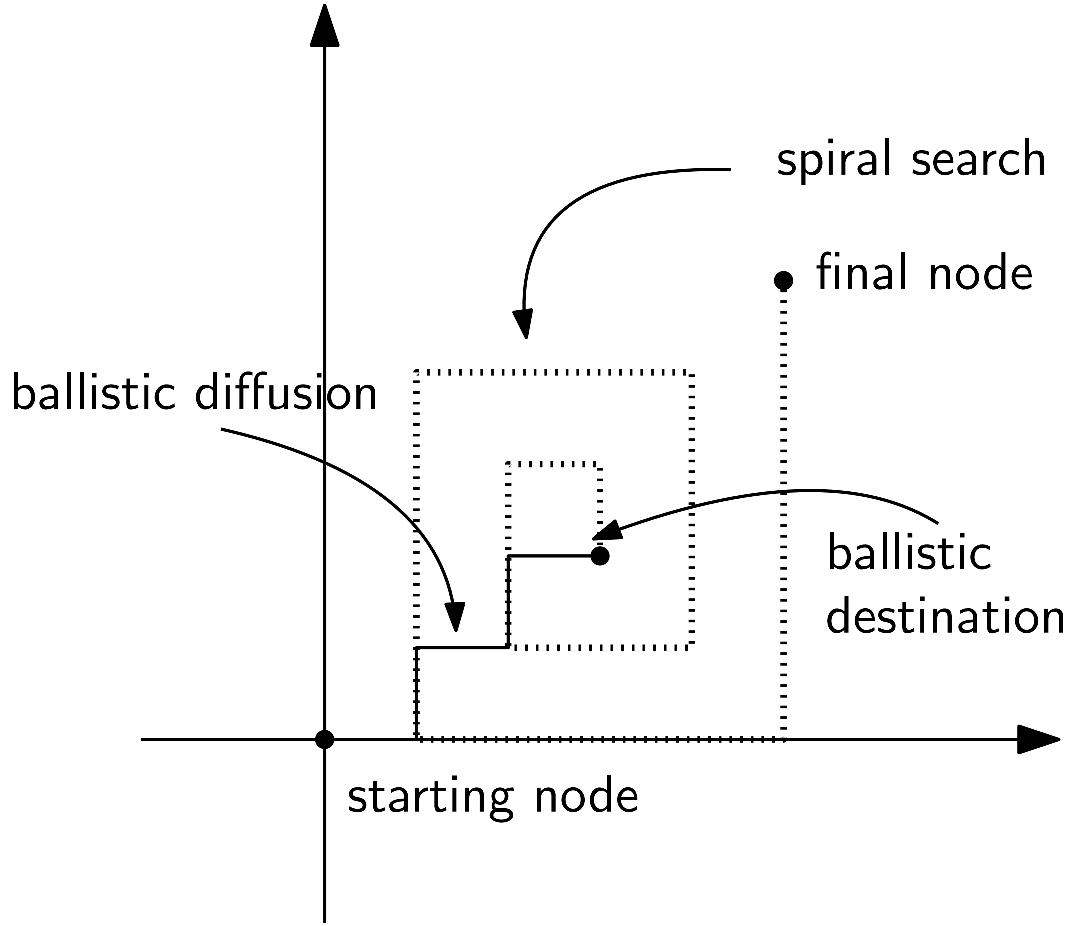

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The Harmonic Search Algorithm

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Let $\alpha > 1$ be a real constant

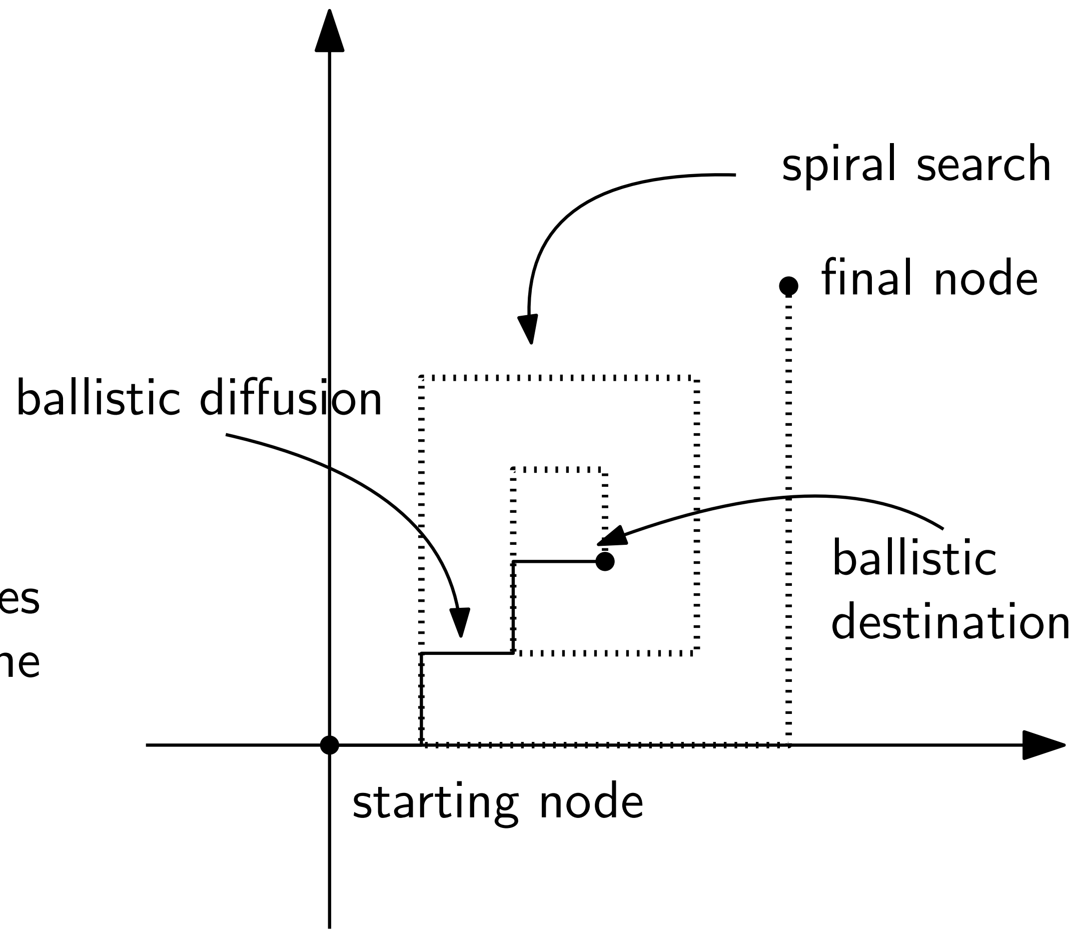
The Harmonic Search algorithm: each agent performs the following instructions

a) it samples a Lévy jump-length d with probability c_α / d^α

b) (ballistic diffusion) in d steps, it moves to a destination at distance d from the starting node chosen u.a.r.

c) (normal diffusion) once at the destination, it starts exploring the around area with a spiral search for $d^{\alpha+1}$ steps

d) it returns in the origin and repeats



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Results [*Korman et al.*, PODC, '12] (informal):

- the smaller $\delta > 0$ (or $\alpha > 1$), the better the performances
- the work made by k walkers is $\mathcal{O}\left(\ell^{2+\delta}\right)$ with probability $\geq 1 - \epsilon$, for any $\epsilon > 0$ and $k \geq \Theta\left(f(\epsilon)\ell^{\delta}\right)$

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Reminder: the lower bound on the work is $\Omega\left(\ell^2\right)$ with constant probability

Our Work

We give the first definition of Lévy walk in the discrete setting in $\mathbb{Z}^2$, the *Pareto walk*, which is *natural* and *time-homogeneus*

- the jump-length distribution we choose is a common variant of the Pareto distribution, which is a power-law

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Task:
- minimize the work to find the treasure
- estimate the distribution of the hitting time

Some Preliminaries

Definition: we say that an event E depending on a parameter $n \in \mathbb{N}$ holds *with high probability* (w.h.p. in short) w.r.t. n if
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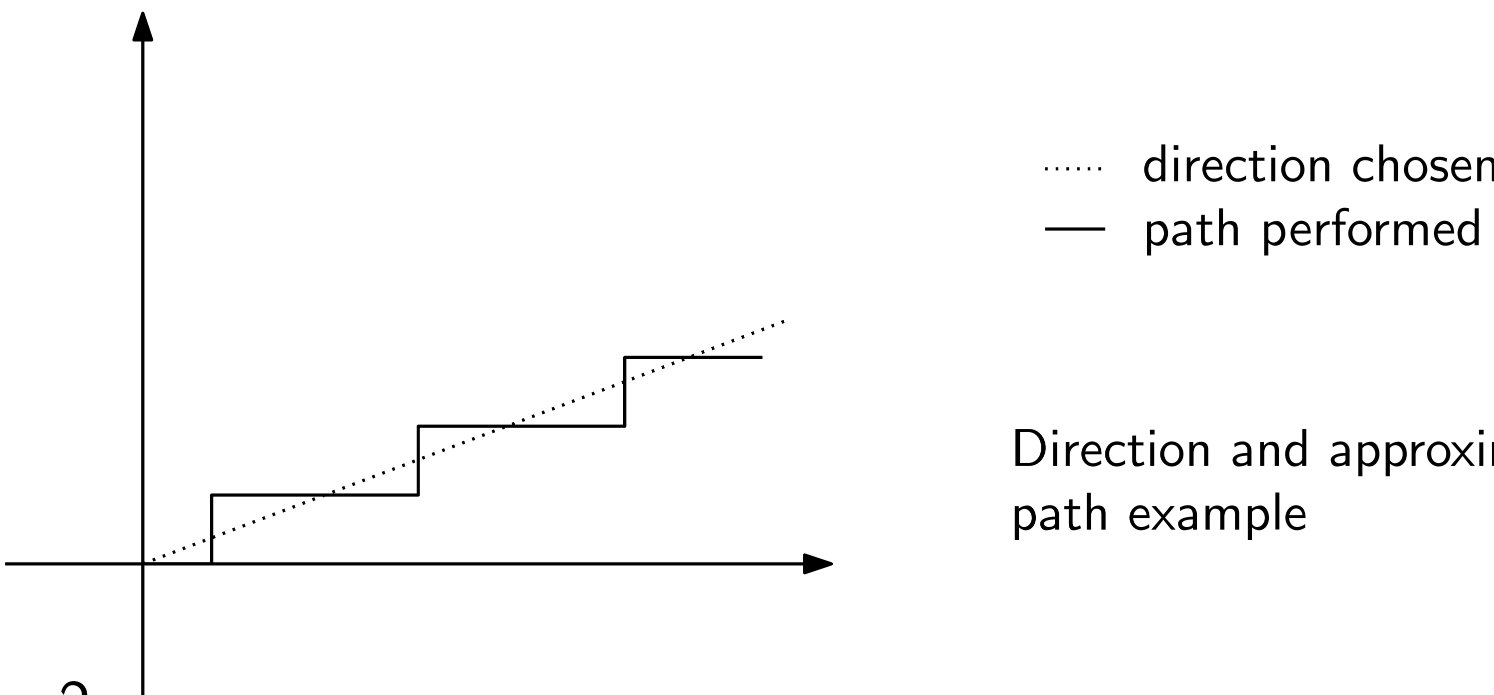
Two notions:
- choice of a direction u.a.r. in $\mathbb{Z}^2$
- selection of a "direction-approximating" path



Direction and approximating path example

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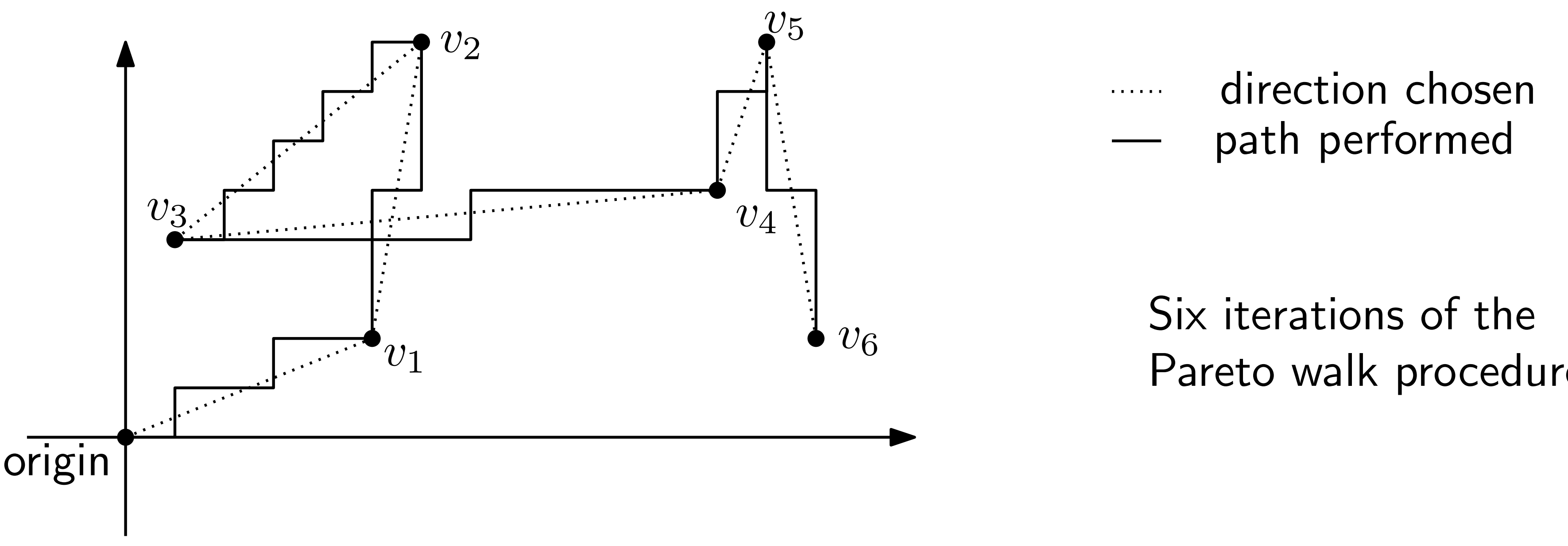
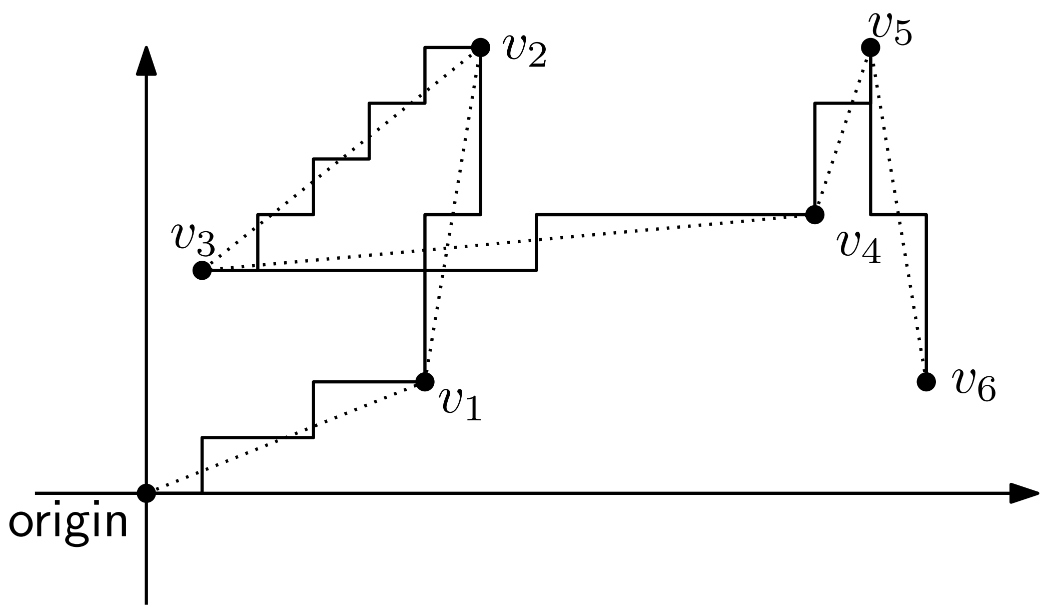


Six iterations of the Pareto walk procedure

Some Preliminaries



direction chosen
path performed

Six iterations of the Pareto walk procedure

Let $\alpha > 1$ be a real constant

Pareto walk: each agent performs the following instructions

a) it chooses a distance $d \in \mathbb{N}$ with probability $c_\alpha/(1+d)^\alpha$
b) it chooses a direction u.a.r.
c) it walks along the corresponding direction-approximating path for d steps, one edge at a time, crossing d nodes
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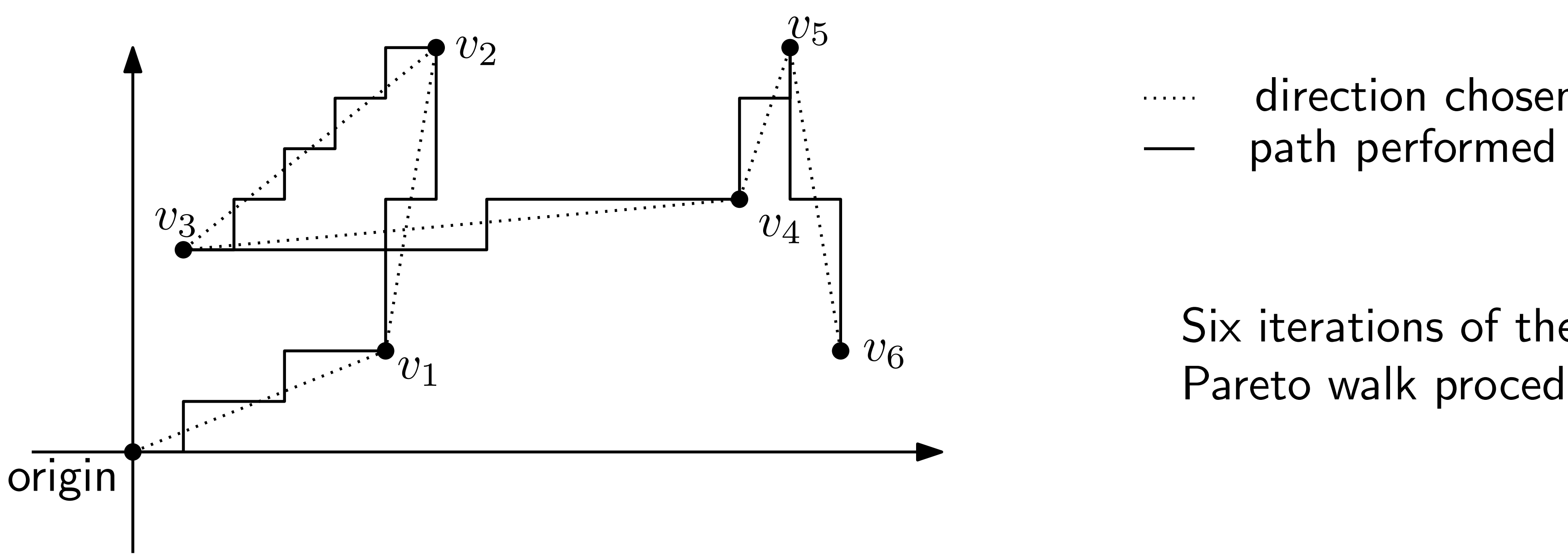

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Remark: the probability distribution in (a) is a known variant of the Pareto distribution [Wiley StatsRef, '15]

Our Results

Reminder: the lower bound on the work is $\Omega\left(\ell^2\right)$ with constant probability

Result (up to polylogarithms): for each choice of $\alpha > 1$ there is just one polynomial value (in ℓ) for k such that, w.h.p., the work is equal to ℓ^2, thus optimal

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Exponent	Value of k	Hitting time	Work
$\alpha \geq 3$	$\log^{\Theta(1)} \ell$	$\tilde{\Theta}\left(\ell^2\right)$	$\tilde{\Theta}\left(\ell^2\right)$
$2 < \alpha < 3$	$\tilde{\Theta}\left(\ell^{1-(\alpha-2)}\right)$	$\tilde{\Theta}\left(\ell^{1+(\alpha-2)}\right)$	$\tilde{\Theta}\left(\ell^2\right)$
$1 < \alpha \leq 2$	$\tilde{\Theta}\left(\ell\right)$	$\tilde{\Theta}\left(\ell\right)$	$\tilde{\Theta}\left(\ell^2\right)$

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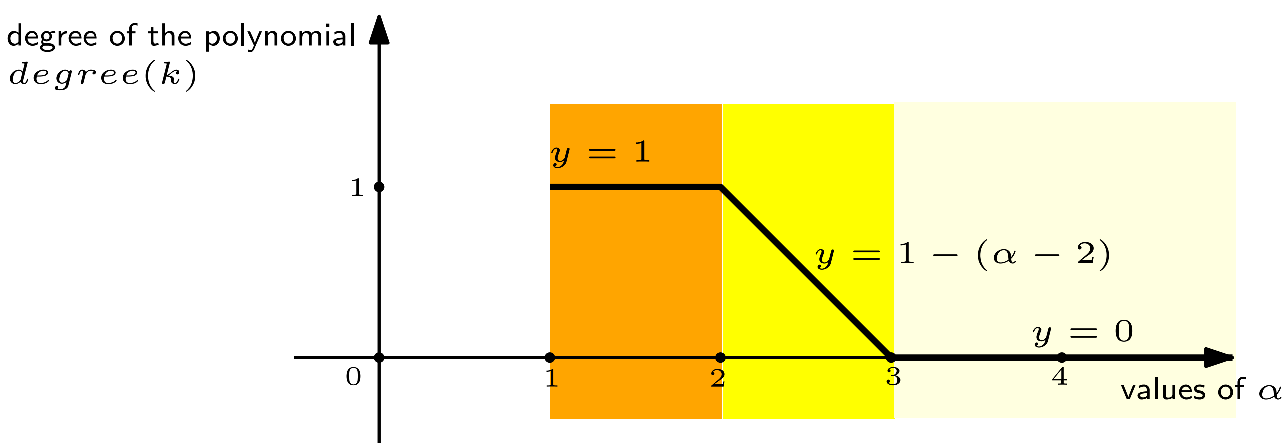

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The results in the previous slide are *almost-tight*:

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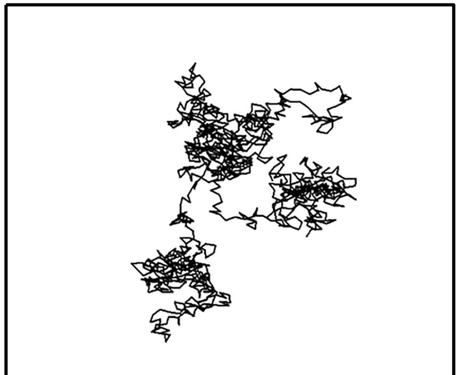
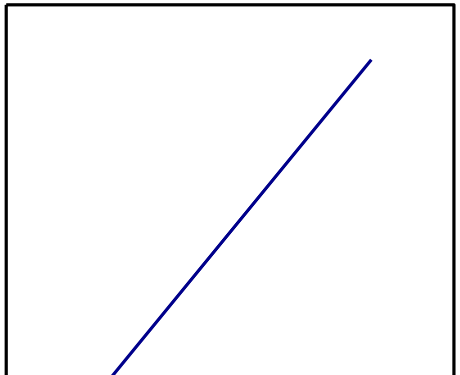
We also prove the following equivalences, in terms of work-efficiency

- $\alpha \geq 3 \sim$ simple random walk (normal diffusion)
- $1 < \alpha \leq 2 \sim$ ballistic walk (ballistic diffusion)



normal diffusion



ballistic diffusion

Some Considerations

The exponent $\alpha = 2$ does not play any crucial role in our setting

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Hint: the optimal search strategy depends on the chosen setting (i.e., the environment)

Now, some details on how we prove the upper bound on the hitting time for the super-diffusive regime...

Some Analysis: $2 < \alpha < 3$

Remark: if $2 < \alpha < 3$, the expected jump-length of the Pareto walk is constant

Proof: indeed, the expectation is
$\sum_{d \geq 0} c_\alpha d/(1+d)^\alpha \sim \sum_{d \geq 0} c_\alpha/(1+d)^{\alpha-1} < +\infty$

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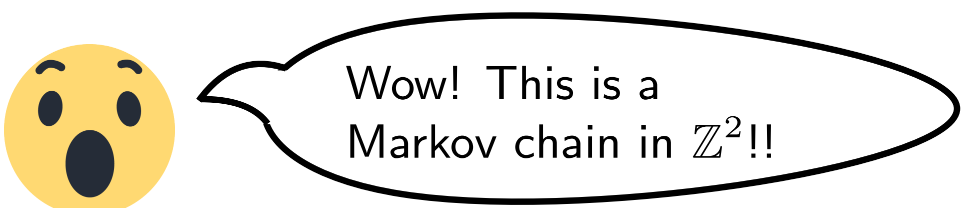
Pareto flight: the Pareto flight is a Pareto walk in which the agent takes just one step/time unit to reach a jump-destination, without visiting intermidiate nodes



Coupling Result

Coupling result: if one single Pareto flight finds the treasure within t steps with probability $p(t)$ conditional on the event that all the performed jump lengths are less than $(t \log t)^{\frac{1}{\alpha-1}}$, then one Pareto walk finds the treasure within $\Theta(t)$ steps with probability at least $[p(t) - \exp\left(-t^{\Theta(1)}\right)]/2$, without any conditional event

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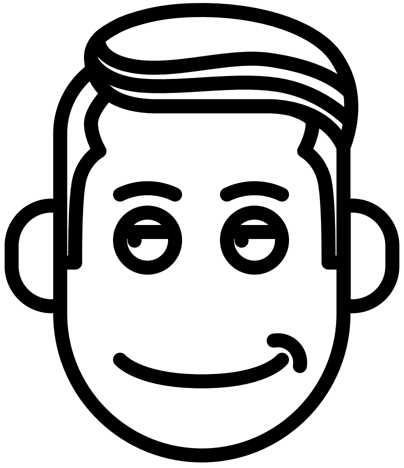
Basically, we reduce ourselves to study the hitting time distribution for the treasure of the Pareto flight to get the same result on the Pareto walk



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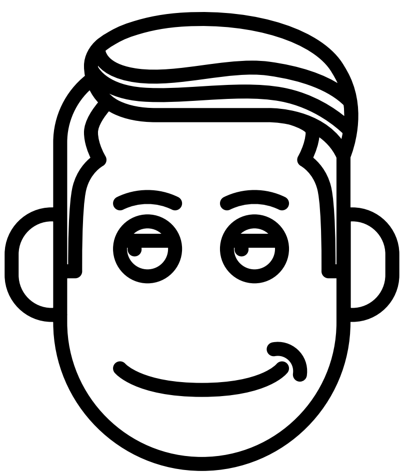
We look at one single Pareto flight to determine $p(t)$ in order to use the coupling result...



Trying to get $p(t)$...

Let

- $\mathcal{P}$ be the treasure
- $|\mathcal{P}|_1 = \ell$ its Manhattan distance from the origin
- $Z_{\mathcal{P}}(t) =$ random variable of number of visits in $\mathcal{P}$ until time t
- $\mathcal{E}_t =$ the event first t jumps have length $\leq (t \log t)^{\frac{1}{\alpha - 1}}$
- $a_t = \mathbb{E}\left[Z_{(0,0)}(t) \mid \mathcal{E}_t\right]$

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For the Pareto flight, it holds

Lemma: $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}(t) > 0 \mid \mathcal{E}_t\right) \geq \mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right] / a_t$

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We now look for $\mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right]$ and a_t...

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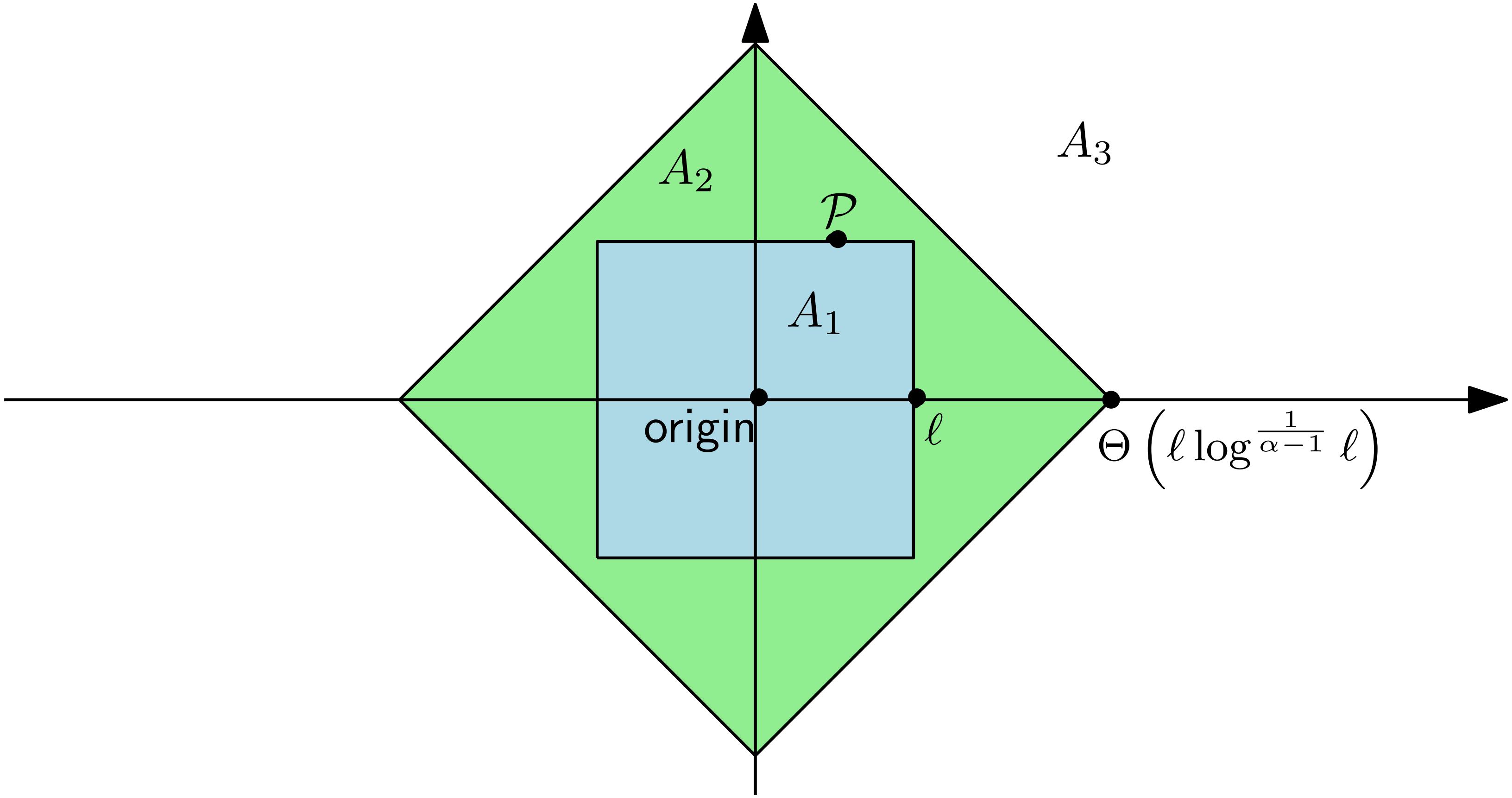
We partition $\mathbb{Z}^2$ in three areas in the following way



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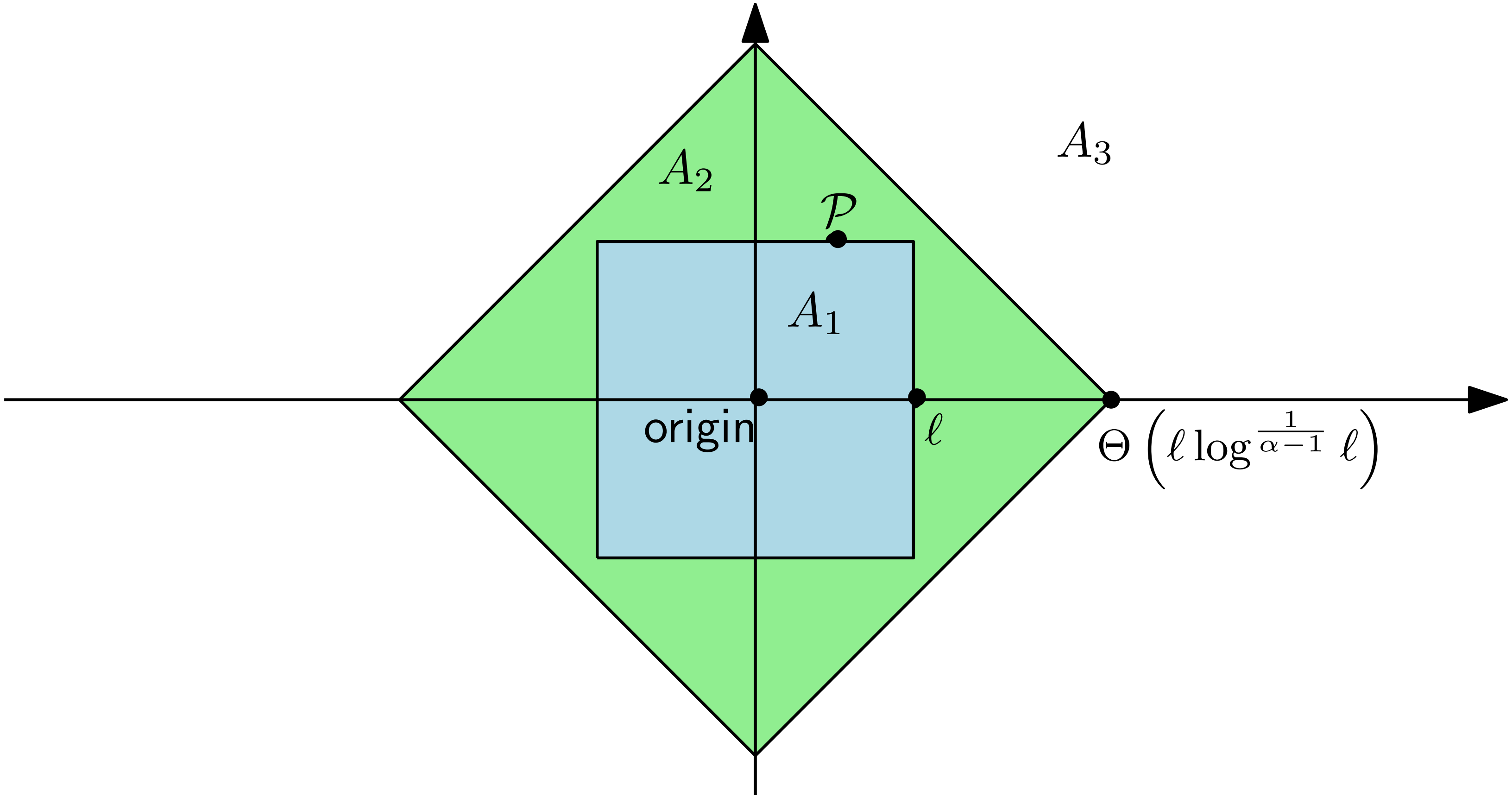

- $A_1 = Q(\ell) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq \ell\}$
- $A_2 = B_{\Theta\left(\ell \log^{\frac{1}{\alpha-1}} \ell\right)}((0,0)) \setminus A_1$
- $A_3 = \mathbb{Z}^2 \setminus (A_1 \cup A_2)$

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For some $t = \Theta\left(\ell^{1+(\alpha-2)}\right)$, we prove that:

b) $\mathbb{E}\left[Z_{A_1}\left(t\right) \mid \mathcal{E}_t\right] \leq \frac{3}{4}t$

c) $\mathbb{E}\left[Z_{A_2}\left(t\right) \mid \mathcal{E}_t\right] \leq \mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right] \cdot \Theta\left(\ell^2 \log^{\frac{2}{\alpha-1}} \ell\right)$

d) $\mathbb{E}\left[Z_{A_3}\left(t\right) \mid \mathcal{E}_t\right] = \mathcal{O}\left(t/\log t\right)$

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Combine (a) with (b), (c), and (d) to get
$\mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right] = \tilde{\Omega}\left(1/\ell^{1-(\alpha-2)}\right)$

Back to $p(t)$...

Reminder: $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}\left(t\right) > 0 \mid \mathcal{E}_t\right) \geq \mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right]/a_t$

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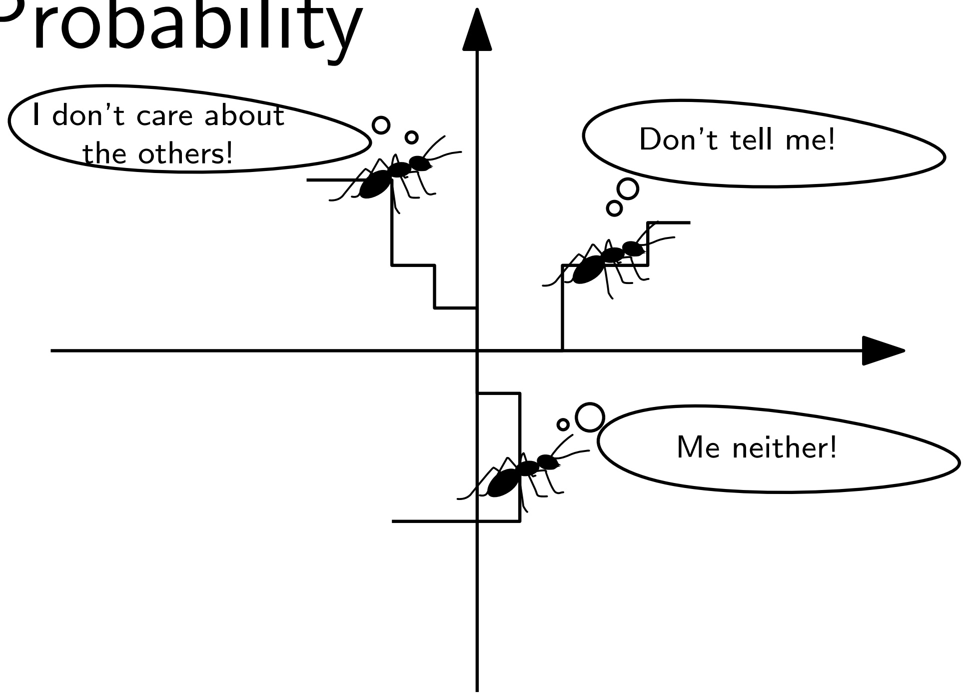
Question: how to get the high probability?

The High Probability

We exploit independence!



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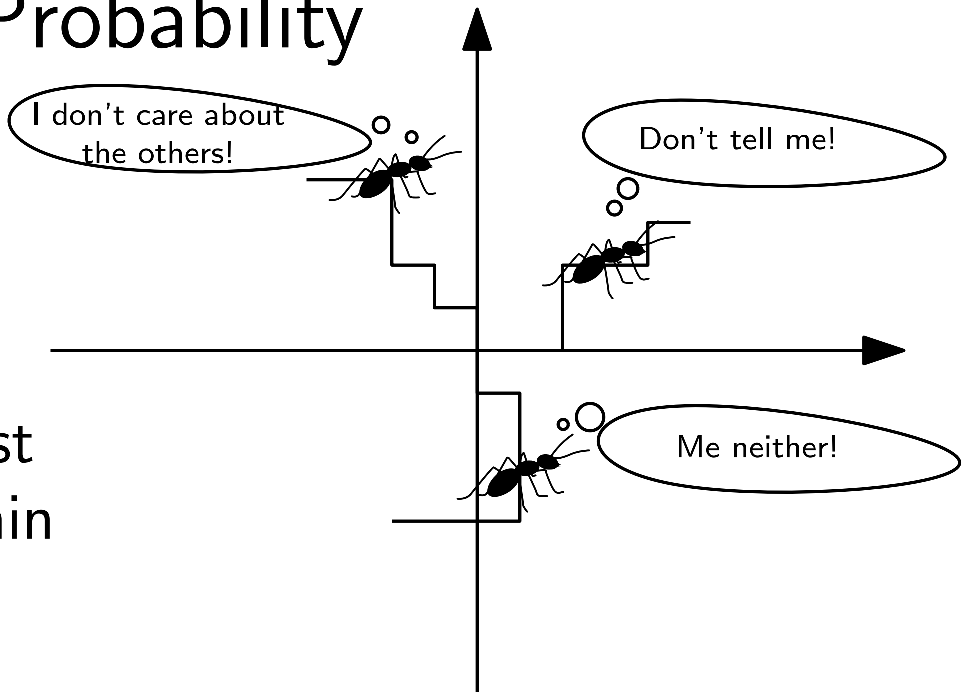

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k walkers, each has probability at least $p(t) \xrightarrow[t\to\infty]{} 0$ to find the treasure within time $t = \Theta\left(\ell^{1+(\alpha-2)}\right)$

They don't find the treasure within time t with probability $[1-p(t)]^k$

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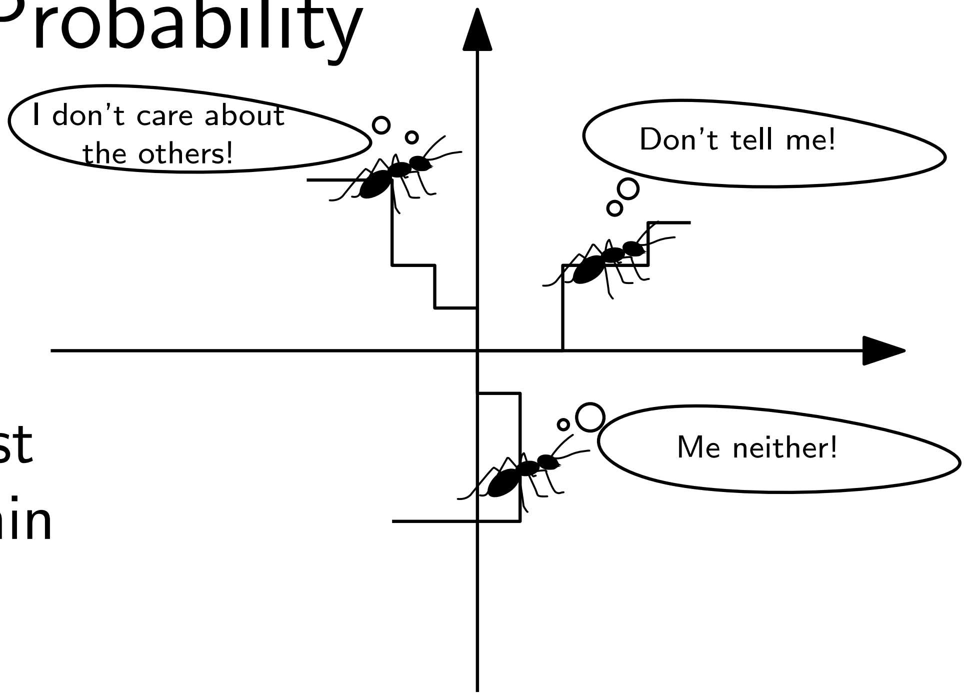

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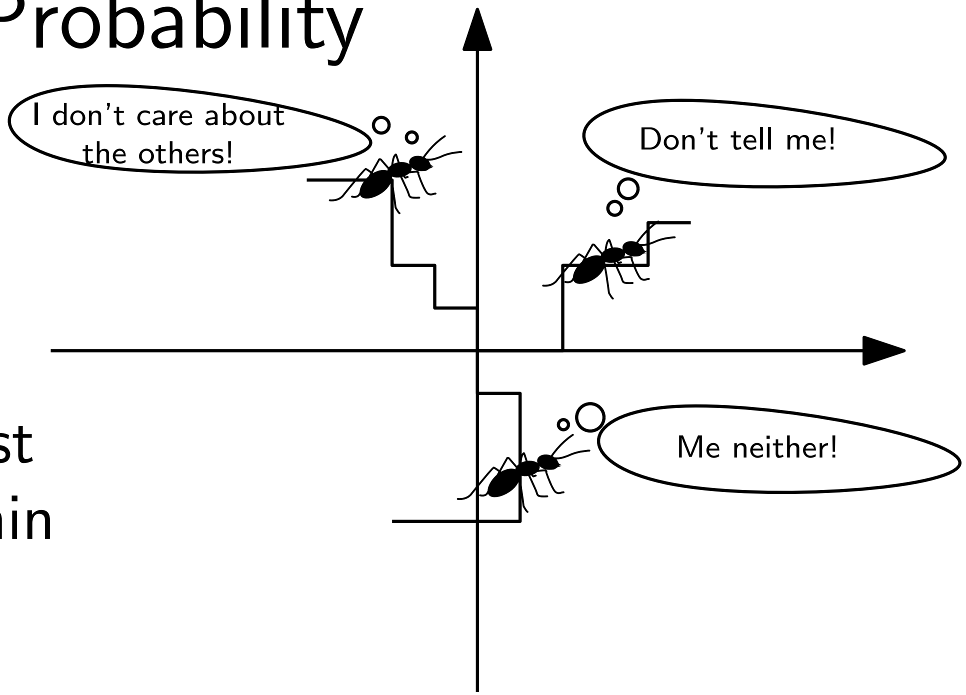

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The probability that at least one walker finds the treasure within time t is

$$1 - [1 - p(t)]^{\frac{\log \ell}{p(t)}} \sim 1 - e^{\log \ell} = 1 - \frac{1}{\ell}$$

We thus need $\log \ell / p(t) = \tilde{\mathcal{O}}\left(\ell^{1-(\alpha-2)}\right)$ walkers to find the treasure within time $t = \Theta\left(\ell^{1+(\alpha-2)}\right)$, making a work equal to $\tilde{\mathcal{O}}\left(\ell^2\right)$, w.h.p.

Recap

In this work, we

- provide a definition of a discrete version of the Lévy walk
- analyze k Pareto walks in the ANTS Problem setting
- show that for any $\alpha > 1$ there is a choiche of k such that k Pareto walks achieve optimal search efficiency
- argue the non (universal) optimality of exponent $\alpha = 2$

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- argue the non (universal) optimality of exponent $\alpha = 2$

Furthermore, we show that the Pareto walk is "equivalent" to

- the simple random walk when $\alpha \geq 3$
- the discrete ballistic walk for $1 < \alpha \leq 2$

THANK YOU FOR YOUR ATTENTION

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