Sparse Temporal Spanners with Low Stretch

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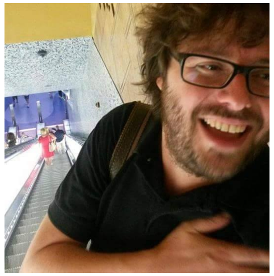
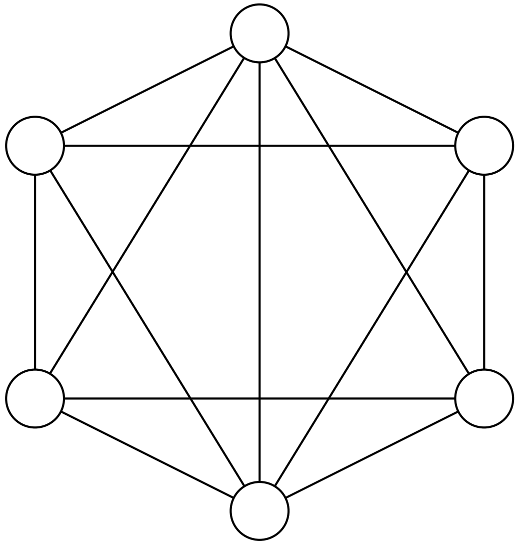


Temporal Graphs



A temporal graph is a graph $G = (V, E)$ in which each edge $e \in E$ has an associated **time-label** $\lambda(e) \in \mathbb{N}^+$

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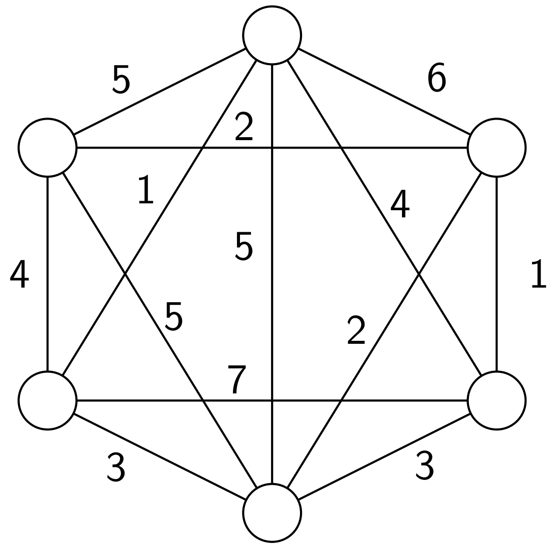

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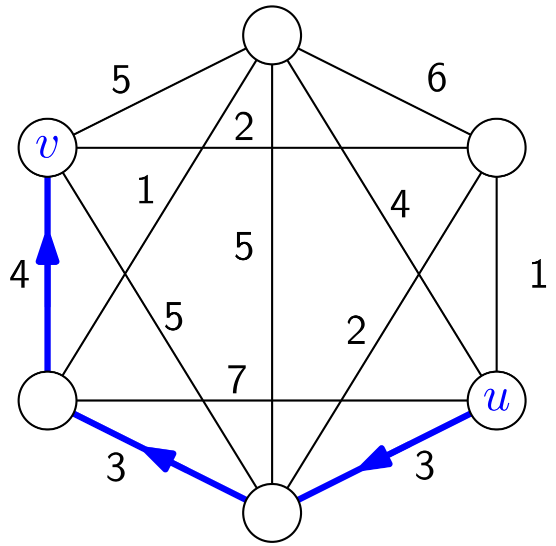

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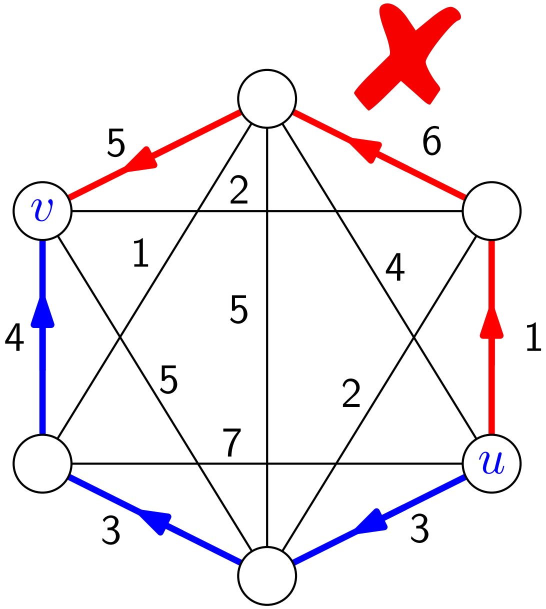

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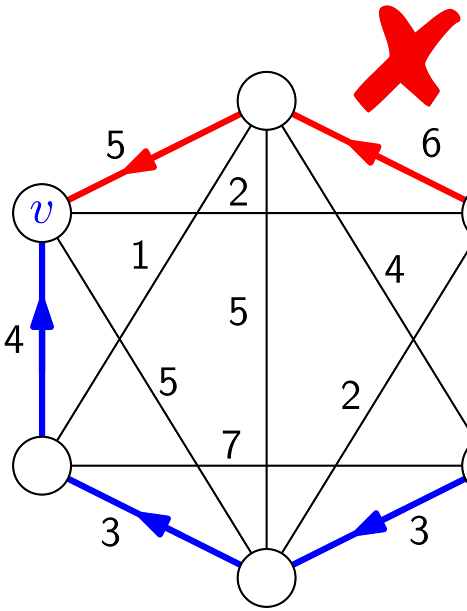

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A **temporal path** from u to v is a path from u to v in which the traversed edges have non-decreasing time-labels

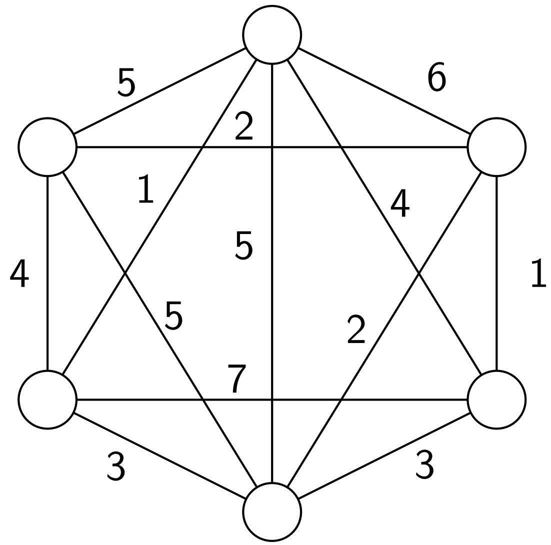
A vertex u is **temporally connected** to a vertex v if there is a temporal path from u to v

(not symmetric)

Temporal Spanners

A **temporal spanner** of a temporal graph G is a subgraph H of G that preserves the temporal connectivity between all pairs of vertices

$$u \text{ is temporally connected to } v \text{ in } G \iff u \text{ is temporally connected to } v \text{ in } H$$



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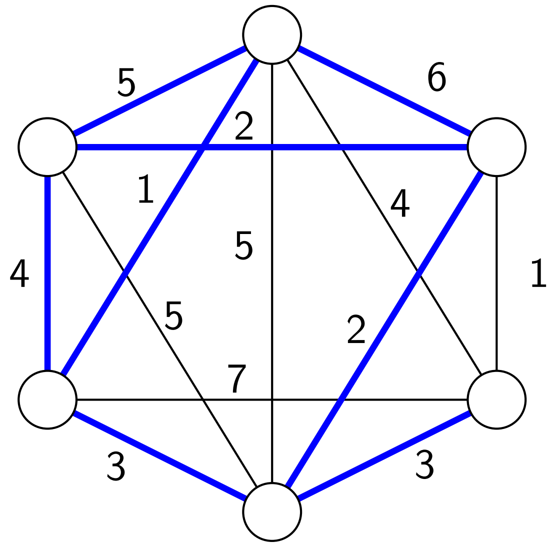

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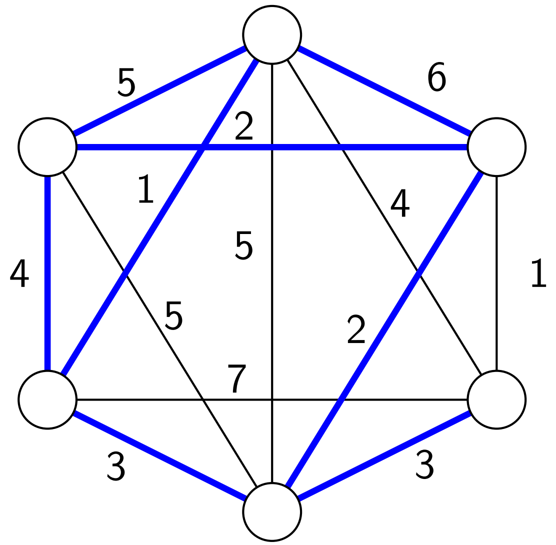

The **size** of a temporal spanner is the number of its edges

Kempe et al. [STOC 2000]: Are there sparse (small size) temporal spanners?

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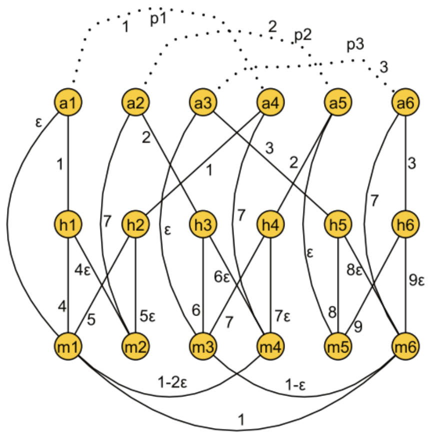

Lower Bound of $\Omega(n^2)$.

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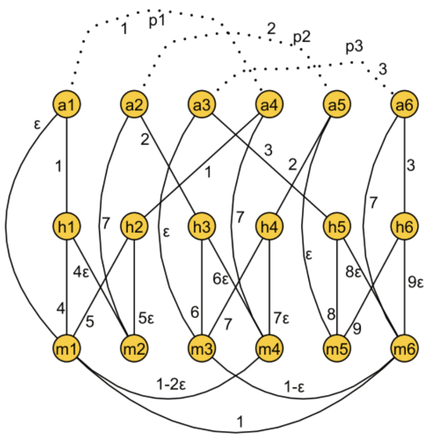
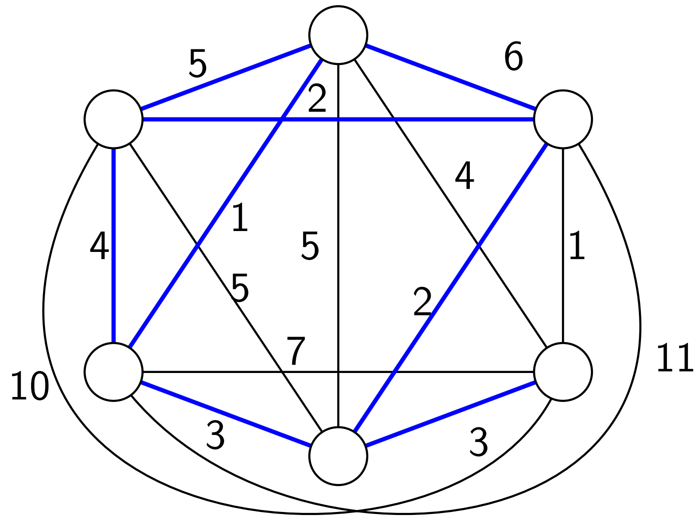


Lower Bound of $\Omega(n^2)$**.**

Casteigts et al. [ICALP 2019]:
Yes, on **temporal cliques**.



Upper Bound of $O(n \log n)$**.**

Temporal Spanners

Kempe et al. [STOC 2000]: Are there sparse (small size) temporal spanners?

The temporal spanner of Casteigts et al. only preserves temporal reachability

No guarantee on how "**temporal distances**" between vertices are affected

High "**temporal stretch**"

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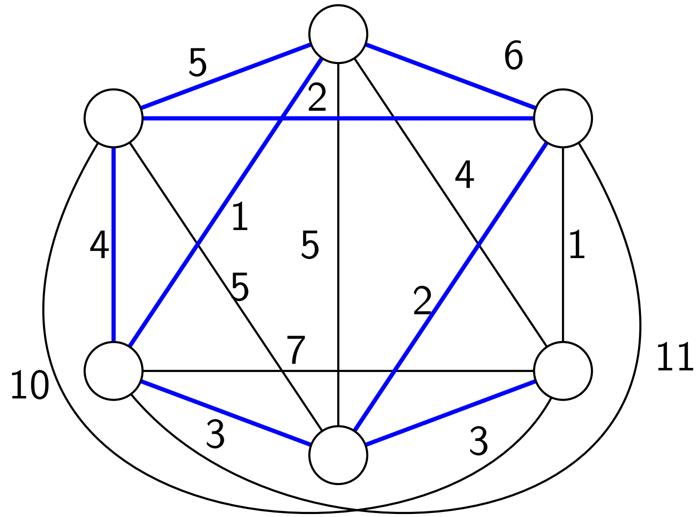

Upper Bound of $O(n \log n)$.

Temporal Spanners with Stretch

A **temporal spanner with stretch** α of a temporal graph G is a temporal spanner H of G such that, for every pair of vertices u, v:

$$\mathsf{dist}_H(u,v) \leq \alpha \cdot \mathsf{dist}_G(u,v)$$

$\mathsf{dist}(u,v) = +\infty$ ff u is not temporally connected to v

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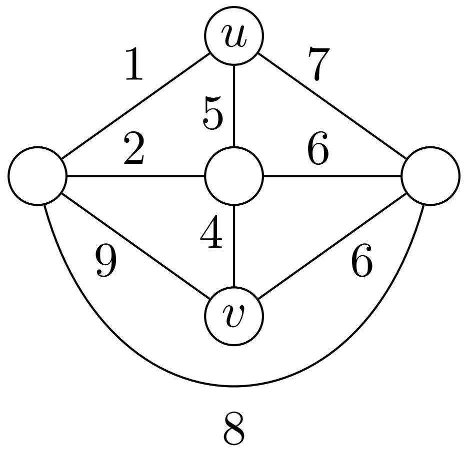
How do we define **dist**?

Unlike static graphs, there are several natural notions of distance for temporal graphs:



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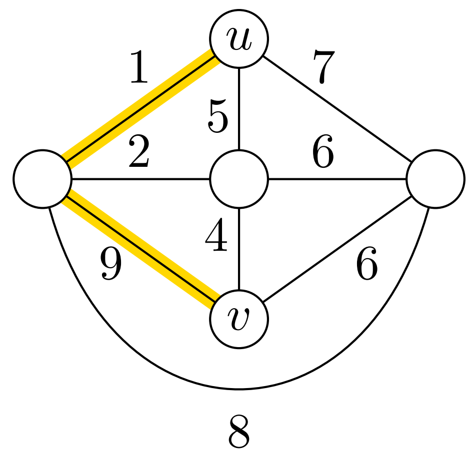

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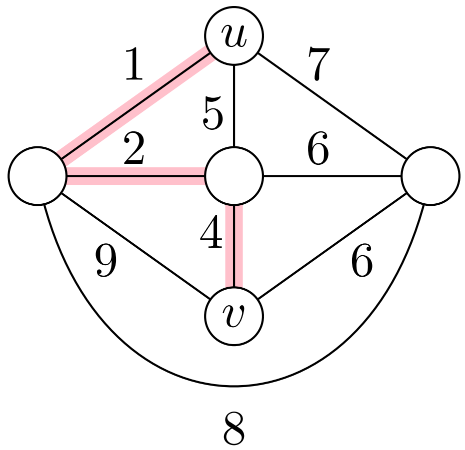

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Unlike static graphs, there are several natural notions of distance for temporal graphs:

- smallest number of edges in a temporal path from u to v — 2
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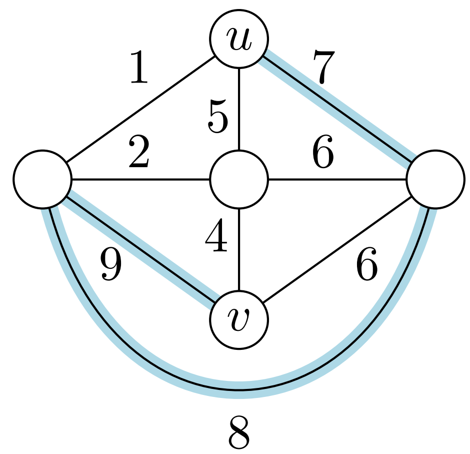

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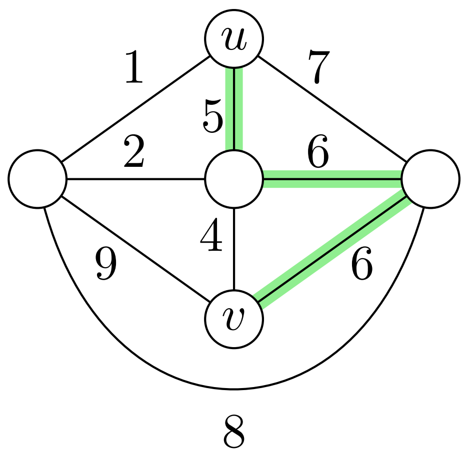

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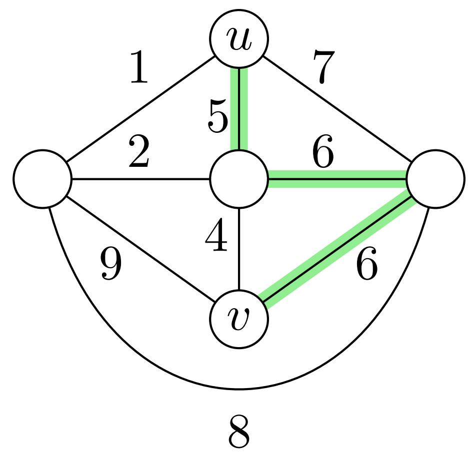
Lower bounds of $\Omega(n^2)$ edges



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Our choice. Generalizes distance on static graphs.

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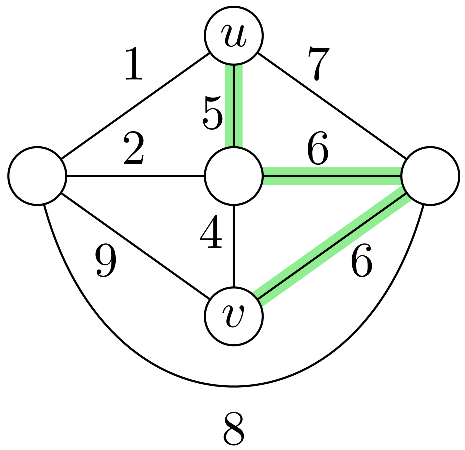
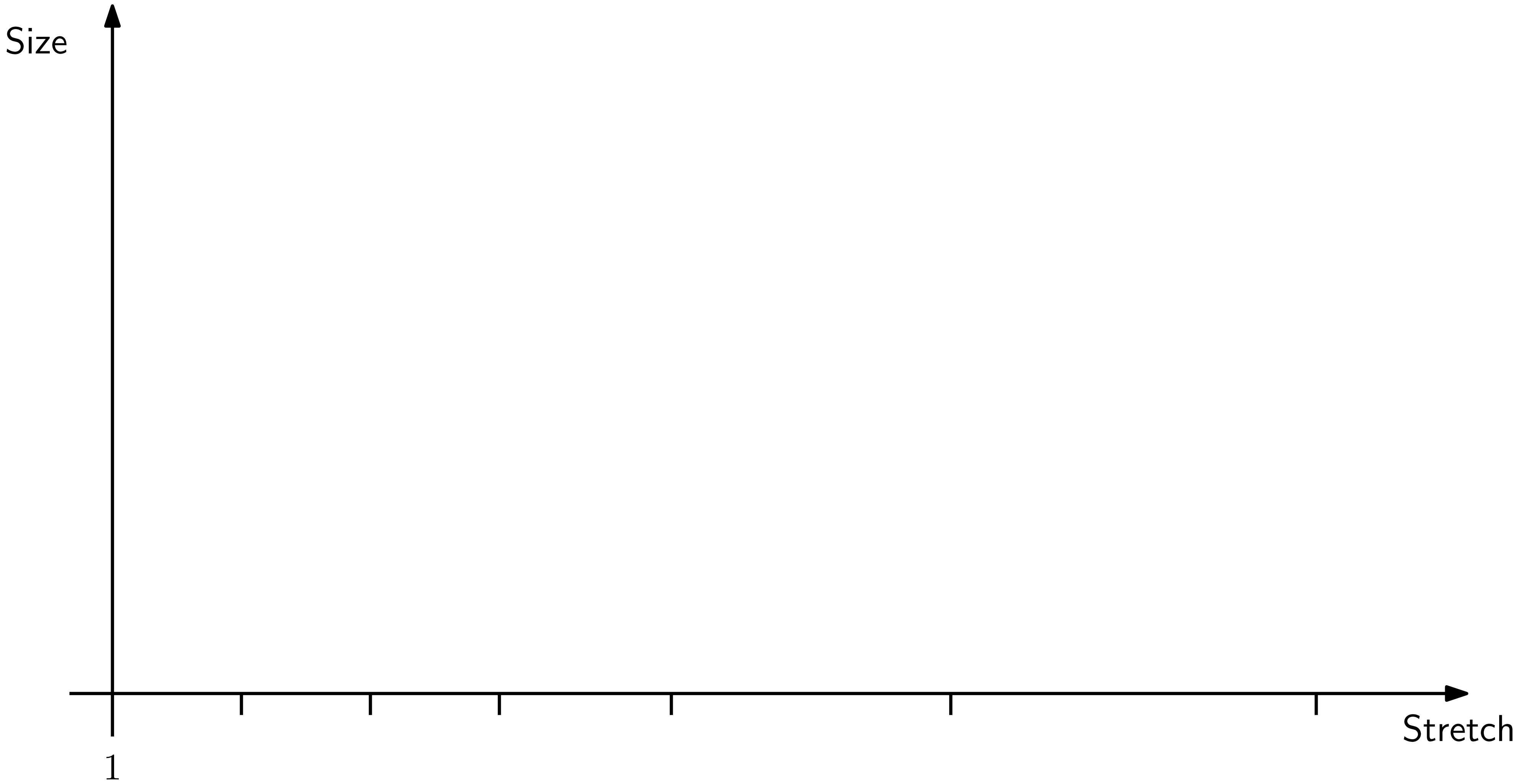


Our Results: Temporal Cliques



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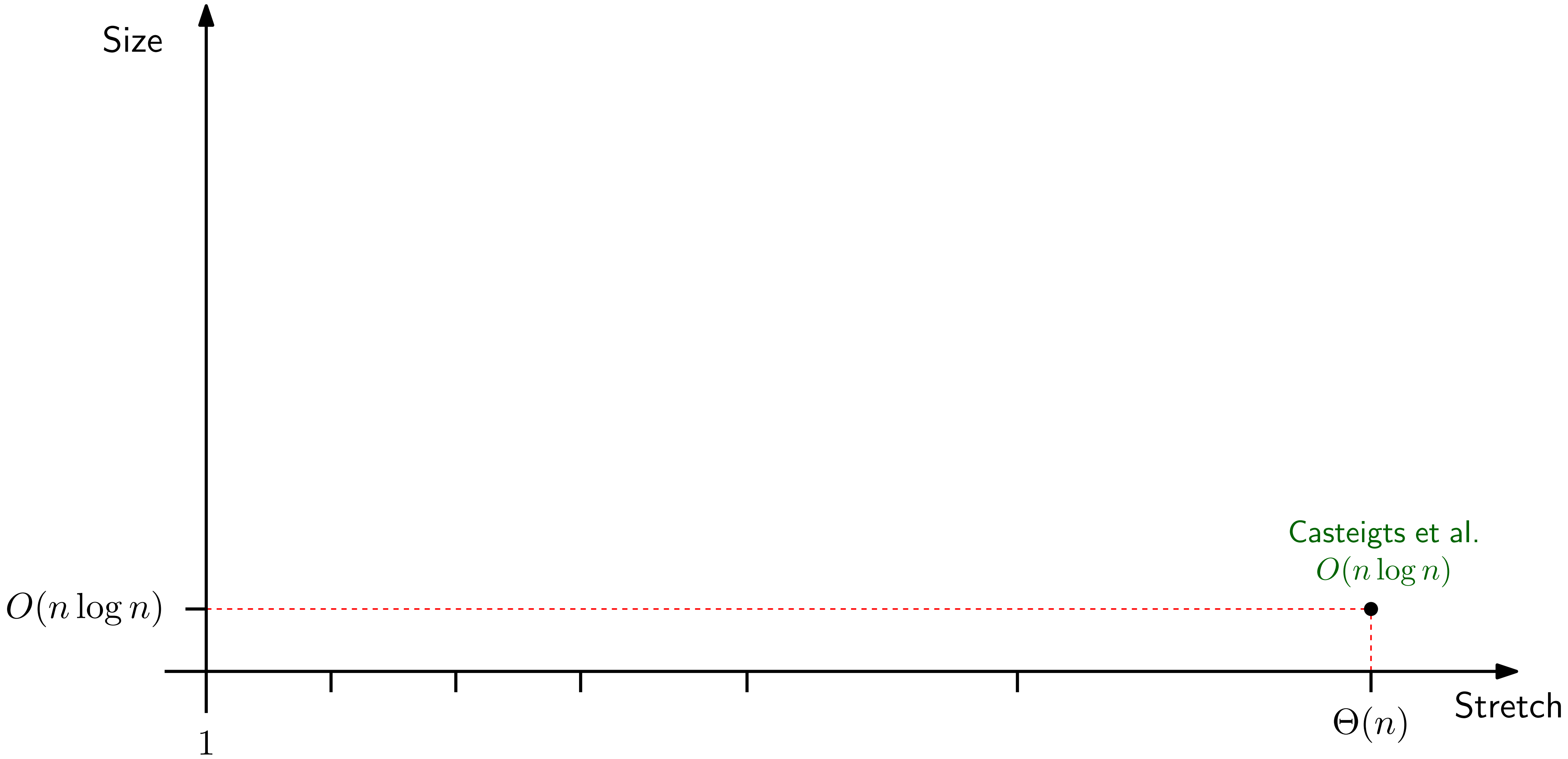

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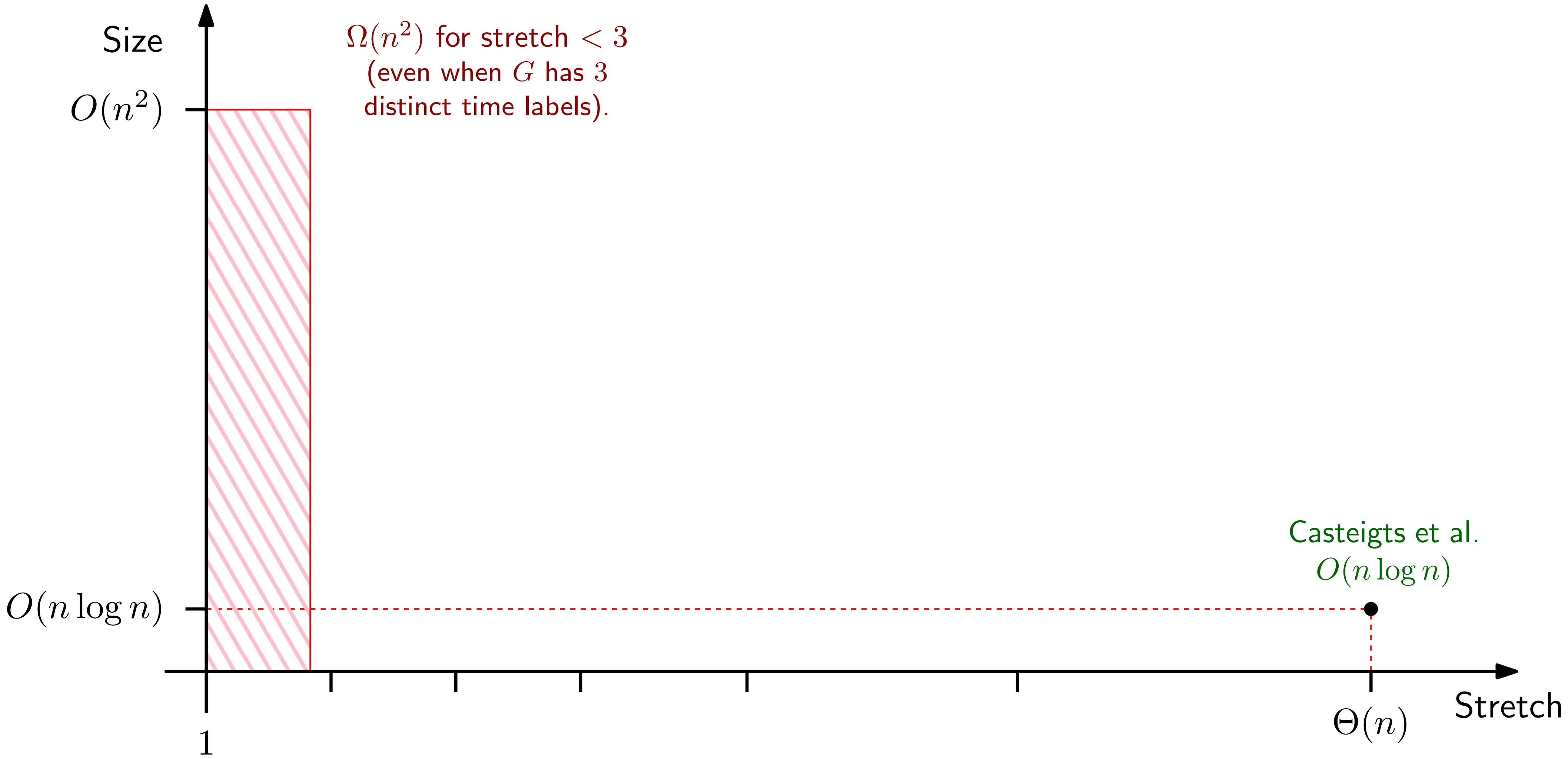

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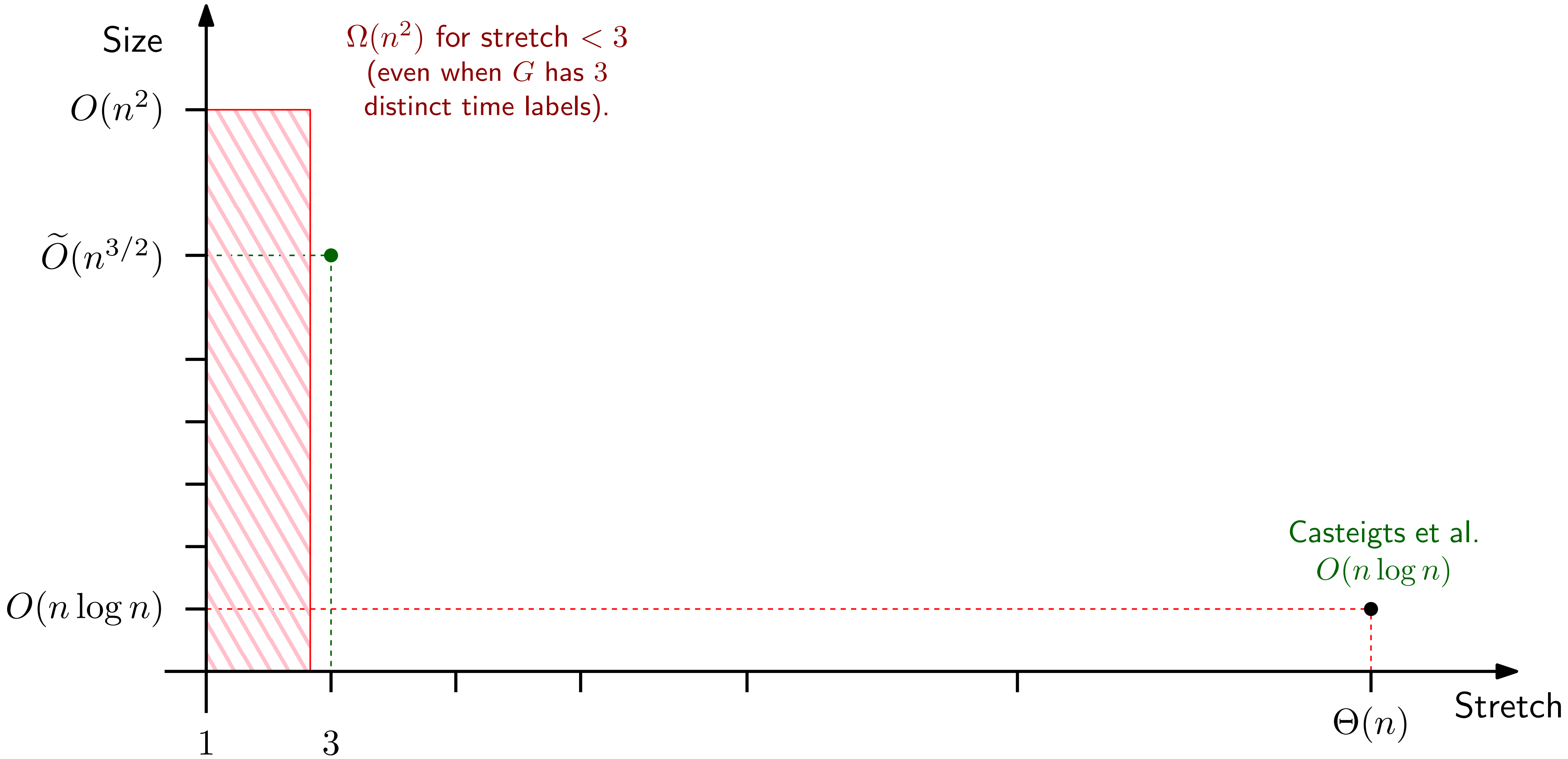

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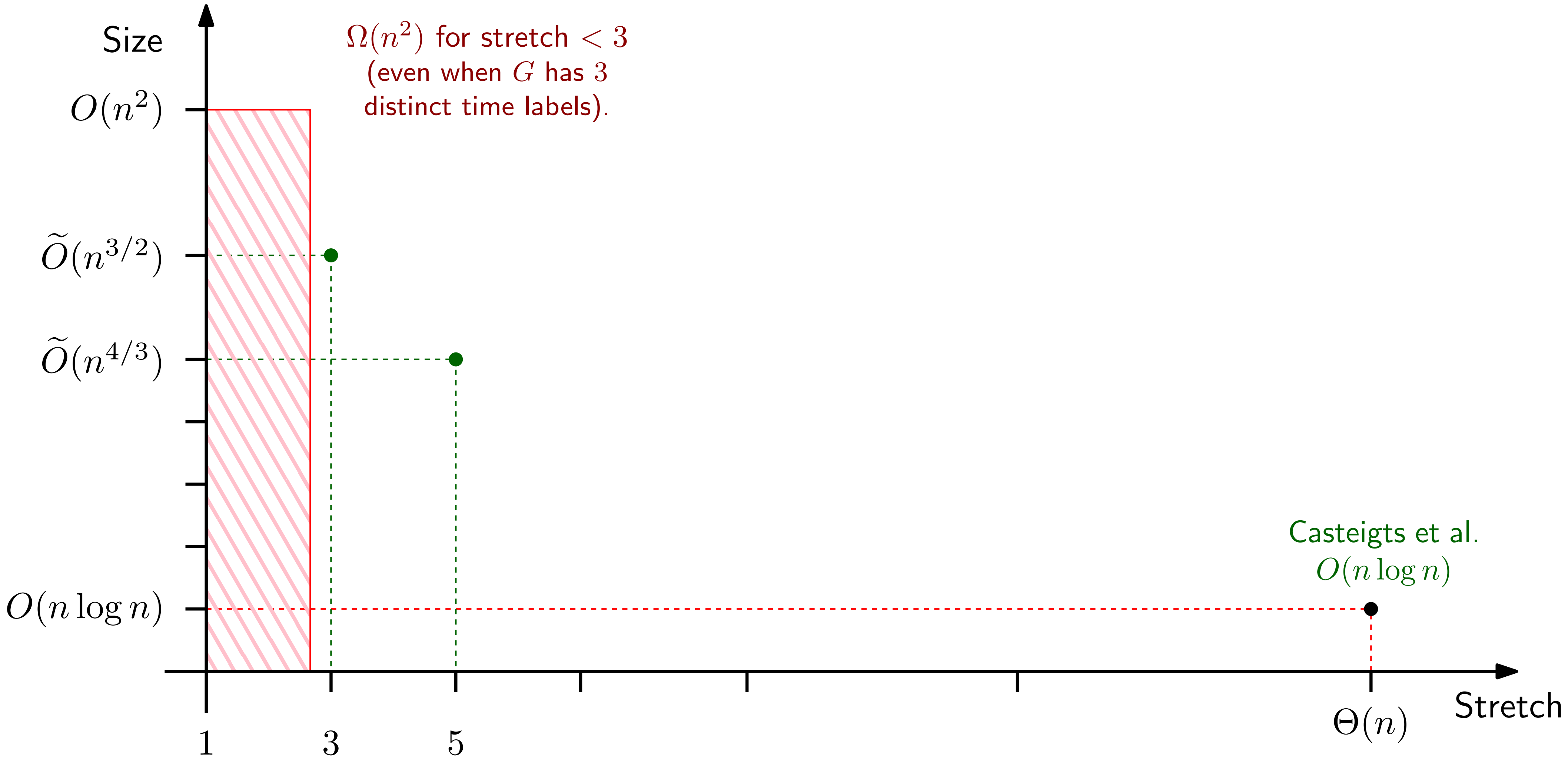

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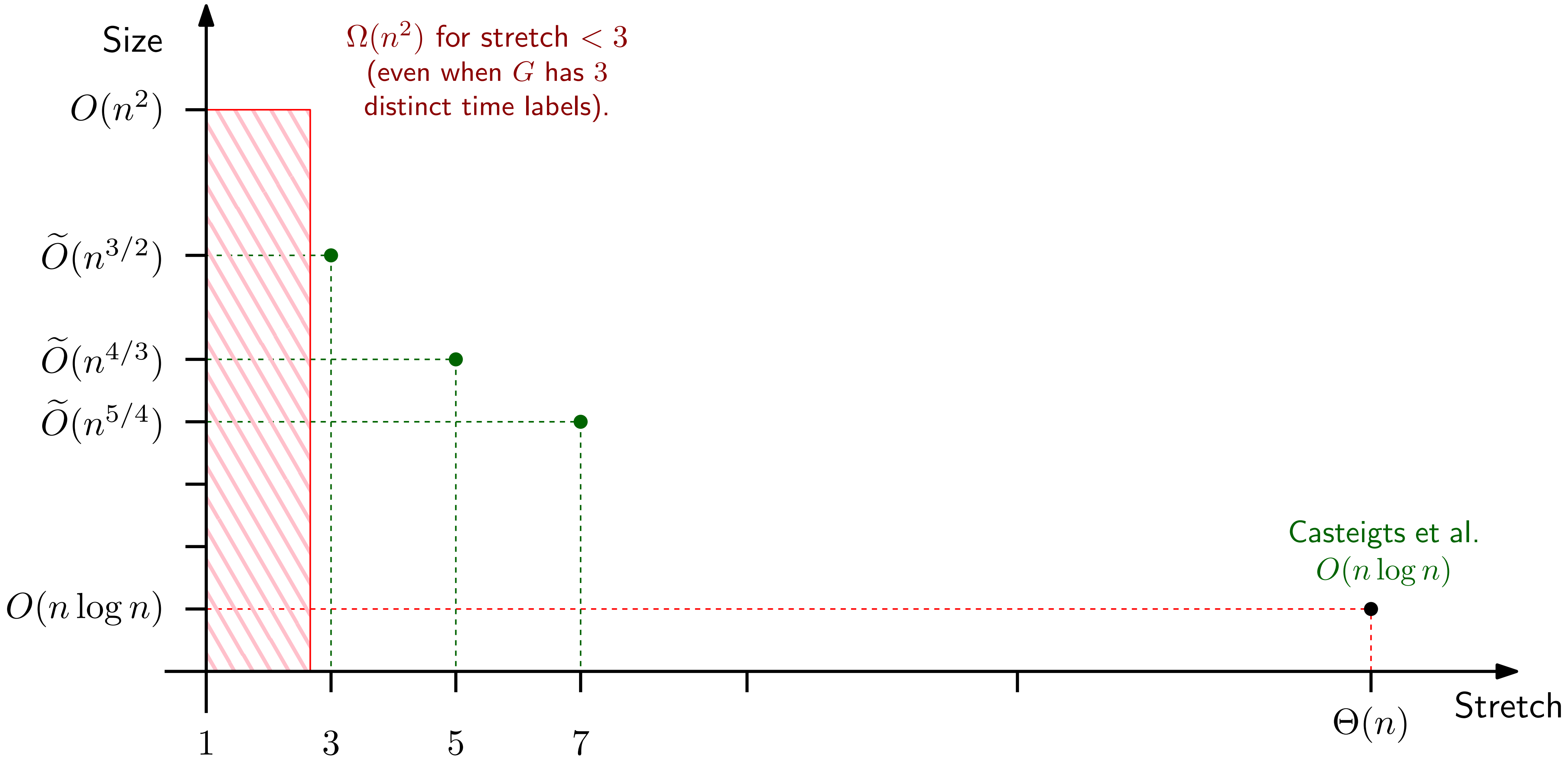

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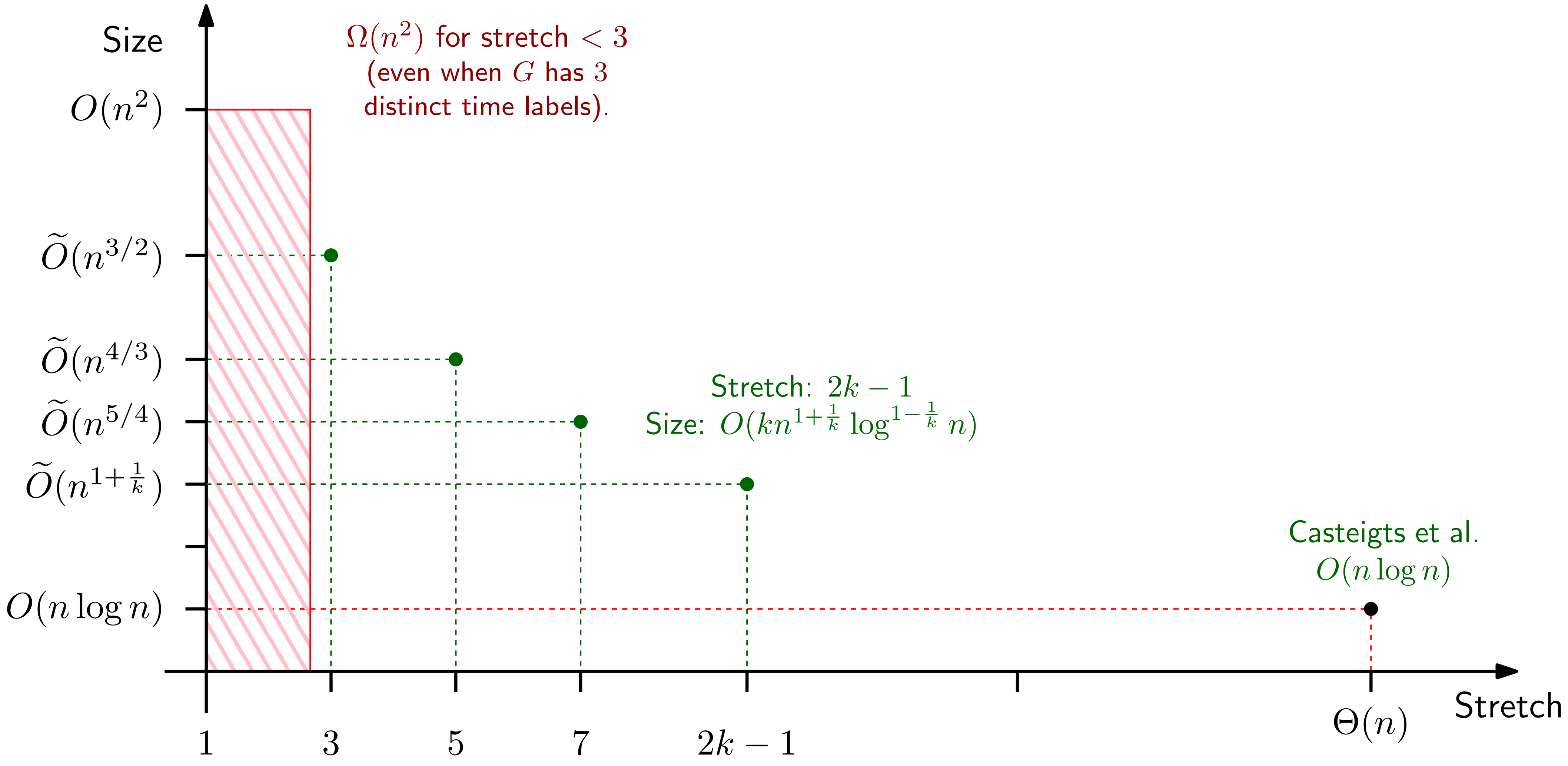

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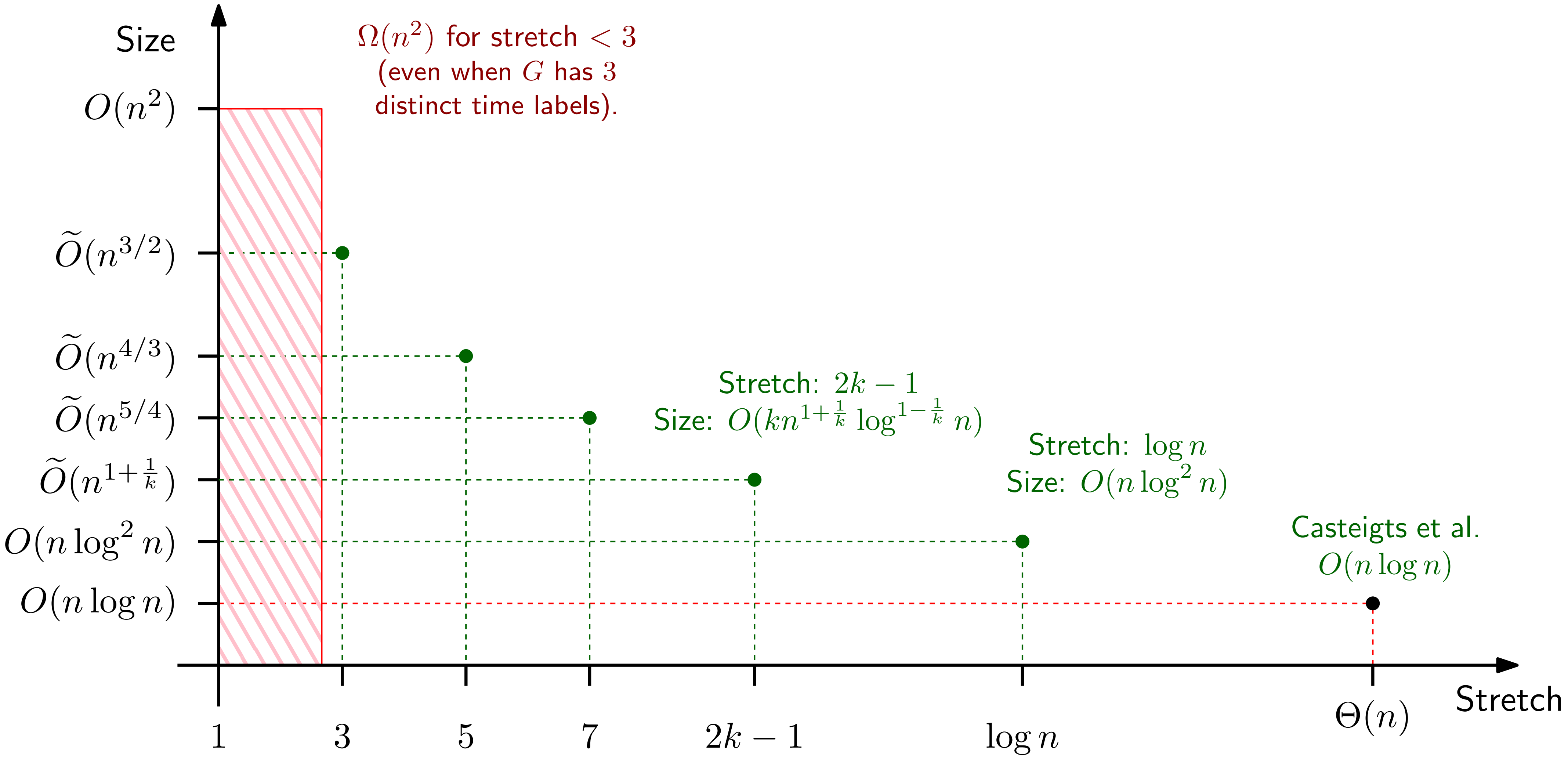

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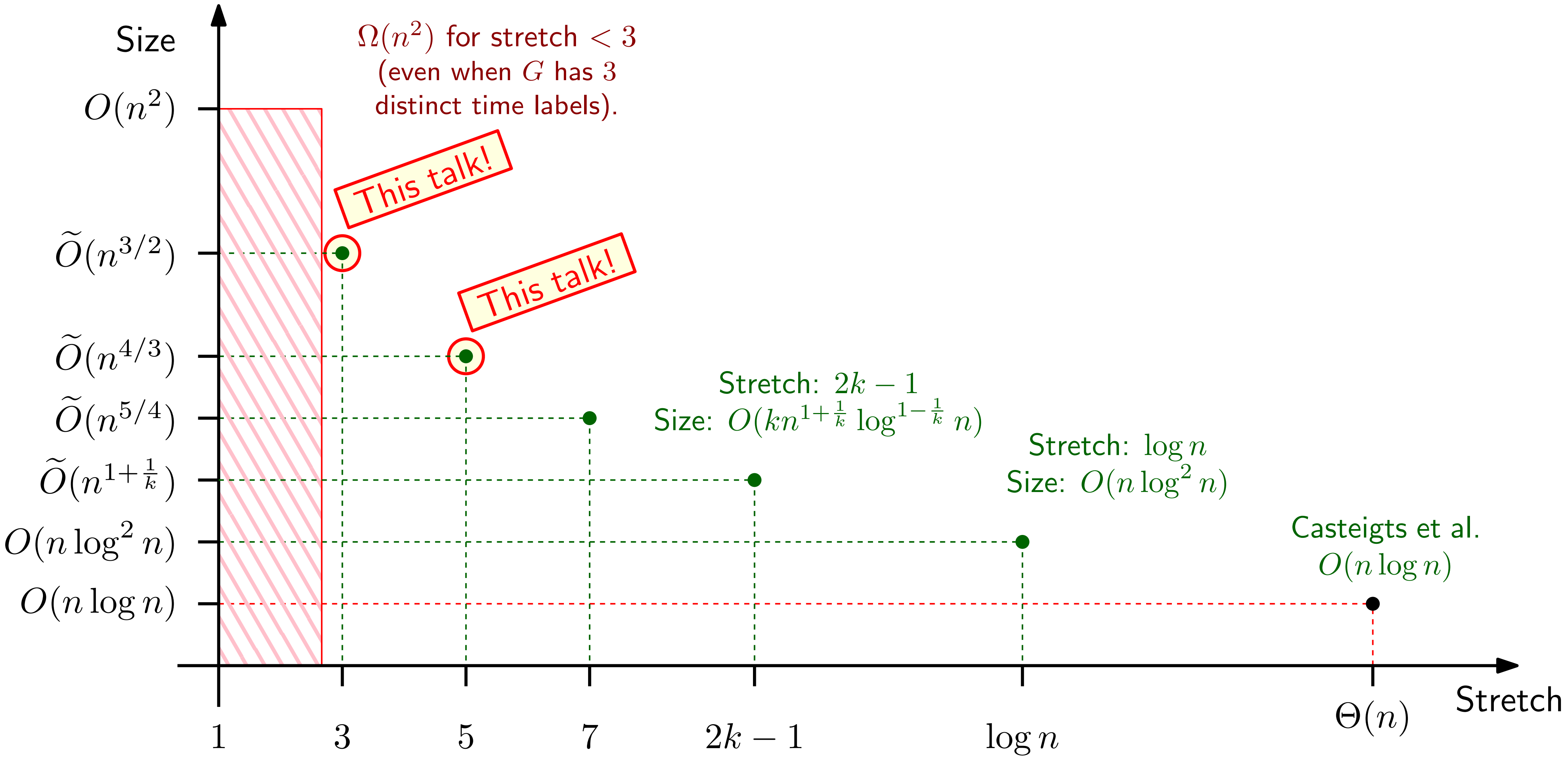

Our Results: General Graphs

Axiotis and Fotakis [ICALP 2016]: lower bound of $\Omega(n^2)$ for temporal connectivity

What about single-source spanners (with stretch α)?

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For some fixed source s, we want a subgraph H of G such that:

$$\mathsf{dist}_H(s, v) \le \alpha \cdot \mathsf{dist}_G(s, v) \qquad \forall v$$

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- Size: $O(n \cdot \frac{\log^4 n}{\log(1+\varepsilon)})$

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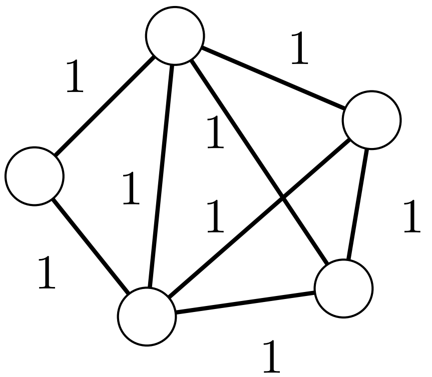
$\boxed{\text{This talk!}}$

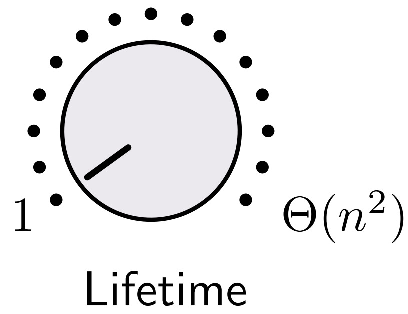
Our Results: The Role of Lifetime

The **lifetime** L of a temporal graph is the number of its distinct time-labels

W.l.o.g. the time-labels are in $1, \ldots, L$

Intuitively, the lifetime measures "how temporal" is the graph





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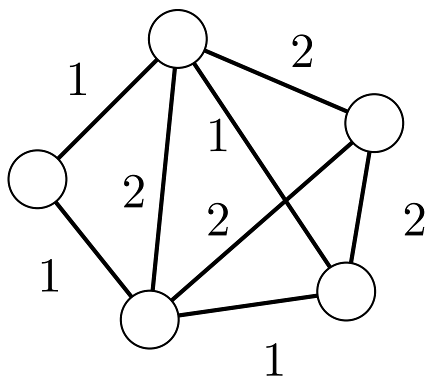

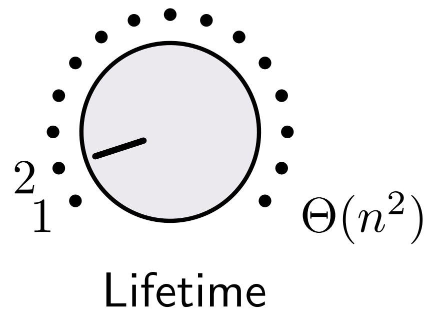

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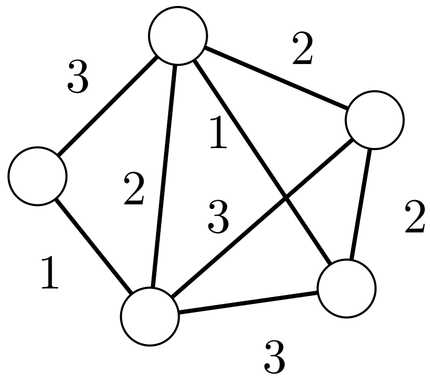

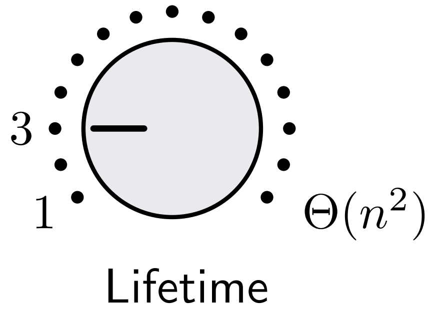

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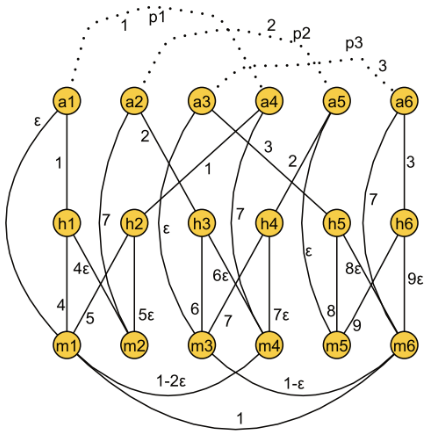

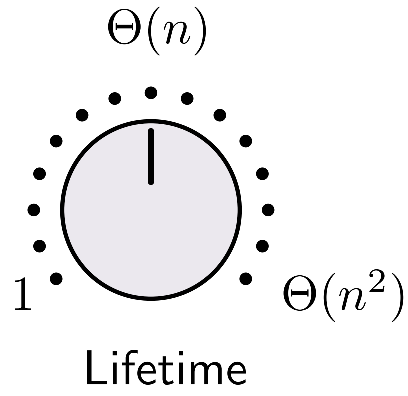

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- For $L = \Theta(n)$ the lower bound of $\Omega(n^2)$ on temporal connectivity applies Axiotis and Fotakis [ICALP 2016]

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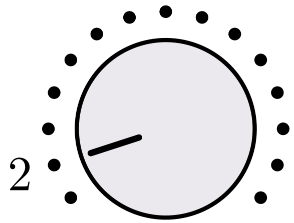
On temporal cliques:

Lifetime	**Stretch**	
	$\alpha = 2$	$\alpha = 3$
$L = 2$	$O(n \log n)$	



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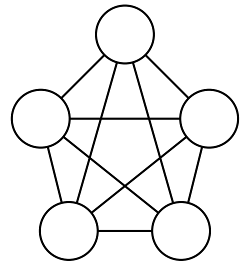

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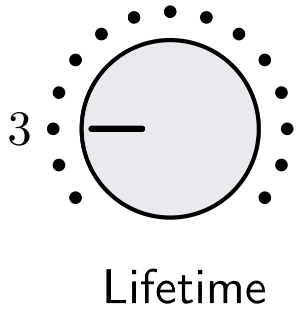

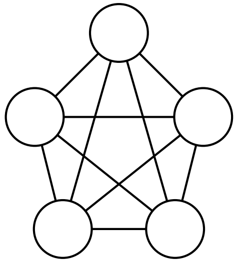

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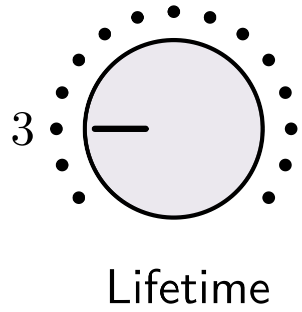
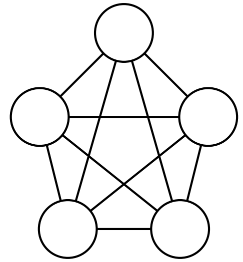
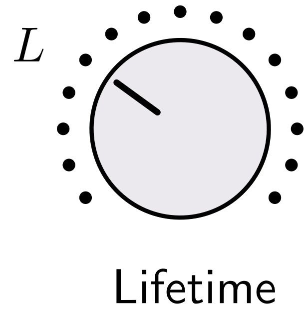


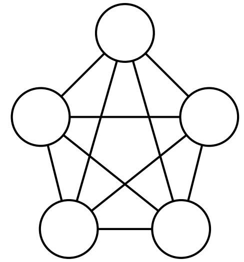


Our Results: The Role of Lifetime

On temporal cliques:

Lifetime	**Stretch** $\alpha = 2$	$\alpha = 3$
$L = 2$	$O(n \log n)$	
$L = 3$	$\Omega(n^2)$	$O(n \log n)$
L		$O(2^L \cdot n \log n)$





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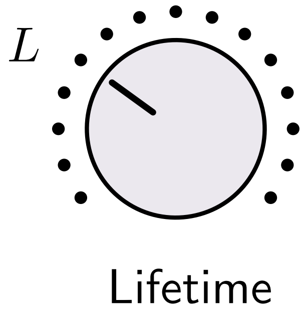
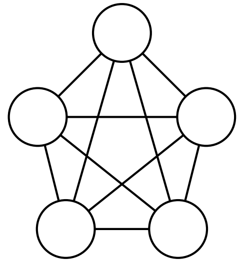


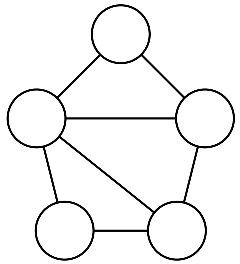


On general temporal graphs:

α-spanner of size $f(n)$ for static graphs

$\Longrightarrow$

temporal spanner of size $O(L \cdot f(n))$ for temporal graphs with lifetime L



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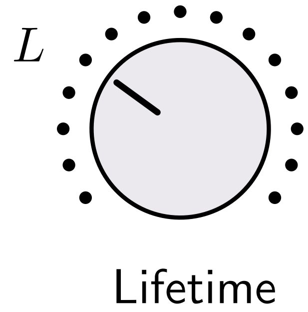

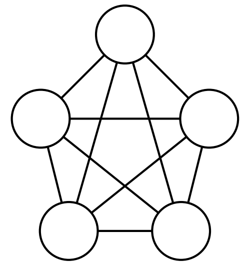

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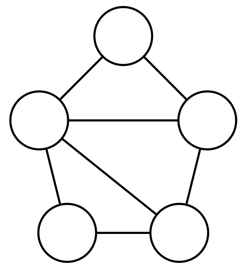

Consequence: any temporal graph with lifetime $L = o(n)$ admits a temporal spanner with stretch $\log n$ and subquadratic size.

Our Results: The Role of Lifetime

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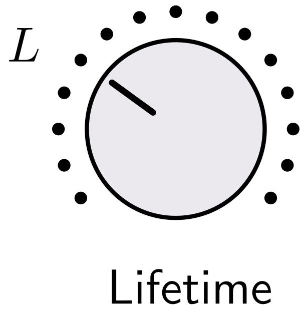

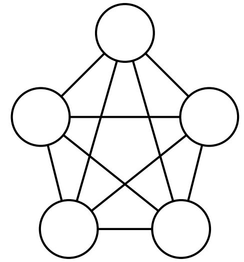

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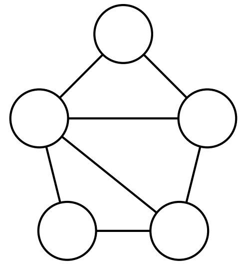


Consequence: any temporal graph with lifetime $L = o(n)$ admits a temporal spanner with stretch $\log n$ and subquadratic size.

vs

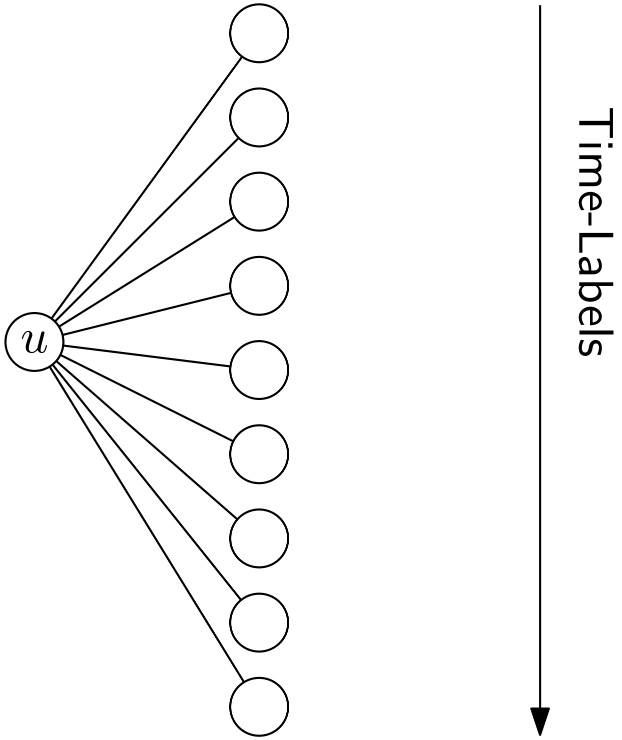
Lower bound of $\Omega(n^2)$ for $L = \Theta(n)$, regardless of stretch.

Axiotis and Fotakis [ICALP 2016]

Our Temporal 3-Spanner of Size $\widetilde{O}(n\sqrt{n})$ (for temporal cliques)

Our Temporal 3-Spanner

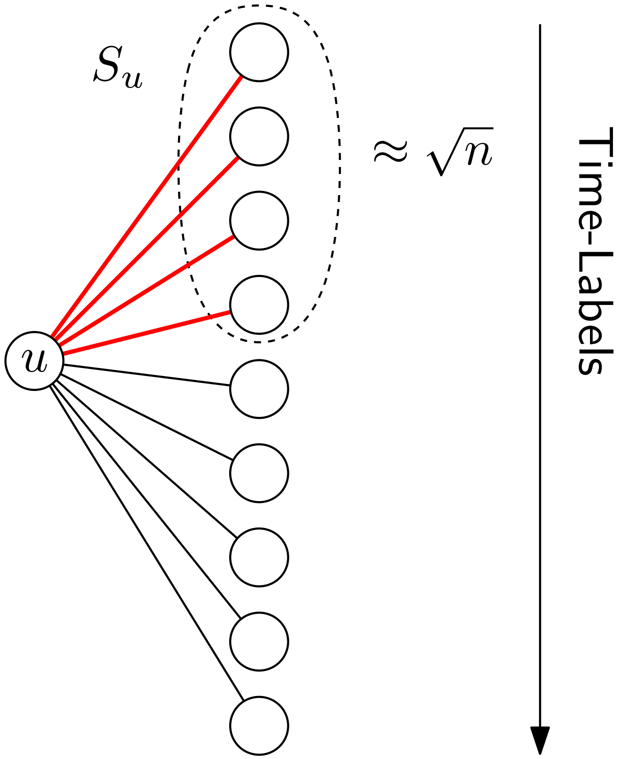
For every $u \in V$:



Our Temporal 3-Spanner

For every $u \in V$:

S_u = set of neighbors v of u such that (u, v) is one the $\approx \sqrt{n}$ edges incident to u with the smallest label



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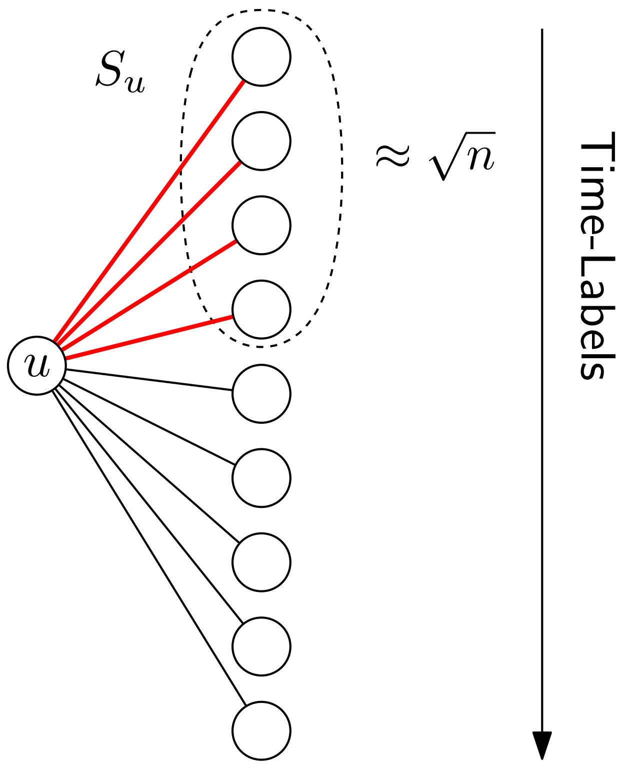

#red edges $= O(n\sqrt{n})$

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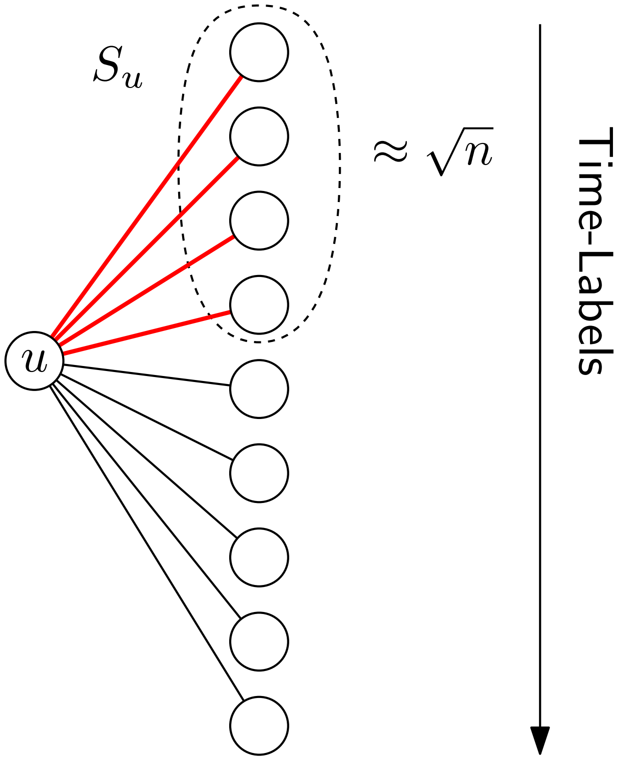
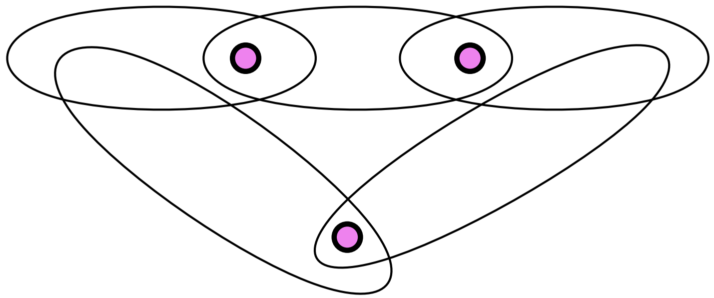


#red edges $= O(n\sqrt{n})$

Compute a hitting set R of the collection $\mathcal{C} = \{S_u \mid u \in V\}$



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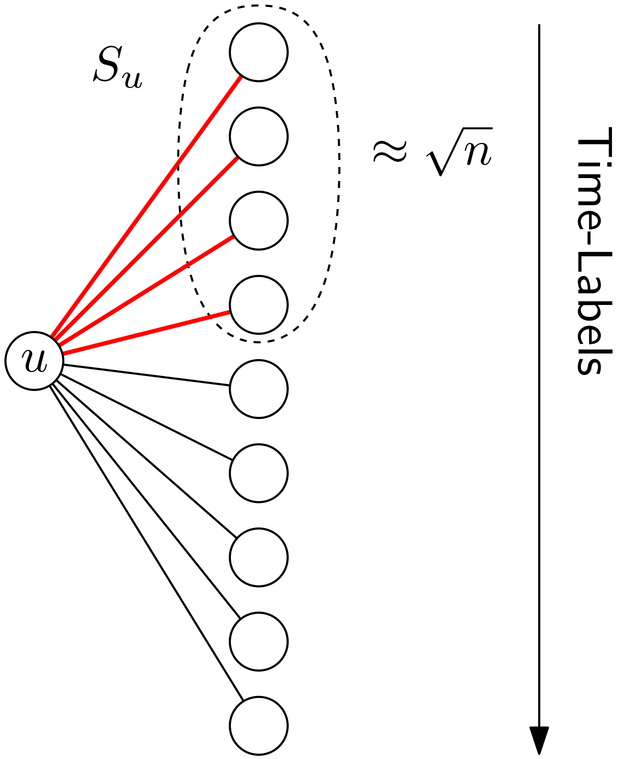

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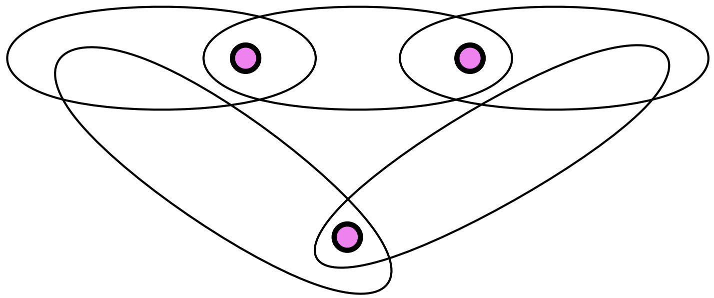


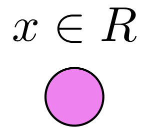
Lemma: If every set S_u has size at least k, an hitting set of $\mathcal{C}$ of size $\widetilde{O}(n/k)$ can be found in polynomial time.

$$|R| = \widetilde{O}\left(\frac{n}{\sqrt{n}}\right) = \widetilde{O}(\sqrt{n})$$

Our Temporal 3-Spanner

Cluster the vertices around R:

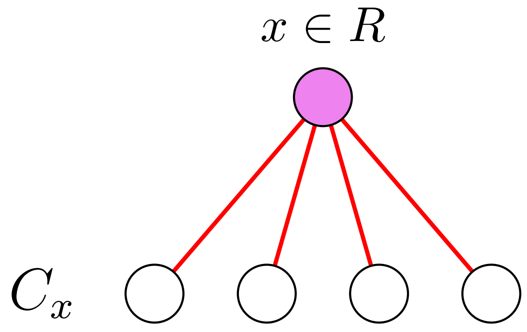
- Each vertex $x \in R$ is a center



Our Temporal 3-Spanner

Cluster the vertices around R:

- Each vertex $x \in R$ is a center
- Arbitrarily assign each vertex u to a neighboring center $x \in S_u$
- Each center x, along with the set C_x of its assigned nodes, forms a **cluster**



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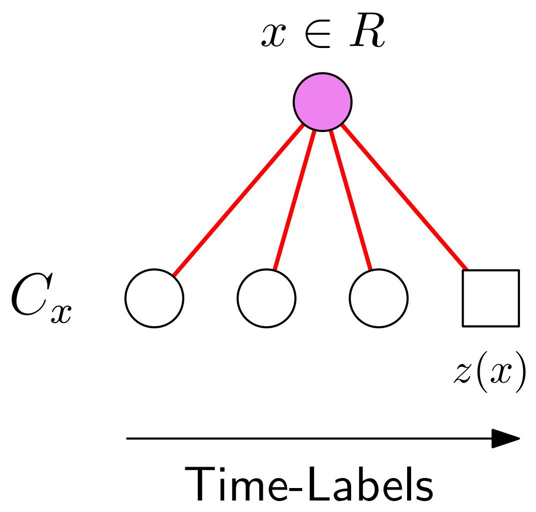

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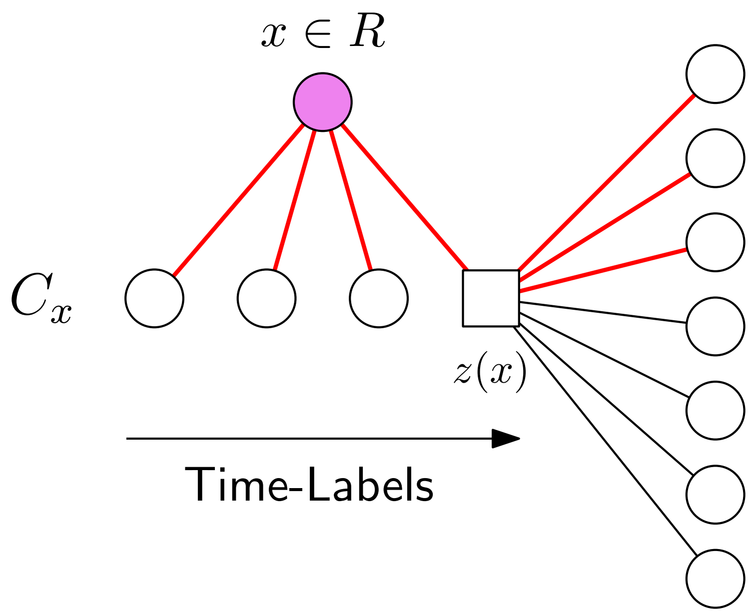

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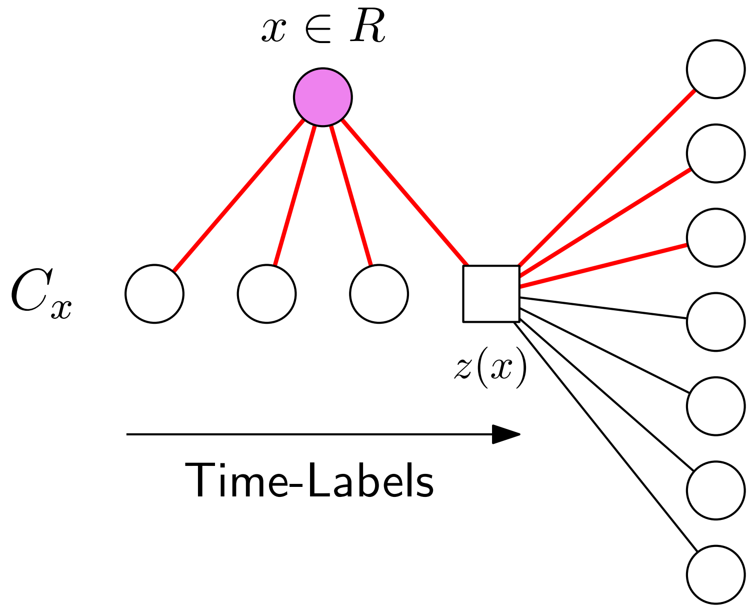
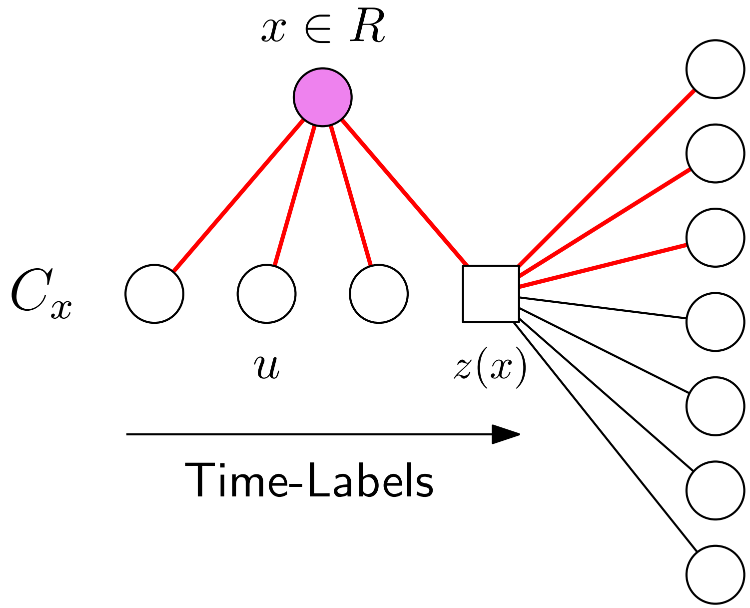


H = all red edges

Our Temporal 3-Spanner

Consider two vertices u, v and focus on the cluster of u with center x



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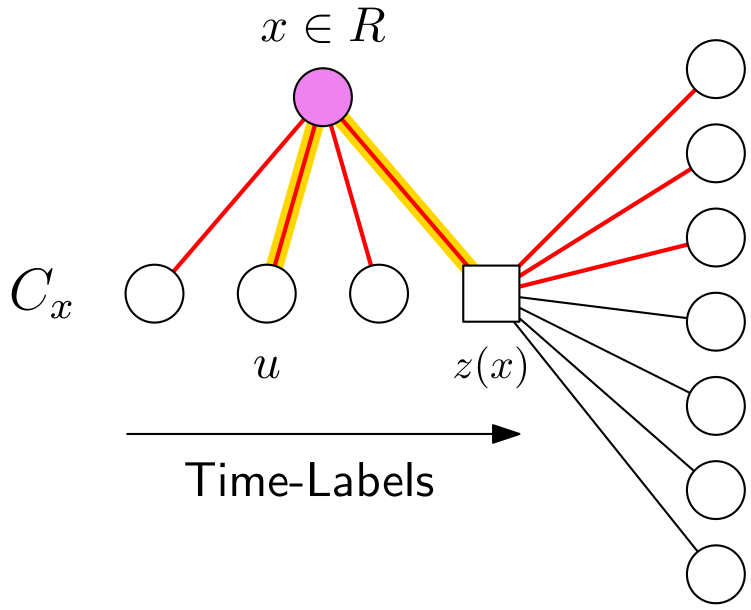
Consider two vertices u, v and focus on the cluster of u with center x

Case 1: $v = z(x)$

- There is a red path of length ≤ 2 between u and v



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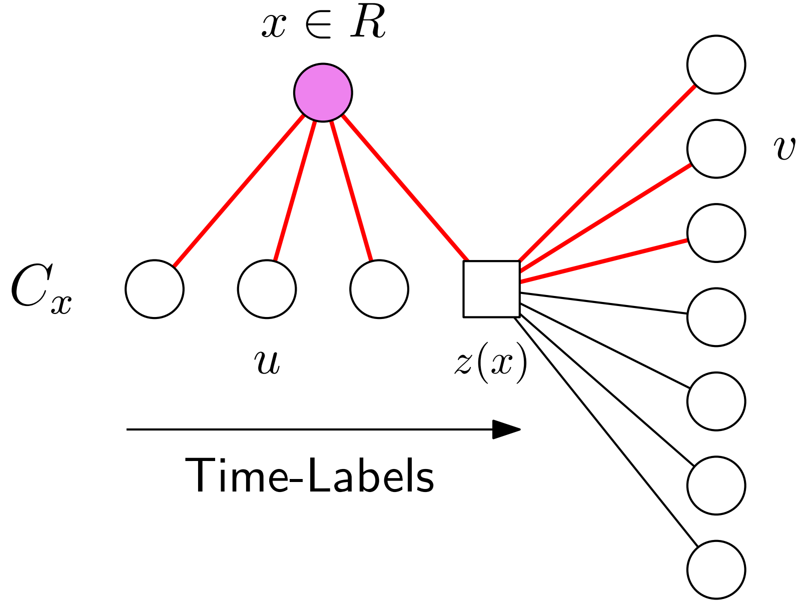

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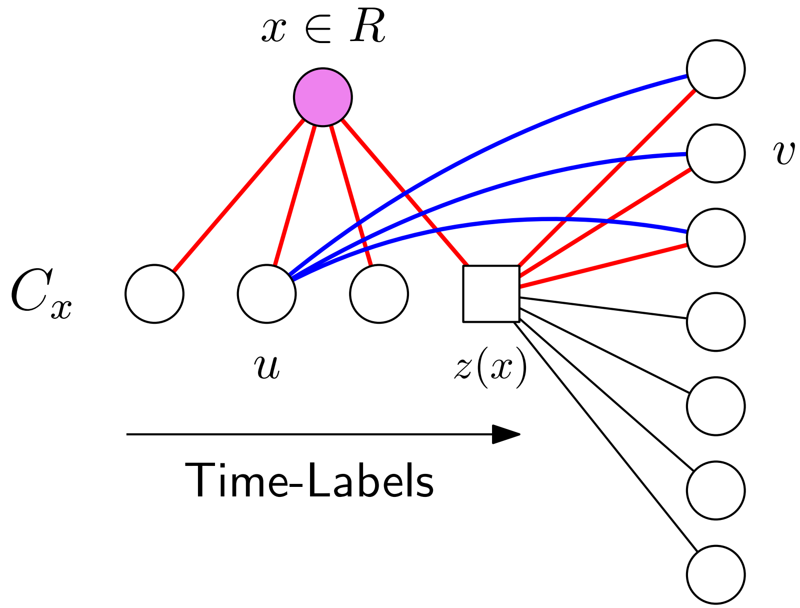

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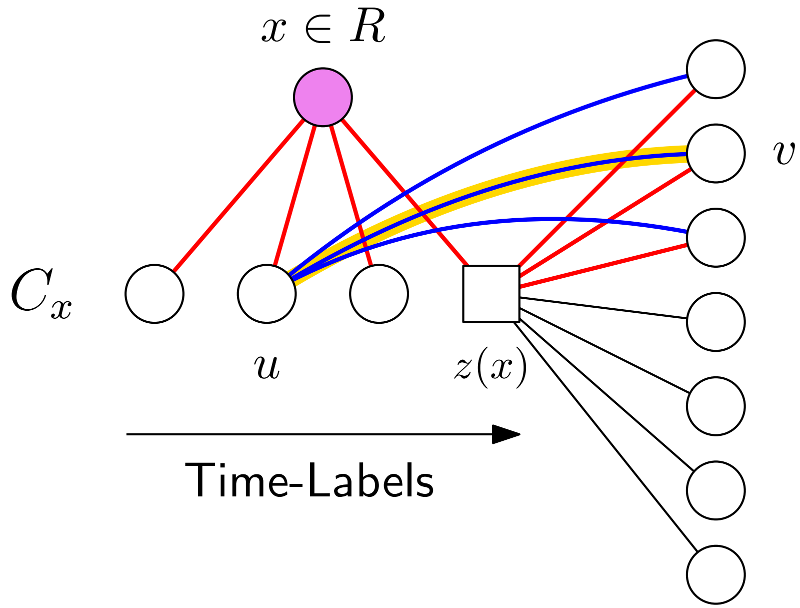

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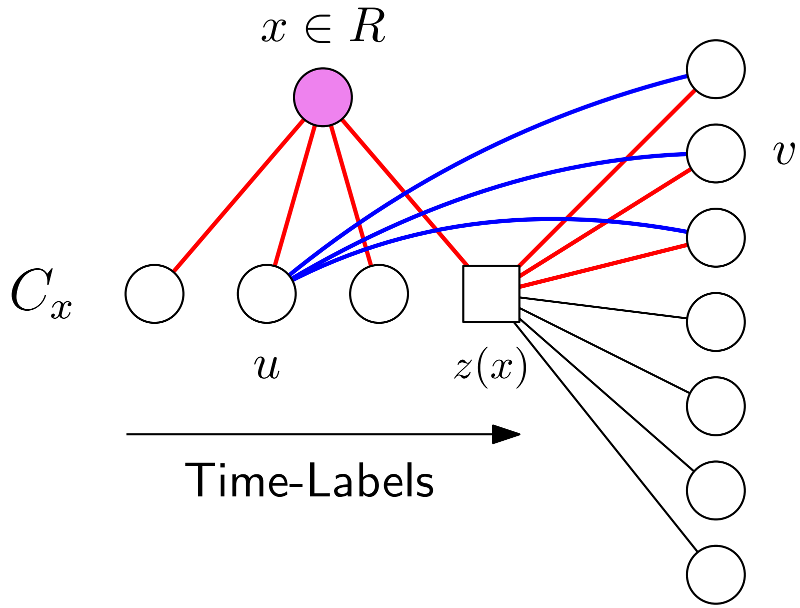

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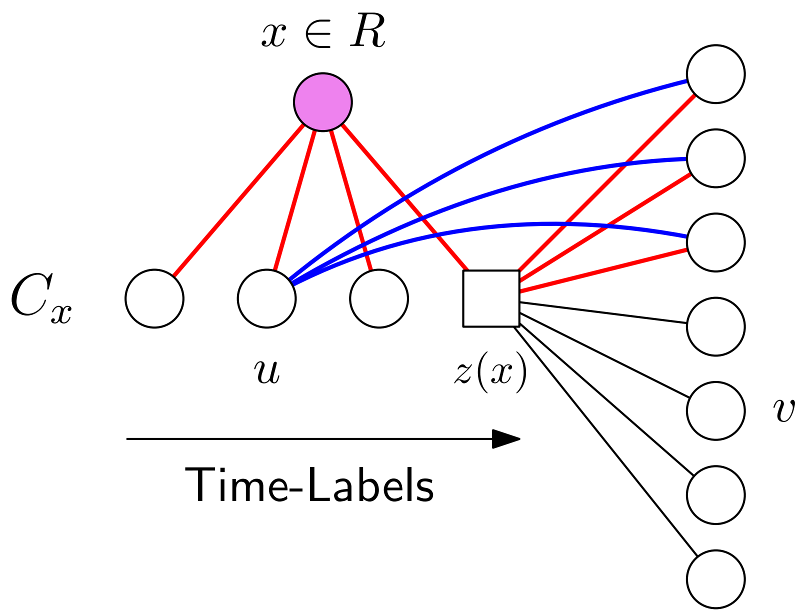

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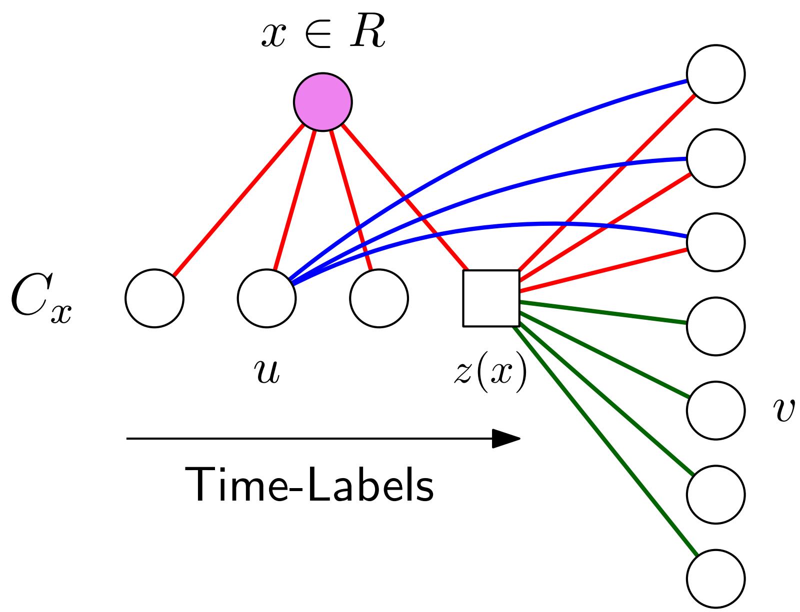

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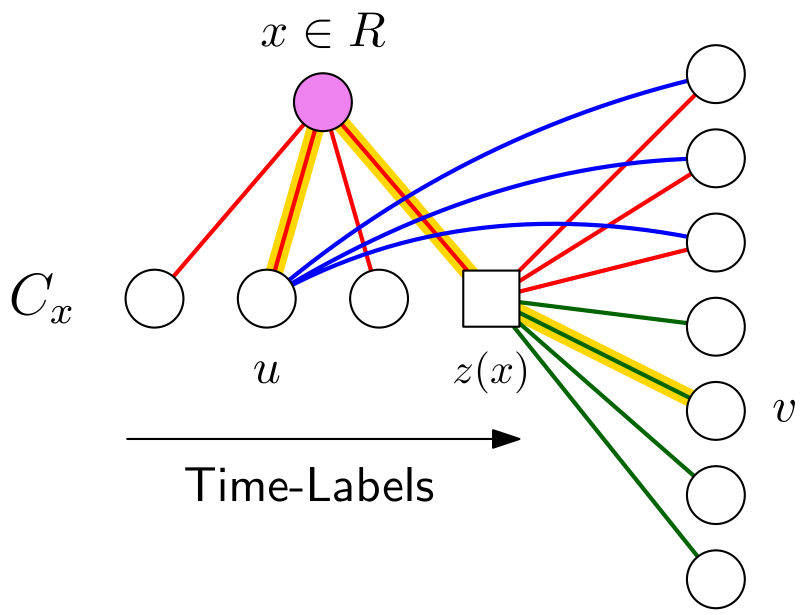

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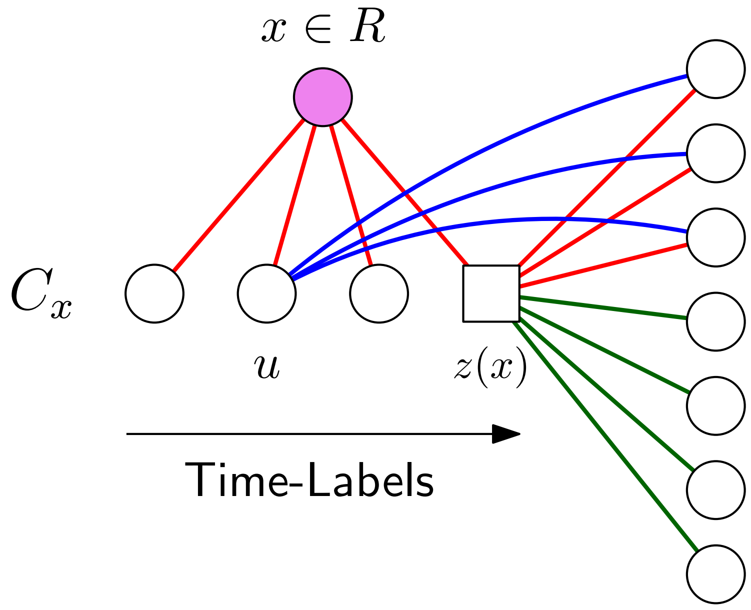

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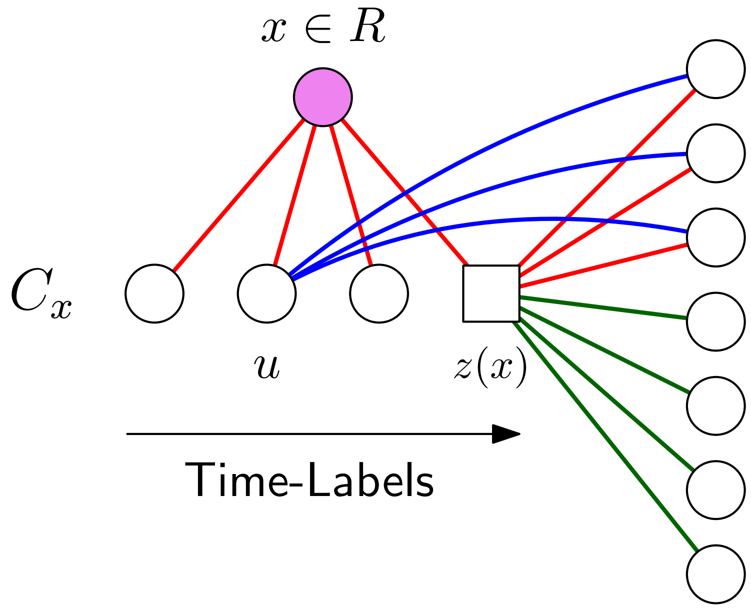

Size?

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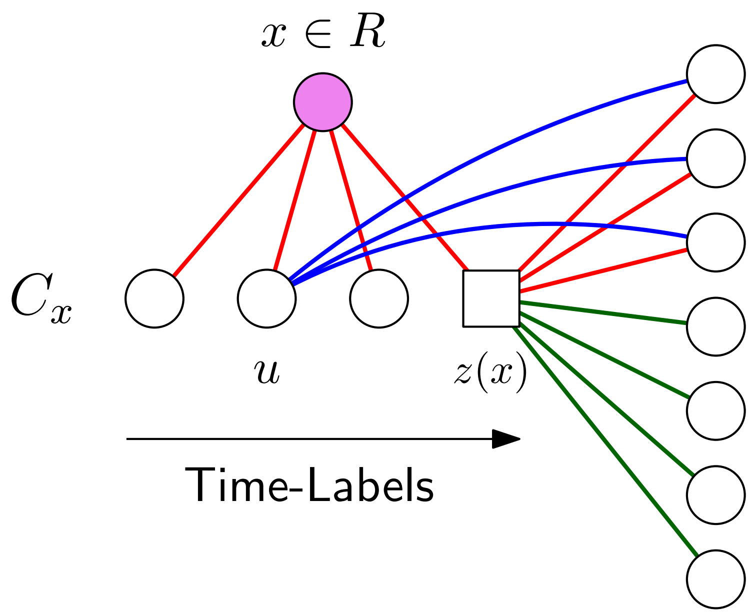

Size?

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$O(n\sqrt{n})$

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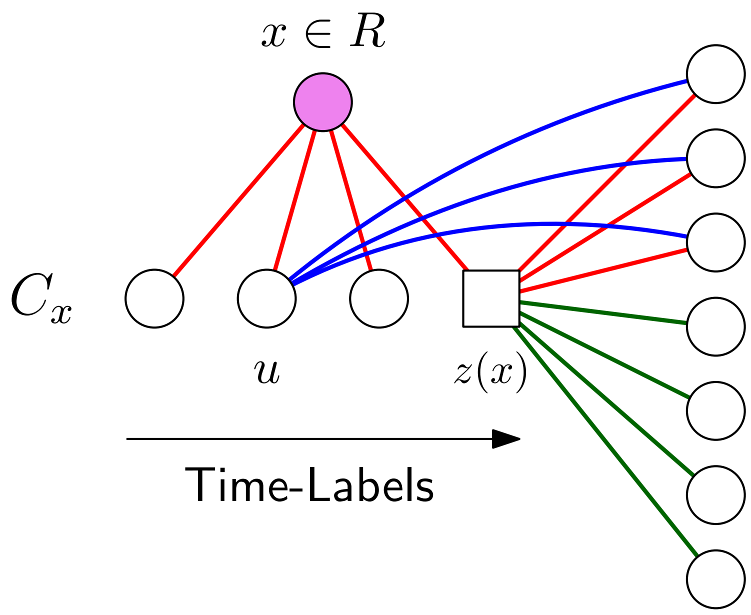

Size?

$H =$ all red edges $\cup$ all blue edges $\cup$ all green edges

$$O(n\sqrt{n}) \quad + \quad n \cdot O(\sqrt{n})$$

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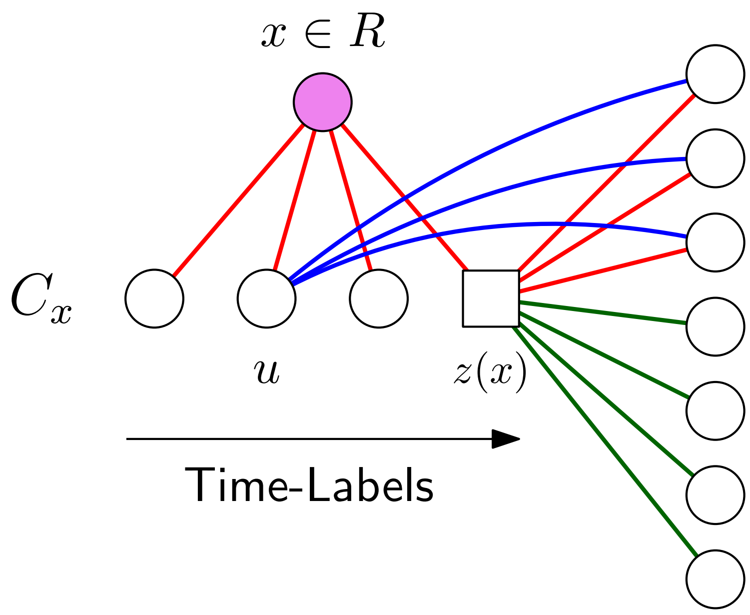

Size?

H = all red edges $\cup$ all blue edges $\cup$ all green edges

$$O(n\sqrt{n}) \quad + \quad n \cdot O(\sqrt{n}) \quad + \quad |R| \cdot O(n)$$

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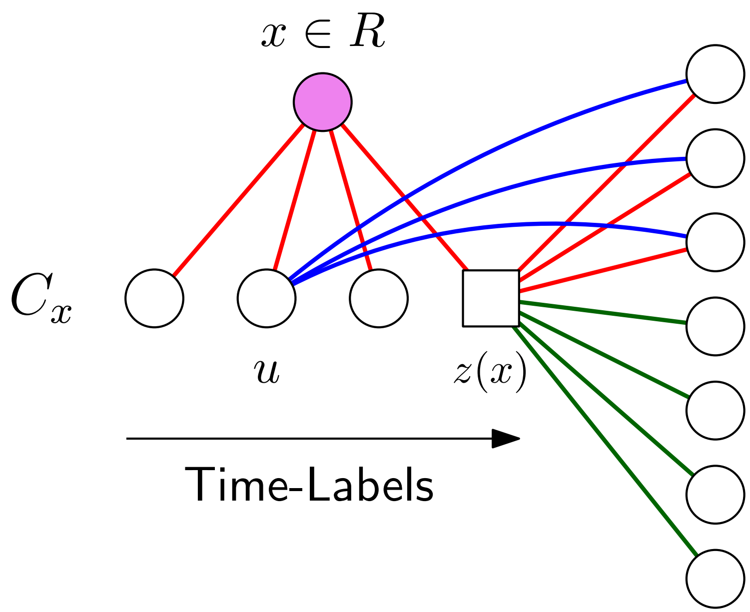

Size?

$H =$ all red edges $\cup$ all blue edges $\cup$ all green edges

$$O(n\sqrt{n}) \quad + \quad n \cdot O(\sqrt{n}) \quad + \quad \widetilde{O}(\sqrt{n}) \cdot O(n)$$

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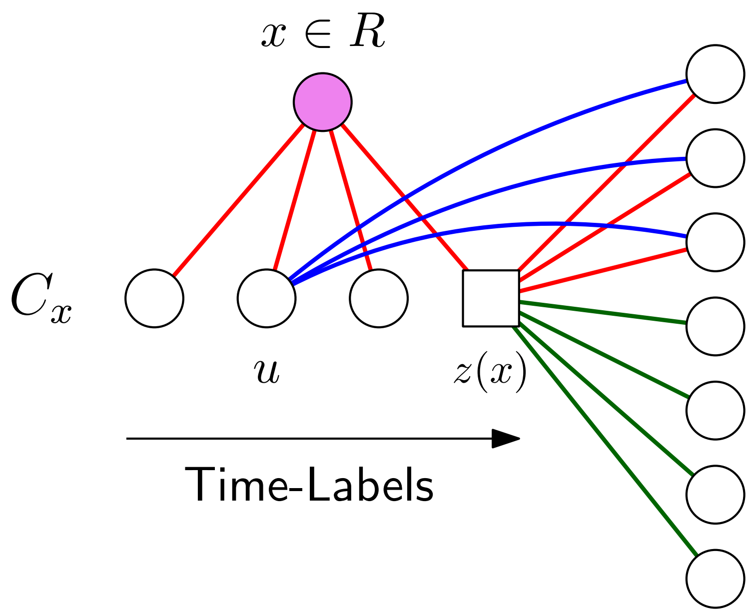

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$$O(n\sqrt{n}) \quad + \quad n \cdot O(\sqrt{n}) \quad + \quad \widetilde{O}(\sqrt{n}) \cdot O(n)$$

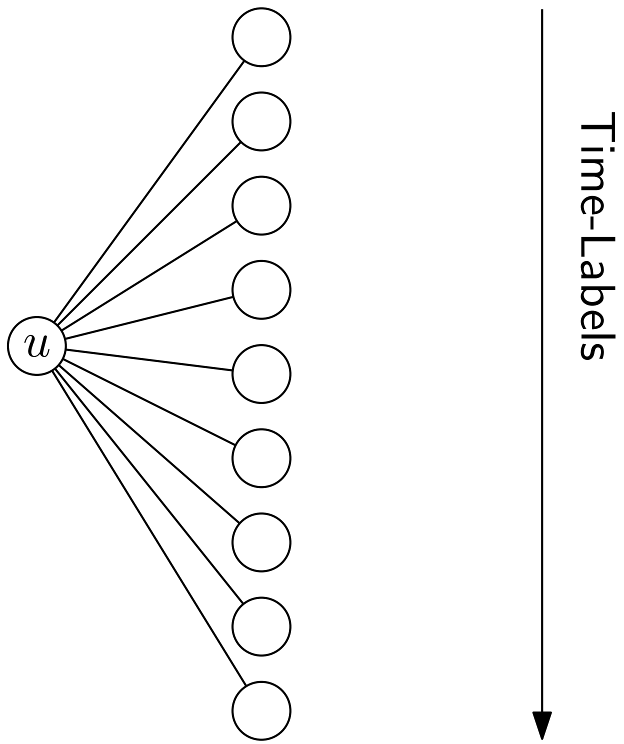
Size?

H is a temporal spanner with stretch 3 and size $\widetilde{O}(n^{3/2})$

Our Temporal 5-Spanner of Size $\widetilde{O}(n\sqrt[3]{n})$
(for temporal cliques)

Our Temporal 5-Spanner

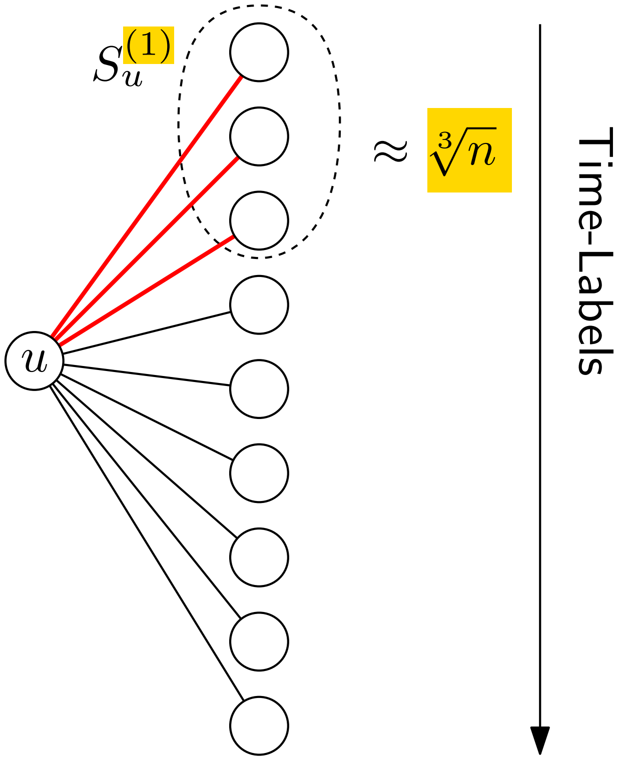
For every $u \in V$:



Our Temporal 5-Spanner

For every $u \in V$:

$S_u^{(1)}$ = set of neighbors v of u such that (u, v) is one the $\approx \sqrt[3]{n}$ edges incident to u with the smallest label



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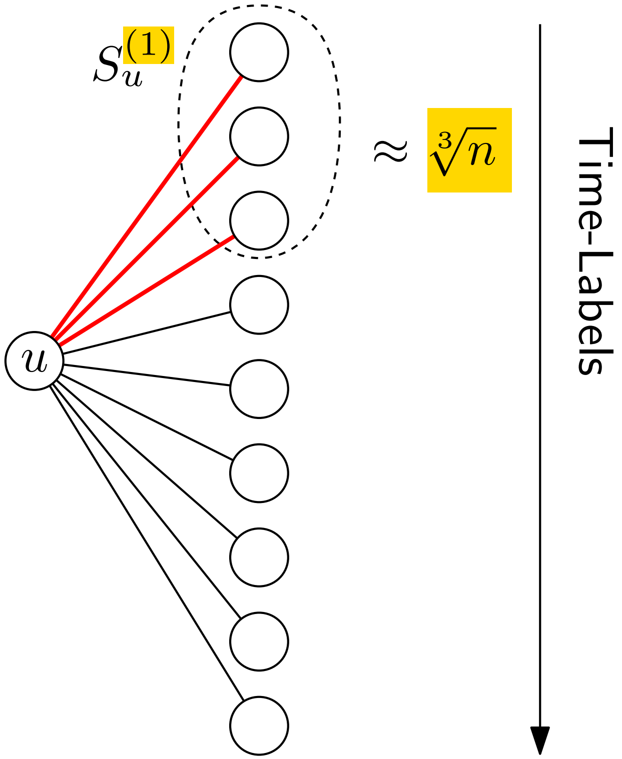

#red edges $= O(n\sqrt[3]{n})$

Our Temporal 5-Spanner

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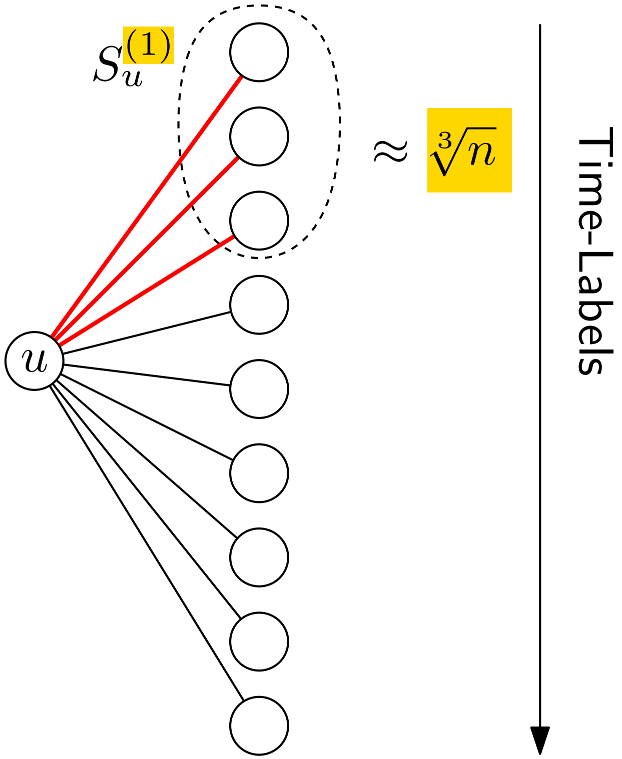
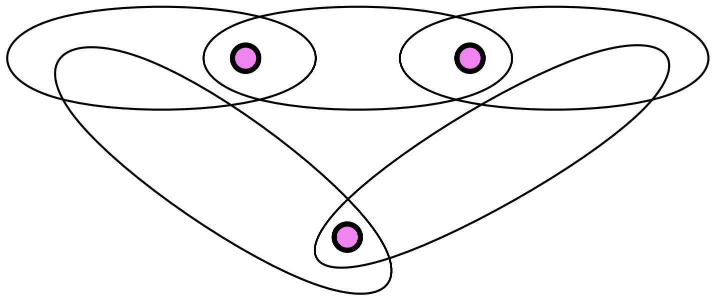


#red edges $= O(n\sqrt[3]{n})$

Compute a hitting set $R^{(1)}$ of the collection $\mathcal{C} = \{S_u^{(1)} \mid u \in V\}$



Our Temporal 5-Spanner

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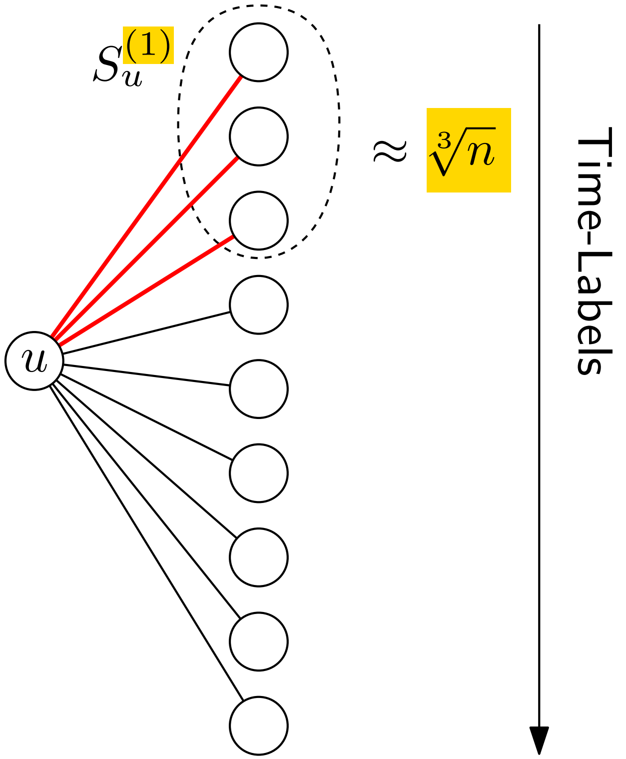

$\#$red edges $= O(n\sqrt[3]{n})$

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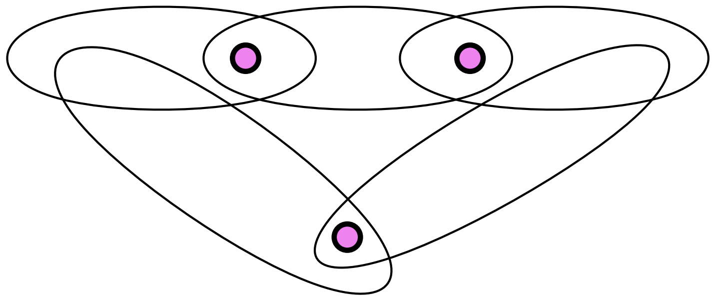


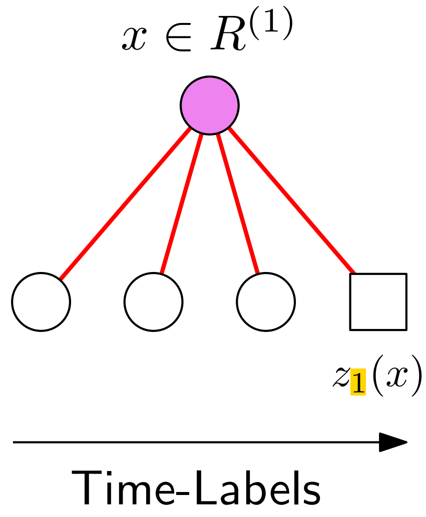
Lemma: If every set S_u has size at least k, an hitting set of $\mathcal{C}$ of size $\widetilde{O}(n/k)$ can be found in polynomial time.

$$|R^{(1)}| = \widetilde{O}\left(\frac{n}{\sqrt[3]{n}}\right) = \widetilde{O}(n^{2/3})$$

Our Temporal 5-Spanner

Cluster the vertices in V vertices around $R^{(1)}$:

- For each center $x \in R^{(1)}$, choose a **special vertex** $z_1(x)$ that is assigned to x and maximizes the time-label of $(x, z_1(x))$
- $Z = \{z_1(x) \mid x \in R^{(1)}\}$



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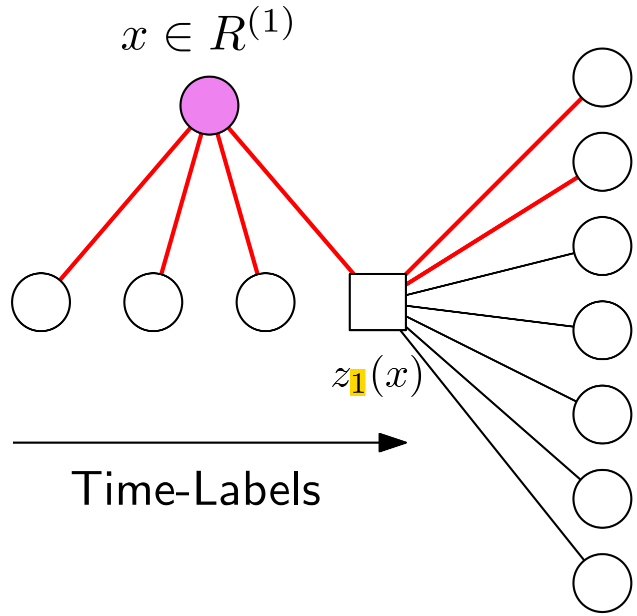

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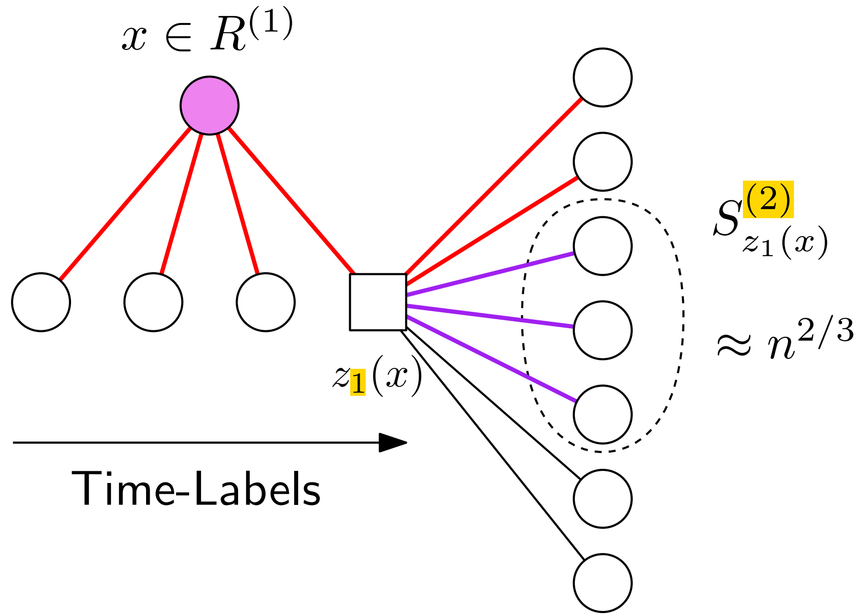
$S_z^{(2)}$= set of neighbors v of z such that (u, v) is one the $\approx n^{2/3}$ edges between z and $V \setminus S_z^{(1)}$ with the smallest label



Our Temporal 5-Spanner

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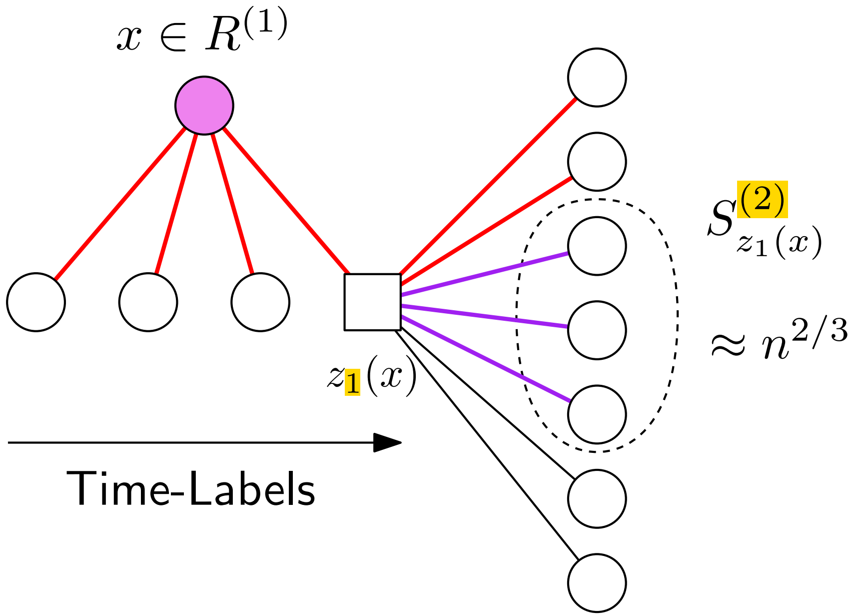

$S_z^{(2)}$= set of neighbors v of z such that (u, v) is one the $\approx n^{2/3}$ edges between z and $V \setminus S_z^{(1)}$ with the smallest label

#purple edges $= \widetilde{O}(n^{2/3}) \cdot O(n^{2/3}) = \widetilde{O}(n\sqrt[3]{n})$

Our Temporal 5-Spanner

Cluster the vertices in V vertices around $R^{(1)}$:

- For each center $x \in R^{(1)}$, choose a **special vertex** $z_1(x)$ that is assigned to x and maximizes the time-label of $(x, z_1(x))$
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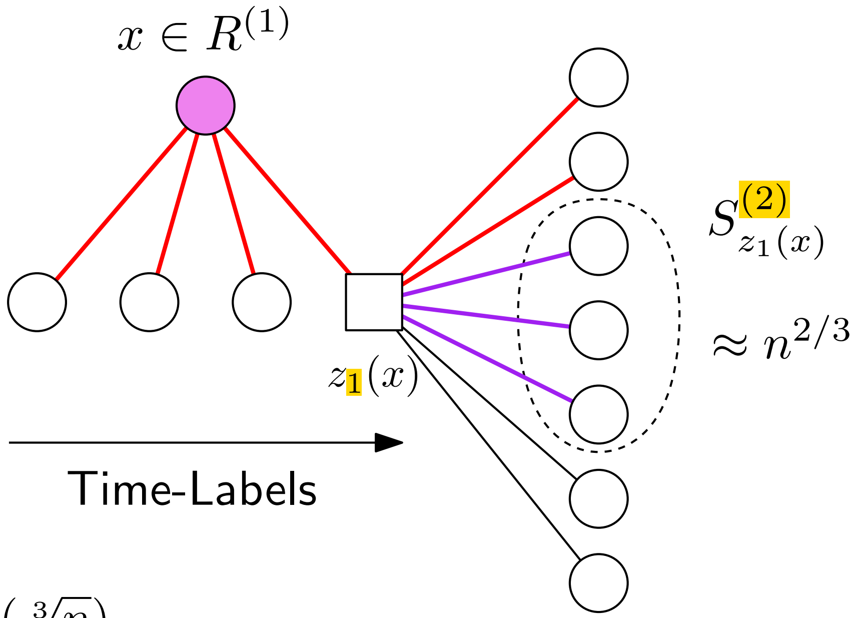


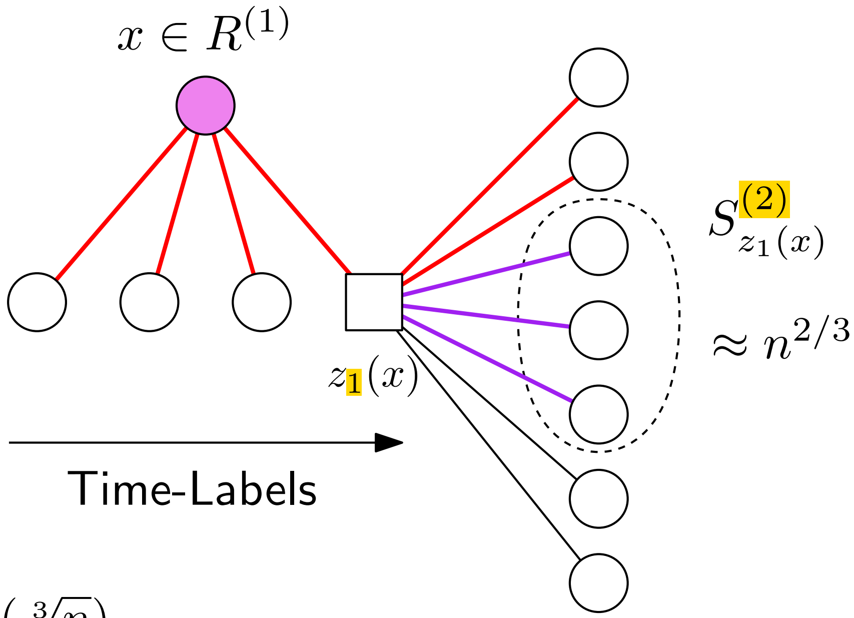
Compute a hitting set $R^{(2)}$ of $\{S_z^{(2)} \mid z \in Z\}$ of size $\widetilde{O}(\sqrt[3]{n})$

#purple edges $= \widetilde{O}(n^{2/3}) \cdot O(n^{2/3}) = \widetilde{O}(n\sqrt[3]{n})$

Our Temporal 5-Spanner

Cluster the vertices in V vertices around $R^{(1)}$:

- For each center $x \in R^{(1)}$, choose a **special vertex** $z_1(x)$ that is assigned to x and maximizes the time-label of $(x, z_1(x))$
- $Z = \{z_1(x) \mid x \in R^{(1)}\}$



$S^{(2)}_z$= set of neighbors v of z such that (u, v) is one the $\approx n^{2/3}$ edges between z and $V \setminus S^{(1)}_z$ with the smallest label

Compute a hitting set $R^{(2)}$ of $\{S^{(2)}_z \mid z \in Z\}$ of size $\widetilde{O}(\sqrt[3]{n})$

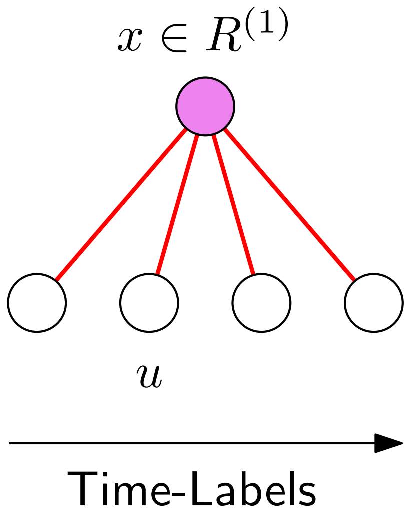
Cluster the vertices in Z around $R^{(2)}$:

- For each center $y \in R^{(2)}$, choose a **2nd-level special vertex** $z_2(y)$ that is assigned to y and maximizes the time-label of $(x, z_2(y))$

#purple edges $= \widetilde{O}(n^{2/3}) \cdot O(n^{2/3}) = \widetilde{O}(n\sqrt[3]{n})$

Our Temporal 5-Spanner

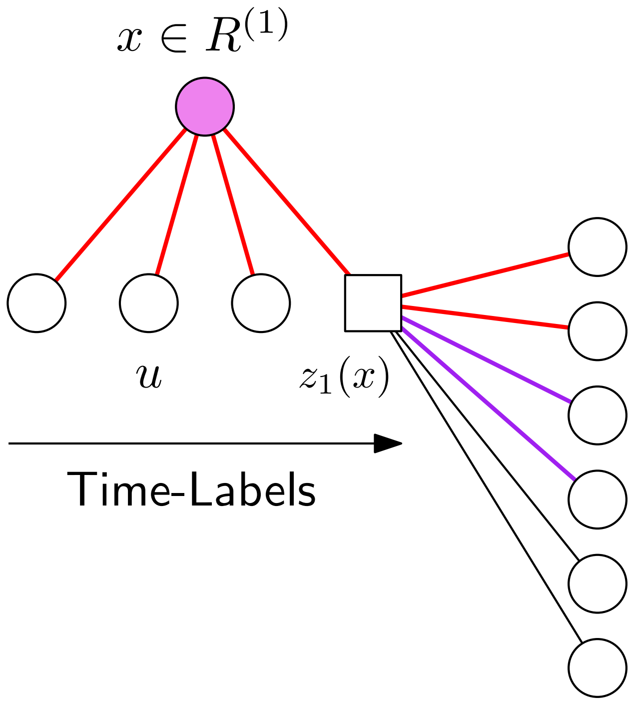
For every $u \in V$:



H = all red edges $\cup$ all purple edges

Our Temporal 5-Spanner

For every $u \in V$:



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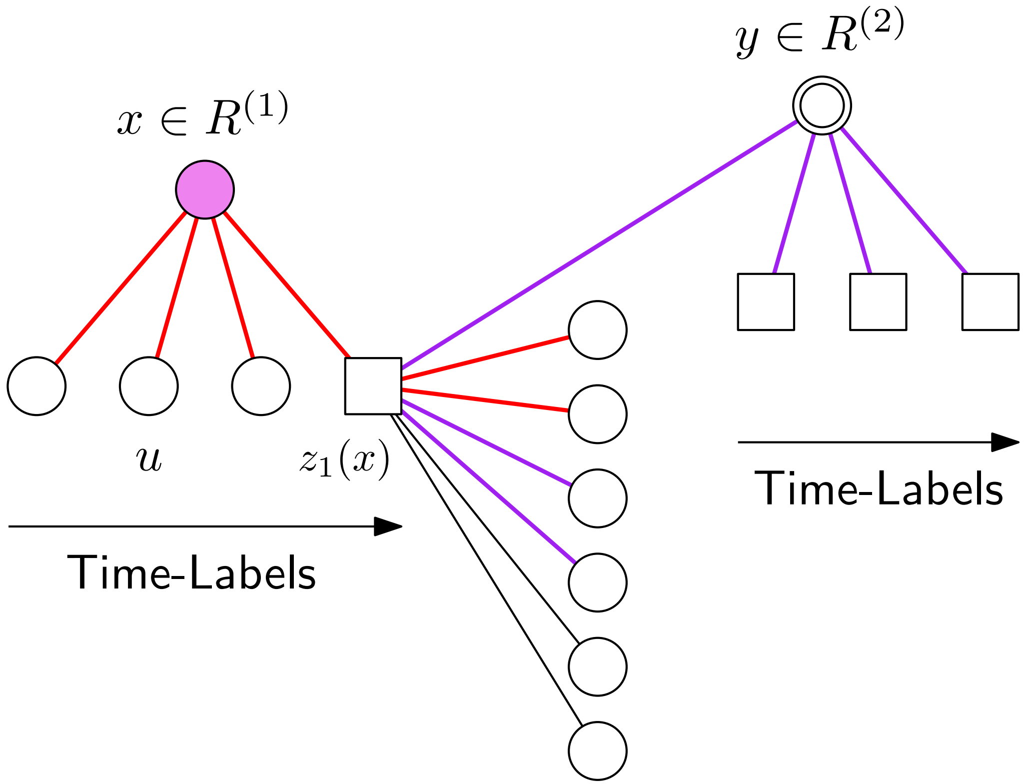
For every $u \in V$:

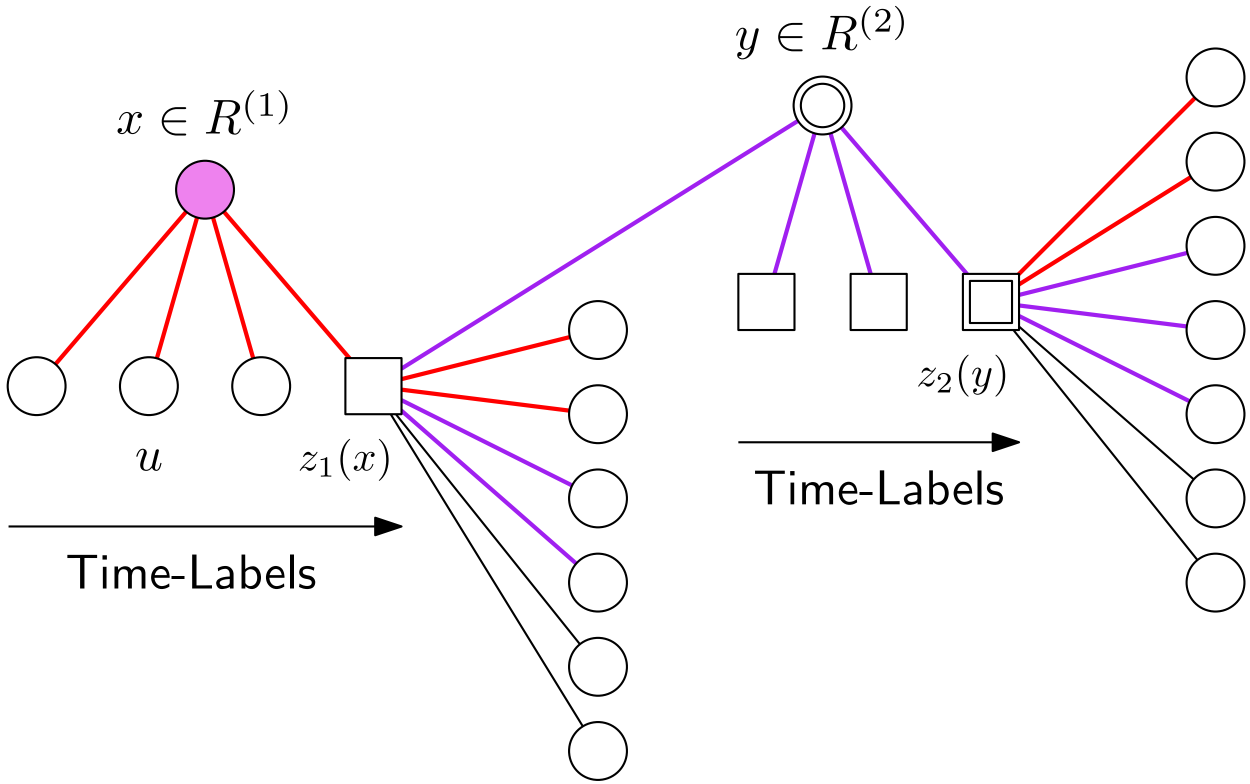


H = all red edges $\cup$ all purple edges

Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$



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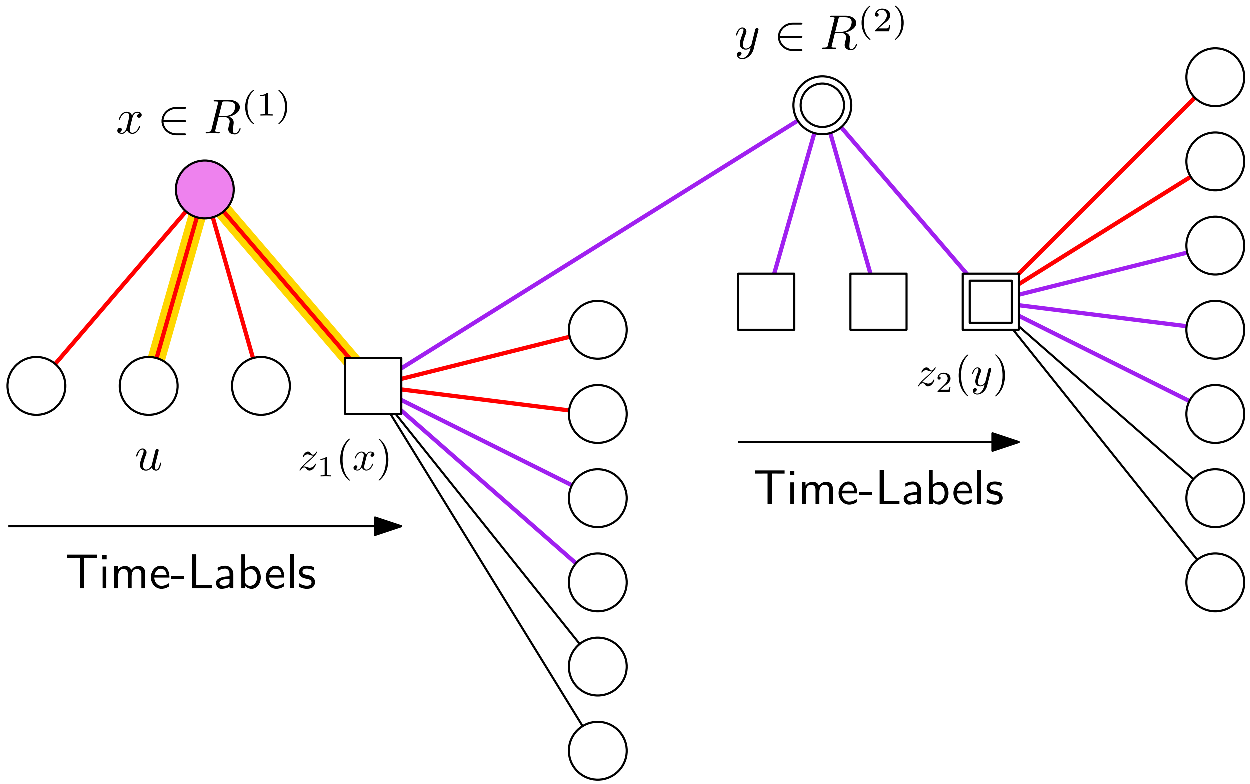

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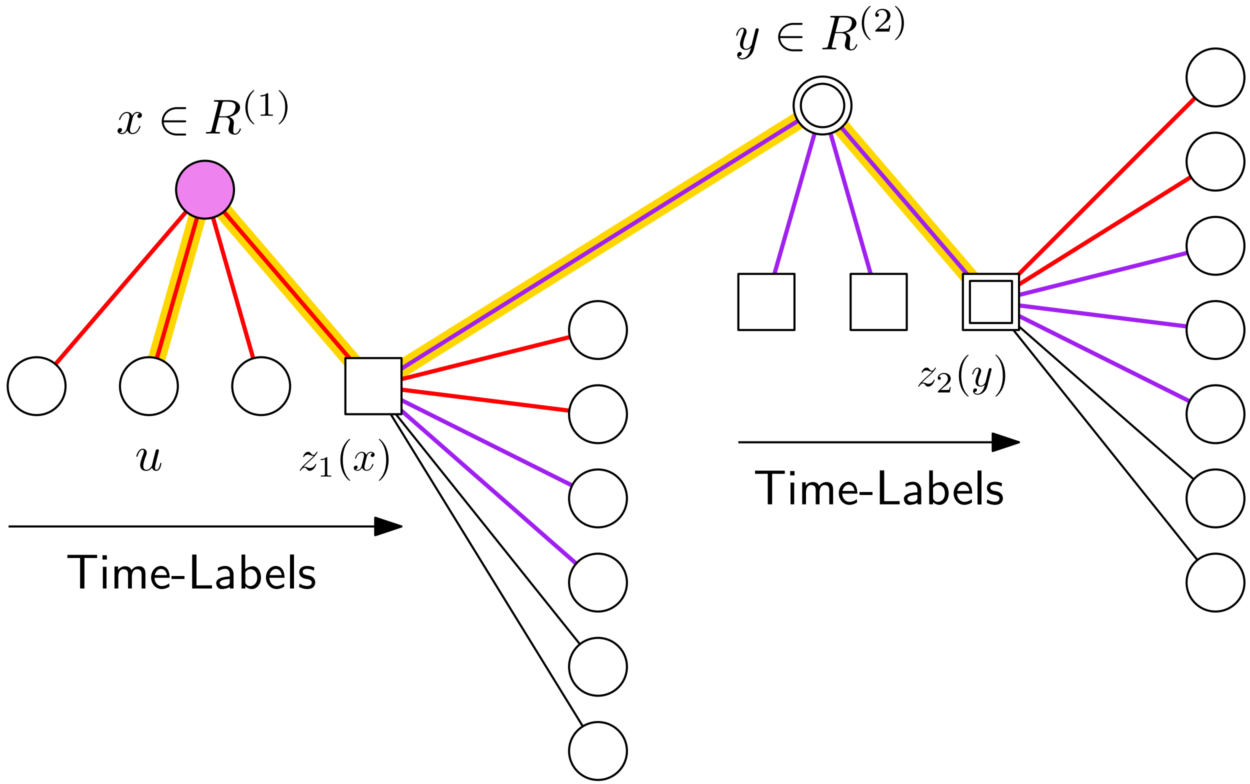


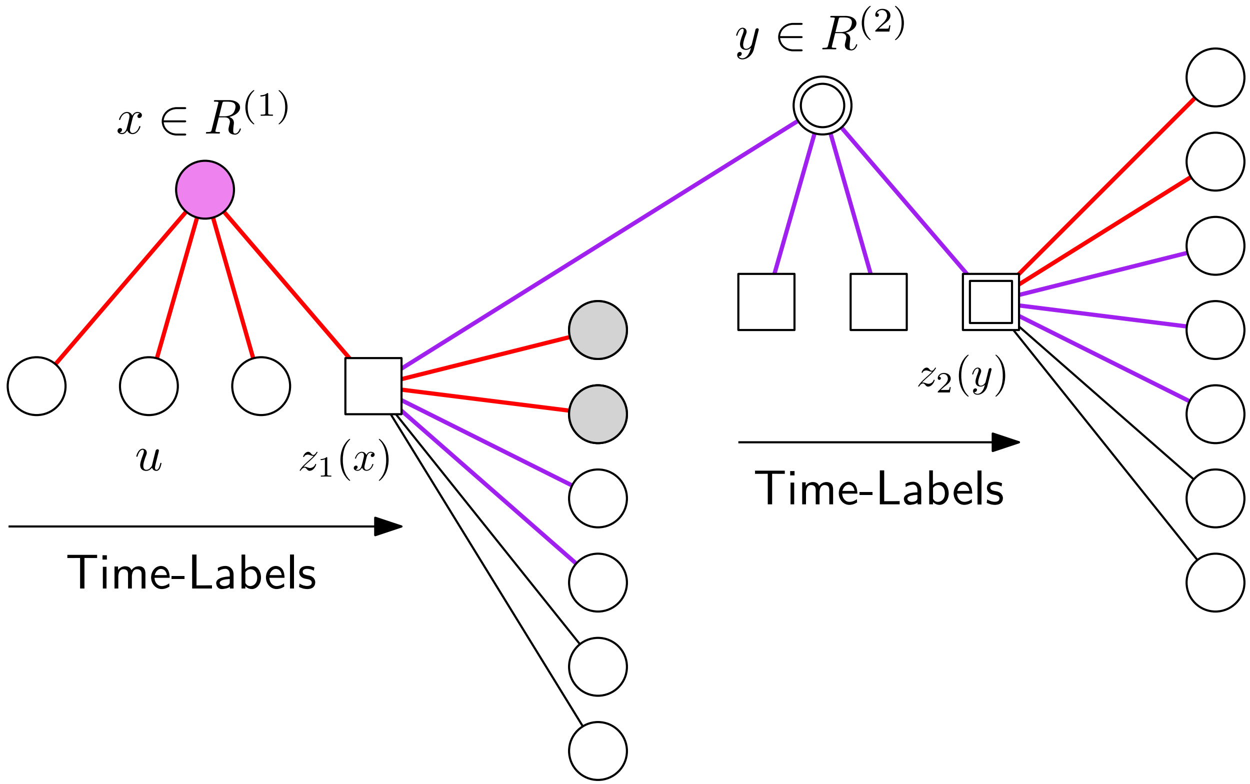
H = all red edges $\cup$ all purple edges

Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

Case 2a: $v \in S^{(1)}_{z_1(x)}$



H = all red edges $\cup$ all purple edges

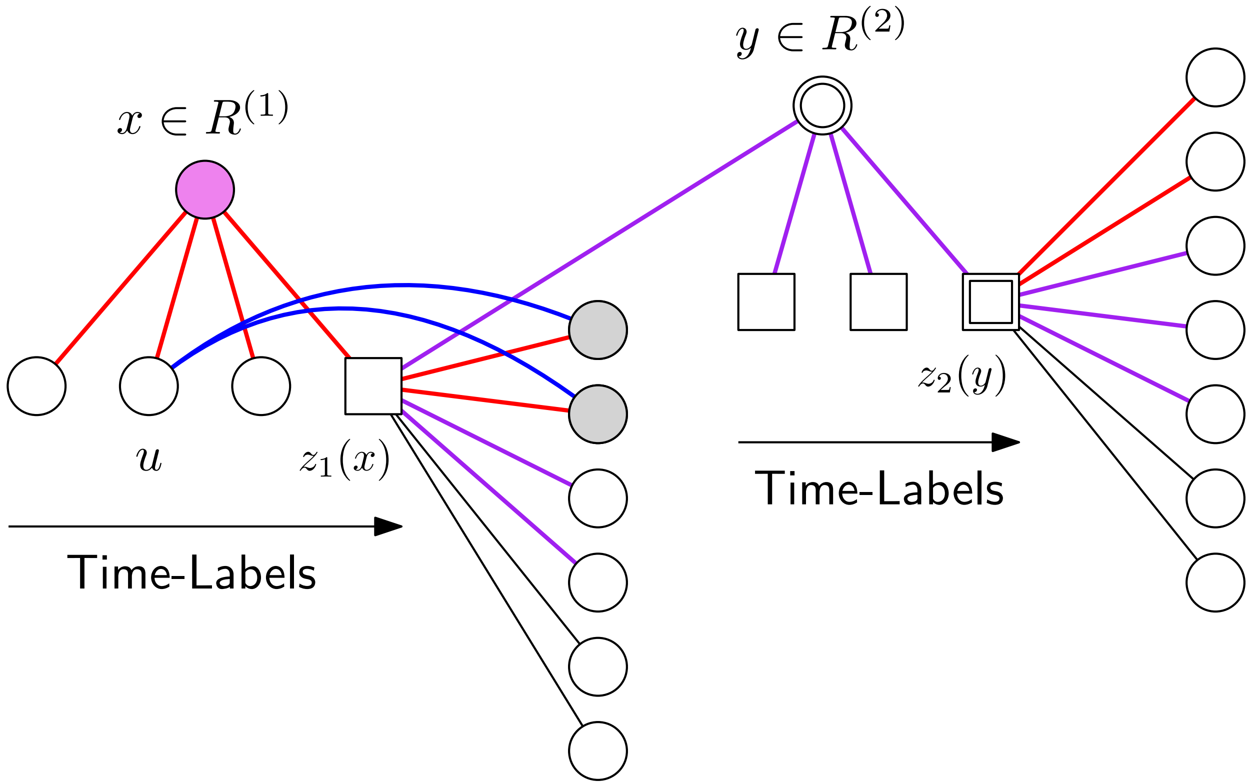
Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

Case 2a: $v \in S^{(1)}_{z_1(x)}$

- Add all blue edges between u and $S^{(1)}_{z_1(x)}$



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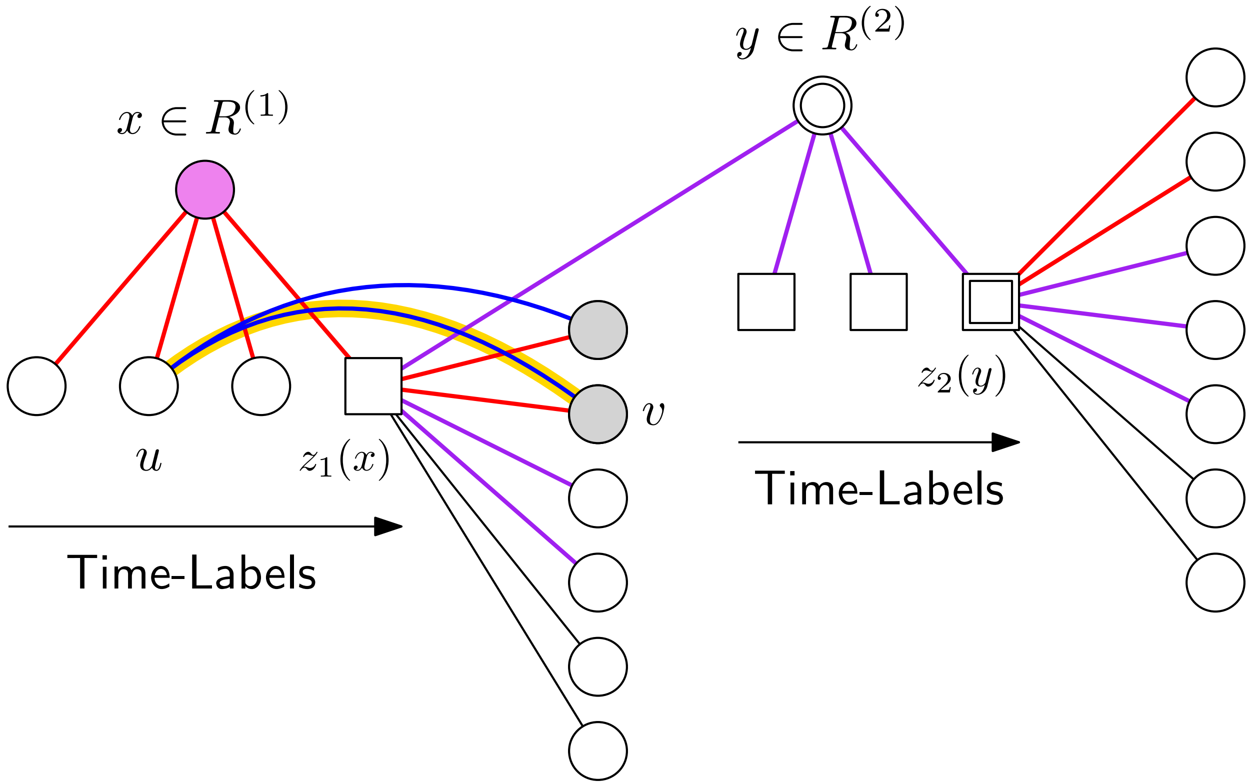

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For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

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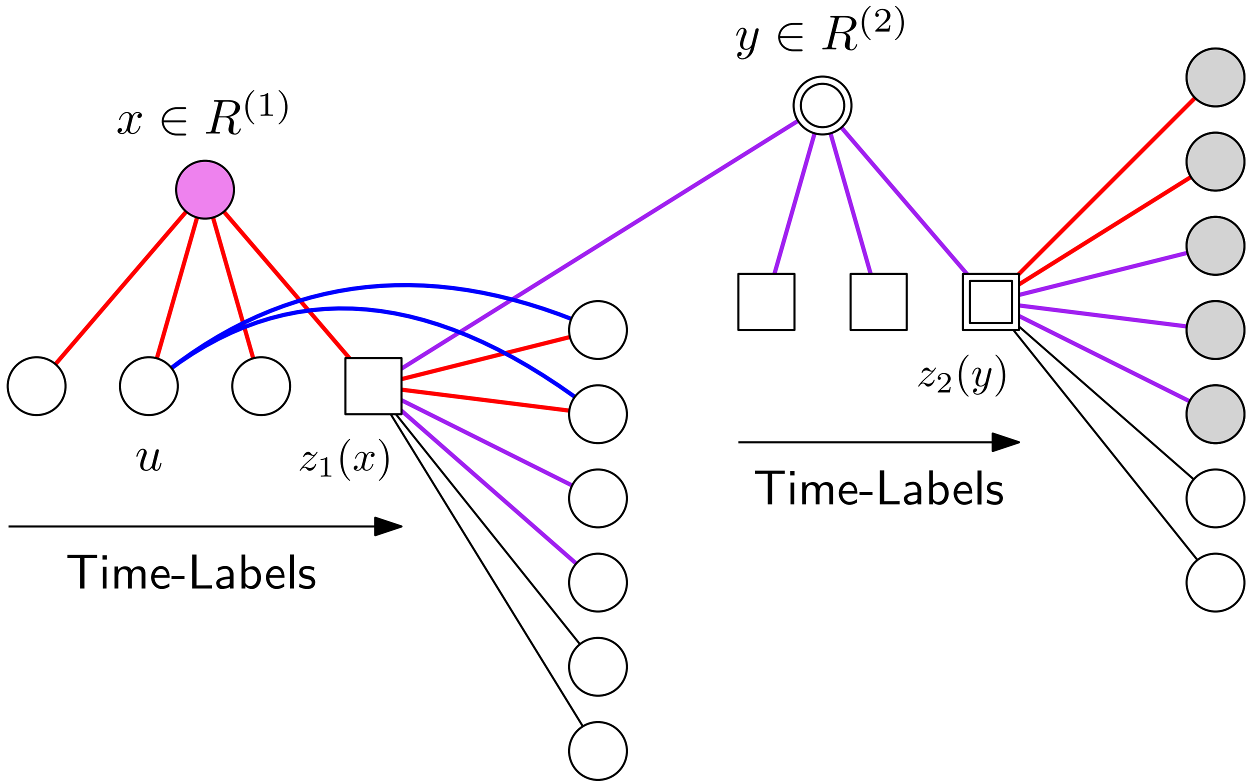
- Add all blue edges between u and $S^{(1)}_{z_1(x)}$

Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$



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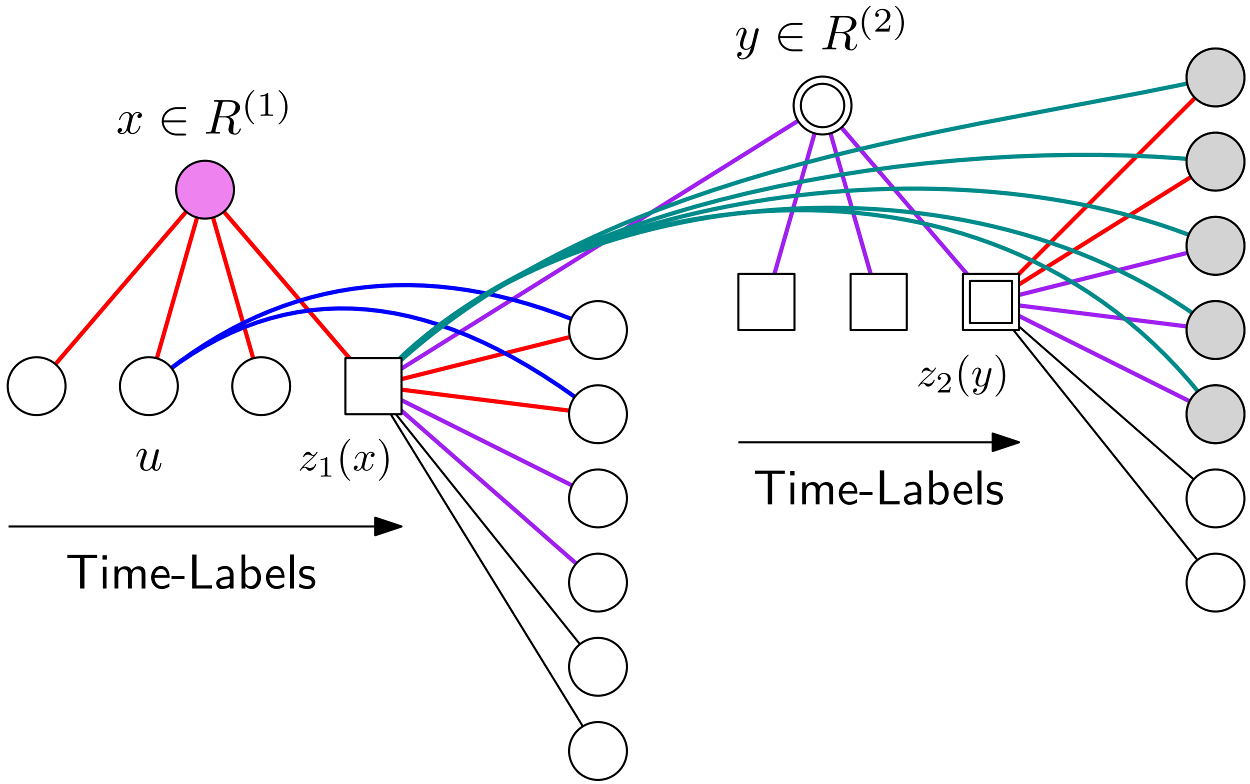
- Add all blue edges between u and $S^{(1)}_{z_1(x)}$

Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$



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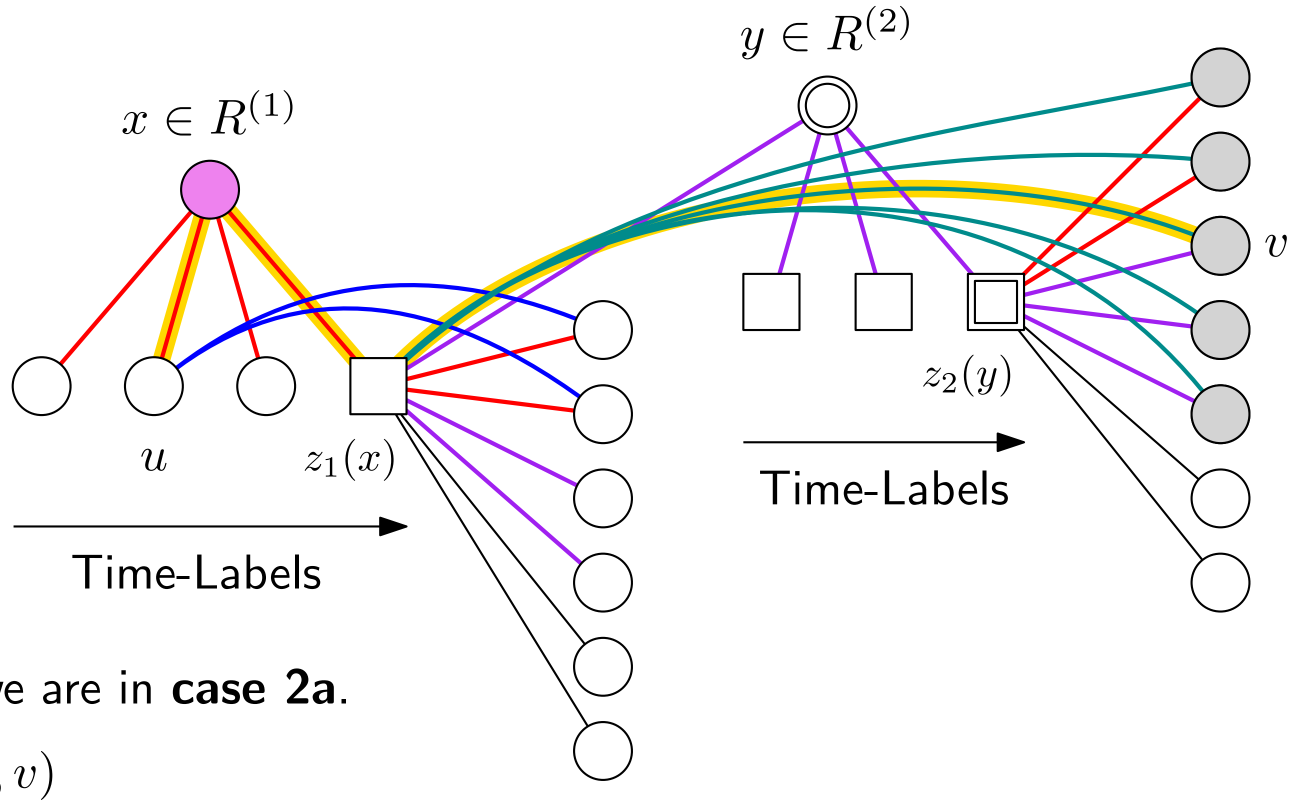
- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$

— If $(z_1(x), v)$ is a red edge of z_1 we are in **case 2a**.

— Otherwise $\lambda(x, z_1(x)) \leq \lambda(z_1(x), v)$



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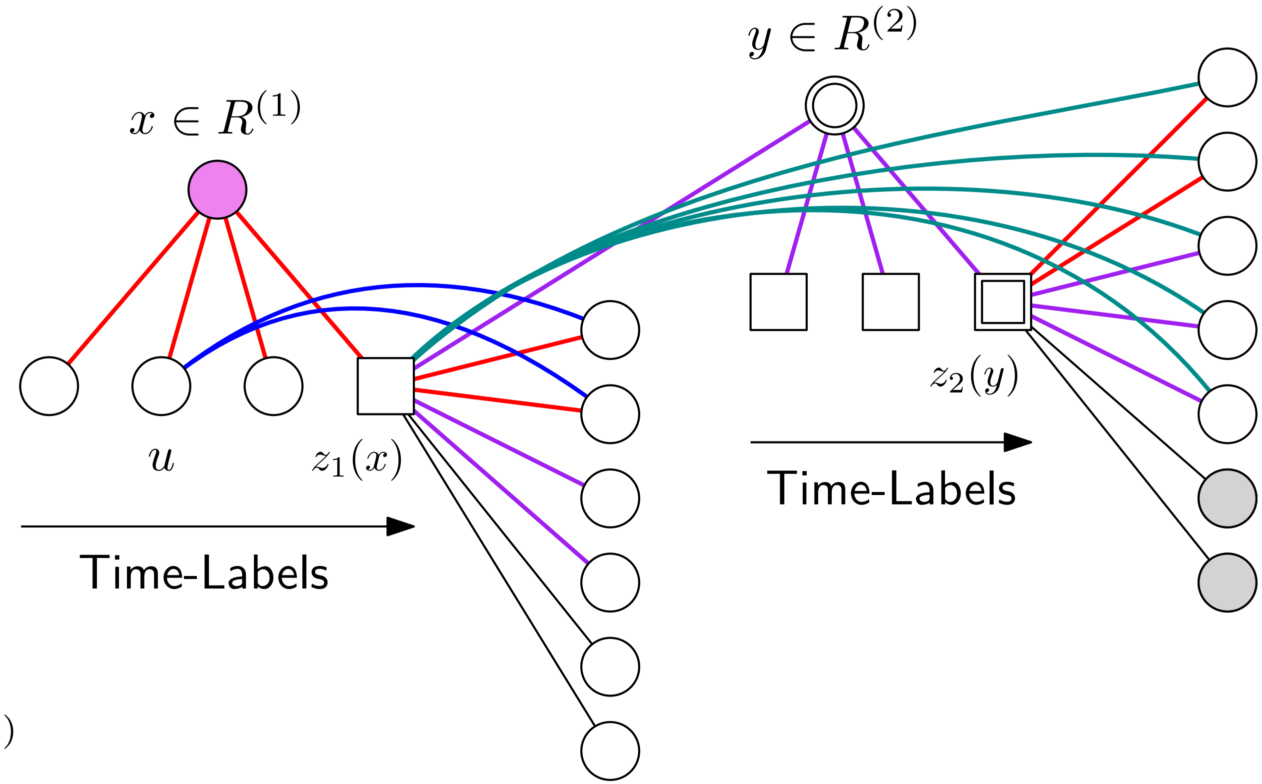
Case 3: $v \neq\in S^{(1)}_{z_2(y)}$ and $v \neq\in S^{(2)}_{z_2(y)}$



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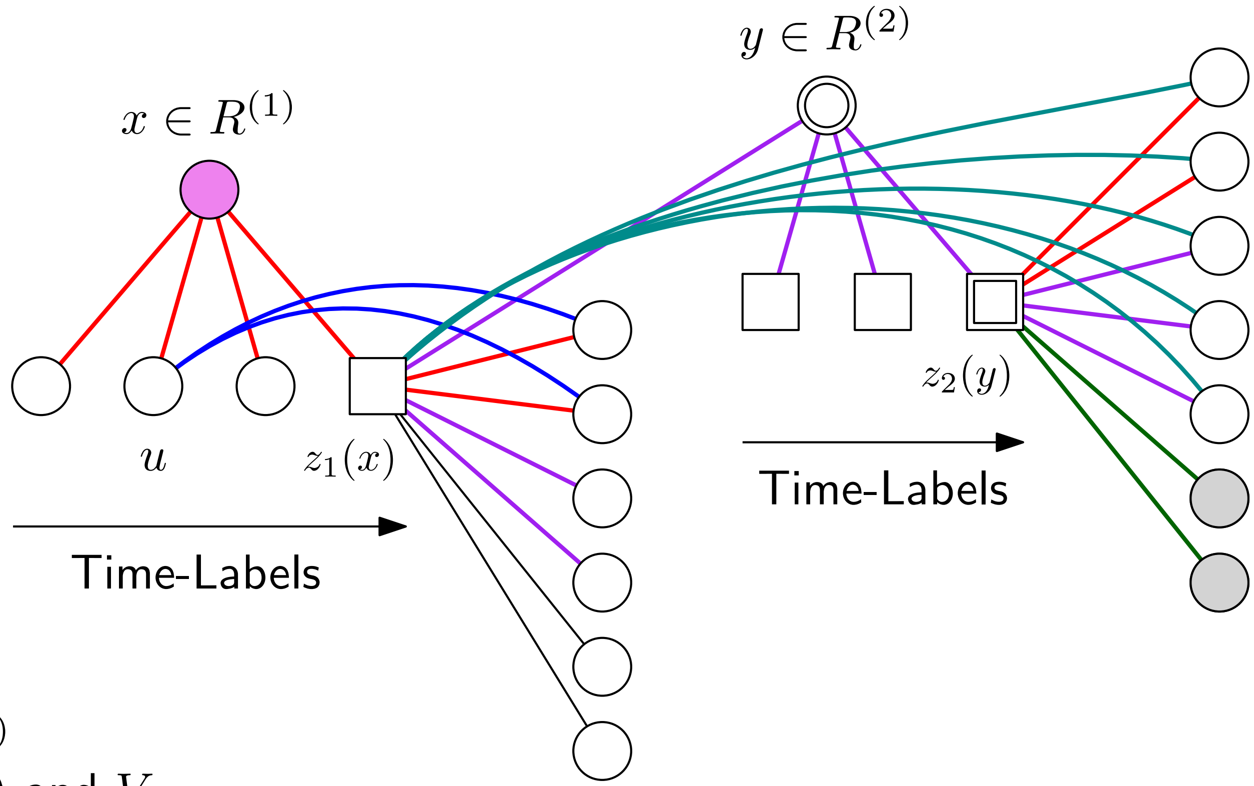
Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$

Case 3: $v \neq\in S^{(1)}_{z_2(y)}$ and $v \neq\in S^{(2)}_{z_2(y)}$

- Add all green edges between $z_2(y)$ and V



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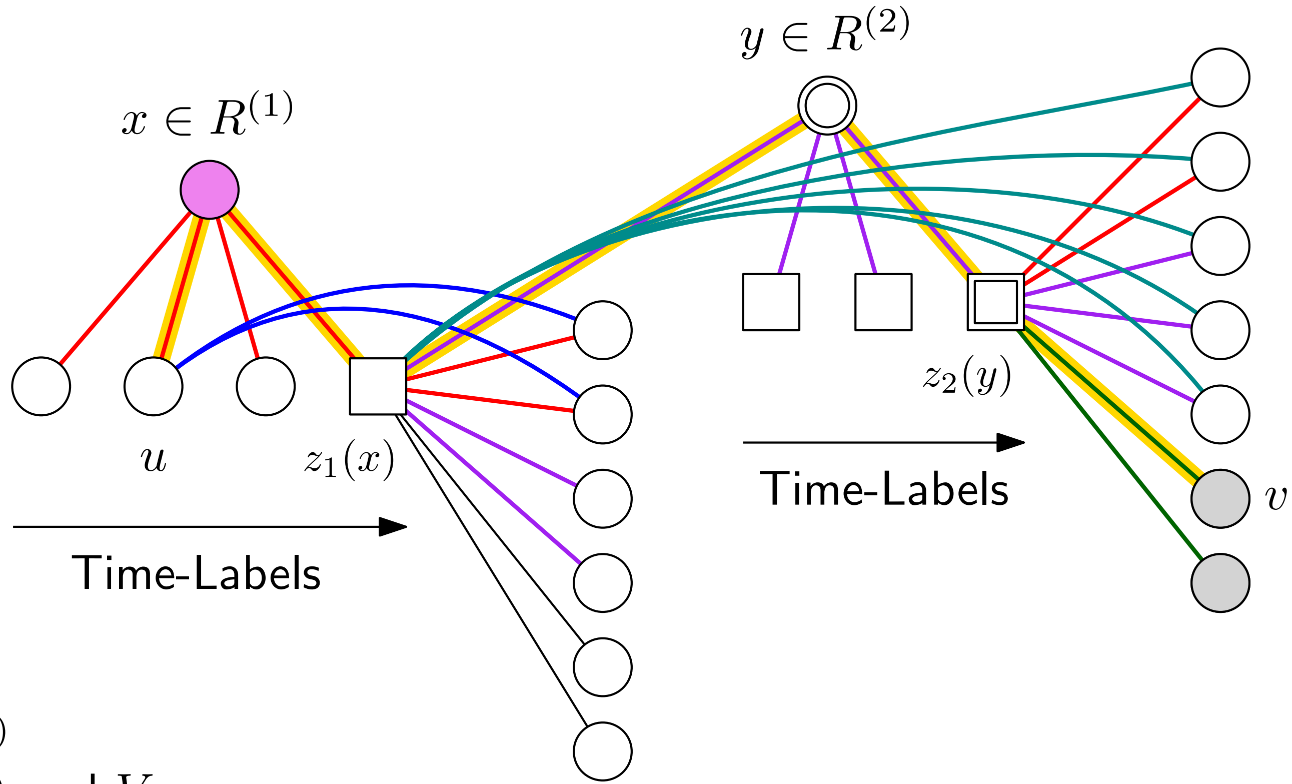

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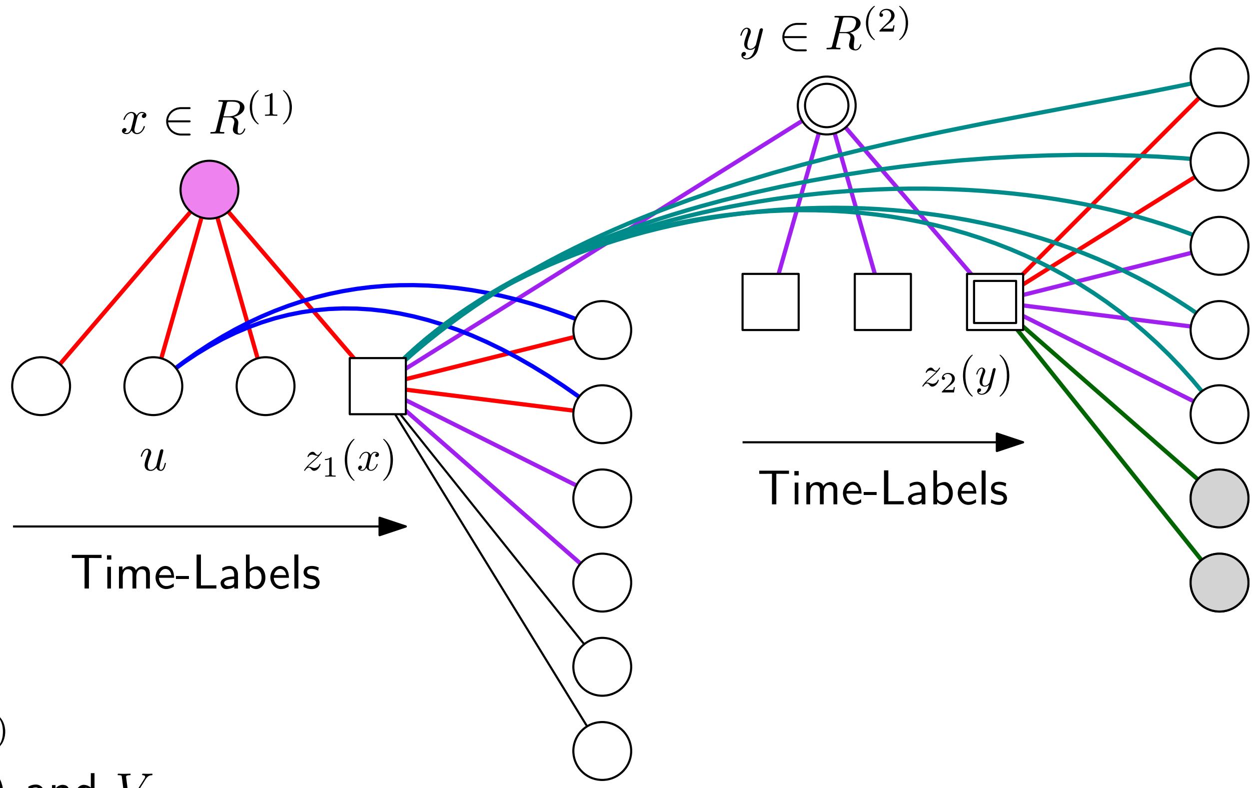

Size?

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Our Temporal 5-Spanner

For every $u \in V$:

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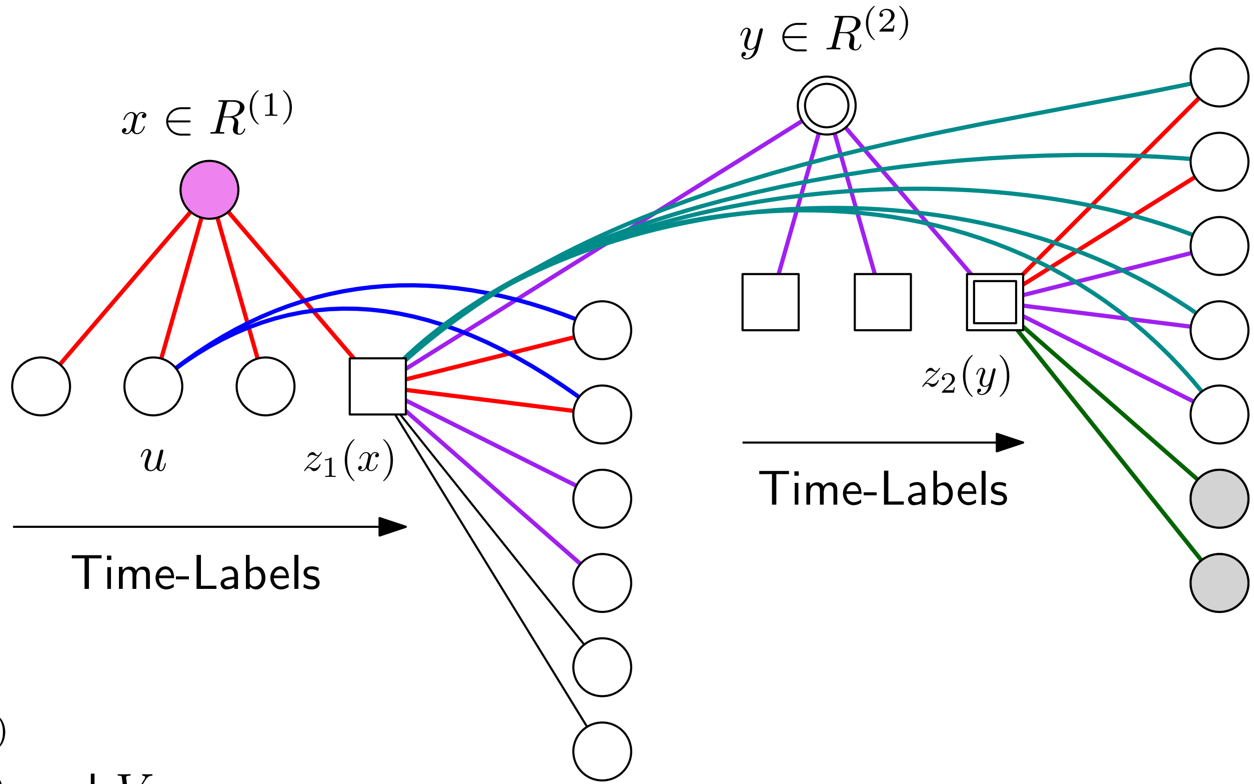
Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$

Case 3: $v \neq\in S^{(1)}_{z_2(y)}$ and $v \neq\in S^{(2)}_{z_2(y)}$

- Add all green edges between $z_2(y)$ and V



Size?

$H =$ all red edges $\cup$ all purple edges $\cup$ all blue edges $\cup$ all teal edges $\cup$ all green edges

$O(n\sqrt[3]{n}) \quad + \quad O(n\sqrt[3]{n})$

Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

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Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$

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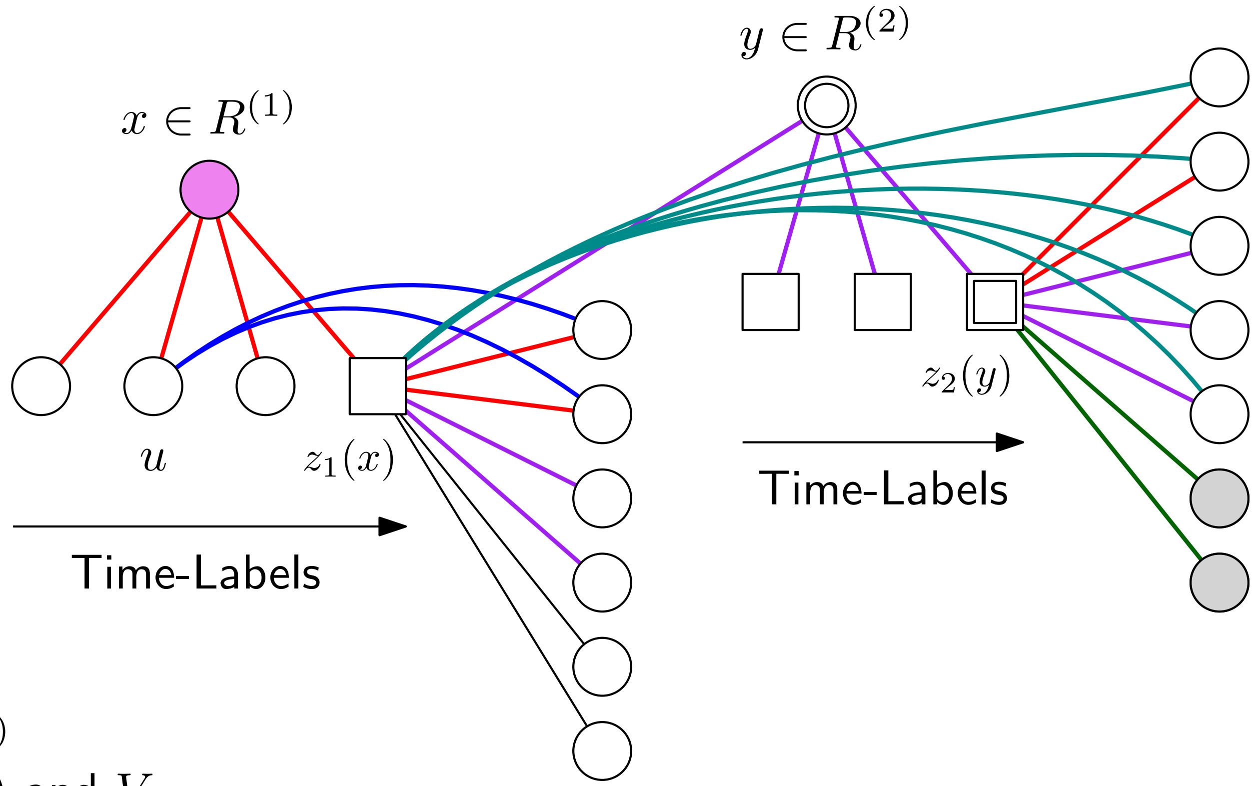
- Add all green edges between $z_2(y)$ and V



Size?

$H =$ all red edges $\cup$ all purple edges $\cup$ all blue edges $\cup$ all teal edges $\cup$ all green edges

$O(n\sqrt[3]{n}) \quad + \quad O(n\sqrt[3]{n}) \quad + \quad n \cdot O(\sqrt[3]{n})$

Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

Case 2a: $v \in S^{(1)}_{z_1(x)}$

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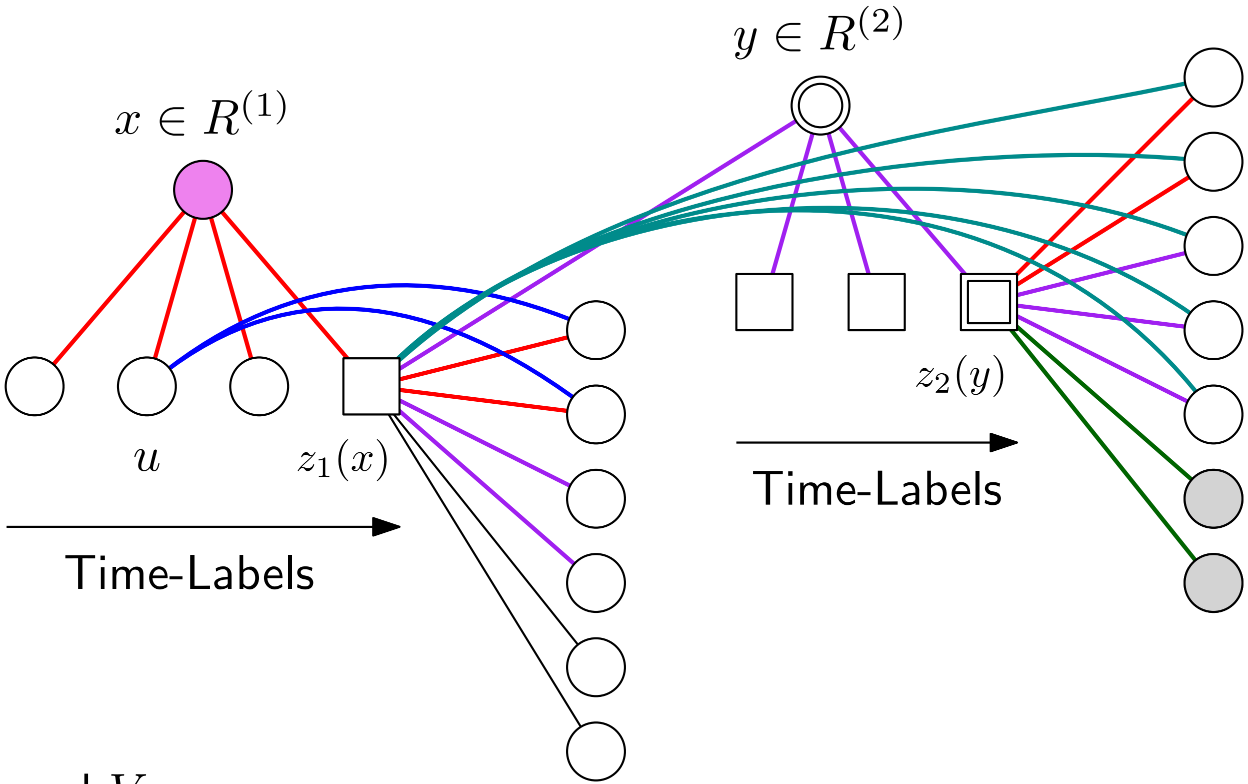
Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$

Case 3: $v \neq\in S^{(1)}_{z_2(y)}$ and $v \neq\in S^{(2)}_{z_2(y)}$

- Add all green edges between $z_2(y)$ and V



Size?

$H =$ all red edges $\cup$ all purple edges $\cup$ all blue edges $\cup$ all teal edges $\cup$ all green edges

$O(n\sqrt[3]{n}) \quad + \quad O(n\sqrt[3]{n}) \quad + \quad n \cdot O(\sqrt[3]{n}) \quad + \quad |R^{(1)}| \cdot O(n^{2/3})$

Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

Case 2a: $v \in S^{(1)}_{z_1(x)}$

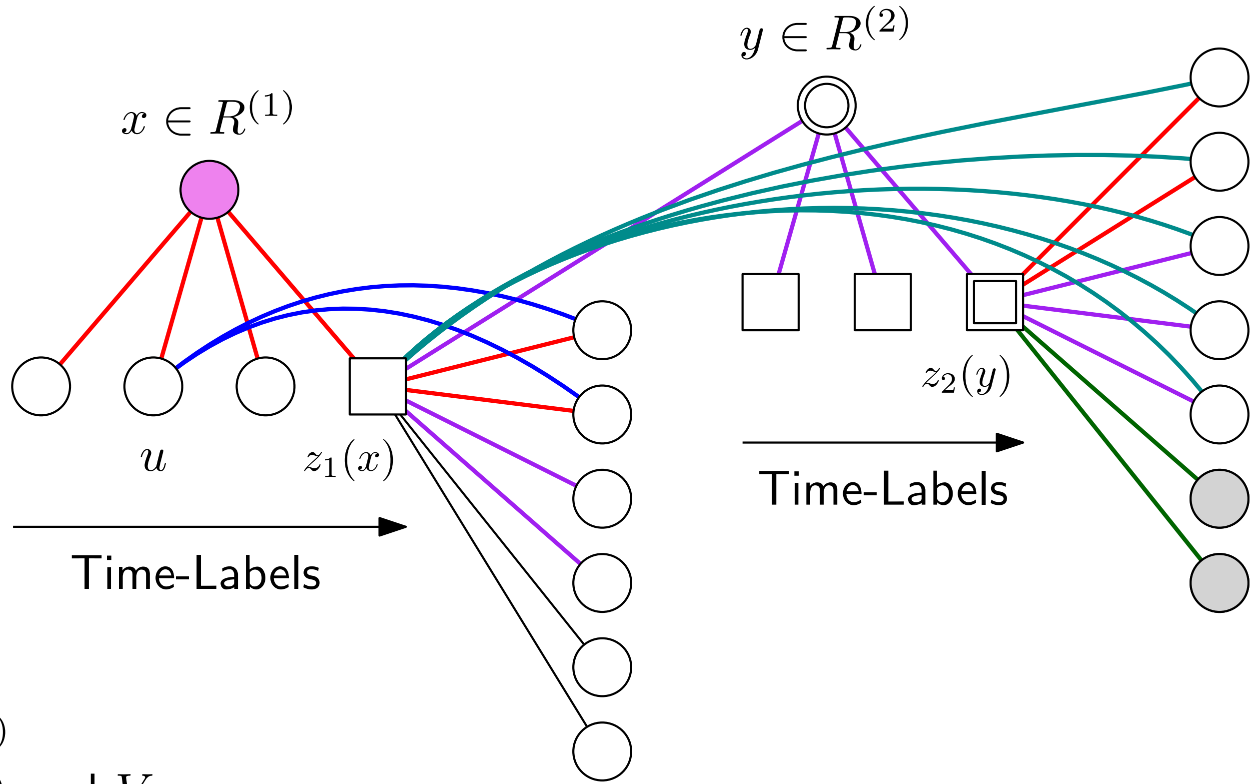
- Add all blue edges between u and $S^{(1)}_{z_1(x)}$

Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

- Add all teal edges between $z_1(x)$ and $S^{(1)}_{z_2(y)} \cup S^{(2)}_{z_2(y)}$

Case 3: $v \neq\in S^{(1)}_{z_2(y)}$ and $v \neq\in S^{(2)}_{z_2(y)}$

- Add all green edges between $z_2(y)$ and V



Size?

$H =$ all red edges $\cup$ all purple edges $\cup$ all blue edges $\cup$ all teal edges $\cup$ all green edges

$O(n\sqrt[3]{n}) \quad + \quad O(n\sqrt[3]{n}) \quad + \quad n \cdot O(\sqrt[3]{n}) \quad + \quad \widetilde{O}(n^{2/3}) \cdot O(n^{2/3})$

Our Temporal 5-Spanner

For every $u \in V$:

Case 1: $v = z_1(x)$ or $v = z_2(y)$

Case 2a: $v \in S^{(1)}_{z_1(x)}$

- Add all blue edges between u and $S^{(1)}_{z_1(x)}$

Case 2b: $v \in S^{(1)}_{z_2(y)}$ or $v \in S^{(2)}_{z_2(y)}$

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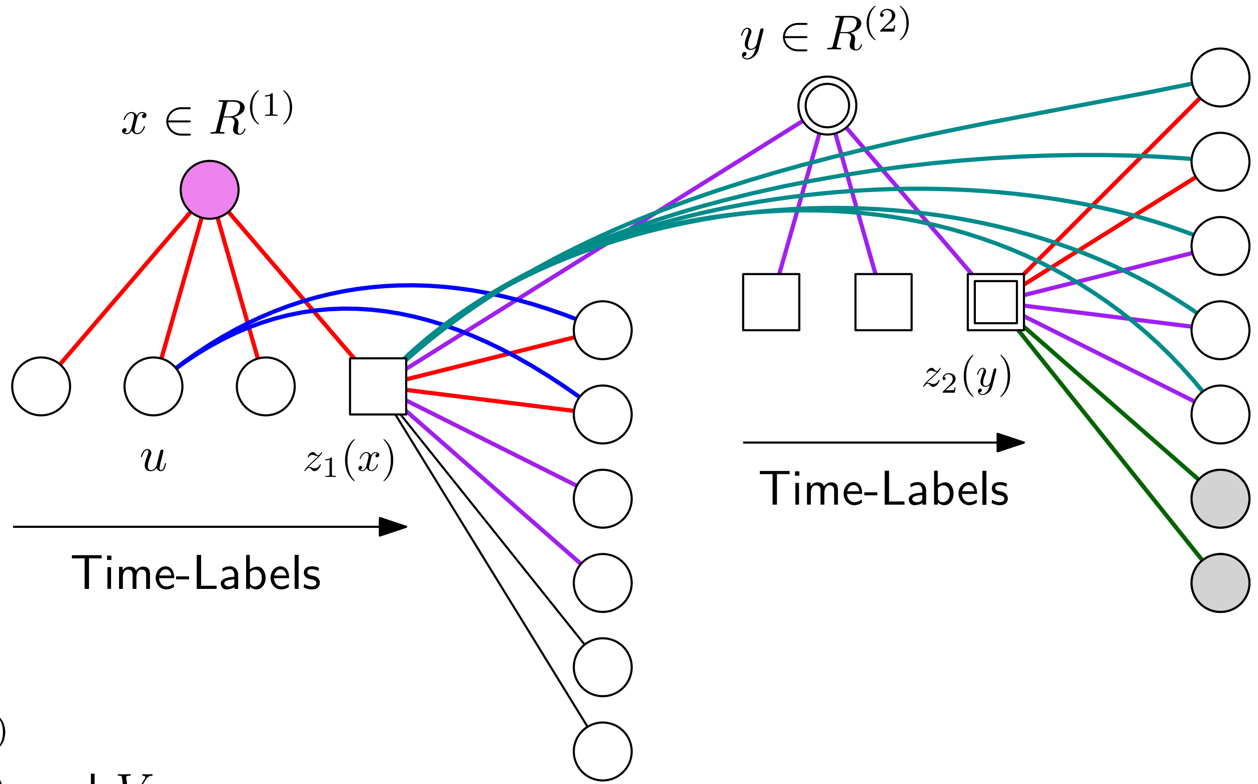

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$$O(n\sqrt[3]{n}) + O(n\sqrt[3]{n}) + n \cdot O(\sqrt[3]{n}) + \widetilde{O}(n^{2/3}) \cdot O(n^{2/3}) + |R^{(2)}| \cdot O(n)$$

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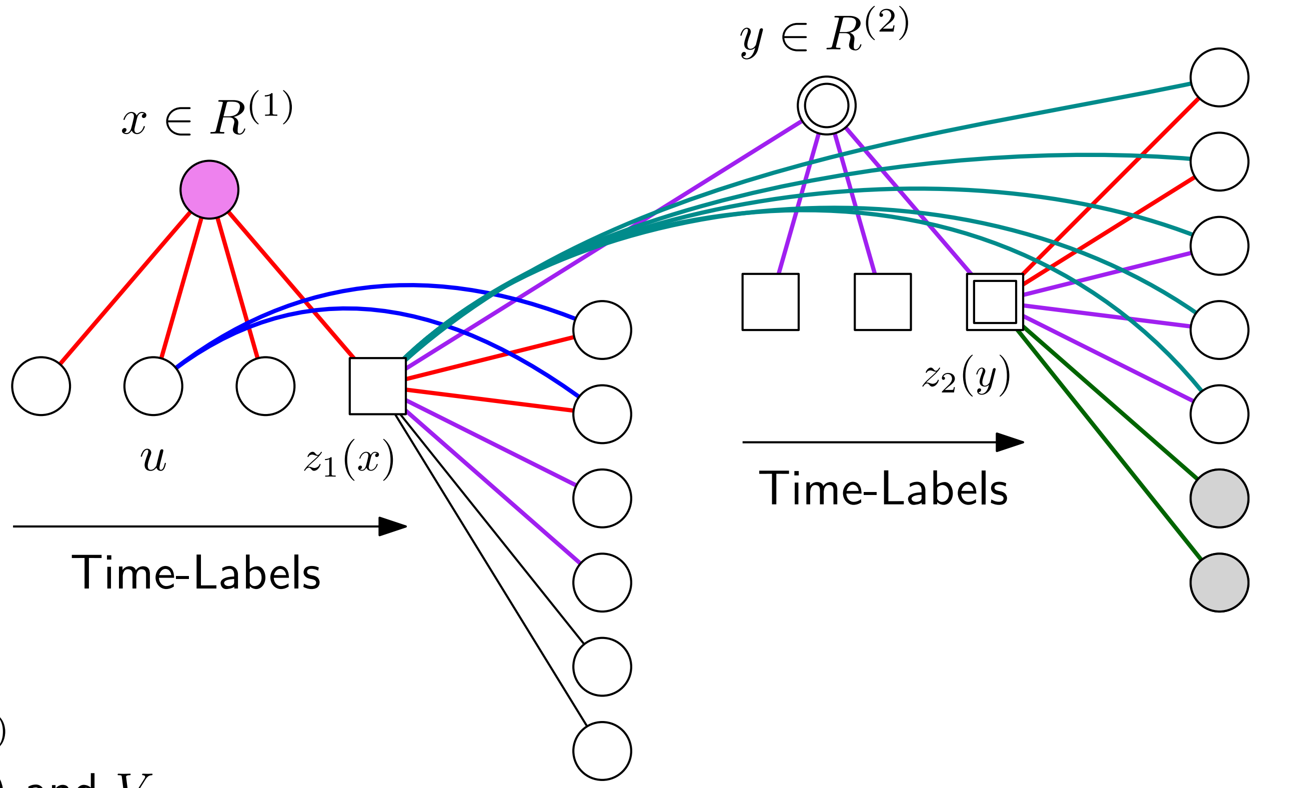

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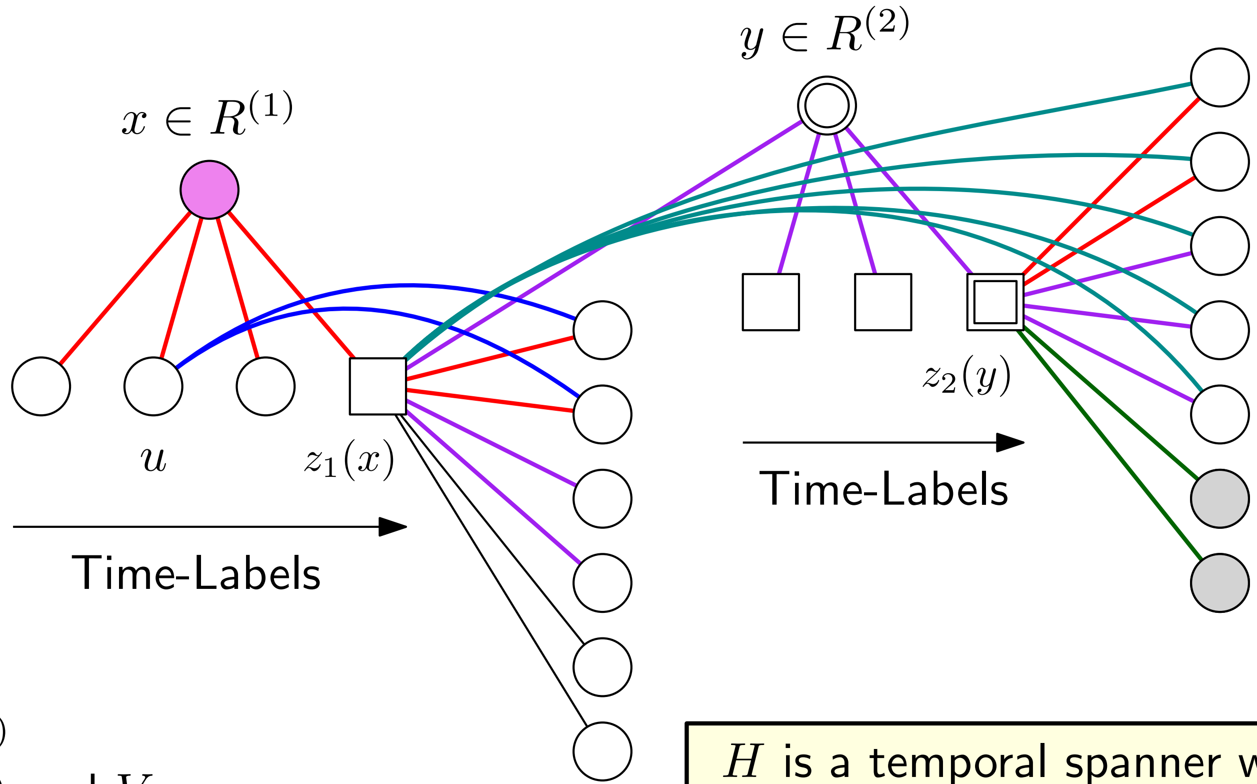

H is a temporal spanner with stretch 5 and size $\widetilde{O}(n^{4/3})$

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Our Single-Source Temporal Spanner
(for general graphs)

Our Single-Source Temporal Spanner

For a given source s. we will build a **single-source** temporal spanner with an additional guarantee:

- Let $\text{dist}^{\leqslant\tau}(s, v)$ be the length of the shortest temporal path from s to v **with arrival time** $\leq \tau$.
- For every arrival time τ and every vertex v

$$\text{dist}_H^{\leqslant\tau}(s, v) \leq \alpha \cdot \text{dist}_G^{\leqslant\tau}(s, v) \qquad \forall v$$

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General result:

Stretch: $1+\varepsilon$

Size: $O\left(\frac{n\log^4 n}{\log(1+\epsilon)}\right)$

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To get stretch $(1+\delta)$ choose $\varepsilon=\delta/3$ $(1+\delta/3)^2 \leq (1+\delta)$ for $\delta \leq 3$

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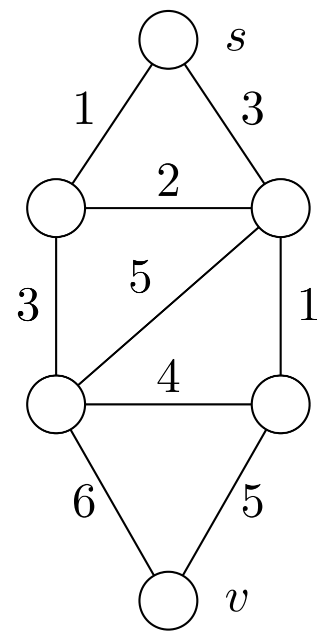
Consider a fixed source s:

A τ-**restricted** temporal path from s to v is a temporal path from s to v with arrival time at most τ



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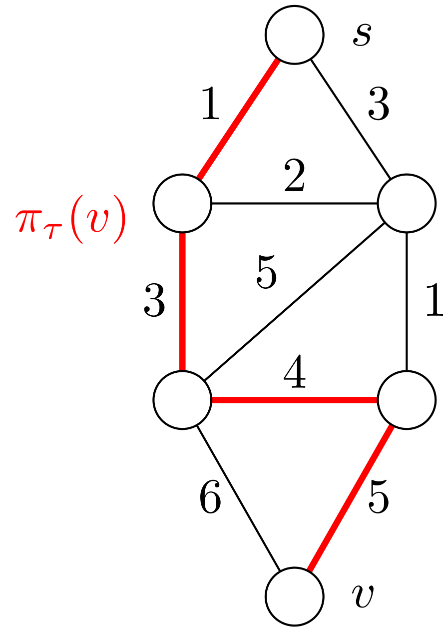

$\mathsf{dist}_G^{\leqslant 5}(s,v) = 4$

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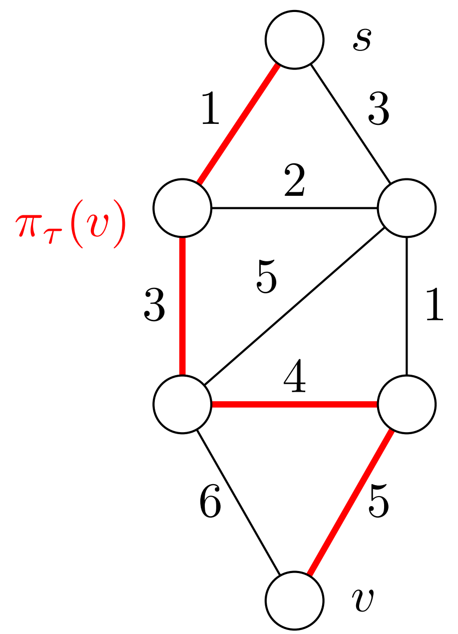
Can be found in polynomial-time
Wu et al. [VLDB 2014]



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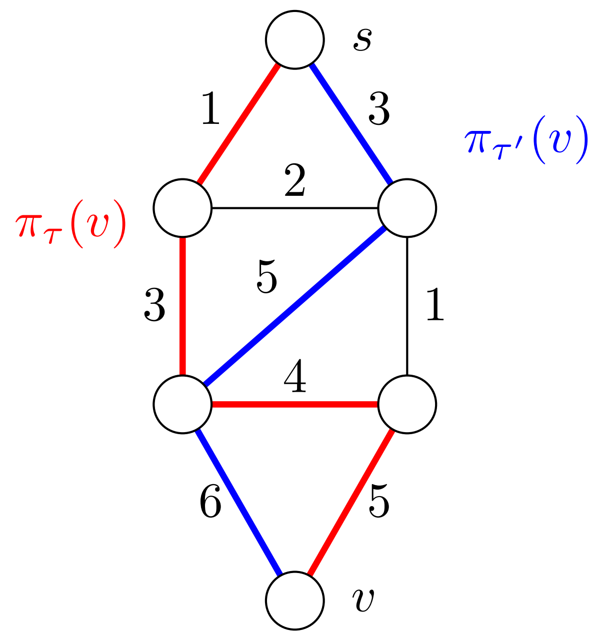

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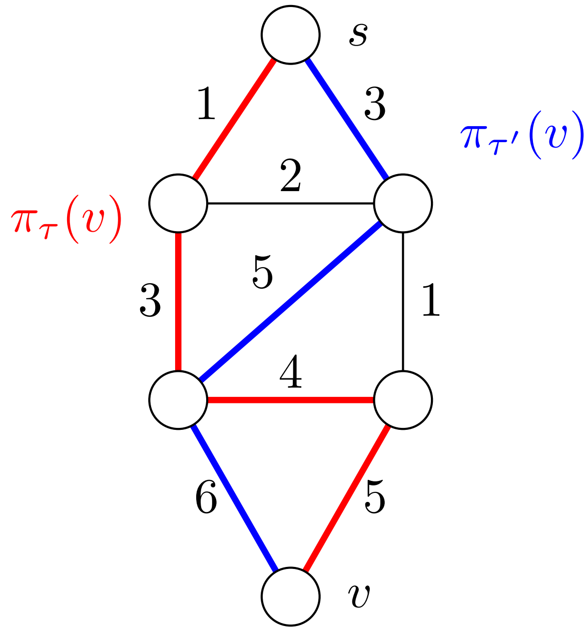

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For each $v \in V$ we build a set Π_v of paths from s to v such that, for every τ there is some path $\pi \in \Pi_v$ that satisfies:

$$|\pi| \leq (1+\epsilon) \cdot \text{dist}_G^{\leqslant \tau}(s,v)$$

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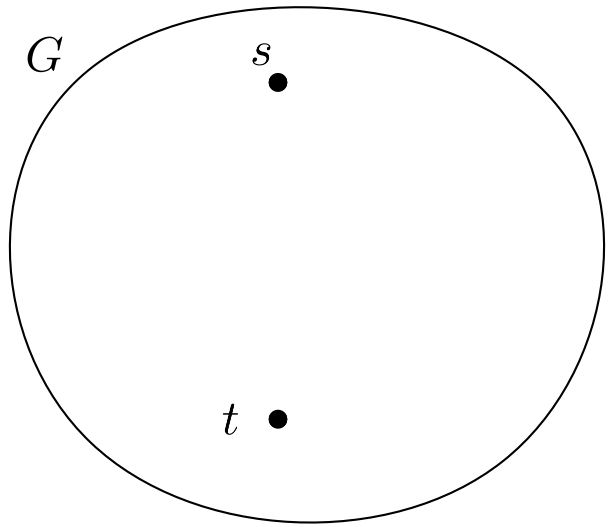
$D = +\infty$

For $\tau = 1, 2, \ldots, L$:

If $D > (1+\varepsilon) \cdot \mathsf{dist}_G^{\leq \tau}(s, v)$:

$\Pi_v \leftarrow \Pi_v \cup \{\pi_\tau(v)\}$

$D = |\pi_\tau(v)|$



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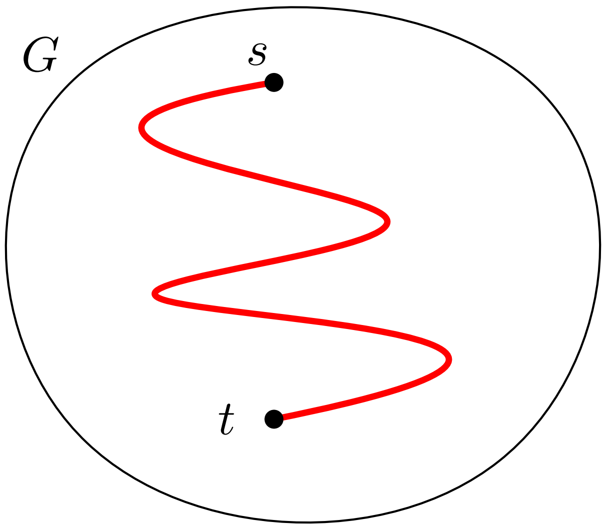

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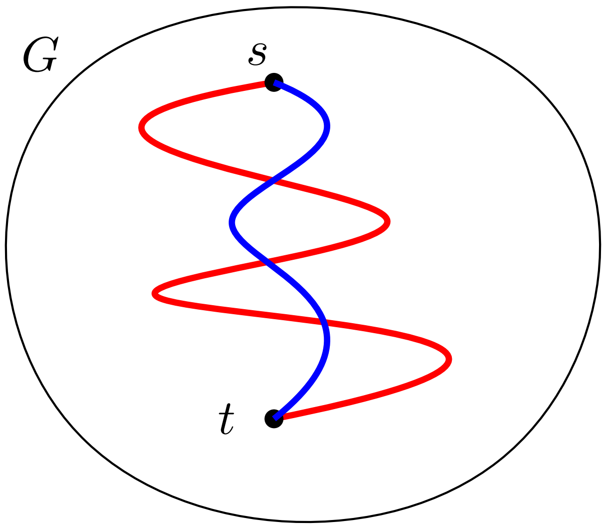

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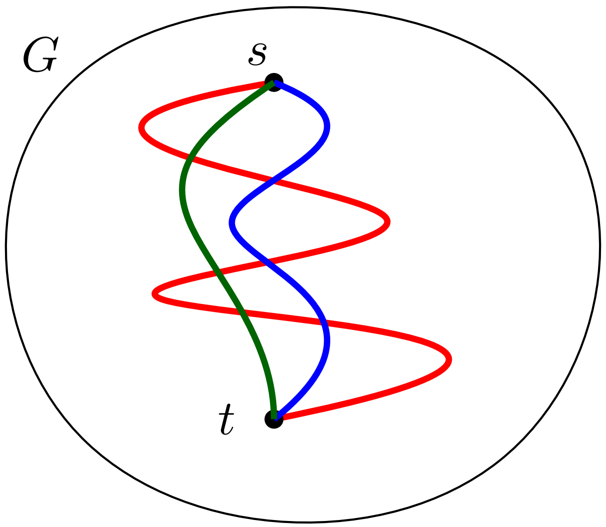

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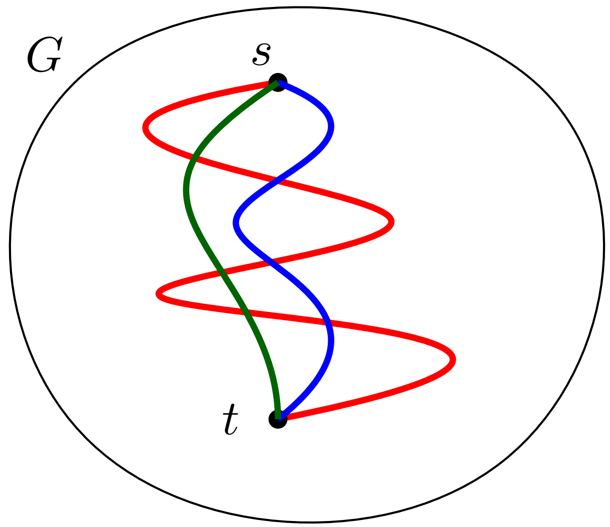

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$$\implies |\Pi_v| = O\left(\frac{\log n}{\log(1+\epsilon)}\right)$$

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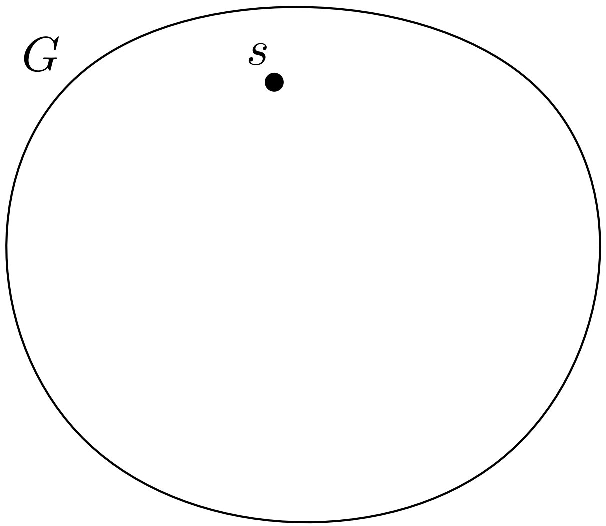
A path is **long** if it has length at least $\sqrt{n}$

A vertex x hits a long path π if it is one of the last $\sqrt{n}$ vertices of π



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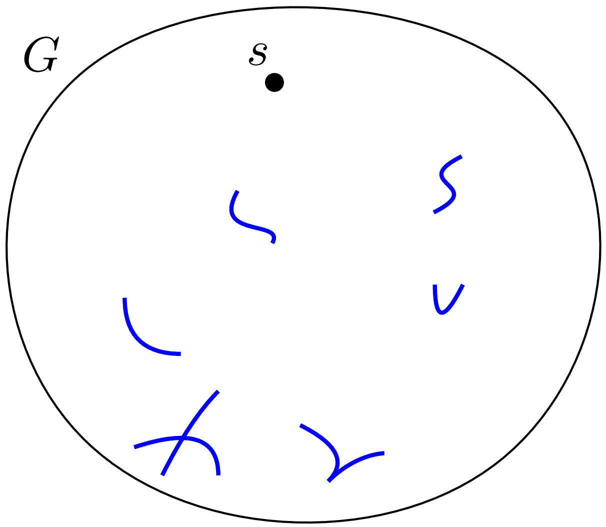

The spanner H contains:

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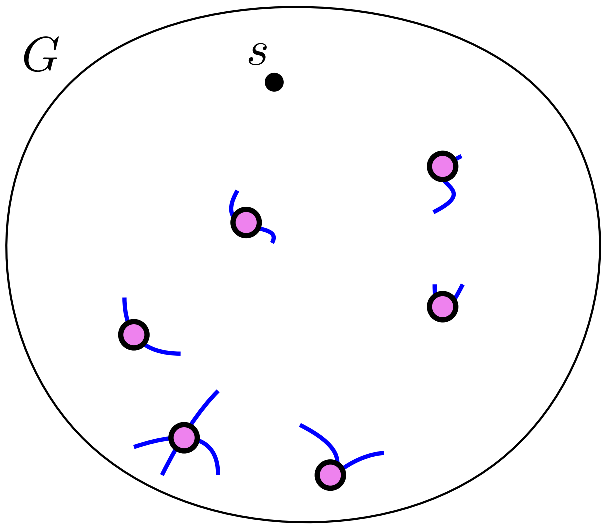
Choose a set R of $\widetilde{O}(\sqrt{n})$ vertices that hits all long paths in $\cup_v \Pi_v$



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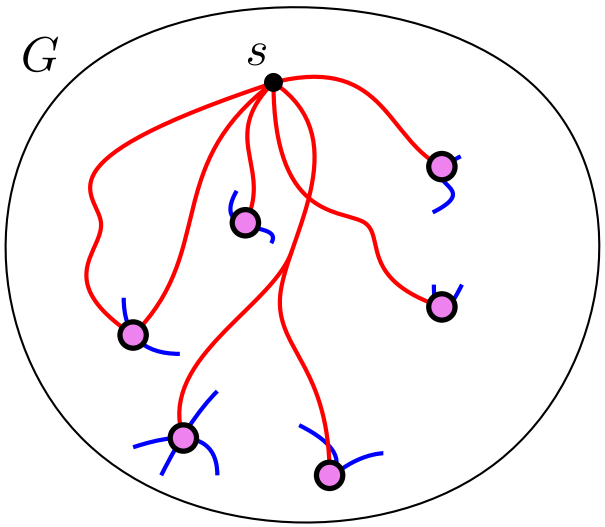

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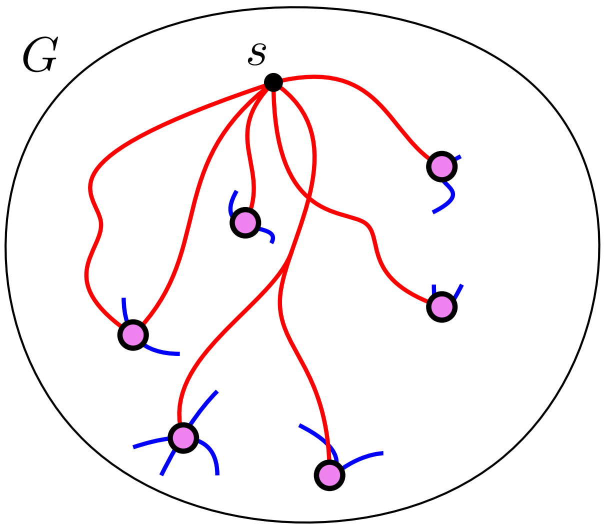

The spanner H contains:

Size

- The last $\sqrt{n}$ edges of each path in $\bigcup_{v \in V} \Pi_v$ — $n \cdot O(\frac{\log n}{\log(1+\varepsilon)}) \cdot O(\sqrt{n})$
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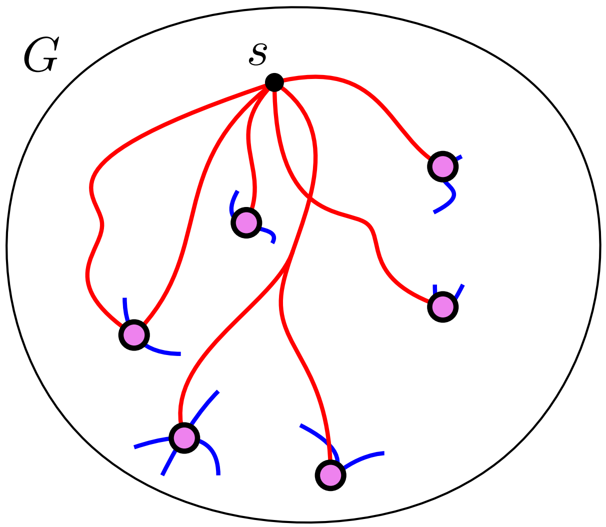

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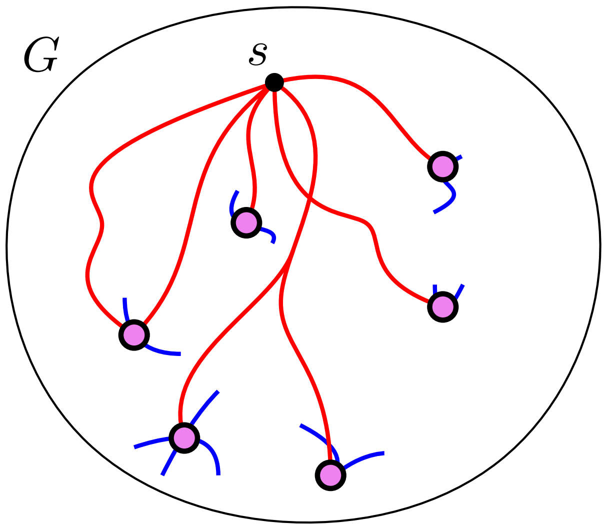

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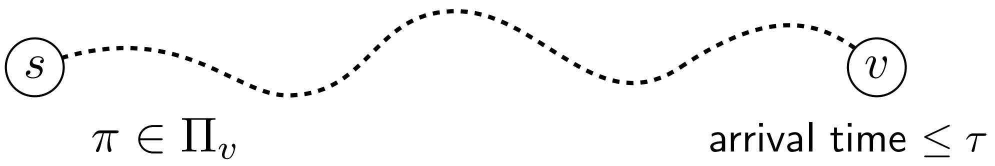
Stretch factor?



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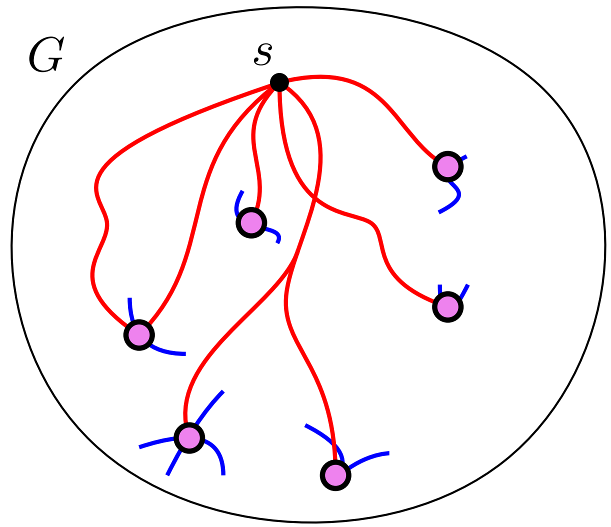

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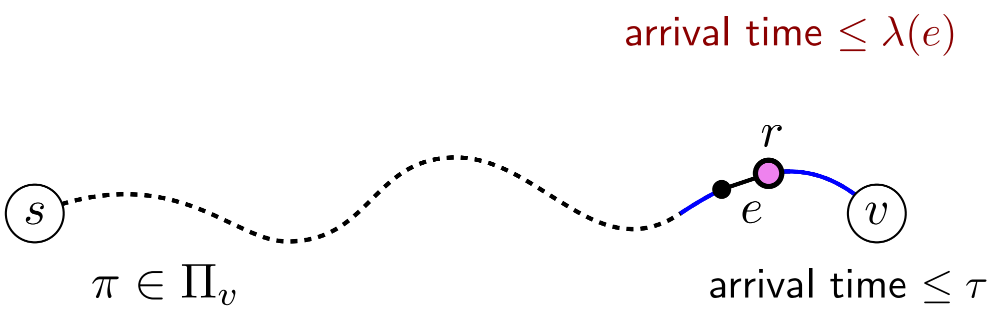

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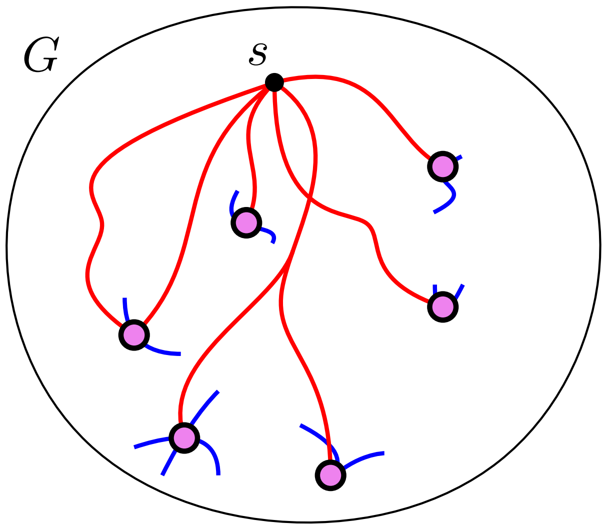

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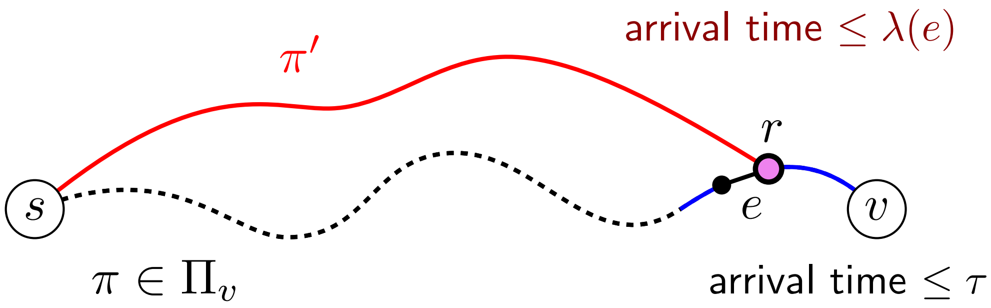

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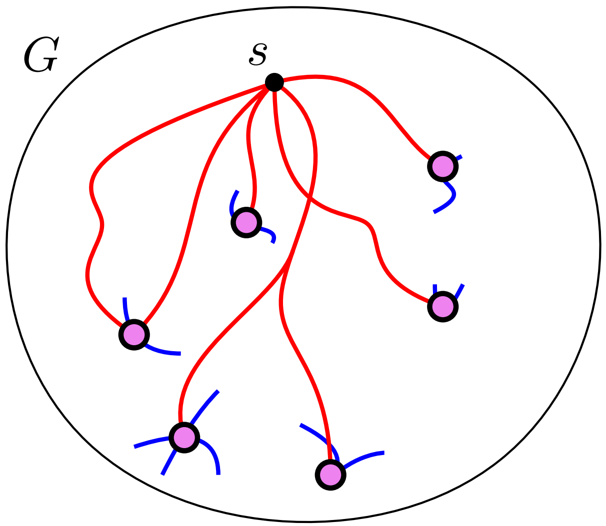

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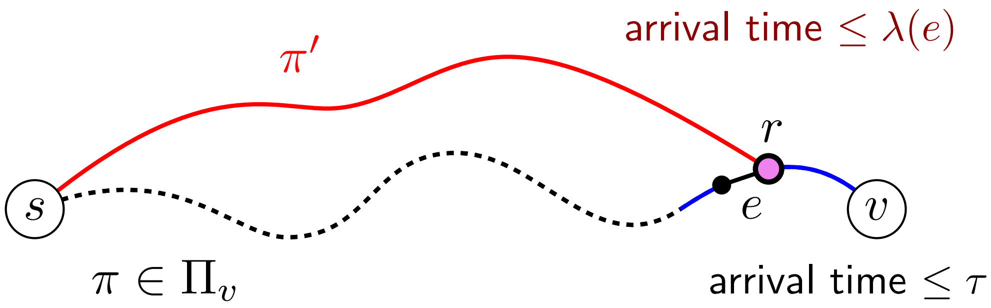

$|\pi'| \leq (1+\varepsilon) \cdot \mathrm{dist}_G^{\leq \lambda(e)}(s, r)$

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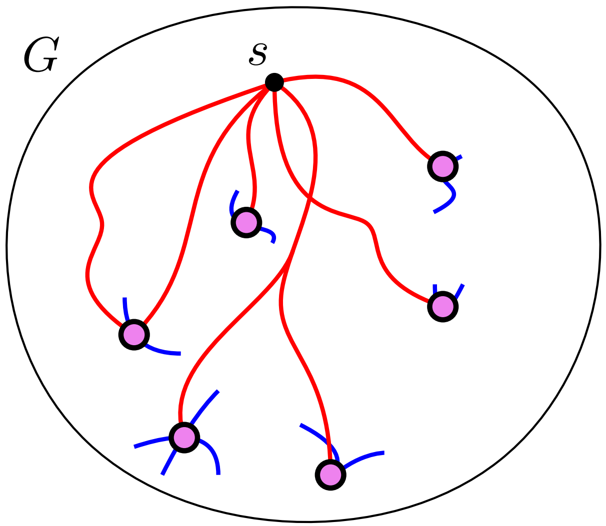
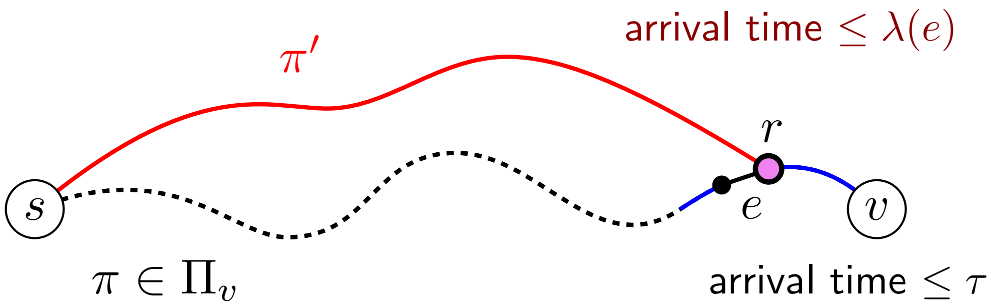


The spanner H contains:	**Size**
• The last $\sqrt{n}$ edges of each path in $\bigcup_{v \in V} \Pi_v$	$n \cdot O(\frac{\log n}{\log(1+\varepsilon)}) \cdot O(\sqrt{n})$
• All paths in $\bigcup_{r \in R} \Pi_r$	$\widetilde{O}(\sqrt{n}) \cdot O(\frac{\log n}{\log(1+\varepsilon)}) \cdot n$

Stretch factor?



$|\pi'| \leq (1+\varepsilon) \cdot \text{dist}_G^{\leq \lambda(e)}(s, r) \leq (1+\varepsilon) \cdot |\pi[s:r]|$

Our Single-Source Temporal Spanner

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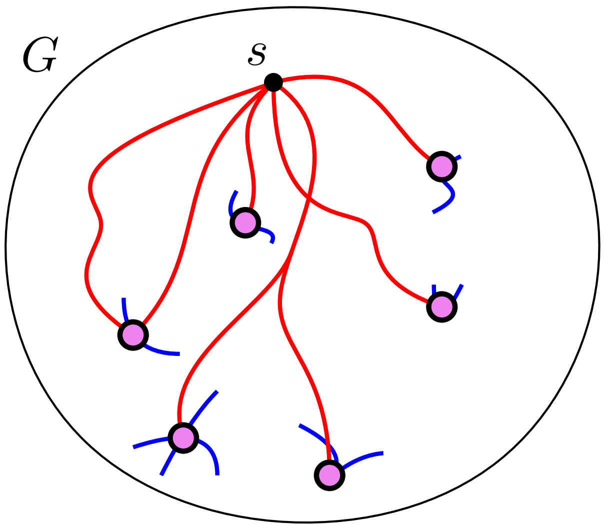

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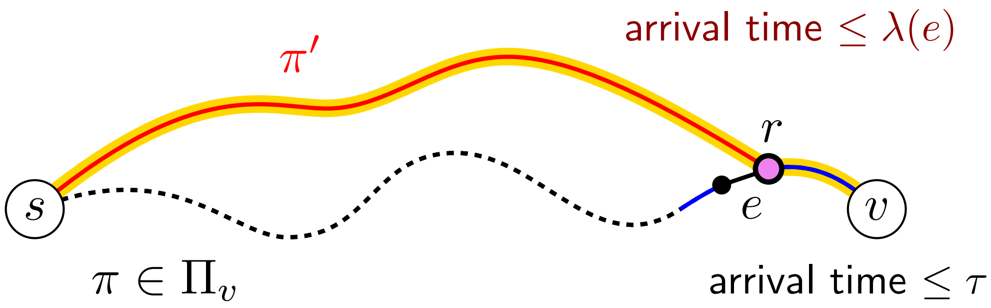

$$|\pi'| \leq (1+\varepsilon) \cdot \text{dist}_G^{\leq \lambda(e)}(s, r) \leq (1+\varepsilon) \cdot |\pi[s:r]|$$

$$\text{dist}_H^{\leq \tau}(s, v) \leq (1+\varepsilon) \cdot |\pi[s:r]| + |\pi[r:v]|$$

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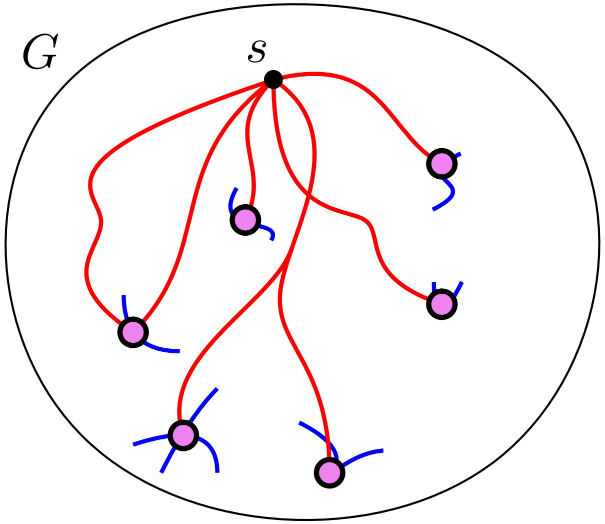

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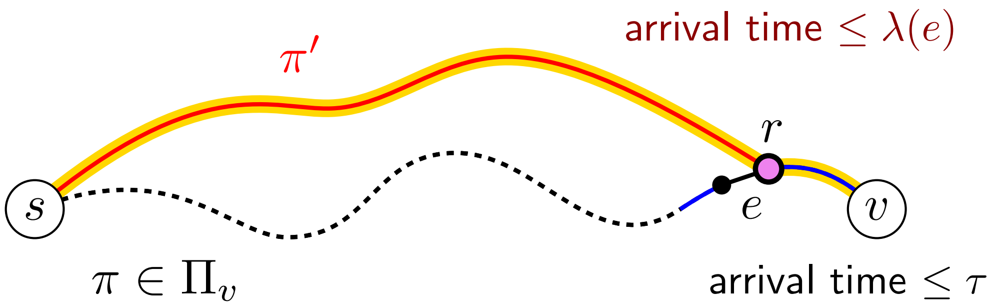

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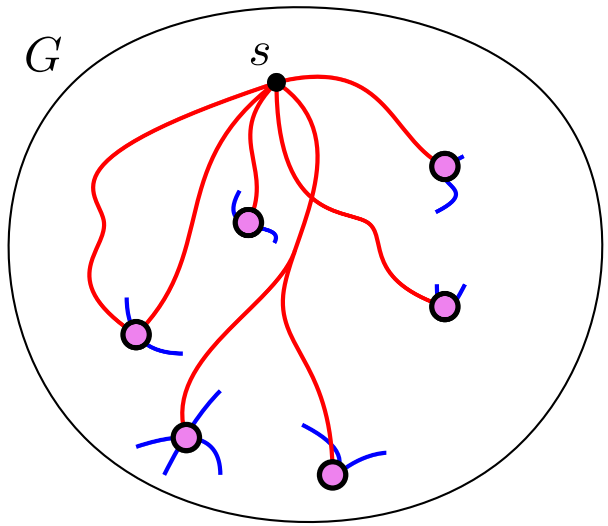

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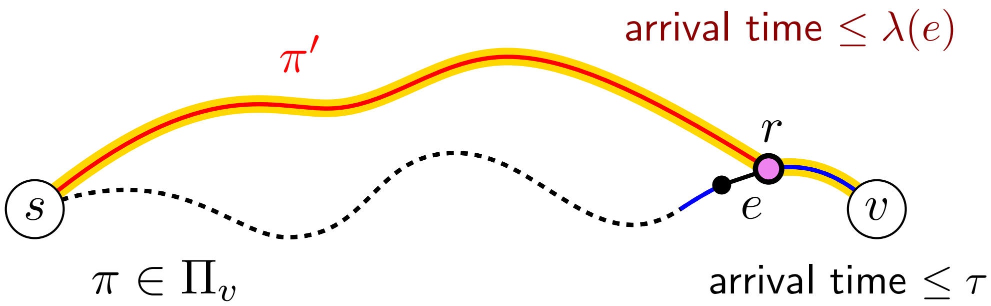
Stretch factor?



$$|\pi'| \leq (1+\varepsilon) \cdot \mathsf{dist}_G^{\leq \lambda(e)}(s,r) \leq (1+\varepsilon) \cdot |\pi[s:r]|$$

$$\begin{aligned} \mathsf{dist}_H^{\leqslant \tau}(s,v) &\leq (1+\varepsilon) \cdot |\pi[s:r]| + |\pi[r:v]| \\ &\leq (1+\varepsilon)|\pi| \\ &\leq (1+\varepsilon)(1+\varepsilon) \cdot \mathsf{dist}_G^{\leqslant \tau}(s,t) \end{aligned}$$

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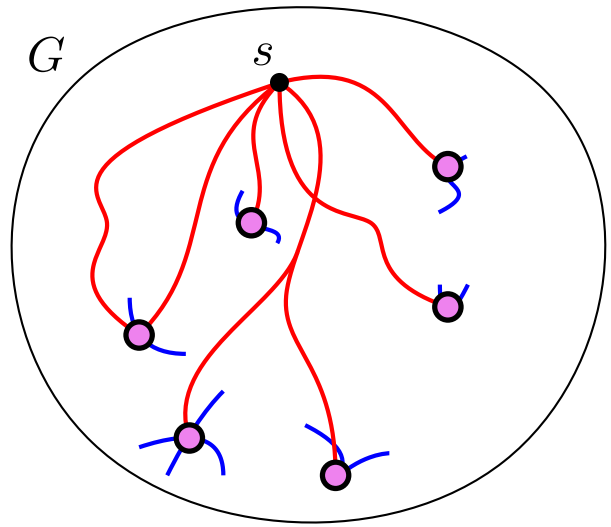

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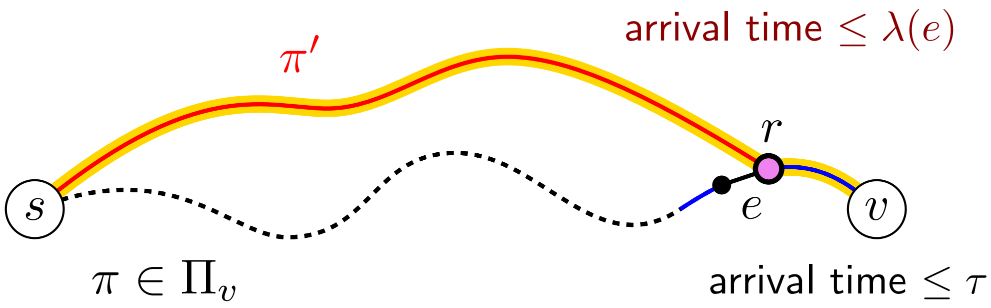

H is a single-source temporal spanner with stretch $(1+\varepsilon)^2$ and size $\widetilde{O}(n^{3/2})$

Open Problems

Open Problems

Better Size-Stretch trade-offs

Temporal 3-Spanner on Cliques

$\widetilde{O}(n)$	vs.	$\Omega(n^{1+\varepsilon})$ for some $\varepsilon > 0$

Open Problems

Better Size-Stretch trade-offs

Temporal 3-Spanner on Cliques

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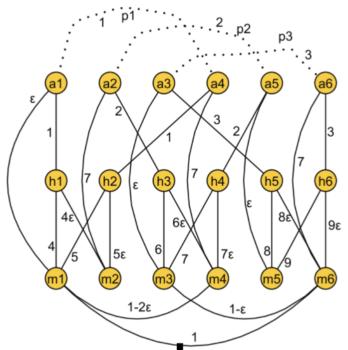
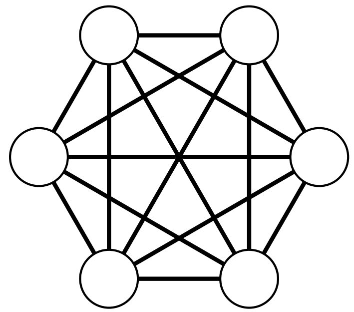
Beyond Temporal Cliques

$\Omega(n^2)$
temporal
connectivity





$\widetilde{O}(n)$
temporal
connectivity

$\widetilde{O}(n^{1+1/k})$
stretch
$2k-1$

(dense)
General Graphs

Cliques

Open Problems

Better Size-Stretch trade-offs

Temporal 3-Spanner on Cliques

$\widetilde{O}(n)$

vs.

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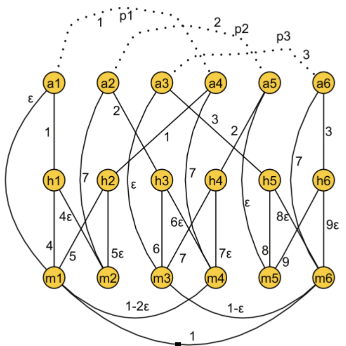
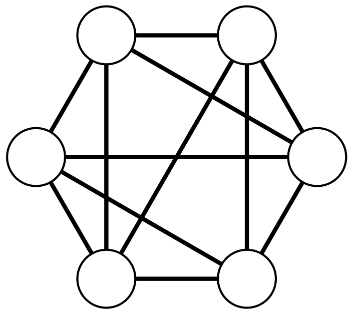
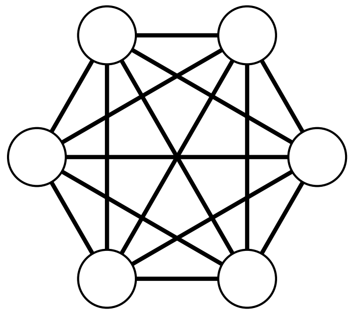
Beyond Temporal Cliques

$\Omega(n^2)$ temporal connectivity







$\widetilde{O}(n)$ temporal connectivity

$\widetilde{O}(n^{1+1/k})$ stretch $2k-1$

(dense) General Graphs

$(n-2)$-regular graphs

Cliques

Open Problems

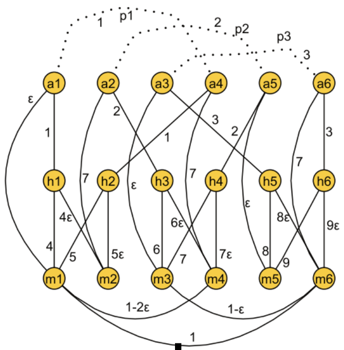
Better Size-Stretch trade-offs

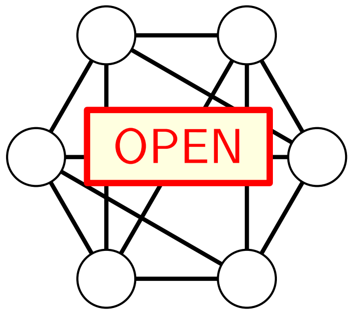
Temporal 3-Spanner on Cliques

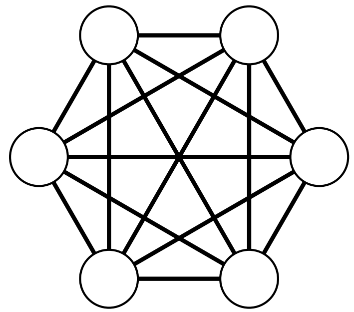
$\widetilde{O}(n)$ vs. $\Omega(n^{1+\varepsilon})$ for some $\varepsilon > 0$

Beyond Temporal Cliques

$\Omega(n^2)$ temporal connectivity







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Cliques

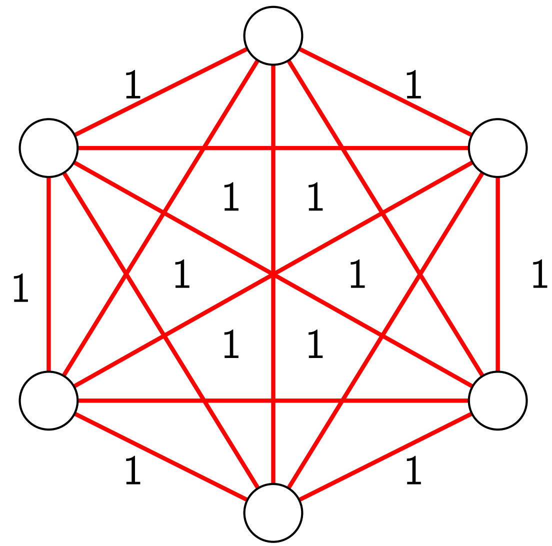
Variants and Extensions

Variants and Extensions

Strict Paths

A **temporal path** from u to v is a path from u to v in which the traversed edges have ~~non-decreasing~~ time-labels

increasing



Variants and Extensions

Strict Paths

A **temporal path** from u to v is a path from u to v in which the traversed edges have ~~non-decreasing~~ time-labels

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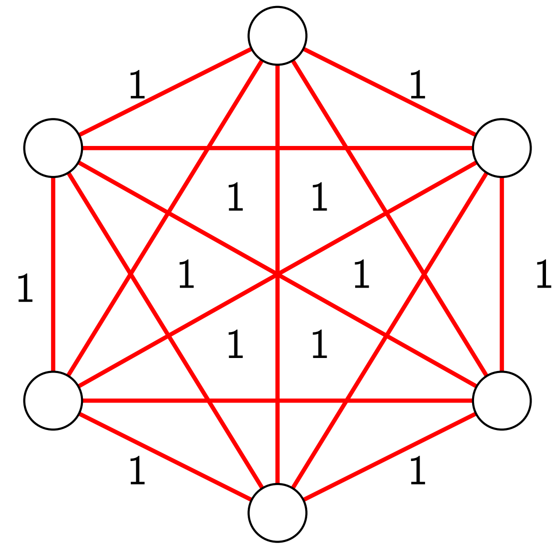
Trivial Lower-bound of $\Omega(n^2)$ on cliques



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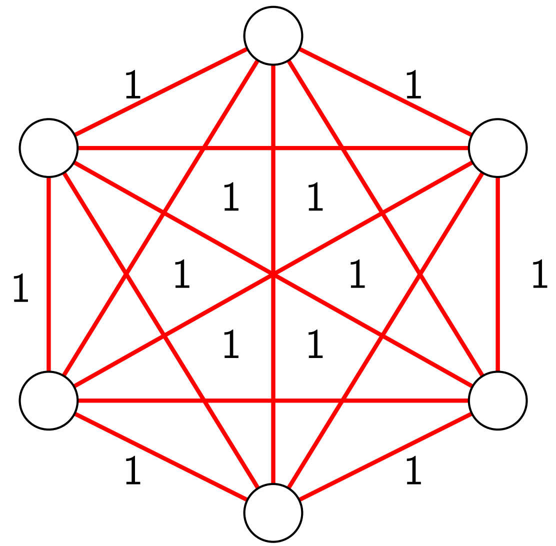
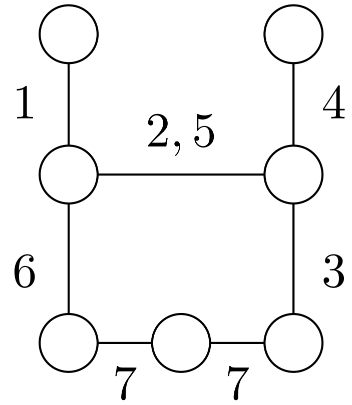


Trivial Lower-bound of $\Omega(n^2)$ on cliques

Multiple Time-Labels

Edges can have more than one time label and a path can use **any** of them



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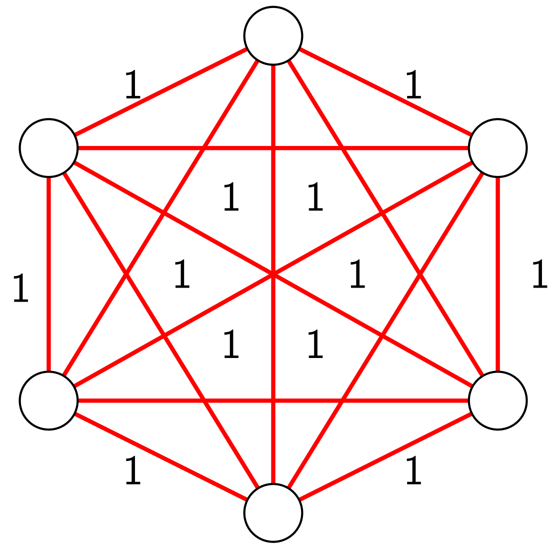

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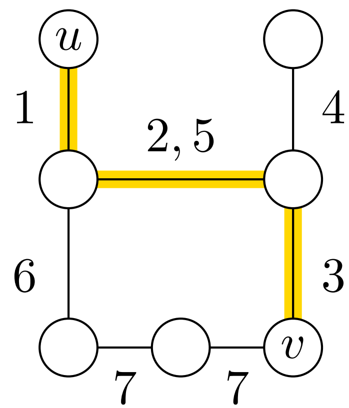

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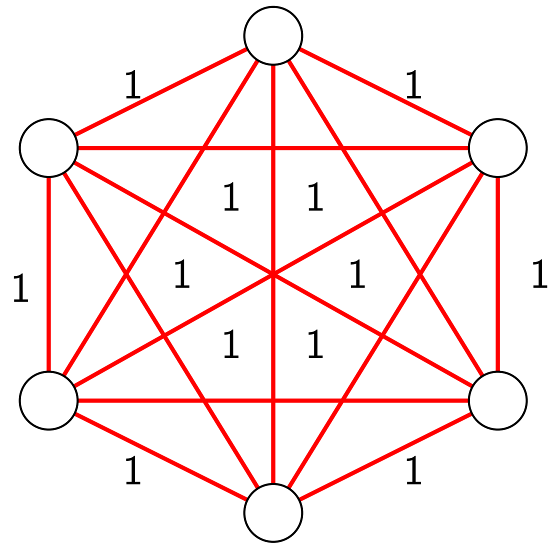

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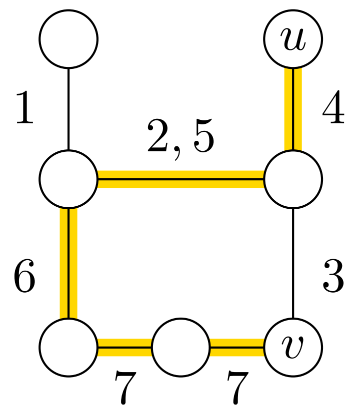

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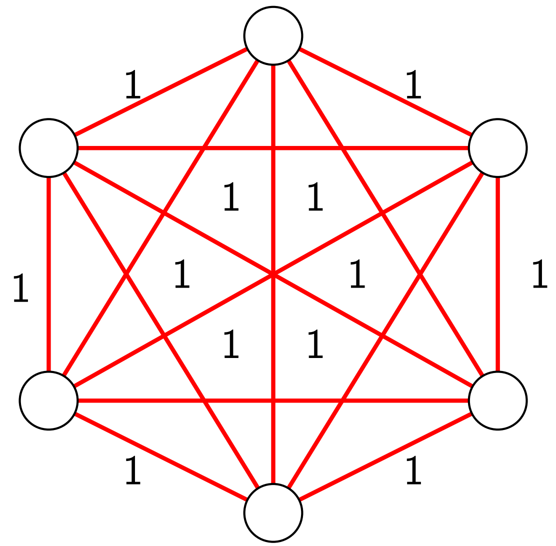

Trivial Lower-bound of $\Omega(n^2)$ on cliques

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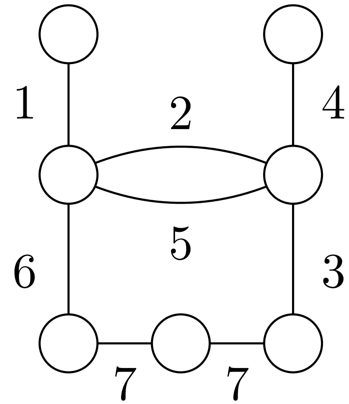
Edges can have more than one time label and a path can use **any** of them

Can be simulated with parallel edges



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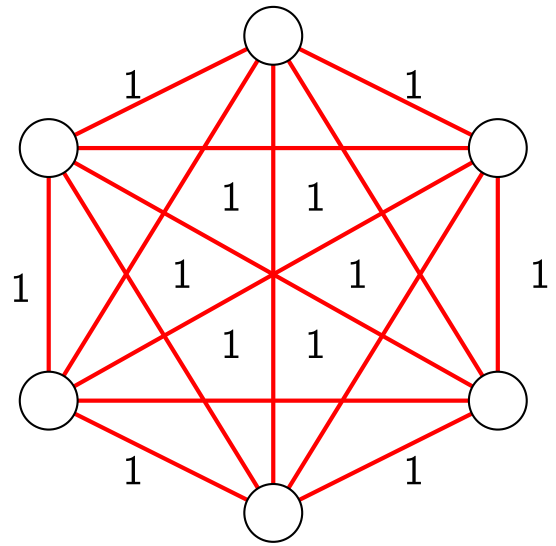

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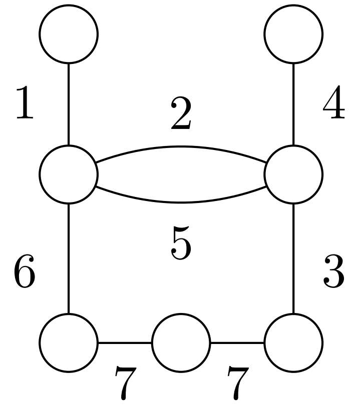


Can be simulated with parallel edges

The results in this talk also hold for temporal graphs with multiple time-labels

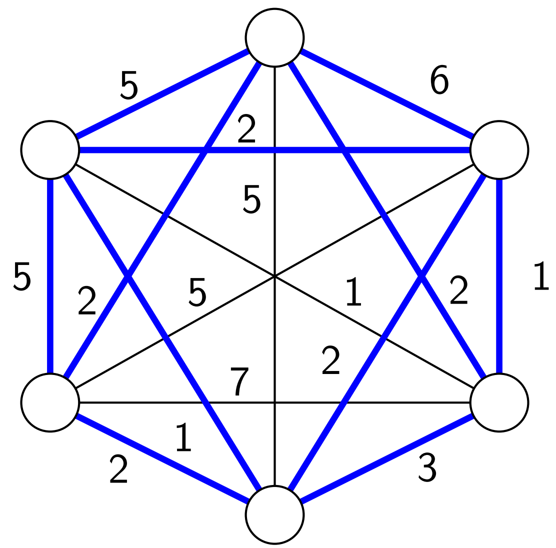
Variants and Extensions

Edge Failures

A temporal f-**edge fault-tolerant** spanner with stretch α of G is a subgraph H such that, for every set $F \subseteq E(G)$ of at most f edges:

$$\mathsf{dist}^{\leqslant \tau}_{H-F}(u,v) \leq \alpha \cdot \mathsf{dist}^{\leqslant \tau}_{G-F}(u,v) \qquad \forall u,v$$

H remains a temporal spanner with stretch α even when up to f edges fail.



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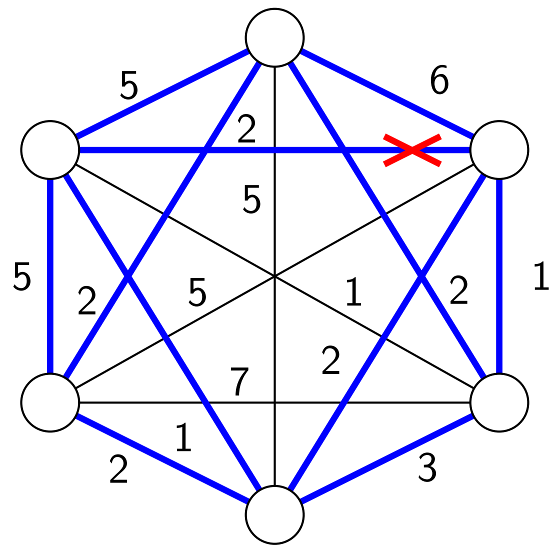

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	1-EFT clique			1-EFT		2-EFT clique		2-EFT	3-EFT clique
	single-pair	single-source	all-pairs	single-pair	single-source	single-pair	single-source	single-pair	single-pair
temporal spanner	$\uparrow$	$O(n)$		$O(n)$	$\Omega(n^2)$	$O(n)$	$\Omega(n^2)$	$\Omega(n^2)$ $\Rightarrow$	
temporal $(1+\varepsilon)$-spanner	$\uparrow$			$O\left(\frac{n\log^4 n}{\log(1+\varepsilon)}\right)$	$\downarrow$	$O\left(\frac{n\log^4 n}{\log(1+\varepsilon)}\right)$	$\downarrow$	$\downarrow$	$\downarrow$
temporal 1-spanner	$O(n)$	$\Omega(n^2)$ $\rightarrow$		$\Omega(n^2)$	$\downarrow$	$\Omega(n^2)$	$\downarrow$	$\downarrow$	$\downarrow$

= sparse (almost linear)   = dense (quadratic)

[ALGOSENSORS 2022]

Variants and Extensions

Edge Failures

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	1-EFT clique single-pair	1-EFT clique single-source	1-EFT clique all-pairs	1-EFT single-pair	1-EFT single-source	2-EFT clique single-pair	2-EFT clique single-source	2-EFT single-pair	3-EFT clique single-pair
temporal spanner	↑	$O(n)$	OPEN	$O(n)$	$\Omega(n^2)$	$O(n)$	$\Omega(n^2)$	$\Omega(n^2)$ ⟹	
temporal $(1+\varepsilon)$-spanner	↑			$O\left(\frac{n\log^4 n}{\log(1+\varepsilon)}\right)$	↓	$O\left(\frac{n\log^4 n}{\log(1+\varepsilon)}\right)$	↓	↓	↓
temporal 1-spanner	$O(n)$	$\Omega(n^2)$ →		$\Omega(n^2)$	↓	$\Omega(n^2)$	↓	↓	↓

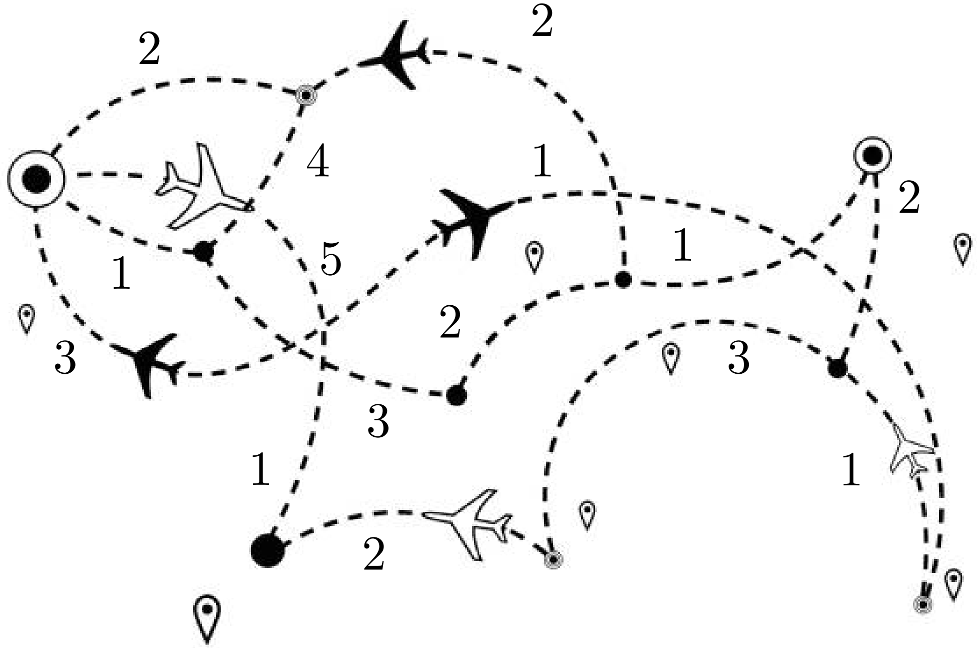
= sparse (almost linear) = dense (quadratic)

[ALGOSENSORS 2022]

Variants and Extensions

Blackout Failures

All edges with the same time-label fail simultaneously



Variants and Extensions

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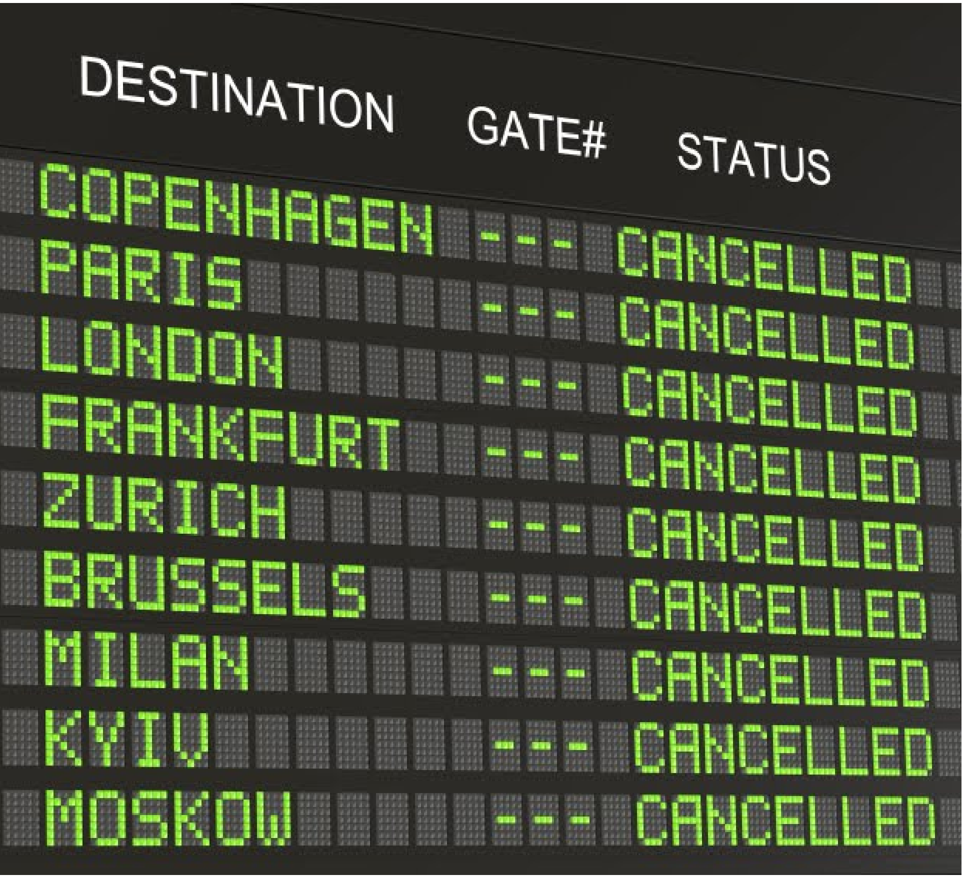

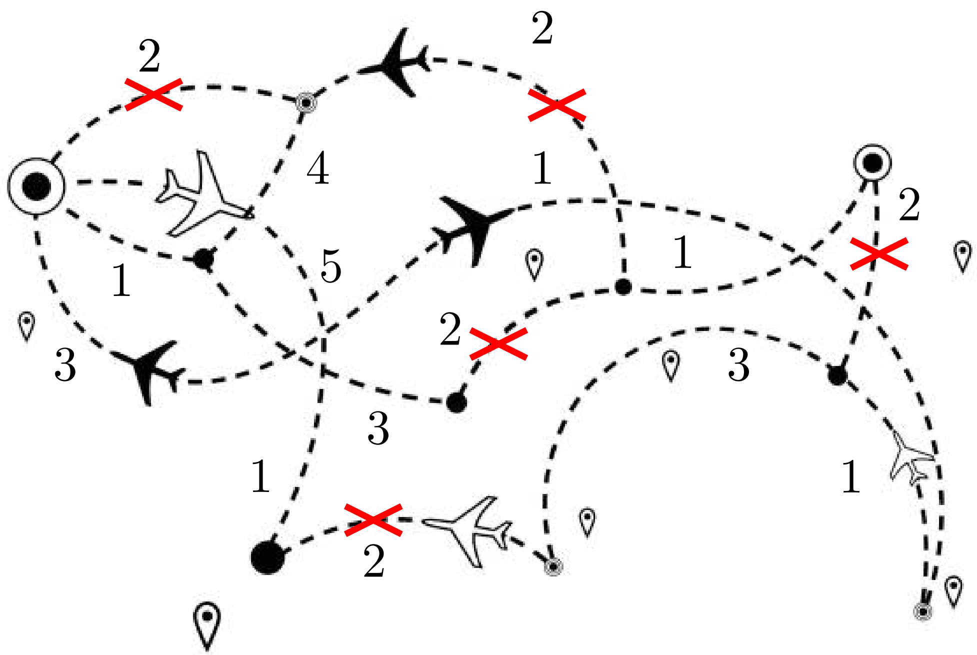

Variants and Extensions

Blackout Failures

All edges with the same time-label fail simultaneously

	1-blackout temporal clique			1-blackout	2-blackout t. clique	2-blackout	3-blackout t. clique
	single pair	single source	all pairs	single pair	single pair	single pair	single pair
Reachability	$\uparrow$	$\boldsymbol{\Omega(\frac{n^2}{\log^2 n})}$	$\Omega(n^2)$	$\boldsymbol{O(n)}$	$\rightarrow$	$\boldsymbol{\Omega(\frac{n^2}{\log^2 n})}$	$\rightarrow$
$(1+\varepsilon)$-apx distances	$\uparrow$	$\downarrow$	$\downarrow$	$\boldsymbol{O(\frac{n \log^4 n}{\log(1+\varepsilon)})}$	$\rightarrow$	$\downarrow$	$\downarrow$
exact distances	$O(n)$	$\Omega(n^2)$	$\downarrow$	$\boldsymbol{\Omega(\frac{n^2}{\log^2 n})}$	$\rightarrow$	$\downarrow$	$\downarrow$

= sparse (almost linear) = dense (almost quadratic) [ALGOSENSORS 2022]

Thank you!