Biased opinion dynamics: when the devil is in the details

Based on

Aris Anagnostopoulos [a] - **Luca Becchetti** [a] – Emilio Cruciani [b] - Francesco Pasquale [c] - Sara Rizzo. Biased opinion dynamics: when the devil is in the details. Proc. of IJCAI 2020

Luca Becchetti [a], Vincenzo Bonifaci [d], Emilio Cruciani [b], Francesco Pasquale [c]. On a Voter Model with Context-Dependent Opinion Adoption. Proc. of IJCAI 2023

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Motivation and summary

Motivation

1970's

Technological status quo: typewriters

Motivation

1970’s

Technological status quo:
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1980’s

Techonologically
superior alternative:
personal computers

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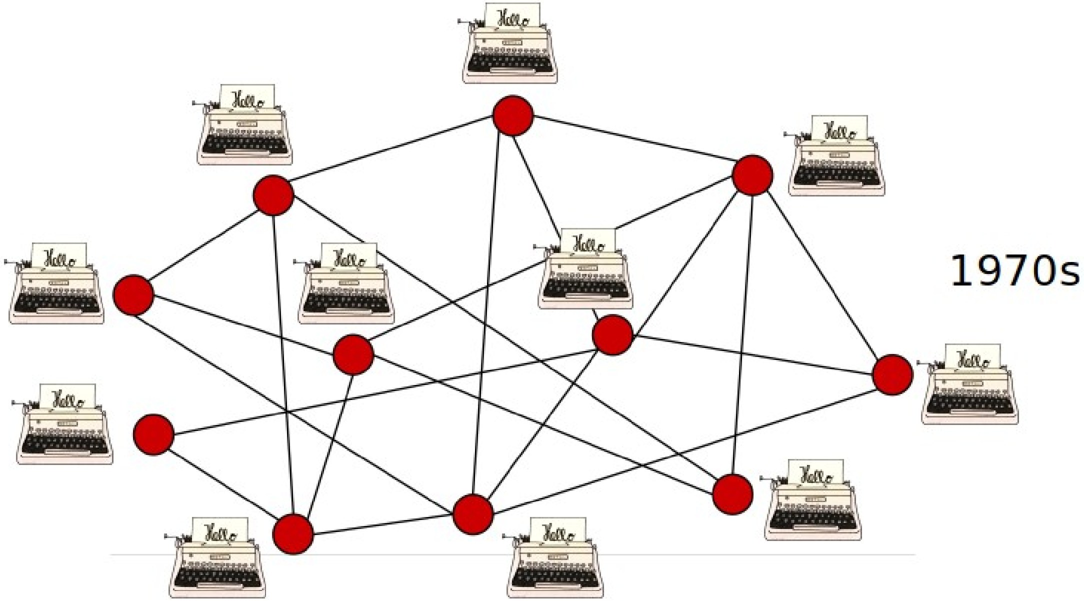


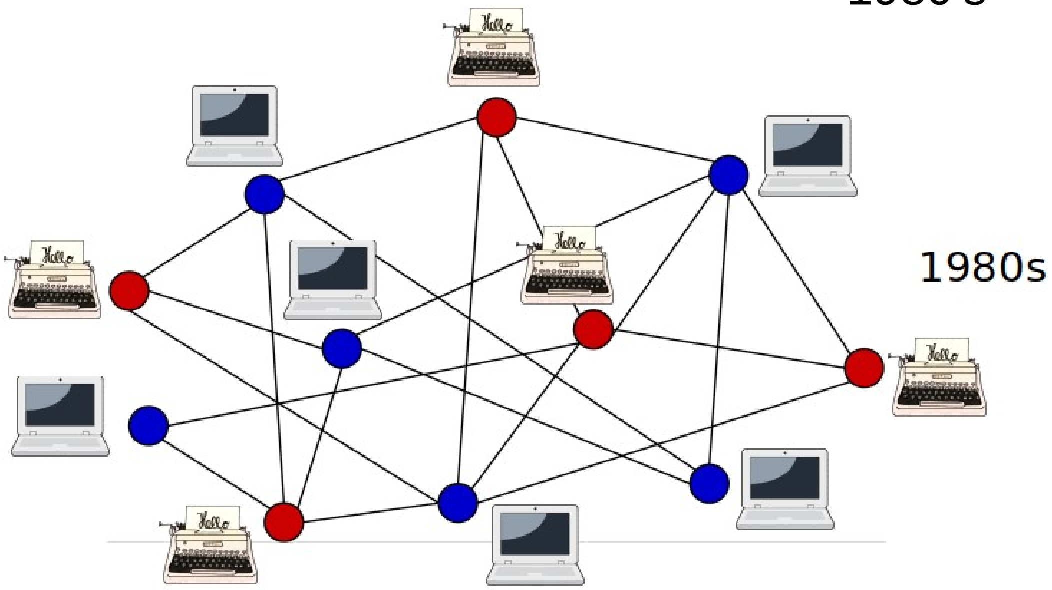
Motivation

1970’s

Technological status quo: typewriters

1980’s

Techonologically superior alternative: personal computers



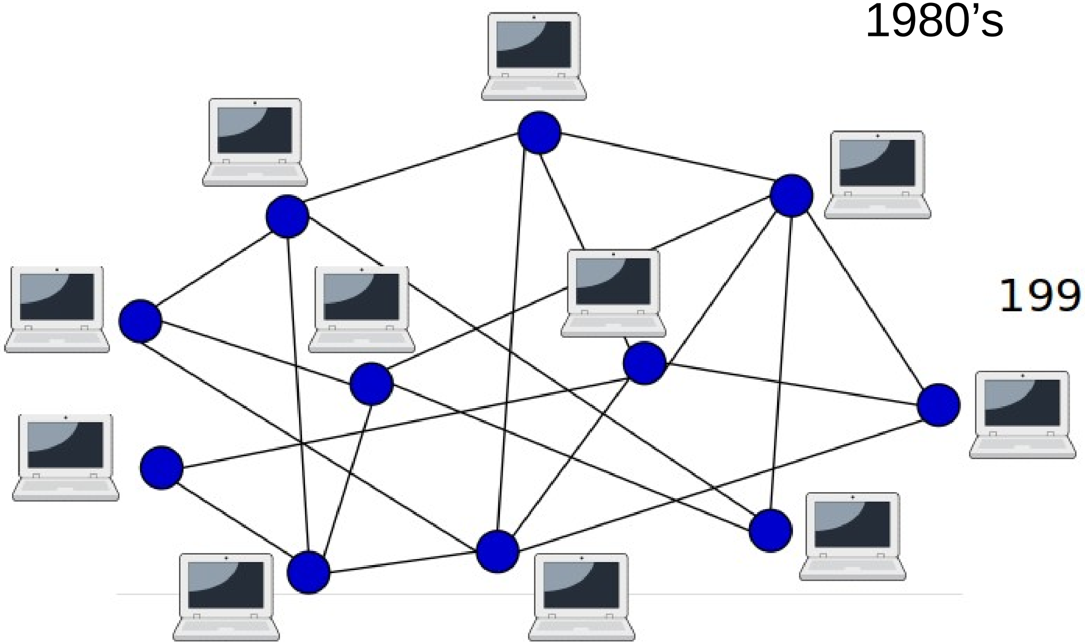
Motivation

1970’s

Technological status quo: typewriters

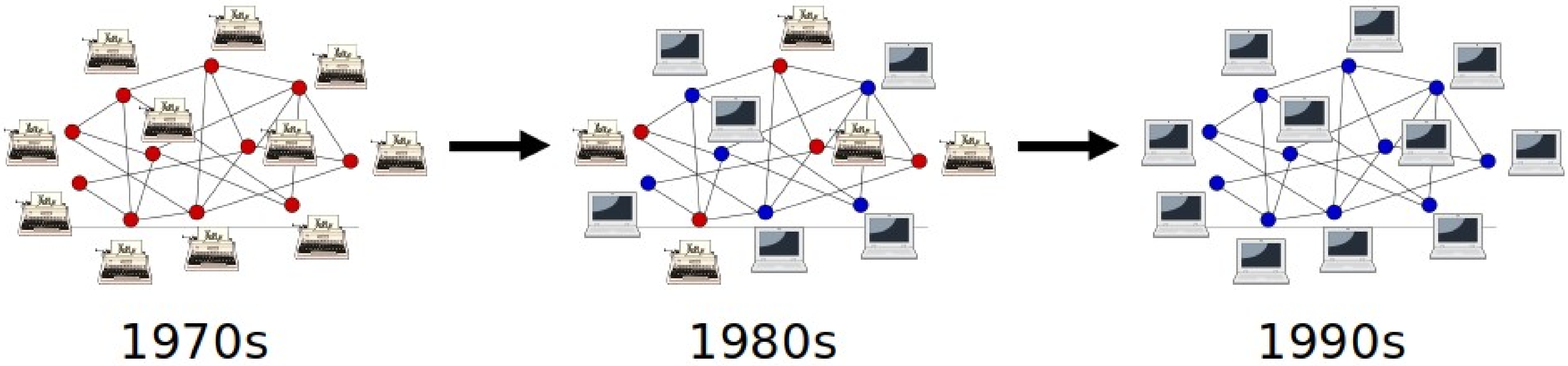
1980’s

Techonologically superior alternative: personal computers



1990s

Motivation



How does the combination of network structure and opinion dynamics affect convergence to *global adoption* of a *superior alternative*?

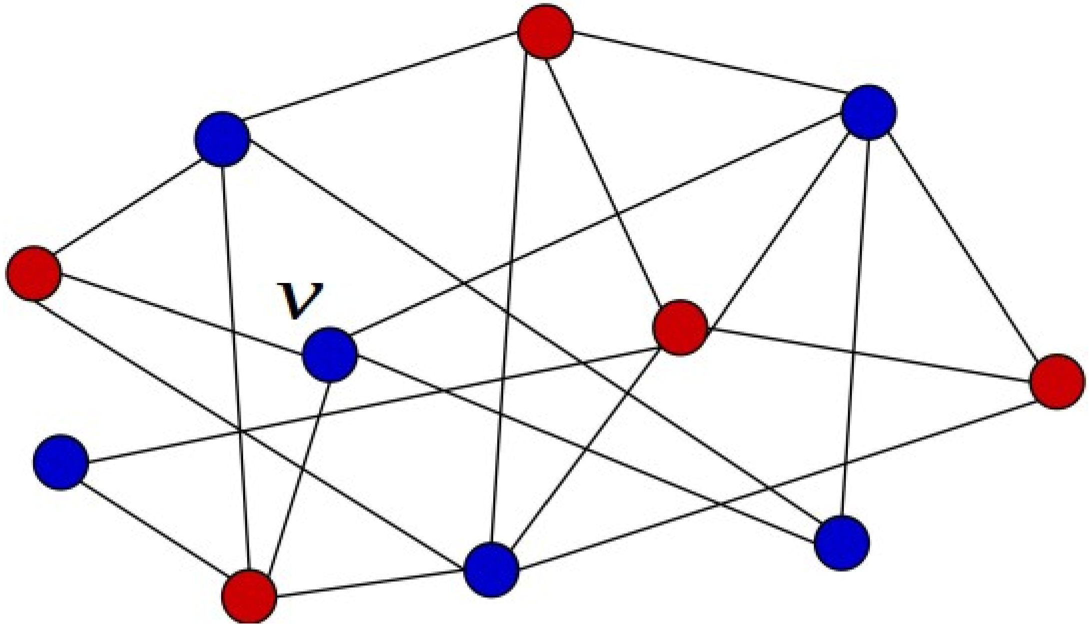
Qualitatively speaking: any recurring patterns?

Summary

- Goal: investigate interplay between
  - Network structure
  - Opinion dynamics
  - Bias
- Main message(s)
  - First part
    - Different opinion dynamics can result in radically different behavior
    - Also *qualitatively* speaking and in very simple settings
  - Second part
    - Seemingly minor changes to opinion update rule can greatly affect behavior
    - May require adoption of different techniques of analysis

Part 1: as simple as it can be

Model

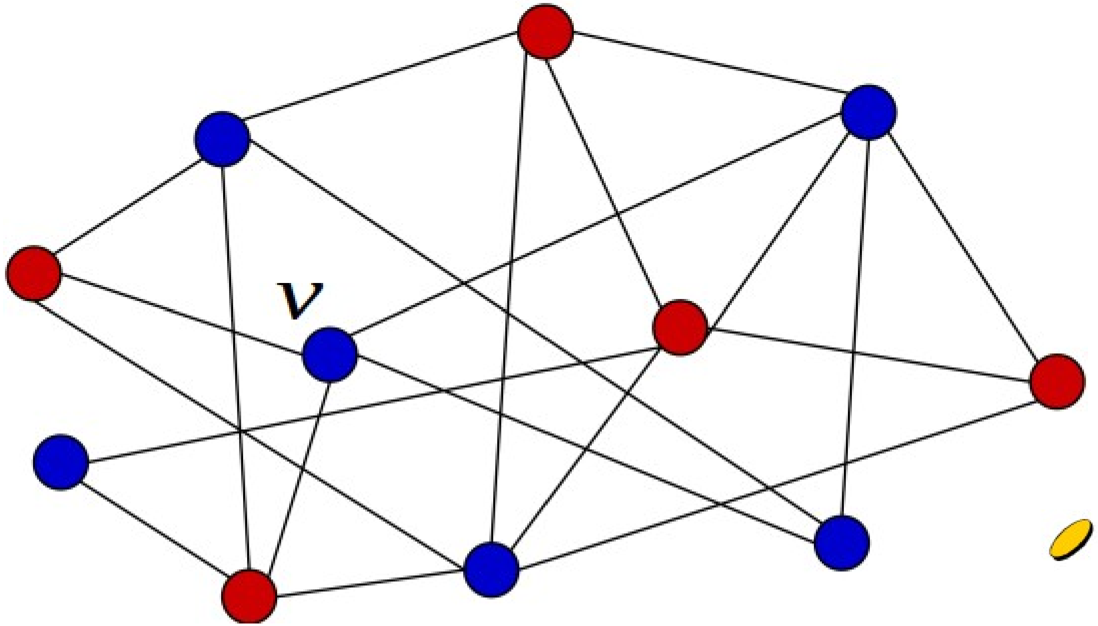


In every round:

- node v sampled u.a.r.
- v updates its opinion

Each node has an opinion 0/1

Model



Each node has an opinion **0**/**1**

In every round t:

node v sampled u.a.r.

v updates its opinion

<u>*HEADS*</u> (prob. α)

$$x_v^{(t)} = 1$$

<u>*TAILS*</u> (prob. 1 - α)

$$x_v^{(t)} = f_G(v, \mathbf{x})$$

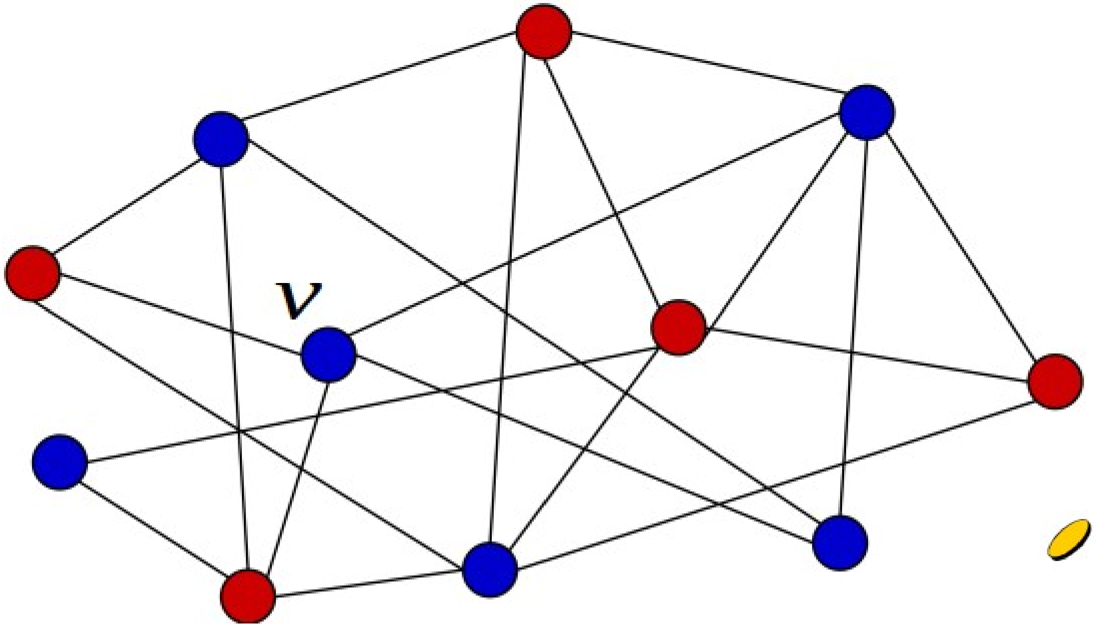
function of the *opinions* of v's *neighbors* in the previous round

This paper:
Majority Voter

Pros and cons

- Cons
  - Simplistic model $\Rightarrow$ 0 is “doomed” from the start
  - More sophisticated models proposed (few analyzed)
    - E.g., [Montanari, Saberi 2010]: best-response dynamics, payoff depends on degree of agreement, bias present
- Pros
  - Bias decoupled from opinion dynamics
  - Highlight interplay between opinion dynamics and network structure
  - Plug & play: plug in a new opinion dynamics, analyze resulting biased model
    - Benchmark to investigate interplay between network structure and opinion dynamics in the presence of bias
  - Still rich enough to highlight interesting phenomena (some in accordance with previous work)

Biased Majority



Each node has an opinion 0/1

In every round:

node v sampled u.a.r.

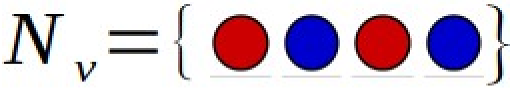
v updates its opinion

HEADS (prob. α)

$$x_v^{(t)} = 1$$

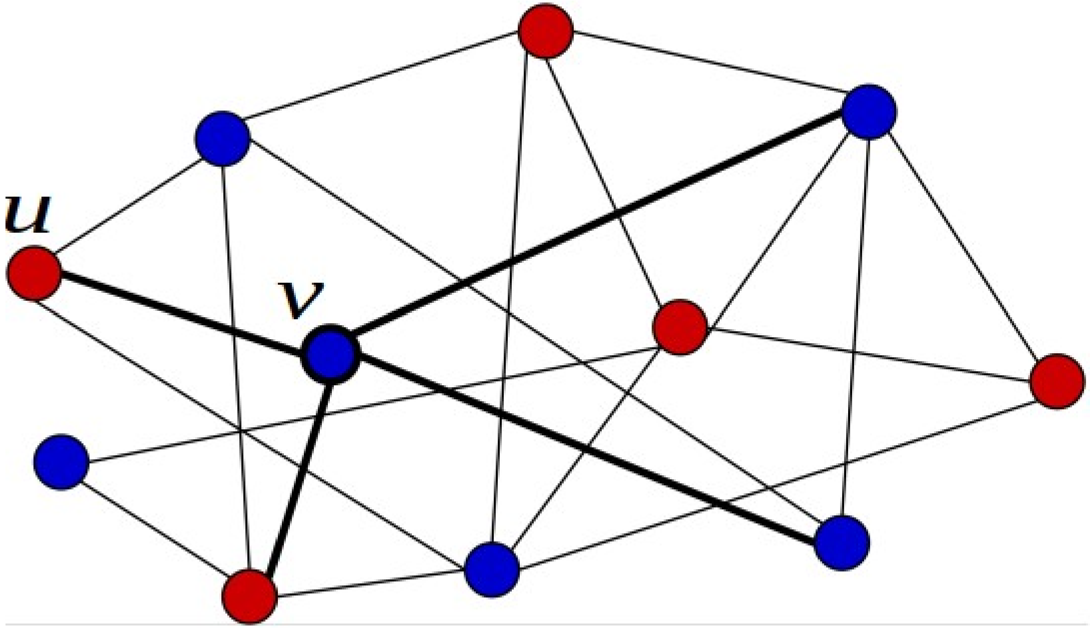
TAILS (prob. 1 - α)

Most frequent opinion among v’s *neighbors* in the previous round, TIES broken u.a.r.



$$x_v^{(t)} = 0$$

Biased Voter



Each node has an opinion 0/1

In every round:

node v sampled u.a.r.

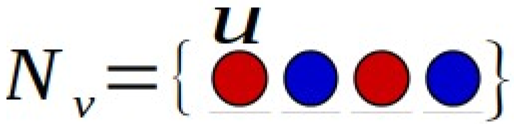
v updates its opinion

HEADS (prob. α)

$$x_v^{(t)} = 1$$

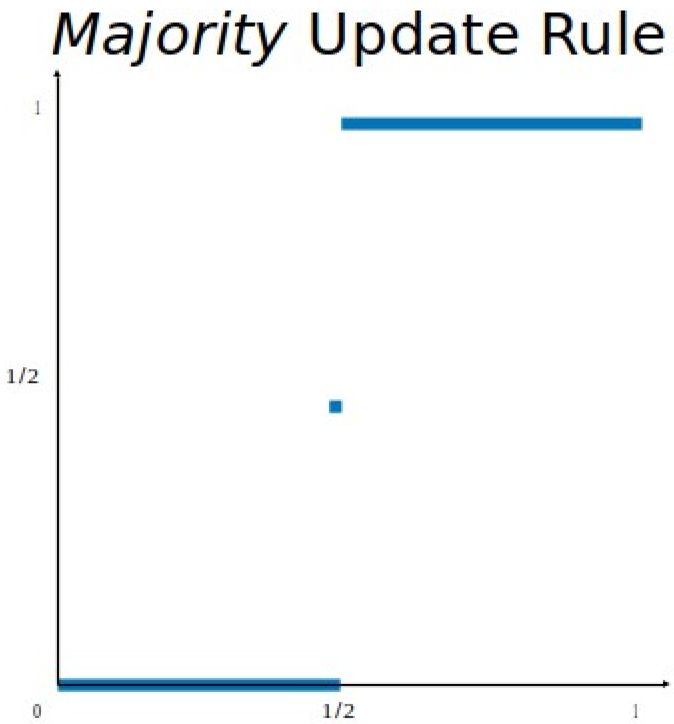
TAILS (prob. 1 - α)

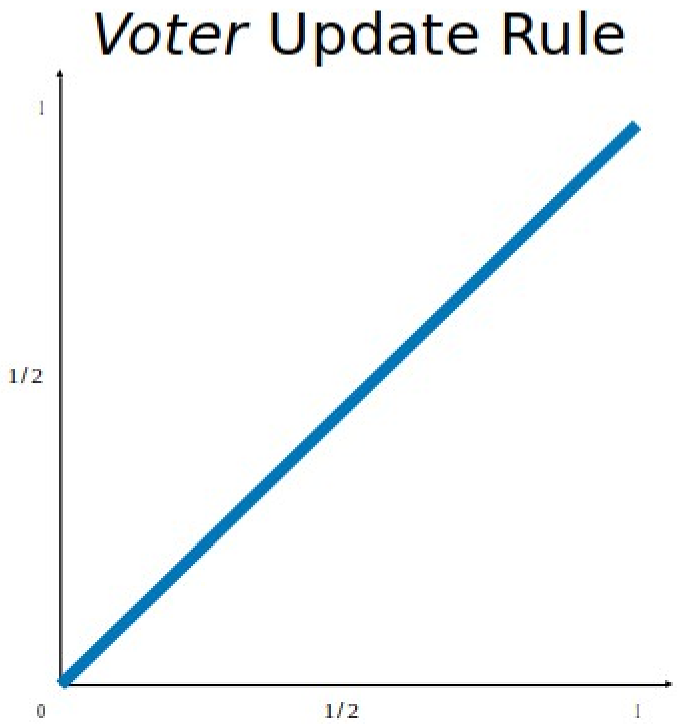
Copy opinion of neighbor sampled u.a.r.



$$x_v^{(t)} = x_u^{(t-1)} = 0$$

Majority vs Voter

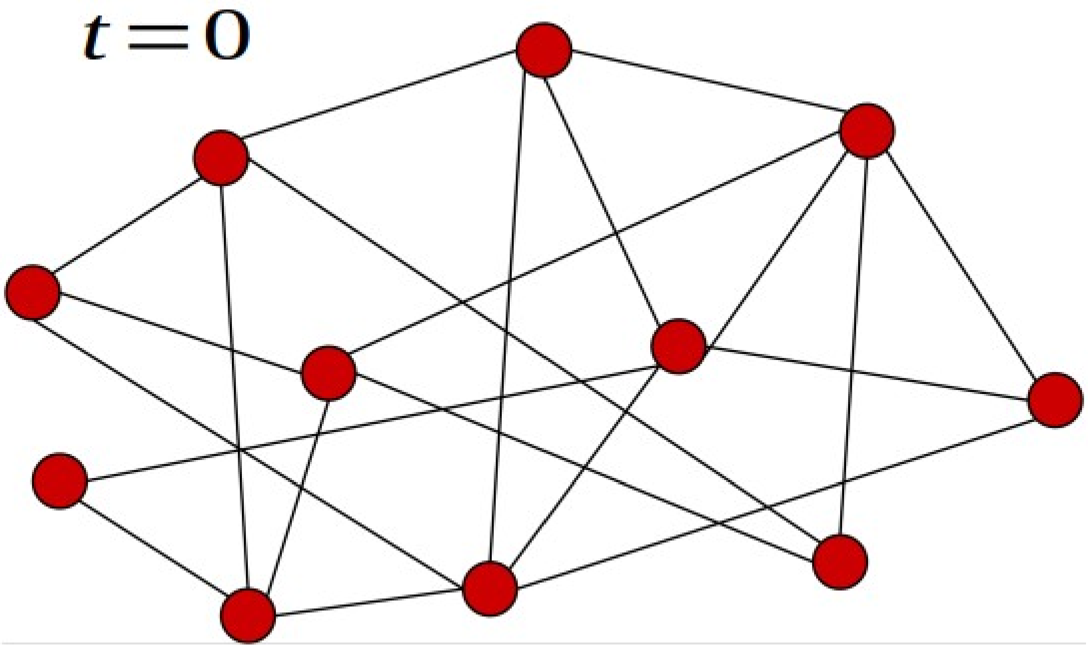




X-axis: fraction of neighbors with given opinion

Y-axis: probability of adoption

Evolution of the process



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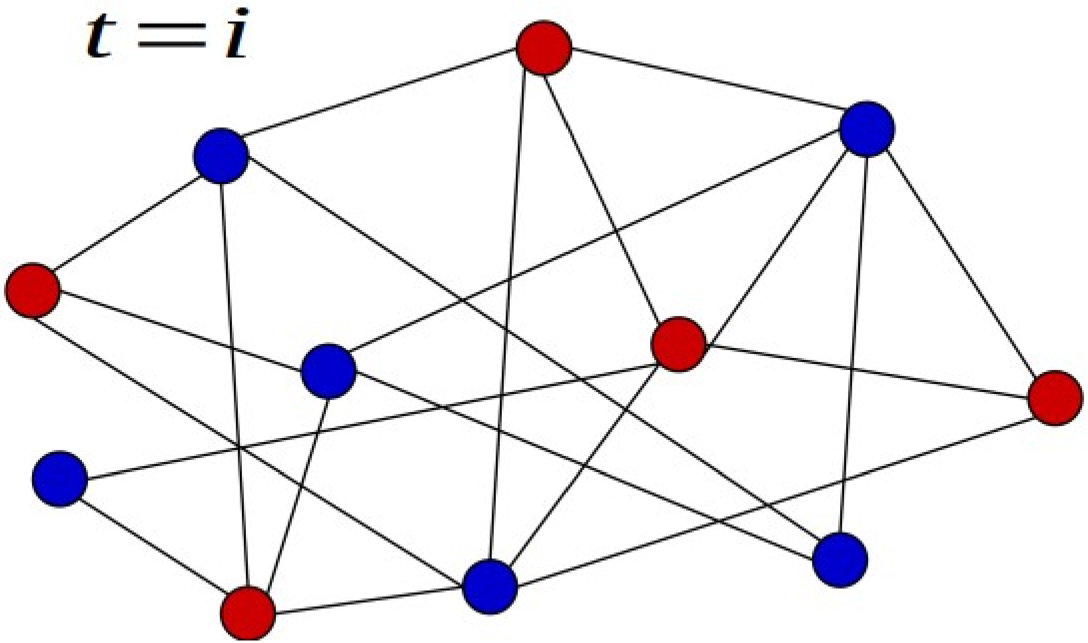

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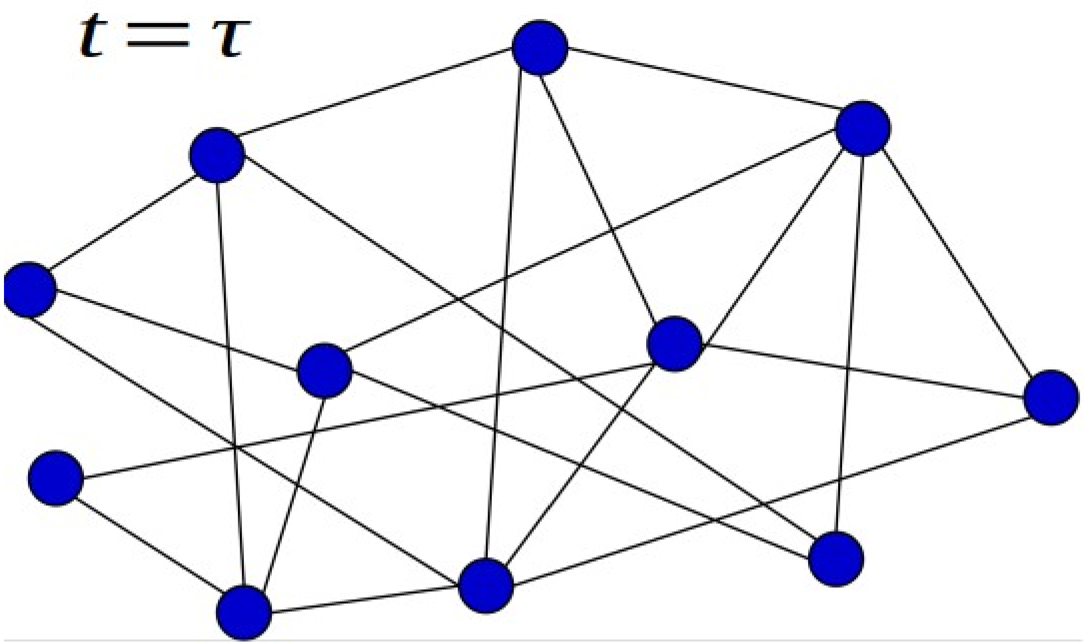
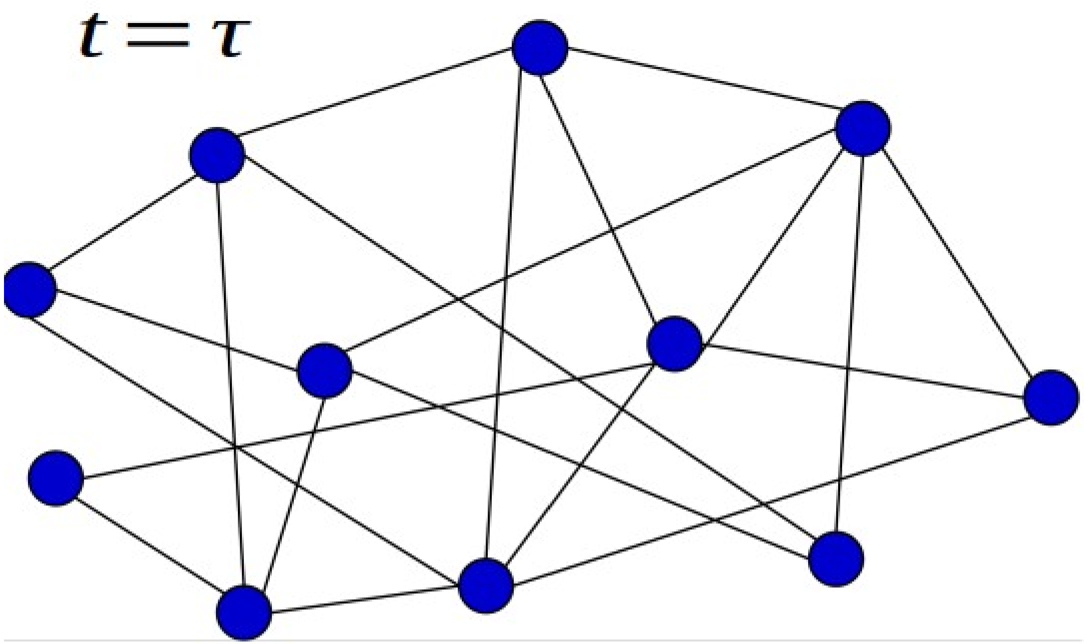


Evolution of the process



The process is a Markov Chain with a *single absorbing state* corresponding to consensus on the 1 opinion

Questions

- What is the expected time to reach the absorbing state (consensus on 1)?
- High probability (?)
- How does expected time to consensus depend on network structure and bias?

Overview of results

Biased majority

Lower bound on absorption time τ

$\mathbb{E}[\tau] = \Omega(e^{\epsilon^2 \delta_{\min}}/n)$ for $\alpha \leq (1-\epsilon)/2$

Note that $\delta_{\min} = \omega(\log n) \Rightarrow \mathbb{E}[\tau] = n^{\omega(1)}$

Biased majority

Upper bounds for specific graph families

Cycles: $\tau = \mathrm{O}(\alpha^{-1} n \log n)$ w.h.p.

$\mathrm{O}(\log n)$-degree trees: $\mathbb{E}[\tau] = \mathrm{O}(n^c)$

$\mathrm{O}(\log n)$ disc. cliques: $\mathbb{E}[\tau] = \mathrm{O}(n^c)$

Biased majority

Upper bound on absorption time τ

$\tau = \mathrm{O}(n \log n)$ w.h.p.

Holds for $\alpha \geq 1/2$ and $\delta_{\min} = \omega(\log n)$

Phase transition for $\alpha = 1/2$

Biased voter

Upper bound on absorption time τ

$\tau = \mathrm{O}(\alpha^{-1} n \log n)$ w.h.p.

Note that $\alpha = \Theta(n^{-s}) \Rightarrow \tau = \mathrm{O}(n^{s+1} \log n)$

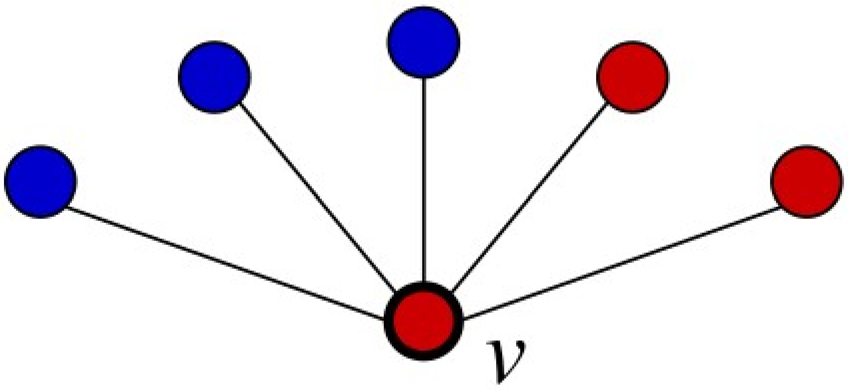
Majority: Lower bound (1)

Consider first time $\hat{\tau}$ in which $\exists$ v $\in$ V with at least half of its neighbors in state 1

$\hat{\tau}$ is a stopping time

Lower bound on $\hat{\tau}$ implies lower bound on τ

Majority **never** applies for t < $\hat{\tau}$



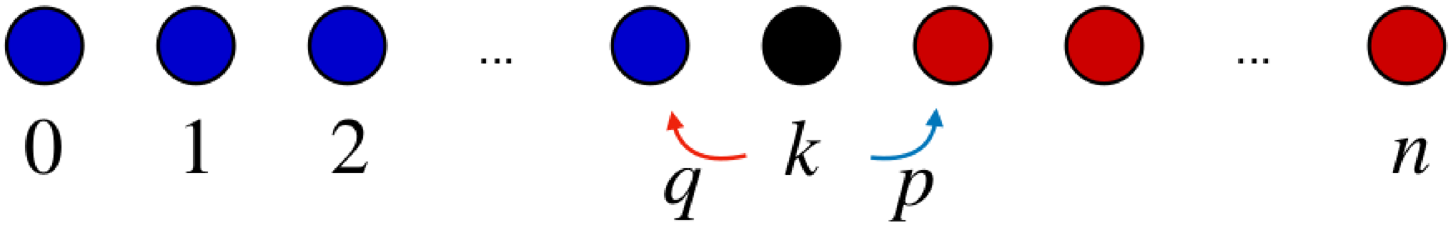
TAILS (probability $1 - \alpha$):

$$x_v^{(t)} = 0$$

Majority of N_v is always in state 0

Majority: Lower bound (2)

Known result on Birth-and-Death Markov Chains:

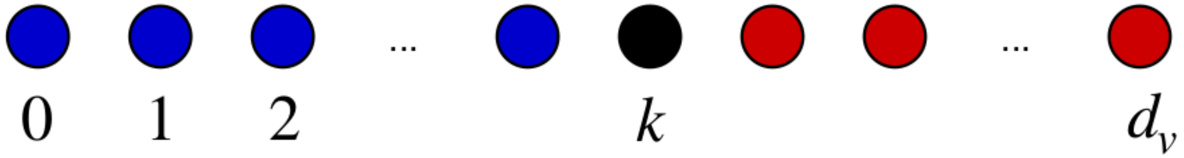


$$\Pr(\tau_n < \tau_0) \leq (p/q)^{n-k}$$

where τ_i is the first time such that the chain is in state i

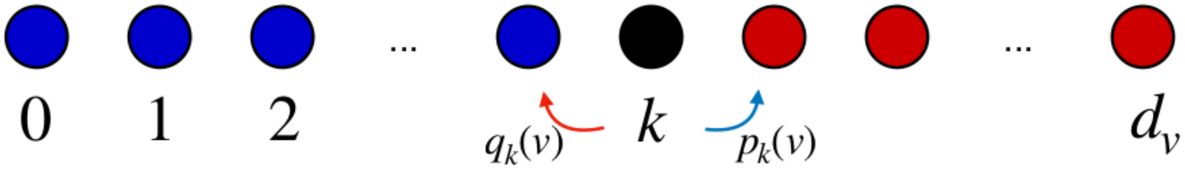
Majority: Lower bound (3)

Number of neighbors k in state 1 for node v



Majority: Lower bound (3)

Number of neighbors k in state 1 for node v



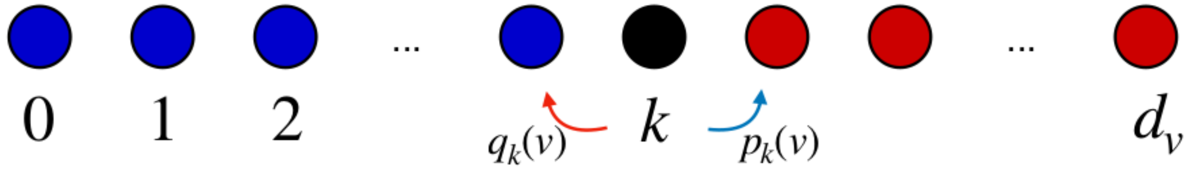
$$\boxed{q_k(v) = \frac{k}{n}(1-\alpha)}$$

$$\boxed{p_k(v) = \frac{d_v - k}{n}\alpha}$$

Before stopping time $\hat{\tau}$

Majority: Lower bound (3)

Number of neighbors k in state 1 for node v



$$q_k(v) = \frac{k}{n}(1-\alpha)$$

$$p_k(v) = \frac{d_v - k}{n}\alpha$$

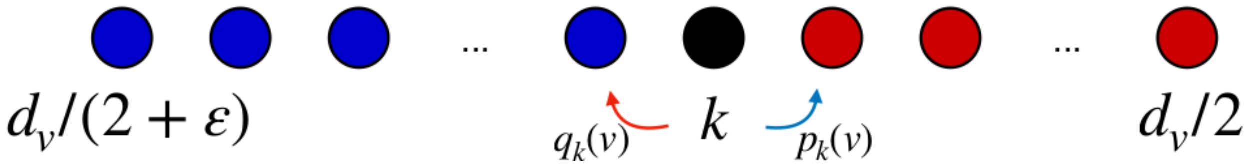
$\forall v$ and $\forall k \geq d_v/(2+\varepsilon)$

$$\frac{p_k(v)}{q_k(v)} \leq 1 - \varepsilon$$

since $\alpha \leqslant (1-\varepsilon)/2$

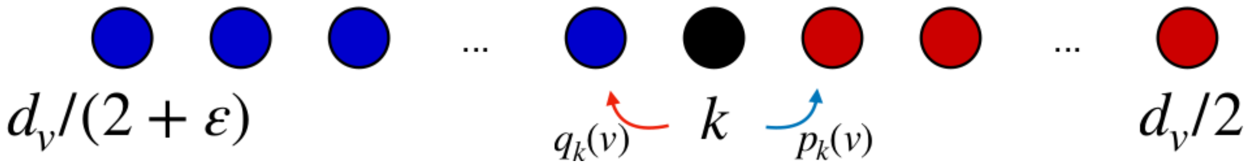
Majority: Lower bound (4)

Consider k in the range $(d_v/(2+\varepsilon),\ d_v/2)$



Majority: Lower bound (5)

Consider k in the range $(d_v/(2+\varepsilon),\, d_v/2)$

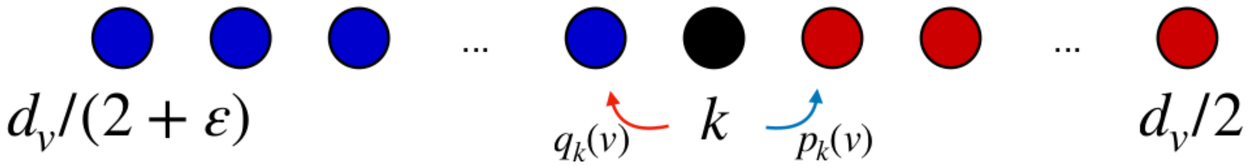


Application of the known result from Birth-and-Death Markov Chains:

$$\Pr(\tau_n < \tau_0) \le (p/q)^{n-k} \implies \boxed{\Pr(\tau_{d_v/2} < \tau_{d_v/(2+\varepsilon)}) = O(e^{-\varepsilon^2 \delta_{\min}})}$$

Majority: Lower bound (6)

Consider k in the range $(d_v/(2+\varepsilon),\ d_v/2)$



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Note that this implies $\mathrm{E}[\tau] = \Omega(e^{\epsilon^2\delta_{\min}}/n) = n^{\omega(1)}$ when $\delta_{\min} = \omega(\log n)$

Biased majority – a closer look

- Dense networks (min-degree = $\omega(\log n)$)
  - Expected convergence time becomes super-polynomial when $\alpha < ½ – \epsilon$
  - It is $O(n \log n)$ for $\alpha > ½ + \epsilon$
  - Phase transition for $\alpha = ½$
- Special case of more general phenomenon
  - Simulations show non-obvious dependence of expected absorption time from min-degree and bias
  - Q.[Anagnostopoulos et al. 2020]: does “low” degree afford polynomial convergence? [Lesfari et al. IJCAI 2022] proved exponential convergence for classes of regular graphs of constant degree, thus *negatively answering the above question we posed*

Biased majority: Upper bound for cycles (1)

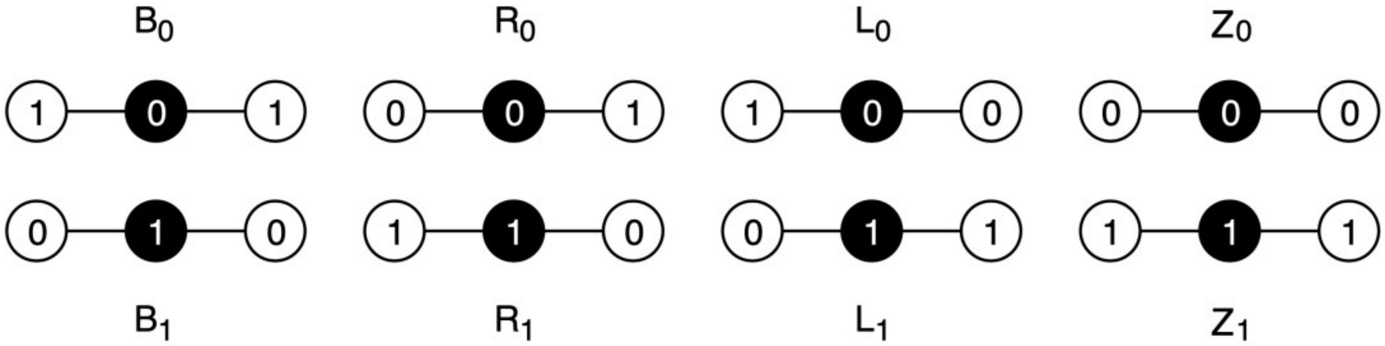


Figure 1: Categories of a node v in C_n; node v is black and its left and right neighbors are white.

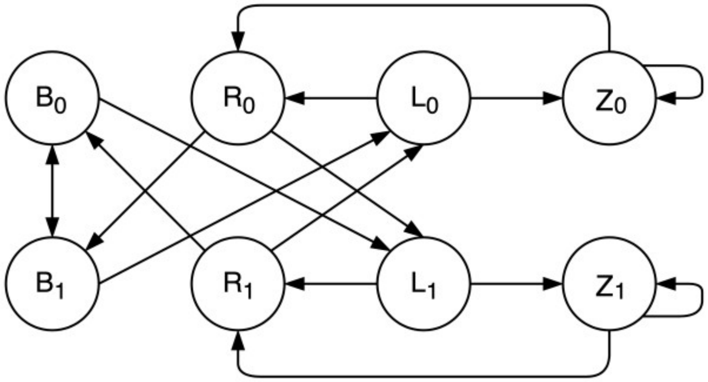


Figure 2: The *cycle binary configuration graph* H_C.

Structural lemma

$$|B_0| + \frac{|S_0|}{2} = |B_1| + \frac{|S_1|}{2}$$

Biased majority: Upper bound for cycles (2)

X_t = # 1's ⇒ Birth – death chain with

$$p_k = \alpha\frac{n-k}{n} + (1-\alpha)\left(\frac{|B_0|}{n} + \frac{1}{2}\frac{|S_0|}{n}\right)$$

$$q_k = (1-\alpha)\left(\frac{|B_1|}{n} + \frac{1}{2}\frac{|S_1|}{n}\right)$$

$$r_k = 1 - p_k - q_k$$

$$\mathbb{E}[X_t \mid X_{t-1} = k] = k+\alpha\frac{n-k}{n}+\frac{1-\alpha}{n}\left(|B_0| + \frac{|S_0|}{2} - |B_1| - \frac{|S_1|}{2}\right) = k+\alpha\frac{n-k}{n}$$

This implies $\mathbb{E}[n - X_t \leq \frac{1}{n}]$ for $t \geq \frac{2}{\alpha} n \ln n$ ⇒ whp

Voter model: upper bound (1)

$$\mathbb{E}[\mathbf{x}^{(t)} \mid \mathbf{x}^{(t-1)} = \mathbf{x}] = \frac{\alpha}{n} + \frac{1}{n}\big((1-\alpha)P + (n-1)I\big)\mathbf{x}.$$

- $P = D^{-1}A$ is (possibly weighted) graph's transition matrix
- Above implies:

$$\mathbb{E}[\mathbf{1} - \mathbf{x}^{(t)}] = \frac{1}{n}\big((1-\alpha)P + (n-1)I\big)\,\mathbb{E}[\mathbf{1} - \mathbf{x}^{(t-1)}]$$

Voter model: upper bound (2)

$$\mathbb{E}[\mathbf{1} - \mathbf{x}^{(t)}] = \left(1 - \frac{\alpha}{n}\right) \hat{P}\mathbb{E}[\mathbf{1} - \mathbf{x}^{(t-1)}],$$

$$\text{with } \hat{P} = \frac{n-1}{n-\alpha}\left(\left(\frac{1-\alpha}{n-1}\right)P + I\right) \text{ stochastic}$$

- Main eigenvalue of $(1 - \alpha/n)\hat{P}$ is 1 – α/n ⇒
  - O(n/α) in expectation
  - O((n/α)log n) whp
- High probability follows from Markov's inequality

Summary

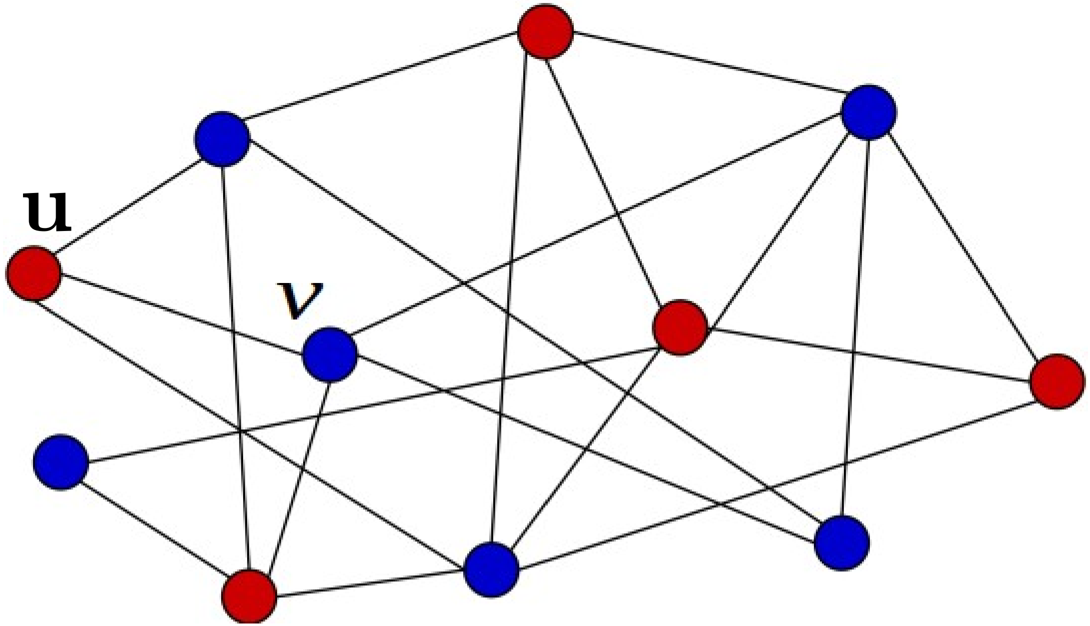
α	**Majority**	**Voter**
$\geq 1/2$	Fast absorption on *dense graphs* $\delta_{\min} = \omega(\log n)$	Fast absorption on *every topology*
$< 1/2$	Slow absorption on *dense graphs* $\delta_{\min} = \omega(\log n)$ Fast absorption on some *sparse topologies* with bounded degree $\delta_{\max} = O(\log n)$	Fast absorption on *every topology*

Part 2: voter model and evolutionary dynamics on graphs

Voter and models of natural selection

- Voter model well characterized
  - Continuous time
  - Discrete time, asynchronous case (considered here)
  - Discrete time, synchronous case
    - In each round, all nodes update using voter rule
  - [Hassin & Peleg, ‘99 and 2001], [Aldous’ & Fill’s book], [Oliveira 2012] …
  - Main ingredient: connection to coalescing random walks
- Wright-Fisher model without drift (all nodes have same fitness)
  - On graphs: this is the synchronous voter model
    - Classical Wright-Fisher (no mutations): clique with self-loops
  - Not the topic of this talk

Biased voter



Each node has an opinion **0**/**1**

In every round t:

node v sampled u.a.r.

v updates its opinion

1. Sample neighbor u.a.r.

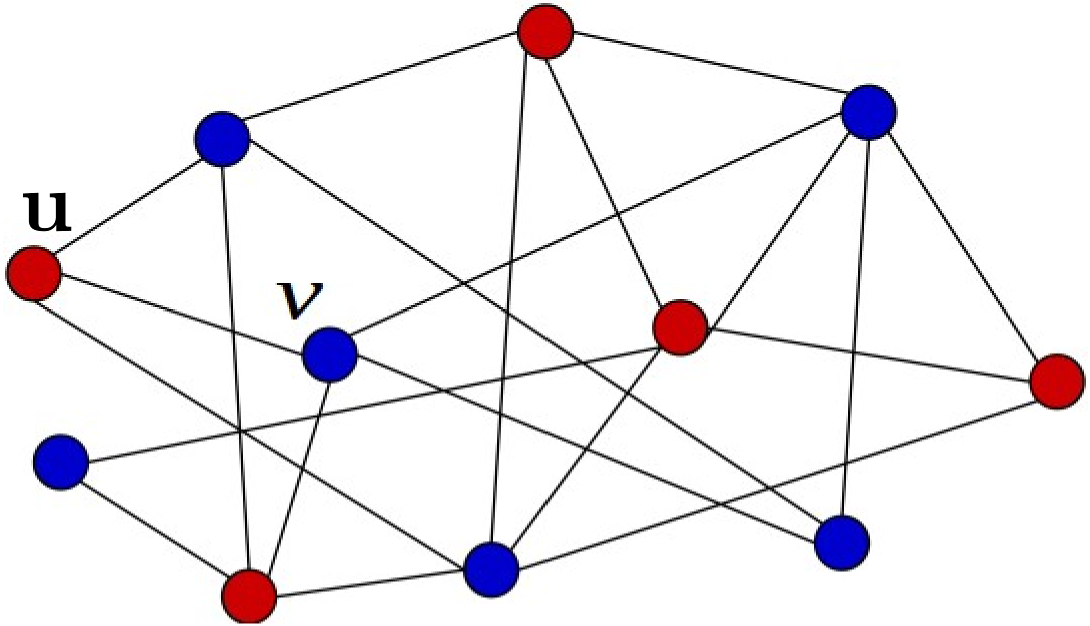
$$u \leftarrow sample(\mathcal{N}(v))$$

$$x_v^{(t)} = c \text{ and } x_u^{(t)} = c'$$

2. Opinion update

$\Pr(x_u^{(t+1)} = c' \mid v \text{ is sampled})$	$x_v^{(t)} = 0$	$x_v^{(t)} = 1$
$x_u^{(t)} = 0$	1 (wlog)	α_{01}
$x_u^{(t)} = 1$	α_{10}	1 (wlog)

Biased voter



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__2. Opinion update__

$\Pr(x_u^{(t+1)} = c' \mid v \text{ is sampled})$	$x_v^{(t)} = 0$	$x_v^{(t)} = 1$
$x_u^{(t)} = 0$	1 (wlog)	α_{01}
$x_u^{(t)} = 1$	α_{10}	1 (wlog)

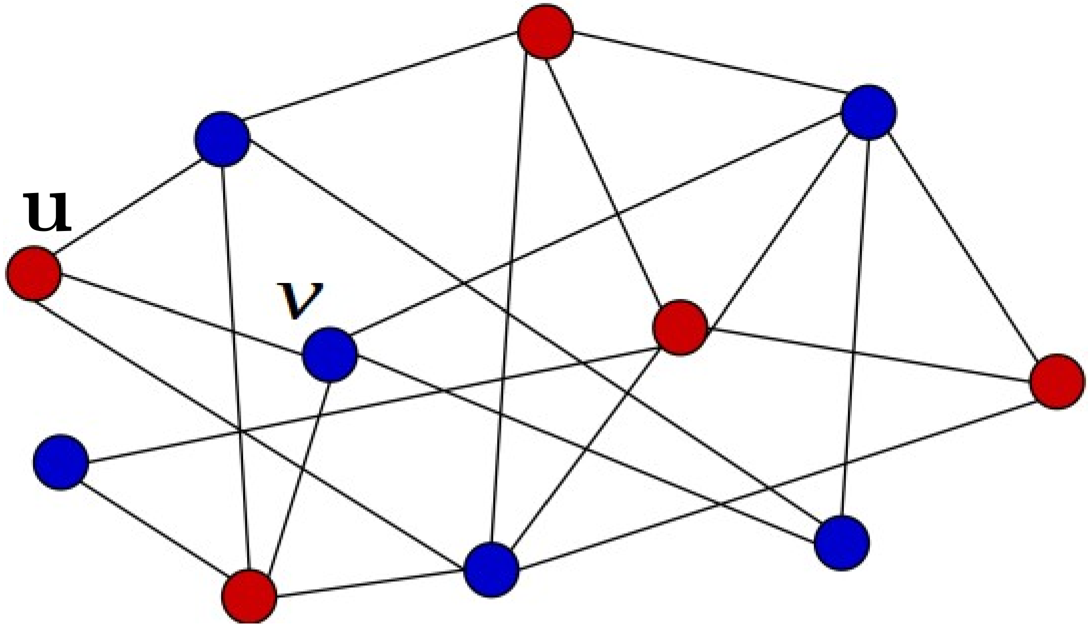
Each node has an opinion **0**/**1**

Obviously **voter model** when $\alpha_{cc'} = 1$

Biased voter

- This time, corresponding Markov chain has two absorbing states
- Similar to model considered in [Berenbrink et al., ICALP 2016], but not quite the same
- Questions
  - Absorption/fixation probability: probability of consensus in either state given initial configuration
  - (Expected) number of rounds to consensus

Discrete Moran process



Each node has a **type 0/1**
Non-mutants have fitness 1
Mutants have fitness r > 1
At t = 0: k mutants

In every round t:

1. One node v sampled for reproduction proportionally to its fitness

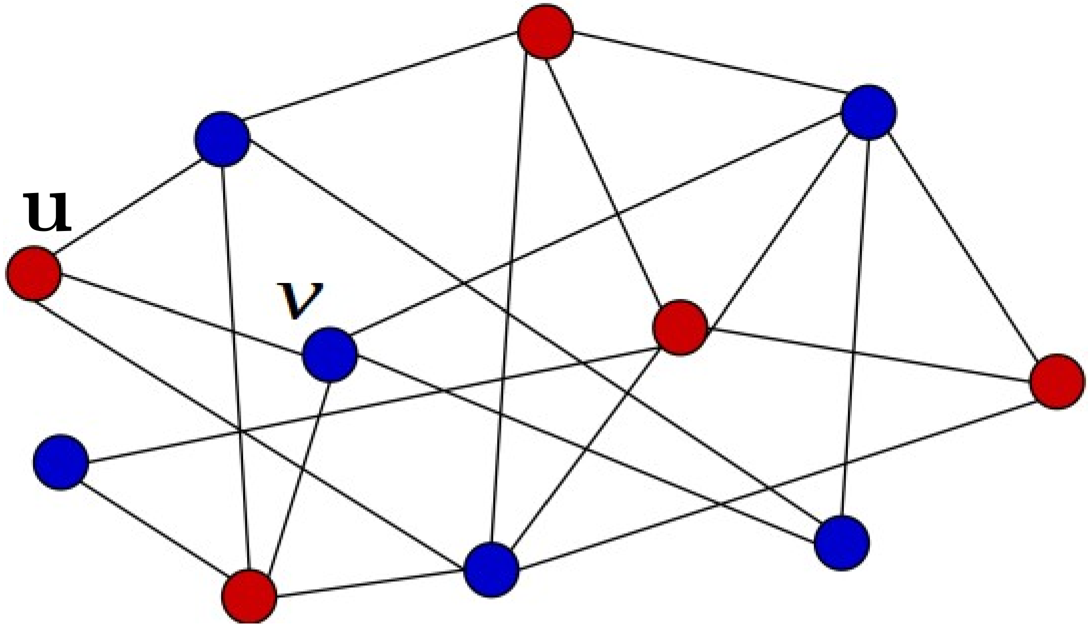
2. v chooses neighbor u u.a.r. for replacement

Discrete Moran process
[Lieberman et al. Nature 2005]
[Diaz et al. RSA, 2016]
[Diaz and Mitsche, Elsevier CSR 2021]

Discrete Moran process

$f_v^{(t)}$: v's fitness at t

$k^{(t)}$: # mutants at t



Each node has a **type** **0**/**1**

Non-mutants have fitness 1

Mutants have fitness r > 0

At t = 0: k mutants

In every round t:

1. Node sampled for reproduction

$$\mathbb{P}[v \ sampled] = \frac{f_v^{(t-1)}}{rk^{(t-1)} + n - k^{(t-1)}}$$

2.1 v chooses neighbor u u.a.r. for replacement

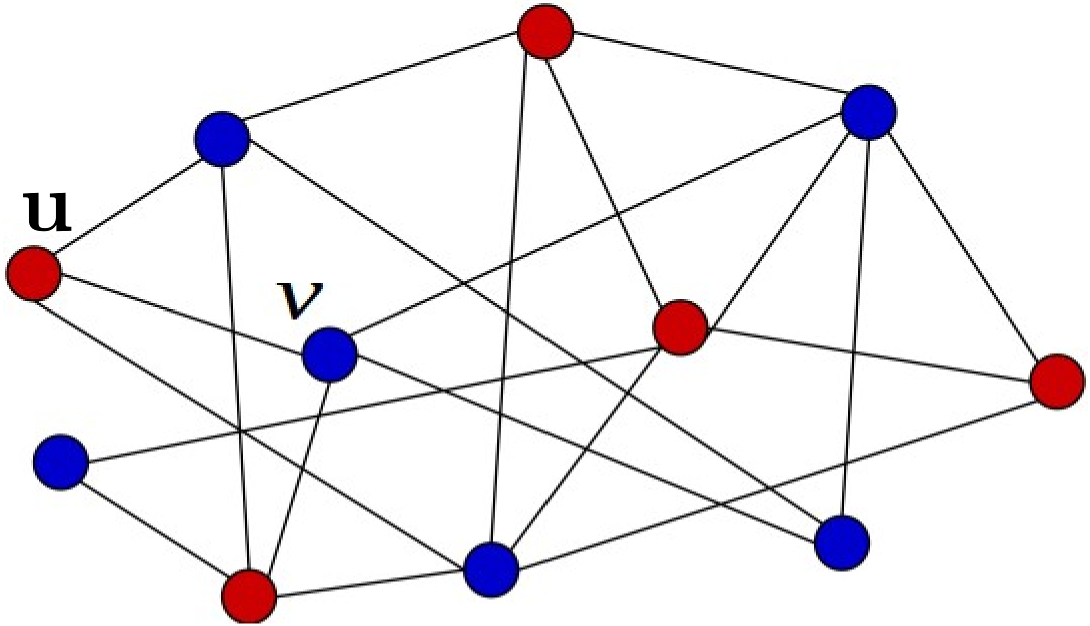
2.2.

$$x_u^{(t)} = x_v^{(t-1)}$$

Discrete Moran process

$f_v^{(t)}$: v's fitness at t

$k^{(t)}$: # mutants at t



This is a **push** model
(Biased) voter is a **pull** model

In every round t:

1. Node sampled for reproduction

$$\mathbb{P}[v \ sampled] = \frac{f_v^{(t-1)}}{rk^{(t-1)} + n - k^{(t-1)}}$$

2.1 v chooses neighbor u u.a.r. for replacement

2.2.

$$x_u^{(t)} = x_v^{(t-1)}$$

Discrete Moran process

- Questions:
  - Absorption: Prob. mutants take over (fixation)
  - (Expected) time to absorption
- Problem well studied
  - Fixation probability well-known for class of graphs that includes regular graphs [Lieberman et al. 2005]

$$\mathbb{P}[fixation] = \frac{1 - r^{-k}}{1 - r^{-n}} \text{ with } k \text{ initial mutants}$$

$$\mathbb{P}[fixation] \to \frac{k}{n} \text{ as } r \to 1$$

  - In other cases simple closed-form not possible but bounds available
  - Bounds on expected time to fixation also known
  - [Lieberman et al. Nature 2005], [Diaz et al., Algorithmica 2014], [Diaz et al. RSA, 2016], [Diaz and Mitsche, Elsevier CSR 2021], [Goldberg et al., RSA, 2020] ...

Biased voter – symmetric case

- In this case: $\alpha_{01} = \alpha_{10}$
- Voter vs Moran process with neutral drift (r = 1)

$$\mathbb{E}_{Voter}[x_u^{(t+1)} \mid \mathbf{x}^{(t)}] = \left(1 - \frac{\alpha}{n}\right) x_u^{(t)} + \frac{\alpha}{n} \sum_{v \sim u} \frac{x_v^{(t)}}{d_u}$$

$$\mathbb{E}_{Moran}[x_u^{(t+1)} \mid \mathbf{x}^{(t)}] = \left(1 - \frac{1}{n}\right) x_u^{(t)} + \frac{1}{n} \sum_{v \sim u} \frac{x_v^{(t)}}{d_v}$$

Biased voter – symmetric case

- In this case: $\alpha_{01} = \alpha_{10}$
- Voter vs Moran process with neutral drift ($r = 1$)
- On regular graphs

$$\mathbb{E}_{Voter}[x_u^{(t+1)} \mid \mathbf{x}^{(t)}] = \left(1 - \frac{\alpha}{n}\right) x_u^{(t)} + \frac{\alpha}{n} \sum_{v \sim u} \frac{x_v^{(t)}}{d_u}$$

$$\mathbb{E}_{Moran}[x_u^{(t+1)} \mid \mathbf{x}^{(t)}] = \left(1 - \frac{1}{n}\right) x_u^{(t)} + \frac{1}{n} \sum_{v \sim u} \frac{x_v^{(t)}}{d_u}$$

Biased voter – symmetric case

- In this case: $\alpha_{01} = \alpha_{10}$
- Voter vs Moran process with neutral drift (r = 1)
- On regular graphs

$$\mathbb{E}_{Voter}[x_u^{(t+1)} \mid \mathbf{x}^{(t)}] = \left(1 - \frac{\alpha}{n}\right) x_u^{(t)} + \frac{\alpha}{n} \sum_{v \sim u} \frac{x_v^{(t)}}{d_u}$$

$$\mathbb{E}_{Moran}[x_u^{(t+1)} \mid \mathbf{x}^{(t)}] = \left(1 - \frac{1}{n}\right) x_u^{(t)} + \frac{1}{n} \sum_{v \sim u} \frac{x_v^{(t)}}{d_u}$$

  - Models related in this case
  - Can reuse results for one model to analyze the other

Biased voter - symmetric case – connected graphs

- This boils down to known results
  - Process equivalent to a lazy asynchronous voter model
  - Voter models studied through the lens of coalescing random walks [Hassin & Peleg 1999, Aldous' and Fill's book, Oliveira 2012]
- Alternatively: look at expected evolution of the process

$$\mathbb{E}[\mathbf{x}^{(t+1)}] = \left(\left(1 - \frac{\alpha}{n}\right) I + \frac{\alpha}{n} P\right) \mathbb{E}[\mathbf{x}^{(t)}], \; P \text{ random walk matrix of } G$$

- Either way:

$$\mathbb{P}[\mathbf{x}^{(\infty)} = \mathbf{1}] = \frac{k}{n} \text{ with } k \text{ initial mutants}$$

$$\text{Convergence of } \mathbb{E}[\mathbf{x}^{(t)}] \text{ depends on } \lambda_2(P)$$

Biased voter - general case ($\alpha_{01} \neq \alpha_{10}$)

- Problem becomes considerably harder
  - Probabilities of updating now depend on which nodes hold which opinion
  - For regular graphs
    - Processes are no longer equivalent
    - Still related in more subtle way

$$\mathbb{P}_{Moran}[v\ sampled] = \frac{f_v^{(t-1)}}{rk^{(t-1)} + n - k^{(t-1)}}$$

Biased voter – general case - regular graphs

- Assume set S of nodes with opinion 1
- Probability p_S of transition to a state with |S| + 1 nodes with opinion 1

$$p_S = \sum_{u \notin S} \mathbb{P}[u\ sampled]\mathbb{P}[x_u \leftarrow x_v, v \in S] = \sum_{u \notin S} \frac{1}{n} \frac{|N_u \cap S|}{d} \alpha_{01} = \frac{\alpha_{01}}{nd} |\partial(S)|$$

d is degree, N_u is set of u's neighbors, $\partial(S)$ cut through S

$$\textbf{Push:}\ p_S = \sum_{u \in S} \mathbb{P}[u\ sampled]\mathbb{P}[x_u \rightarrow x_v, v \in V - S] = \sum_{u \notin S} \frac{\alpha_{01}}{n} \frac{|N_u \cap V - S|}{d}$$

$$= \frac{\alpha_{01}}{nd} |\partial(S)|,\ u \text{ sampled with prob. } \alpha_{01}/n$$

Regular graphs

- Assume set S of nodes with opinion 1
- Probability p_S of transition to a state with |S| + 1 nodes with opinion 1

$$p_S = \sum_{u \notin S} \mathbb{P}[u\ sampled]\mathbb{P}[x_u \leftarrow x_v, v \in S] = \sum_{u \notin S} \frac{1}{n} \frac{|N_u \cap S|}{d} \alpha_{01} = \frac{\alpha_{01}}{nd} |\partial(S)|$$

d is degree, N_u is set of u's neighbors, $\partial(S)$ cut through S

- Probability q_S of transition to state with |S| - 1 nodes with opinion 1

$$q_S = \frac{\alpha_{10}}{nd} |\partial(S)|$$

$$p_S / q_S = \alpha_{01} / \alpha_{10} \doteq r$$

Regular graphs

- Absorption probabilities
  - Trace back to birth-death chain with constant ratio p_k/q_k
  - See Thm. 1 in [Lieberman et al., Nature 2005]

$$\mathbb{P}[fixation] = \frac{1 - r^{-k}}{1 - r^{-n}} \text{ with } k \text{ initial mutants}$$

$$\mathbb{P}[fixation] \to \frac{k}{n} \text{ as } r \to 1$$

- Absorption time
  - Cannot directly use same trick (not even for regular graphs)
  - Reason: # 1's may not change in a step
  - Probability of change depends on subset of vertices holding opinion 1
  - In the clique: $\Theta(n \log n)$ for constant α_{01}/α_{10}, $\alpha_{01} > \alpha_{10}$

Outlook

- More results for synchronous variant in our IJCAI 2023 paper
- Synchronous case harder
- Analysis in [Berenbrink et al. 2016] when initial number k of nodes with popular opinion is at least c log n.
  - We consider case where k is arbitrary
  - Fixation probability at least k/n in clique
  - Absorption time at most linear, depending on $\alpha_{01} - \alpha_{10}$
- **Q.:** what can we say about more general topologies?
- Each round of the process is described by a Markov chain, but depend on current state and are thus correlated

THANK YOU