# Causal Limits of Distributed Computation

**Francesco d'Amore**

Based on the works [Coiteux-Roy et al. STOC '24], [Akbari et al. STOC '25], [Balliu et al. STOC '25], [Balliu et al. '25].

**INdAM - RomaTre**
07 May 2025

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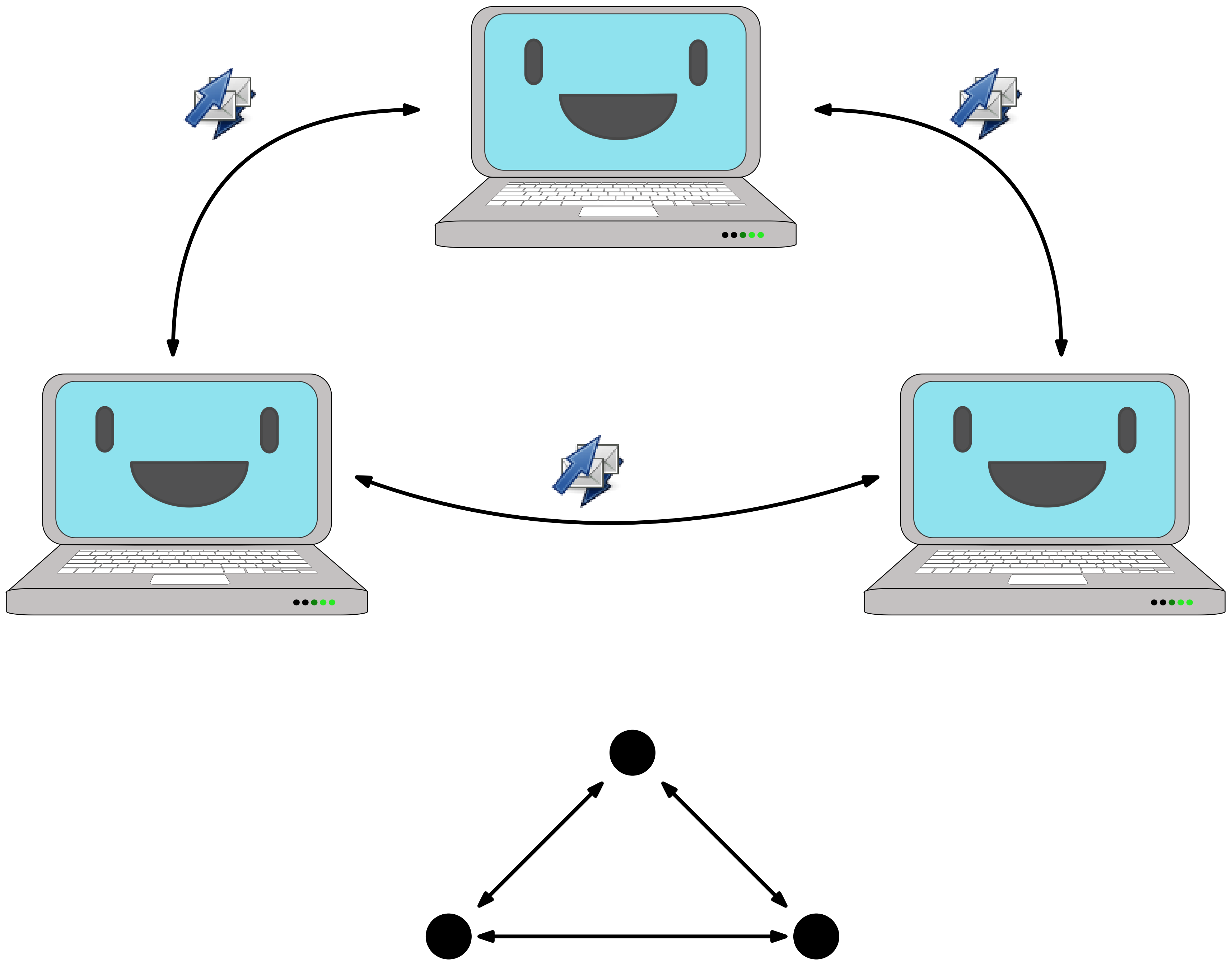
# Distributed computation

- **Synchronous distributed network**
  - graph $G = (V, E)$ with $|V| = n$
  - $E$: communication links
  - discrete communication rounds: $t = 1, 2, \dots$

  - each node in $V$ runs the same algorithm
  - each round = local computation + communication



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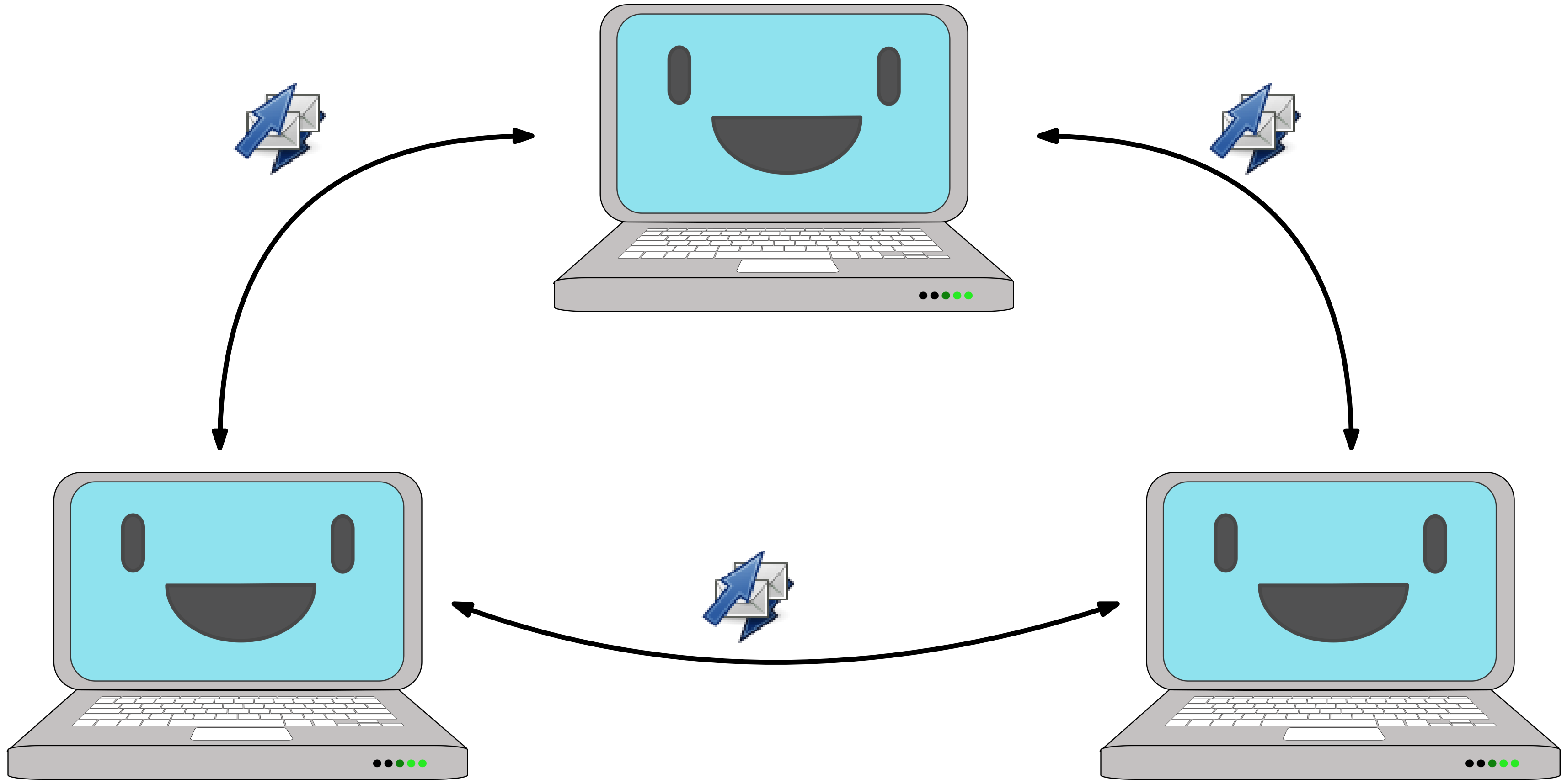

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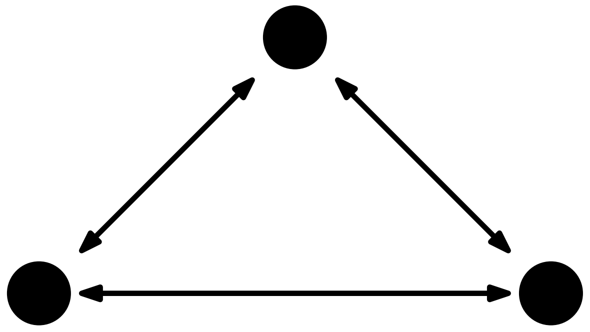

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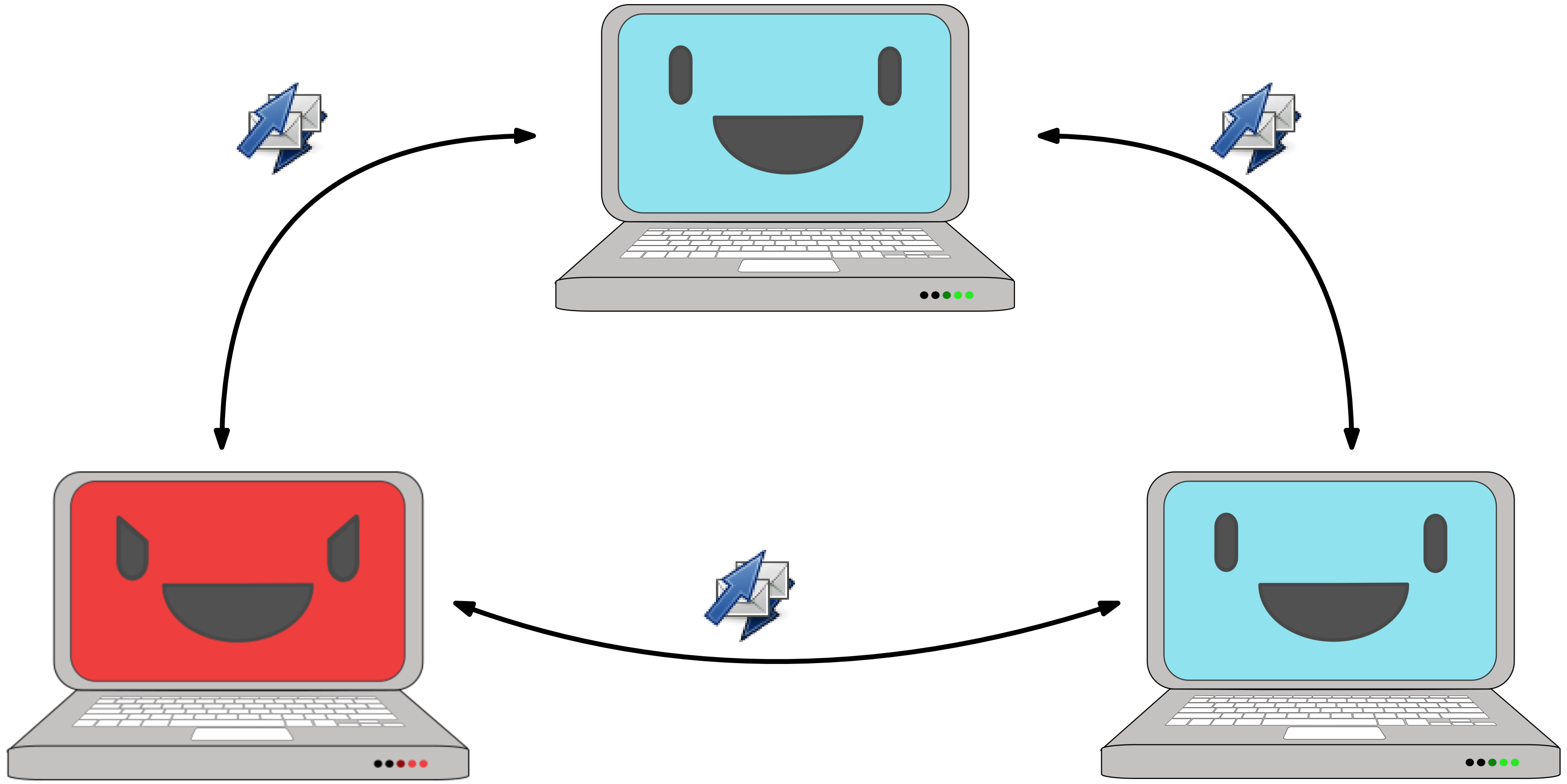

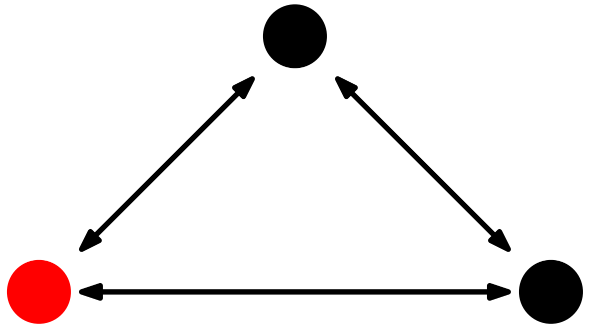

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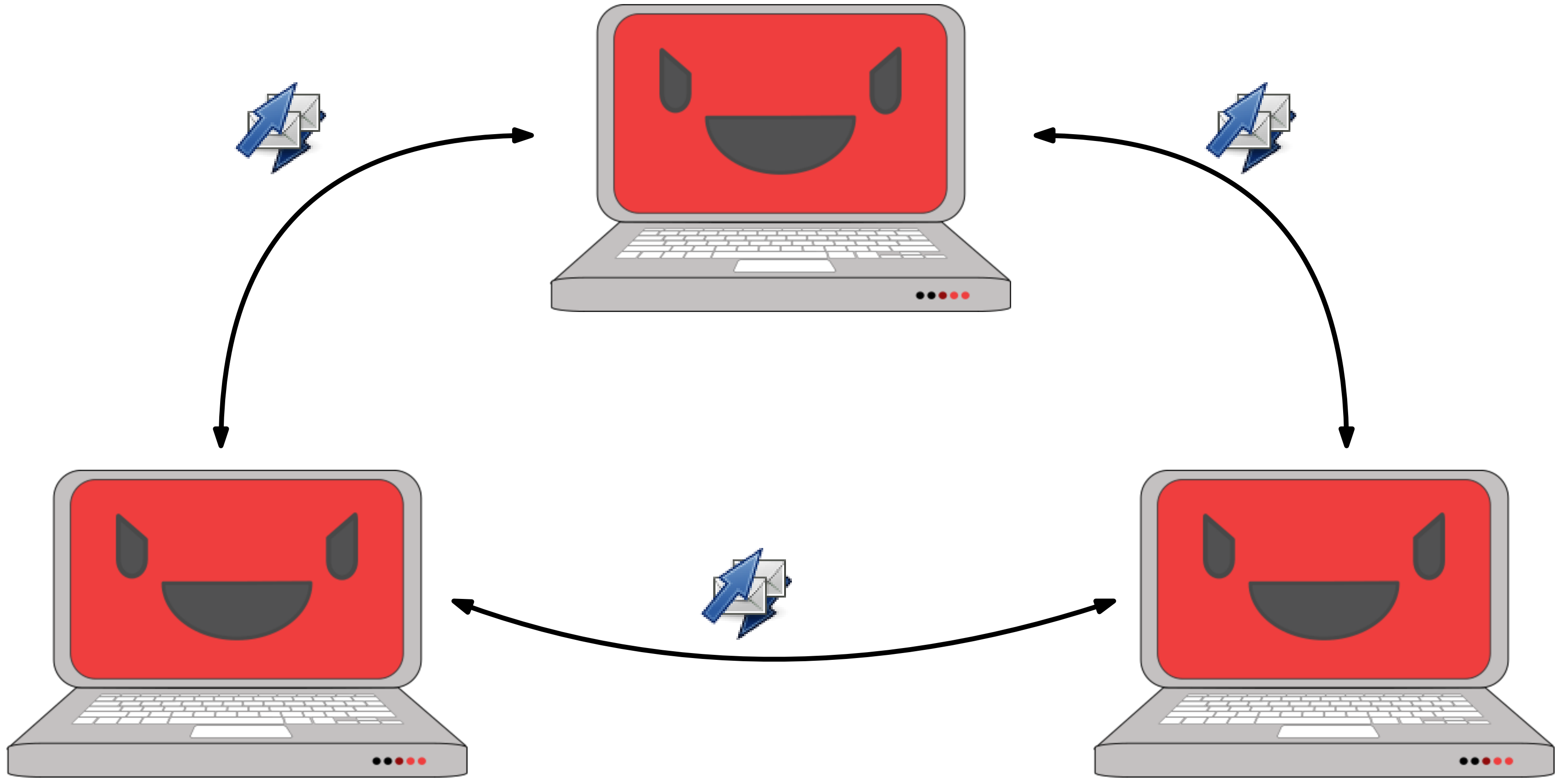

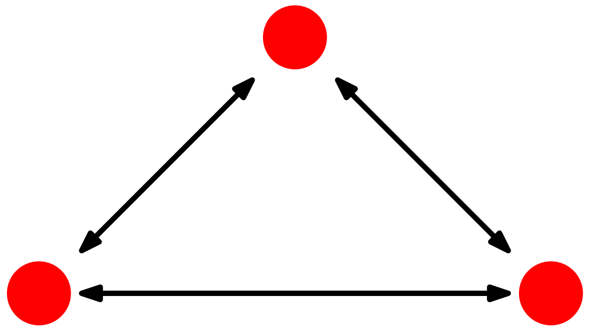

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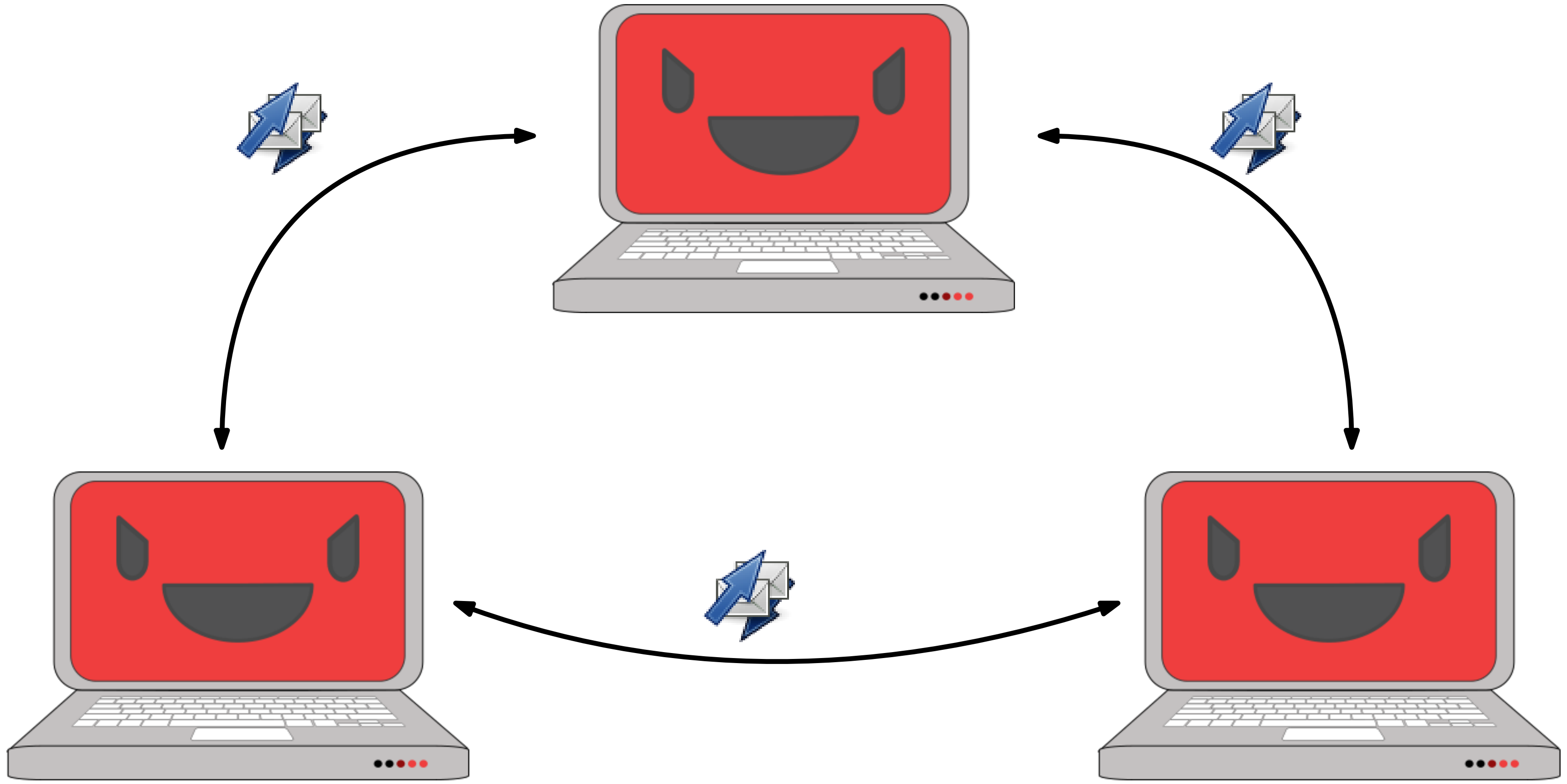

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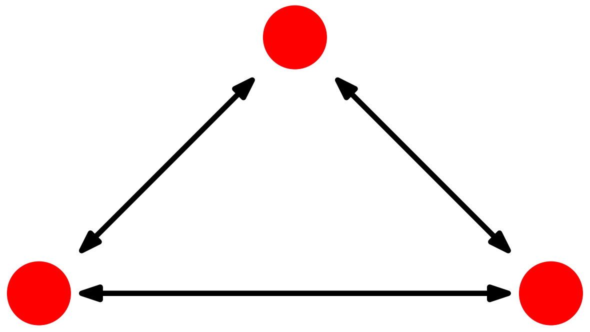

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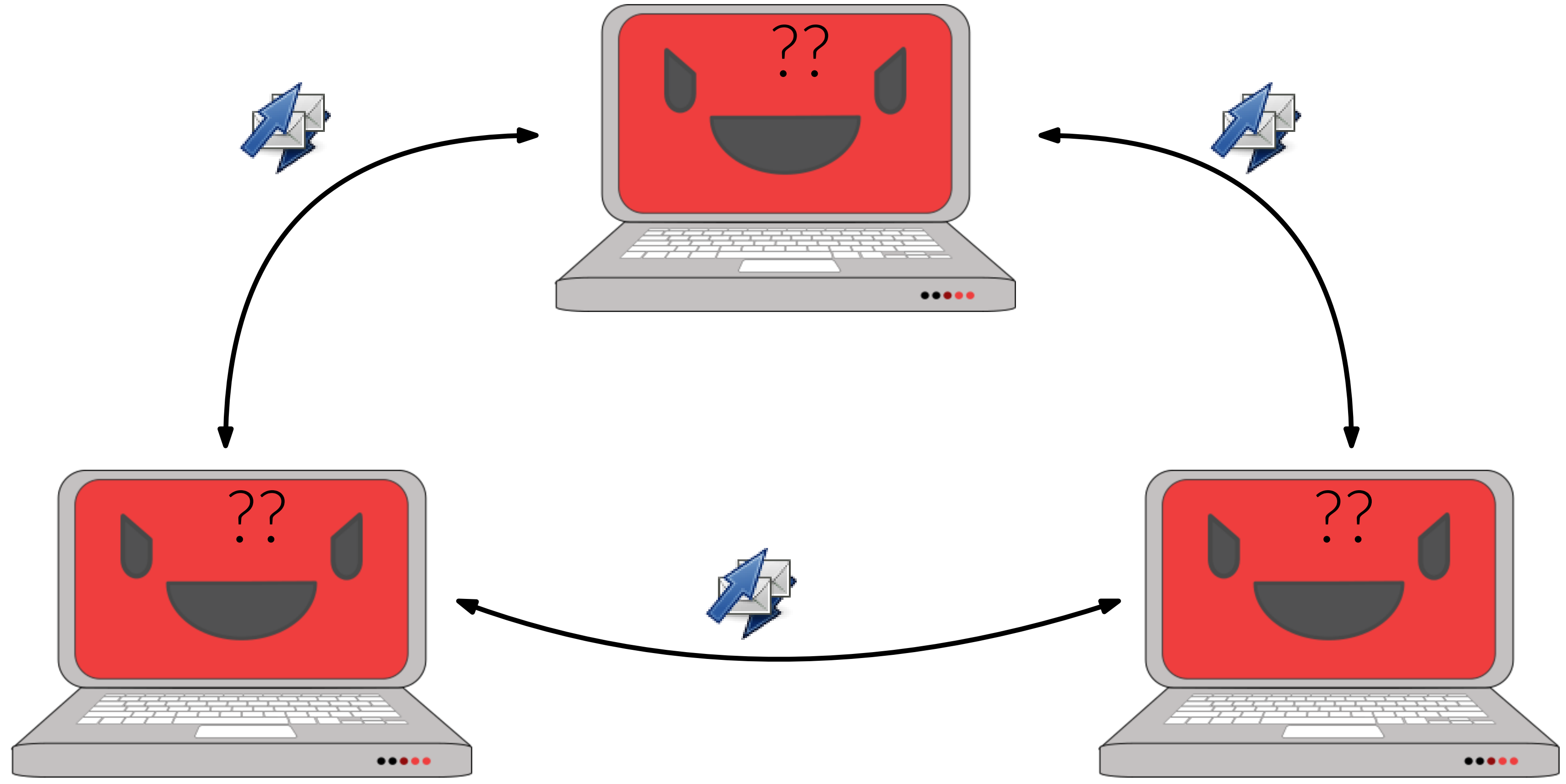


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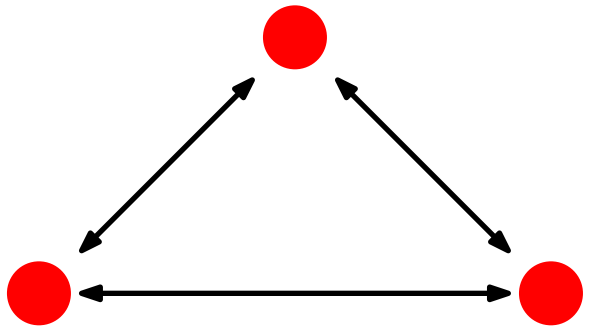
    - algorithm: ?? we need to break symmetry



# The LOCAL model

[Linial FOCS ’87 & SICOMP ’92]

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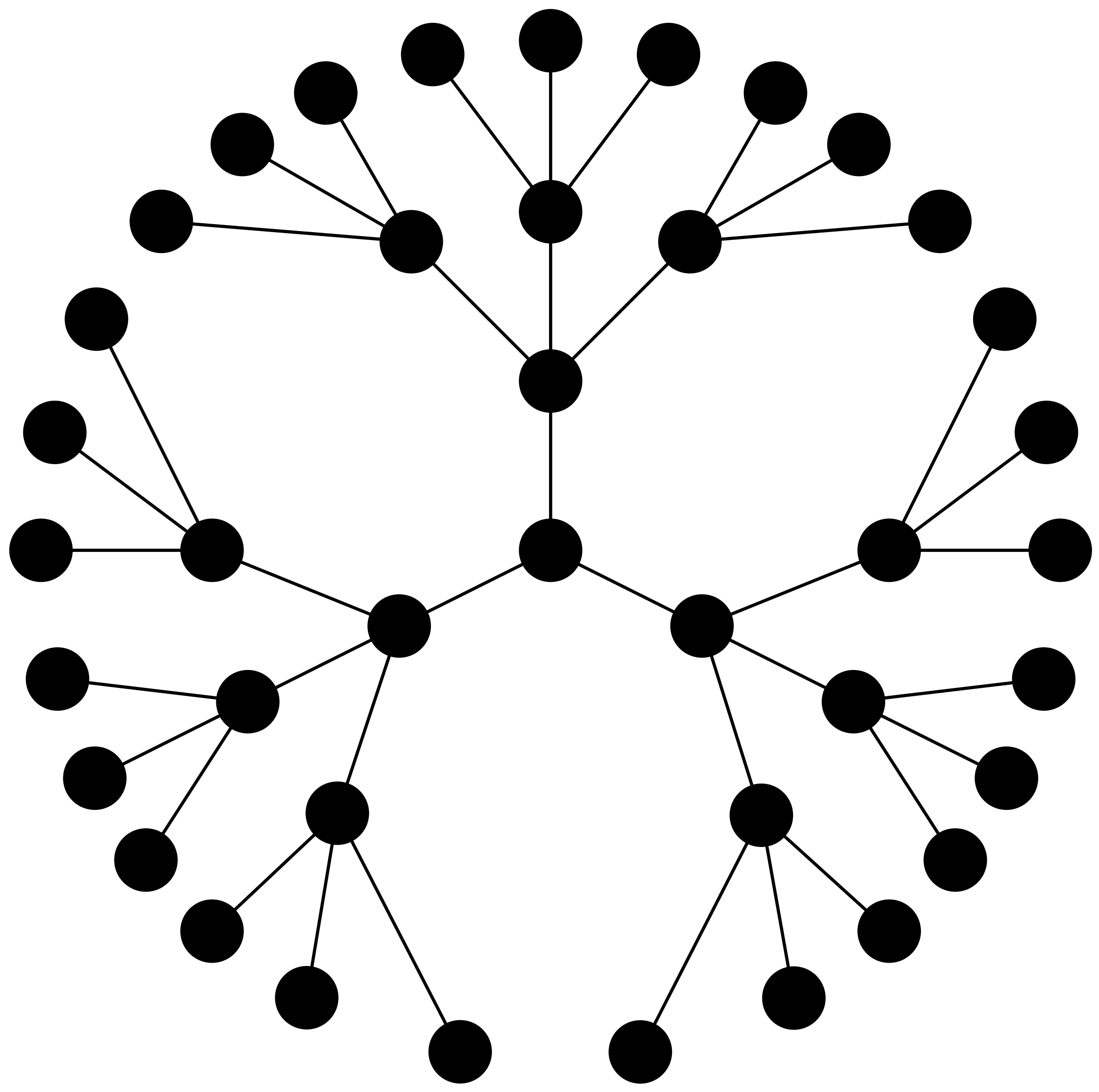

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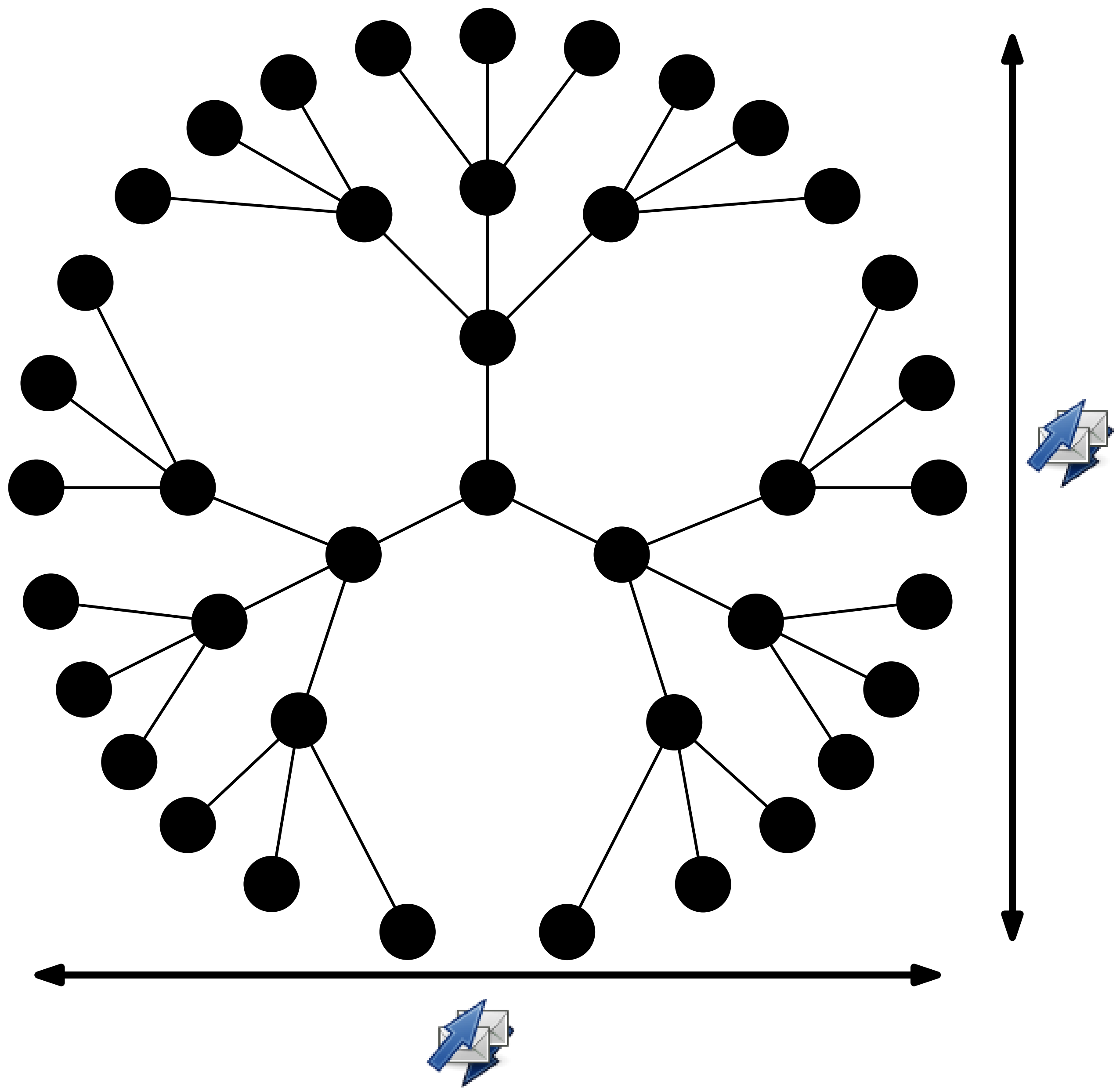

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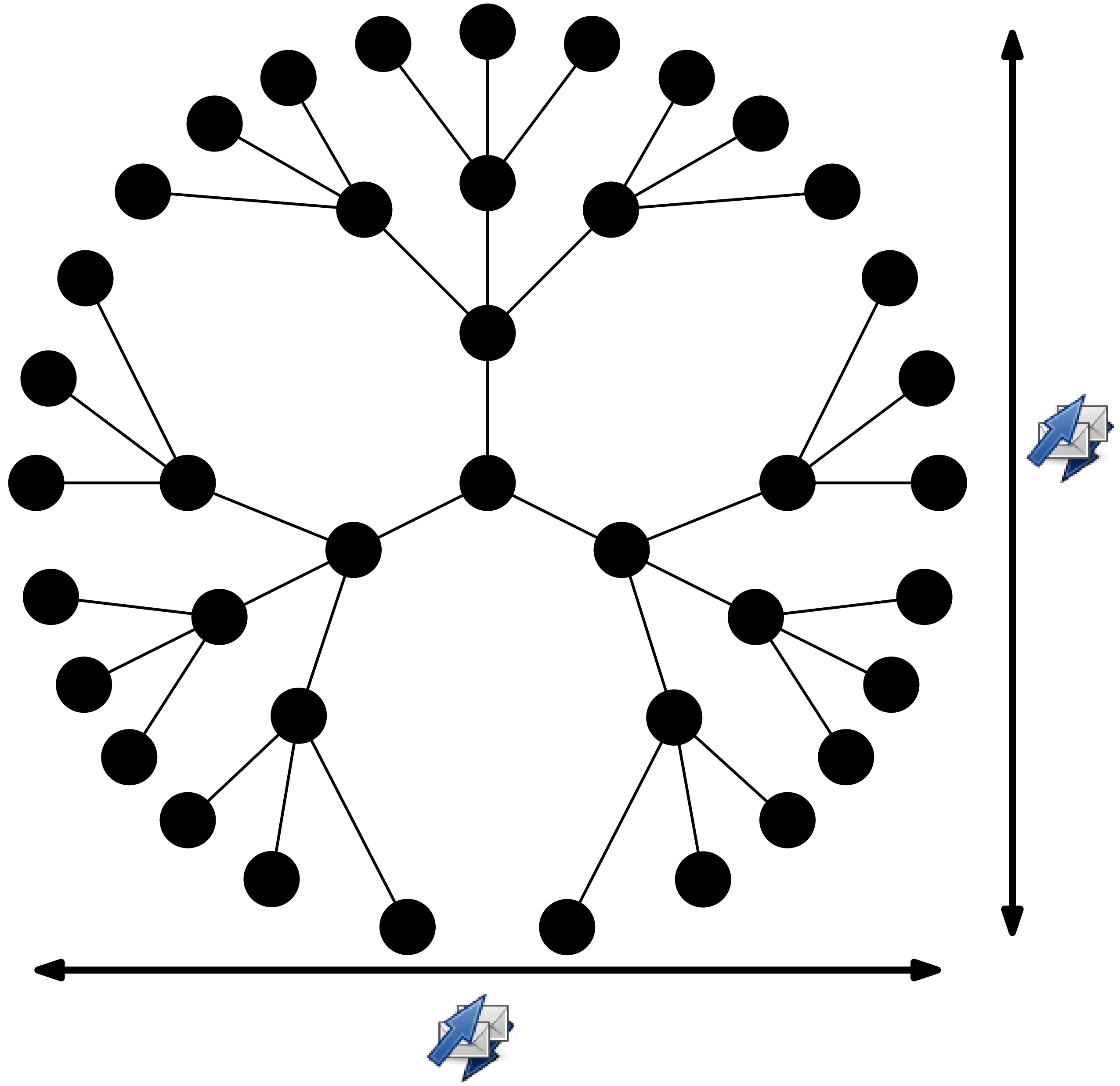

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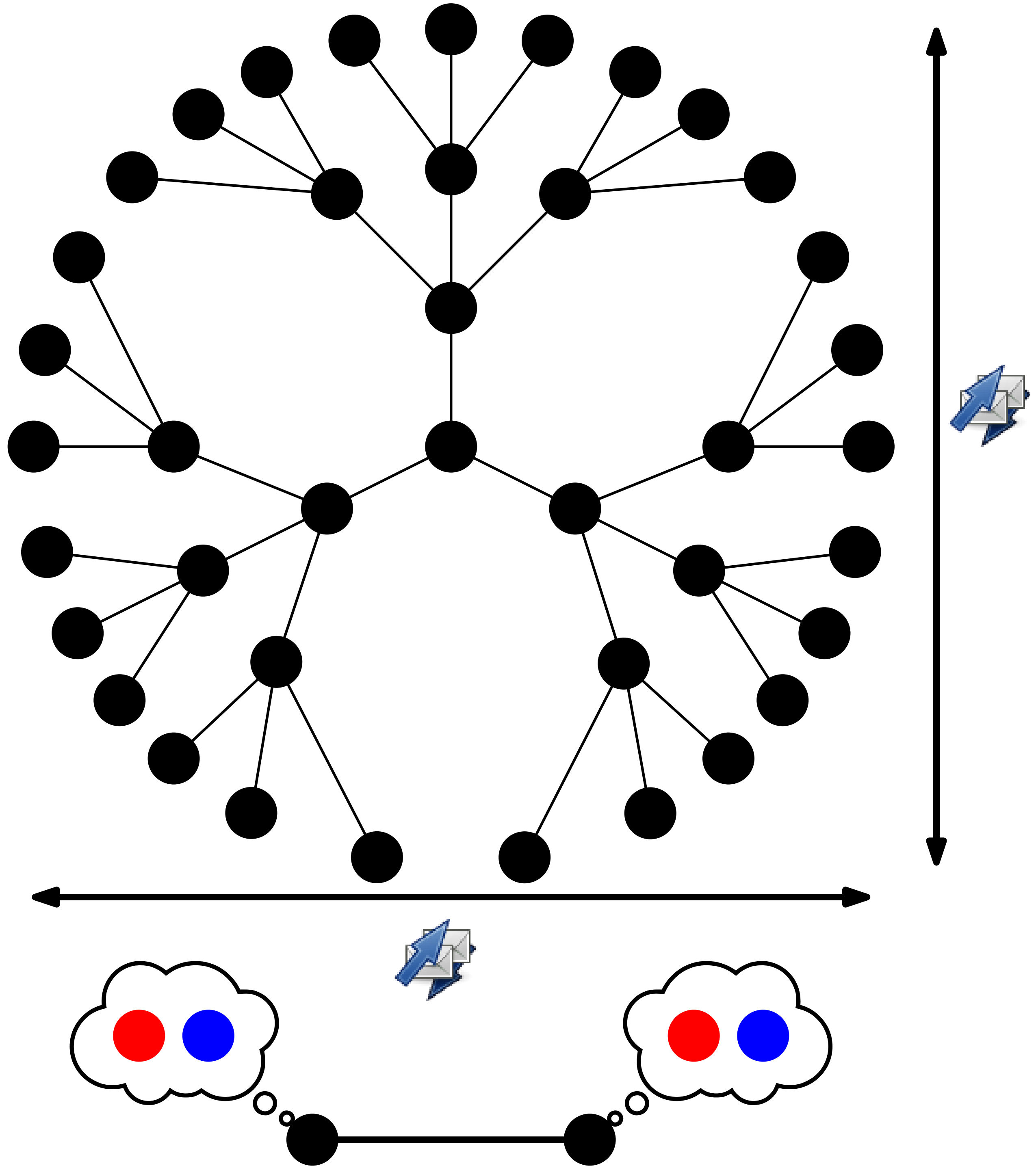

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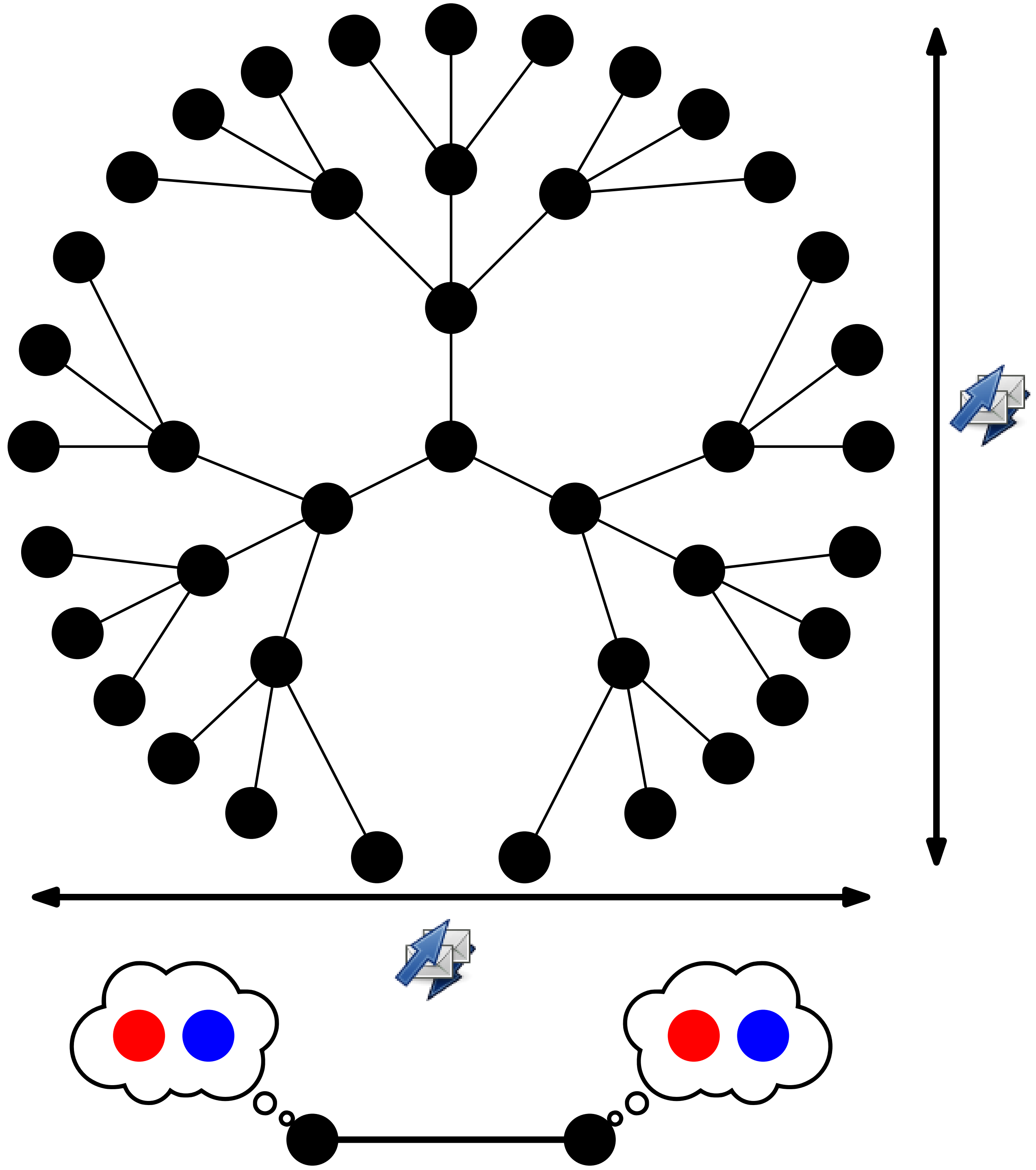

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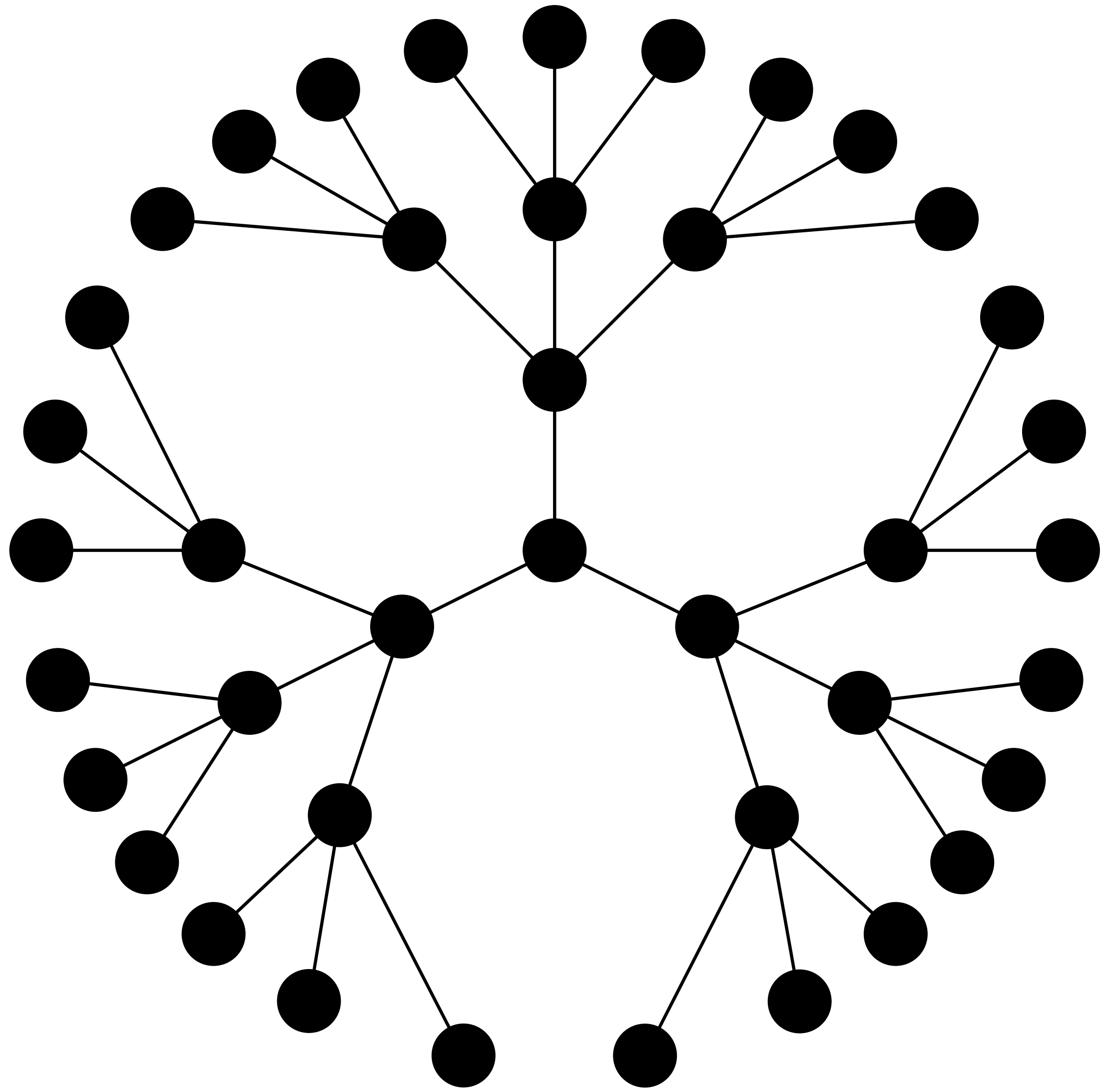
- $A$: $T(n)$-round LOCAL algorithm



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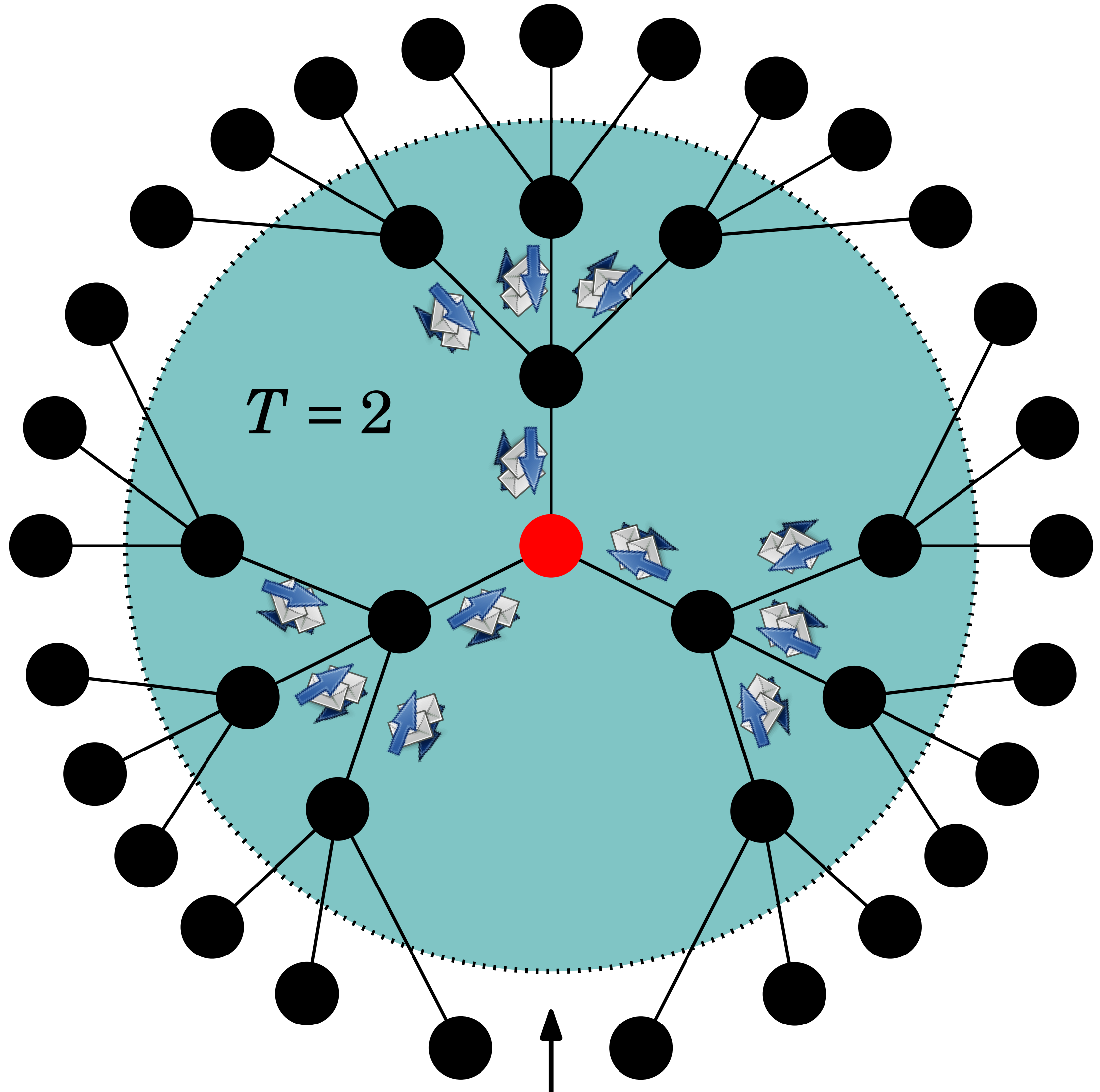


knowledge after $\mathbf{2}$ rounds of communication

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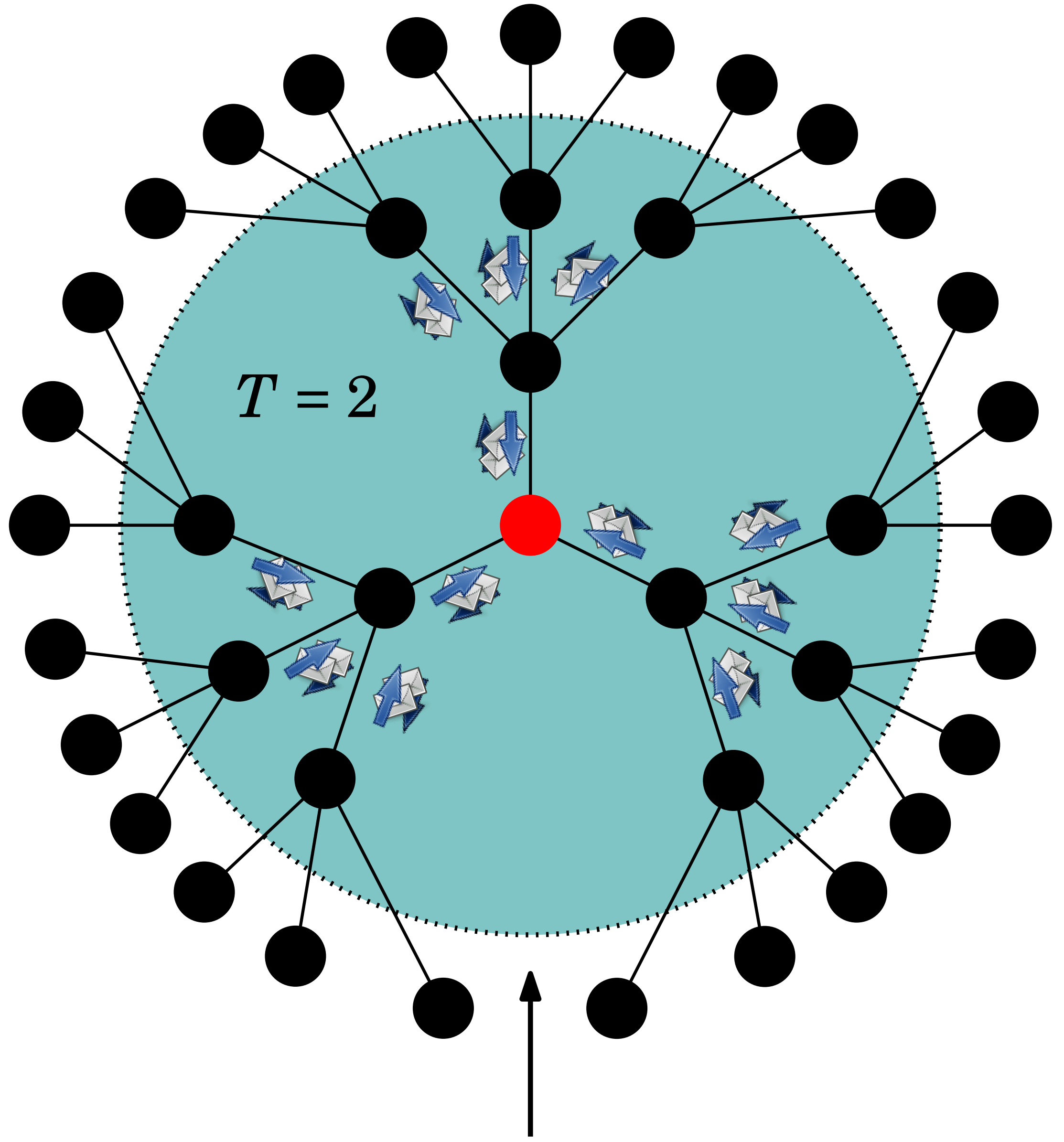


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- given a LOCAL algorithm $A$, we can construct the mapping $f$
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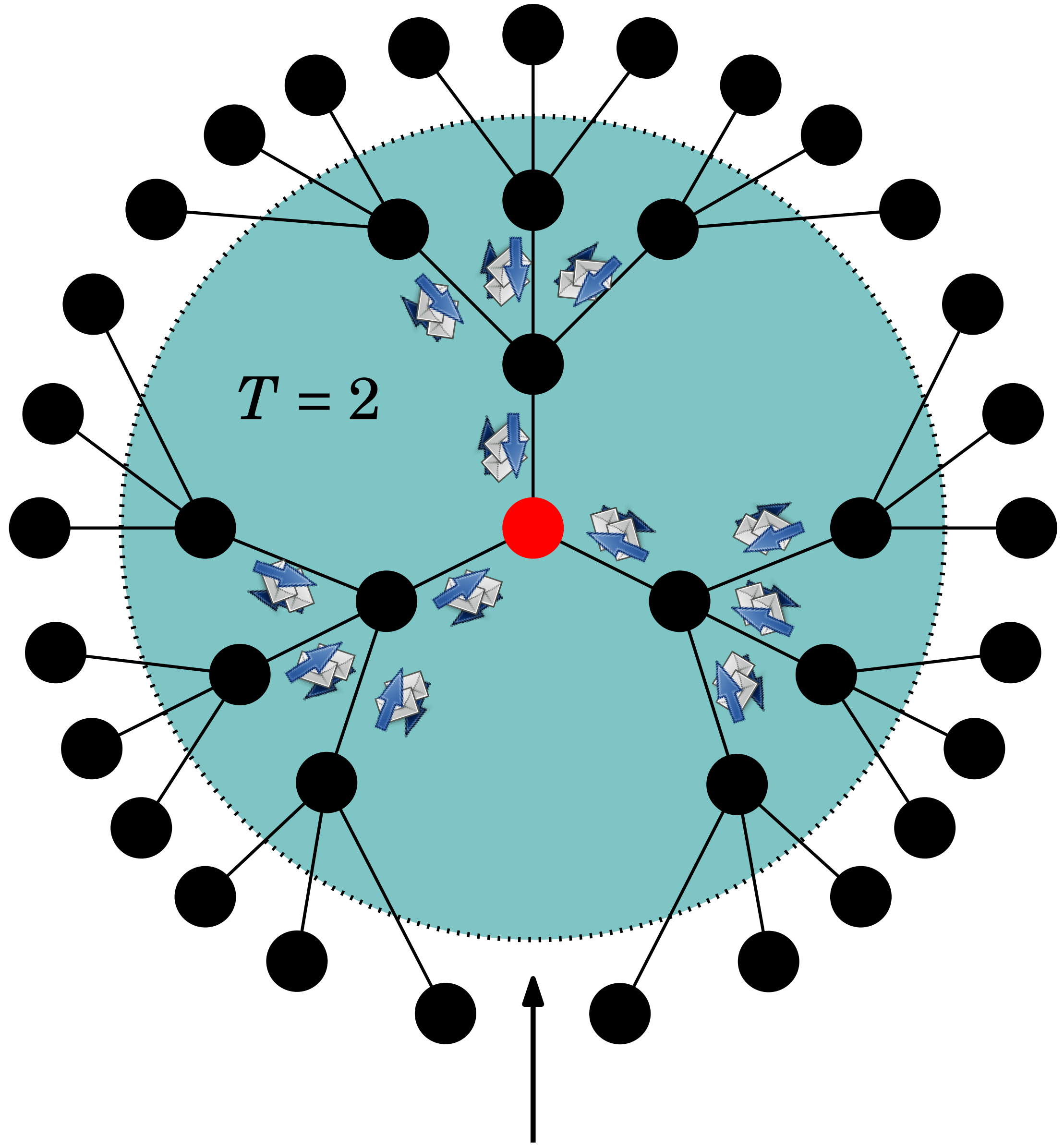


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# Local view

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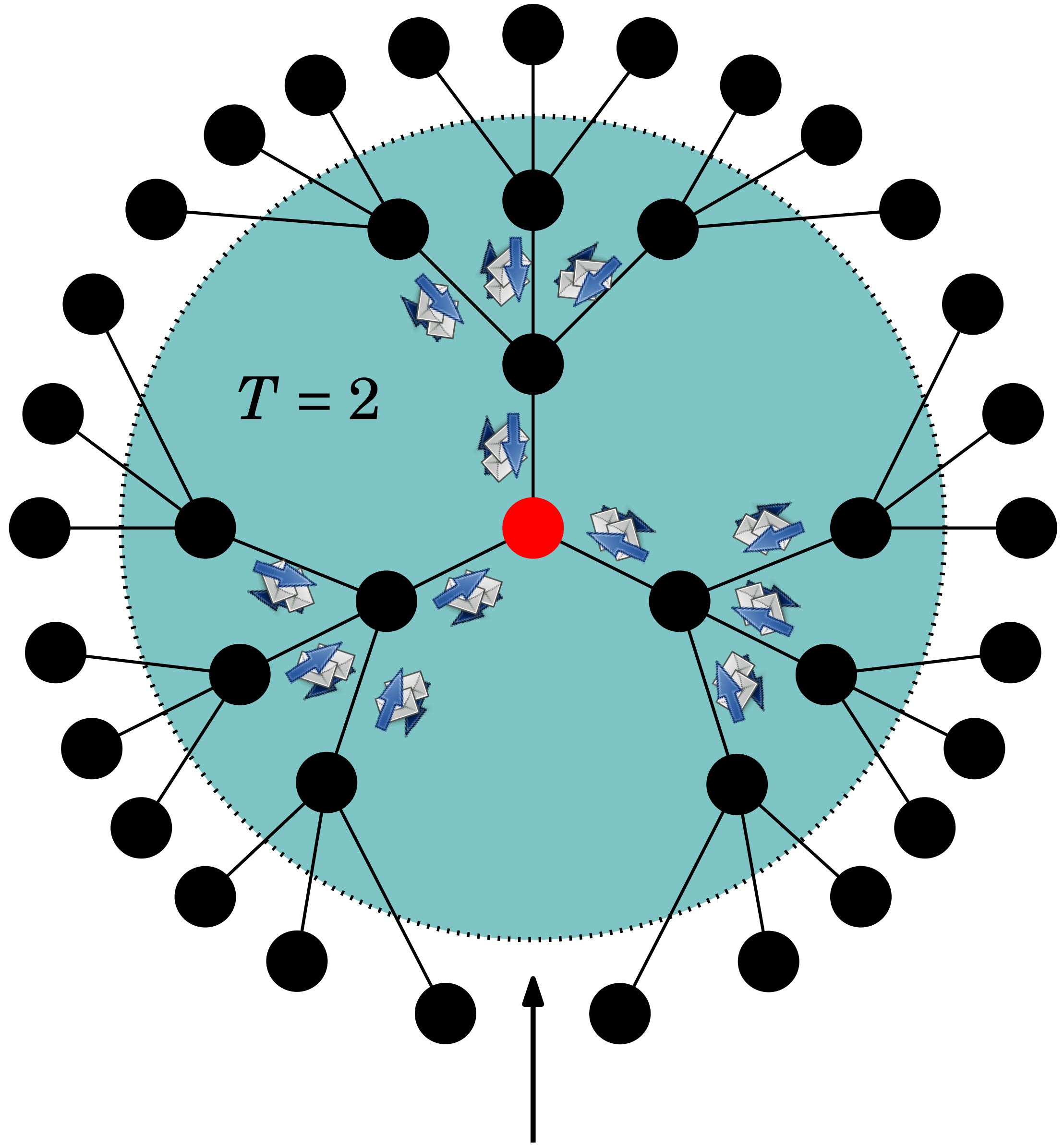
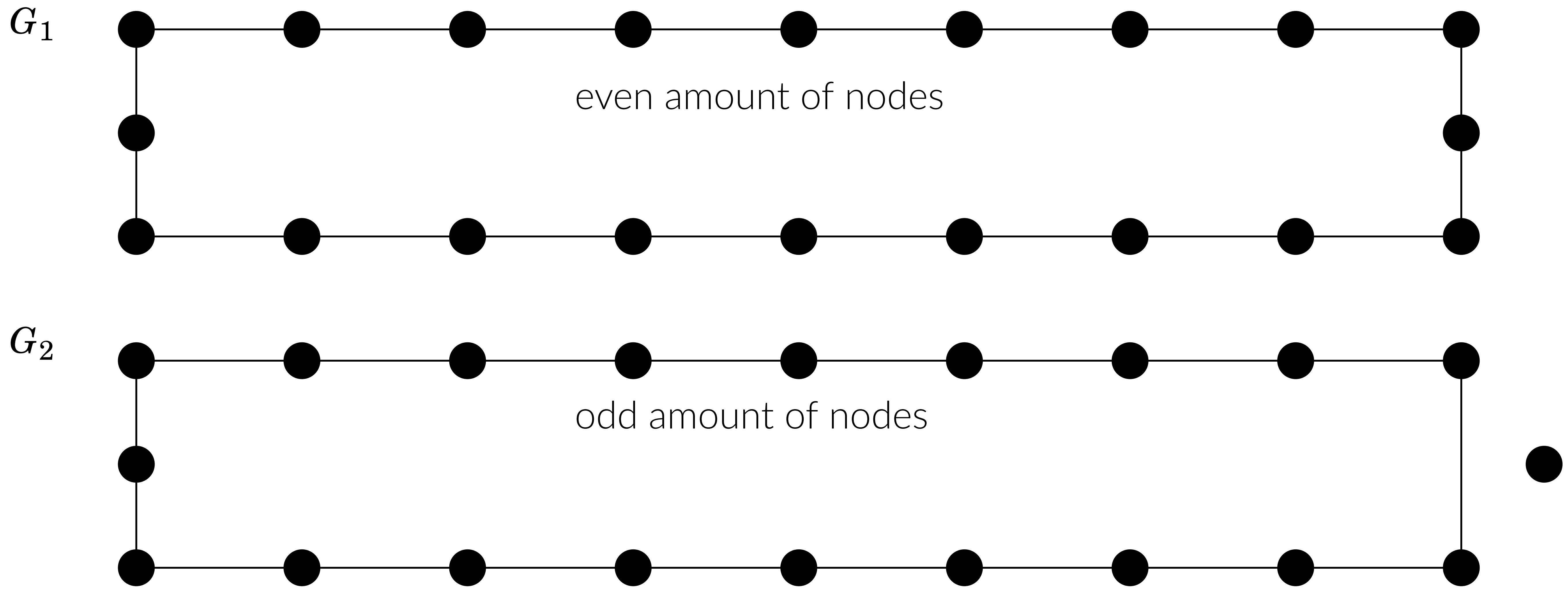



knowledge after $2$ rounds of communication

- Locality $T = \text{diam}(G) + 1$ is always sufficient to solve any problem: gathering algorithm

# Lower bound: indistinguishability argument

- **Problem**: 2-coloring even cycles. Assume we have a $T$-round LOCAL algorithm with $T \leq \frac{n}{2} - 2$



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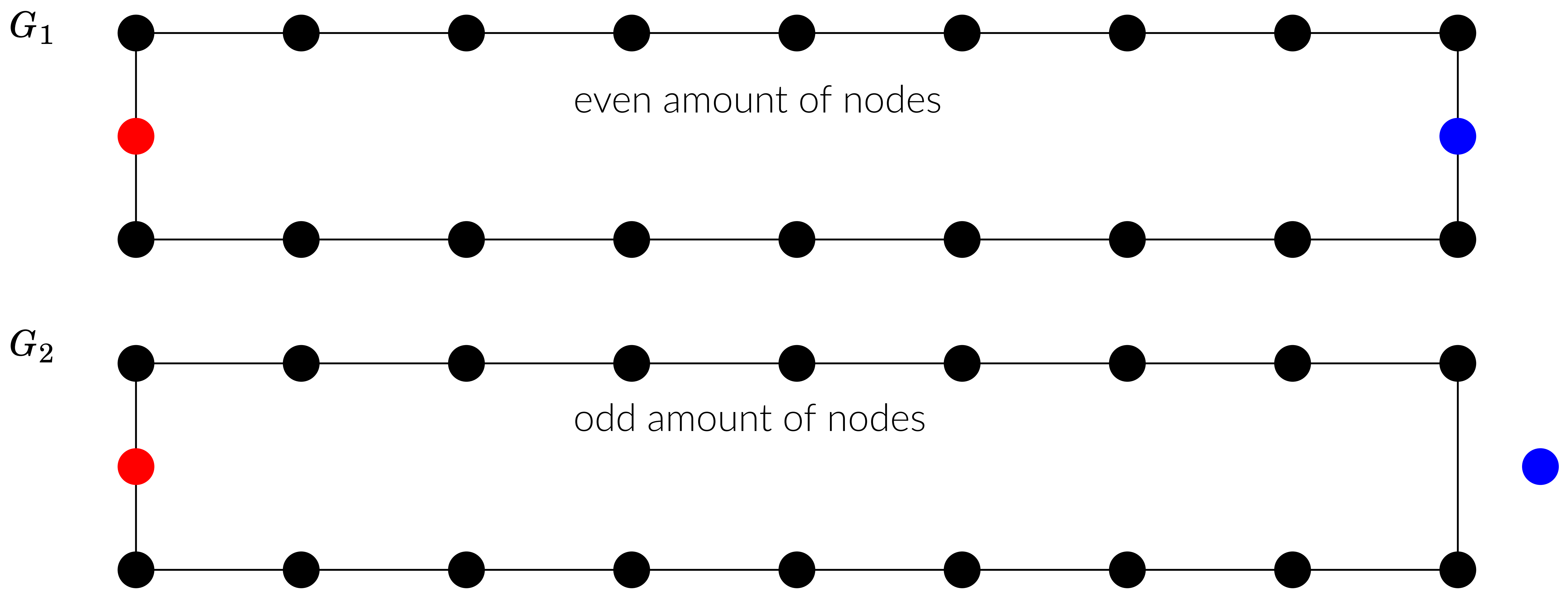

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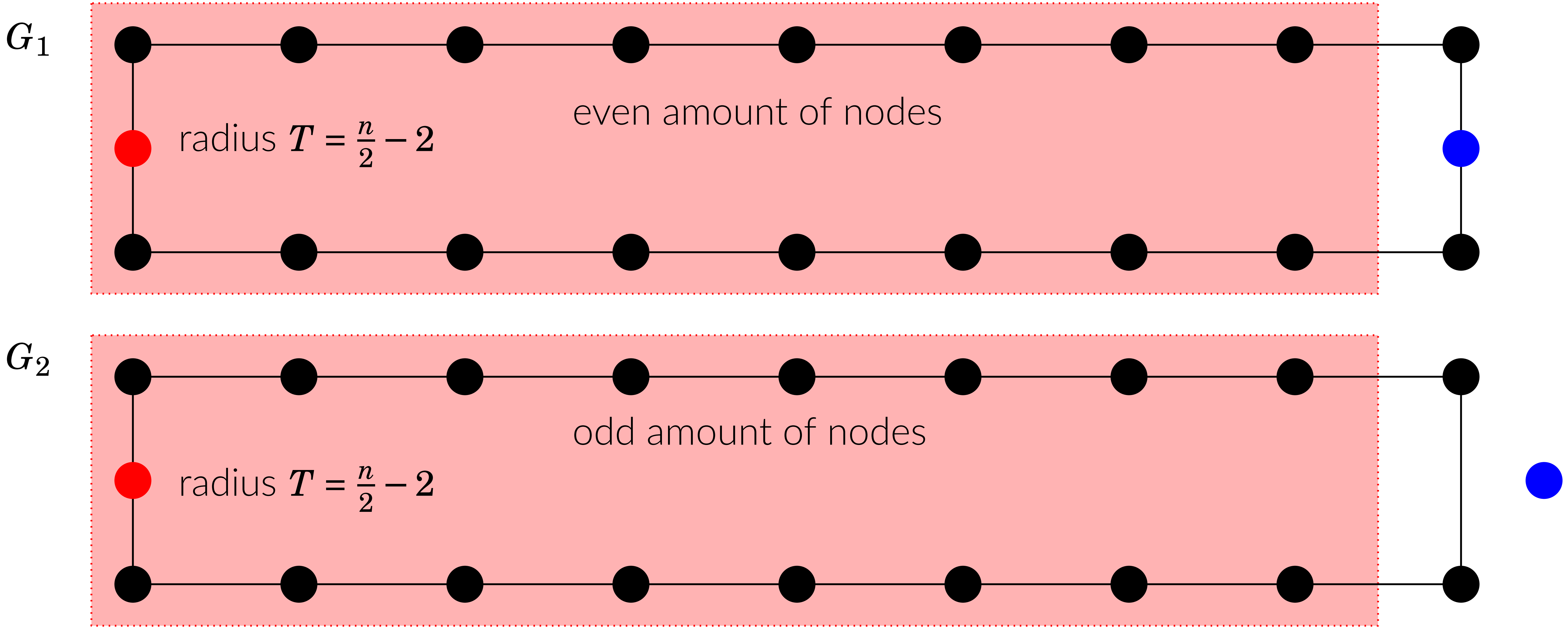

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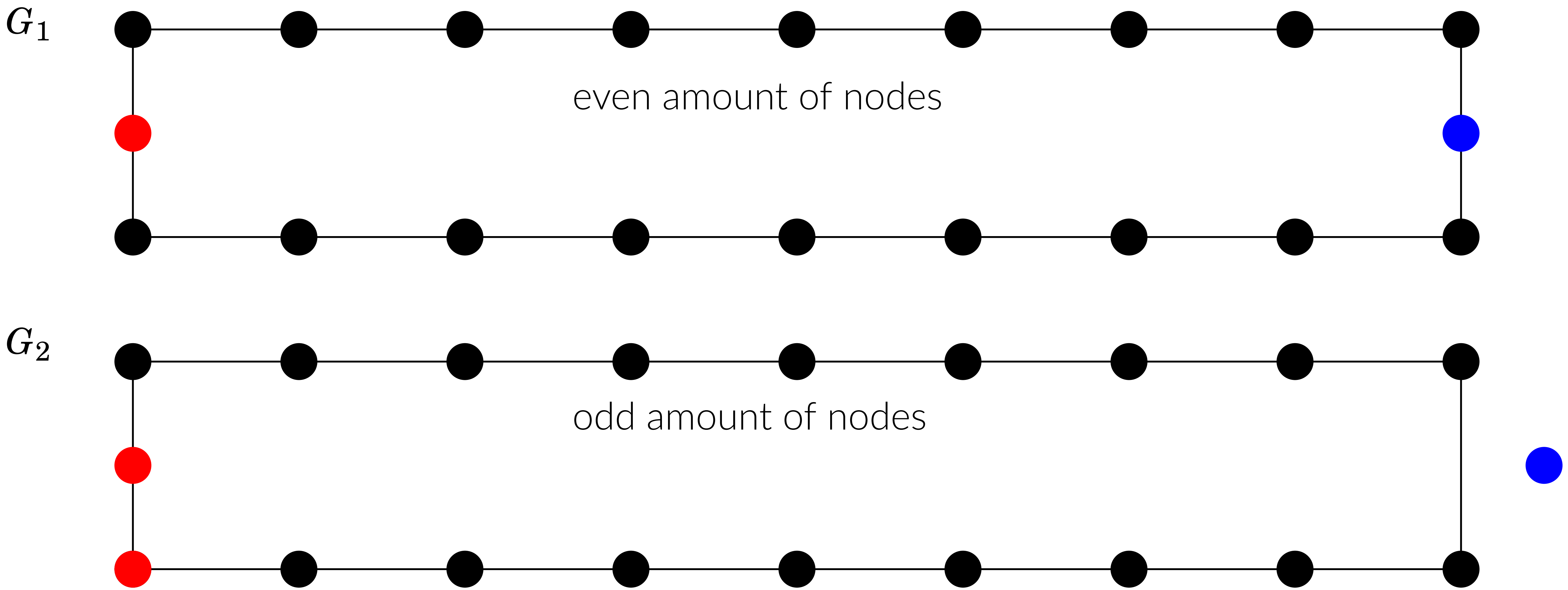

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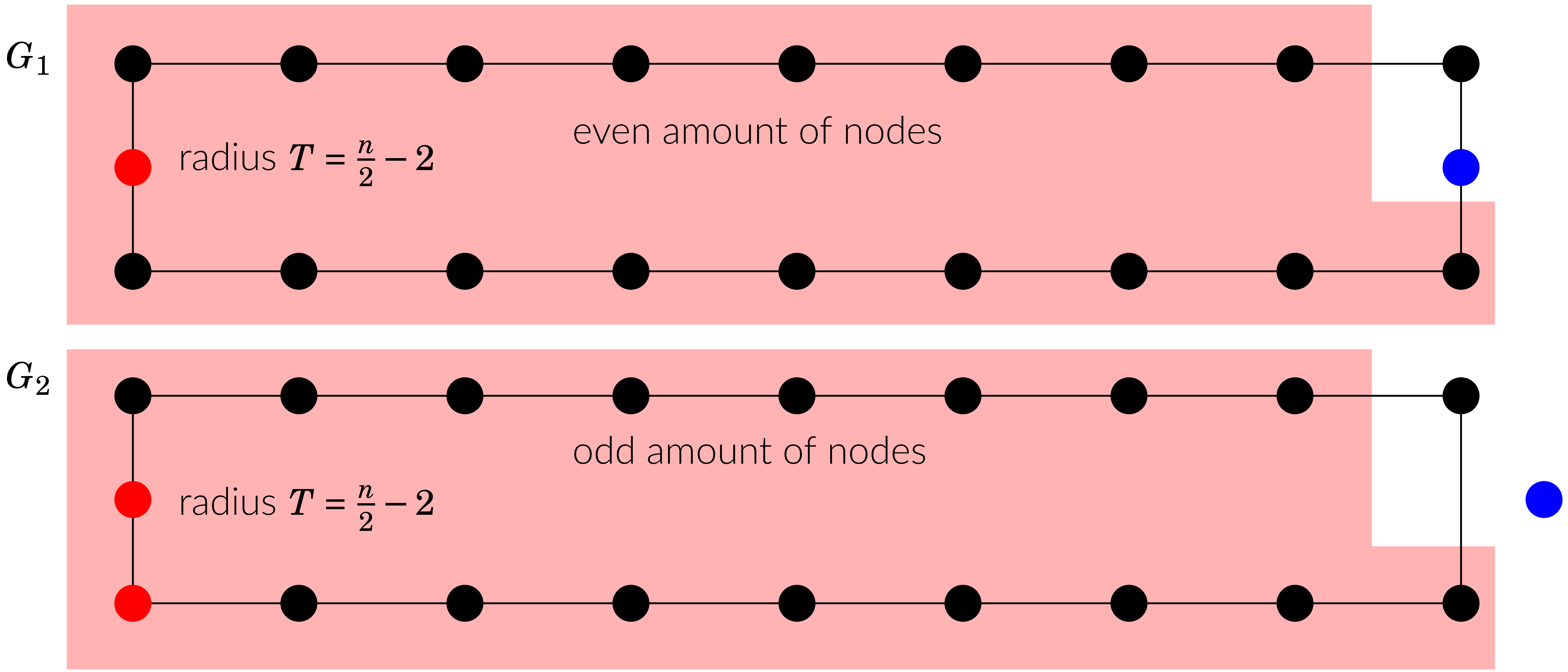

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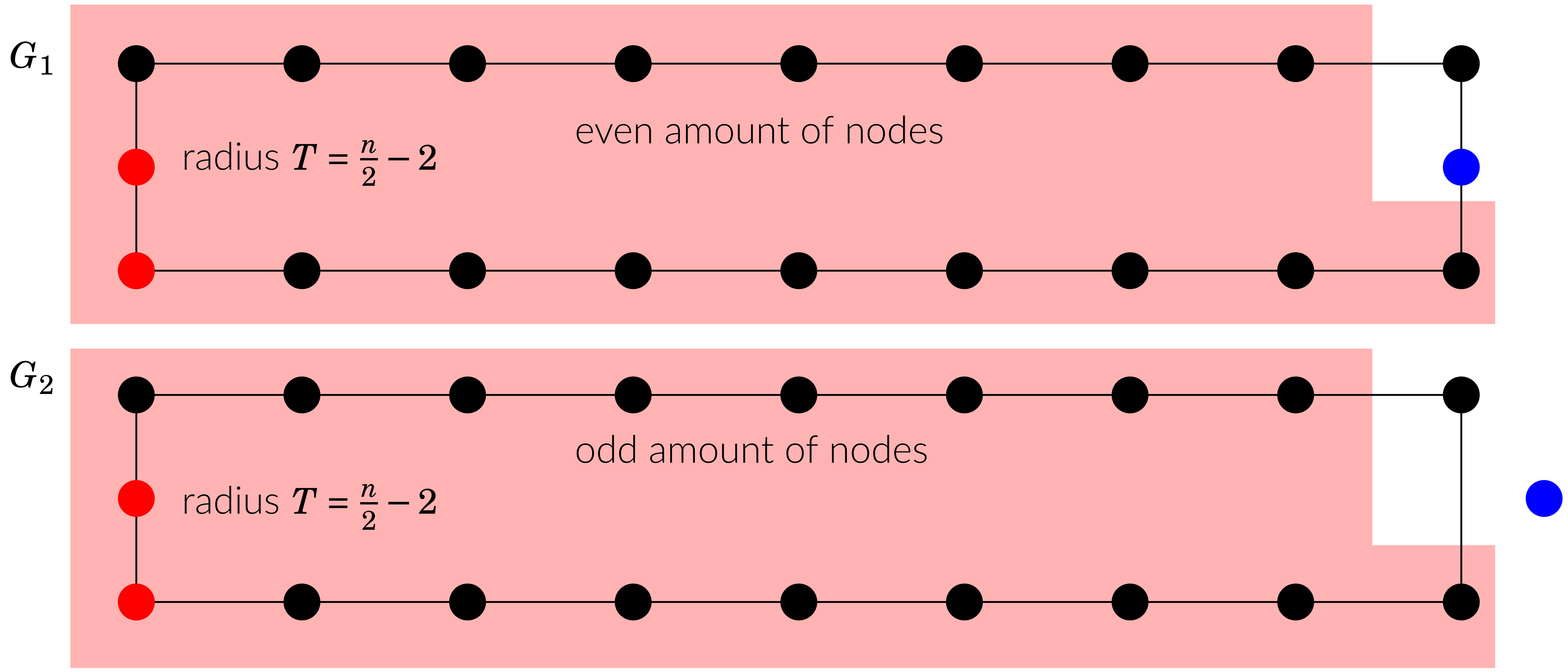

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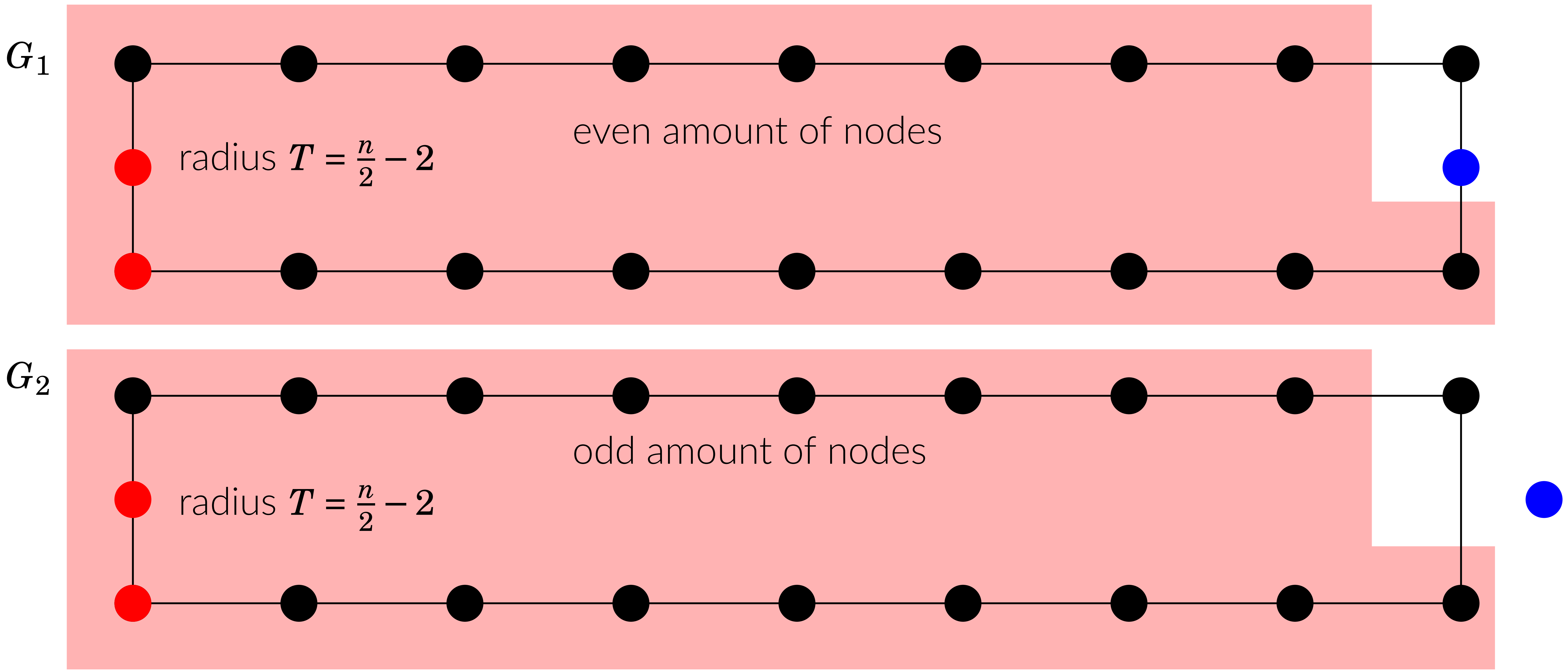


existence of $o(n)$-round LOCAL algorithm that 2-colors even cycles $\Longrightarrow$ 2-coloring of odd cycles

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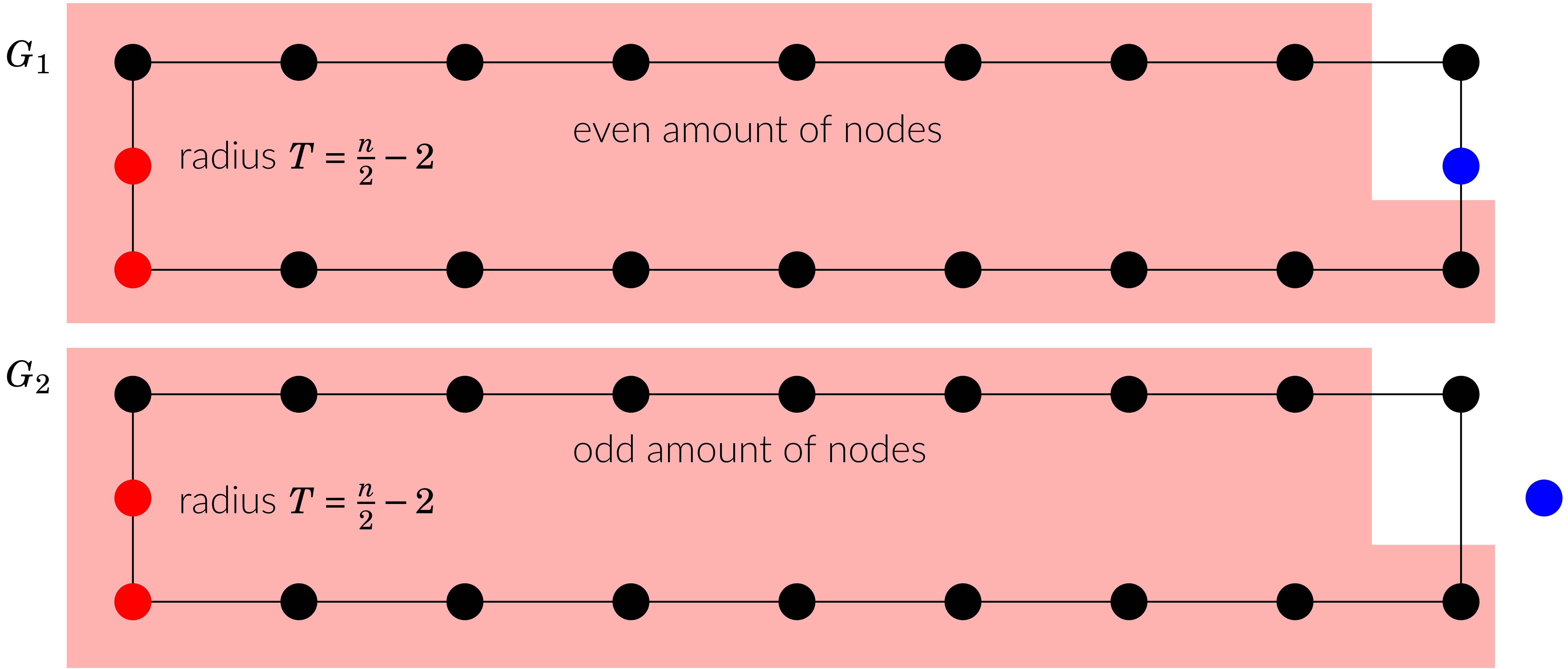
- **Problem**: 2-coloring even cycles requires locality $\Omega(n)$

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[Naor and Stockmeyer STOC '93 & SICOMP '95]

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  - "analogue" of NP in the distributed setting
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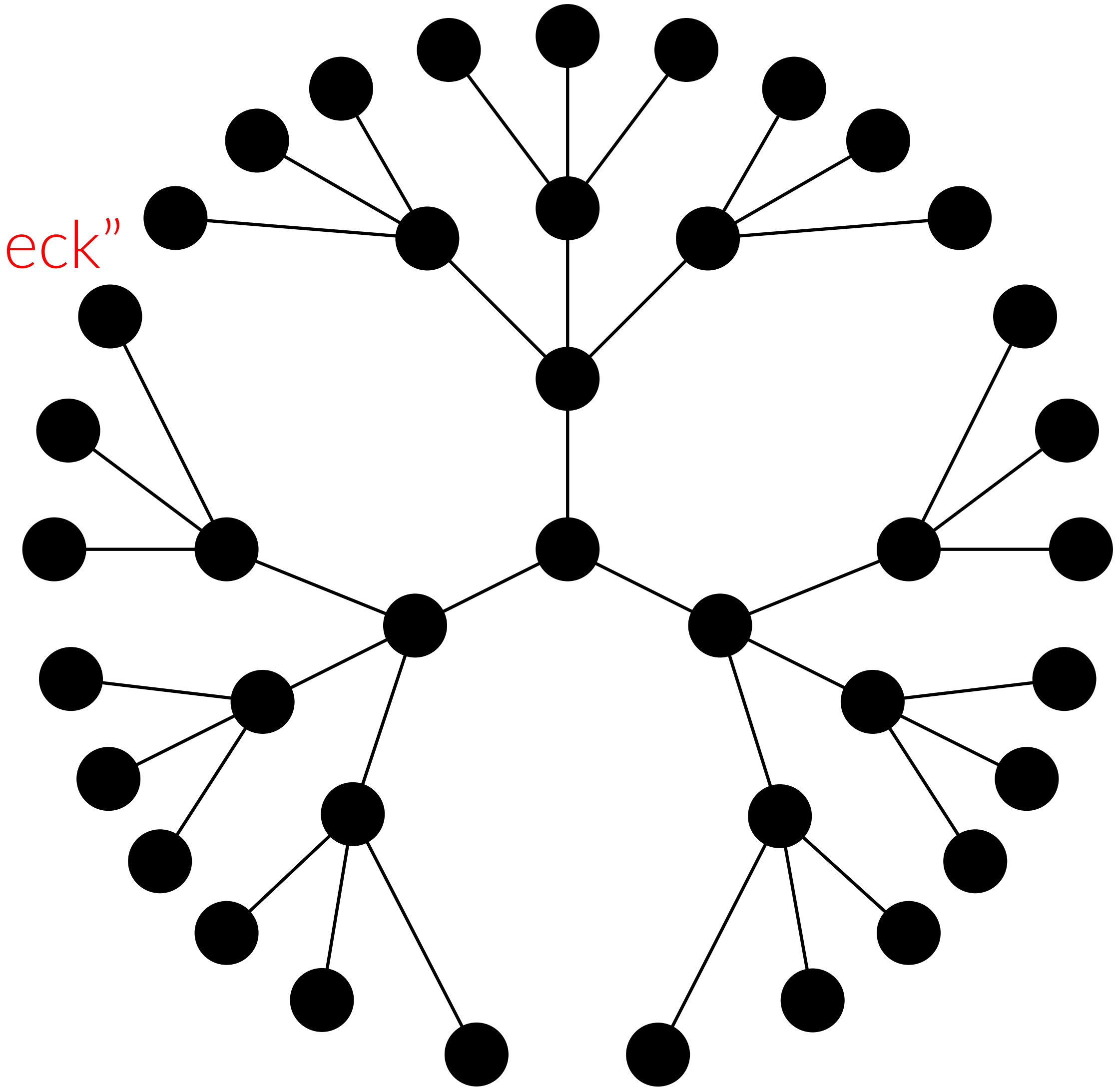

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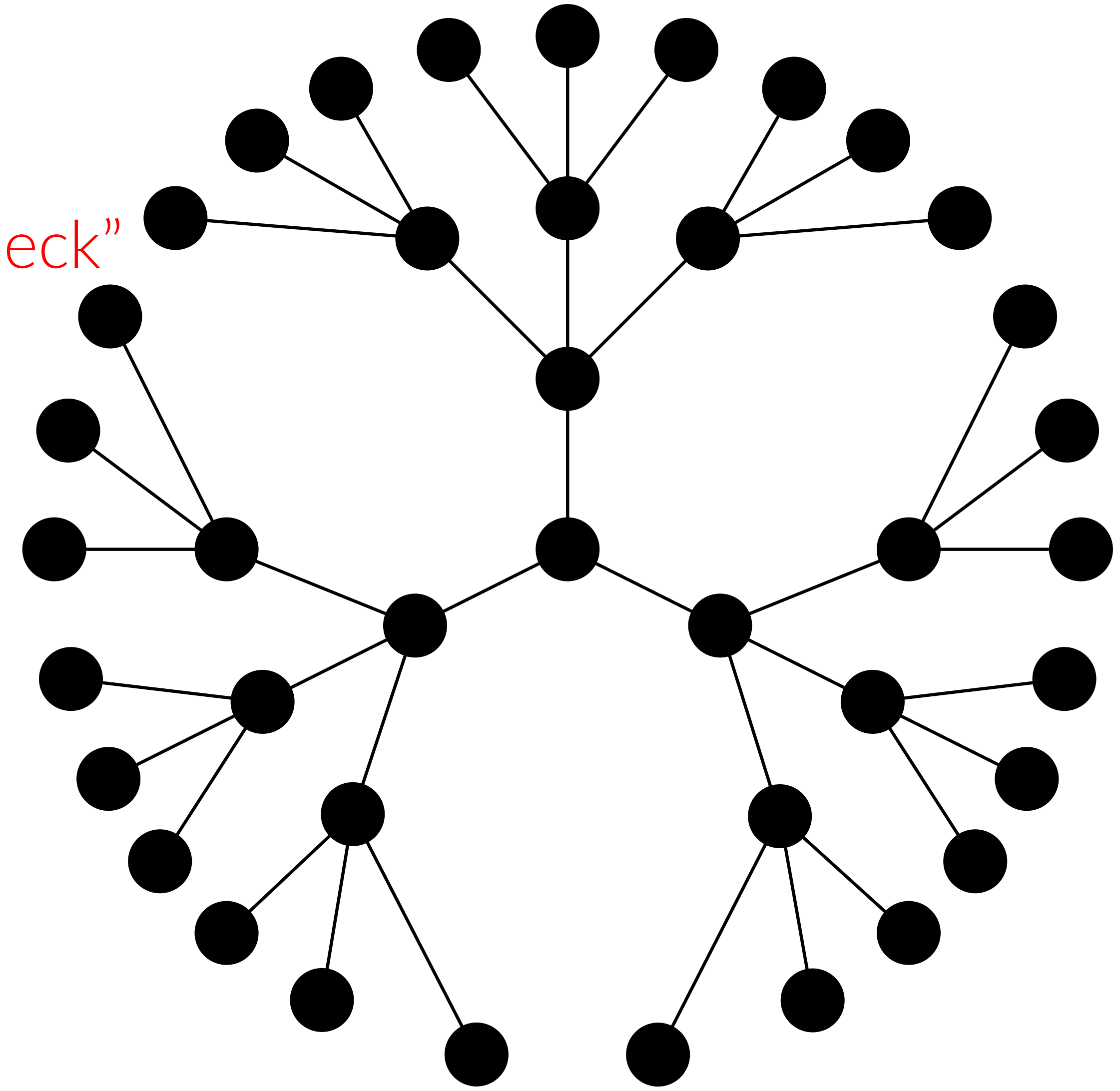

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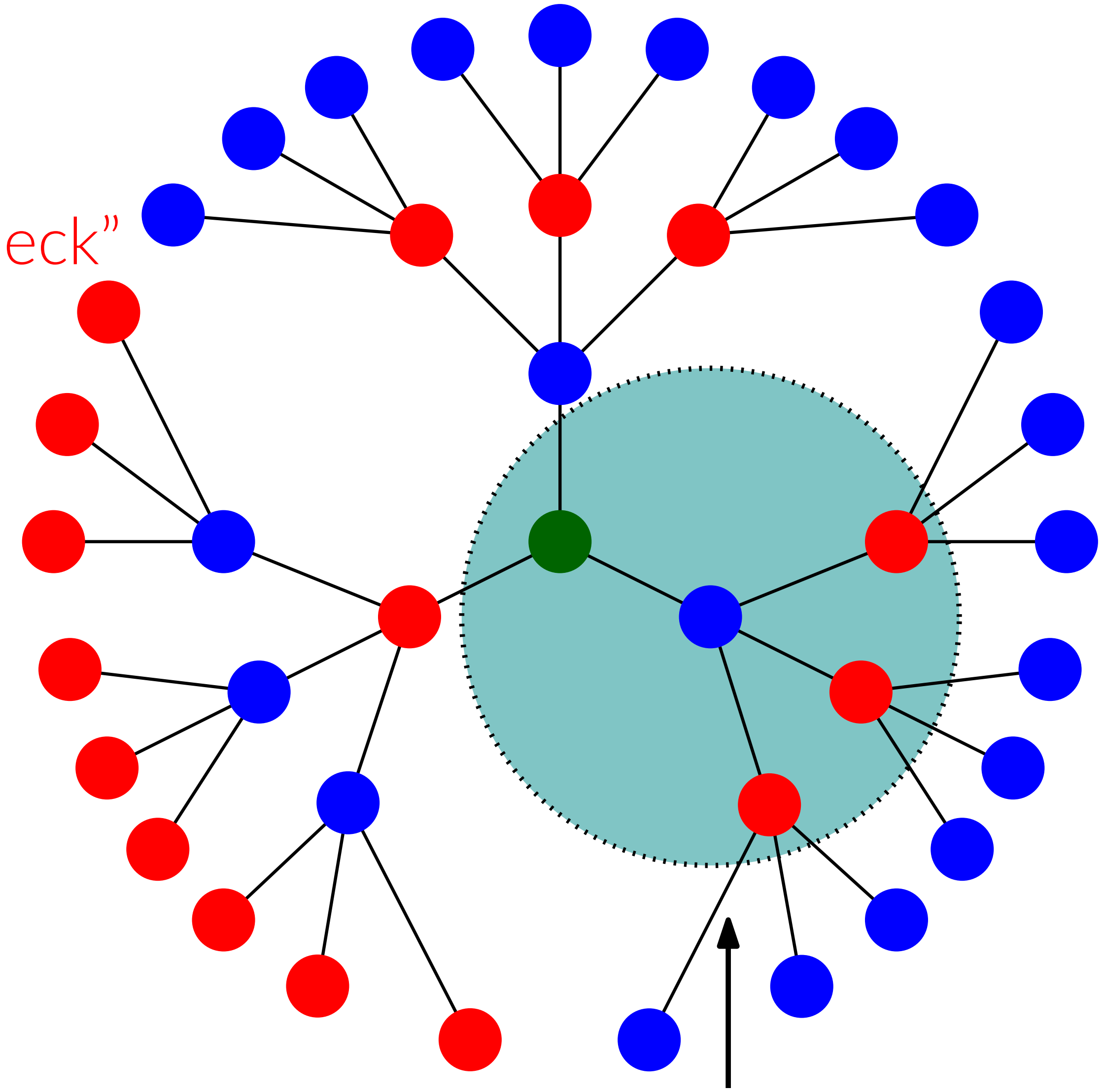

**3-coloring**: the blue node checks if its color is different from those of its neighbors

valid LCL

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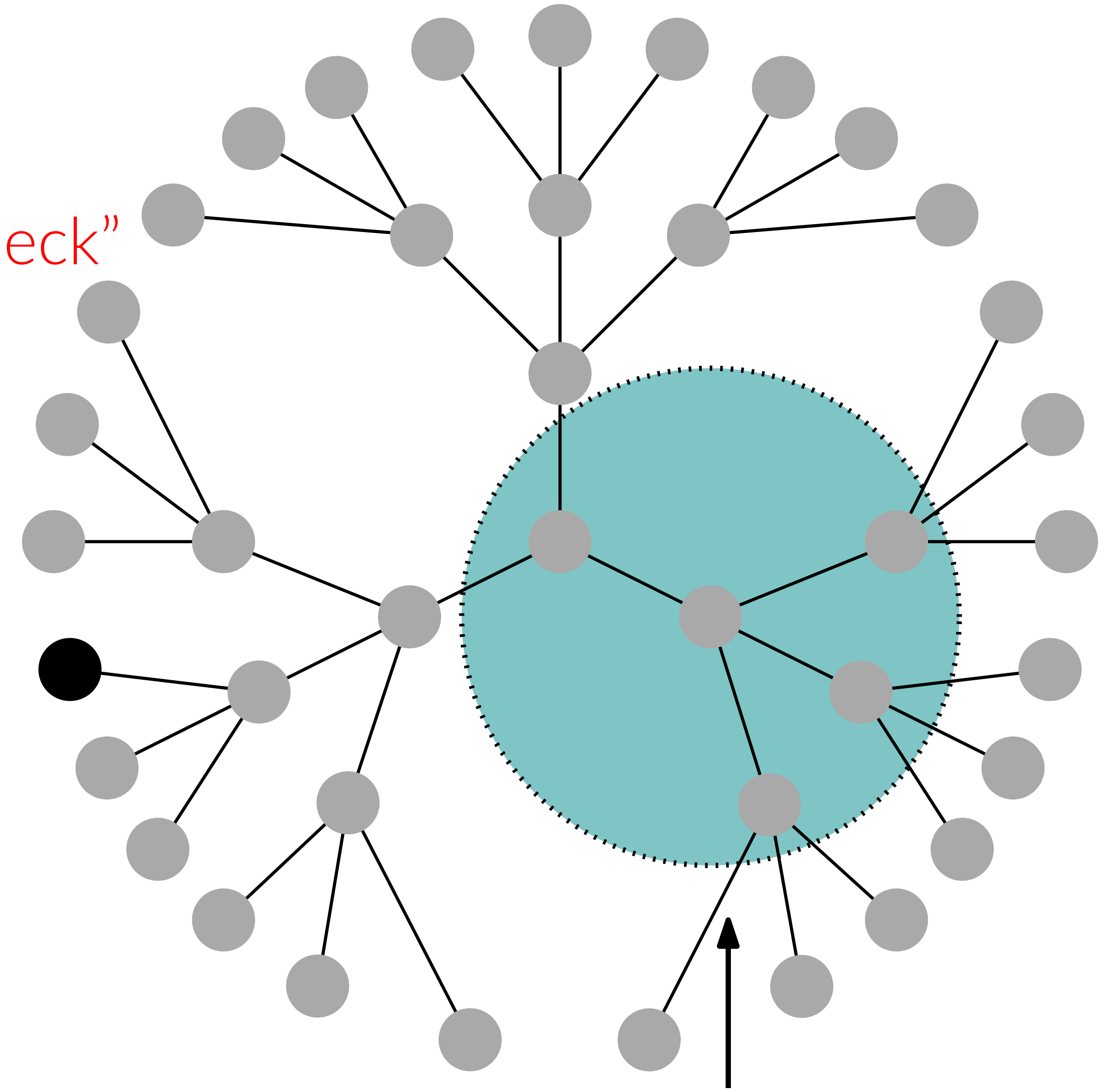

**Leader election**: the checking radius should be $r = \mathrm{diam}(G)$

not an LCL

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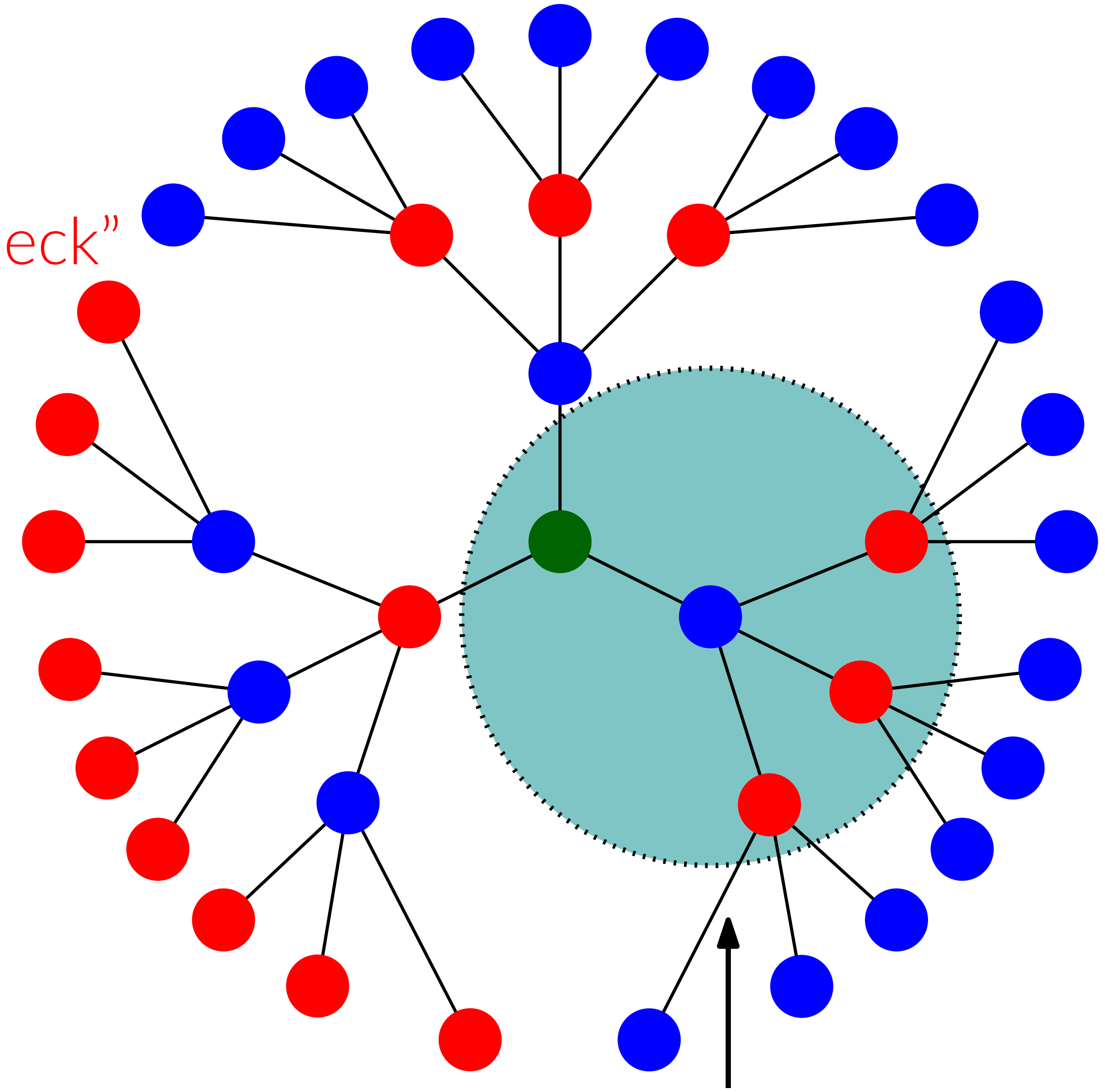

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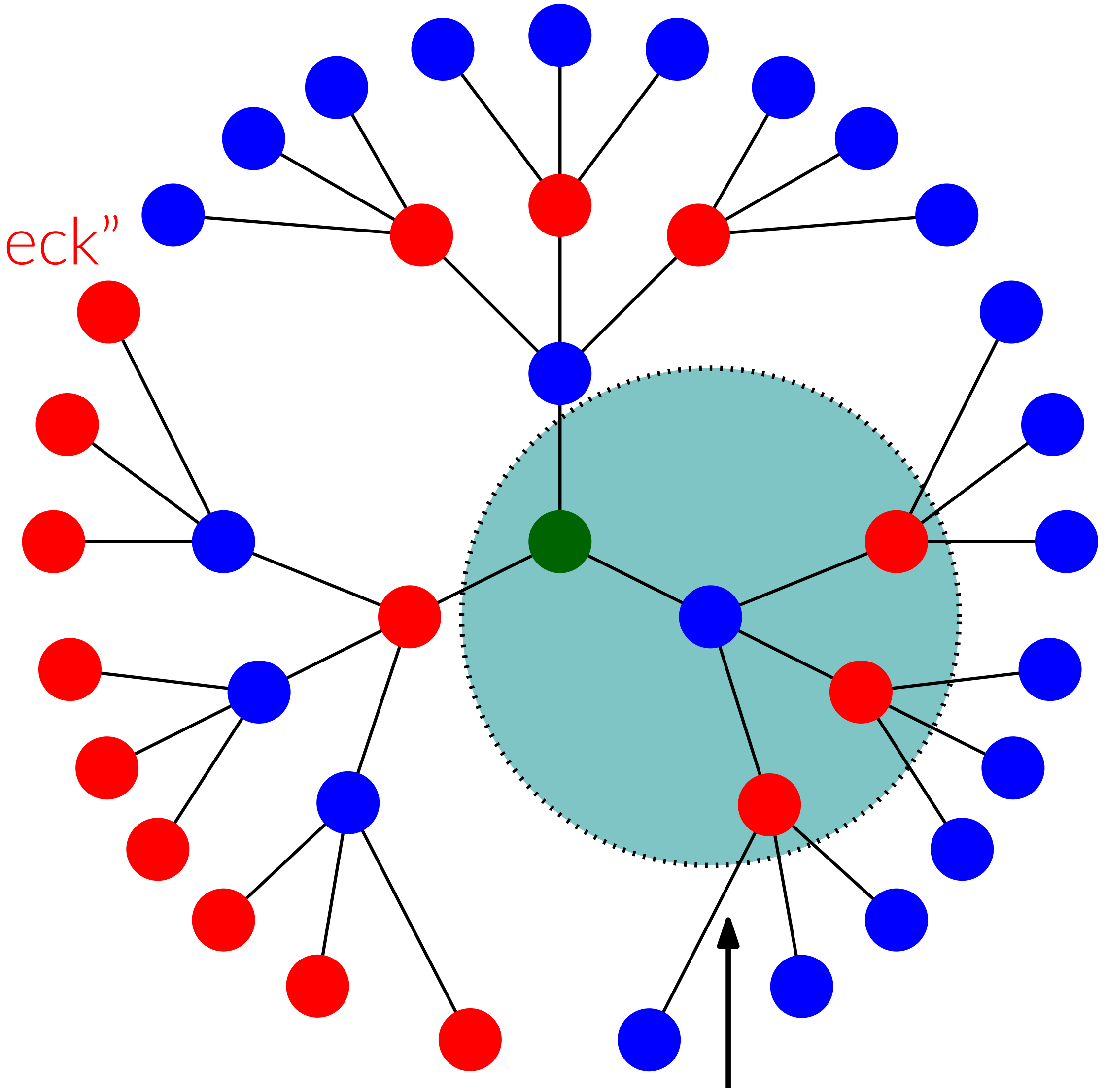

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- A **lot of literature** studying LCLs:
  - classification of LCLs based on complexity (locality)
  - e.g.: complexity $T(n)$ in randomized-LOCAL $\implies O(T(2^{n^2}))$ in deterministic-LOCAL [Chang et al. SICOMP '19]

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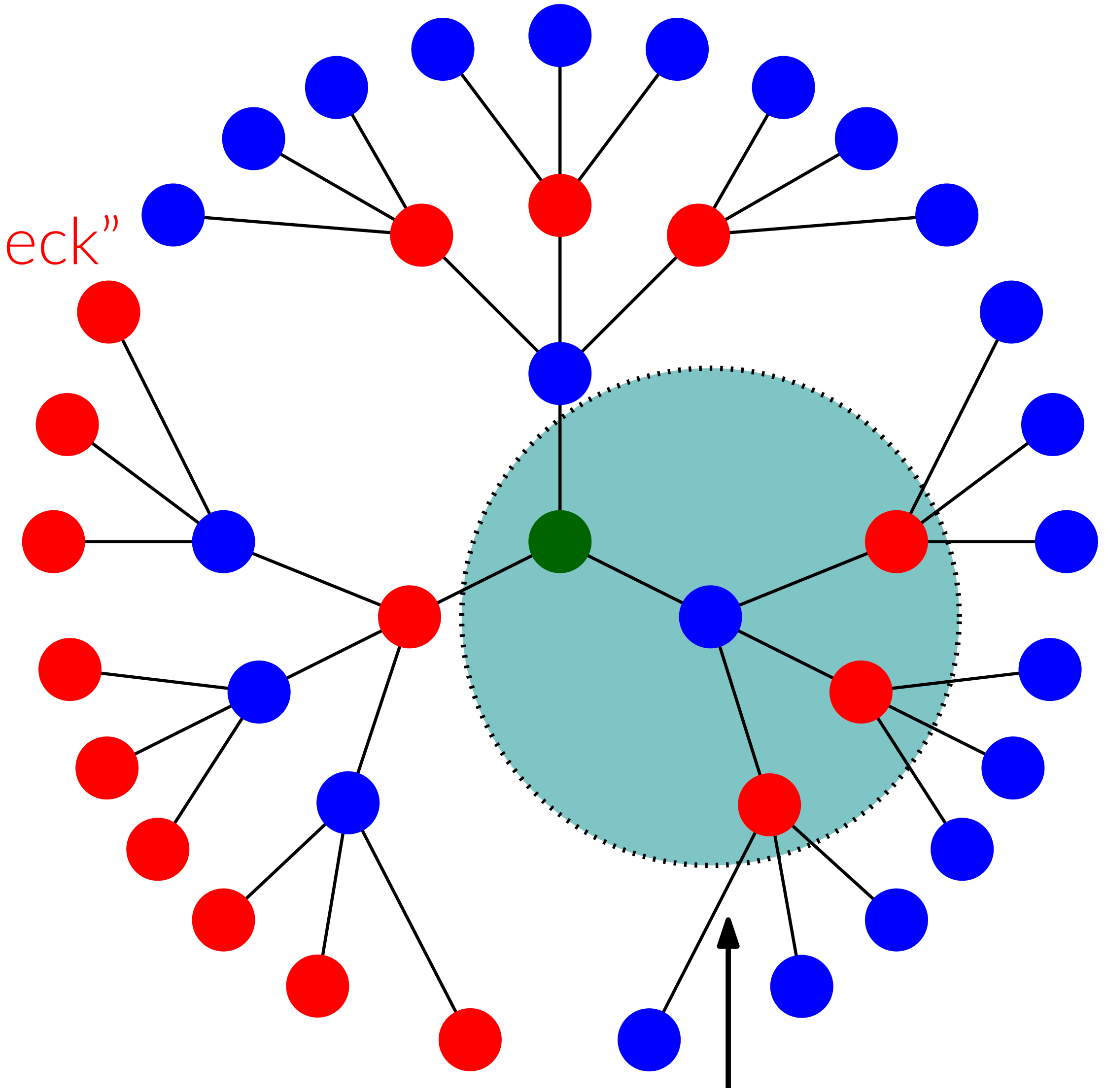

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  - [BFHKLRSU STOC '16; BHKLOPRSU PODC'17; GKM STOC '17; GHK FOCS '18; CP SICOMP '19; BHKLOS STOC '18; BBCORS PODC '19; BBOS PODC '20; BBHORS JACM '21; BBCOSS DISC '22; AELMSS ICALP '23; etc.]

# Complexity landscape of LCLs

- **Paths and cycles**

| | | | |
|---|---|---|---|
| det-LOCAL | $O(1)$ | $\Theta(\log^\star n)$ | $\Theta(n)$ |
| rand-LOCAL | $O(1)$ | $\Theta(\log^\star n)$ | $\Theta(n)$ |

- **Balanced $d$-dimensional toroidal grids**

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- **Bounded-degree trees**

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|---|---|---|---|
| det-LOCAL | $O(1)$ | $\Theta(\log^\star n)$ | $\Theta(n)$ |
| rand-LOCAL | $O(1)$ | $\Theta(\log^\star n)$ | $\Theta(n)$ |

- **Balanced $d$-dimensional toroidal grids**

| | | | |
|---|---|---|---|
| det-LOCAL | $O(1)$ | $\Theta(\log^\star n)$ | $\Theta(n^{1/d})$ |
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→ randomness does not help

- **General graphs??**

- **Bounded-degree trees**

<table>
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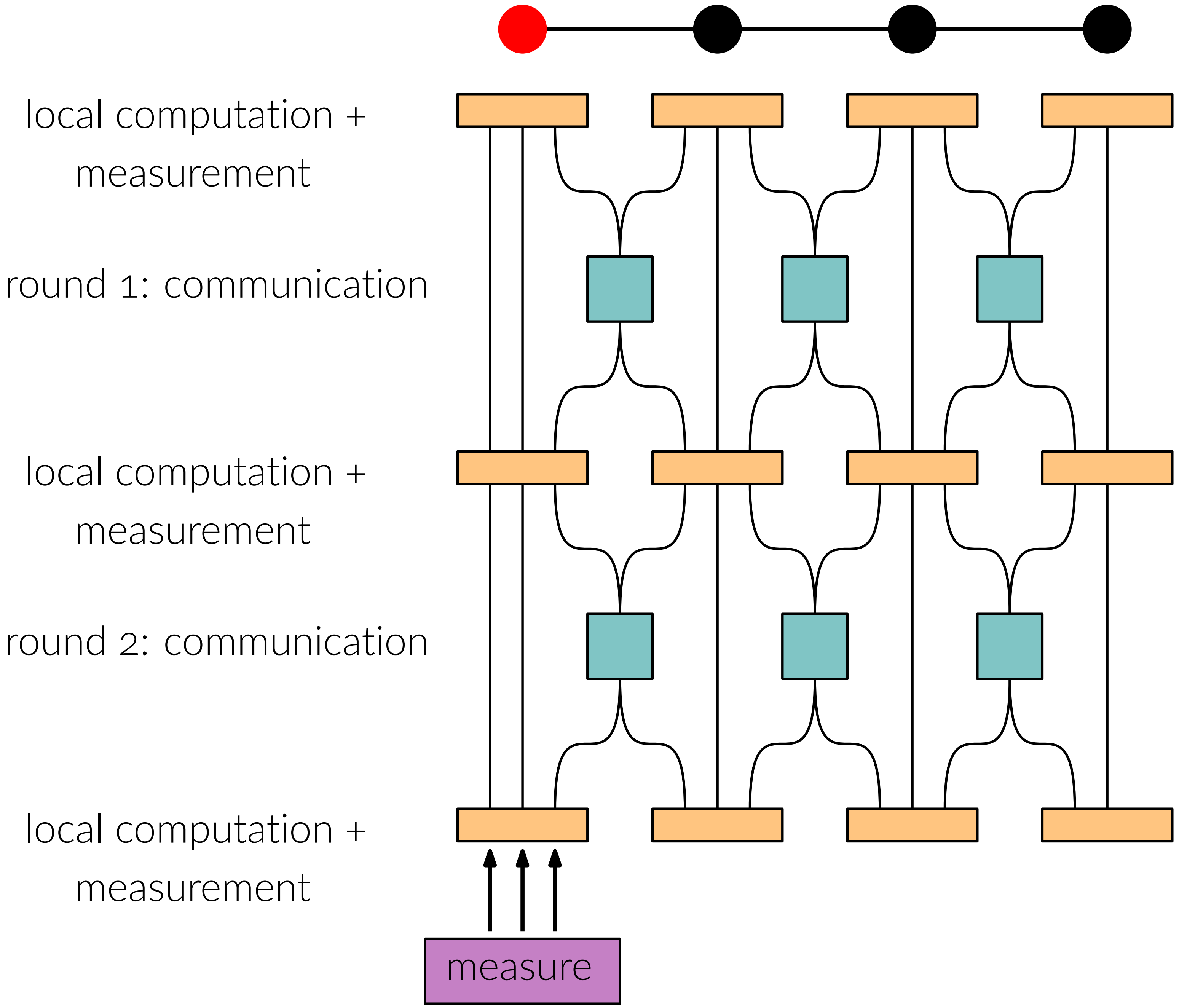
# Quantum-LOCAL

[Gavoille et al. DISC '09]

- **Distributed system** of $n$ quantum processors/nodes
  - quantum computation
  - quantum communication (qubits)
  - output: measurement of qubits
- **Complexity measure**: number of communication rounds



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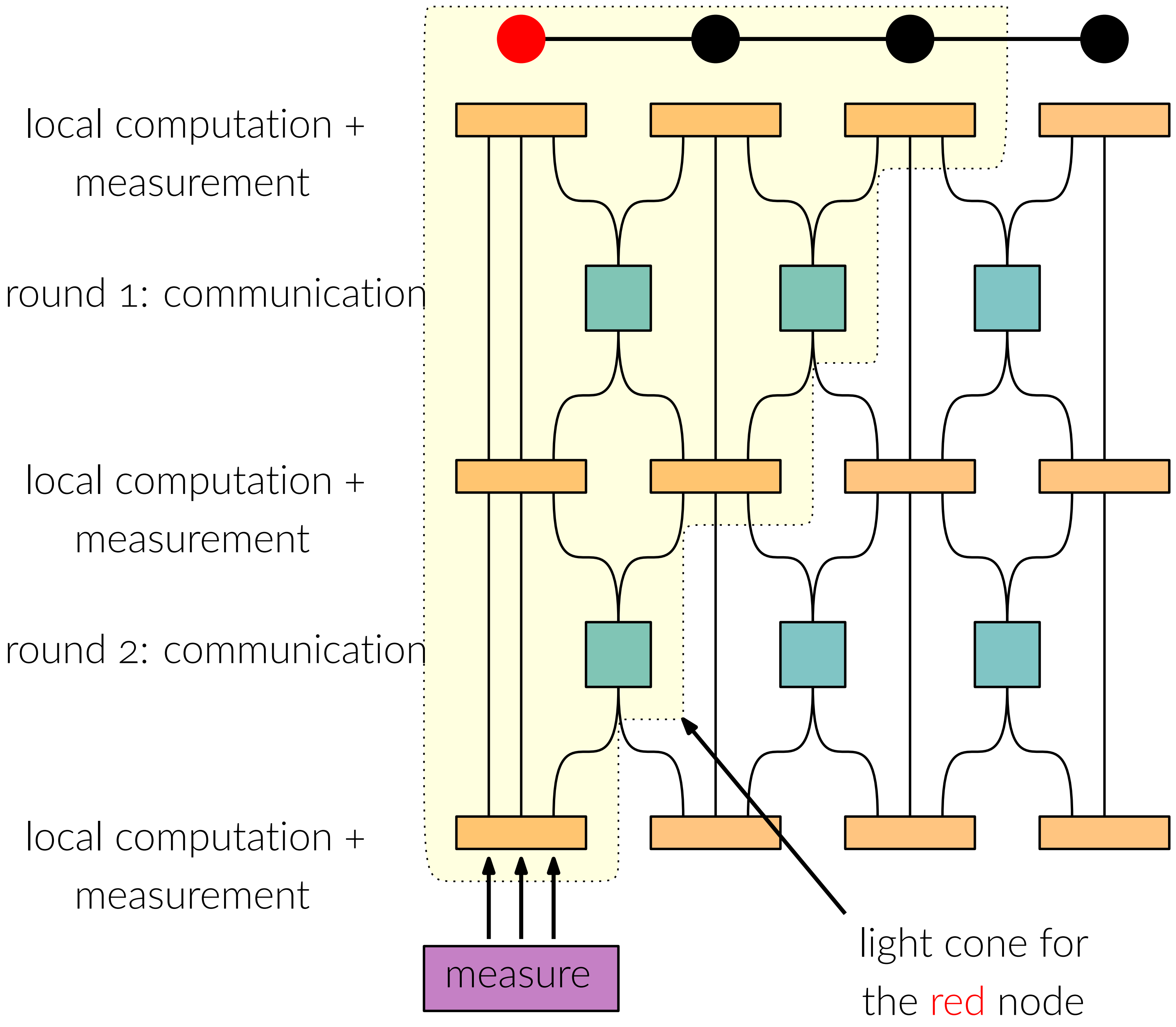

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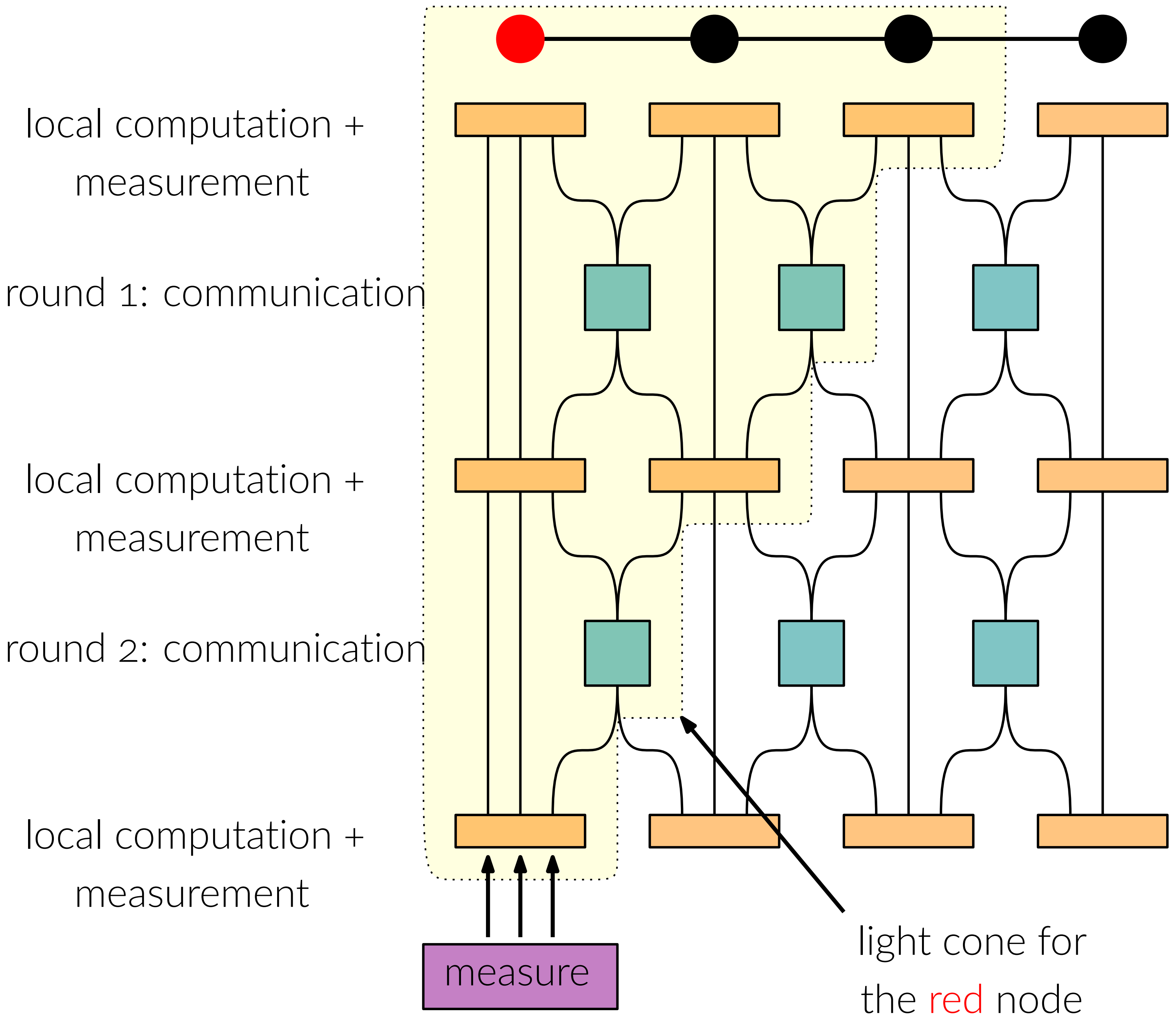


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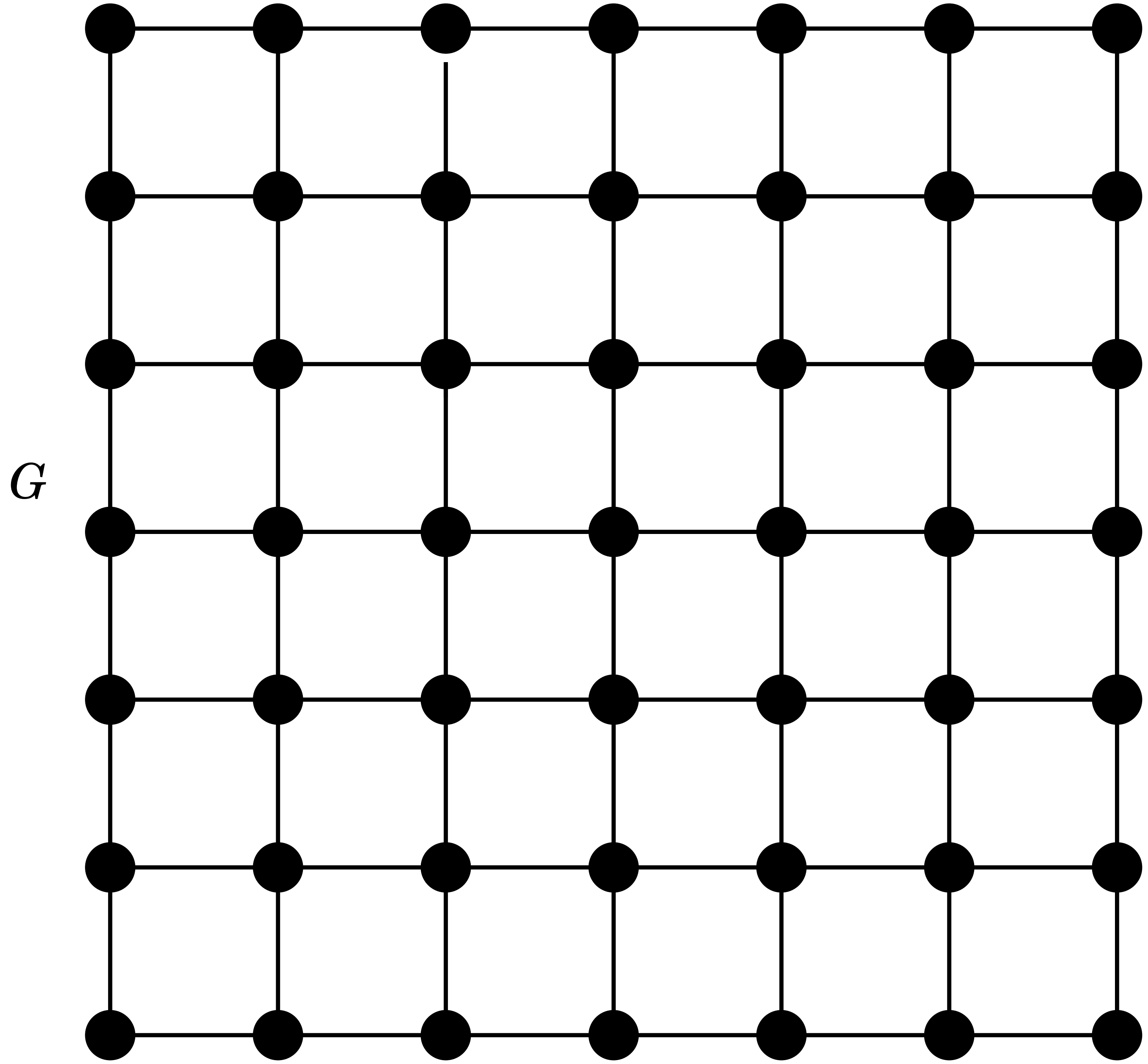
- *What do we know?*
  - focus on LCLs
  - input graph degree is bounded by a constant $\Delta$ [Naor and Stockmeyer SICOMP '95]

- **Run** a $2$-round algorithm $A$ in $G$



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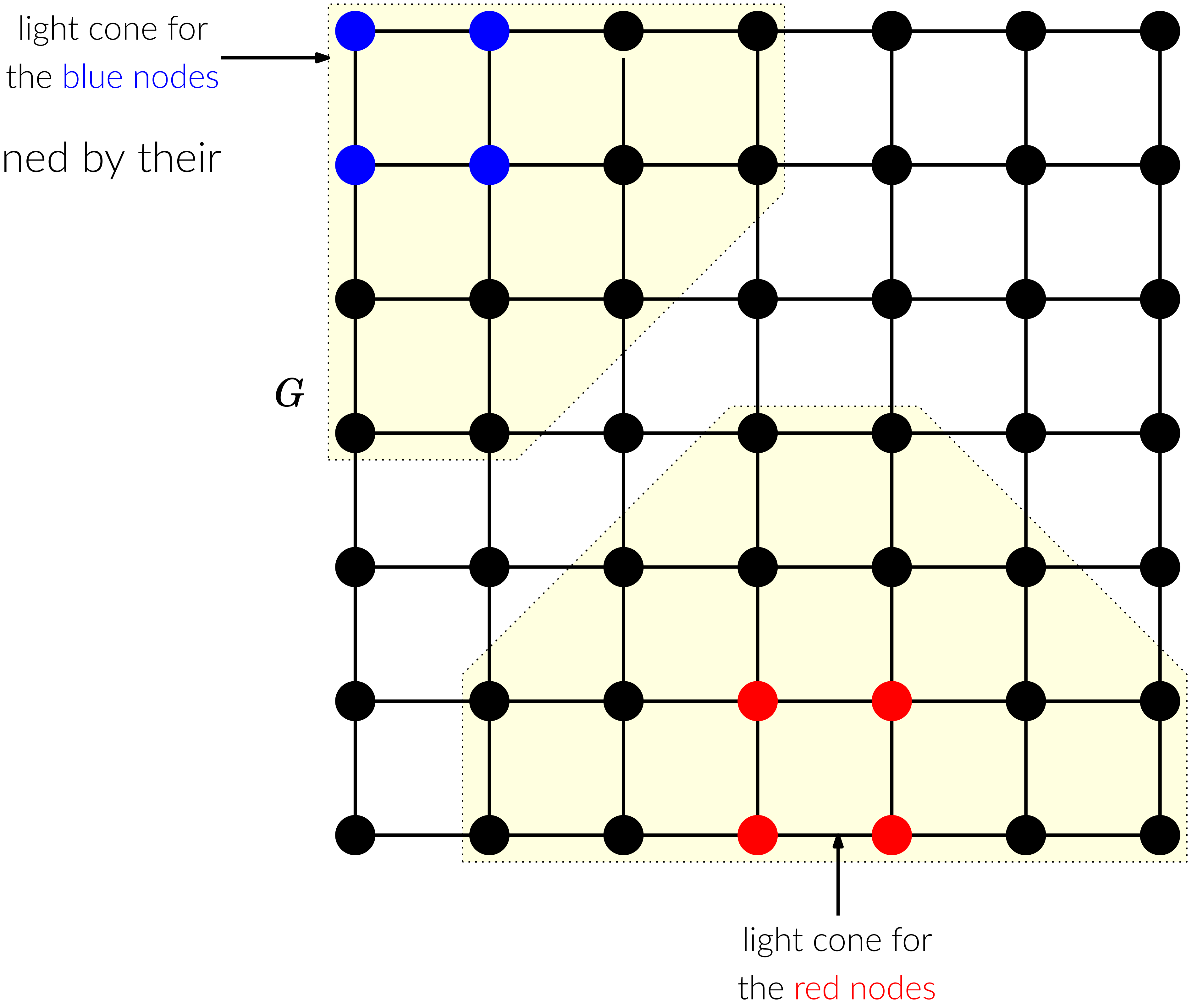

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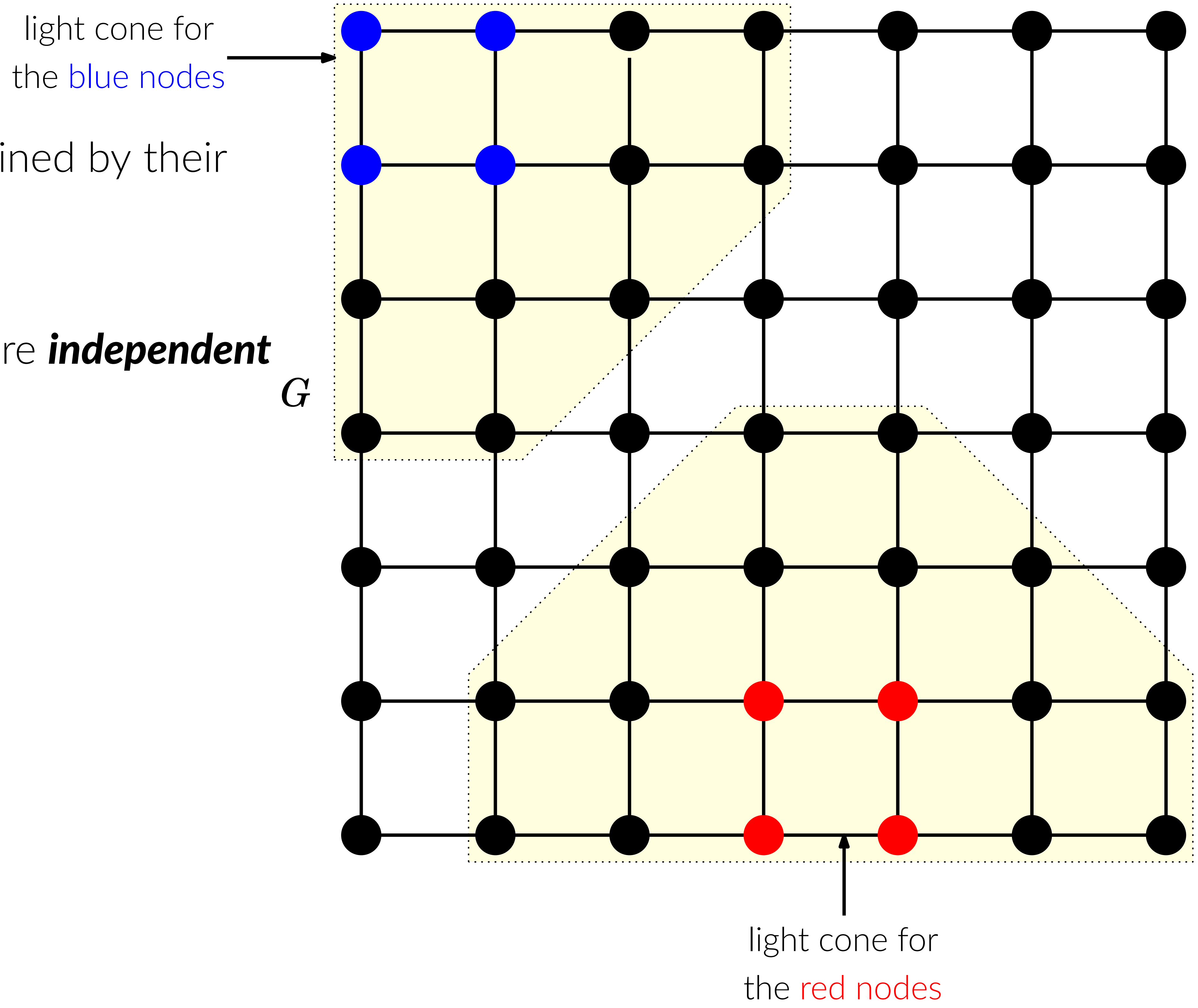

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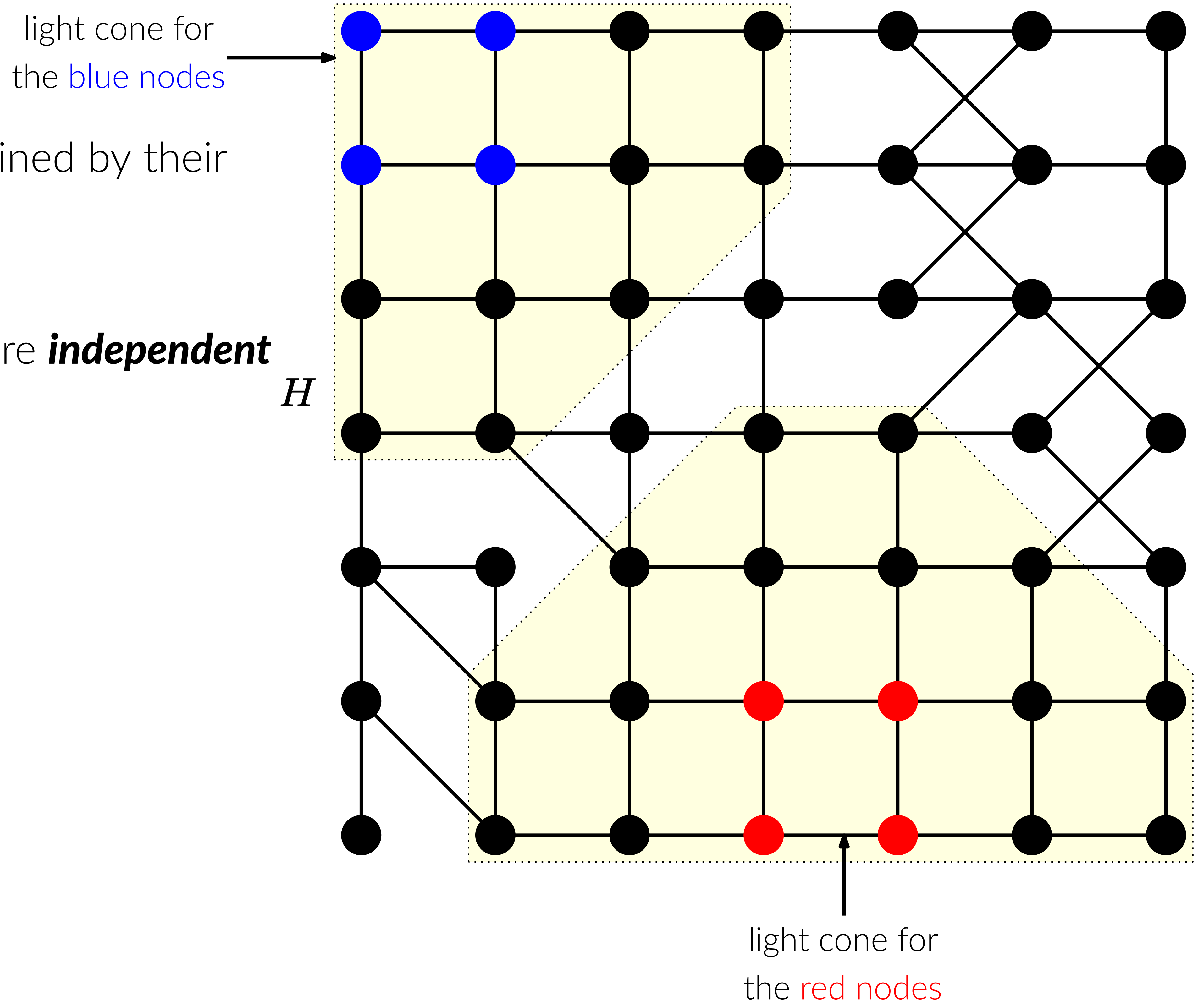

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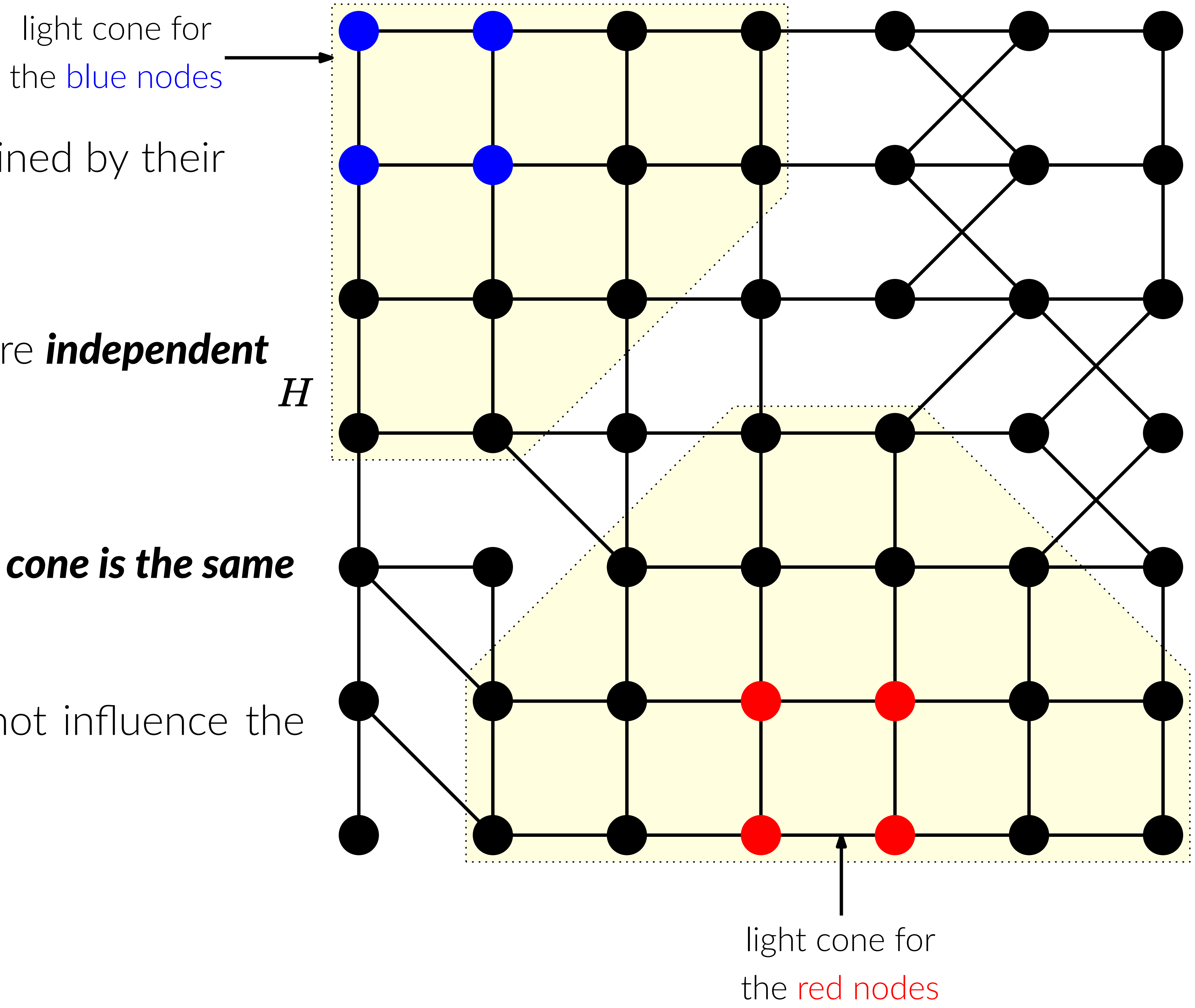
- **Output distributions** remains ***the same*** if ***light cone is the same***
  - non-signaling property
  - changes that are beyond $2$-hops away do not influence the output distribution
  - also known as causality



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bounded-dependence model $\longrightarrow$ non-signaling model

bounded-dependence model: locality $T$ = independence at distance $2T + 1$ plus non-signaling beyond distance $T$

[Holroyd and Liggett Forum of Mathematics Pi '14]
[Akbari et al. STOC '24]* finitely-dependent distributions if $T = O(1)$

non-signaling model: locality $T$ = non-signaling beyond distance $T$

[Gavoille et al. DISC '09]
[Arfaoui and Fraigniaud PODC '12 & SIGACT News '14]

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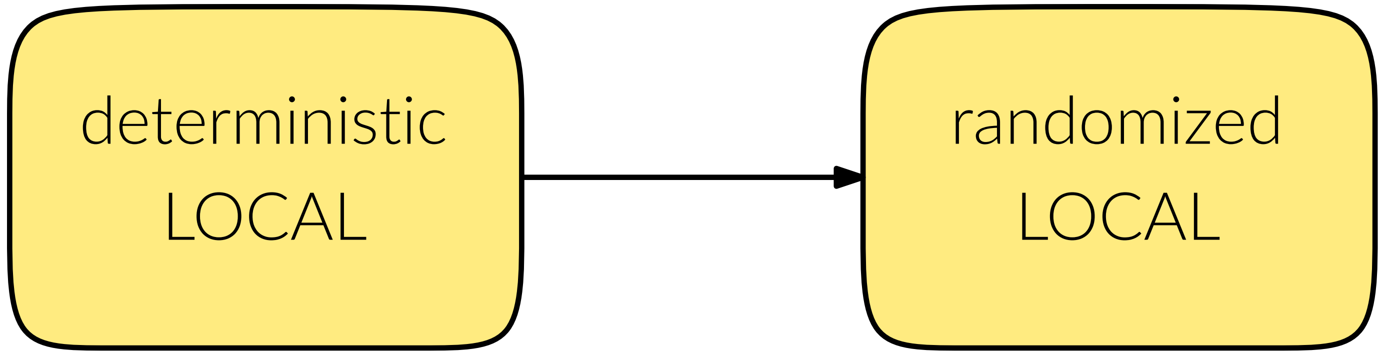

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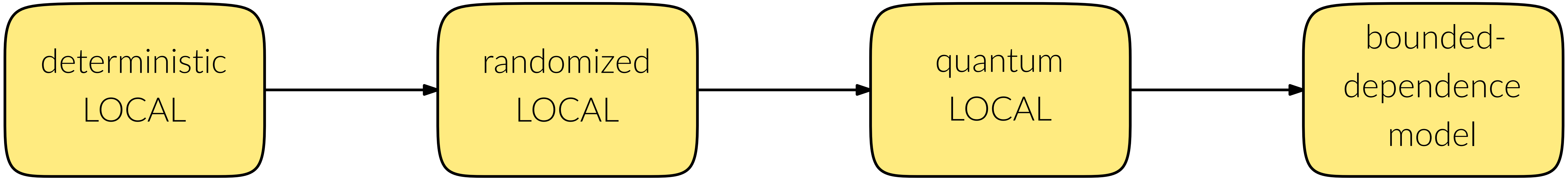

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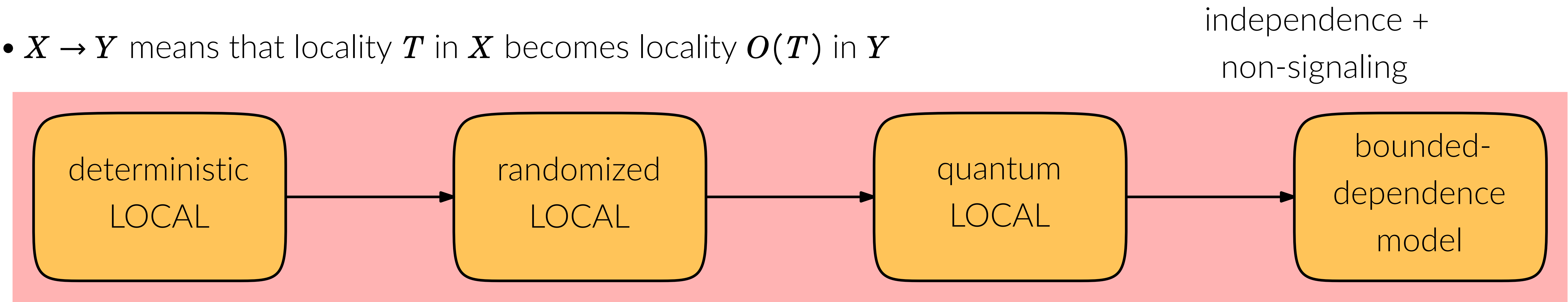

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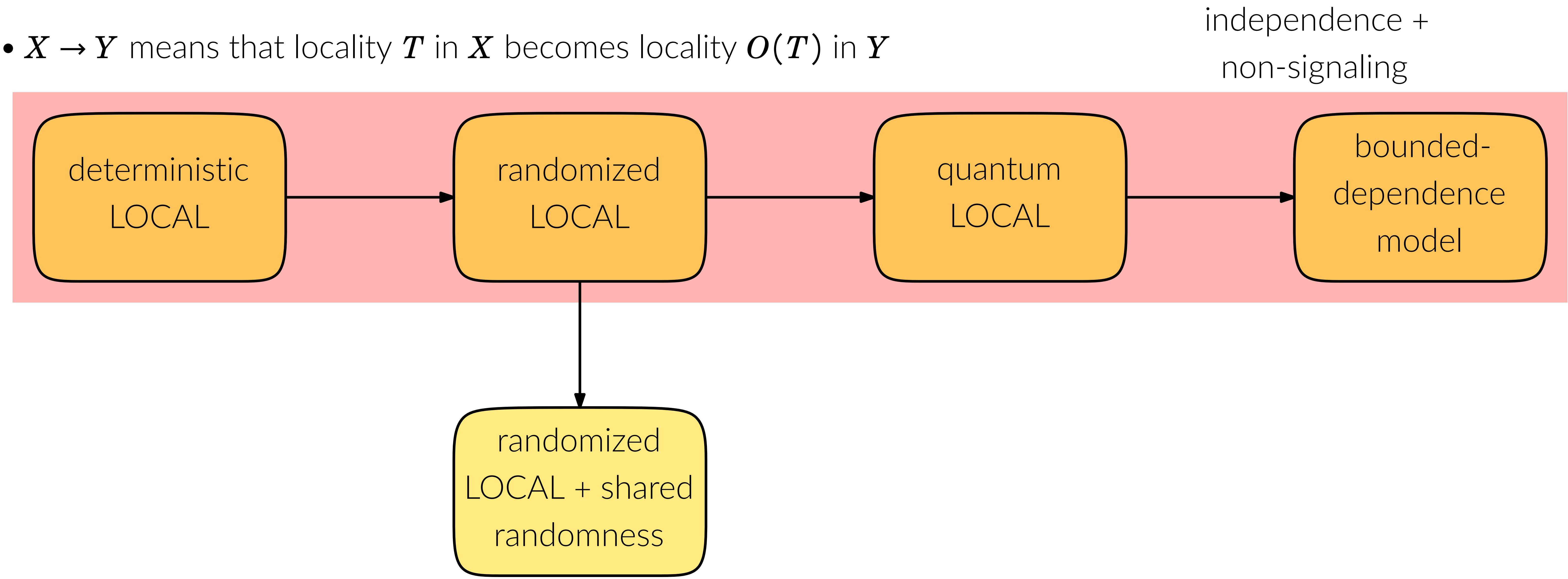

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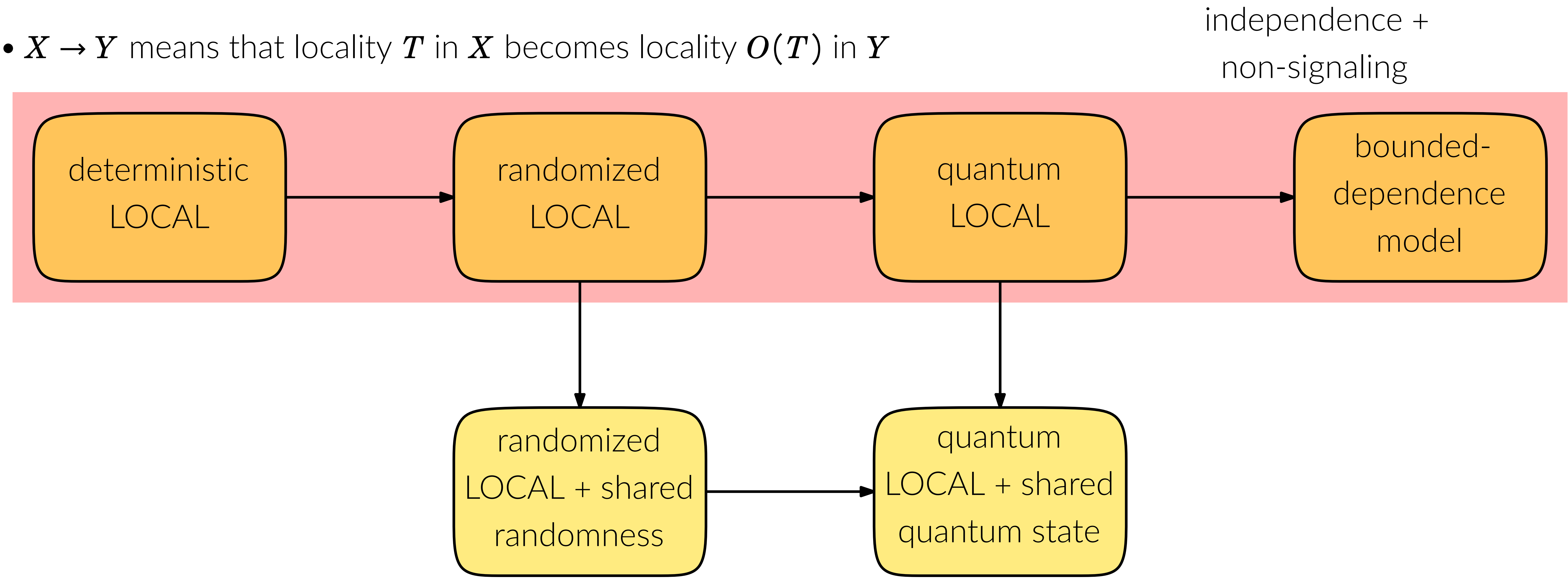

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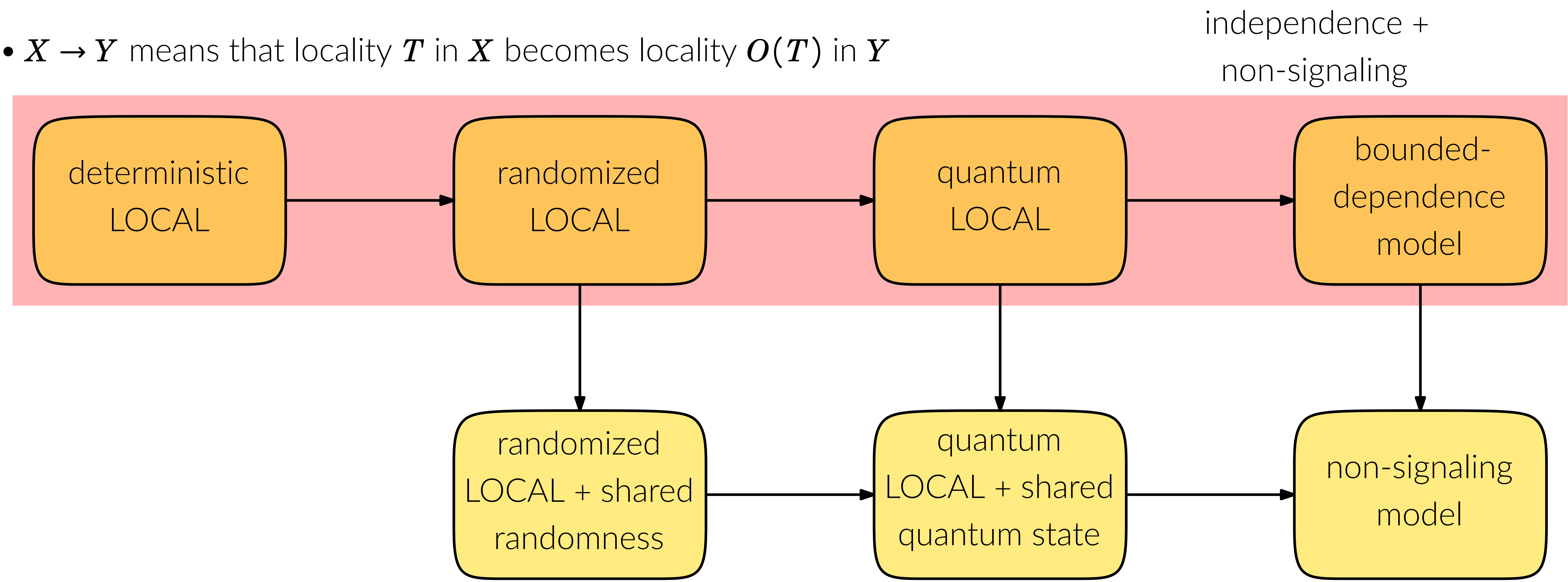

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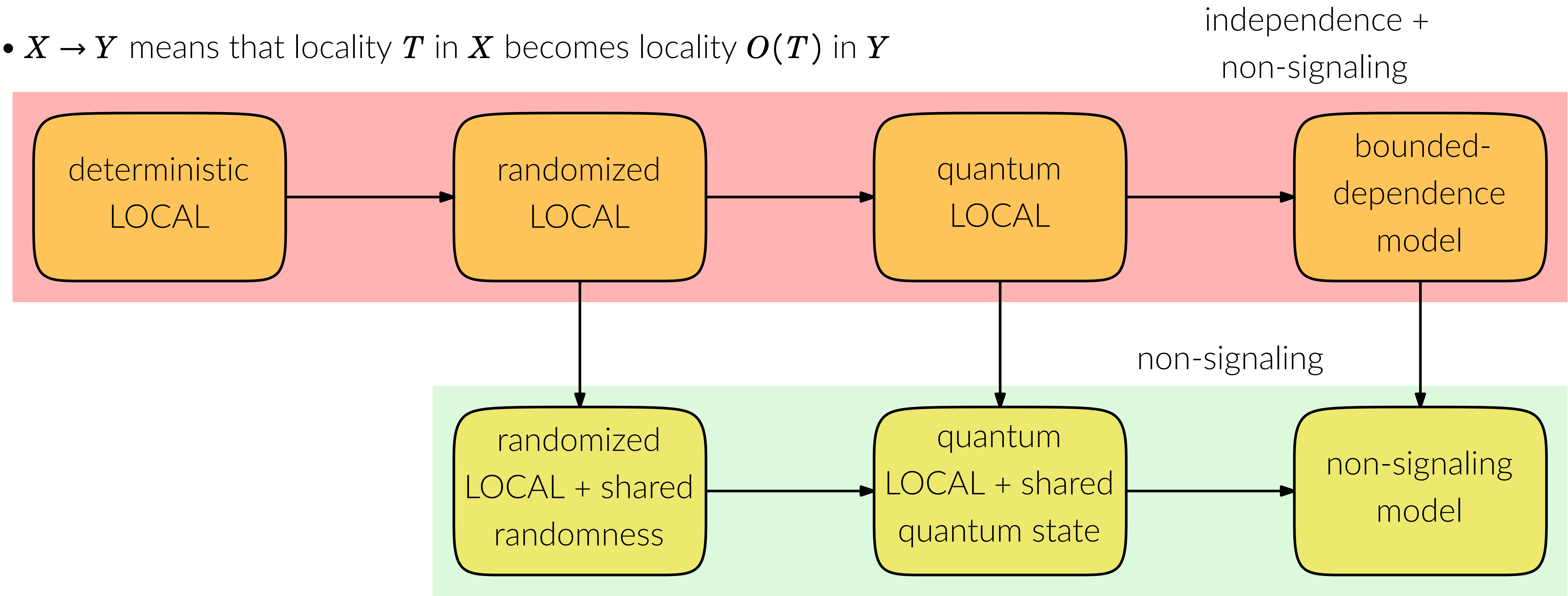

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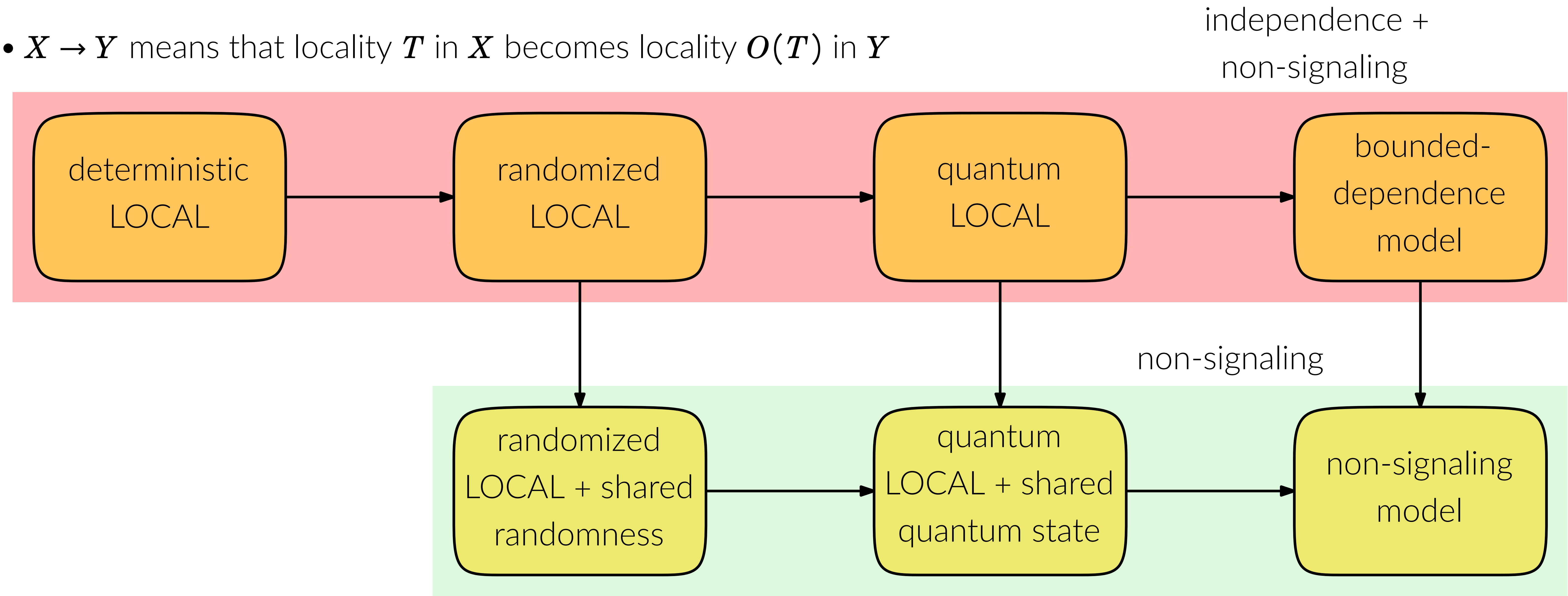


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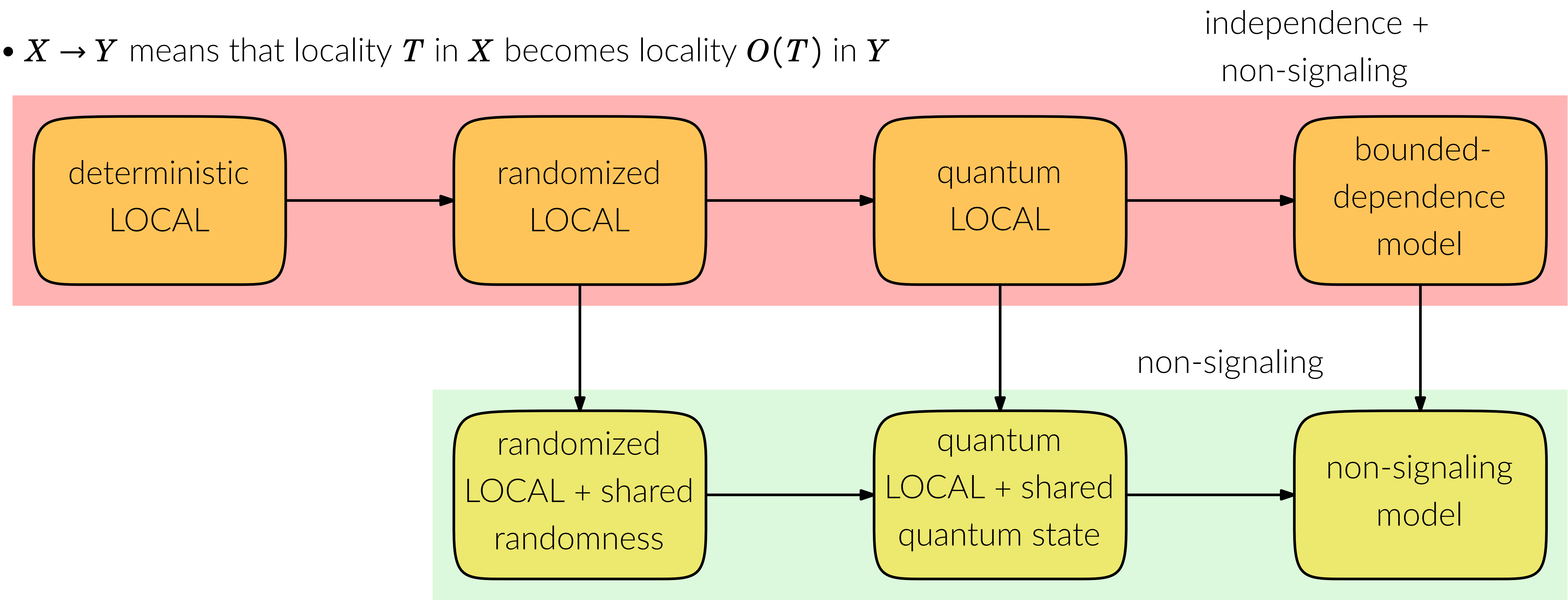


- Is it possible to **"sandwich" quantum-LOCAL** between weaker and stronger models?
  - yes! *for some problems*

# Some results: non-signaling model

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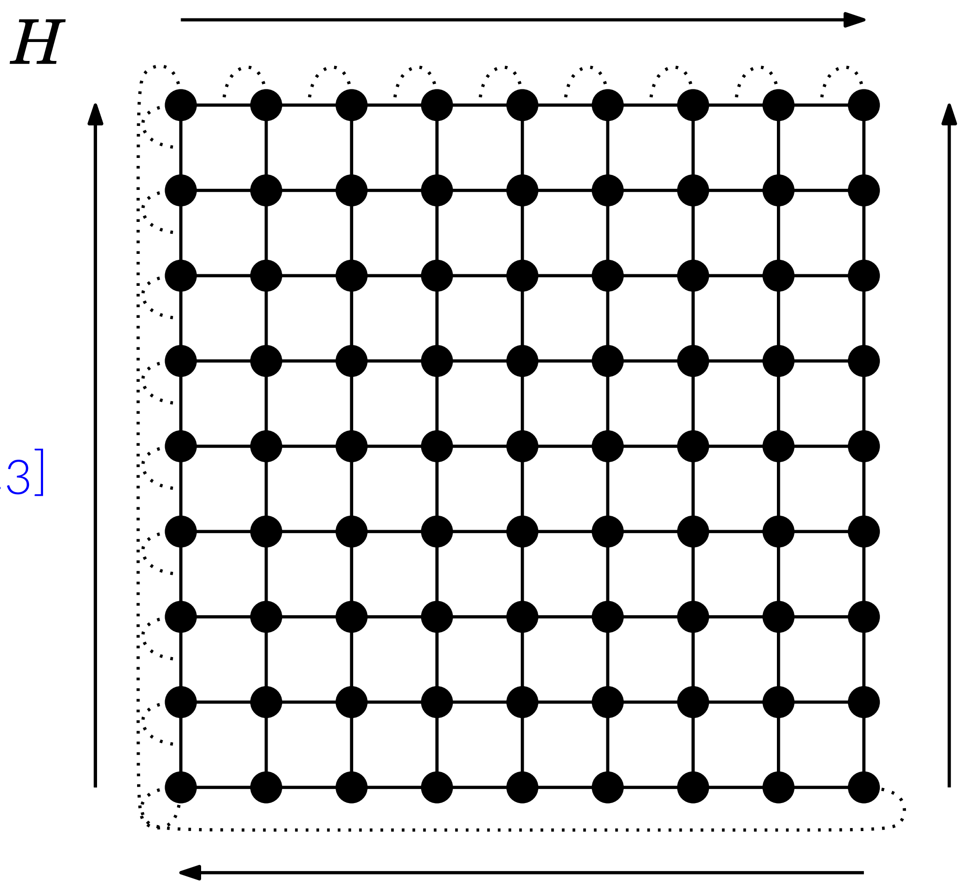

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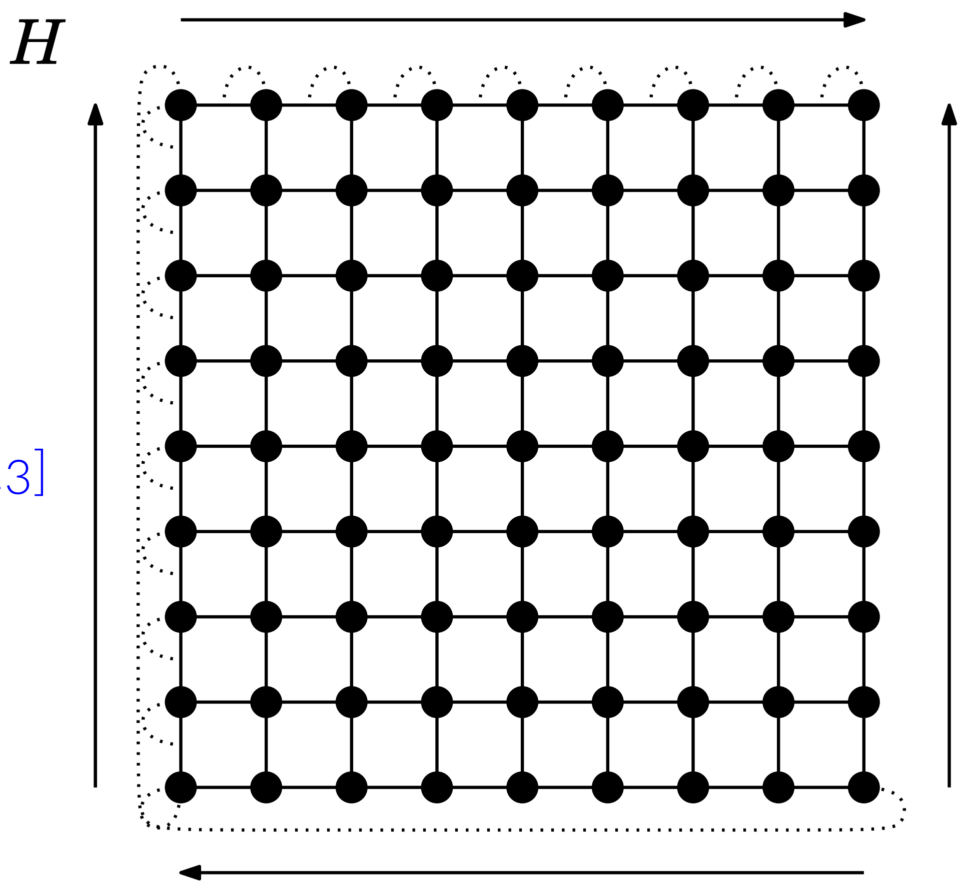

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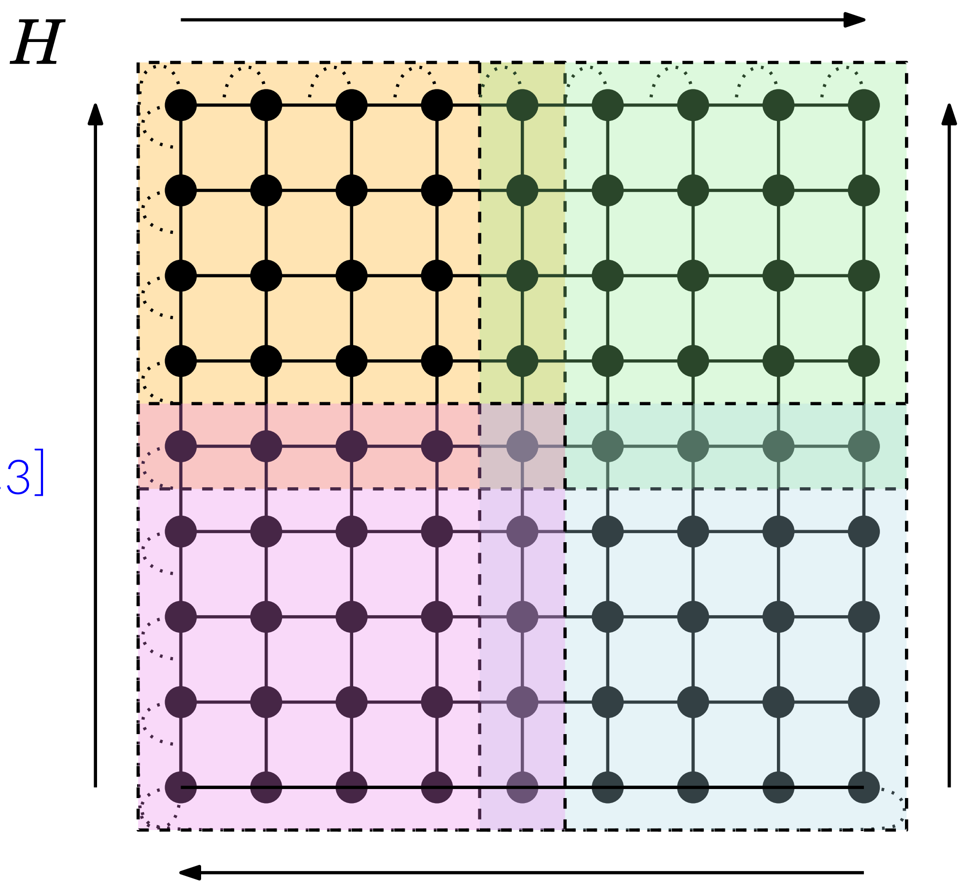

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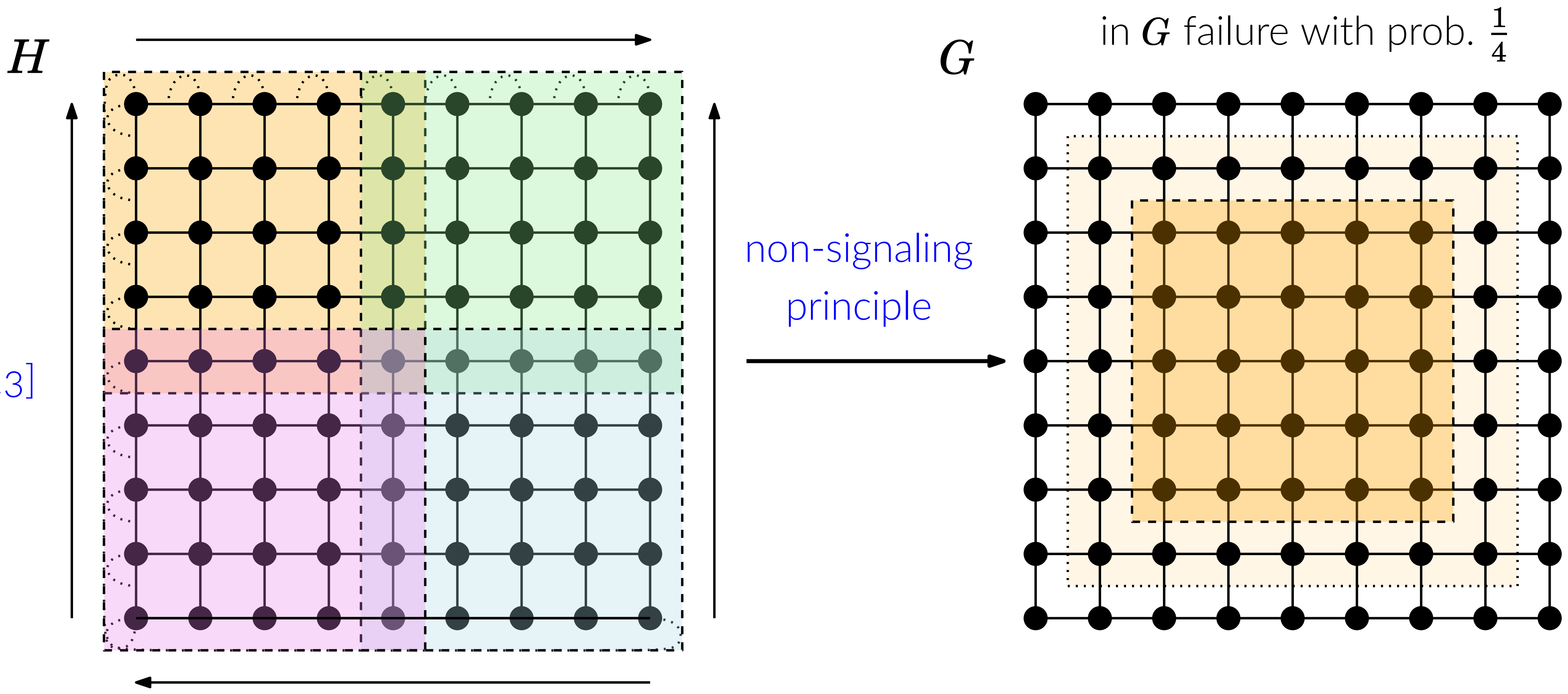

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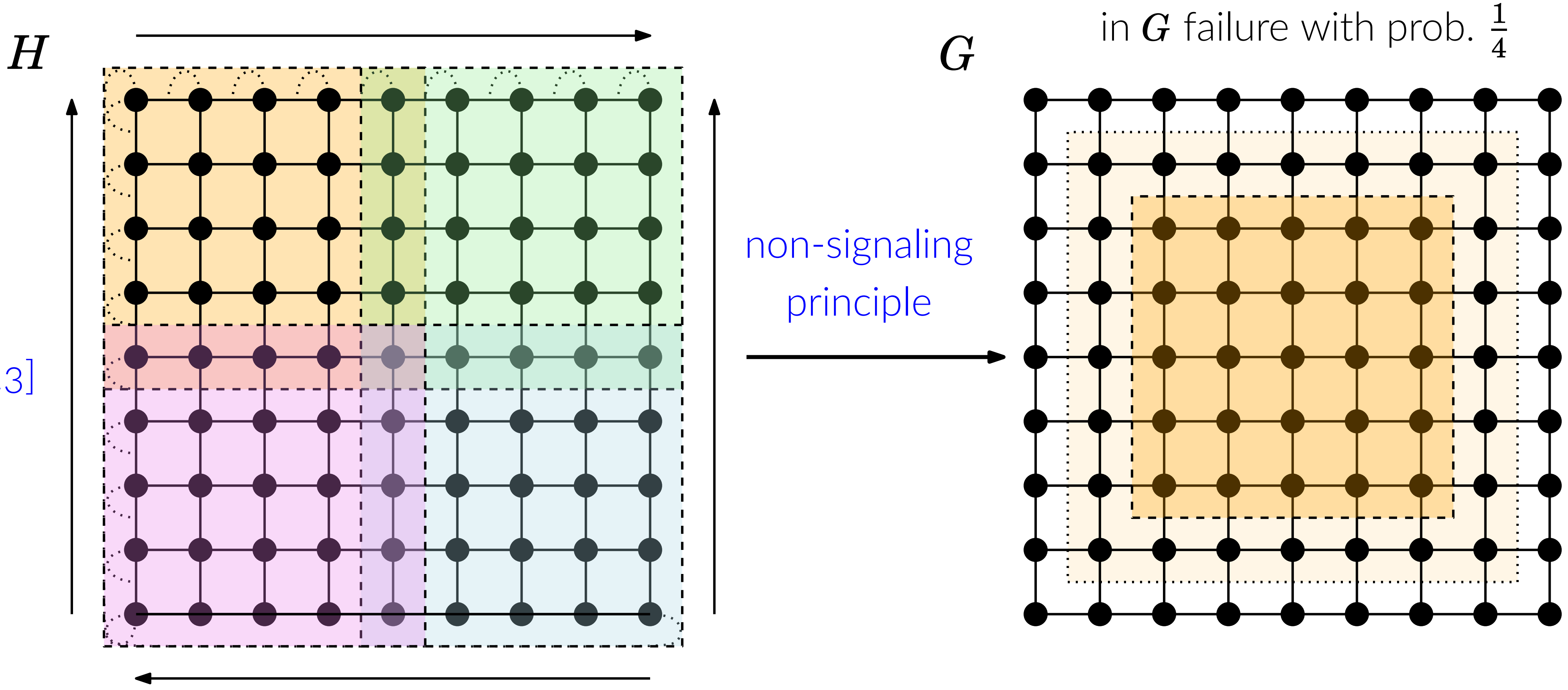


- **Boosting failure prob.** is also possible

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**What about other known lower bounds?** E.g., $3$-coloring cycles has complexity $\Theta(\log^\star n)$ [Linial FOCS '87]

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  - For any $\Theta(\log^\star n)$ LCL $\Pi$ on bounded degree graphs, *there is a bounded-dependent distribution ($T = O(1)$) solving $\Pi$*

    [Akbari et al. STOC '24]

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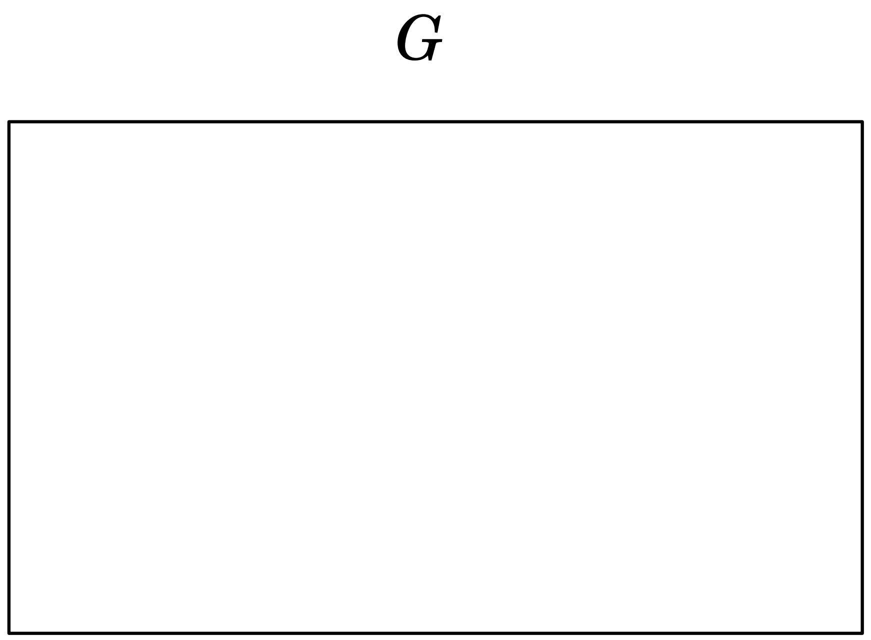

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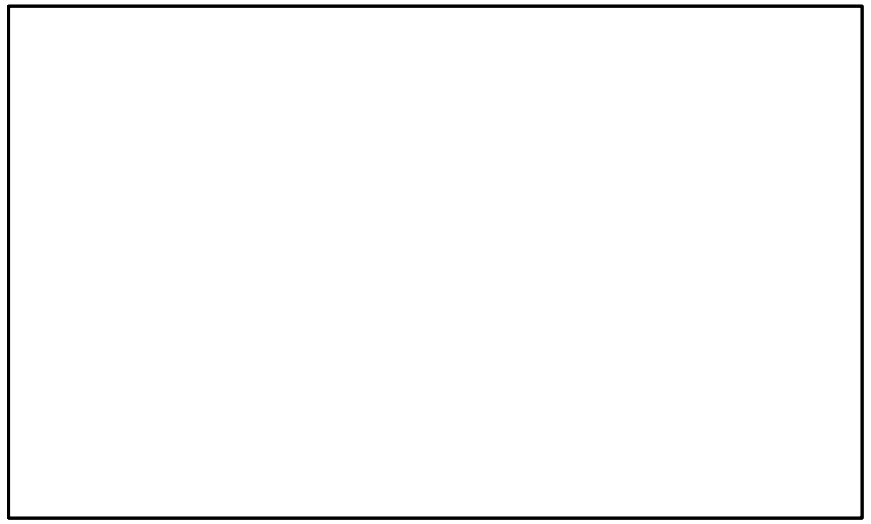
$G$



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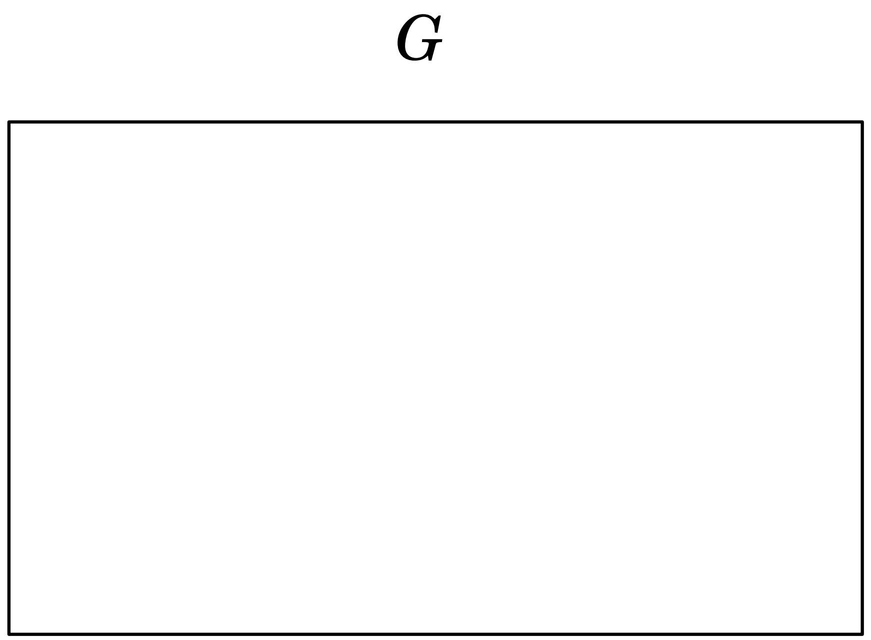

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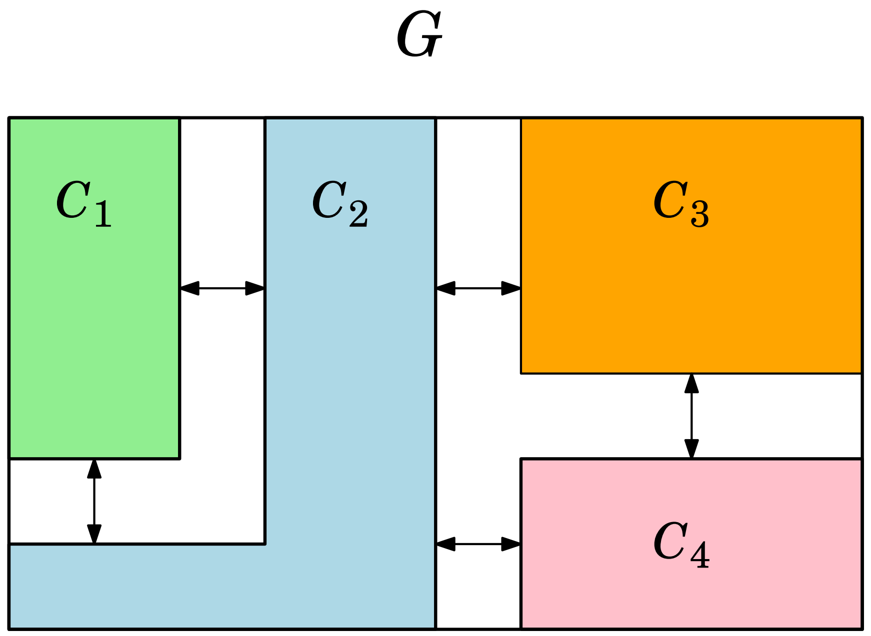

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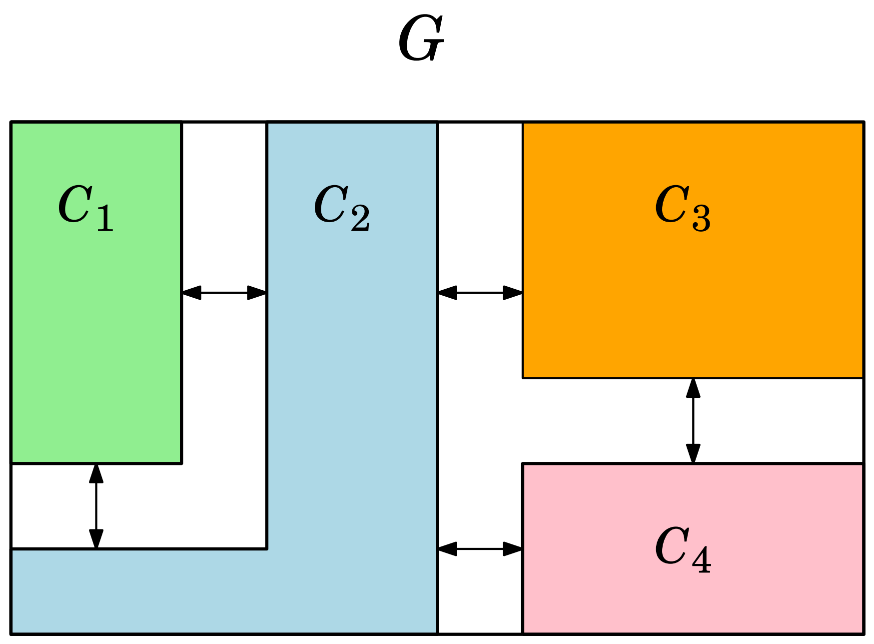
  - locality $T$ = independence at distance $2T+1$ plus non-signaling beyond distance $T$

- **Observation**: LOCAL algorithms $A_1$ and $A_2$ with localities $T_1$ and $T_2$

  - merged into a single LOCAL algorithm $B$ with locality $T = O(T_1 + T_2)$

- We can "nicely" decompose the graph with a LOCAL algorithm

  - locality $T = \tilde{O}(\sqrt{nT})$




diam$(C_i) = \tilde{O}(\sqrt{nT})$
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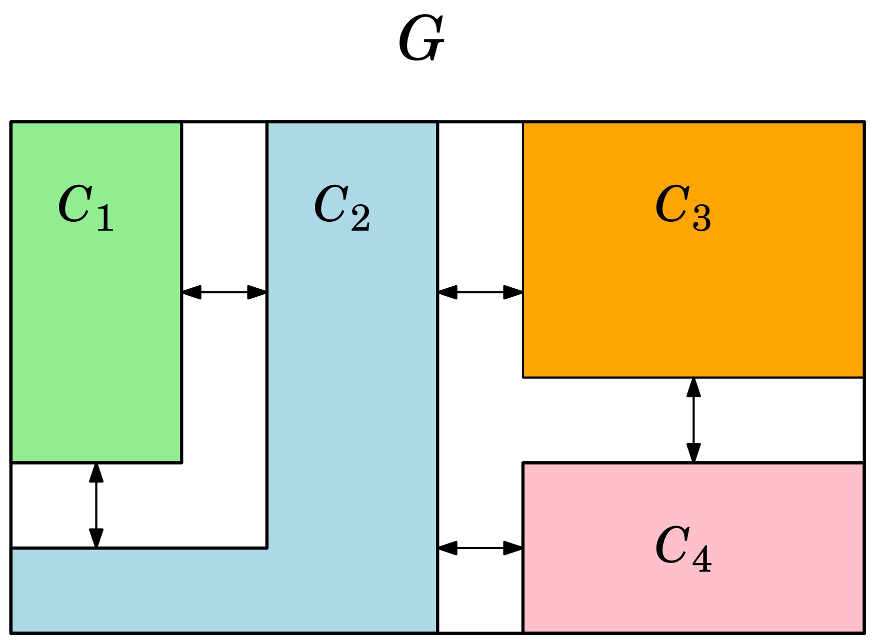


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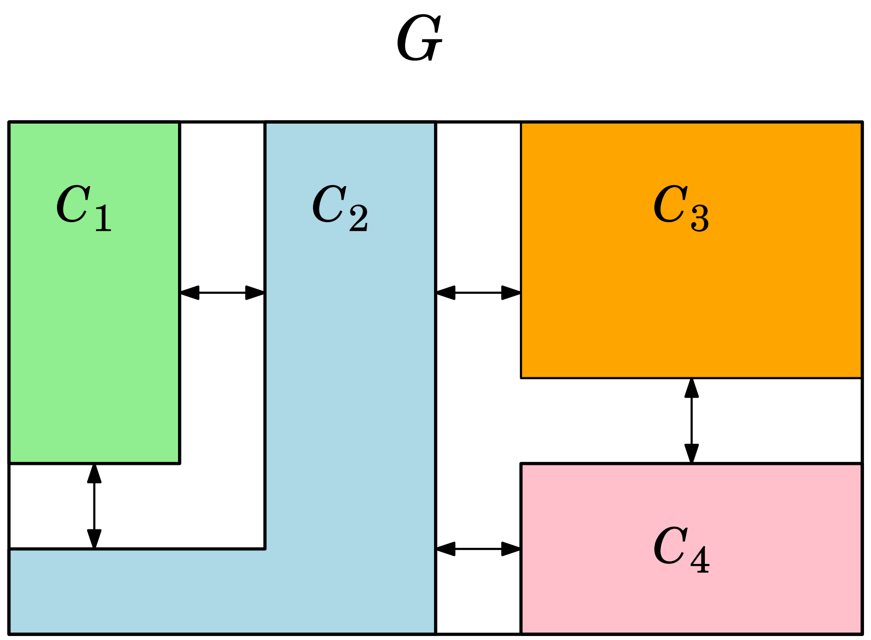


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- **Bounded dependent distribution** with locality $T \implies$ **LOCAL algorithm** with locality $\tilde{O}(\sqrt{nT})$

  - LOCAL complexity is $\Theta(n) \implies$ bounded-dependence complexity $\tilde{\Omega}(n)$ (same for quantum-LOCAL)

  - bounded-dependence (or quantum-LOCAL) complexity $O(1) \implies$ LOCAL complexity is $\tilde{O}(\sqrt{n})$

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# Conclusions

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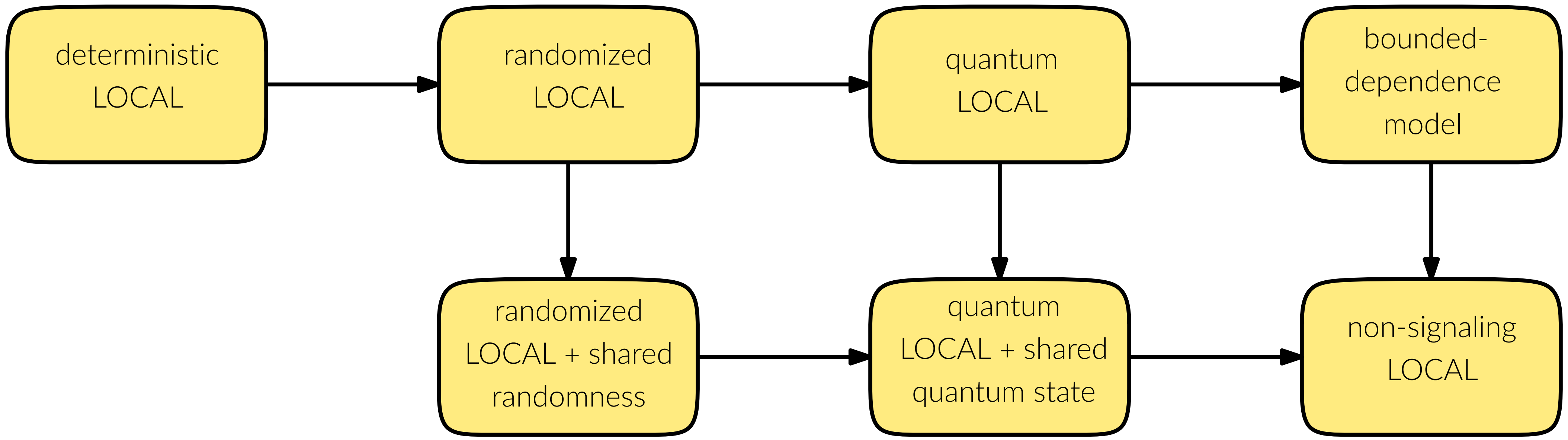

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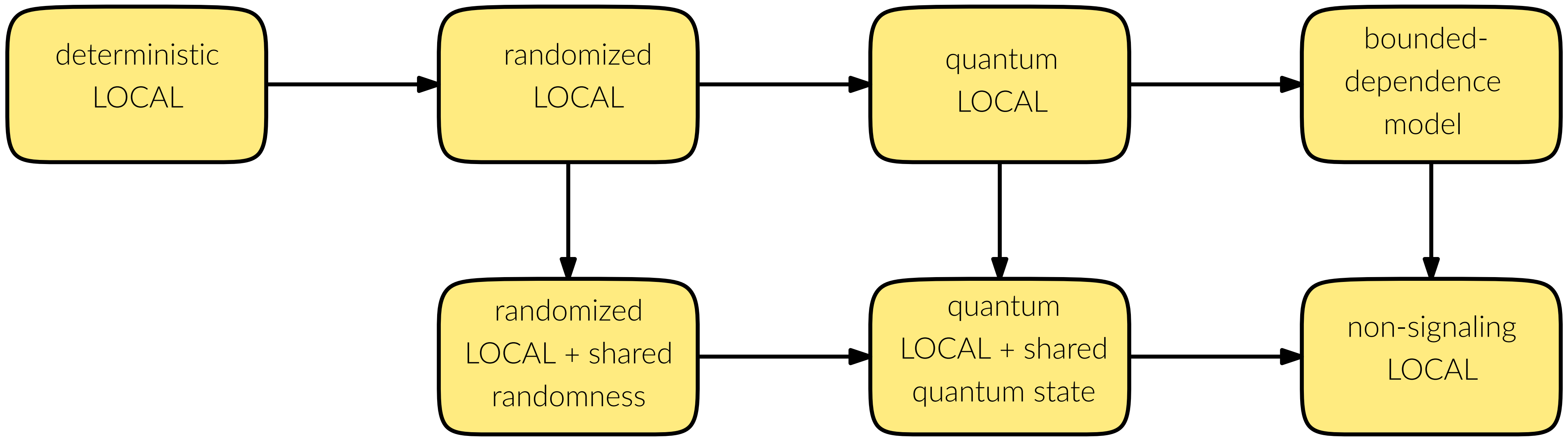

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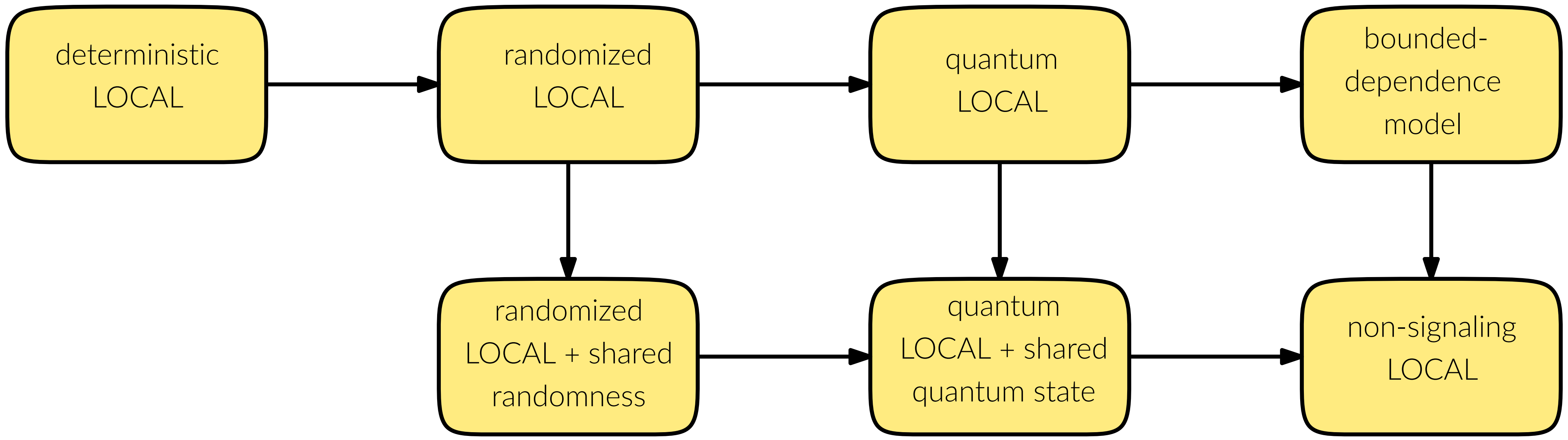

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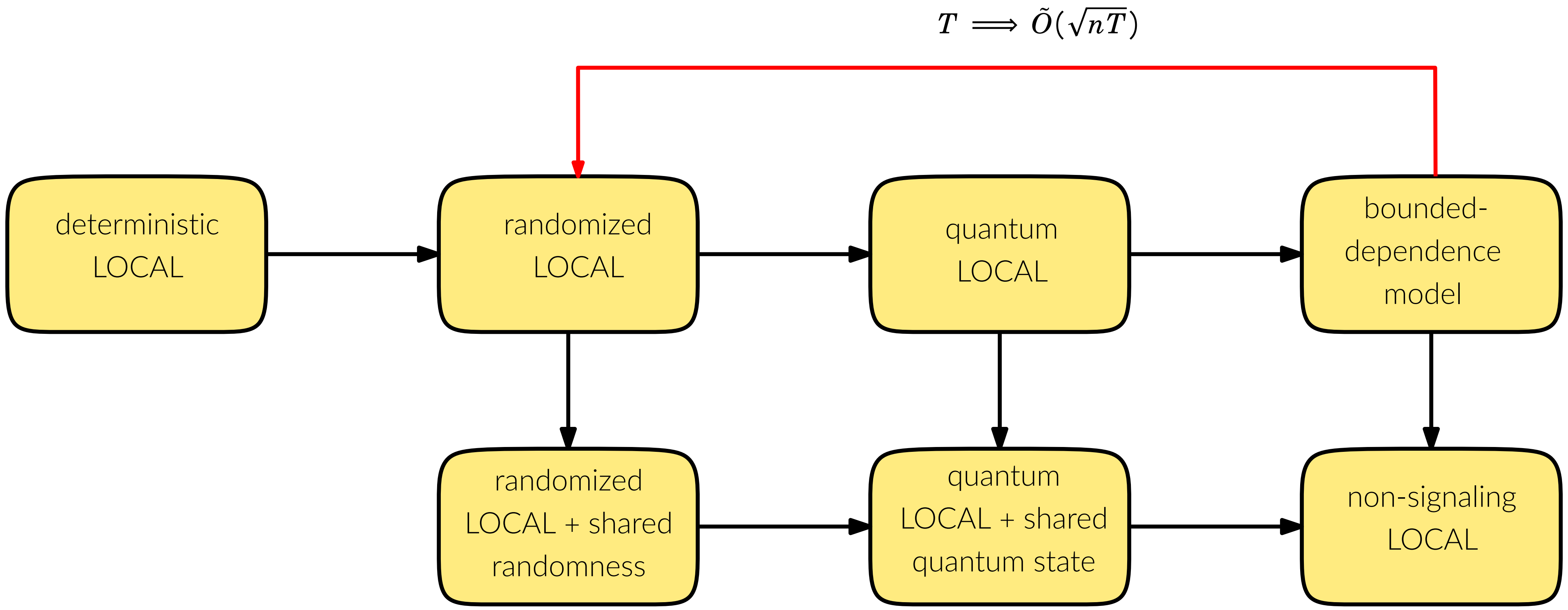

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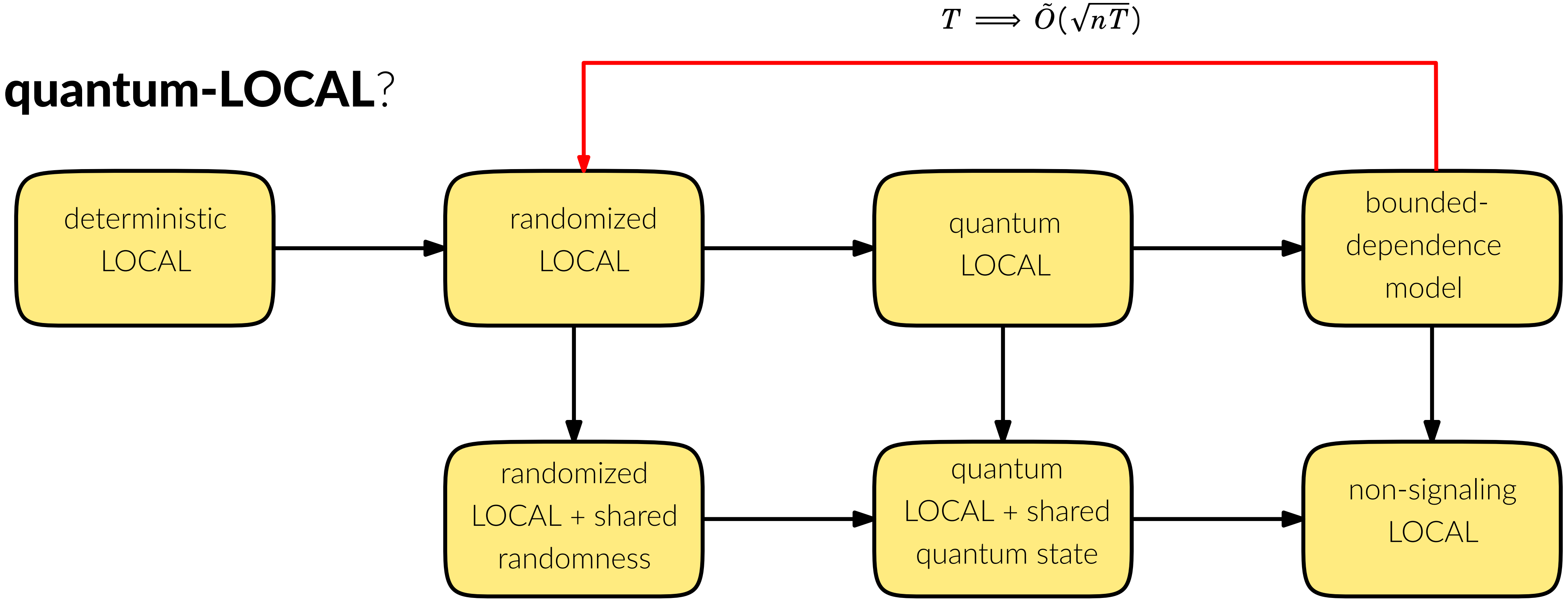

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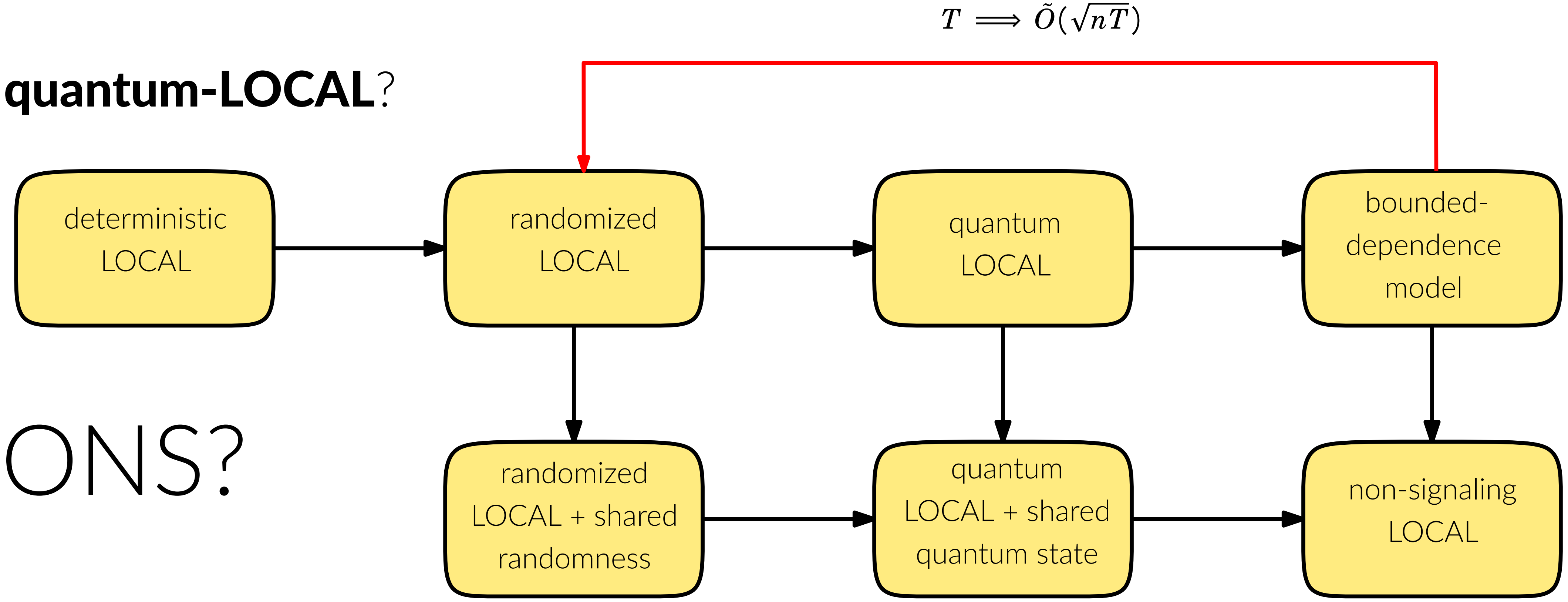


THANKS! QUESTIONS?