Search via Parallel Lévy Walks on $\mathbb{Z}^2$

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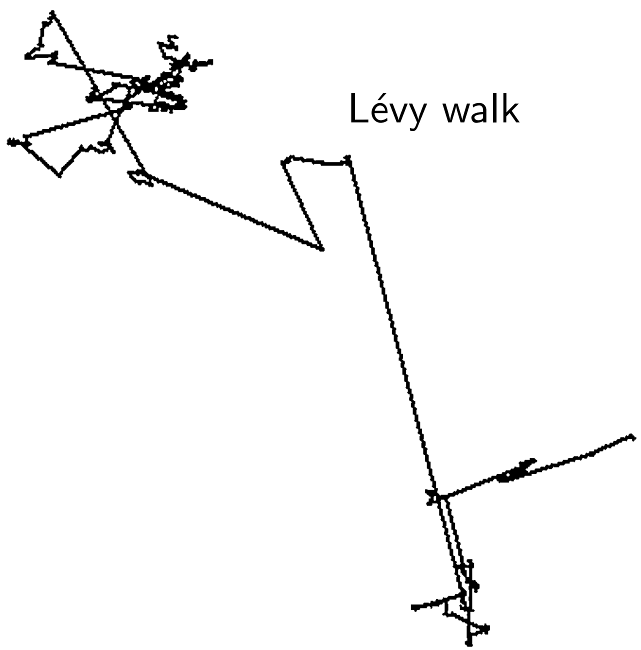
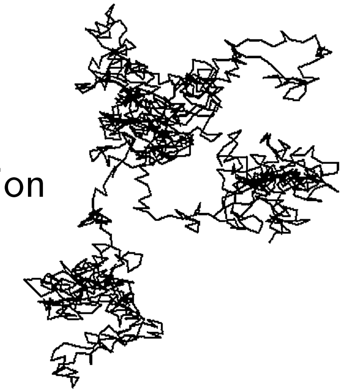
What are Lévy walks?





Lévy walk (informal):
A Lévy walk is a random walk whose step-length density distribution is proportional to a power-law, namely, for each $d \in \mathbb{R}$, $f(d) \sim 1/d^{\alpha}$, *for some* $\alpha > 1$

Note: the speed of the walk is constant

Movement models and foraging theory

Lévy walks are used to model **movement patterns** [Reynolds, Biology Open 2018]

Examples:

- T cells within the brain
- swarming bacteria
- midge swarms
- termite broods
- schools of fish
- Australian desert ants
- a variety of molluscs

Austrialian desert ants

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Widely employed in the Foraging theory

Some fun: mussels Lévy walk video [de Jager et al., Science 2011]

Foraging theory

Setting:
- a density distribution ρ in $\mathbb{R}^n$ describing food locations
- an uninformed walker searching for food

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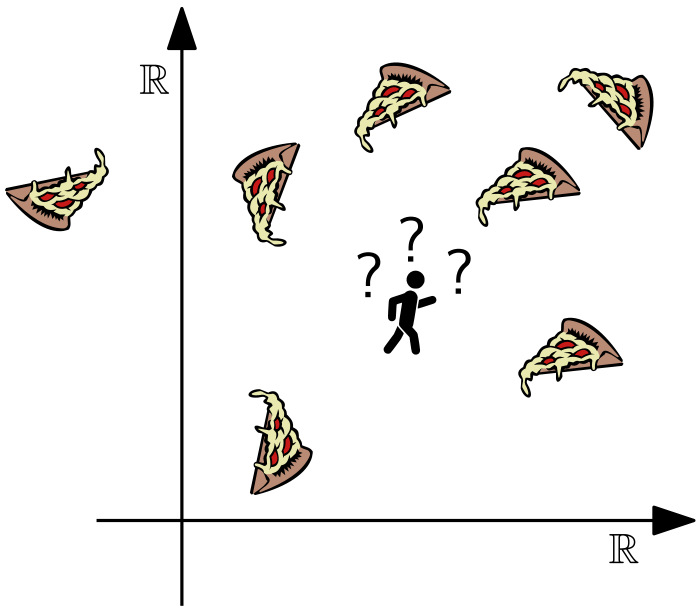

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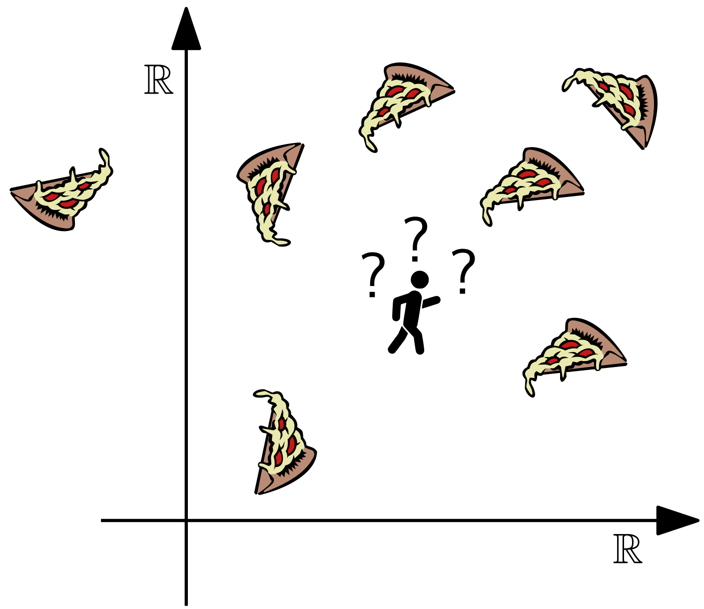

Task: find a strategy which maximizes the expected food discovery rate

Lévy walk optimality

[Viswanathan et al., Nature 1999] takes into account two different settings:

- non-destructive foraging (the food regenerates once found)
- destructive foraging (the food does not regenerate once found)

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- non-destructive foraging (the food regenerates once found)
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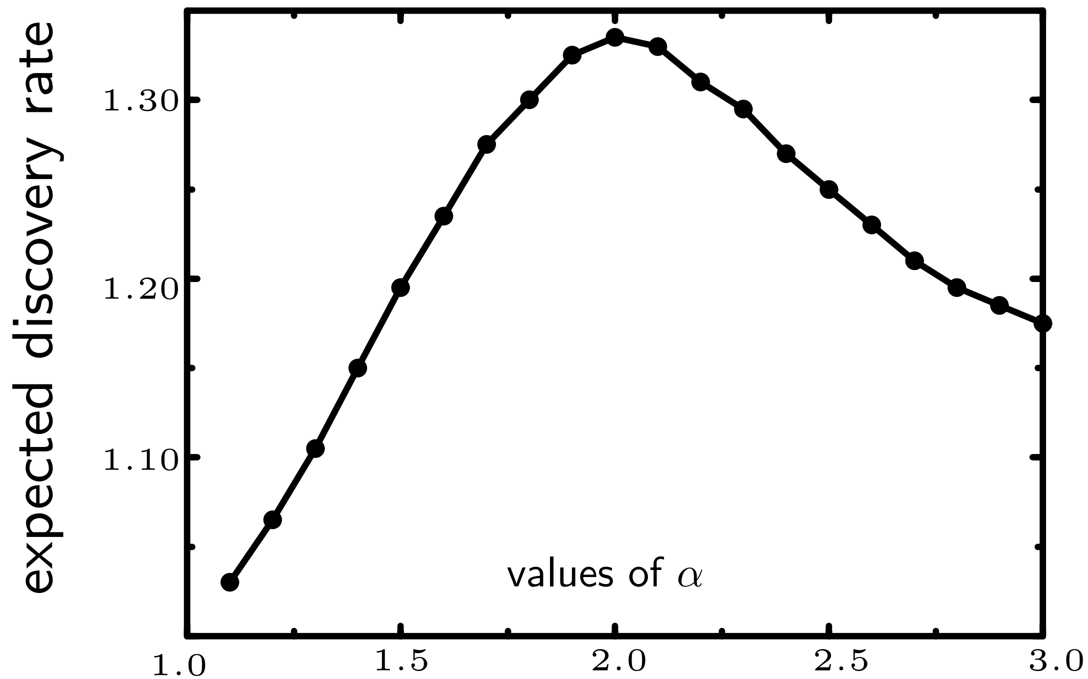
Result: in order to maximize the expected food discovery rate, the walker should perform

- a Lévy walk with exponent $\alpha = 2$, for non-destructive foraging
- a ballistic walk, for destructive foraging

Simulations by [Viswanathan et al., 1999]



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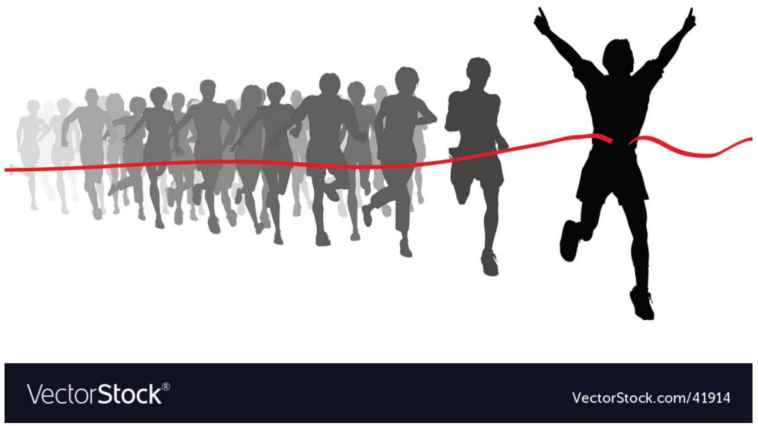
These results shaped much of subsequent research

The performance of Lévy walks has been analyzed in a wide range of search problems



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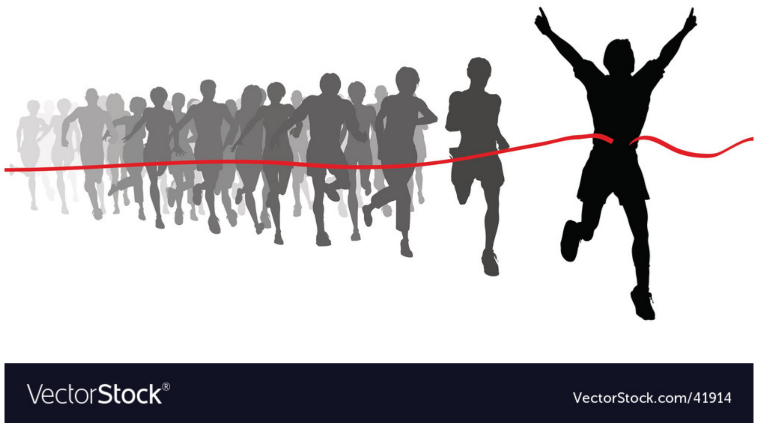

We focus on the ANTS problem

The ANTS problem

Introduced by [Feinerman et al., PODC 2012]

Setting:
- k (mutually) independent walkers (agents) start moving on $\mathbb{Z}^2$ from the origin
- time is synchronous and marked by a global clock
- one special node $\mathcal{P} \in \mathbb{Z}^2$, the *target*, placed by an adversary at unknown (Manhattan) distance ℓ from the origin

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Task: find the target as fast as possible

A lower bound on the hitting time

[Feinerman et al., PODC 2012] shows the following:

Lemma: for any $k \geq 1$, and for any search algorithm $\mathcal{A}$, the hitting time to find $\mathcal{P}$ is $\Omega\left(\ell^2/k + \ell\right)$ both with constant probability and in expectation

Proof:

A lower bound on the hitting time

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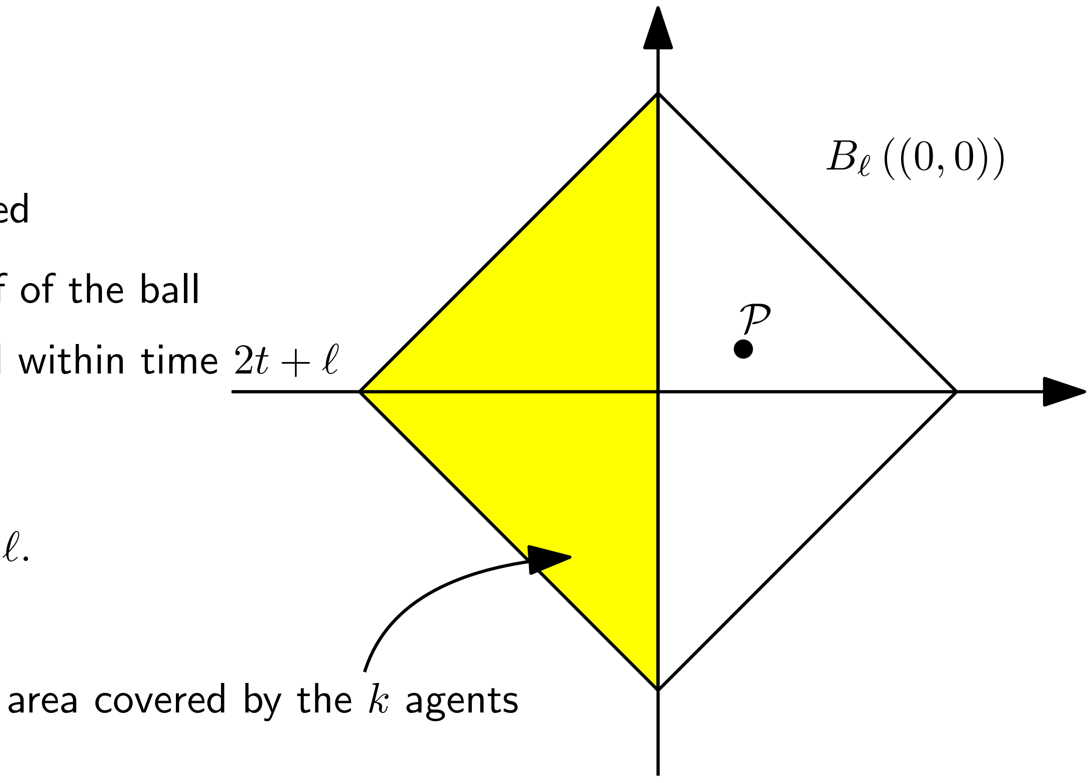
Proof:

- $|B_\ell\left((0,0)\right)| = \ell^2$
- set $t = \ell^2/(4k)$
- within time $2t$, at most $2kt = \ell^2/2$ nodes covered
- the adversary locates the target in the other half of the ball
- probability at least $1/2$ the treasure is not found within time $2t + \ell$
- $H =$ first hitting time for the treasure, then

$$\mathbb{E}\left[\mathsf{H}\right] \geq 2t \cdot \frac{1}{2} + \ell = \ell^2/(4k) + \ell.$$



No advice, no communication

[Feinerman et Korman, DC 2017] proposes many solutions to the problem

Many considered settings, in which

- agents exchange information at the source node
- agents receive some advice on the number of agents k
- there is no communication and no advice

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We focus on the case **no advice, no communication**

Their best algorithm in this case achieves expected hitting time

$$\mathcal{O}\left(\left(\ell^2/k+\ell\right)\log^{1+\epsilon}\ell\right),$$

for any fixed constant $\epsilon > 0$

No advice, no communication

Uniform algorithm proposed in [Feinerman et Korman, DC 2017]

(idea)

i. fix a subset of k_i agents to be moved
ii. fix a ball of some radius ℓ_i
iii. agents go to random nodes in the ball
iv. agents perform a spiral search of length d_i around the chosen nodes
v. agents return to the source node
vi. increase k_i, ℓ_i, d_i and repeat (i)-(v)

However, the above algorithm is not that natural

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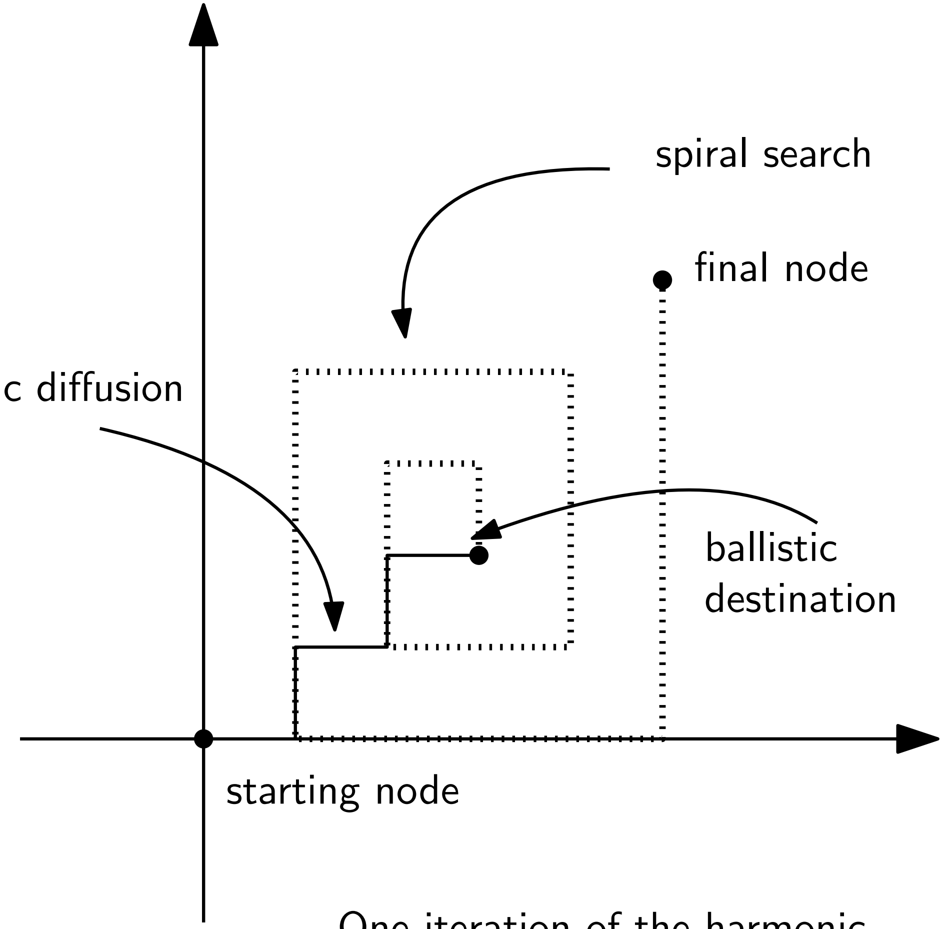
[Feinerman et Korman, DC 2017] proposes a more natural algorithm, the Harmonic search algorithm (HSA)

The Harmonic search algorithm

HSA worsens performance, but increases probability: the hitting time is

$$\mathcal{O}\left(\ell^{2+\delta}/k + \ell\right)$$

with probability $1-\epsilon$ for any fixed constants $0 < \delta$, $\epsilon < 1$



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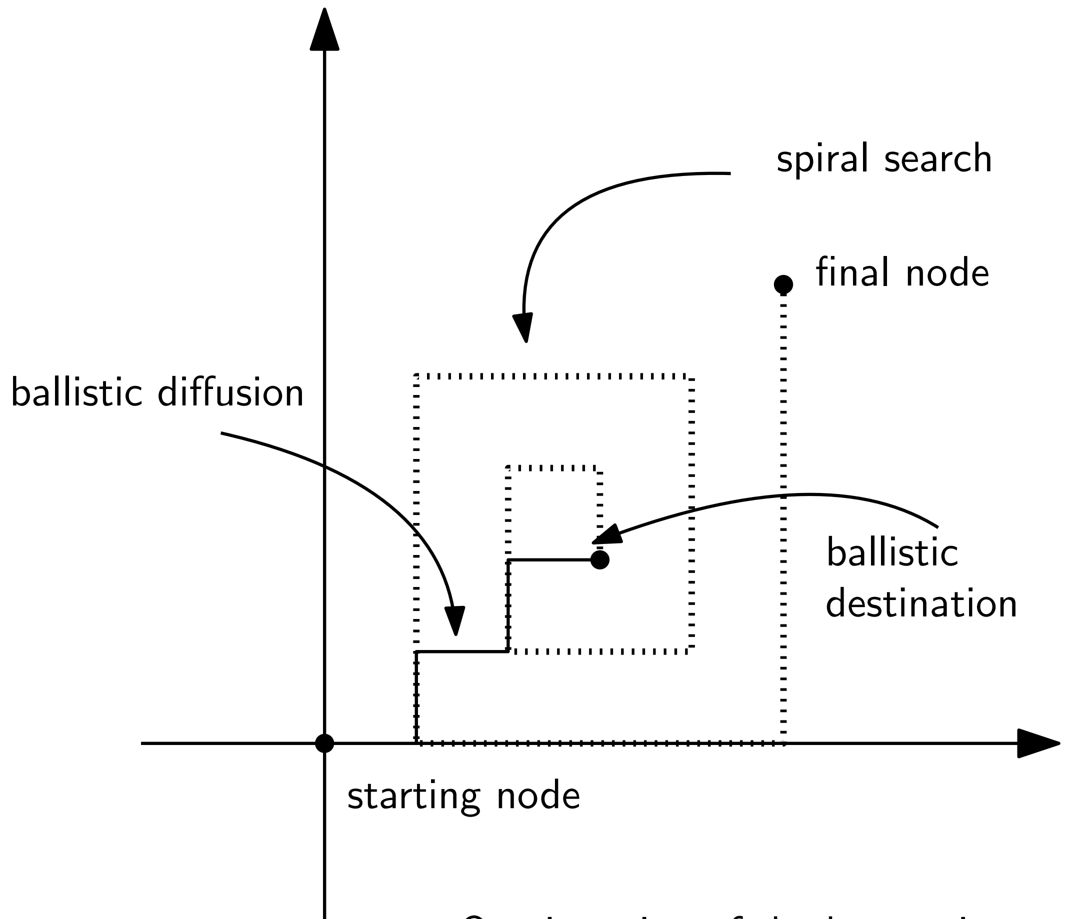
HSA: each agent

a) samples a jump-length d with a power-law distribution with exponent $\alpha = 1 + \delta$ (small)
b) (ballistic diffusion) moves to a destination at distance d chosen u.a.r.
c) (normal diffusion) starts a spiral search for $d^{\delta+2}$ steps
d) returns in the origin and repeats



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Defining the discrete Lévy walk

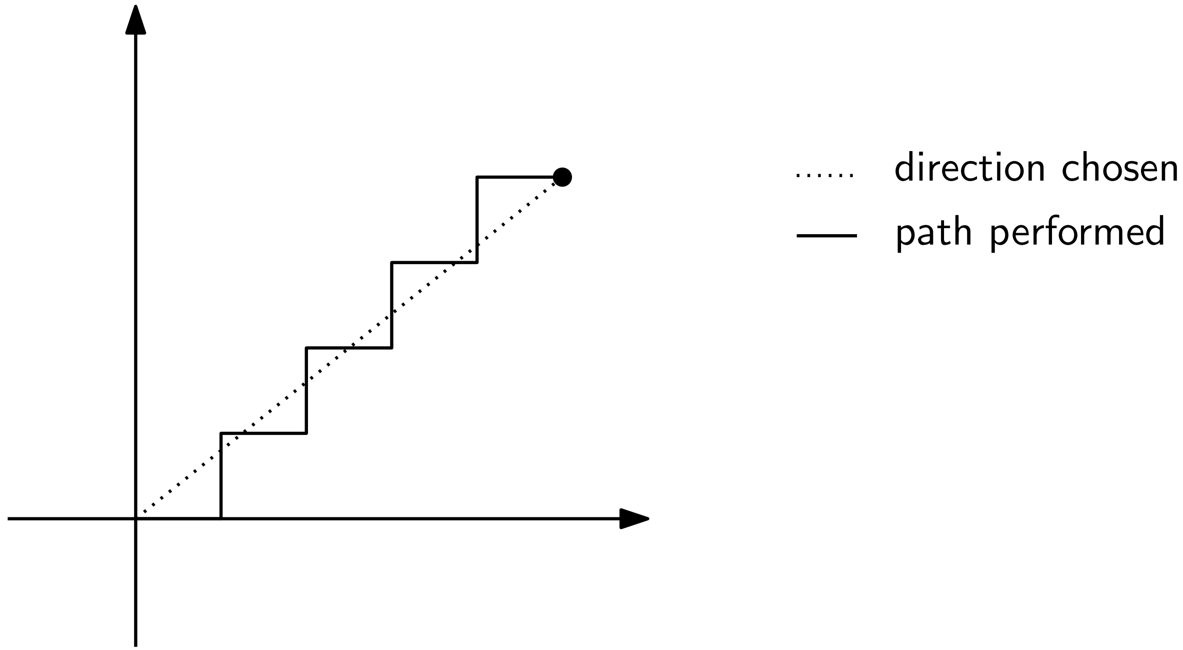
Two choices to make:

- define the jump-length distribution
- define a notion of approximating a line-segment

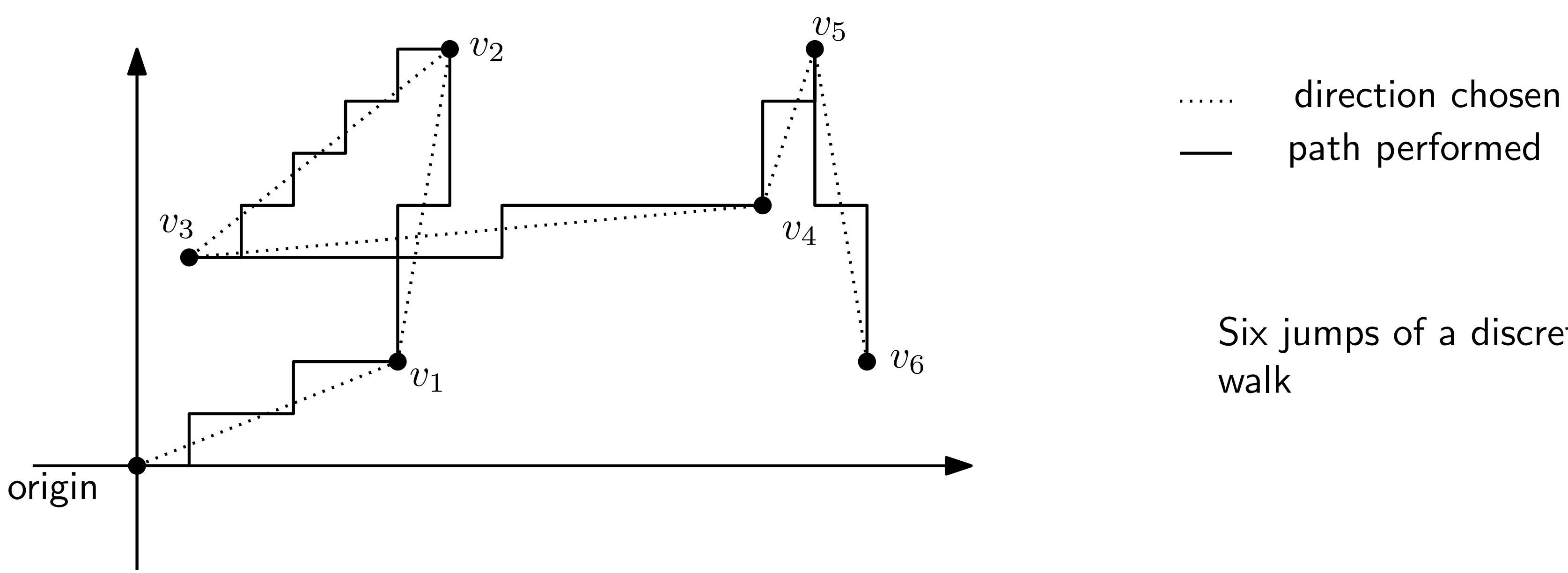
Jump length distribution

- $d = 0$ with probability $1/2$
- $d \geq 1$ with probability c_α / d^α

Approximation of a line-segment



Discrete Lévy walk



Six jumps of a discrete Lévy walk

Let $\alpha > 1$ be a real constant

Lévy walk: the agent

a) chooses a distance $d \in \mathbb{N}$ as follows: $d = 0$ w.p. $1/2$, and $d \geq 1$ w.p. c_α / d^α
b) chooses a destination u.a.r. among those at distance d
c) walks along an approximating path for d steps, one edge at a time, crossing d nodes
d) repeats the procedure

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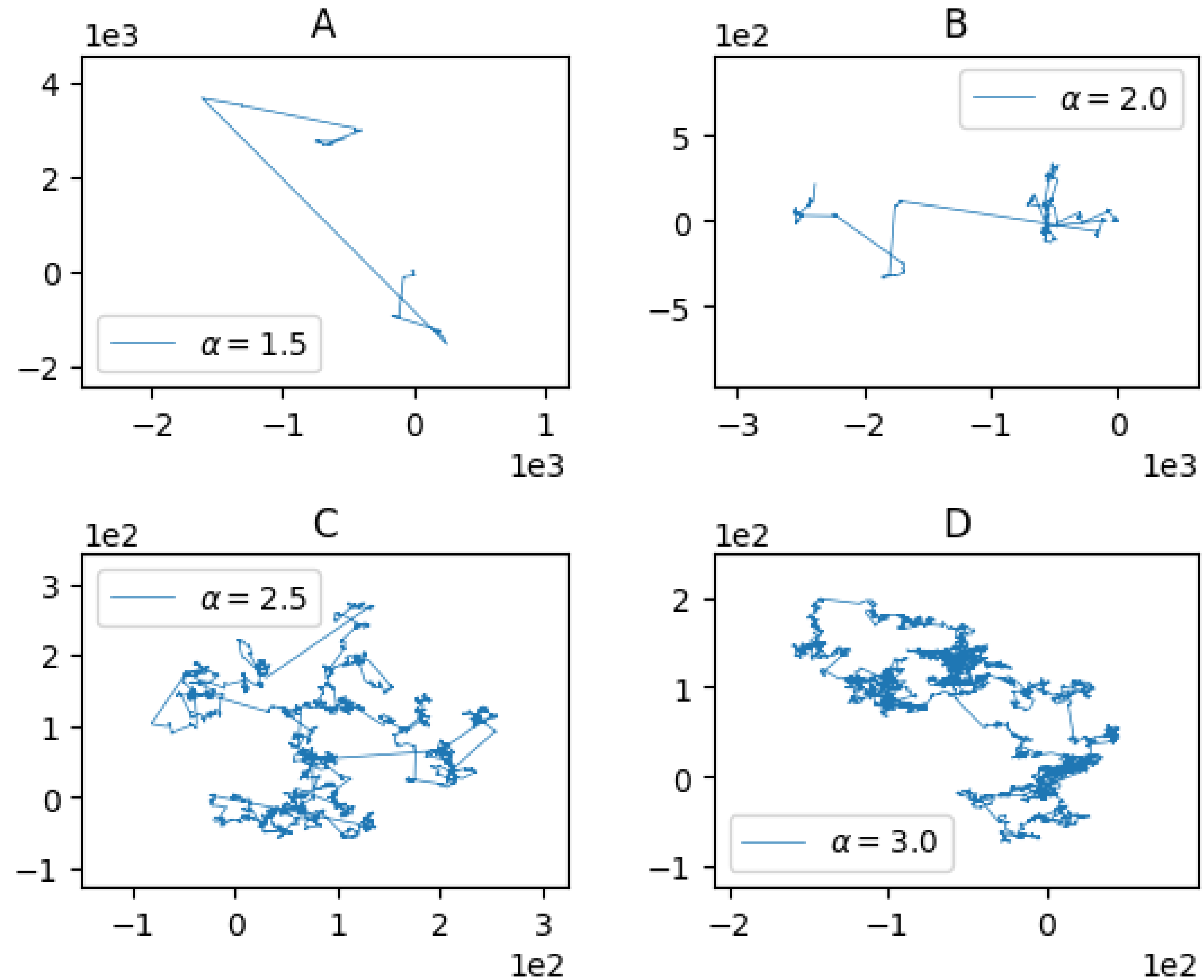
Known facts about the continuous Lévy walk

- $1 < \alpha \leq 2$ ballistic diffusion (fig.s A and B)
- $2 < \alpha < 3$ super diffusion (fig. C)
- $3 \leq \alpha$ normal diffusion (fig. D)



Other known facts

Expected jump-length

- $1 < \alpha \leq 2$: $\int_1^\infty x^{-\alpha+1} \mathsf{d}x = \infty$
- $2 < \alpha$: $\int_1^\infty x^{-\alpha+1} \mathsf{d}x = \Theta(1)$

Jump-length second moment

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The secret lies in the range $2 < \alpha < 3$...

Three ranges for k and ℓ

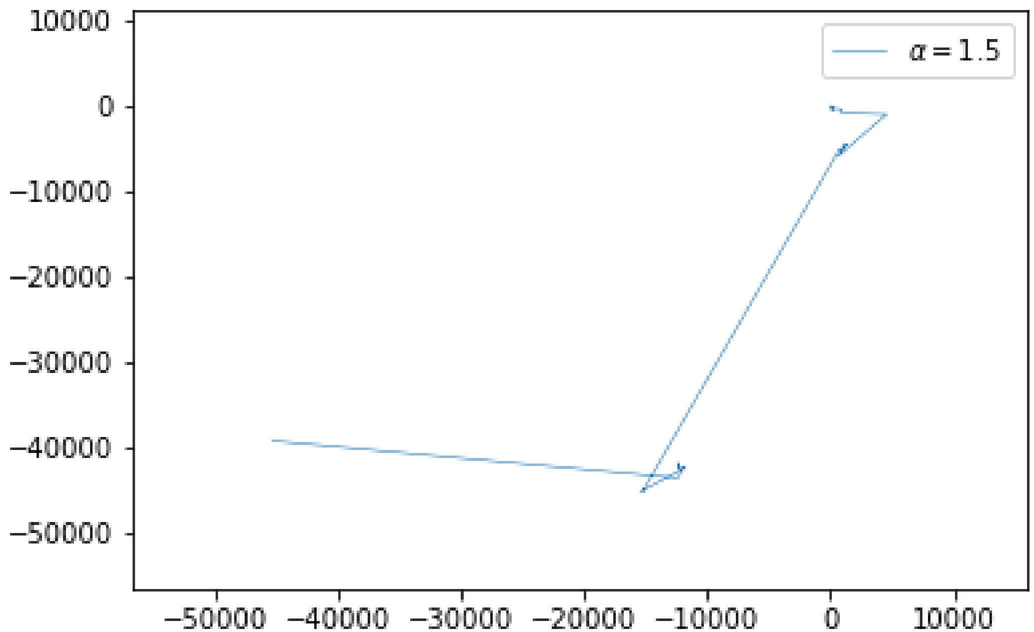
Recall: ℓ target distance, k number of agents

Three different possible settings:

1. close target: $\ell \leq k/\text{polylog}\,(k)$
2. far target: $k/\text{polylog}\,(k) \leq \ell \leq \exp\left(k^{\Theta(1)}\right)$
3. very far target: $\exp\left(k^{\Theta(1)}\right) \leq \ell$

Close target: $\ell \leq k/\text{polylog}\,(k)$

Best strategy = ballistic walks: any α in $(1,2]$



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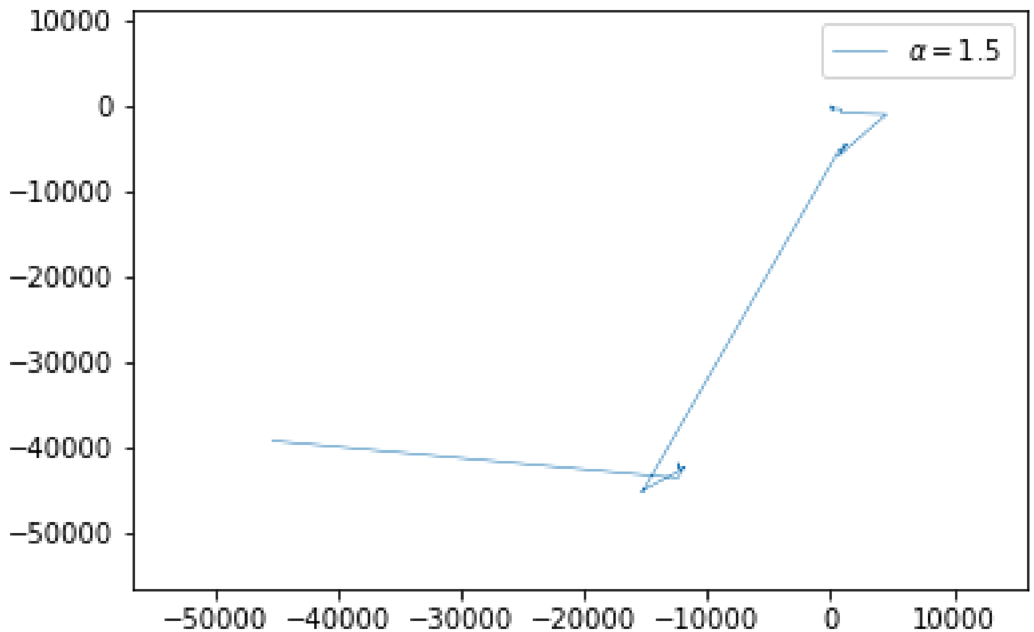
With high probability in ℓ, the hitting time is

$$\mathcal{O}\left(\ell\,\text{polylog}\,(\ell)\right)$$



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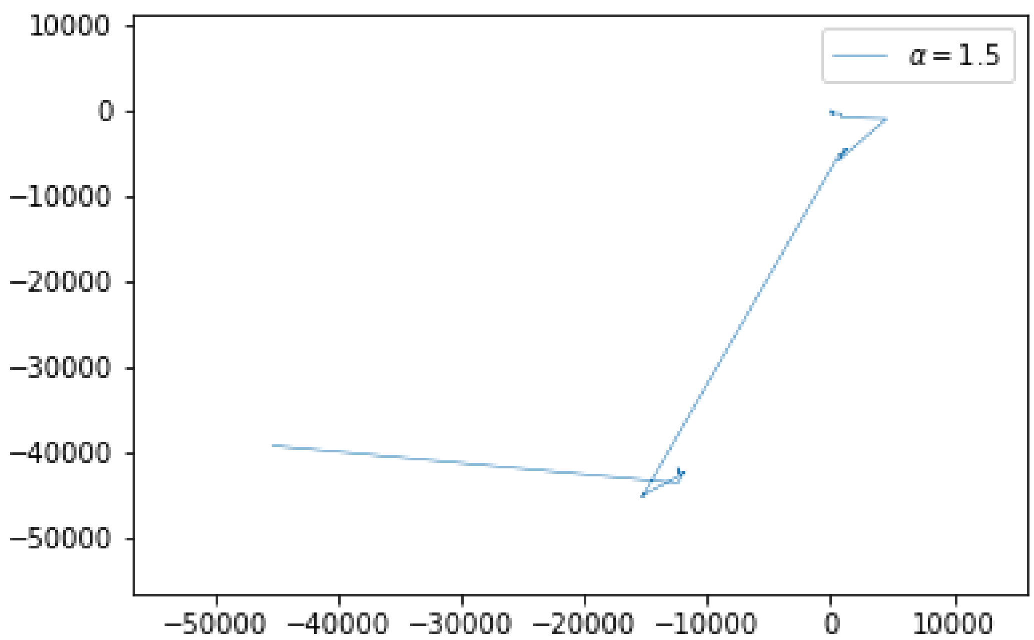
Recall: an event E depending on a parameter ℓ holds with high probability in ℓ if $\mathbb{P}\left(E\right) \geq 1 - \ell^{-\Theta(1)}$



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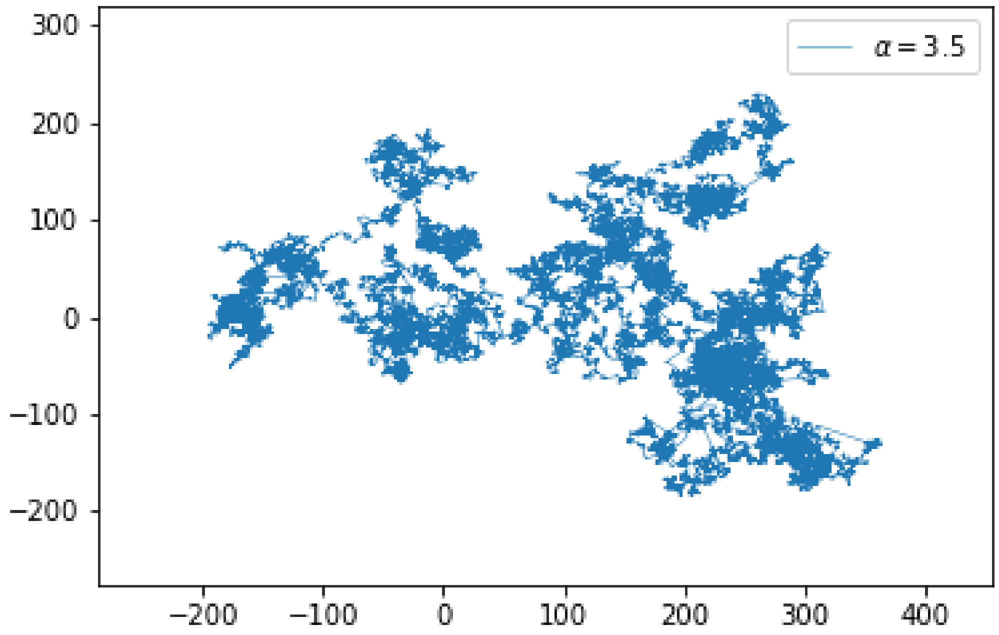
Best strategy = diffusive walks: any α in $[3, +\infty)$ (brownian-like behavior)

With probability 1, the walks will eventually find the target



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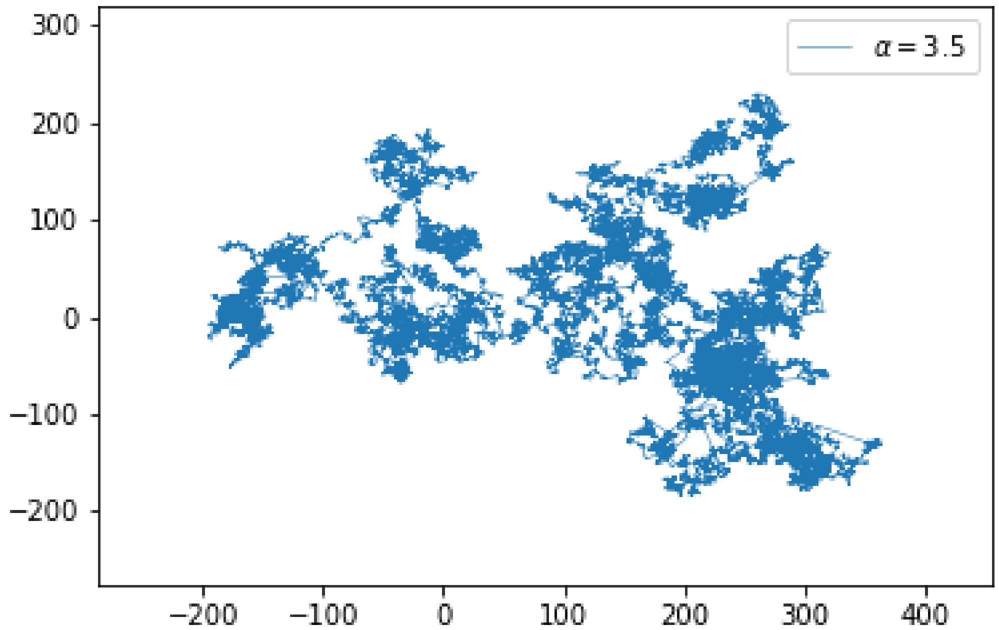

If $\alpha = 3 - \epsilon$, with high probability the target is not found

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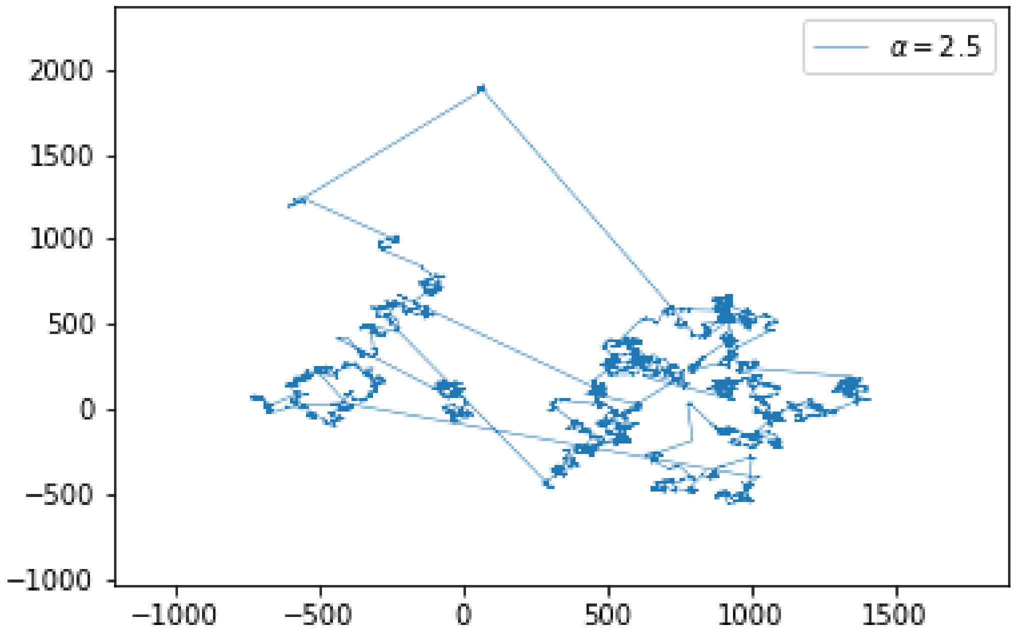

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The followings hold w.h.p. in ℓ

- if $\alpha = \alpha^\star + \mathcal{O}\left(\log\log \ell / \log \ell\right)$, the hitting time is

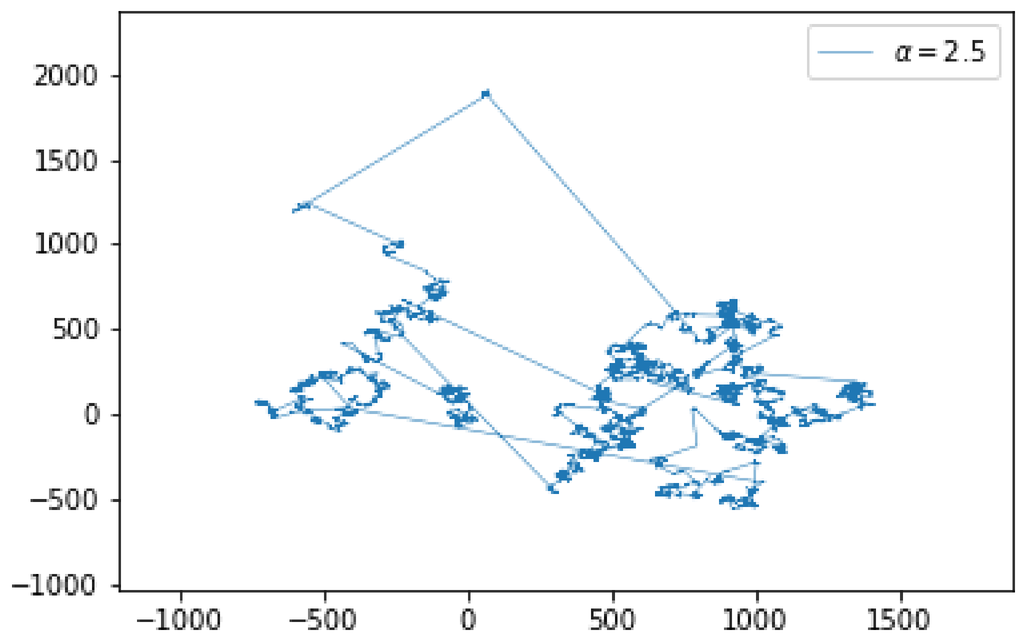
$$\mathcal{O}\left(\left(\ell^2/k + \ell\right)\text{polylog}\,(\ell)\right)$$

- if $\alpha = \alpha^\star + \epsilon$, the hitting time is

$$\Omega\left(\left(\ell^2/k + \ell\right)\ell^c\right),$$

  for some constant $c > 0$

- if $\alpha = \alpha^\star - \epsilon$ the hitting time is *infinite*



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Algorithm: *each agent u samples u.a.r. a real number $\alpha_u \in (2, 3)$. Then, it performs a discrete Lévy walk with exponent α_u*

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If $\ell \leq \exp\left(k^{\Theta(1)}\right)$, the hitting time is $\mathcal{O}\left(\left(\ell^2/k + \ell\right) \text{polylog}\left(\ell\right)\right)$ w.h.p.

The idea behind the algorithm

Fix some $\epsilon = \mathcal{O}\left(\log\log \ell / \log \ell\right)$

We use: $\ell < \exp\left(k^{\Theta(1)}\right)$ ($\iff k \geq$ polylog (ℓ)) + Chernoff bound

$\implies$ at least $\Theta\left(\epsilon k\right)$ agents choose an exponent in the range $\left(\alpha^\star - \epsilon, \alpha^\star + \epsilon\right)$ w.h.p.

$\Theta\left(\epsilon k\right)$ agents are sufficient to ensure high probability to find the target fast enough

Recap

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  - improves the HSA (just polylog factor worse than optimum, not polynomial)
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  - very natural and time-homogeneus random process
  - improves the HSA (just polylog factor worse than optimum, not polynomial)
  - does not improve their optimal solution
- argue the non (universal) optimality of exponent $\alpha = 2$

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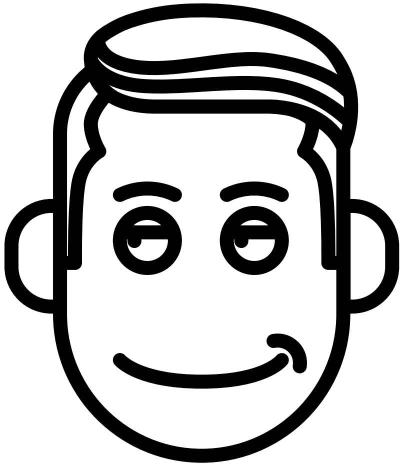

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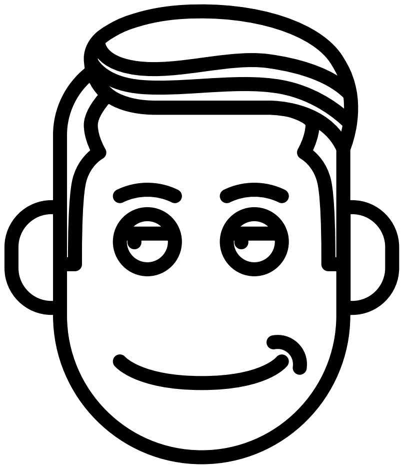

Remark: Here, we DON'T show why other values for α are worse than $\alpha^\star$. We just show how to find $\alpha^\star$

Analyzing a single Lévy flight

Let

- $Z_u\left(t\right) =$ random variable of number of visits in u until time t
- $\mathcal{E}_t =$ the event first t jumps have length $\leq (t\log t)^{\frac{1}{\alpha-1}}$
- $a_t = \mathbb{E}\left[Z_{(0,0)}\left(t\right) \mid \mathcal{E}_t\right]$
- $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}\left(t\right) > 0|\mathcal{E}_t\right)$

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Lemma: $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}(t) > 0 \mid \mathcal{E}_t\right) \geq \mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right] / a_t$

Comes from two facts

(i) $\mathbb{E}\left[Z_{\mathcal{P}}(t) | Z_{\mathcal{P}}(t) > 0, \mathcal{E}_t\right] \leq a_t$

(ii) $\mathbb{E}\left[Z_{\mathcal{P}}(t) | Z_{\mathcal{P}}(t) > 0, \mathcal{E}_t\right] \cdot \mathbb{P}\left(Z_{\mathcal{P}}(t) > 0 | \mathcal{E}_t\right) = \mathbb{E}\left[Z_{\mathcal{P}}(t) | \mathcal{E}_t\right]$

Analyzing a single Lévy flight

Let
- $Z_u(t)$ = random variable of number of visits in u until time t
- $\mathcal{E}_t$ = the event first t jumps have length $\leq (t \log t)^{\frac{1}{\alpha-1}}$
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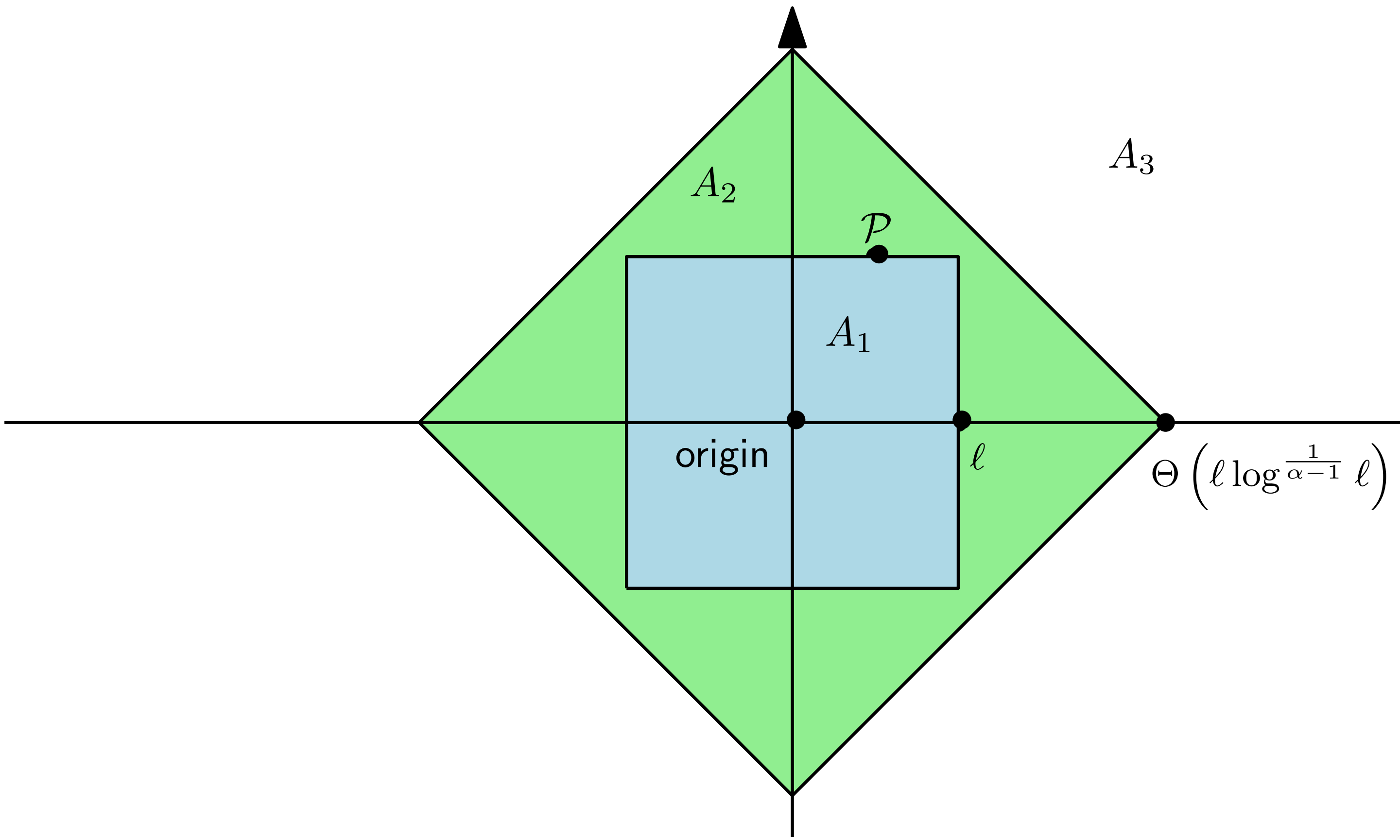
We now look for $\mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right]$ and a_t...

Partition of the space

We partition $\mathbb{Z}^2$ in three areas in the following way



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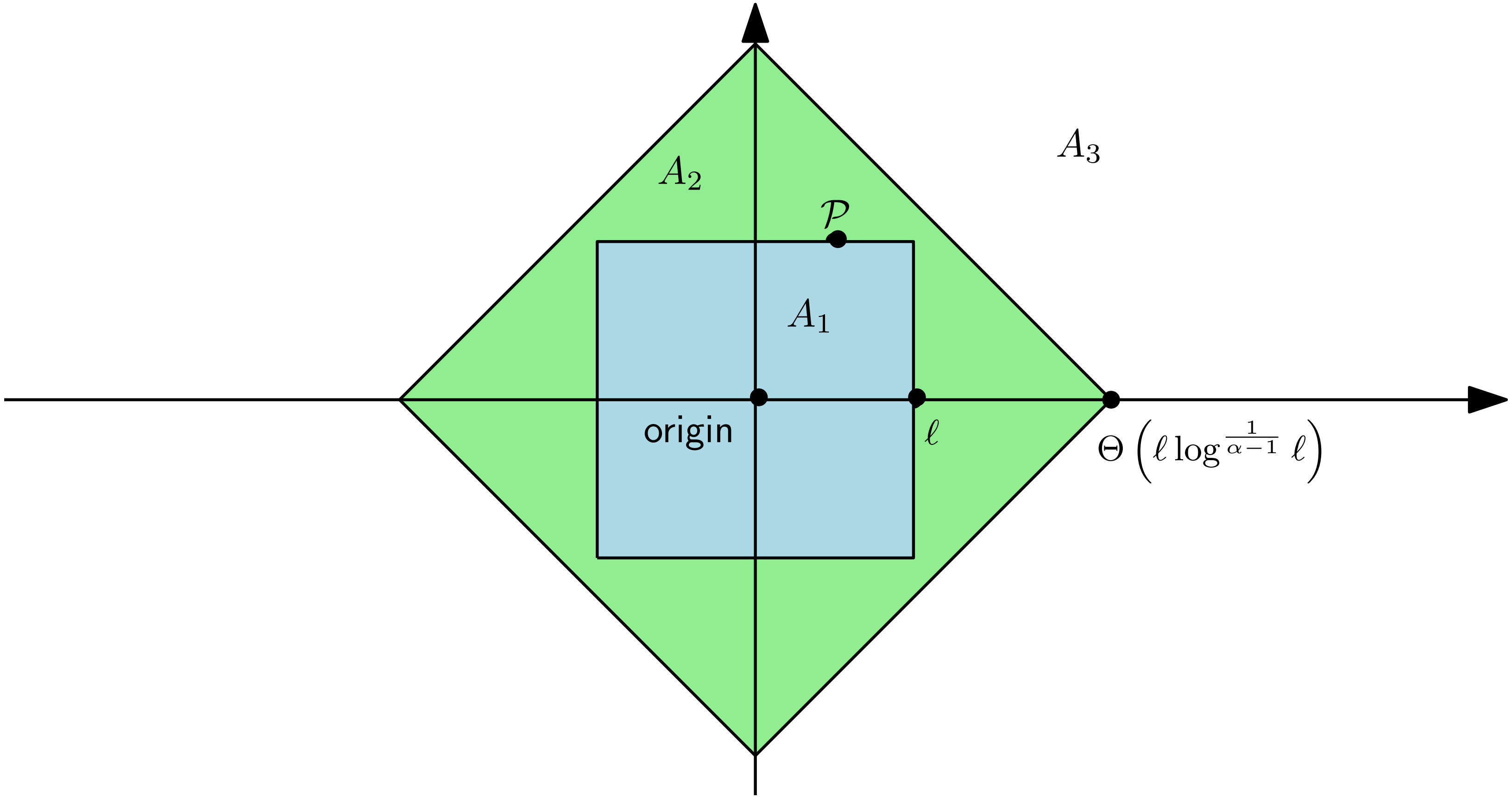

- $A_1 = Q(\ell) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq \ell\}$
- $A_2 = B_{\ell \mathsf{polylog}(\ell)}((0, 0)) \setminus A_1$
- $A_3 = \mathbb{Z}^2 \setminus (A_1 \cup A_2)$

Getting $\mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) | \mathcal{E}_t\right]$

Denote by $Z_S\left(t\right)$ the total number of visits in the set S until time t

Getting $\mathbb{E}\left[Z_{\mathcal{P}}\left(t\right)|\mathcal{E}_t\right]$

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a) $\mathbb{E}\left[Z_{A_1}\left(t\right) \mid \mathcal{E}_t\right] + \mathbb{E}\left[Z_{A_2}\left(t\right) \mid \mathcal{E}_t\right] + \mathbb{E}\left[Z_{A_3}\left(t\right) \mid \mathcal{E}_t\right] = t$

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For some $t = \Theta\left(\ell^{\alpha-1}\right)$, we prove that:

b) $\mathbb{E}\left[Z_{A_1}\left(t\right) \mid \mathcal{E}_t\right] \leq \frac{3}{4}t$

c) $\mathbb{E}\left[Z_{A_2}\left(t\right) \mid \mathcal{E}_t\right] \leq \mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right] \cdot \Theta\left(\ell^2 \text{polylog}\left(\ell\right)\right)$

d) $\mathbb{E}\left[Z_{A_3}\left(t\right) \mid \mathcal{E}_t\right] = \mathcal{O}\left(t / \log t\right)$

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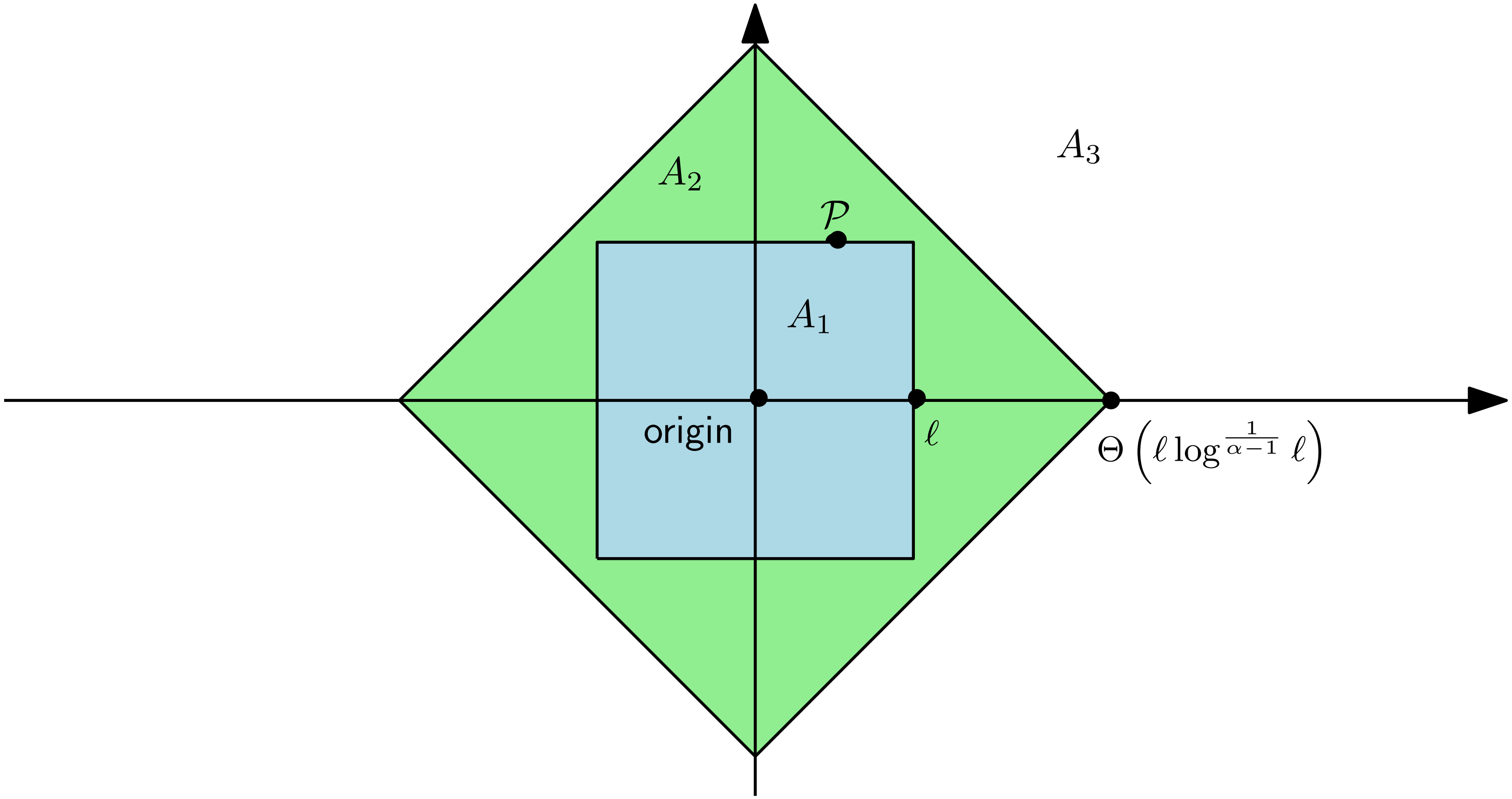
d) $\mathbb{E}\left[Z_{A_3}\left(t\right) \mid \mathcal{E}_t\right] = \mathcal{O}\left(t/\log t\right)$

Combine (a) with (b), (c), and (d) to get

$$\mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right] = \Omega\left(1/\left(\ell^{3-\alpha}\text{polylog}\left(\ell\right)\right)\right)$$

Getting $\mathbb{E}\left[Z_{\mathcal{P}}(t) | \mathcal{E}_t\right]$



b) $\mathbb{E}\left[Z_{A_1}(t) \mid \mathcal{E}_t\right] \leq \frac{3}{4} t$

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d) $\mathbb{E}\left[Z_{A_3}(t) \mid \mathcal{E}_t\right] = \mathcal{O}\left(t / \log t\right)$

c) comes from a monotonicity property

Getting $p(t)$

Reminder: $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}\left(t\right) > 0 \mid \mathcal{E}_t\right) \geq \mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right]/a_t$

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We prove that $a_t = \mathbb{E}\left[Z_{(0,0)}\left(t\right) \middle| \mathcal{E}_t\right]$ is constant w.r.t. t

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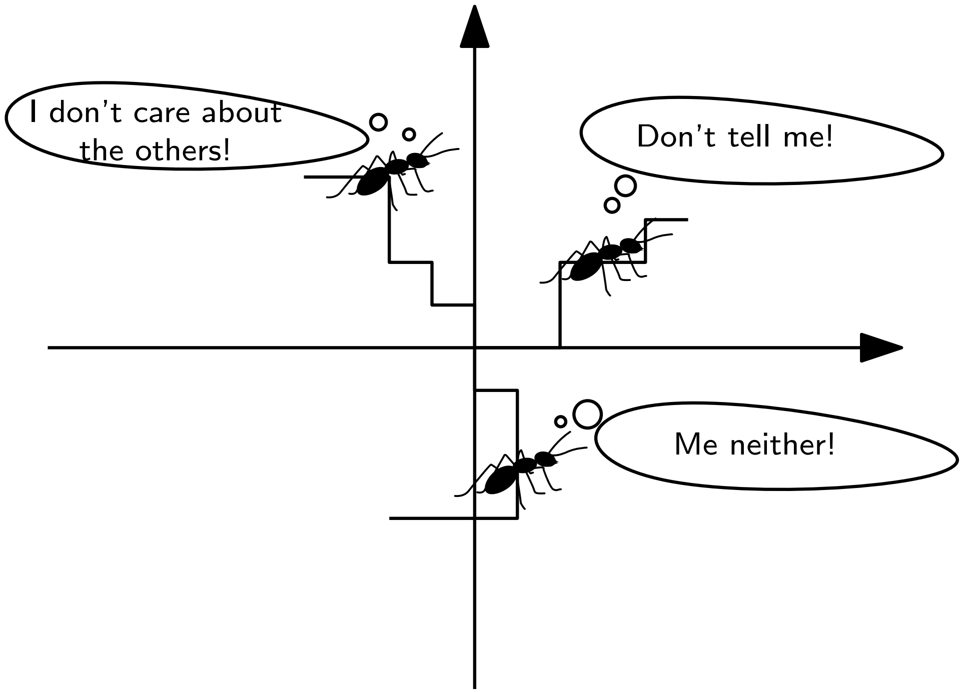
Lemma: for $t = \Theta\left(\ell^{\alpha-1}\right)$, it holds that

$$p(t) = \Omega\left(1/\left(\ell^{3-\alpha}\text{polylog}\left(\ell\right)\right)\right)$$

Note: the coupling result gives us the same asymptotic bound for the Lévy walk

The parallel search

We exploit *independence*!



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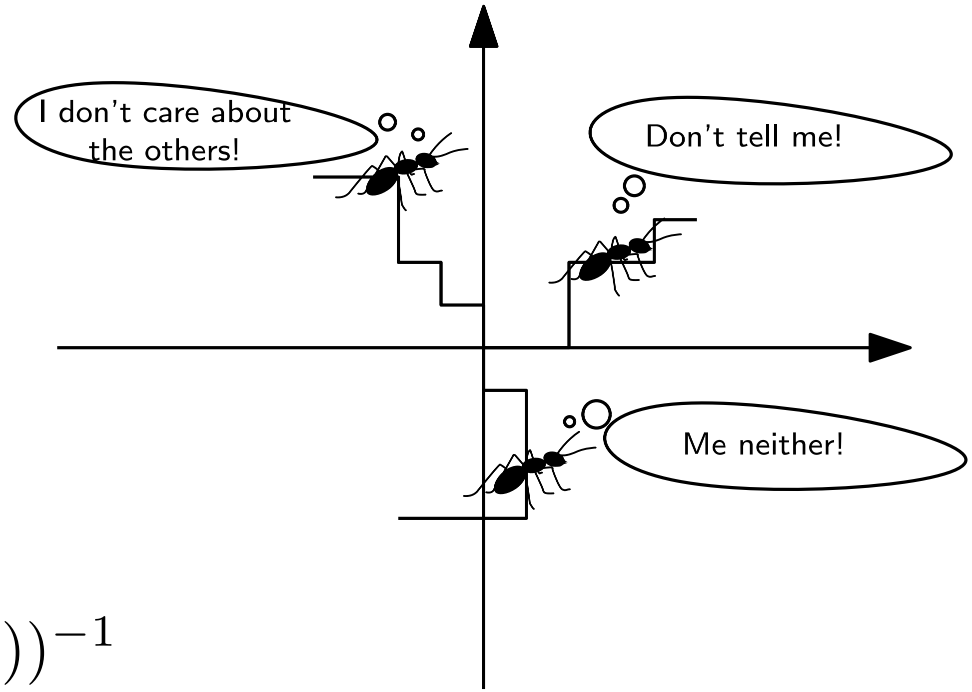
For $t = \mathcal{O}\left(\ell^{\alpha-1}\right)$, set $k = \log \ell \cdot (p(t))^{-1}$



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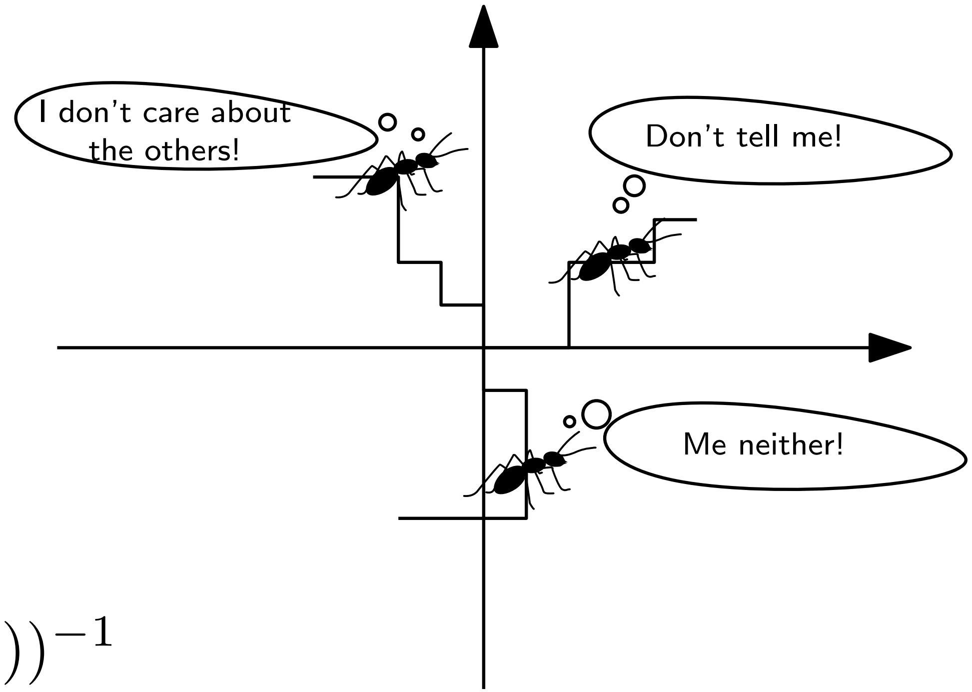

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The probability that at least one walker finds the treasure within time t is

$$1 - [1 - p(t)]^{\frac{\log \ell}{p(t)}} \sim 1 - e^{\log \ell} = 1 - \frac{1}{\ell}$$

k walkers find the target within time $t = \Theta\left(\ell^{\alpha-1}\right)$, w.h.p.

Finding $\alpha^\star$

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$$\alpha = 3 - \log k / \log \ell + \mathcal{O}\left(\log \log \ell / \log \ell\right)$$

Finding $\alpha^\star$

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$$\boxed{\alpha = 3 - \log k / \log \ell + \mathcal{O}\left(\log\log \ell / \log \ell\right)}$$

$$\Longrightarrow \boxed{\alpha^\star = 3 - \log k / \log \ell}$$

Questions?

THANK YOU FOR YOUR ATTENTION