Multidimensional Random Subset Sum Problem

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COATI team

Based on joint work with L. Becchetti, A. Carvalho Walraven da Cunha, A. Clementi, H. Lesfari, E. Natale, and L. Trevisan

Aalto University
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Example:

- Sequence: -9, -7, -1, -1, 0, $+3$, $+4$, $+5$, $+9$, $+11$
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- $-9 + 11 = 2$
- $-7 + 9 = 2$
- $-1 + 3 = 2$
- $-1 - 1 + 4 = 2$
- $-7 + 4 + 5 = 2$
- etc.

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- **Applications**:

- combinatorial number theory [Zhi-Wei, 2003]
- cryptography [Gemmel et Johnston, 2001; Kate et Goldberg, 2011]

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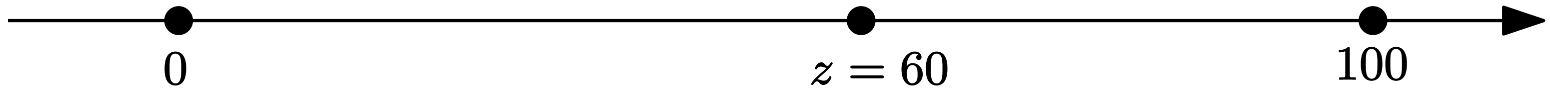

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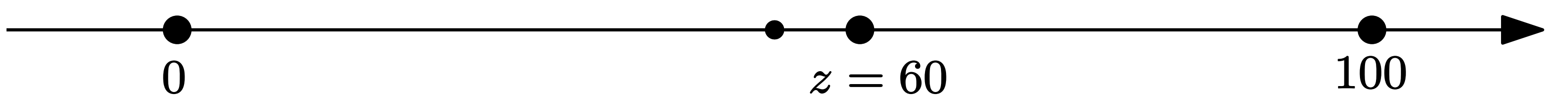

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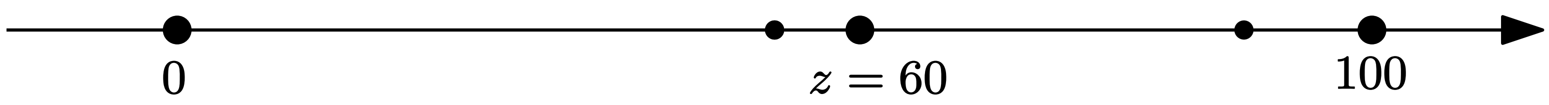

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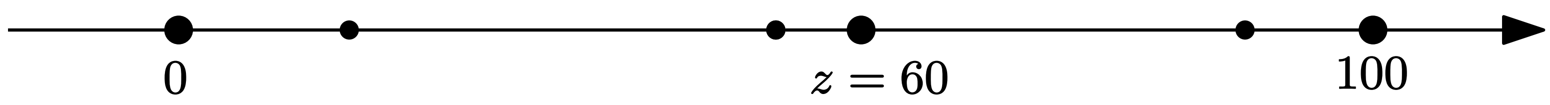

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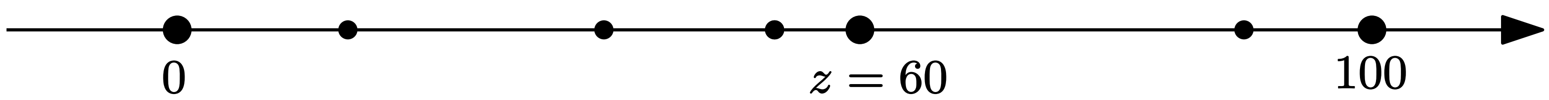

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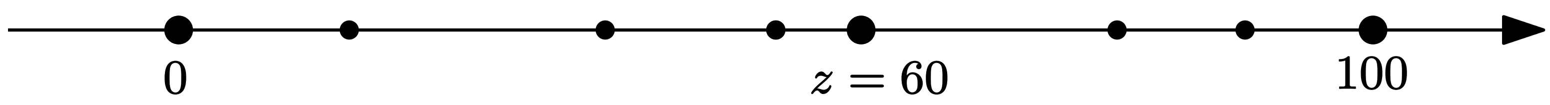

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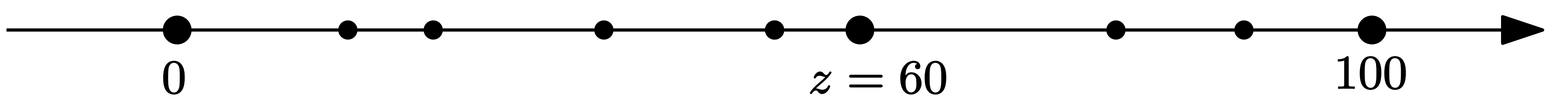

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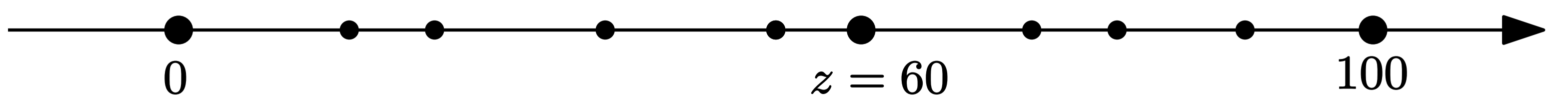

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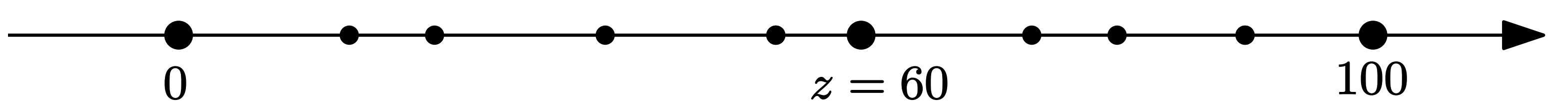

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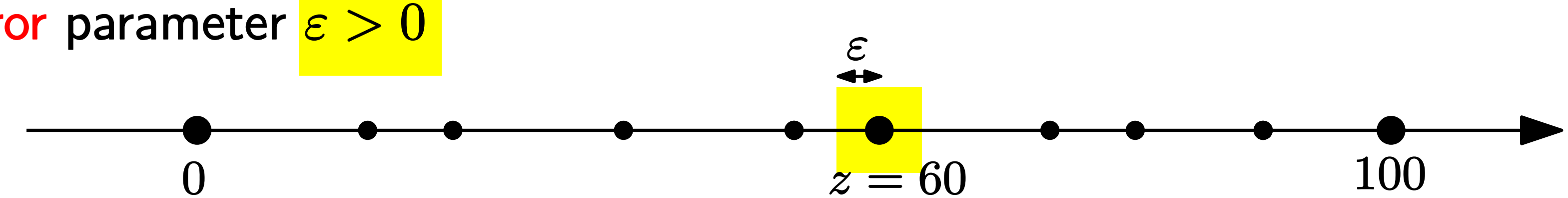

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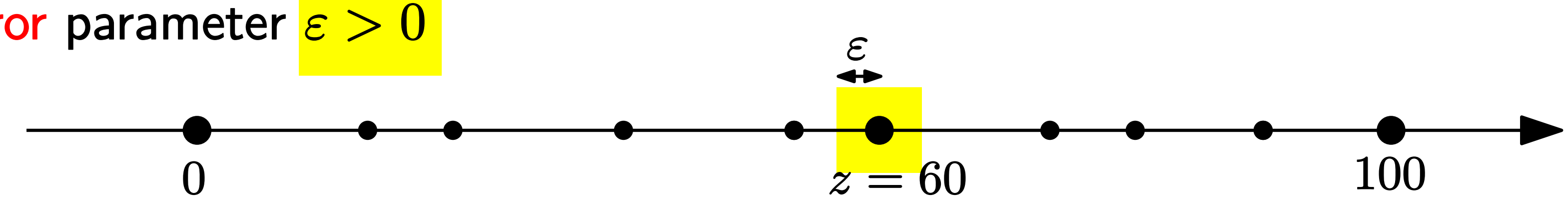

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We say that S 2ε-approximates z

Related literature

- Mainly, results from Lueker [Lueker, 1982; Lueker, 1998]
- Proved in a simpler way in [Carvalho, d'Amore, Giroire, Lesfari, Natale, and Viennot, 2022]

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Specific instance of RSSP

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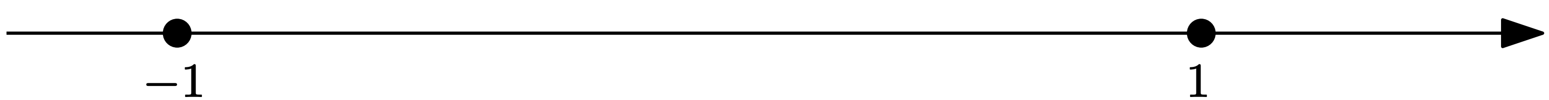

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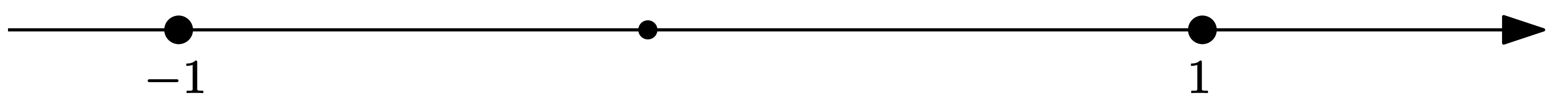

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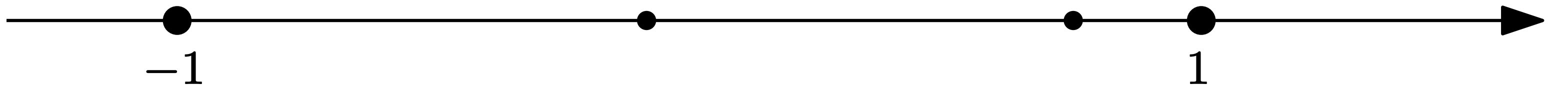

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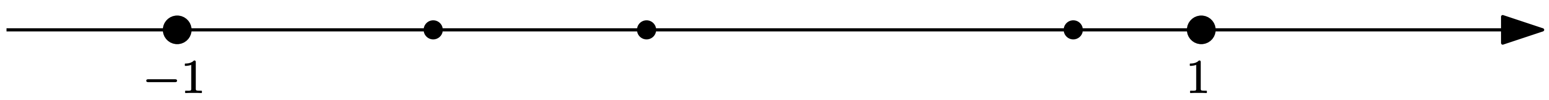

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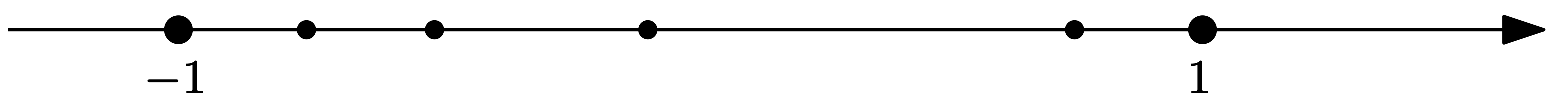

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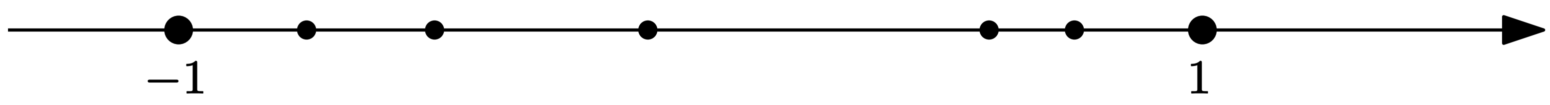

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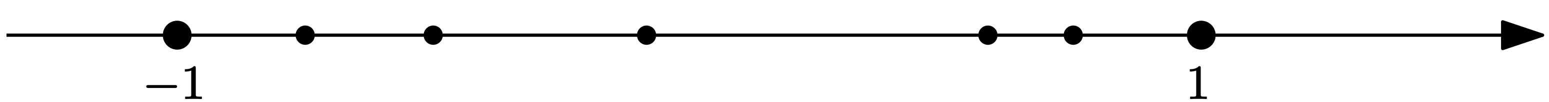

Stronger **question**:

- How large n such that, with h. p., for any $z \in [-1, 1]$ a subset $S_z \subseteq [n]$ exists, with $|z - \sum_{i \in S_z} X_i| \leq 2\varepsilon$?

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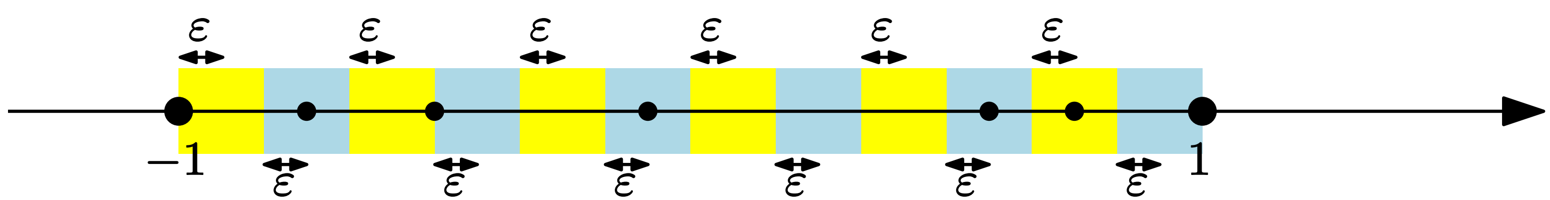

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Theorem: [Lueker, 1998]

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Corollaries:

- the result applies to a wider class of distributions: any density $f \geq b > 0$ for $x \in [-a, a]$

- upper bound on the expectation of

- the $[a, b]$-Subset Sum gap: minimum value of 2ε such that any real in $[a, b]$ can be 2ε-approximated by some subset S of n variables

- the $[a, b]$-Number Partition gap: minimum value of 2ε such that any real in $[a, b]$ can be 2ε-approximated by using coefficients $\{-1, +1\}$ with n variables

Applications of the RSSP

Machine learning:

- Proof of the Strong Lottery Ticket Hypothesis [Pensia et al., NeurIPS 2020]
  - *any target network of width d and depth ℓ can be approximated by pruning a random network that is a factor $\mathcal{O}(\log d\ell)$ wider and twice as deep*
  - feed-forward, fully connected, ReLU activation
- Related results [Carvalho et al., ICLR 2022; Burkoholz et al., ICLR 2022]
- Pruning in Federated Learning [Wang et al., EMNLP 2021]

Proof sketch of the RSSP result

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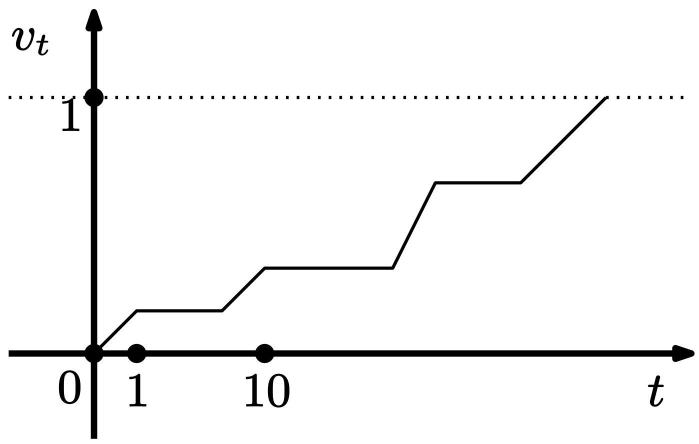

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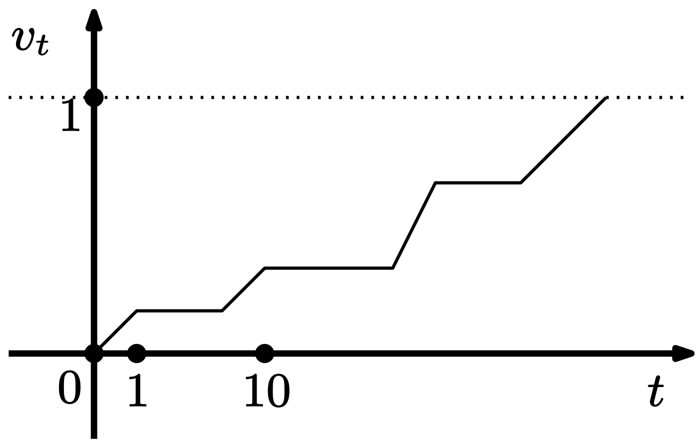

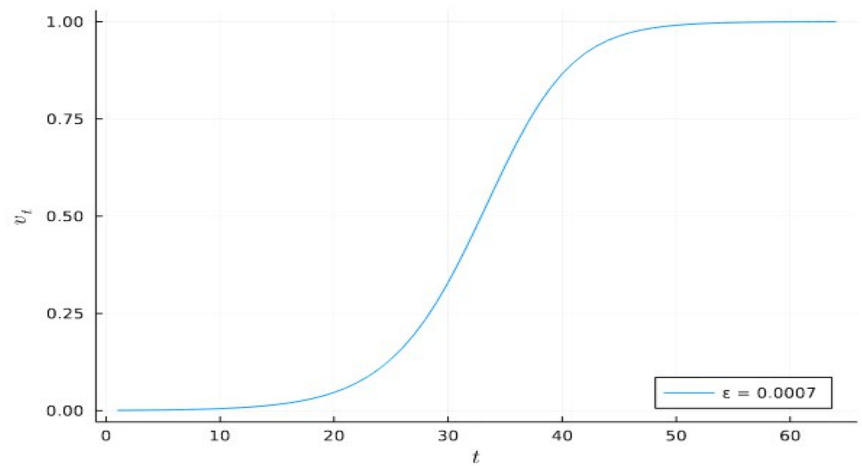

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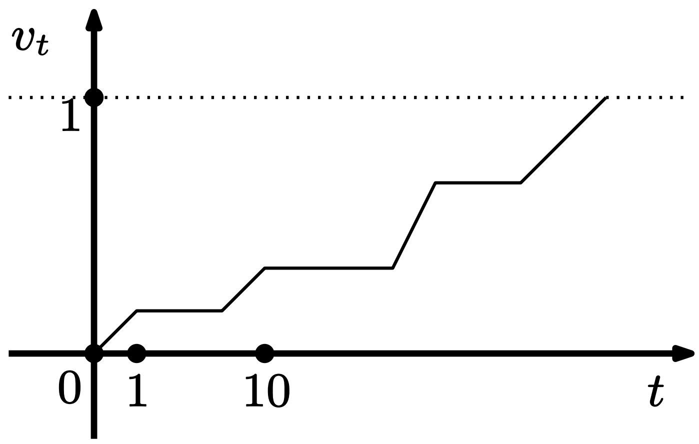

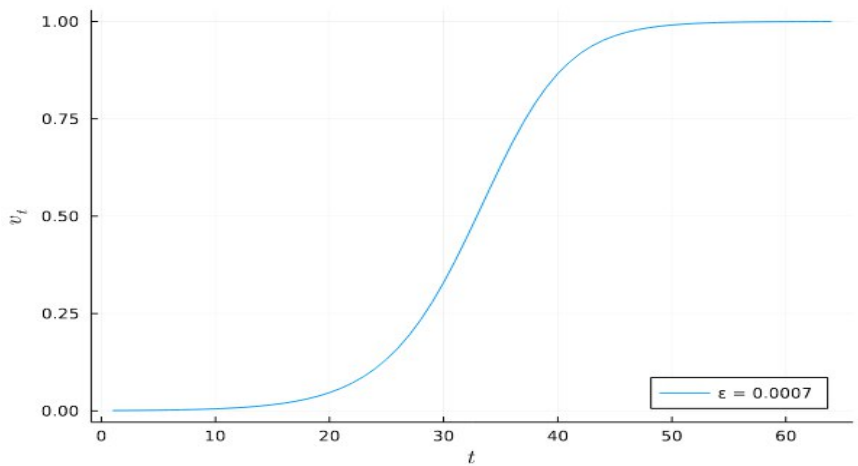

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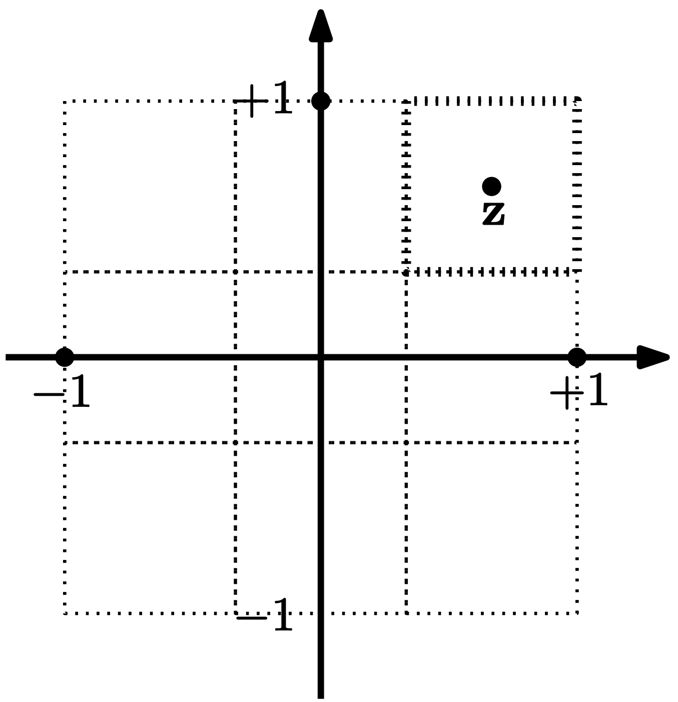

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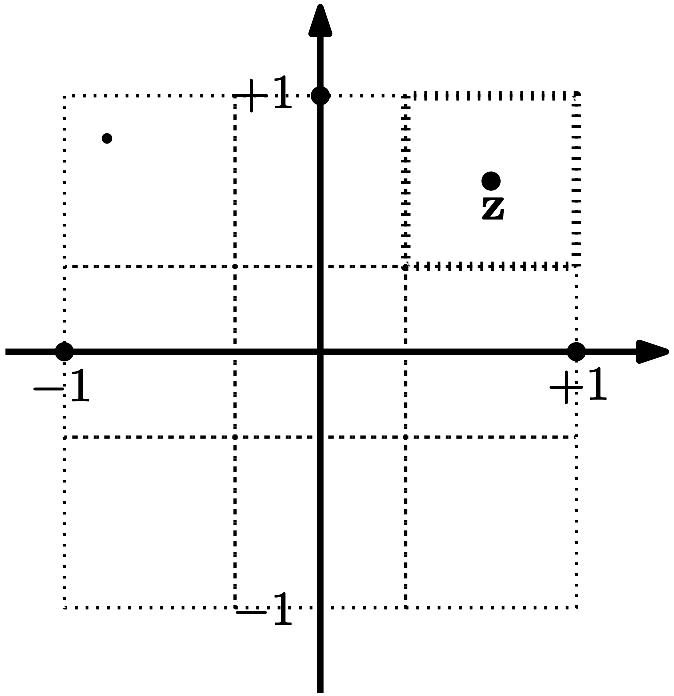

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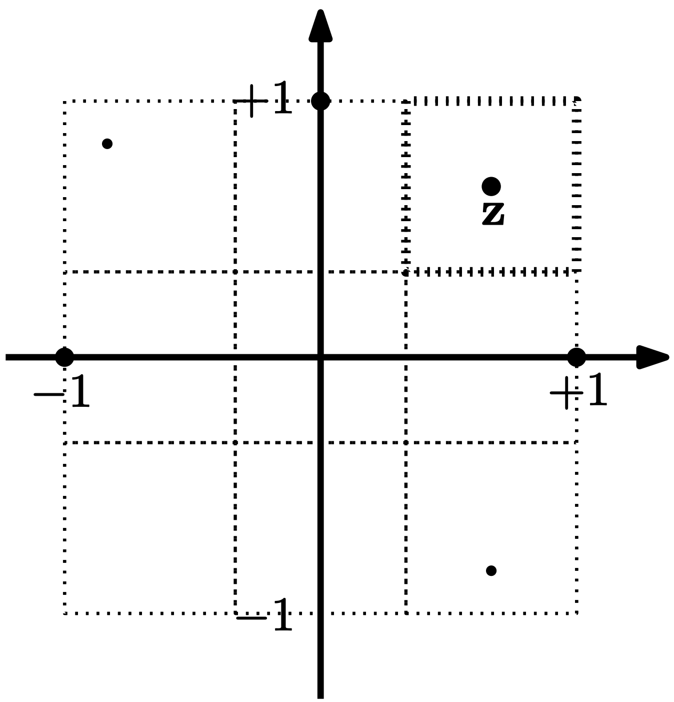

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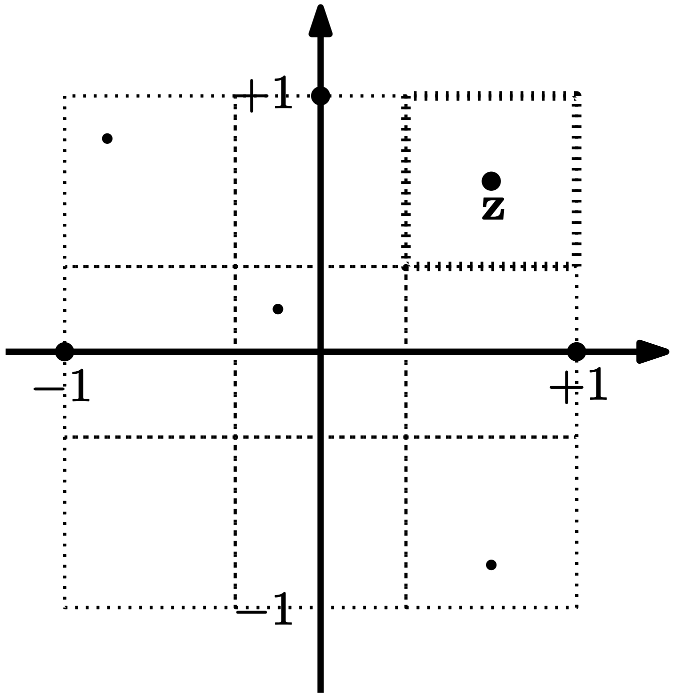

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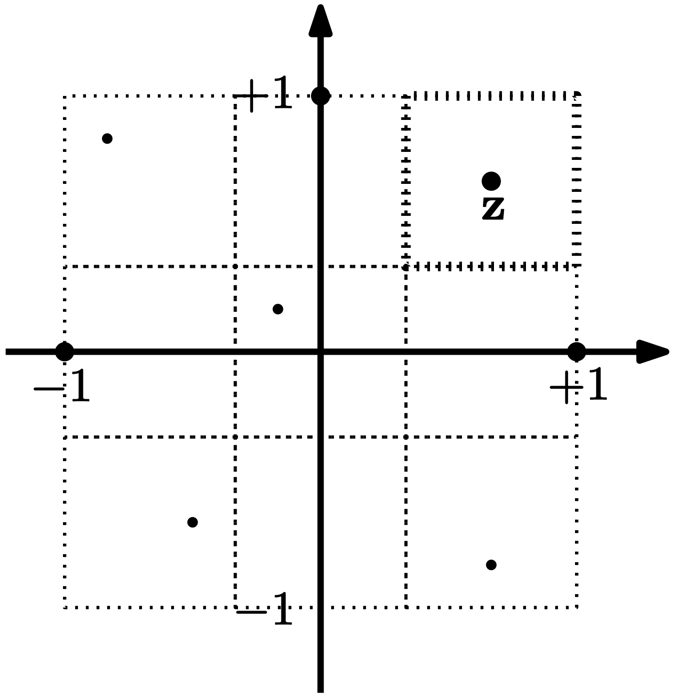

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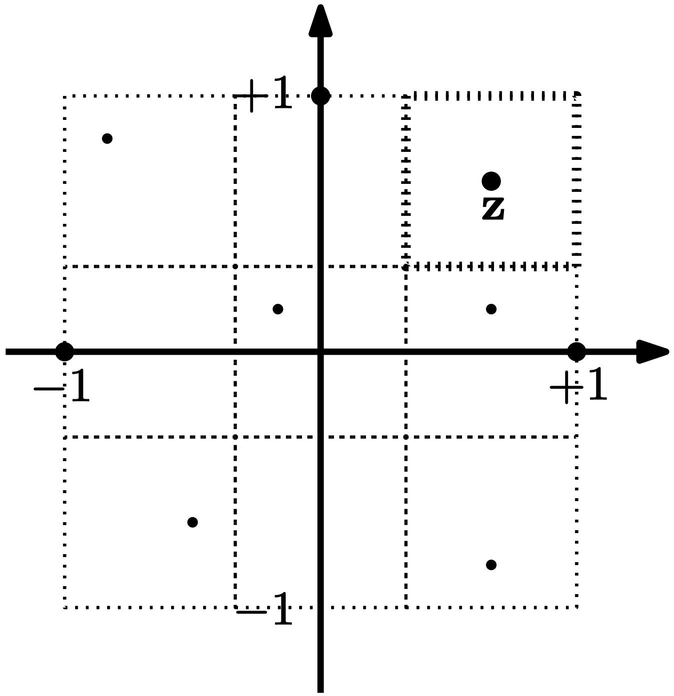

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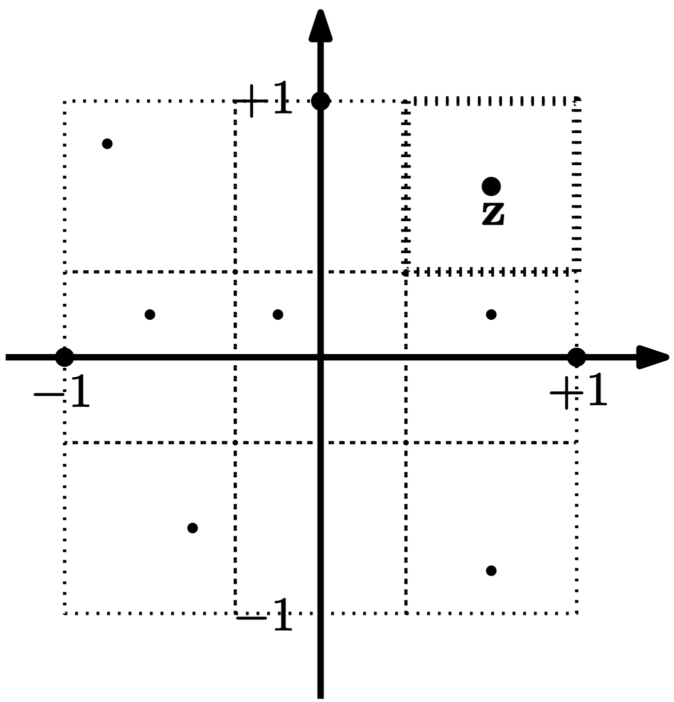

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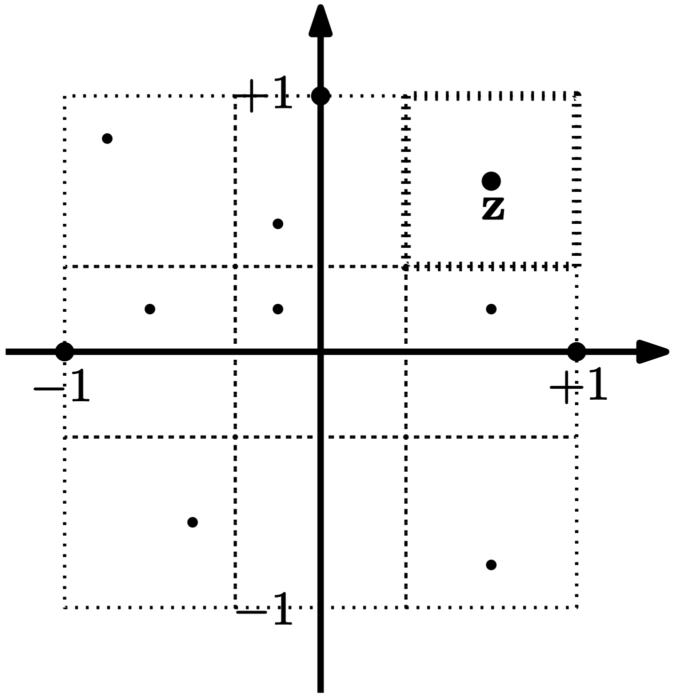

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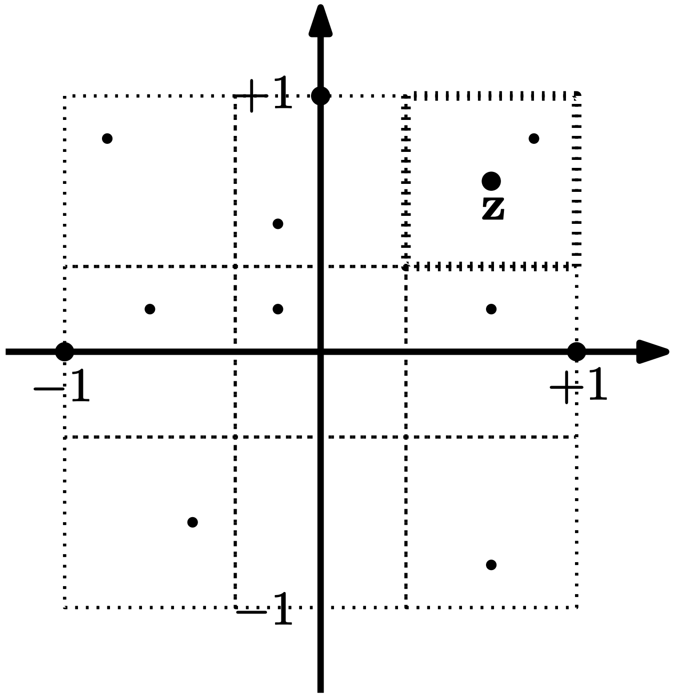

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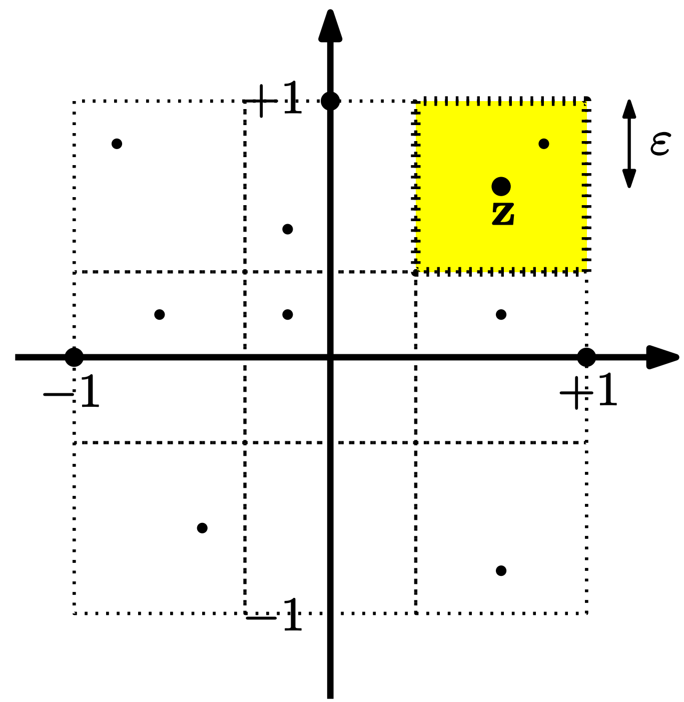

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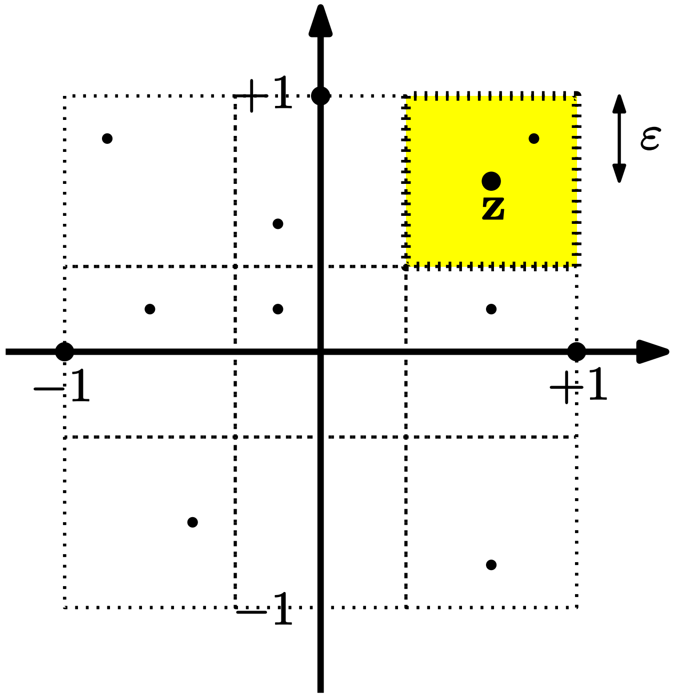

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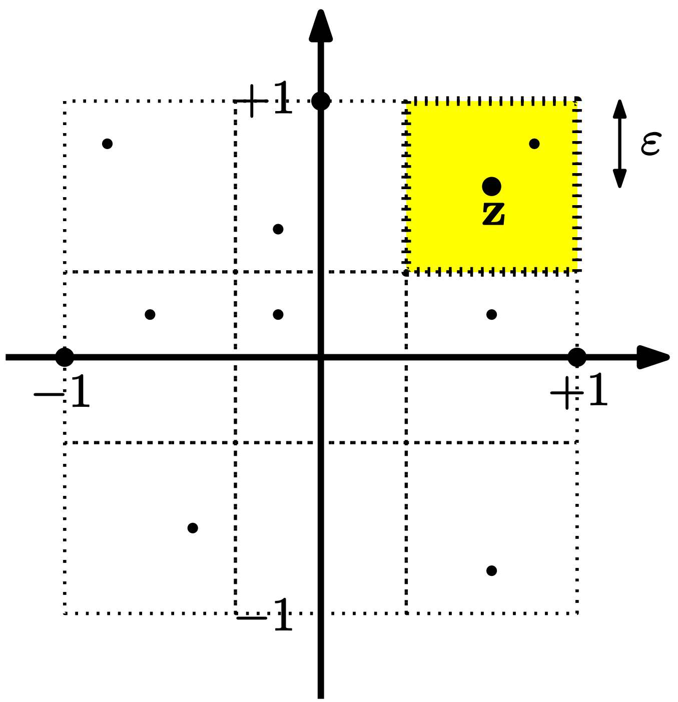

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- If $X_i \sim Unif([-1, 1]^d)$, same proof as before leads to $n \geq \exp(d^{\Omega(1)}) \log \frac{1}{\varepsilon}$ variables
- No success with method of average bounded differences, or Janson's variant of Chernoff bound

MRSSP: definition of the problem

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- Sequence of n independent **standard normal** random vectors $X_1, \ldots, X_n \sim \mathcal{N}(\mathbf{0}, I_d)$
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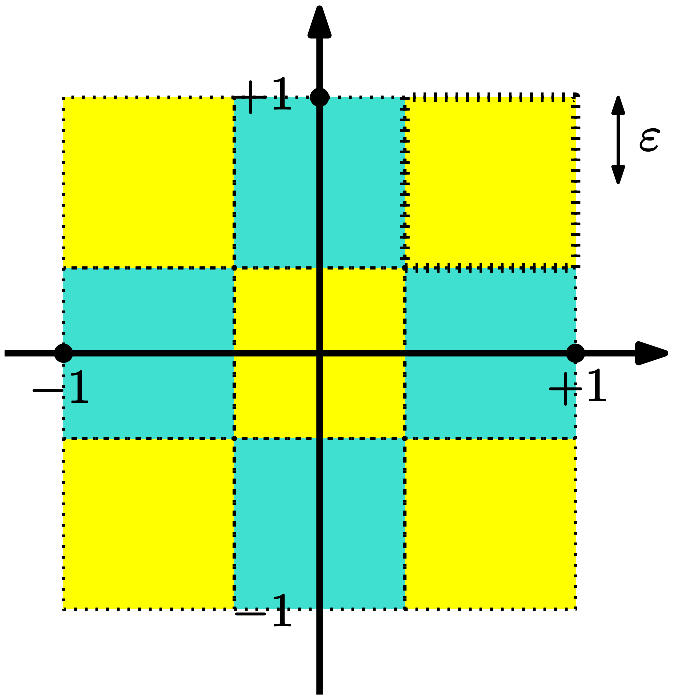

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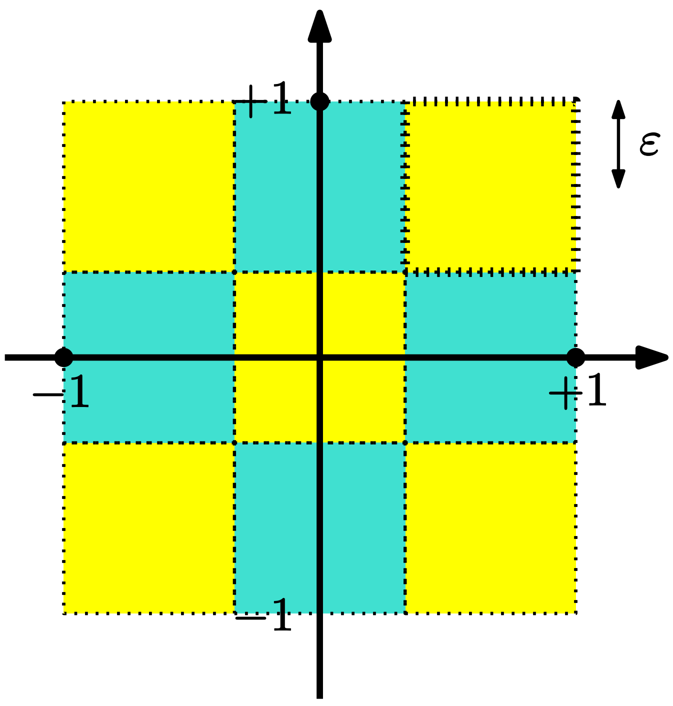

Observations:

- $\Theta\left(\frac{1}{\varepsilon^d}\right) = 2^{\Theta\left(d \log \frac{1}{\varepsilon}\right)}$ ∞-norm balls of radius ε are necessary to cover the hypercube

- at least $\Omega\left(d \log \frac{1}{\varepsilon}\right)$ vectors

MRSSP: upper bound in expectation

Negletting constants

- Fix $\mathbf{z} \in [-1, 1]^d$ and a 2ε-side hypercube $B_\varepsilon(\mathbf{z})$ around $\mathbf{z}$ — the ∞-norm ball of radius ε.

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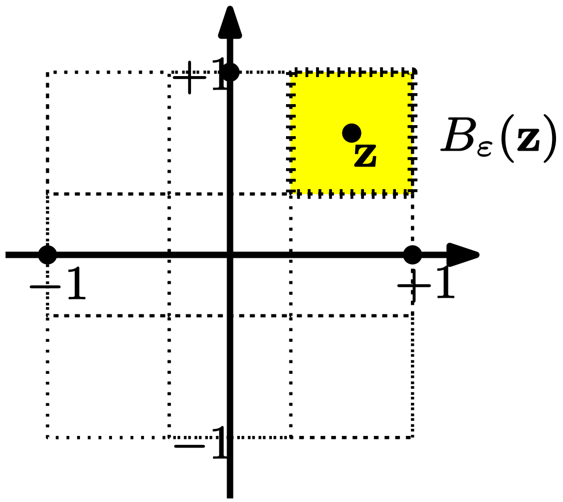

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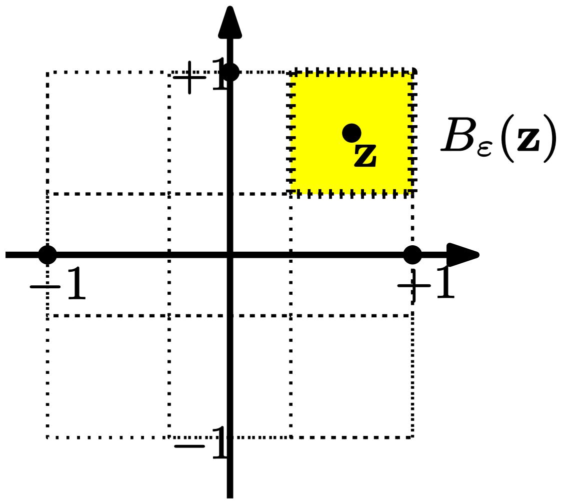

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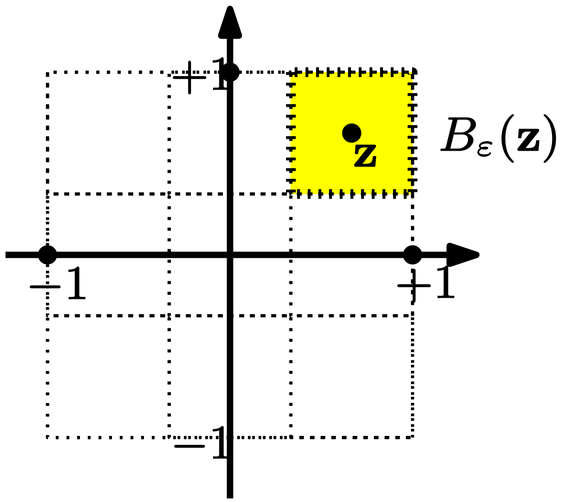

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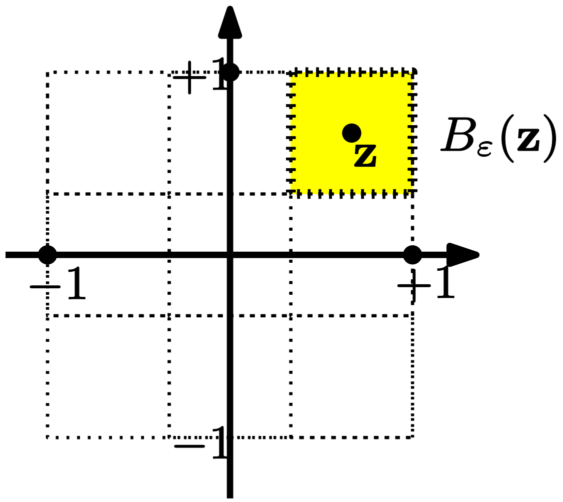

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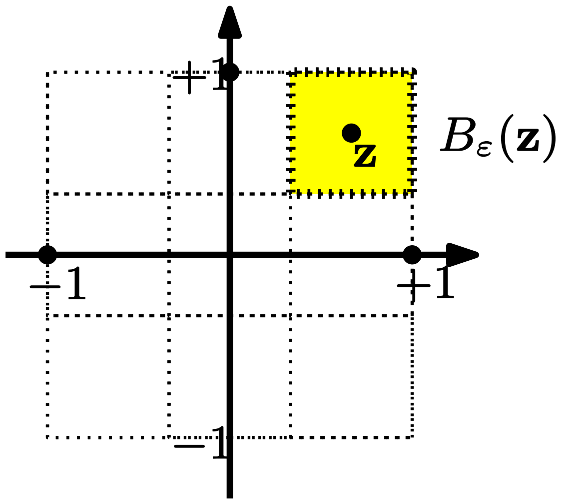

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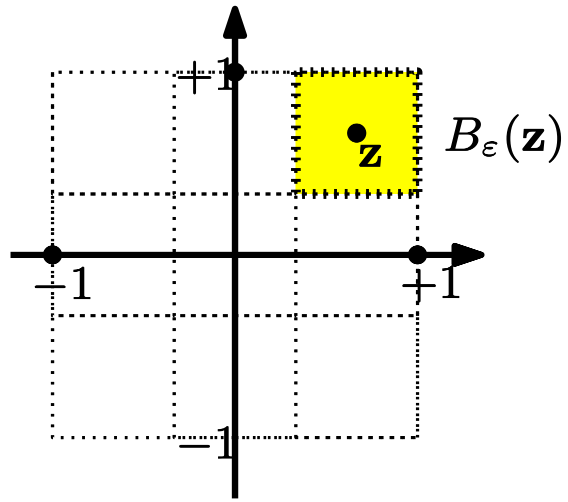

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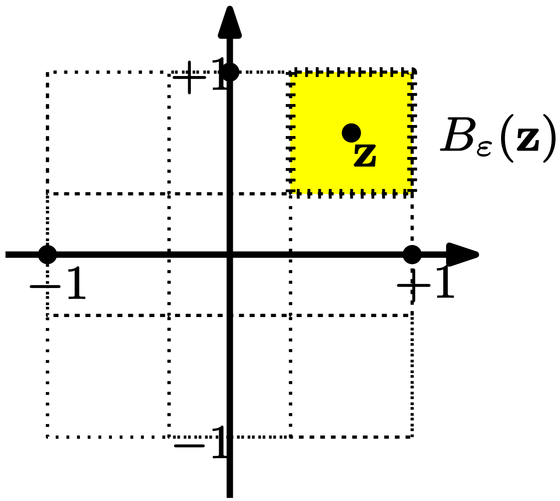

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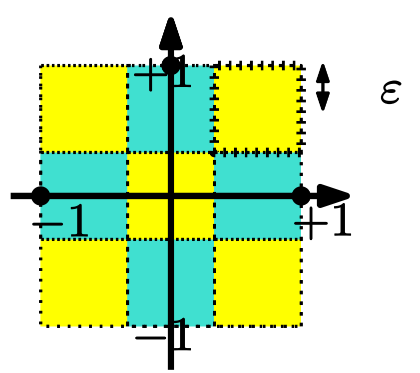

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Generalizations:

- The domain $[-1,1]^d$ can be widened to $[-\sqrt{\frac{n}{d\sqrt{d}}}, \sqrt{\frac{n}{d\sqrt{d}}}]^d$
- The distribution class is larger: every distribution that contains a gaussian

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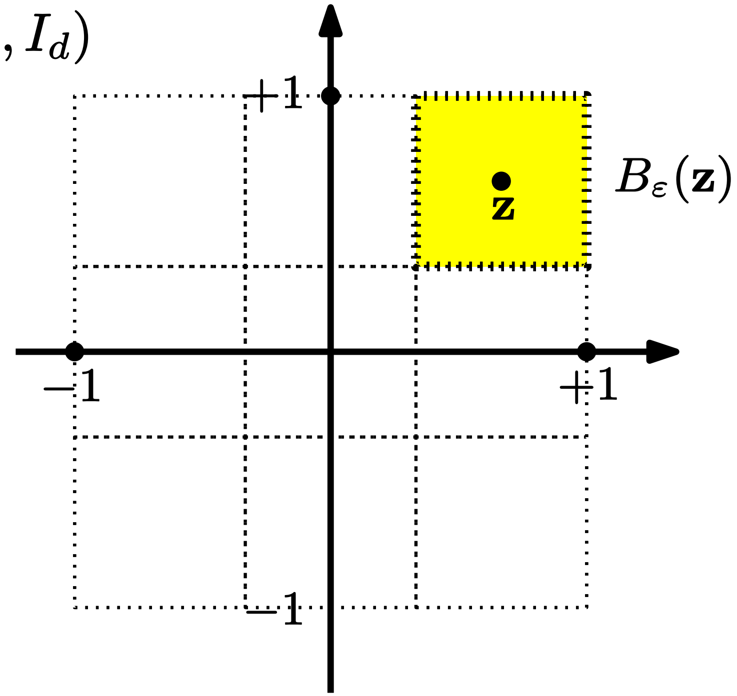

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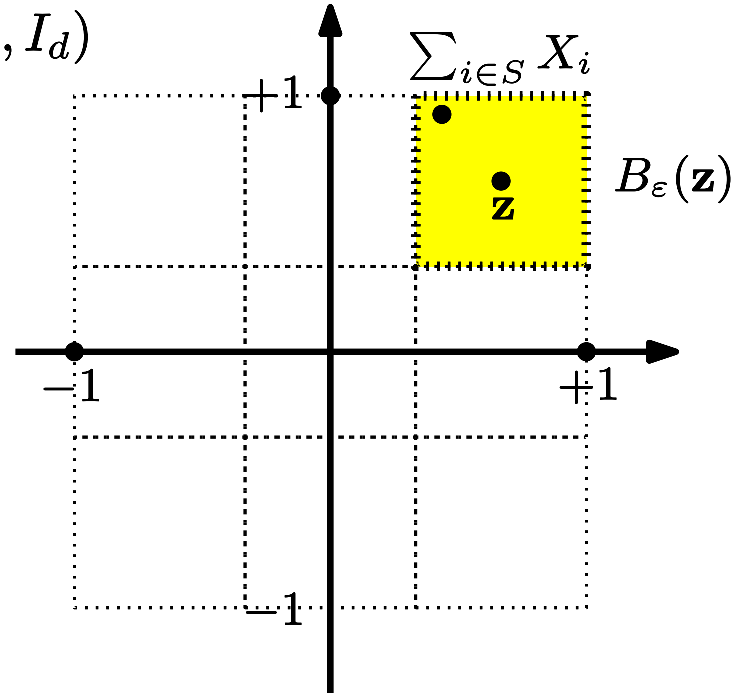

- Fix any subset $S \subseteq [n]$ of size αn

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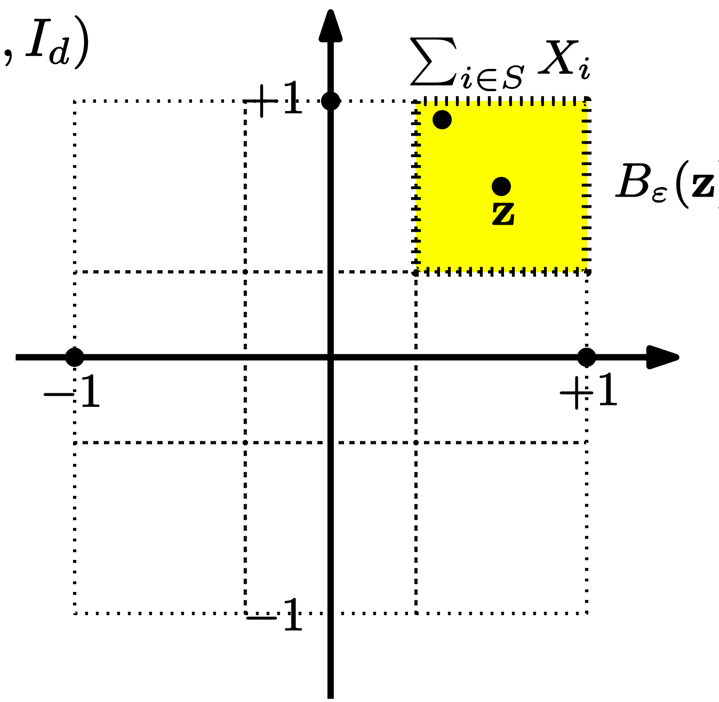

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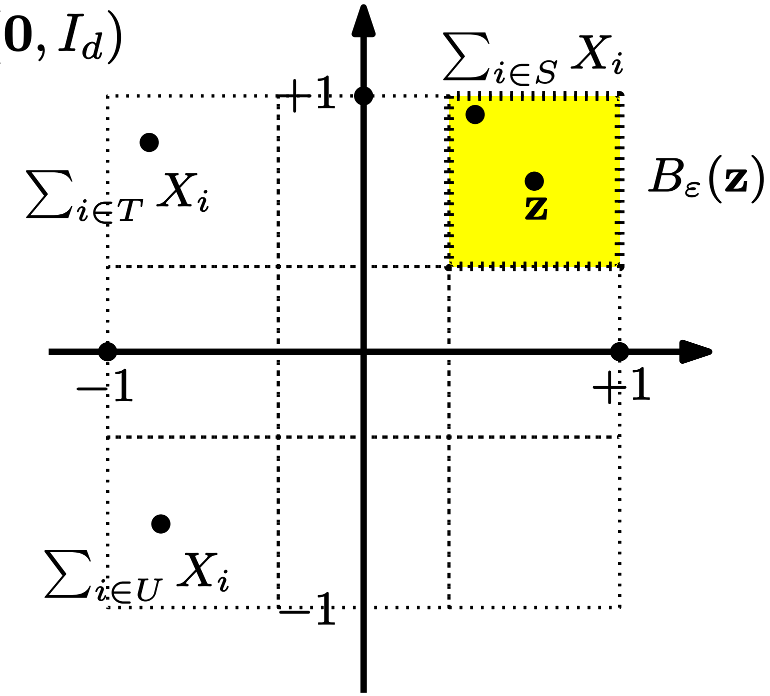

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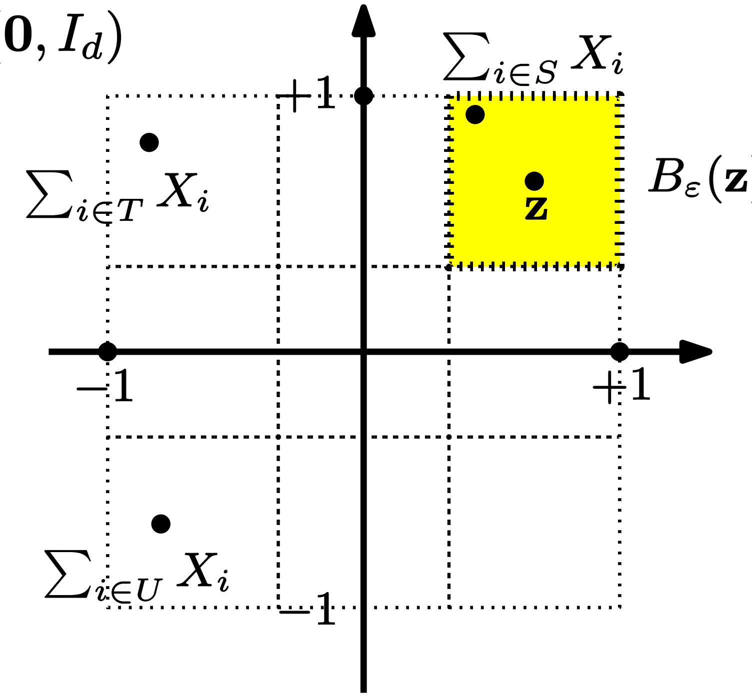

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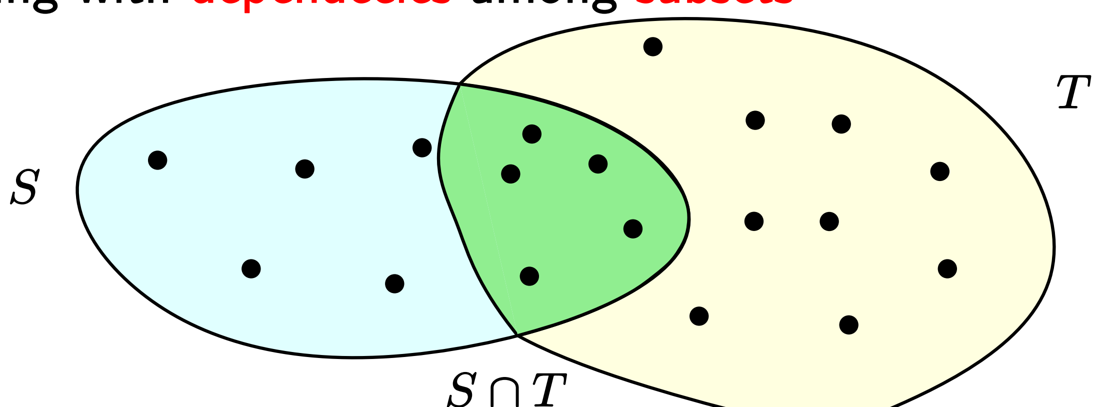
Main difficulty: counting and dealing with dependecies among subsets

- joint probability of Y_S **and** Y_T



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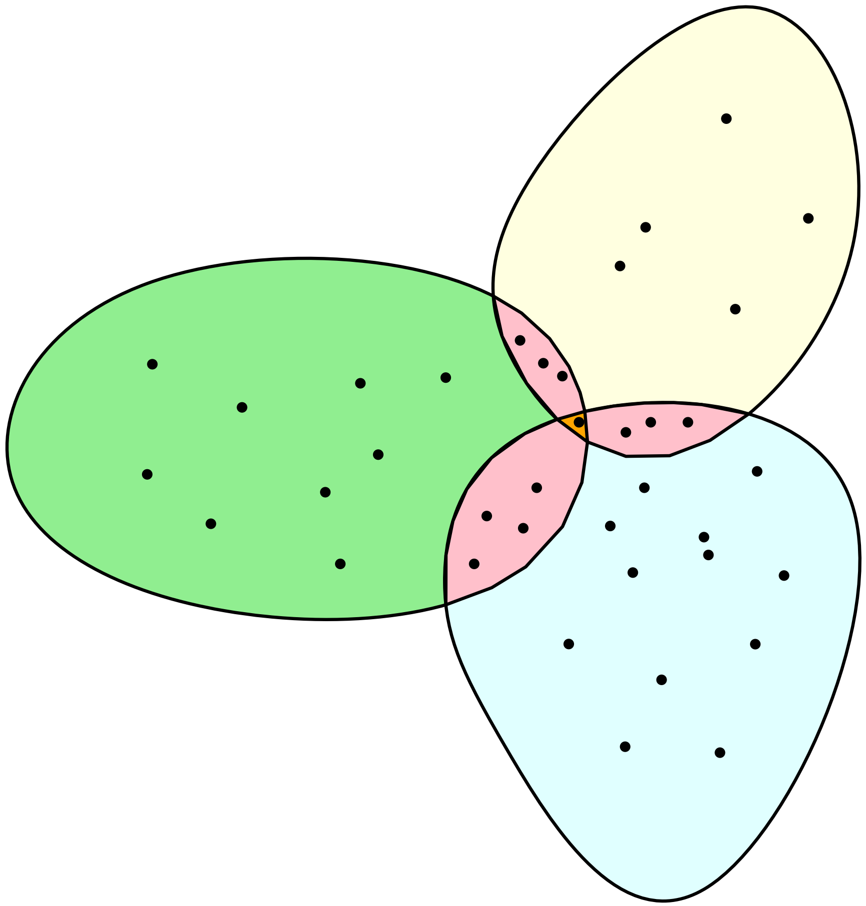

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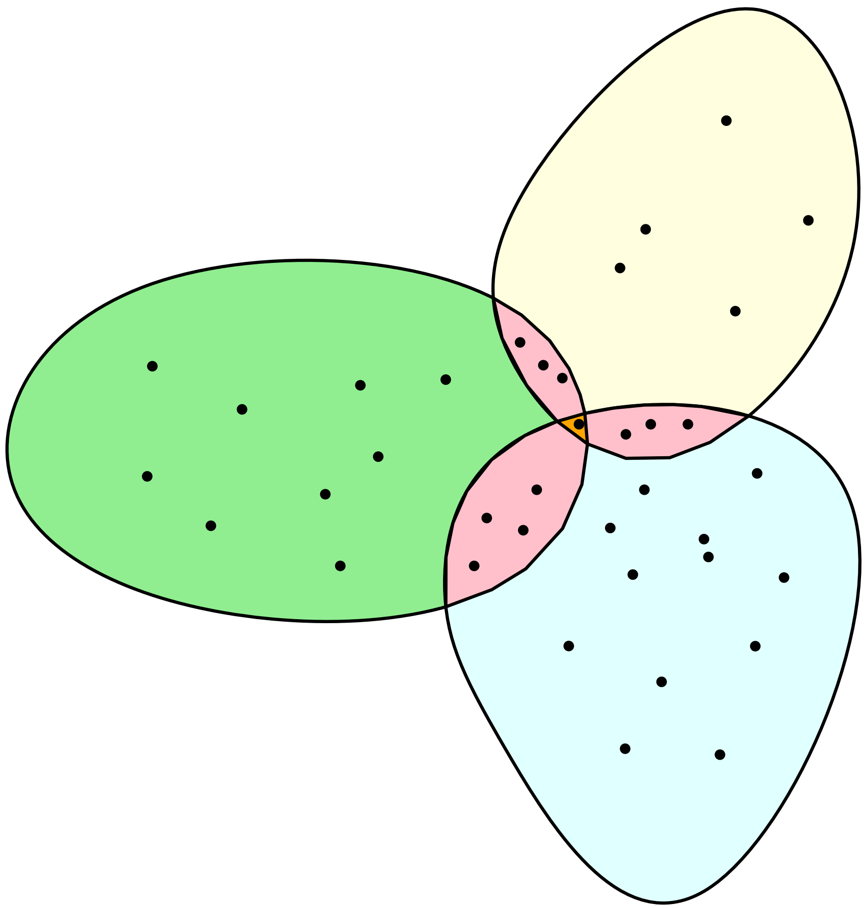

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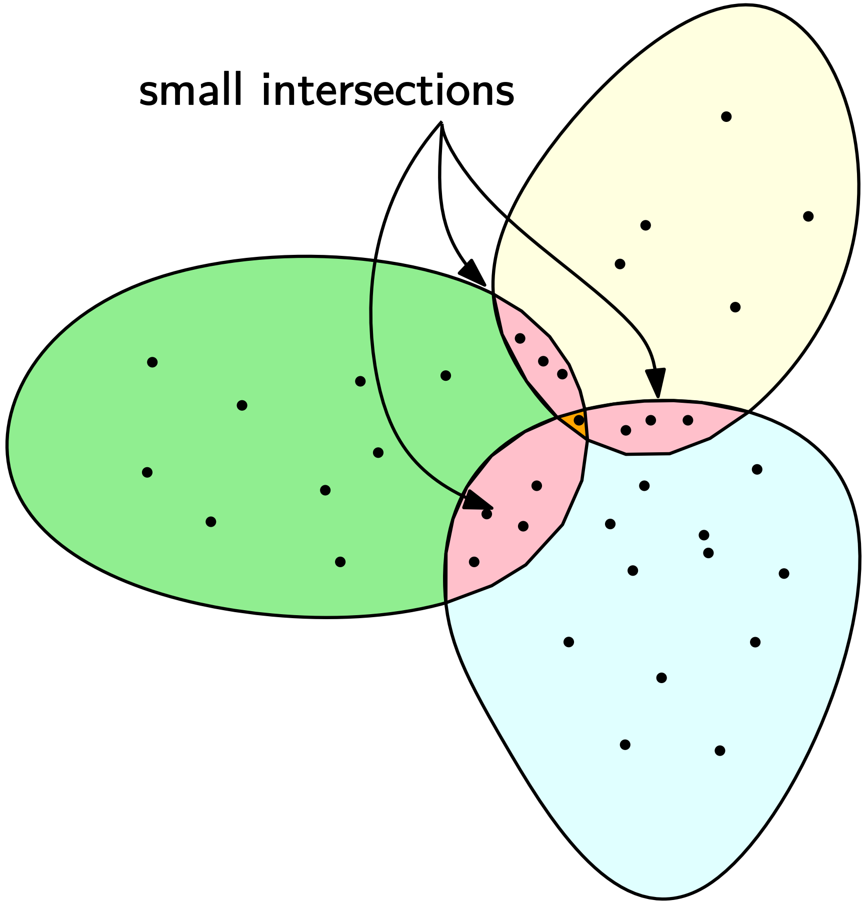

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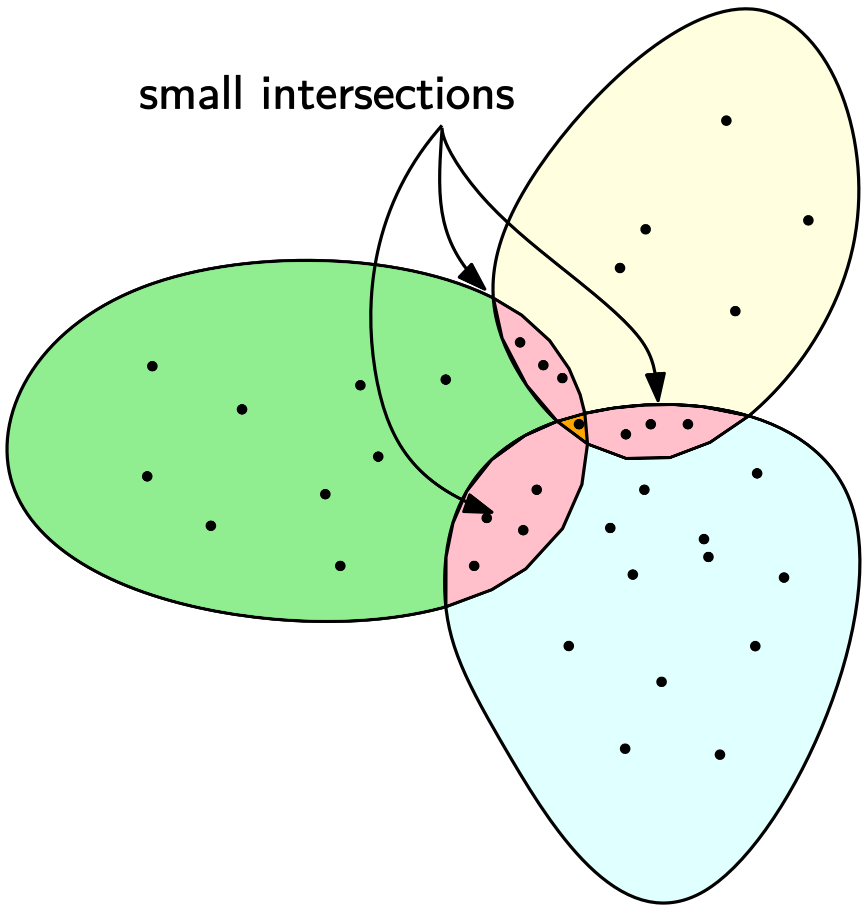

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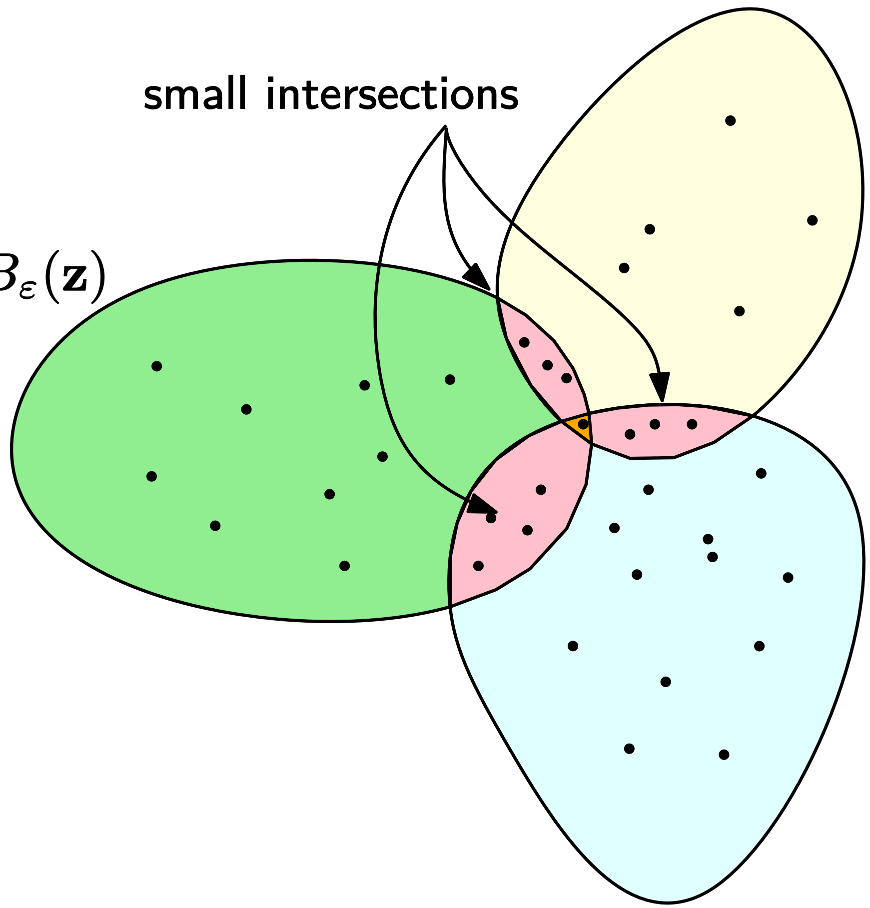

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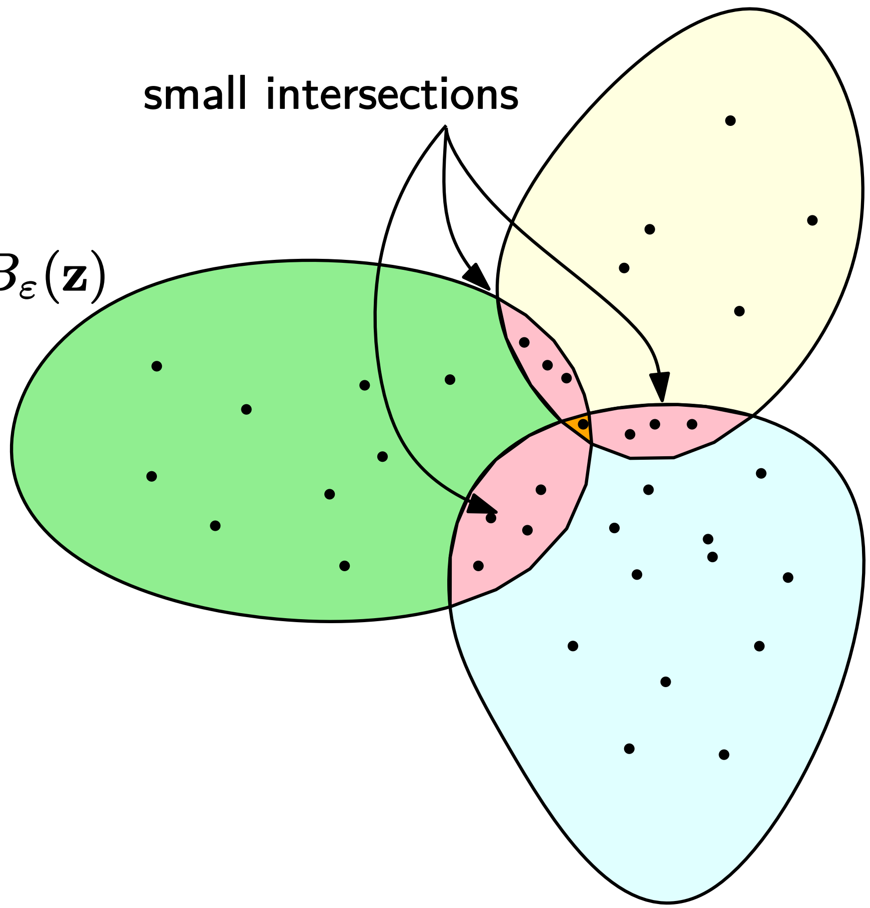

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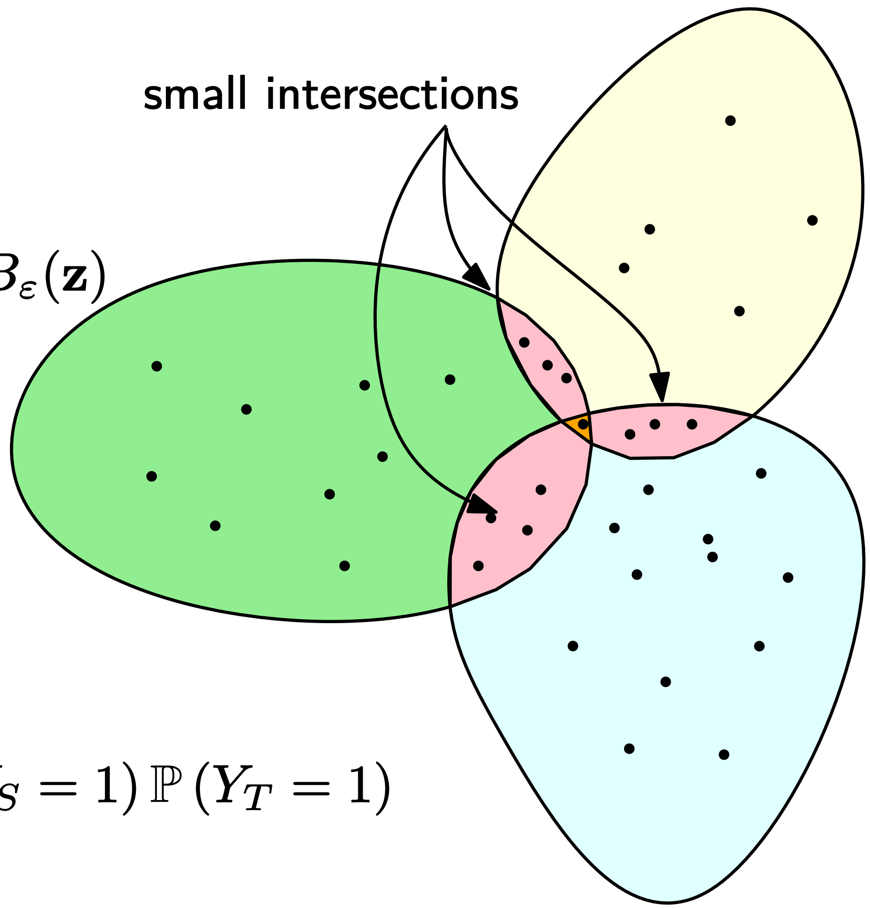

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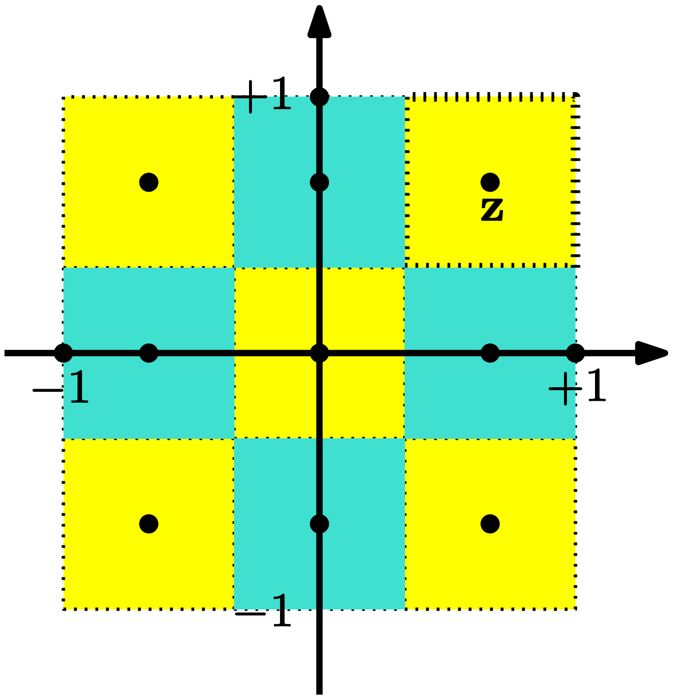

- for all hypercubes?

Amplification argument: union bound

- **Amplification**
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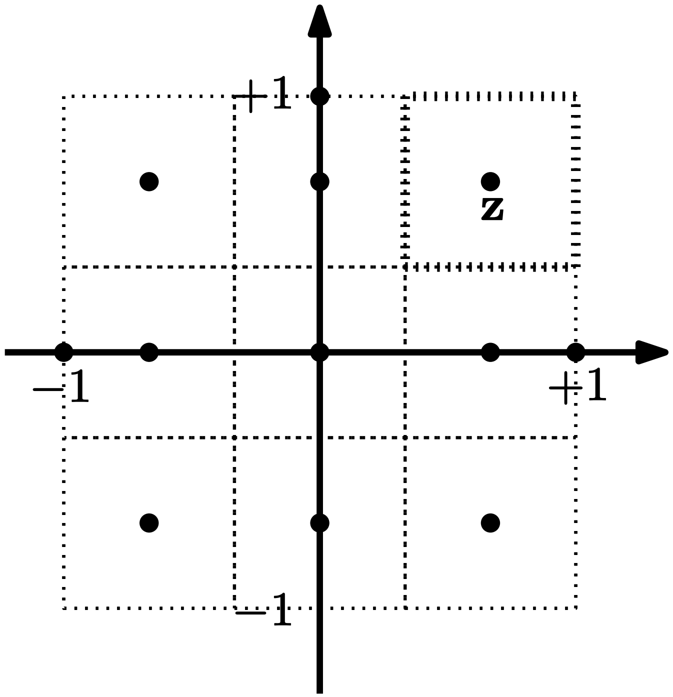

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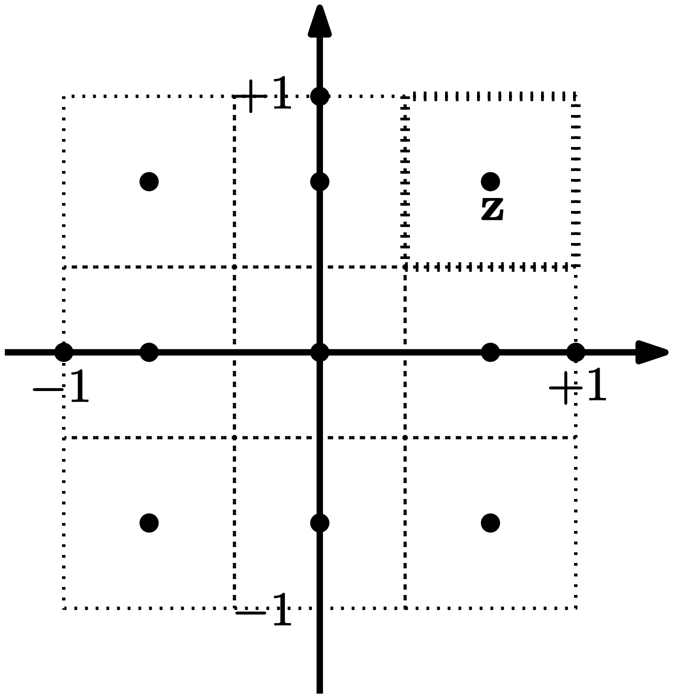

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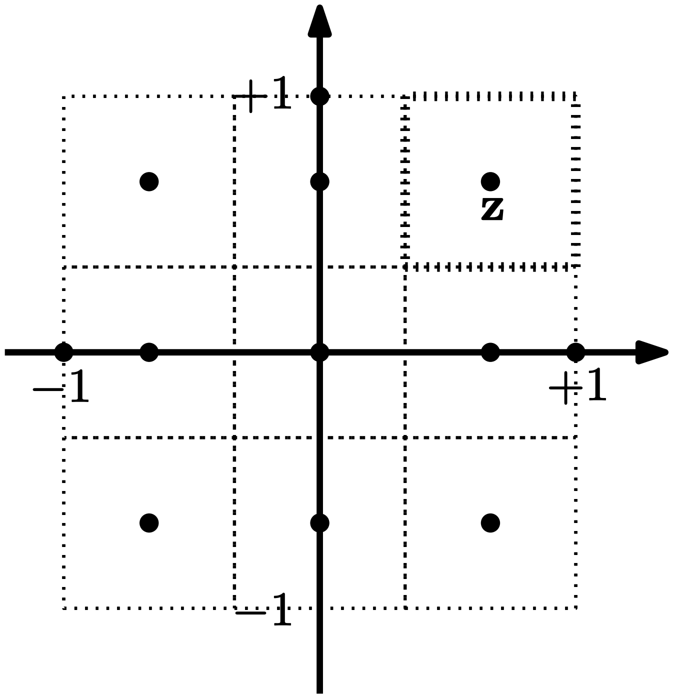

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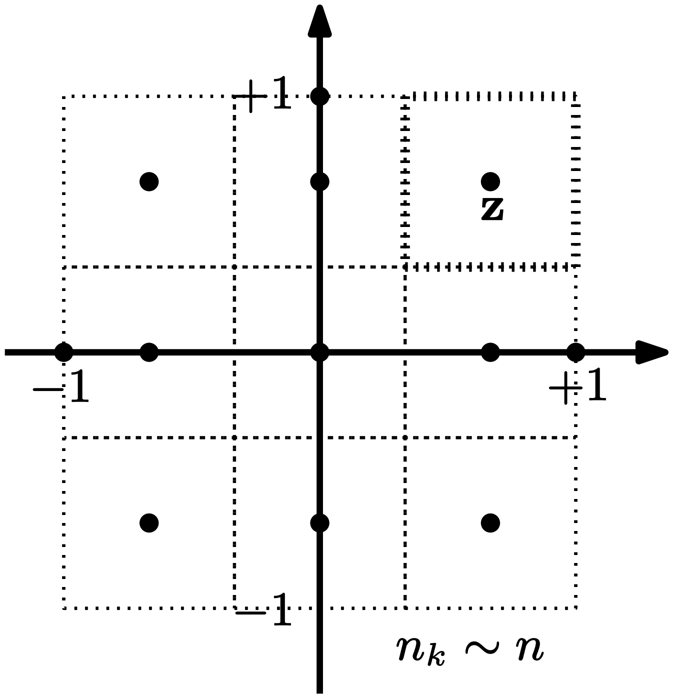

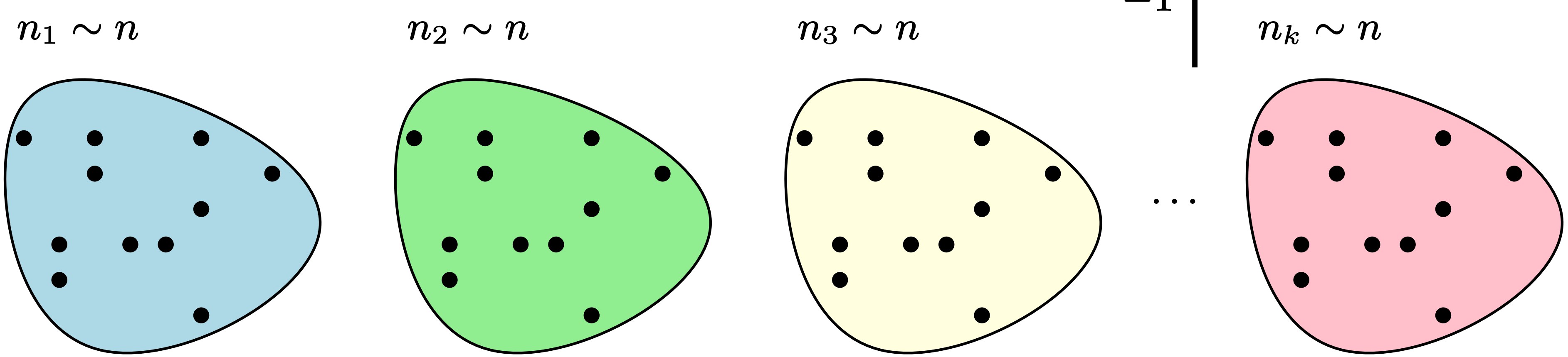

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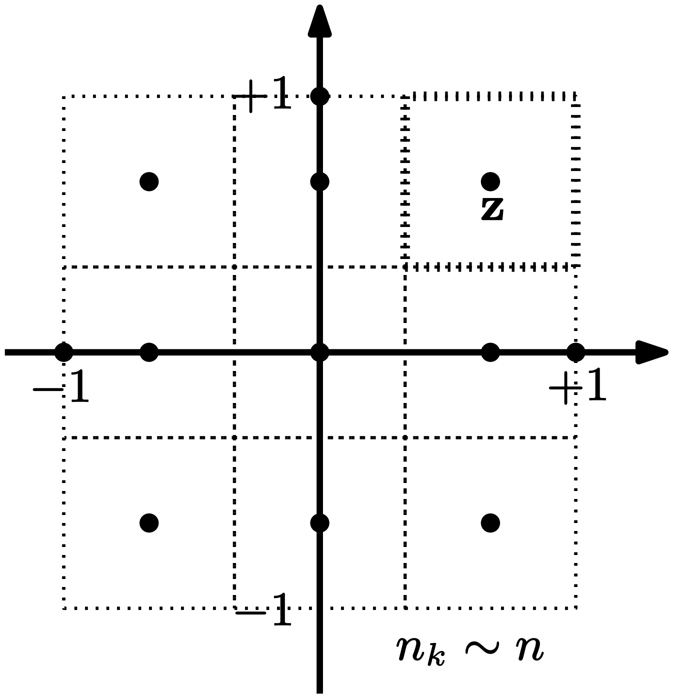

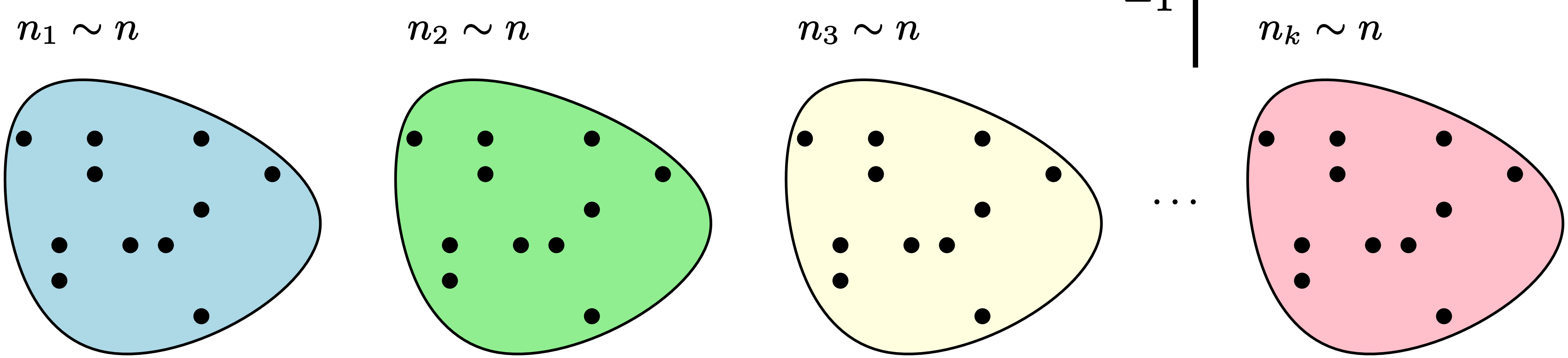

- **Prob. to fail** $\leq \frac{1}{\varepsilon^d} \cdot (\frac{2}{3})^k$
  - $k \sim d \log \frac{1}{\varepsilon}$ to have exponentially small prob. to fail

Recap of results

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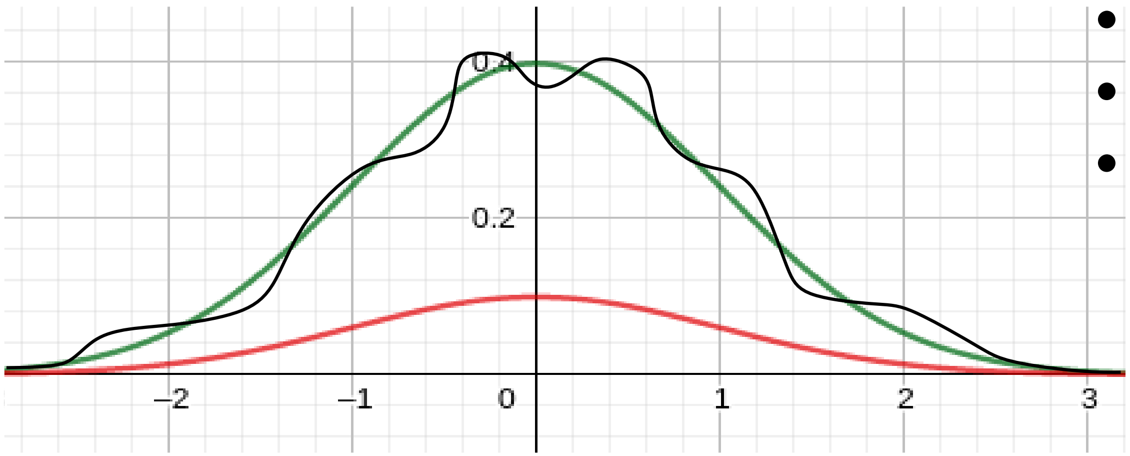

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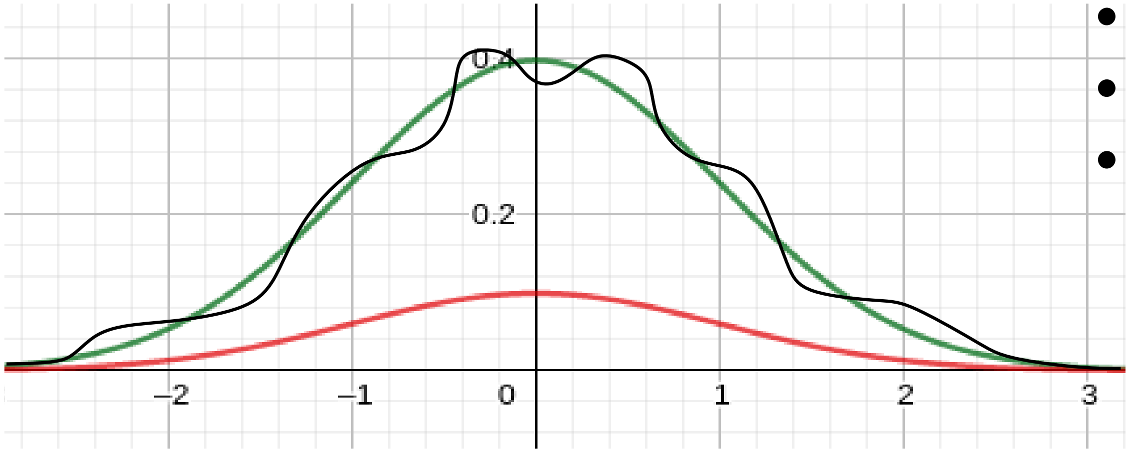

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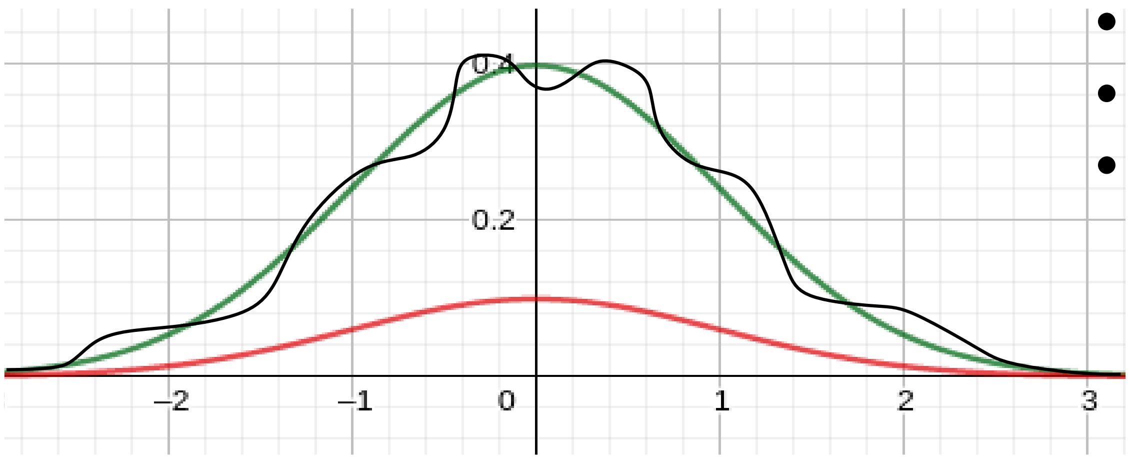

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- Idea from [Lueker, 1998]

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- **Neural network** applications: investigate further structured pruning

The end