Search via Parallel Lévy Walks on $\mathbb{Z}^2$

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Joint work with Andrea Clementi, George Giakkoupis,
and Emanuele Natale

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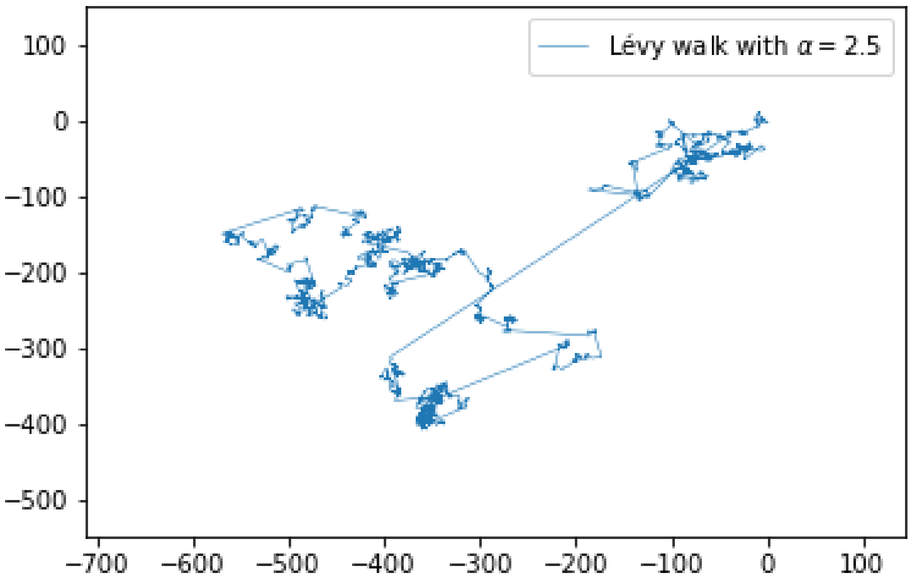
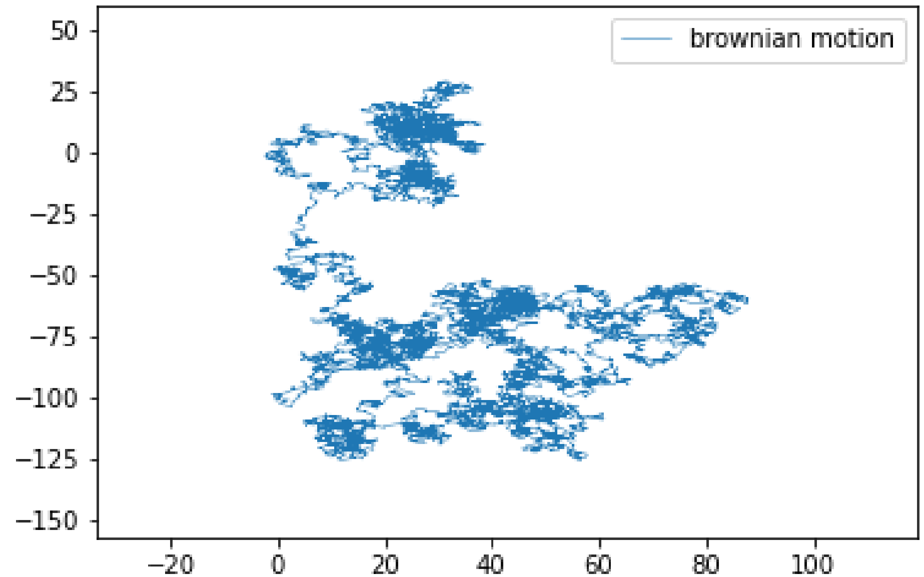
What are Lévy walks?





Lévy walk (informal):
A Lévy walk is a random walk whose step-length density distribution is proportional to a power-law, namely, for each $d \in \mathbb{R}$, $f(d) \sim 1/d^{\alpha}$, *for some* $\alpha > 1$

Note: the speed of the walk is constant

Movement models and foraging theory

Lévy walks are used to model **movement patterns** [Reynolds, Biology Open 2018]

Examples:

- T cells within the brain
- swarming bacteria
- midge swarms
- termite broods
- schools of fish
- Australian desert ants
- a variety of molluscs

Rhytidoponera mayri workers. Credit: Associate Professor Heloise Gibb, La Trobe University

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Widely employed in the Foraging theory

Lévy walk optimality

Foraging theory

- distribution of food locations in $\mathbb{R}^n$
- uninformed walker searching for food

[Viswanathan et al., Nature 1999]: Lévy walk with exponent $\alpha = 2$ is optimal in any dimension, with some assumptions

maximum expected food discovery rate

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Other search problems

- a target in the bidimensional thorus $\mathbf{T}$
- uninformed walker searching for it

[Guinard et Korman, Sciences Advances 2021]: (truncated) Lévy walk with exponent $\alpha = 2$ is optimal

as fast as possible

The Lévy flight foraging hypothesis

Formulation of an evolutionary hypothesis

The Lévy flight foraging hypothesis [Viswanathan et al., Physics of Life Reviews 2008]: since Lévy flights/walks optimize random searches, biological organisms must have therefore evolved to exploit Lévy flights/walks

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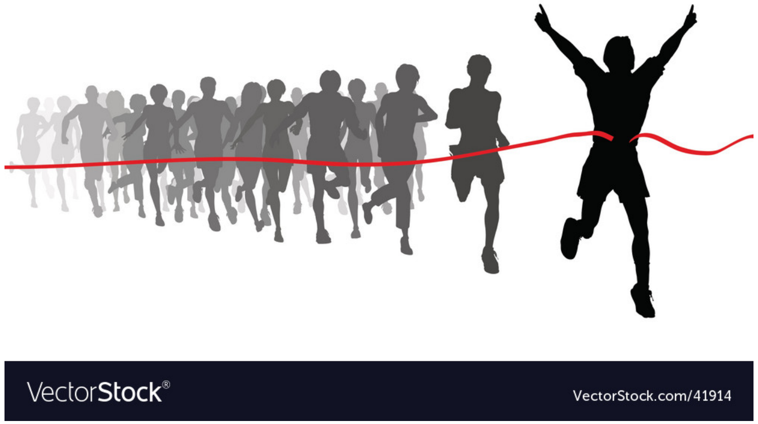
We focus on one search problem:

- the ANTS problem



The ANTS problem

Introduced by [Feinerman et al., PODC 2012]

Setting:

- k (mutually) independent walkers (agents) start moving on $\mathbb{Z}^2$ from the origin
- time is synchronous and marked by a global clock
- one special node $\mathcal{P} \in \mathbb{Z}^2$, the *target*, placed by an adversary at unknown (Manhattan) distance ℓ from the origin

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Task: find the target as fast as possible

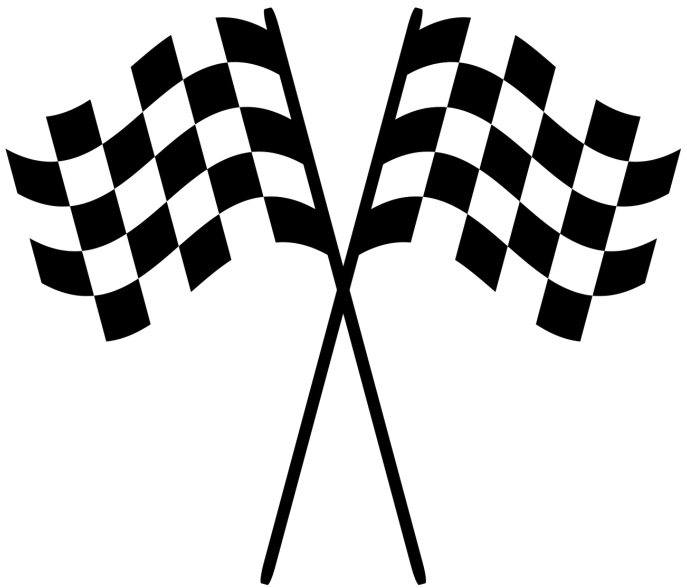

Image by OpenClipart-Vectors from Pixabay

A lower bound on the hitting time

[Feinerman et al., PODC 2012] shows the following:

Lemma: for any $k \geq 1$, and for any search algorithm $\mathcal{A}$, the hitting time to find $\mathcal{P}$ is $\Omega\left(\ell^2/k + \ell\right)$ both with constant probability and in expectation

Proof:

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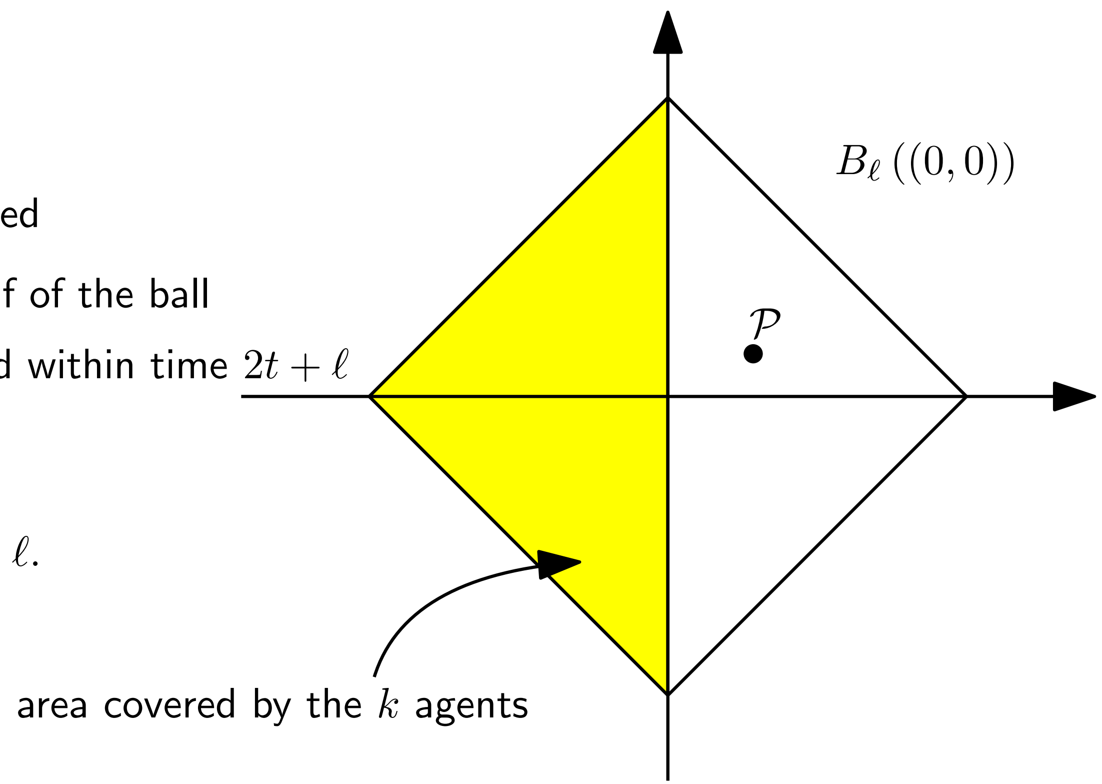
Proof:

- $|B_\ell\left((0,0)\right)| = \ell^2$
- set $t = \ell^2/(4k)$
- within time $2t$, at most $2kt = \ell^2/2$ nodes covered
- the adversary locates the target in the other half of the ball
- probability at least $1/2$ the treasure is not found within time $2t + \ell$
- $H =$ first hitting time for the treasure, then

$$\mathbb{E}\left[\mathsf{H}\right] \geq 2t \cdot \frac{1}{2} + \ell = \ell^2/(4k) + \ell.$$



No advice, no communication

[Feinerman et Korman, DC 2017] proposes many solutions to the problem

Many considered settings, in which

- agents exchange information at the source node
- agents receive some advice on the number of agents k
- there is no communication and no advice

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We focus on the case **no advice, no communication**

Their best algorithm in this case achieves expected hitting time

$$\mathcal{O}\left(\left(\ell^2/k + \ell\right)\log^{1+\epsilon}\ell\right),$$

for any fixed constant $\epsilon > 0$

No advice, no communication

Uniform algorithm proposed in [Feinerman et Korman, DC 2017]

(idea)

i fix a ball of some radius ℓ_i

ii agents go to random nodes in the ball

iii agents perform a spiral search of length d_i around the chosen nodes

iv agents return to the source node

v increase ℓ_i and d_i, and repeat (i)-(v)

However, the above algorithm is not that natural

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[Feinerman et Korman, DC 2017] proposes a more natural algorithm, the Harmonic search algorithm (HSA)

- uses power-law jump length distribution
- worsens performance, but increases probability: for any fixed constants $0 < \delta, \epsilon < 1$, with probability $1 - \epsilon$ the hitting time is

$$\mathcal{O}\left(\ell^{2+\delta}/k + \ell\right)$$

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(ii) to the best of our knowledge, we give the first analysis of the hitting time distribution of k parallel walks

(iii) we show how the Lévy walks can be employed to give a natural, almost-optimal solution to the ANTS problem (no advice, no communication)

Our contributions

(i) DEFINITION OF DISCRETE LÉVY WALK

(ii) ANALYSIS OF THE PARALLEL HITTING TIME

(iii) ALGORITHM FOR THE ANTS PROBLEM

Defining the discrete Lévy walk

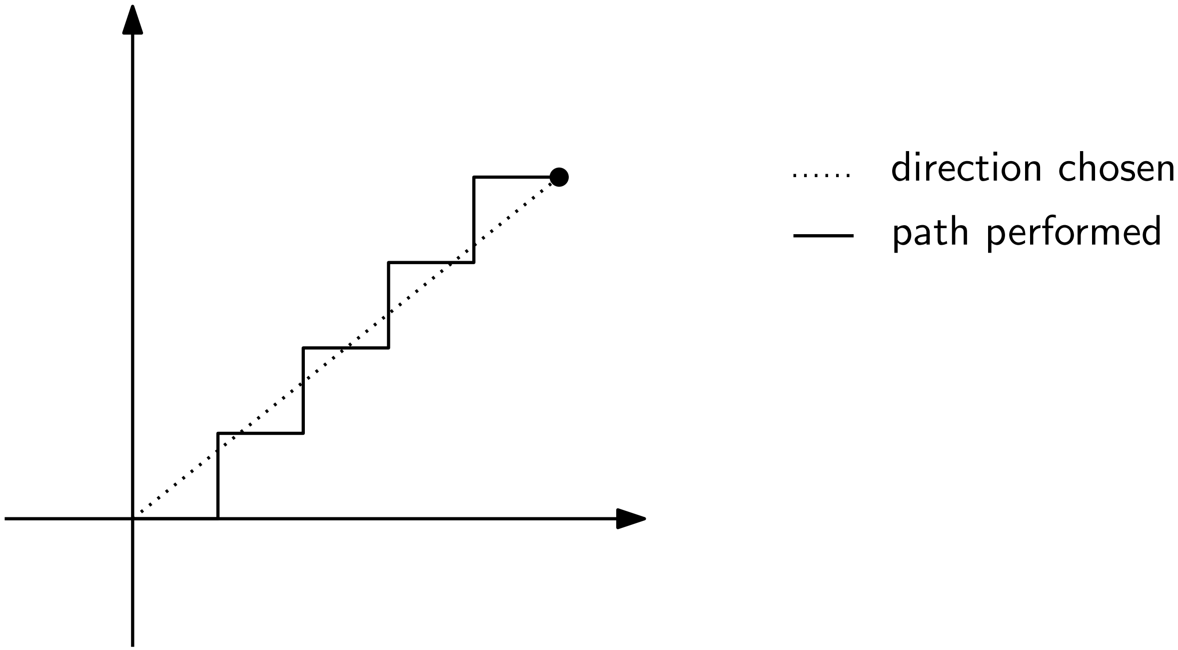
Two choices to make:

- define the jump-length distribution
- define a notion of approximating a line-segment

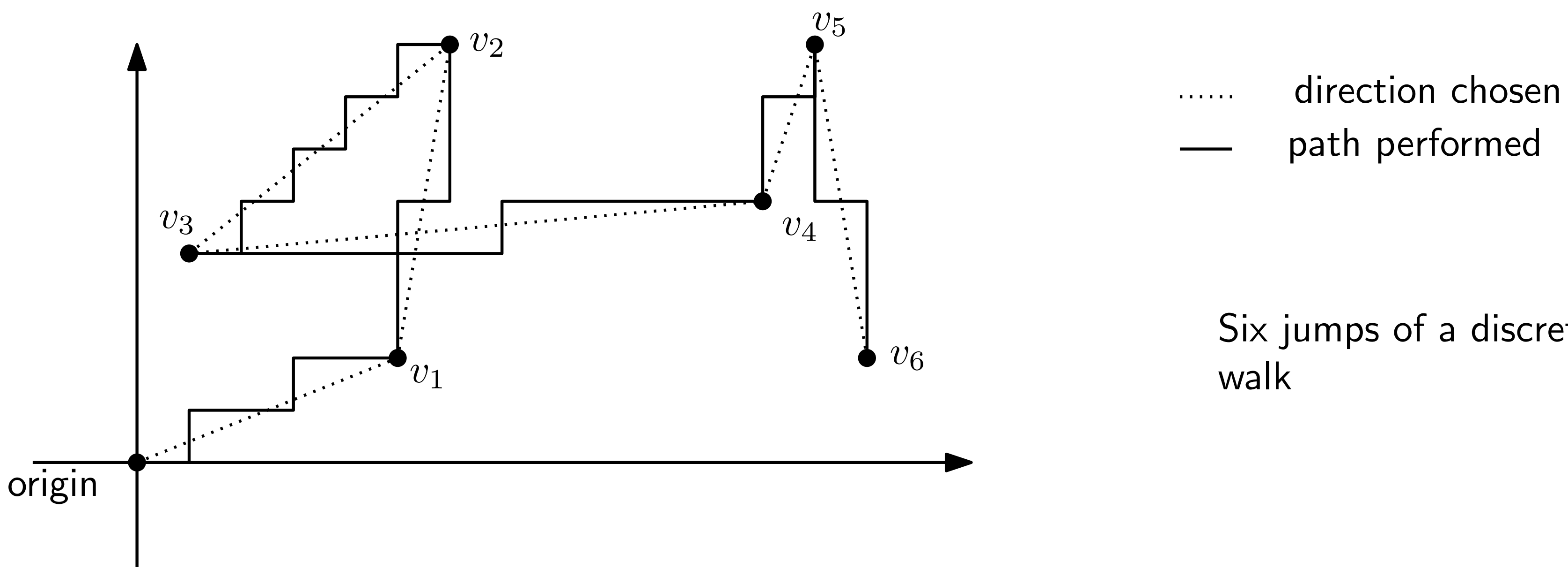
Jump length distribution

- $d = 0$ with probability 1/2
- $d \geq 1$ with probability c_α / d^α

Approximation of a line-segment



Discrete Lévy walk



Six jumps of a discrete Lévy walk

Let $\alpha > 1$ be a real constant

Lévy walk: the agent

a) chooses a distance $d \in \mathbb{N}$ as follows: $d = 0$ w.p. $1/2$, and $d \geq 1$ w.p. c_α / d^α
b) chooses a destination u.a.r. among those at distance d
c) walks along an approximating path for d steps, one edge at a time, crossing d nodes
d) repeats the procedure

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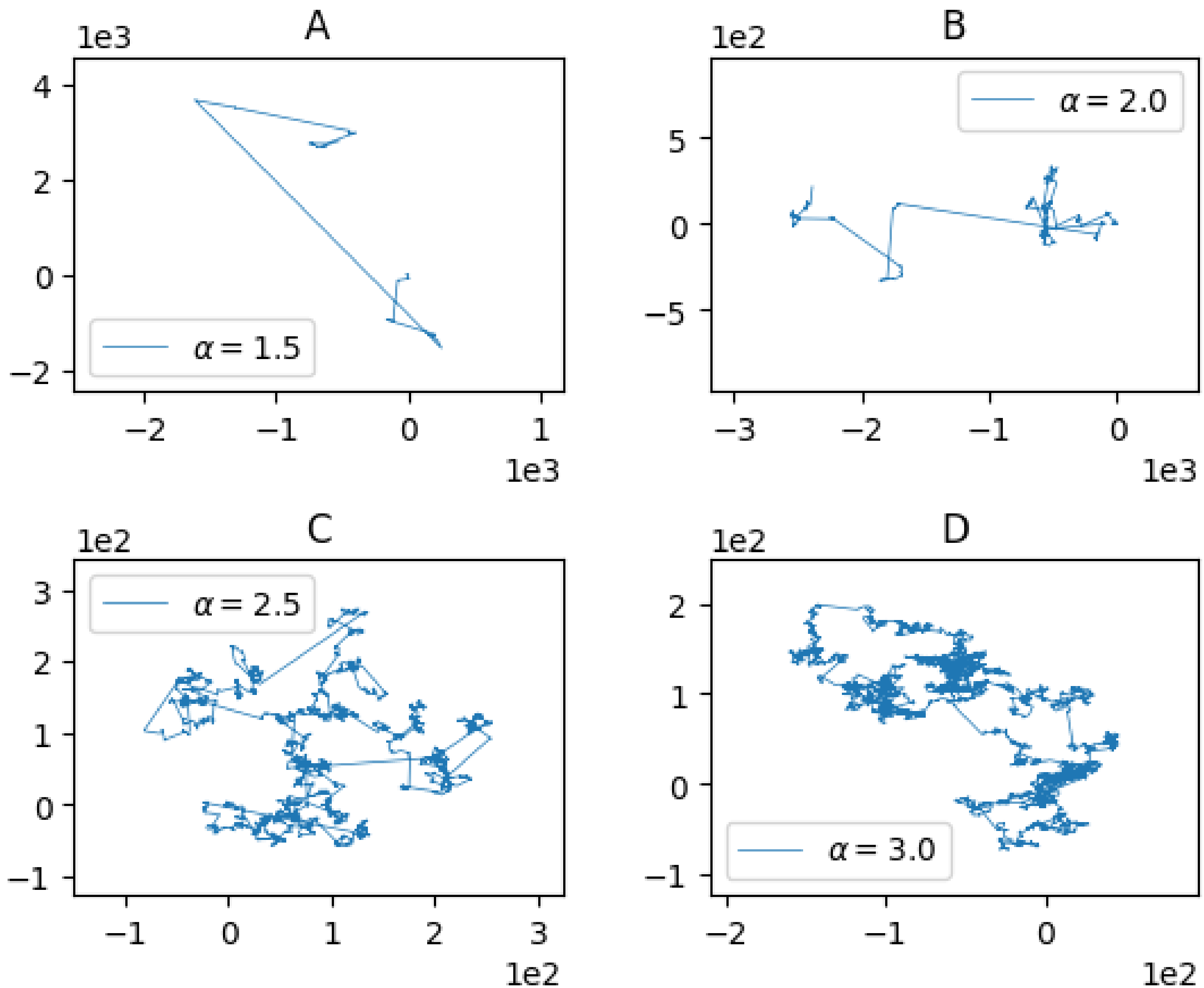
Known facts about the continuous Lévy walk

- $1 < \alpha \leq 2$ ballistic diffusion (fig.s A and B)
- $2 < \alpha < 3$ super diffusion (fig. C)
- $3 \leq \alpha$ normal diffusion (fig. D)



Other known facts

Expected jump-length

- $1 < \alpha \leq 2$: $\int_1^{\infty} x^{-\alpha+1} \mathsf{d}x = \infty$
- $2 < \alpha$: $\int_1^{\infty} x^{-\alpha+1} \mathsf{d}x = \Theta(1)$

Jump-length second moment

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The secret lies in the range $2 < \alpha < 3$...

Three ranges for k and ℓ

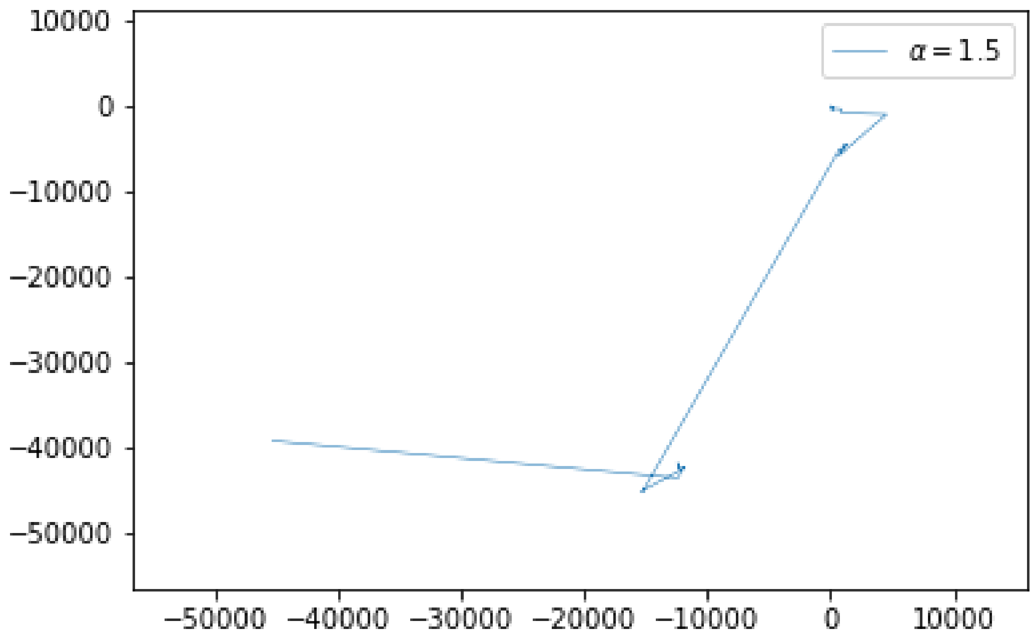
Recall: ℓ target distance, k number of agents

Three different possible settings:

1. close target: $\ell \leq k/\text{polylog}\,(k)$
2. far target: $k/\text{polylog}\,(k) \leq \ell \leq \exp\left(k^{\Theta(1)}\right)$
3. very far target: $\exp\left(k^{\Theta(1)}\right) \leq \ell$

Close target: $\ell \leq k/\text{polylog}\,(k)$

Best strategy $=$ ballistic walks: any α in $(1, 2]$



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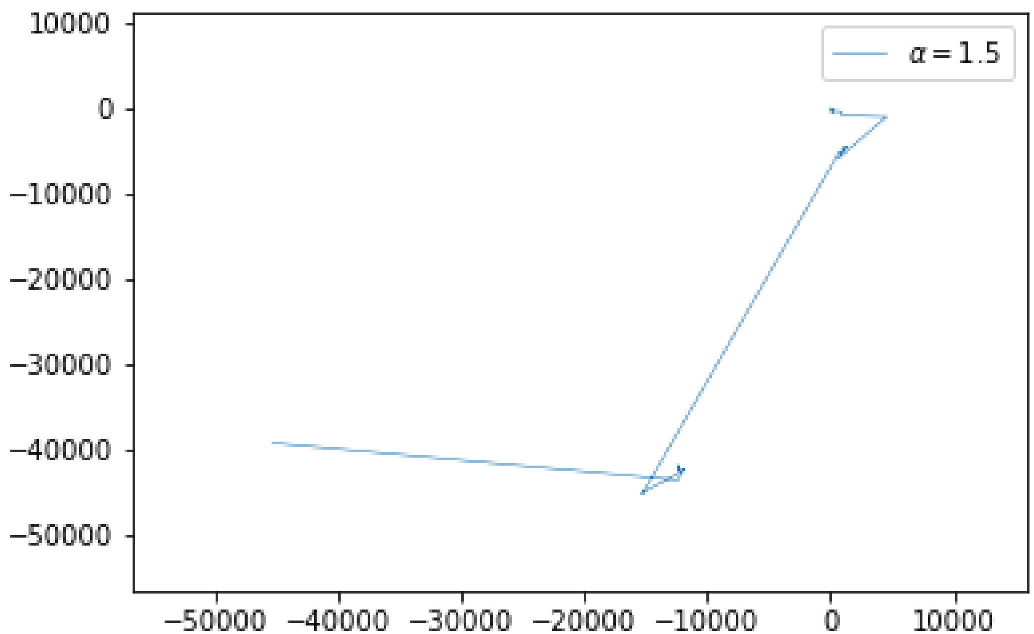
$$\mathcal{O}\left(\ell \text{polylog}\,(\ell)\right)$$



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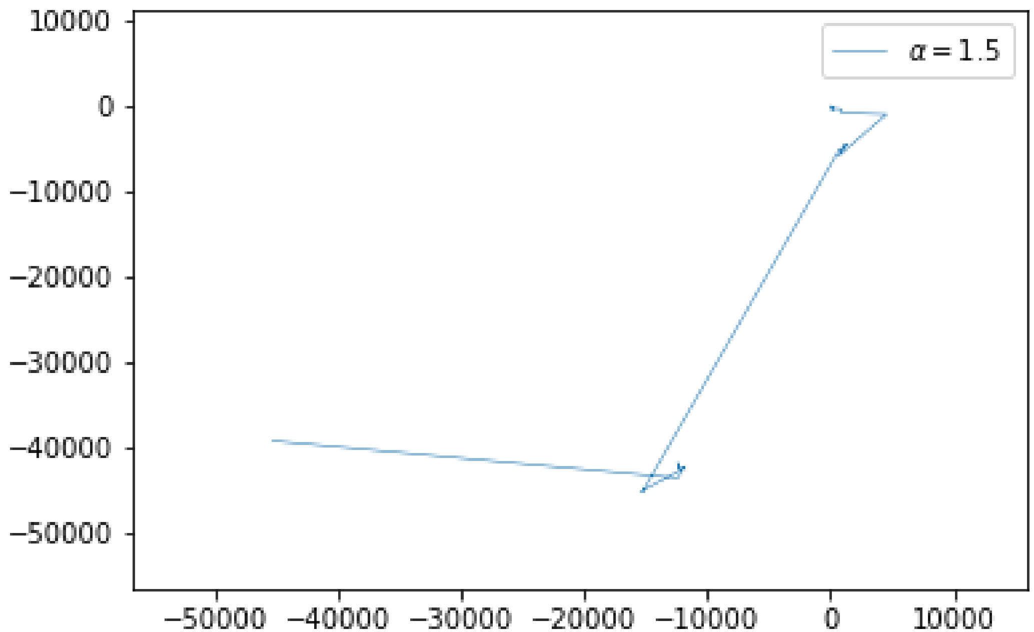
Recall: an event E depending on a parameter ℓ holds with high probability in ℓ if $\mathbb{P}\left(E\right) \geq 1 - \ell^{-\Theta(1)}$



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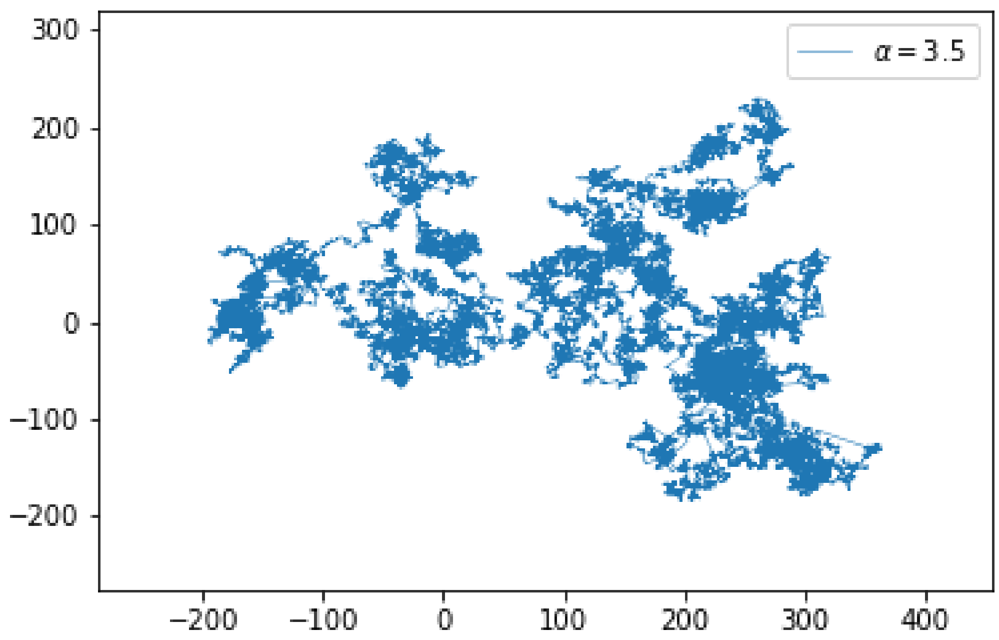
With probability 1, the walks will eventually find the target



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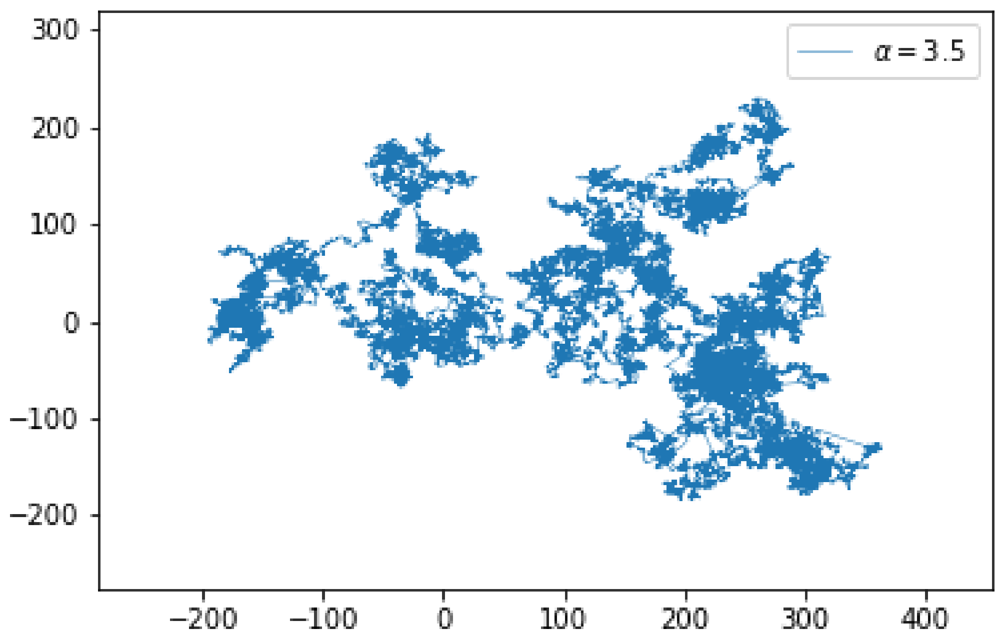

If $\alpha = 3 - \epsilon$, with high probability the target is not found

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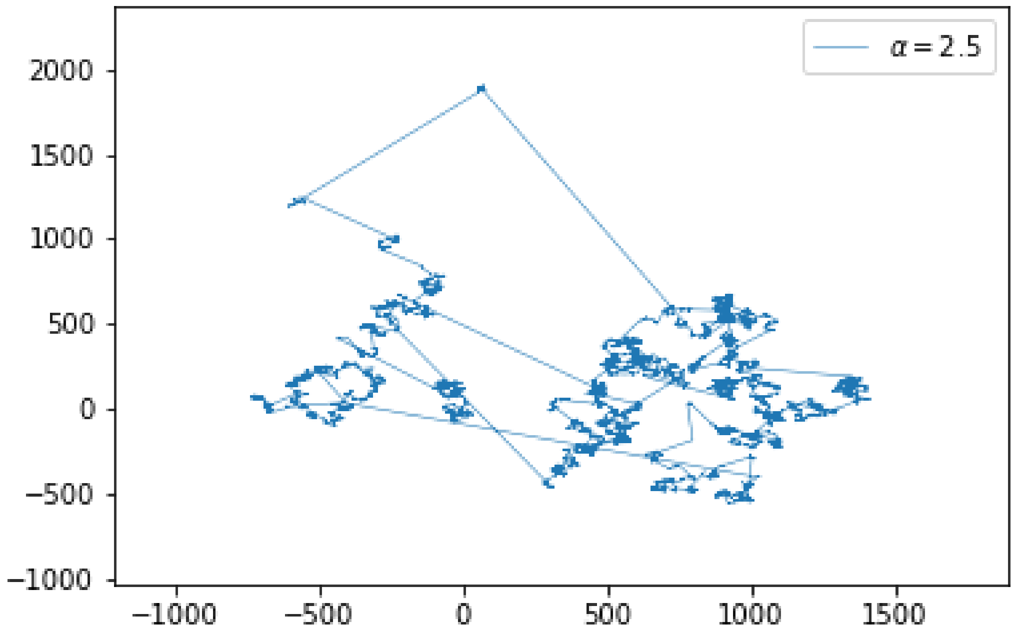
Fix $\alpha^\star = 3 - \log k / \log \ell$: super-diffusive range



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The followings hold w.h.p. in ℓ

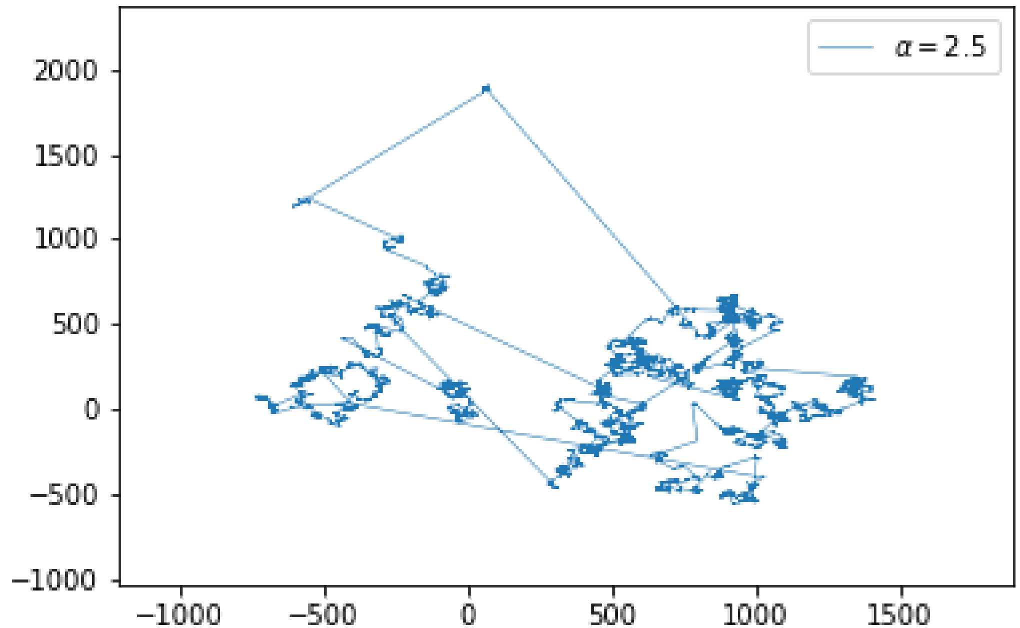
- if $\alpha = \alpha^\star + \mathcal{O}(\log\log \ell / \log \ell)$, the hitting time is

$$\mathcal{O}\left(\left(\ell^2/k + \ell\right) \text{polylog}(\ell)\right)$$

- if $\alpha = \alpha^\star + \epsilon$, the hitting time is

$$\Omega\left(\left(\ell^2/k + \ell\right) \ell^c\right),$$

  for some constant $c > 0$
- if $\alpha = \alpha^\star - \epsilon$ the hitting time is *infinite*



But… No advice, no communication!

How can we find $\alpha^\star$?

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Algorithm: *each agent* u *samples u.a.r. a real number* $\alpha_u \in (2,3)$. *Then, it performs a discrete Lévy walk with exponent* α_u

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If $\ell \leq \exp\left(k^{\Theta(1)}\right)$, the hitting time is $\mathcal{O}\left(\left(\ell^2/k + \ell\right) \text{polylog}\left(\ell\right)\right)$ w.h.p.

The idea behind the algorithm

Fix some $\epsilon = \mathcal{O}(\log\log\ell / \log\ell)$

We use: $\ell < \exp\left(k^{\Theta(1)}\right)$ ($\iff k \geq$ polylog (ℓ)) + Chernoff bound

$\implies$ at least $\Theta(\epsilon k)$ agents choose an exponent in the range $(\alpha^\star - \epsilon, \alpha^\star + \epsilon)$ w.h.p.

$\Theta(\epsilon k)$ agents are sufficient to ensure high probability to find the target fast enough

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  - improves the HSA (just polylog factor worse than optimum, not polynomial)
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  - improves the HSA (just polylog factor worse than optimum, not polynomial)
  - does not improve their optimal solution
- argue the non (universal) optimality of exponent $\alpha = 2$

Questions?

THANK YOU FOR YOUR ATTENTION