Causal limits of distributed computation

Francesco d'Amore

Based on joint work with the Xavier Coiteux-Roy, Rishikesh Gajjala, Fabian Kuhn, François Le Gall, Henrik Lievonen, Augusto Modanese, Marc-Olivier Renou, Gustav Schmid, Jukka Suomela

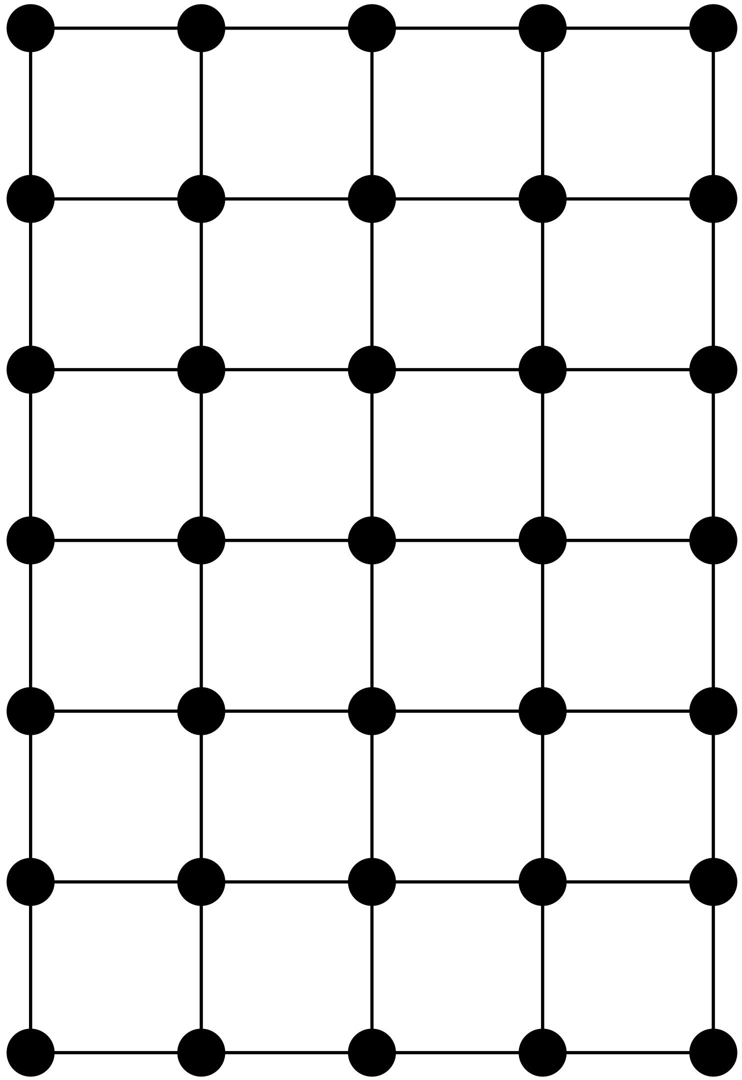
Bocconi University

05 December 2023

The LOCAL model

Introduced by [Linial, FOCS '87]

- **distributed systems** with n nodes/processors
- synchronous rounds
- infinite computational power
- unbounded bandwidth



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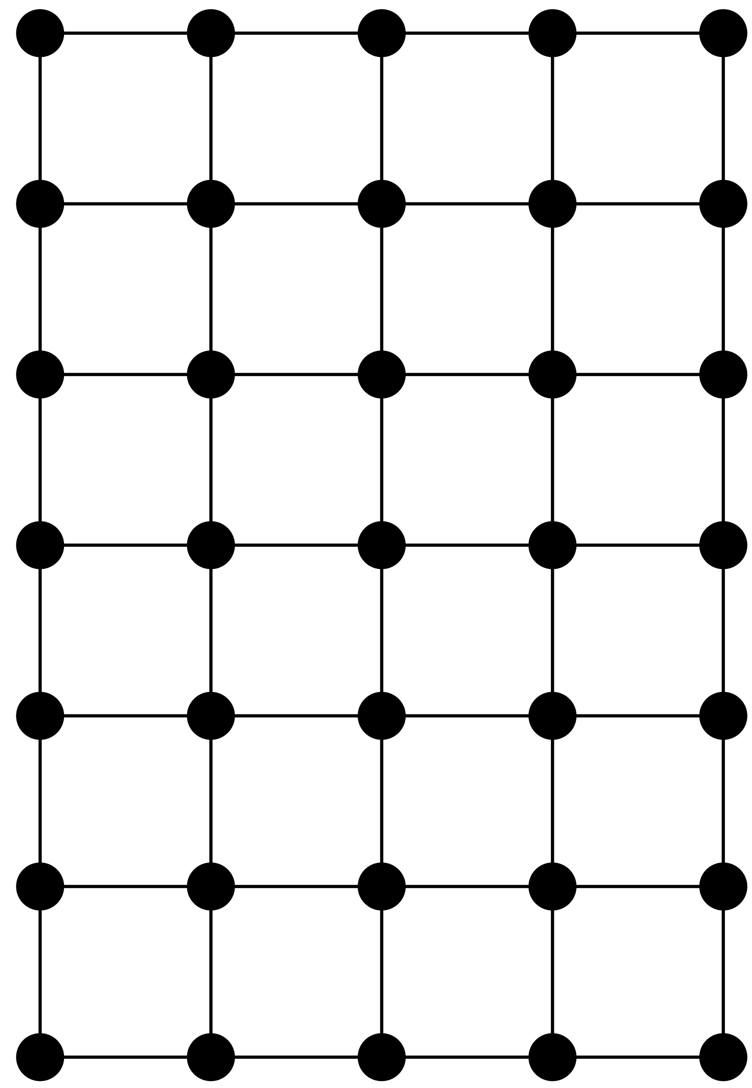

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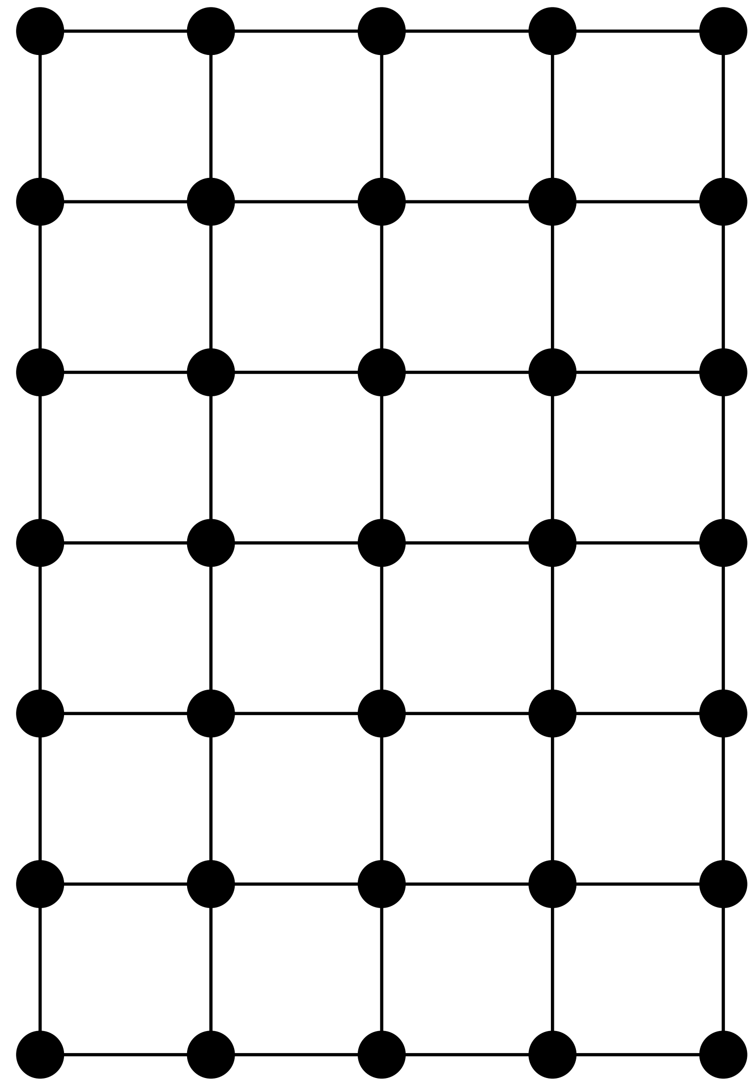

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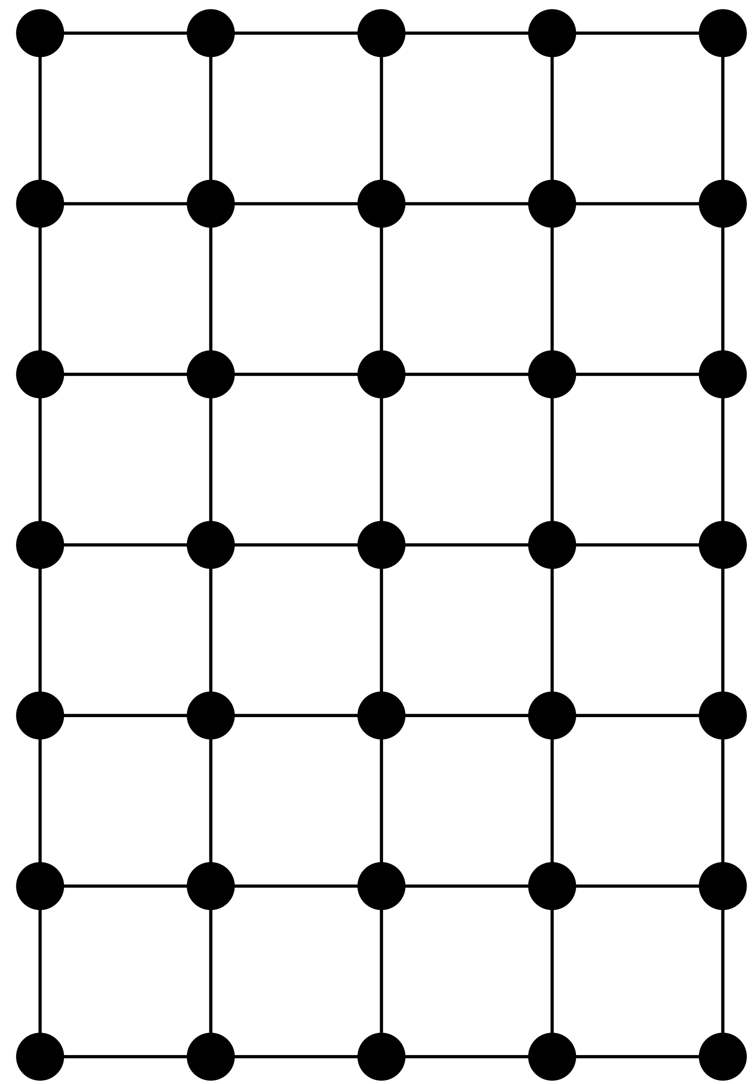

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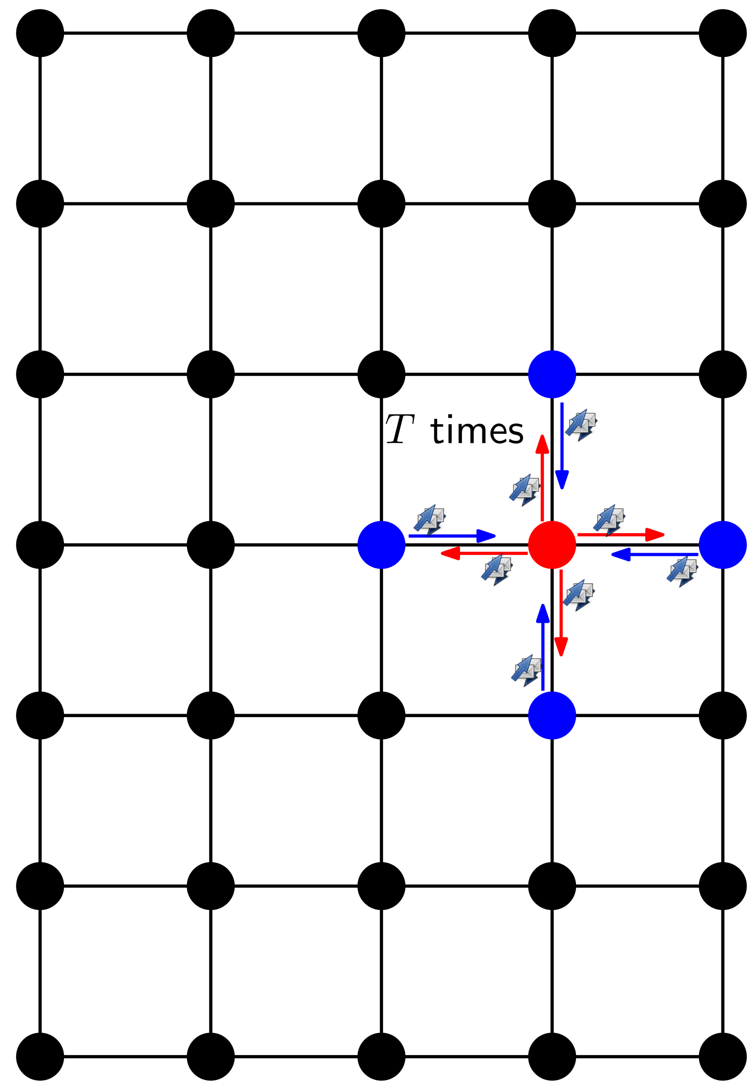

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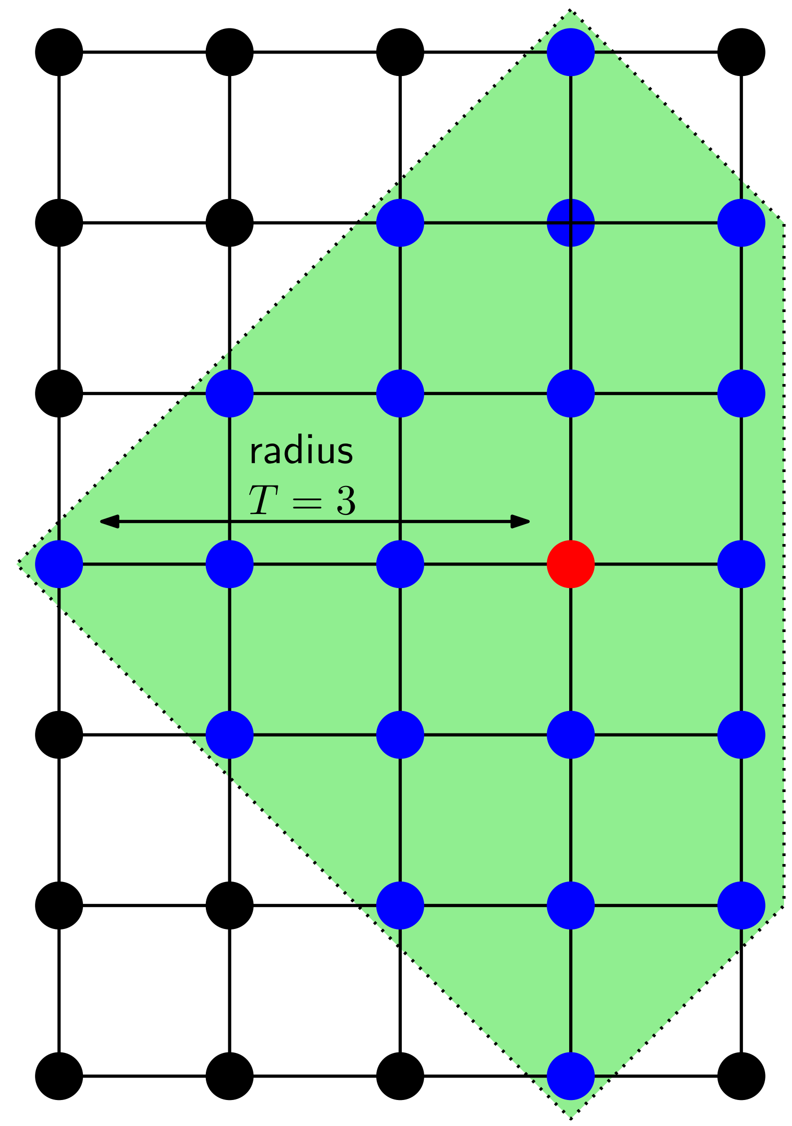

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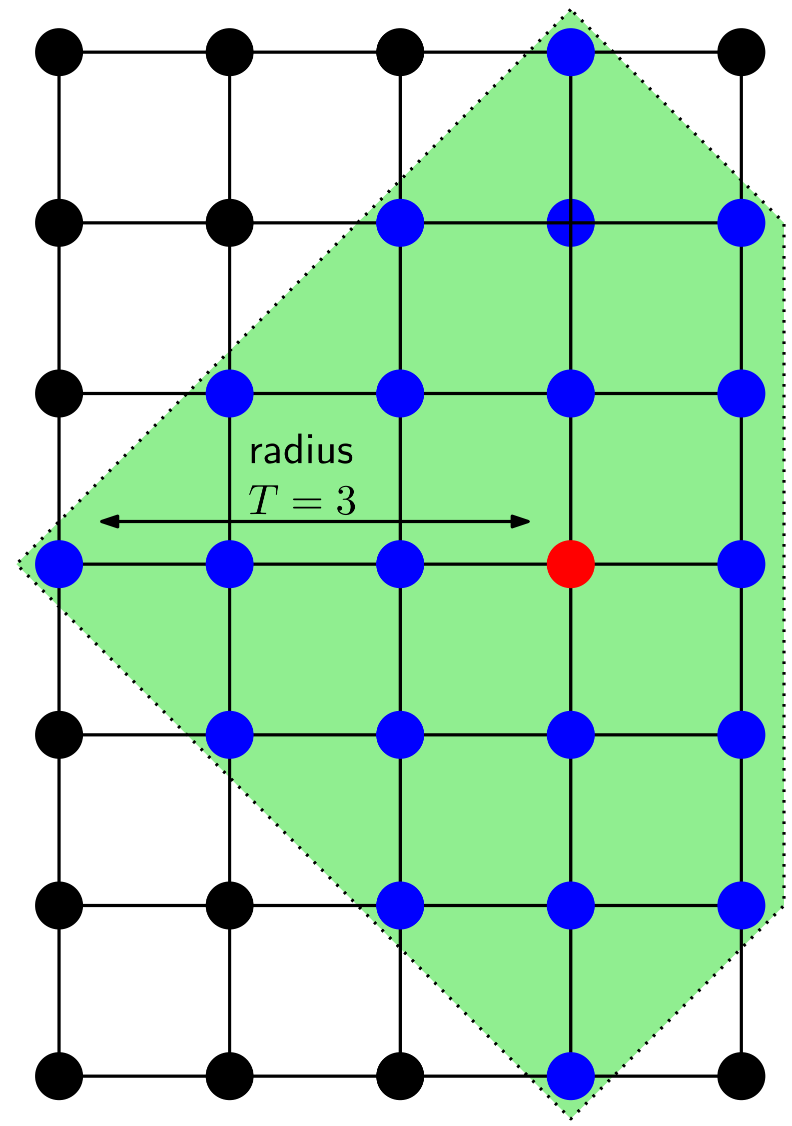
- **randomized**-LOCAL
  - infinite i.i.d. random bit strings
  - error probability $\leq 1/n$



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- [Balliu et al., FOCS '19]
  - MM and MIS cannot be solved in $o(\Delta) + \mathcal{O}(\log^\star n)$

Locally checkable labelling (LCL) problems

[Naor and Stockmeyer, STOC '93]

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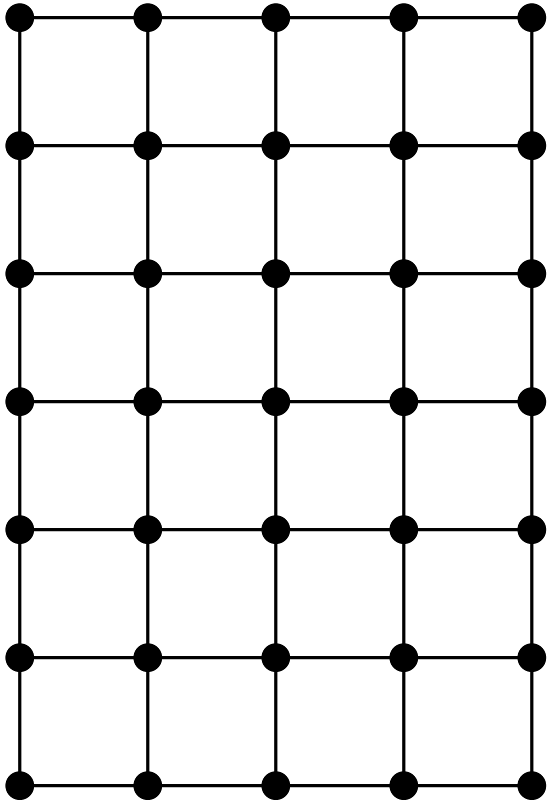

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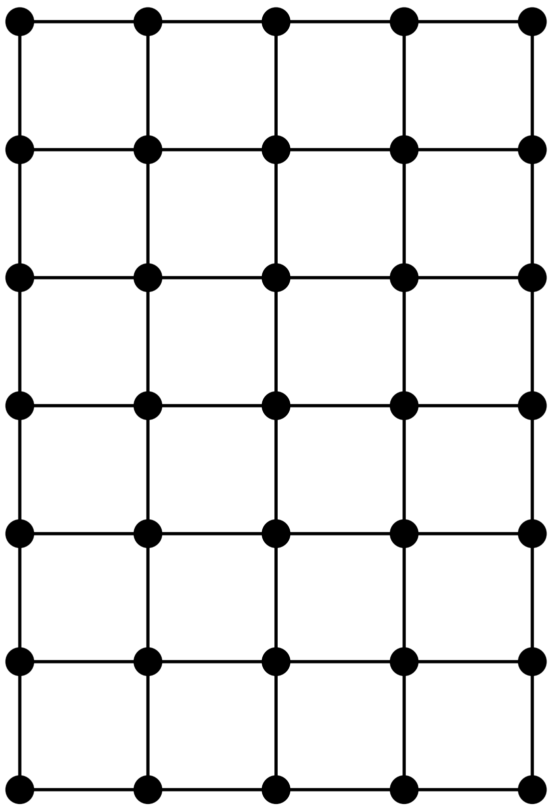

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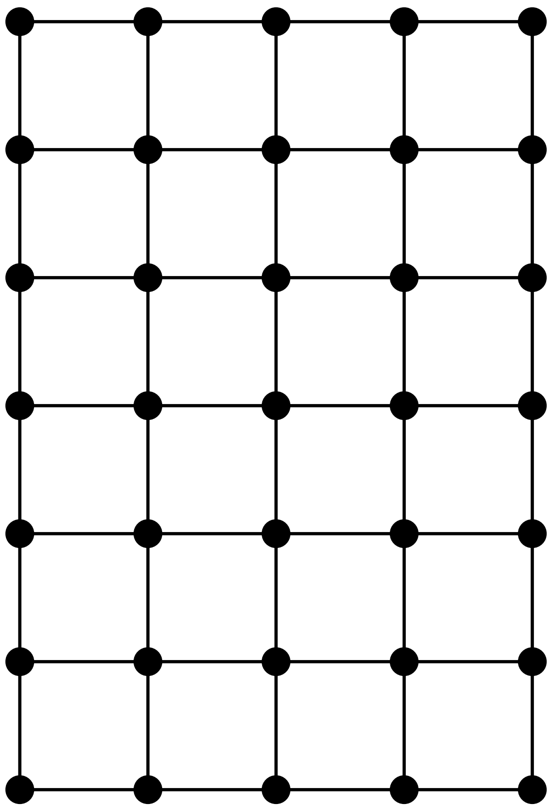

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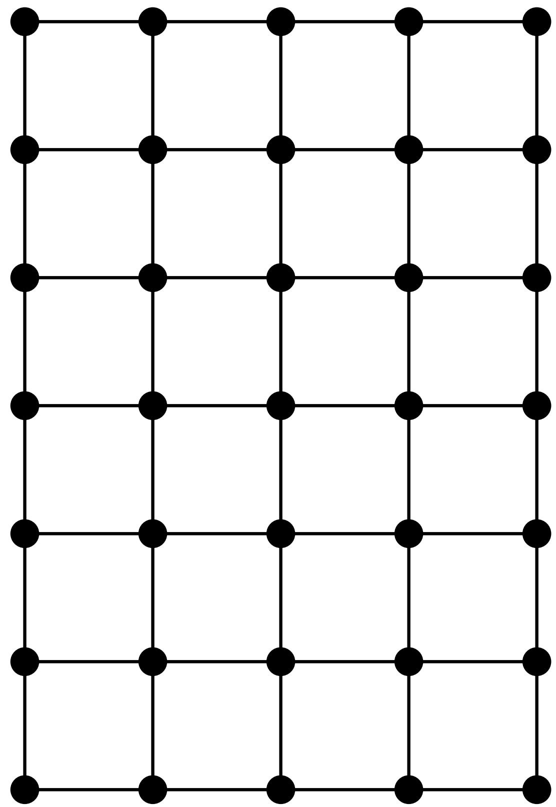

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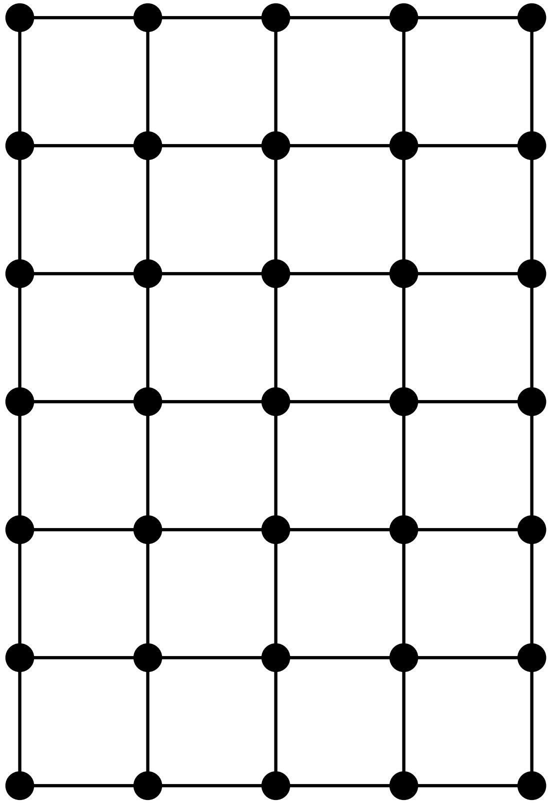

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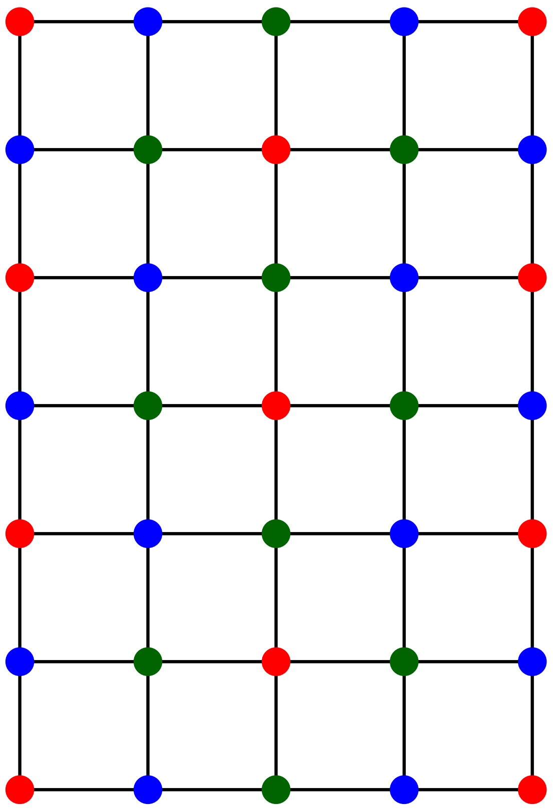

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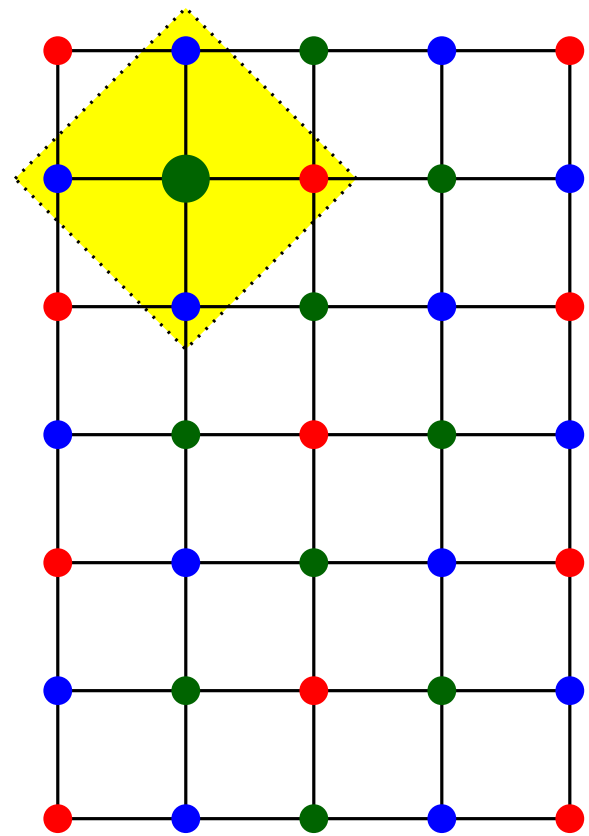

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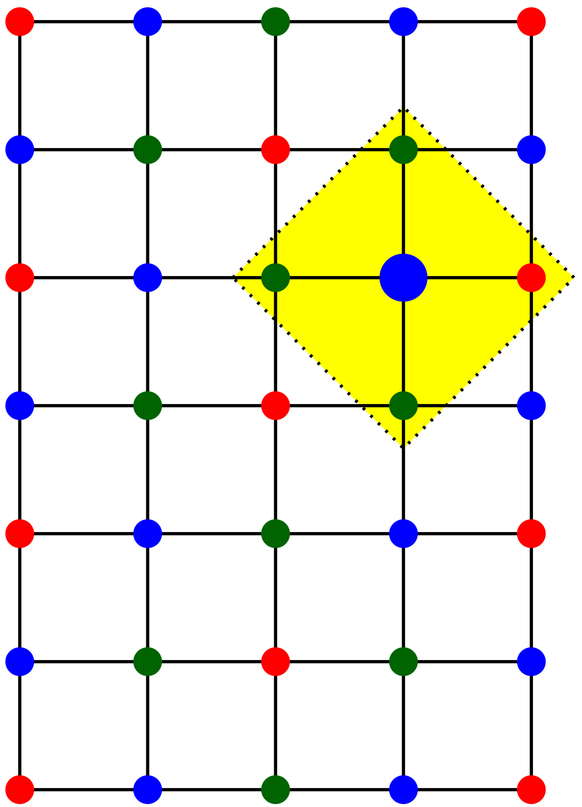

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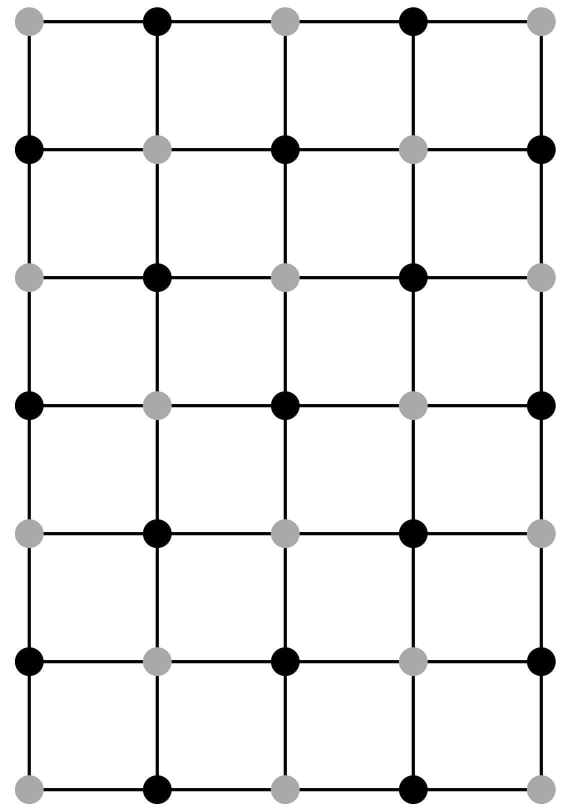

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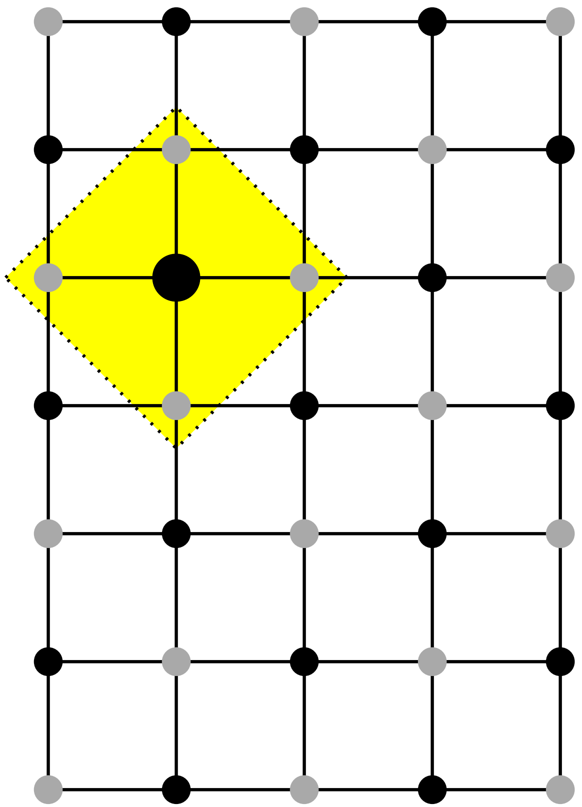

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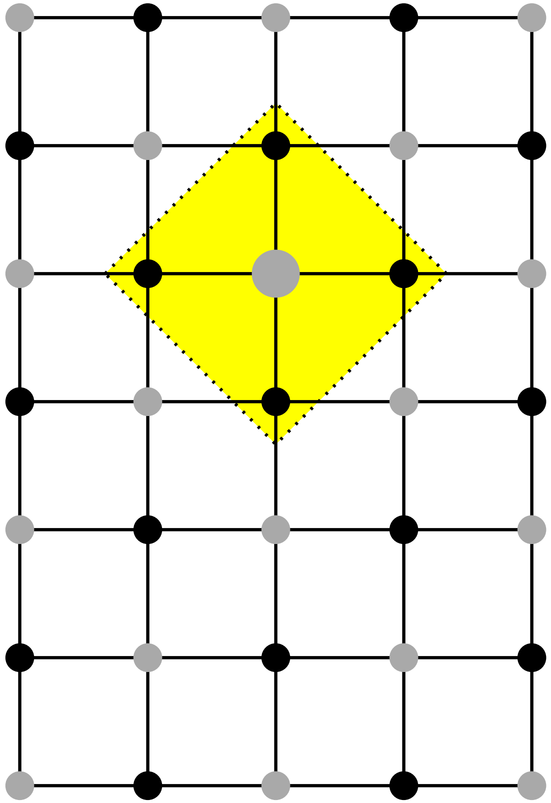

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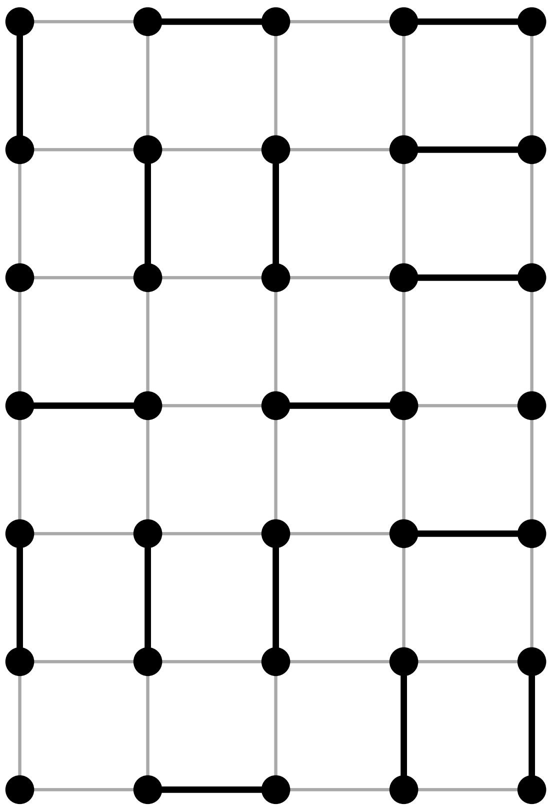

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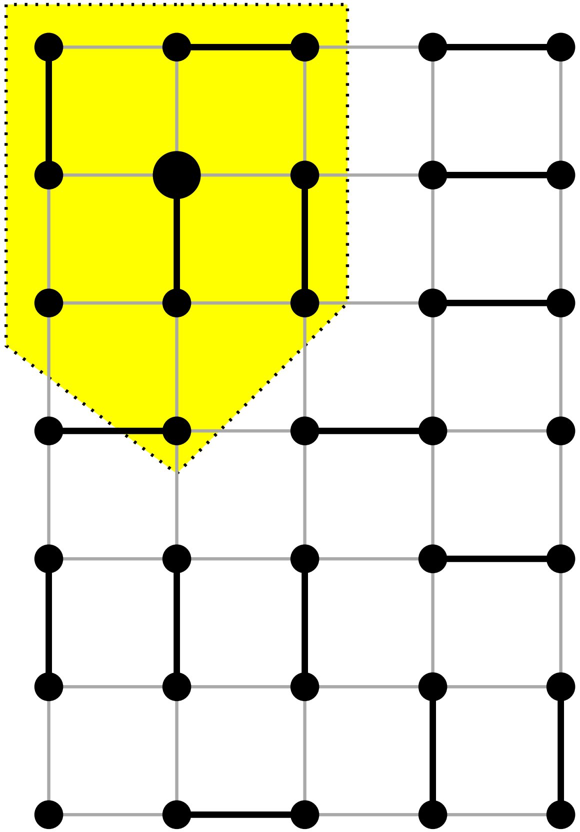

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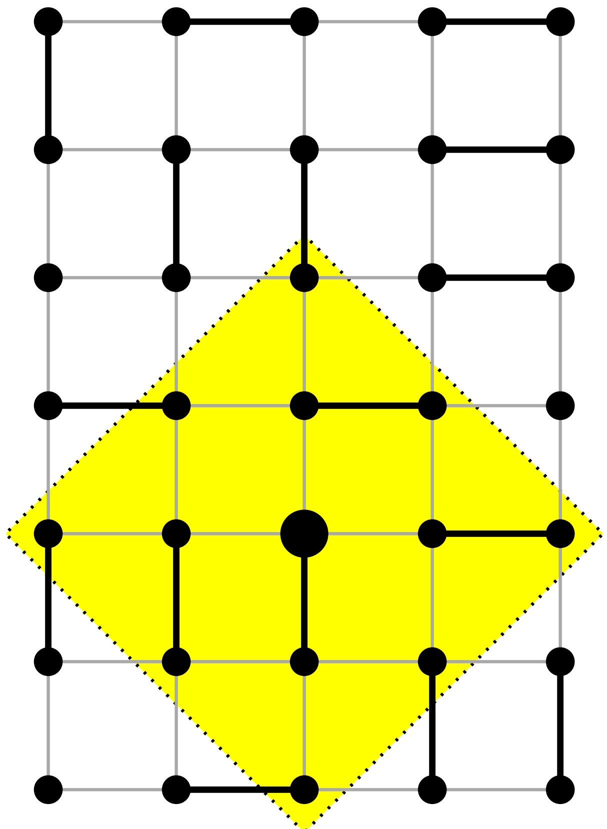

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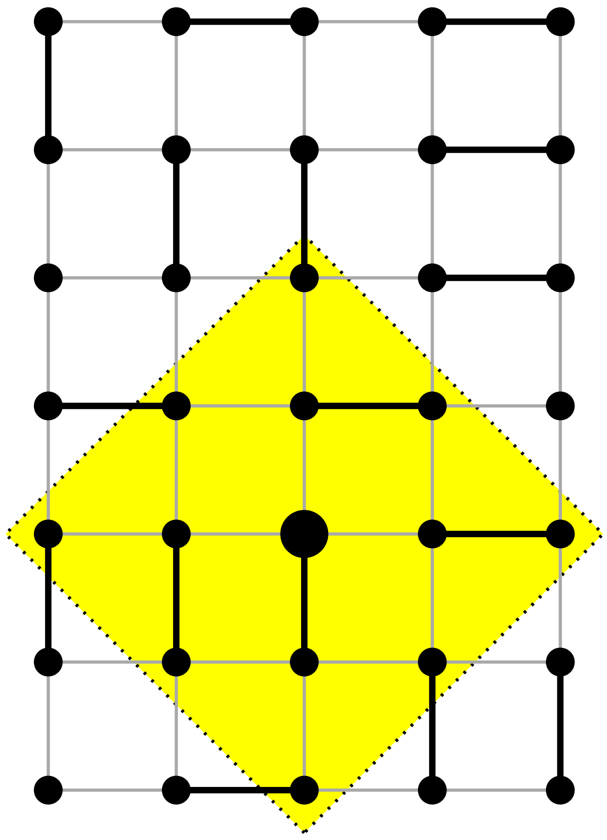

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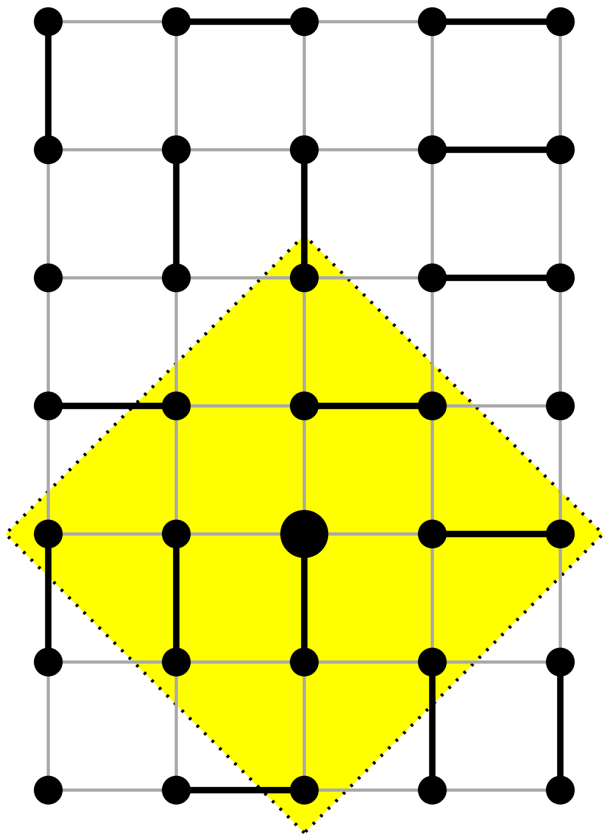
- many others...



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  - in general, randomness helps both exponentially and polynomially

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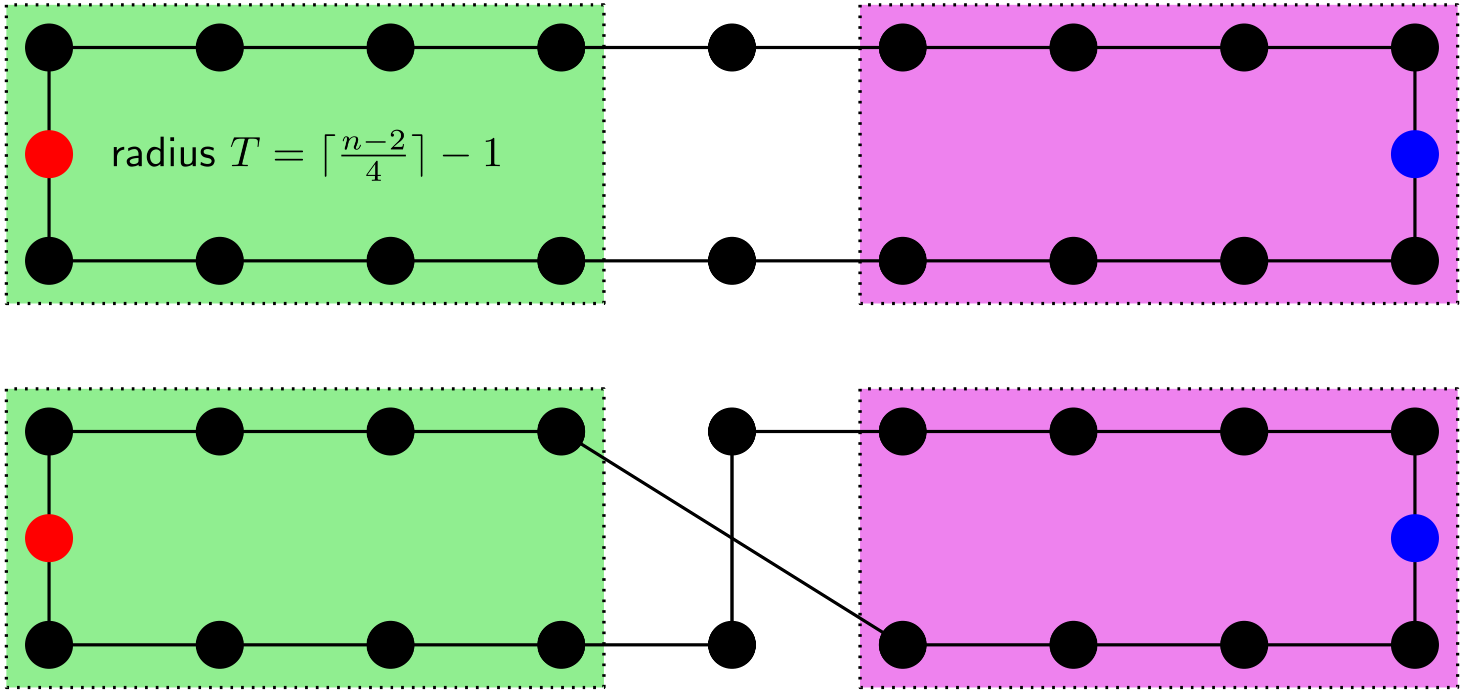

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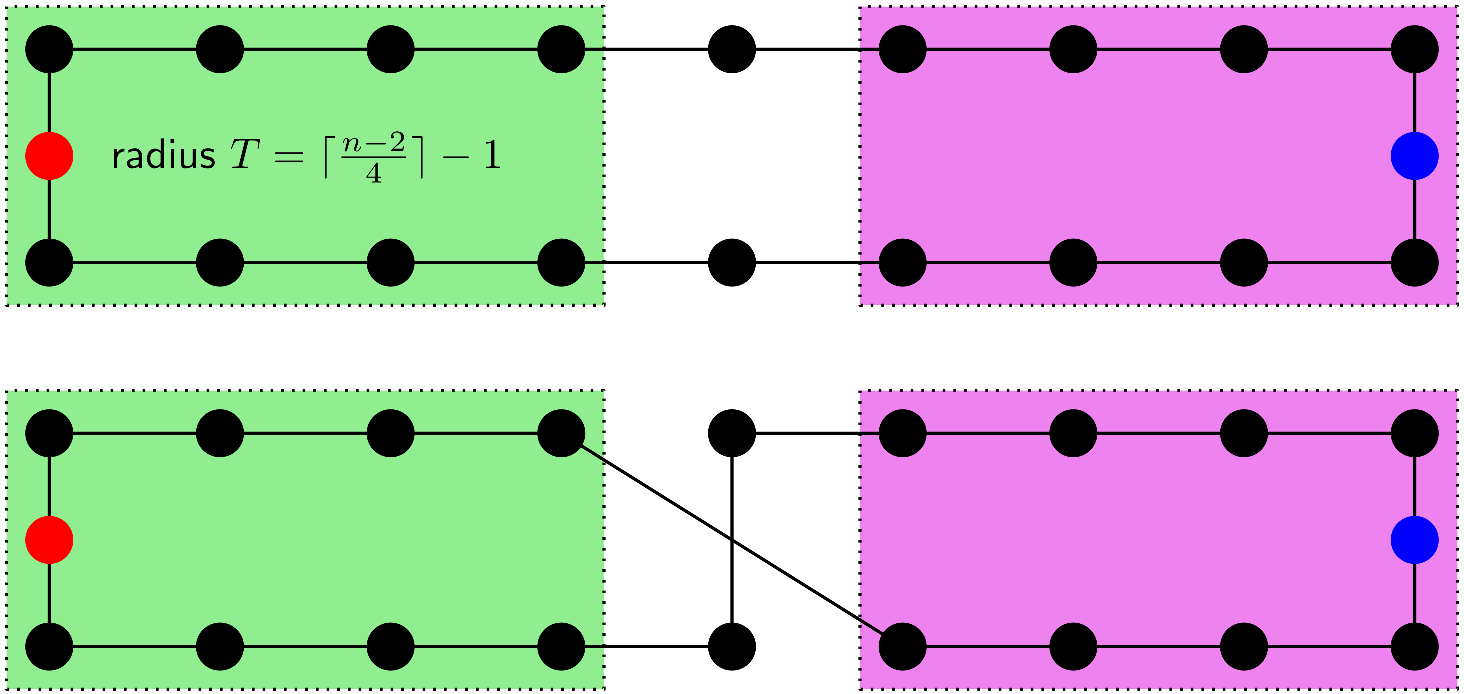

- red nodes must output the same
- blue nodes must output the same
- more complex arguments show $T \geq \lfloor \frac{n}{2} \rfloor - 1$

Quantum advantage

- Natural questions

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- some indistinguishability arguments works! [Gavoille et al., DISC '09]

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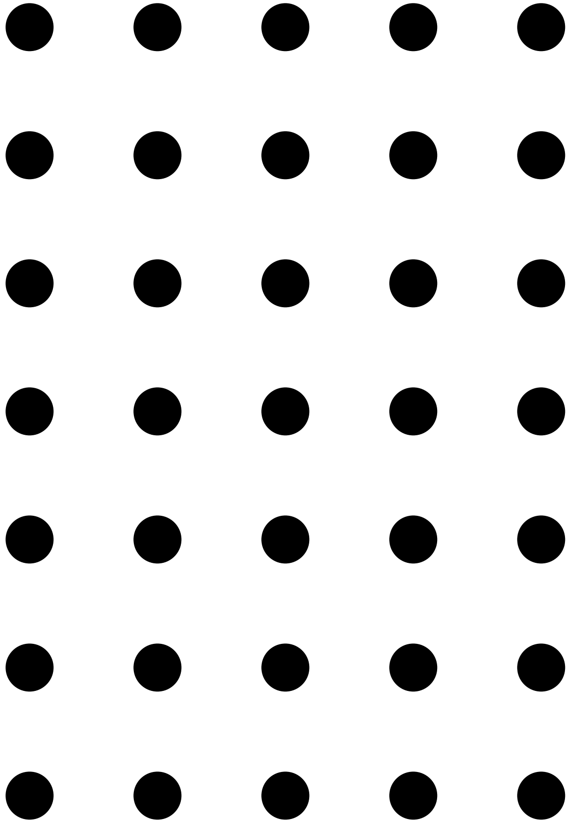

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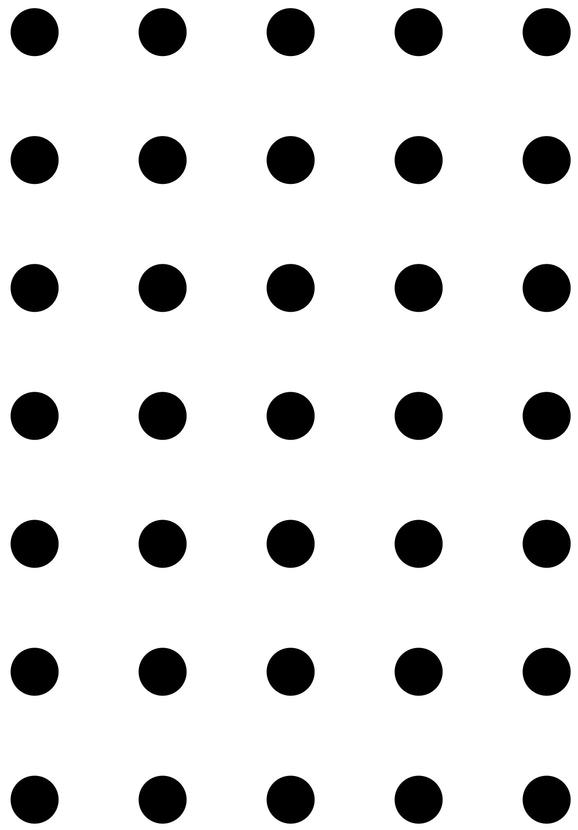

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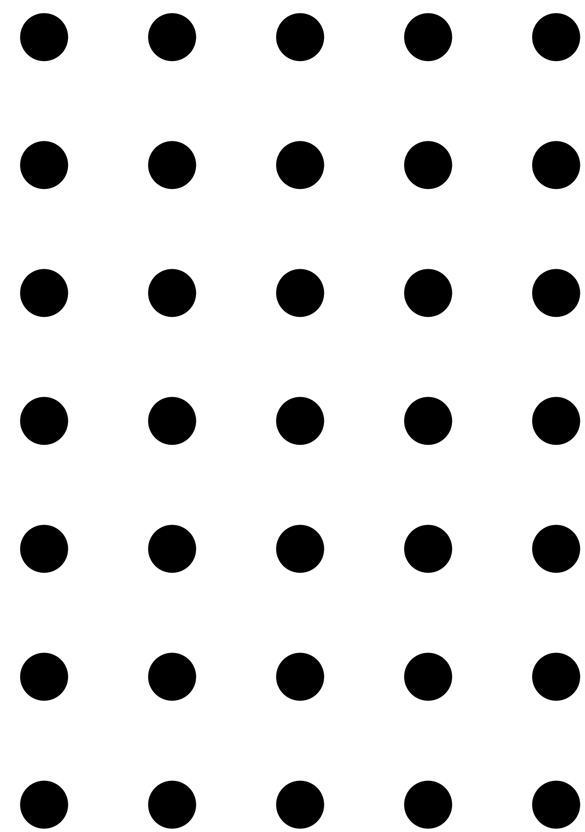

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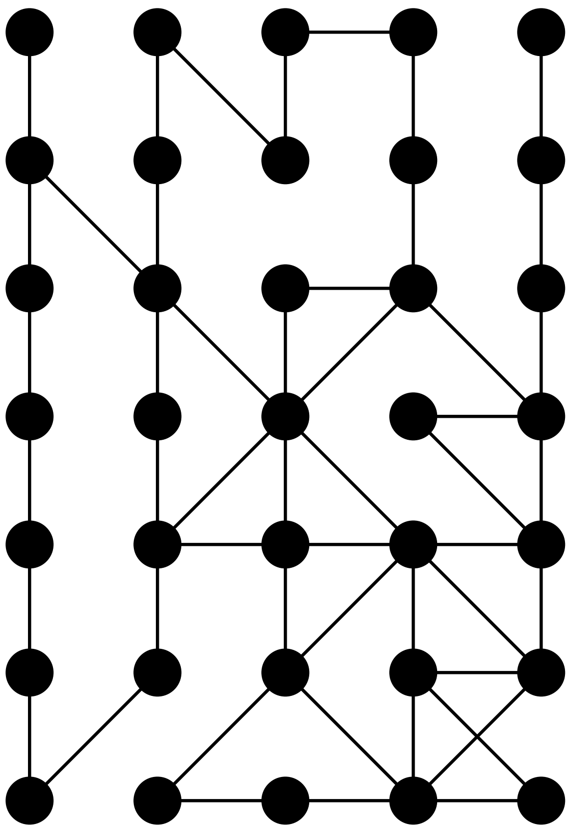

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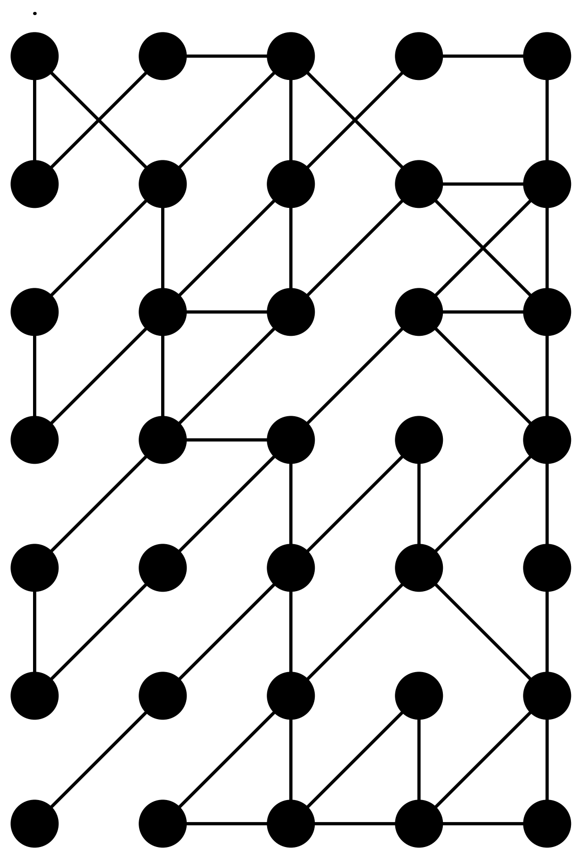

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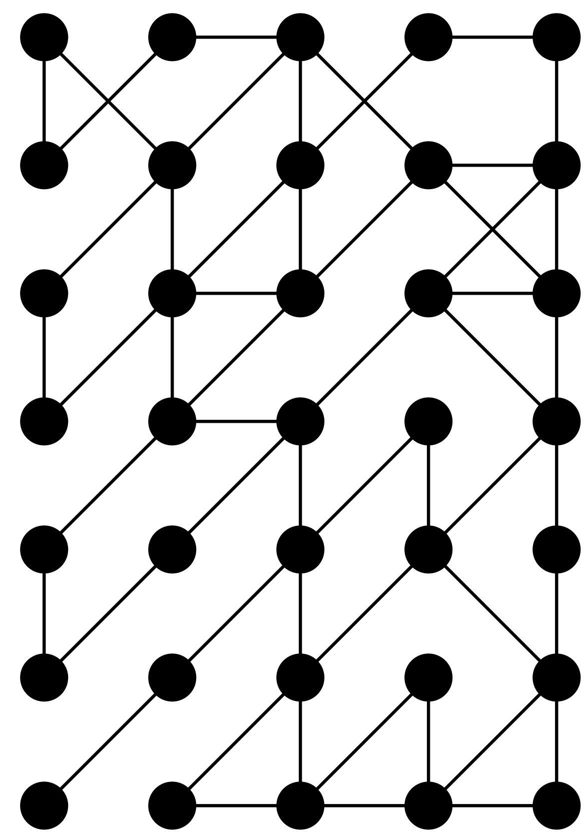

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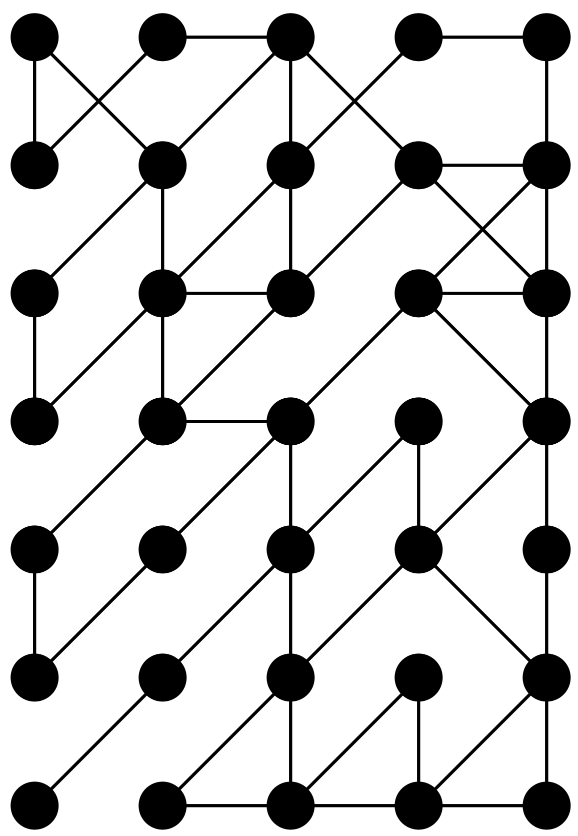
- Wlog, we assume algorithms are defined over **all graphs** of n nodes



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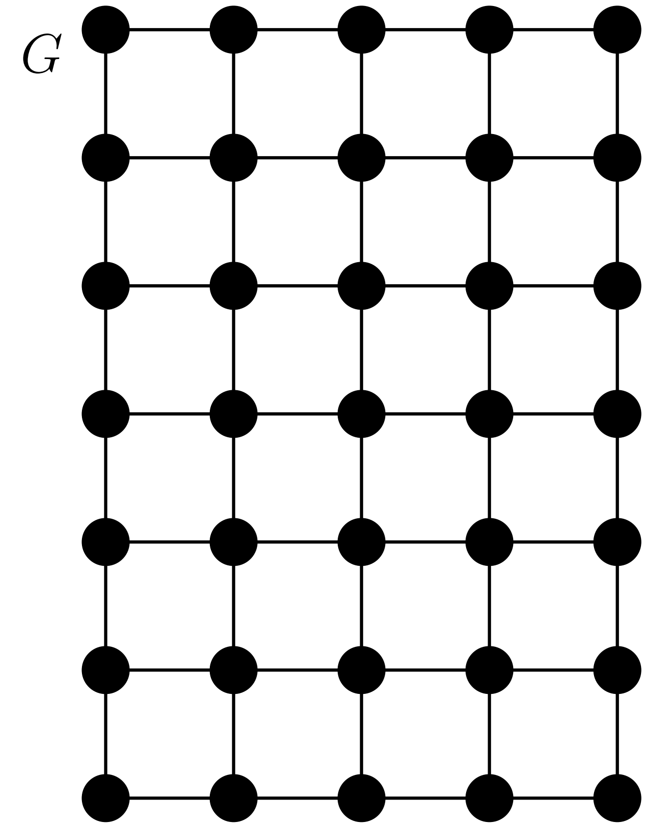

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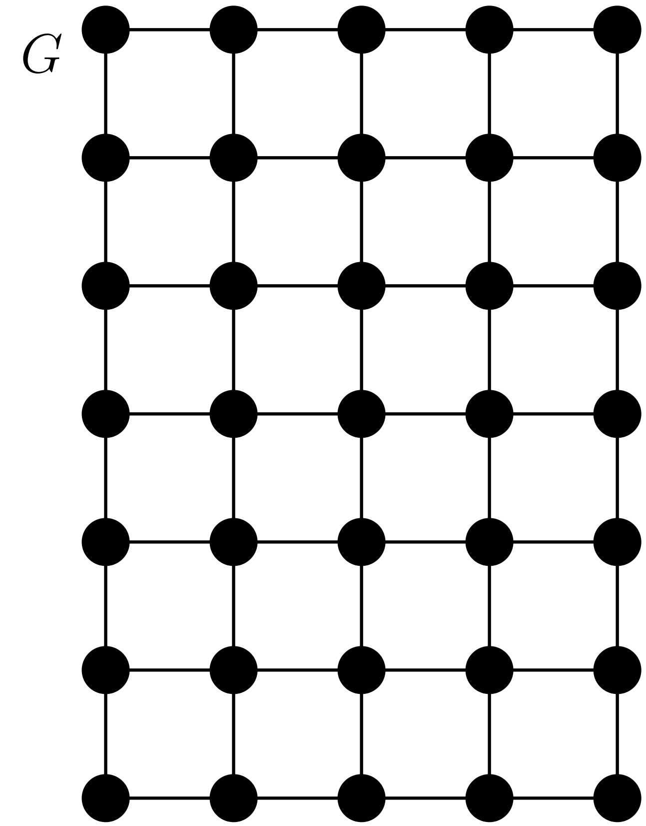

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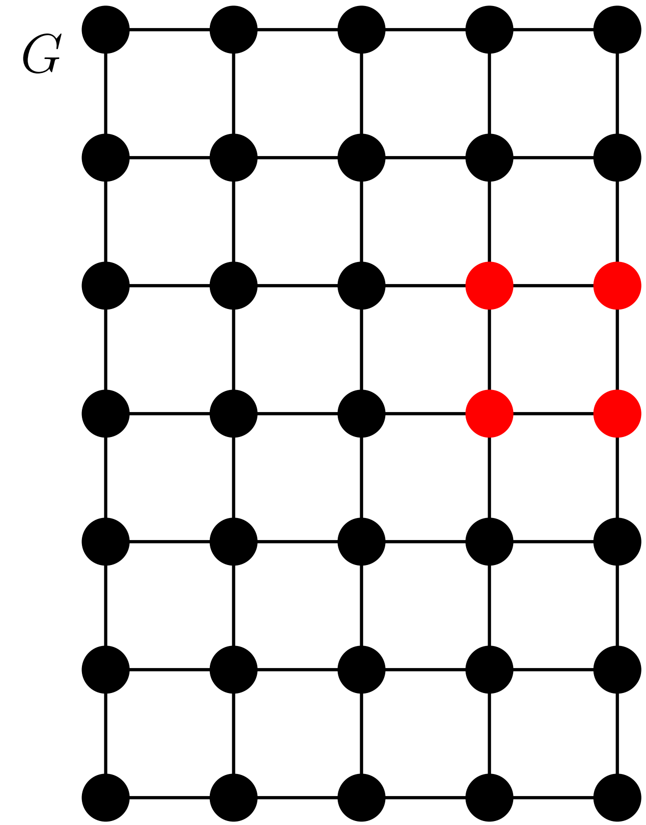

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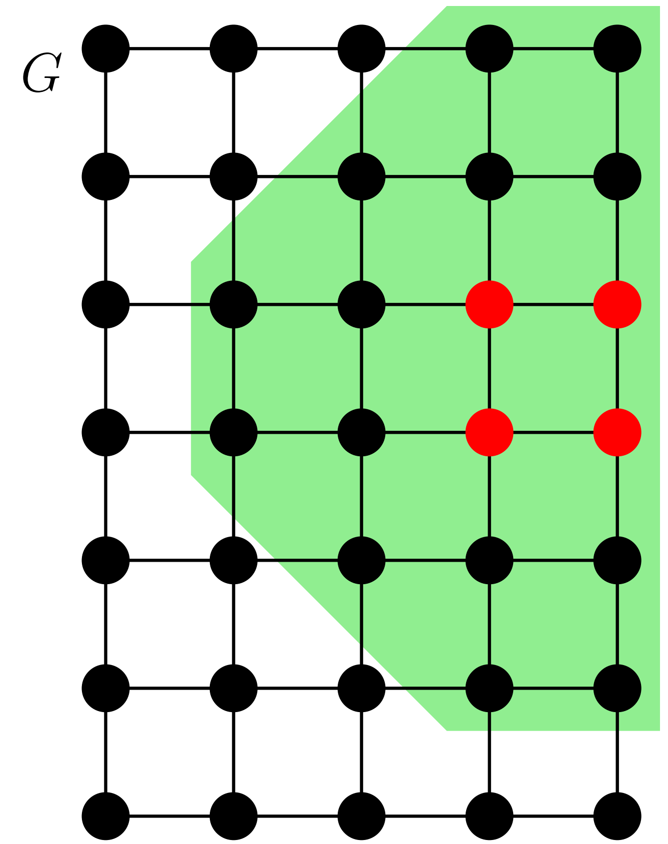

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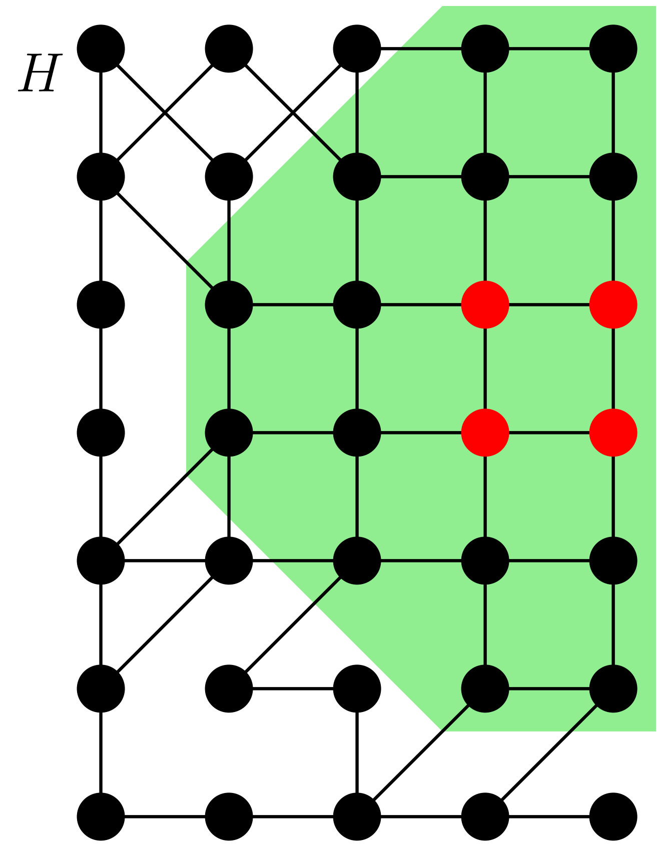

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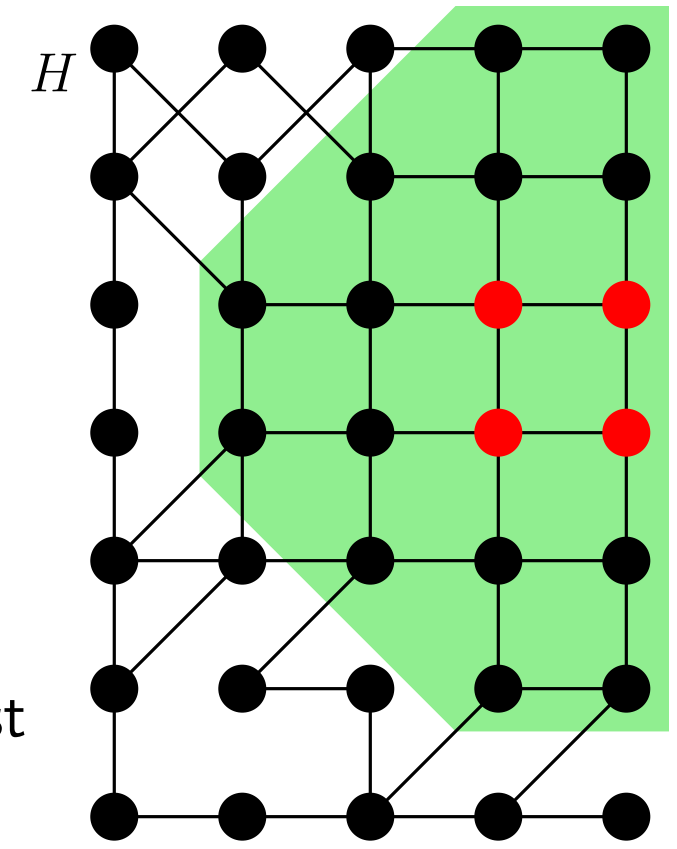

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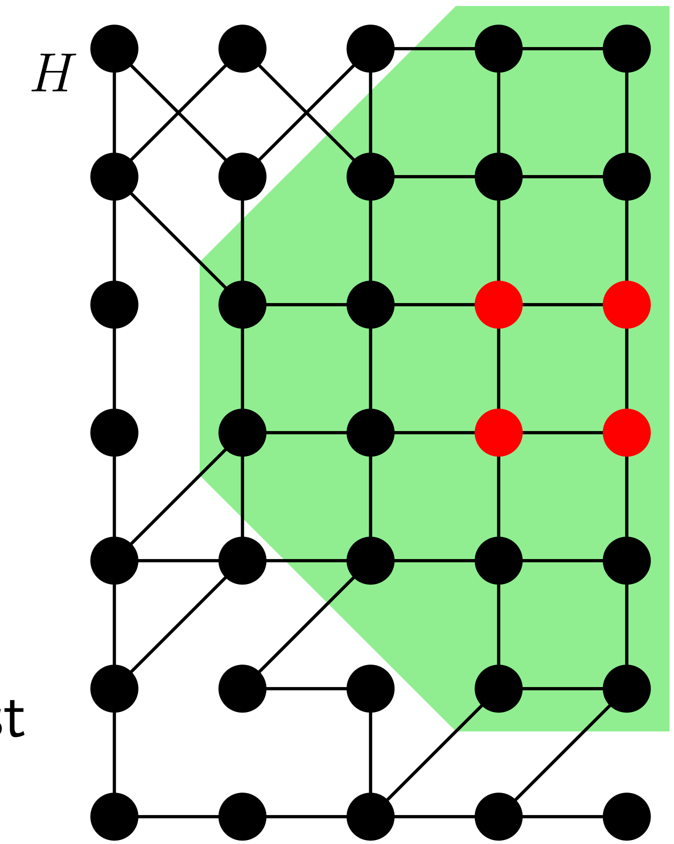

Outcome: "generalization" of algorithm

- An outcome assigns to inputs (G, x) a distribution over outputs $\{(y_i, p_i)\}_{i \in I}$, $y_i : V \to \Sigma_{\text{out}}$

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- An outcome is **non-signaling beyond distance** T if for all $S \subseteq V$, for all inputs $(G, x), (H, y)$ on V with $G[\mathcal{N}_T(S)] \sim H[\mathcal{N}_T(S)]$, the distributions restricted to $G[S]$ and $H[S]$ are the same

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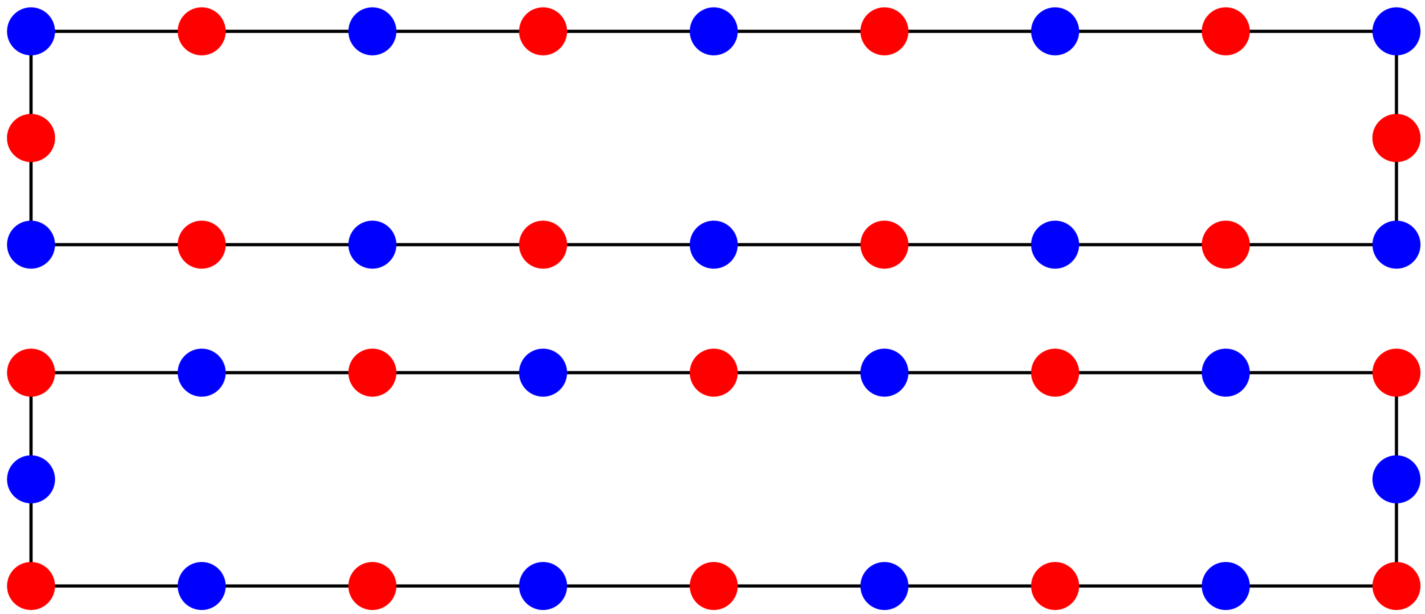

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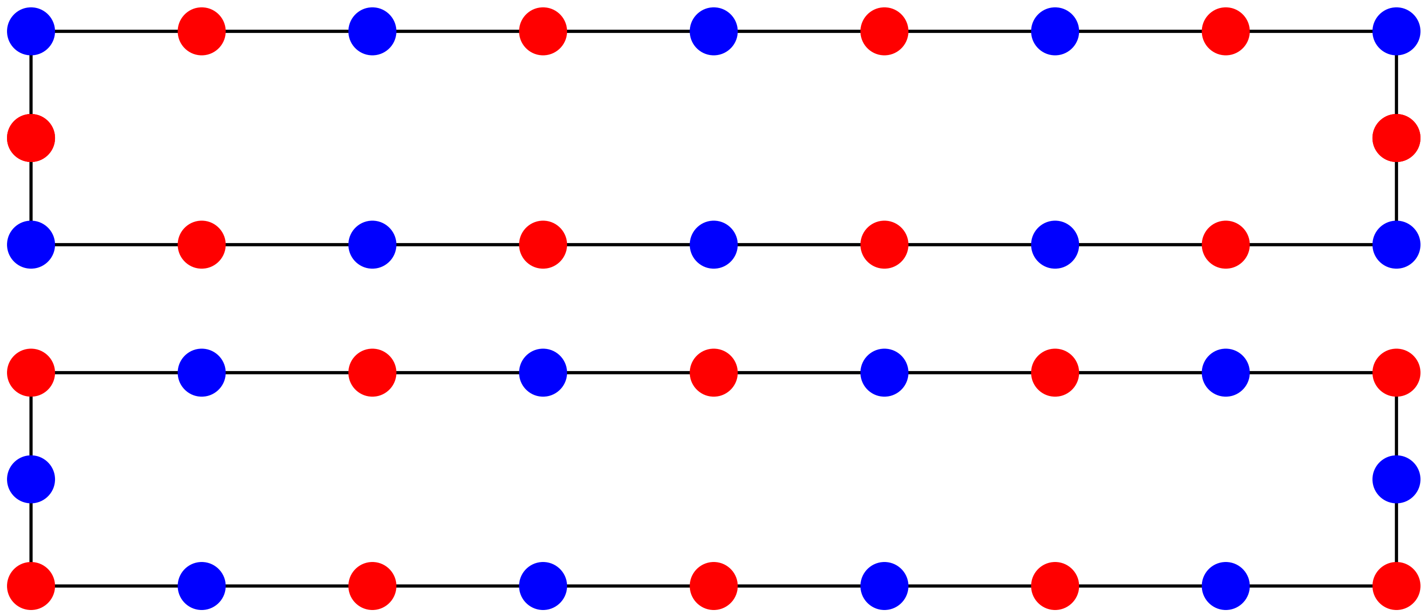

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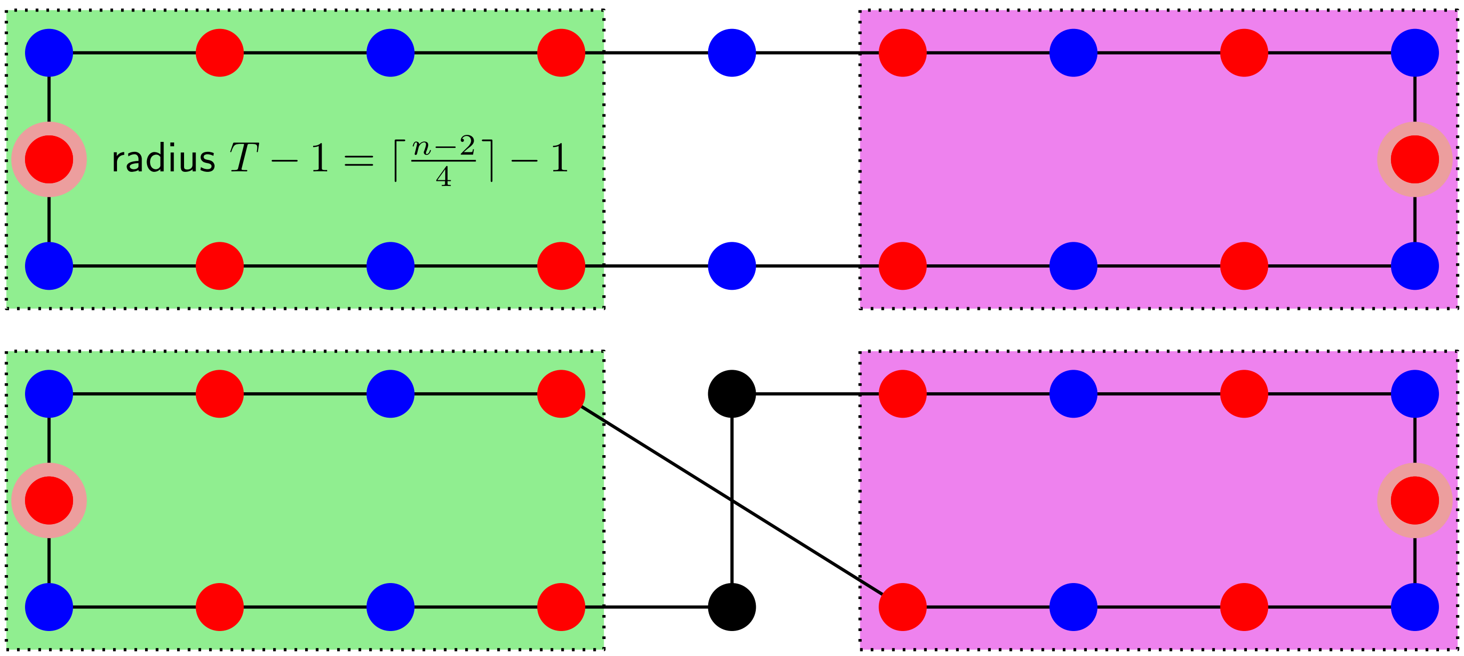

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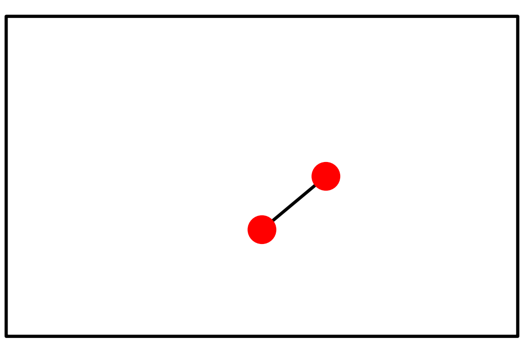

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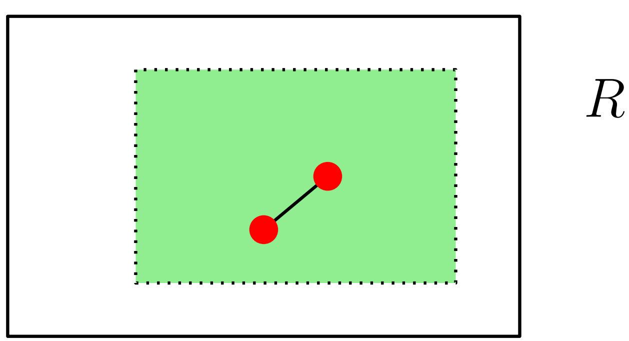

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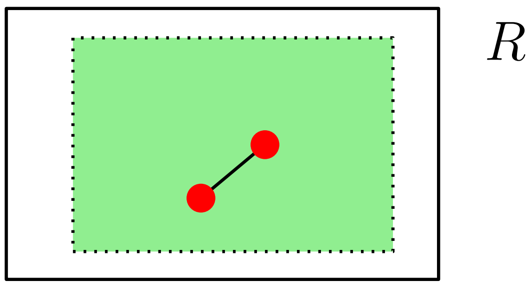

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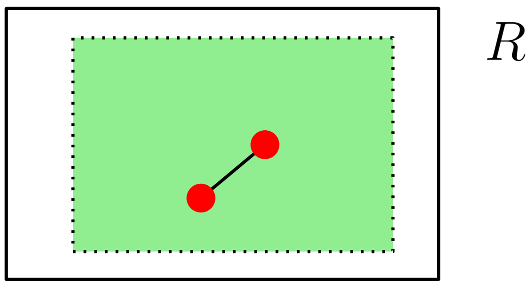

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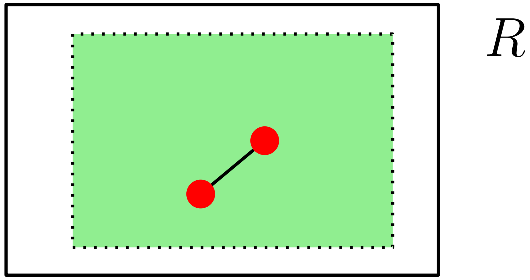

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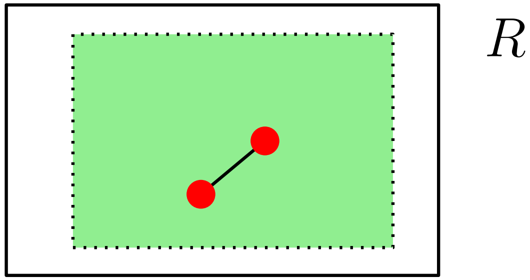

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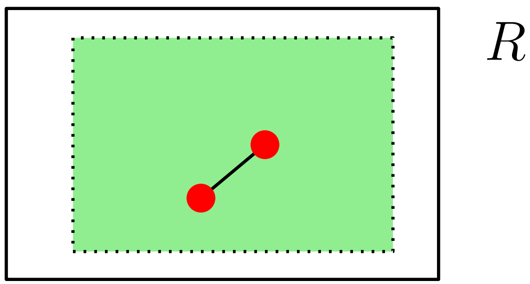

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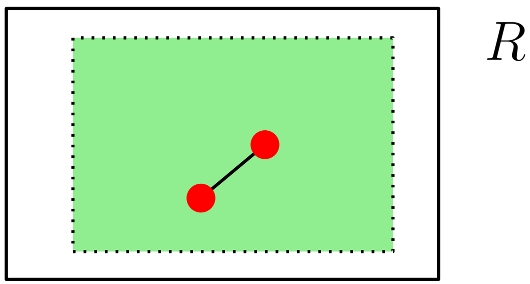

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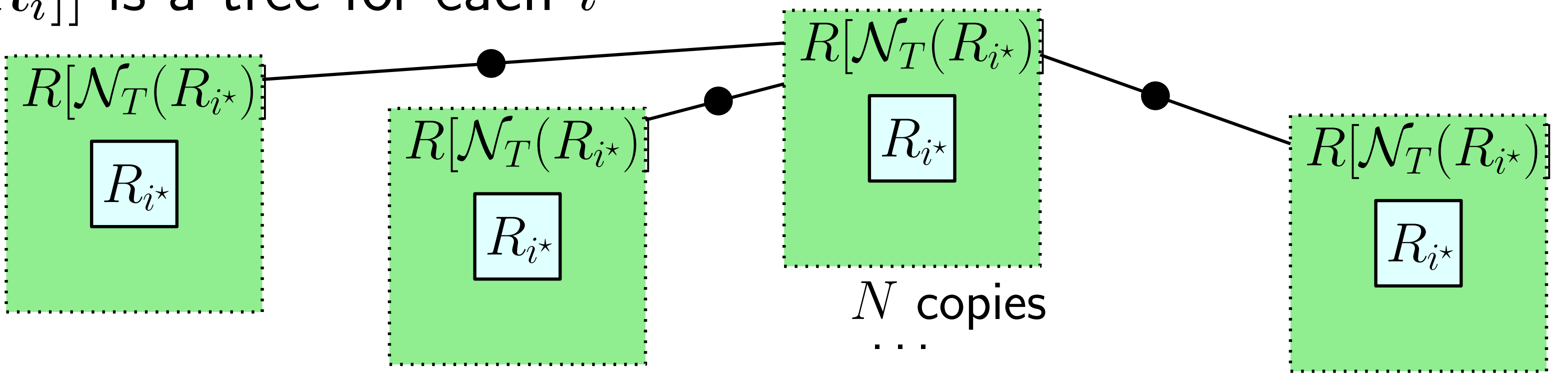

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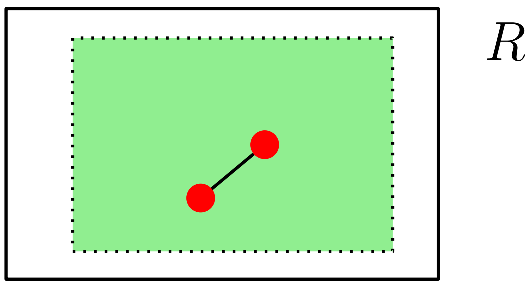

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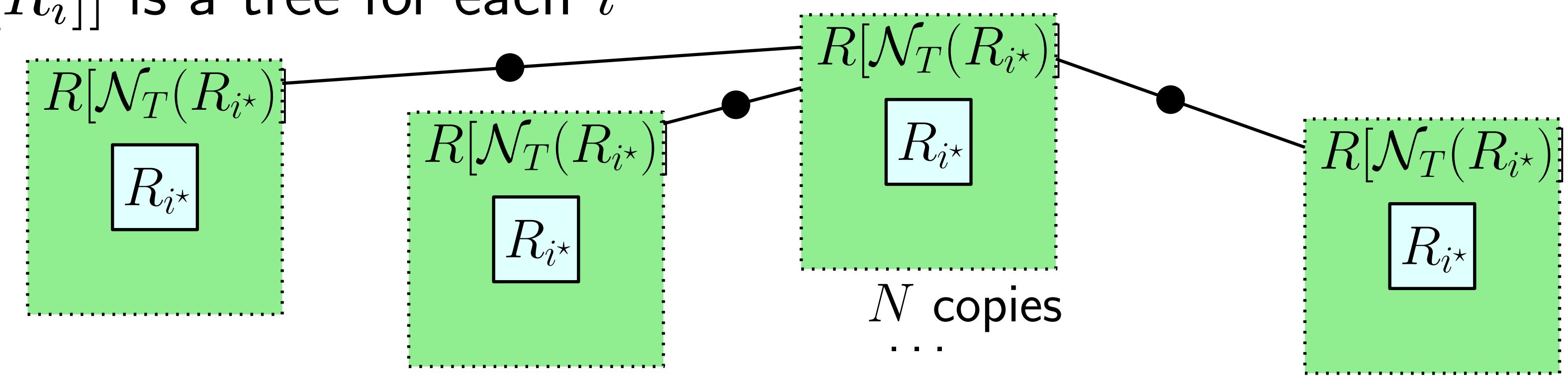

- Failing prob.: $1 - (1 - \frac{1}{k})^N$
- lower bound $T_{\text{new}}(n) = T(n/N)$

Indistinguishability & graph-theory: NS-LOCAL

Issues

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- We need to refine the definition of **cheating graph**

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Theorem: failing probability $\geq 1 - (1 - \frac{1}{k})^N$ and $T_{\text{new}}(n) = T(\frac{n}{N})$

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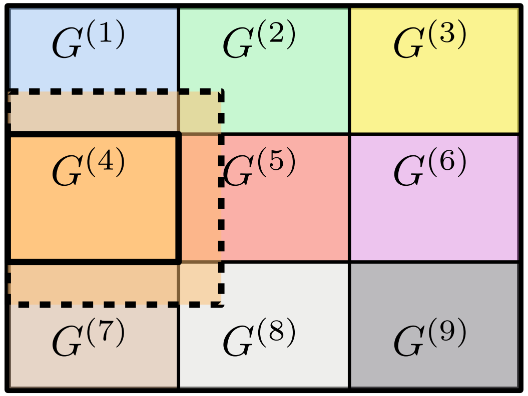
Cheating graph G



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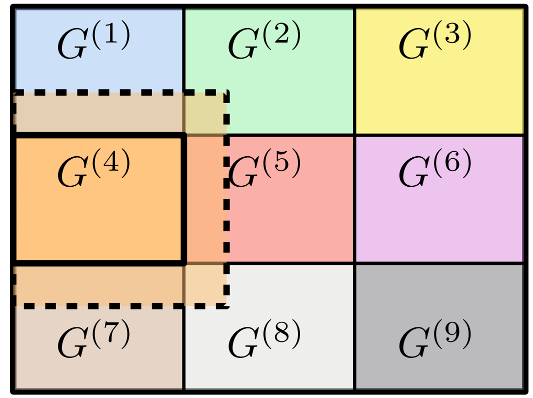

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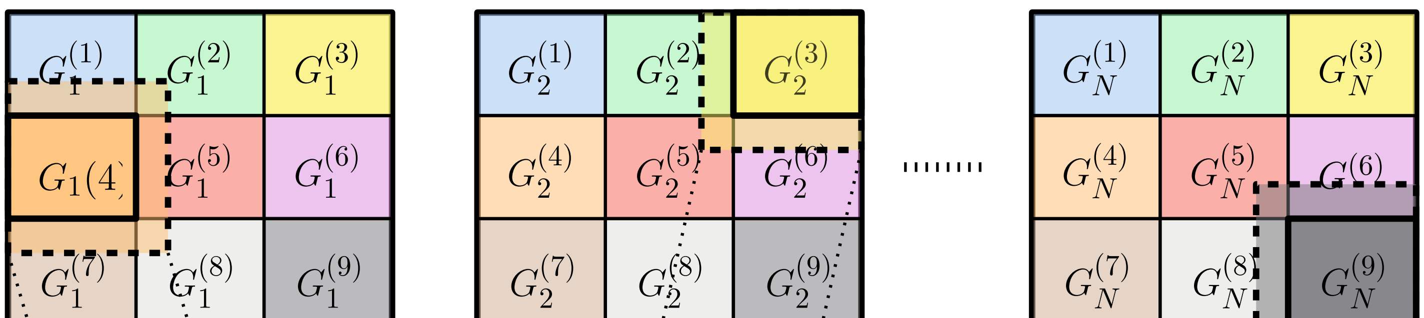
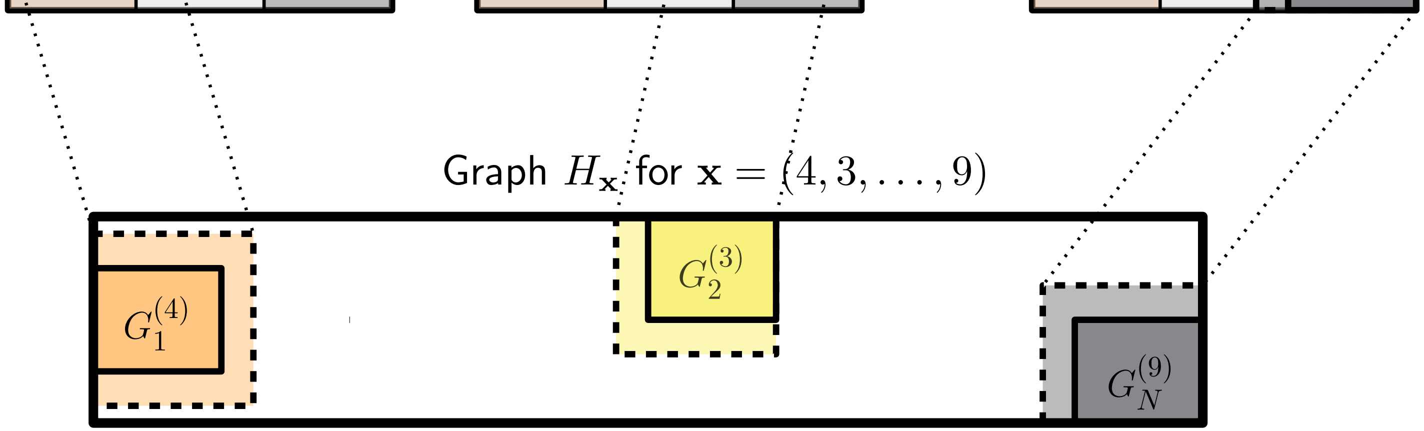
N copies of the cheating graph G

Cheating graph G_1 | Cheating graph G_2 | Cheating graph G_N



Graph $H_{\mathbf{x}}$ for $\mathbf{x} = (4, 3, \ldots, 9)$



Some applications

The following holds in NS-LOCAL

- **Result 1**: c-coloring trees with success probability ε requires locality $T = \Omega(\log_c n - \log_c \log \frac{1}{\varepsilon})$

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Approximate graph coloring

Problem: Let $2 \leq \chi \leq c$. The approximate graph coloring problem asks to c-color χ-chromatic graphs.

- **Previous results** (classical LOCAL model)
  - $c = 2, \chi = 2 \implies T = \Theta(n)$
  - $c = 3, \chi = 2 \implies T = \Omega(\sqrt{n})$ [Brandt et al., PODC '17]
  - $c \geq 4, \chi = 2 \implies T = \Omega(\log(n))$ [Linial, FOCS '87]

- **Our results** (All LOCAL models – from det. to non-signaling)
  - $\alpha = \lfloor \frac{c-1}{\chi-1} \rfloor \implies T = \Omega(n^{\frac{1}{\alpha}}), T = \mathcal{O}(n^{\frac{1}{\alpha}} \text{polylog}(n))$
  - $\implies$ no quantum advantage

Bogdanov's construction

- **Theorem**: Let $\chi \geq 2$, $r \geq 1$, $\alpha \geq 0$ be integers. There exists a graph $G_{\alpha,\chi}$ such that

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$G_{\alpha,\chi}$ is constructed recursively in α

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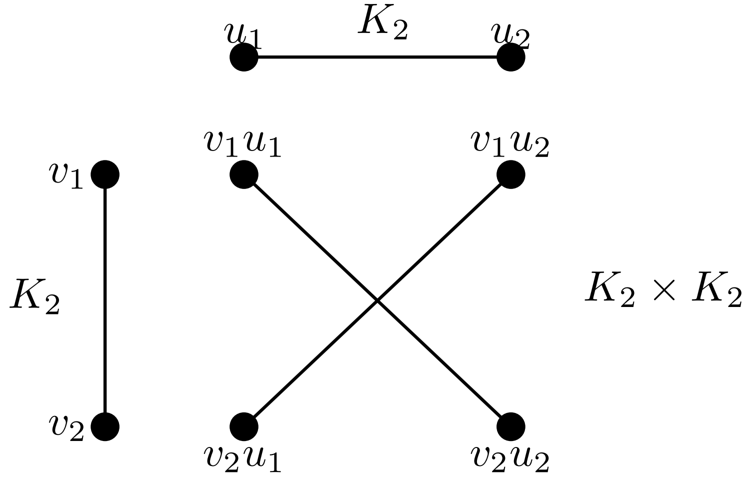
- **Tensor product** of graphs



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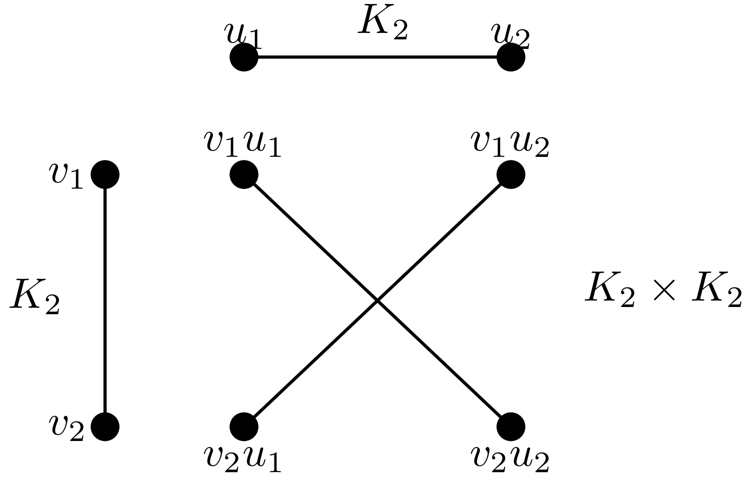

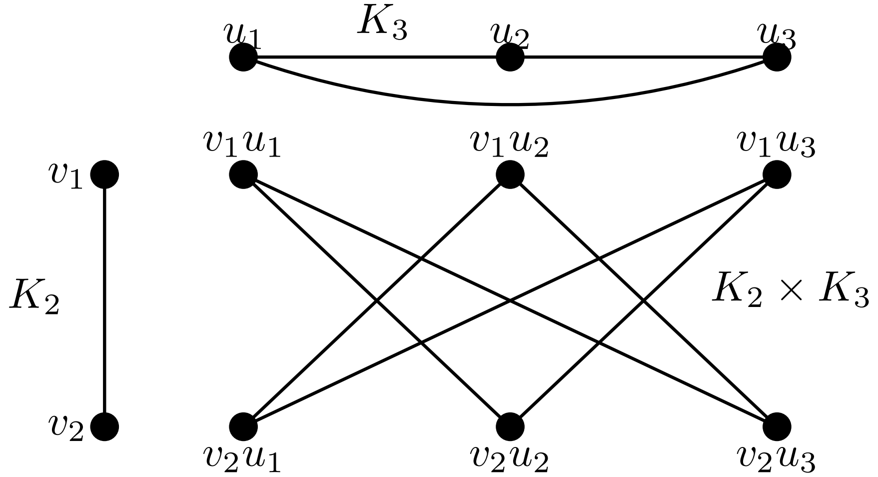

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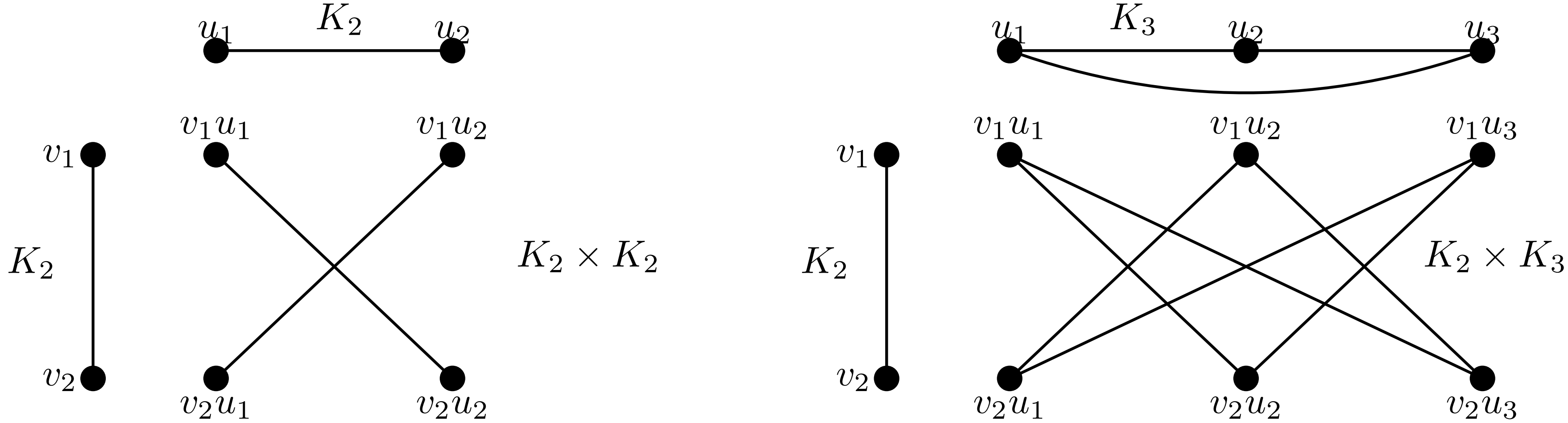

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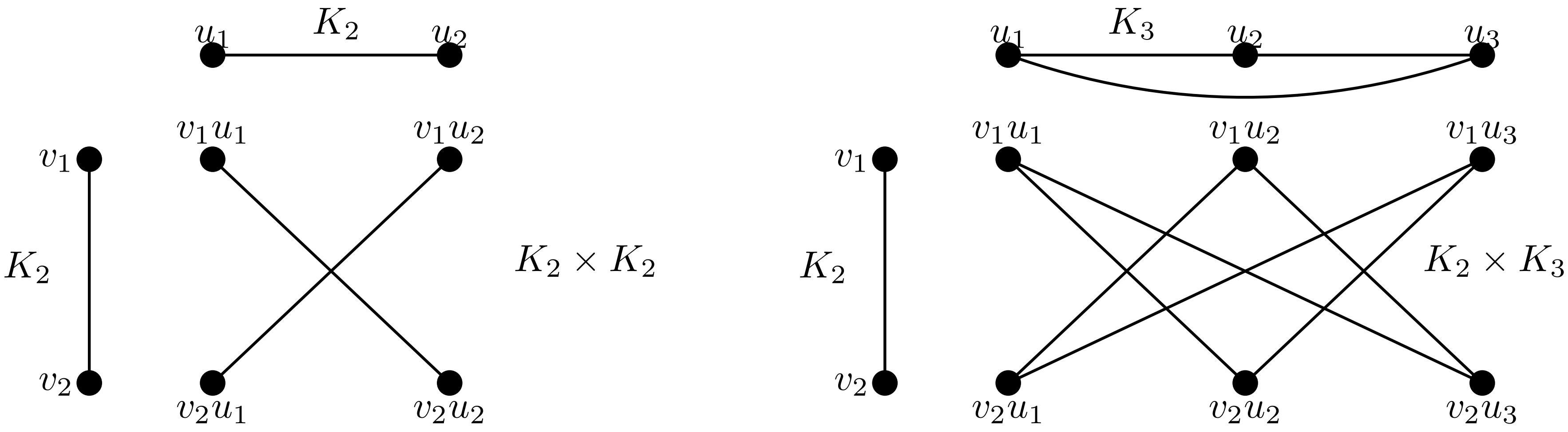

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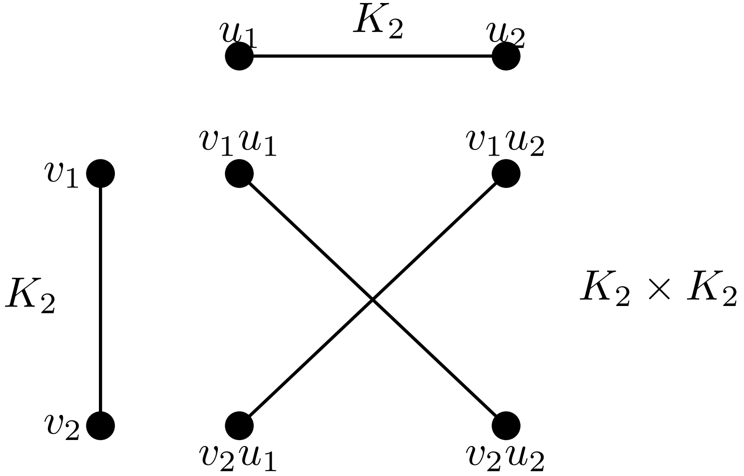

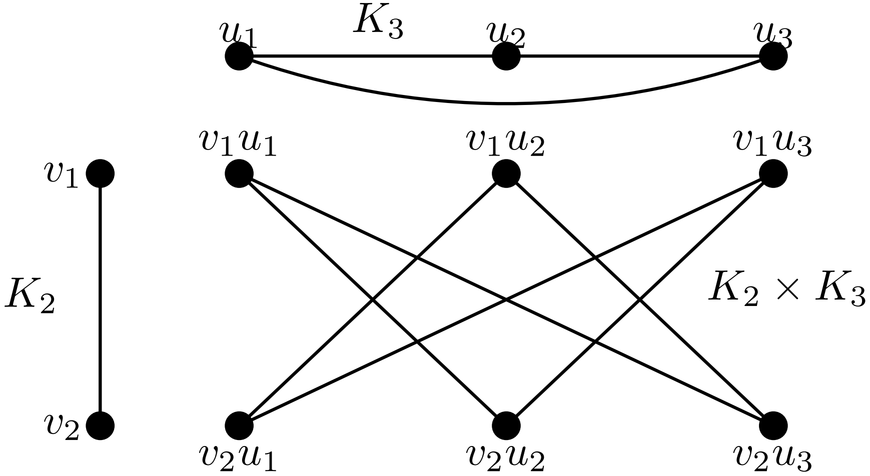

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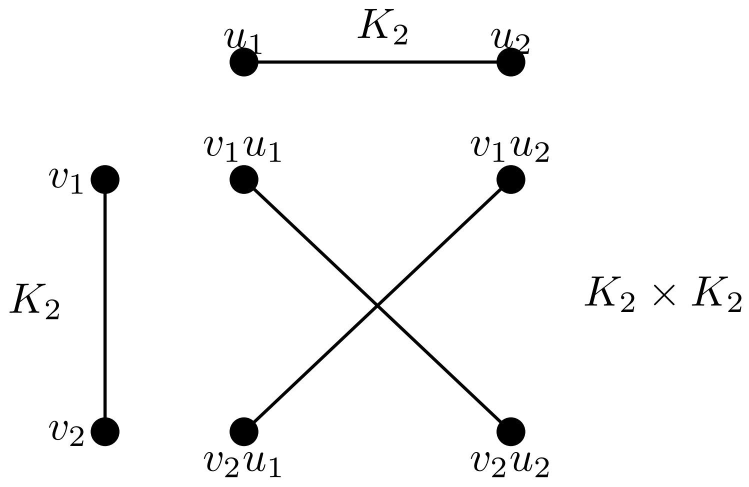

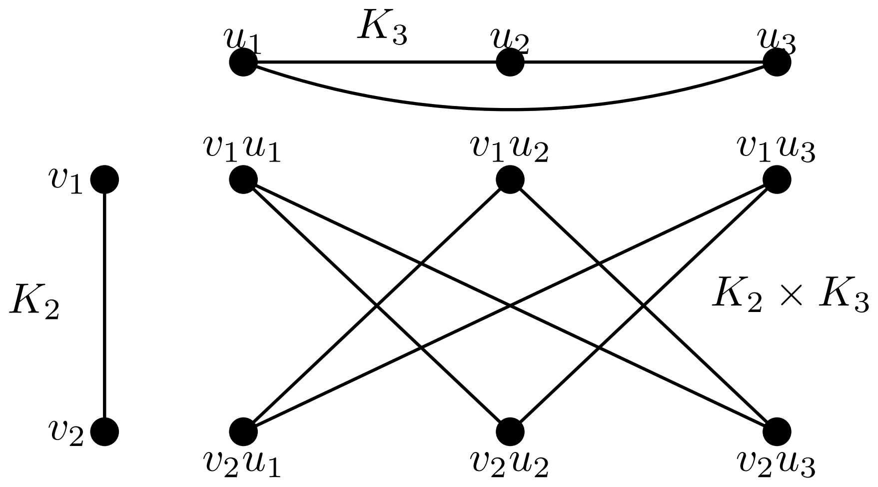

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- **r-join** of graphs $(r = 2)$ $\quad K_2^{(1)} \star_2 K_2^{(2)}$

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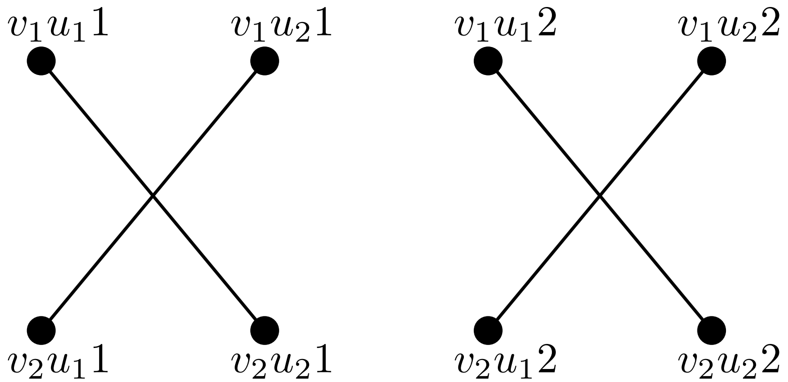

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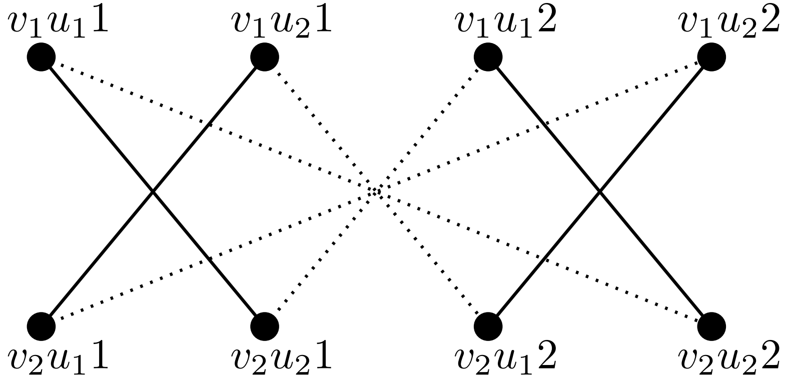

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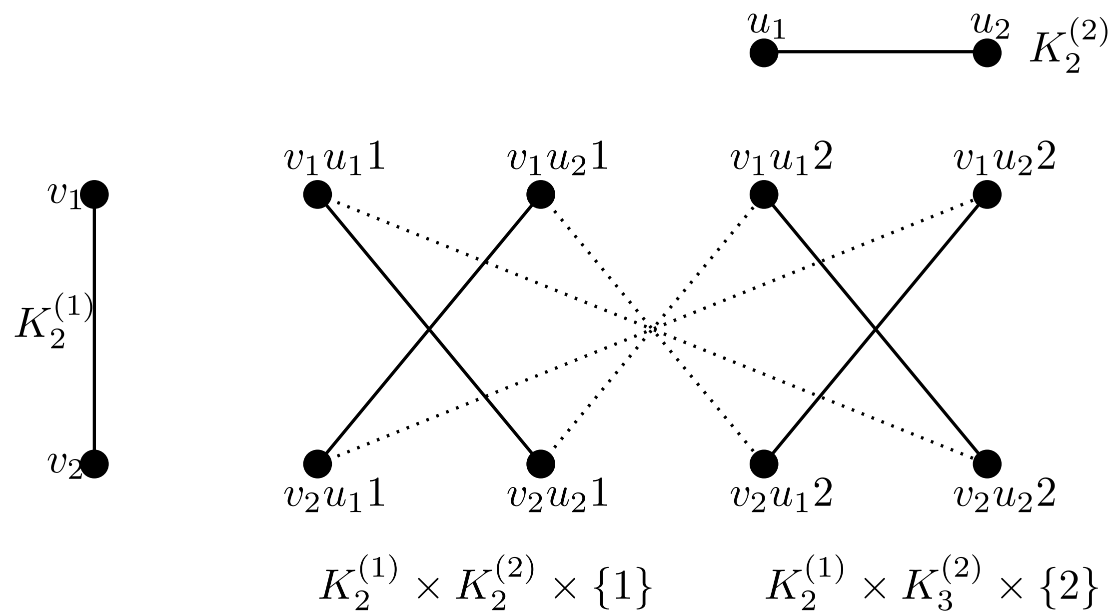

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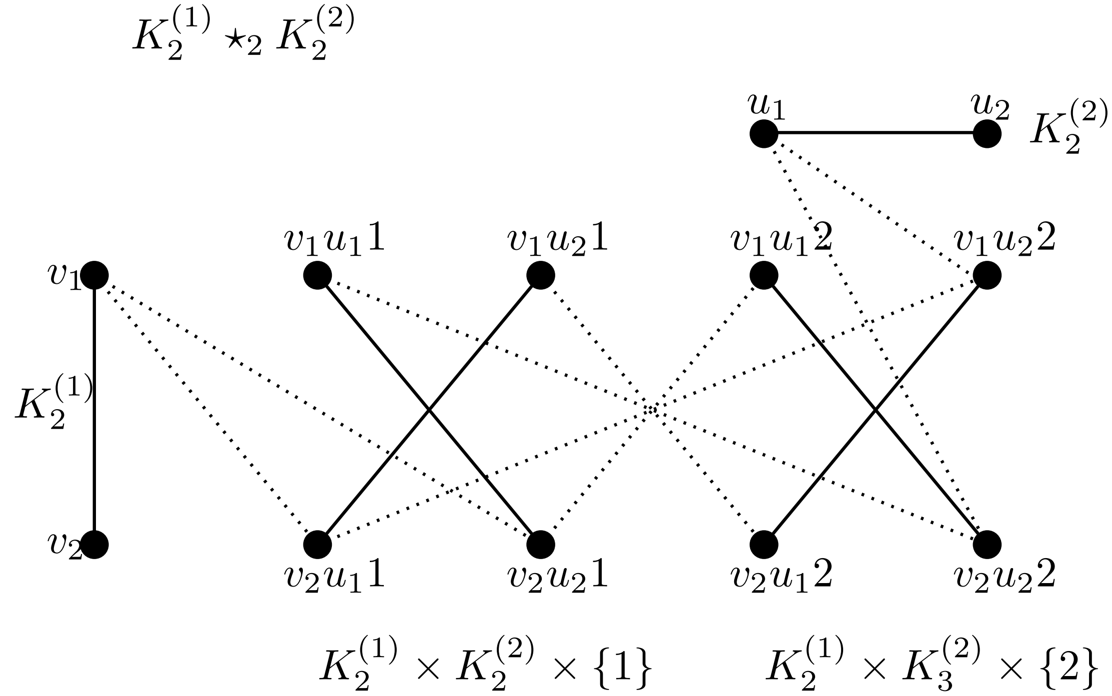

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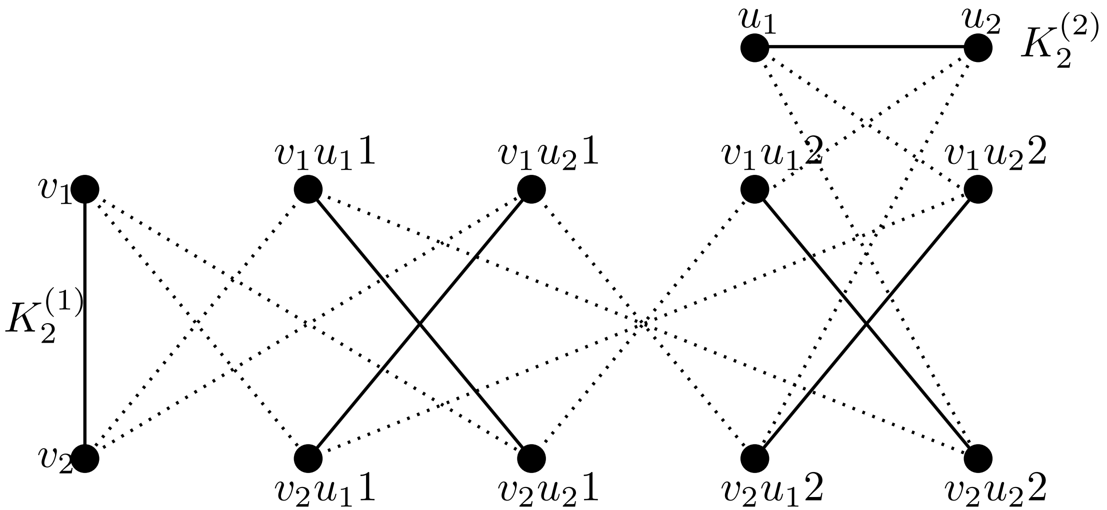

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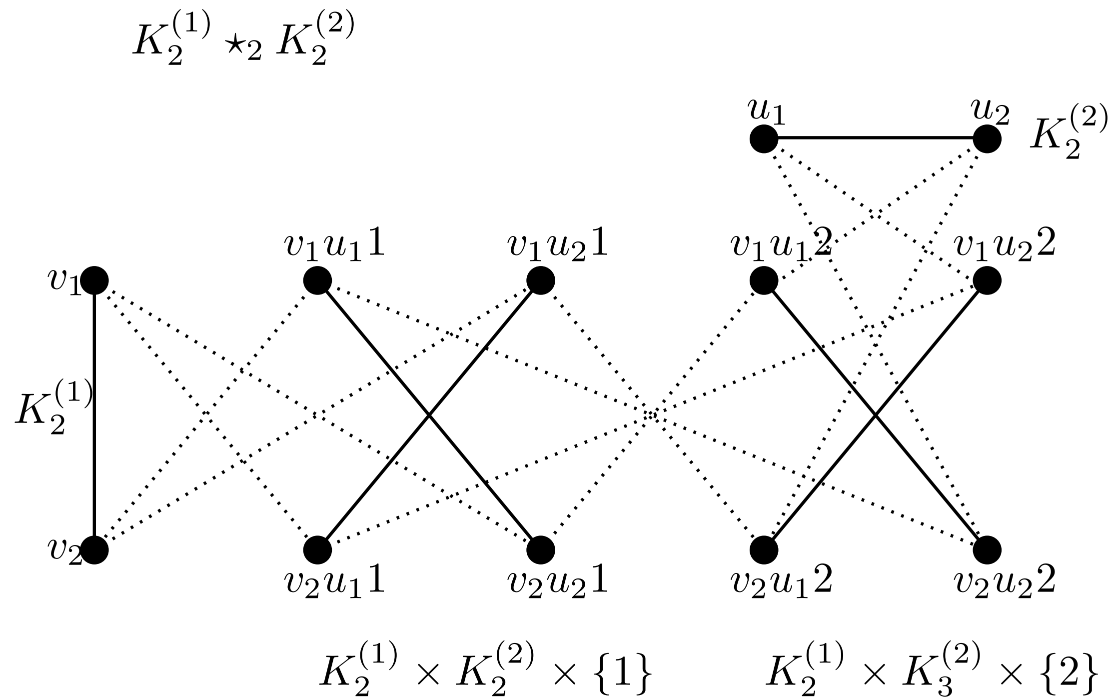

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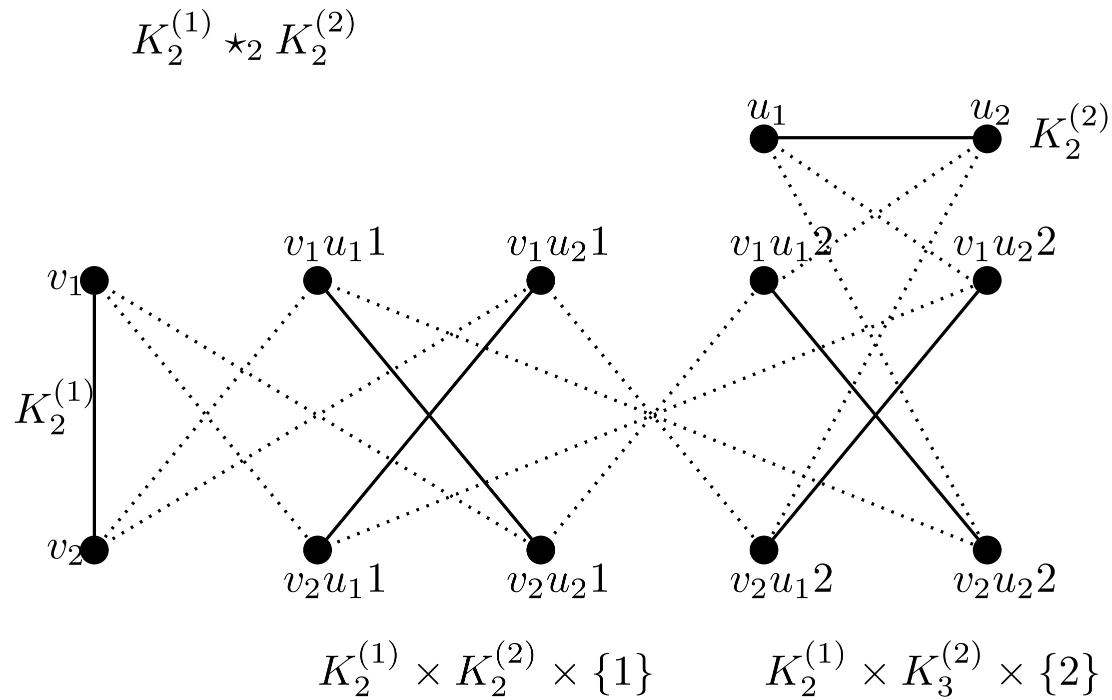

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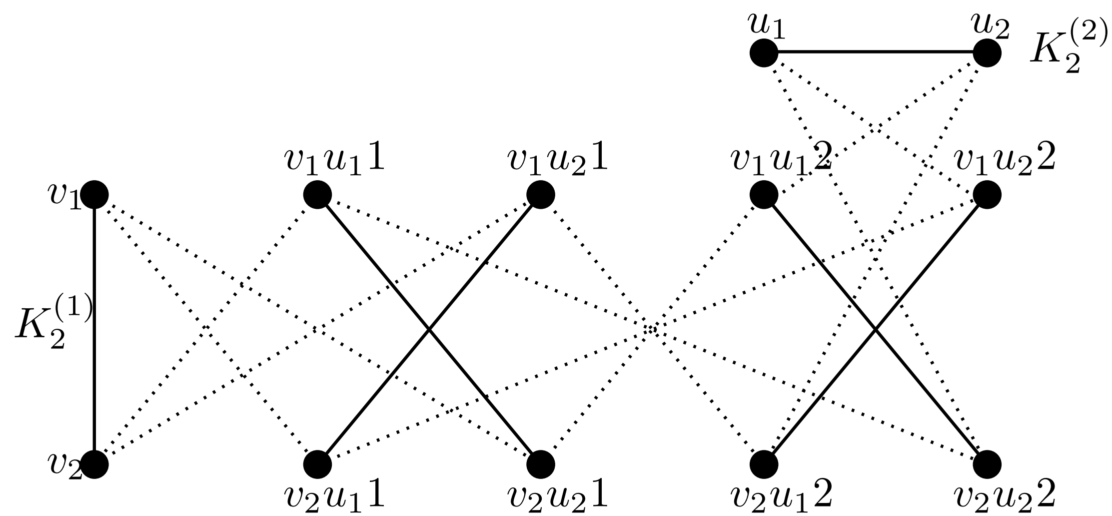

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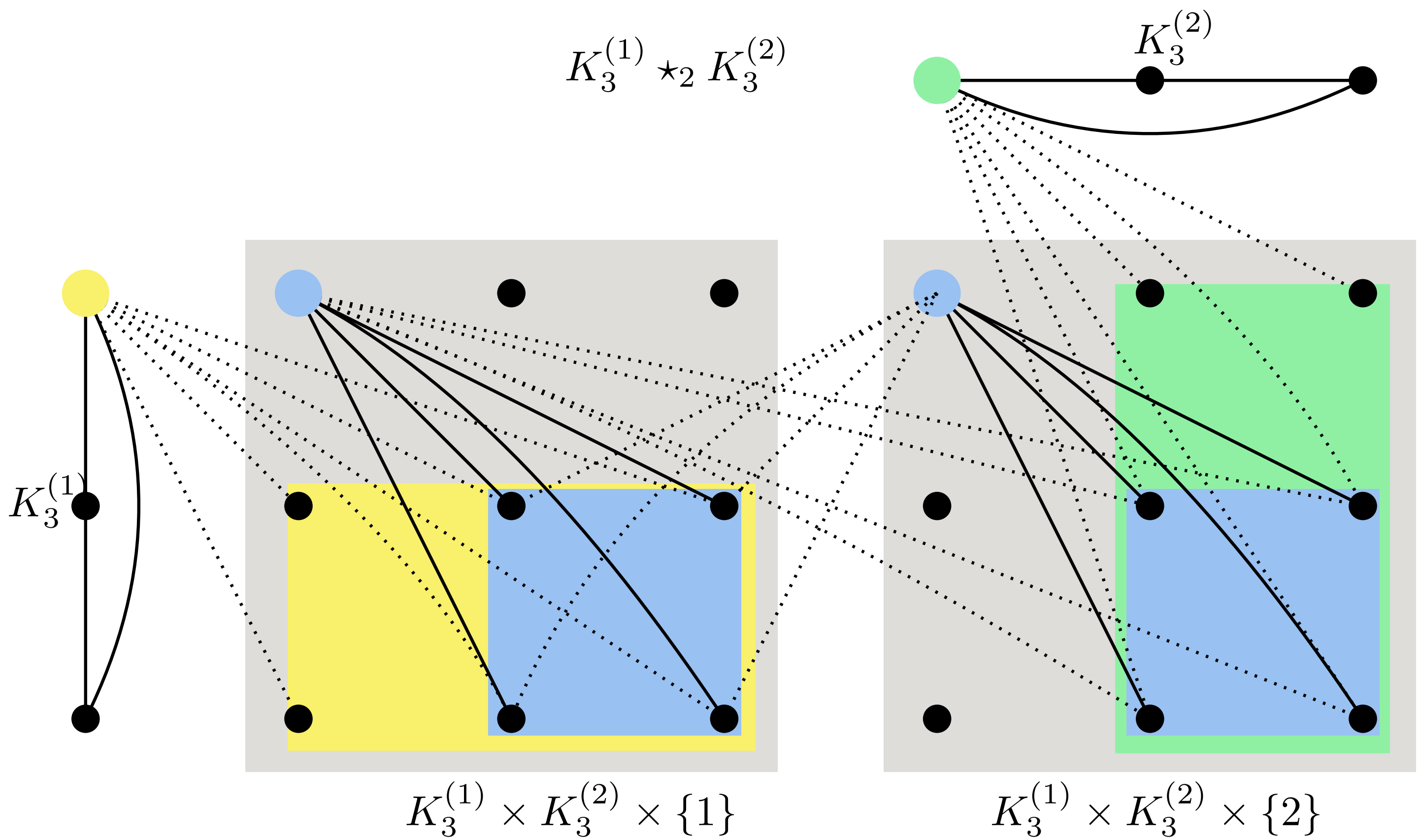

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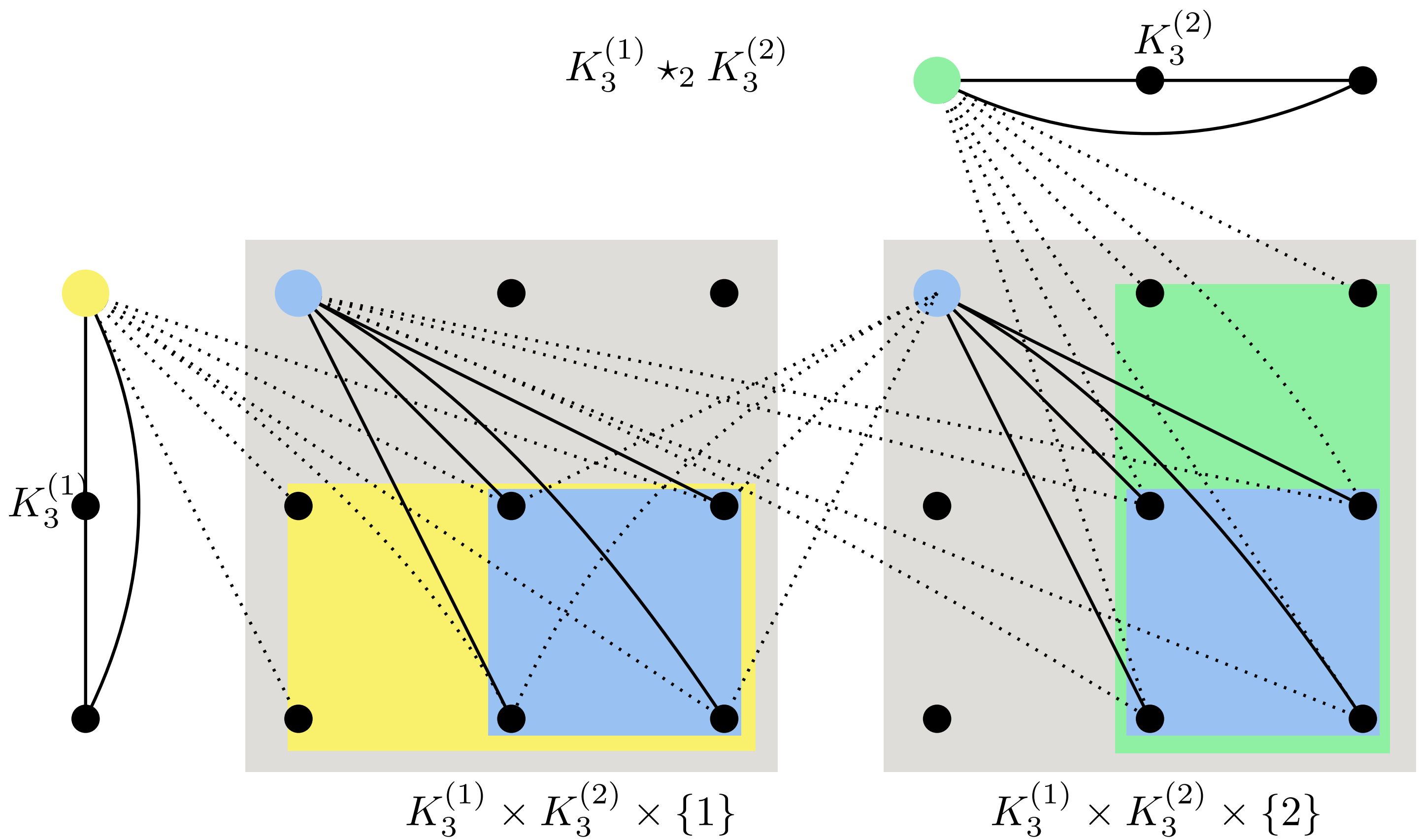

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- Under development (spoiler alert)
- all problems' complexities $\mathcal{O}(\log^\star n)$ in classical LOCAL become $\mathcal{O}(1)$ in NS-LOCAL (plus independence!)
- (paths, cycles, trees): all problems' complexities $\mathcal{O}(1)$ in φ-LOCAL become $\mathcal{O}(\log^\star n)$ in classical LOCAL
- (paths, cycles, rooted trees): there cannot be quantum advantage beyond complexity $\mathcal{O}(\log^\star n)$

THANKS!