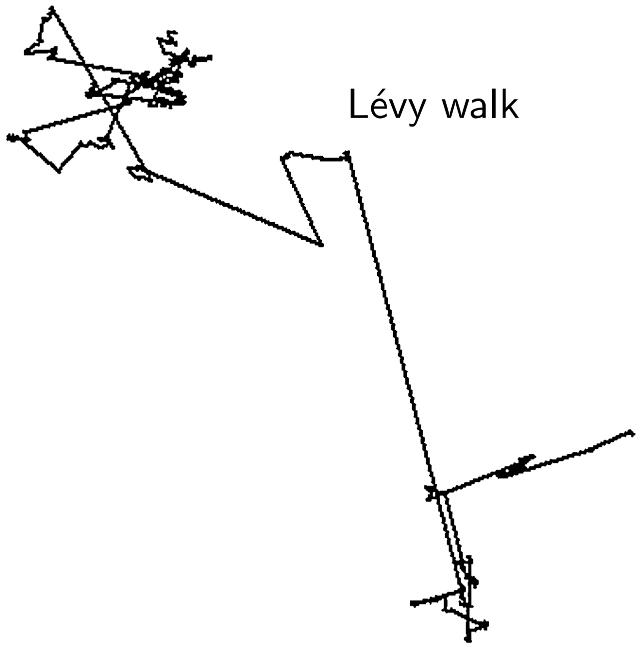
On the Search Efficiency of Parallel Lévy Walks on $\mathbb{Z}^2$

Francesco d'Amore

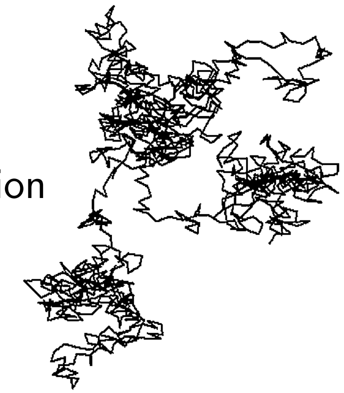
Joint work with Andrea Clementi, George Giakkoupis, and Emanuele Natale

Coati seminar, 30 April 2020

What are Lévy Walks?



Brownian motion



Images from [Reynolds, '15]

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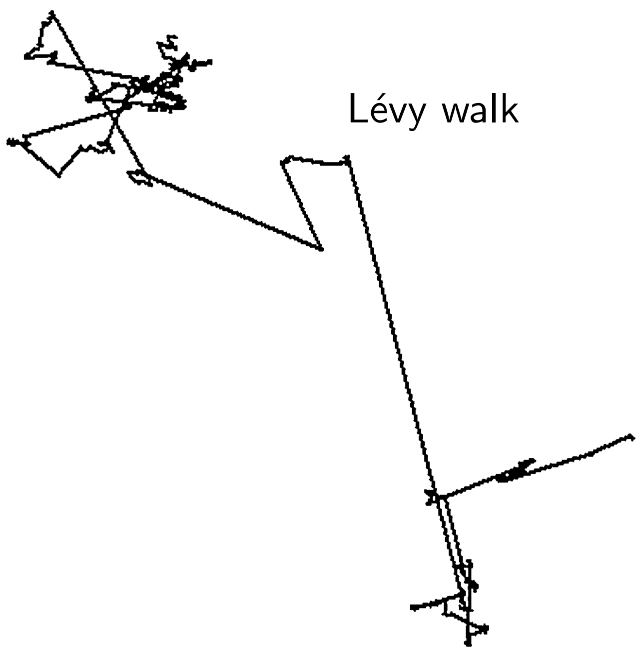
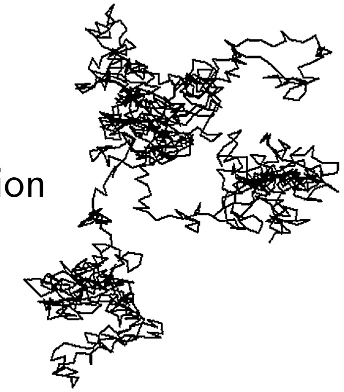


Lévy walk



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Lévy walk (informal):
A Lévy walk is a random walk whose step-length density distribution is proportional to a power-law, namely, for each $d \in \mathbb{R}$, $f(d) \sim 1/d^{\alpha}$, *for some* $\alpha > 1$

Note: the speed of the walk is constant

Why are Lévy walks interesting?

Lévy walks are used to model **movement patterns** [Biology Open, '18]

Examples:
- T cells within the brain
- swarming bacteria
- midge swarms
- termite broods
- fishes
- Australian desert ants
- a variety of molluscs

Austrialian desert ants

Image from professor Gibb, La Trobe University

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Some fun: mussels Lévy walk video [Science, '11]

The Theory of Foraging

Scenario:
- a density distribution ρ in $\mathbb{R}^n$ describing food locations
- an uninformed walker searching for food in $\mathbb{R}^n$

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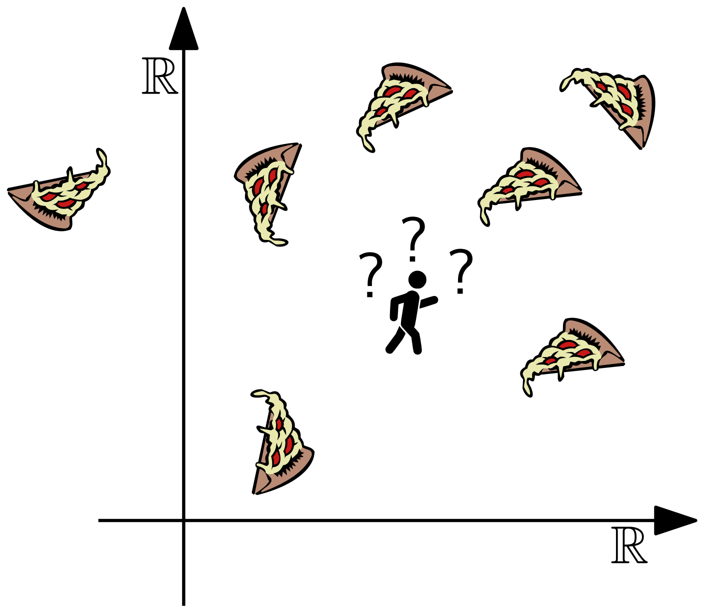

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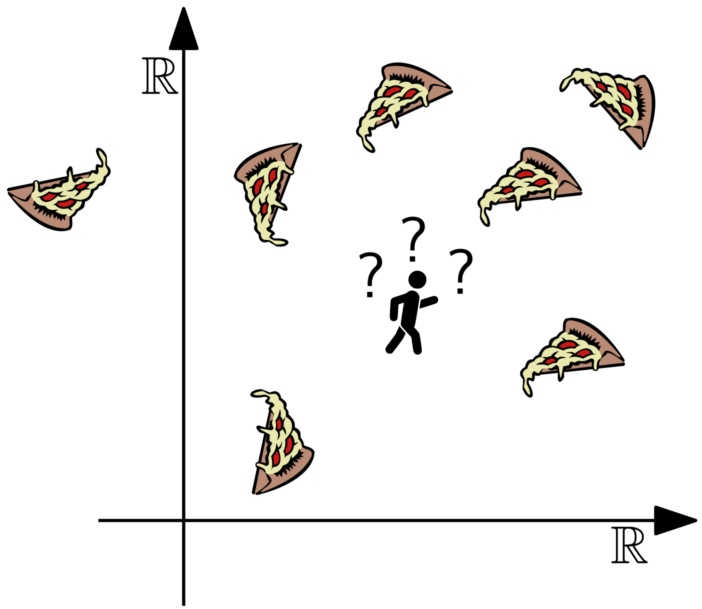

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Question: which strategy maximizes the expected food discovery rate?

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[Nature, '99] analyzes three random search strategies:

(a) normal diffusion **(b)** ballistic diffusion **(c)** super diffusion

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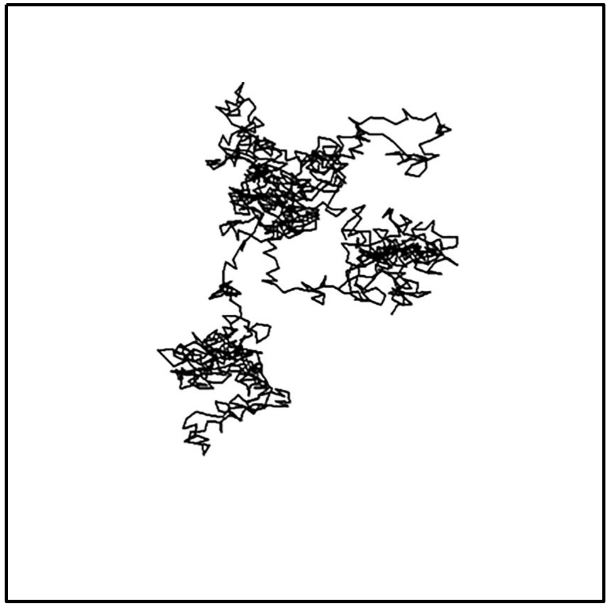

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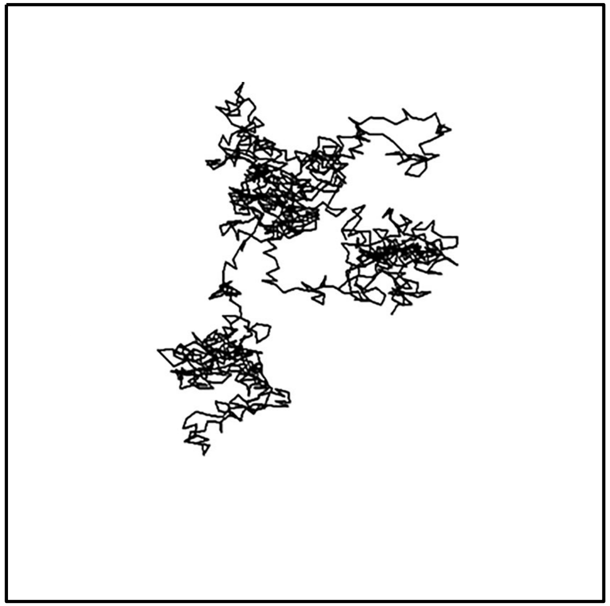

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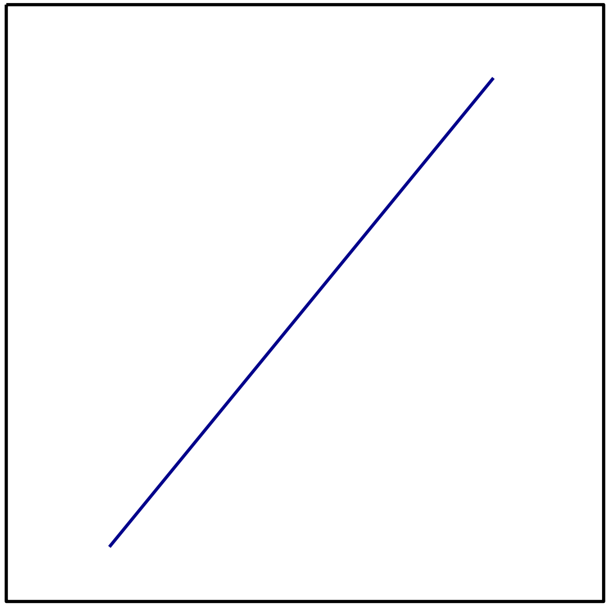

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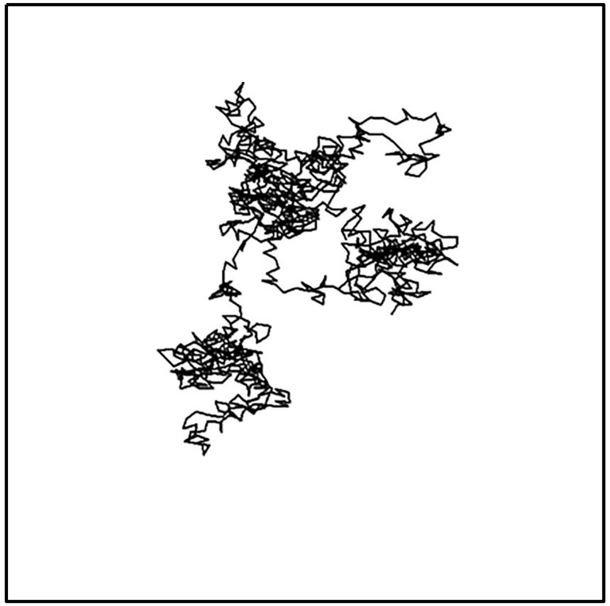

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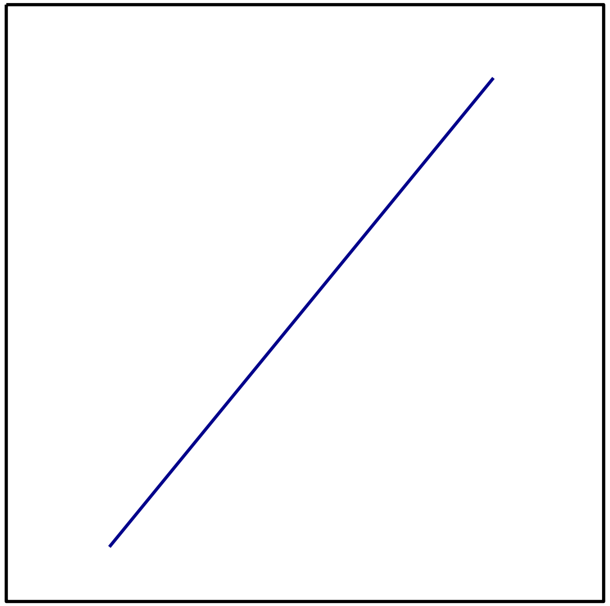
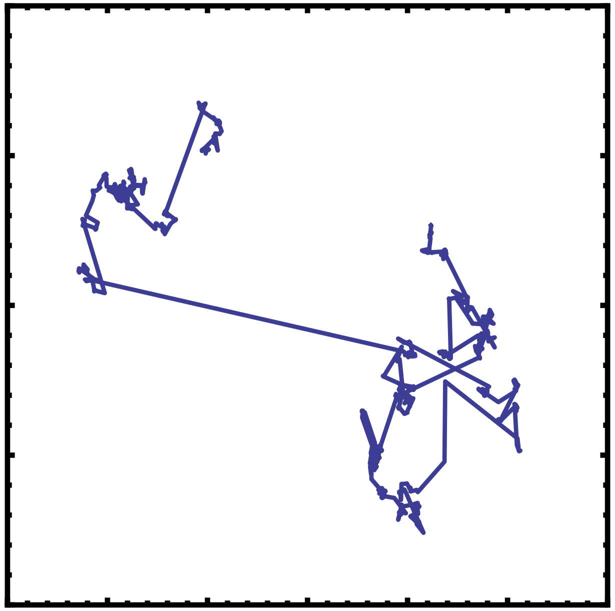


(c) super diffusion

(between (a) and (b))



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The Lévy Walk Regimes

Reminder: the density distribution of the step-length is $f(d) \sim 1/d^{\alpha}$

Case $\alpha \geq 3$: the Lévy walk has normal diffusion

(Idea) In one dimension, and for $\alpha > 3$.

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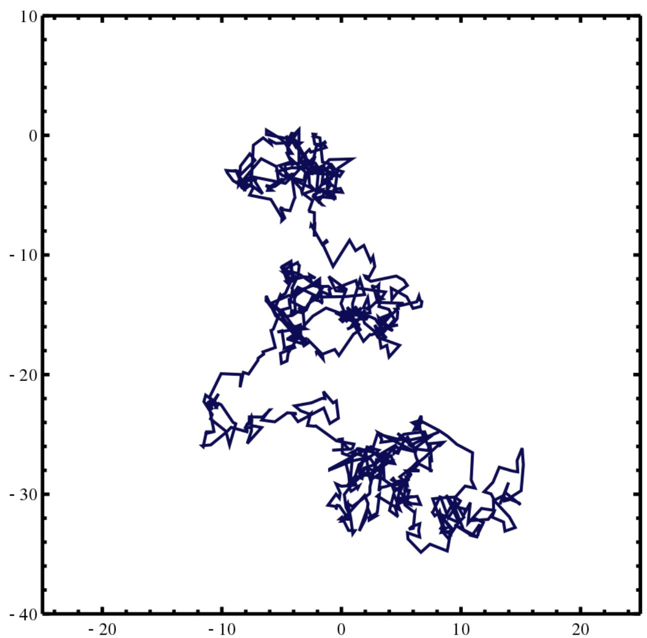

A Lévy walk with parameter $\alpha = 3$ approximates a brownian motion

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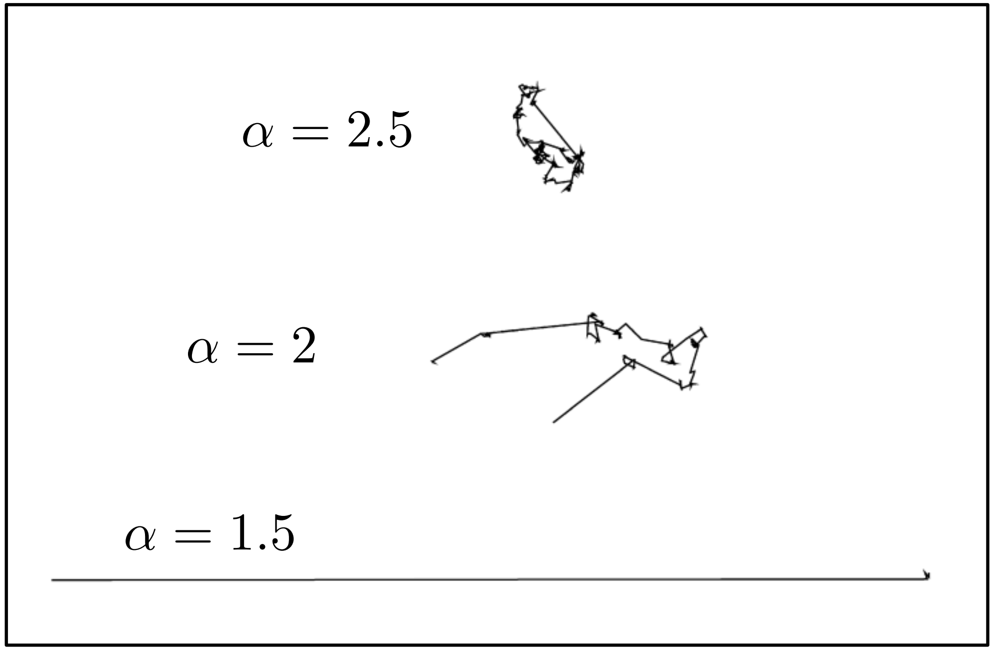

Examples of Lévy walks for different values of α

Images from [Viswanathan et al, '99]

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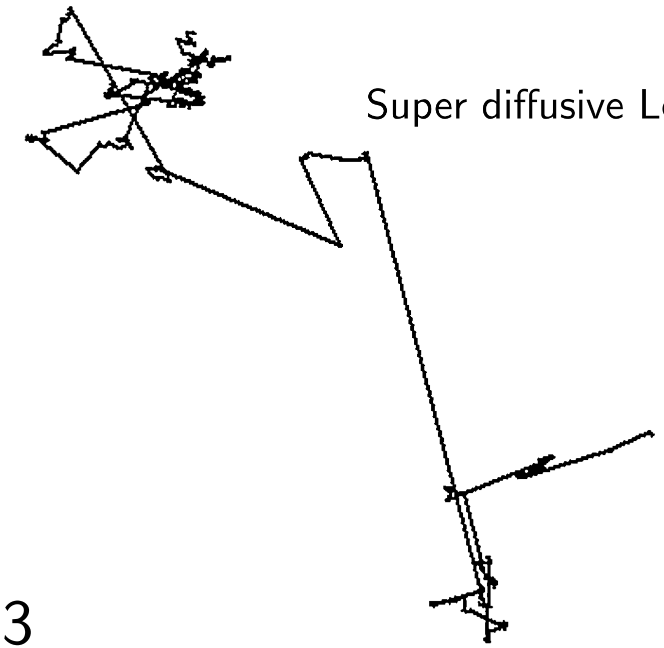
Note: in between normal and ballistic diffusion



Super diffusive Lévy walk

Optimality of Lévy Walk

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- non-destructive foraging (the food regenerates once found)
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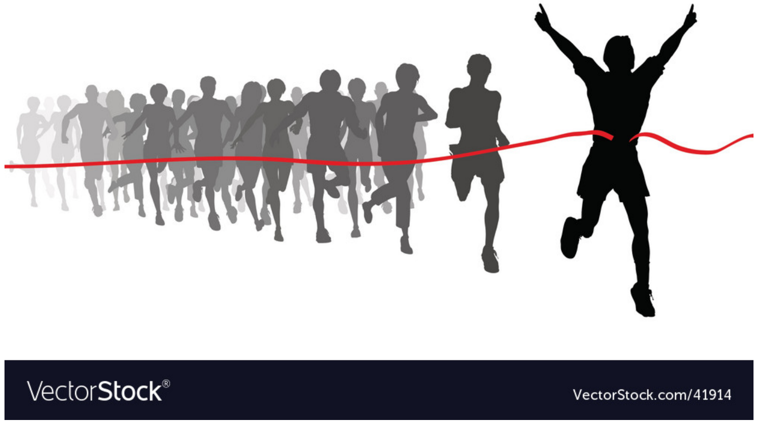
- the Lévy walk with exponent $\alpha = 2$, for non-destructive foraging
- the ballistic walk, for destructive foraging



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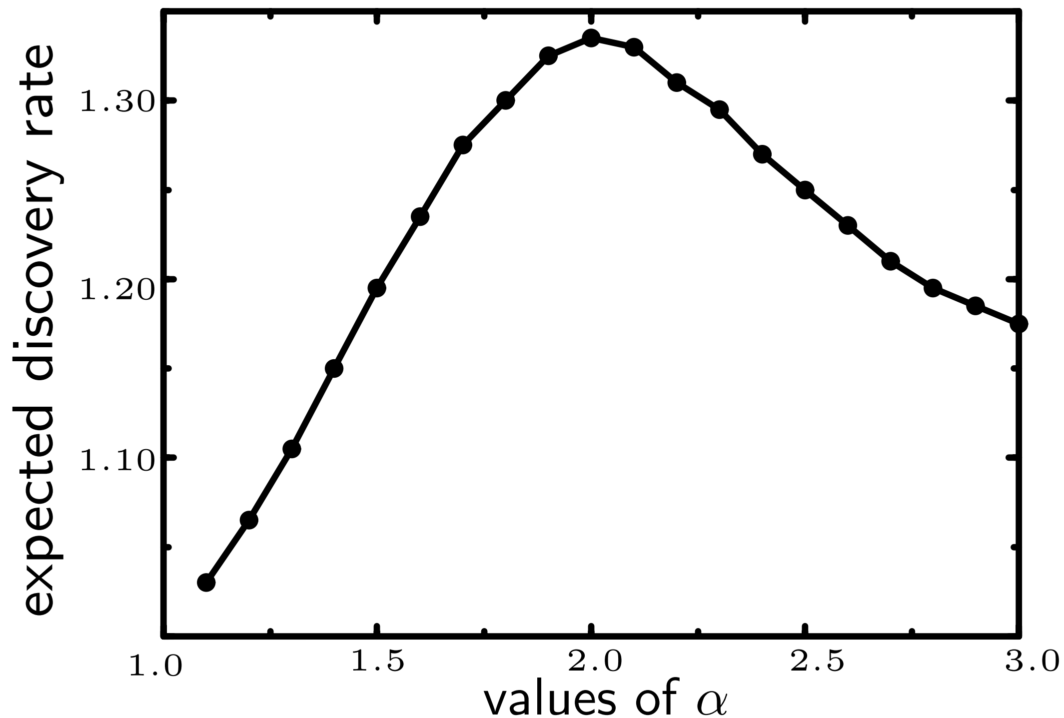

Simulations by [Nature, '99]

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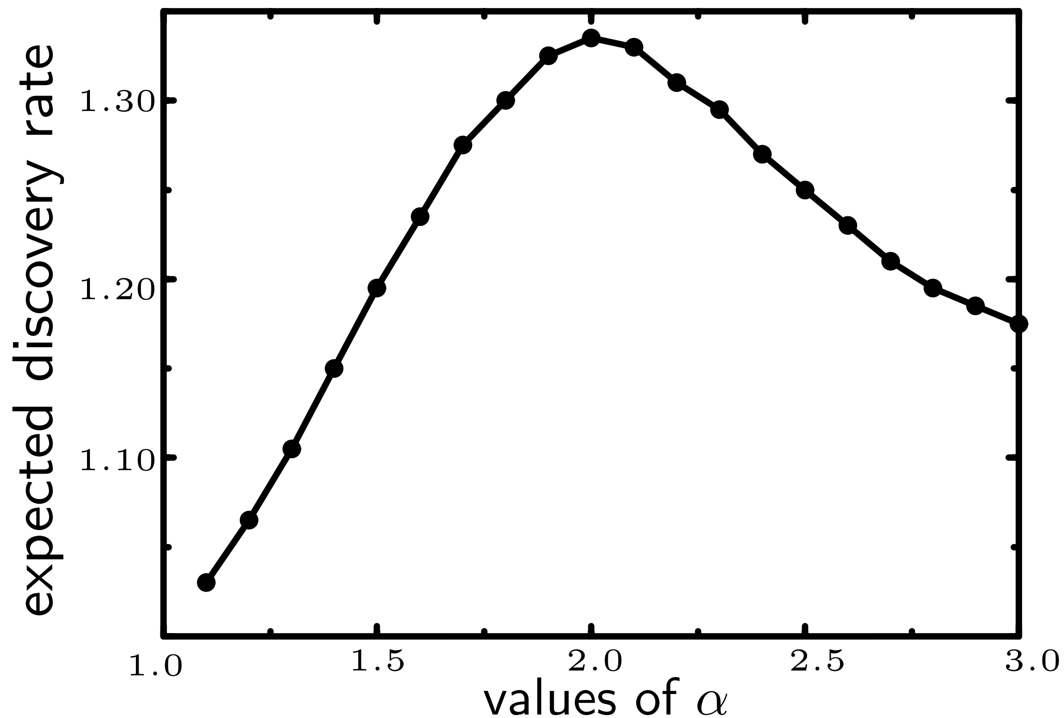

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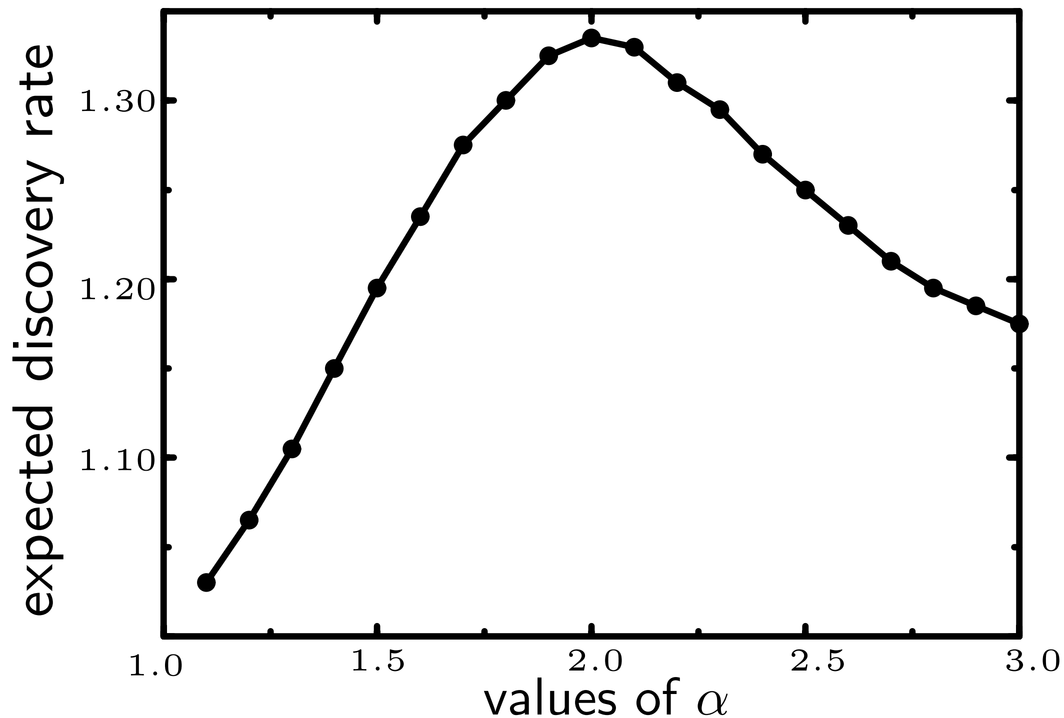

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HOWEVER...

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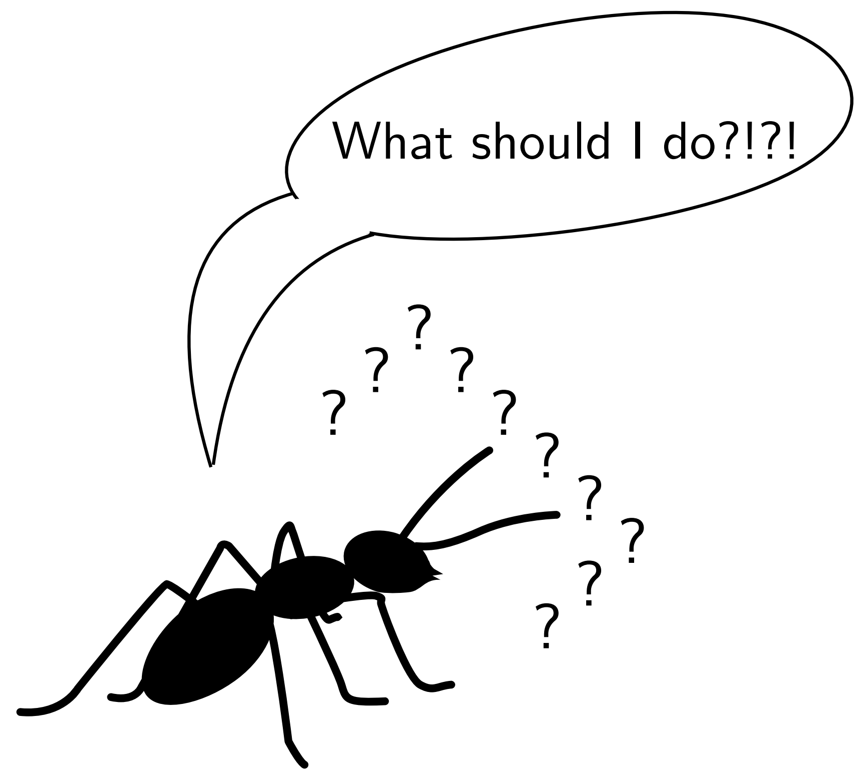

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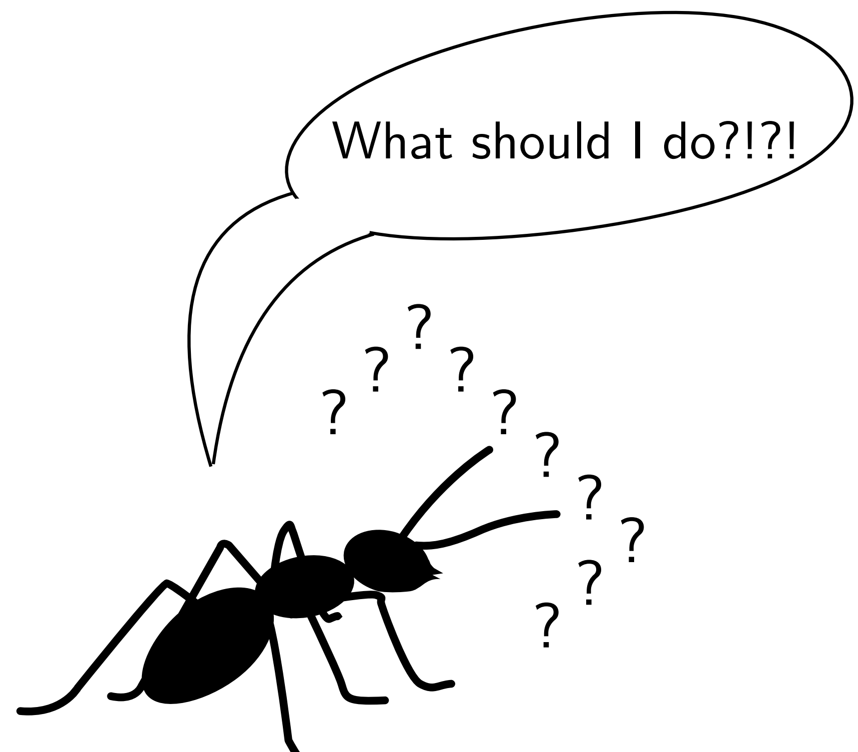
The Lévy walk has never been studied in the discrete setting



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Setting:
- k (mutually) independent walkers (agents) start moving in $\mathbb{Z}^2$ from the origin
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- one special node $\mathcal{P} \in \mathbb{Z}^2$, the *treasure*, at (Manhattan) distance ℓ from the origin

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Question: which strategy is the best one to find the treasure?

image from https://www.youtube.com/watch?v=2M4gXkUhK6M

Some Preliminaries

We denote

- *step* a move that takes one time unit
- $B_d(u)$ the set of nodes of $\mathbb{Z}^2$ having Manhattan distance at most d from u

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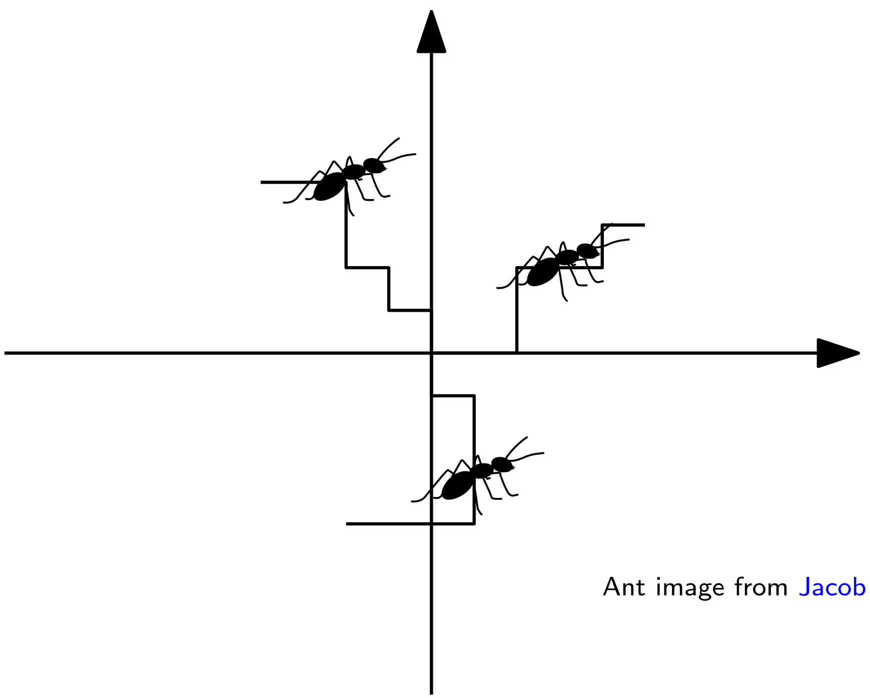
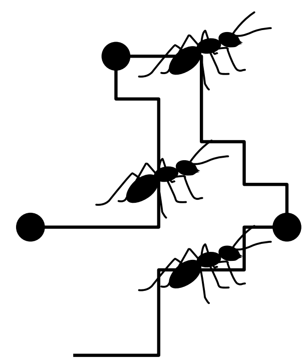
Definition (*work*): k agents moving for t steps make a work equal to $k \cdot t$





work $=$ total path covered

Ant image from Jacob Eckert, from The Noun Project

Lower Bound on the Work

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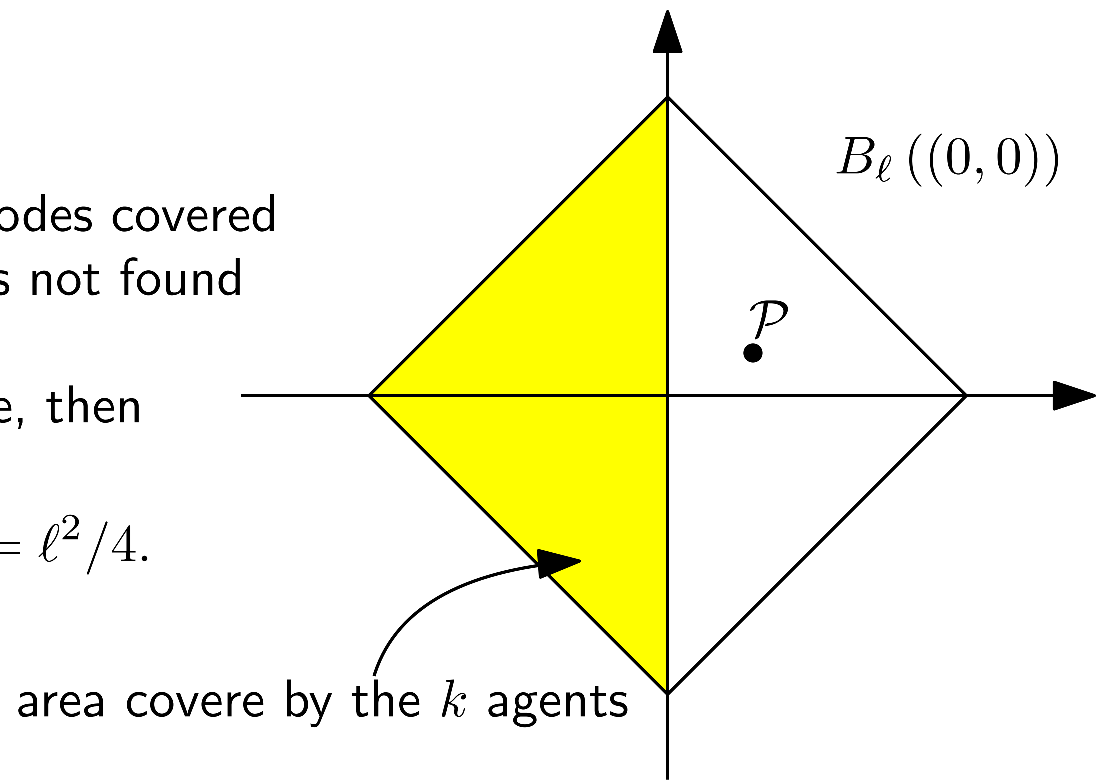
Lemma: locate $\mathcal{P}$ u.a.r. in one node in $B_\ell\left((0,0)\right)$. For any $k \geq 1$, and for any search algorithm $\mathcal{A}$ adopted, the work required to find $\mathcal{P}$ is $\Omega\left(\ell^2\right)$ both with constant probability and in expectation

Proof:

- $|B_\ell\left((0,0)\right)| = \ell^2$
- set $t = \ell^2/(4k)$
- within time $2t$, at most $2kt = \ell^2/2$ nodes covered
- probability at least $1/2$ the treasure is not found within time $2t$
- $H =$ first hitting time for the treasure, then

$$\mathbb{E}\left[\text{work}\right] = \mathbb{E}\left[kH\right] \geq 2kt \cdot \frac{1}{2} = \ell^2/4.$$



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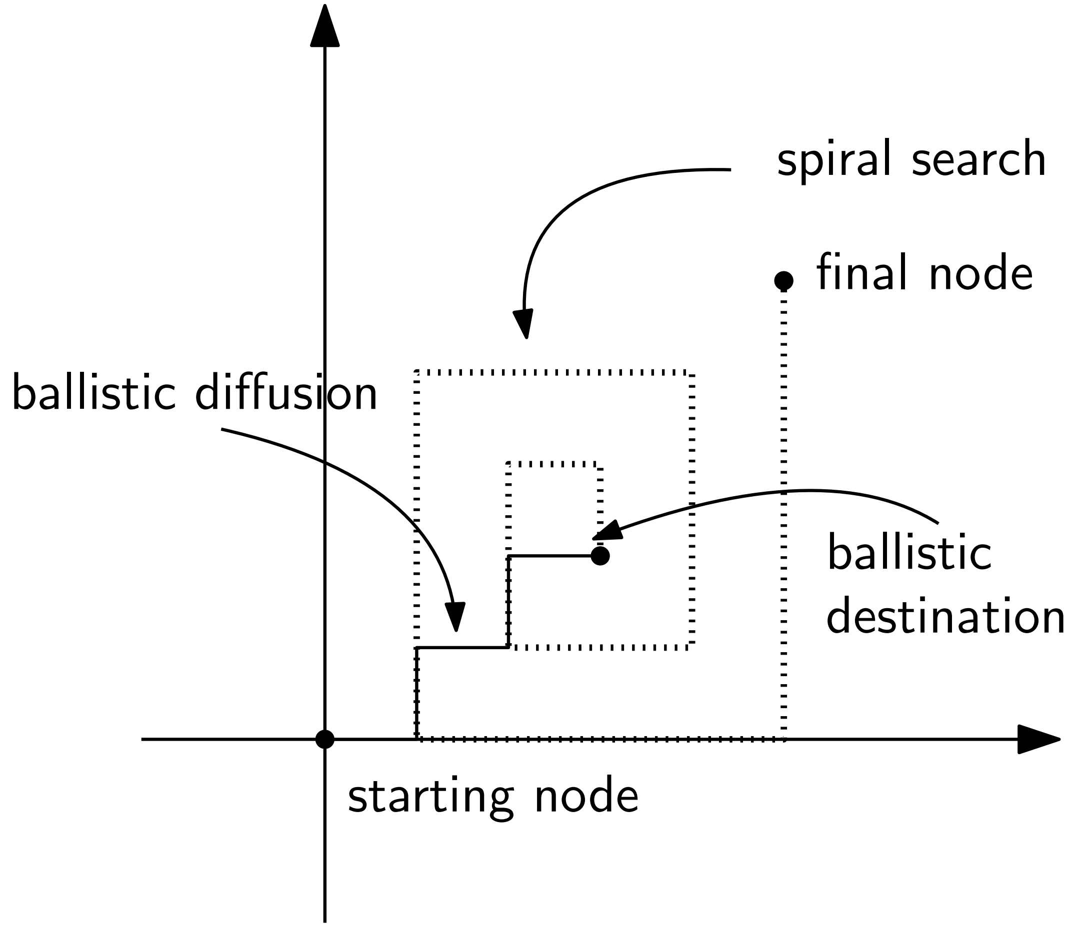

One iteration of the harmonic search algorithm:

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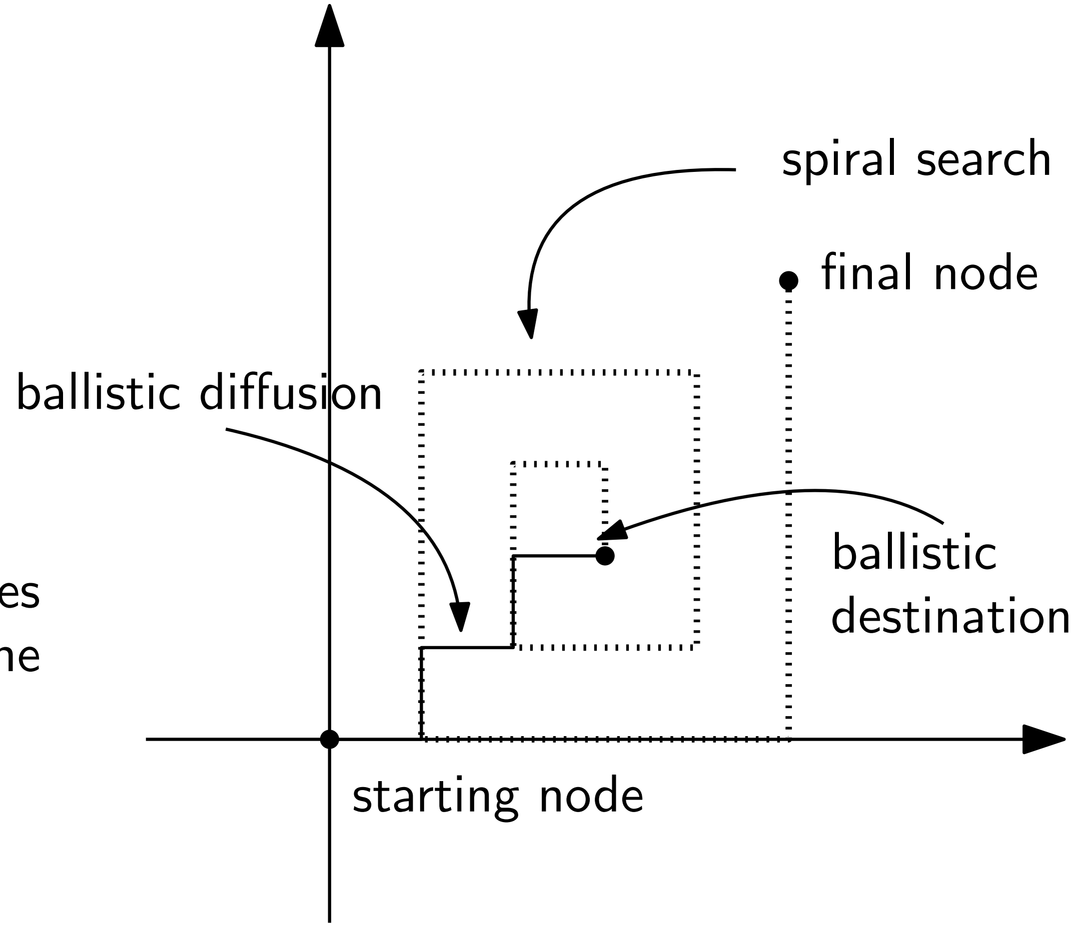
[PODC, '12] proposes a search algorithm which is almost optimal and which is *natural*

Let $\alpha > 1$ be a real constant

The Harmonic Search algorithm: each agent performs the following instructions

a) it samples a Lévy jump-length d with probability c_α / d^α
b) (ballistic diffusion) in d steps, it moves to a destination at distance d from the starting node chosen u.a.r.
c) (normal diffusion) once at the destination, it starts exploring the around area with a spiral search for $d^{\alpha+1}$ steps
d) it returns in the origin and repeats



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Results [PODC, '12] (informal):

- the smaller $\alpha > 1$, the better the performances
- the work made by k walkers is $\mathcal{O}\left(\ell^{\alpha+1}\right) = \mathcal{O}\left(\ell^{2+(\alpha-1)}\right)$ with probability $\geq 1-\epsilon$, for any $\epsilon > 0$ and $k \geq \Theta\left(f(\epsilon)\ell^{\alpha-1}\right)$

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Reminder: the lower bound on the work is $\Omega\left(\ell^2\right)$ with constant probability

Our Work

We give the first definition of Lévy walk in the discrete setting in $\mathbb{Z}^2$, the *Pareto walk*, which is *natural* and *time-homogeneus*

- the jump-length distribution we choose is a common variant of the Pareto distribution, which is a power-law

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Task:

- minimize the work to find the treasure
- estimate the distribution of the hitting time

Some Preliminaries

Definition: we say that an event E depending on a parameter $n \in \mathbb{N}$ holds *with high probability* (w.h.p. in short) w.r.t. n if $\mathbb{P}(E) \geq 1 - 1/n^{\Theta(1)}$

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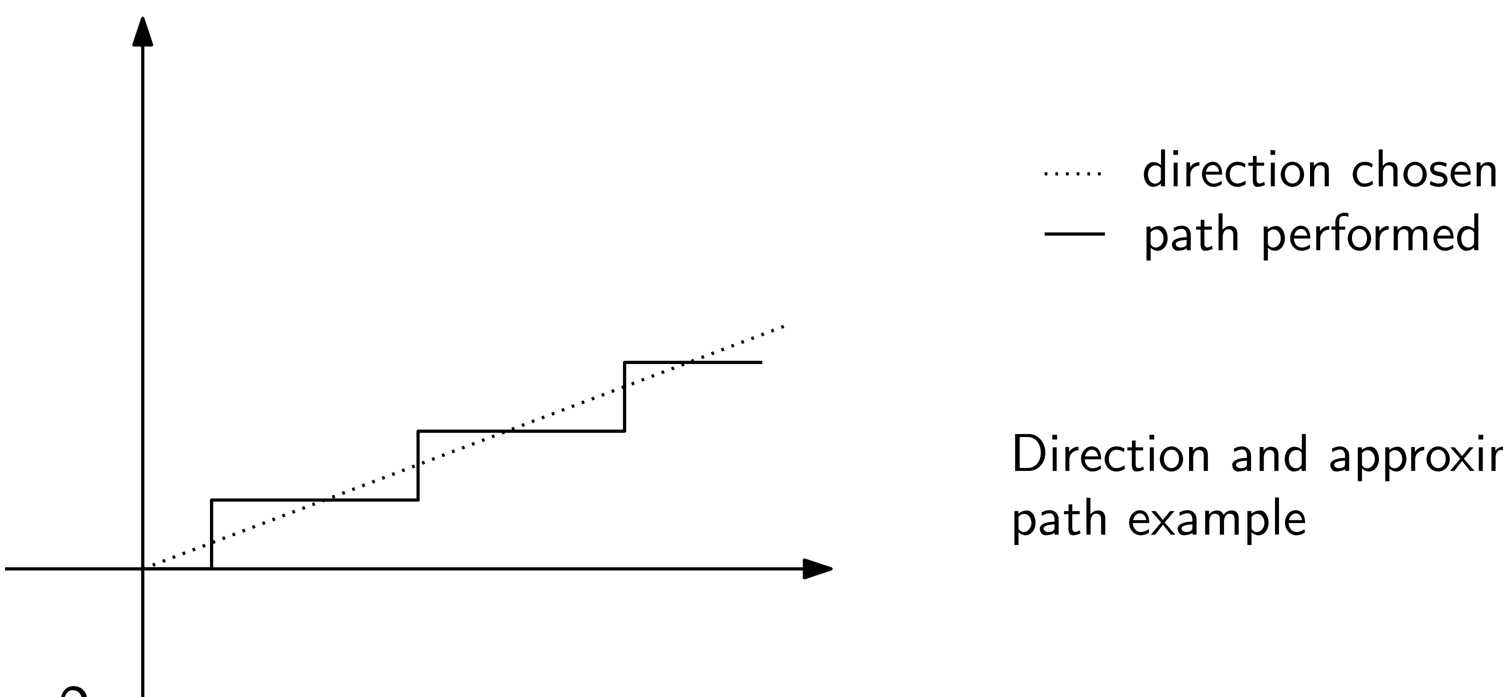
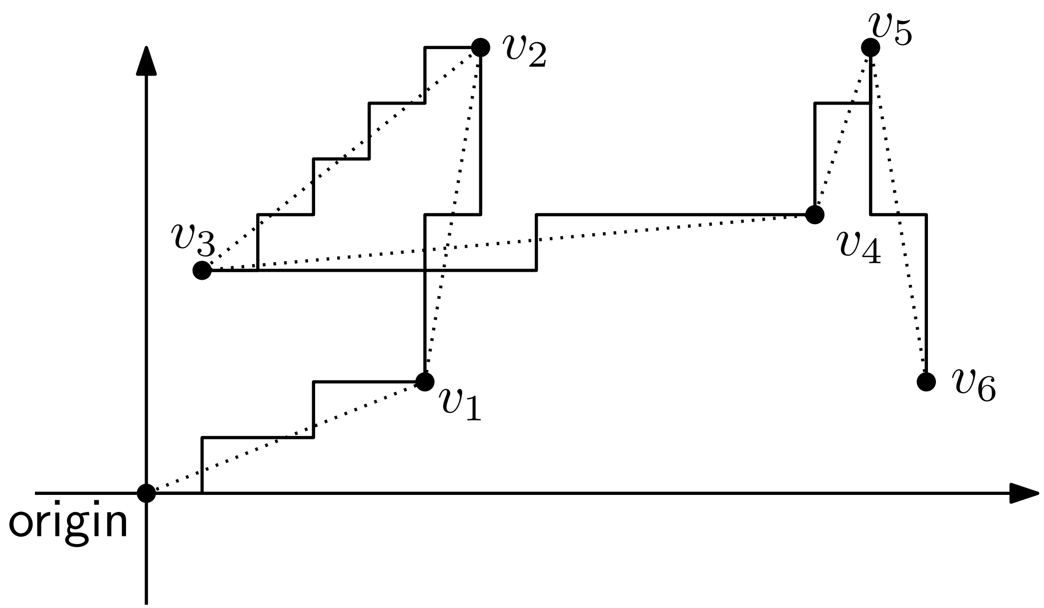
Two notions:
- choice of a direction u.a.r. in $\mathbb{Z}^2$
- selection of a "direction-approximating" path



Direction and approximating path example

Some Preliminaries



direction chosen
path performed

Six iterations of the Pareto walk procedure

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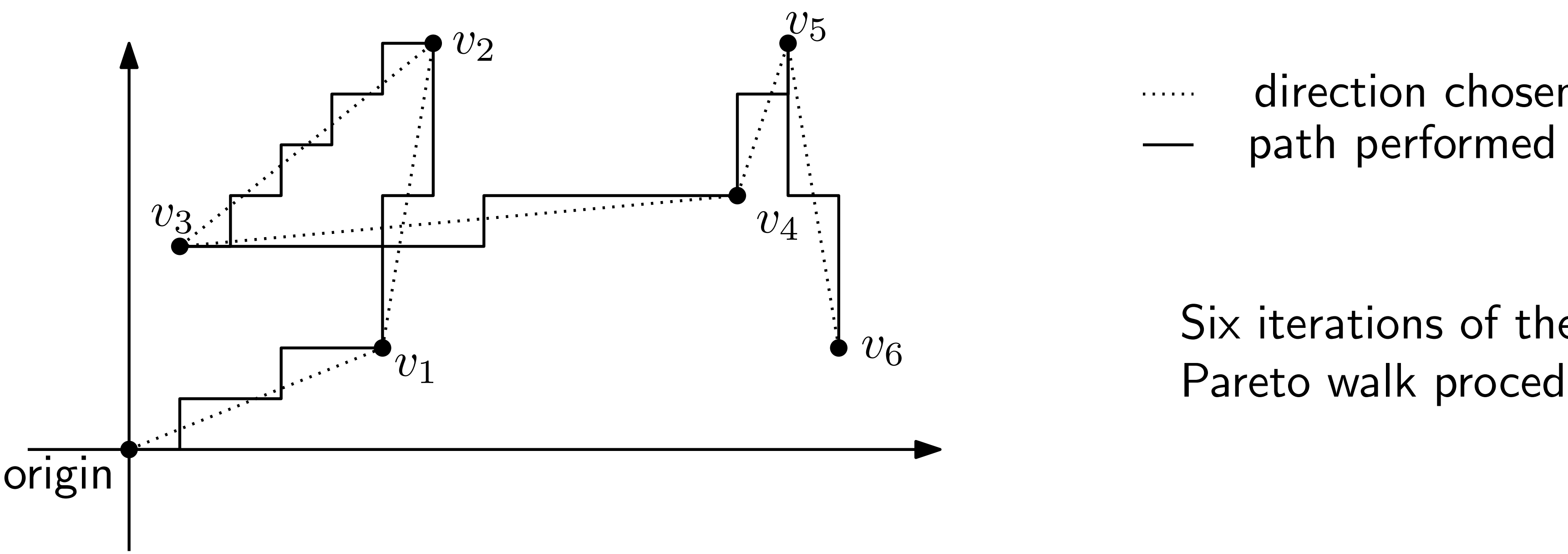

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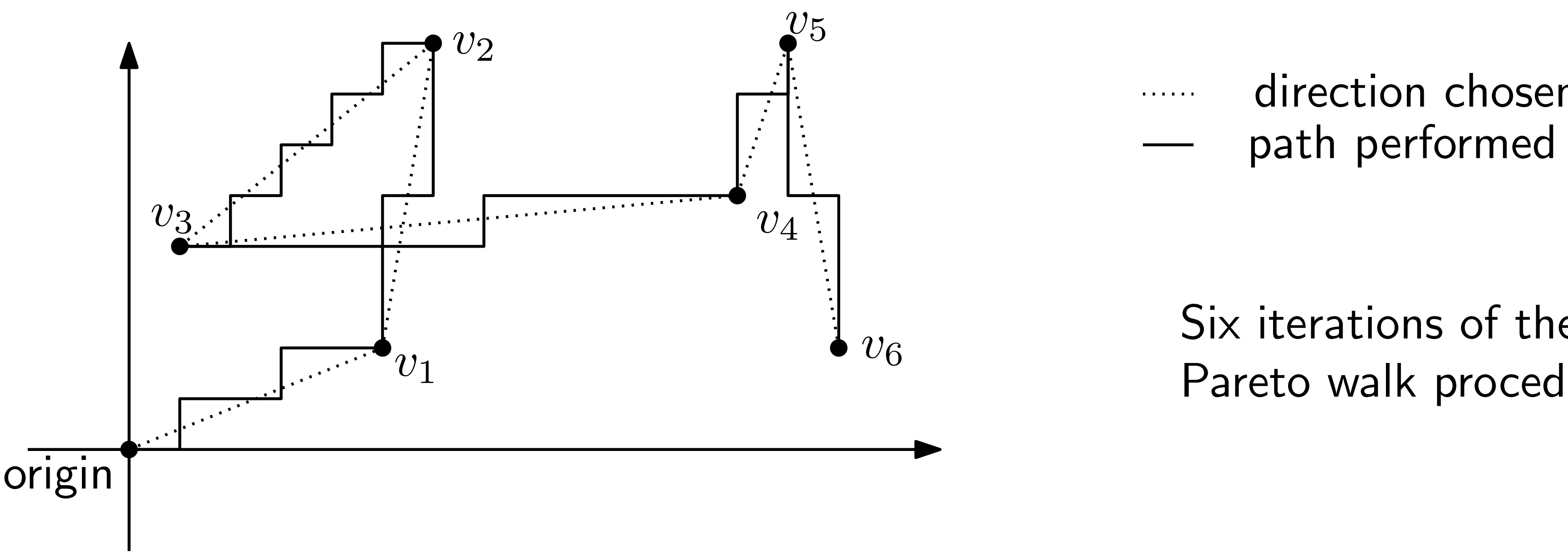

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Remark: the probability distribution in (a) is a known variant of the Pareto distribution

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Result (up to polylogarithms): for each choice of $\alpha > 1$ there is just one polynomial value (in ℓ) for k such that, w.h.p., the work is equal to ℓ^2, thus optimal

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Result (up to polylogarithms): for each choice of $\alpha > 1$ there is just one polynomial value (in ℓ) for k such that, w.h.p., the work is equal to ℓ^2, thus optimal

Exponent	Value of k	Hitting time	Work
$\alpha \geq 3$	$\log^{\Theta(1)} \ell$	$\tilde{\Theta}\left(\ell^2\right)$	$\tilde{\Theta}\left(\ell^2\right)$
$2 < \alpha < 3$	$\tilde{\Theta}\left(\ell^{1-(\alpha-2)}\right)$	$\tilde{\Theta}\left(\ell^{1+(\alpha-2)}\right)$	$\tilde{\Theta}\left(\ell^2\right)$
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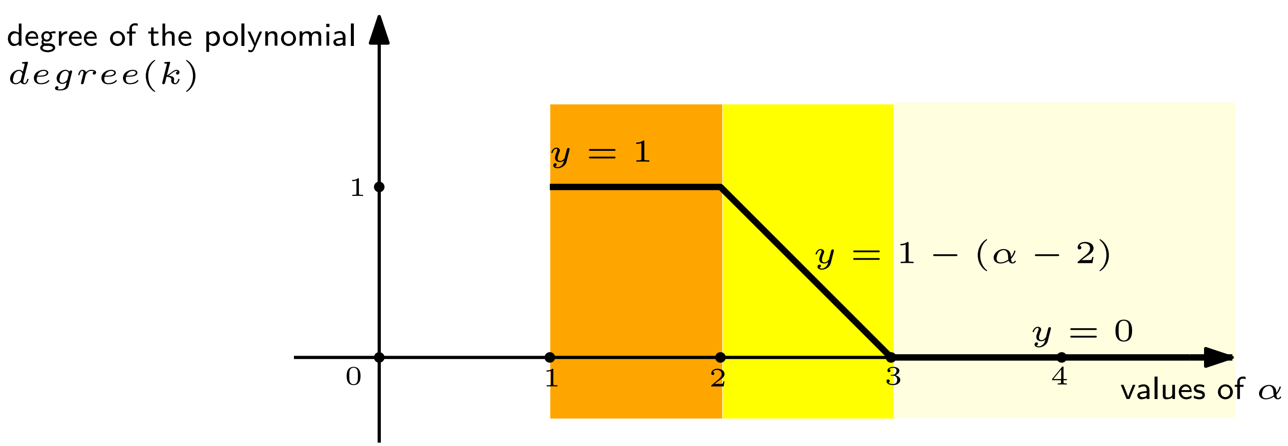

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The results in the previous slide are *almost-tight*:

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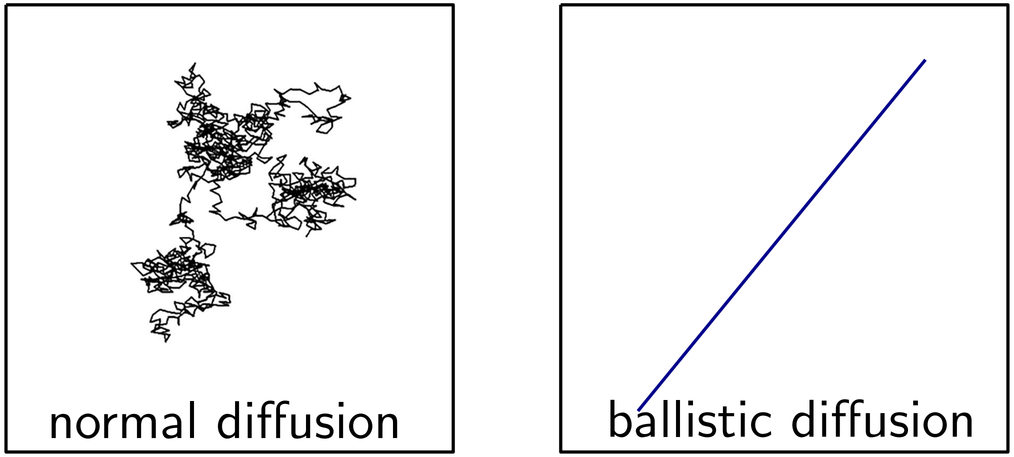
- $\alpha \geq 3 \sim$ simple random walk (normal diffusion)
- $1 < \alpha \leq 2 \sim$ ballistic walk (ballistic diffusion)



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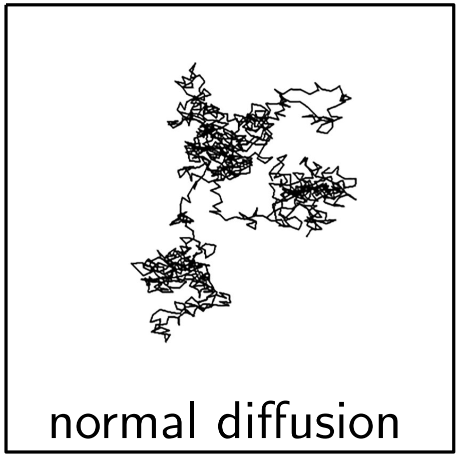

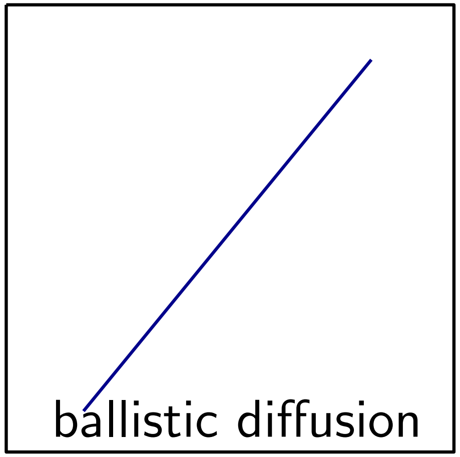

Now, some details on how we prove the upper bound on the hitting time for the super-diffusive regime...

Some Considerations

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Hint: the optimal search strategy depends on the chosen setting (i.e., the environment)

Some Analysis: $2 < \alpha < 3$

Remark: if $2 < \alpha < 3$, the expected jump-length of the Pareto walk is constant

Proof: indeed, the expectation is
$\sum_{d \geq 0} c_\alpha d/(1+d)^\alpha \sim \sum_{d \geq 0} c_\alpha/(1+d)^{\alpha-1} < +\infty$

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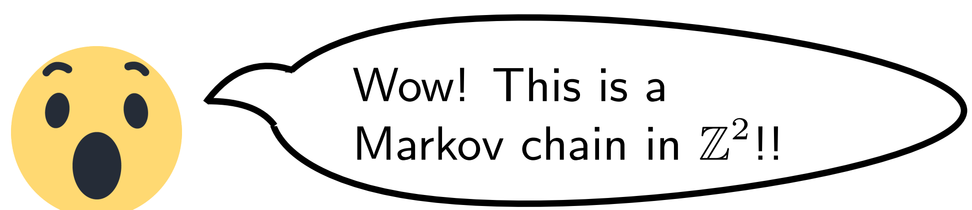
Pareto flight: the Pareto flight is a Pareto walk in which the agent takes just one step/time unit to reach a jump-destination, without visiting intermidiate nodes



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Coupling Result

Coupling result: if one single Pareto flight finds the treasure within t steps with probability $p(t)$ conditional on the event that all the performed jump lengths are less than $(t \log t)^{\frac{1}{\alpha-1}}$, then one Pareto walk finds the treasure within $\Theta(t)$ steps with probability at least $[p(t) - \exp\left(-t^{\Theta(1)}\right)]/2$, without any conditional event

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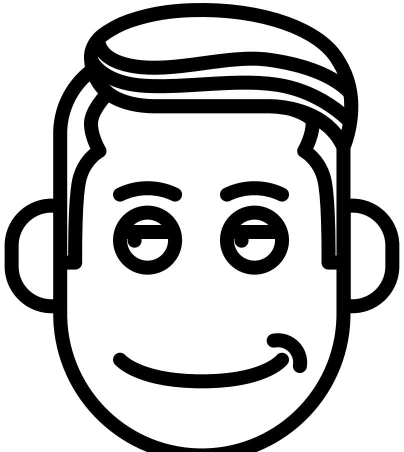

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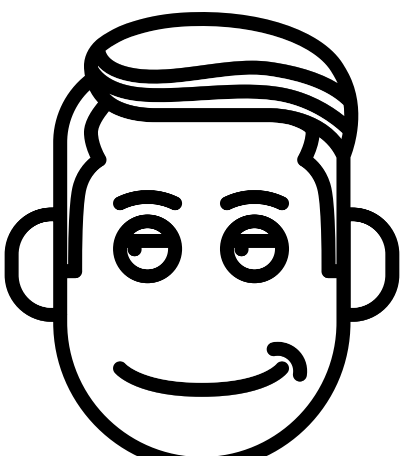
We look at one single Pareto flight to determine $p(t)$ in order to use the coupling result...



Trying to get $p(t)$...

Let
- $\mathcal{P}$ be the treasure
- $|\mathcal{P}|_1 = \ell$ its Manhattan distance from the origin
- $Z_{\mathcal{P}}(t) =$ random variable of number of visits in $\mathcal{P}$ until time t
- $\mathcal{E}_t =$ the event first t jumps have length $\leq (t \log t)^{\frac{1}{\alpha-1}}$
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For the Pareto flight, it holds

Lemma: $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}(t) > 0 \mid \mathcal{E}_t\right) \geq \mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right] / a_t$

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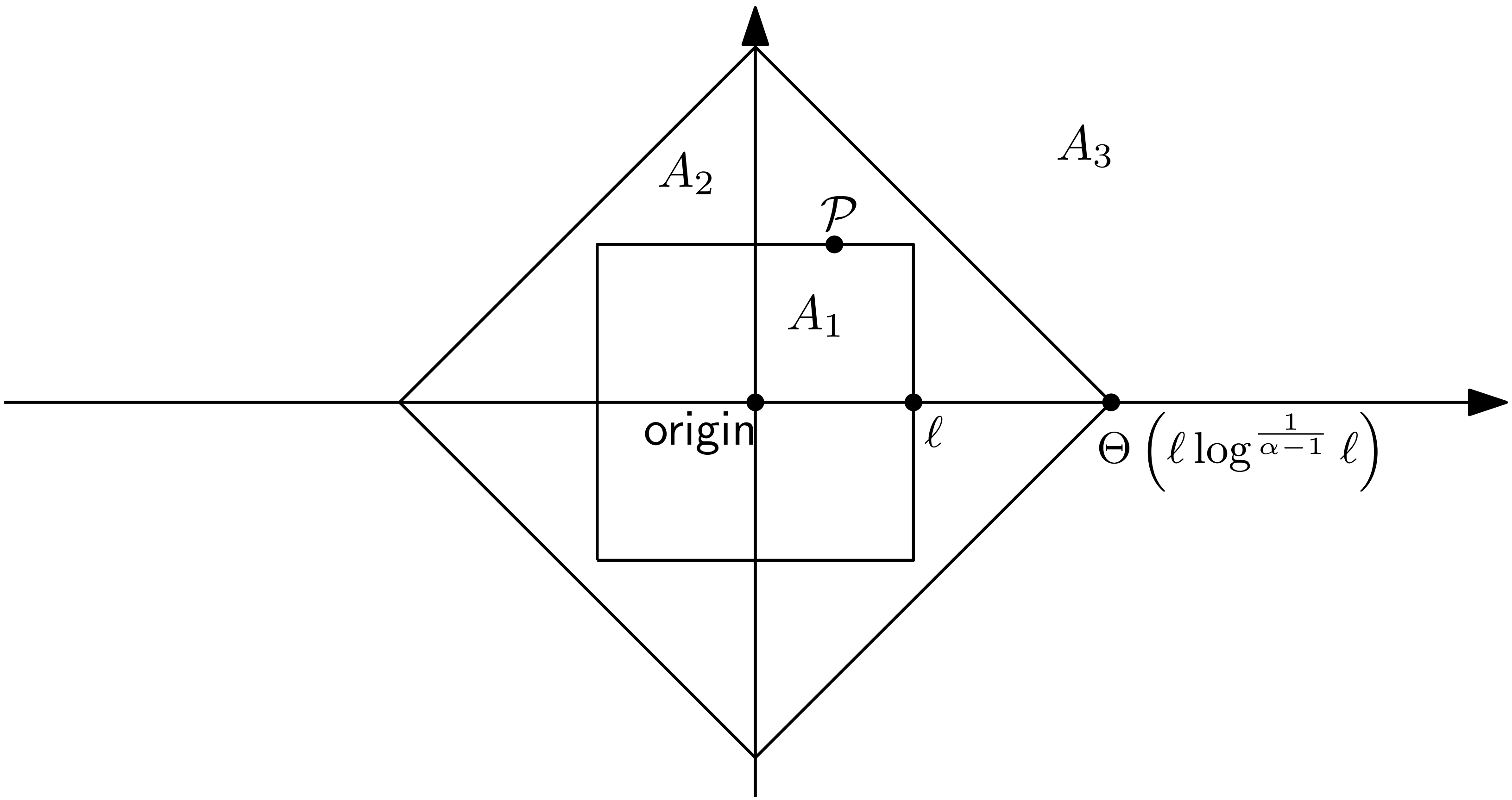
We now look for $\mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right]$ and a_t...

Trying to get $\mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right]$...

We partition $\mathbb{Z}^2$ in three areas in the following way



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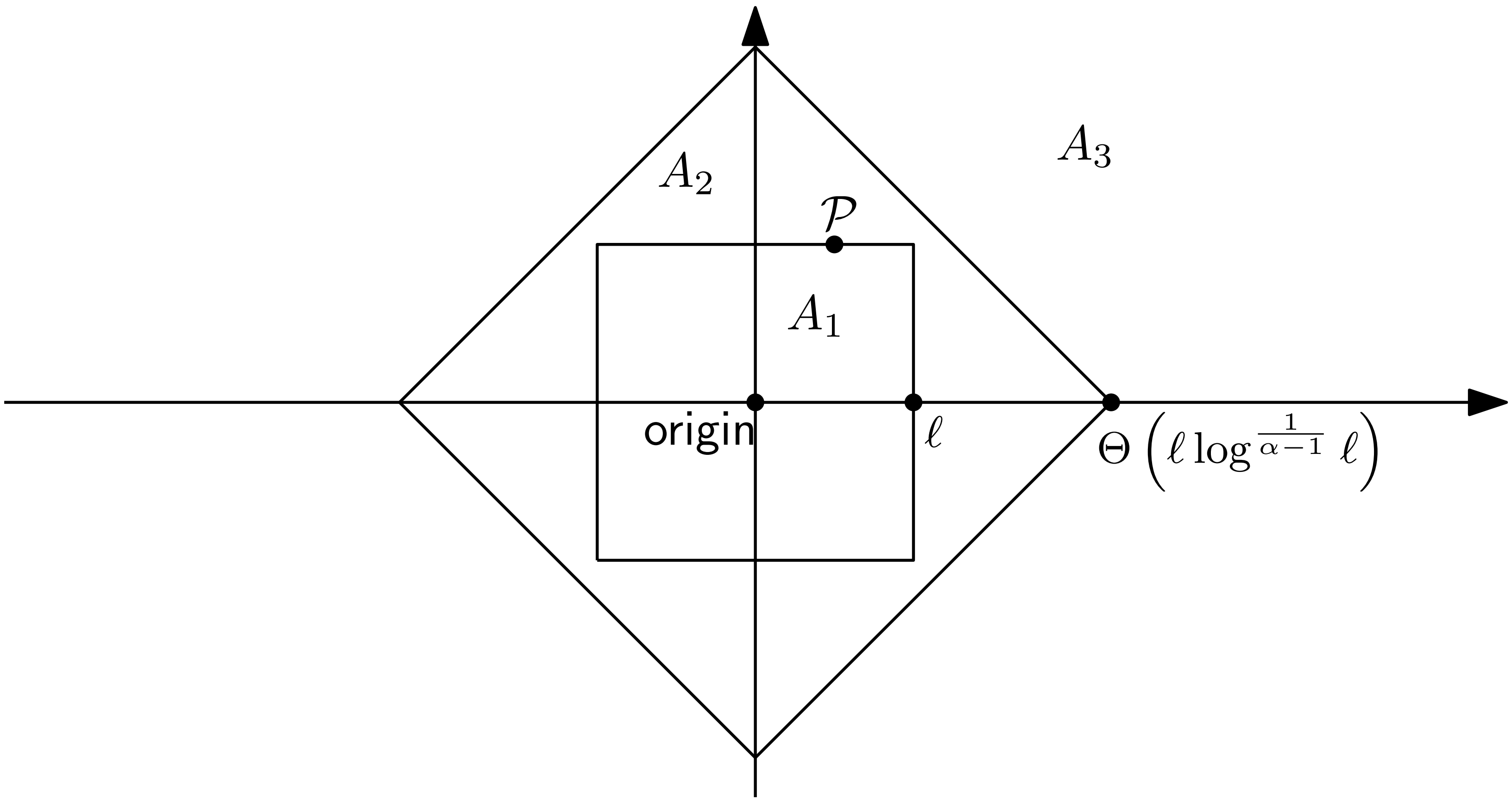

- $A_1 = Q(\ell) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq \ell\}$
- $A_2 = B_{\Theta\left(\ell \log^{\frac{1}{\alpha-1}} \ell\right)}((0,0)) \setminus A_1$
- $A_3 = \mathbb{Z}^2 \setminus (A_1 \cup A_2)$

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(b) $\mathbb{E}\left[Z_{A_1}(t) \mid \mathcal{E}_t\right] \leq \frac{3}{4}t$

(c) $\mathbb{E}\left[Z_{A_2}(t) \mid \mathcal{E}_t\right] \leq \mathbb{E}\left[Z_{\mathcal{P}}(t) \mid \mathcal{E}_t\right] \cdot \Theta\left(\ell^2 \log^{\frac{2}{\alpha-1}} \ell\right)$

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We combine (a) with (b), (c), and (d) to get $\mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right] = \tilde{\Omega}\left(1/\ell^{1-(\alpha-2)}\right)$

Back to $p(t)$...

Reminder: $p(t) = \mathbb{P}\left(Z_{\mathcal{P}}\left(t\right) > 0 \mid \mathcal{E}_t\right) \geq \mathbb{E}\left[Z_{\mathcal{P}}\left(t\right) \mid \mathcal{E}_t\right]/a_t$

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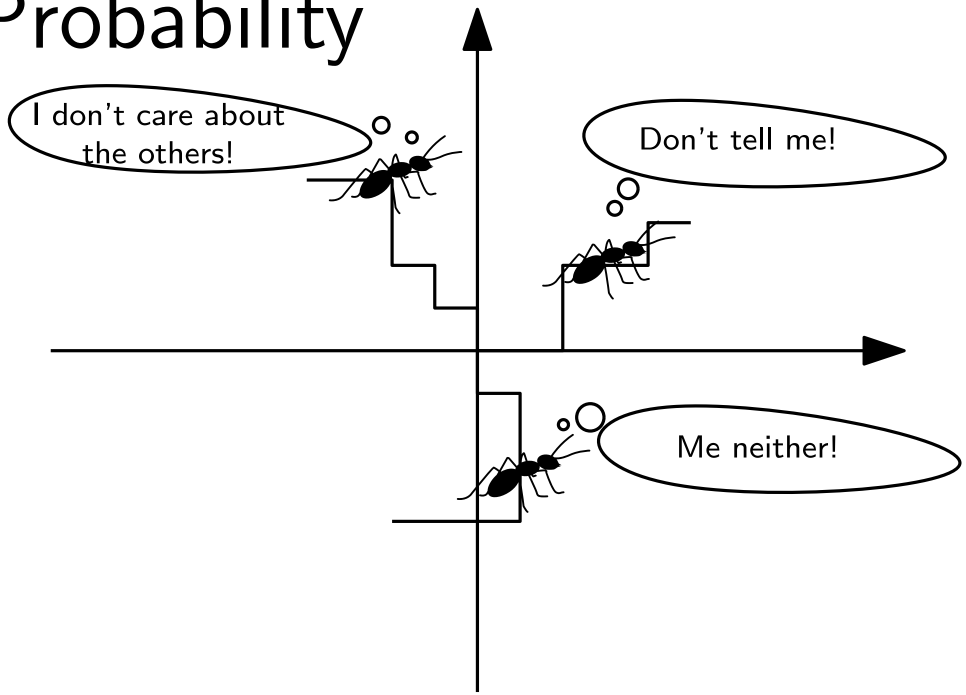
Question: how to get the high probability?

The High Probability

We exploit independence!



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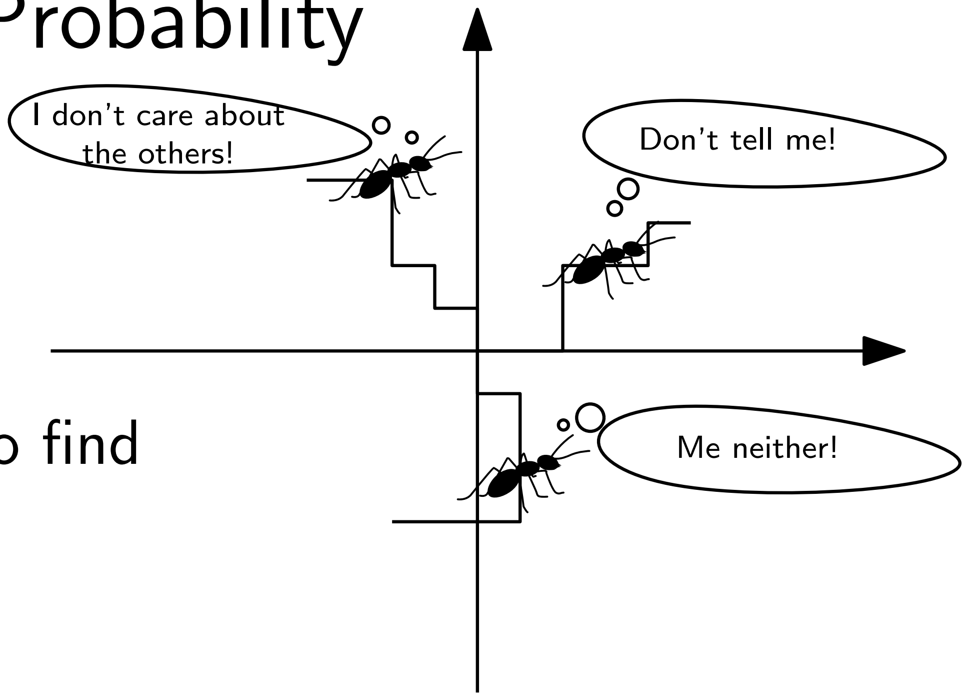

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k walkers don't find the treasure within time t with probability $[1-p(t)]^k$

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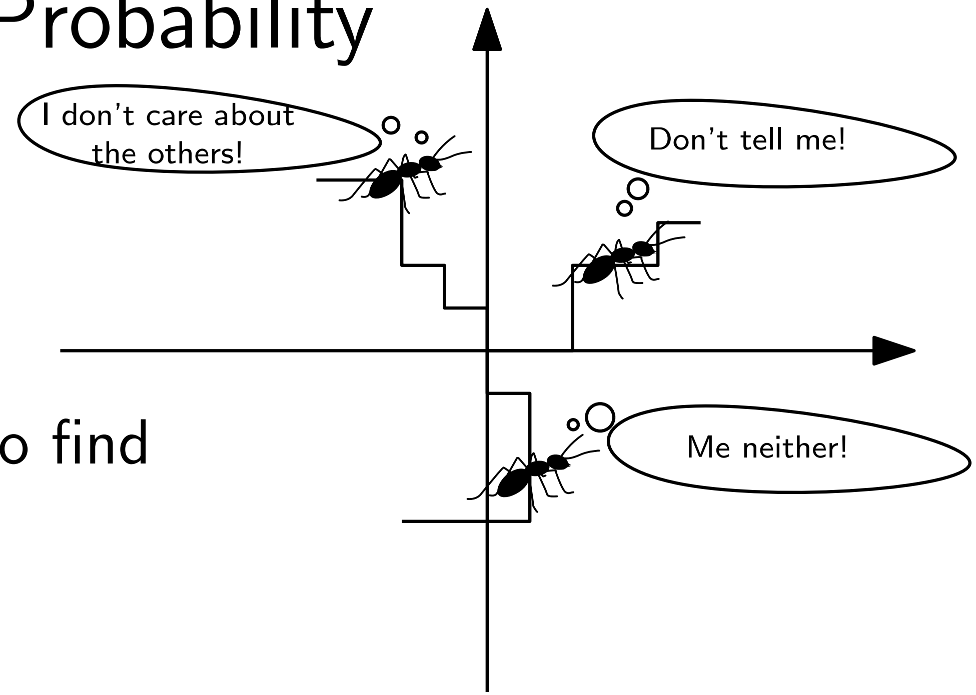

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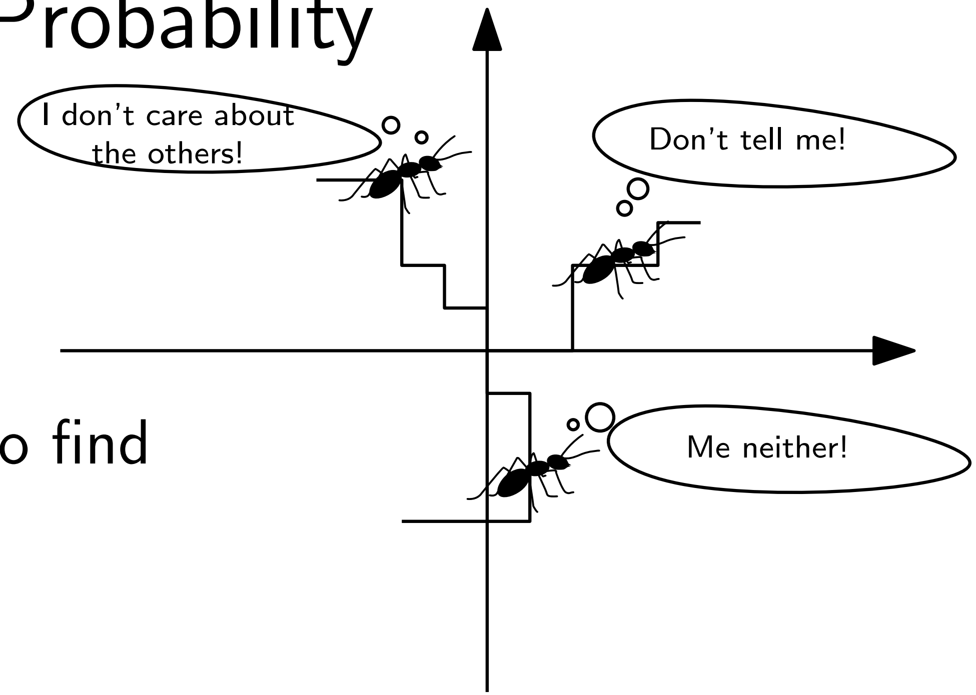

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The probability that at least one walker finds the treasure within time t is

$$1 - [1-p(t)]^{\frac{\log \ell}{p(t)}} \sim 1 - e^{\log \ell} = 1 - \frac{1}{\ell}$$

We thus need $\log \ell / p(t) = \tilde{\mathcal{O}}\left(\ell^{1-(\alpha-2)}\right)$ walkers to find the treasure within time $t = \Theta\left(\ell^{1+(\alpha-2)}\right)$, making a work equal to $\tilde{\mathcal{O}}\left(\ell^2\right)$, w.h.p.

THANK YOU FOR YOUR ATTENTION

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