On Structured Pruning in the Strong Lottery Ticket Hypothesis via Multidimensional Random Subset Sum

Francesco d'Amore

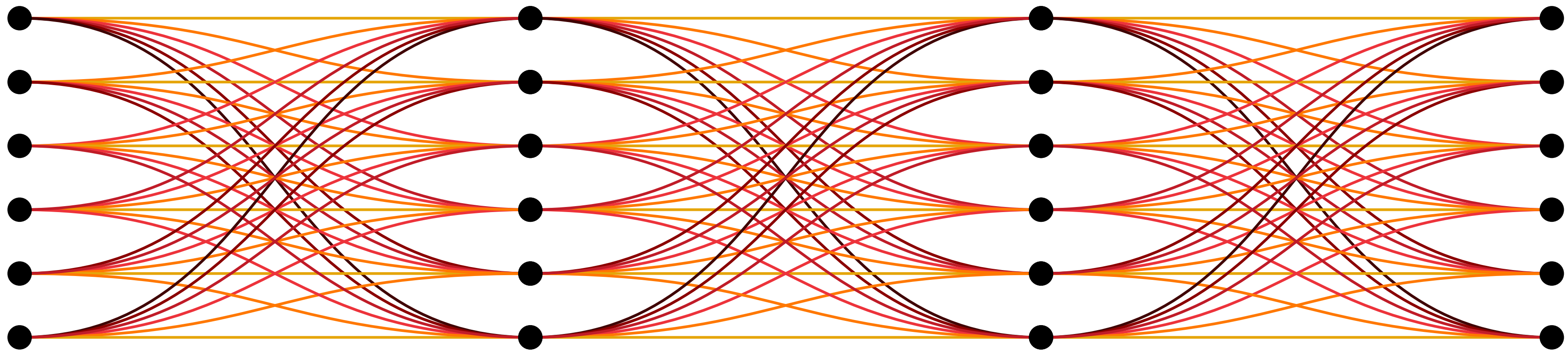
Based on joint work with A. da Cunha and E. Natale [NeurIPS 2023]

Mathematics for AI and ML
19 January 2024

Artificial neural networks are large

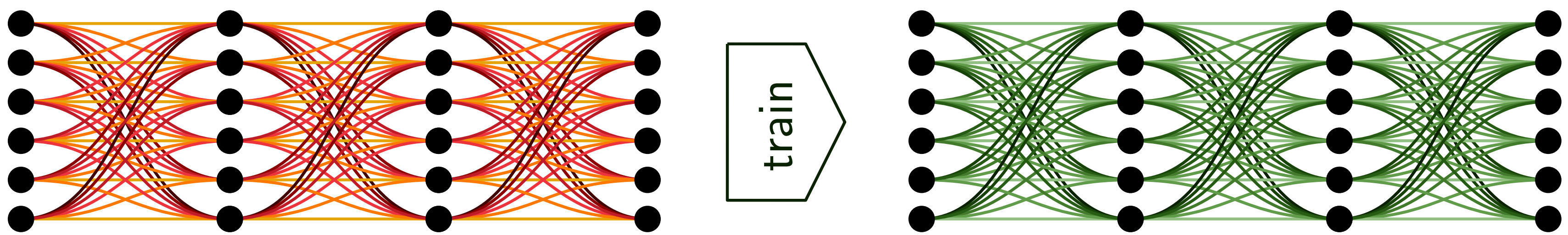
Usually ranging from millions to hundreds of billions parameters

- RESNET-50: > 20 millions parameters [He et al. 2015]
- BERT: > 100 millions parameters [Devlin et al. 2018]
- GPT-3: > 100 billions parameters [Brown et al. 2020]



Training is expensive

- Resource intensive
- Good results
- Resulting network still large



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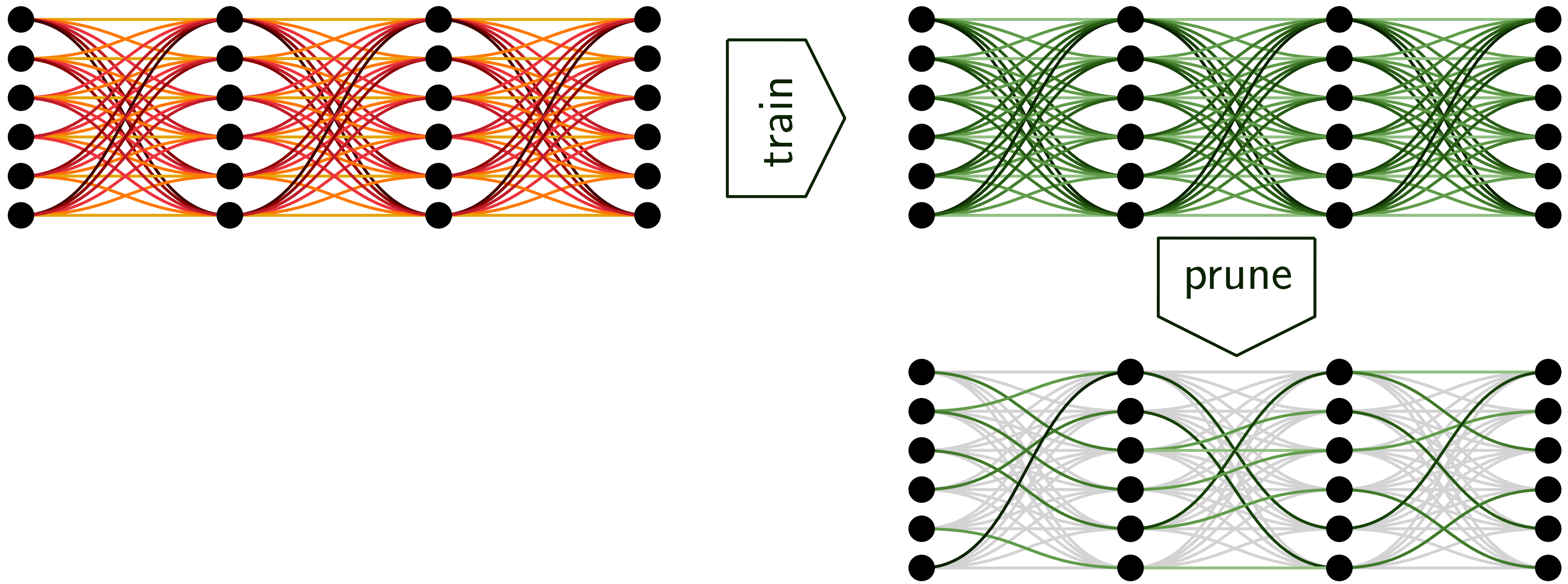

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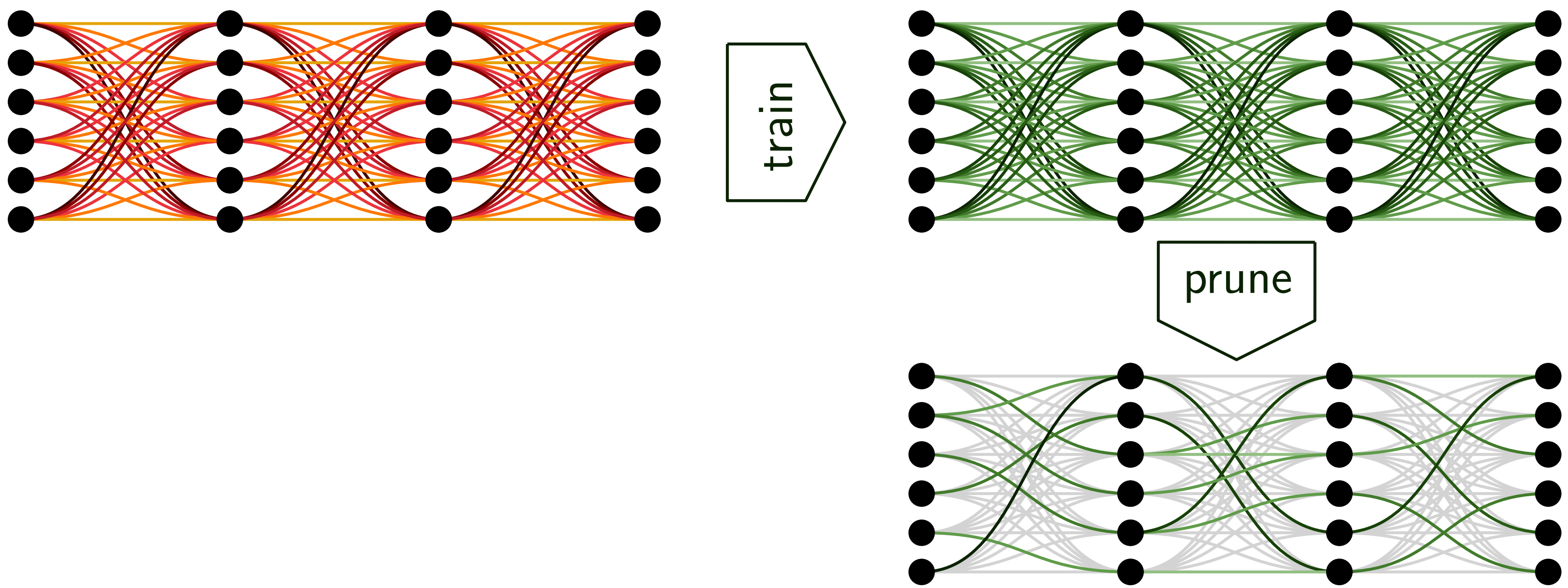

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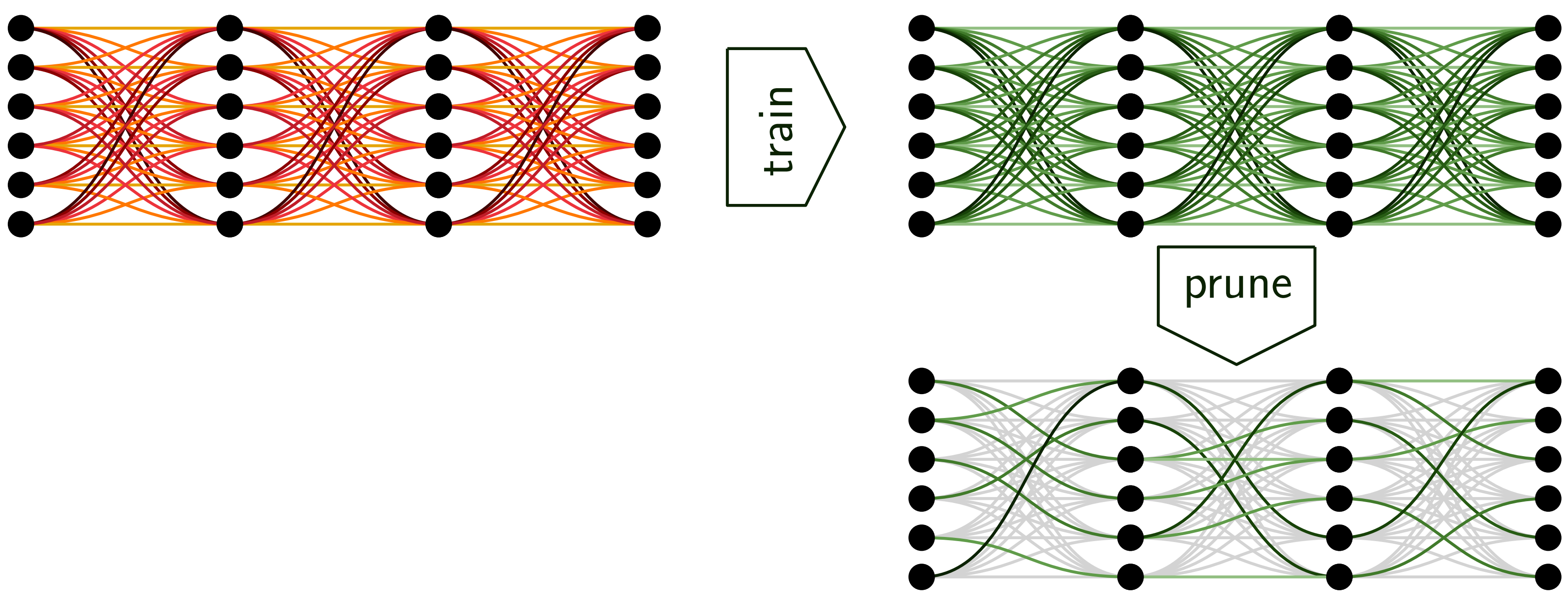
- Pruning $\sim 99\%$ of the edges can perform well [Hoefler et al. 2021]



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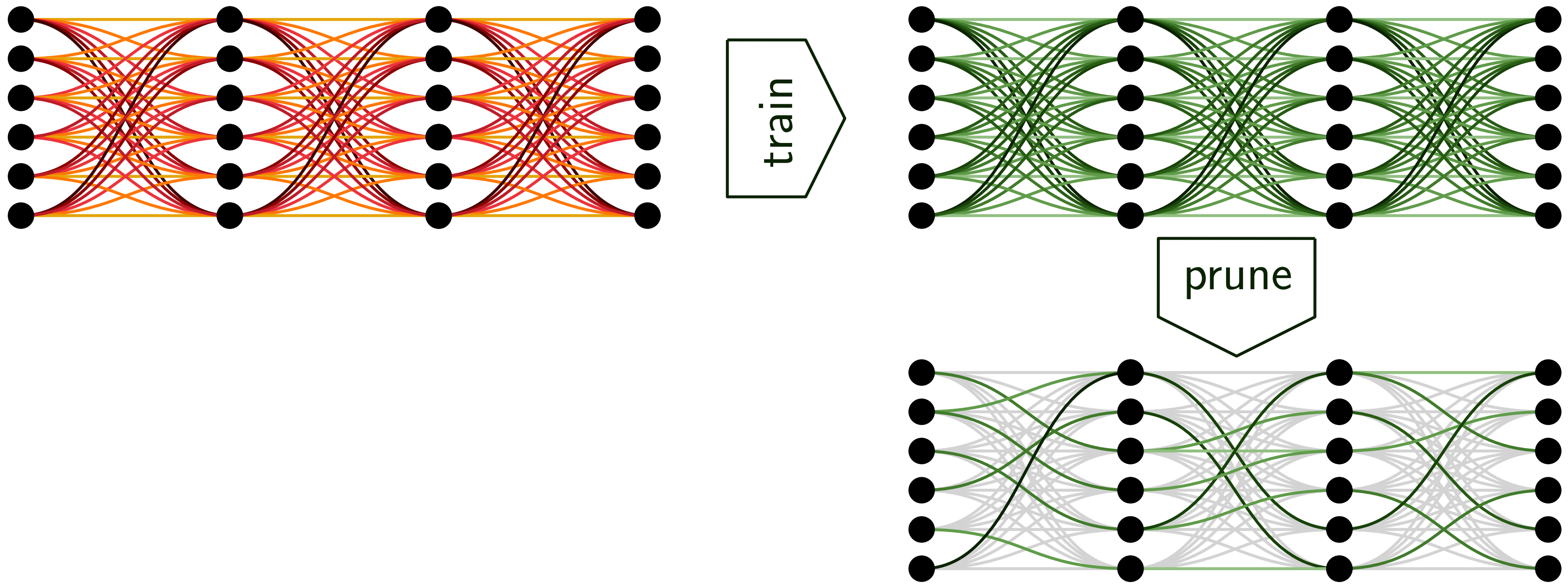

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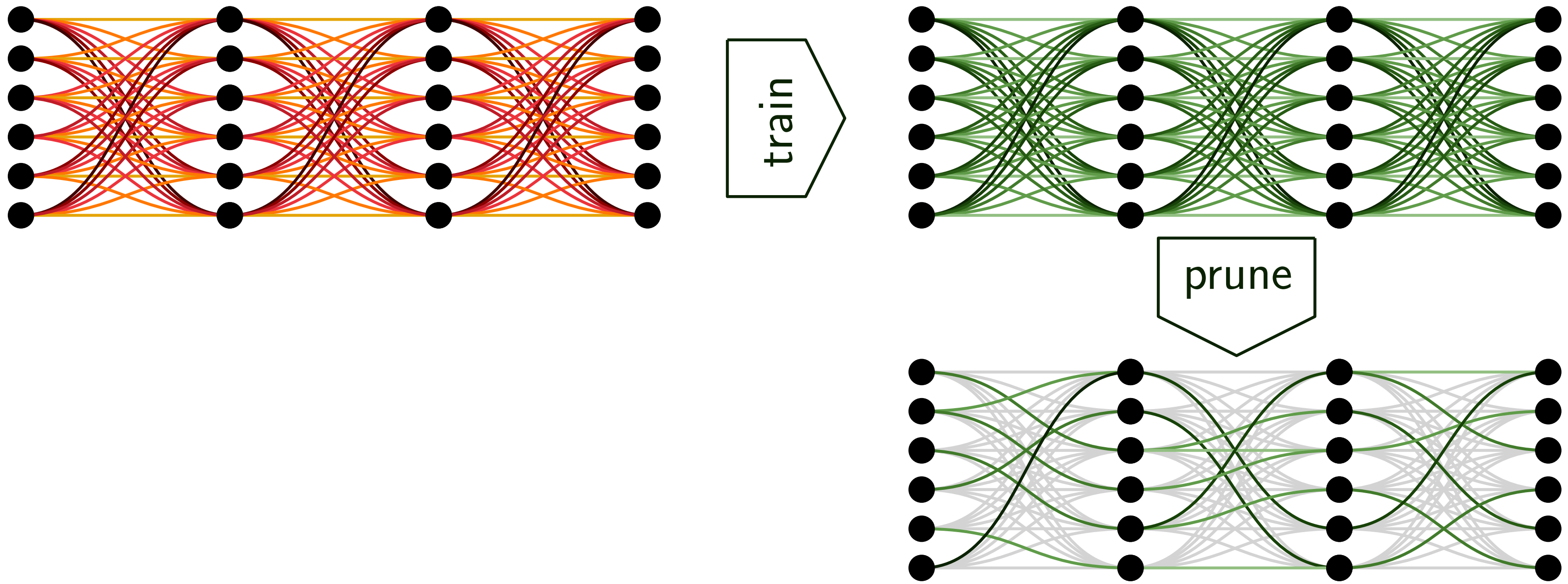

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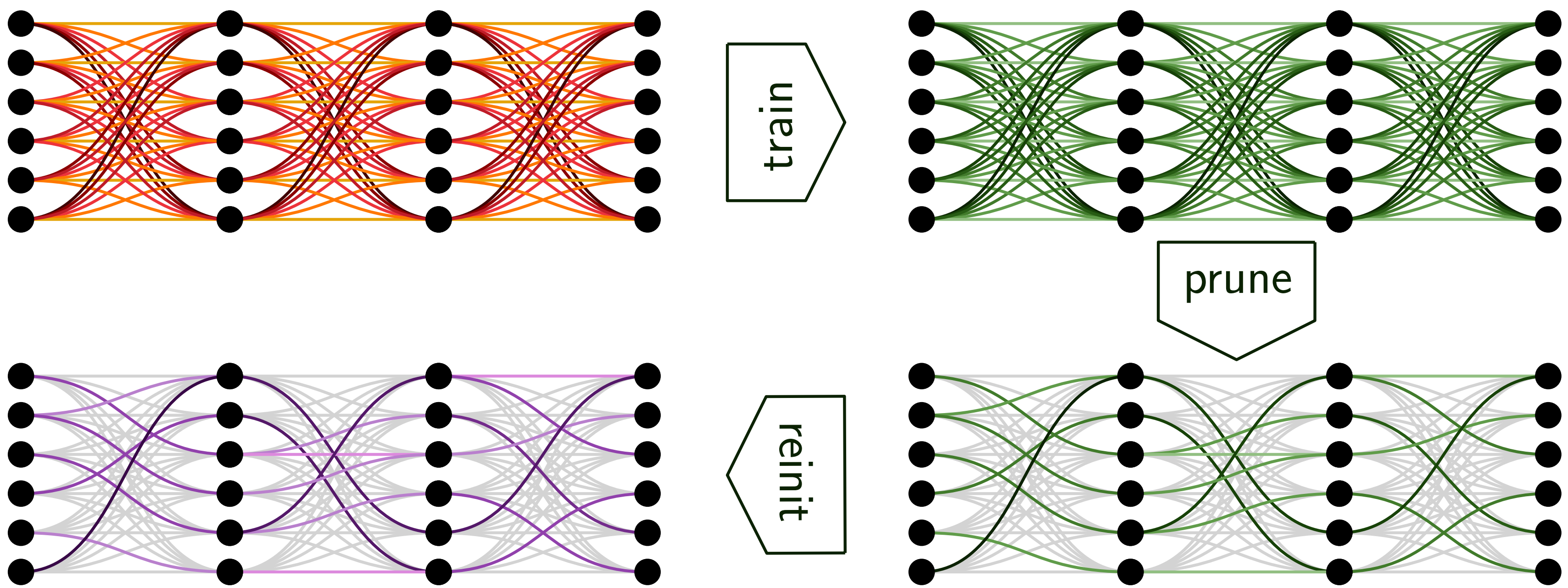

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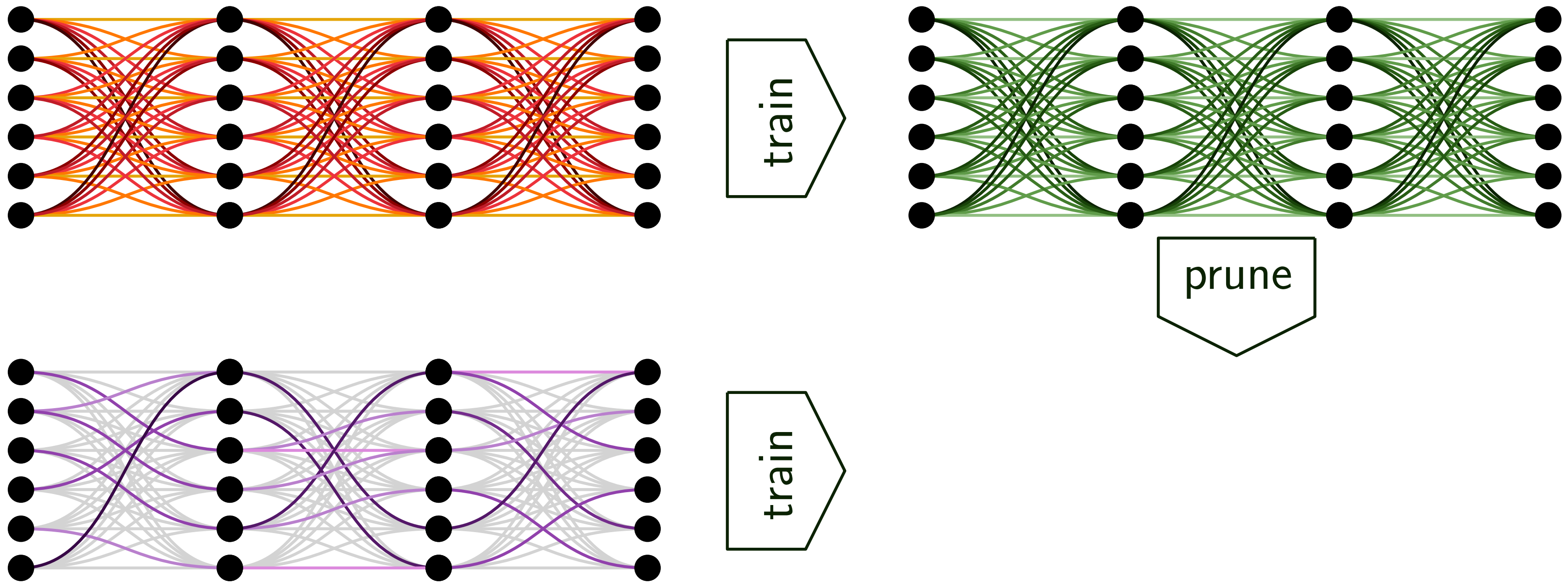

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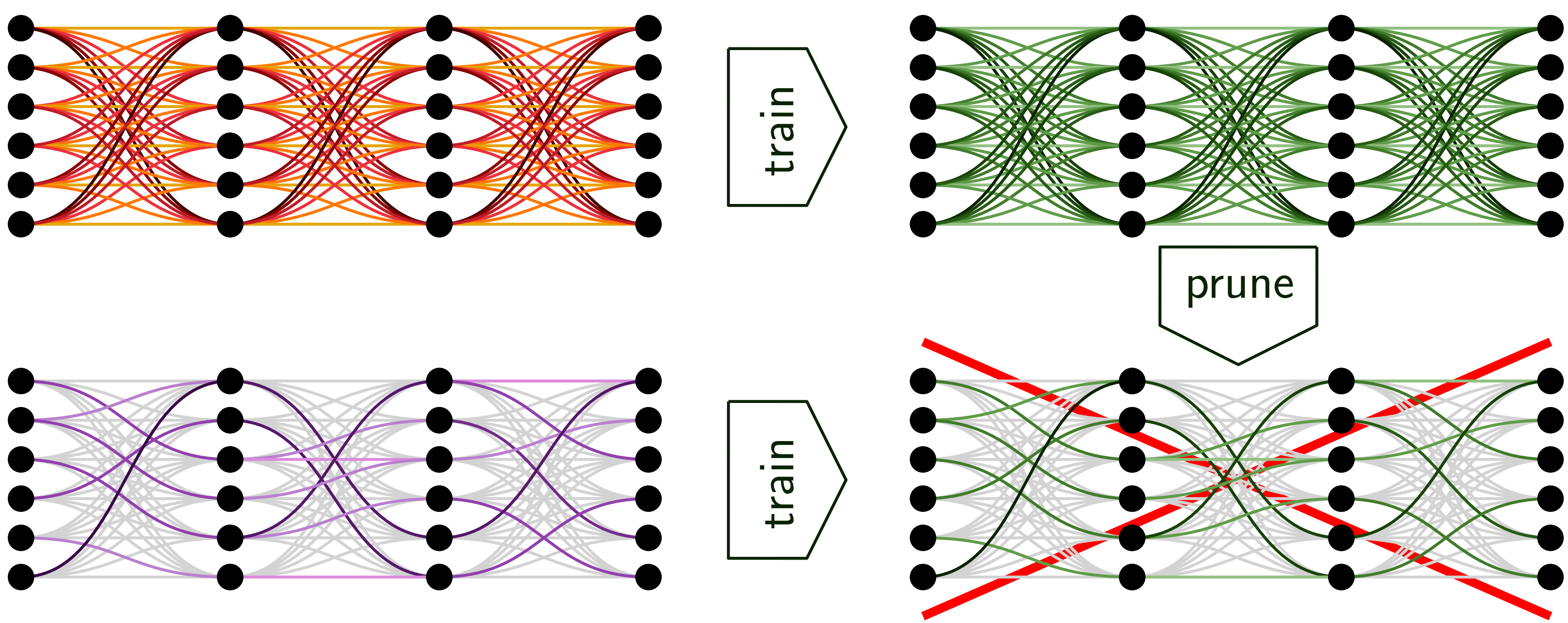

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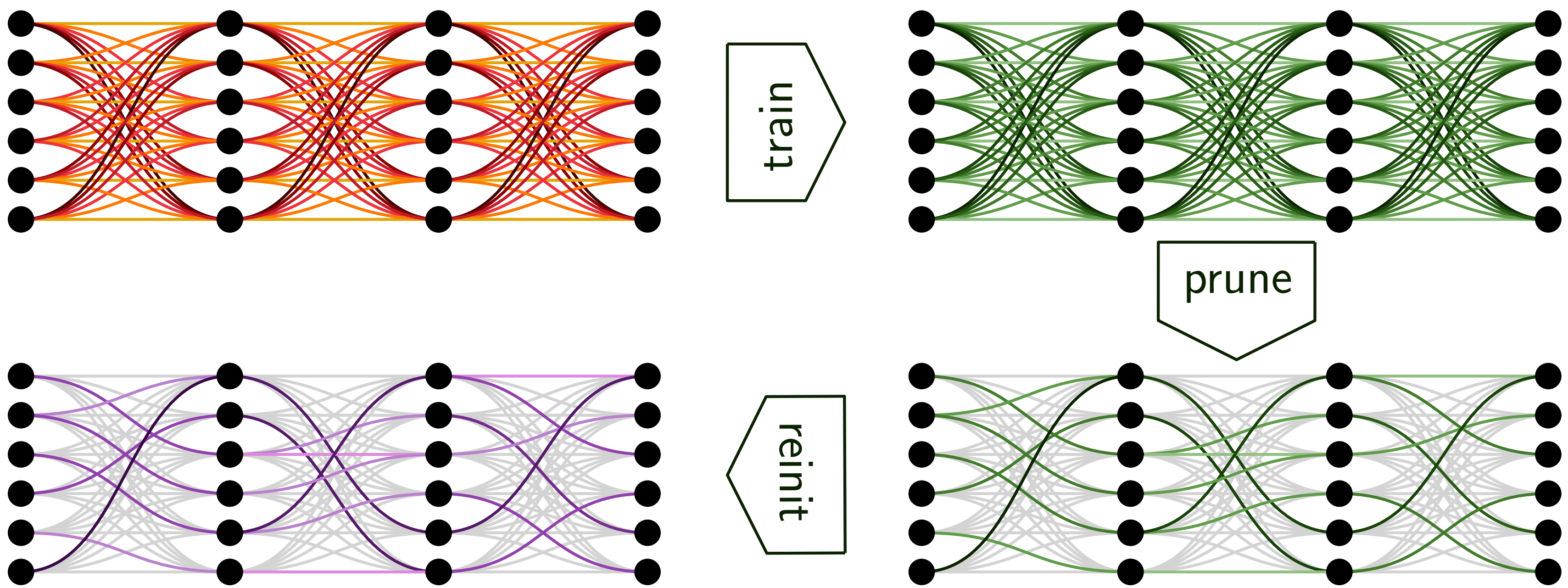

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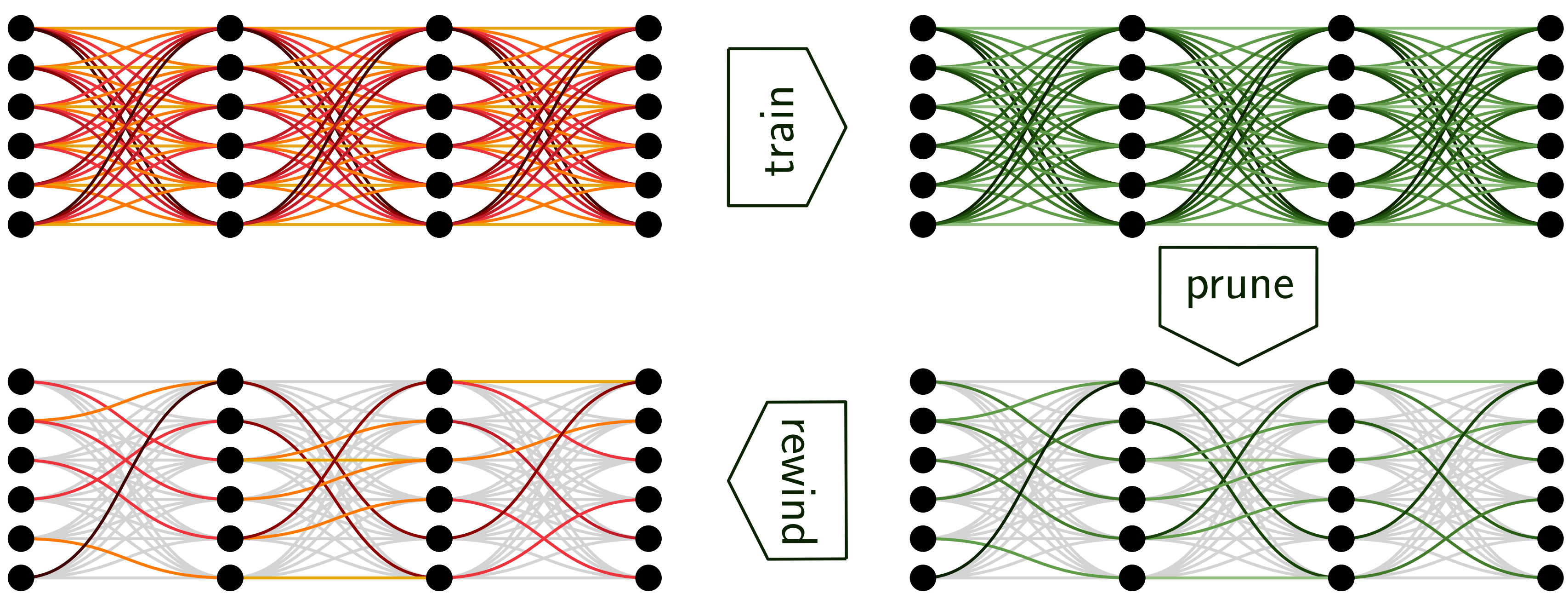

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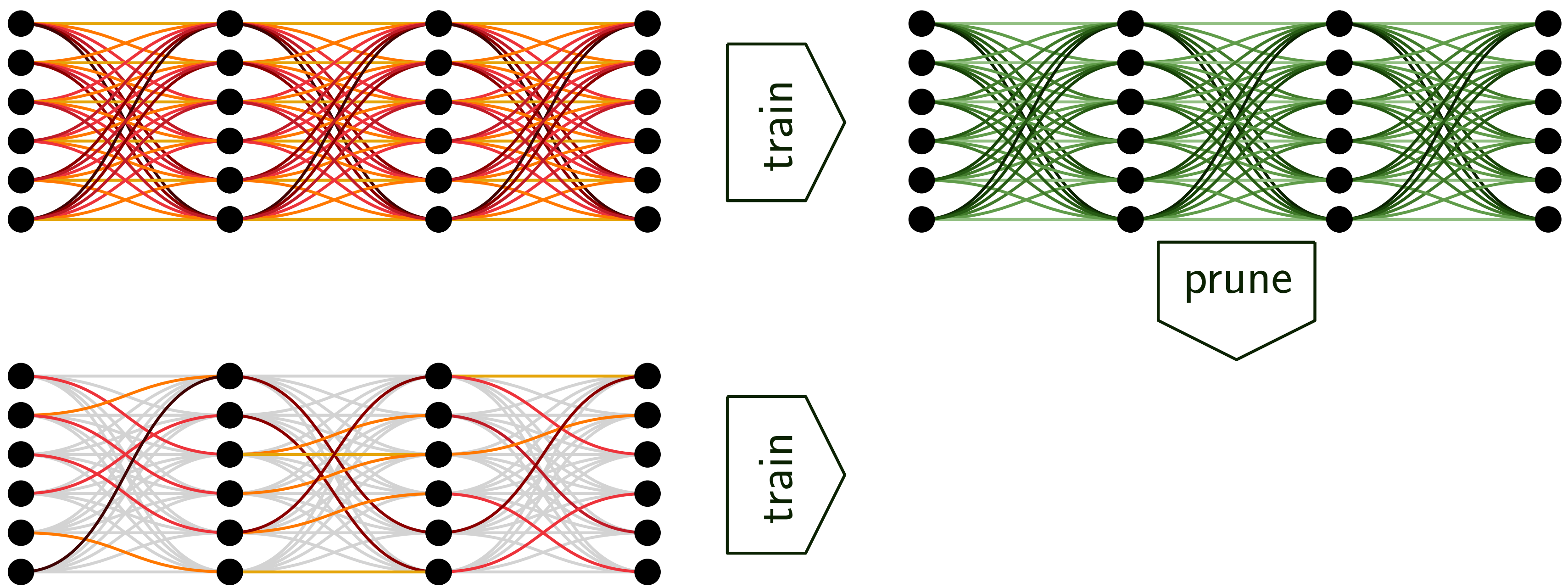

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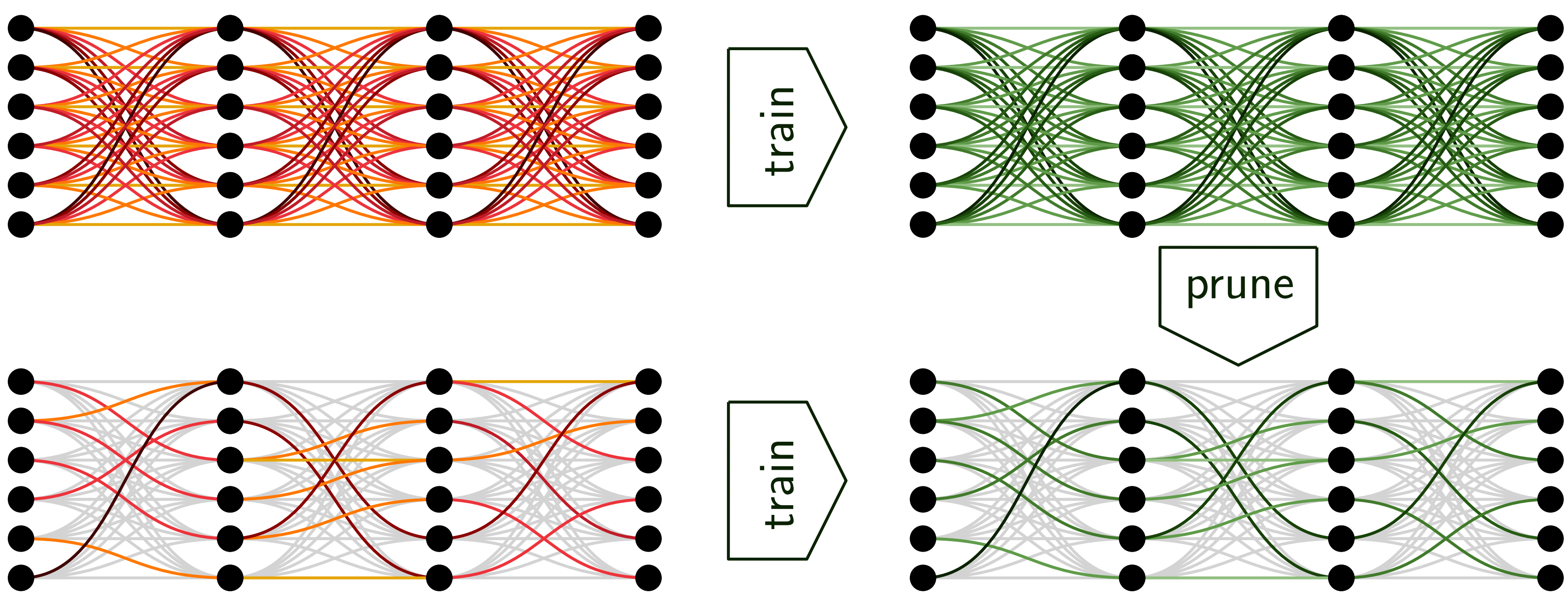

Lottery tickets

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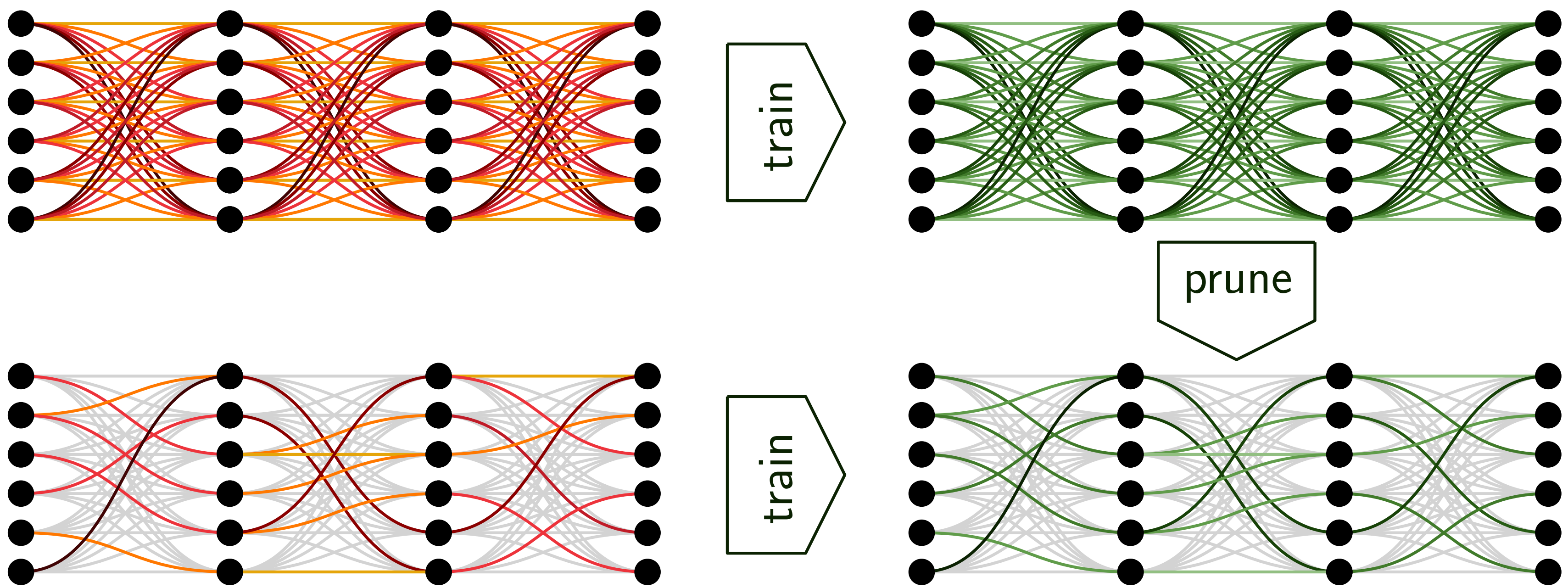

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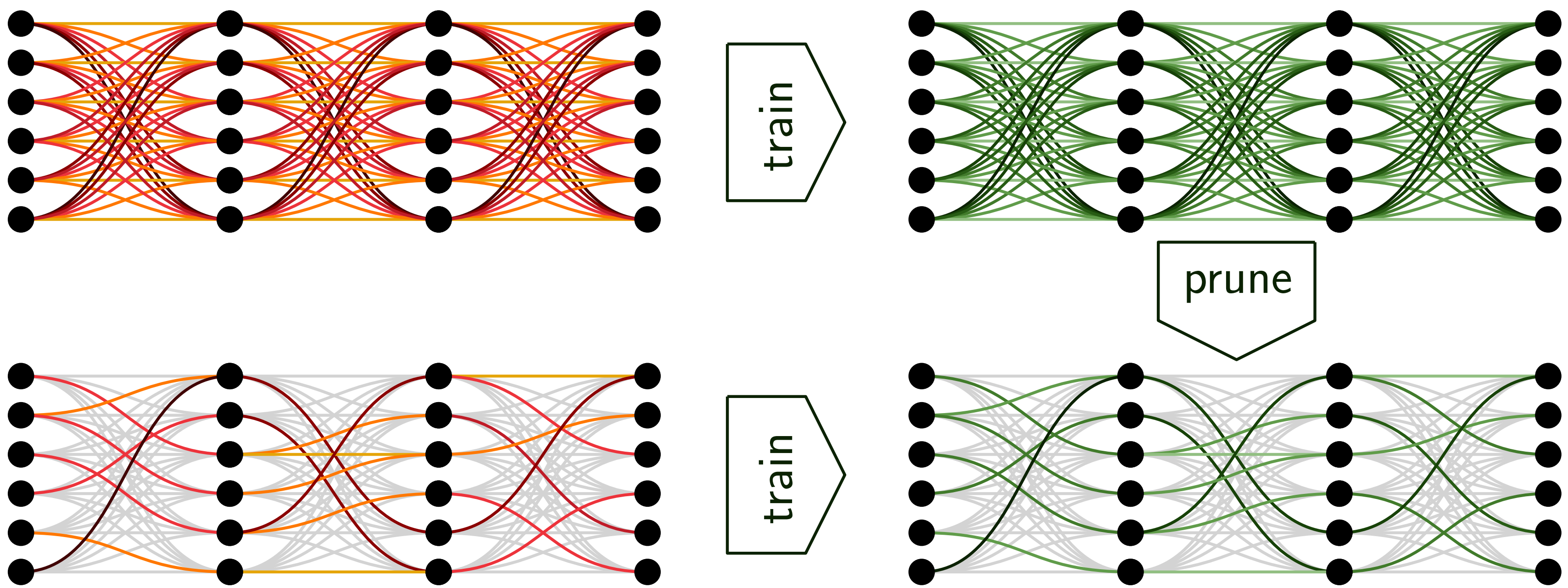
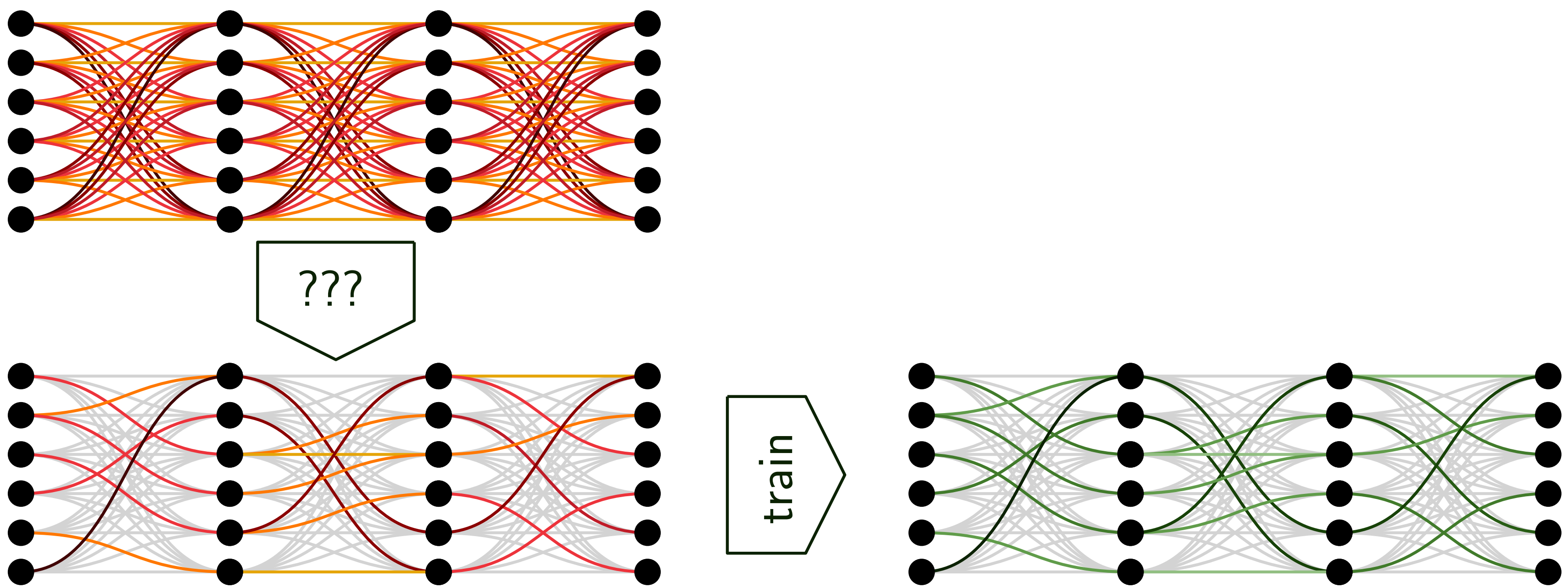


Lottery tickets

- What does it mean?
- This is not a good algorithm
- Existential result
    - Training is about topology + initialization



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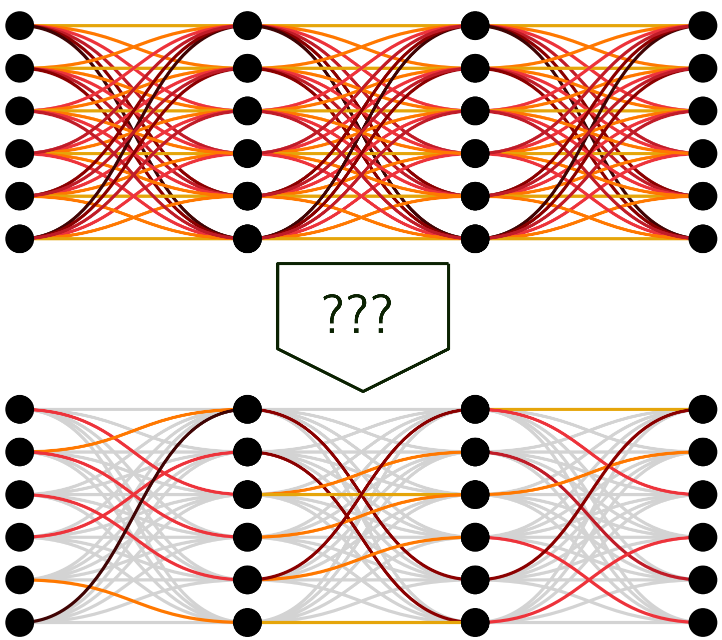
[Frankle and Carbin ICLR '19]: winning lottery tickets always exist



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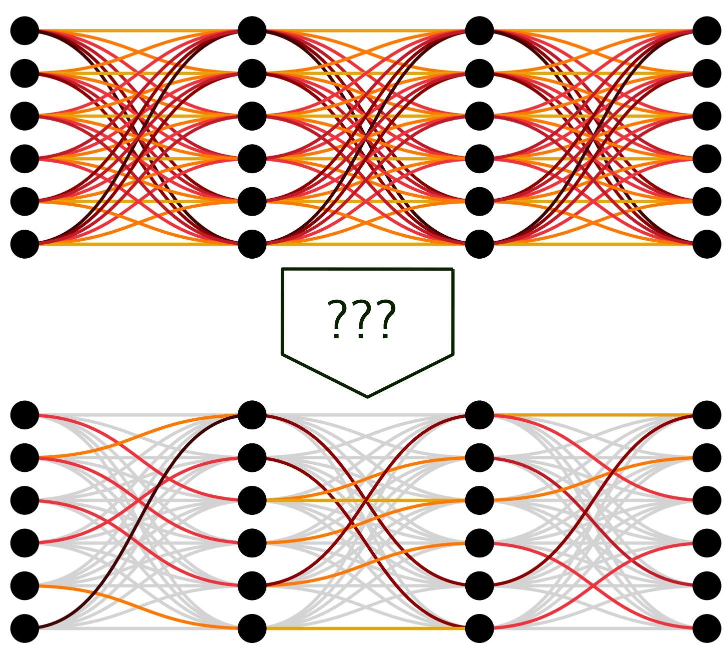

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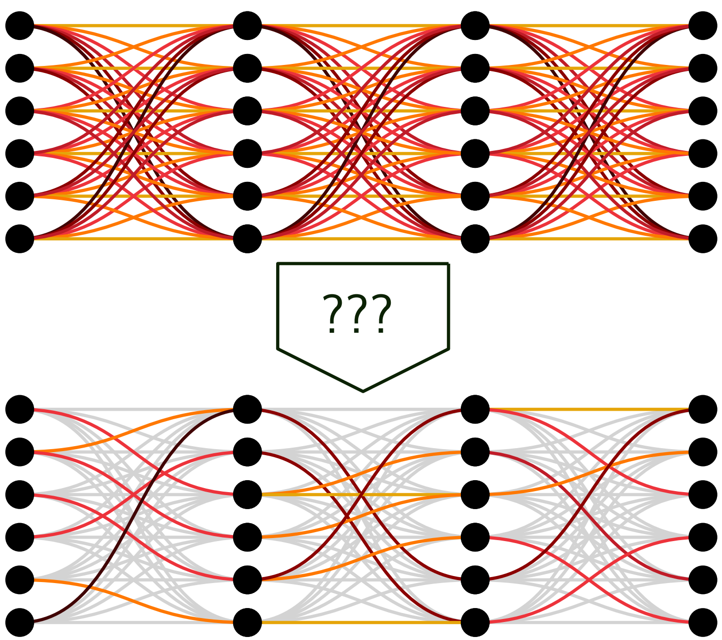
Lot of subsequent work . . .



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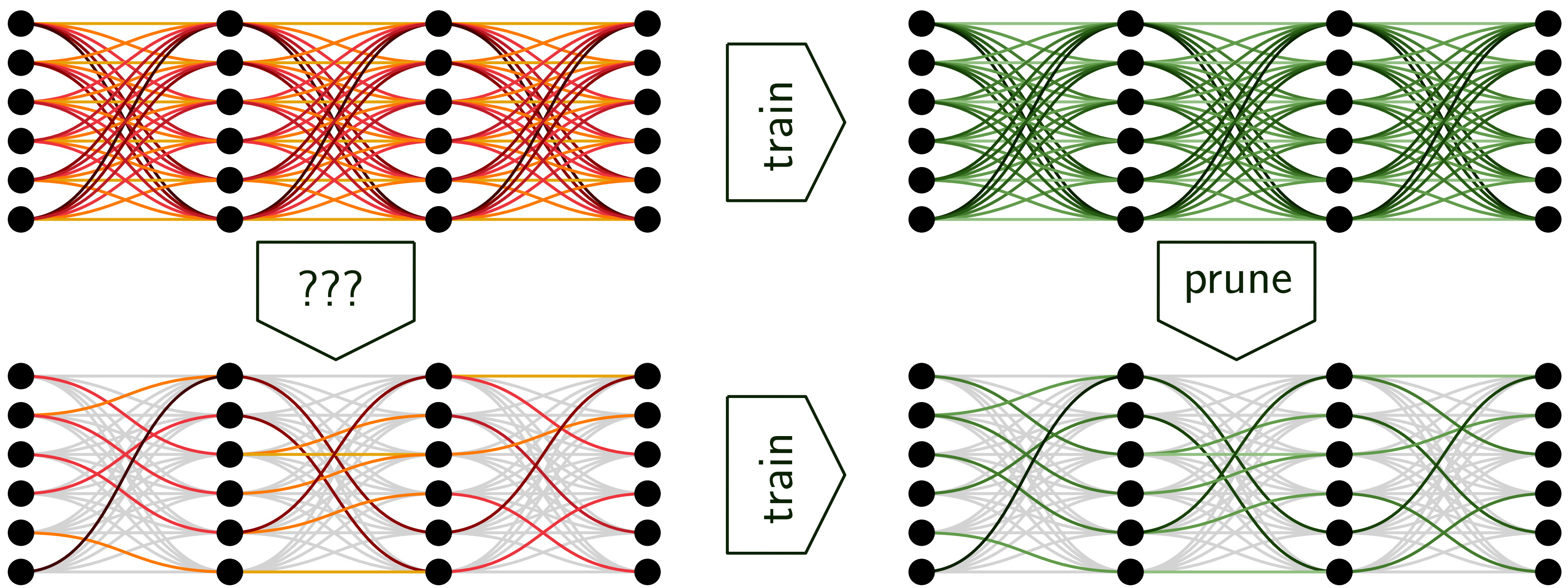

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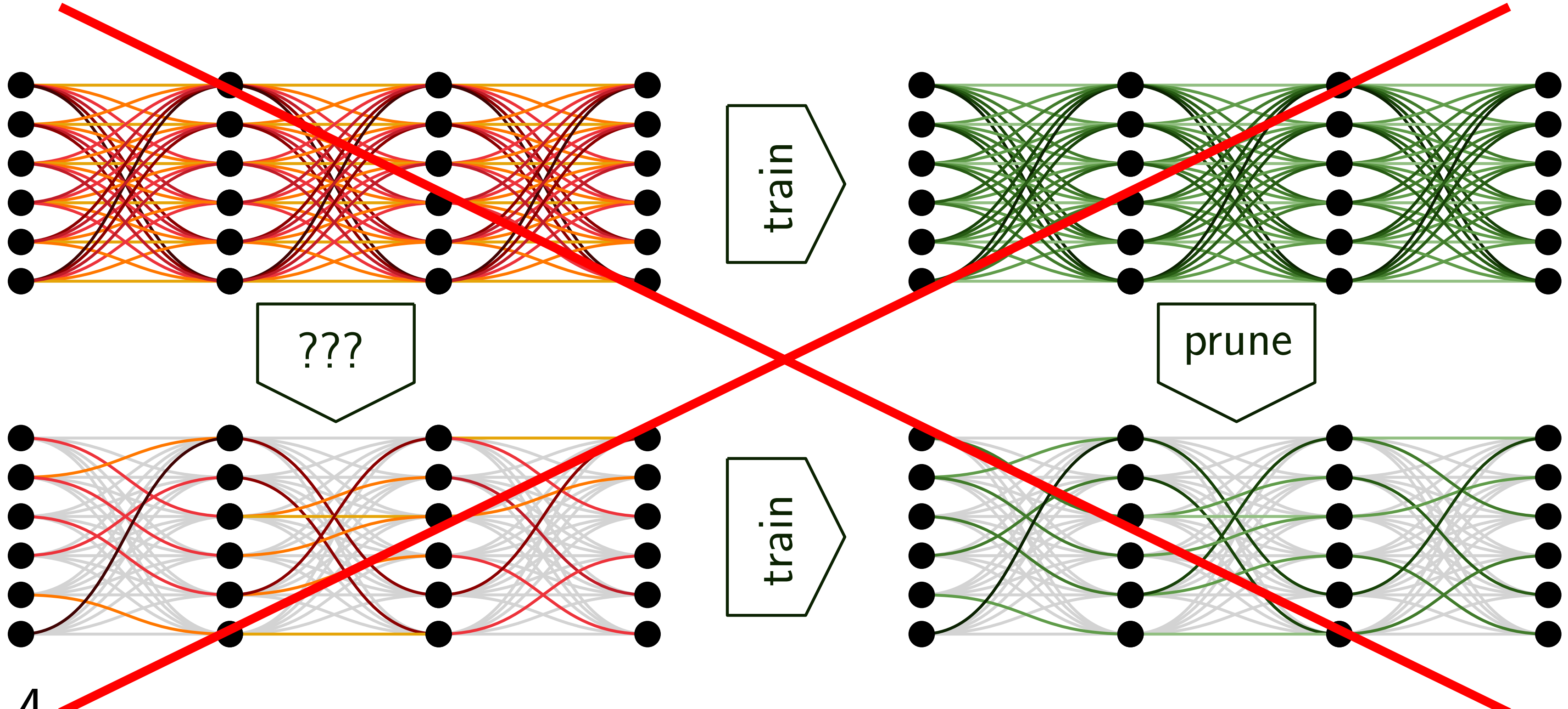

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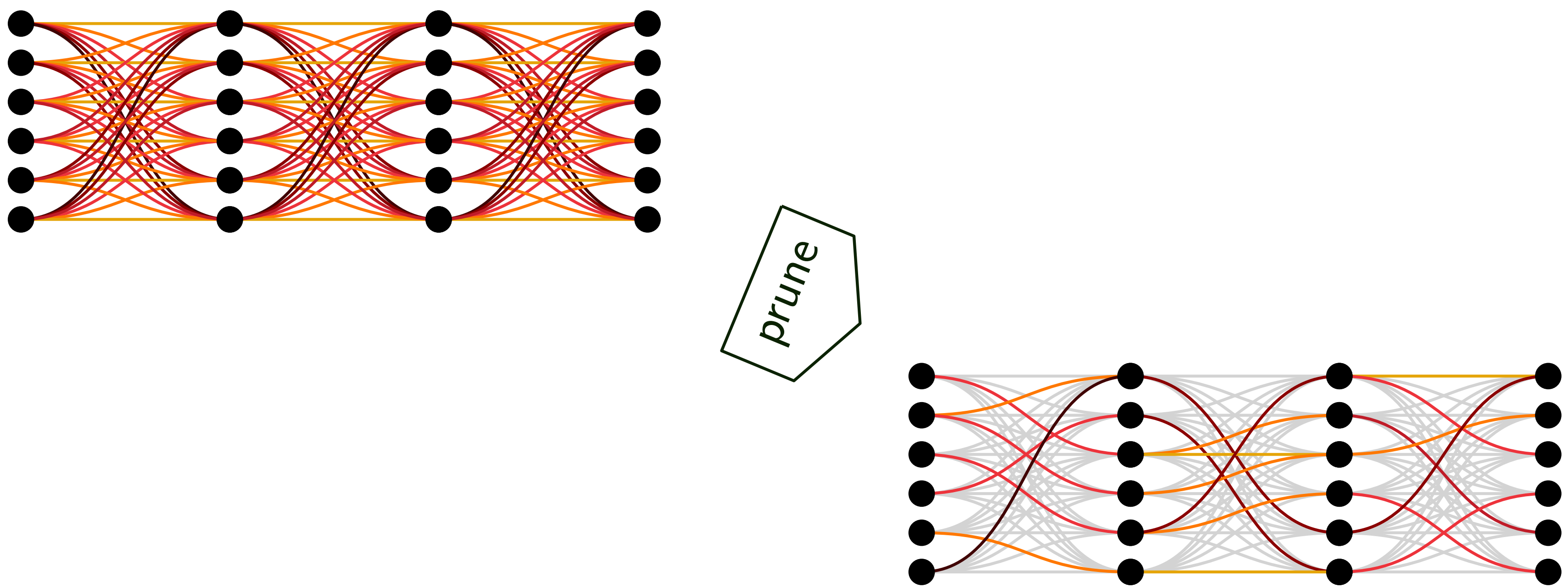

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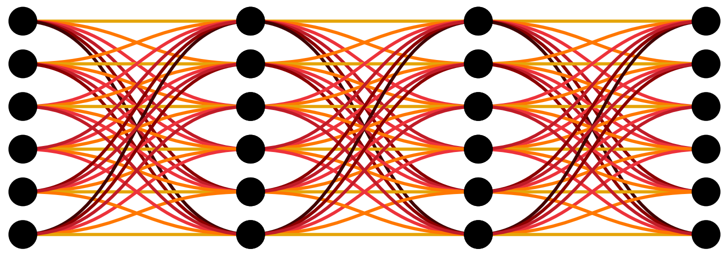

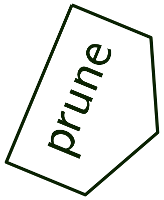

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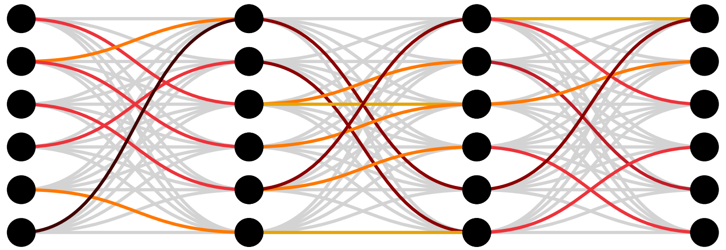

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Dense network: $f(\mathbf{x}) = \mathbf{W}_\ell \sigma(\mathbf{W}_{\ell-1} \dots \sigma(\mathbf{W}_1 \mathbf{x}))$

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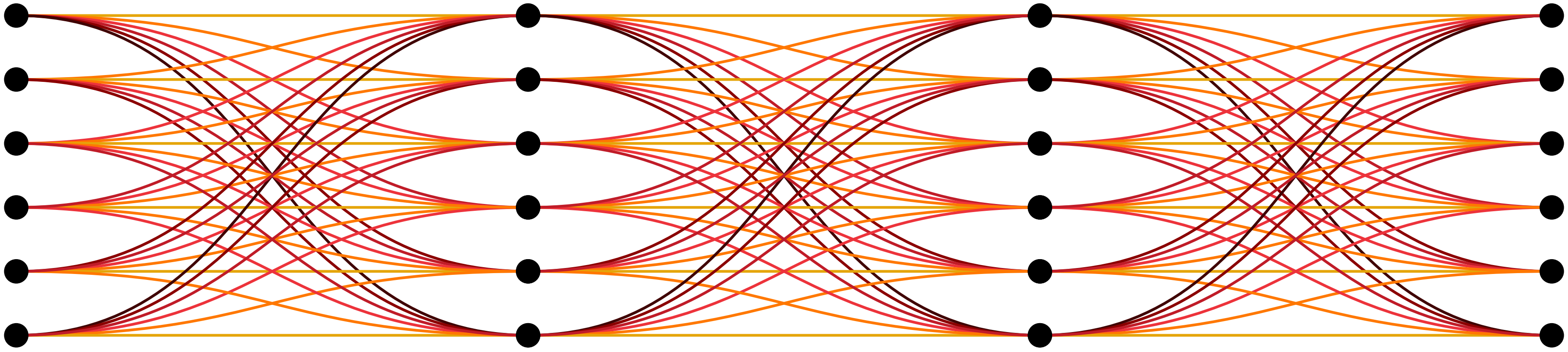

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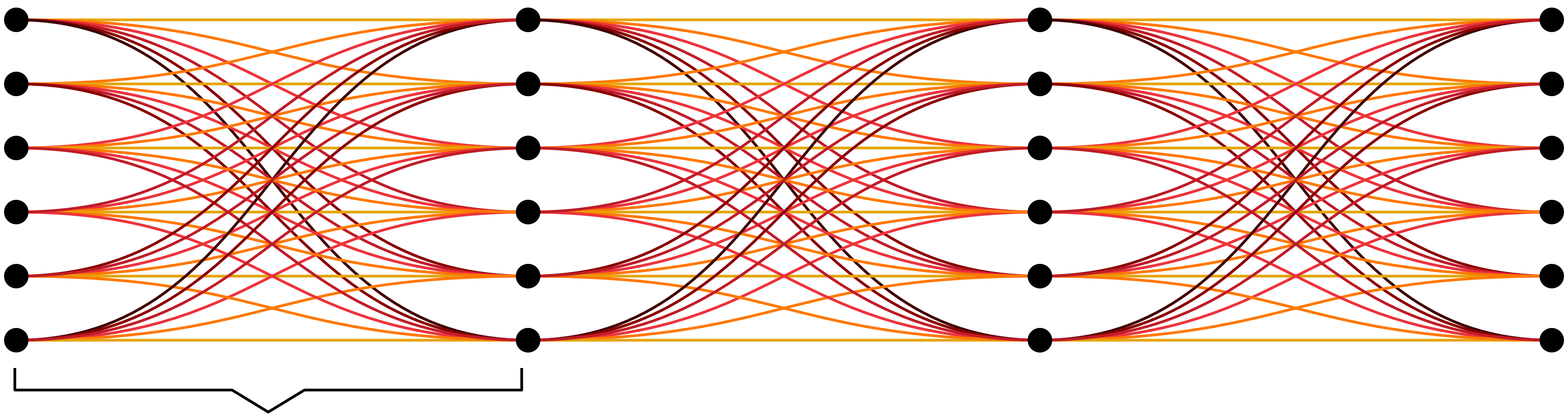

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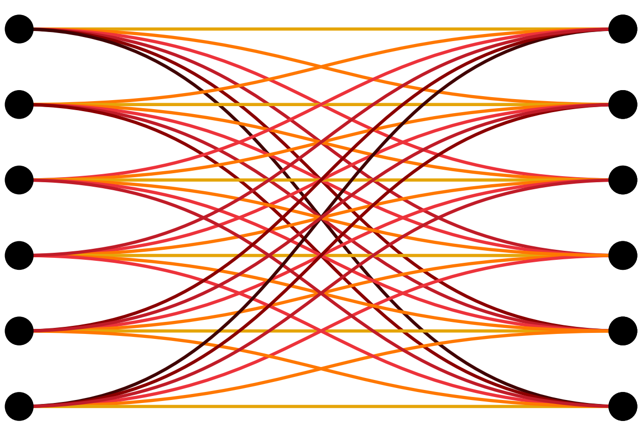

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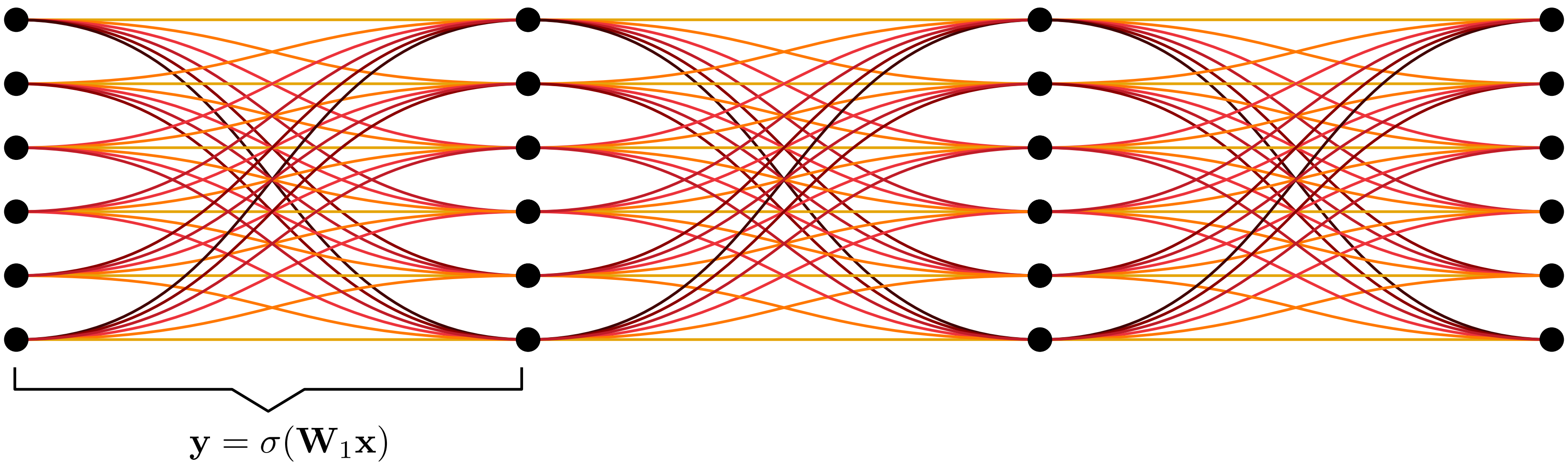

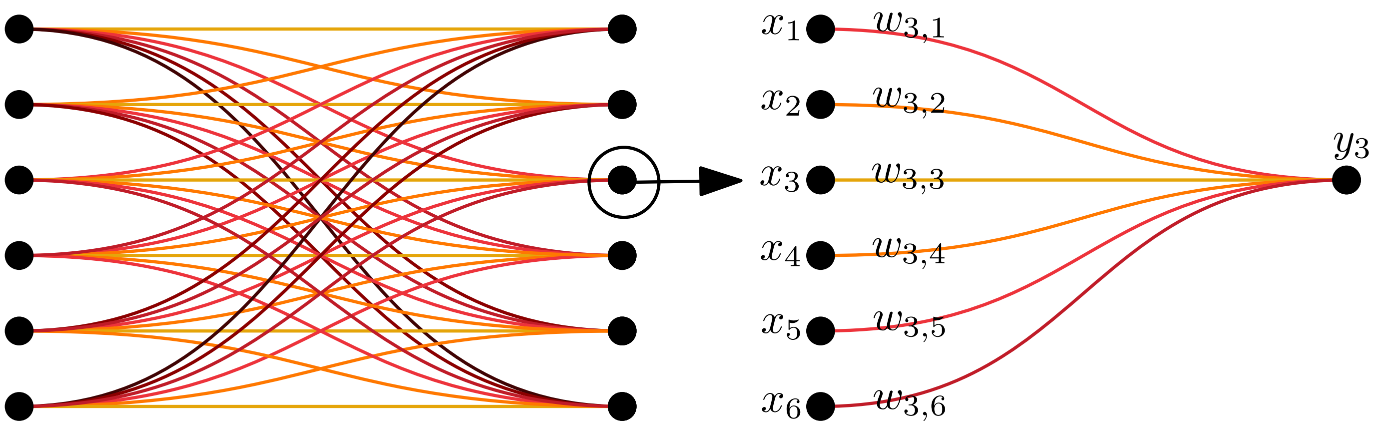

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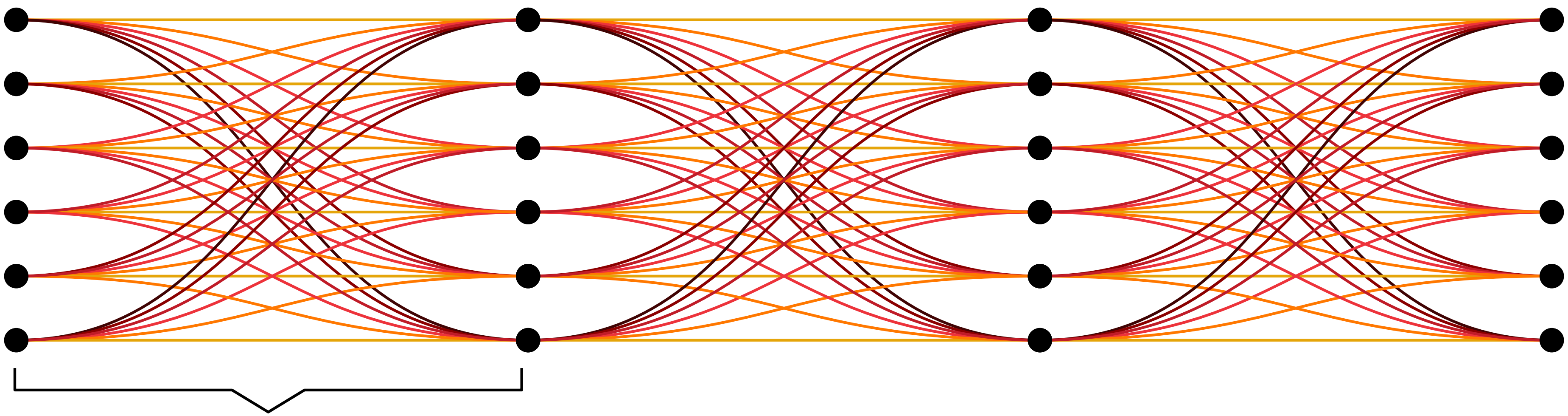

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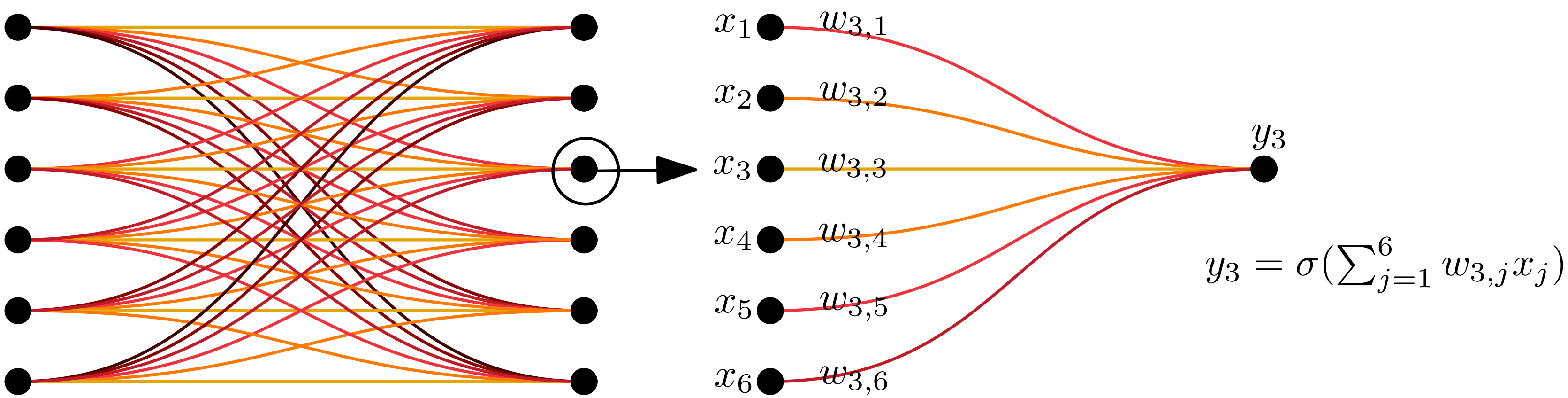

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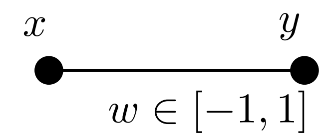

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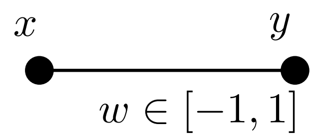

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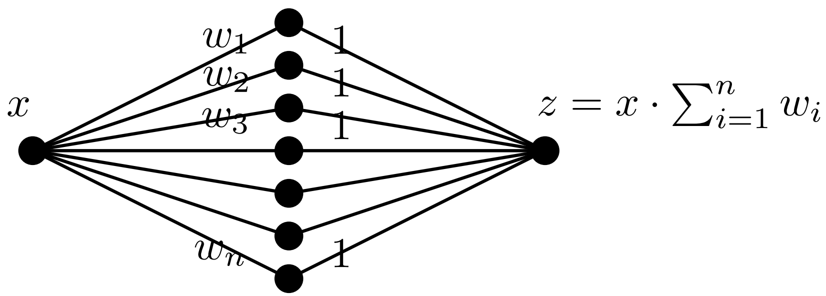

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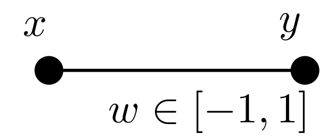

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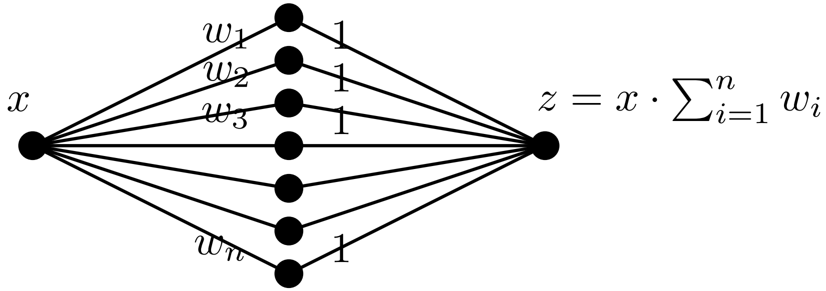

    roughly $1/\varepsilon$ samples

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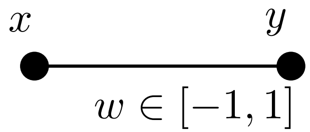
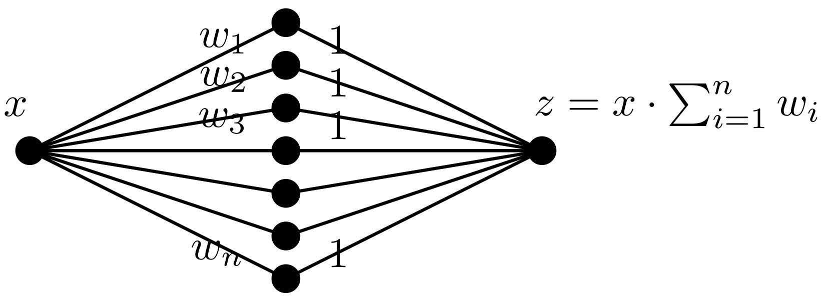


- Original approach

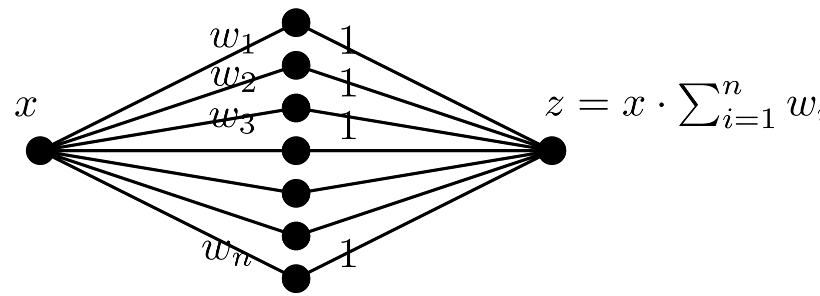
  add intermediate layer, sample
  $w_i \sim$ Unif$[-1, 1]$ until getting $w \pm \varepsilon$



  roughly $1/\varepsilon$ samples

- Random subset sum (RSS) approach:

  add intermediate layer, sample
  $w_i \sim$ Unif$[-1, 1]$ and find a good subset



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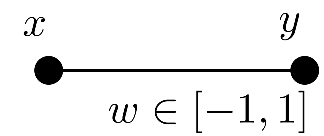

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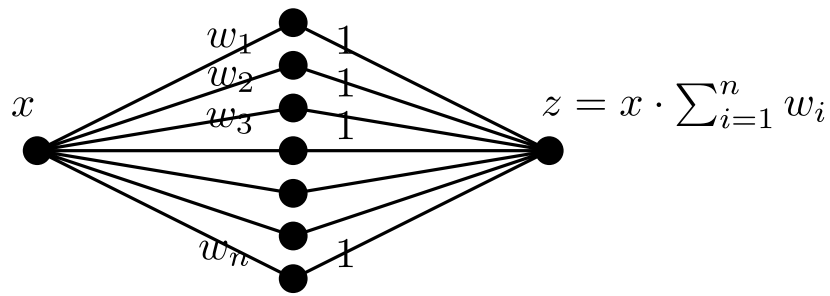

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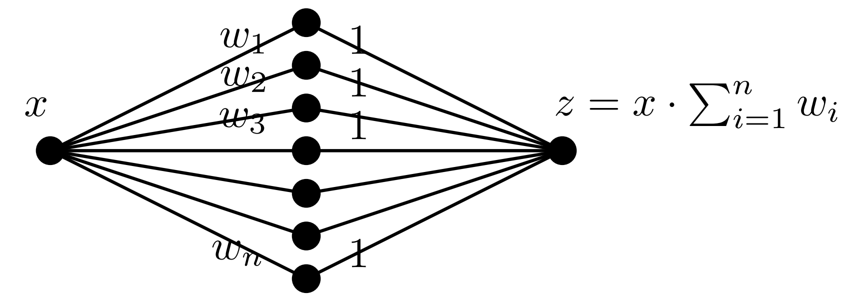

  How many?

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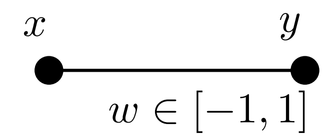

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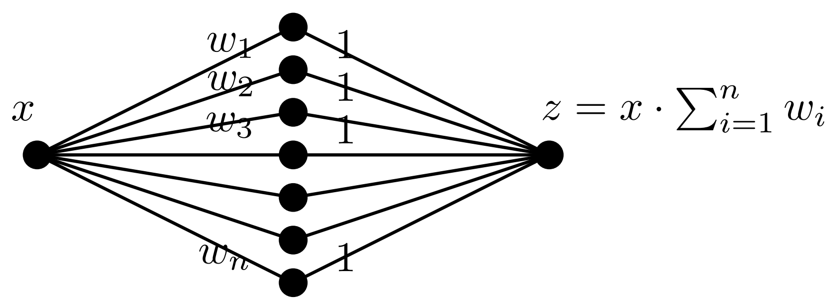

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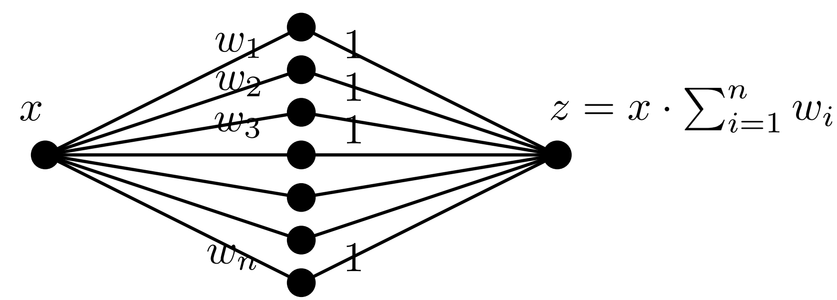

  How many?

Theorem [Lueker 1998; da Cunha et al. ESA '23]: *Let $x_1, \ldots, x_n \in [-1, 1]$ be i.i.d. uniform random variables. Given any error parameter $\varepsilon > 0$, there exists a constant $C > 0$ such that if $n \geq C \log 1/\varepsilon$ then, with probability $1 - \exp\left[(n - C \log 1/\varepsilon)^2 / 4n\right]$, for each $z \in [-1, 1]$ there exists a subset $S \subseteq [n]$ such that $\left|z - \sum_{i \in S} x_i\right| < 2\varepsilon$*

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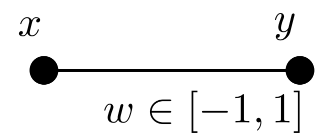

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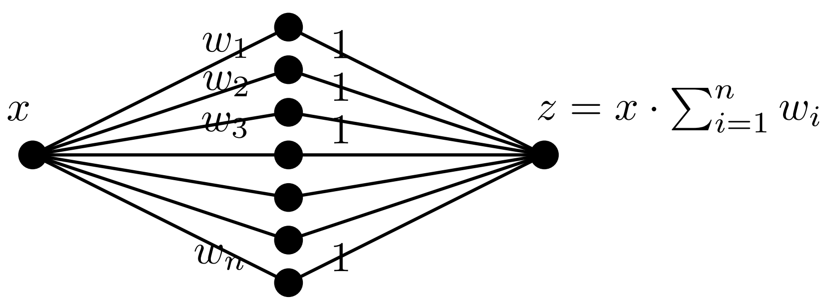

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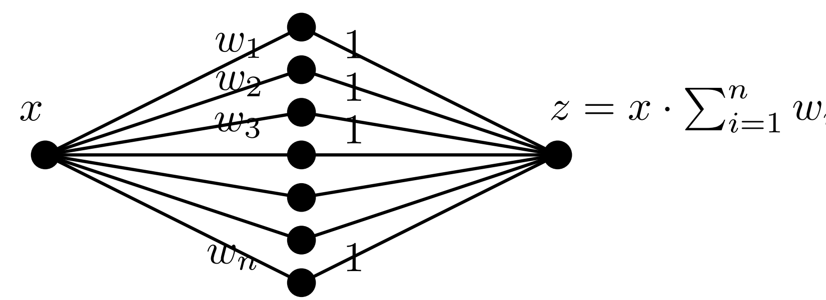

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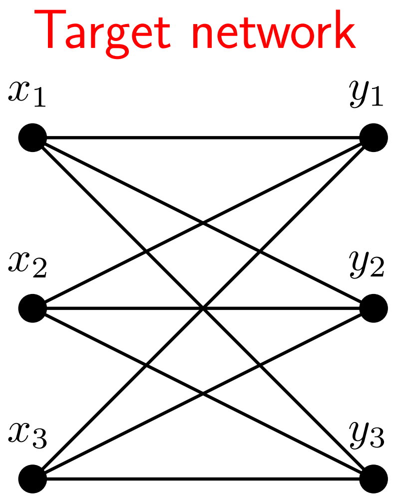
works for all densities $h(x) = pf(x) + (1-p)g(x)$, where f is "uniform"

Putting everything together



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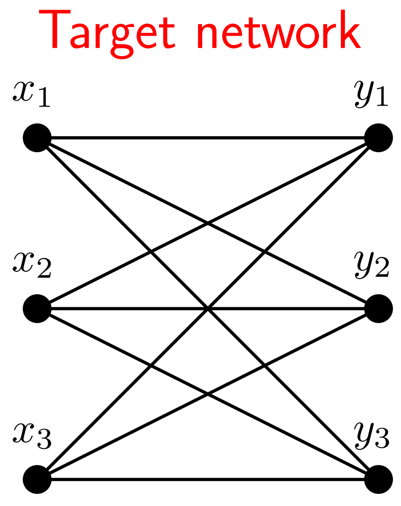

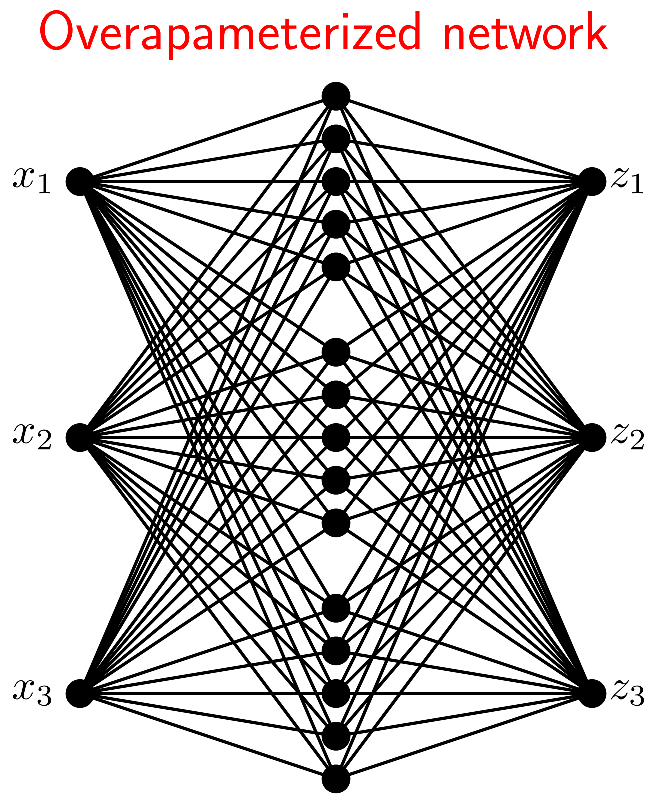

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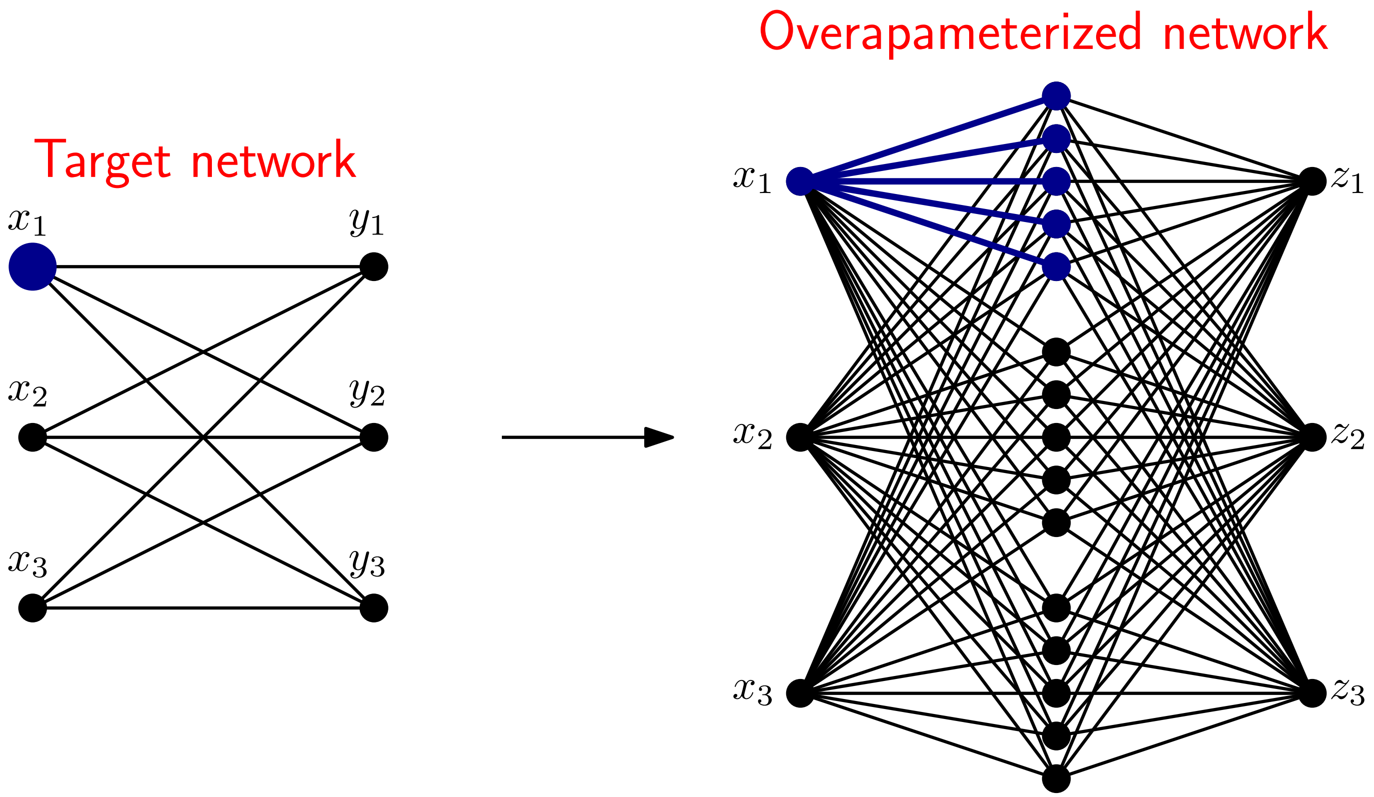

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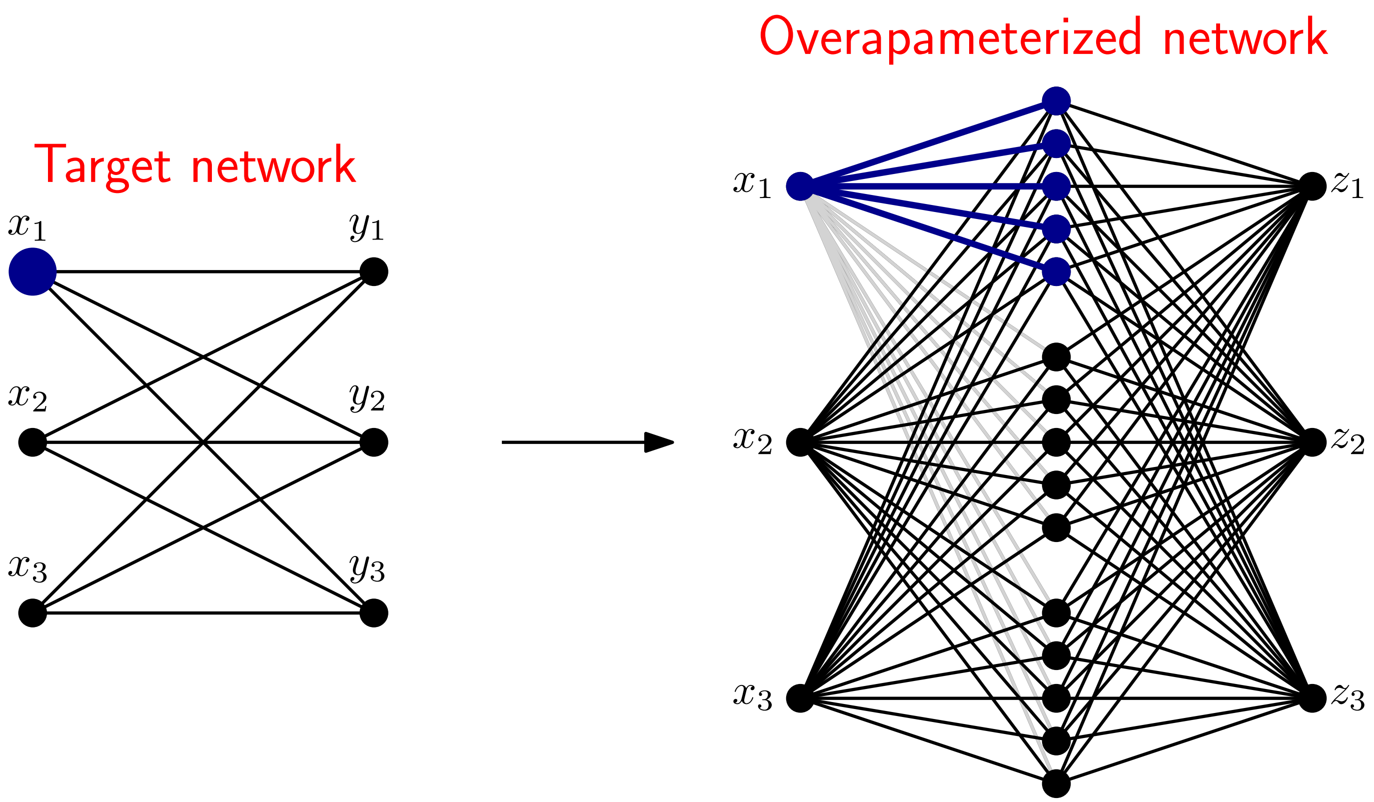

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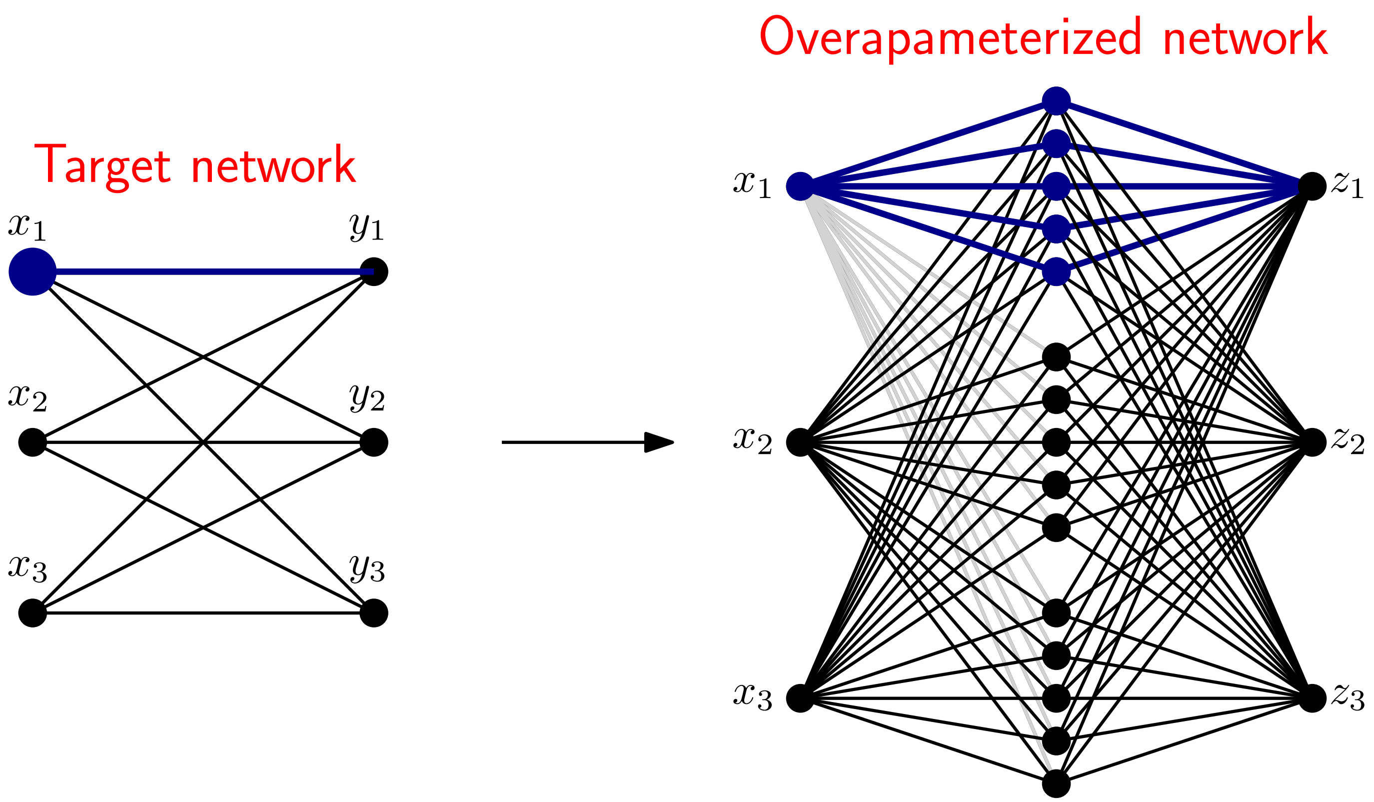

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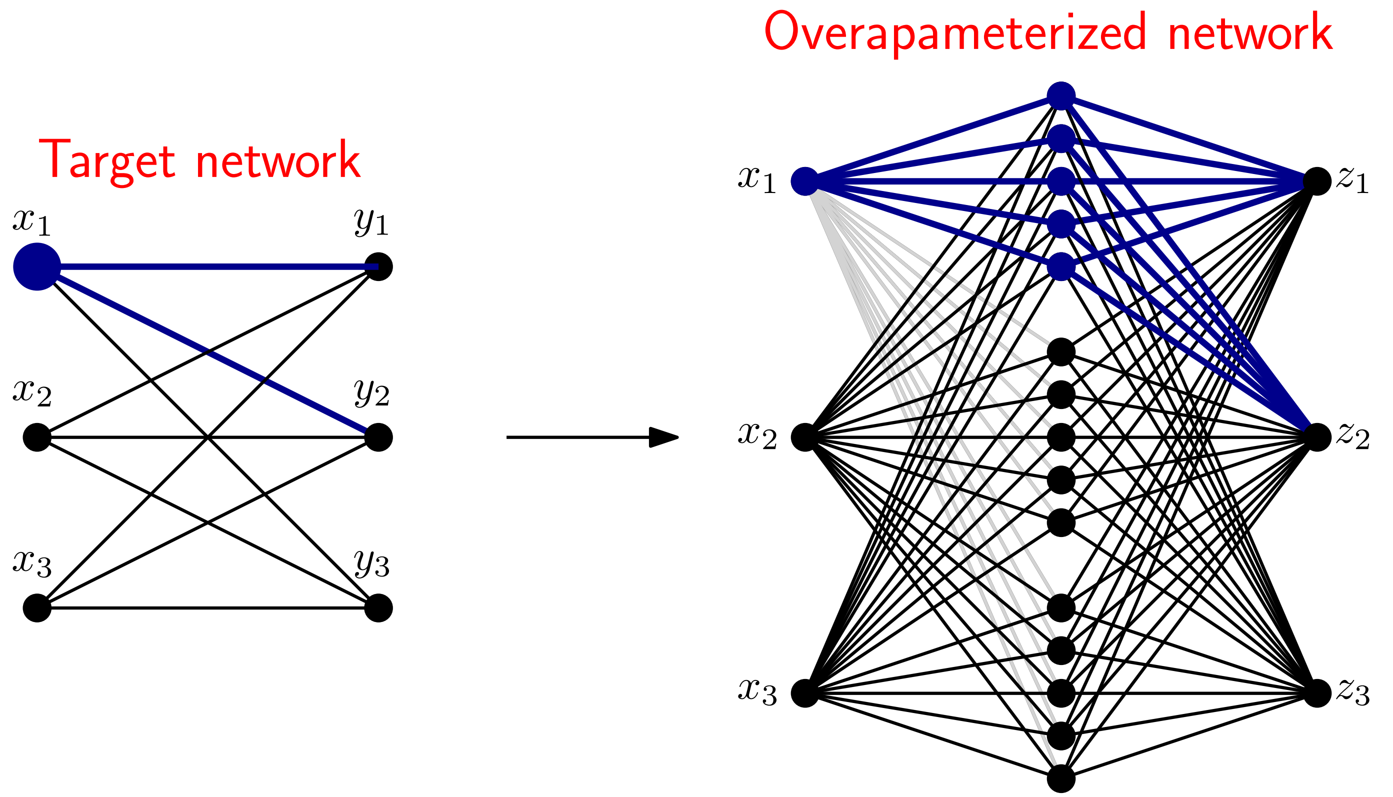

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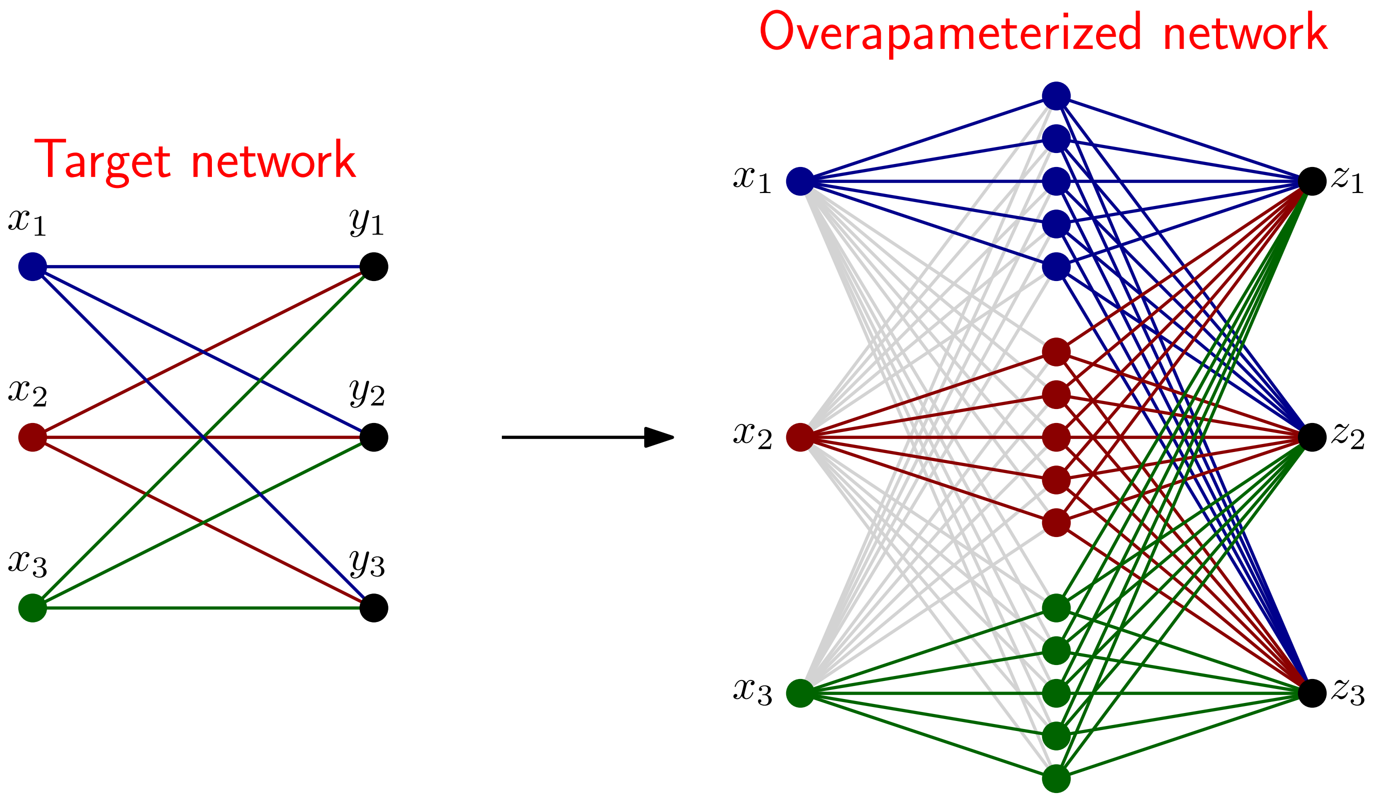
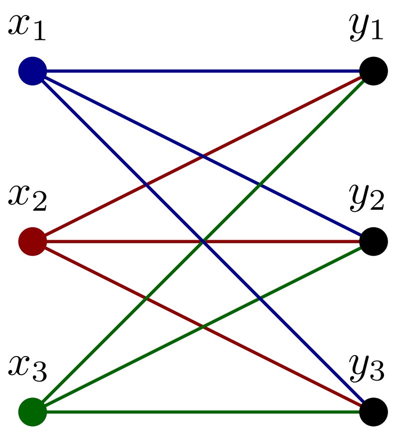
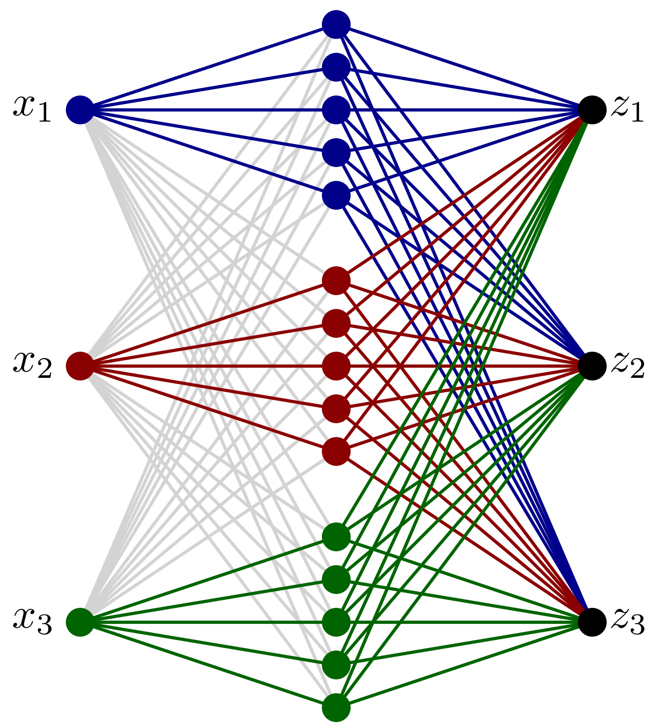


Putting everything together

Target network



Overapameterized network



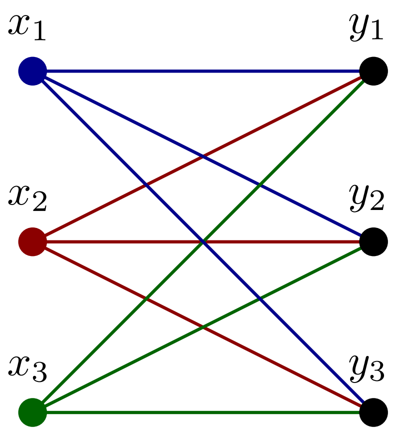
$n \geq C \log d^2 / \varepsilon$

$d = \#$ neurons in the layer

$\implies \|\mathbf{y} - \mathbf{z}\| \leq 2\varepsilon$

Putting everything together

Target network



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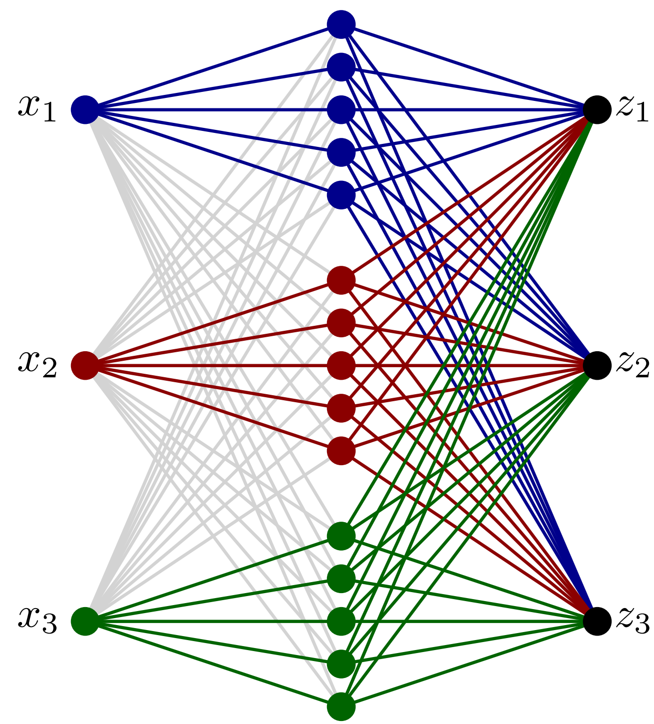

$\boxed{n \geq C \log d^2/\varepsilon}$

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$\longrightarrow$

$\boxed{n \geq C \log \ell d^2/\varepsilon}$

$\ell = \#$ layers

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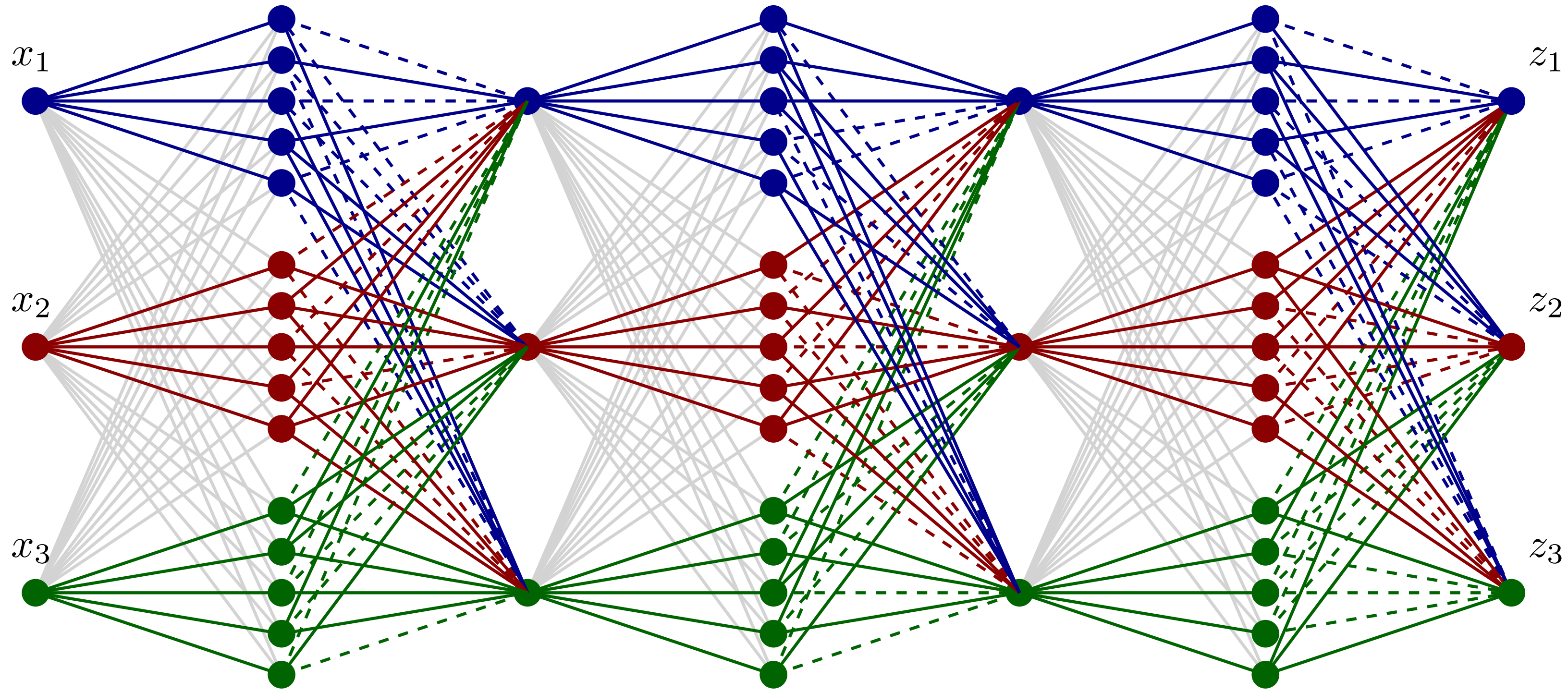
Unstructured pruning

- Removed edges can be everywhere



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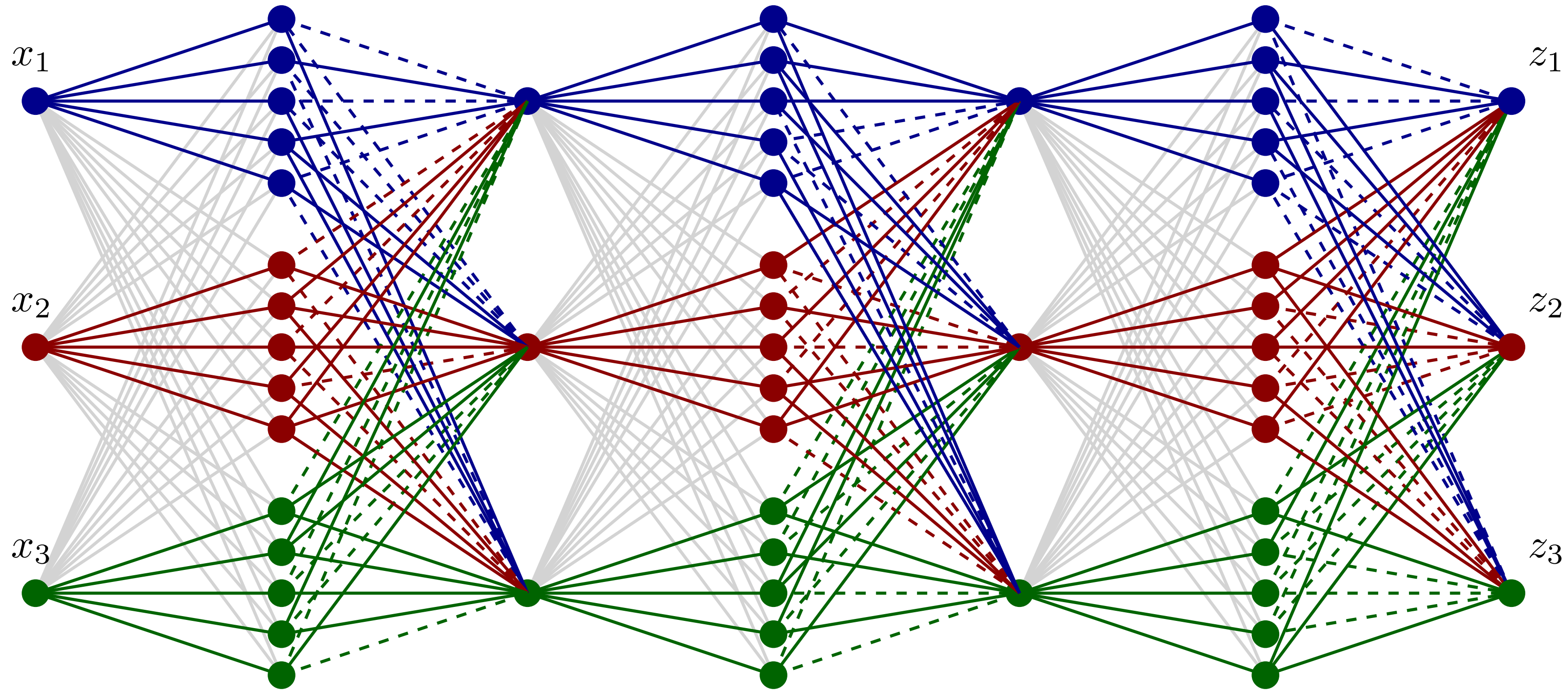

- No structure usually implies slower processes

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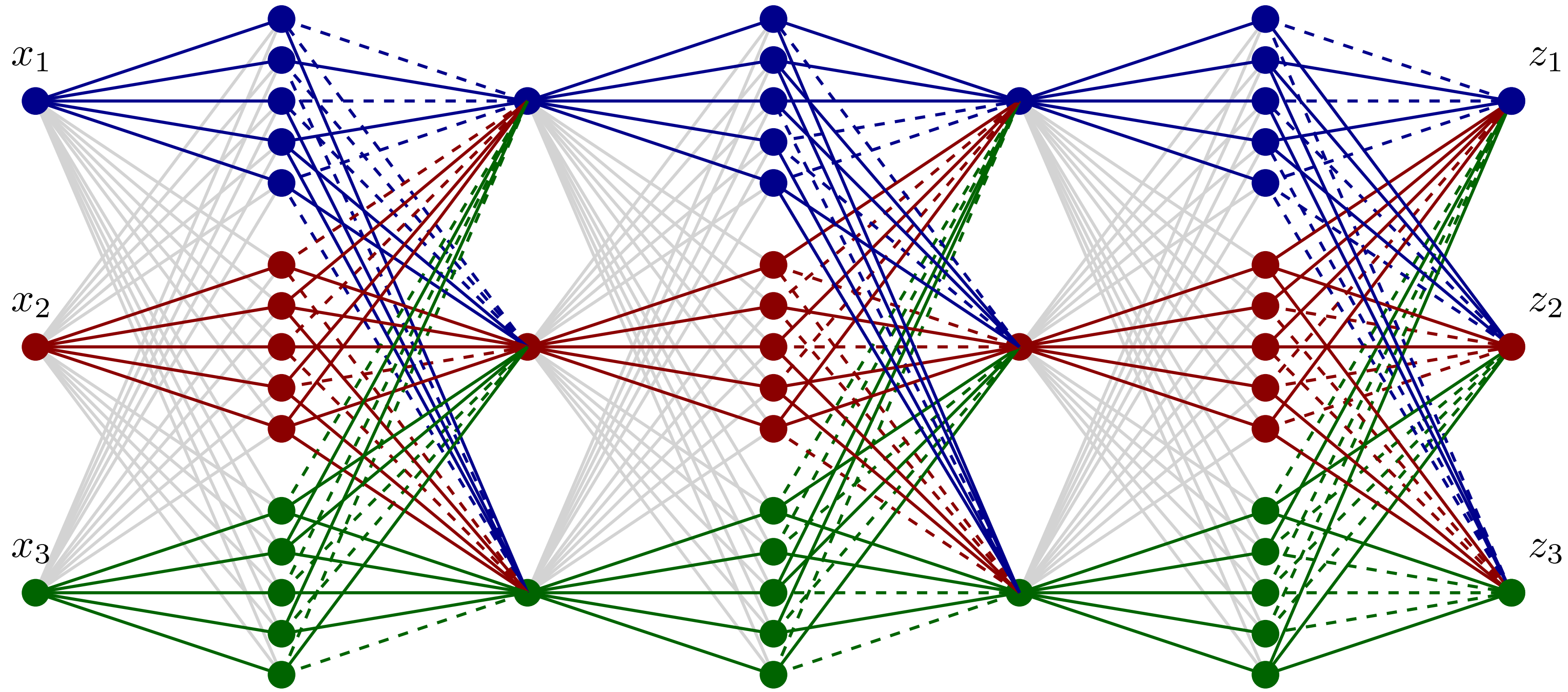

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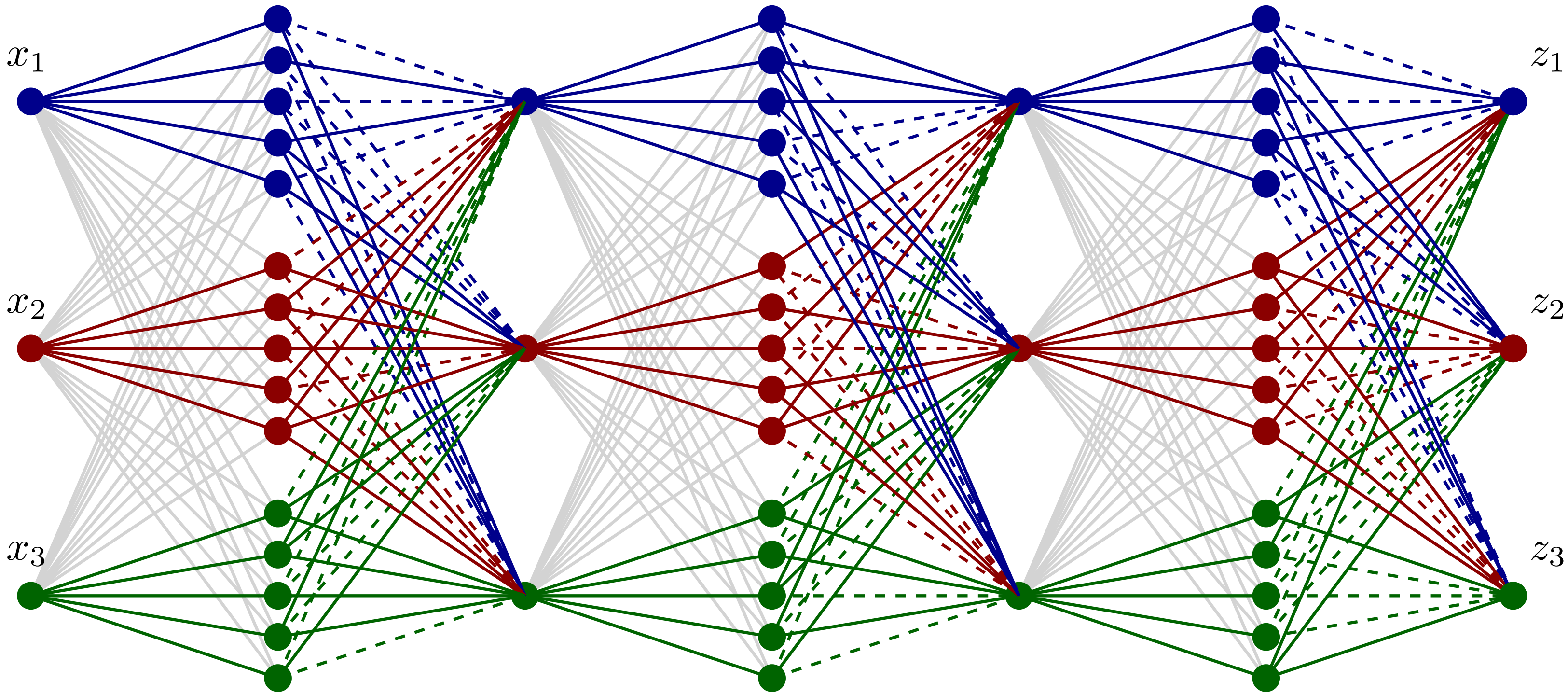

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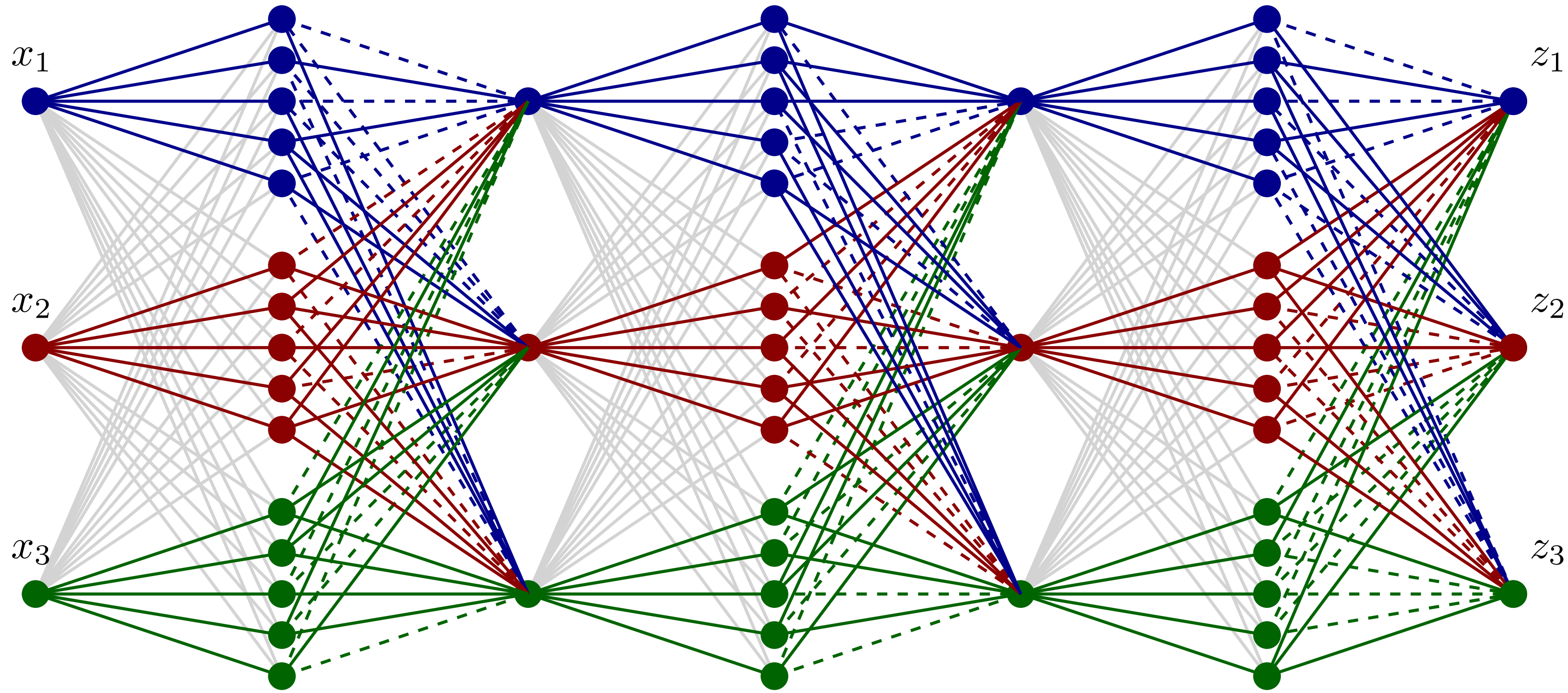

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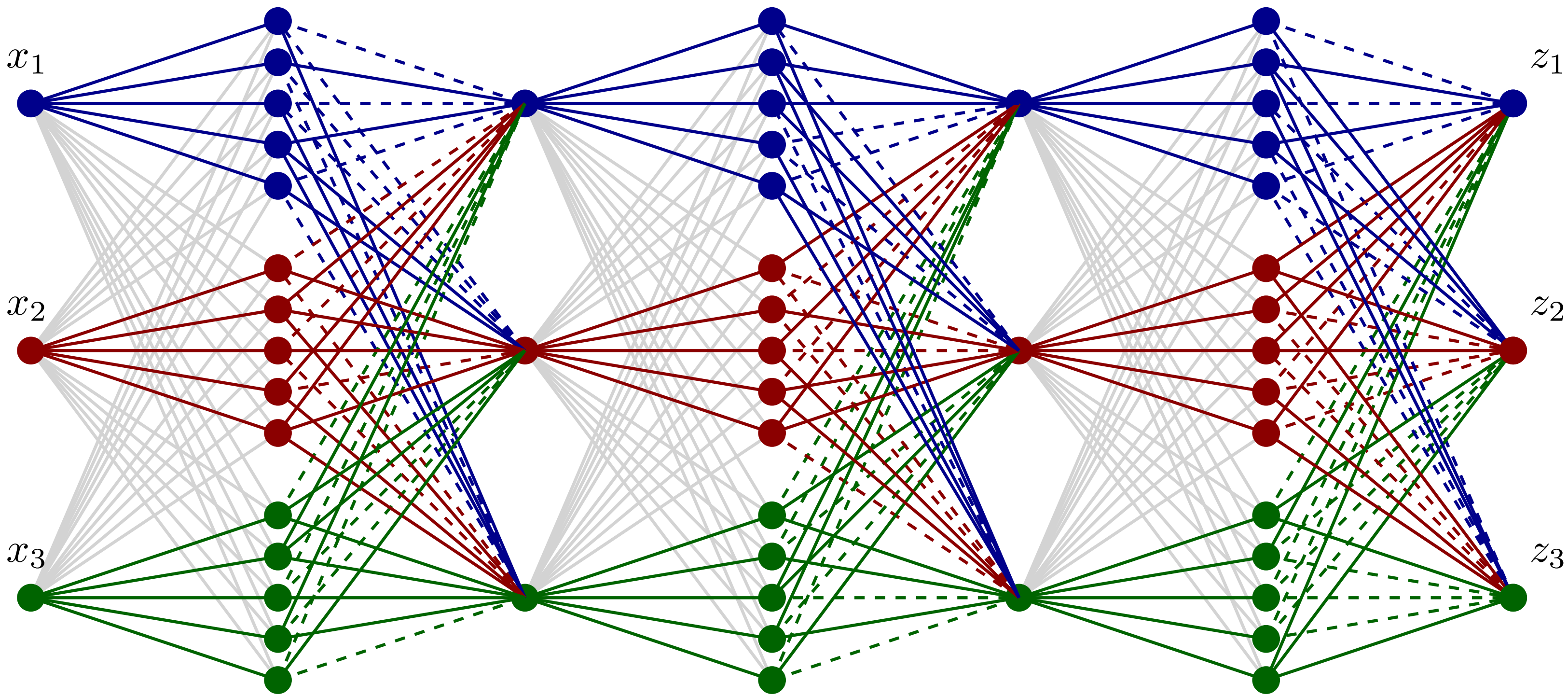

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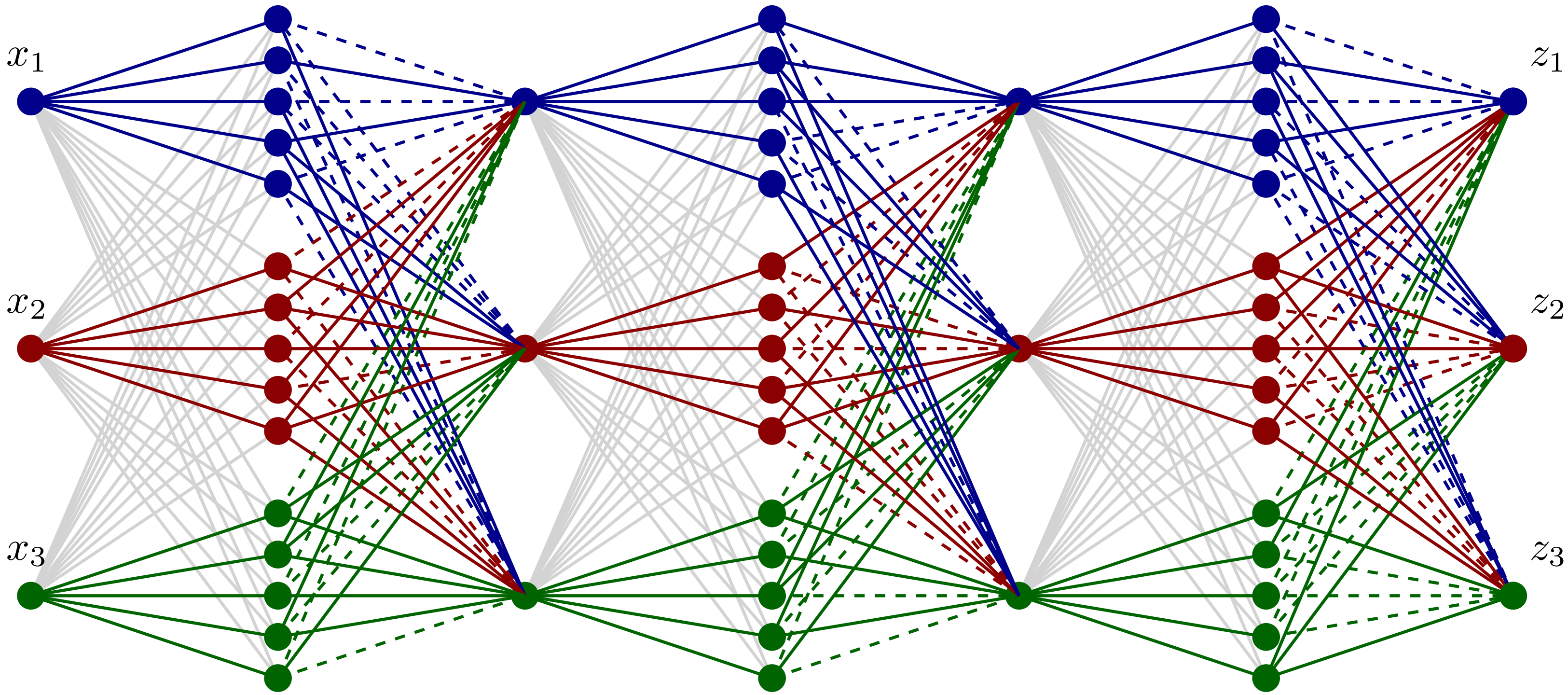

- No structure usually implies slower processes
  - difficulty encoding unstructured sparsity
  - accessing data is more time consuming than processing
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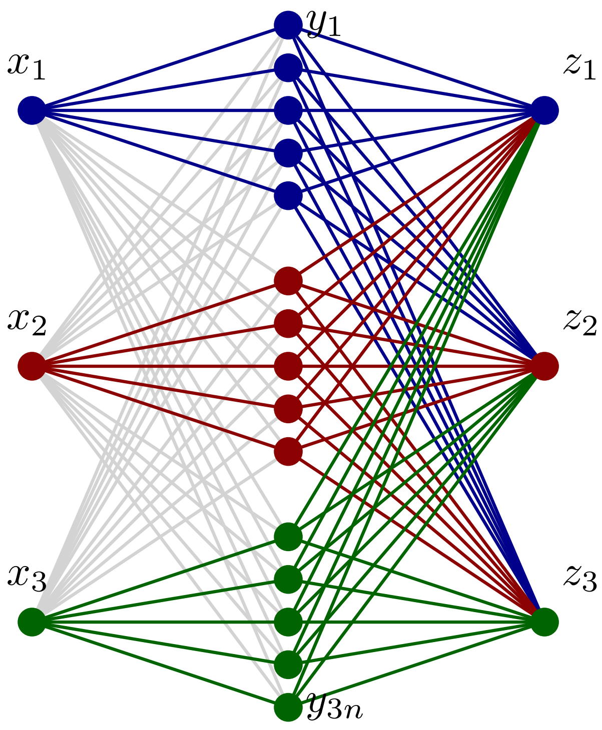
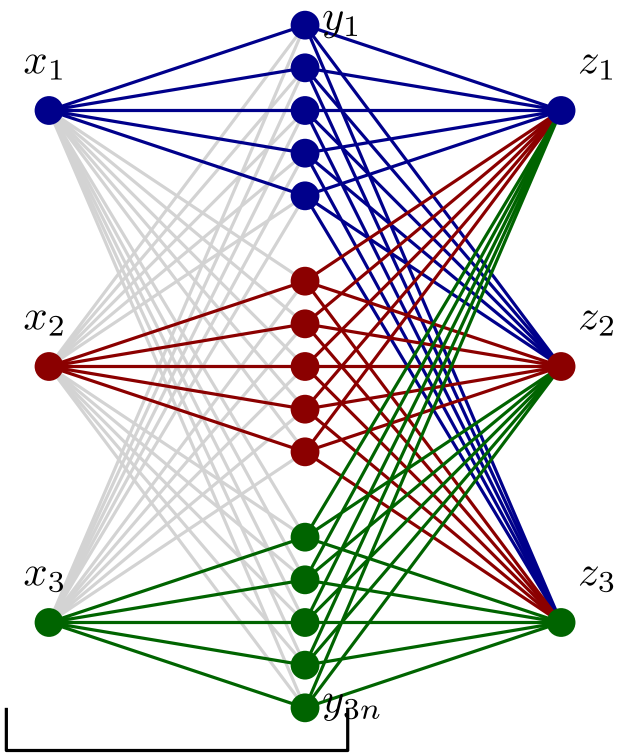
- What about **structured pruning**
  - [Malach et al. ICML '20]: pruning neurons alone requires exponential overparam.

Structured pruning

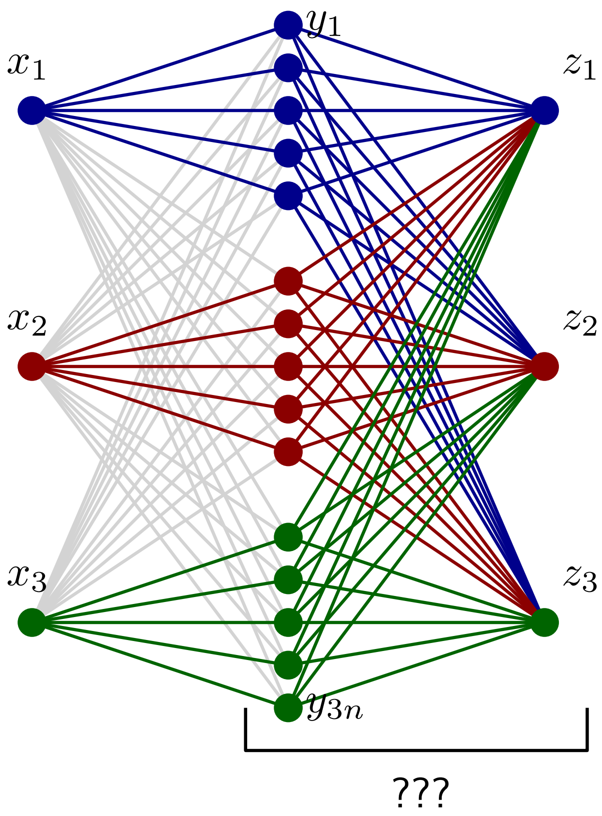


Structured pruning



$$\begin{bmatrix} w_{1,1} & 0 & 0 \\ \vdots & \vdots & \vdots \\ w_{n,1} & 0 & 0 \\ 0 & w_{1,2} & 0 \\ \vdots & \vdots & \vdots \\ 0 & w_{n,2} & 0 \\ 0 & 0 & w_{1,3} \\ \vdots & \vdots & \vdots \\ 0 & 0 & w_{n,3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Structured pruning



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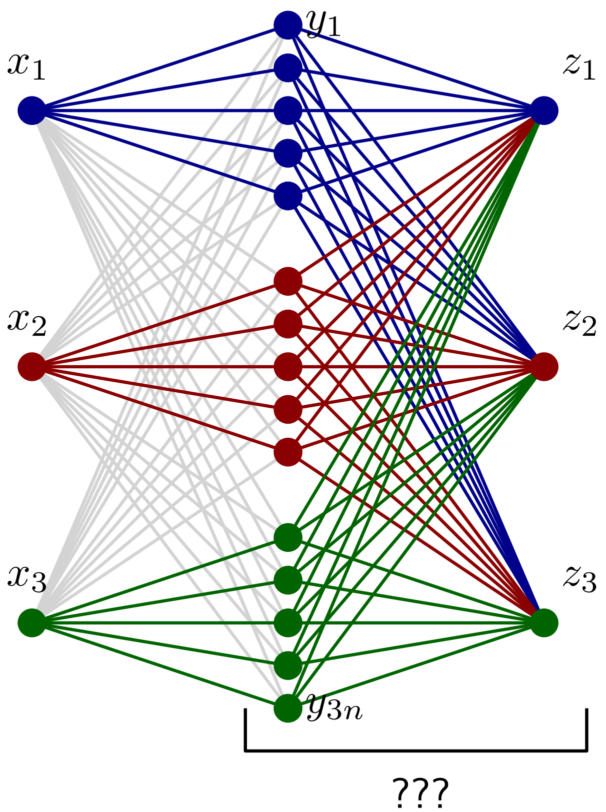
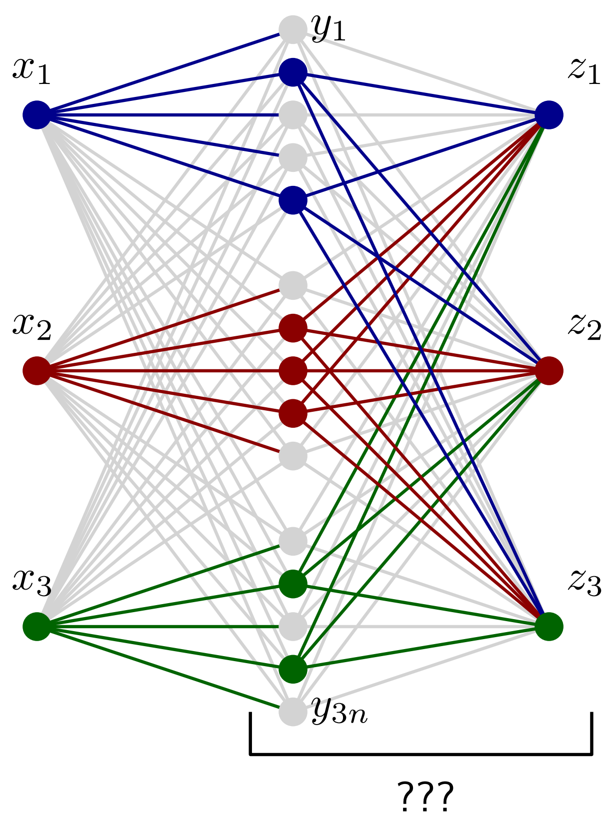


- Removing entire neurons from the middle layer!

Structured pruning



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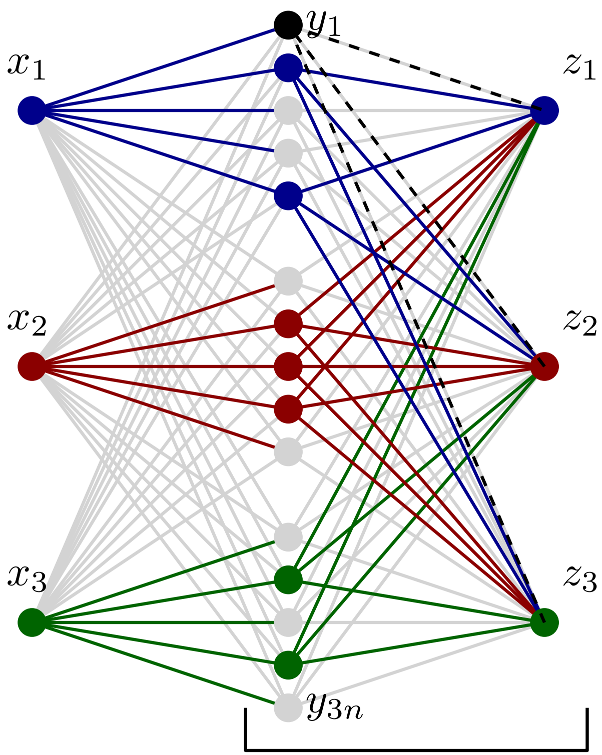
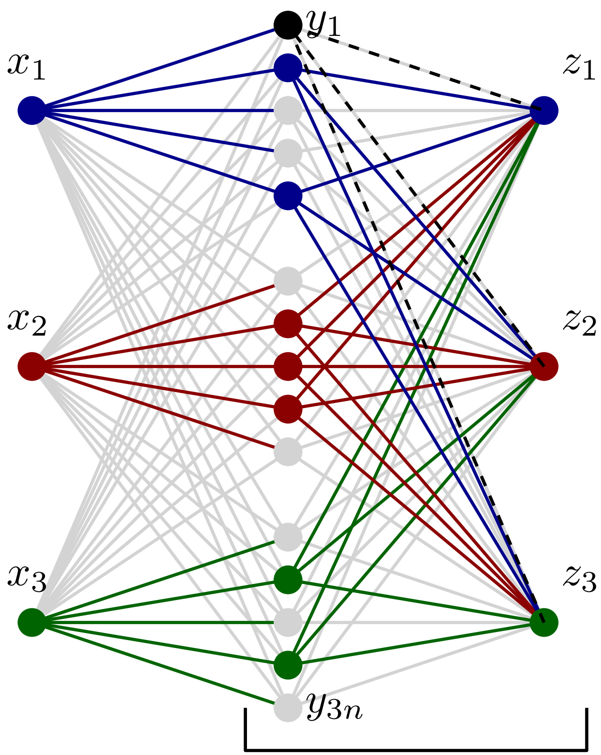


- Removing entire neurons from the middle layer!
  - removes columns!

$$\begin{bmatrix} 0 & v_{1,2} & 0 & \dots & 0 & v_{i,1} & 0 & \dots \\ 0 & v_{2,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \\ 0 & v_{3,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{3n} \end{bmatrix}$$

Structured pruning



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  - leads to exponential bounds

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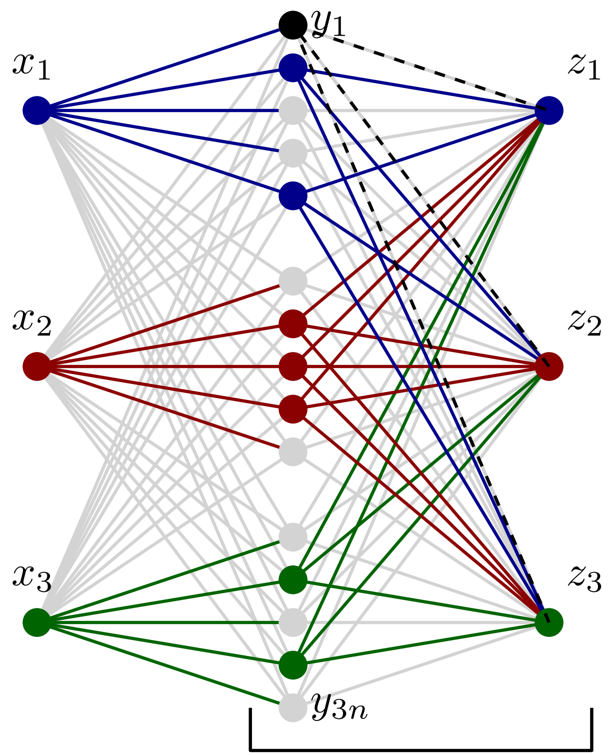
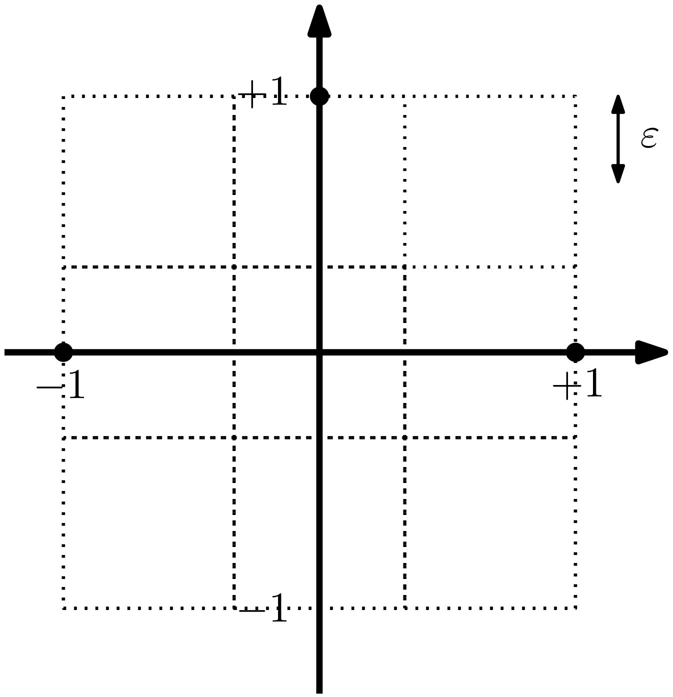


- Removing entire neurons from the middle layer!
  - removes columns!
- The one-dimensional RSS result does not work
  - leads to exponential bounds
- A multidimensional RSS result is required

$$\begin{bmatrix} 0 & v_{1,2} & 0 & \dots & 0 & v_{i,1} & 0 & \dots \\ 0 & v_{2,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \\ 0 & v_{3,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{3n} \end{bmatrix}$$

The multidimensional RSS problem

- Natural generalization



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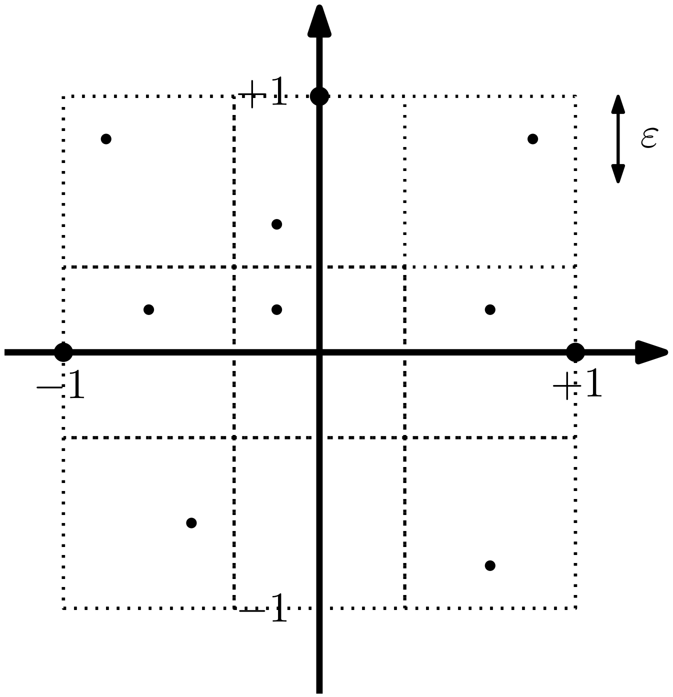

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• Sequence of n i.i.d. random vectors $X_1, \ldots, X_n$

The multidimensional RSS problem

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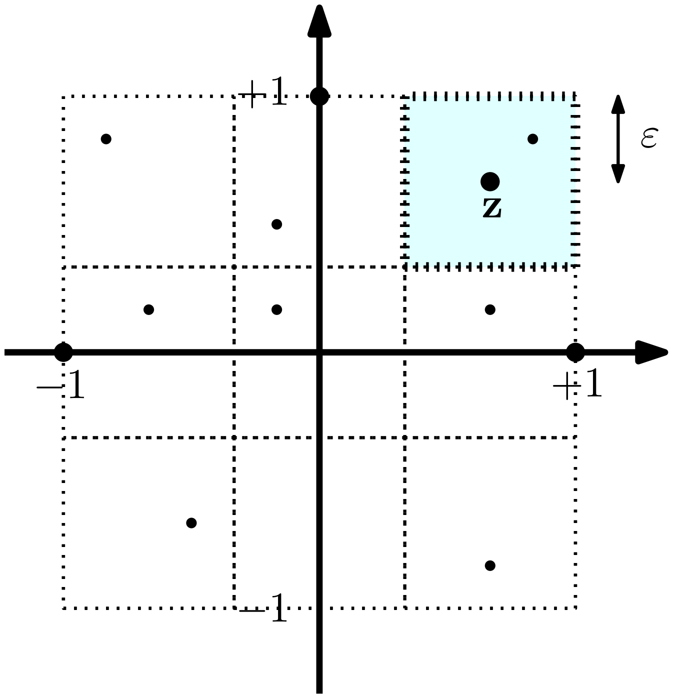

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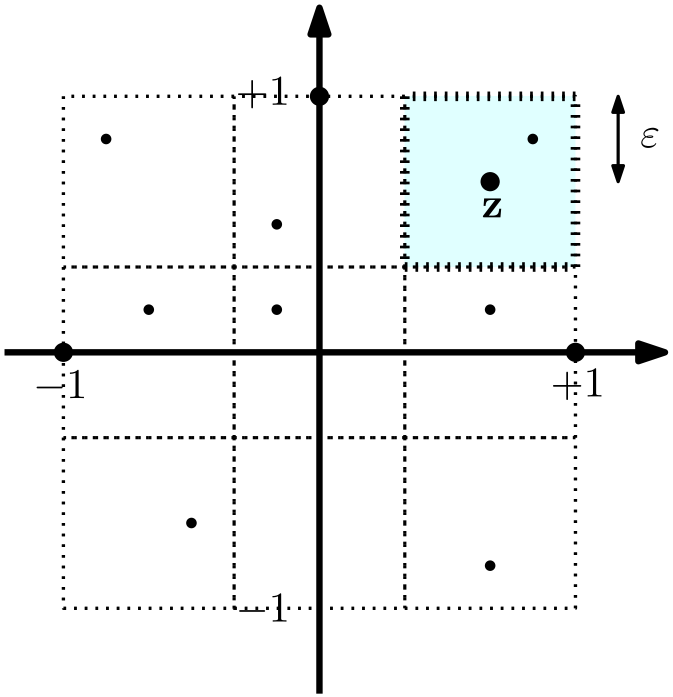

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- Error parameter $\varepsilon > 0$

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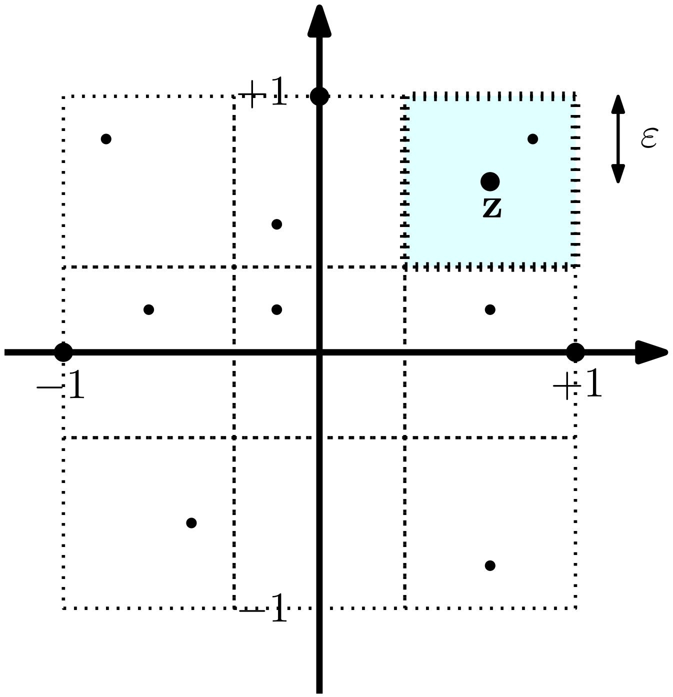

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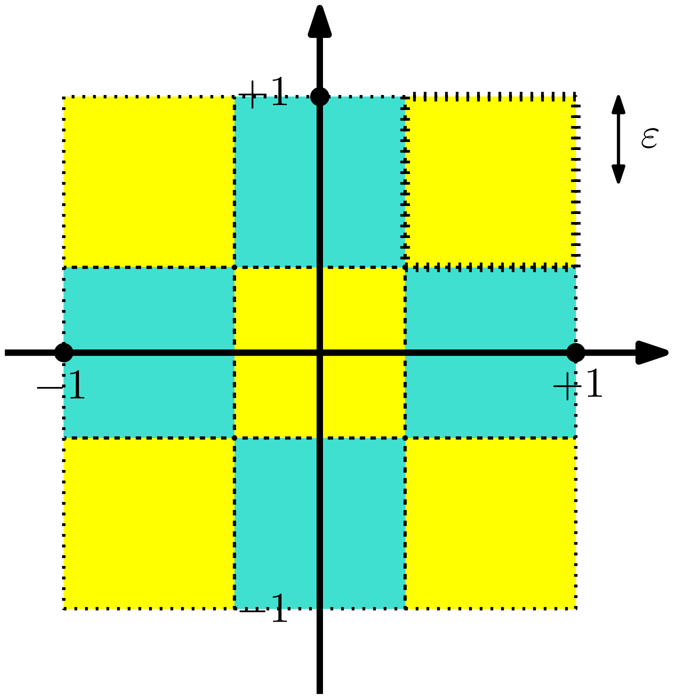
Question:

- Estimate n such that, with high probability, a subset $S \subseteq [n]$ exists with $\|\mathbf{z} - \sum_{i \in S} X_i\|_\infty \leq 2\varepsilon$

MRSS in expectation

- Number of ε-cubes: $1/\varepsilon^d = 2^{d \log 1/\varepsilon}$



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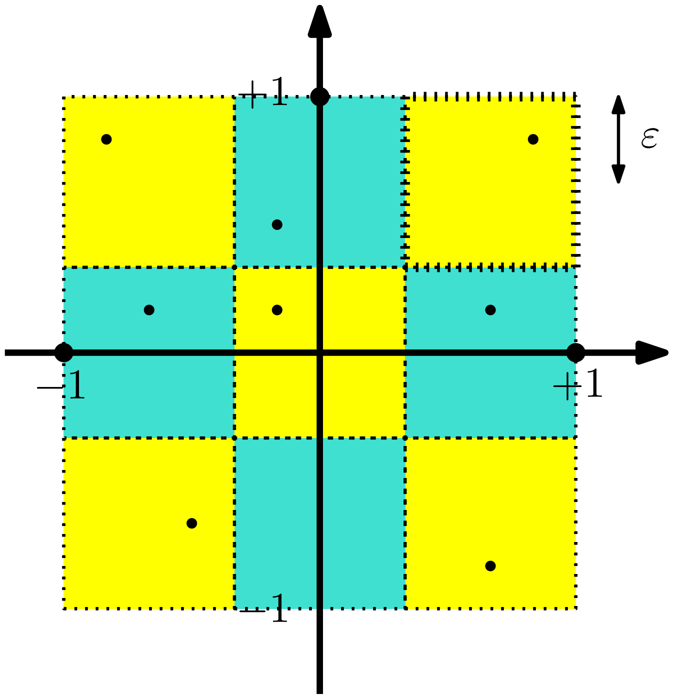

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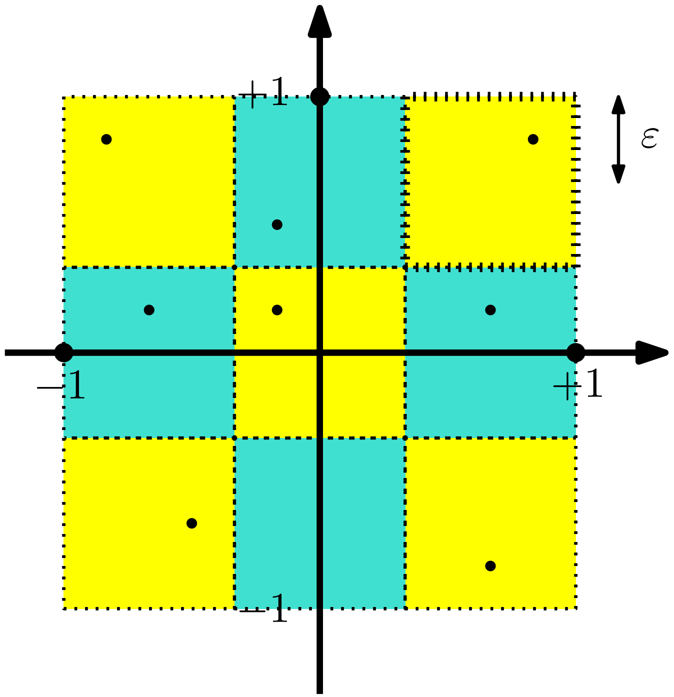

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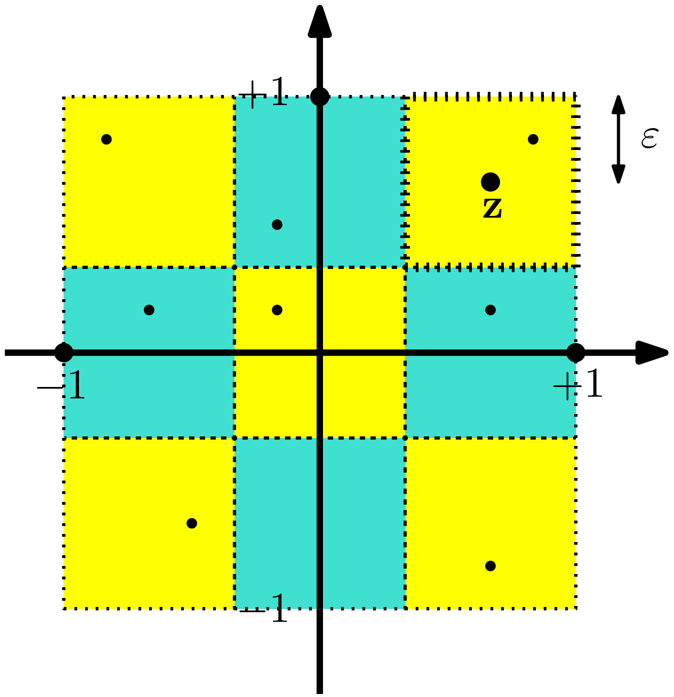
- Target $\mathbf{z} \in [-1, 1]^d$



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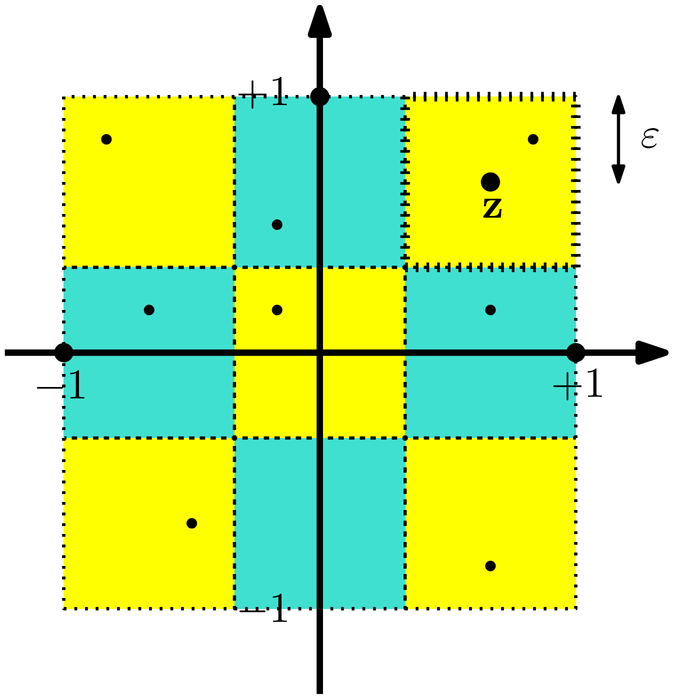
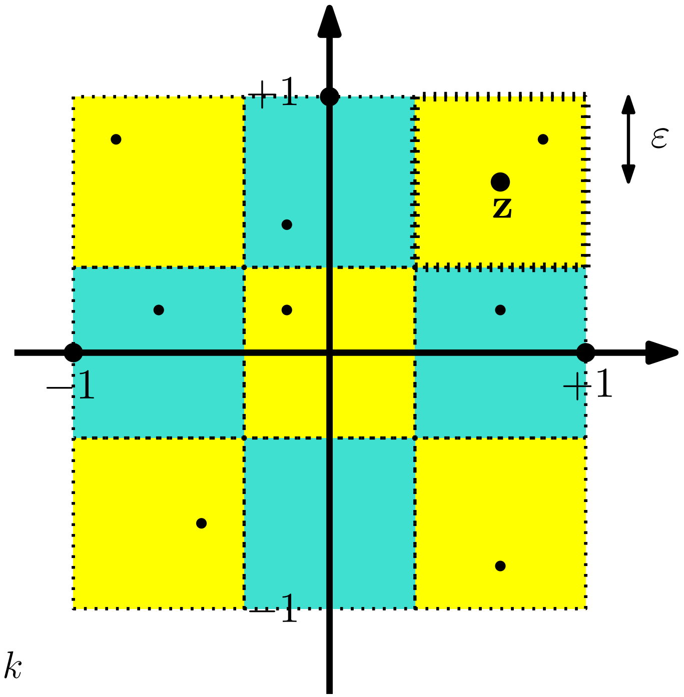


In expectation

MRSS in expectation



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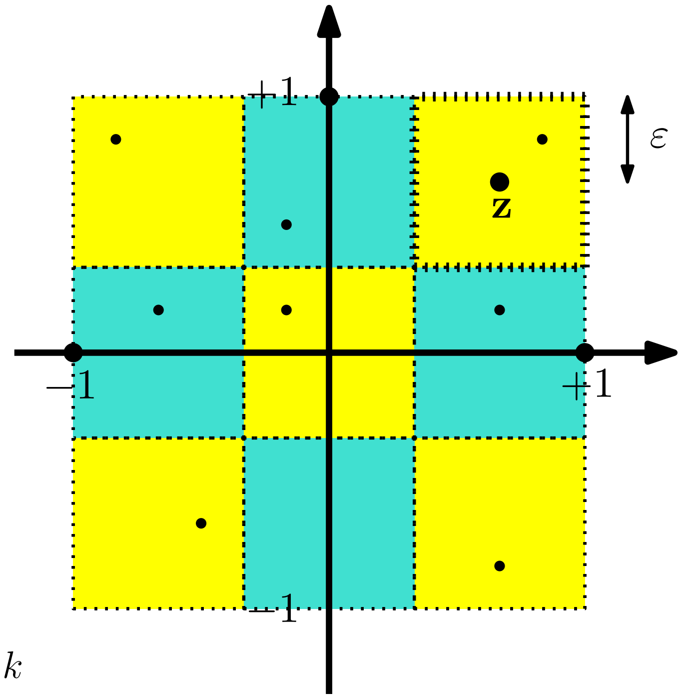

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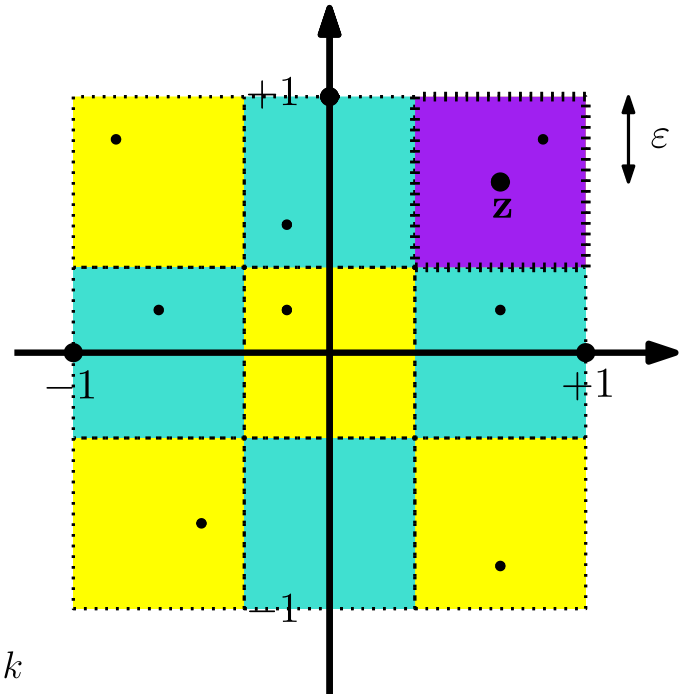

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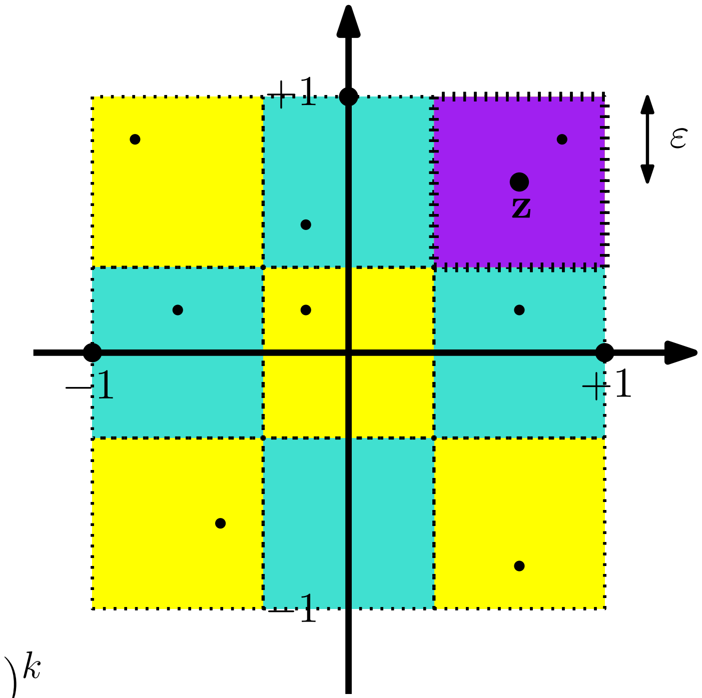

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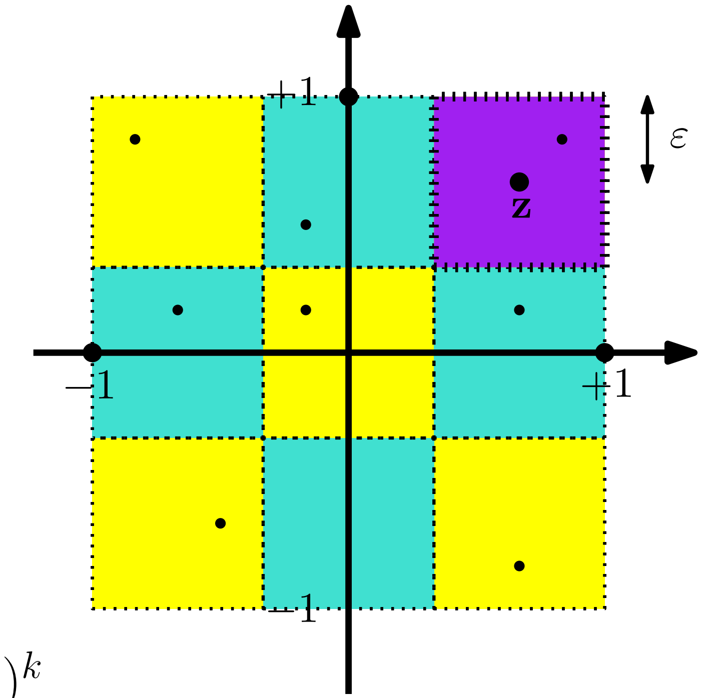

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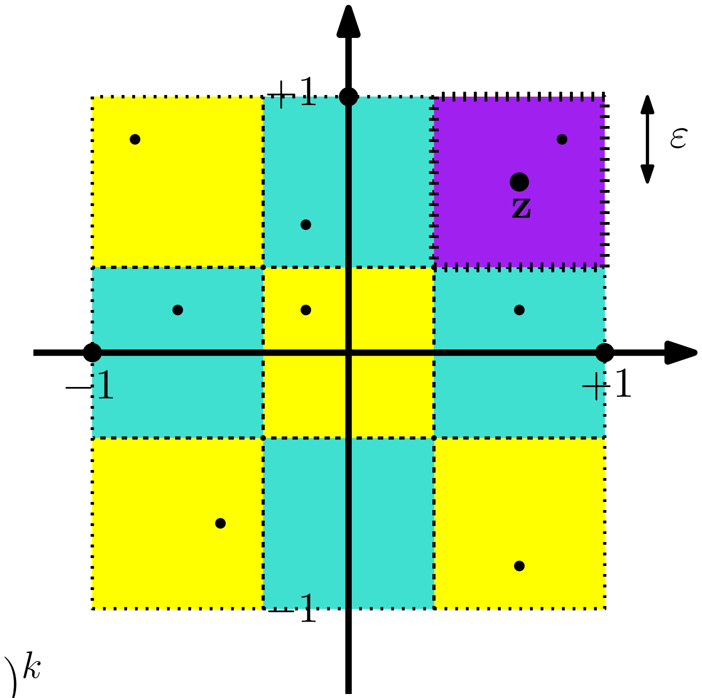

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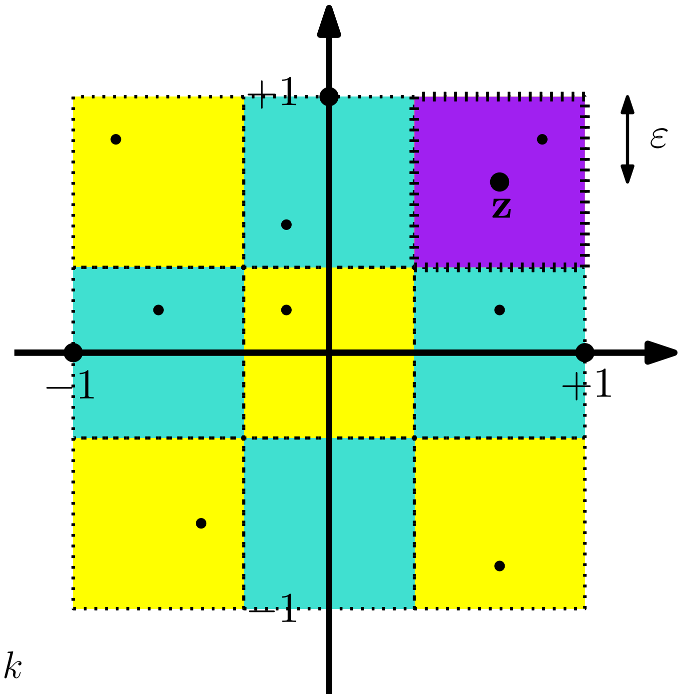

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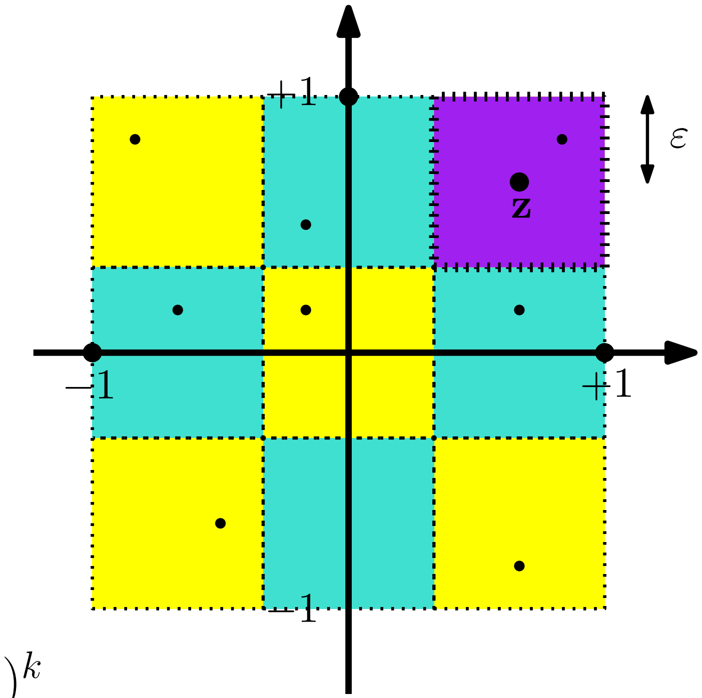

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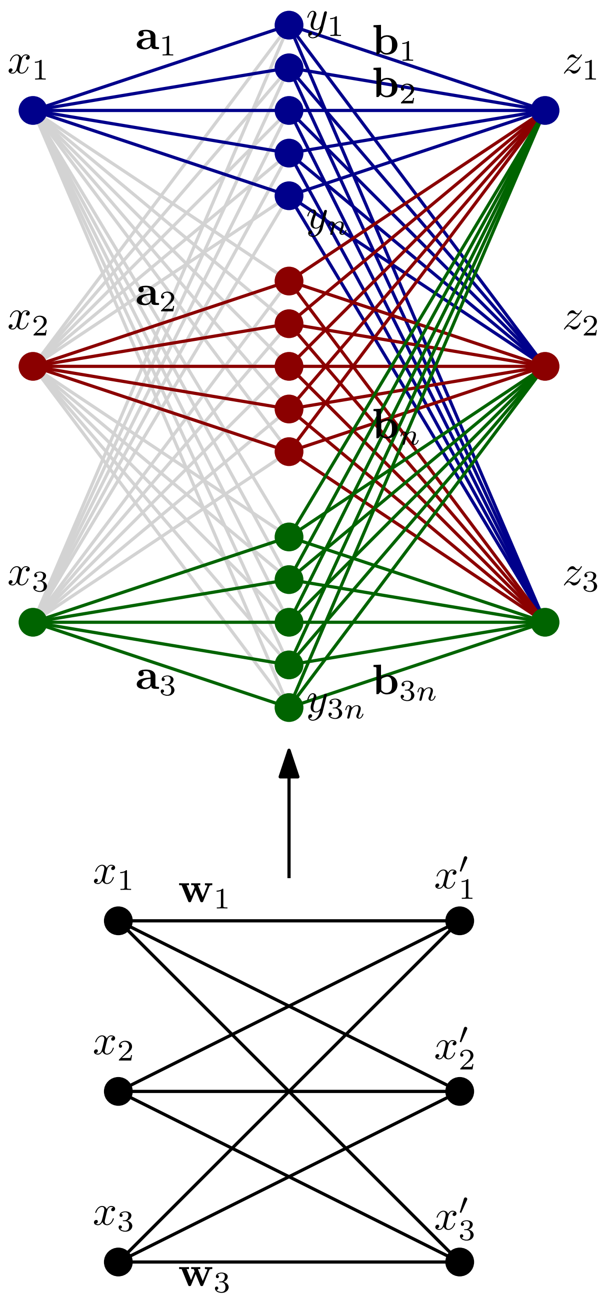
- What about approximating all the hypercube $[-1, 1]^d$? The **union bound** is highly non-optimal

Apply MRSS for structured pruning



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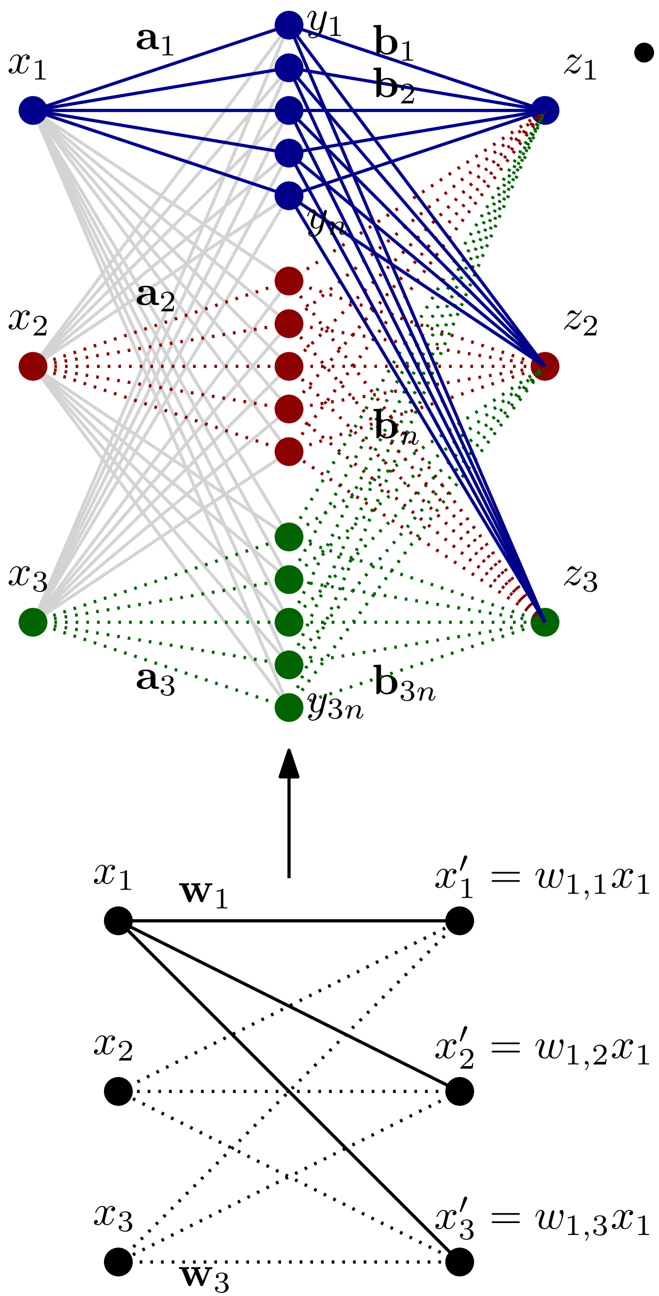

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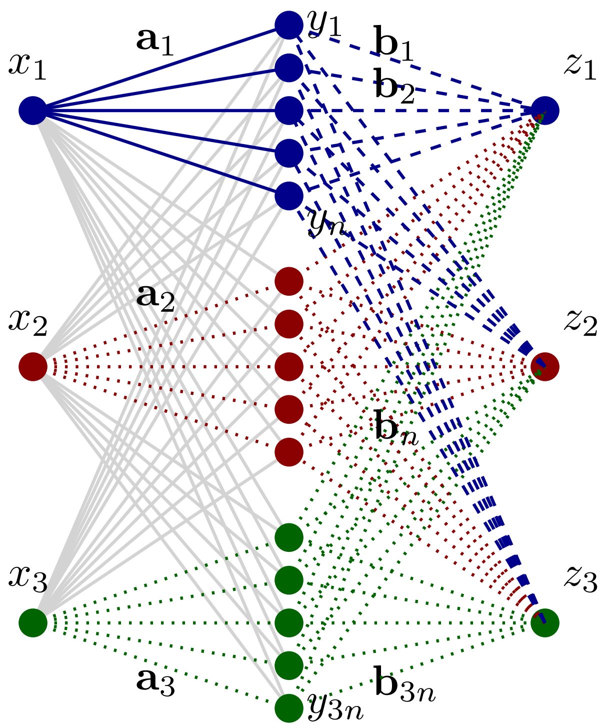

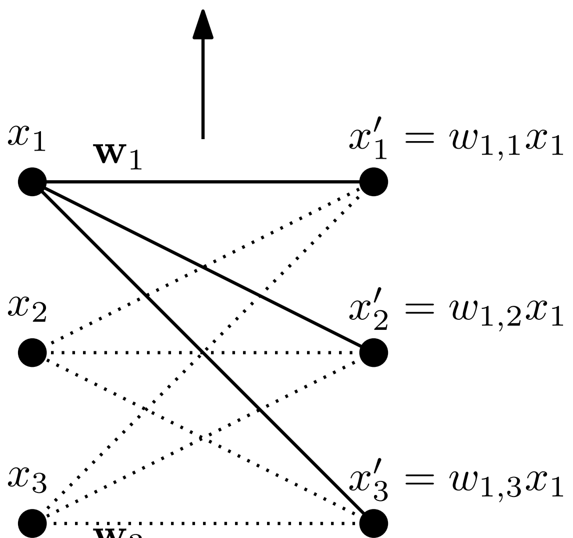

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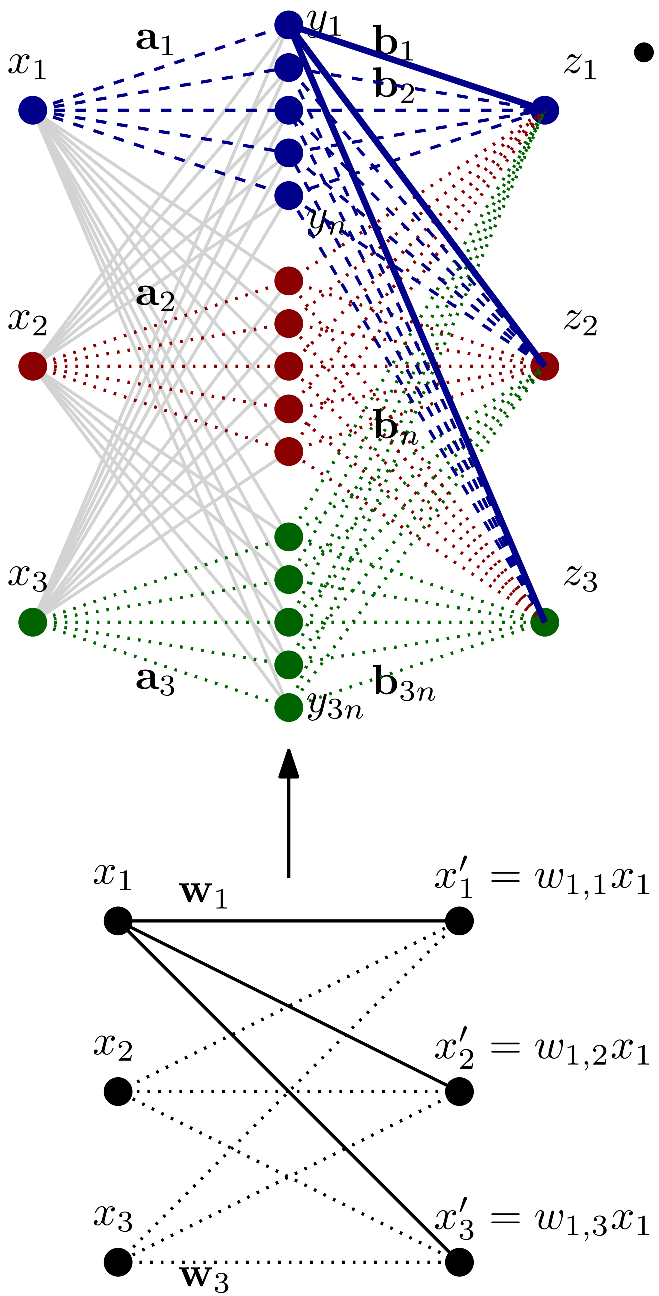

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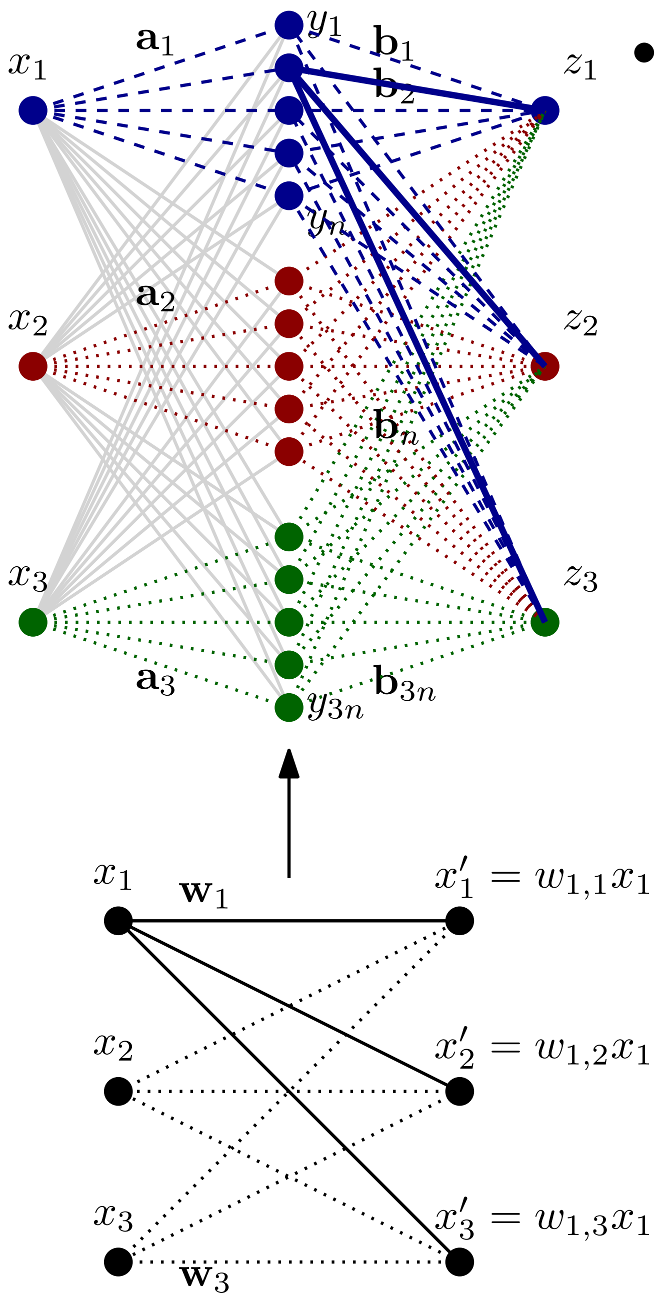

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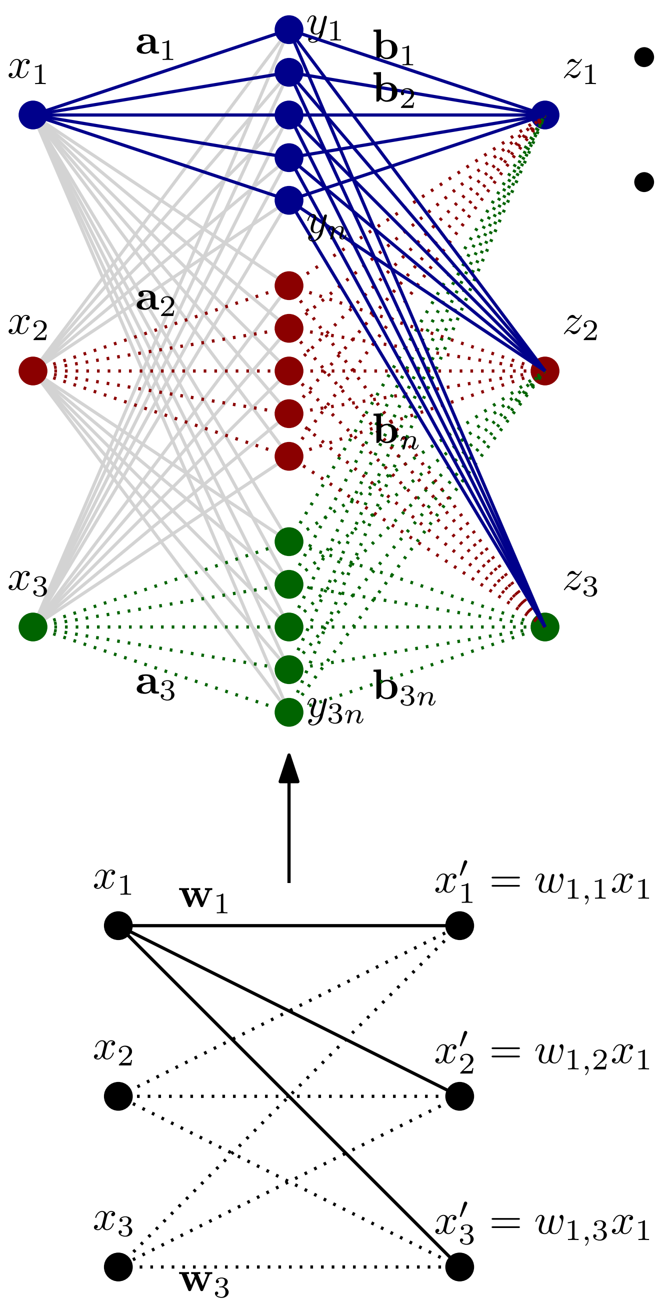

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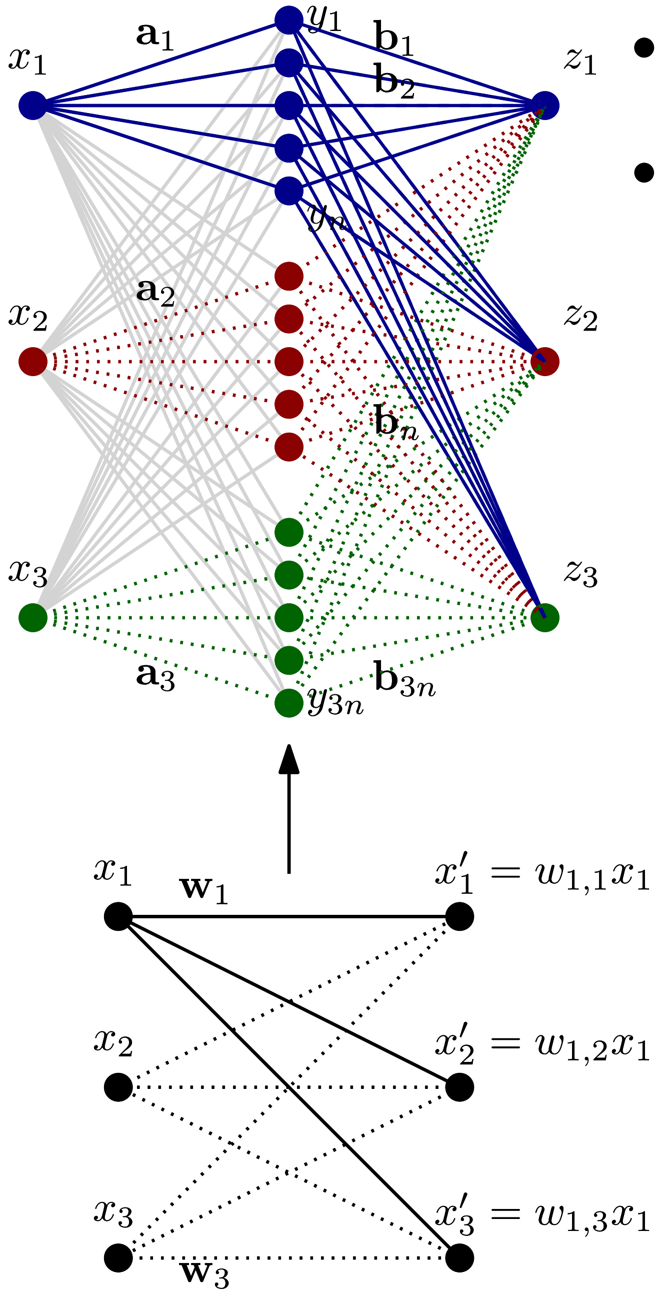

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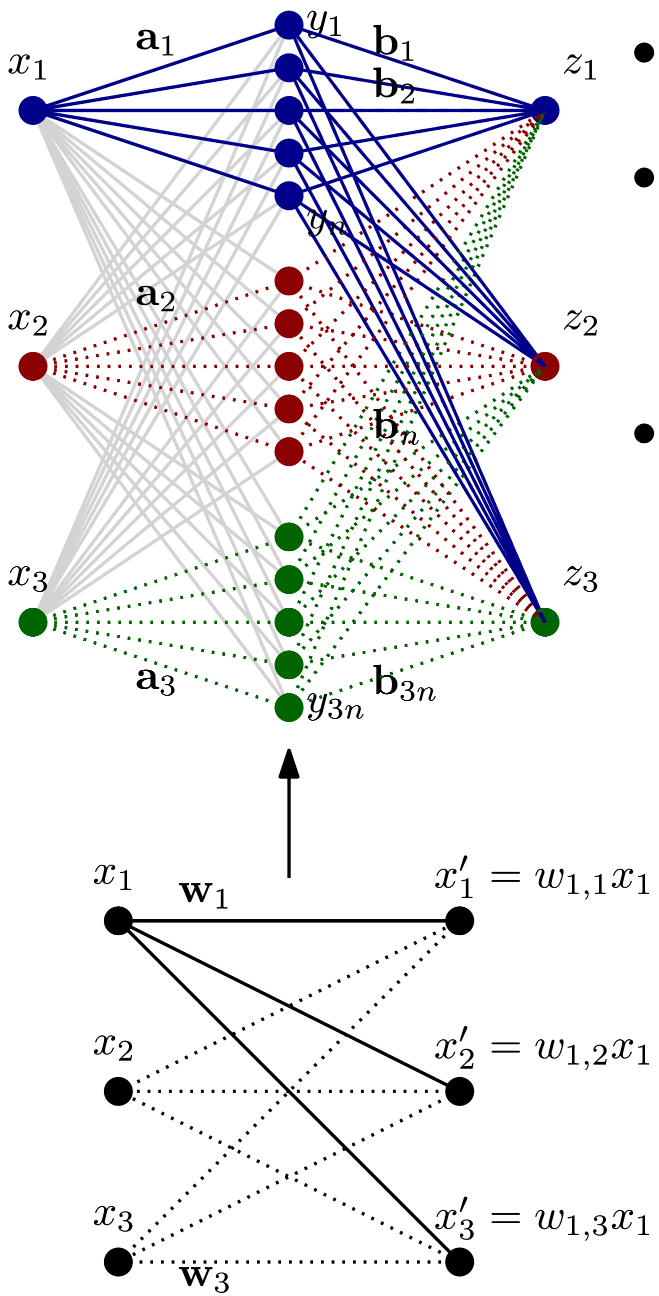

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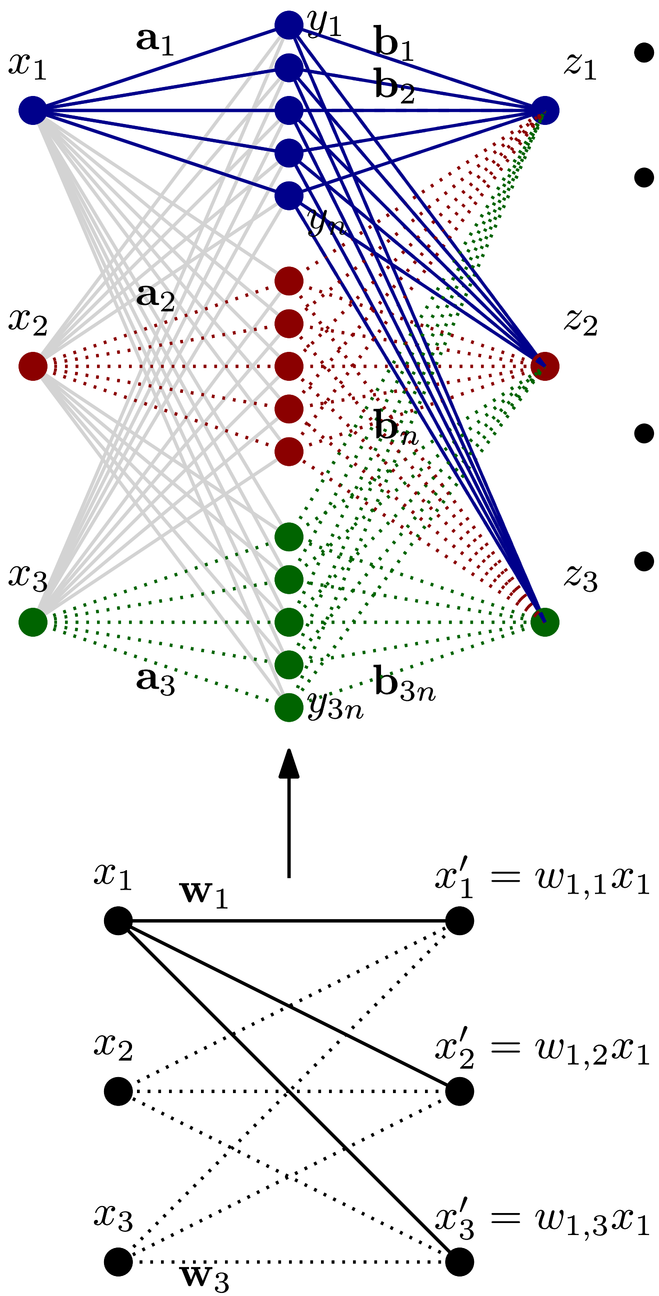

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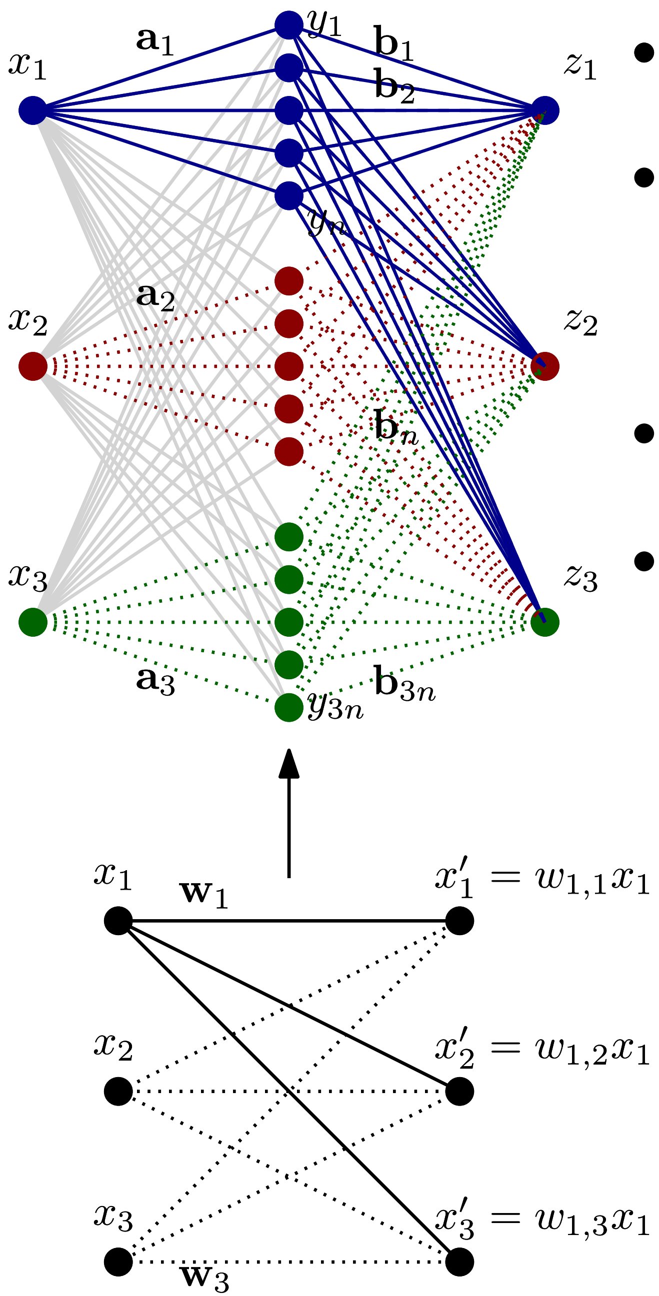

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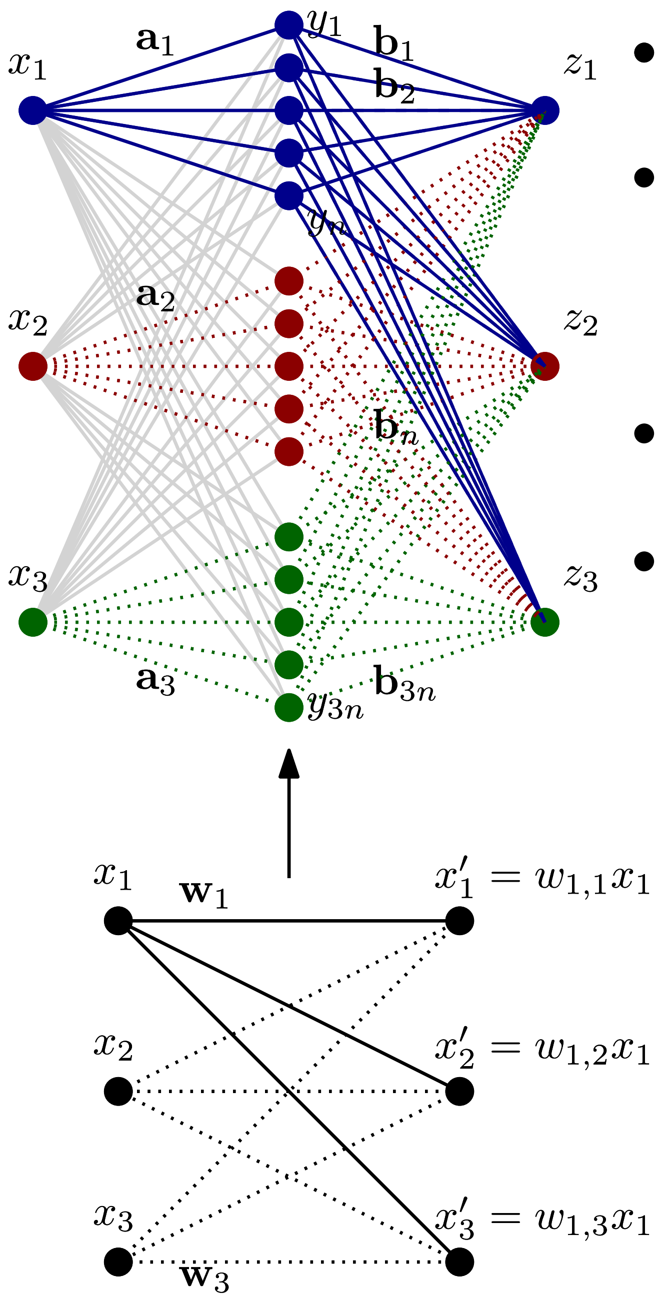

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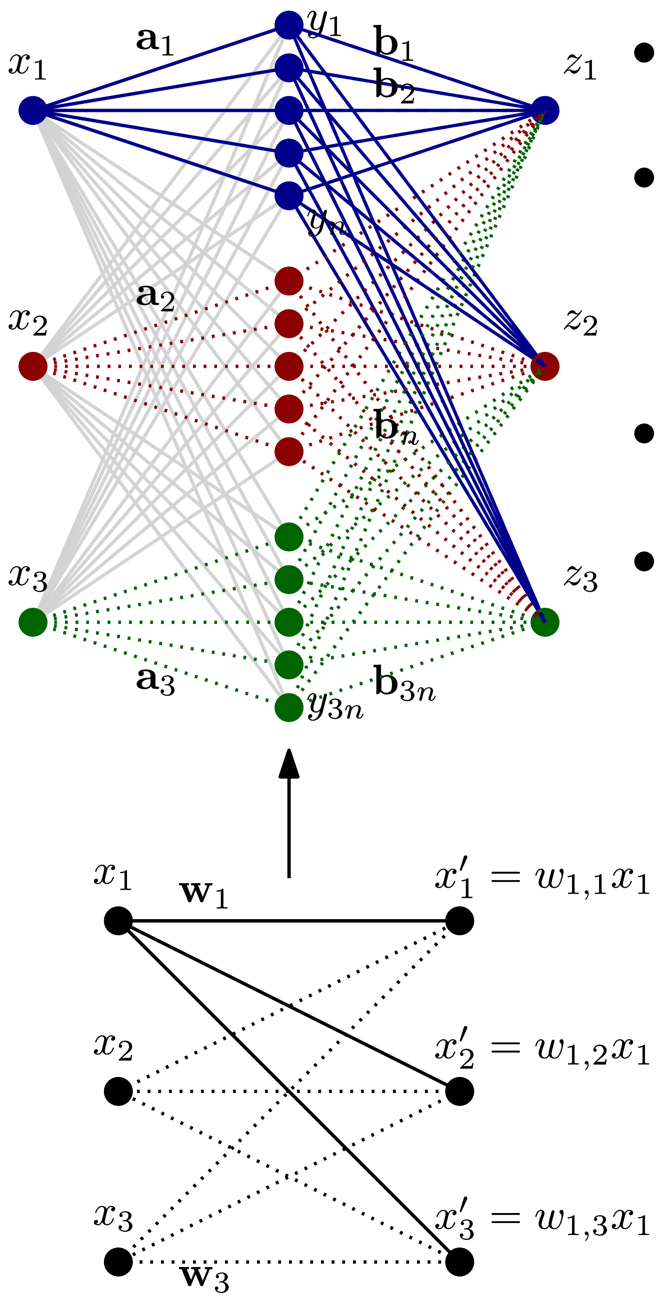

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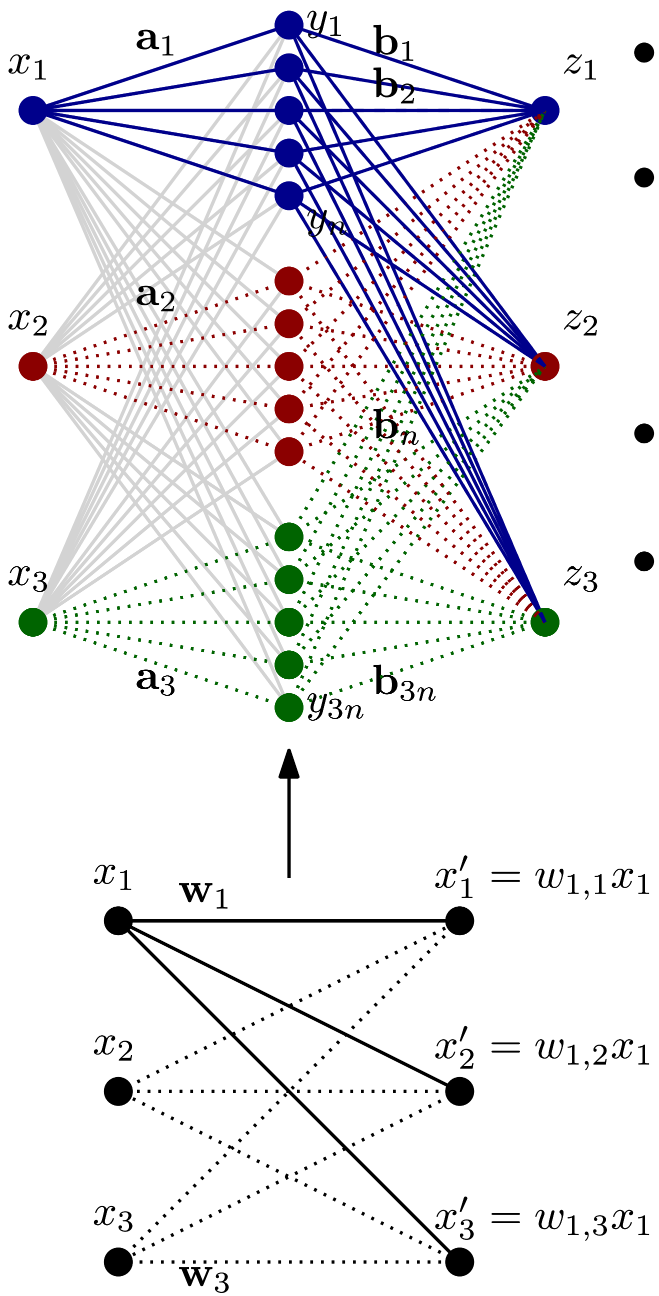

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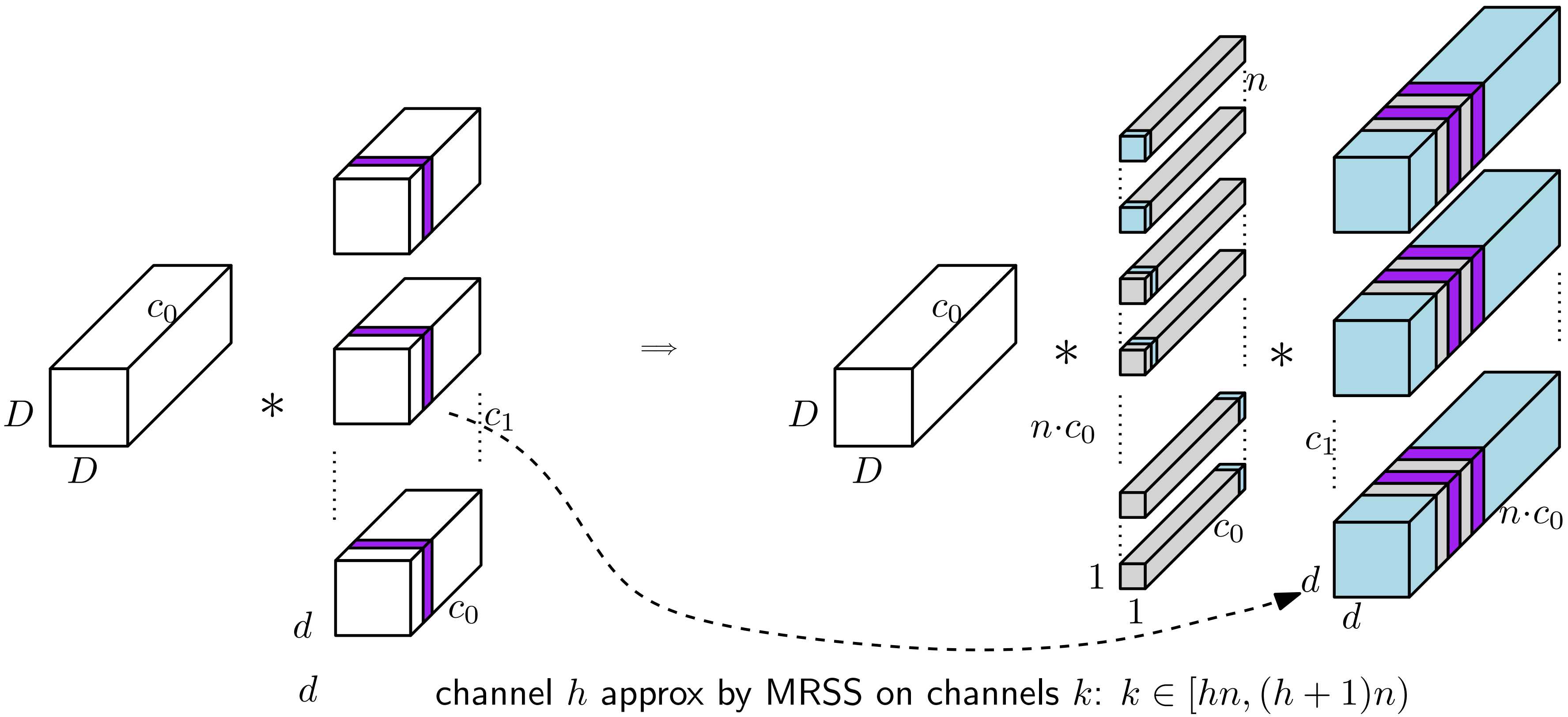
SLTH construction in CNNs



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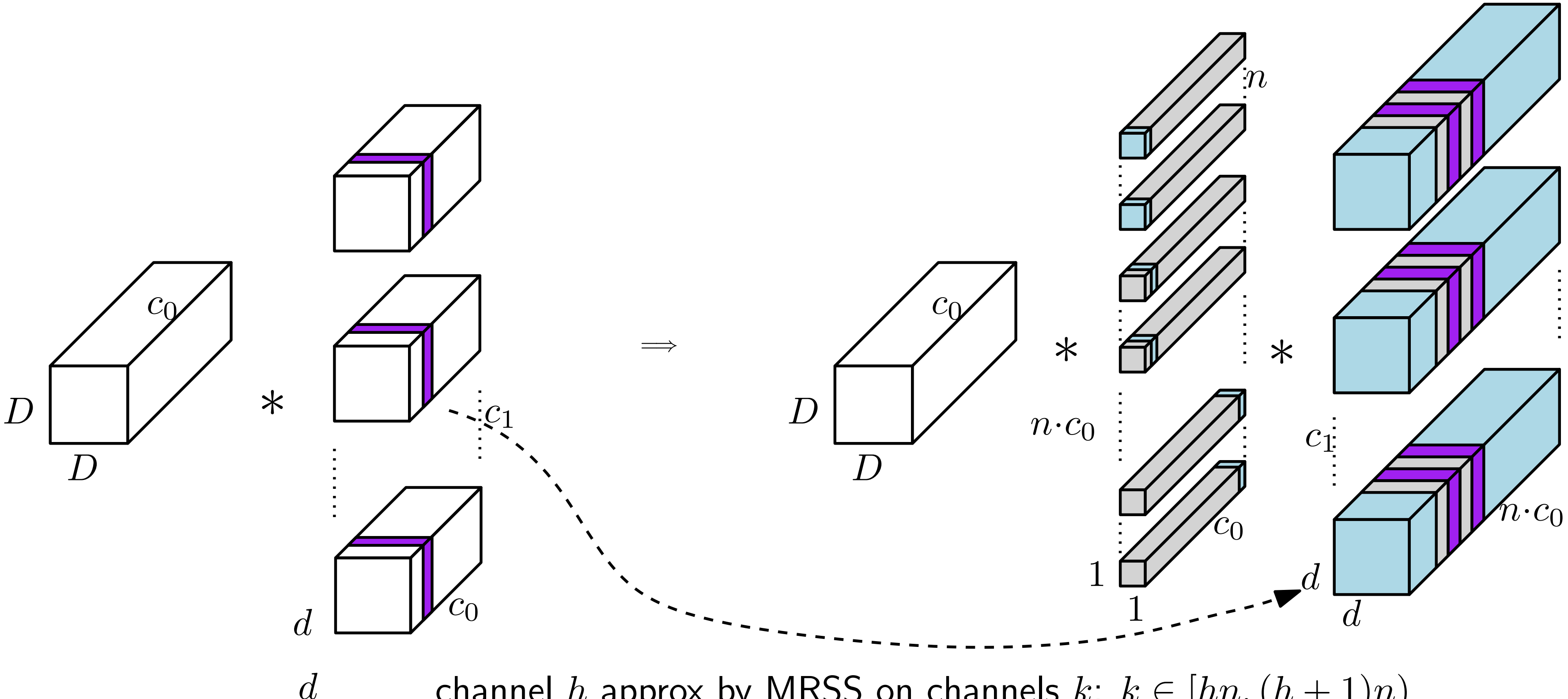

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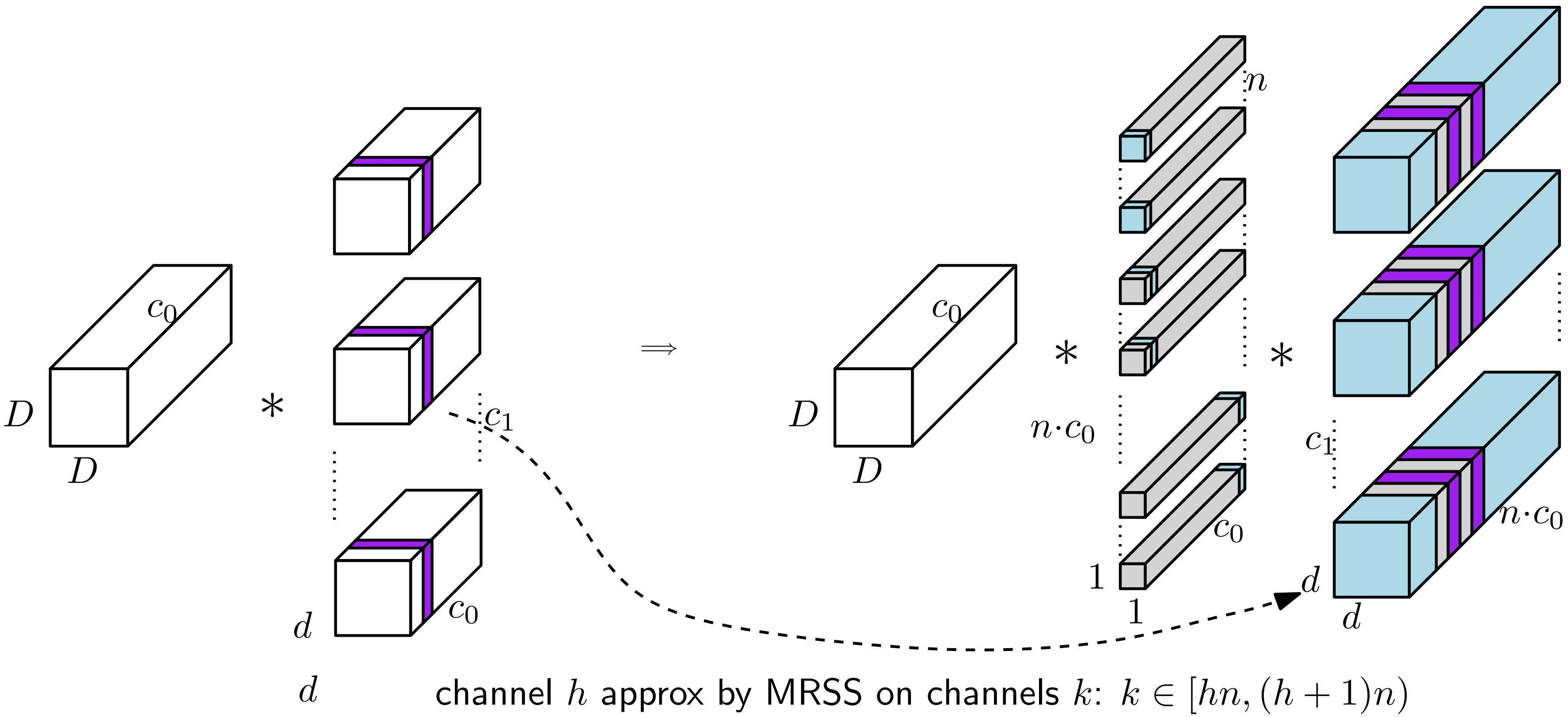
- Restrictions on the structure of the CNN

SLTH construction in CNNs



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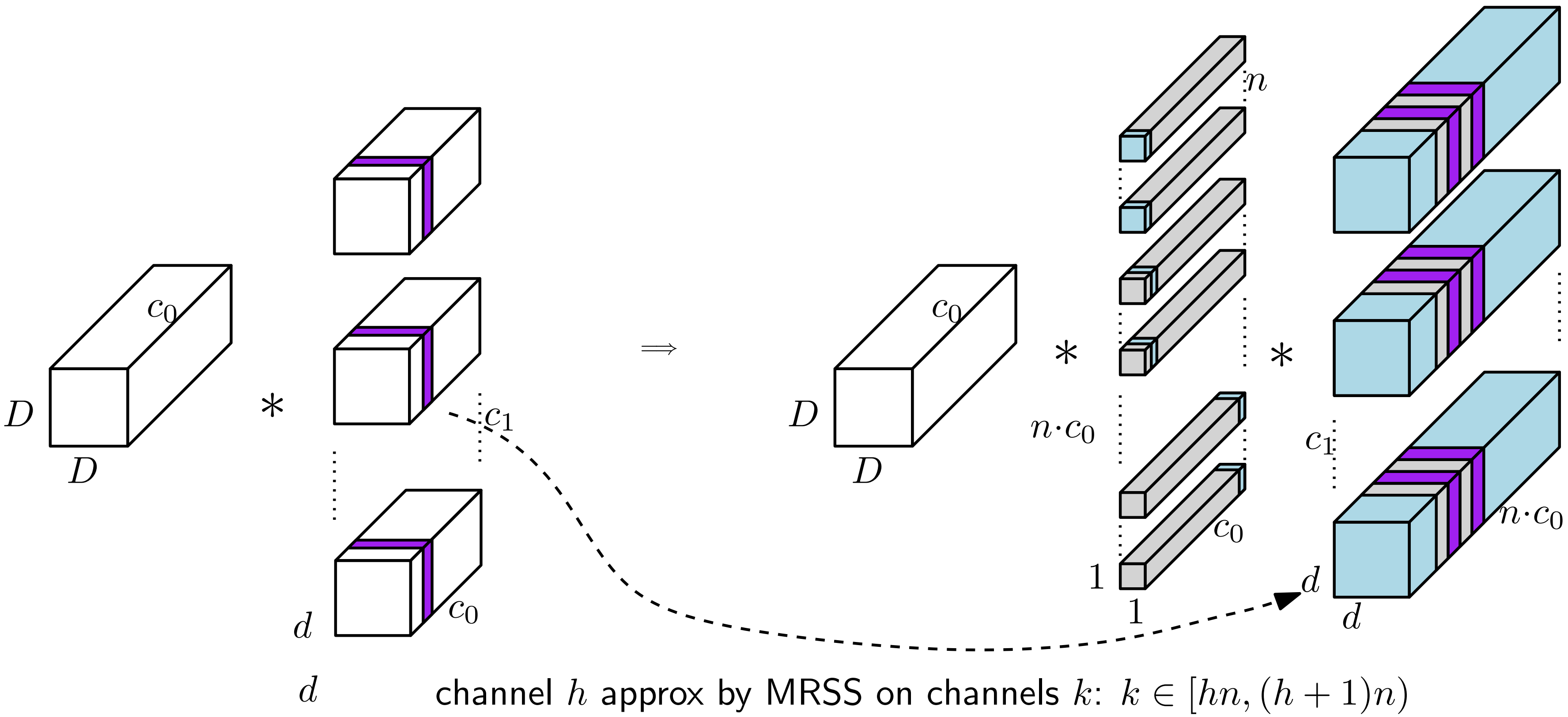

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- $n \geq$ poly$(d) \cdot$ polylog$(d\ell/\varepsilon)$ is sufficient

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Thank you!

RSS proof overview

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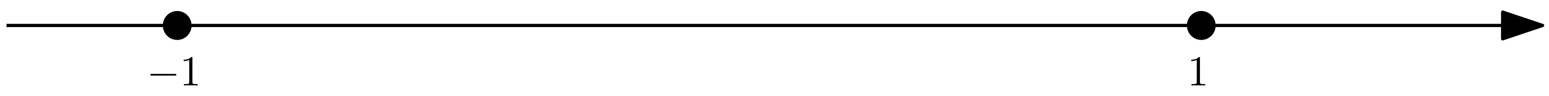
Specific instance of RSSP

- $X_1, \ldots, X_n$ uniform random variables over $[-1, 1]$
- Error parameter $\varepsilon > 0$



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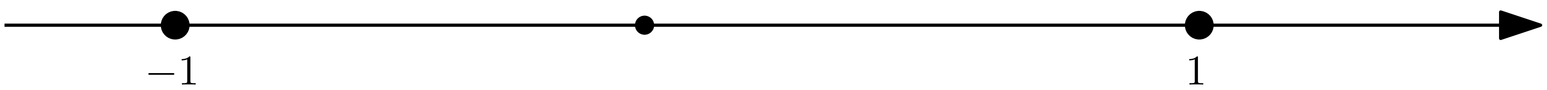

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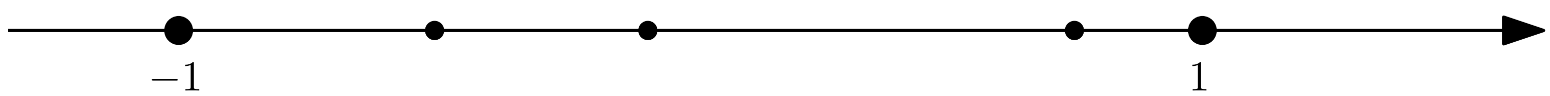

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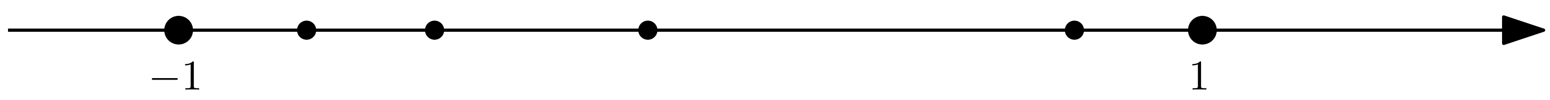

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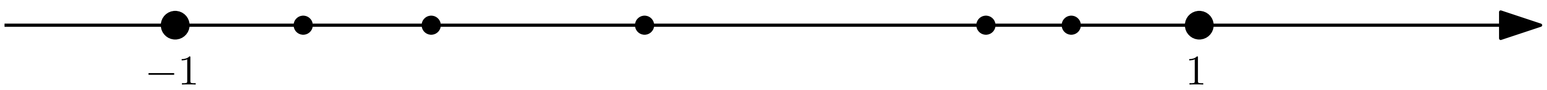

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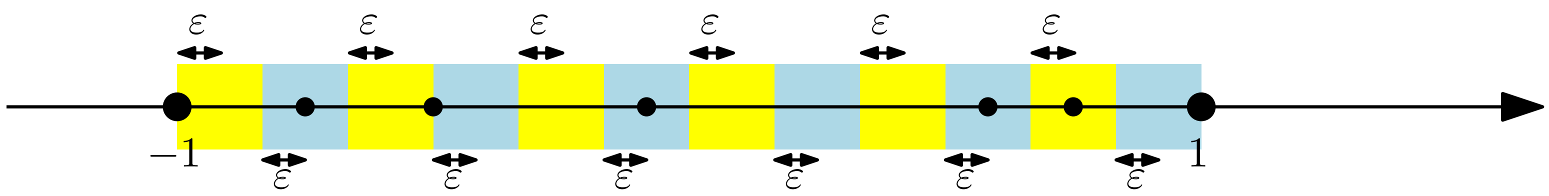

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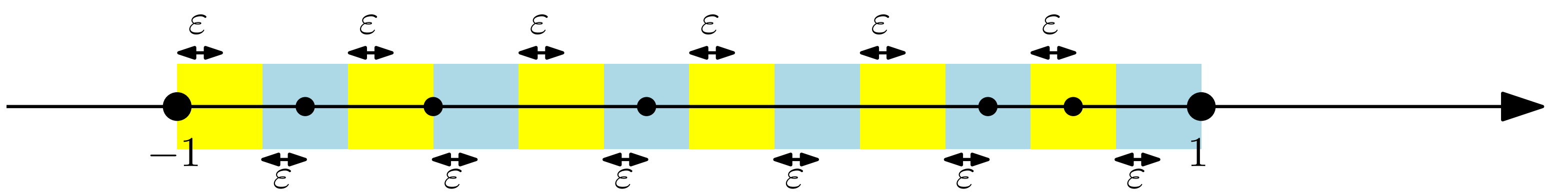

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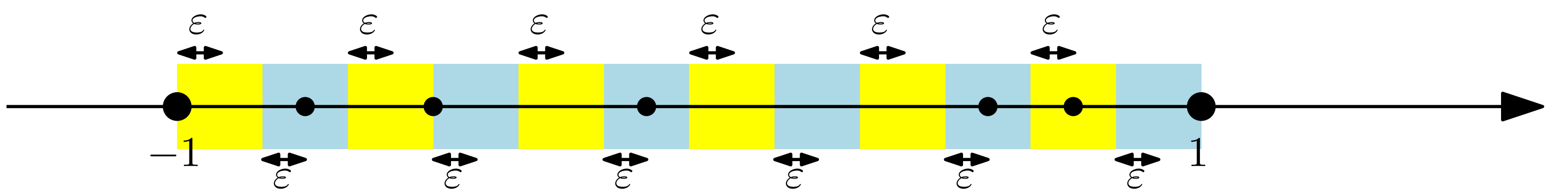

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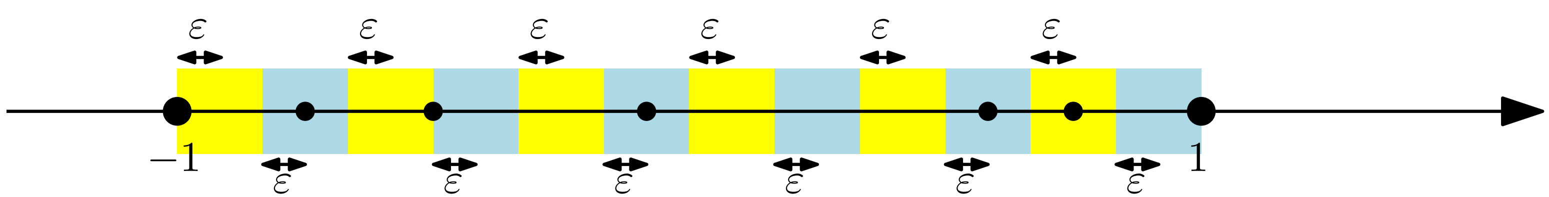

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By restricting f_t, v_t becomes a sub-martingale