Phase Transition of the 3-Majority Dynamics with Uniform Communication Noise

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COATI team

Based on joint work with I. Ziccardi

Bologna
16 June 2022

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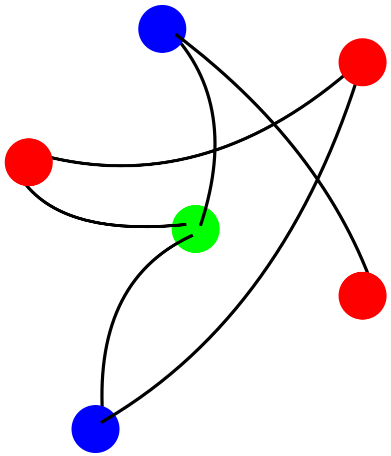
The agreement task

System of agents supporting different opinions



Messages are exchanged upon interaction

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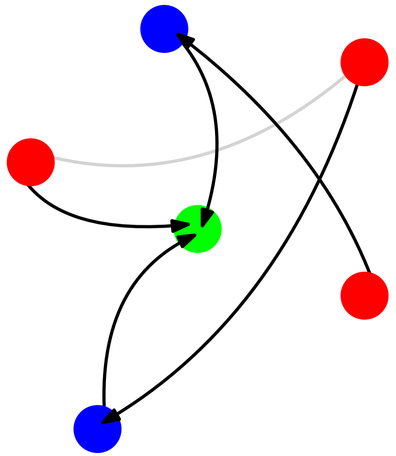

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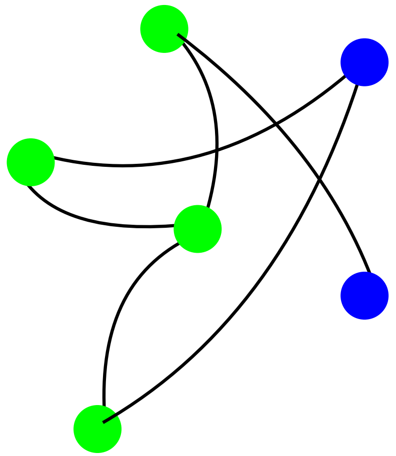

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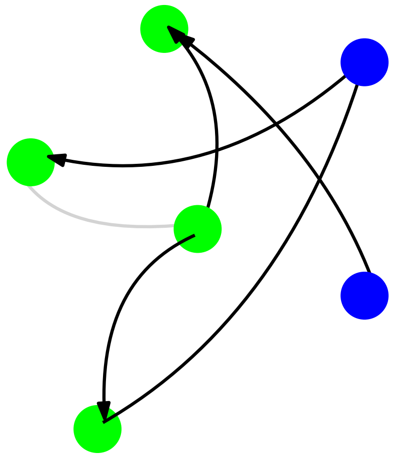

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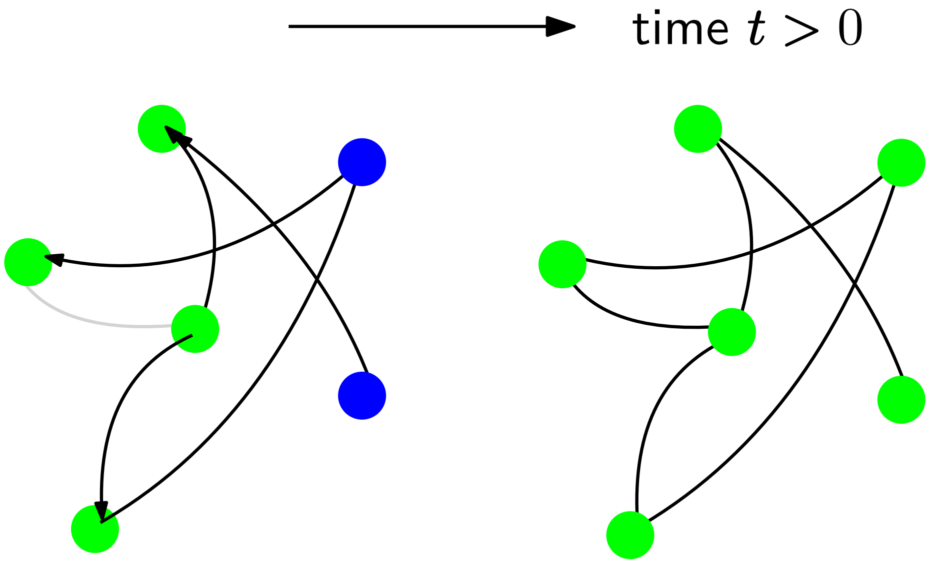

Messages are exchanged upon interaction

Reaching agreement: fundamental task in distributed computing [Becchetti et al., 2020]

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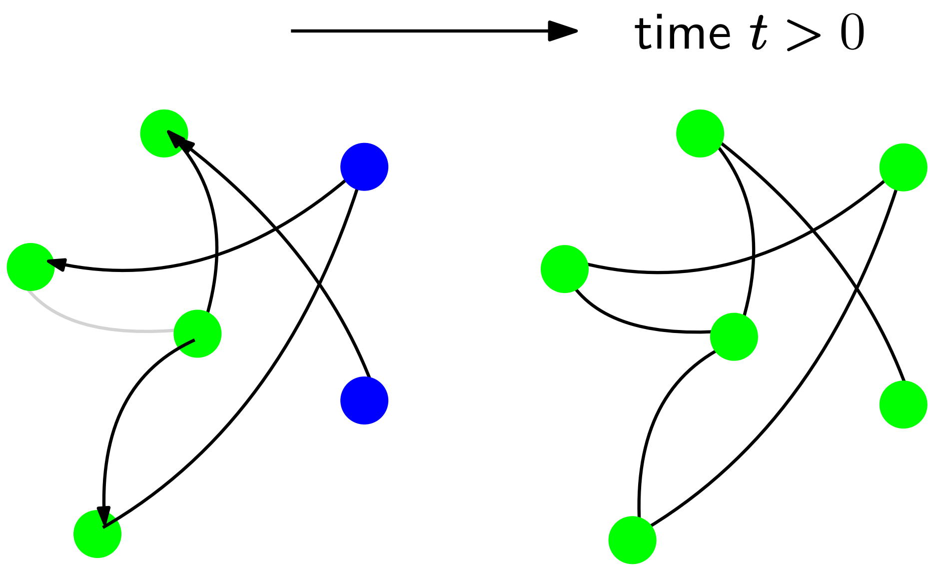

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Reaching agreement: fundamental task in distributed computing [Becchetti et al., 2020]

- social networks [Acemoglu et al., Math. Oper. Res. 2013]
- swarm robotics [Bayindir, Neurocomputing 2016]
- communication networks [Ruan et Mostofi, CDC 2008]
- distributed databases [Dietzfelbinger et al., ICALP 2010]
- biological systems [Feinerman et al., Dis. Comput. 2017]

In biological systems

Often, systems of interacting agents performing collective tasks (swarm-like)

- Reaching agreement: molecules [Carrol, Nature Immunology 2004], bacteria [Bassler, Cell 2002], social insects [Franks et al., 2002] (e.g. bees [Reina et al., Physical Review E 2017])

honey bees

"Queen bee 1" by quisnovu, CC BY-NC 2.0.

The consensus problem

Input: system of n agents supporting opinions, with a communication network

Task: designing a protocol which **brings the system** in finite time to a configuration such that

1. all agents support the same opinion (AGREEMENT)
2. the final opinion is among the initial ones (VALIDITY)
3. the agreement keeps on unless external events occur (STABILITY)

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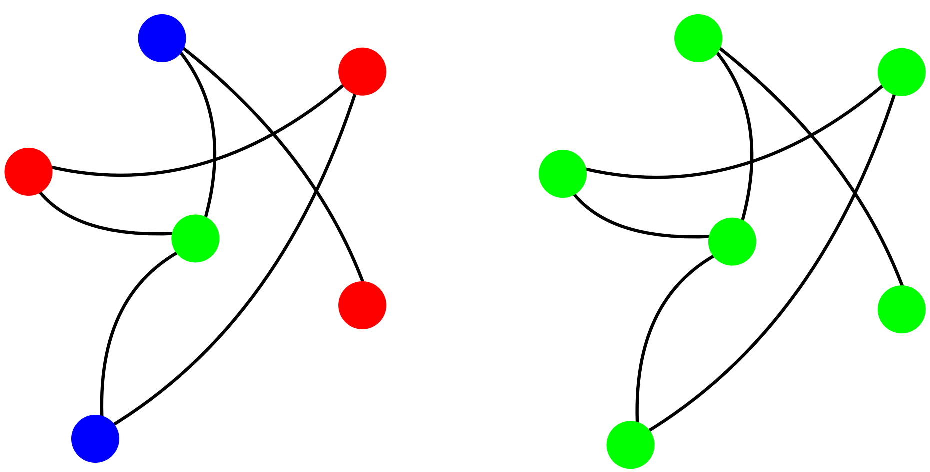

valid consensus

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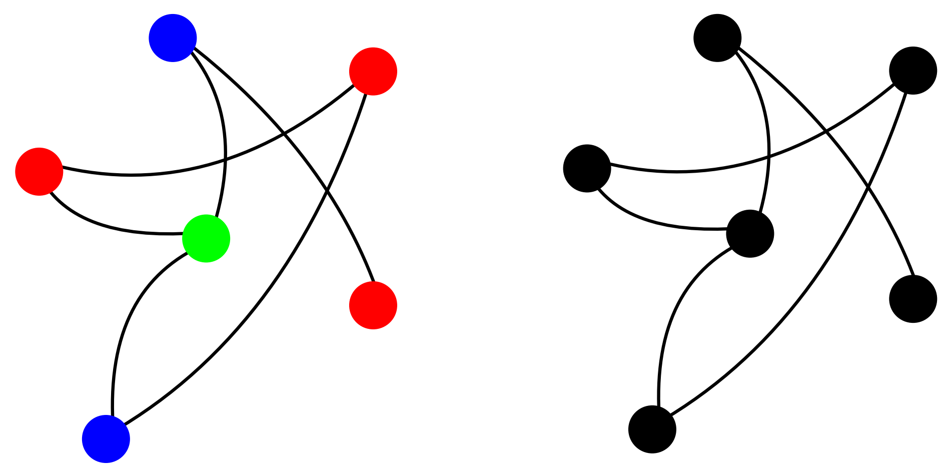


non valid consensus

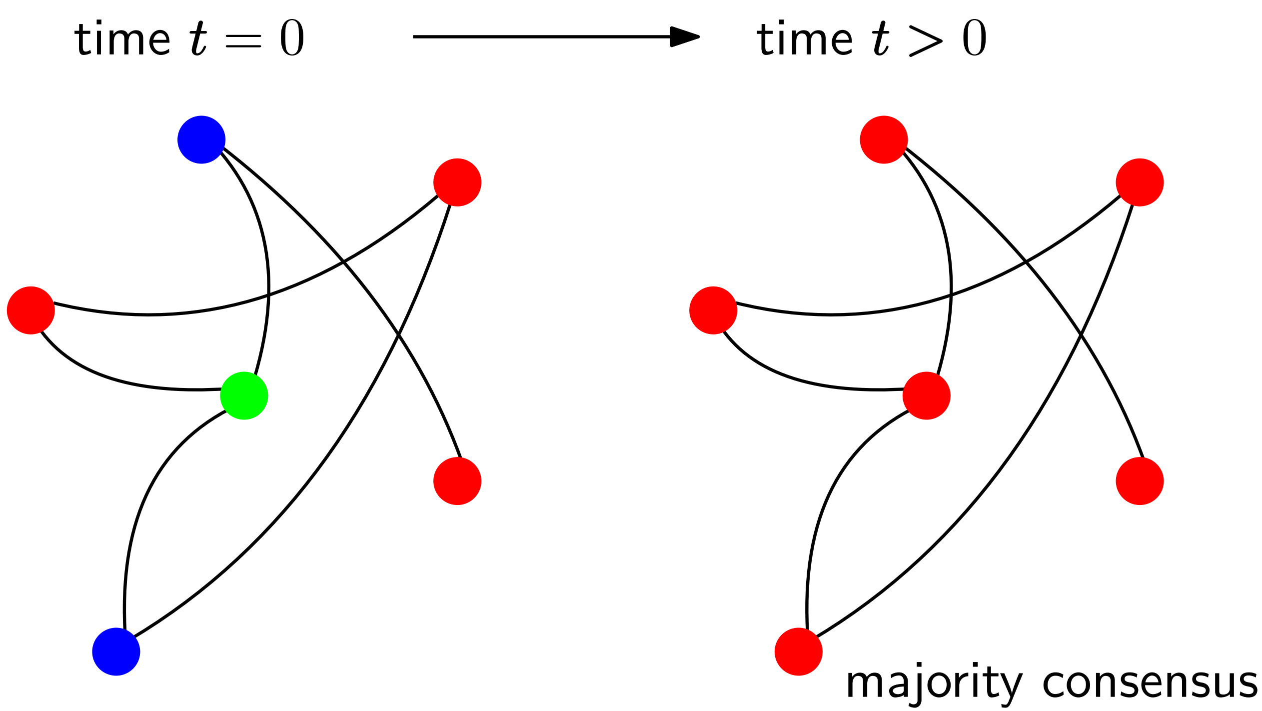
The majority consensus problem

1. AGREEMENT
2. ~~VALIDITY~~
3. STABILITY

$\longrightarrow$

1. AGREEMENT
2. **MAJORITY**
3. STABILITY

2. MAJORITY property: the final opinion is the **initial majority** one



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Opinion dynamics for the consensus problem

Opinion dynamics: class of simple, lightweight parallel protocols for the consensus problems

Very simple update-rules

Many have been investigated, including:

- Voter Model [Hassin and Peleg, Inf. Comput. 2001]
- Averaging dynamics [Becchetti et al., SODA 2017]

linear dynamics

- 3-Majority [Becchetti et al., SODA 2016]
- 2-Choices [Berenbrink et al., PODC 2017]
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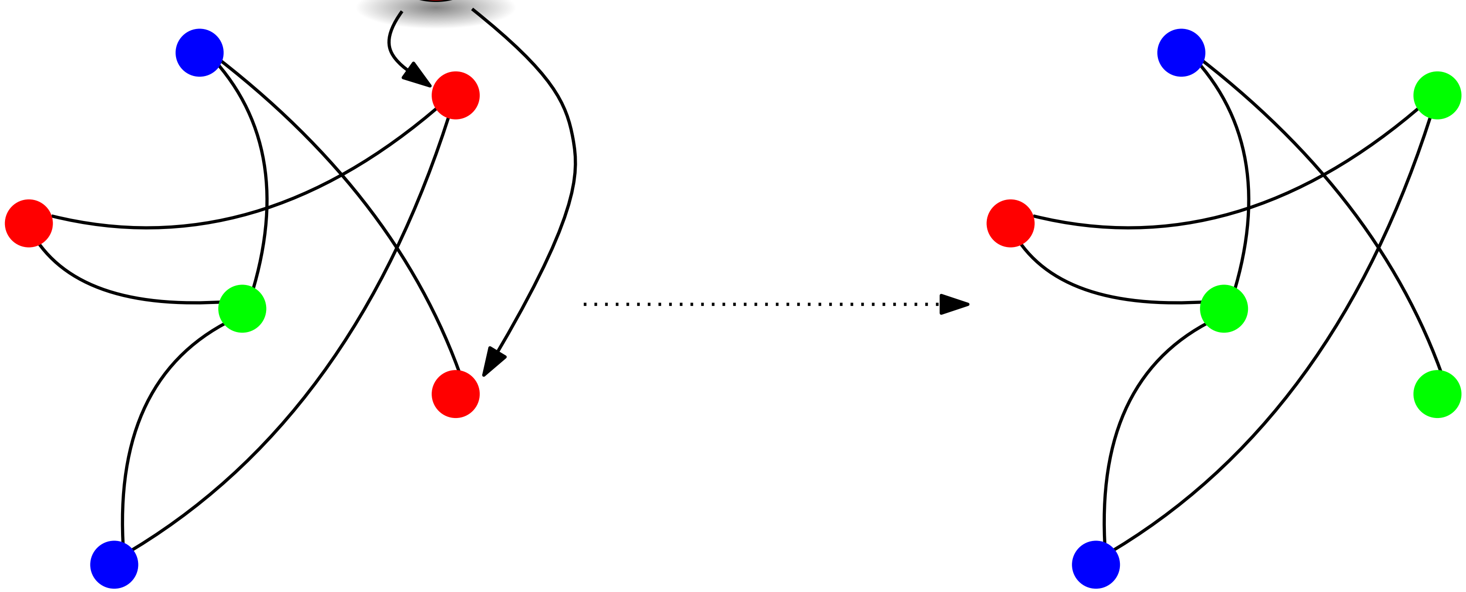
non-linear dynamics

Majority update-rules and the undecided state dynamics have biological inspirations [Reina et al., Physical Review 2017] [Condon et al., Nat. Computing 2020] [Chaouiya et al., PLOS ONE 2013]

Adversarial failures vs. noise

Often, settings with adversarial Byzantine failures are investigated

adversary  changes $\mathcal{O}\left(F\right)$ opinions each round **adversarially**



Not realistic in biological scenarios; rather, uniform noise [Feinerman et al., PODC 2014]

Uniform communication noise

Inspired by [Feinerman et al., PODC 2014], [Freignaud and Natale, PODC 2016]

Σ set of k opinions, $p \in [0, 1]$ constant

When u looks at v's opinion x

a) with probability $1 - p$, u sees x
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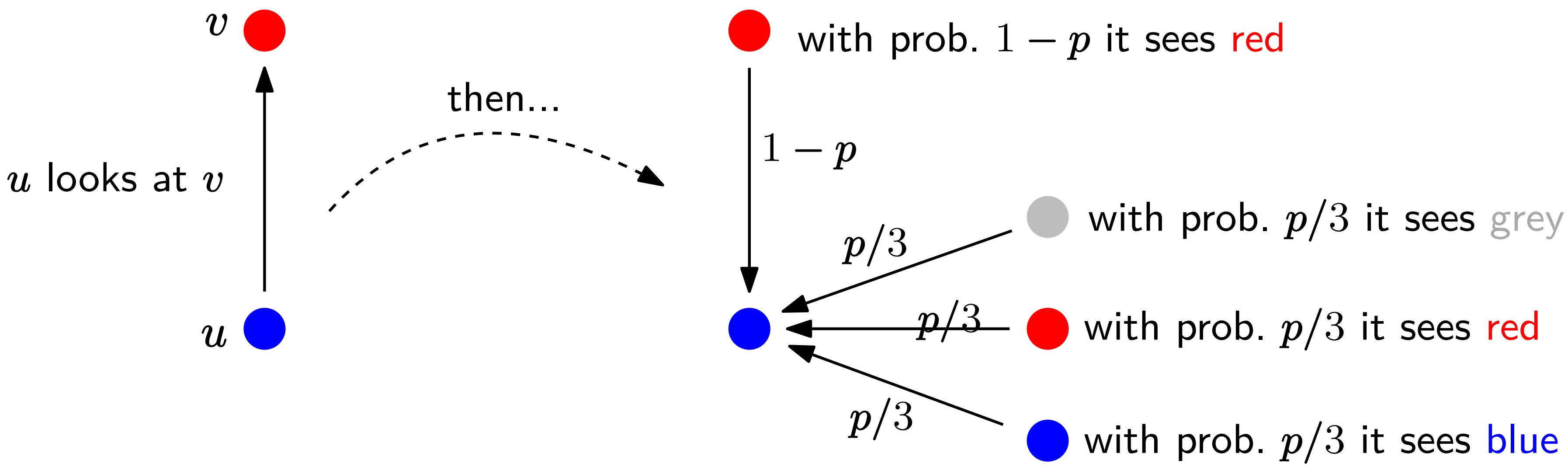
Example: $k = 3$



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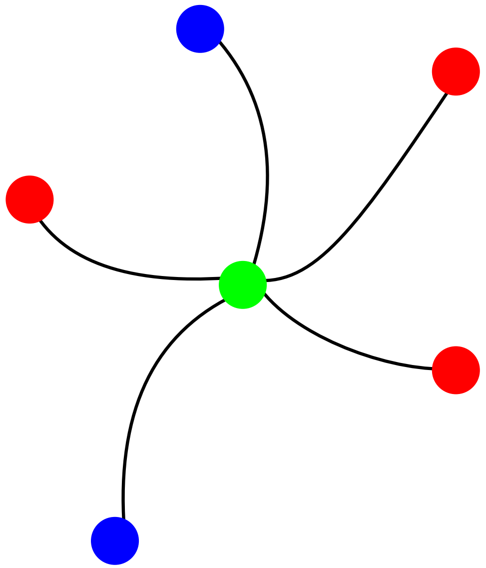
The dynamics

3-Majority dynamics: each node u

1. samples 3 neighbors u.a.r.
2. pulls their opinions
3. updates its opinion to the majority one, if any



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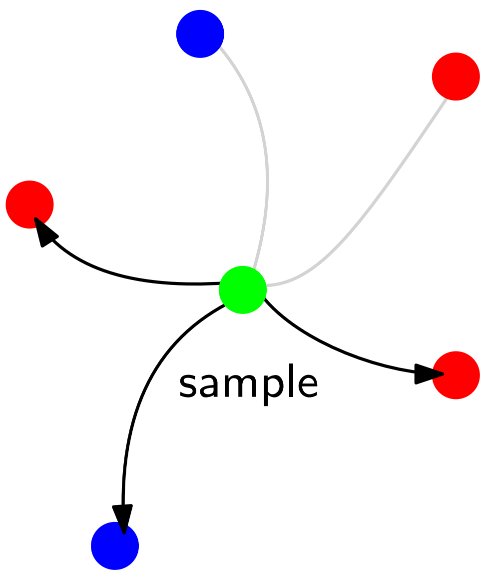

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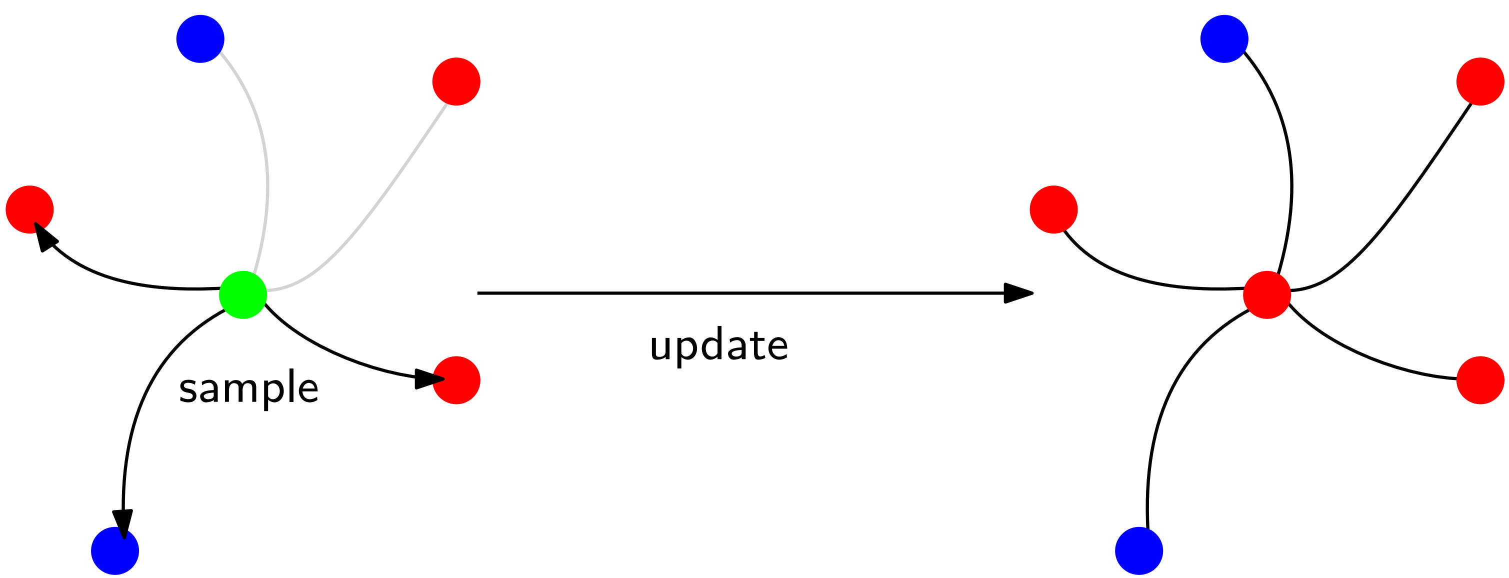

In **parallel**: every agent at each round

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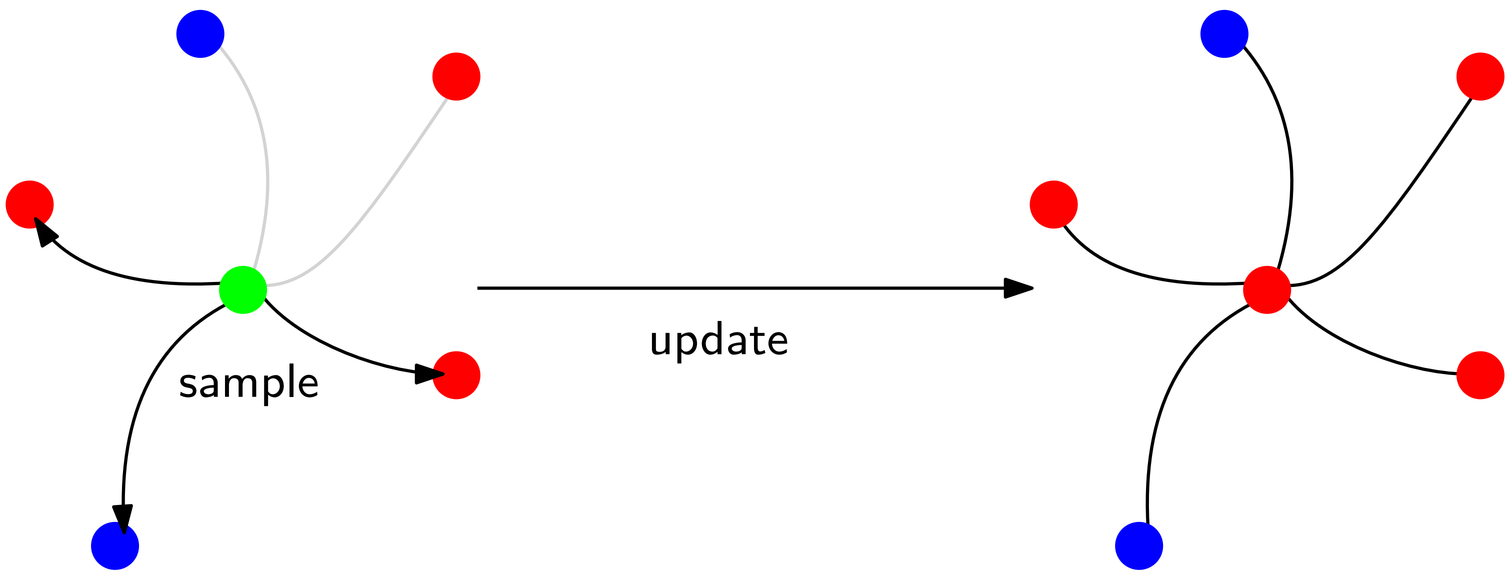

In **parallel**: every agent at each round

Markov chain with k absorbing states

Previous results

Overview of results in noiseless settings [Becchetti et al., SIGACT News 2020]

Binary case: (with high probability)

- in time $\mathcal{O}(\log n)$ the system reaches consensus
- if the initial bias is $\Omega(\sqrt{n \log n})$, we have majority consensus
- robust against adversaries with power $o(\text{initial bias})$

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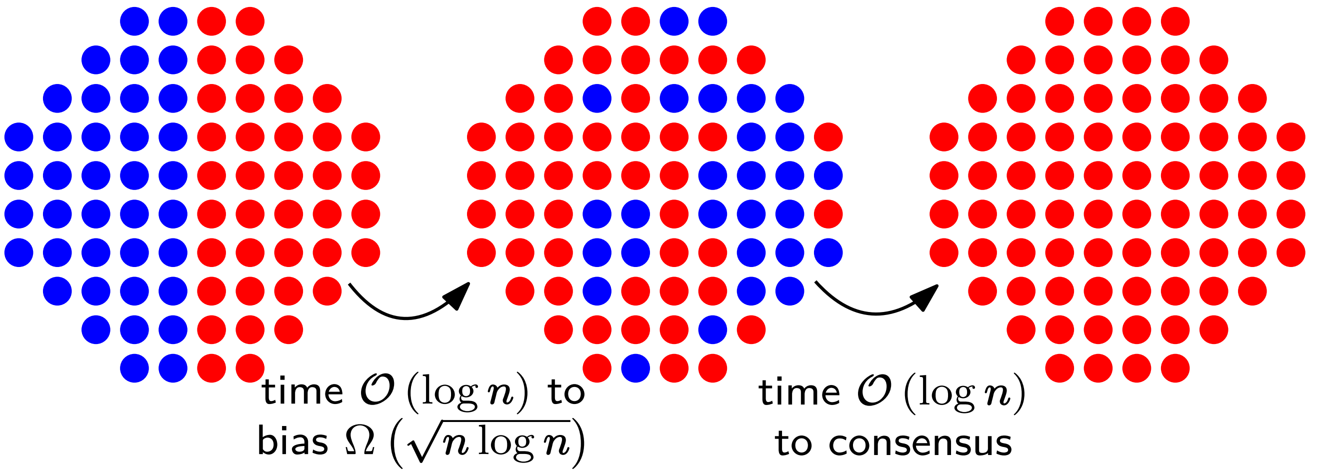

Our contribution

[d'Amore et Ziccardi, SIROCCO 2022]

3-Majority dynamics with $k = 2$ opinions in the presence of uniform noise in the compelete graph

Question: is there any form of metastable consensus?

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For the 3-Majority dynamics: phase-transition

$p < 1/3$:
- a value $\bar{s} = \Theta(n)$ exists such that the bias of the system reaches the interval $I_\varepsilon = [(1-\varepsilon)\bar{s}, (1+\varepsilon)\bar{s}]$ in time $\mathcal{O}(\log n)$ w.h.p., and keeps in I_ε for time poly(n) w.h.p. $\longrightarrow$ **almost-consensus**
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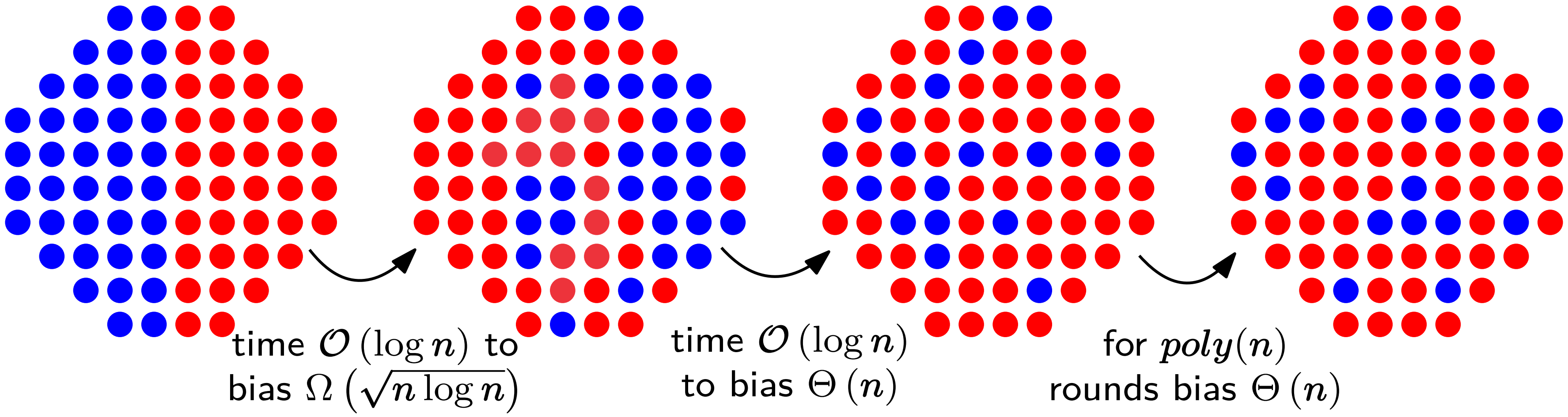
$p > 1/3$:

- in time $\mathcal{O}(\log n)$ the bias becomes bounded by $\mathcal{O}(\sqrt{n \log n})$ and keeps bounded for time poly(n) wh.p. $\longrightarrow$ **victory of noise**
- there is constant probability to switch majority within time $\mathcal{O}(\log n)$

Examples

Almost-consensus



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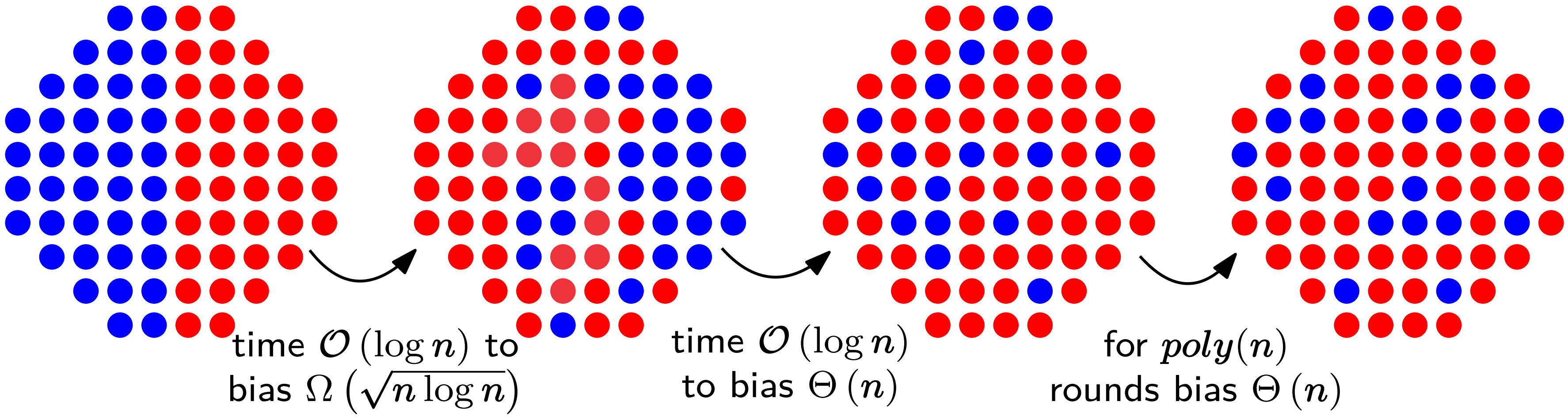
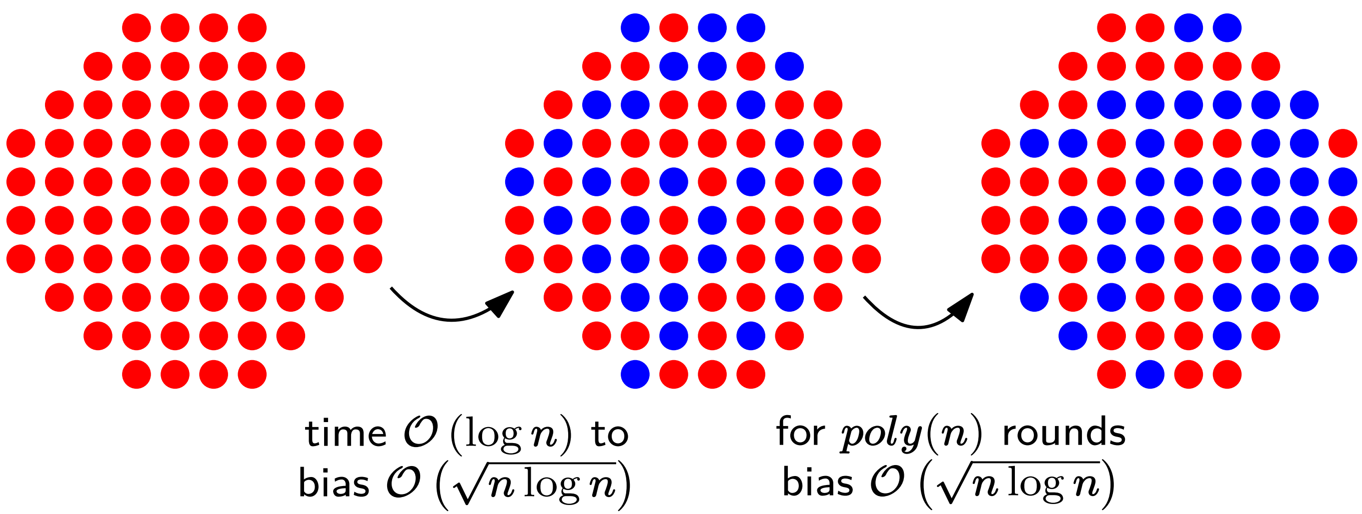


Victory of noise



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Comparison with previous results

[d'Amore et al., SIROCCO 2020]

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$u \backslash v$	opinion i	opinion j	undecided
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- phase-transition at $p = 1/2$, more resilient to noise
- less characterized
  - no precise equilibrium shown
  - no switch of majority shown

Techniques

Equilibrium:

$$\mathbb{E}\left[s_{t+1} \middle| s_t\right] = \frac{s(1-p)}{2}\left(3 - \frac{s_t^2}{n^2}(1-p)^2\right) \implies \bar{s} = \frac{n}{1-p} \cdot \sqrt{\frac{1-3p}{1-p}}$$

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Techniques

Equilibrium:

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For **majority consesus**:

- initial bias $\Omega(\sqrt{n \log n})$
- $M_t = |s_t - \bar{s}|$, $\bar{s}$ equilibrium value for bias
- Chernoff-Hoeffding inequalities to show $M_{t+1} \leq (1-\delta)M_t$ w.h.p. when bias outside $I_\varepsilon = [(1-\varepsilon)\bar{s}, (1+\varepsilon)\bar{s}]$
- chain rule + union bound

Techniques

For **victory of noise**:

- initial bias $s = n$
- super-martingale $N_t \sim s_t$ such that $\mathbb{E}\left[N_{t+1} | \mathcal{F}_t\right] \leq (1-\delta) N_t$
- concentration arguments [Lehre and Witt, ISAAC 2014] $\implies$ majority disappears in time $\mathcal{O}\left(\log(n)\right)$

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For **symmetry breaking**:

- initial bias $o(\sqrt{n \log n})$
- the bias has enough standard deviation to break symmetry (drift analysis results)
  - std roughly $\sqrt{n}$
  - $\mathcal{O}\left(\log n\right)$ trials suffices

Discussion

- **3-Majority** dynamics with noise is not implemented by biological systems
  - despite the bio-inspiration, highly abstract model
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  - $p = 1/3$ (3-Maj) means 2 out of 3 pulled opinions are non-noisy on average

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- Are noise-thresholds independent of k (num. of opinions)?

- What about a generalized noise with different parameters for the two opinions?

- What about sparser topologies, e.g. expanders?

The End

Questions?