Limits of Distributed Quantum Computing

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Based on [STOC '24, STOC '25a, STOC '25b, SODA '26]

Joint works with Amirreza Akbari, Alkida Balliu, Sebastian Brandt, Filippo Casagrande, Xavier Coiteux-Roy, Massimo Equi, Rishikesh Gajjala, Barbara Keller, Fabian Kuhn, François Le Gall, Henrik Lievonen, Darya Melnyk, Augusto Modanese, Dennis Olivetti, Shreyas Pai, Marc-Olvier Renou, Václav Rozhon, Gustav Schmid, Jukka Suomela, Lucas Tendick, Isadora Veeren

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5. **State of the art results**
6. **Quantum advantage**

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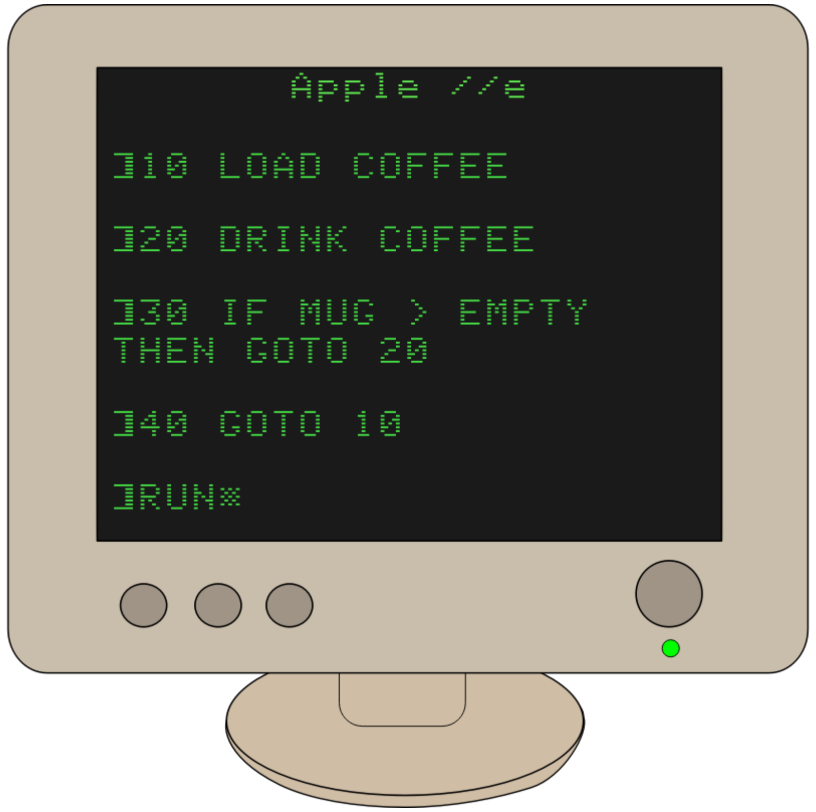
Distributed algorithms

- Write a program $\mathcal{A}$ for a single computer (e.g., for $(\Delta + 1)$-coloring a graph)
  - use commands like *send a message through this communication port*, etc.



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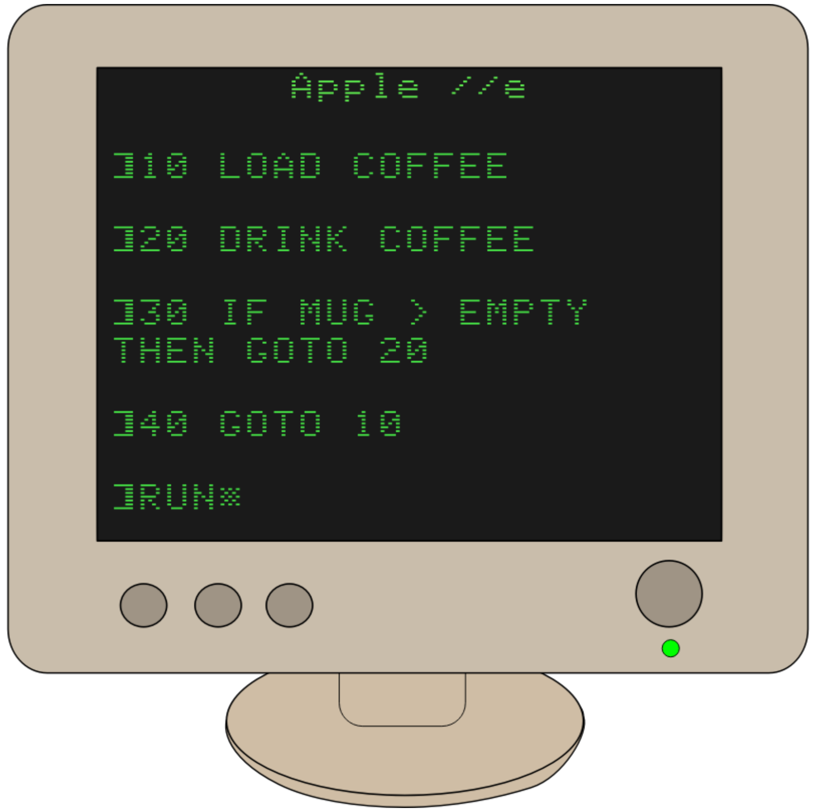

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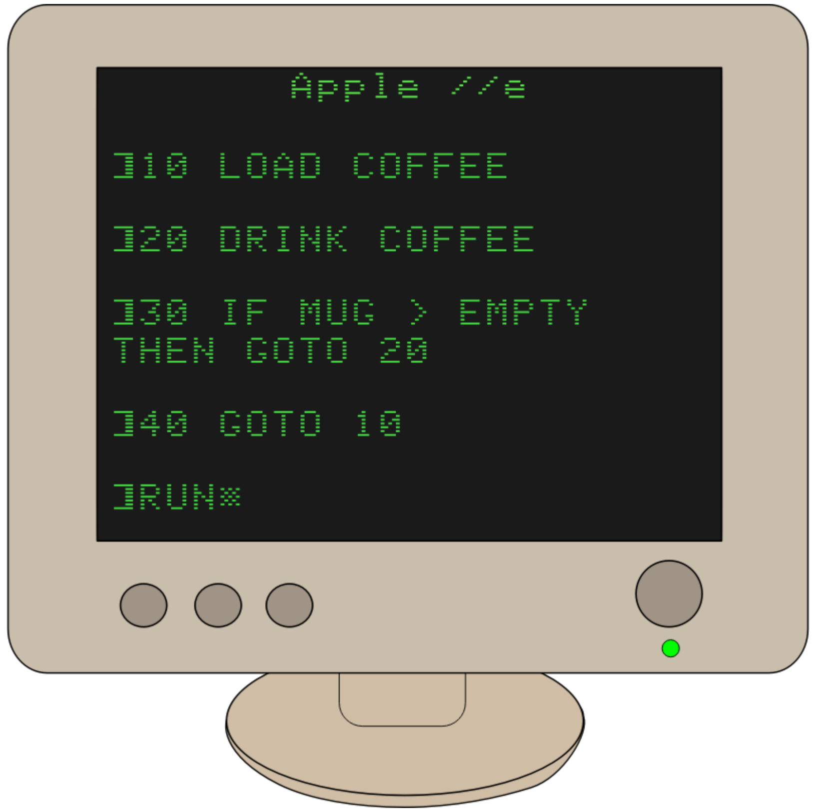

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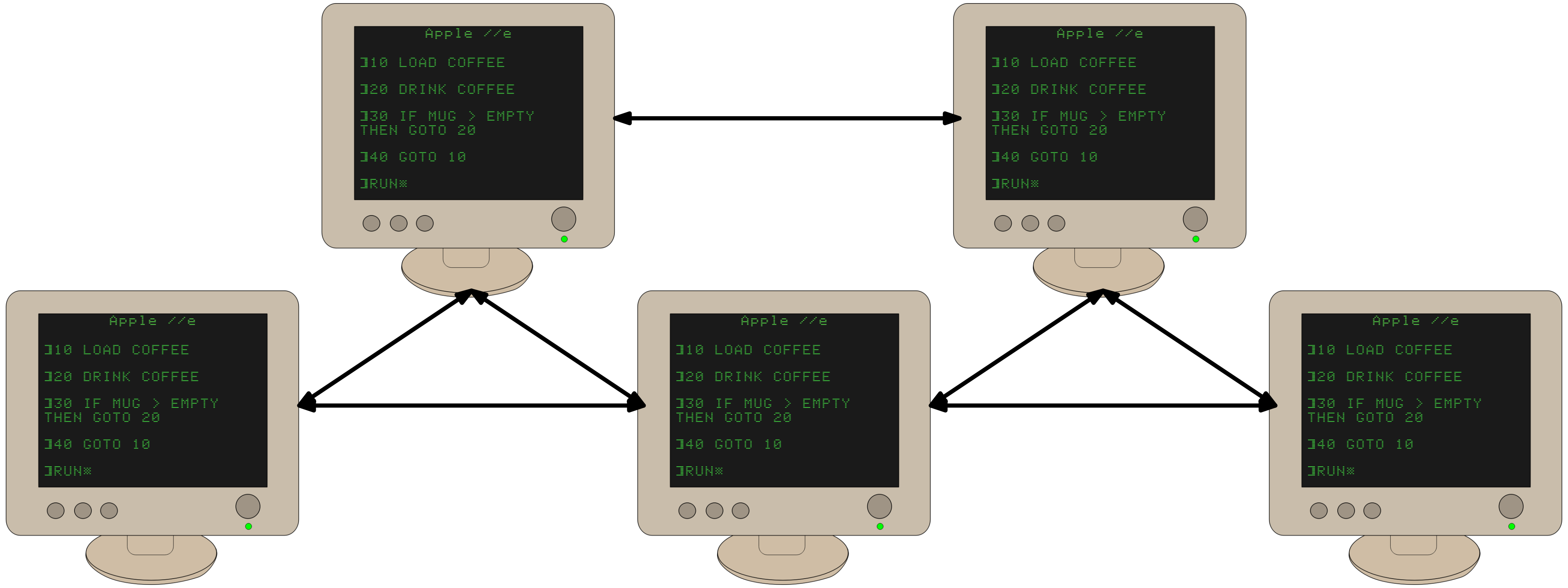

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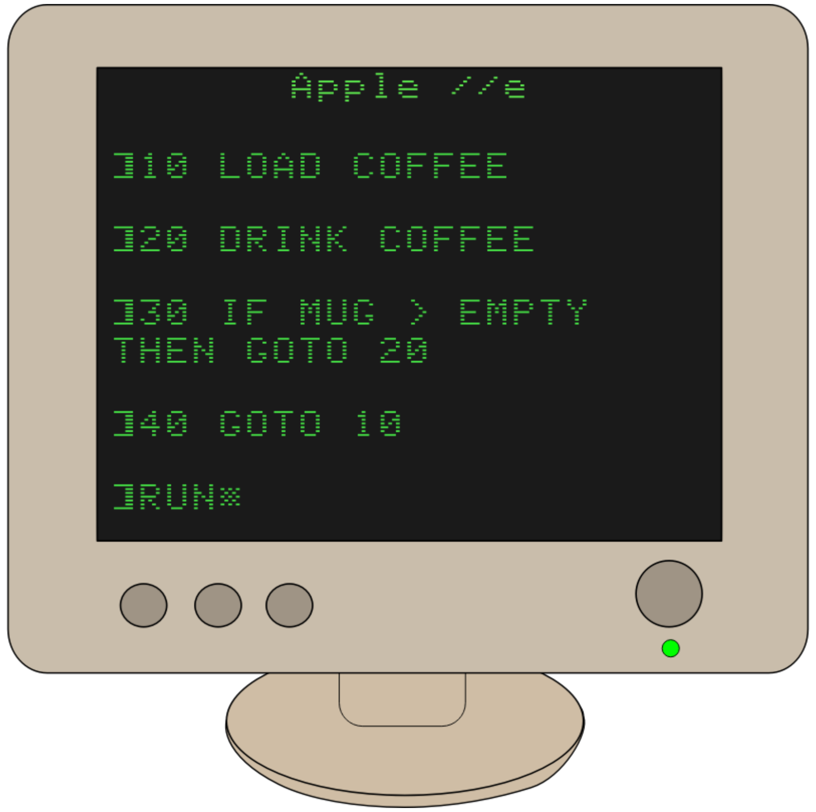
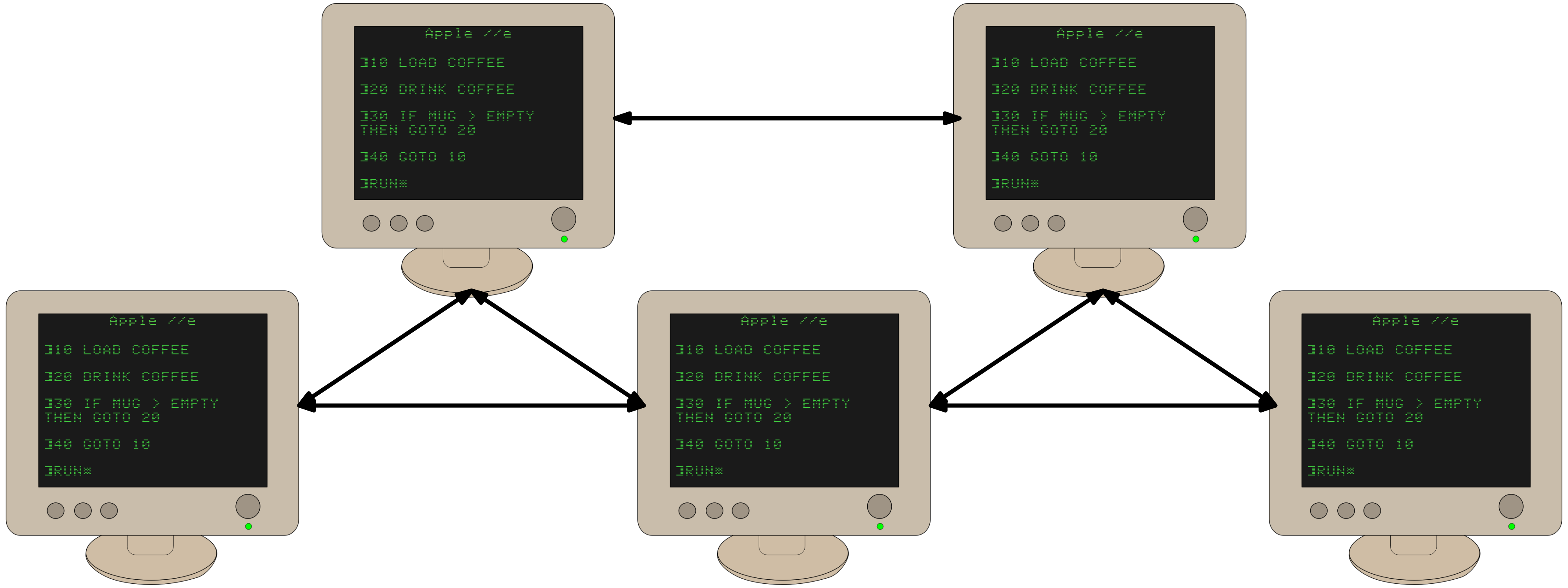
- **Switch everything on**, see what happens





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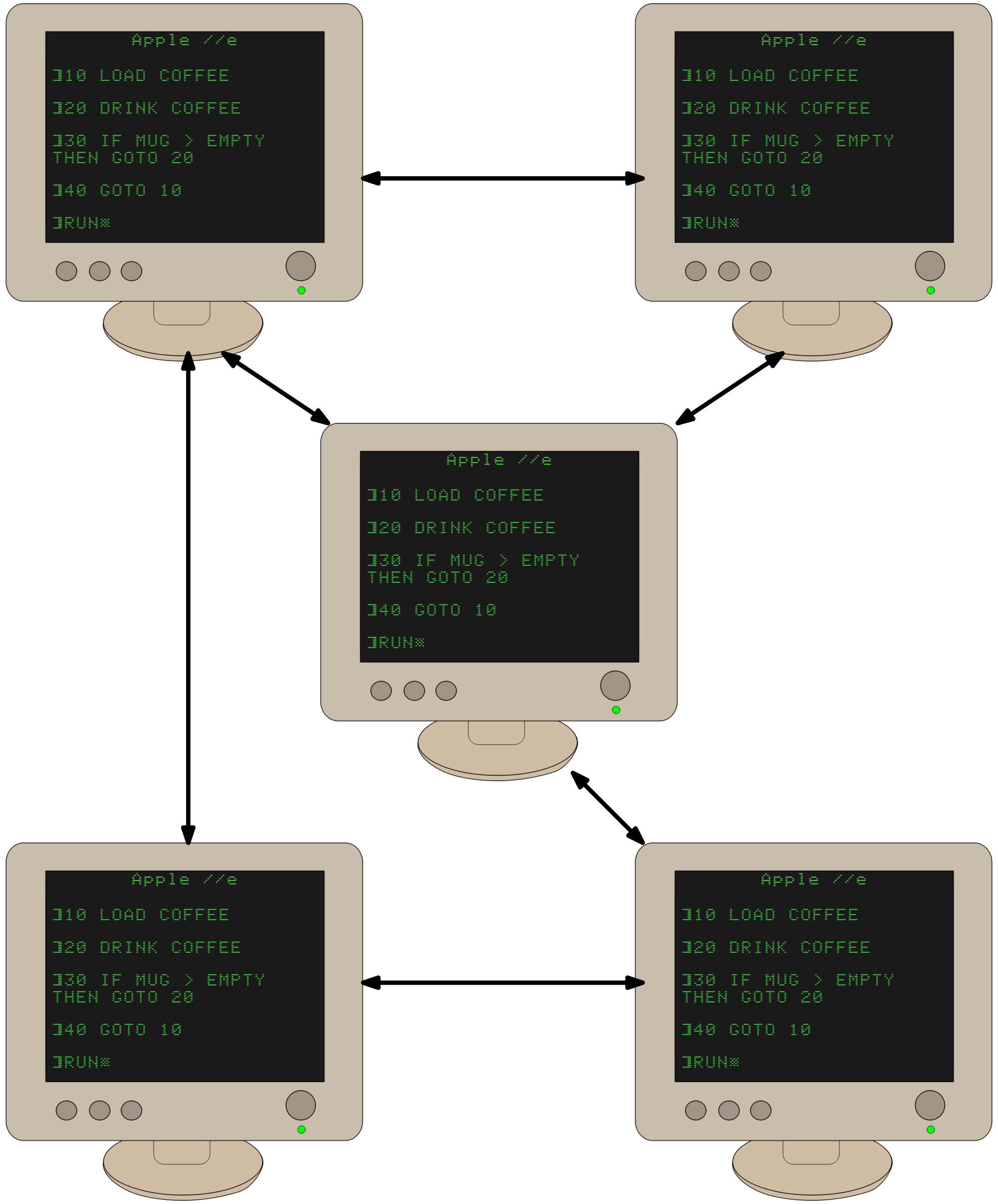

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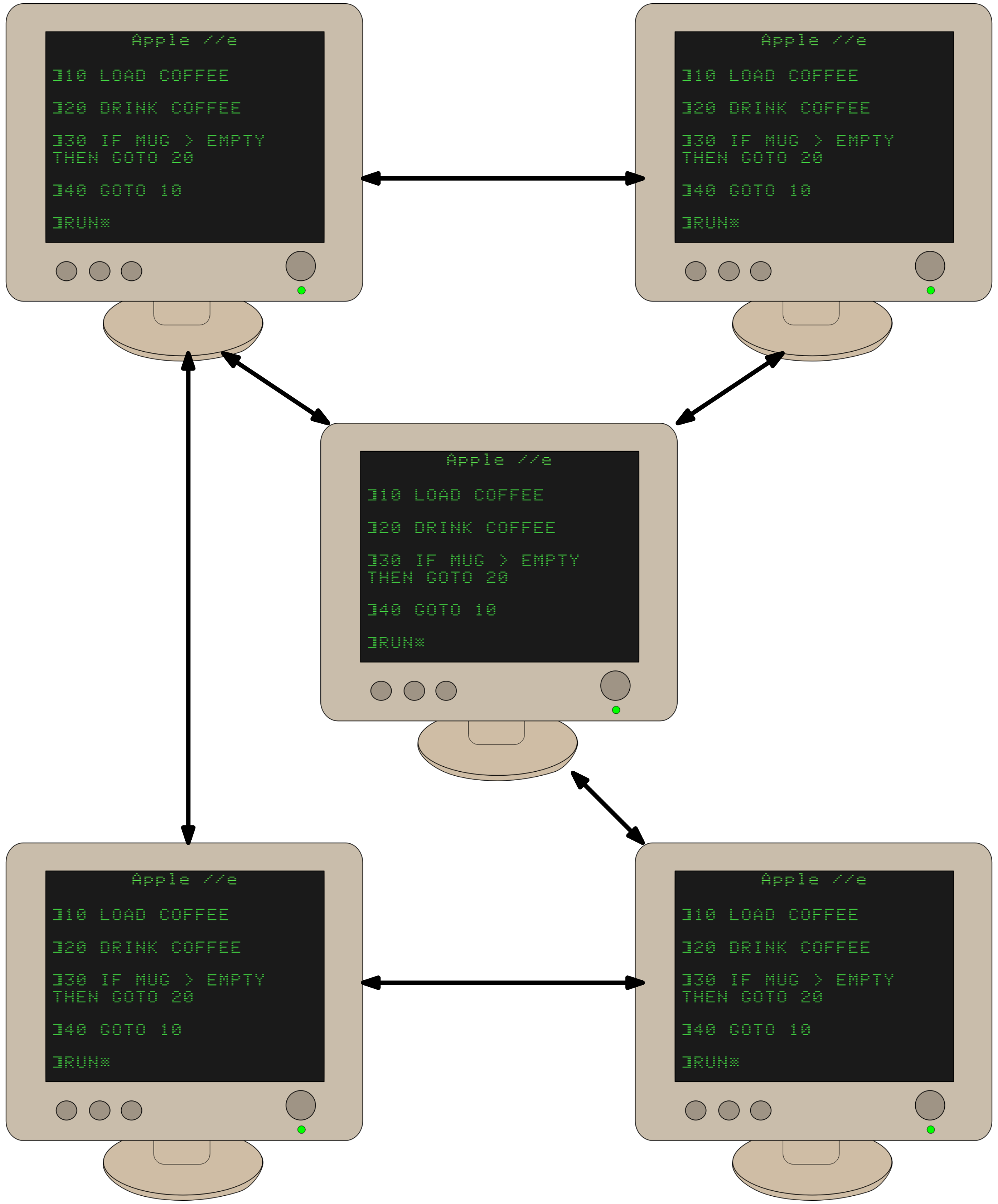

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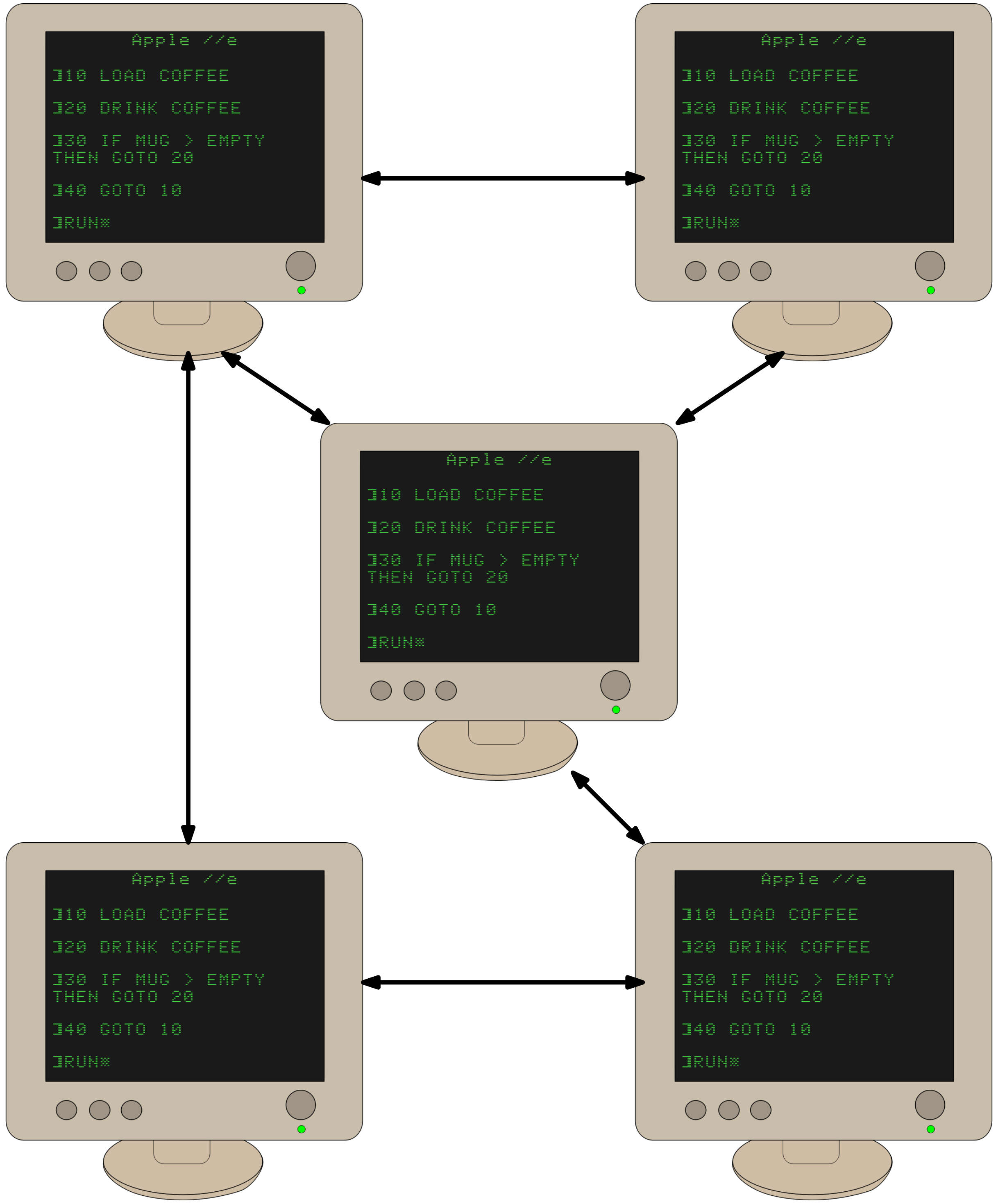

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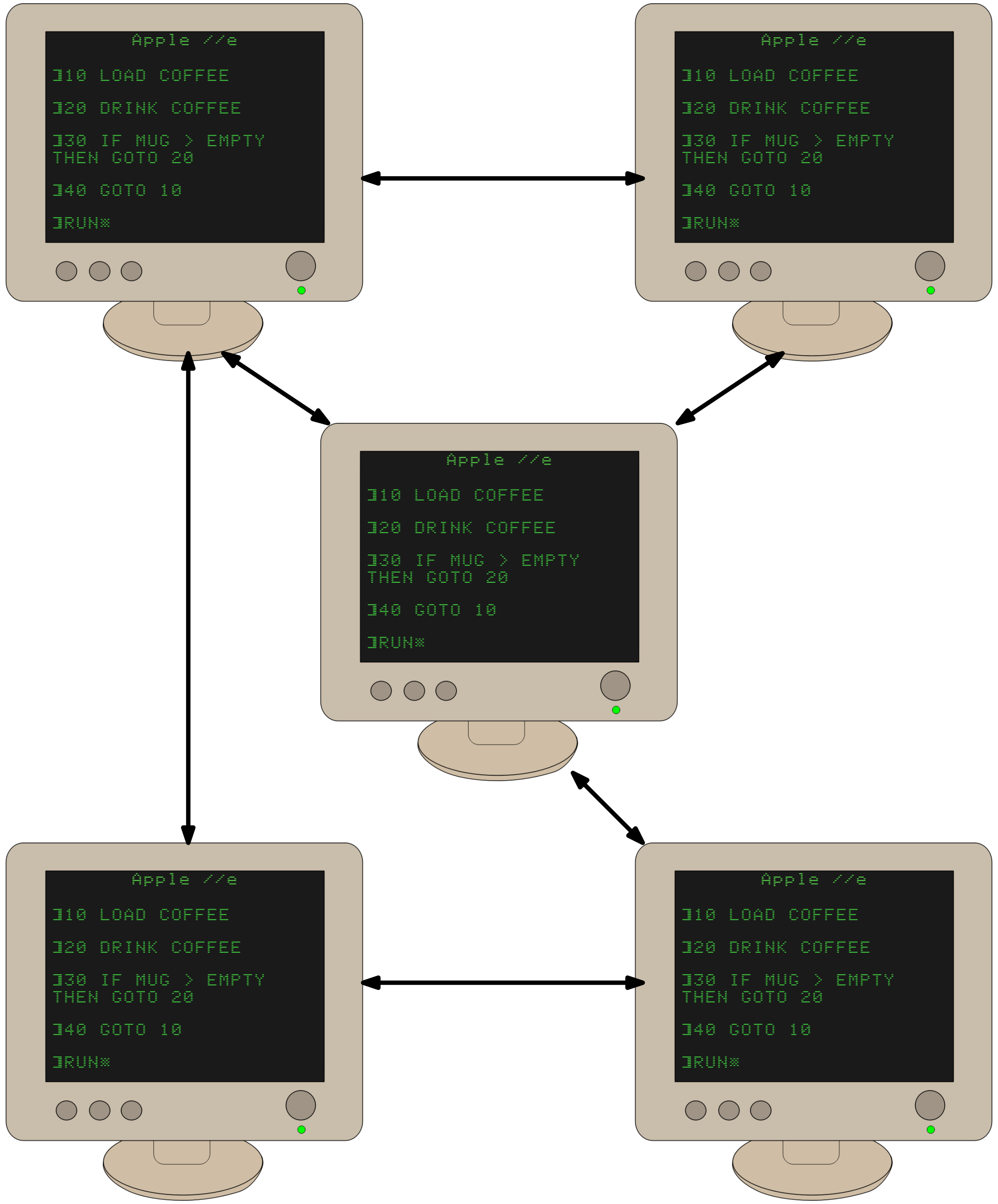

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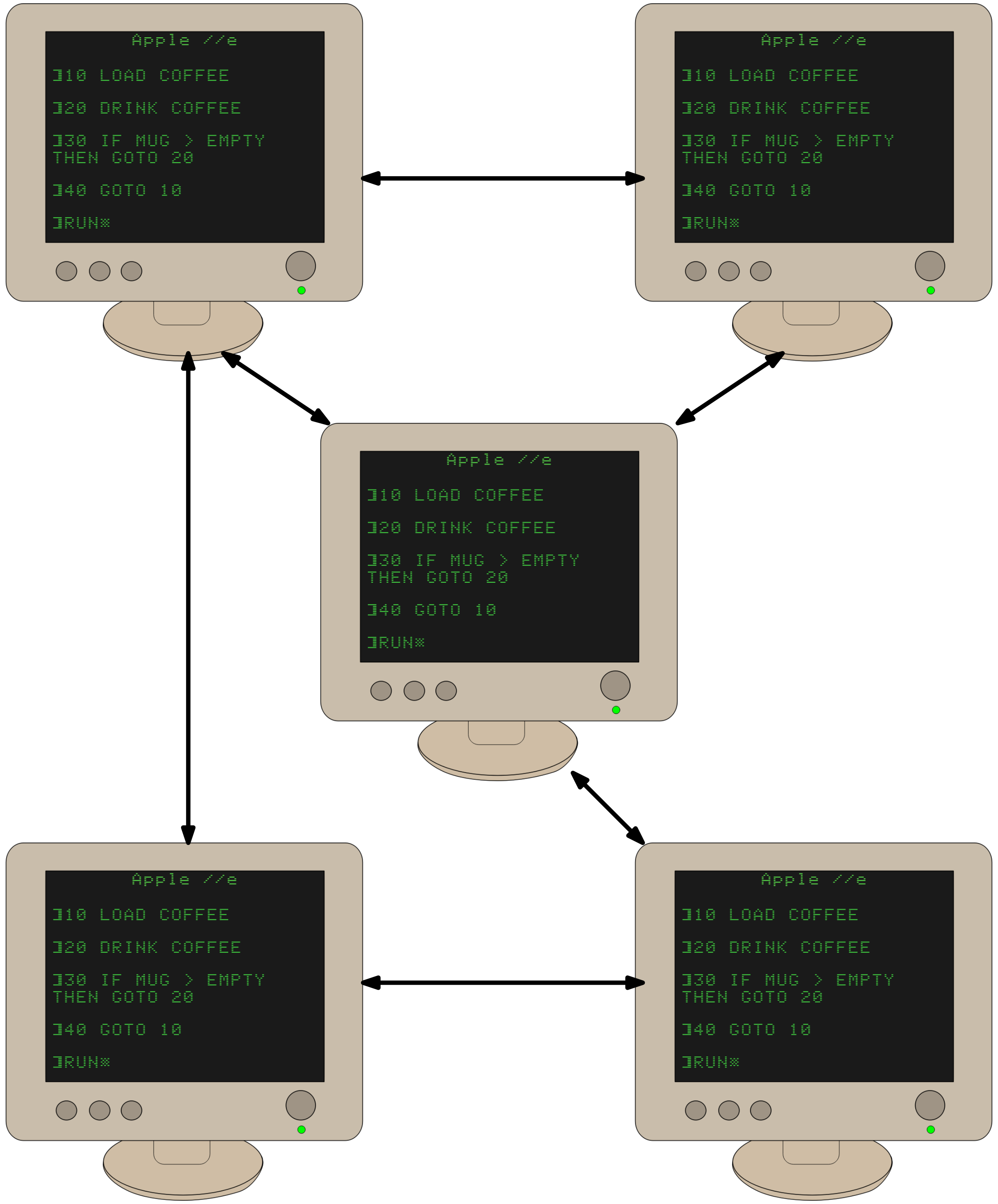
- **Challenge**: what to do in the middle of a network?



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- **Other possibility**: *randomness*. Each node has access to independent source of randomness

The LOCAL model

[Linial FOCS '87 & SICOMP '92]

- **Distributed network** of n processors/nodes

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  - E: communication links
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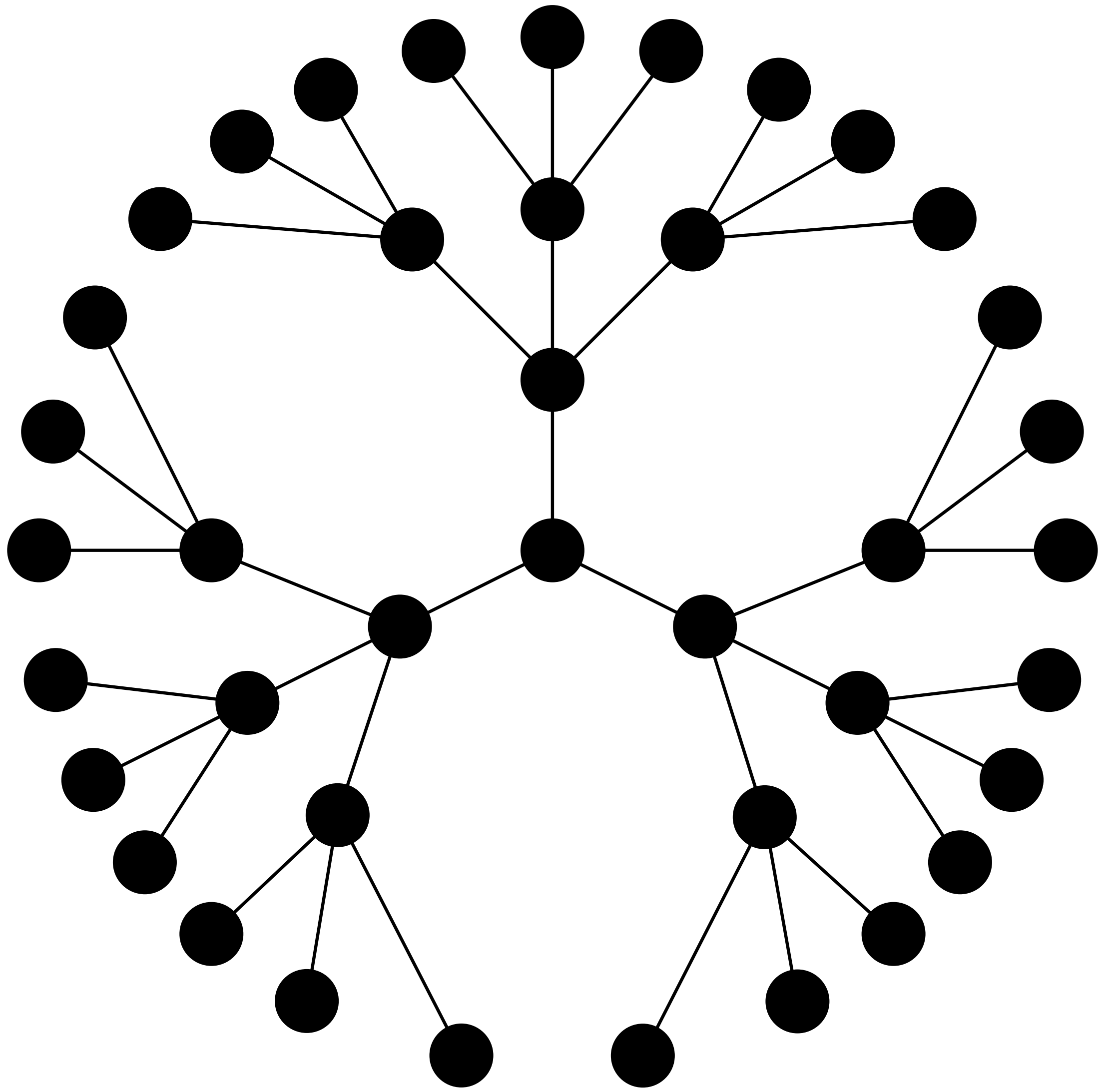

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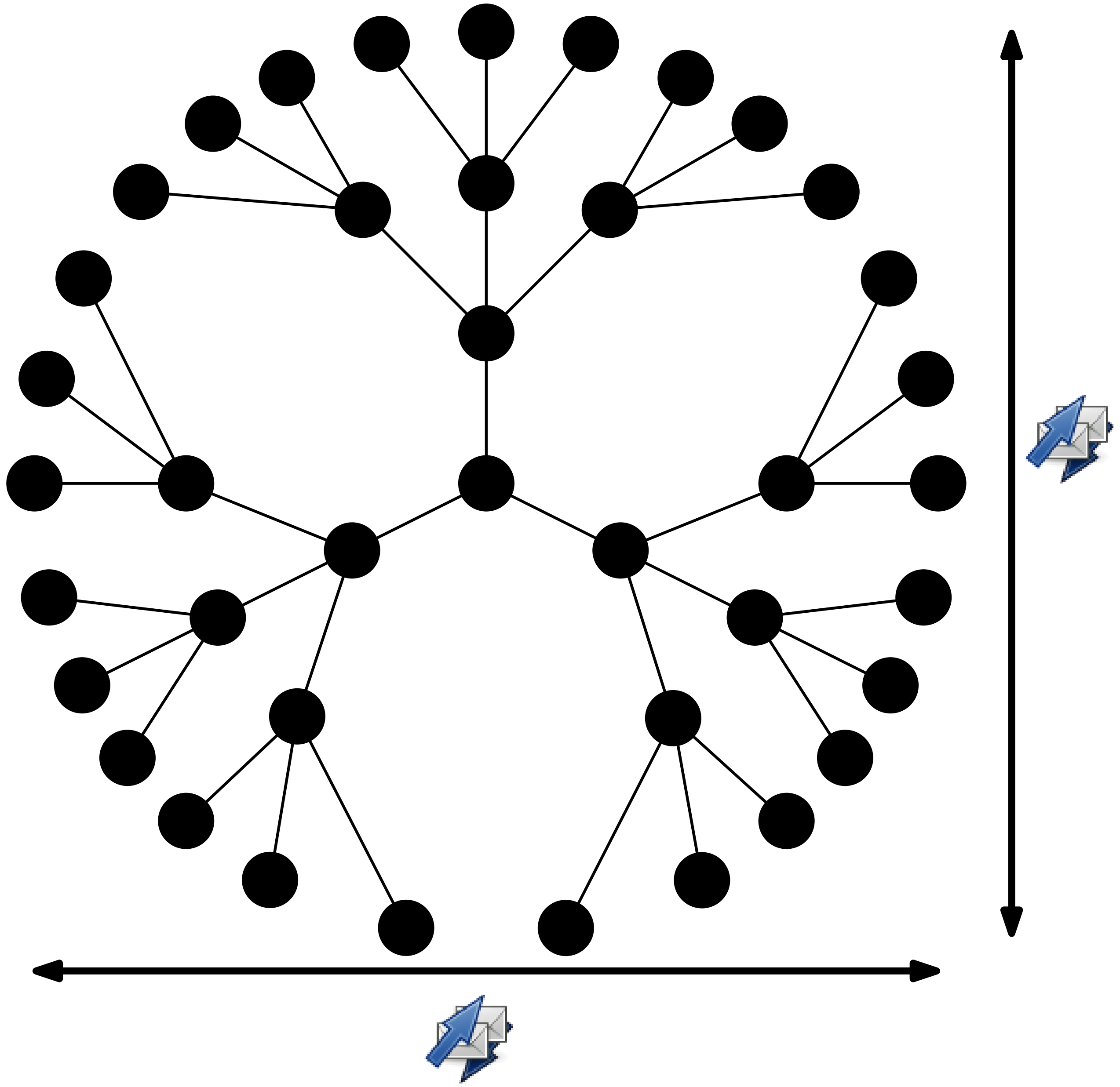

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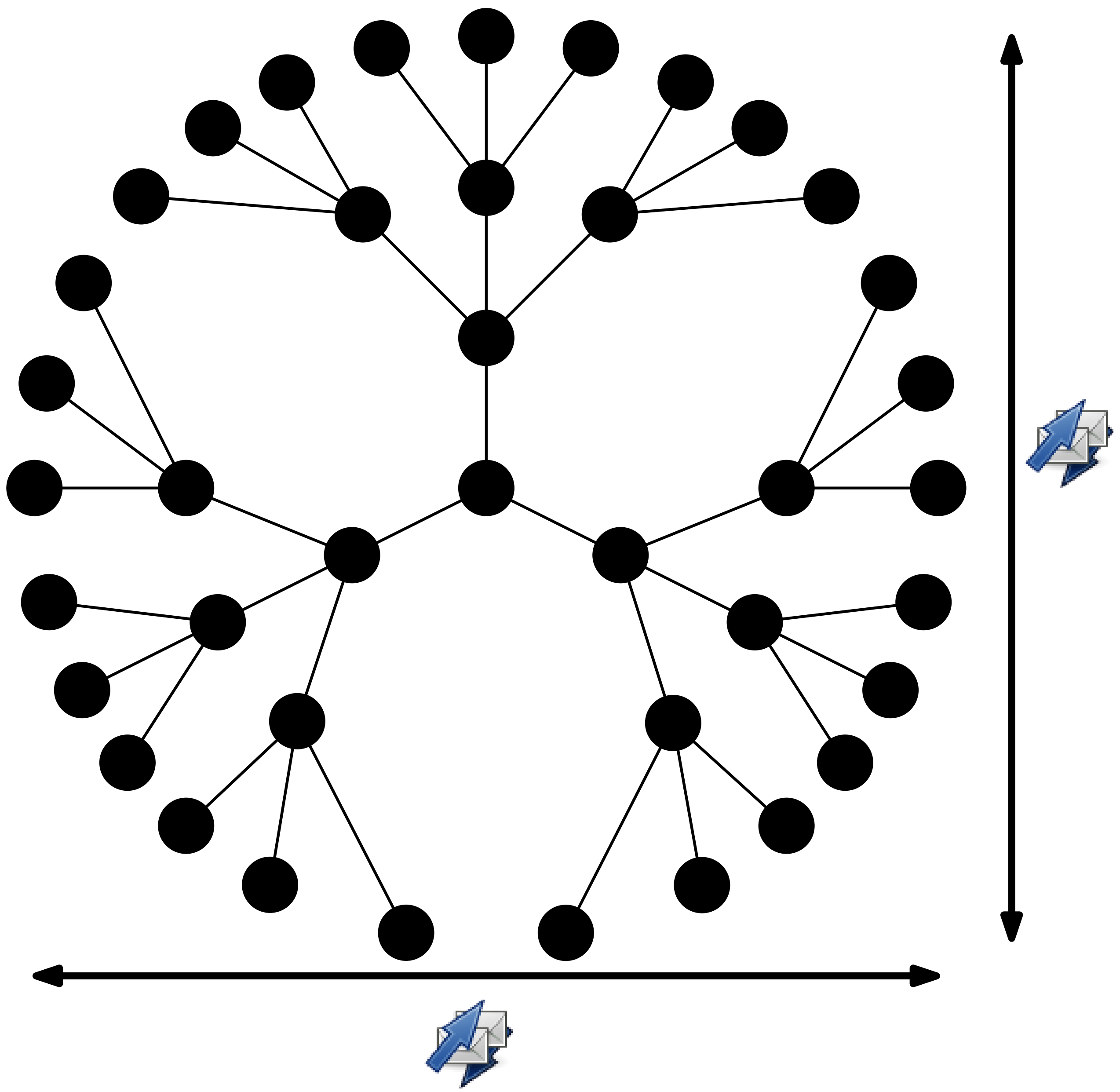

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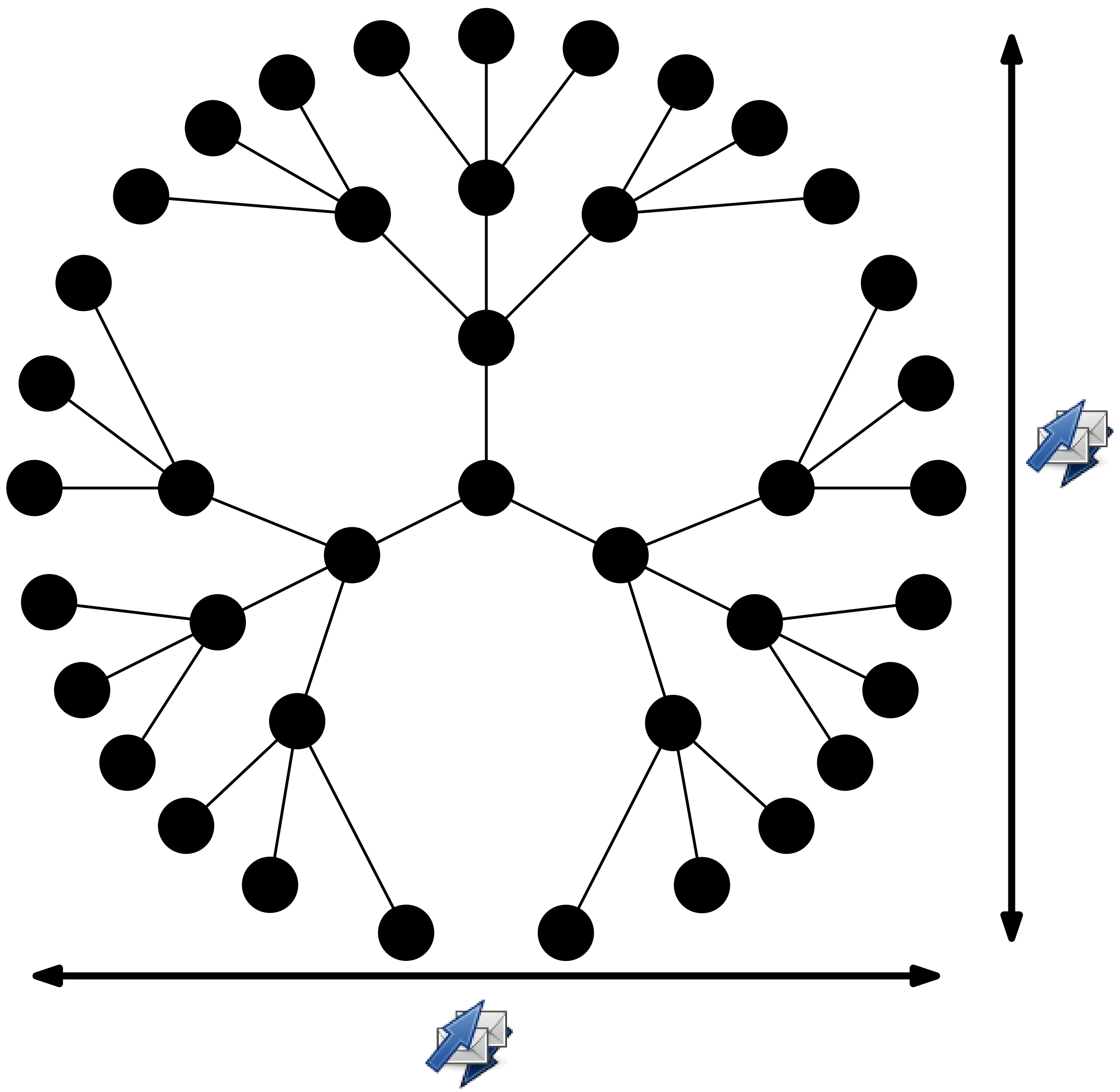

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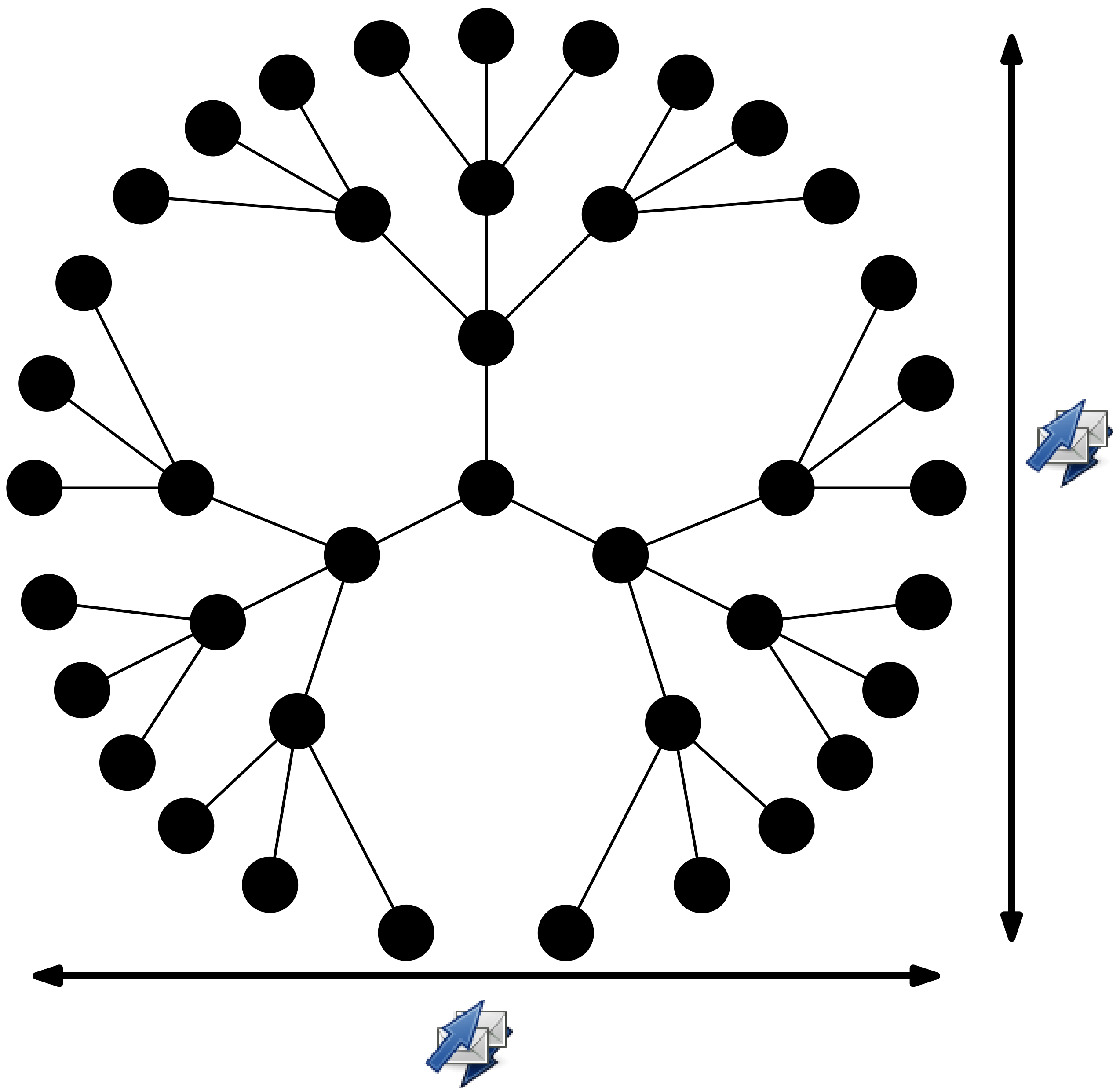

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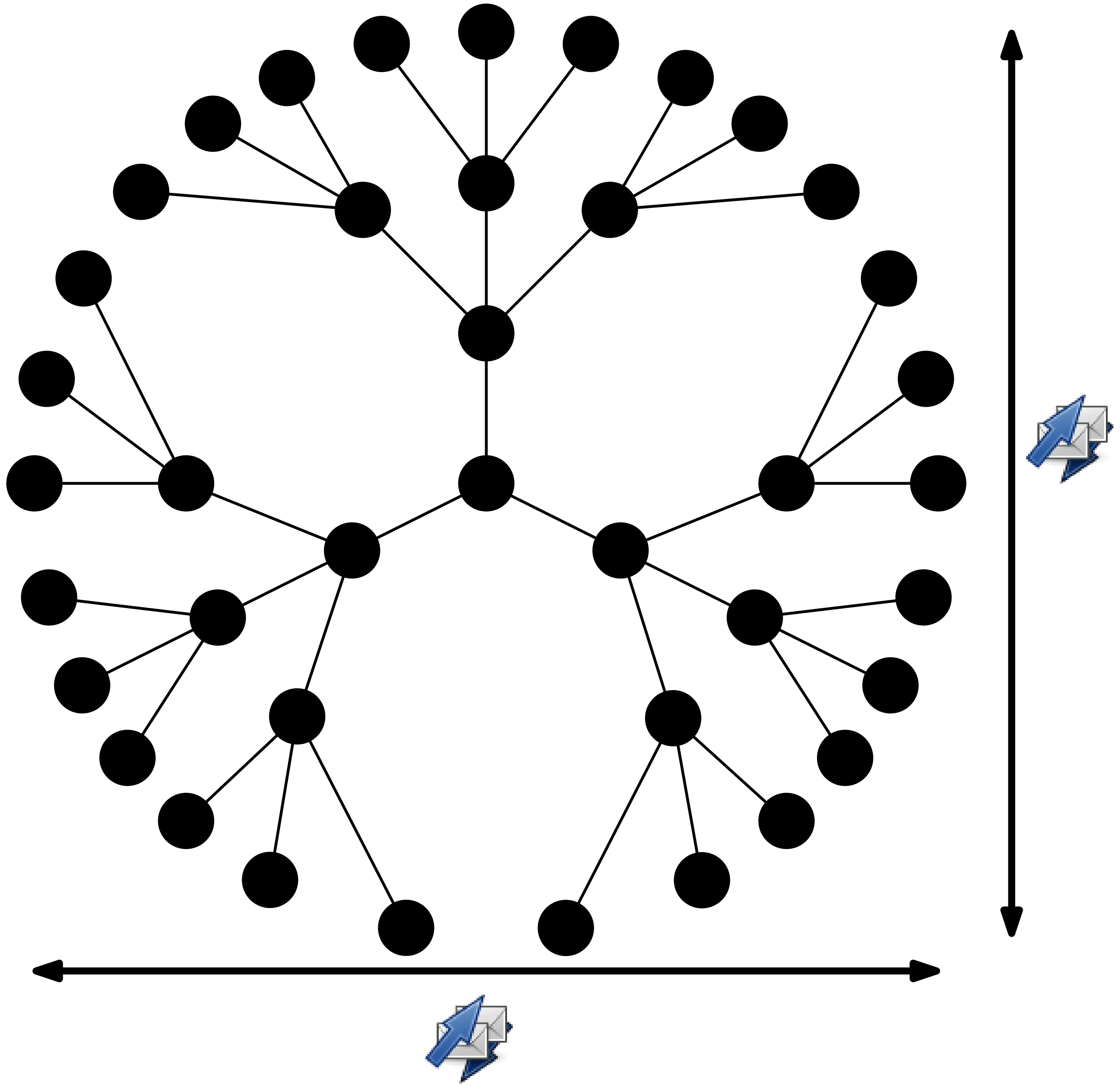

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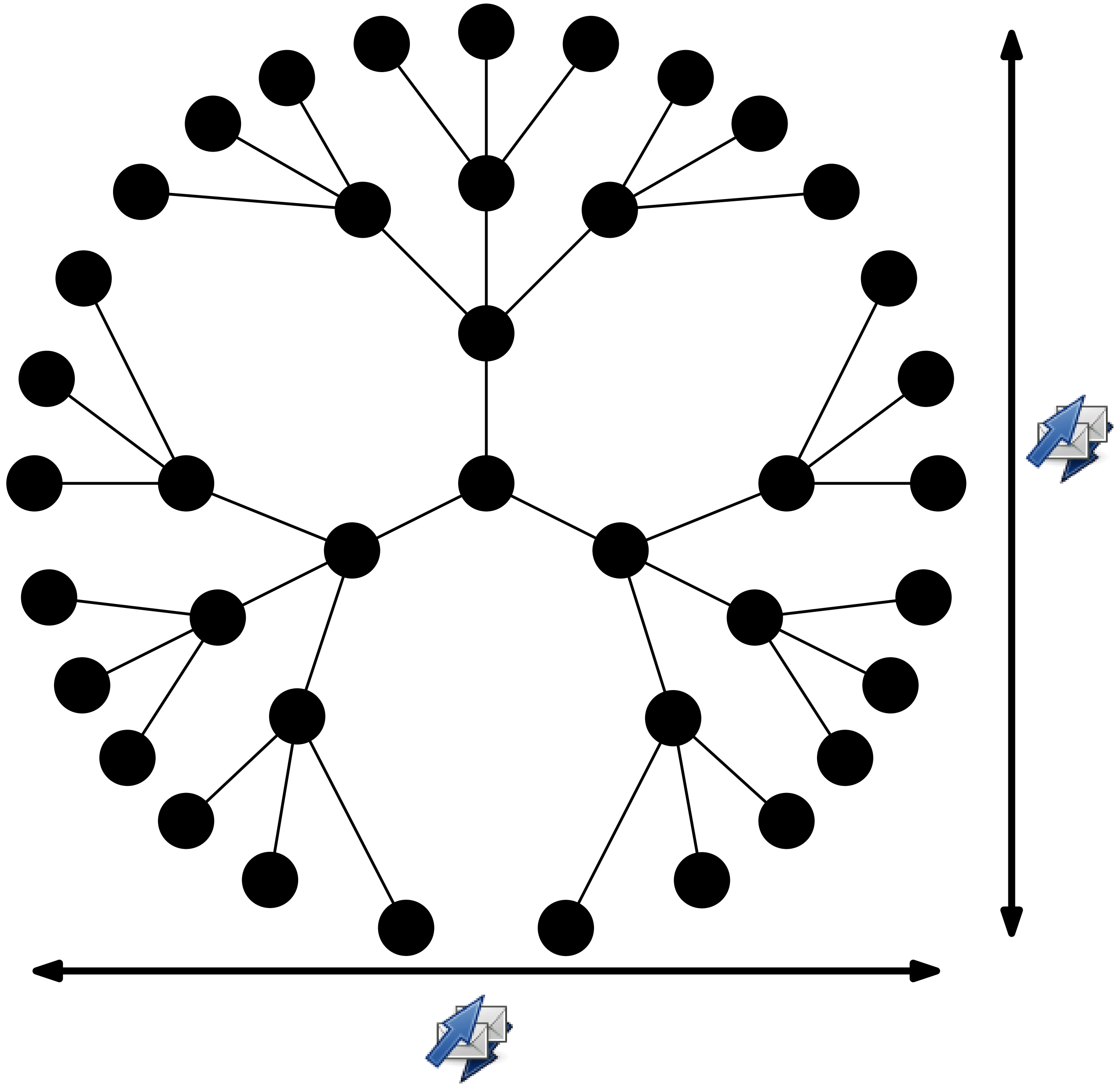
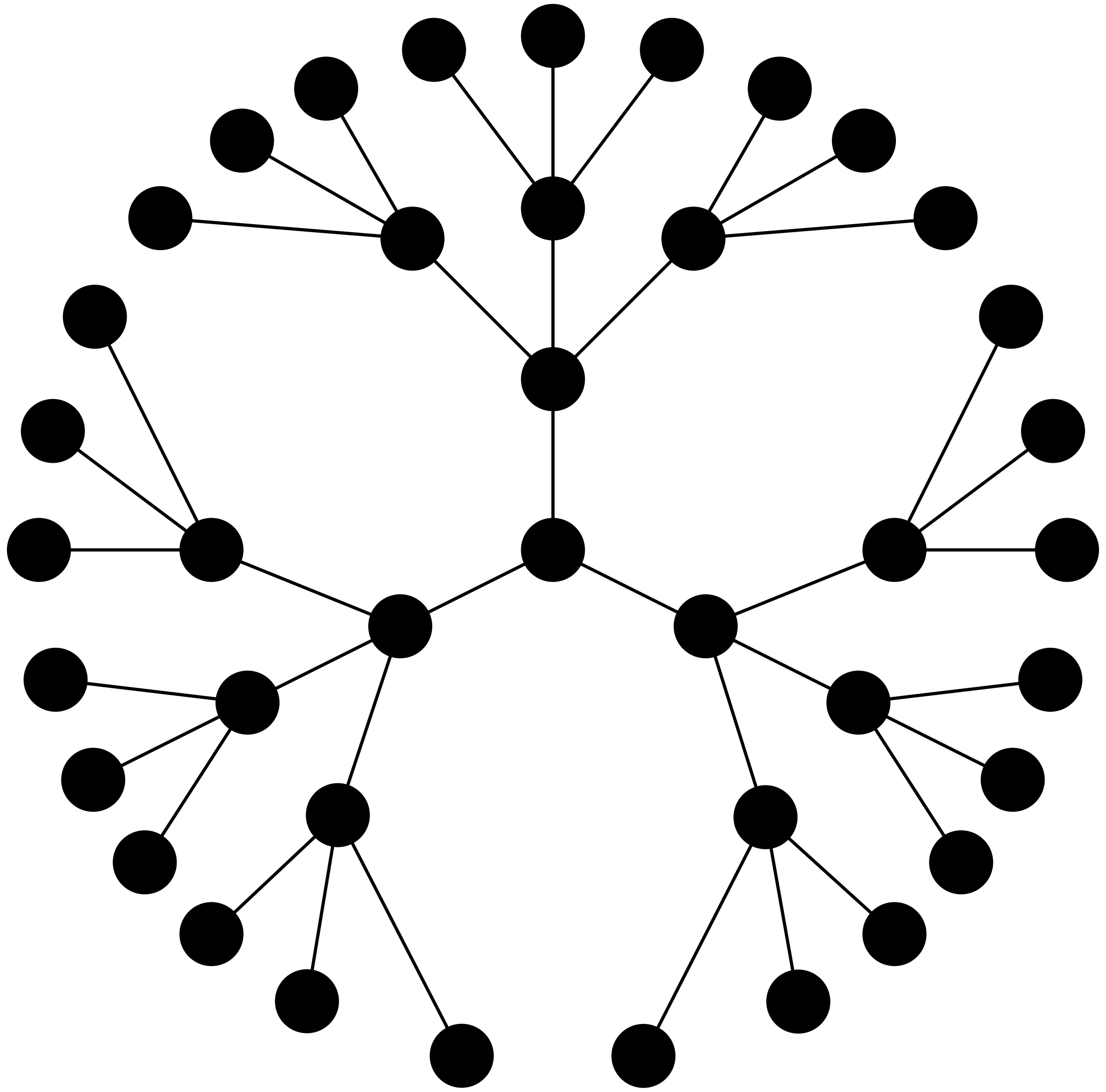


Local view

Complexity measure: number of communication rounds



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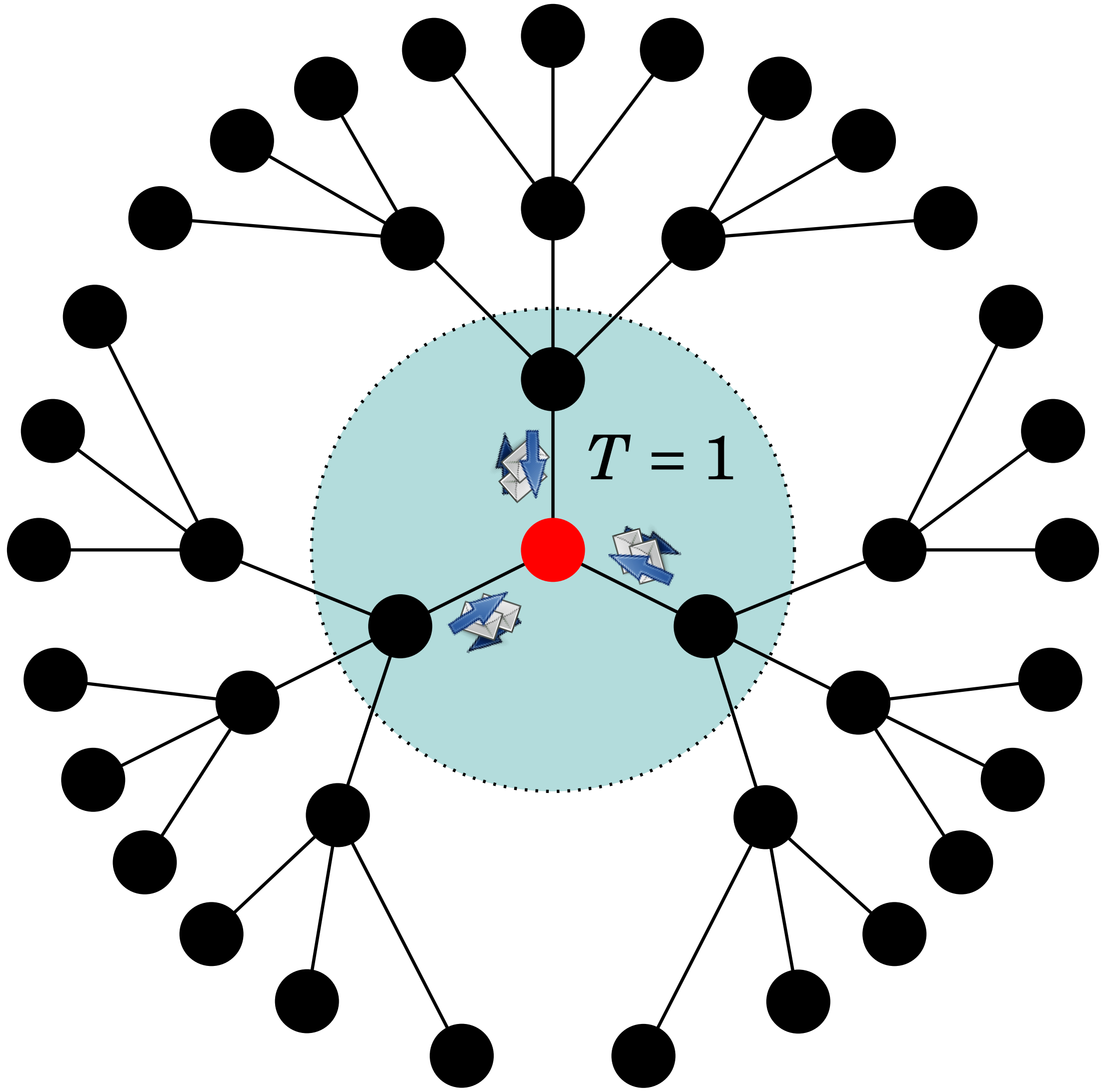

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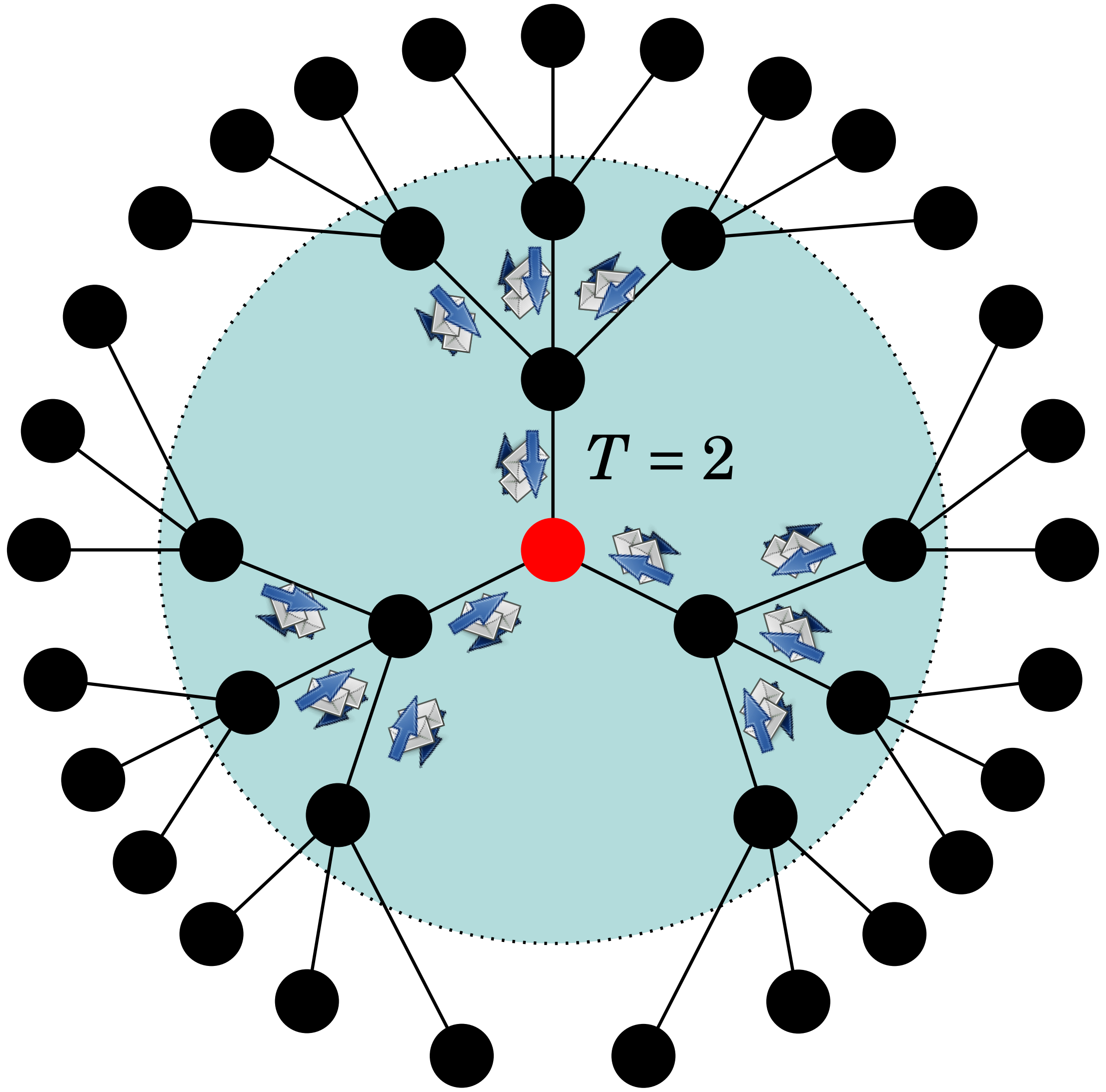

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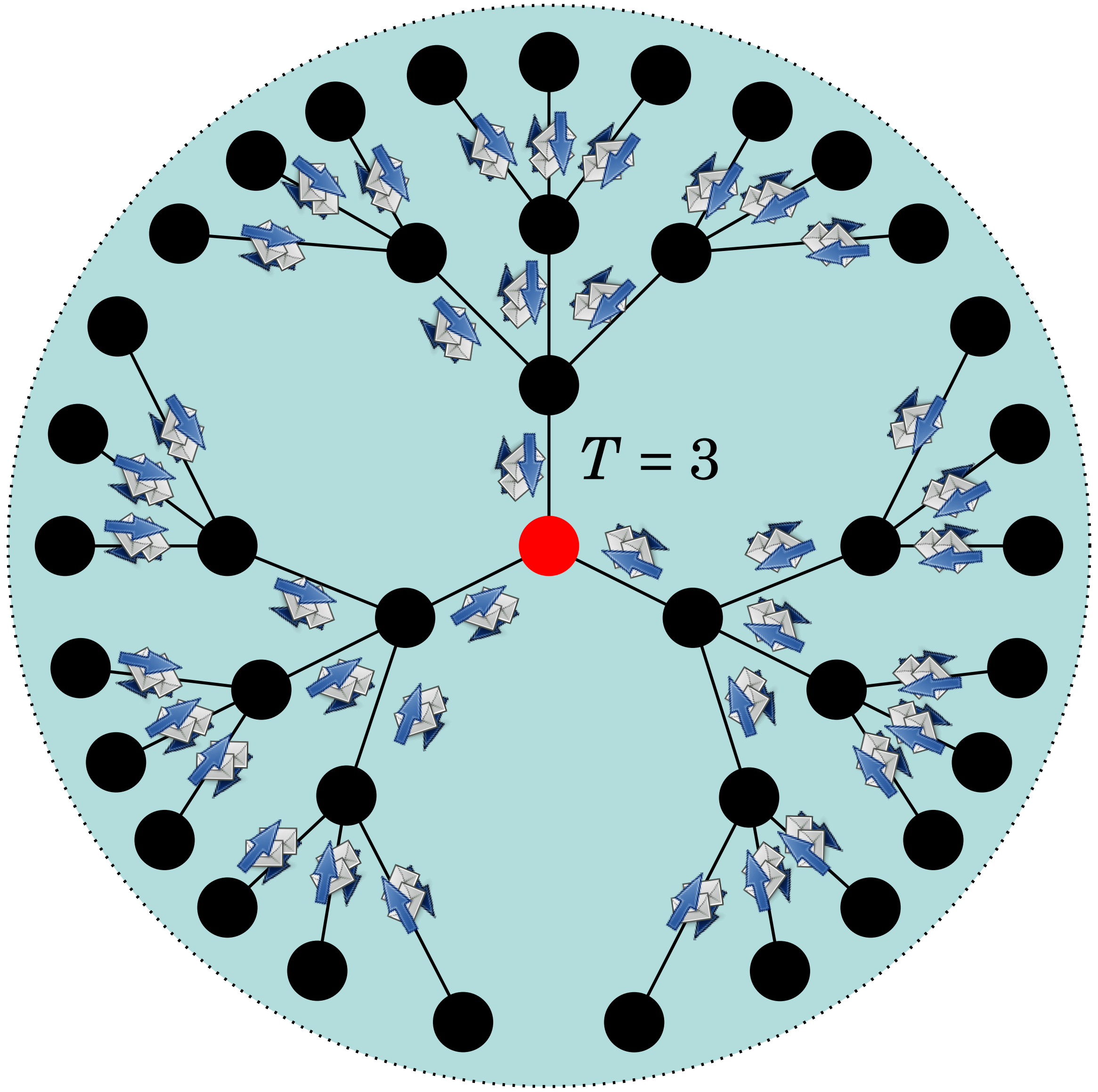

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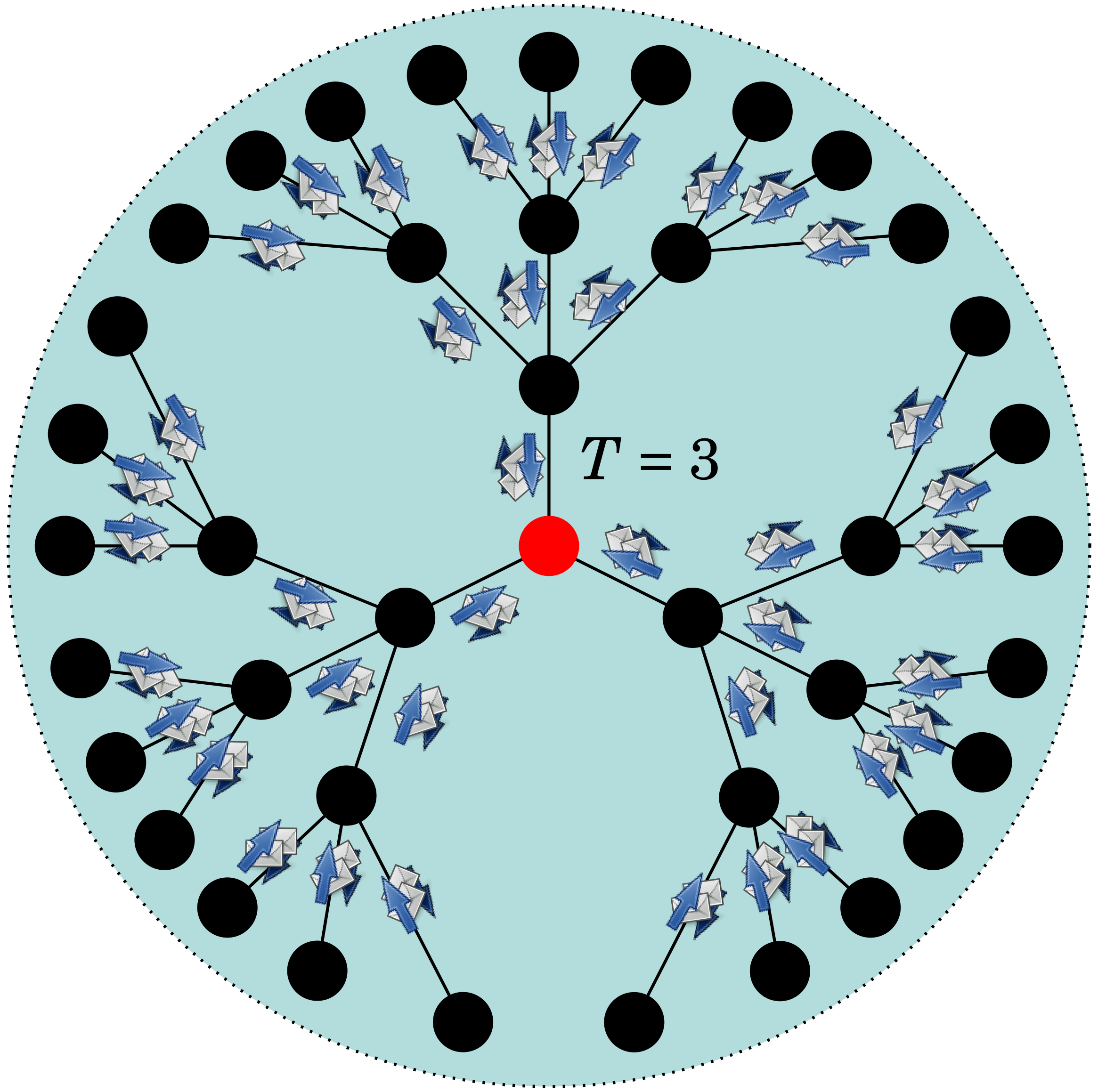

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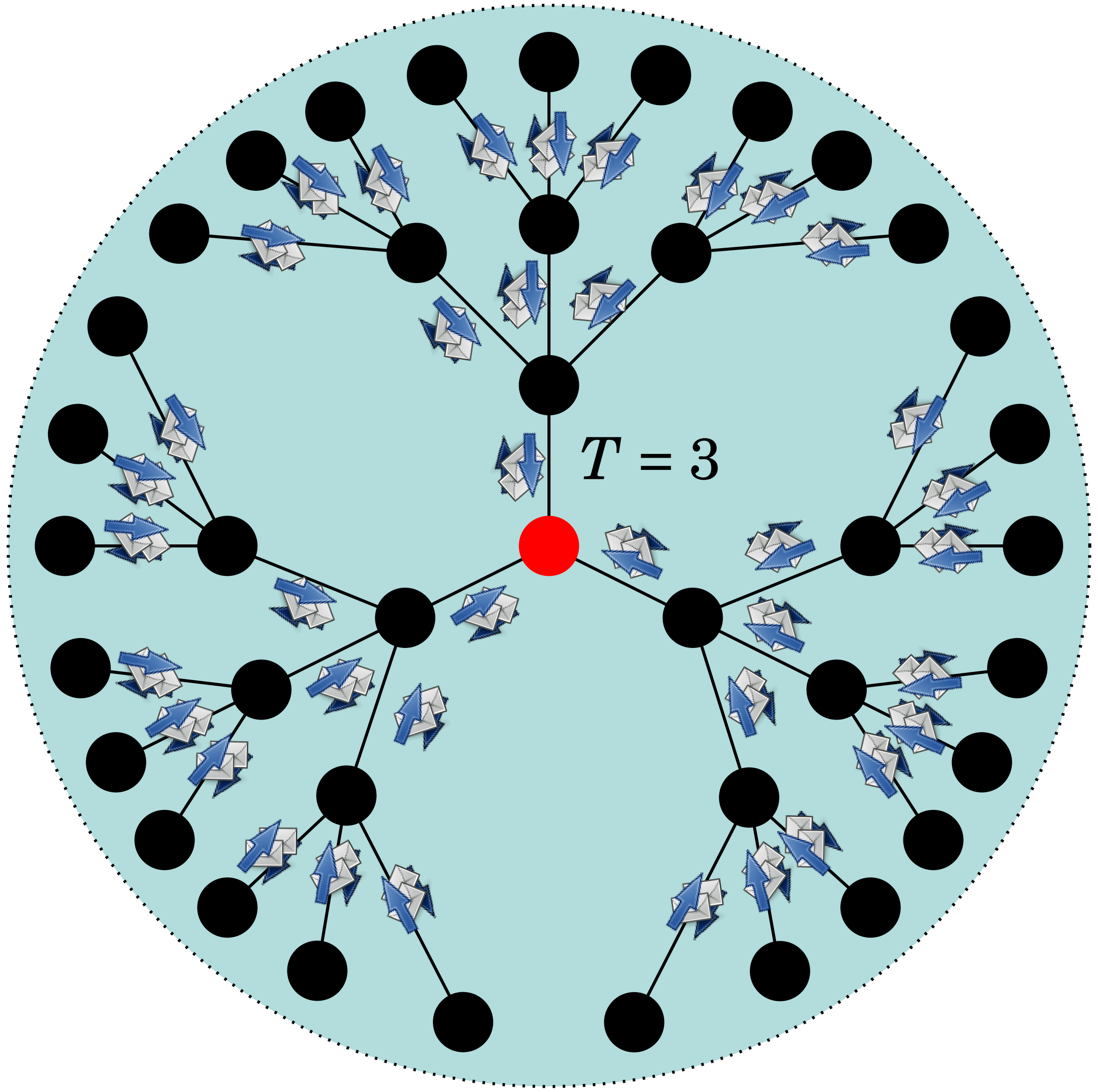

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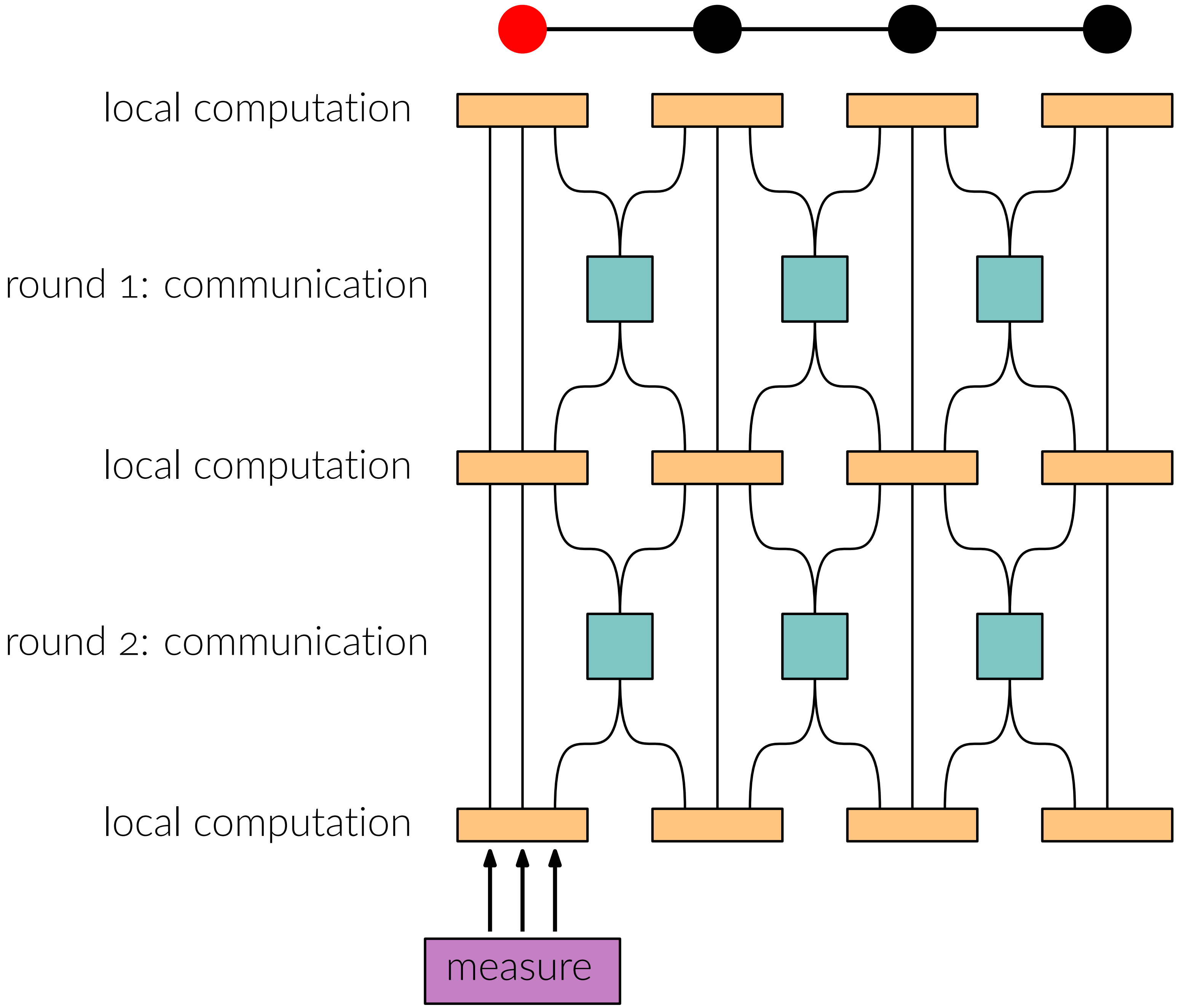
- **Equivalence**: T-round algorithm ≈ funtion mapping radius-T neighborhoods to local outputs
- Locality $T = \text{diam}(G) + 1$ is always sufficient to solve any problem: gathering algorithm

Quantum-LOCAL

[[Gavoille, Kosowski, and Markiewicz, DISC ’09]]

- **Distributed system** of n quantum processors/nodes
  - quantum computation
  - quantum communication (qubits)
  - output: measurement of qubits



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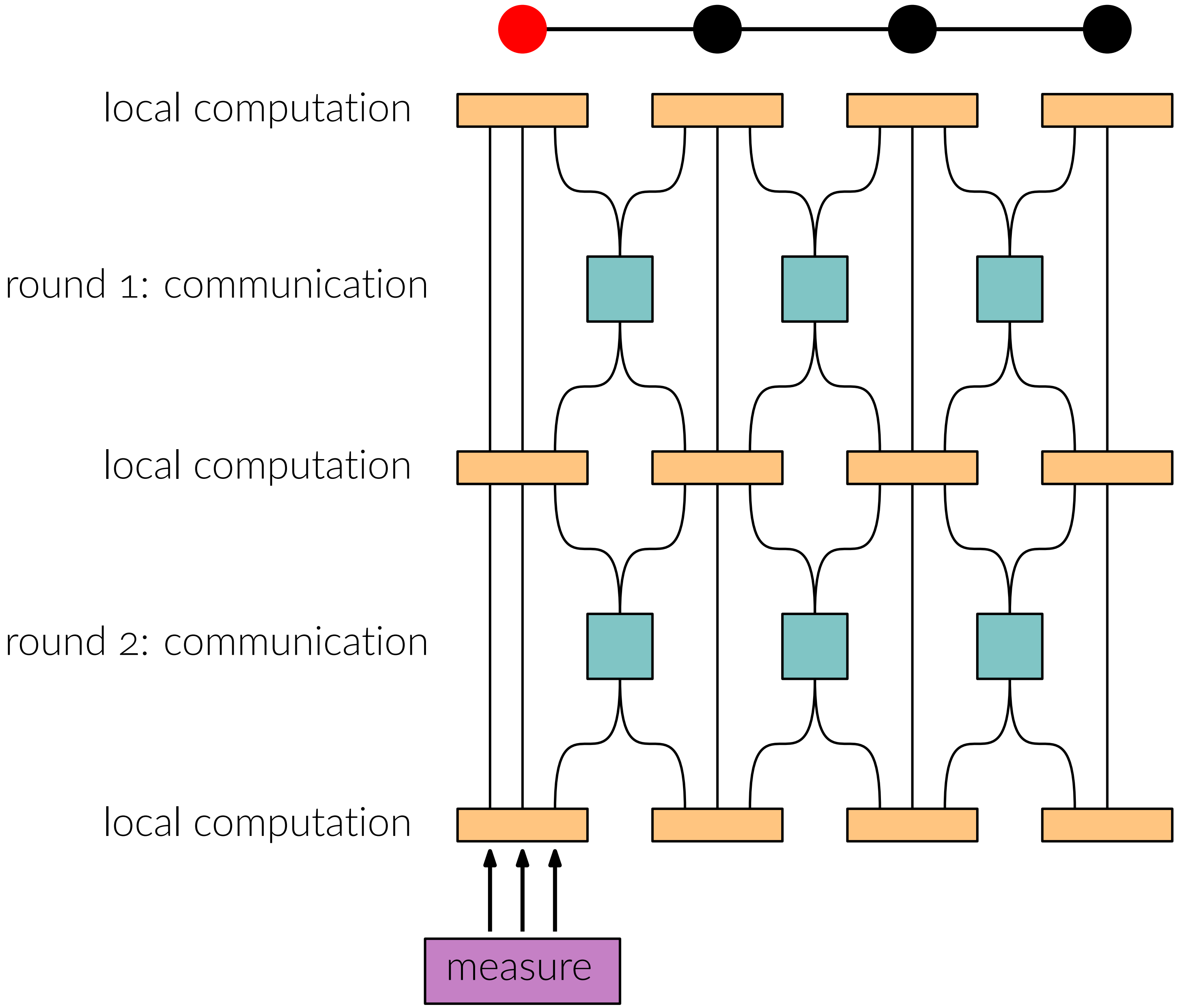

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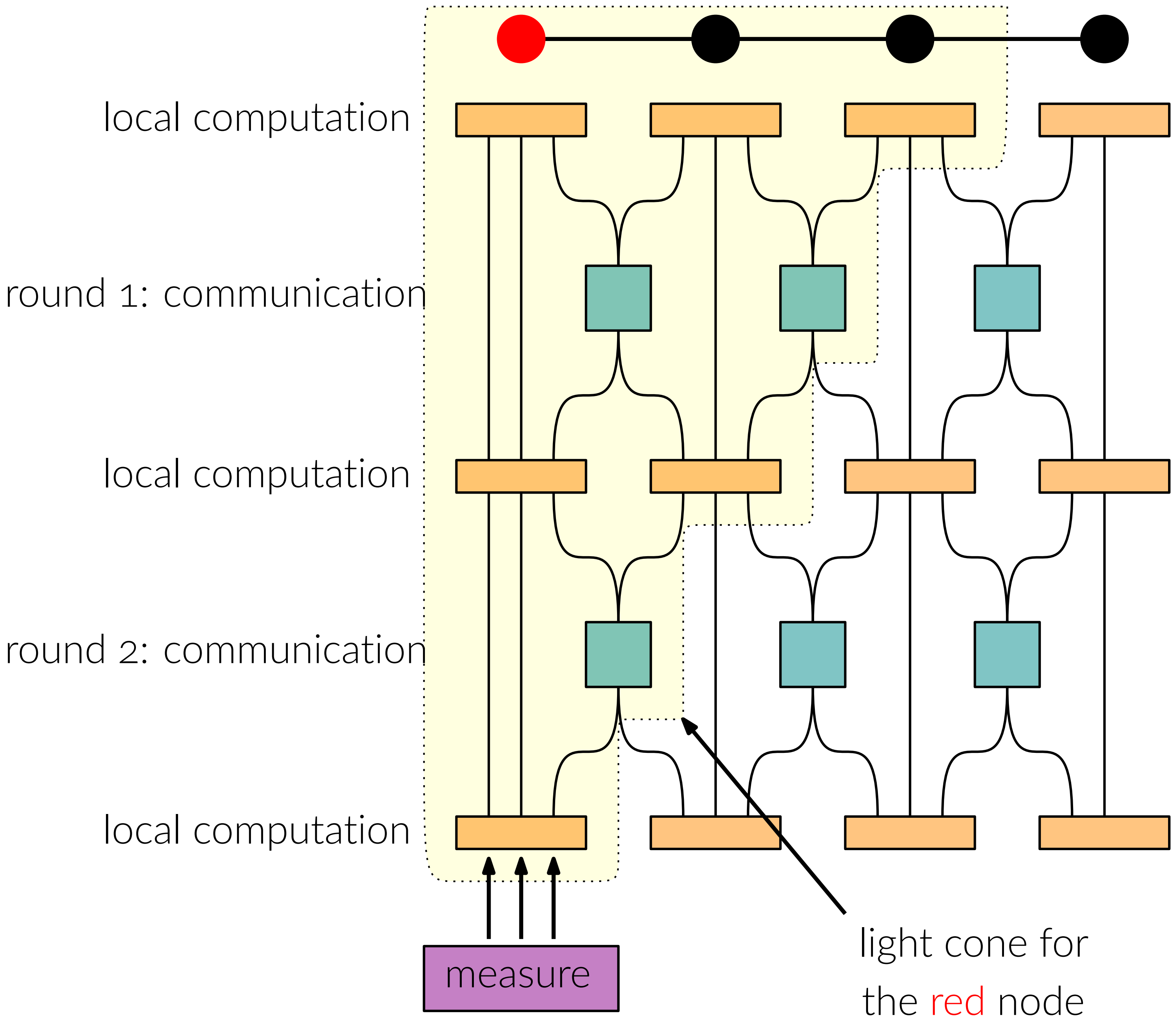

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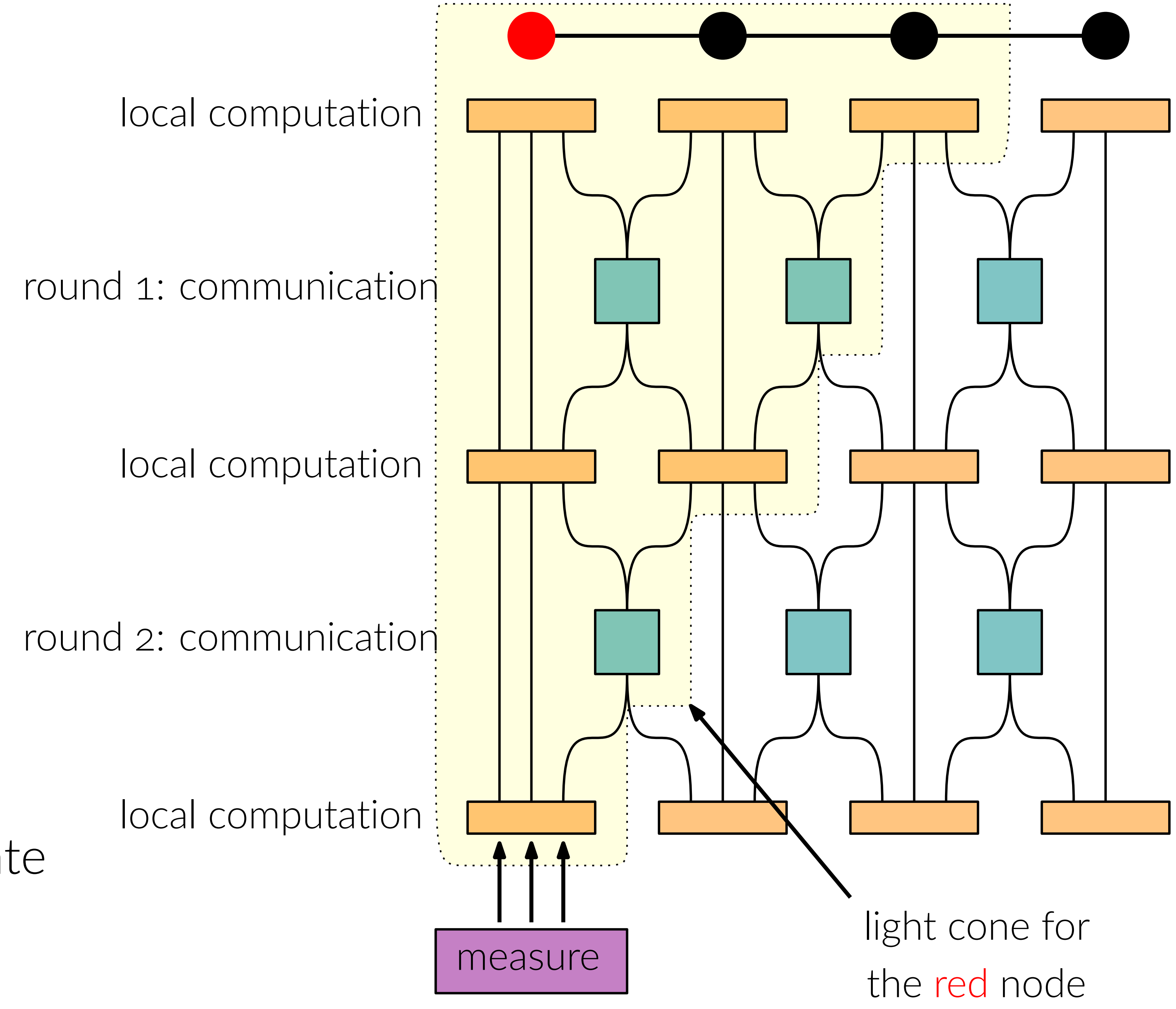
- Still, locality identifies how *far* nodes need to communicate



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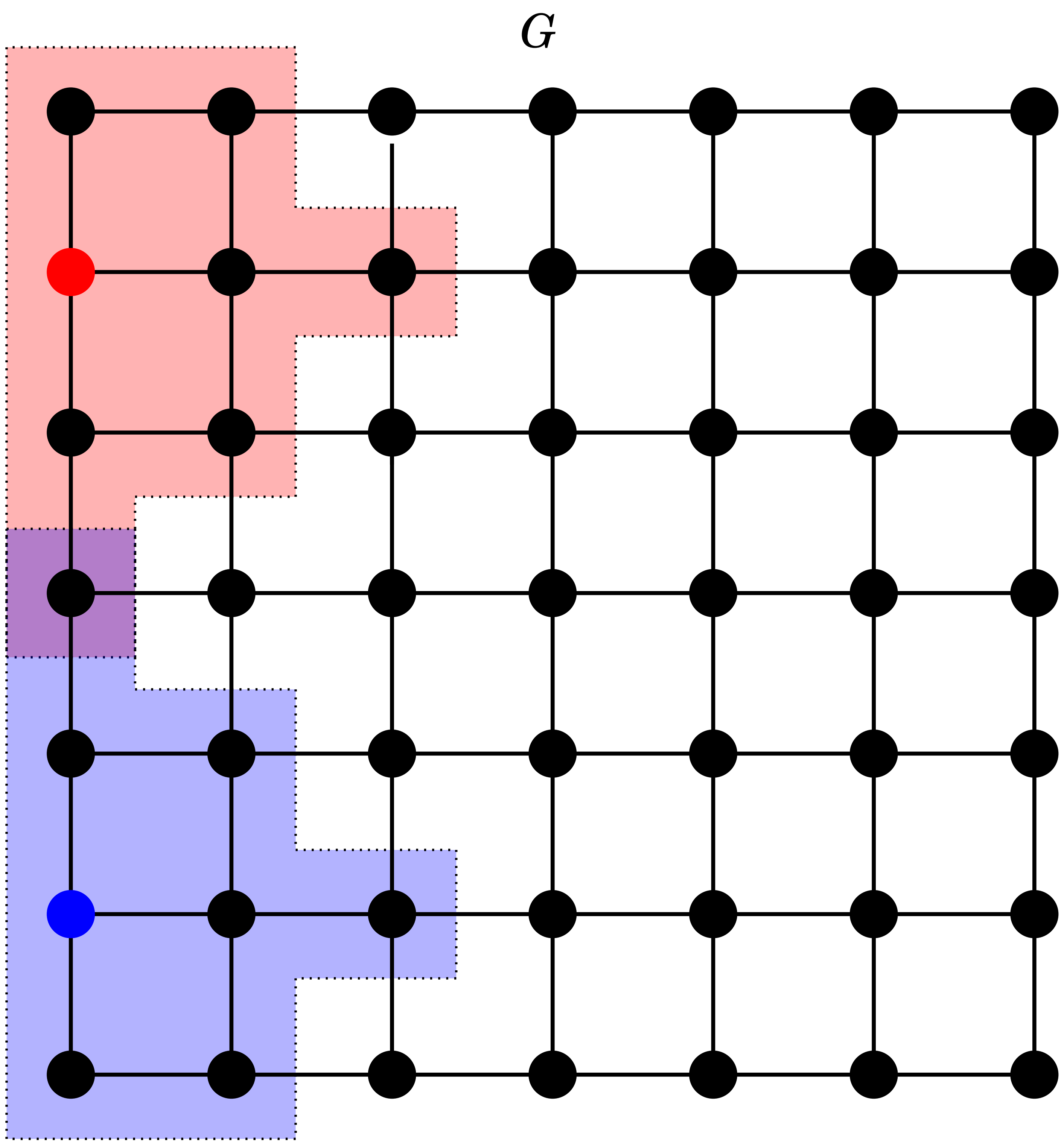

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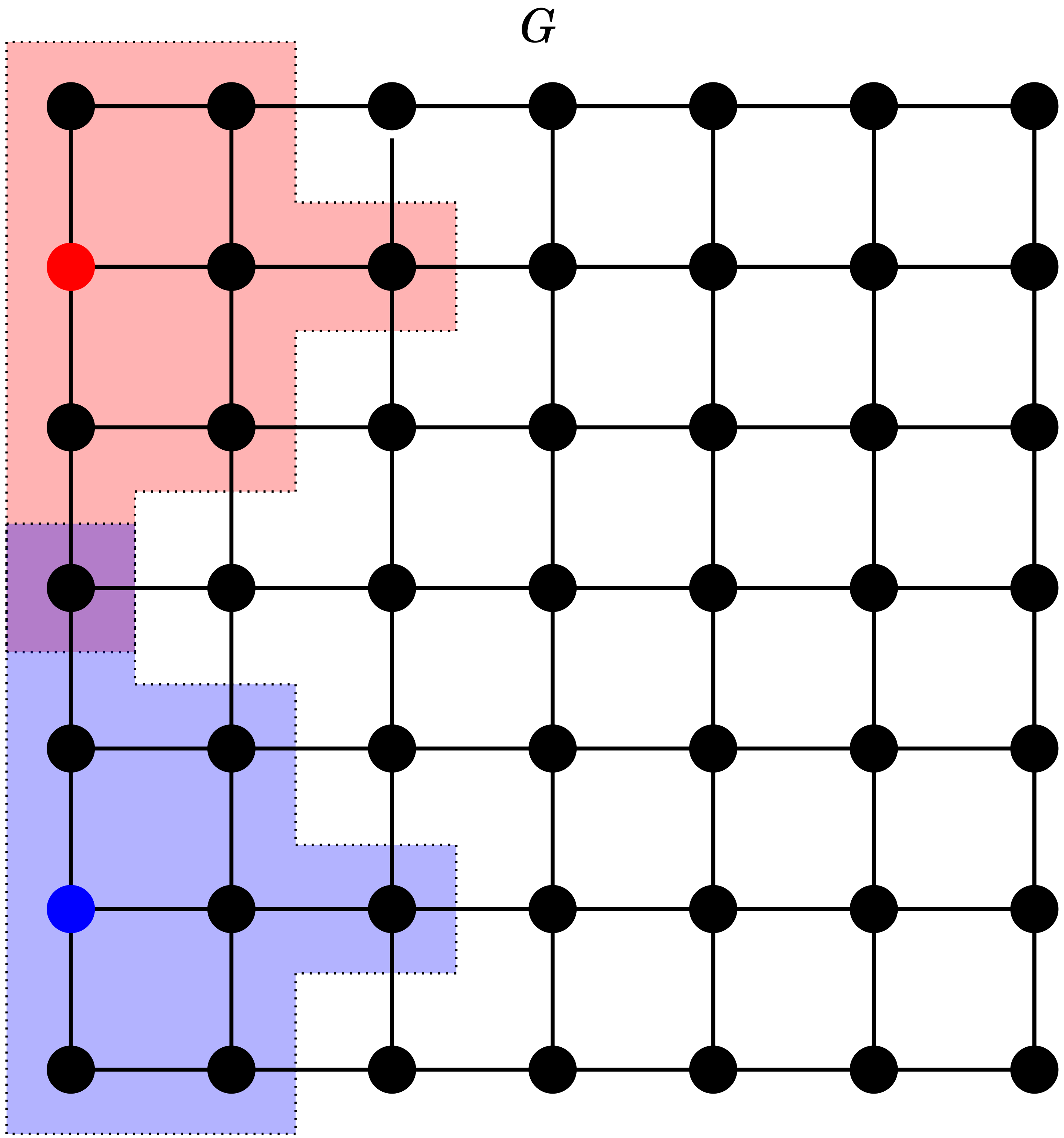

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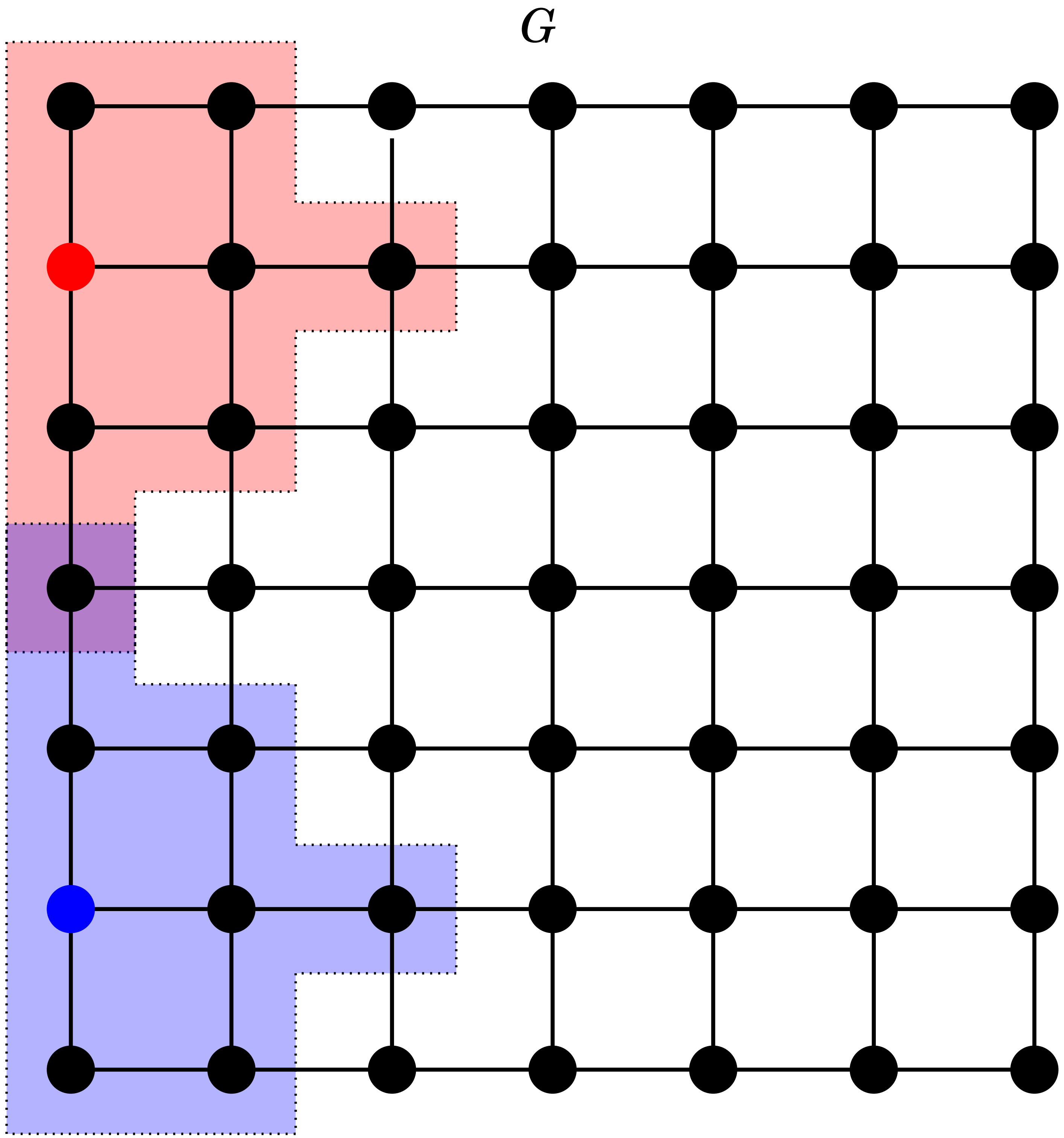

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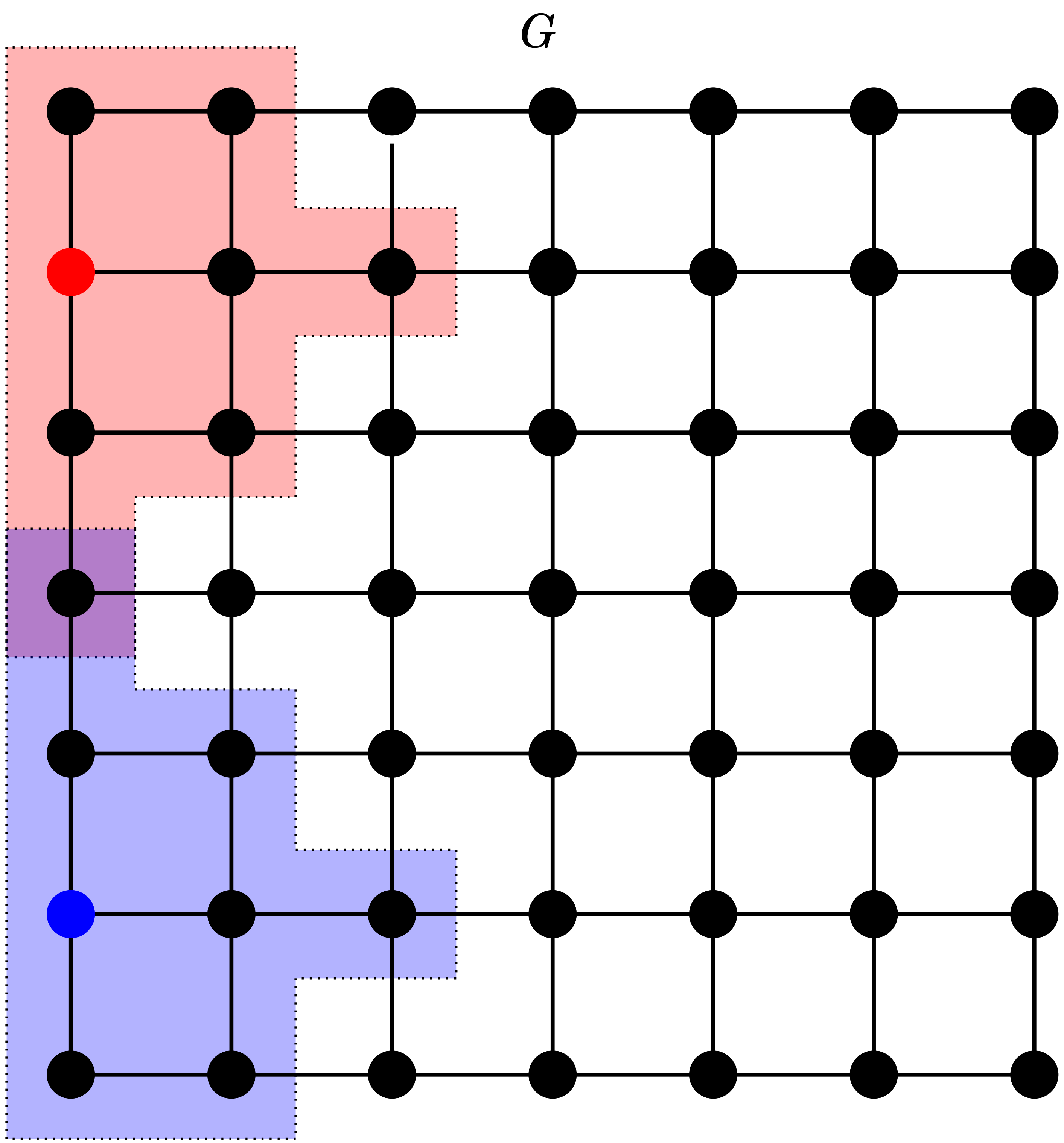

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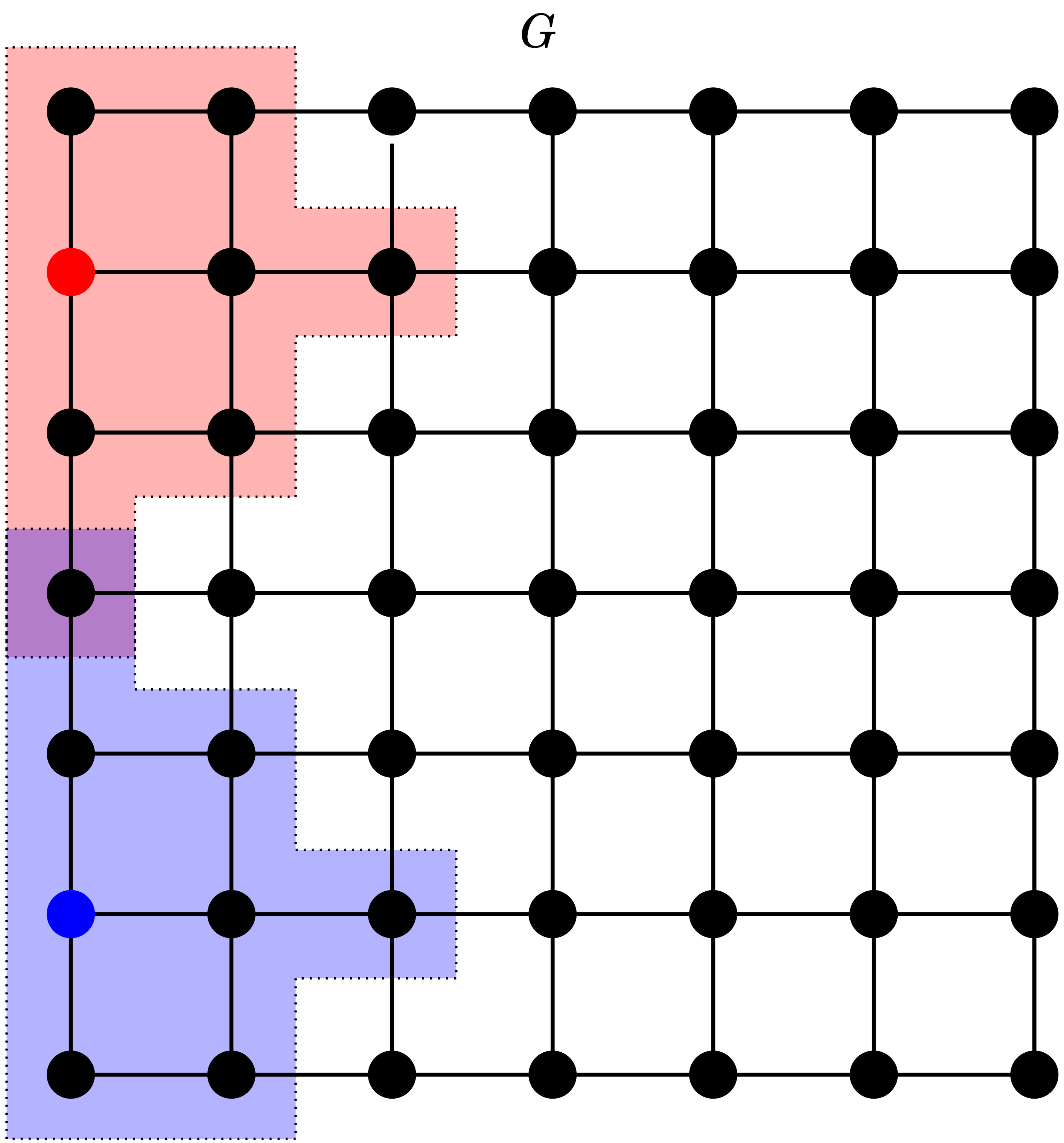

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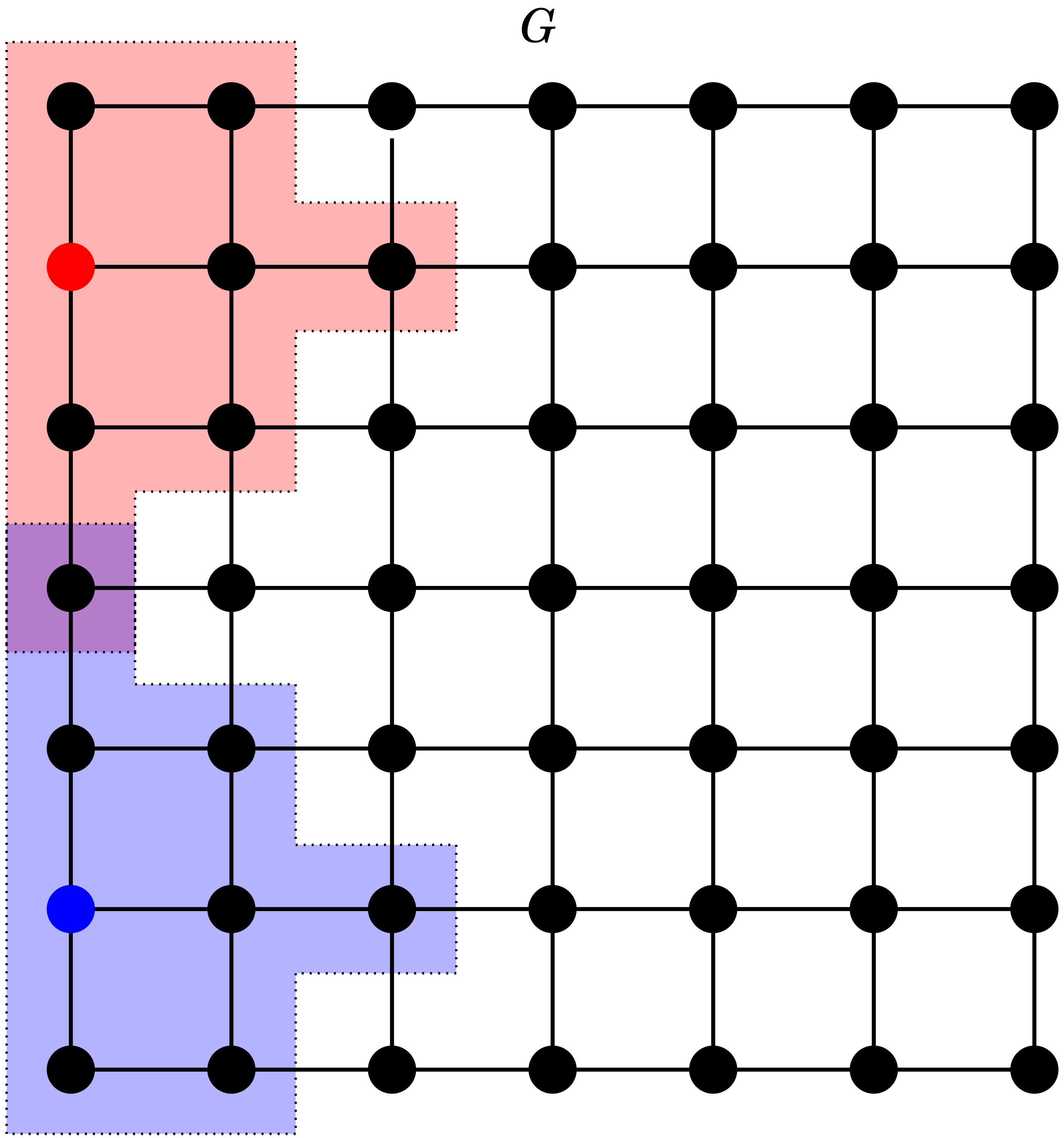
- **Question**: *is there any graph problem that admits quantum advantage?*



Locally checkable labeling (LCL) problems

[Naor and Stockmeyer, STOC '93 & SICOMP '95]

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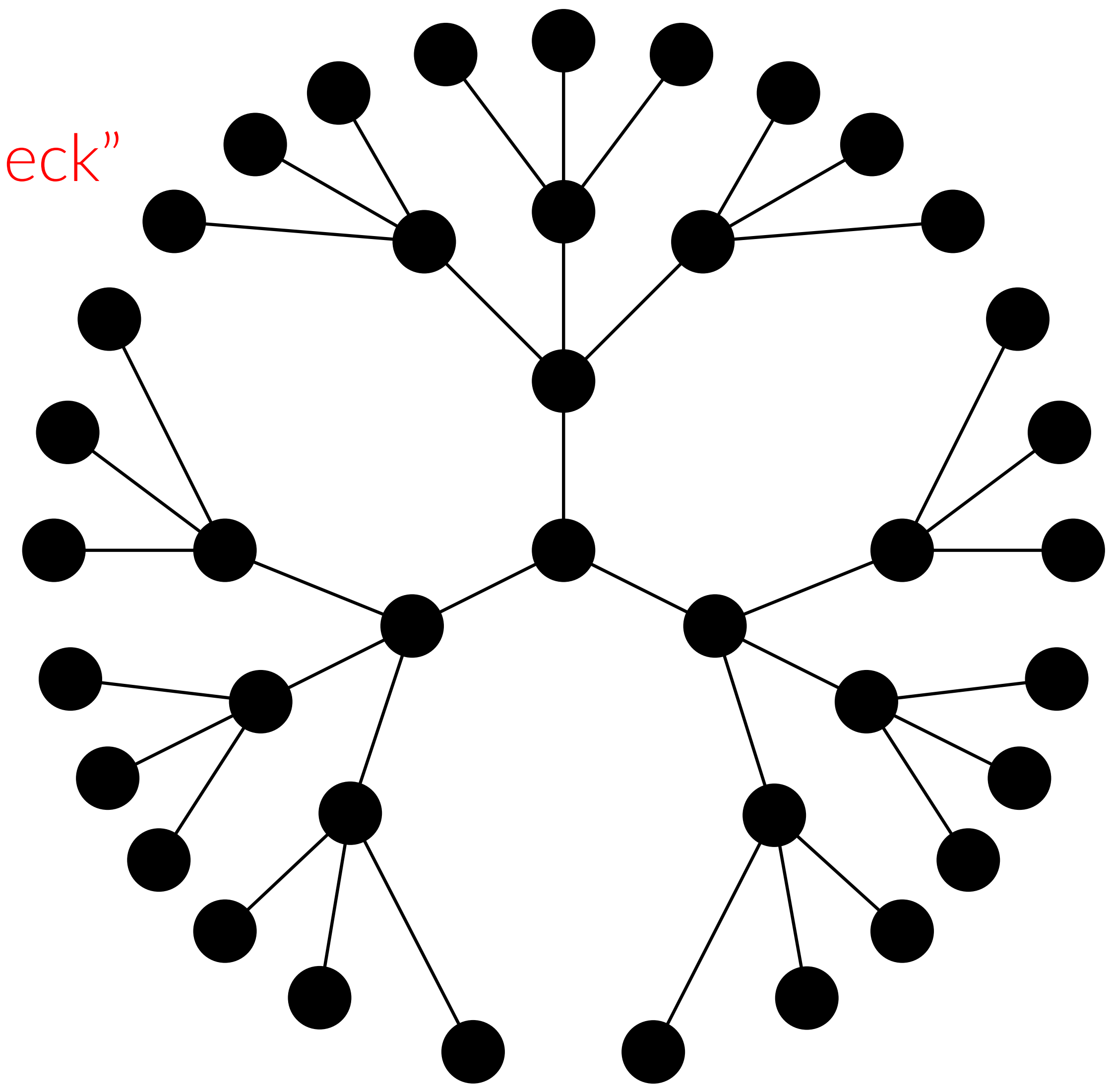
  - coloring, maximal independent set, maximal matching, etc.



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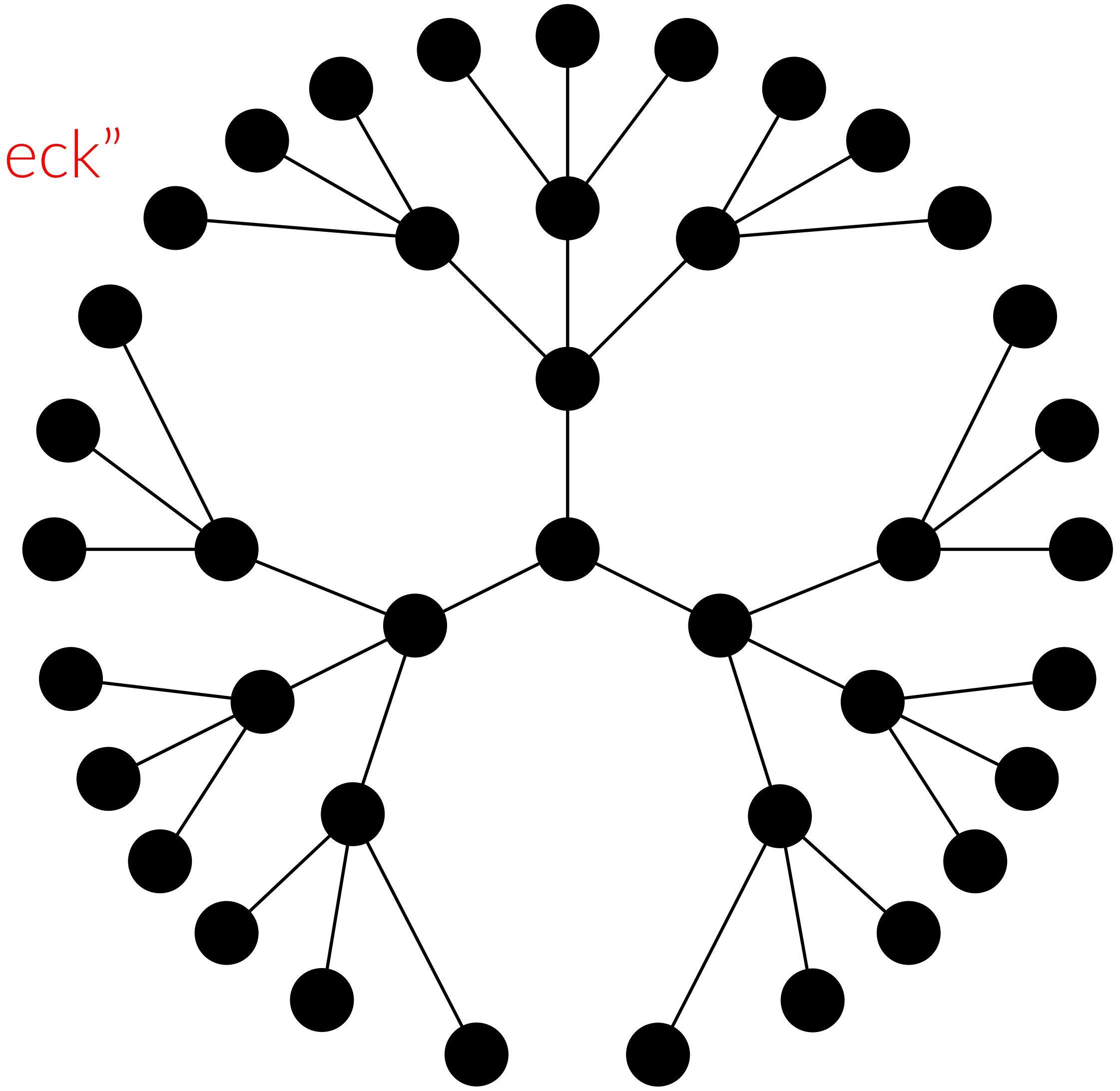

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  - each node can check its solution within its radius-r neighborhood
  - a globally valid iff each node is locally happy

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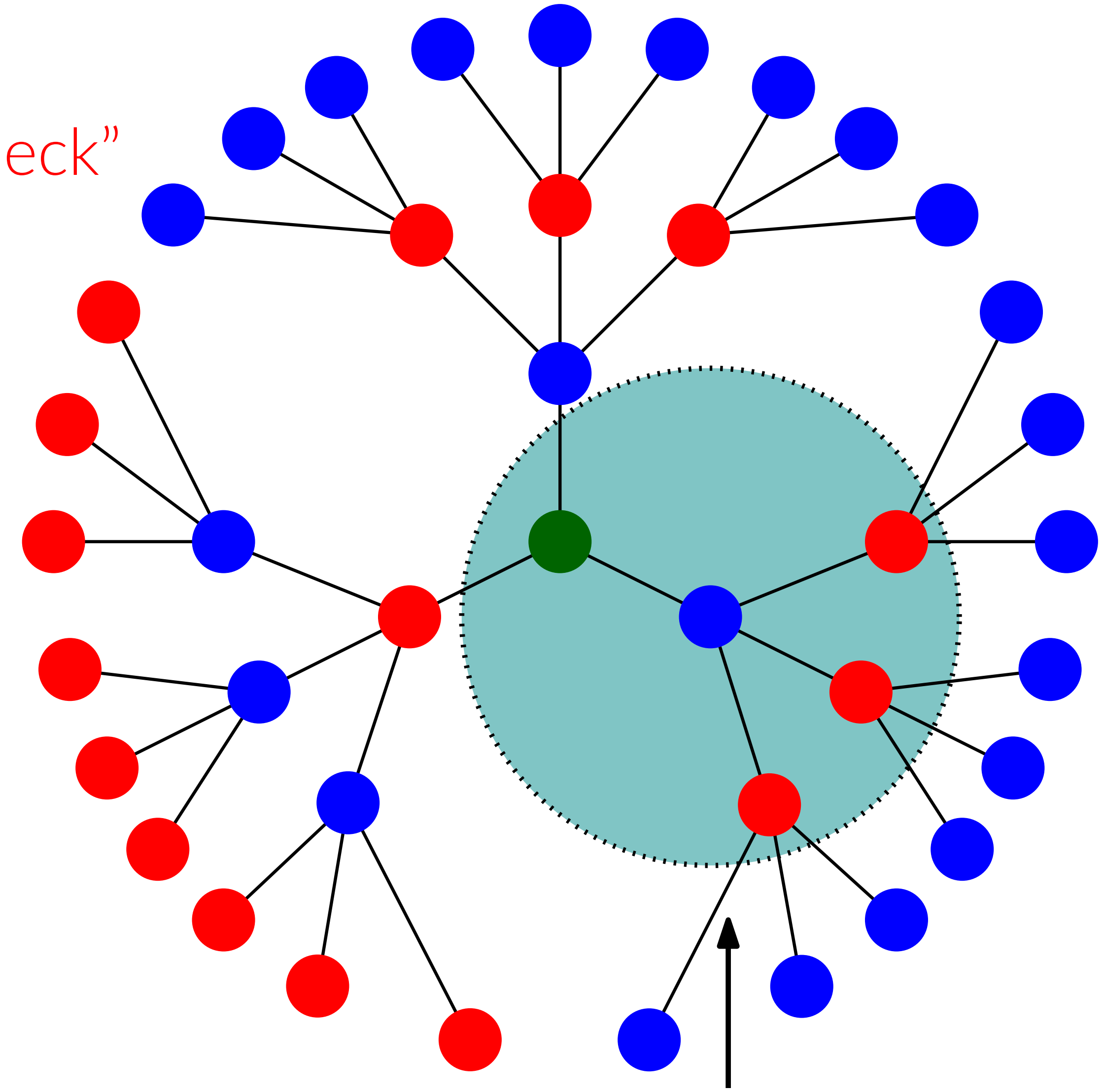

3-coloring: the blue node checks if its color is different from those of its neighbors

valid LCL

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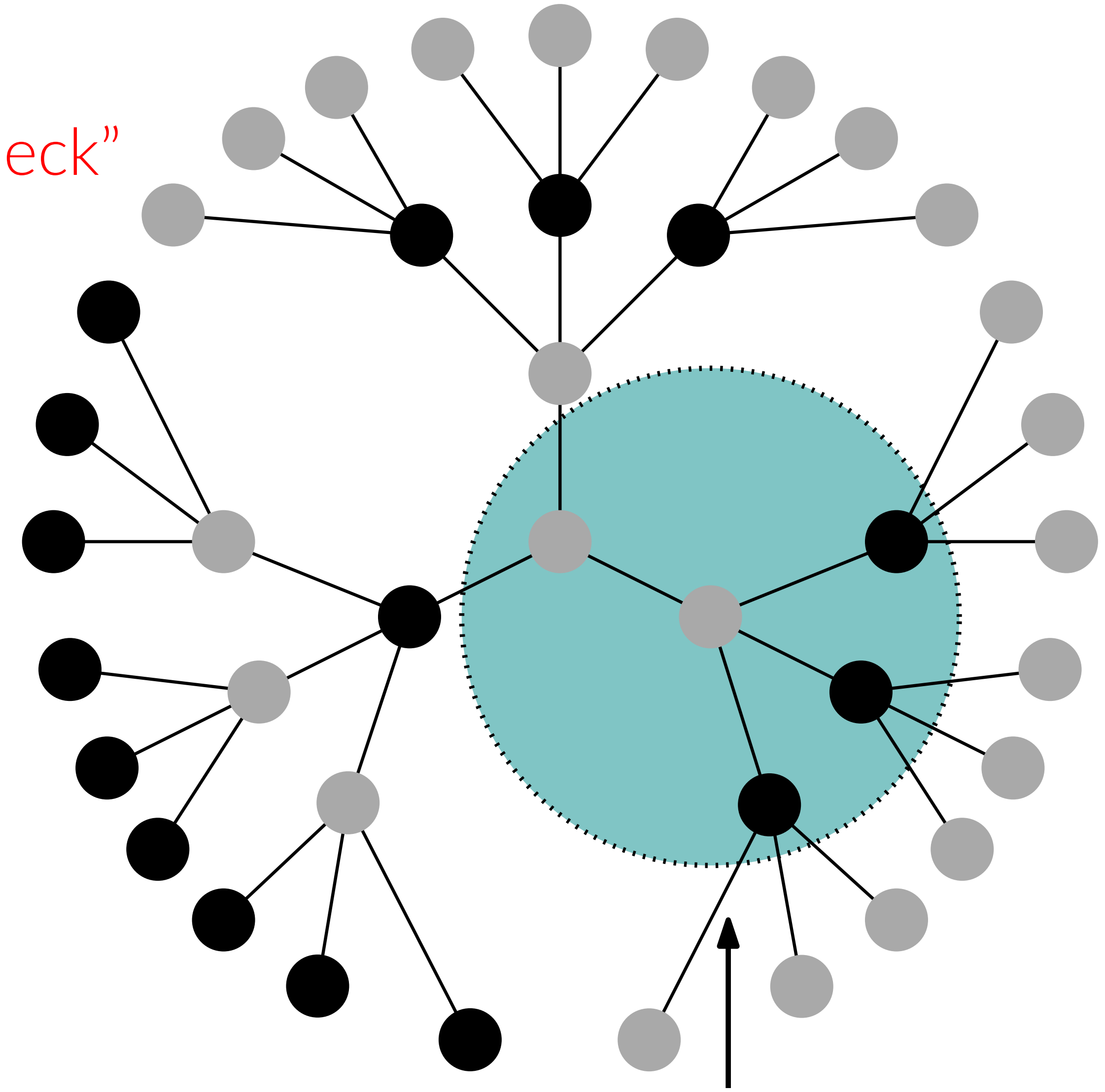

MIS: each node checks if it is in the IS or if it has a neighbor in the IS

valid LCL

Locally checkable labeling (LCL) problems

[Naor and Stockmeyer, STOC '93 & SICOMP '95]

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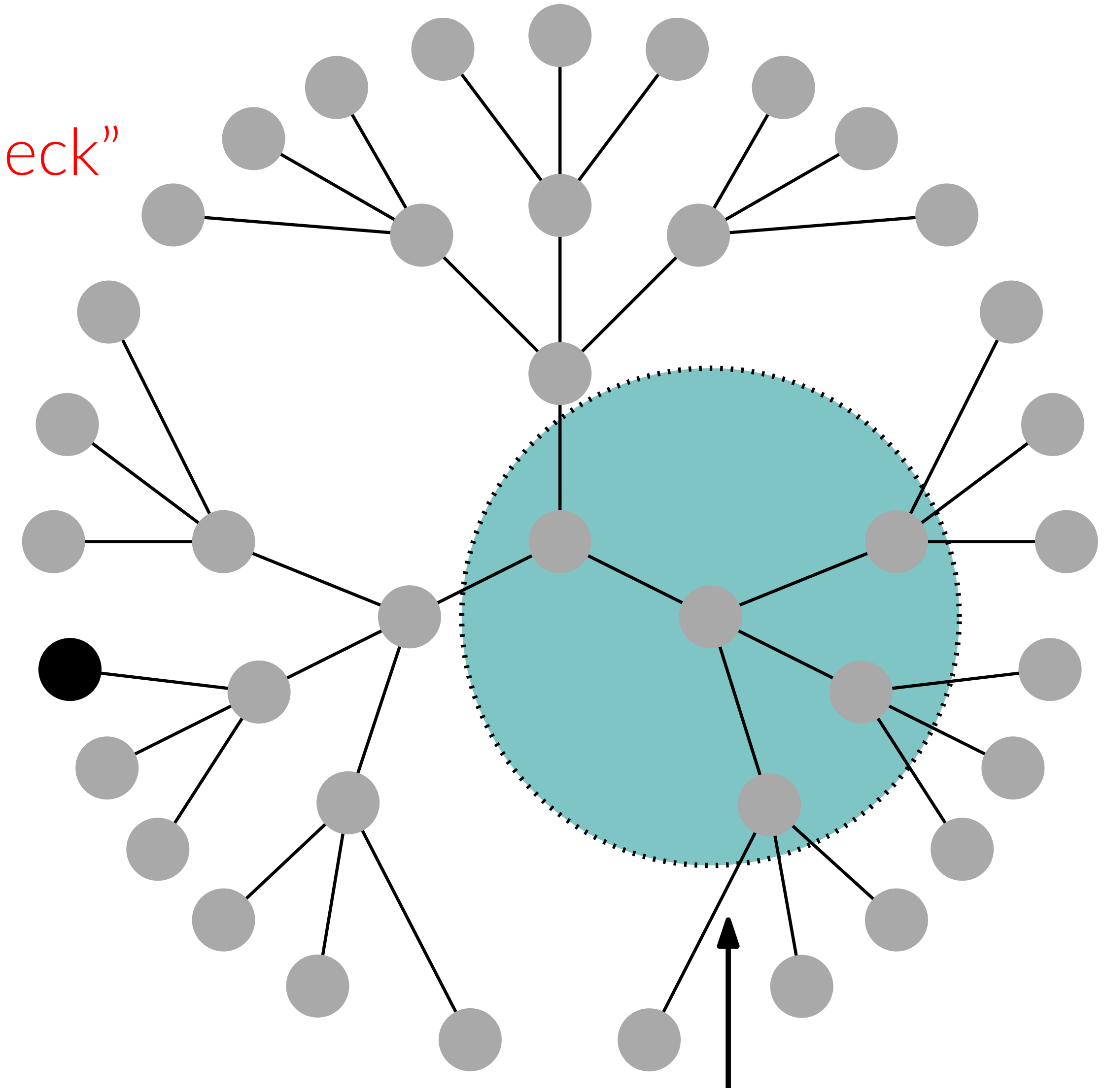

Leader election: the checking radius should be $r = \mathrm{diam}(G)$

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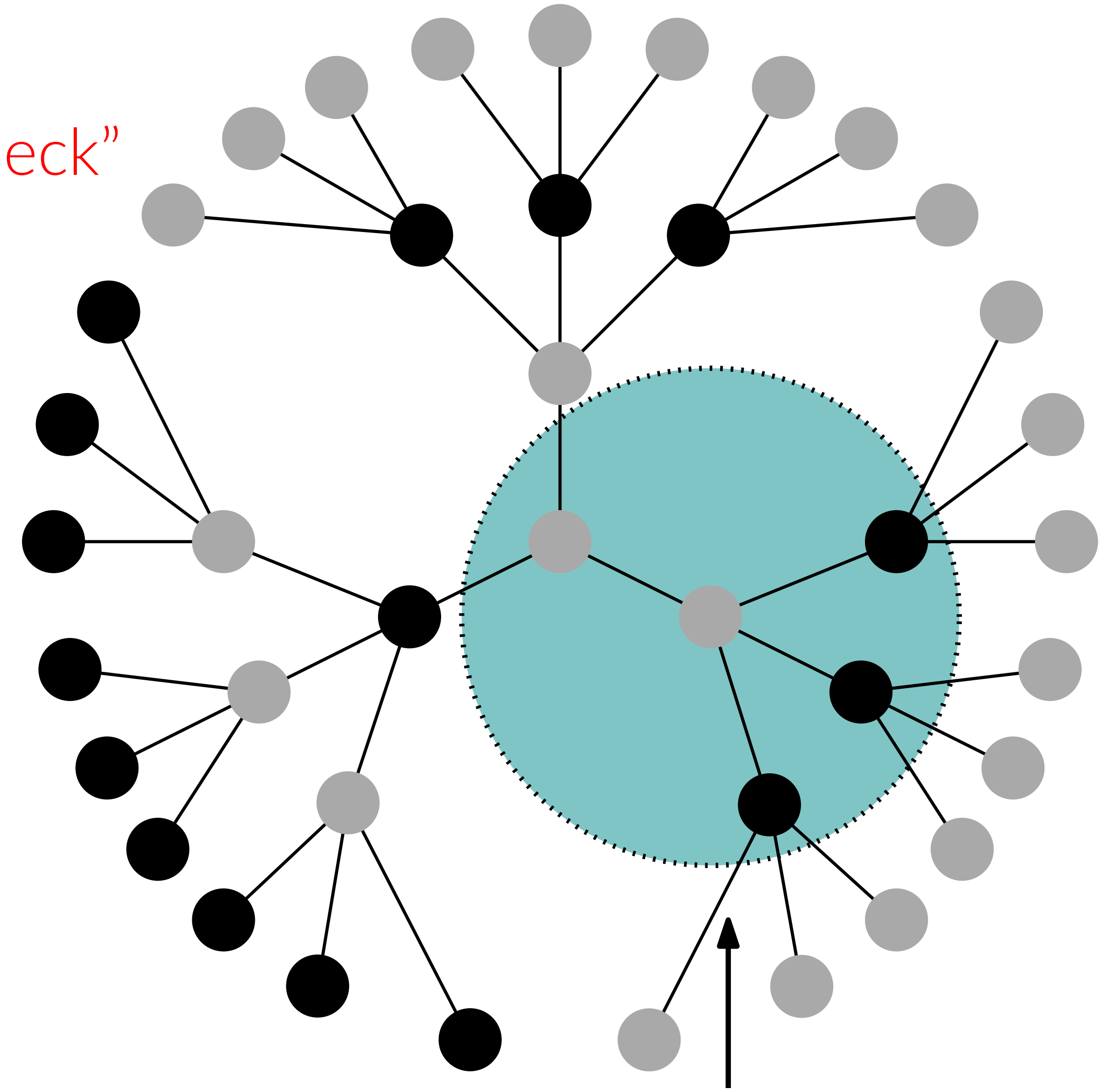

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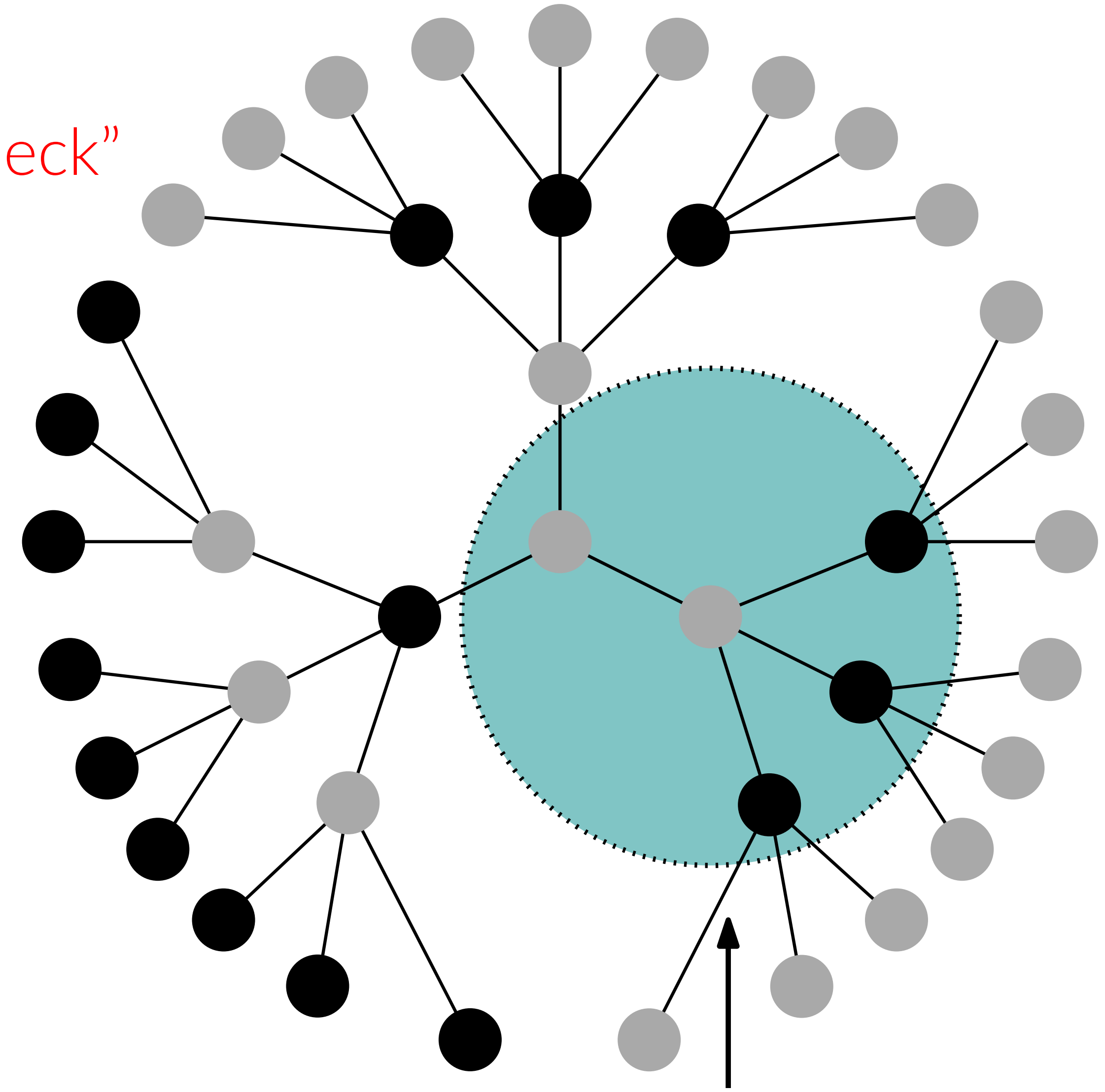

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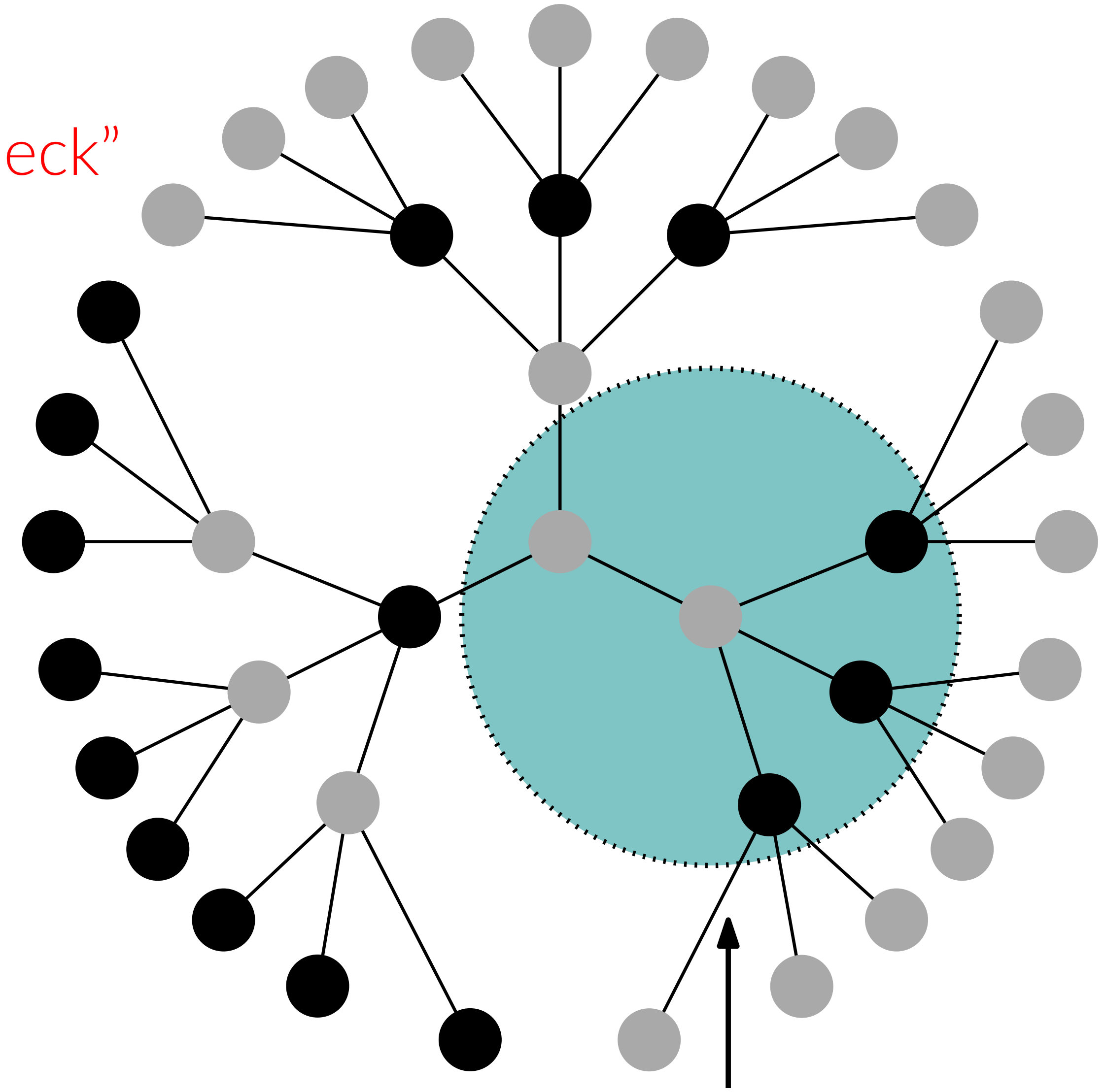

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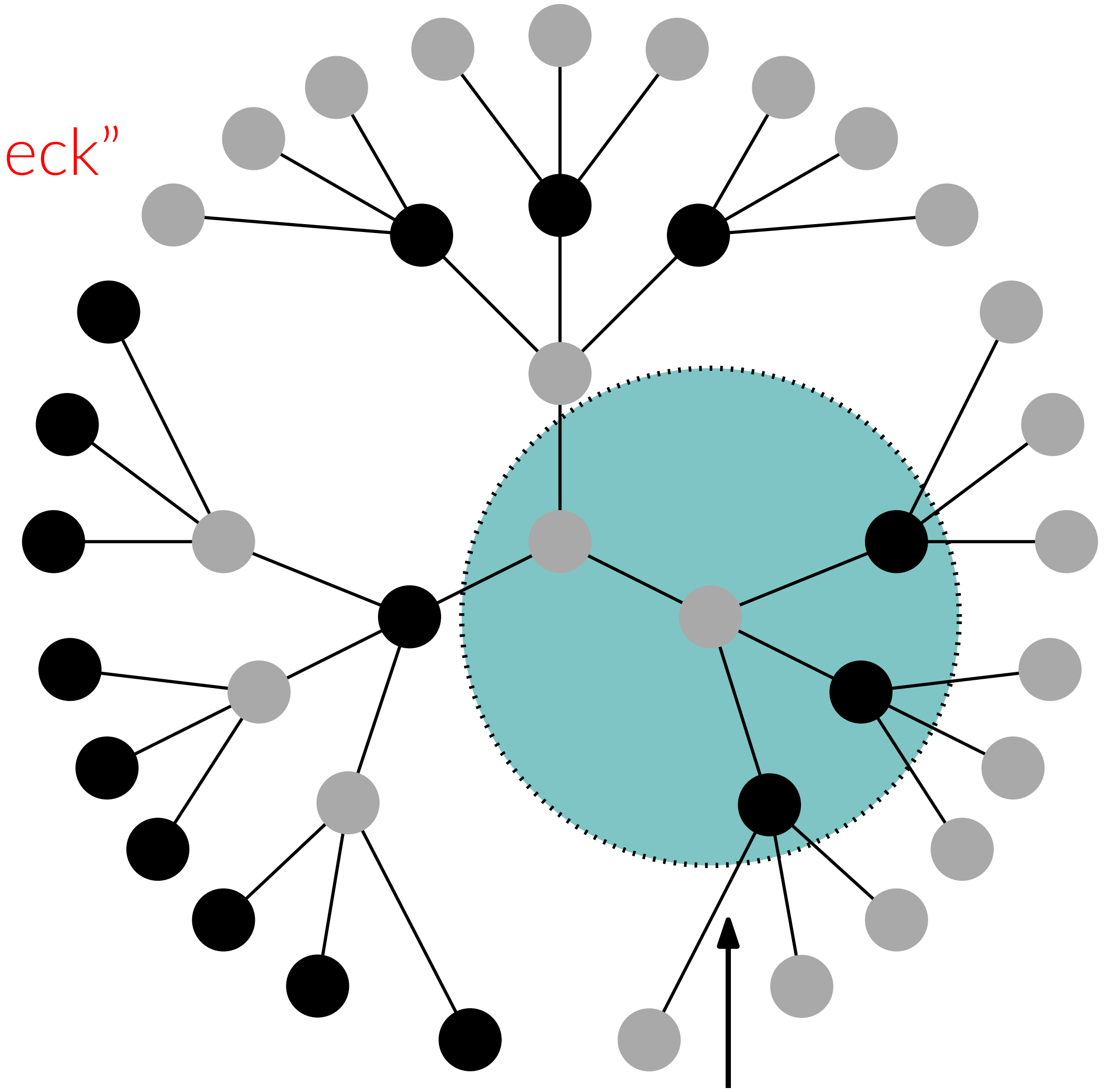

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2. **Classical lower bounds**: the indistinguishability argument

3. **Properties of distributed algorithms**: independence and non-signaling

4. **Super-quantum models**: bounded-dependence and non-signaling model

5. **State of the art results**

6. **Quantum advantage**

Indistinguishability argument

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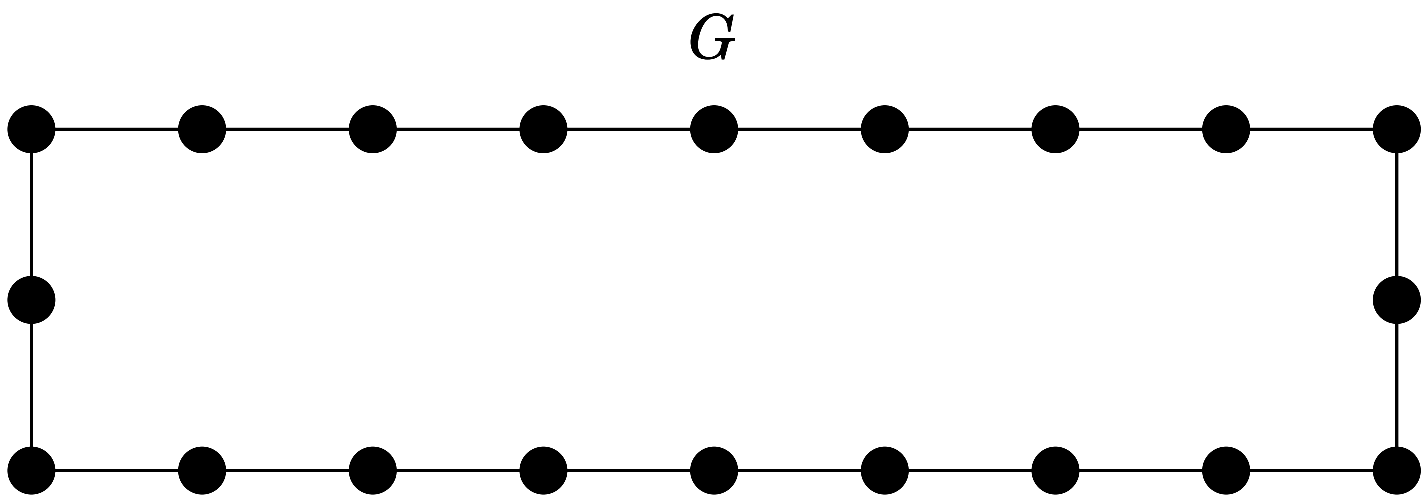

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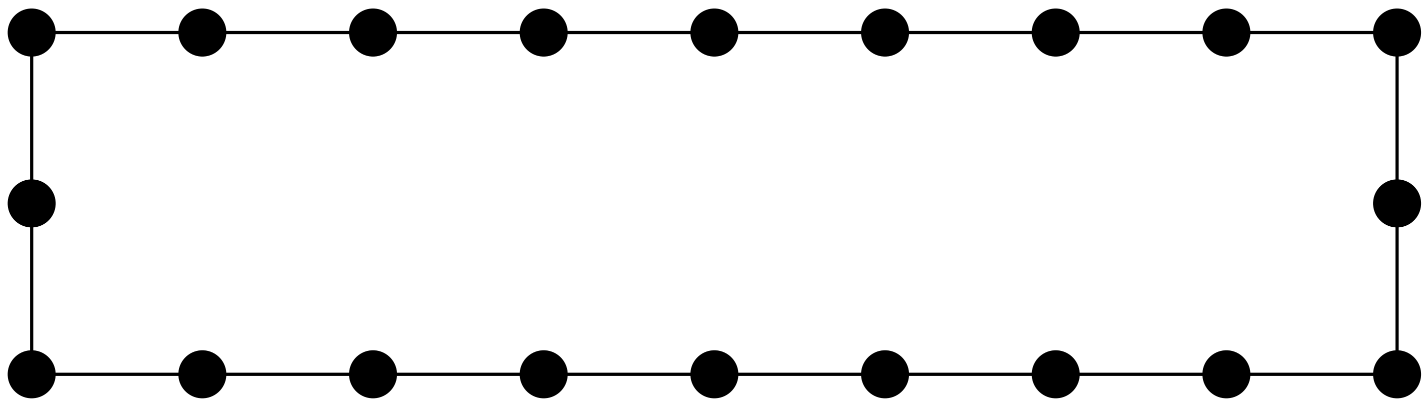

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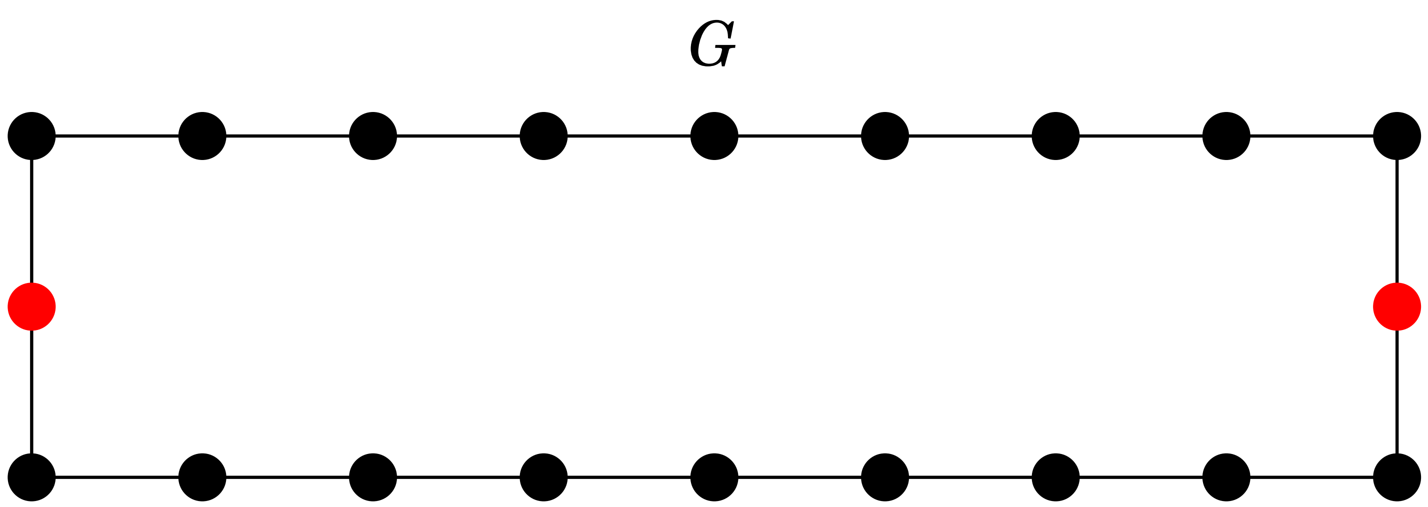

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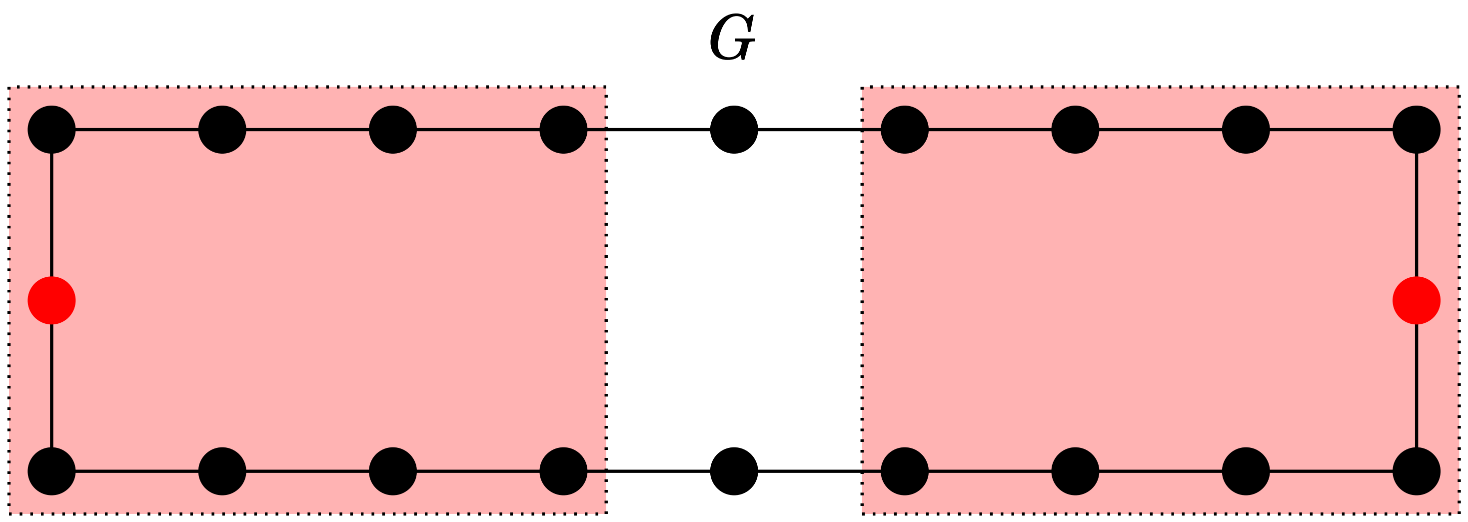

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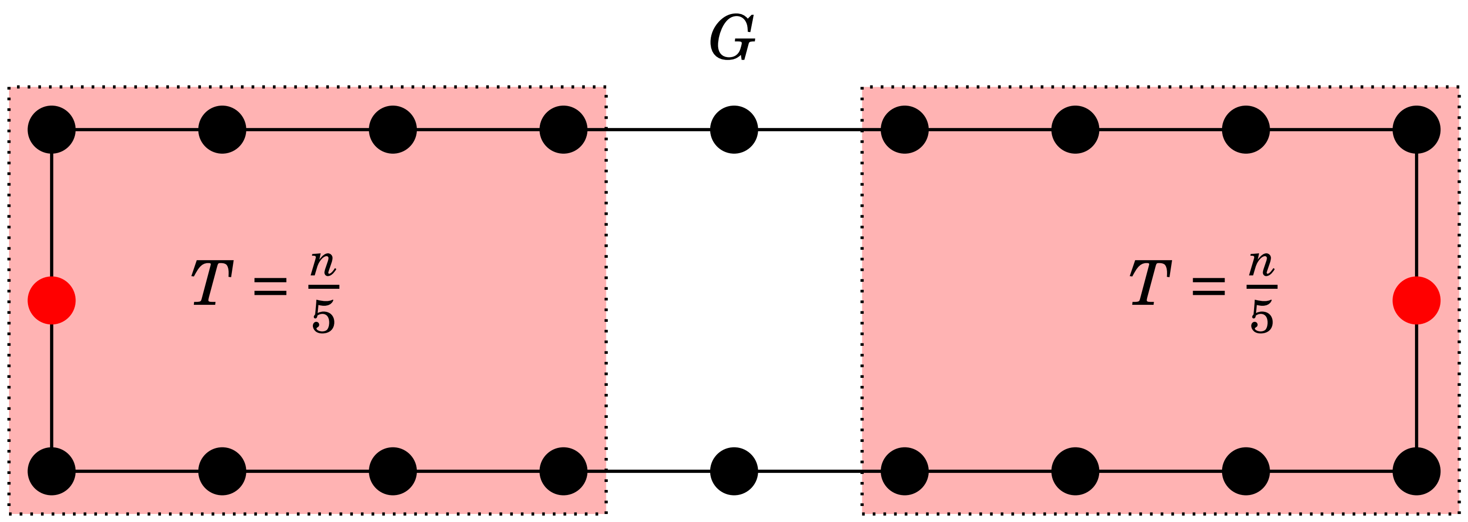

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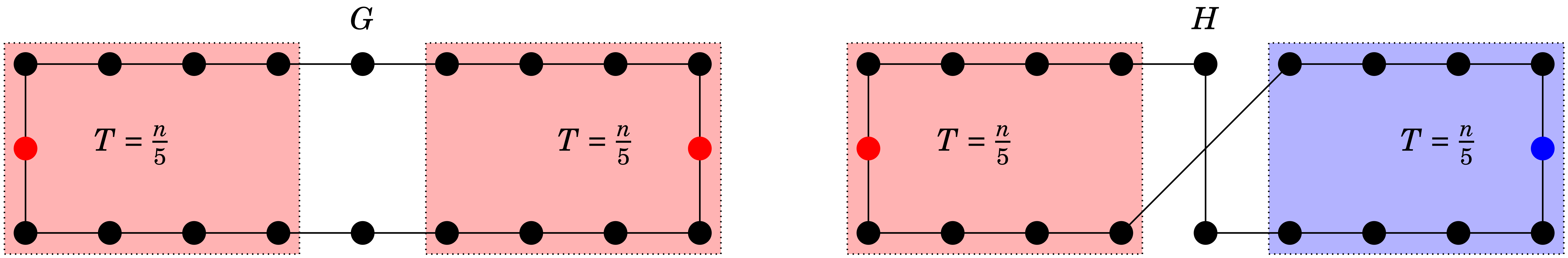

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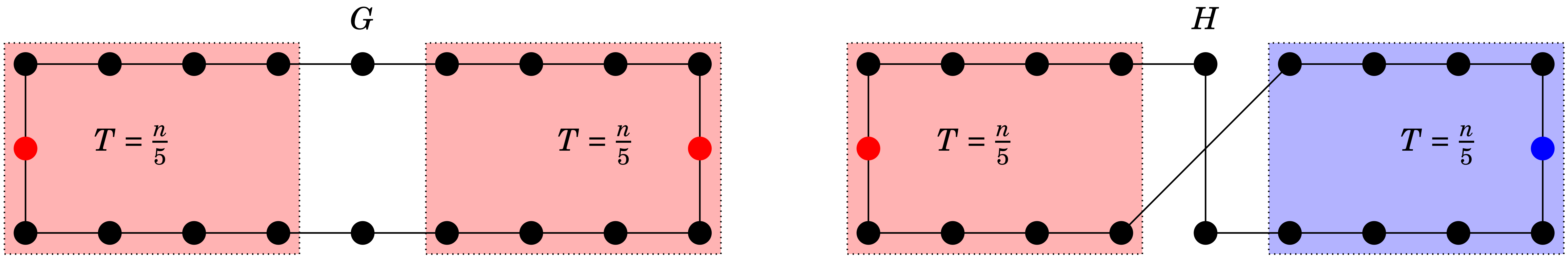

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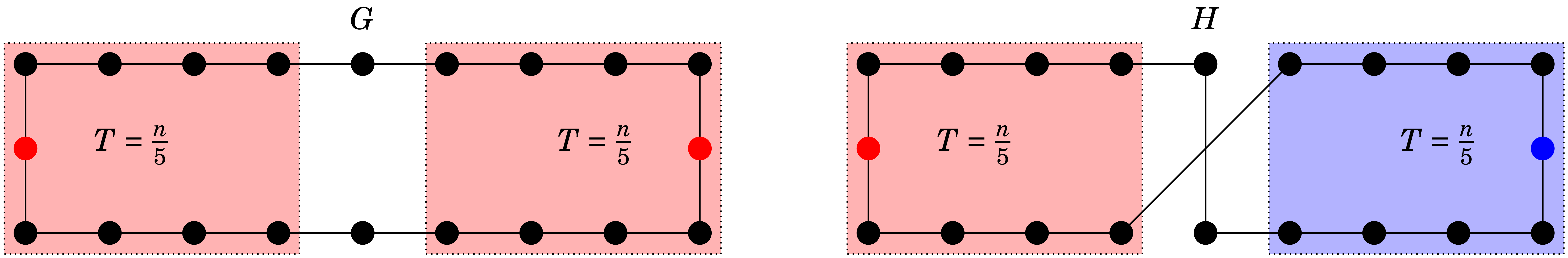

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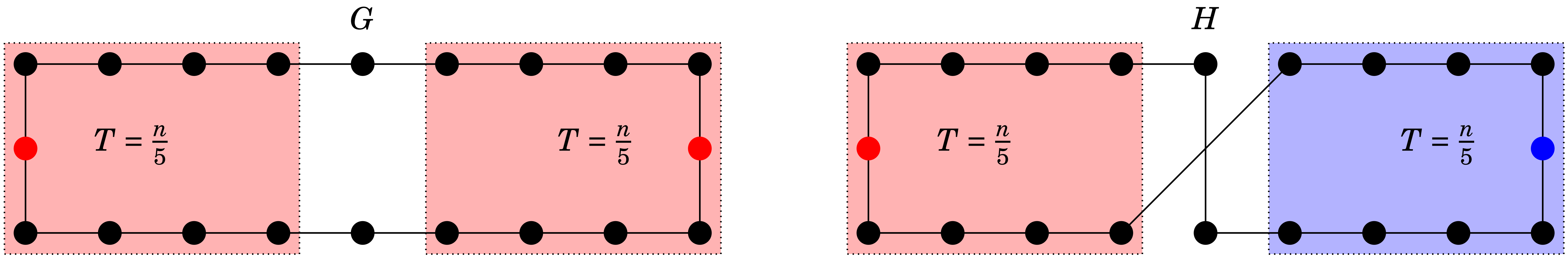

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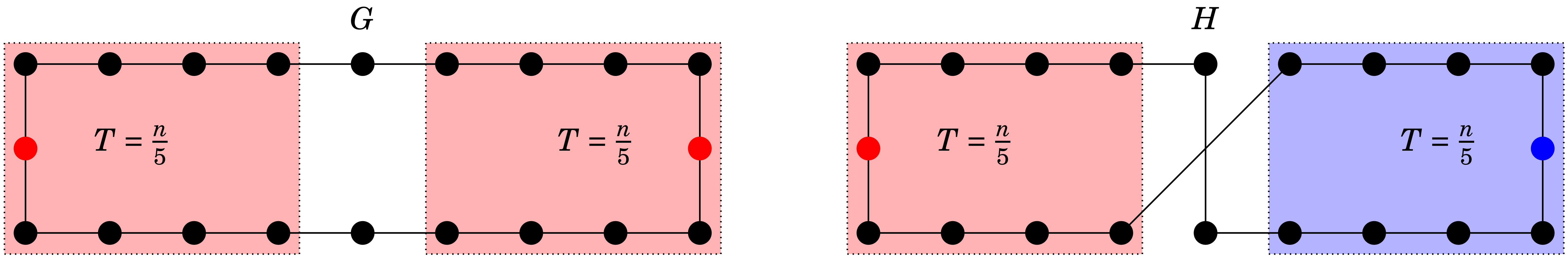

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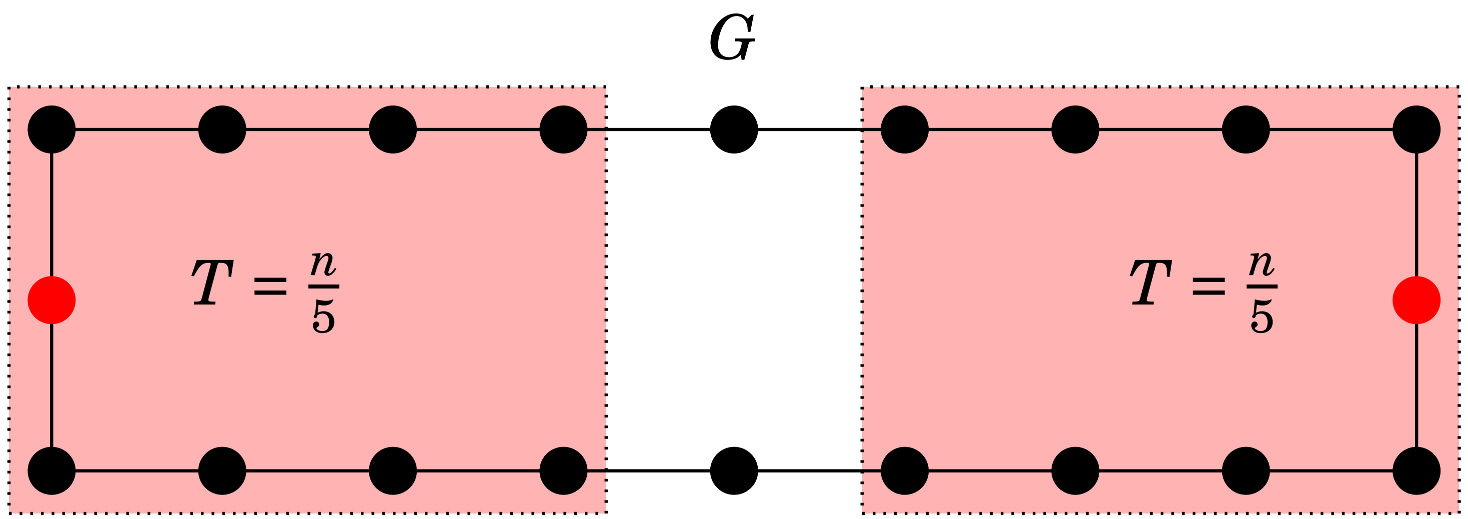

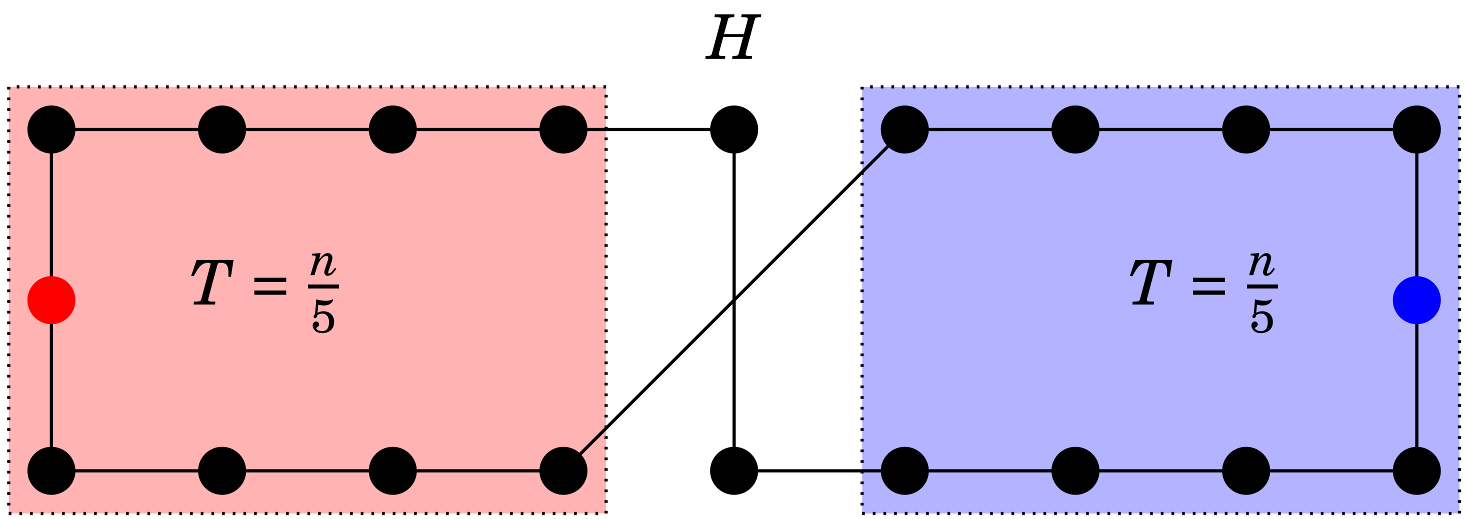

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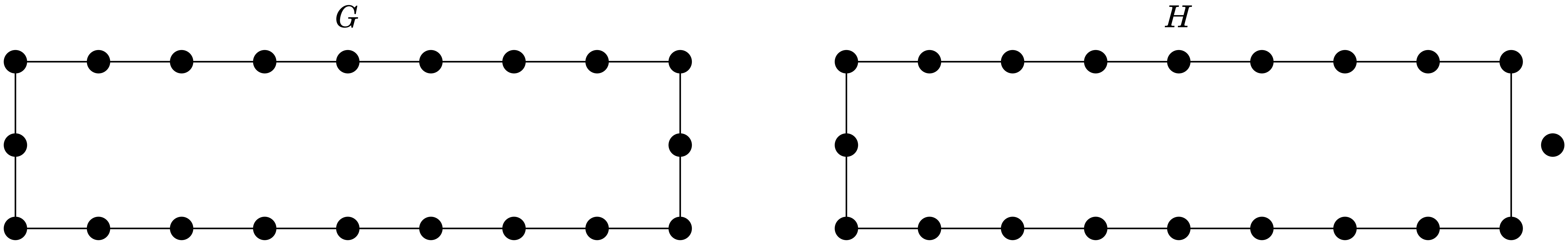

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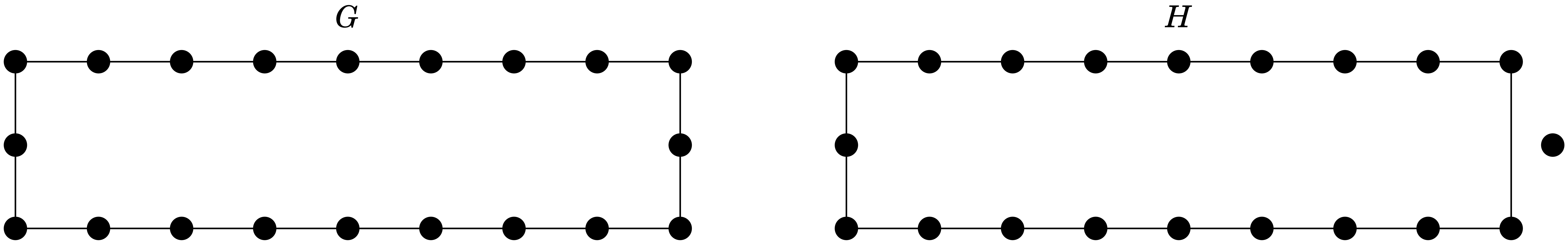

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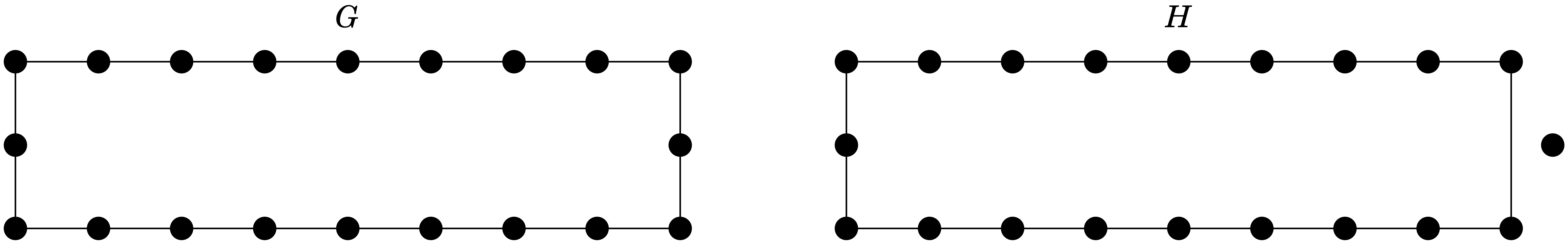

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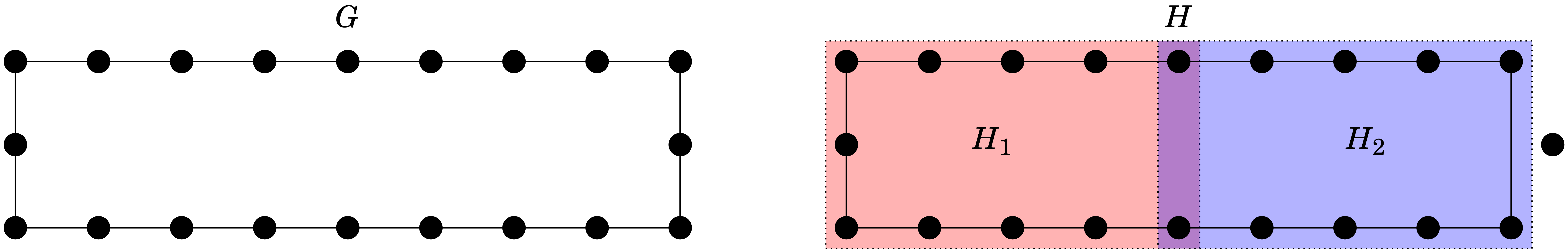

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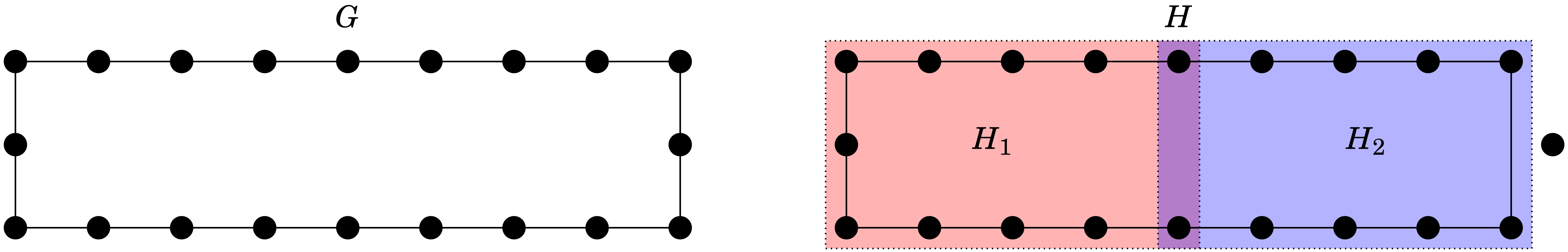

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Graph-existential indistinguishability (randomized)

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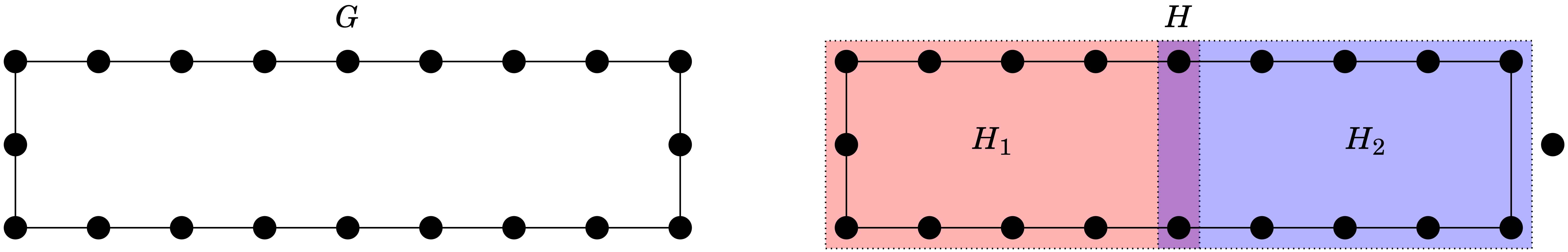

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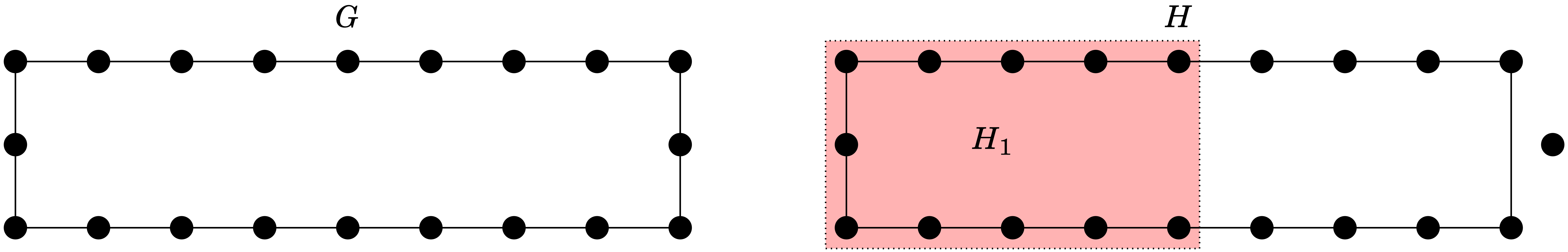

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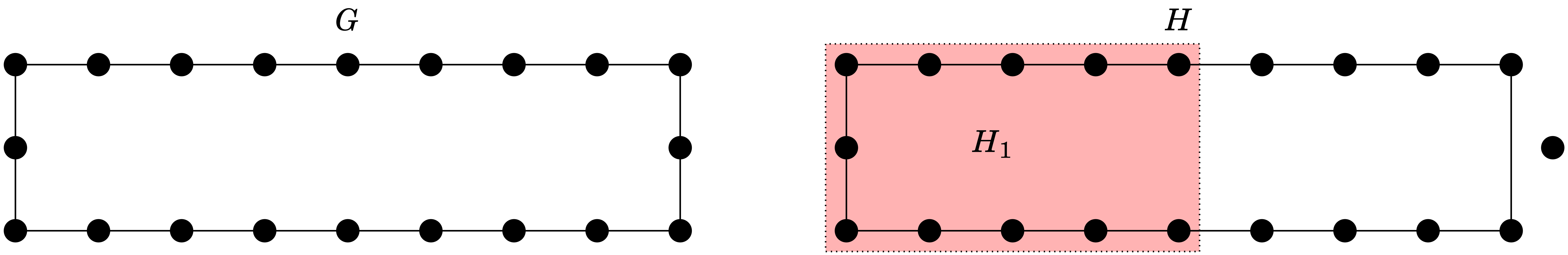

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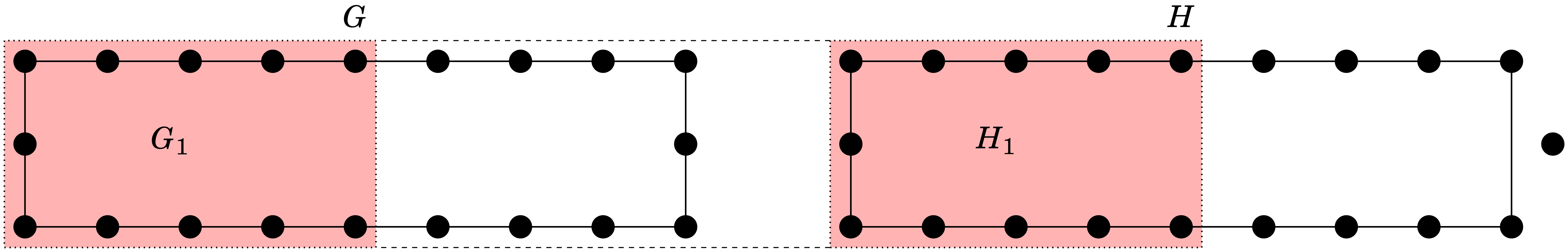

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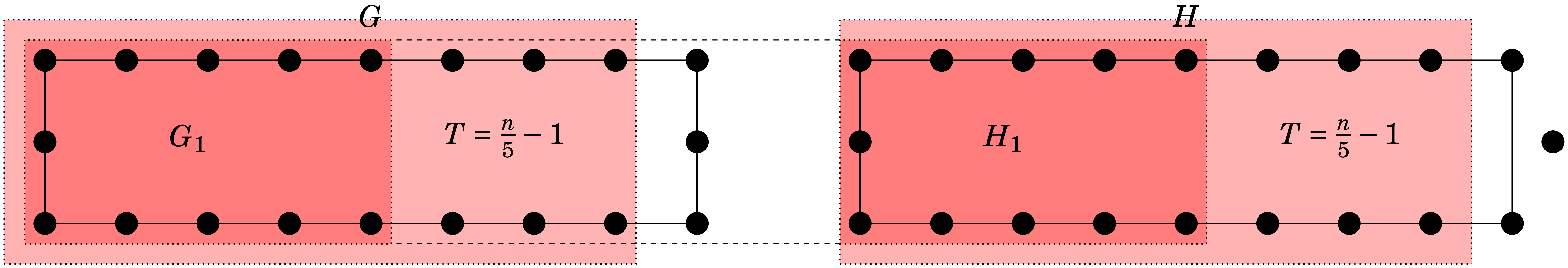

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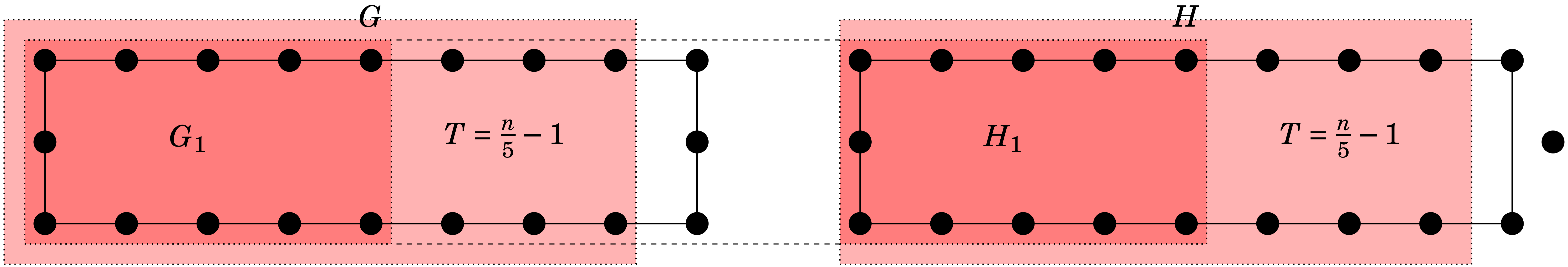

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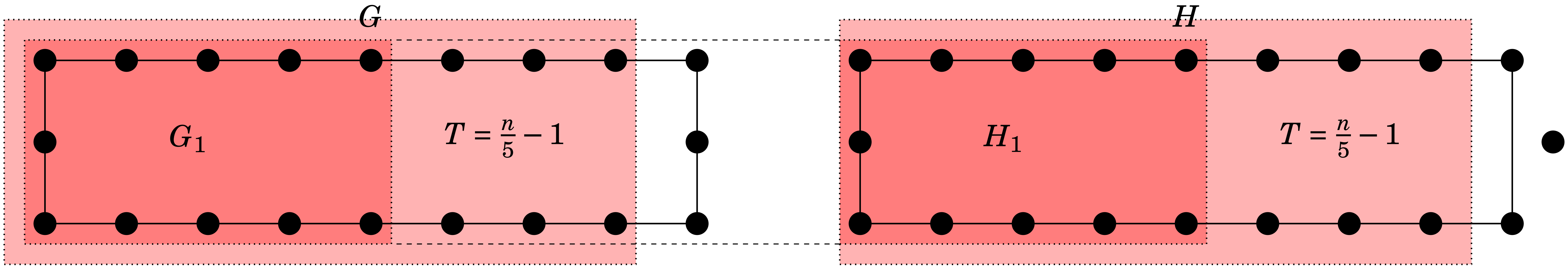

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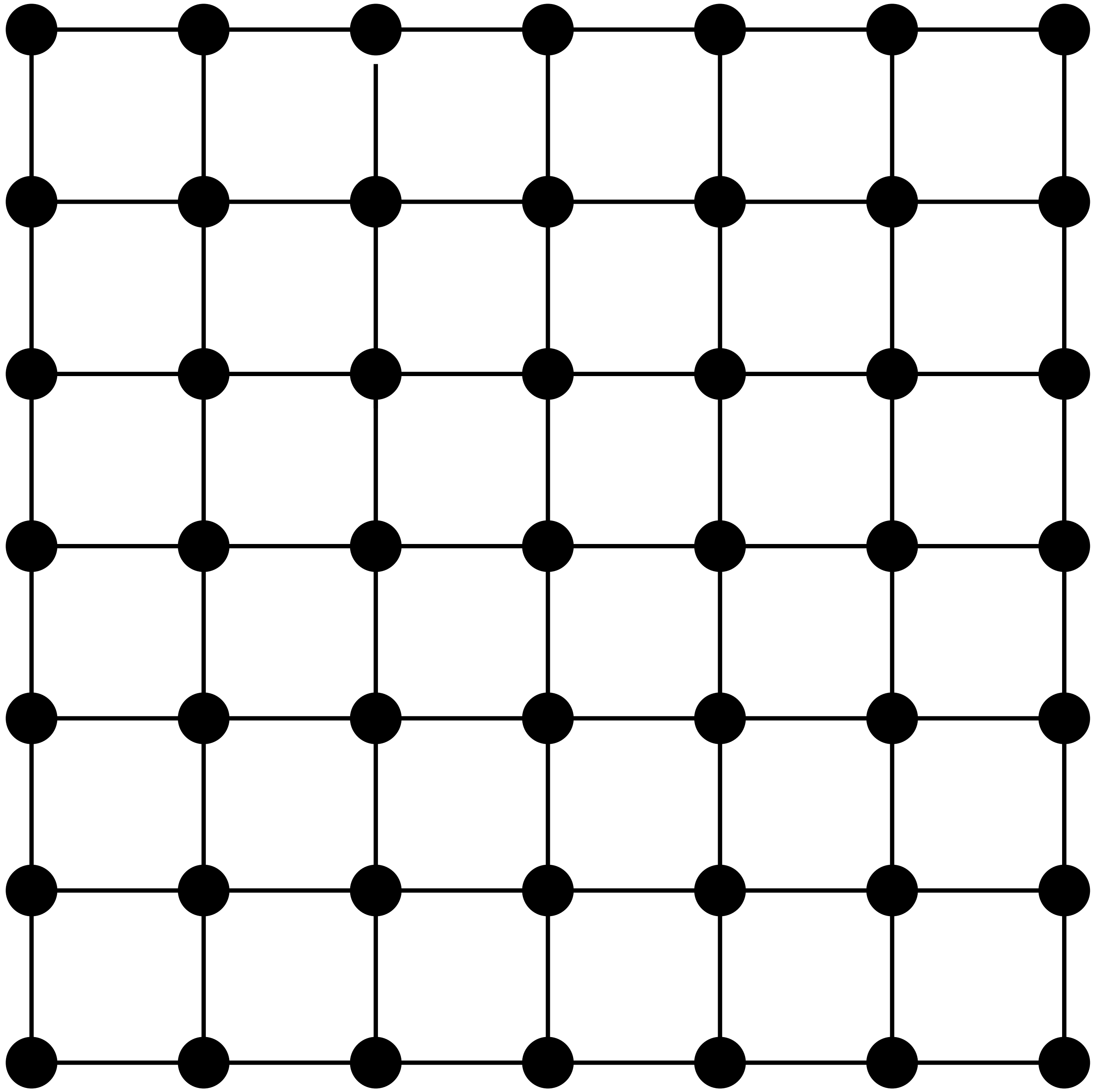
Properties of distributed algorithms

- **Run** a $\mathbf{2}$-round algorithm $\boldsymbol{A}$ in $\boldsymbol{G}$



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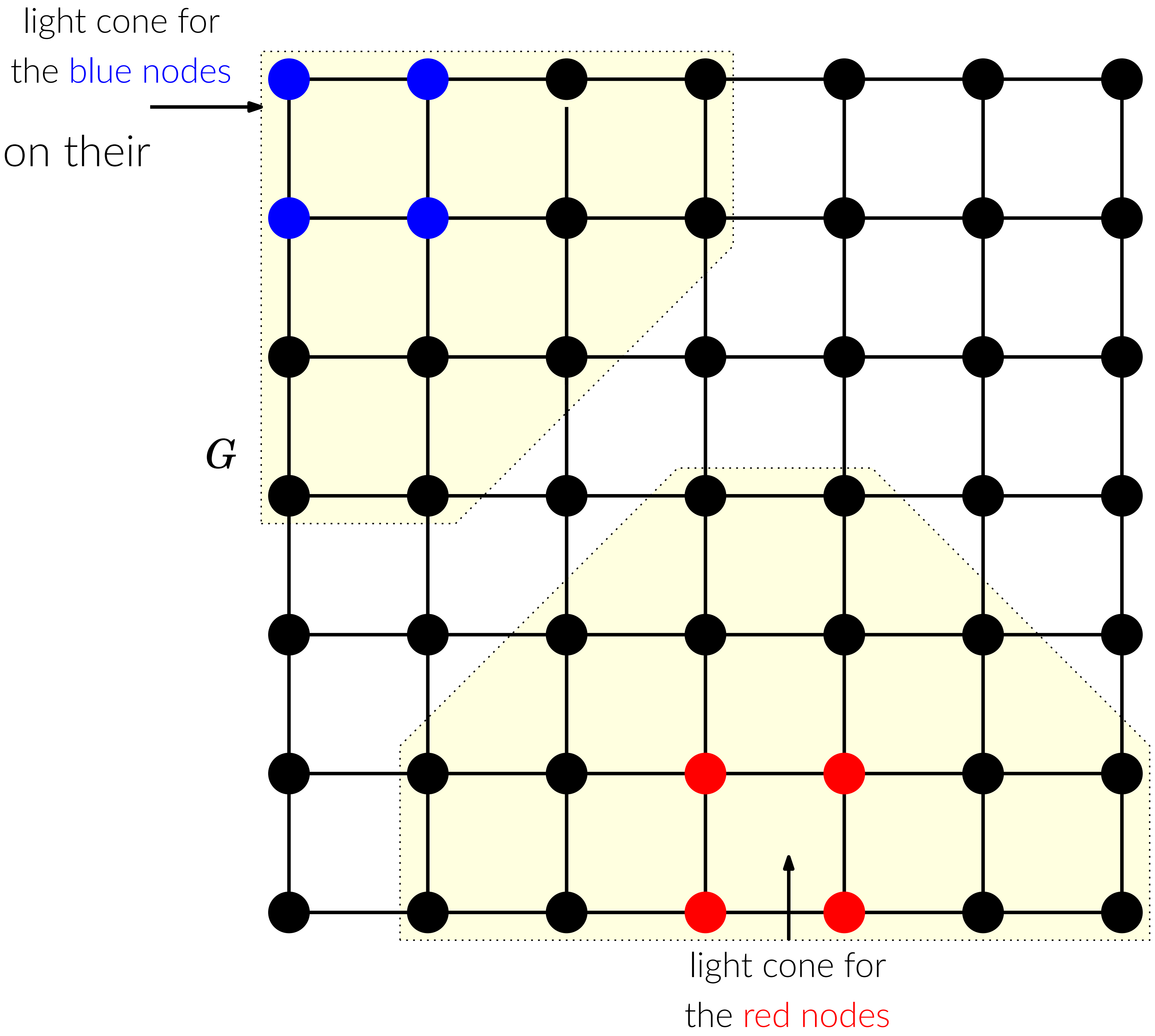
- **Run** a $\mathbf{2}$-round algorithm $\boldsymbol{A}$ in $\boldsymbol{G}$
  - output for the red and blue nodes only depends on their respective light cones



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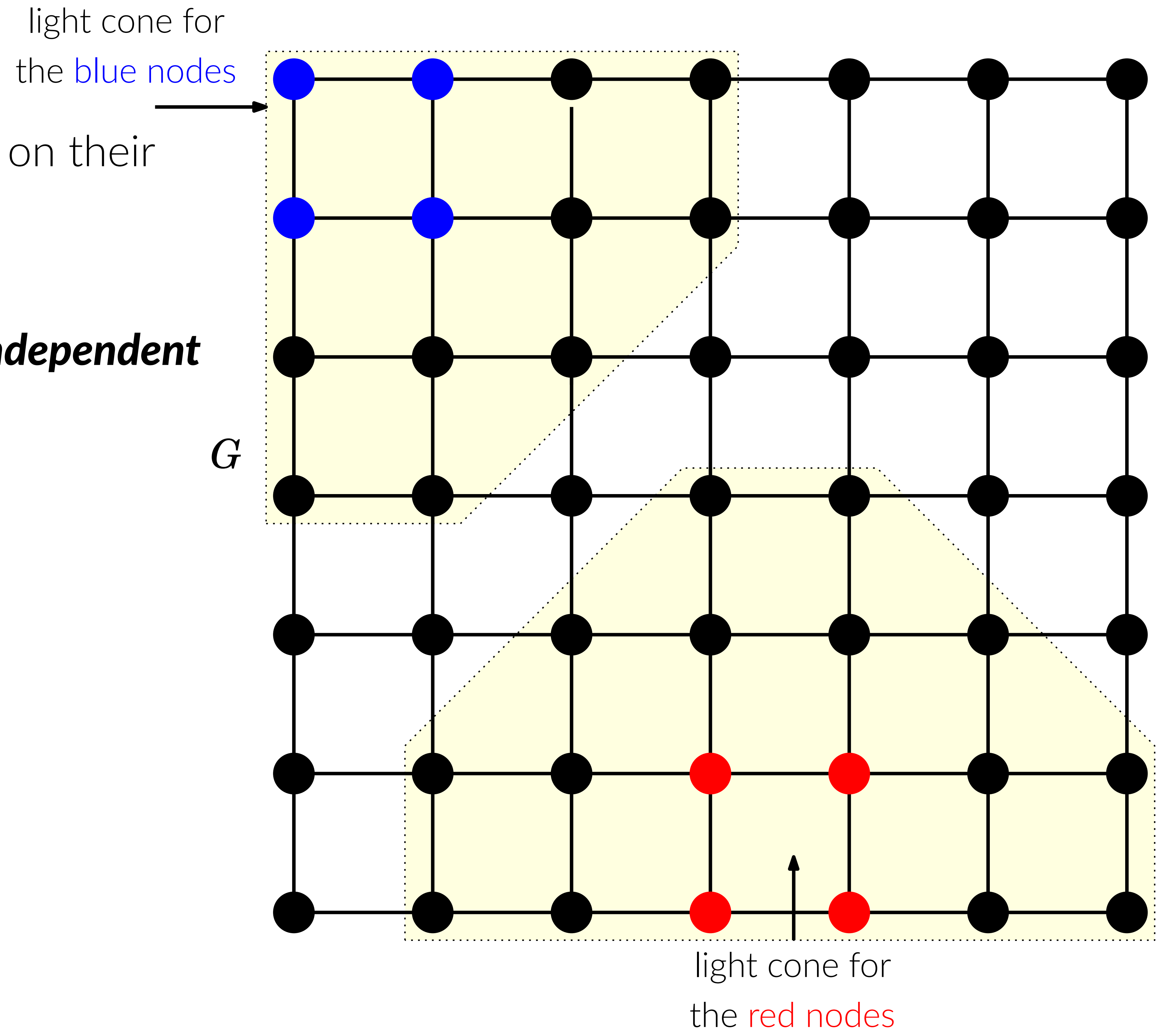

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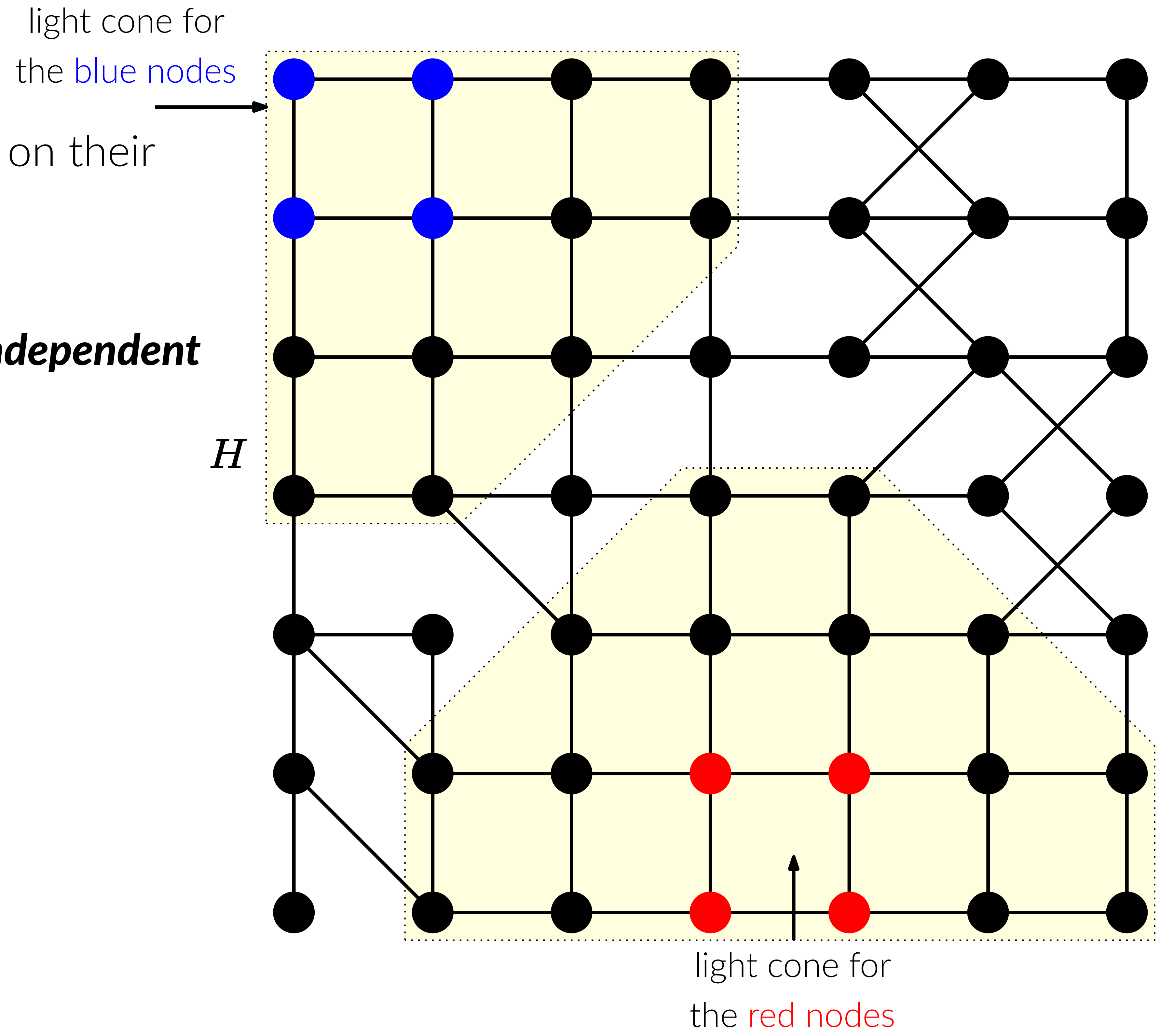

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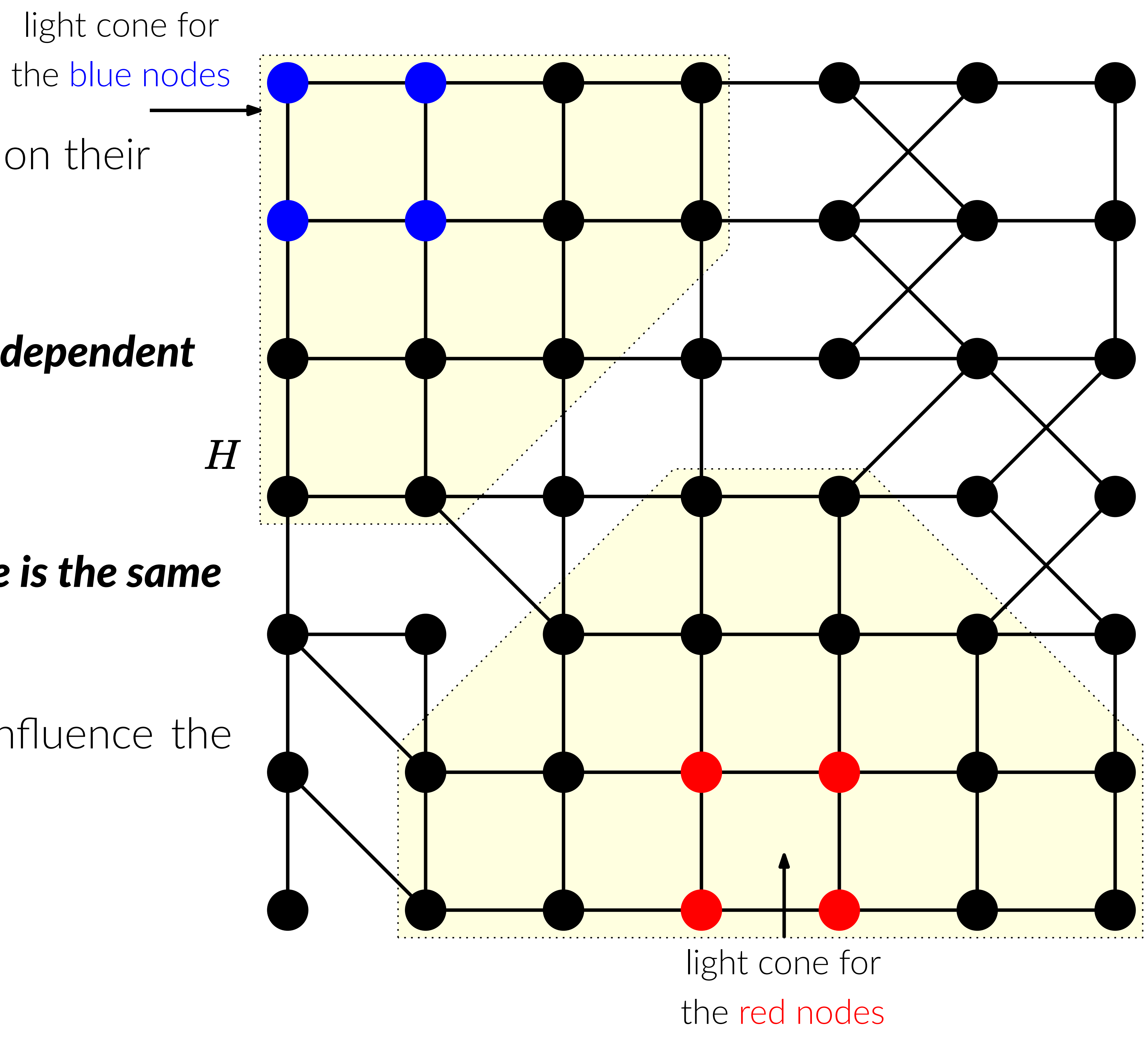
- **Output distributions** for red and blue nodes are ***independent***
  - as long as their distance is at least 5
- **Output distributions** remains ***the same*** if ***light cone is the same***
  - non-signaling property
  - changes that are beyond 2-hops away do not influence the output distribution
  - also known as causality



Abstracting output distributions

- A T-round distributed algorithm yields an **output distribution** with the following **properties**:
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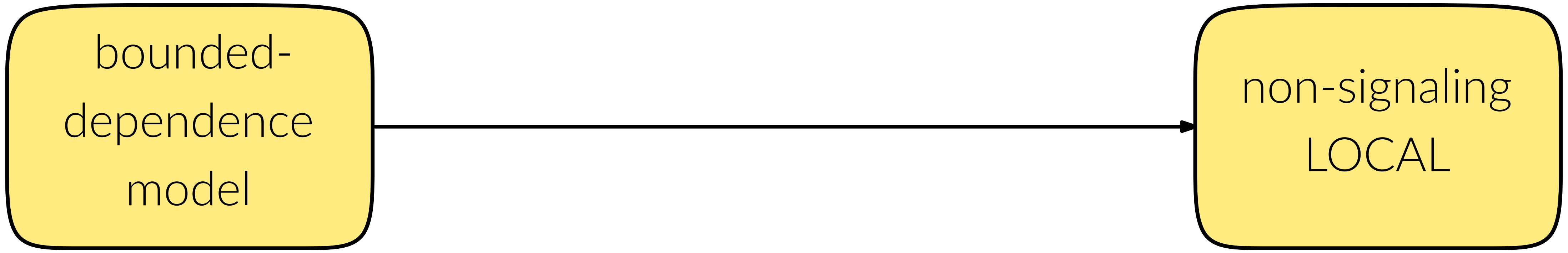

locality T =
independence at distance $2T + 1$ plus
non-signaling beyond distance T

locality T =
non-signaling beyond distance T

[Holroyd and Liggett, Forum of Mathematics, Pi '14]
[STOC '25a]

* finitely-dependent distributions if $T = O(1)$

[Gavoille, Kosowski, and Markiewicz, DISC '09]
[Arfaoui and Fraigniaud, PODC '12 & SIGACT News '14]

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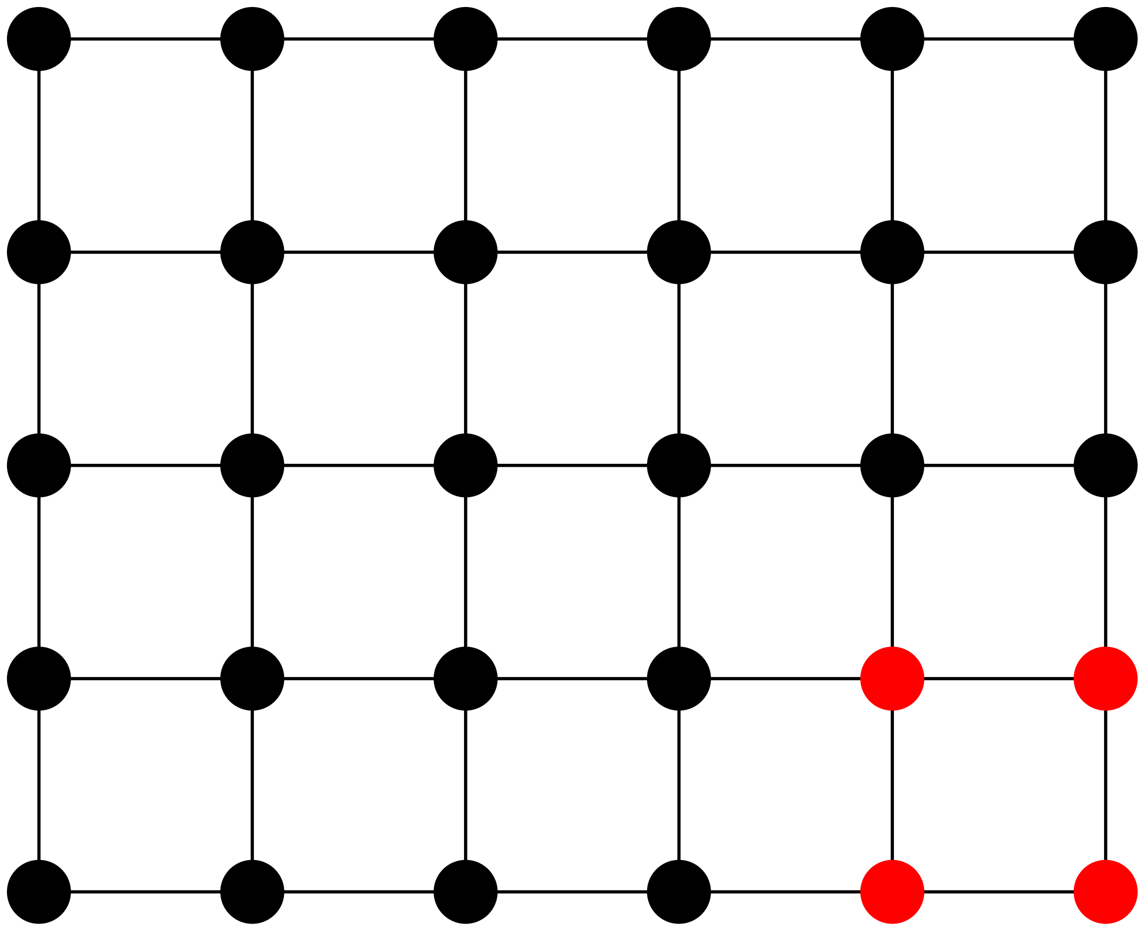
The non-signaling model

- Σ finite set of labels
  - always contains garbage output $\perp$

[STOC '24]



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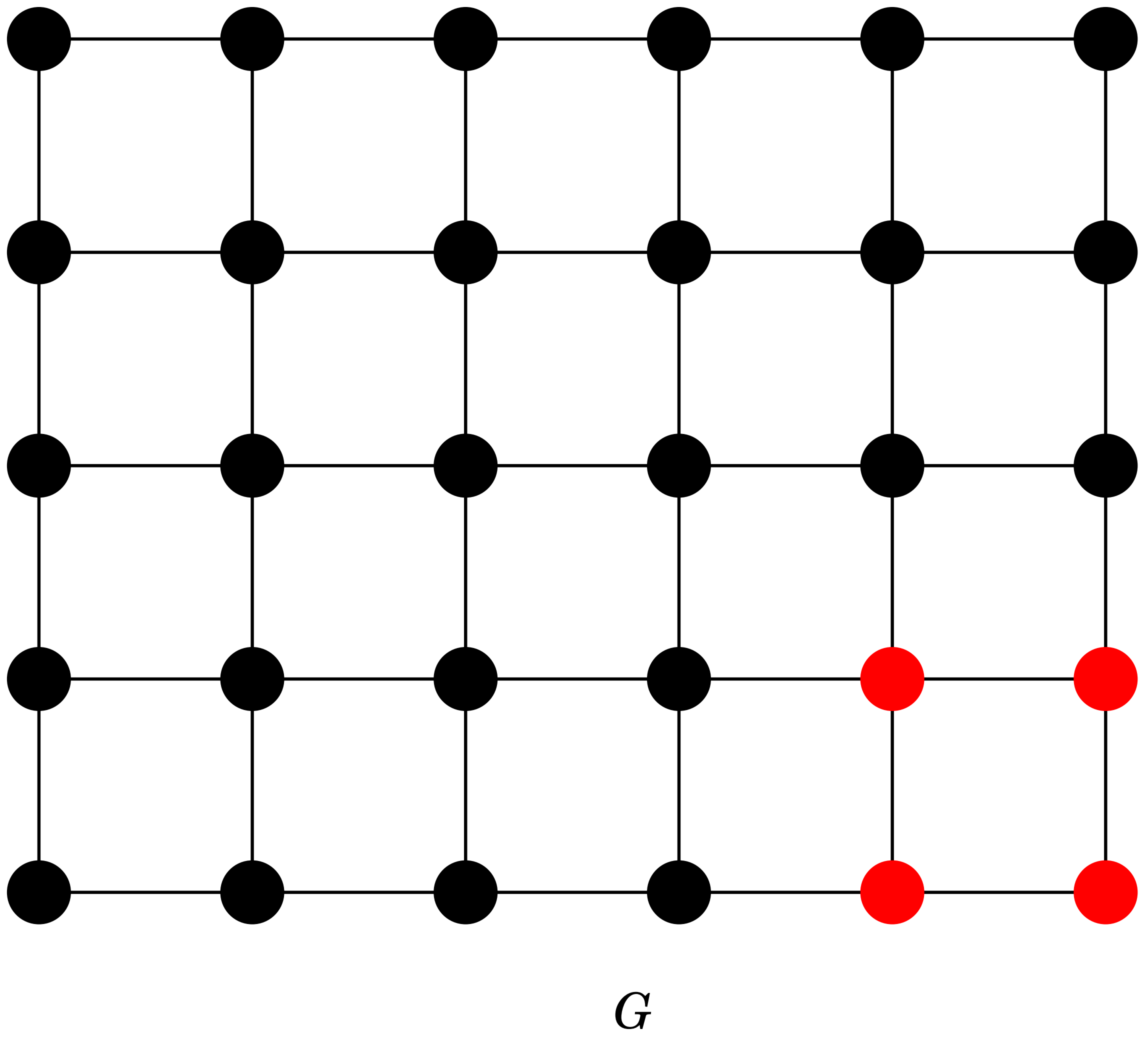

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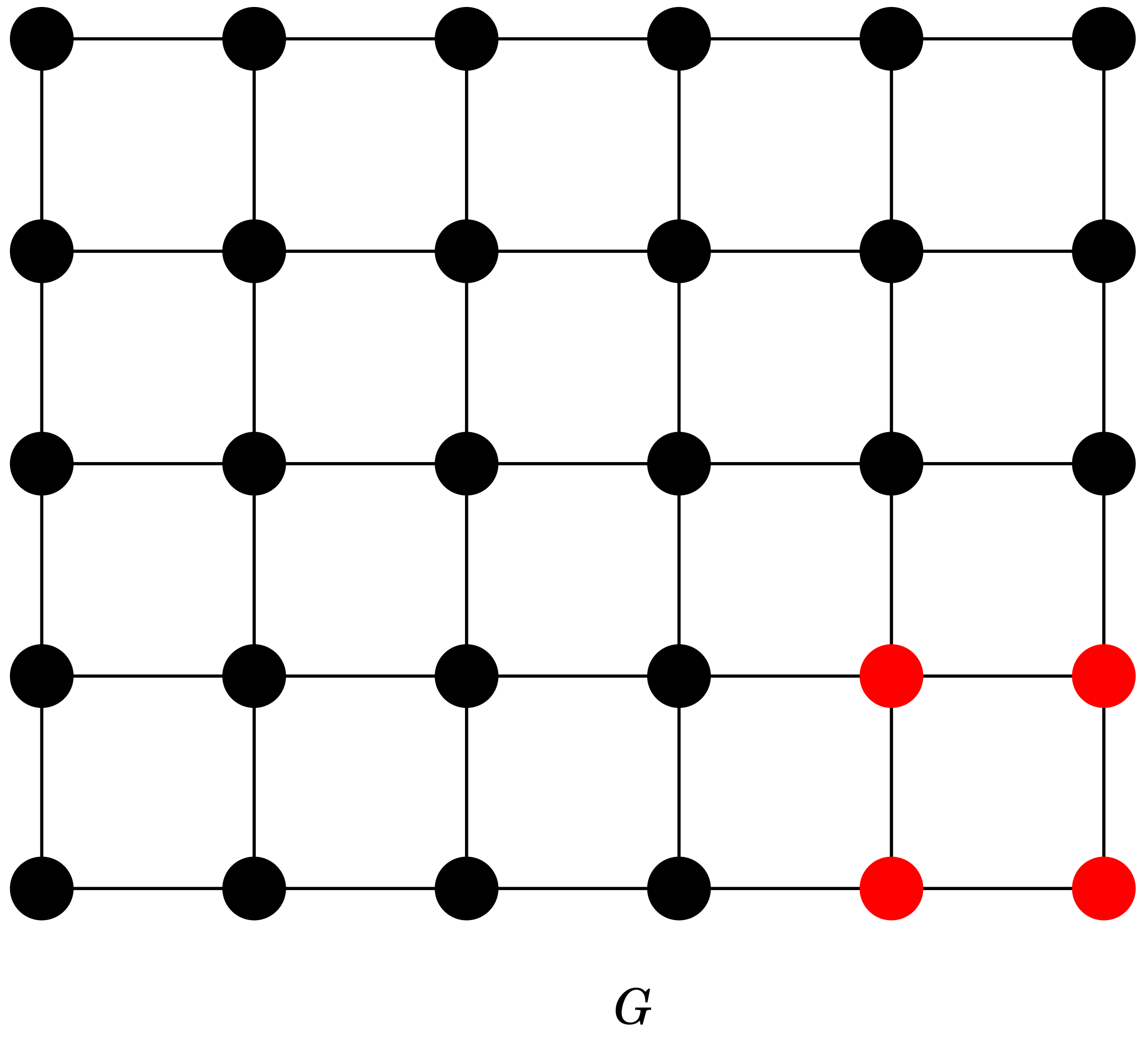

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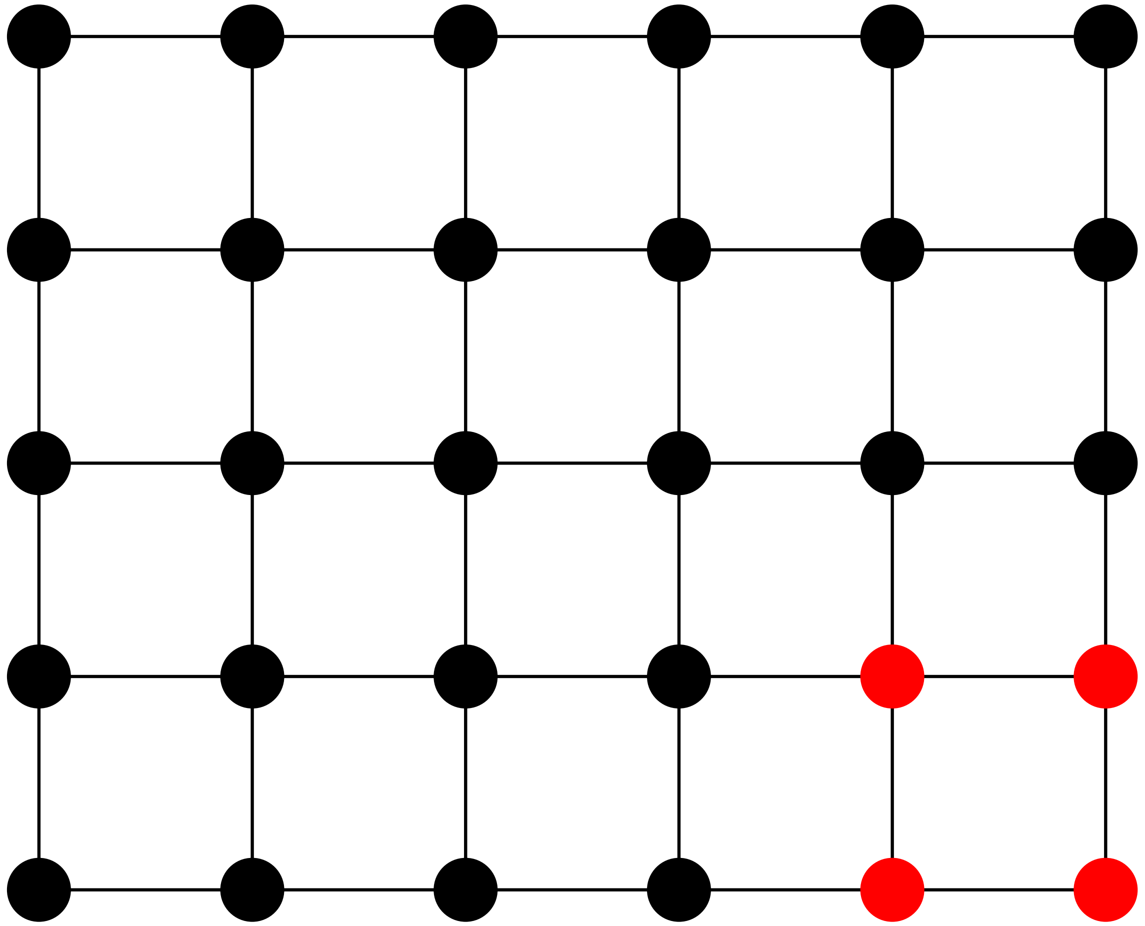

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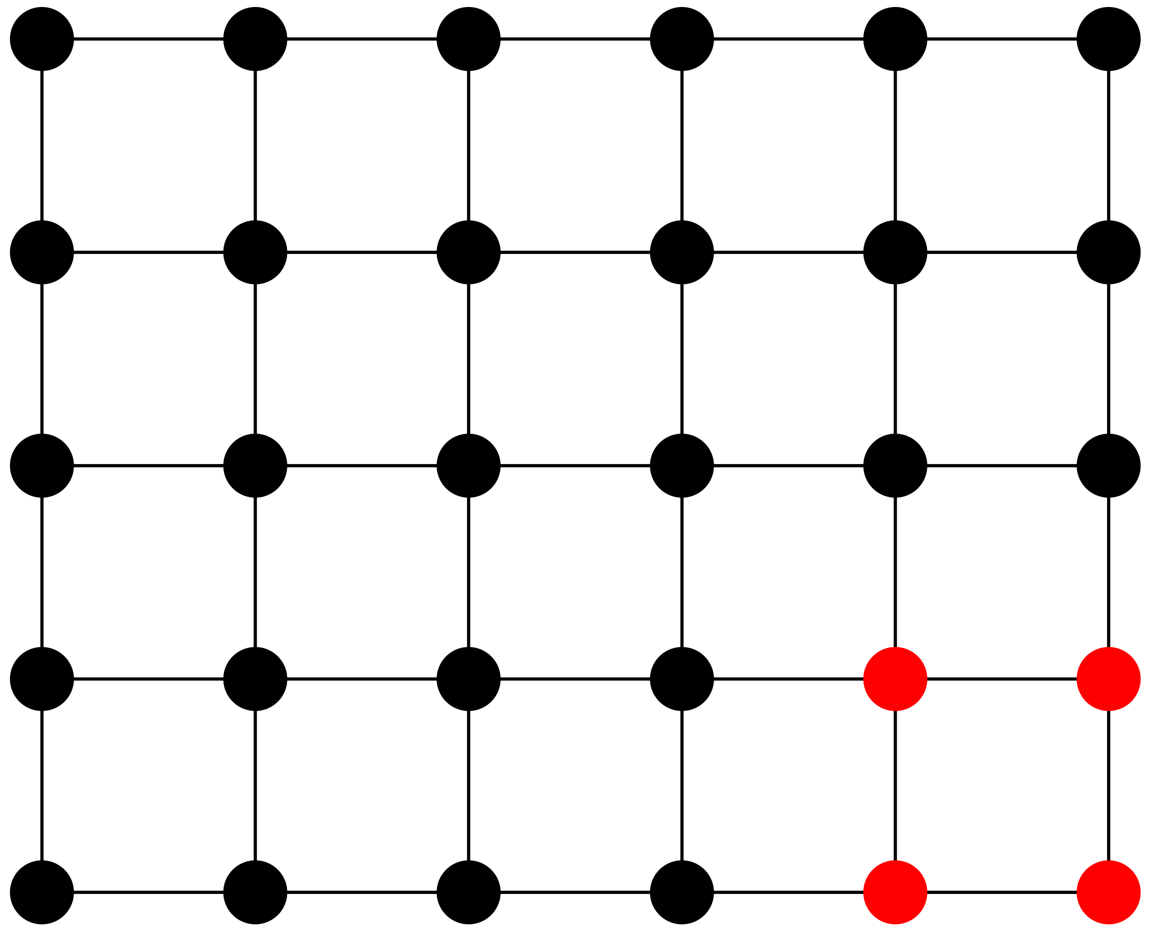

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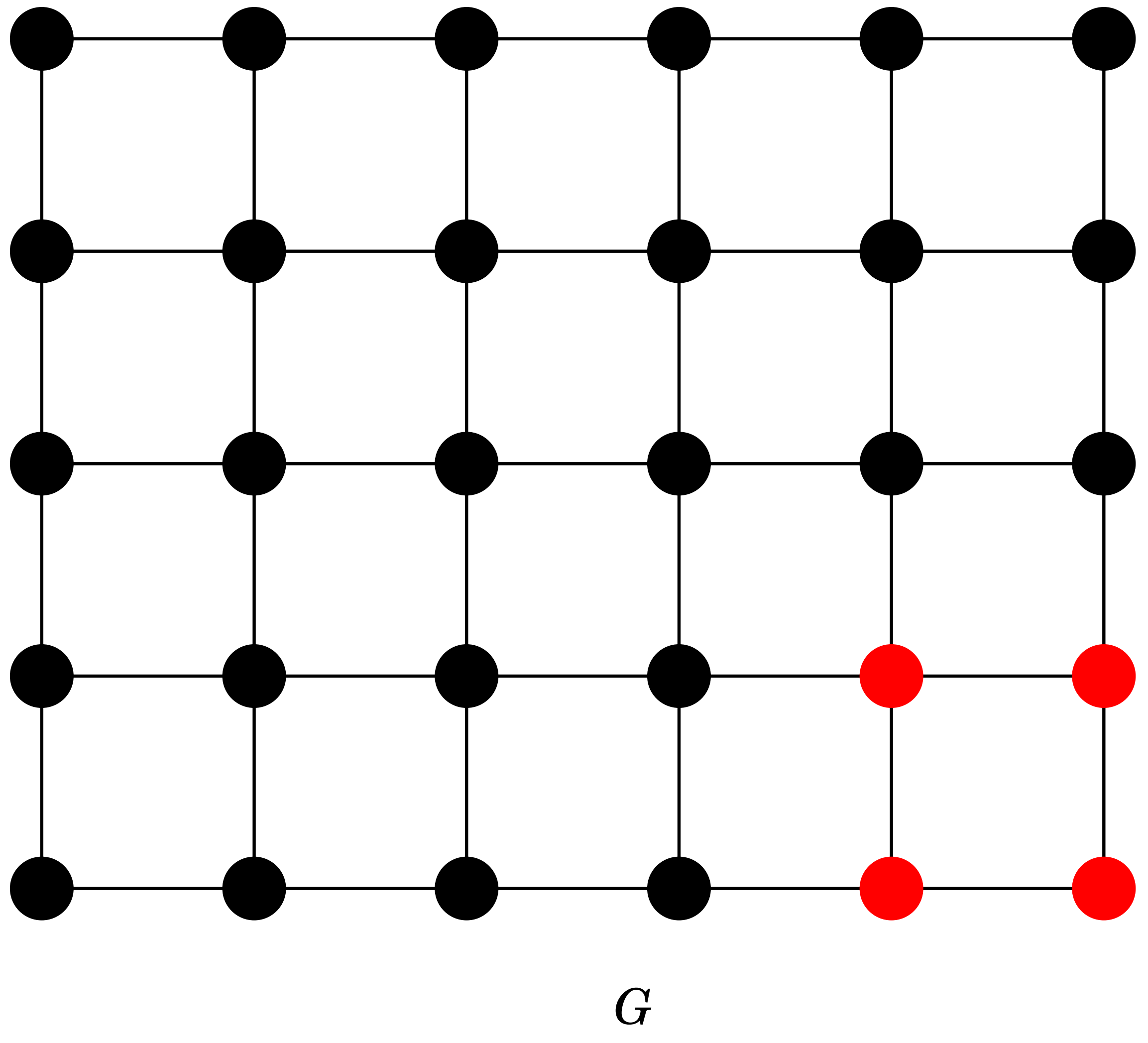

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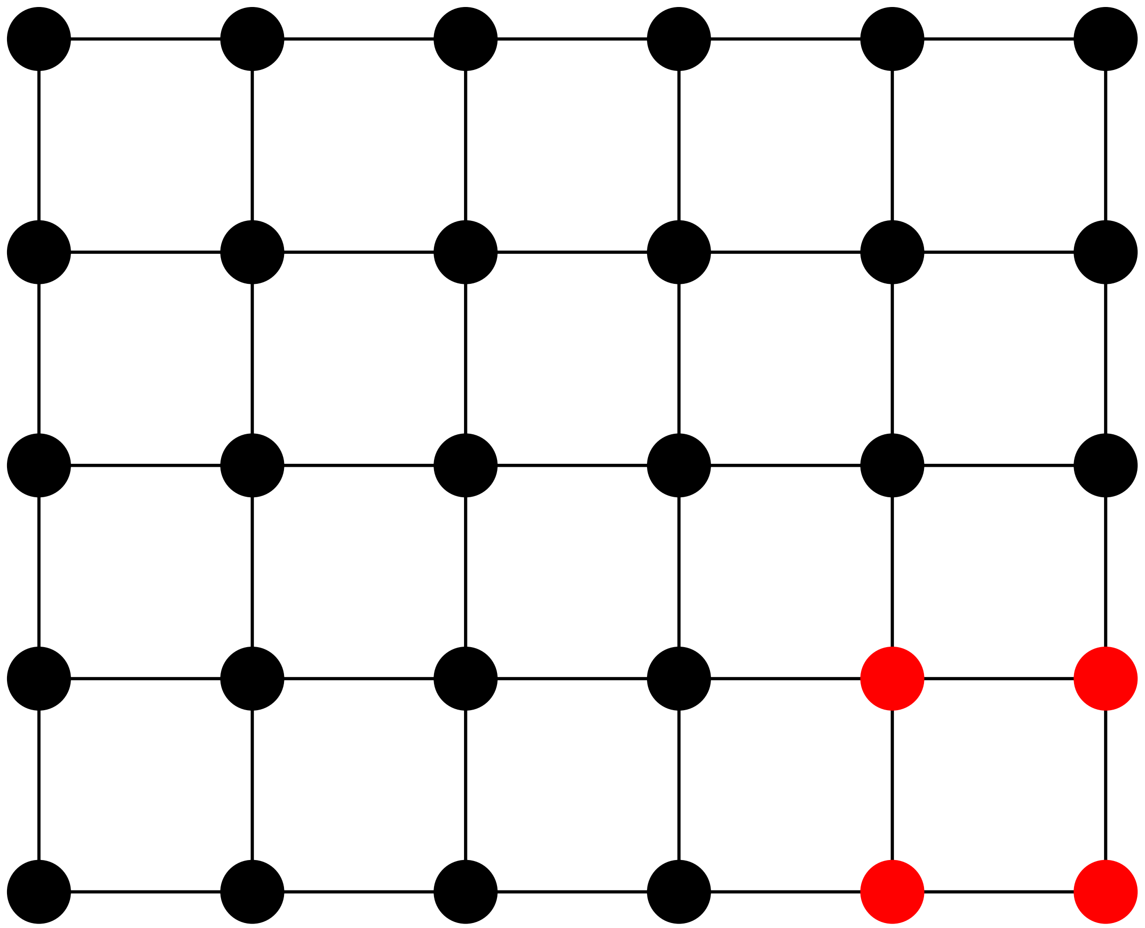

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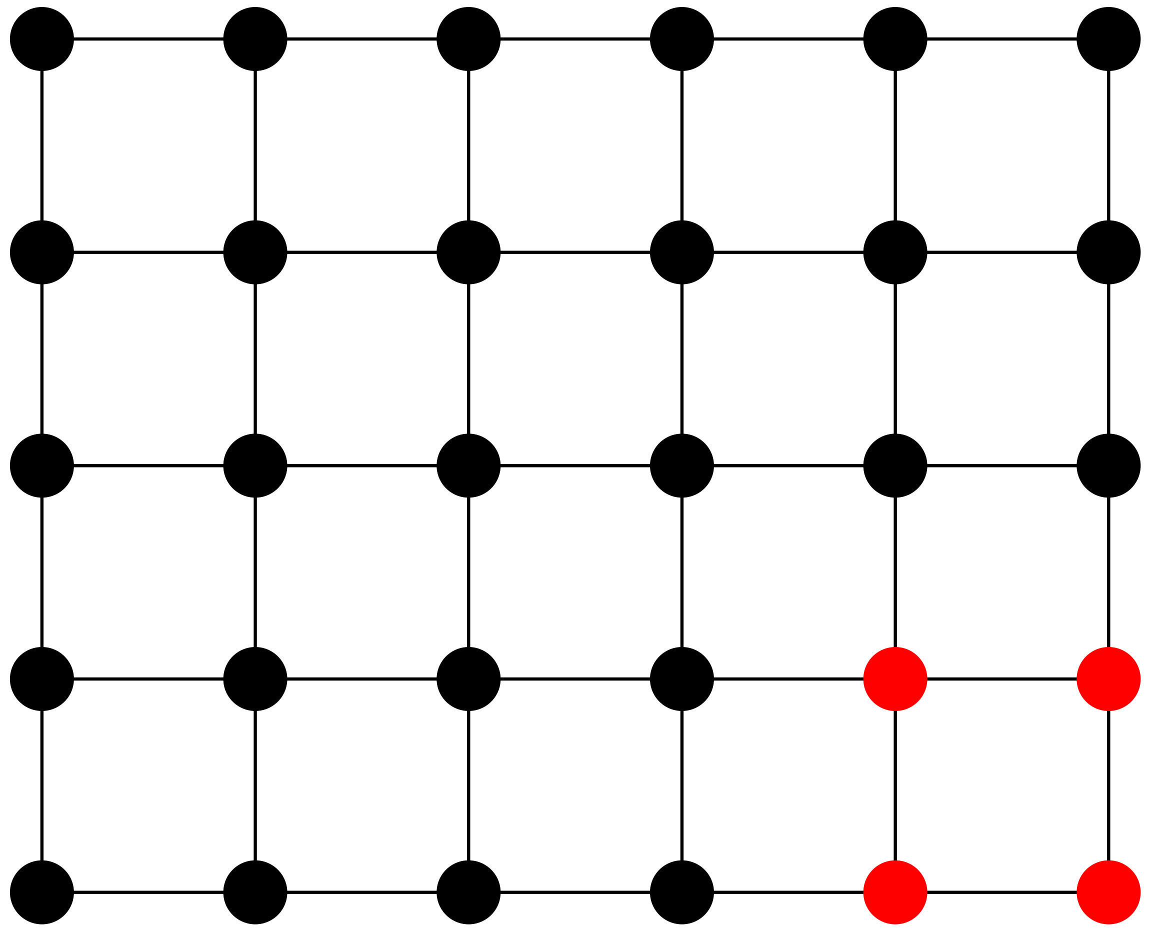

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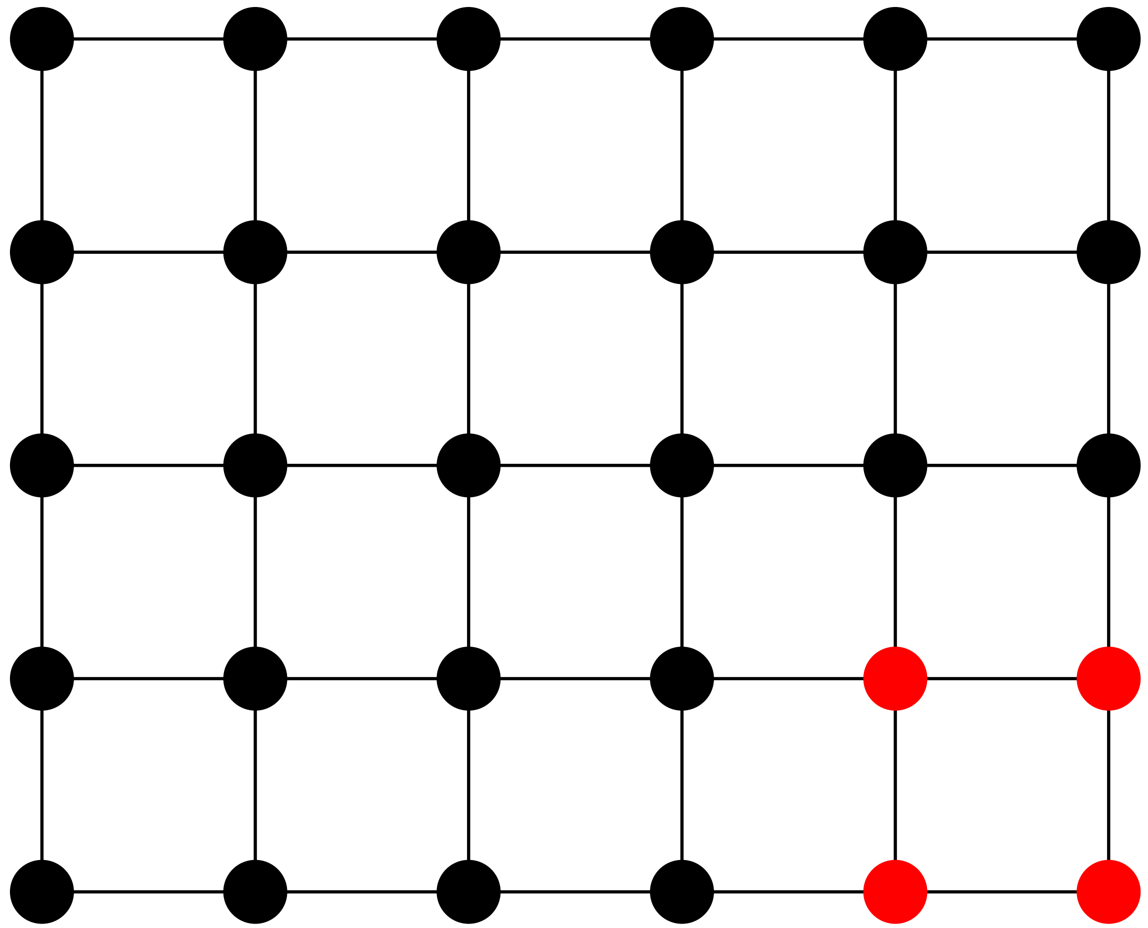

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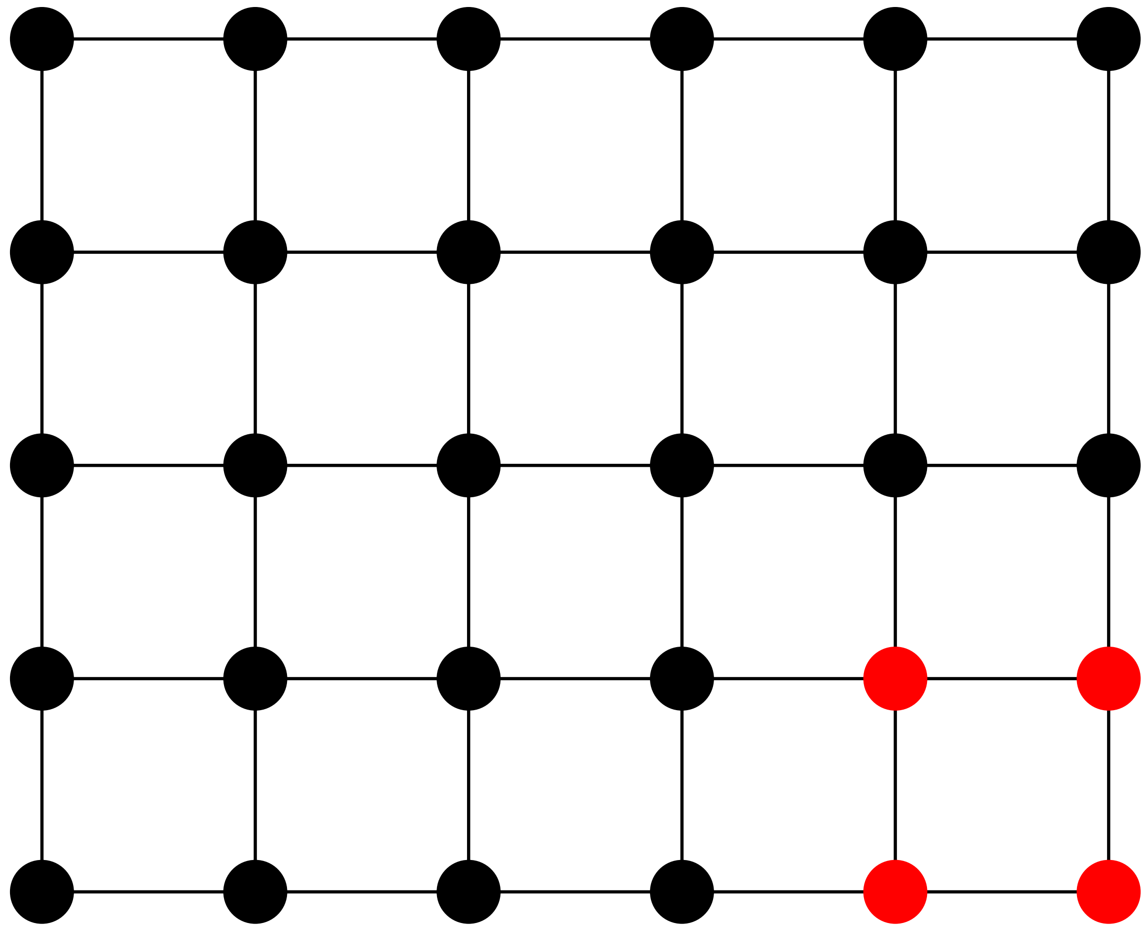

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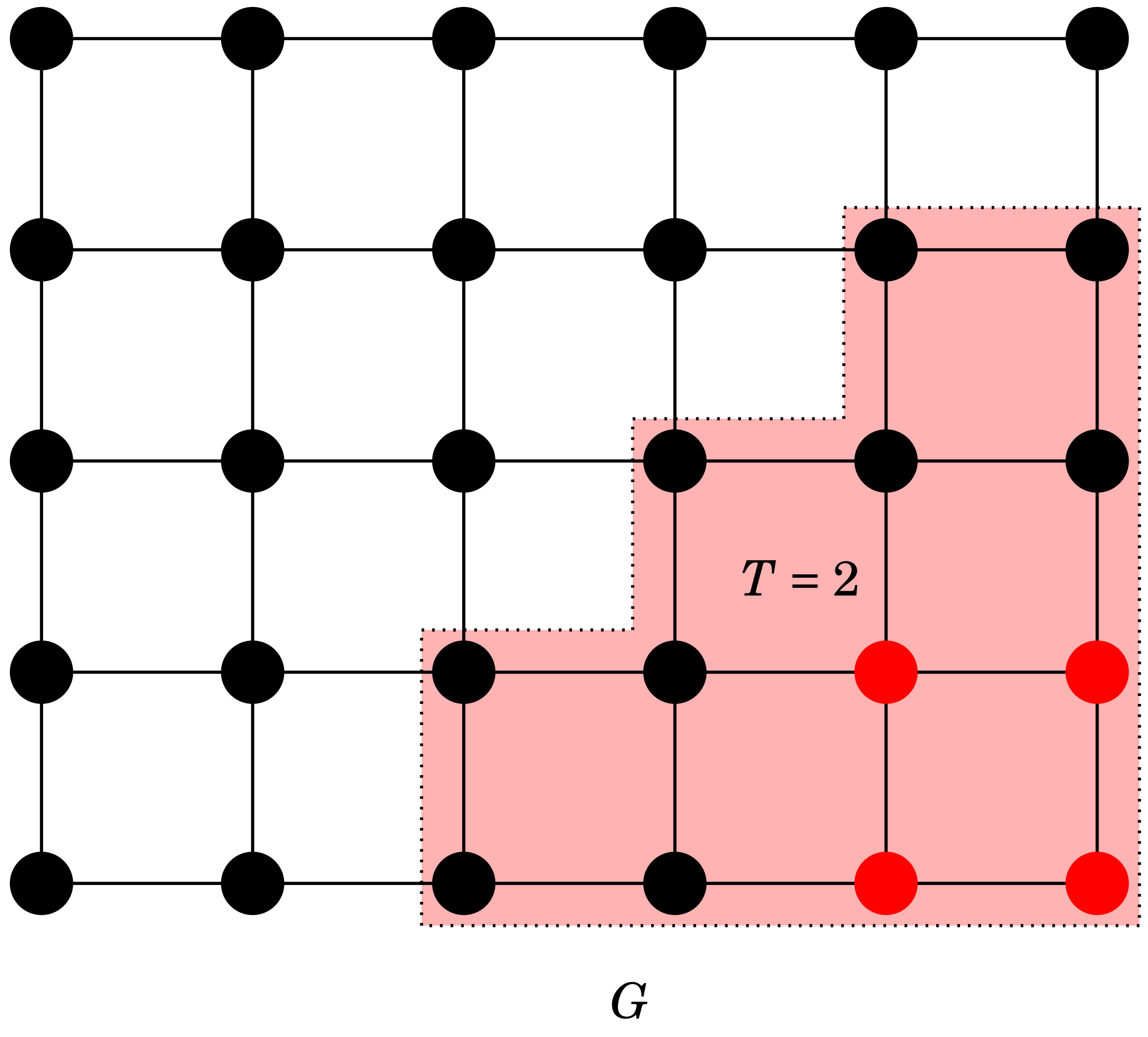

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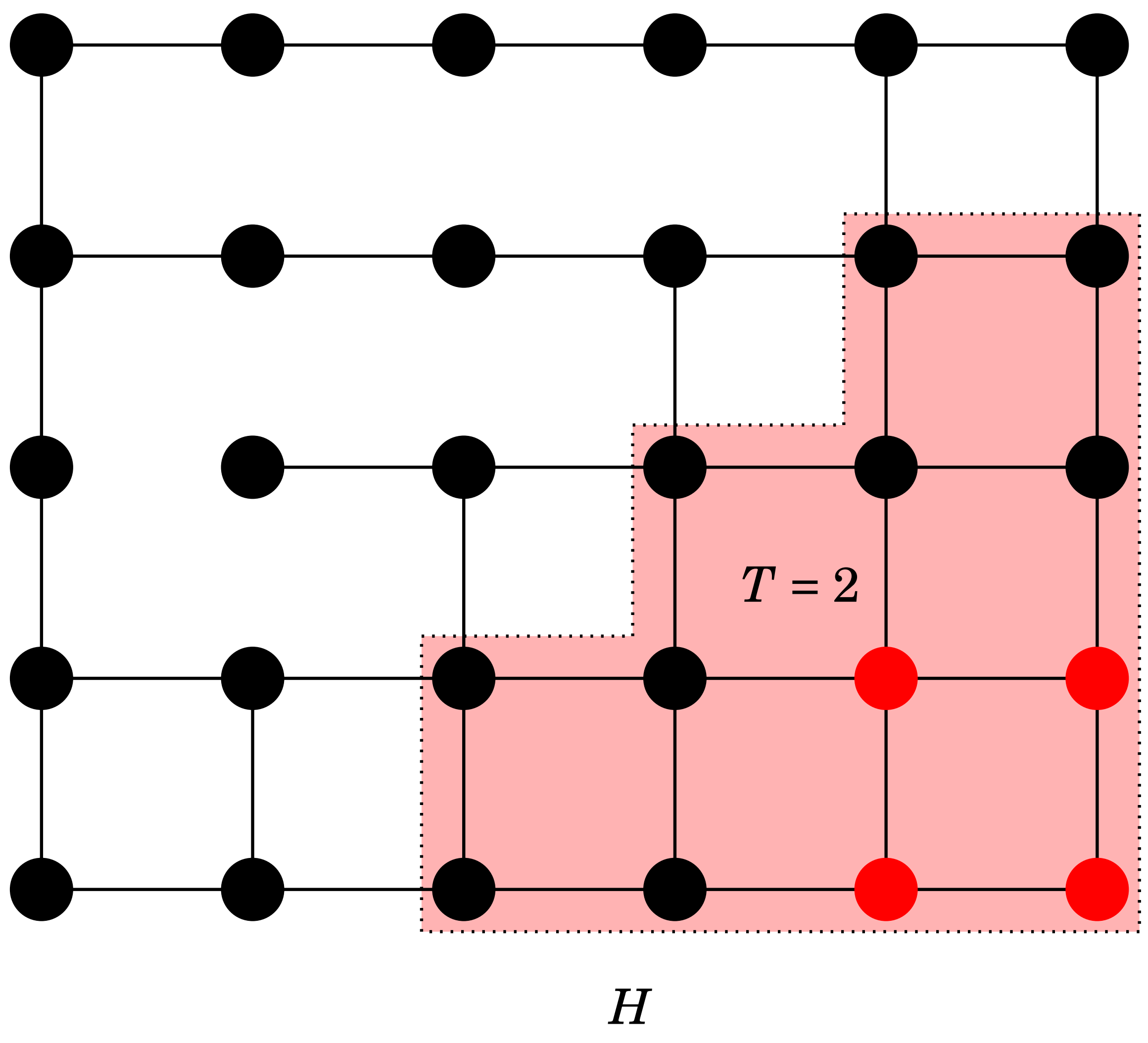

The bounded-dependence model

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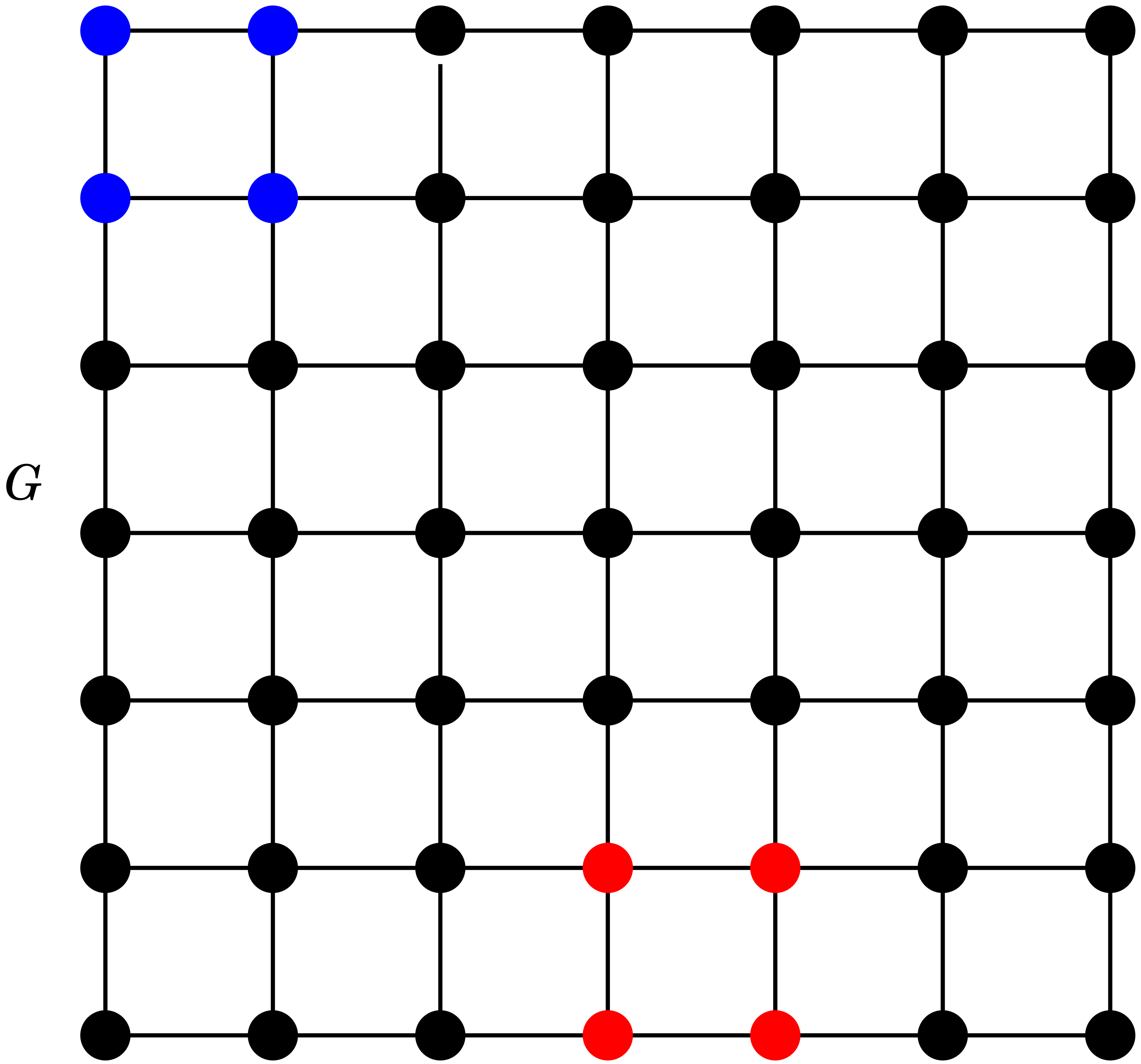

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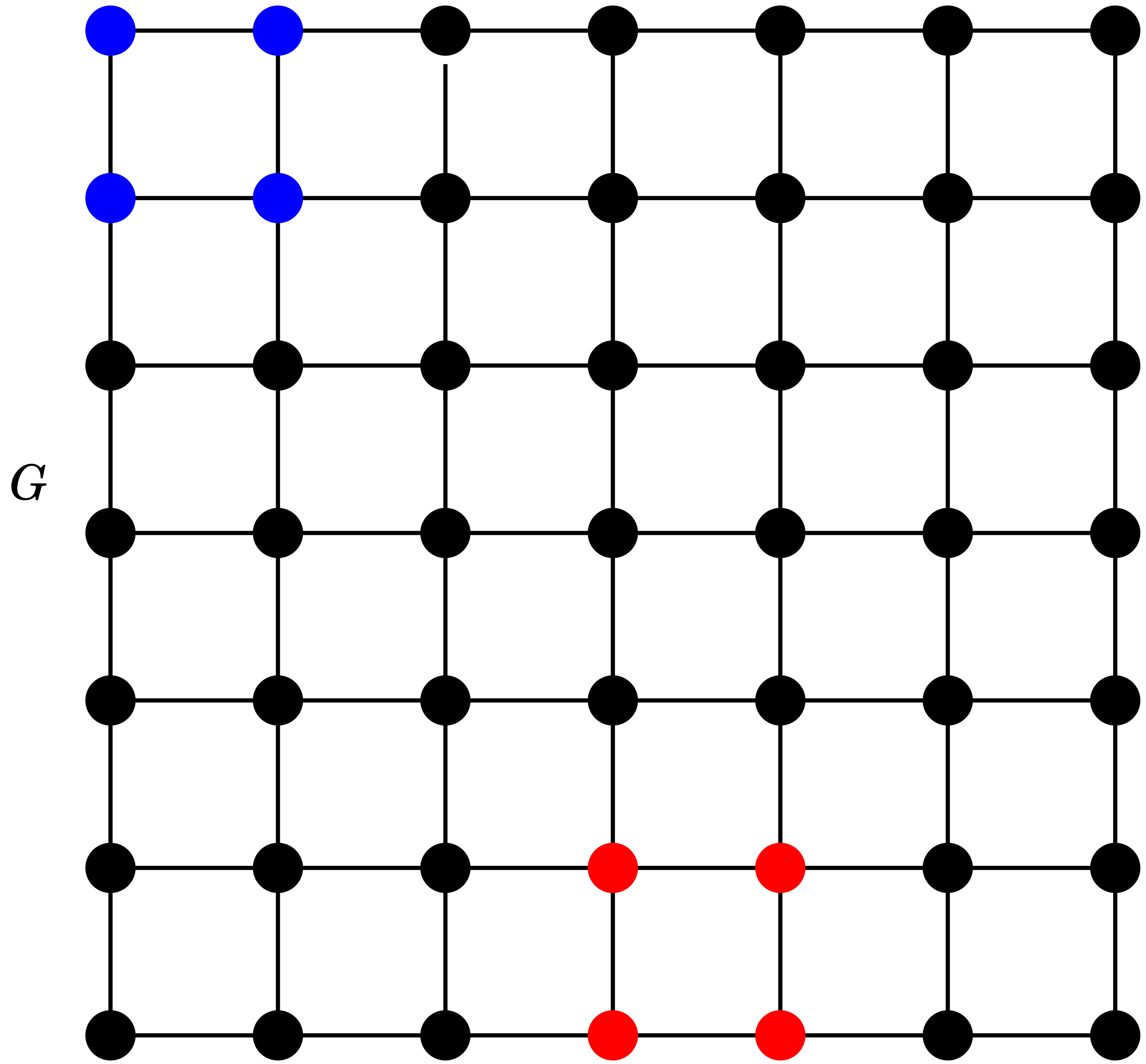

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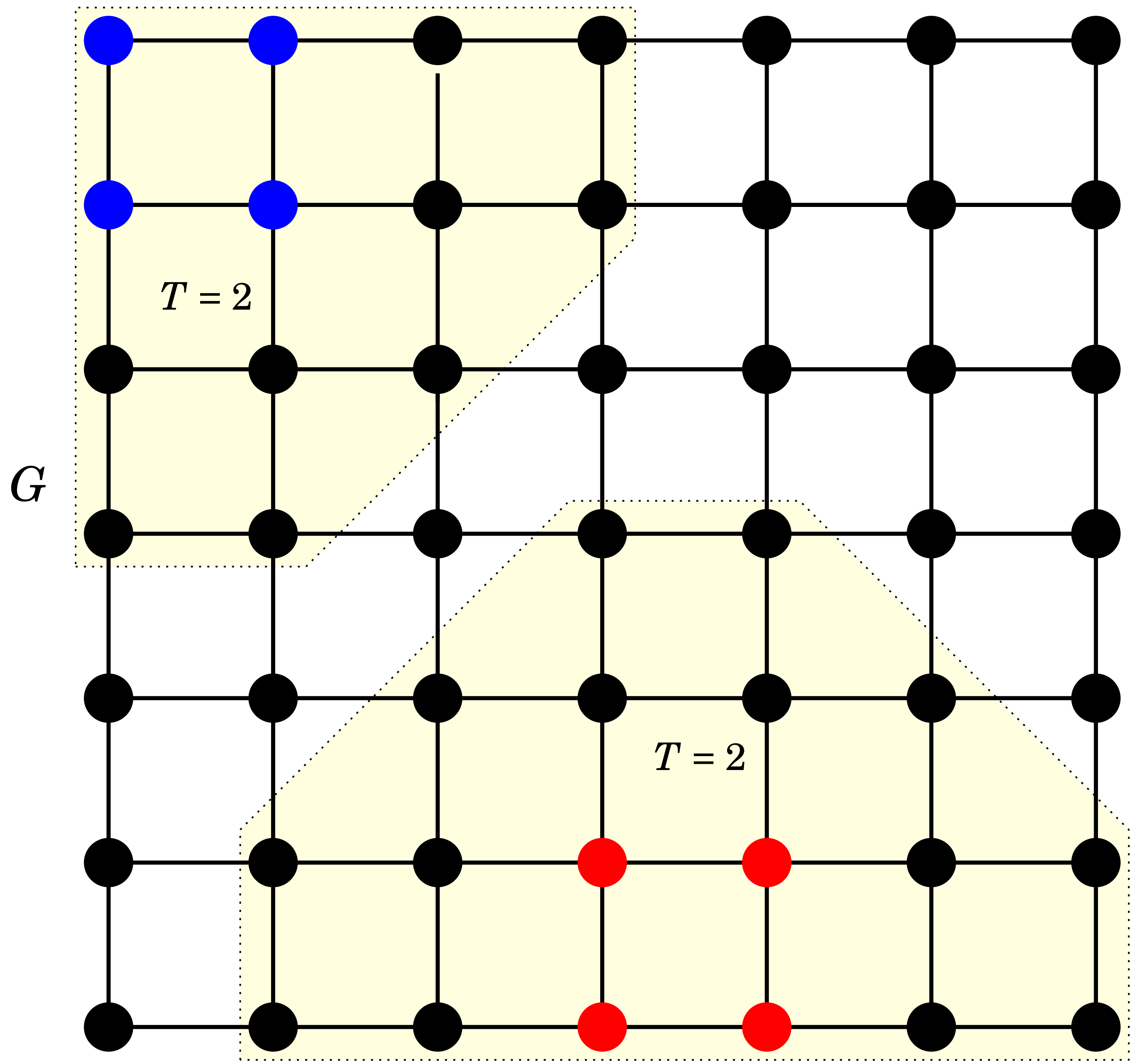

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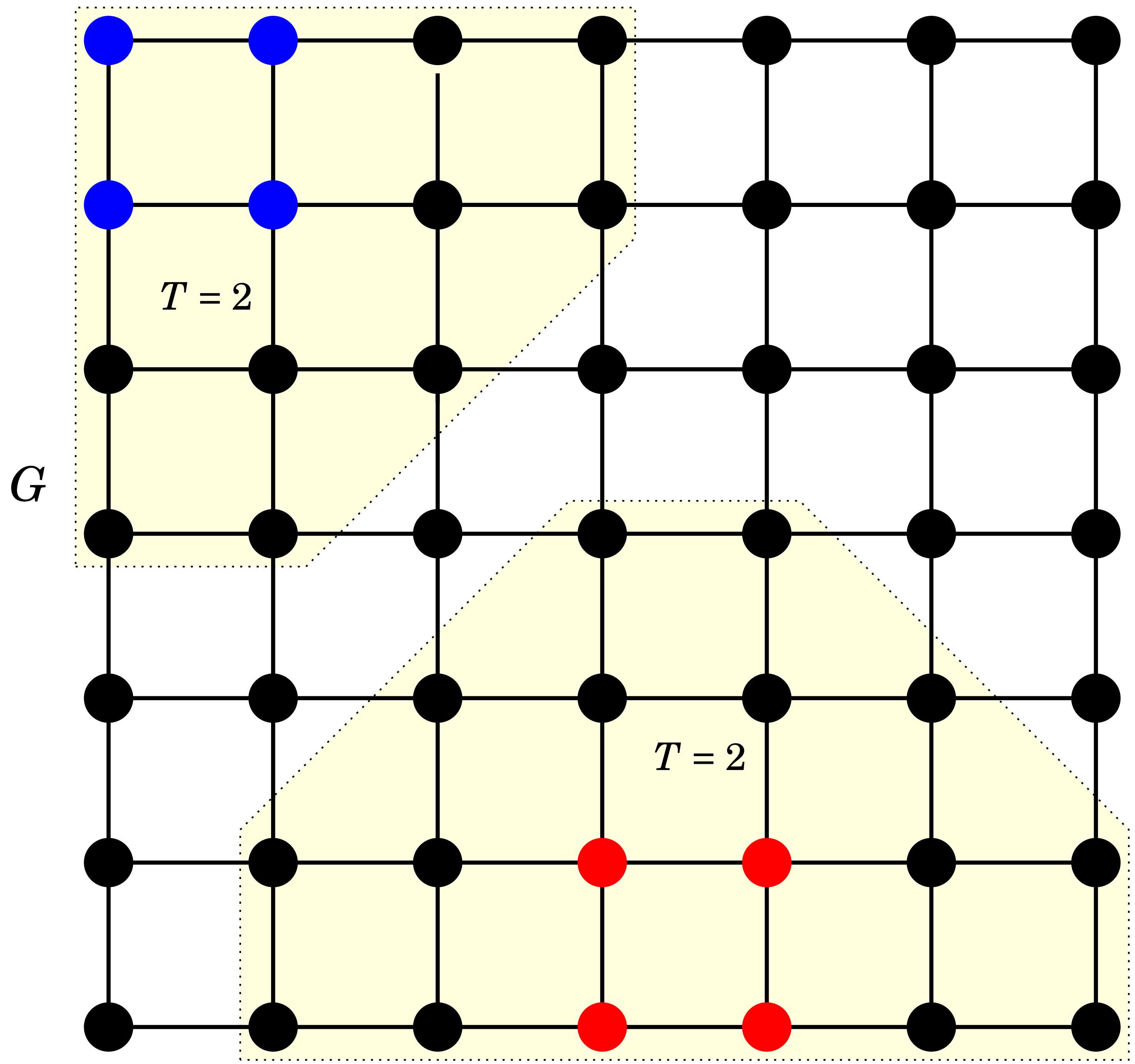

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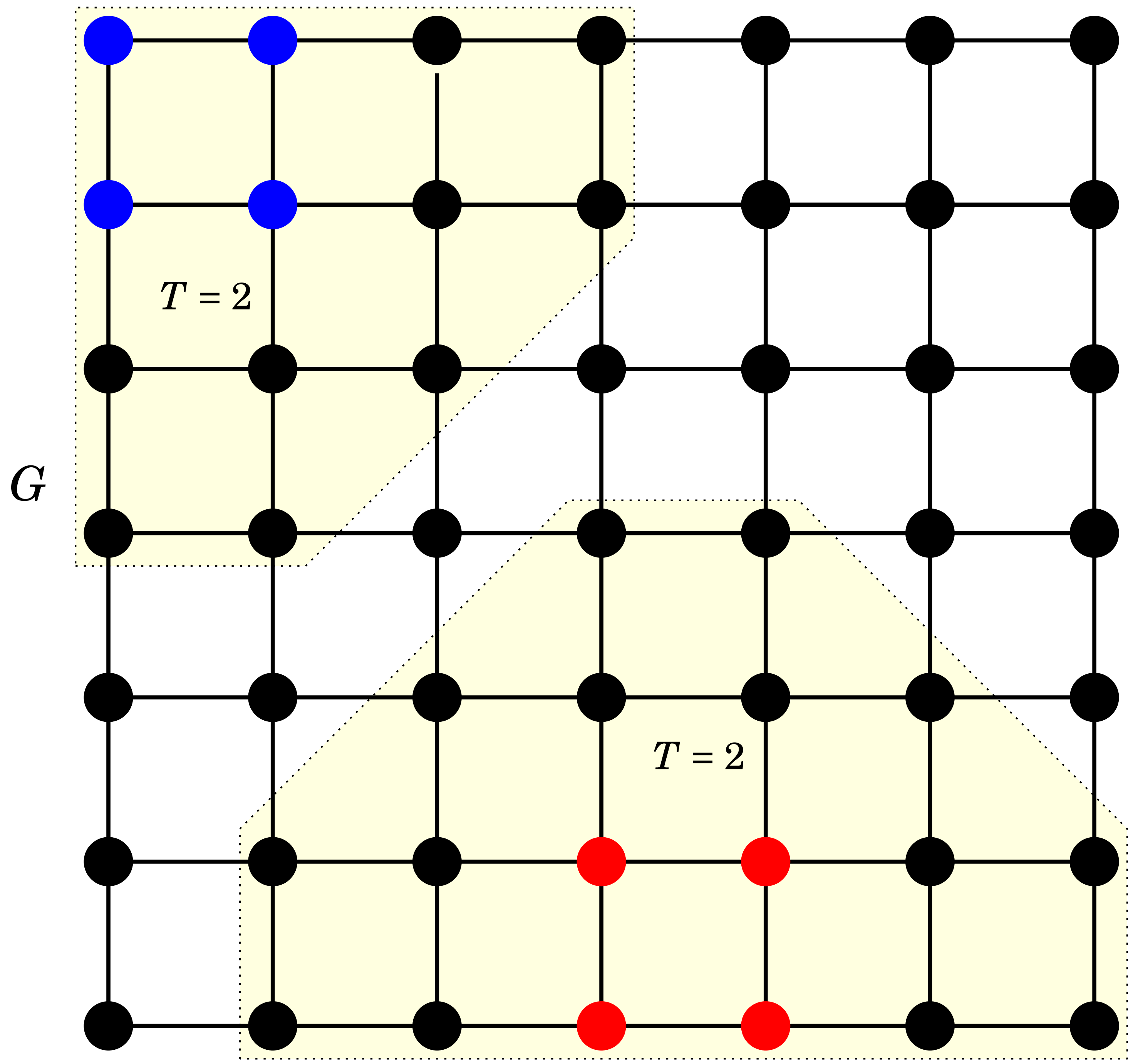
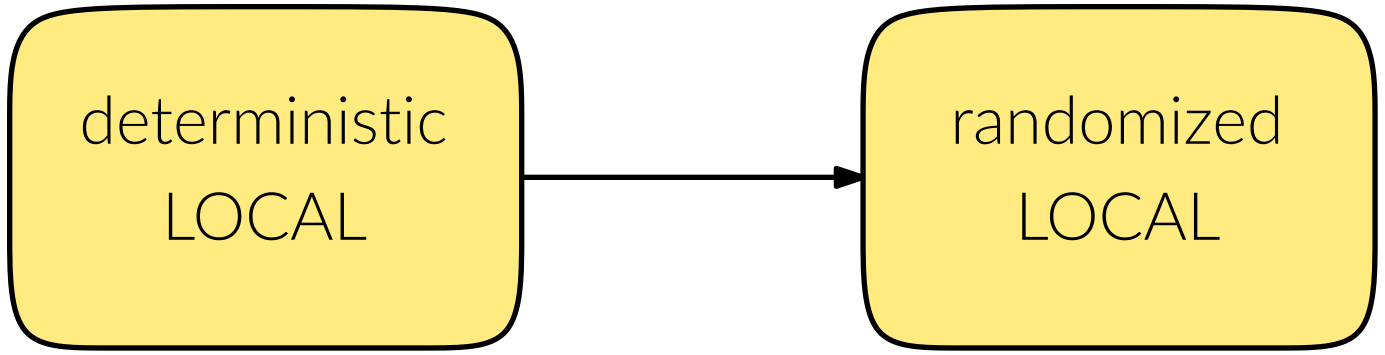


Relations among models

- $X \to Y$ means that locality T in X becomes locality $O(T)$ in Y



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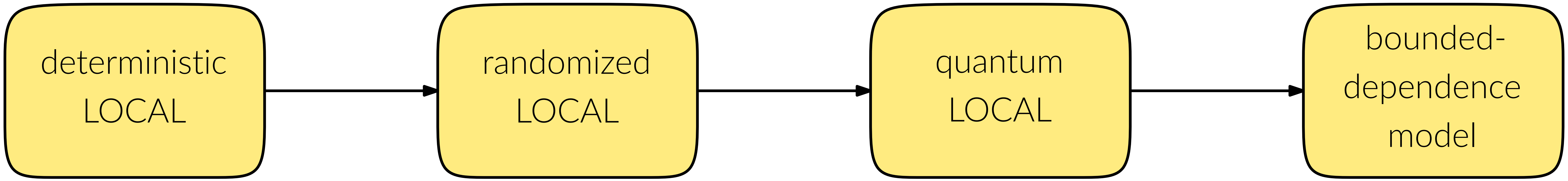

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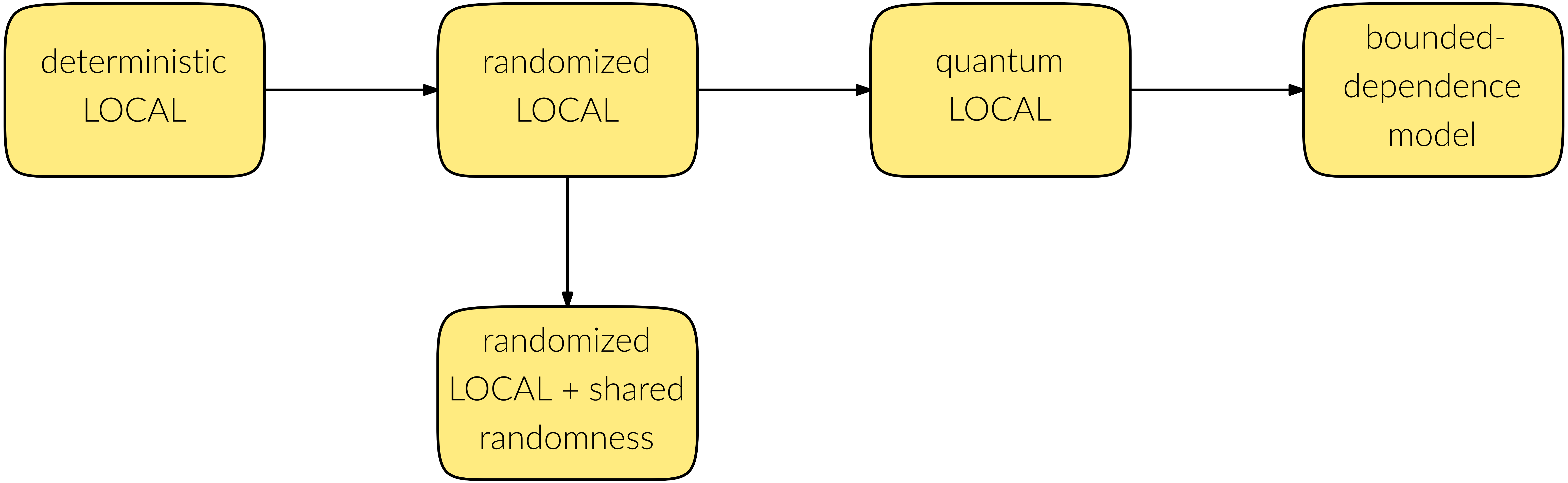

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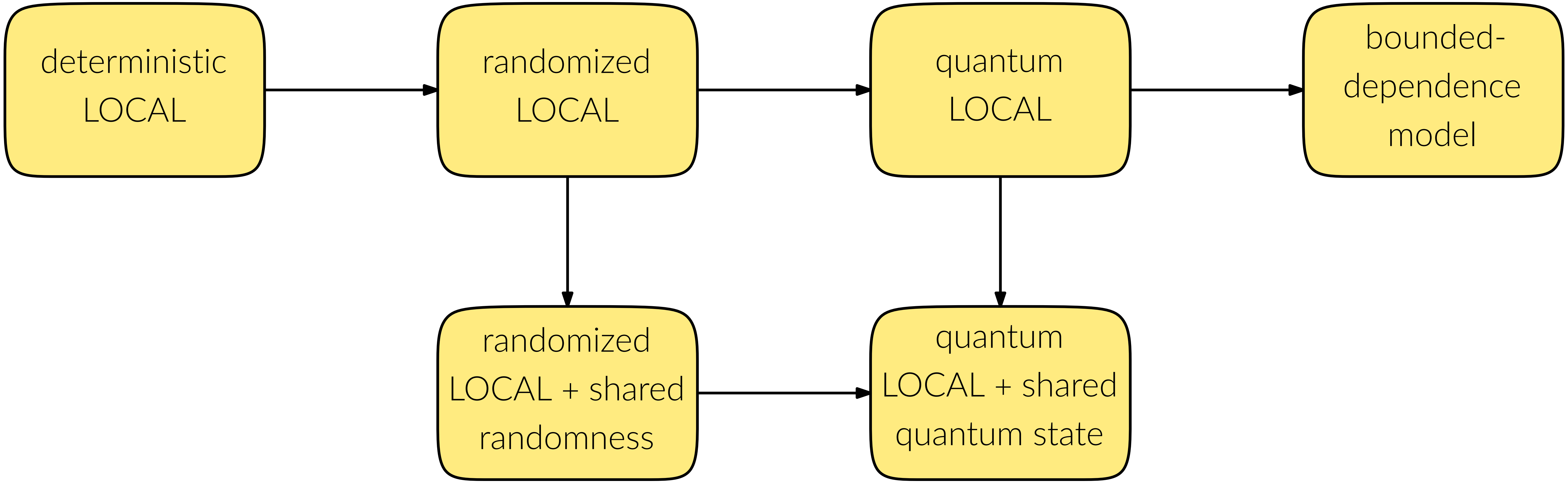

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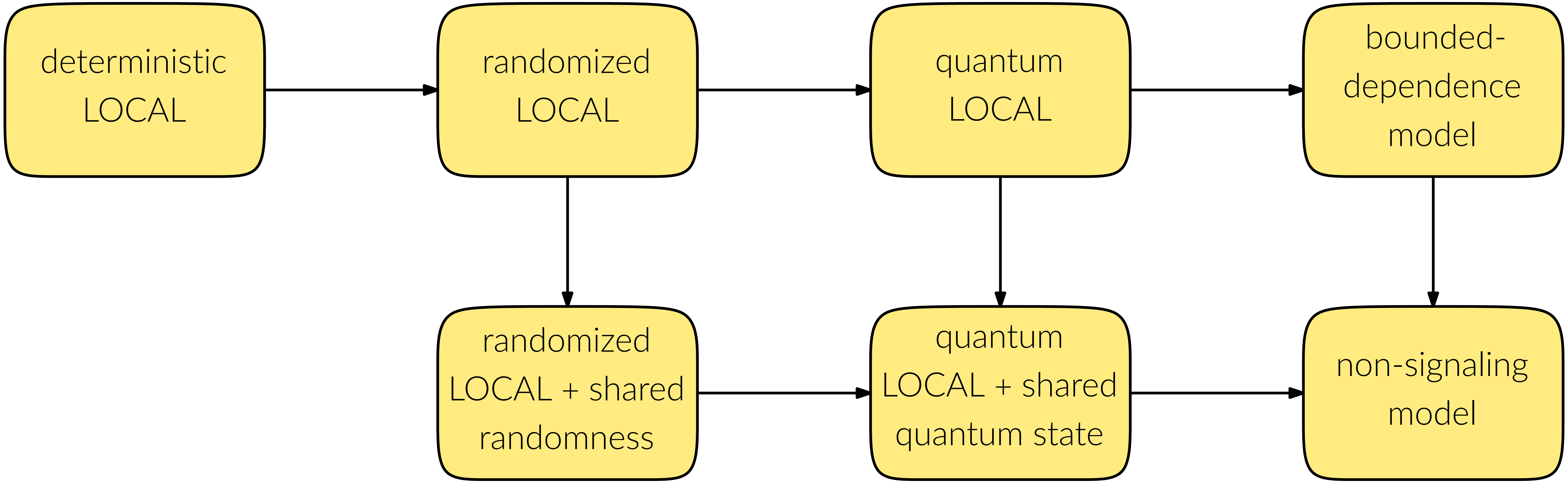

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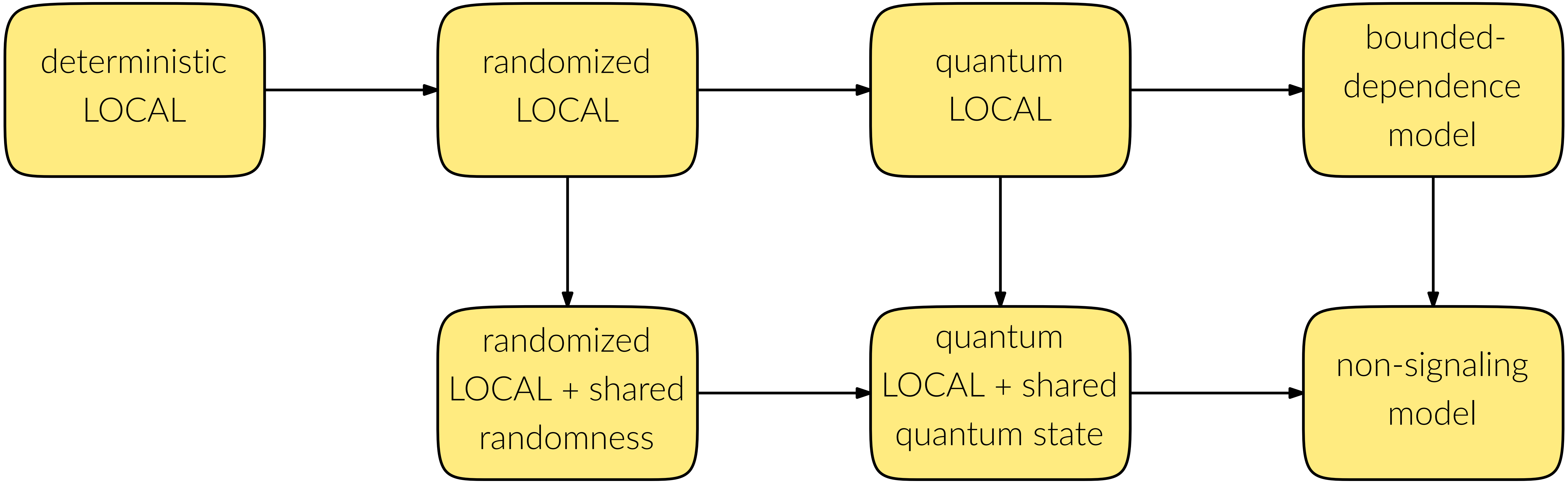

- Is it possible to **“sandwich” quantum-LOCAL** between weaker and stronger models?

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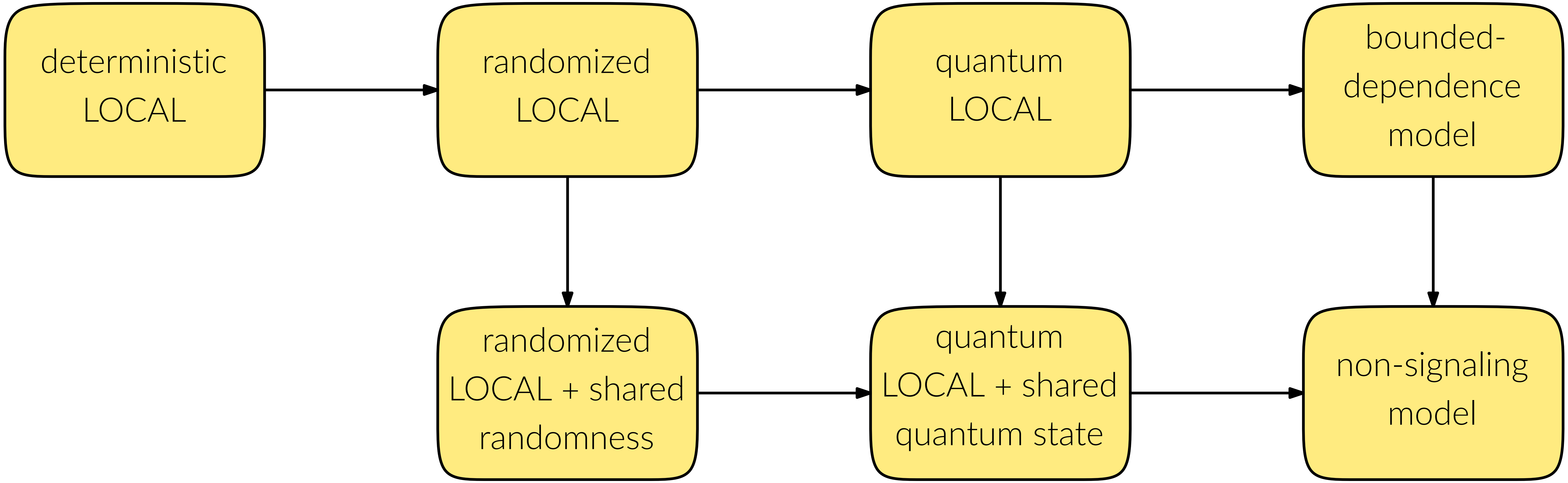
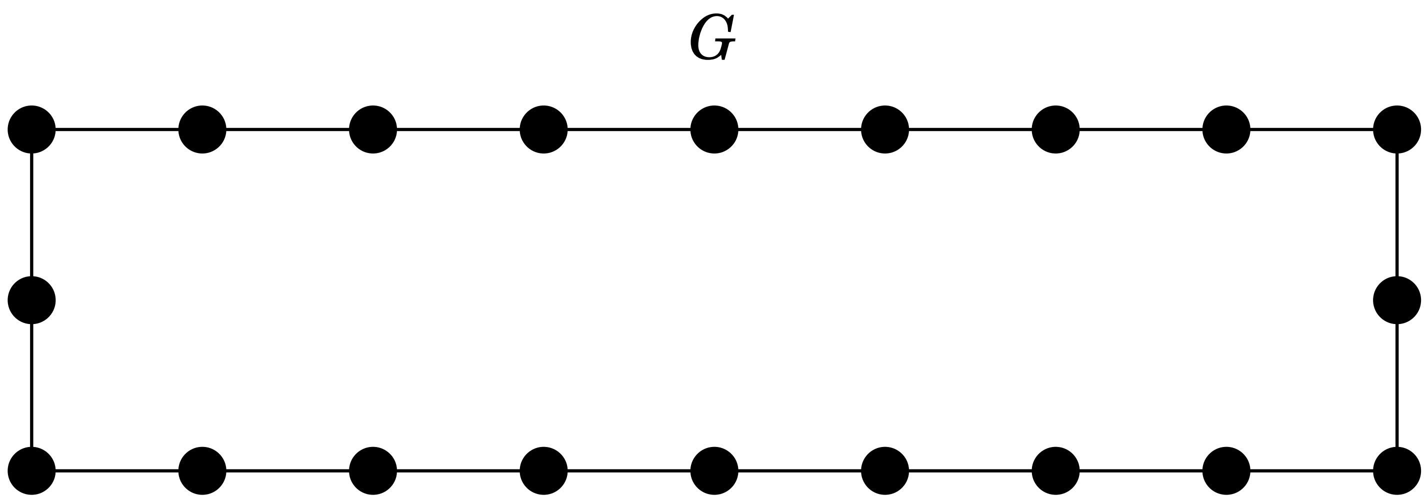


- Is it possible to **"sandwich" quantum-LOCAL** between weaker and stronger models?

  - yes!

Indistinguishability argument

- **Problem**: 2-coloring paths & even cycles in the *non-signaling* model



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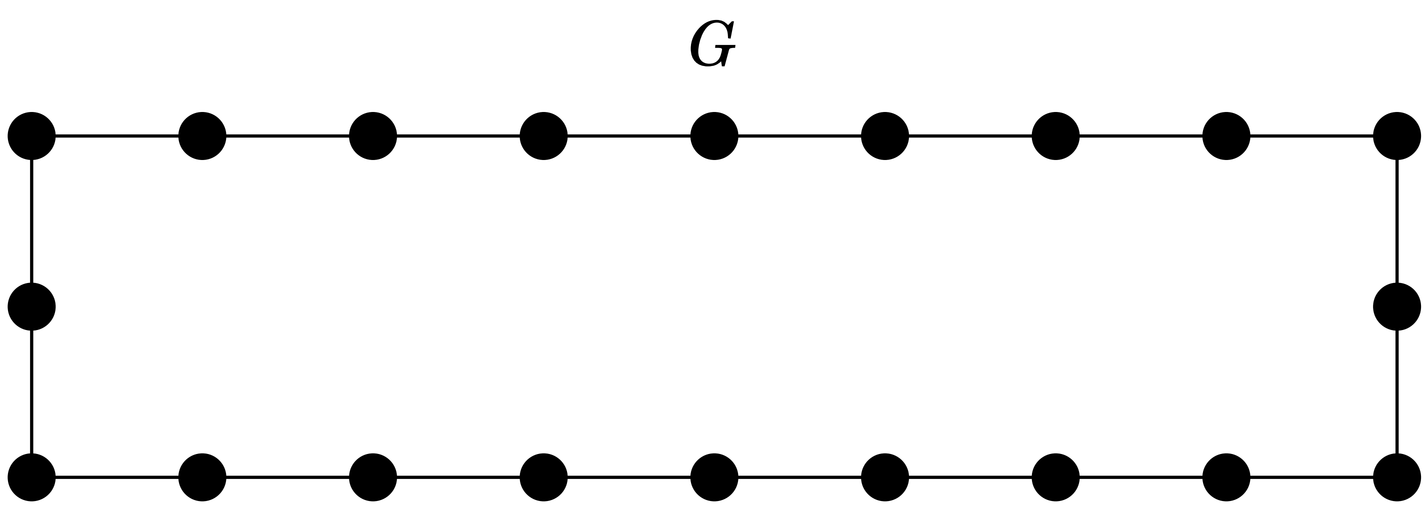

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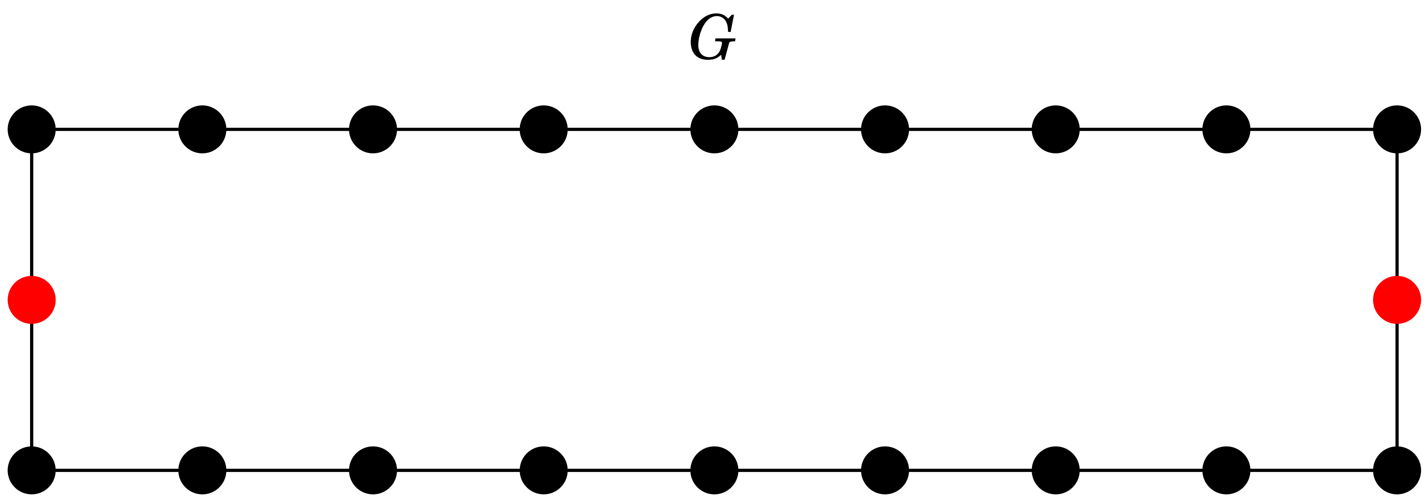

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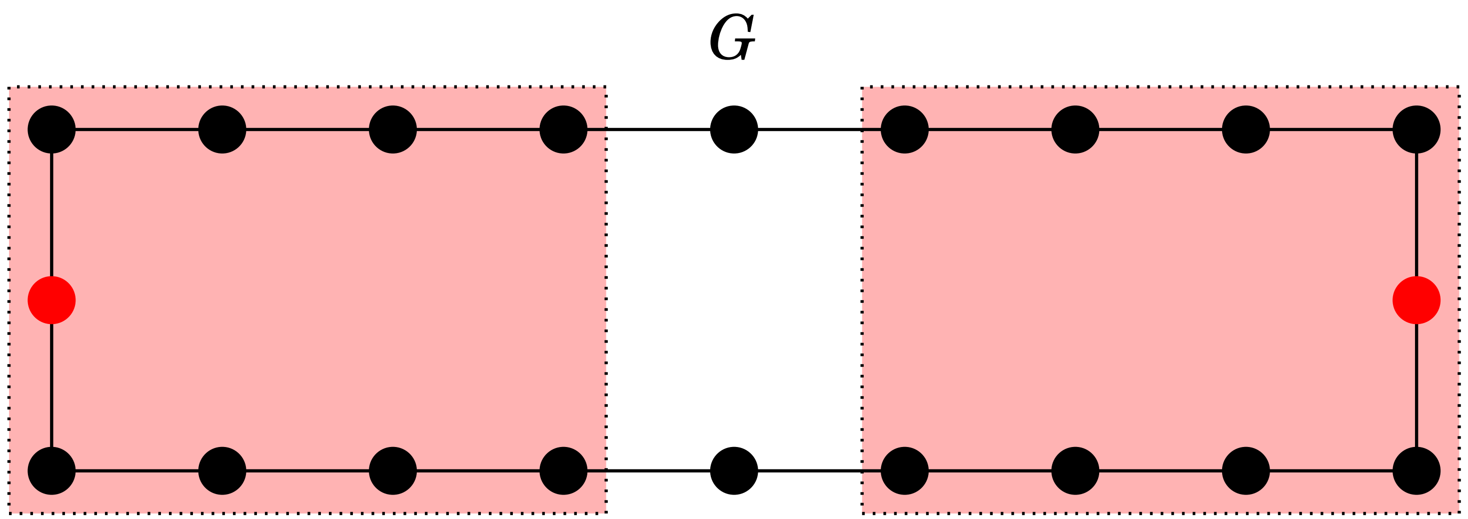

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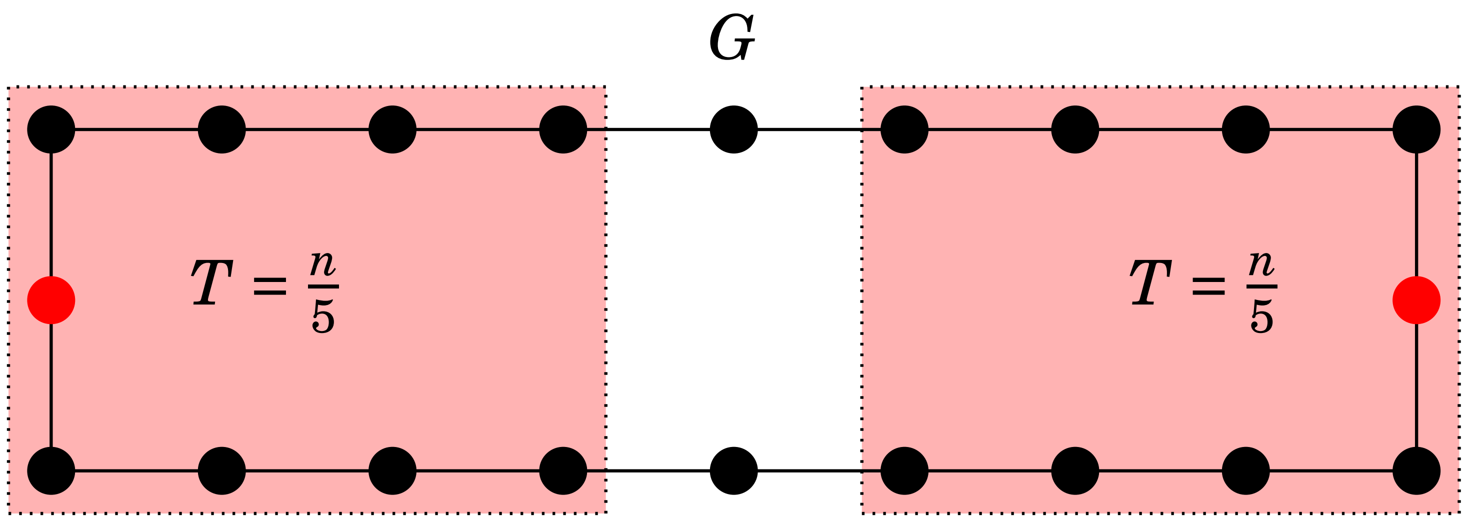

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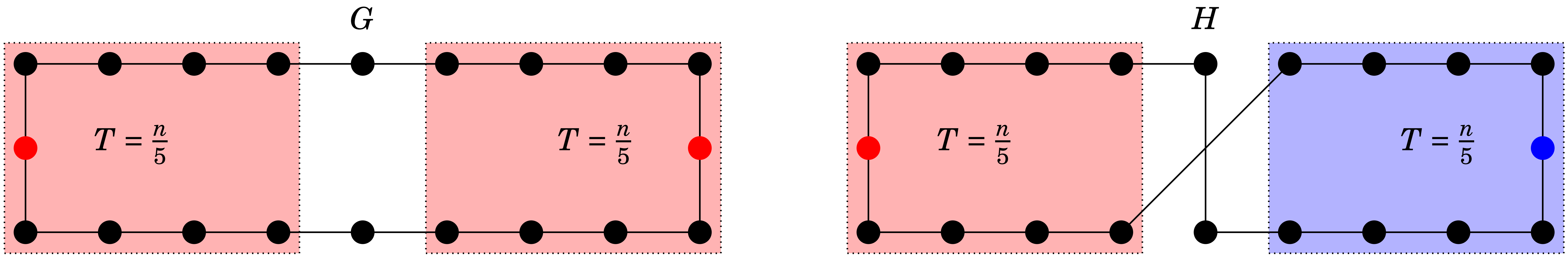

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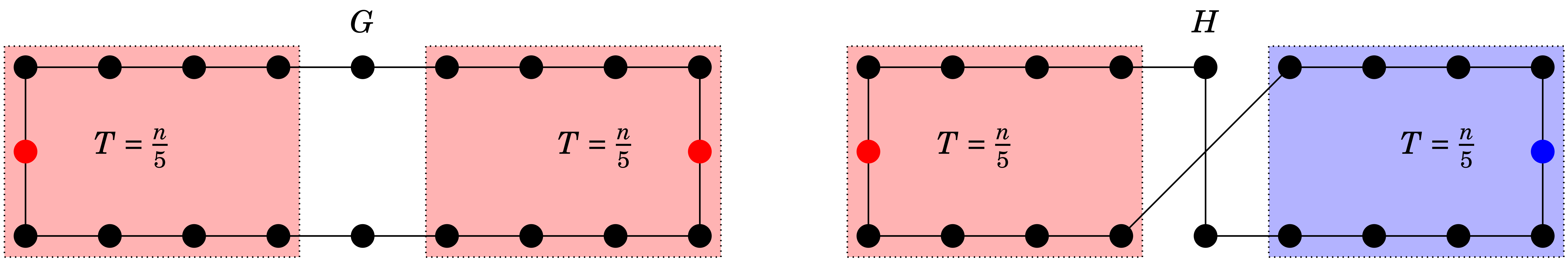

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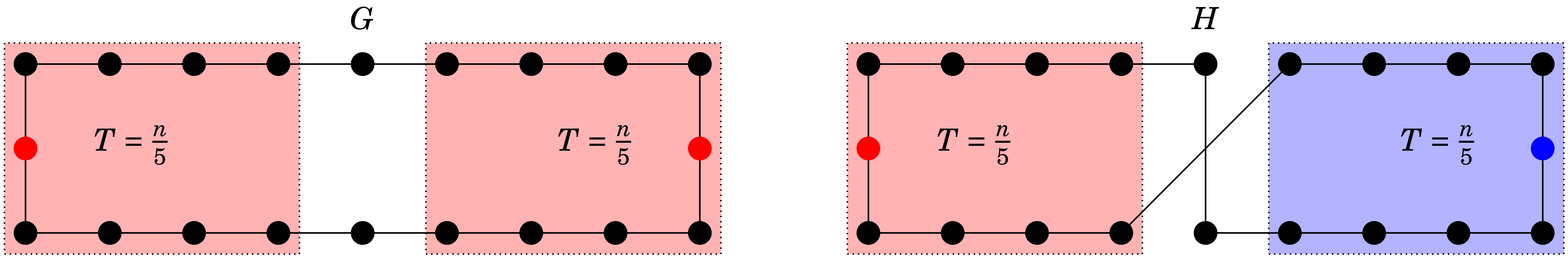

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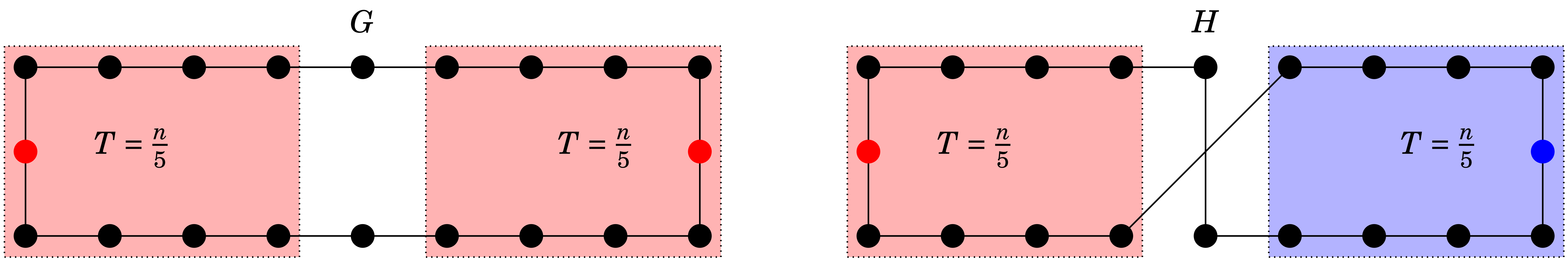

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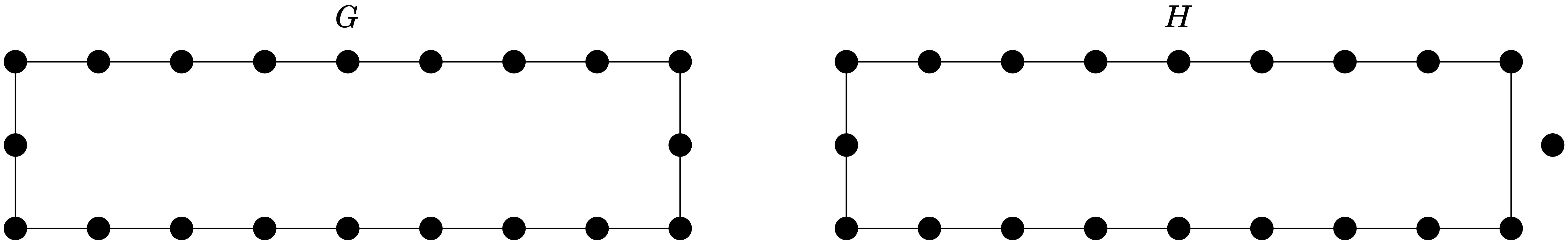
- **Global problem**: complexity $\Theta(n)$

Graph-existential indistinguishability

- **Problem**: $\mathbf{2}$-coloring paths & even cycles



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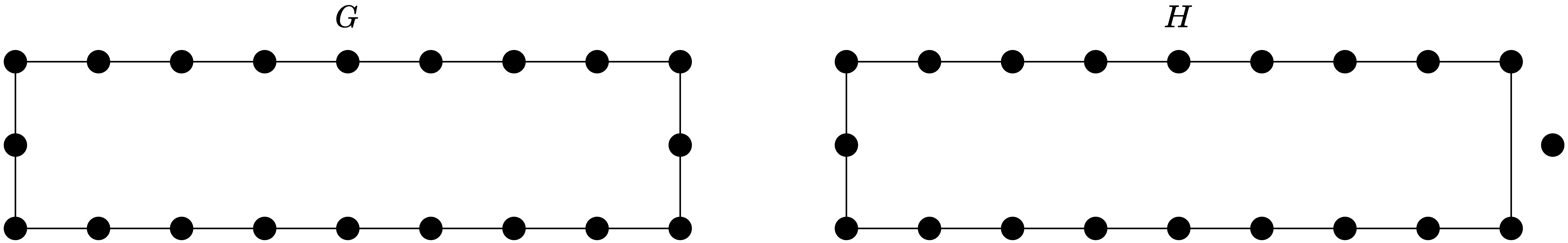

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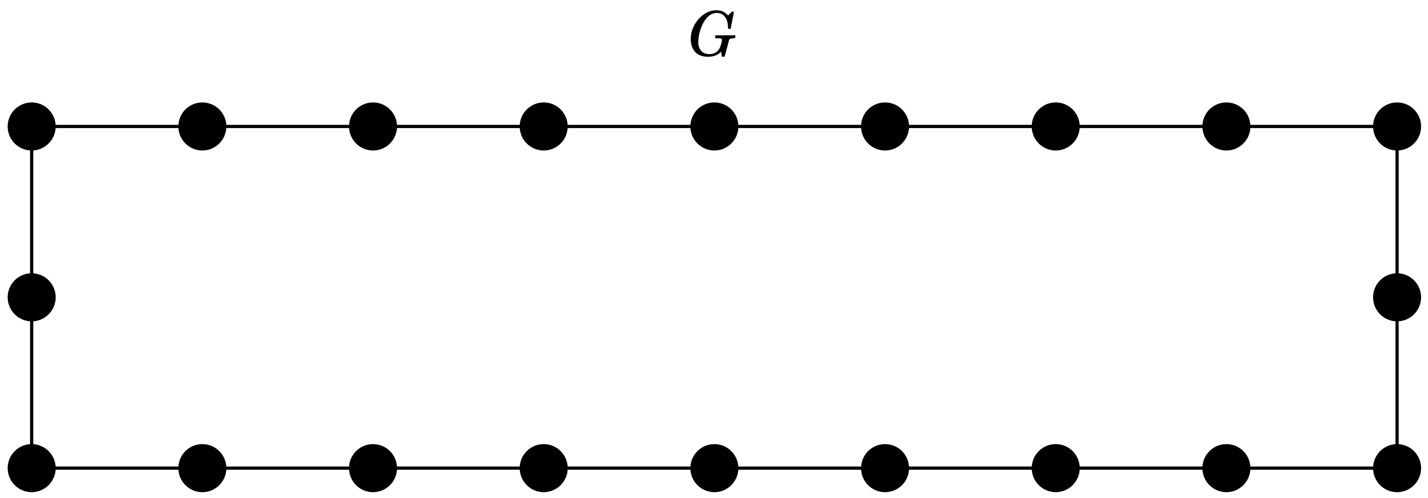

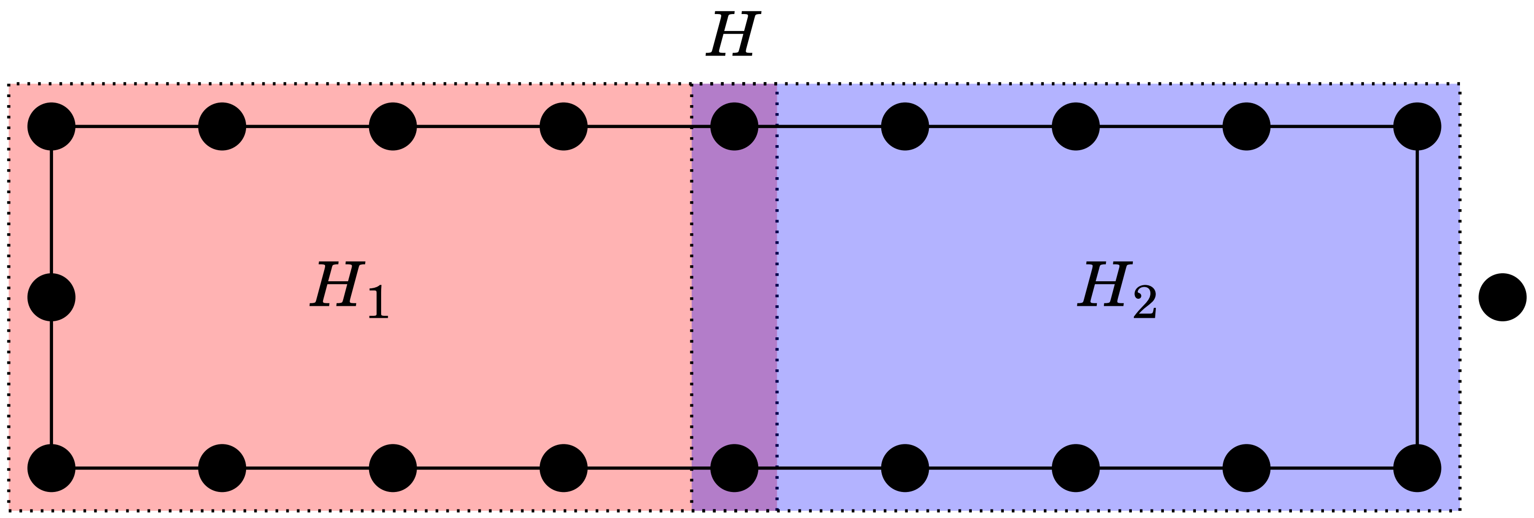

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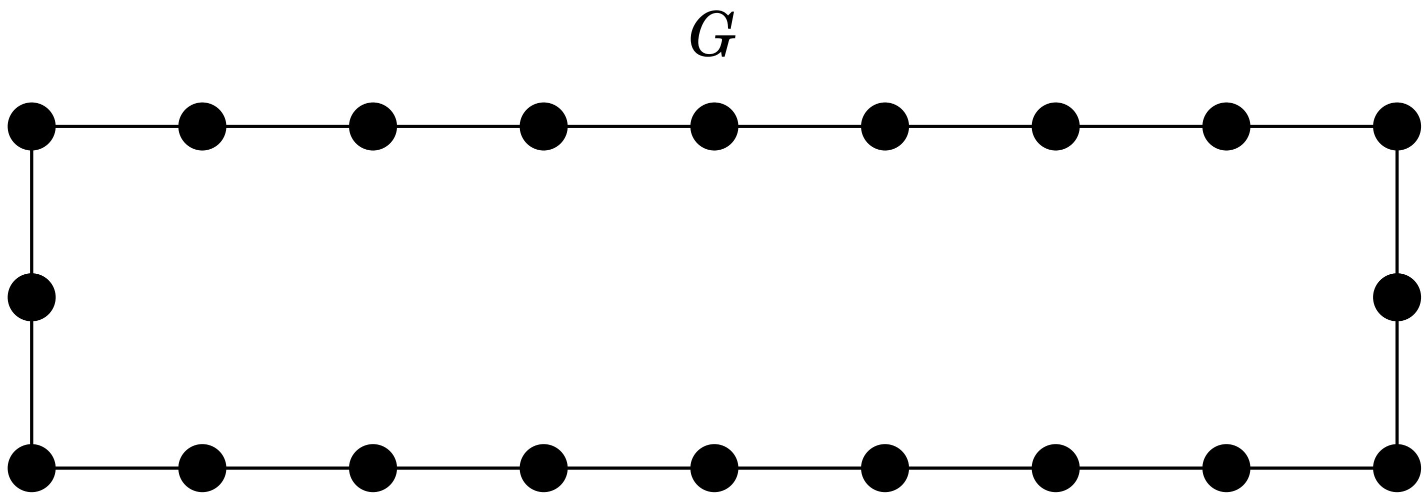

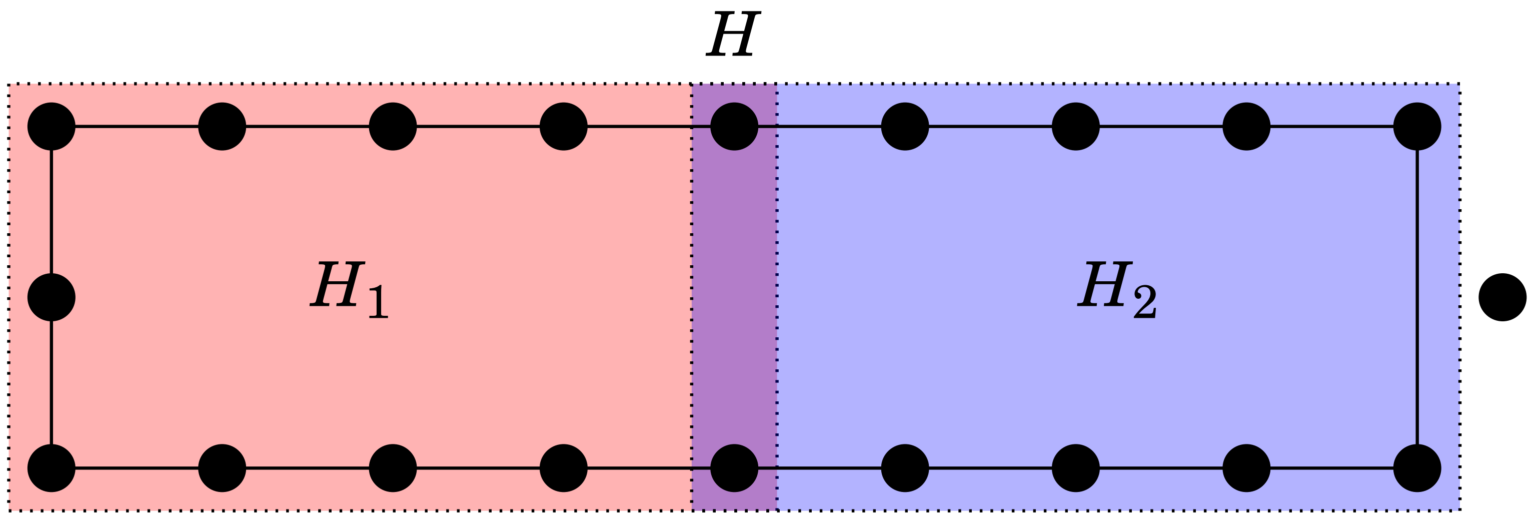

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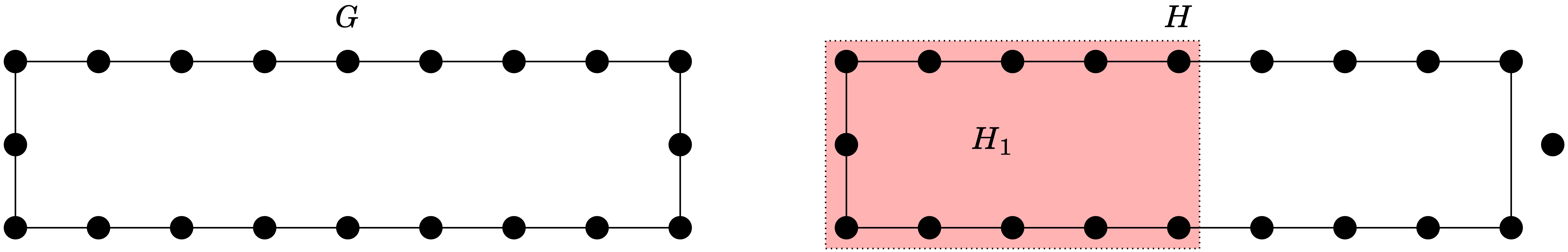

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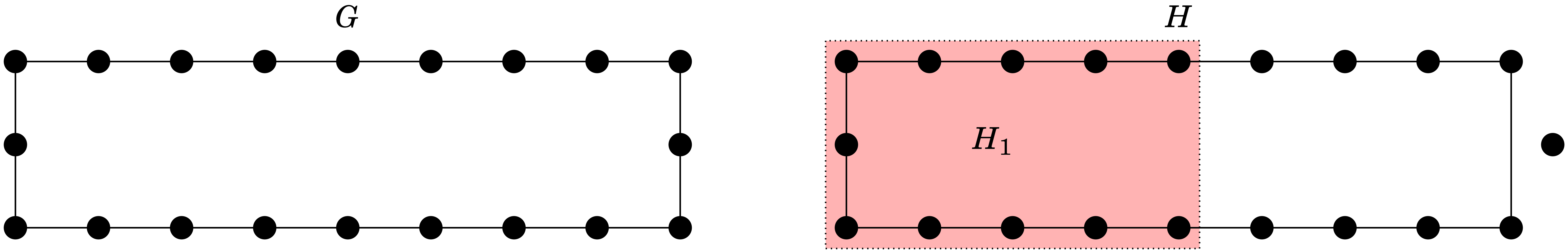

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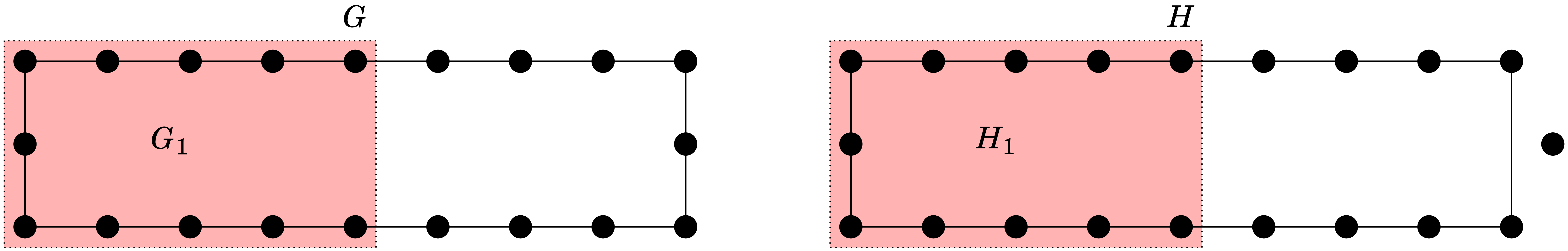

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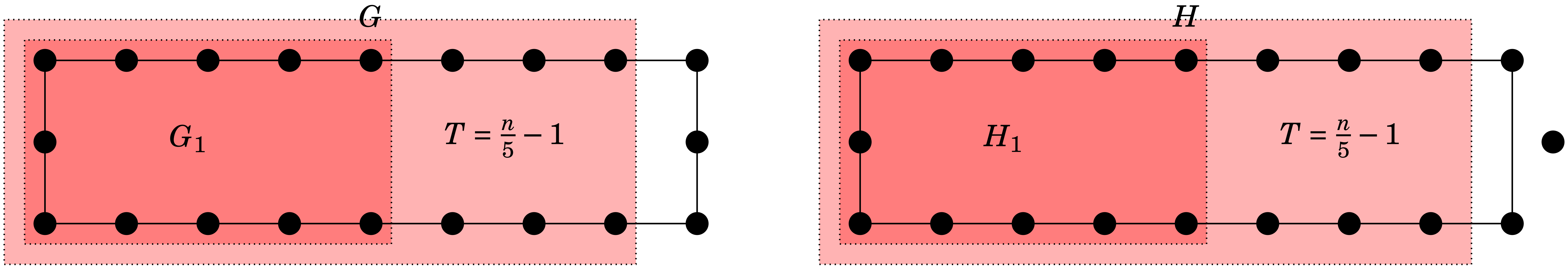

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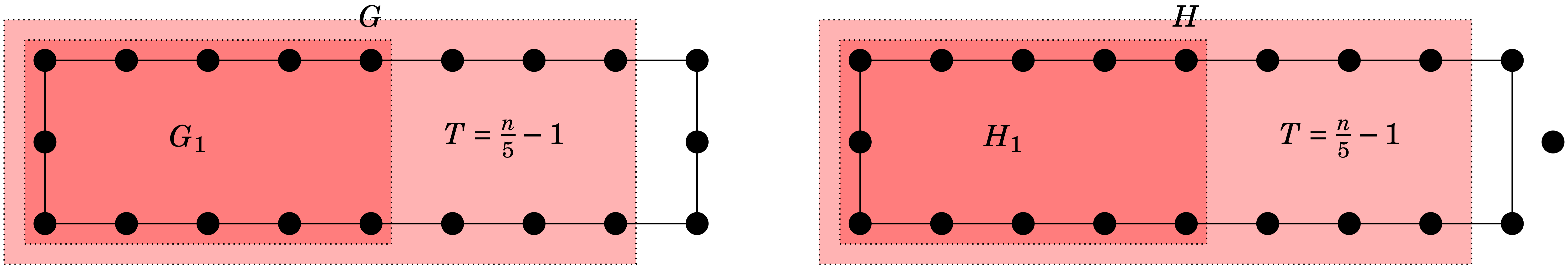

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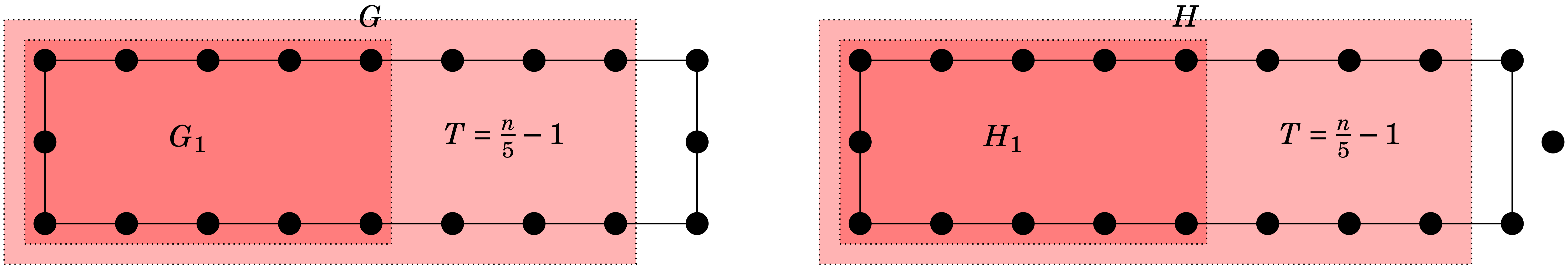

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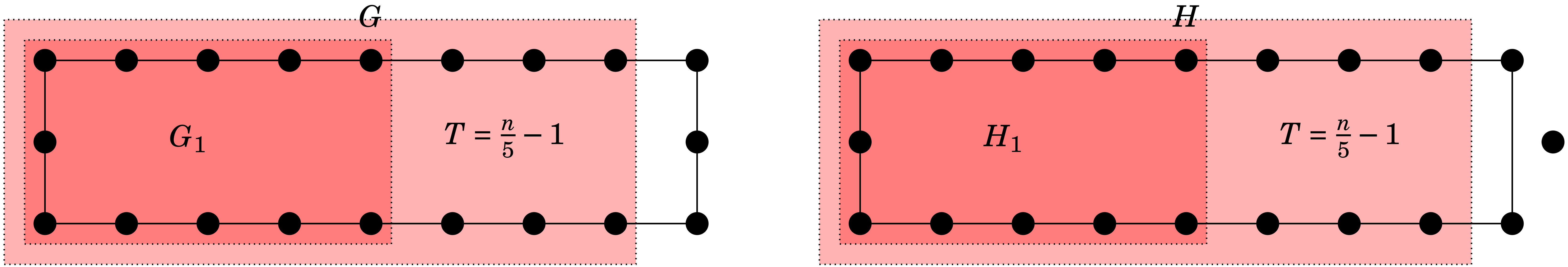

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- **Boosting failure probability**: possibility to boos failure prob. to *any* constant (non-trivial)

Table of content

1. **Intro**: distributed algorithms, the LOCAL model, the quantum-LOCAL model, locally checkable labeling problems
2. **Classical lower bounds**: the indistinguishability argument
3. **Properties of distributed algorithms**: independence and non-signaling
4. **Super-quantum models**: bounded-dependence and non-signaling model
5. **State of the art results**
6. **Quantum advantage**

Complexity of approximate graph coloring

Problem: c-coloring χ-chromatic graphs [STOC '24]

		upper bound		lower bound		
χ	c	old	new	old	new	ref
2	2	$O(n)$	$O(n)$	$\Omega(n)$	$\Omega(n)$	trivial
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2	4					
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- $\alpha = \lfloor\frac{c-1}{\chi-1}\rfloor$ approximation ratio

Some results: non-signaling model

Graph-existential lower bound arguments based on indistinguishability

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What about other known lower bounds? E.g., 3-coloring cycles has complexity $\Theta(\log^{\star} n)$ [Linial, FOCS '87]

Some results: bounded-dependence model

- Can we similarly **rule out quantum advantage for 3-coloring cycles**? Classical complexity $T = \Theta(\log^\star n)$

Some results: bounded-dependence model

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  - no! *There is a finitely-dependent distribution that 3-colors paths and cycles* ($T = O(1)$)
  - [Holroyd and Liggett, Forum of Mathematics, Pi ’14]
  - [Holroyd, Hutchcroft, and Levy, Electronic Communications in Probability ’18]

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  - For any LCL Π on bounded degree graphs, *there is a finitely-dependent distribution* ($\boldsymbol{T} = \boldsymbol{O}(\mathbf{1})$) *solving* Π
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Bounding general quantum advantage

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Relations among models

- $X \to Y$ means that locality T in X becomes locality $O(T)$ in Y

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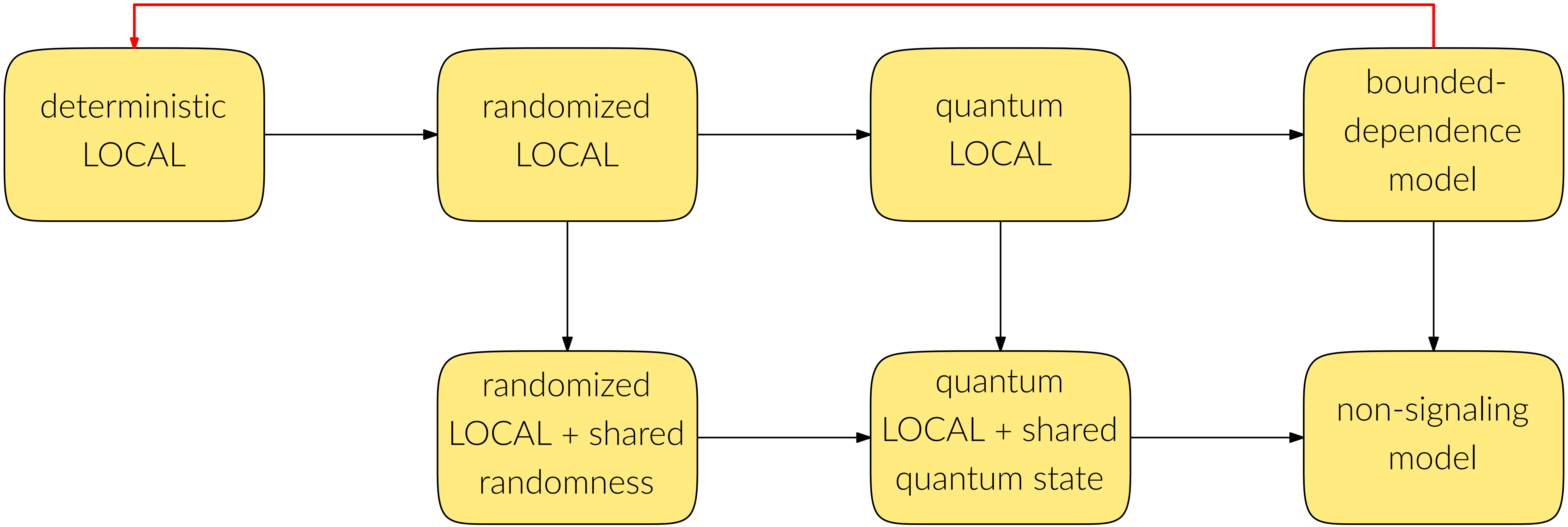

Table of content

1. **Intro**: distributed algorithms, the LOCAL model, the quantum-LOCAL model, locally checkable labeling problems

2. **Classical lower bounds**: the indistinguishability argument

3. **Properties of distributed algorithms**: independence and non-signaling

4. **Super-quantum models**: bounded-dependence and non-signaling model

5. **State of the art results**

6. **Quantum advantage**

Distributed quantum computing

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- **Quantum-LOCAL**: can we do something better? [STOC '25b, SODA '26]

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THANKS! Questions?

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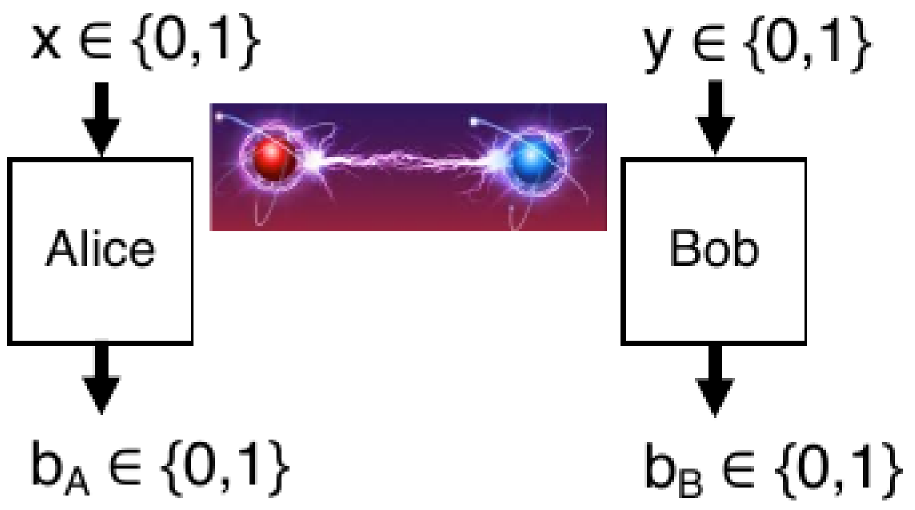
Non-signaling & quantum games

Alice

Bob

- Both Alice and Bob receive an input bit x and y in $\{0,1\}$
- They must output a bit each a and b according to some rule

CHSH Game [Clauser, Horne, Shimony, Holt 1969]



winning condition: $b_A \oplus b_B = xy$

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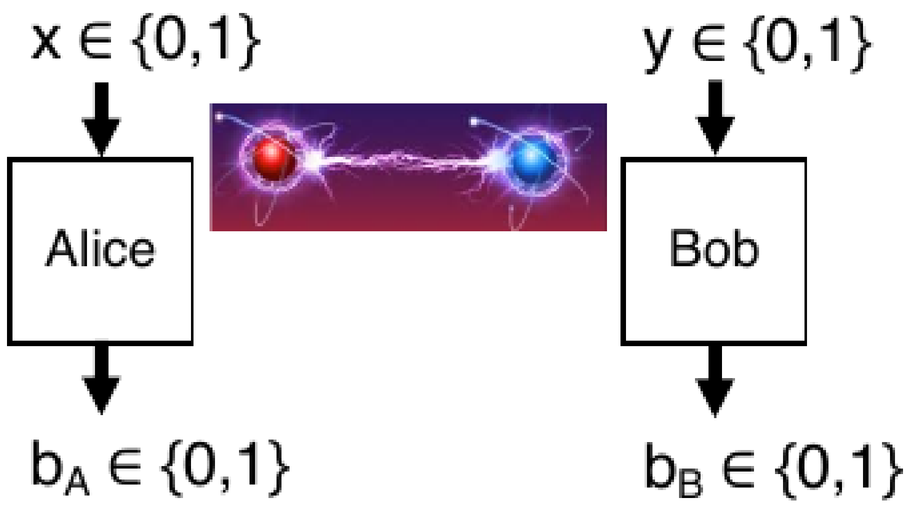

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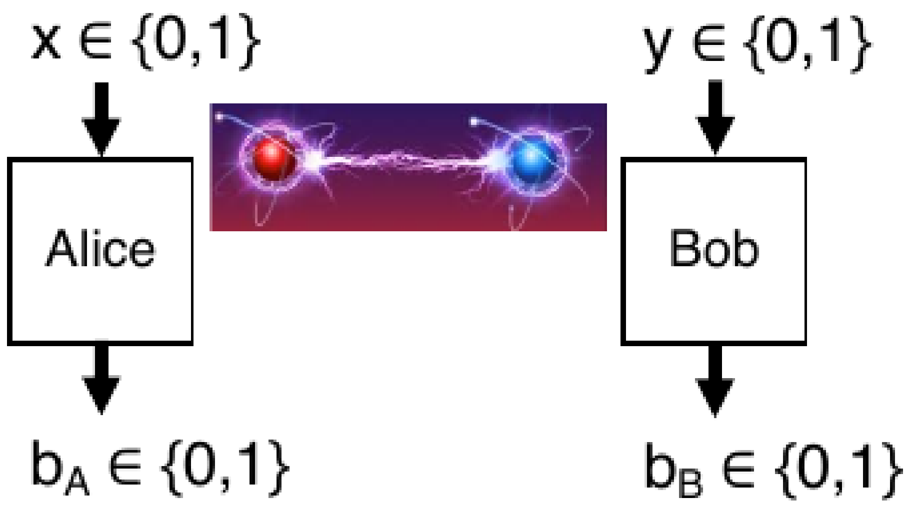

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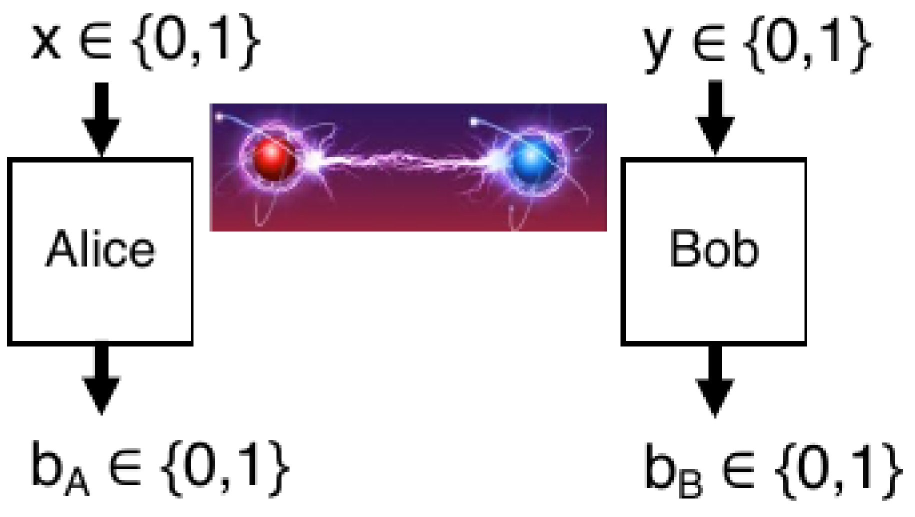

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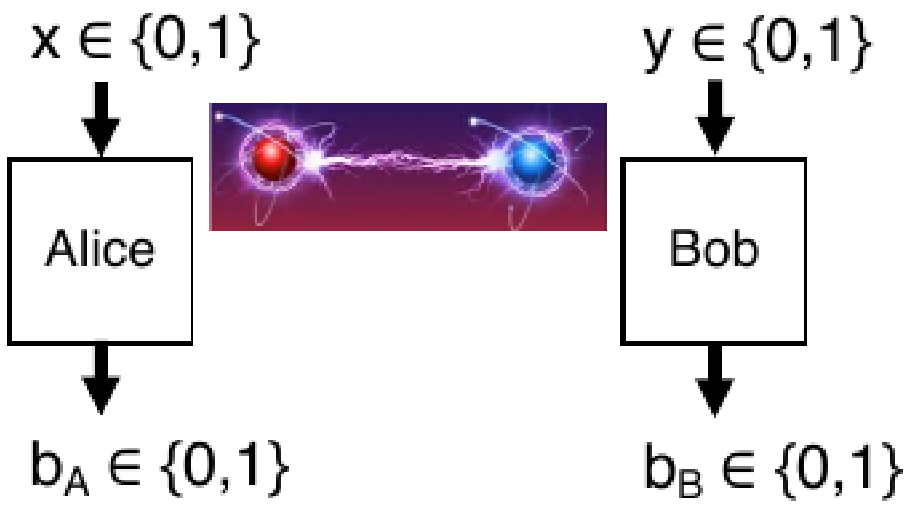

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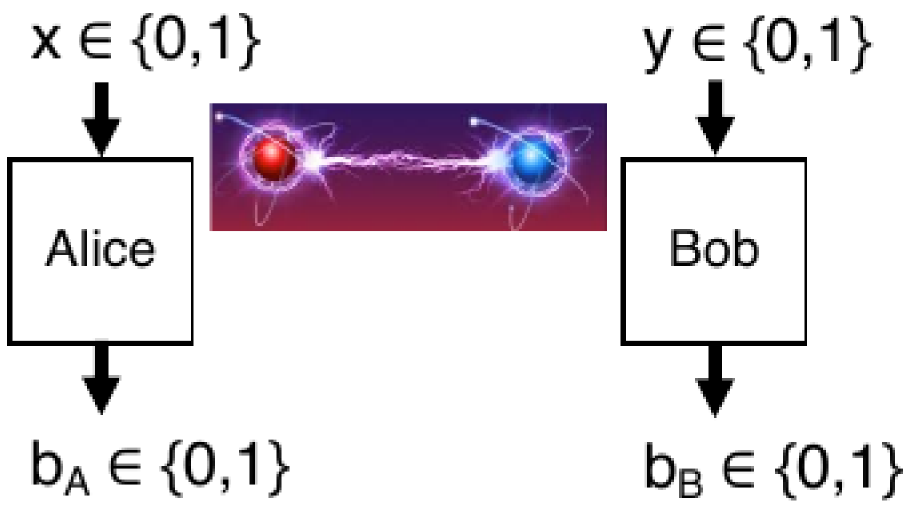

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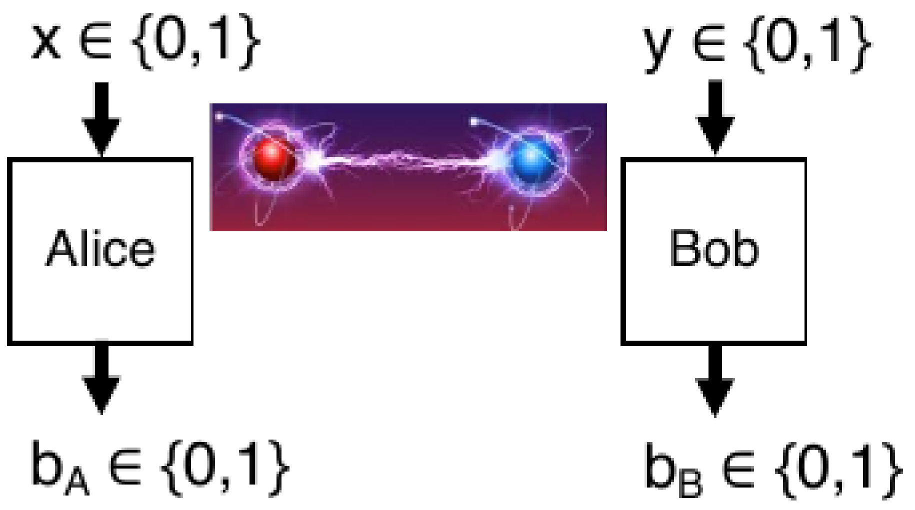

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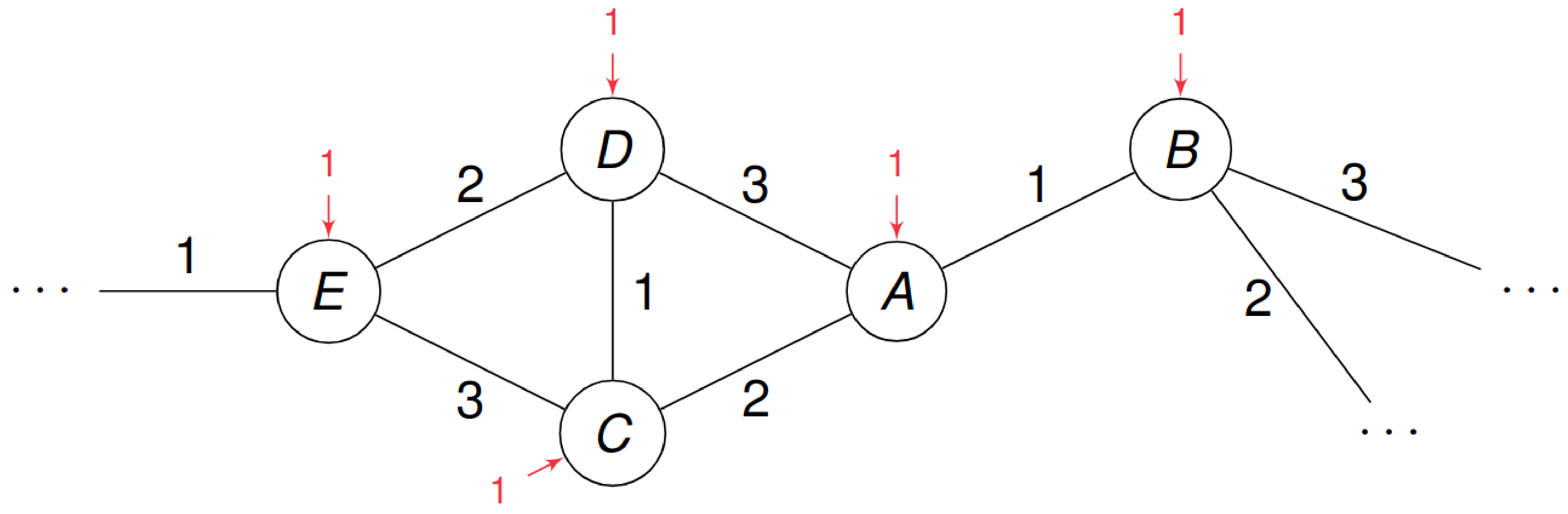

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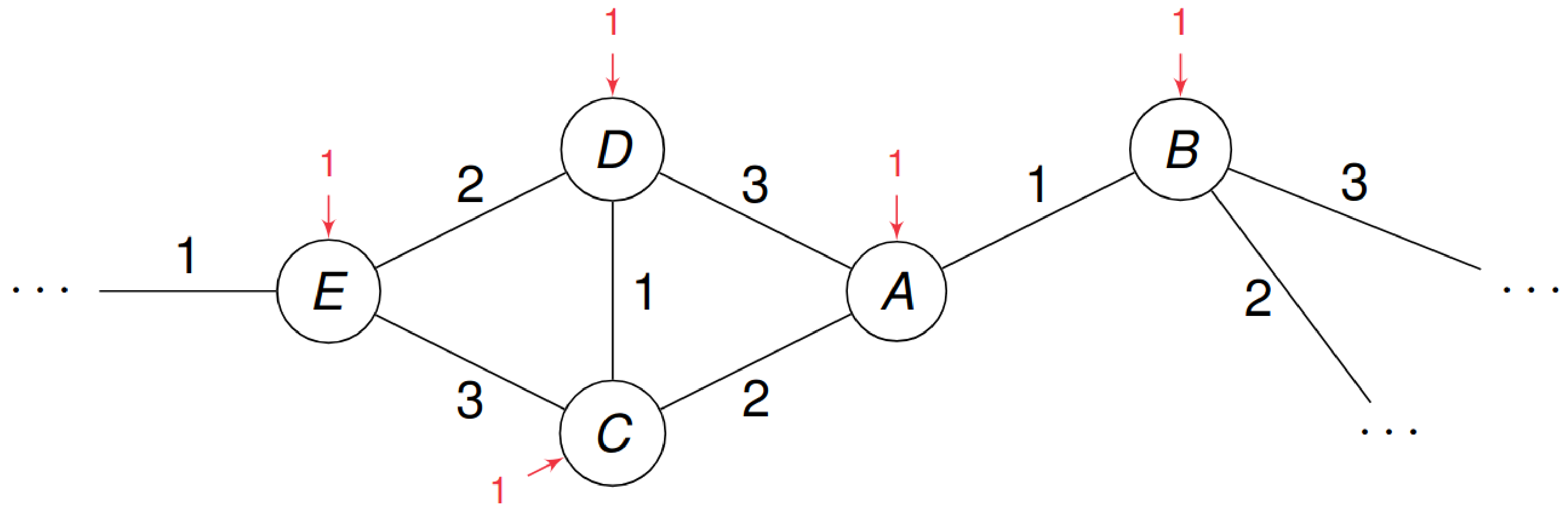

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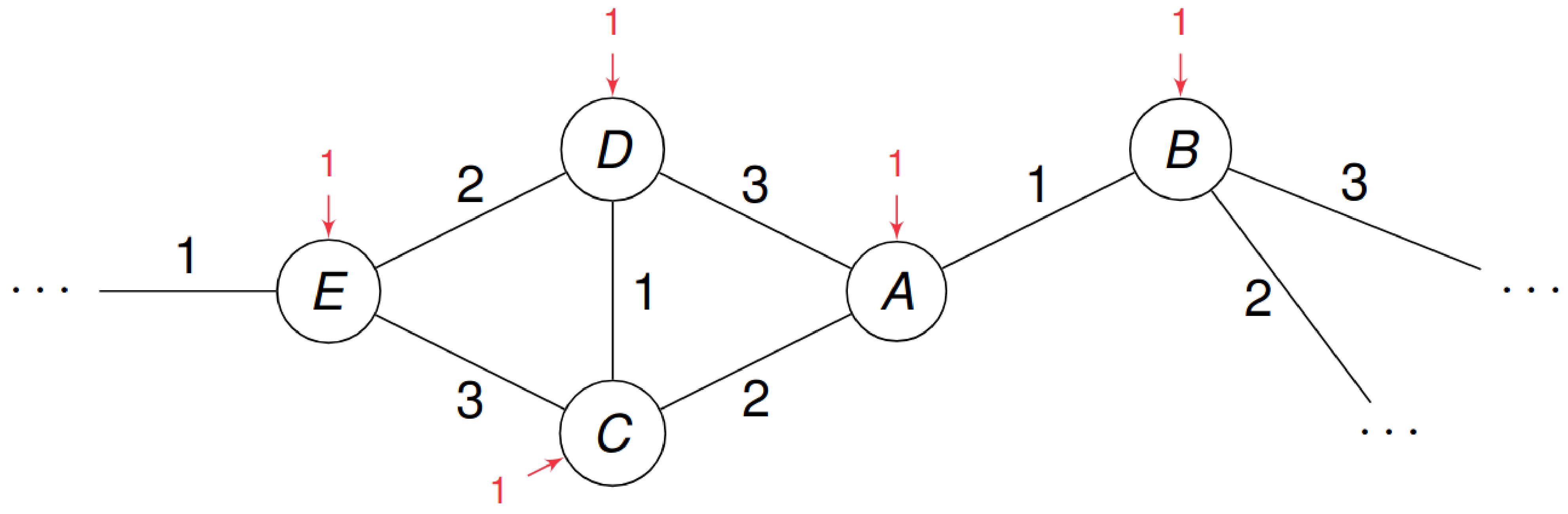

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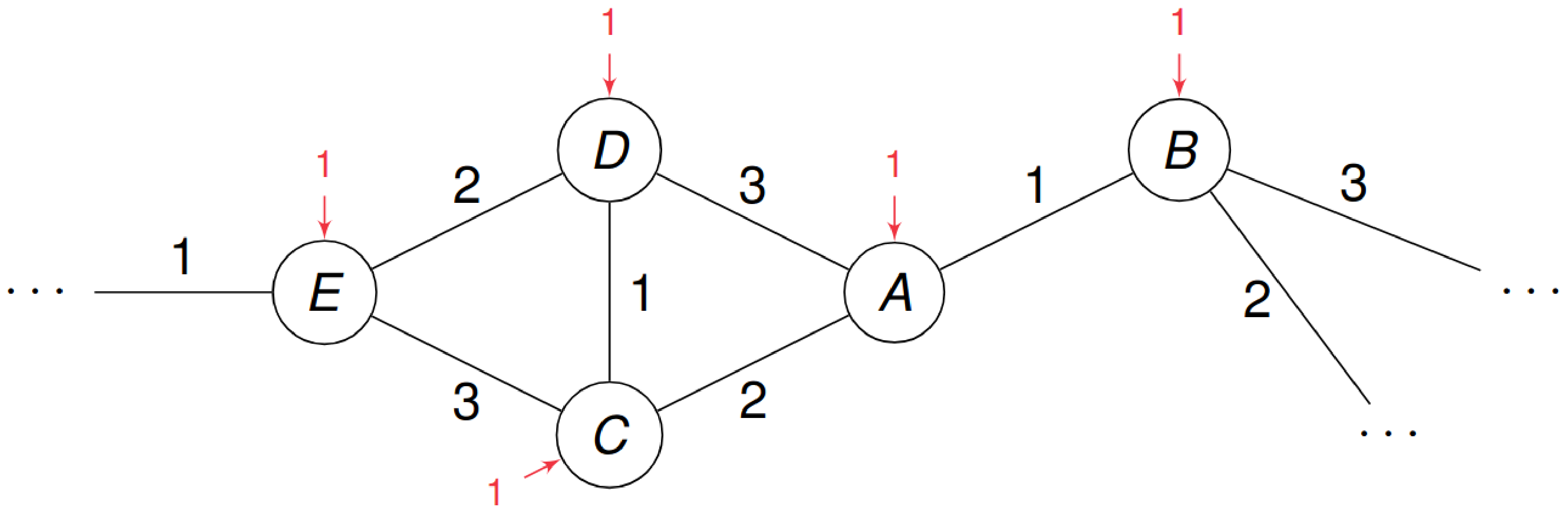

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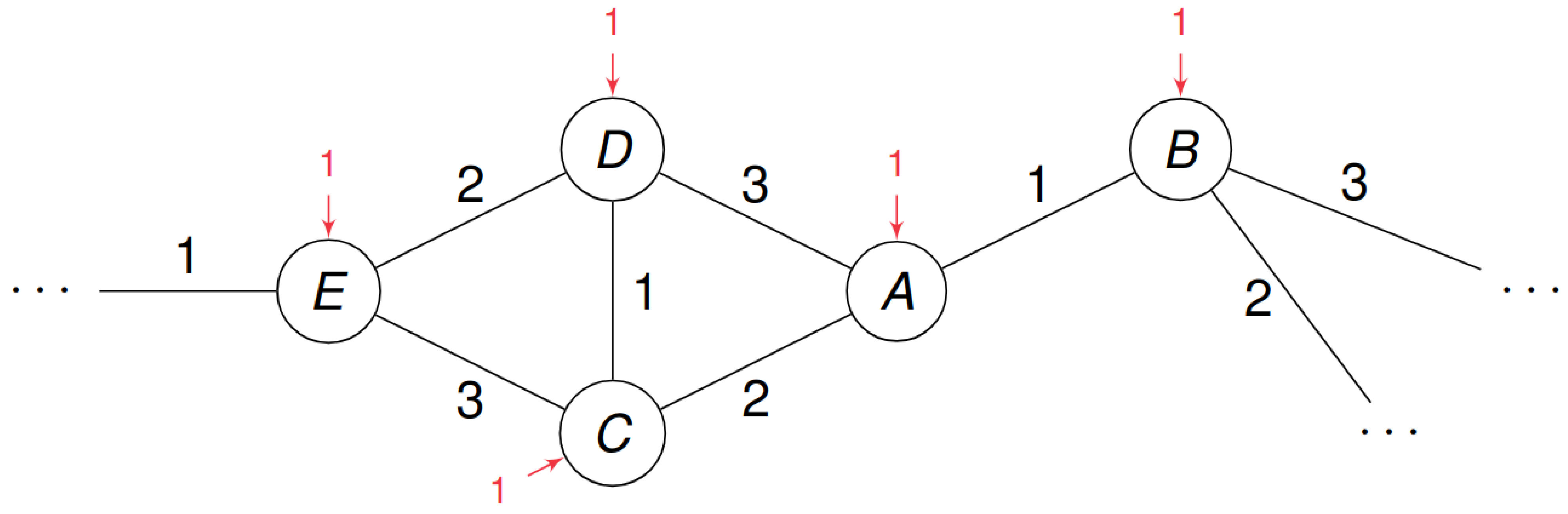

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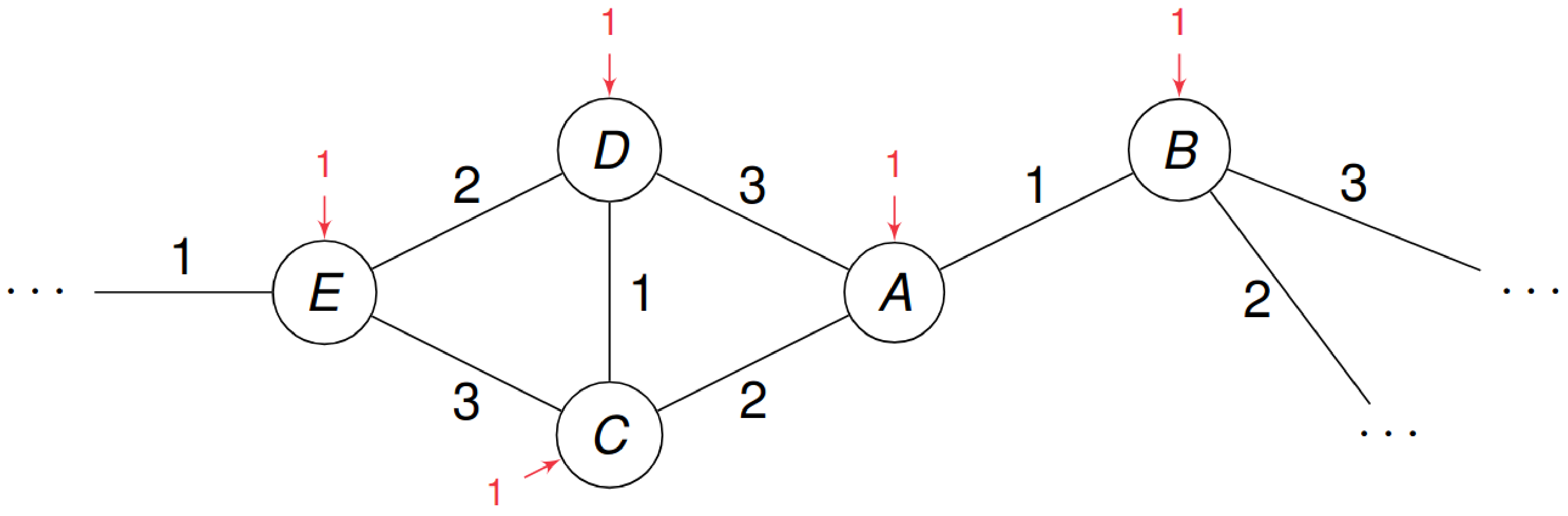

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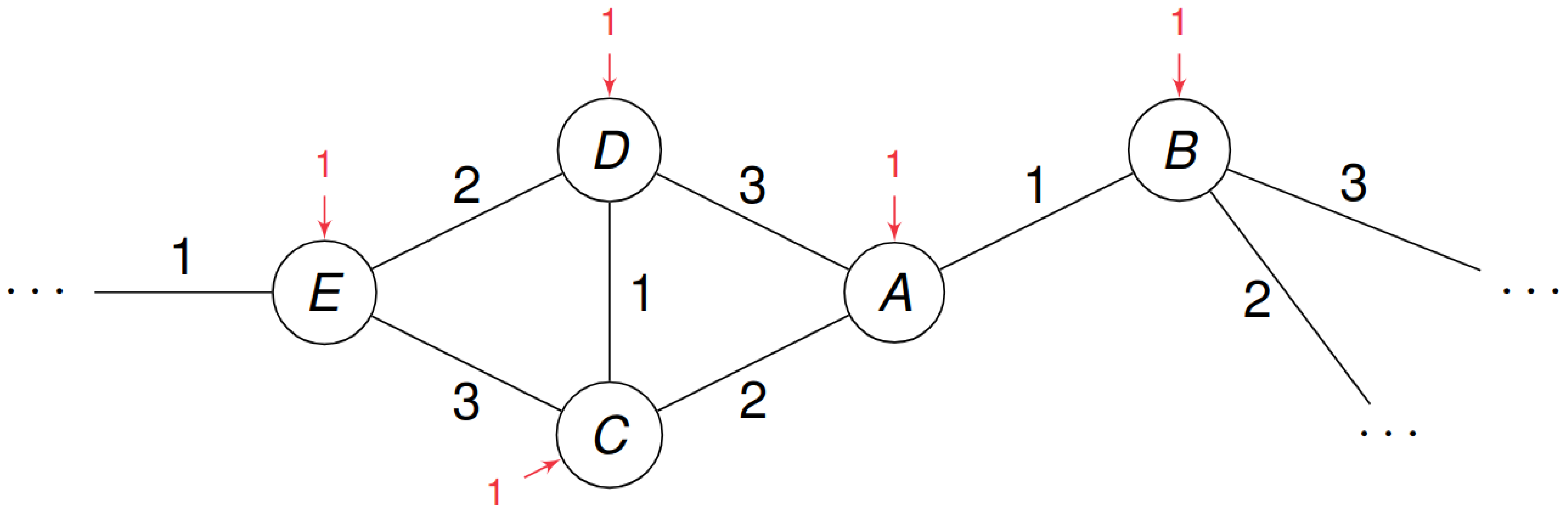

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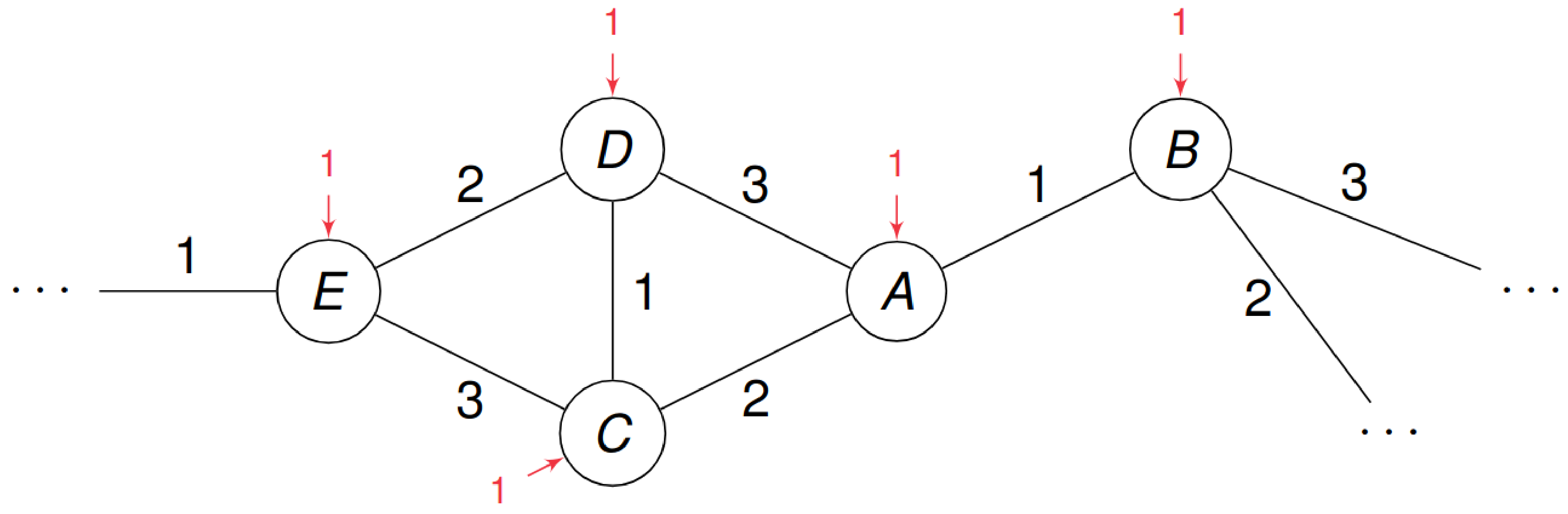

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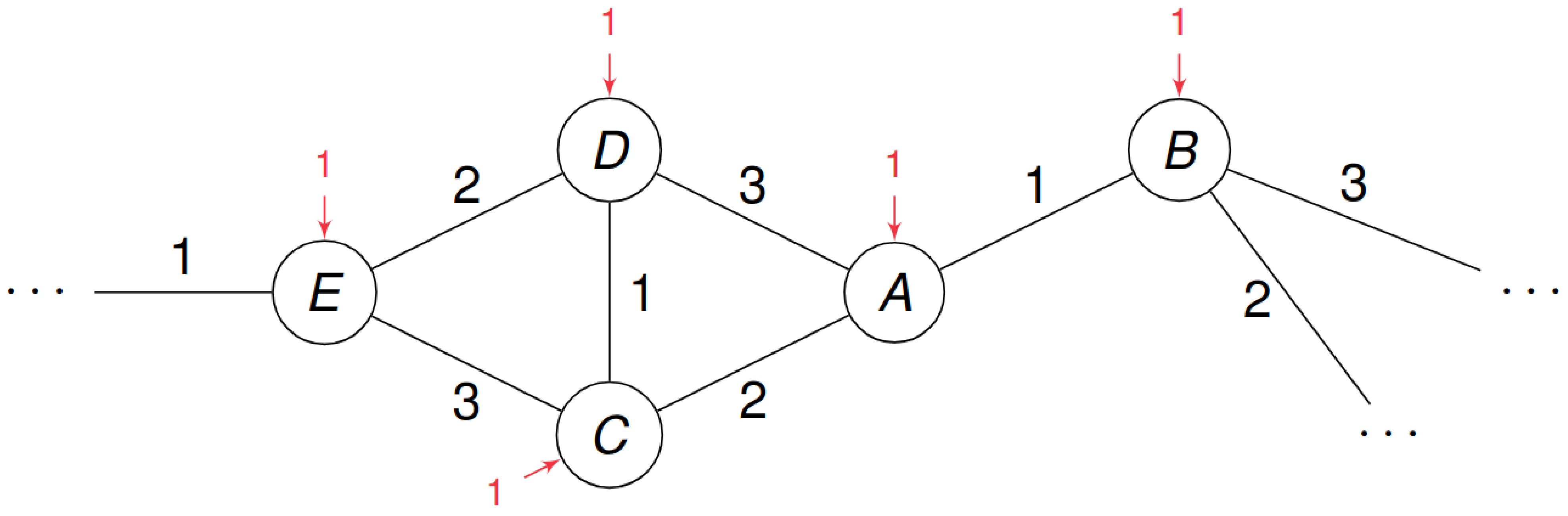

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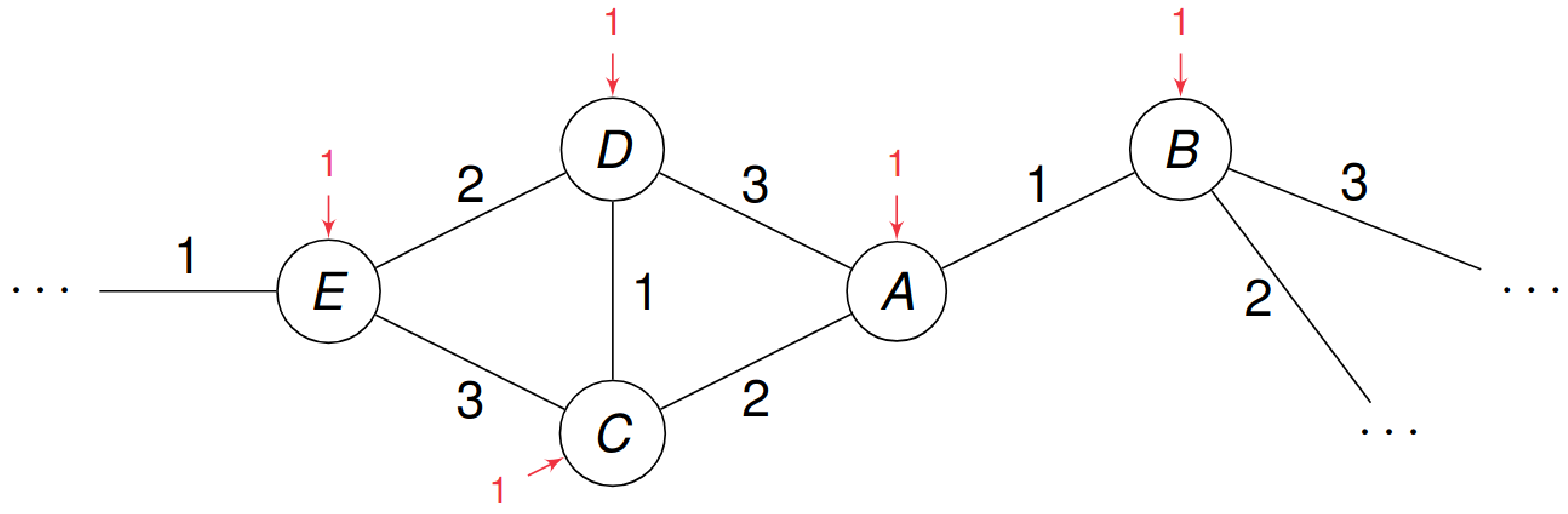

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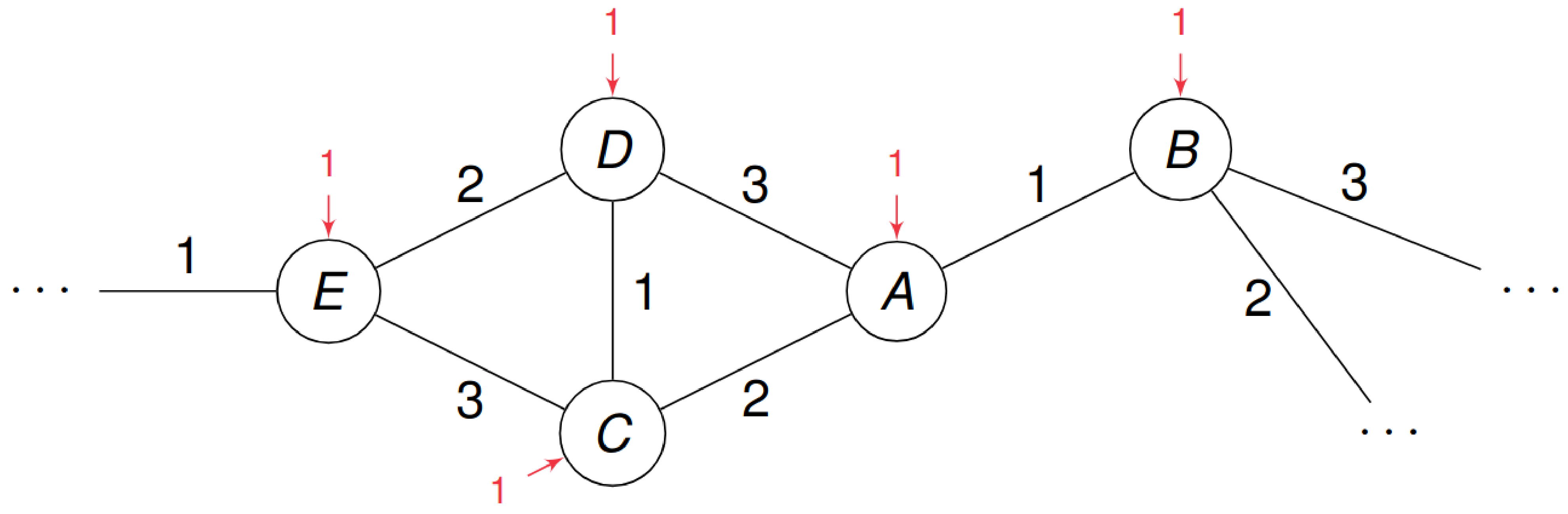

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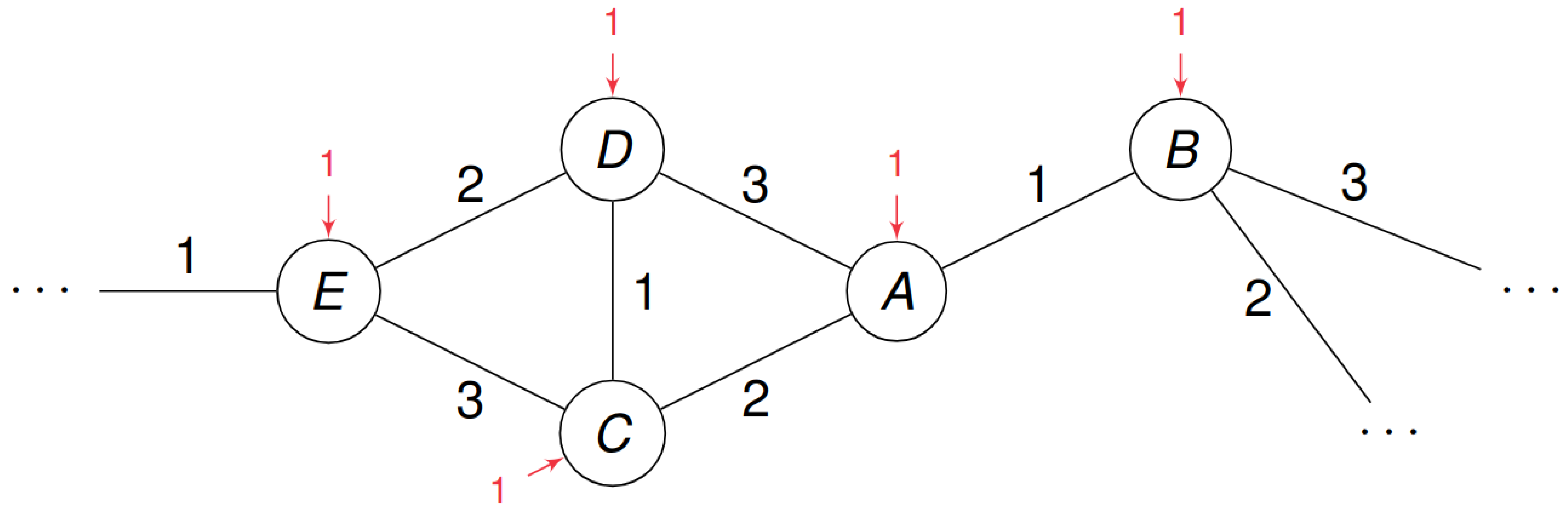

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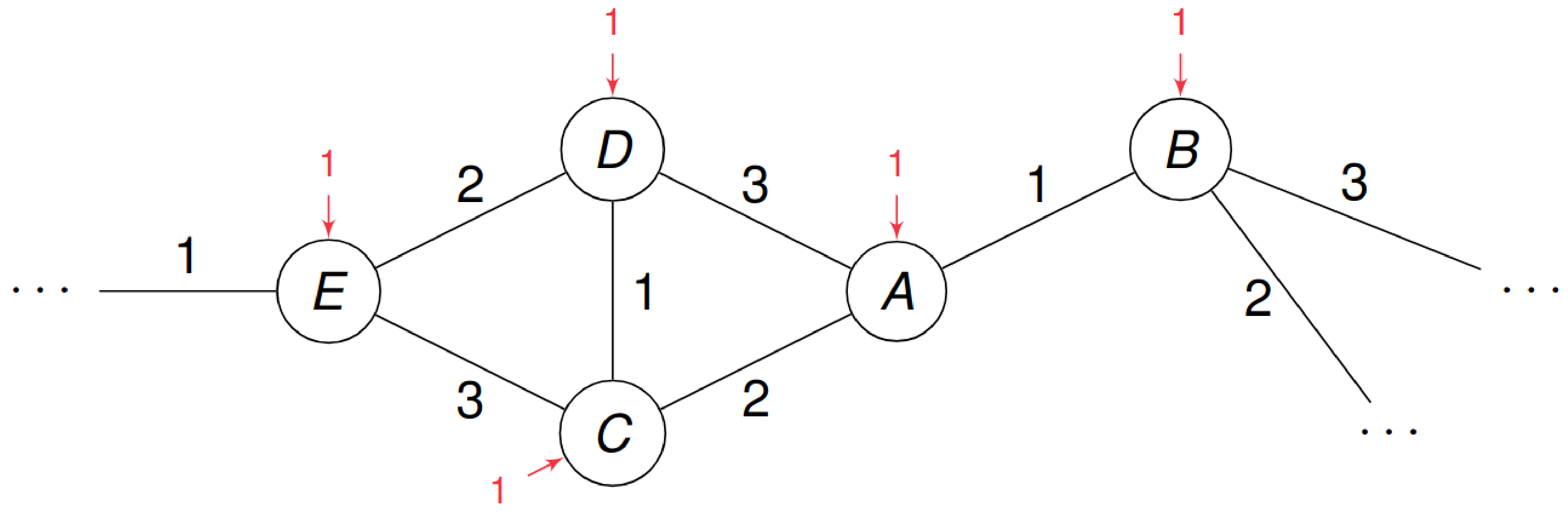

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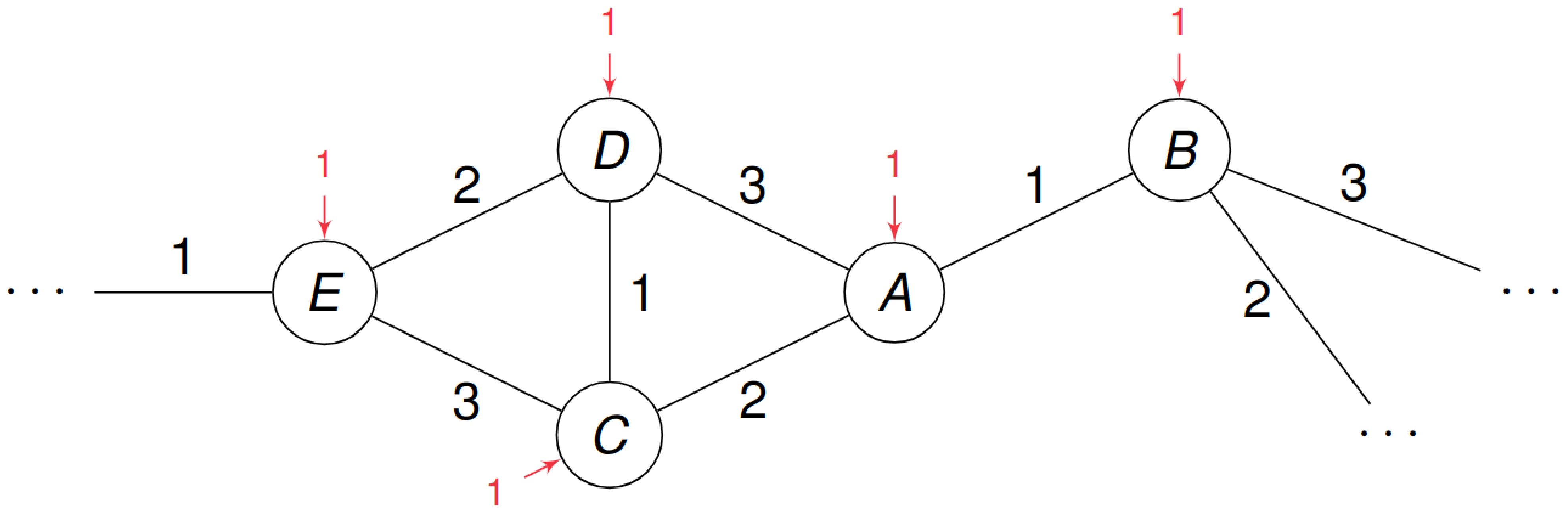

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- Δ-regular graph
- Input Δ-edge coloring
  - each edge is a CHSH game
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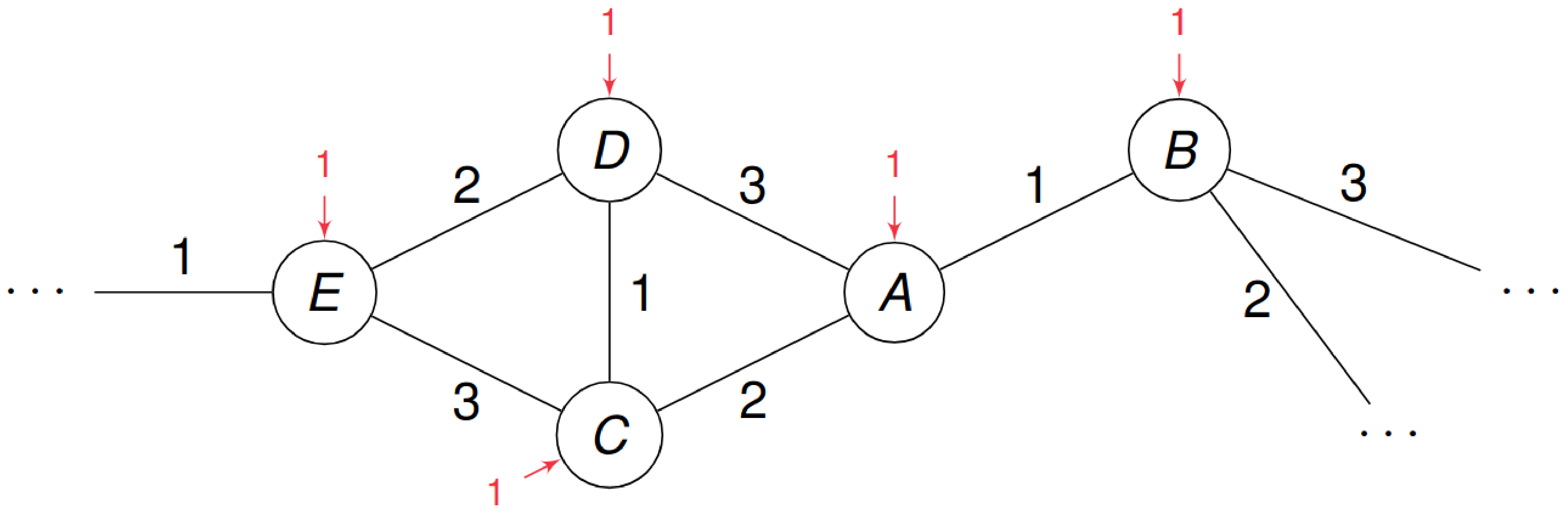

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GHZ game

- **Greenberger-Horne-Zeilinger game**

Alice

Bob

Charlie

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Outputs	$0,0,0$	$1,0,0$	$1,0,0$	$1,0,0$
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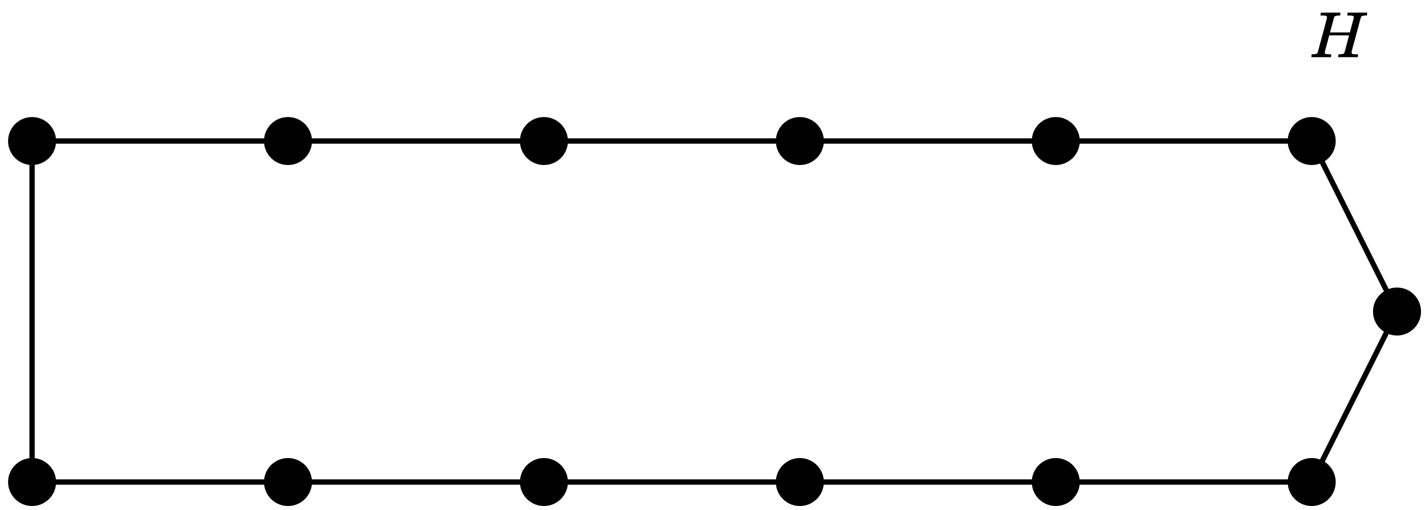
Boosting failure probability: randomized-LOCAL

Problem: $\mathbf{2}$-coloring $\mathbf{2}$-chromatic graphs

- randomized LOCAL



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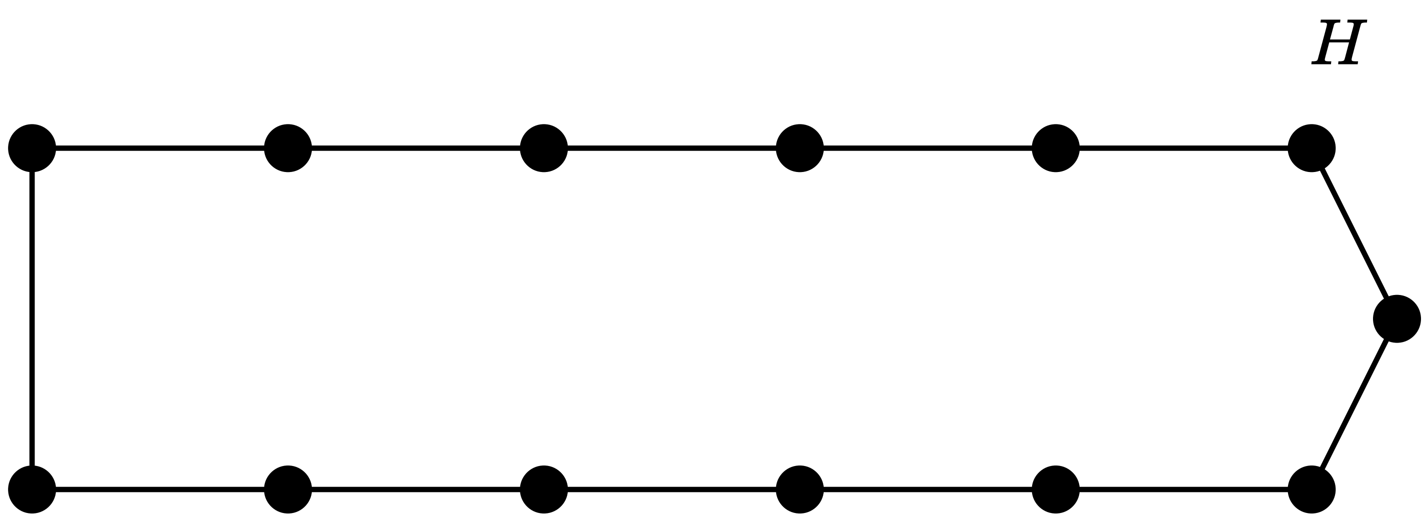

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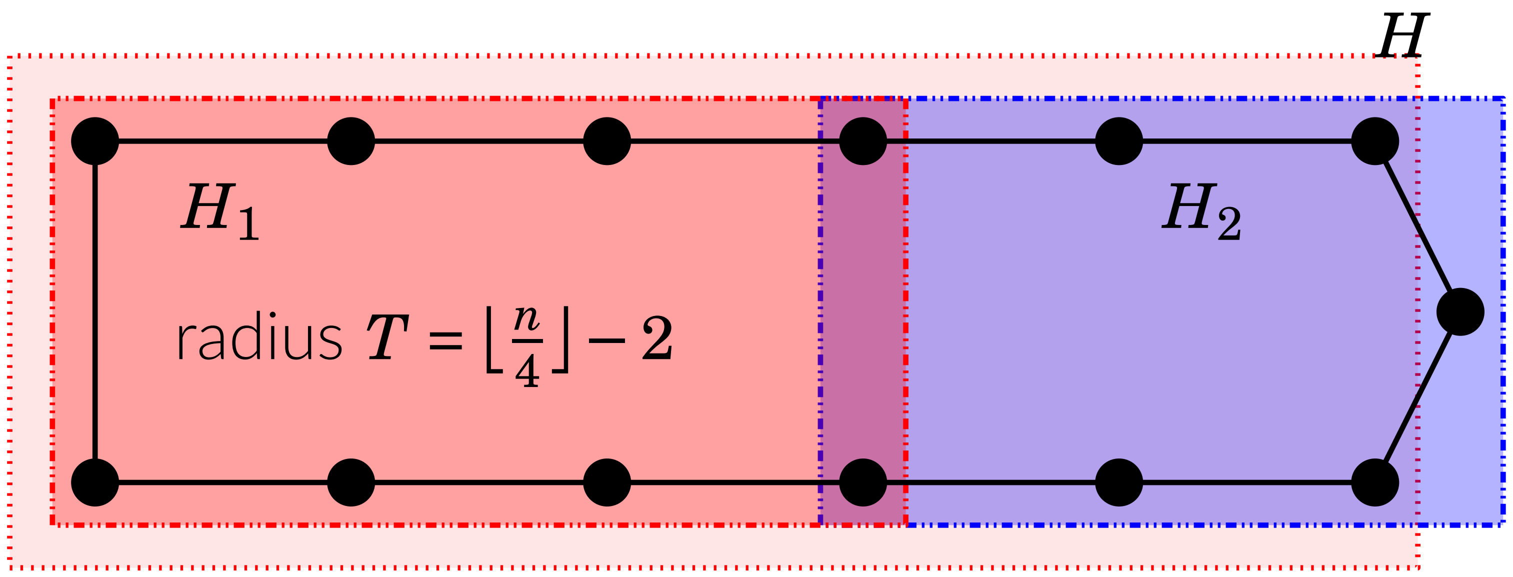

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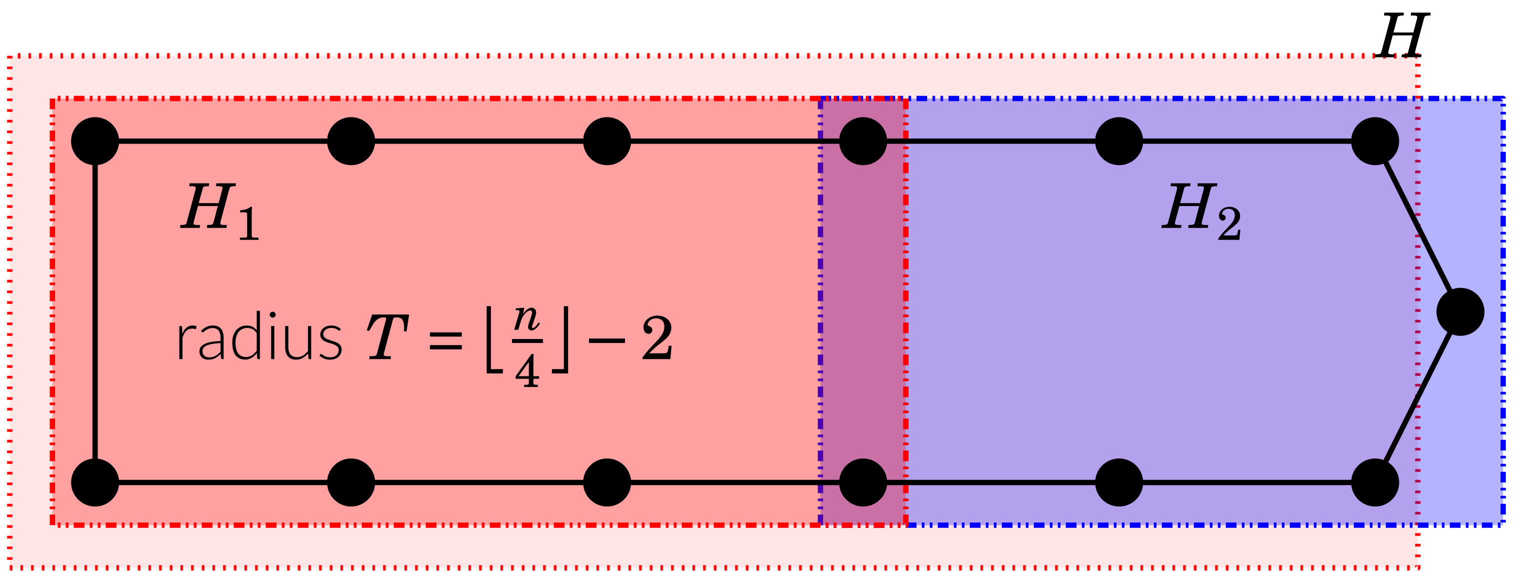

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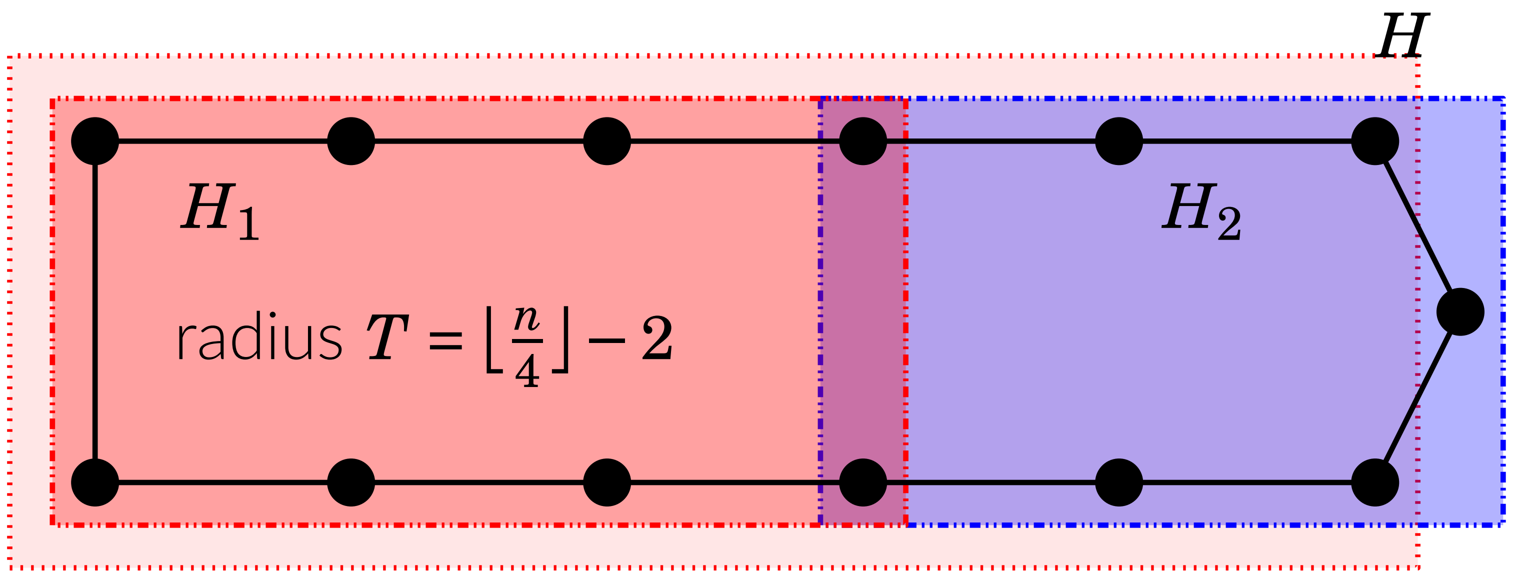

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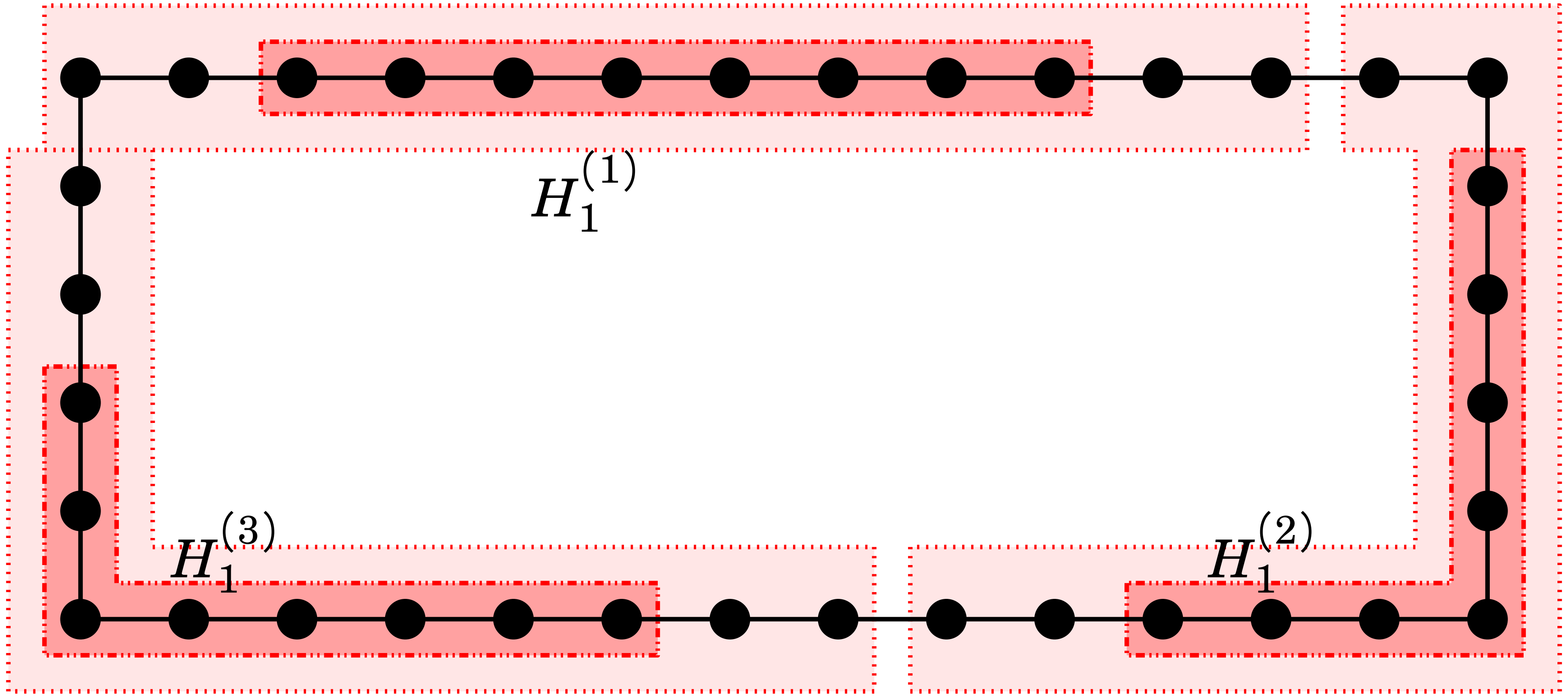

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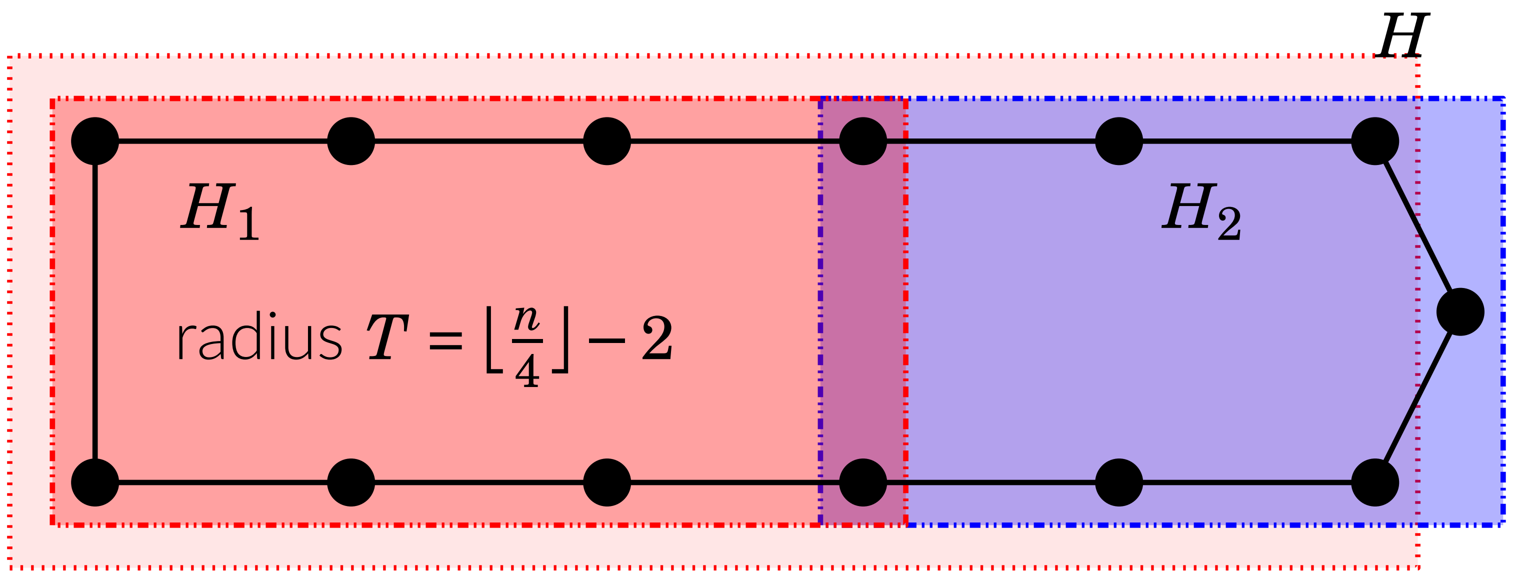

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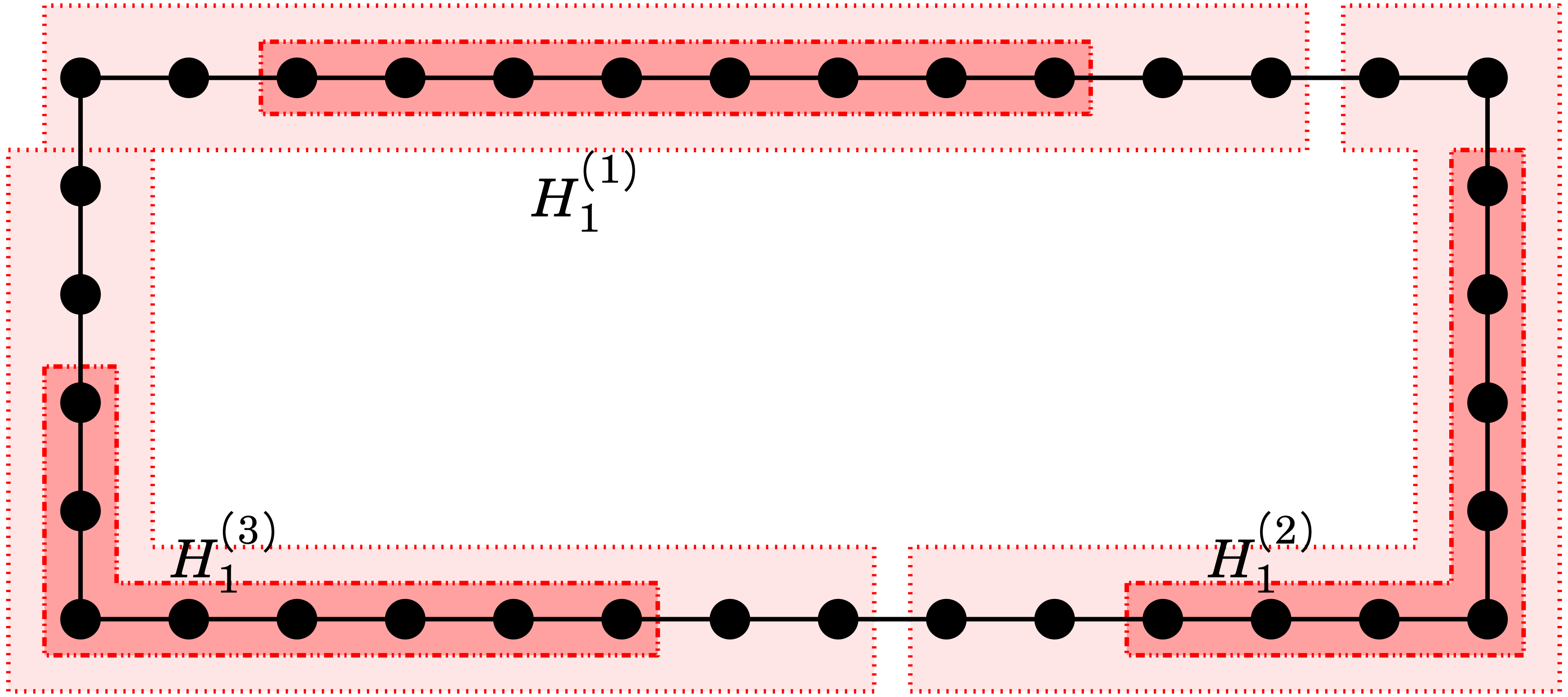

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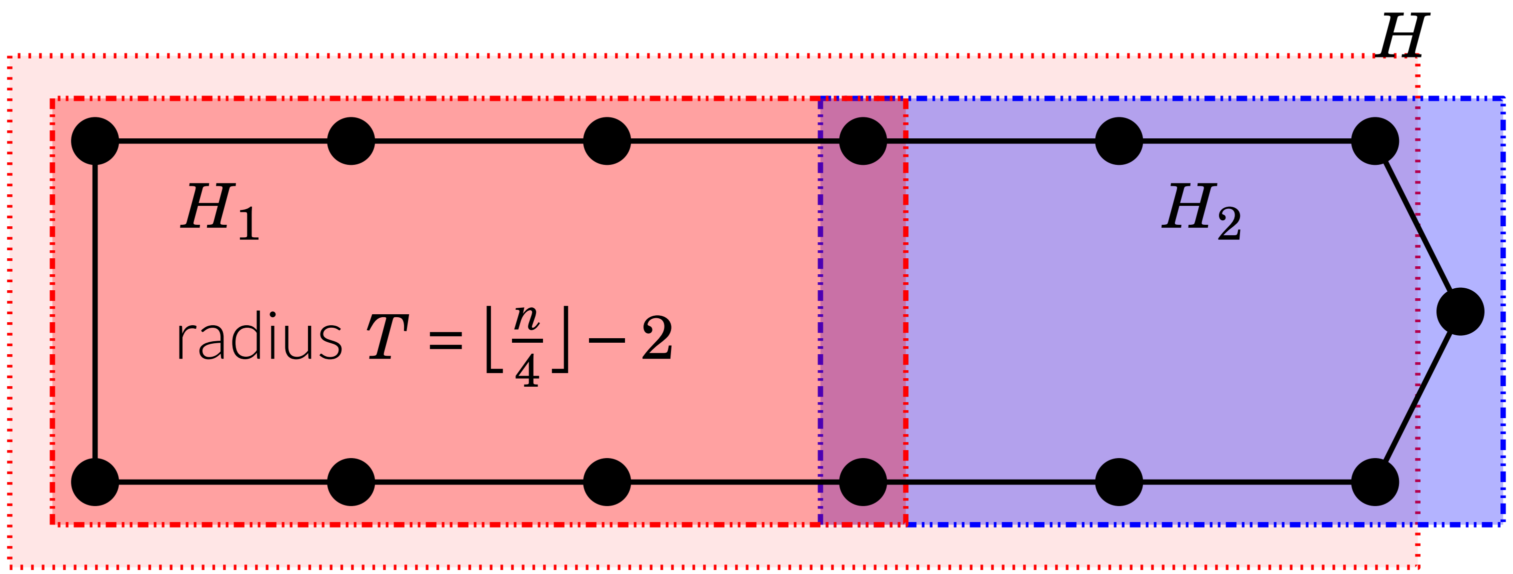

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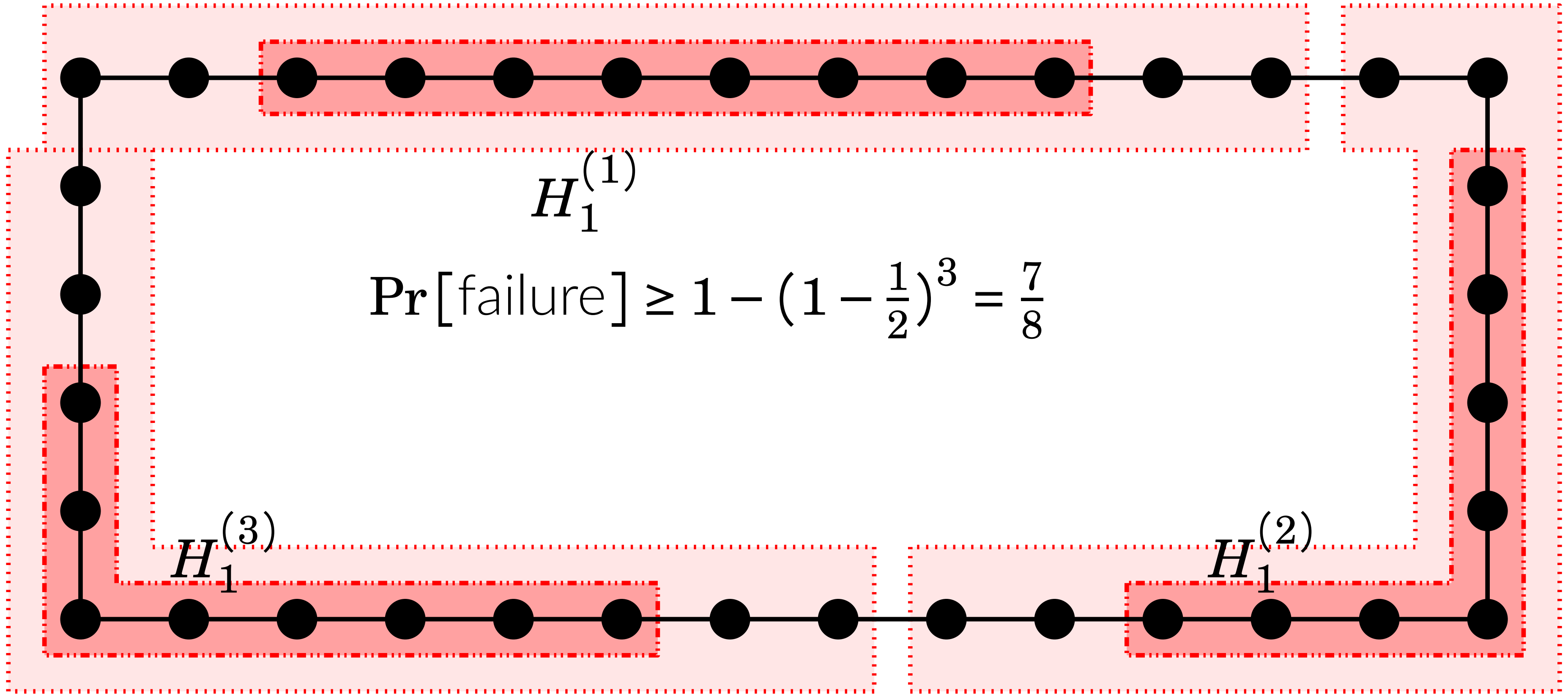

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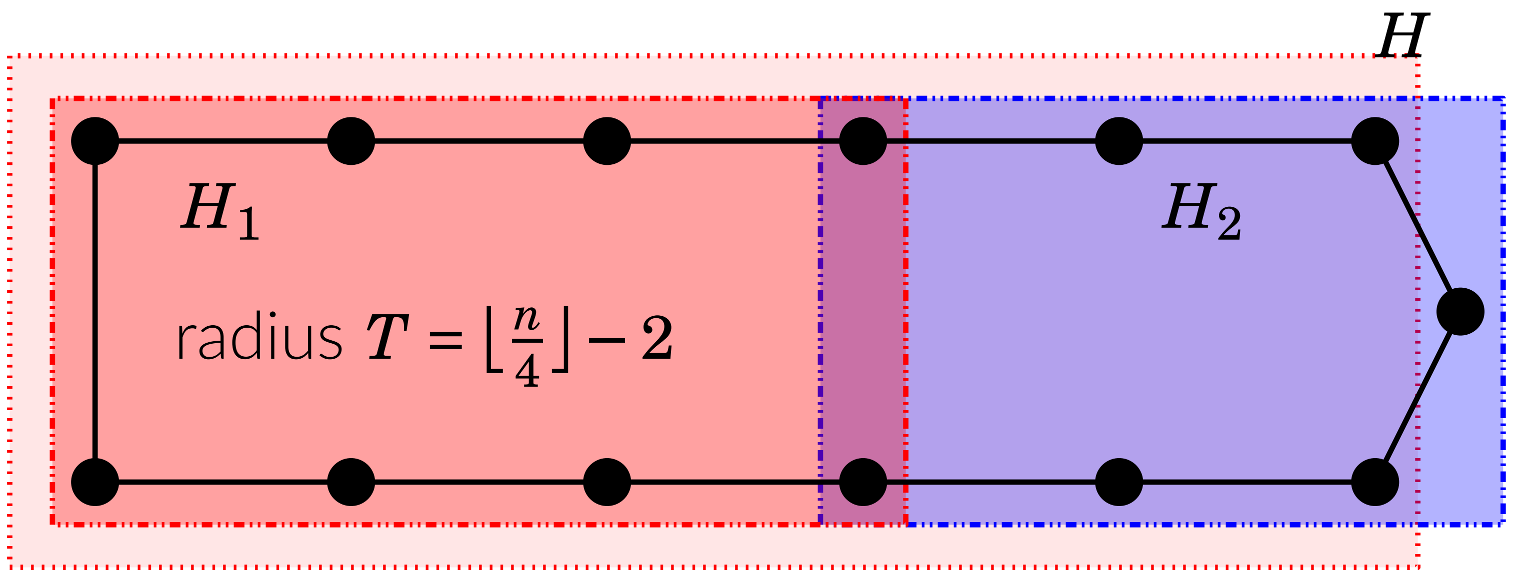

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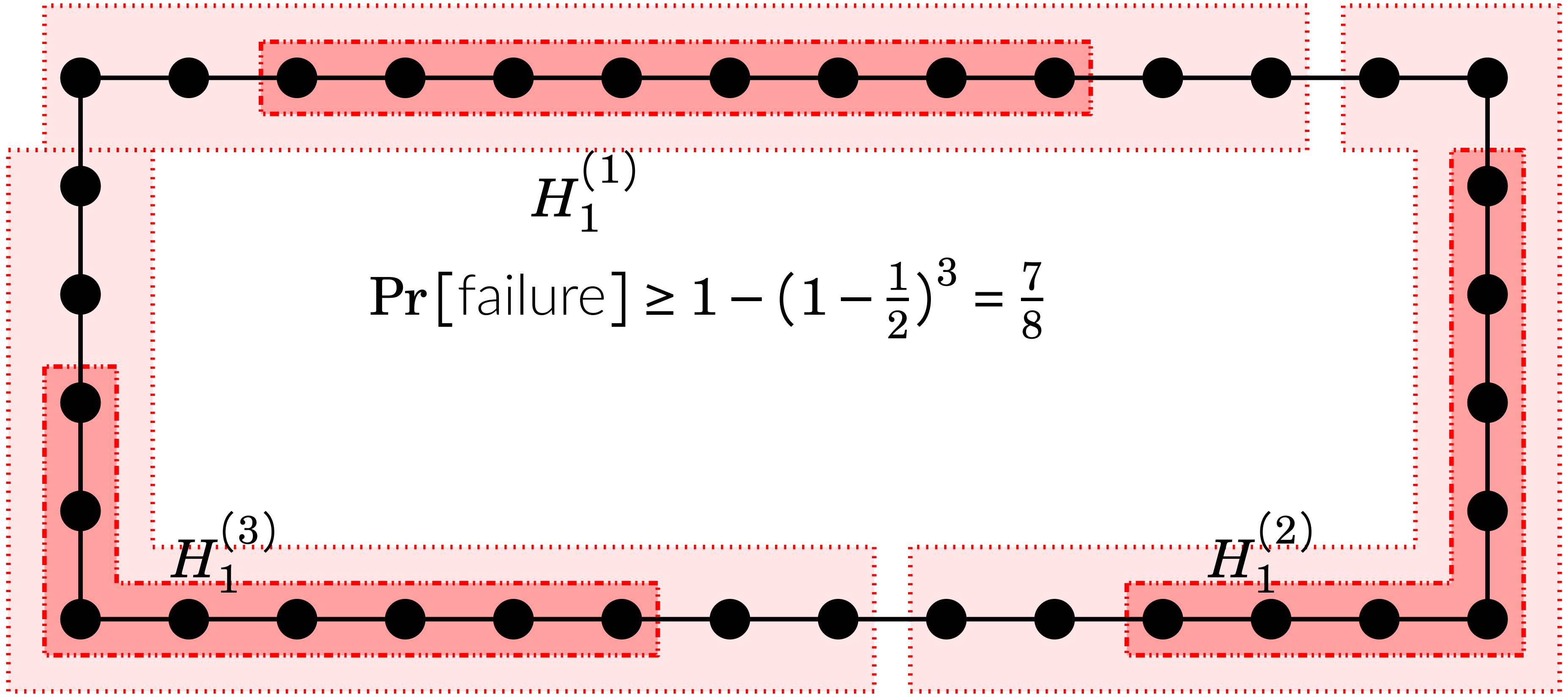

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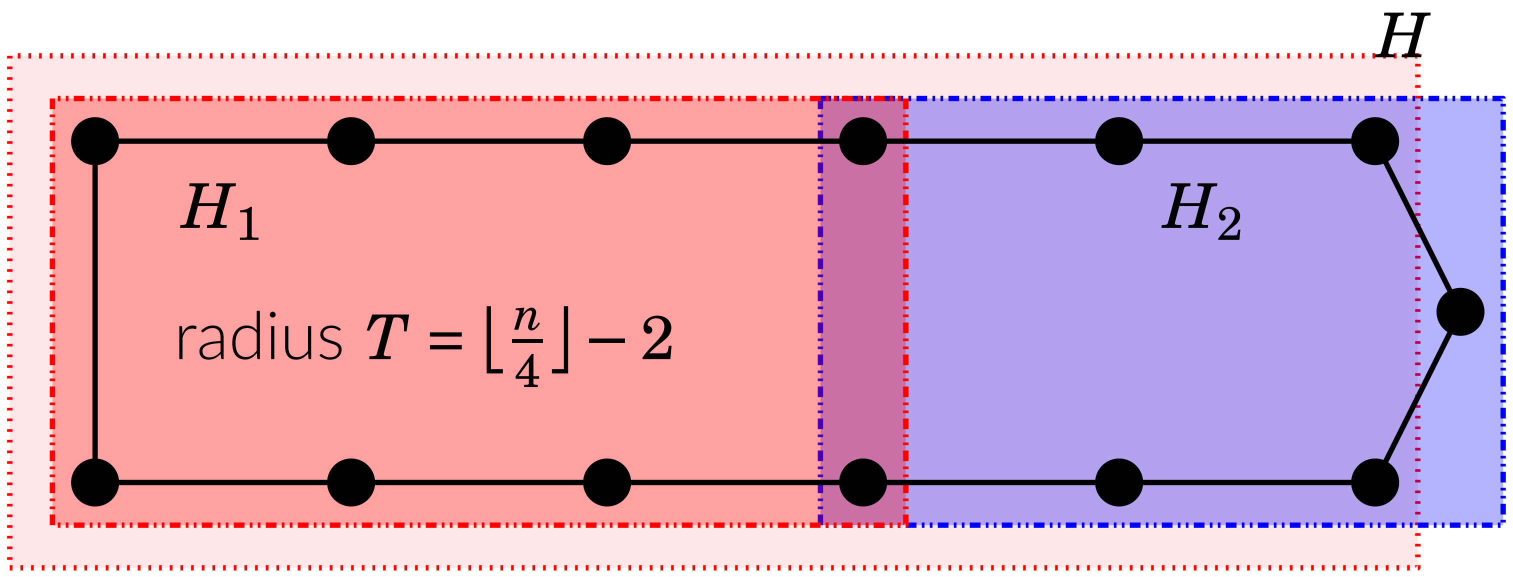

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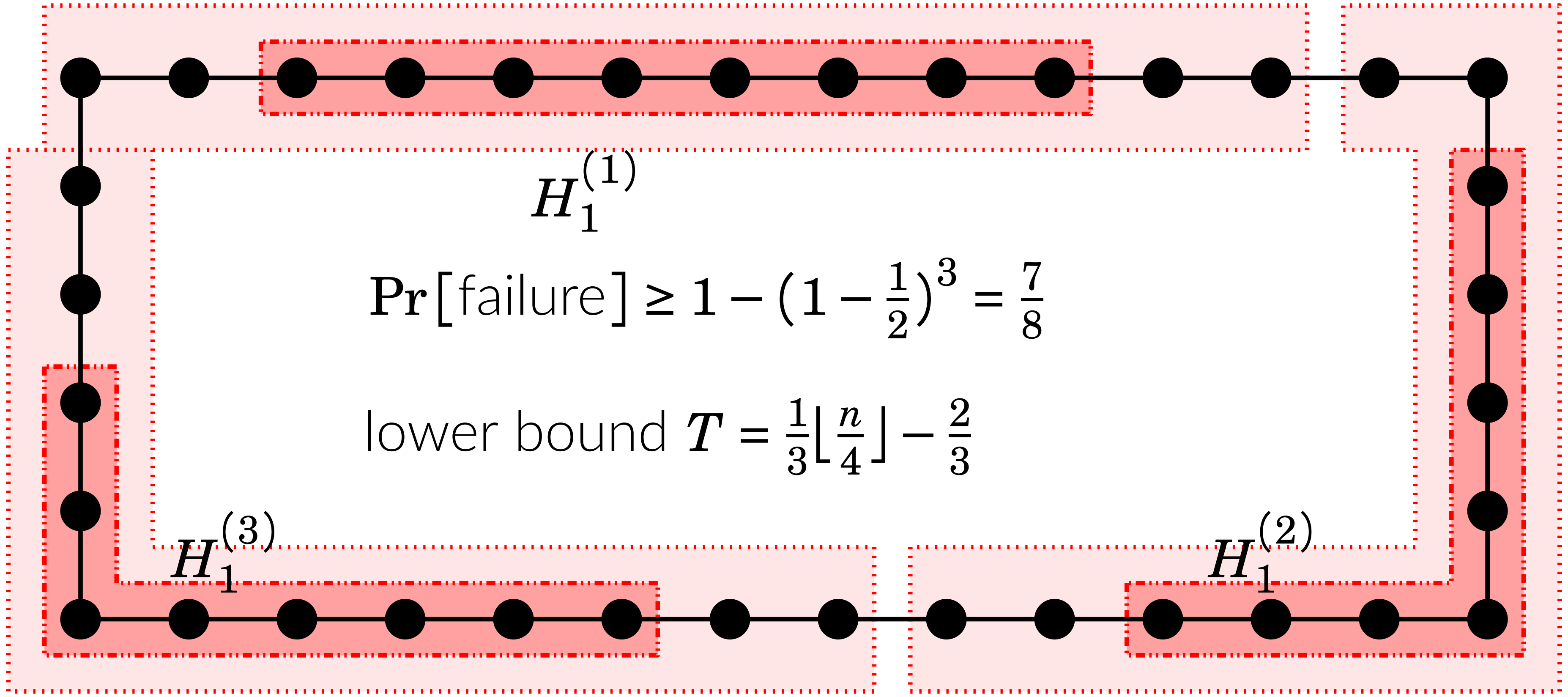

Problems in non-signaling: cloning and dependency

The **non-signaling** model:

- produces non-signaling outcomes

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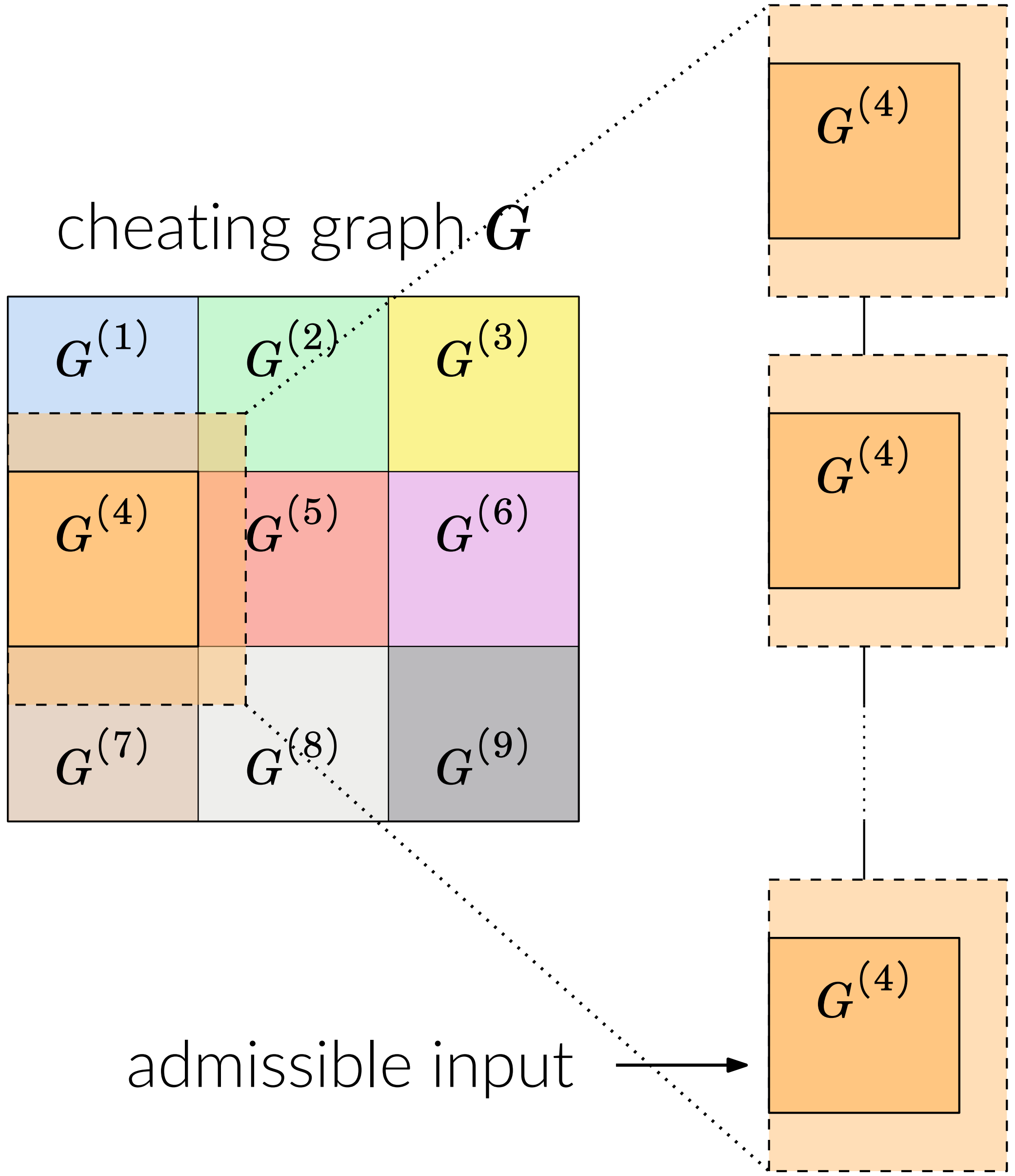

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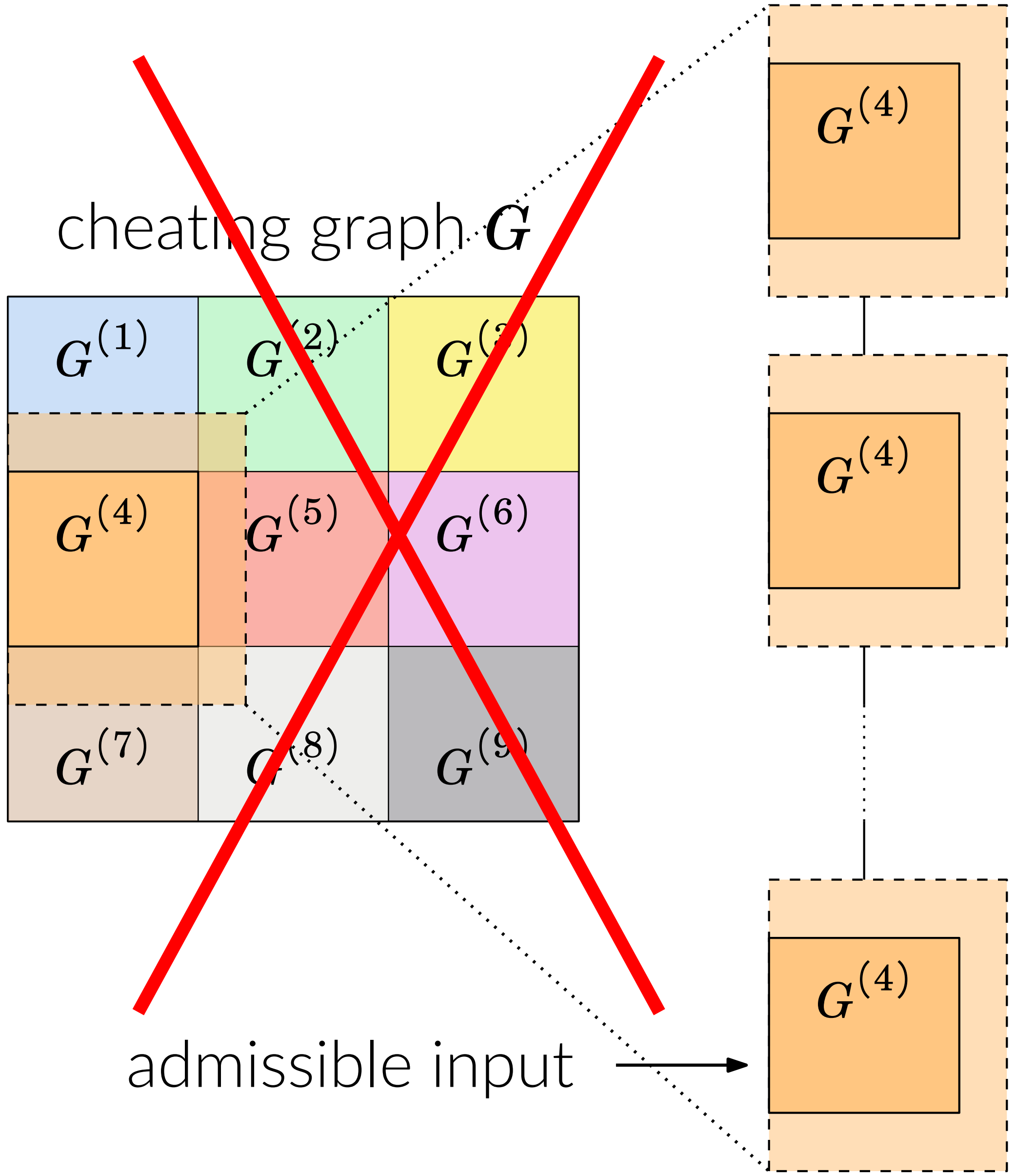

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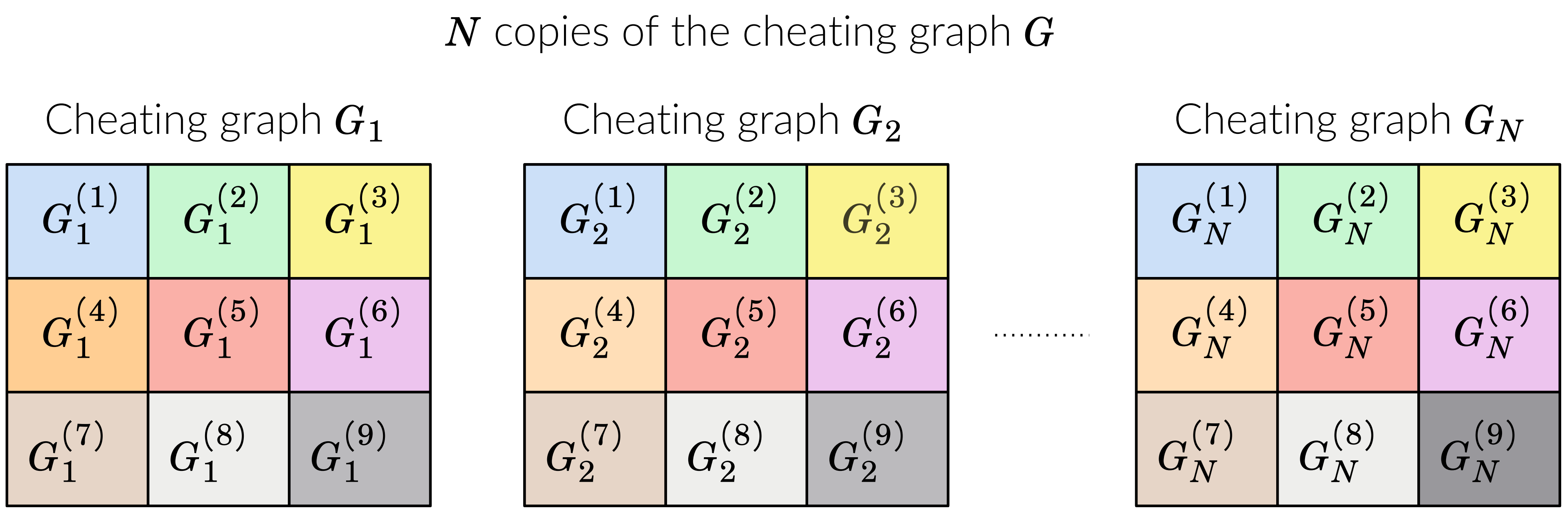

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- Addressing no-cloning
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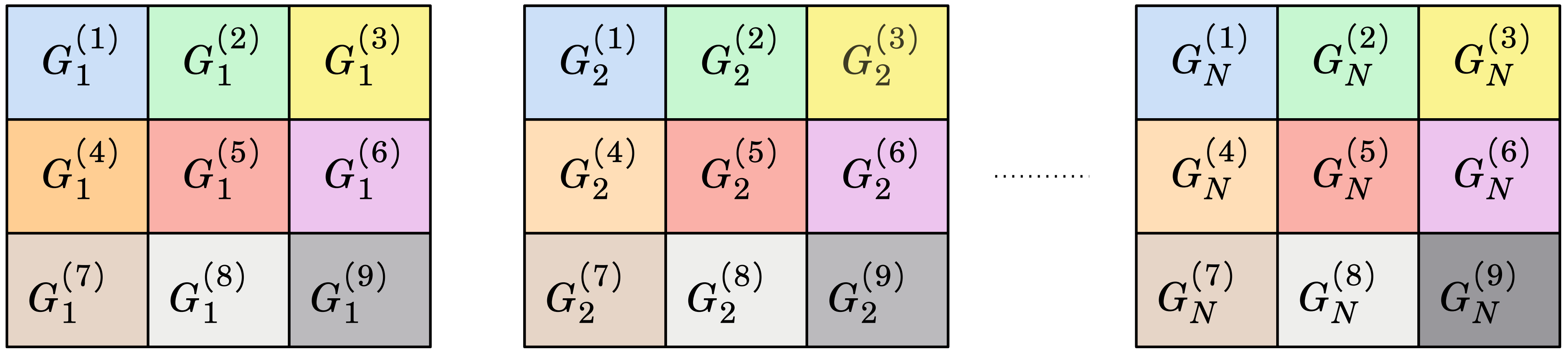
  - consider all nodes, “rearrange” edges

$\boldsymbol{N}$ copies of the cheating graph $\boldsymbol{G}$

Cheating graph $\boldsymbol{G_1}$ Cheating graph $\boldsymbol{G_2}$ Cheating graph $\boldsymbol{G_N}$



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Cheating graph $\boldsymbol{G_2}$

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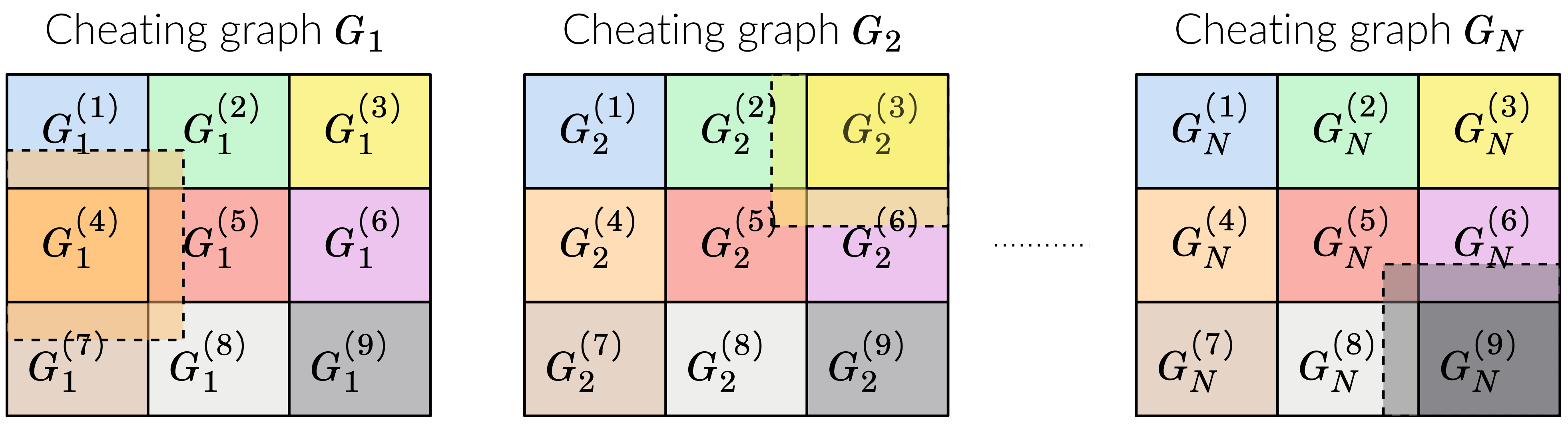

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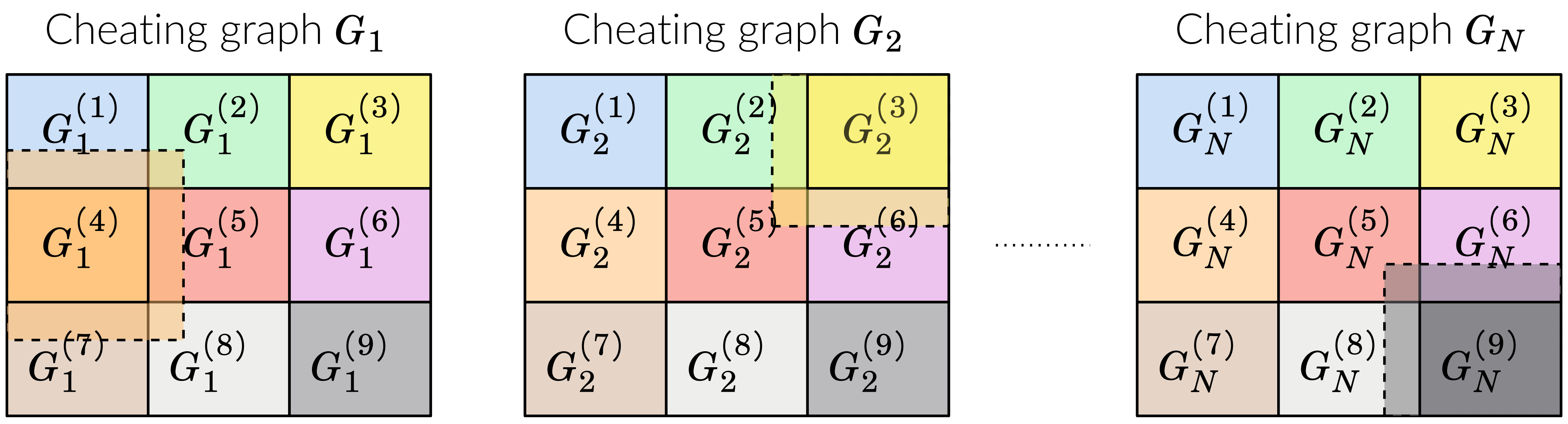

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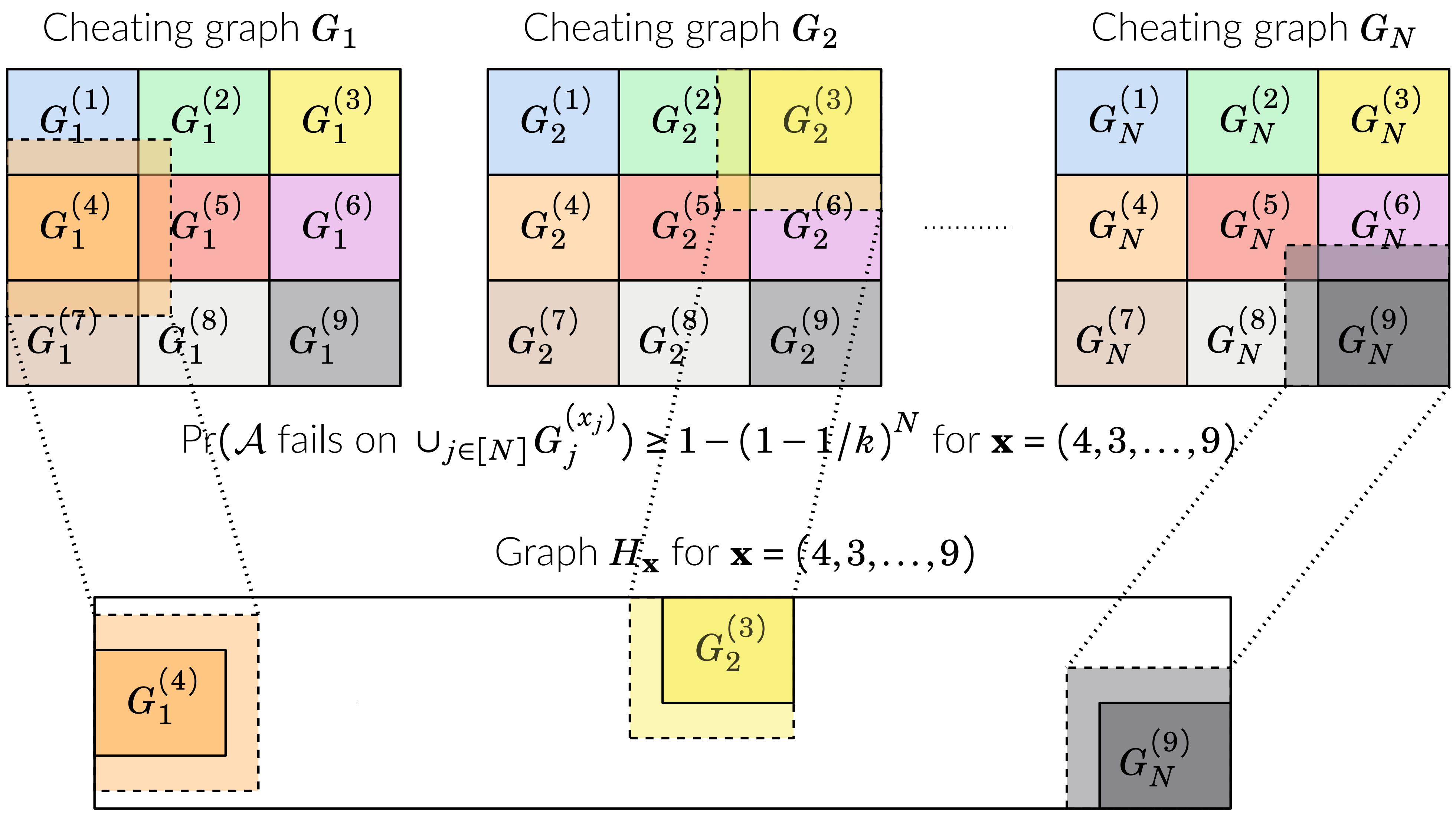

$\Pr(\mathcal{A} \text{ fails on } H_{\mathbf{x}}) \geq 1-(1-1/k)^N$