Balanced Allocations: A refined drift theorem

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Outline

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- Balanced allocations (background and some highlights)
- The exponential and hyperbolic cosine potential functions
- The proof of the drift theorem
- The refinement and its applications
- Open problems

Balanced allocations: Background

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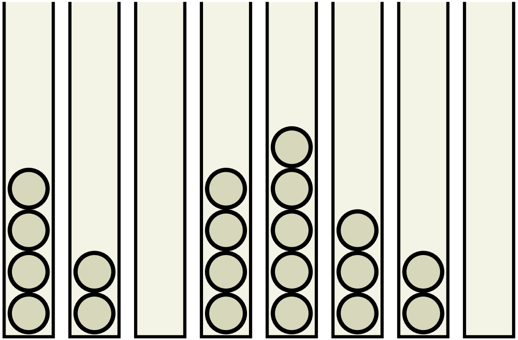

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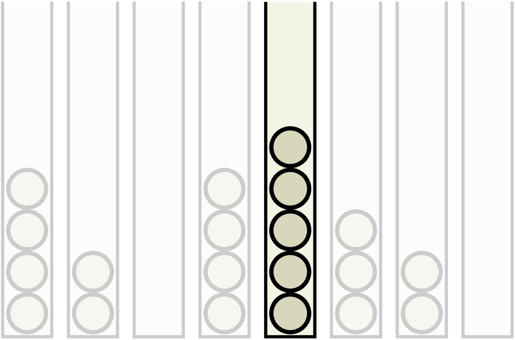

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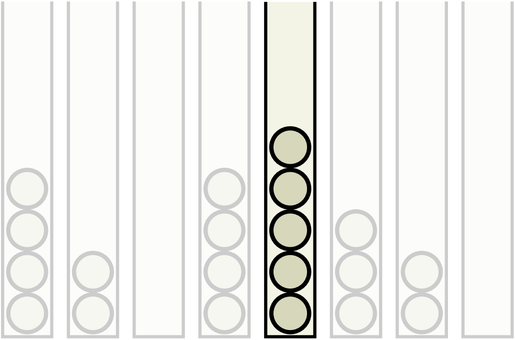

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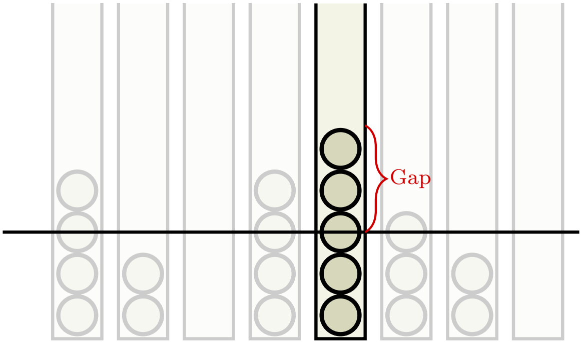

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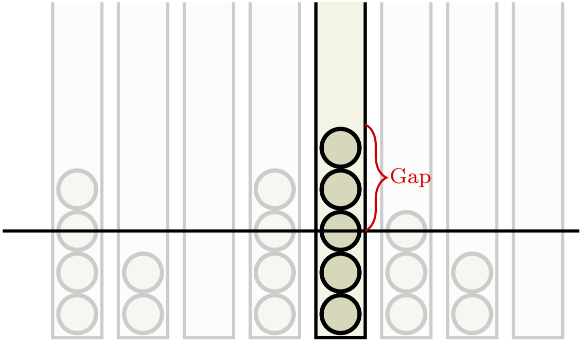

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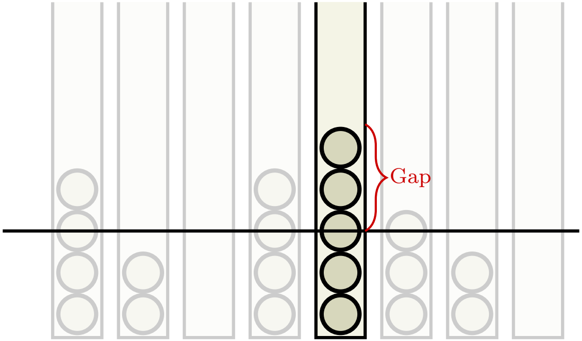

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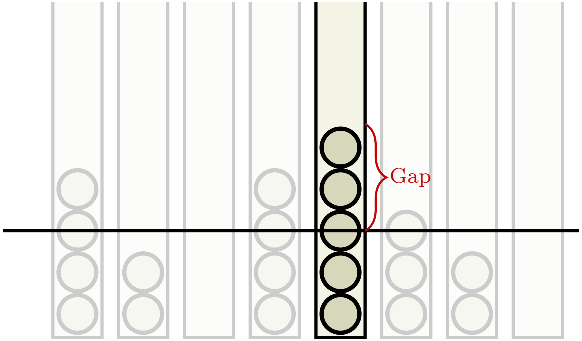

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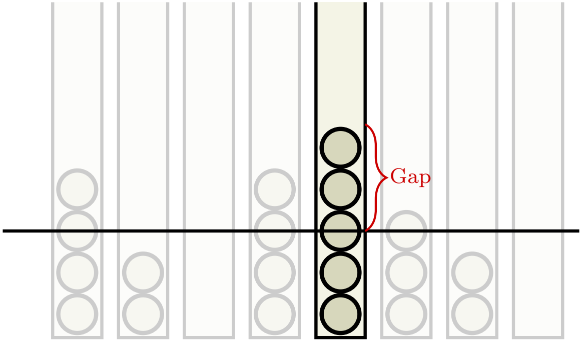

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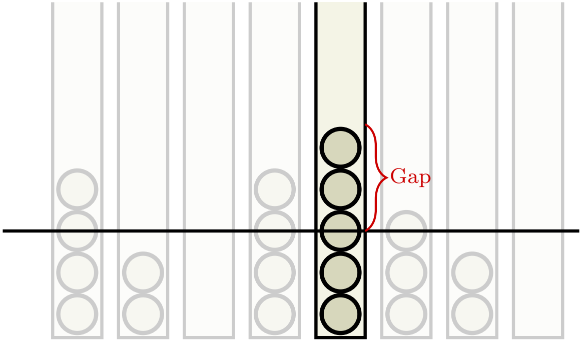

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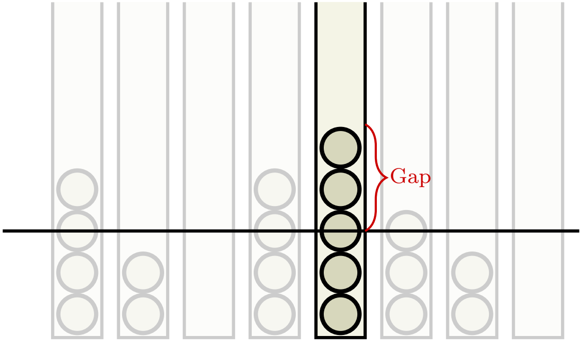

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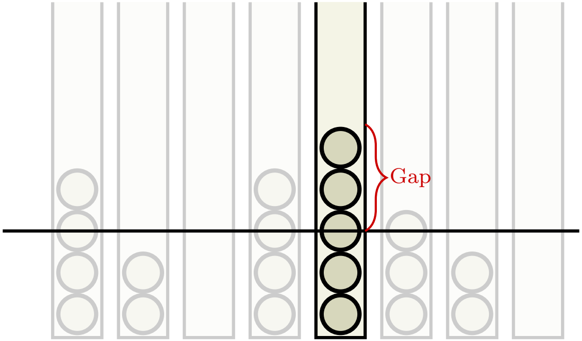

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Meaning with probability
at least $1 - n^{-c}$ for constant $c > 0$.

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Question: Why *variants* and not vanilla TWO-CHOICE?

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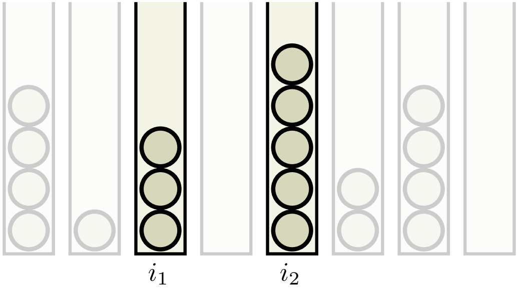

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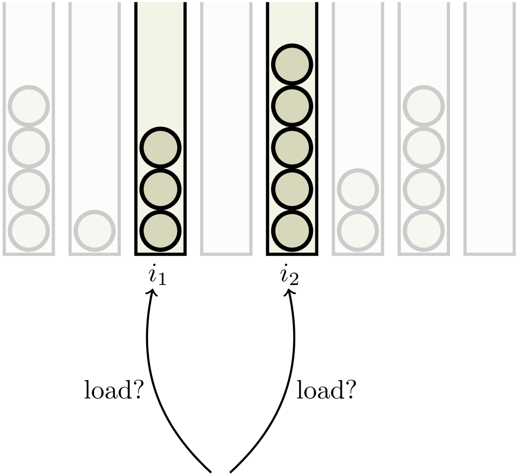

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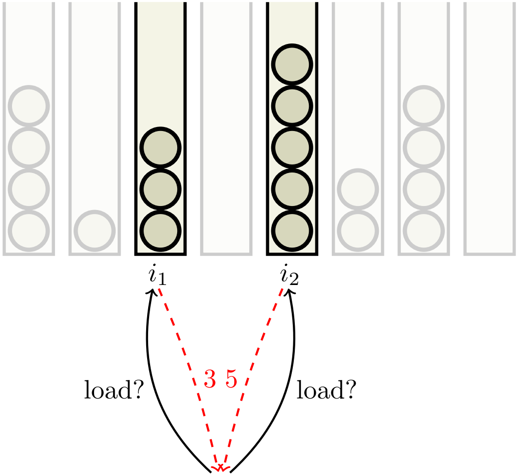

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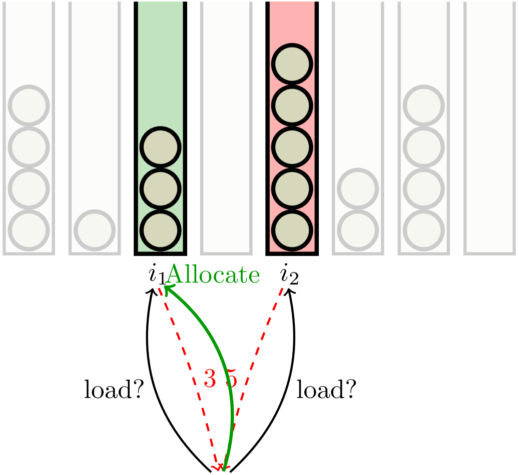

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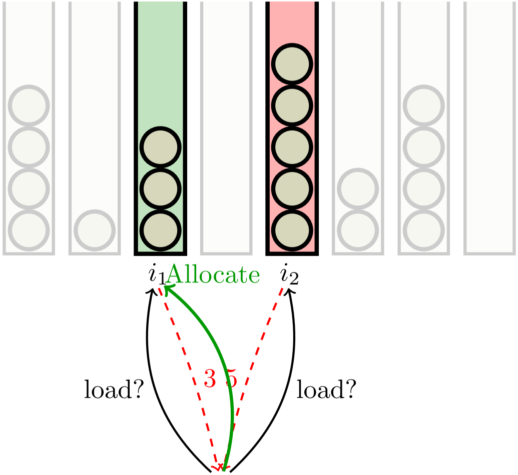

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- In the *queuing setting*, Whitt [Whi86] remarks:

  > *We have shown that several natural selection rules are not optimal in various situations, but we have not identified any optimal rules. Identifying optimal rules in these situations would obviously be interesting, but appears to be difficult.*

An example of a variant of Two-Choice

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Parameter: A *mixing factor* $\beta \in (0,1]$.

Iteration: For each $t \geq 0$, with probability β allocate one ball via the Two-Choice process, otherwise allocate one ball via the One-Choice process.

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- In the heavily-loaded case, [PTW15] proved that the gap is w.h.p. $\mathcal{O}\left(\frac{\log n}{\beta}\right)$ for any $\beta \in (0,1]$.
- It has been used to analyze *population protocols* [AAG18, AGR21], *distributed data structures* [ABK$^+$18, AKLN17, Nad21] and *online carpooling* [GKKS20].

Question: Why choose a $\beta < 1$?

b-BATCHED **Setting**

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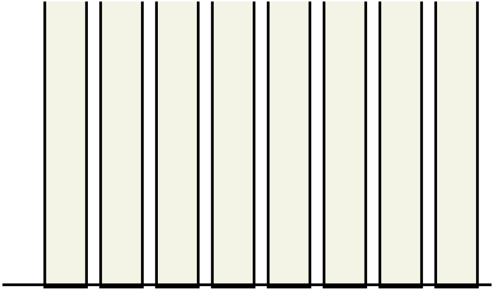

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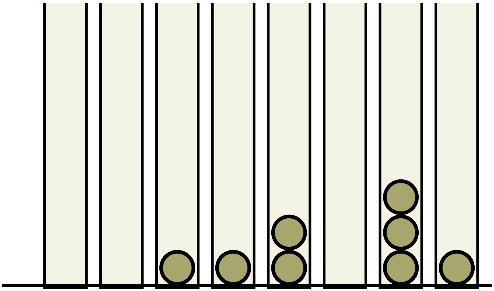

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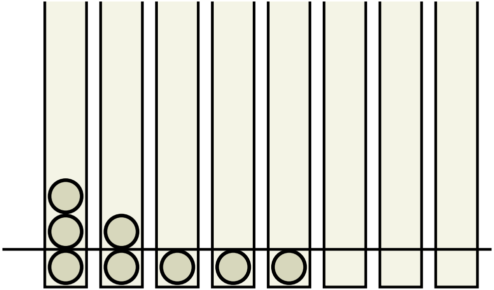

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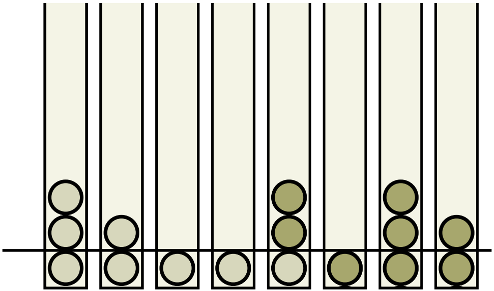

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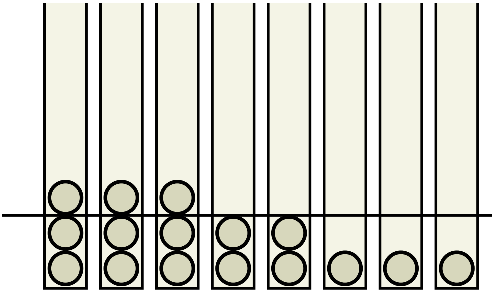

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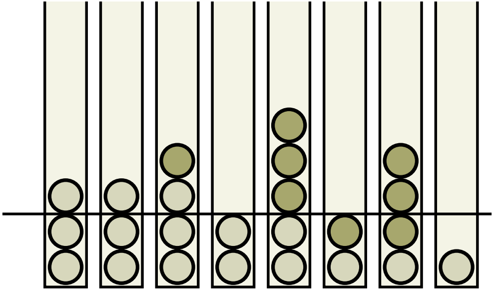

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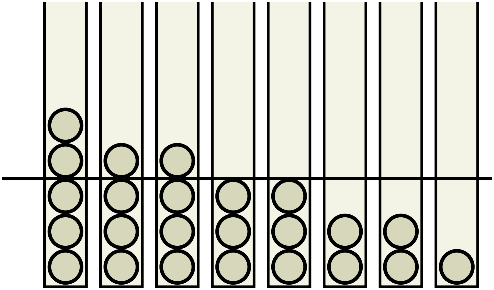

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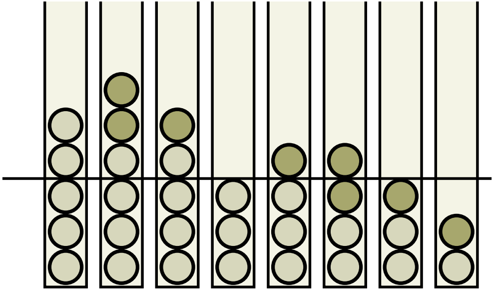

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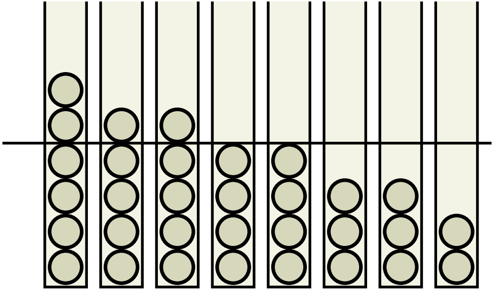

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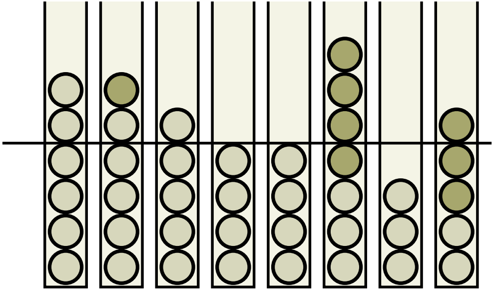

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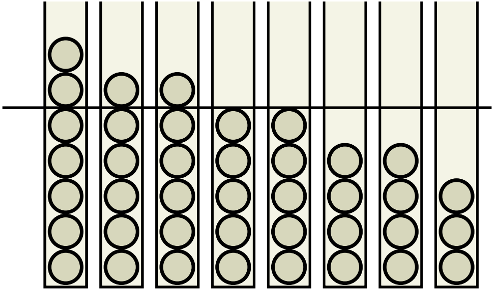

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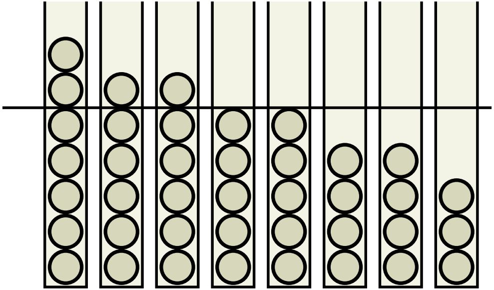

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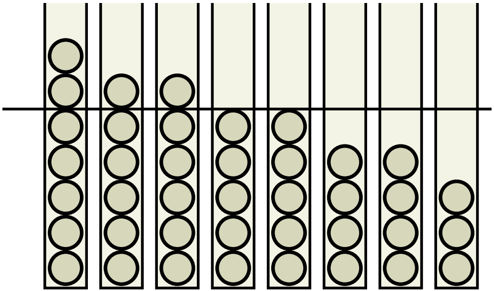

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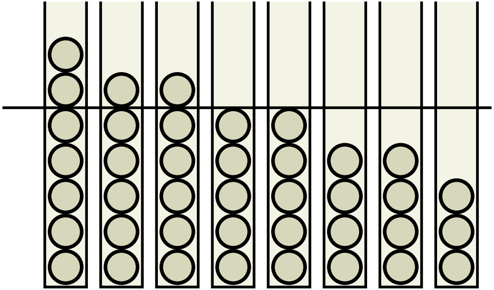

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A closer look at a single batch

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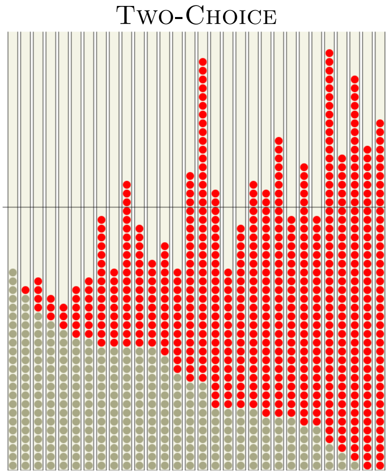

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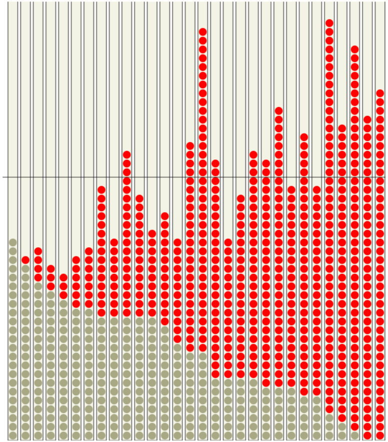

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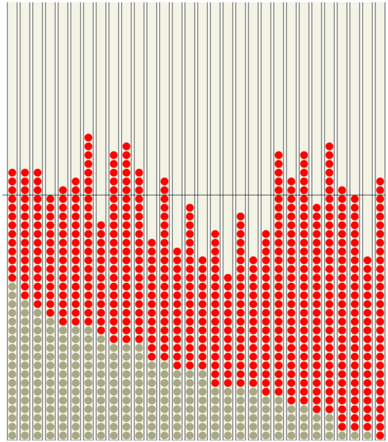

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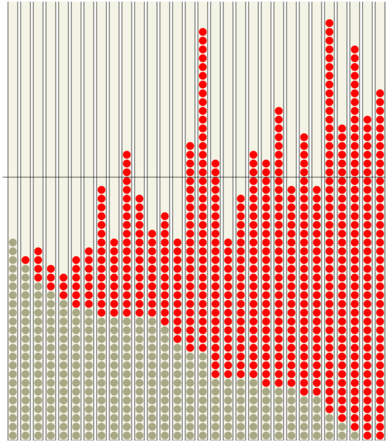

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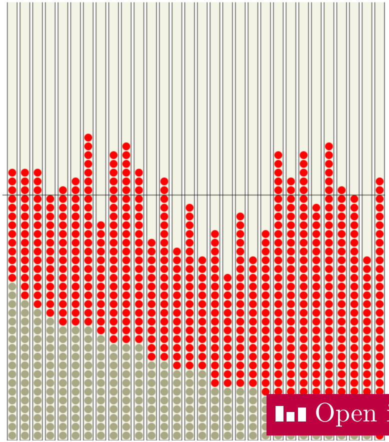

Open in Visualiser.

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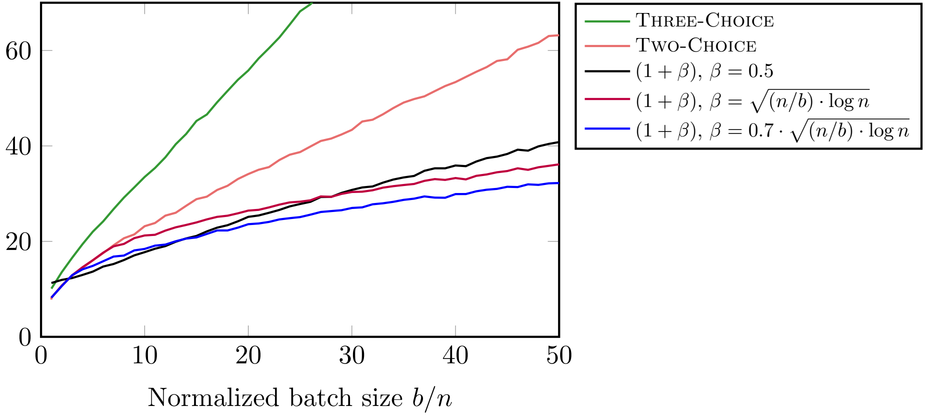
Empirical results for different processes



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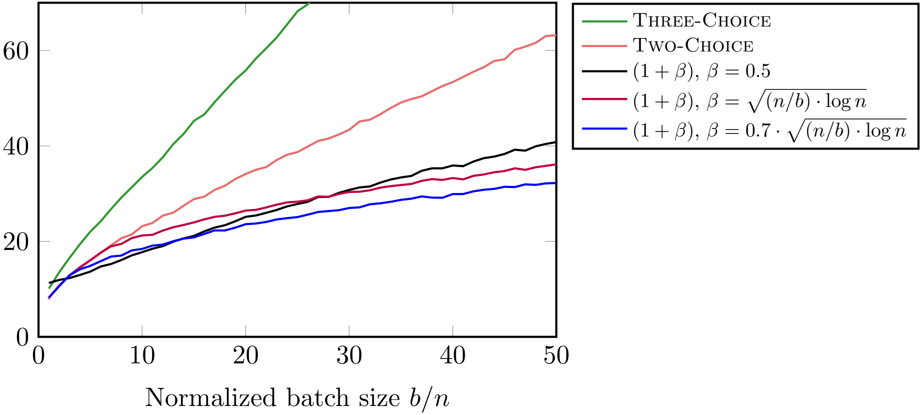

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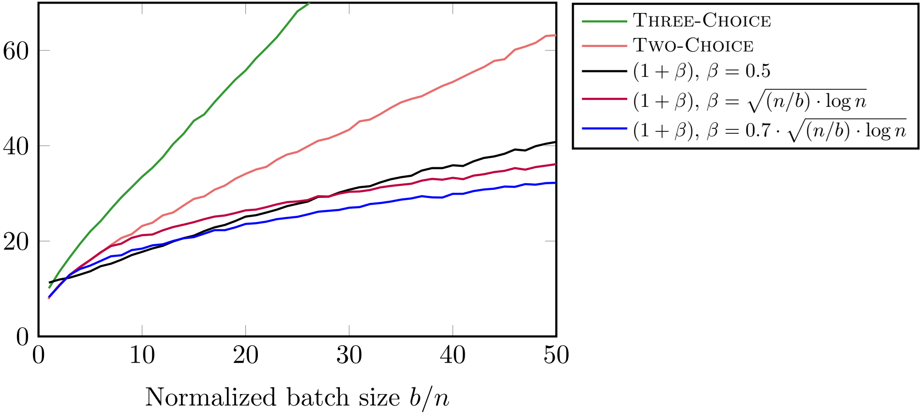

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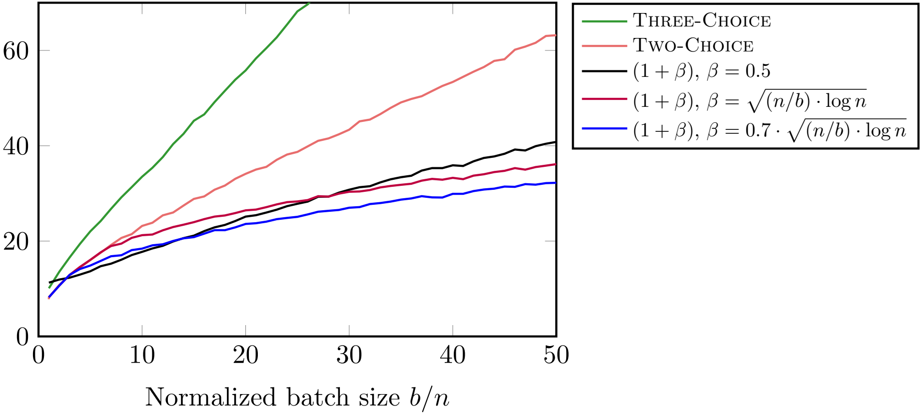

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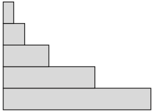
Potential functions

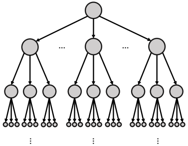
Techniques for analyzing balanced allocations

Layered induction



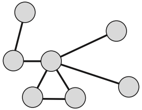
Two-Choice, Memory

Witness trees



Two-Choice, parallel allocations

Graphical processes



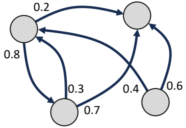
Two-Choice

Poissonisation

$$X_i \sim \mathsf{Poi}(\tfrac{m}{n})$$

Unweighted, time-independent

Markov chains



Some weights, b-Batched, heterogeneous sampling

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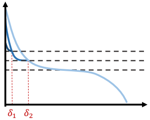

weights, b-Batched, outdated info, noise graphical, heterogeneous sampling

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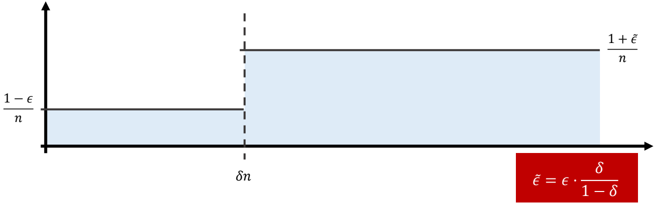

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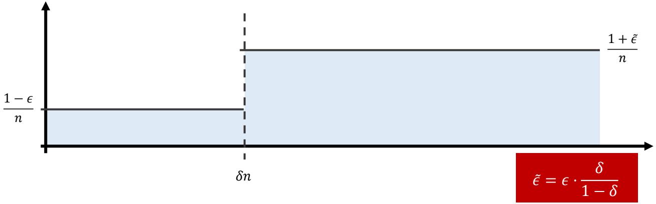

- Our main aim will be to derive the w.h.p. $\mathcal{O}((\log n)/\epsilon)$ gap, for any $\epsilon \in (0,1)$.

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Question: How can we prove this?

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- Then, applying *Markov's inequality* we get that w.h.p. $\mathbf{E}\left[\,\Phi^t\,\right] = \text{poly}(n)$.

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Applies also to weights $\mathcal{W}$ with unit expectation and finite MGF, i.e., $e^{\alpha \mathcal{W}} \leq 1 + \alpha + \alpha^2 \cdot S$.

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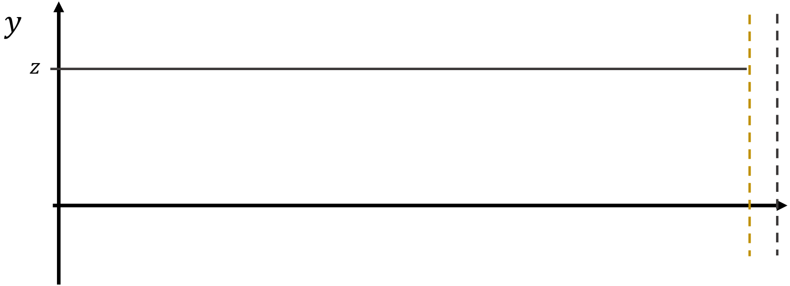

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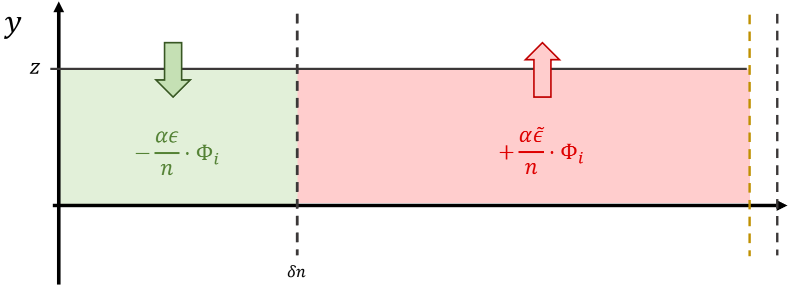

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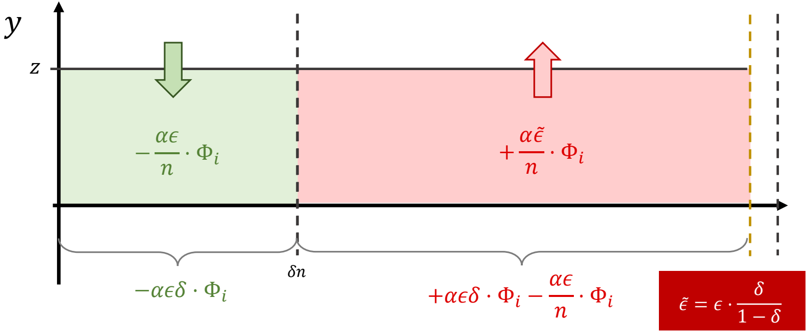

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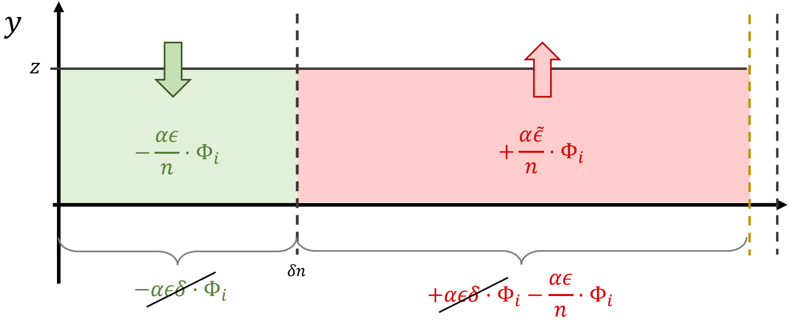

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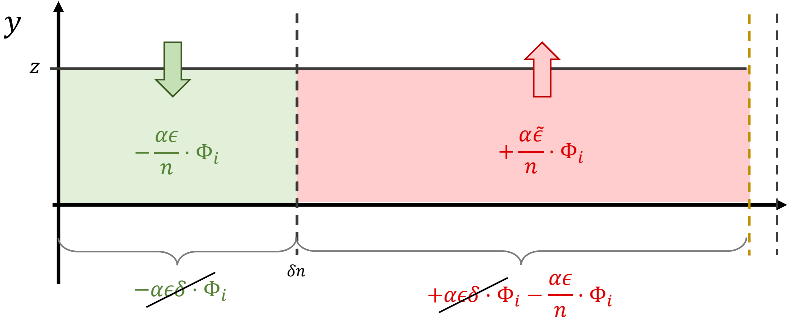

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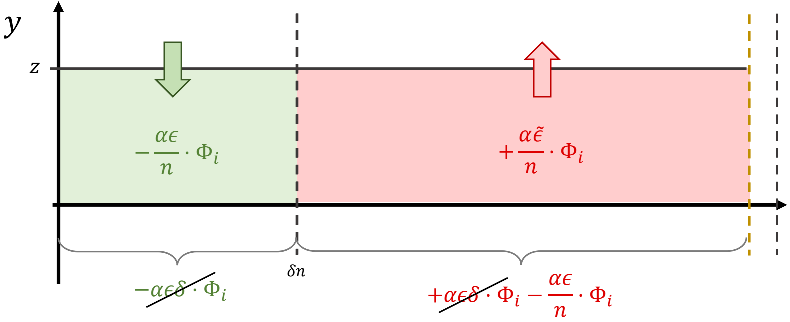

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- The *hyperbolic cosine potential* [PTW15, Spe77] is defined as

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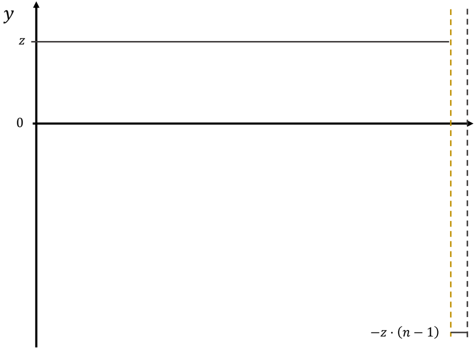

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Drift Theorem

Theorem ([PTW15, Section 2])

Consider any process with *non-decreasing* allocation vector p which is ϵ-biased for some $\epsilon \in (0,1)$ and some constant δ, in the setting with weights sampled from a distribution with finite MGF. Then, for $\Gamma := \Gamma(\alpha)$ with $\alpha := \Theta(\epsilon)$, for any step $t \geq 0$,

$$\mathbf{E}\left[\, \Delta\Gamma^{t+1} \,\middle|\, \mathfrak{F}^t \,\right] \leq -\Gamma^t \cdot \frac{\alpha\epsilon}{4n} + \mathrm{poly}(1/\epsilon),$$

and

$$\mathbf{E}\left[\, \Gamma^t \,\right] \leq n \cdot \mathrm{poly}(1/\epsilon).$$

Refined Drift Theorem

Theorem ([LS22, Corollary 3.2])

Consider any process and a *probability vector* p being ϵ-biased for some $\epsilon \in (0,1)$ and some constant δ. Further assume that it satisfies for some $K > 0$ and for any $t \geq 0$,

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Refined Drift Theorem

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Set	Load	Index	r_i	Dominant Contribution
Good overloaded $\mathcal{G}_+$	$y_i \geq 0$	$i \leq \delta n$	$\frac{1-\epsilon}{n}$	$-\Phi_i \cdot \frac{\alpha\epsilon}{n} + \Psi_i \cdot \frac{\alpha\epsilon}{n}$
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Good underloaded $\mathcal{G}_-$	$y_i < 0$	$i > \delta n$	$\frac{1+\epsilon}{n}$	$+\Phi_i \cdot \frac{\alpha\epsilon}{n} - \Psi_i \cdot \frac{\alpha\epsilon}{n}$
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- **Key Observation 3:** For overloaded bins $\Psi_i^t \leq 1$ and for underloaded bins $\Phi_i^t \leq 1$,
$\rightsquigarrow$ their contribution is $\mathcal{O}(\alpha\epsilon)$.

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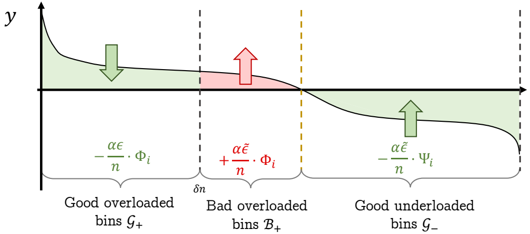

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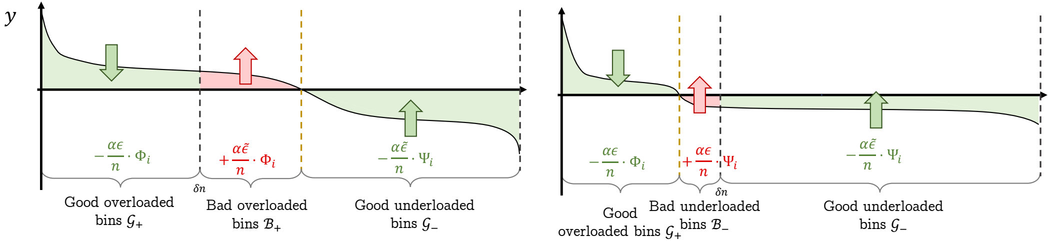

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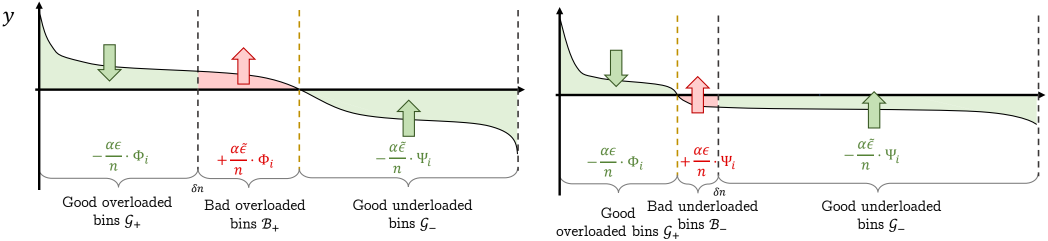

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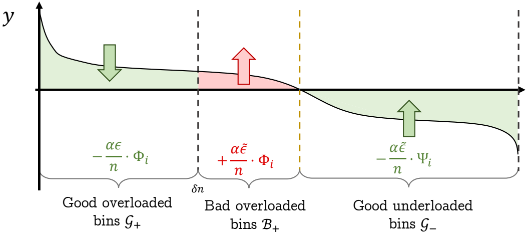

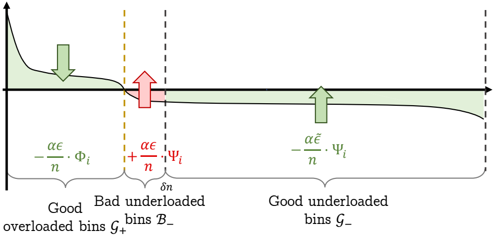

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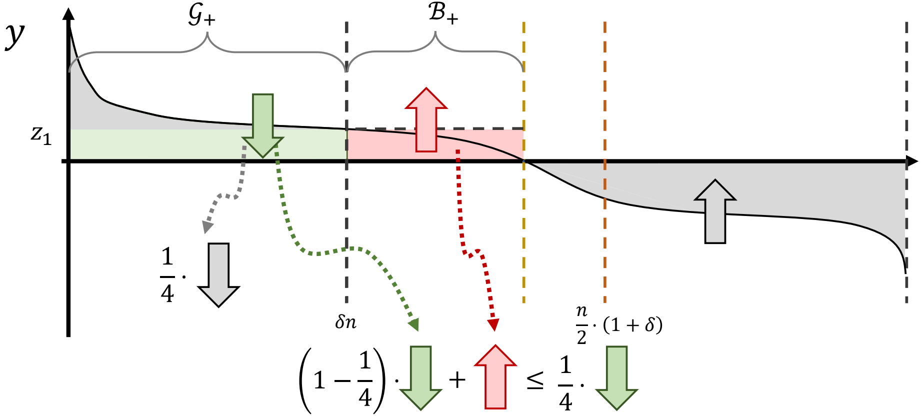

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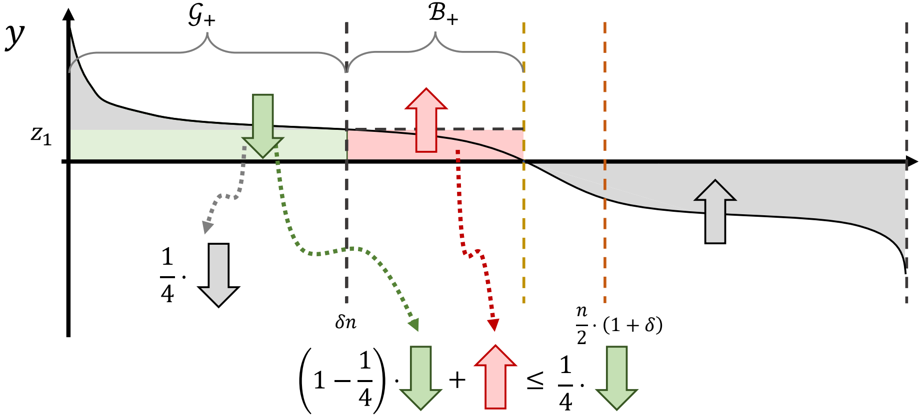

- As with the exponential potential, we counteract the bad bins with a fraction of the decrease of the overloaded good bins. *All* underloaded bins are good.

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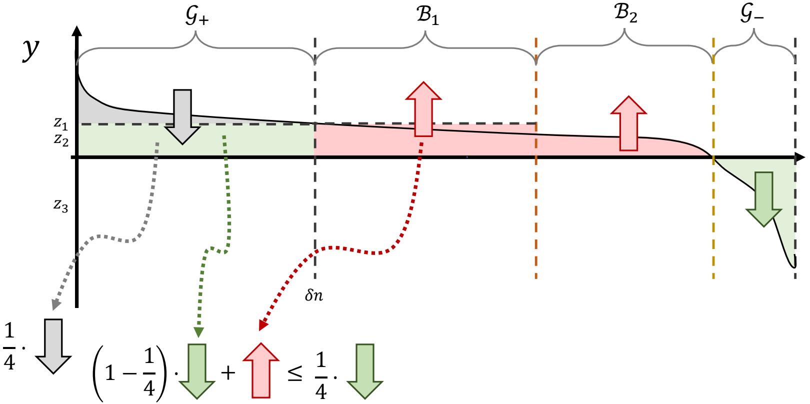

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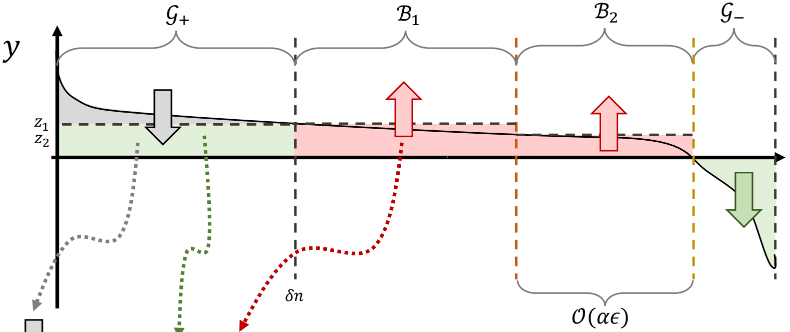

$\frac{1}{4} \cdot \Downarrow \quad \left(1 - \frac{1}{4}\right) \cdot \Downarrow + \Uparrow \leq \frac{1}{4} \cdot \Downarrow$

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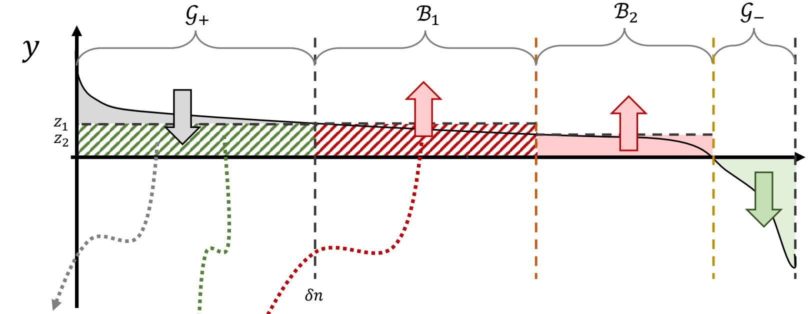

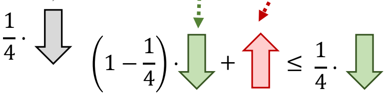

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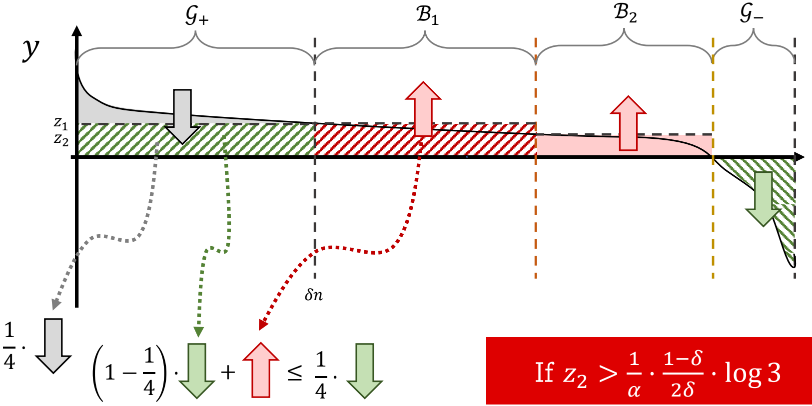

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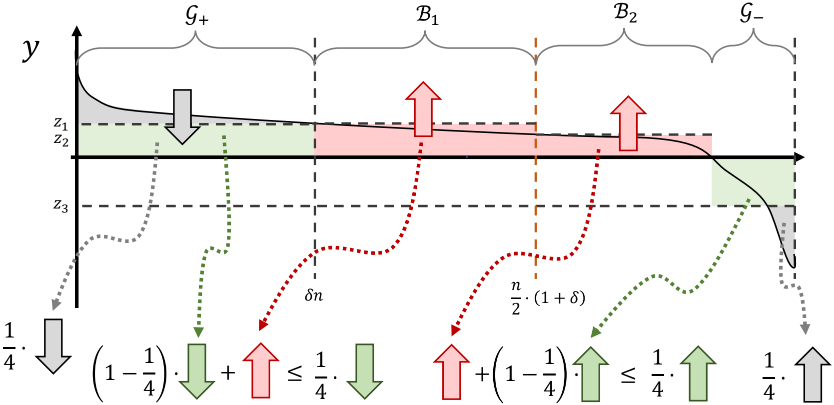

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5. Use the decrease of the underload potential to counteract the increase of bad bins.

The drift theorem

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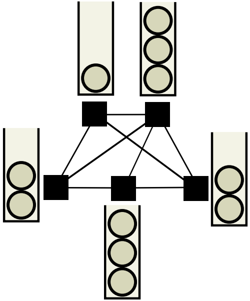

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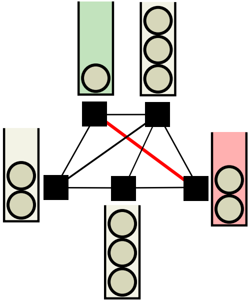

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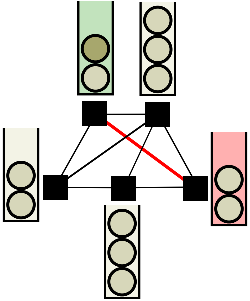

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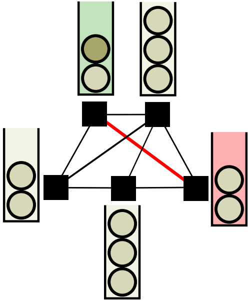

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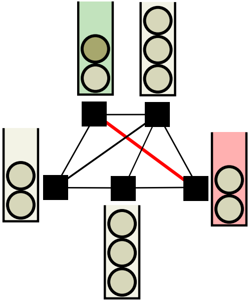

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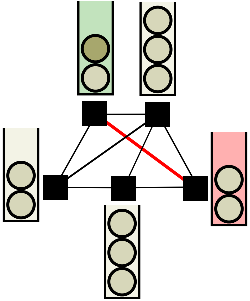

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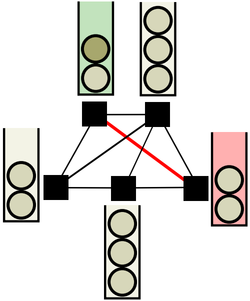

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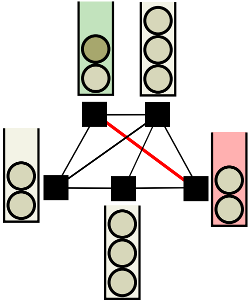

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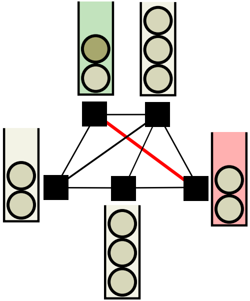

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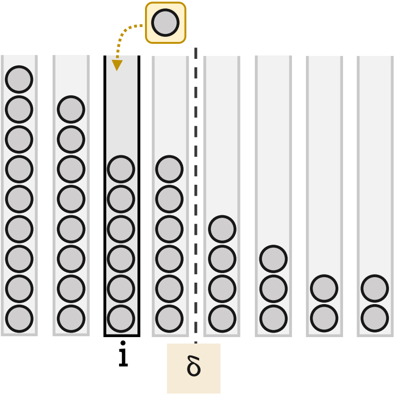

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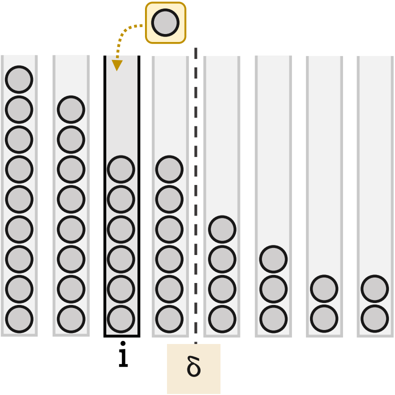

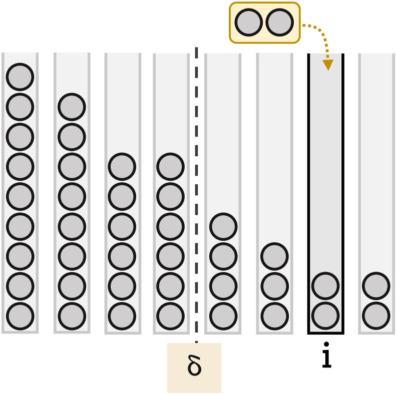

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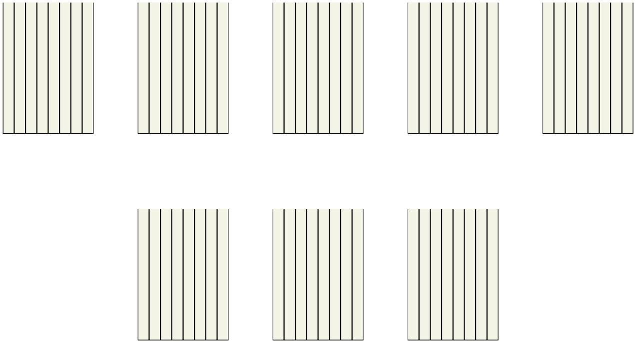
- Can we use a *single potential* to prove sublogarithmic bounds (e.g., the $\log_2 \log n + \Theta(1)$ bound for TWO-CHOICE)?

Questions?



More visualisations: `dimitrioslos.com/phdthesis`

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