Exact and approximate solutions for the Constrained Shortest Path Tour Problem

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Overview

1. Introduction
2. Computational complexity
3. Mathematical formulation
4. Exact approach
5. Heuristic approach
6. Some experimental results

Introduction

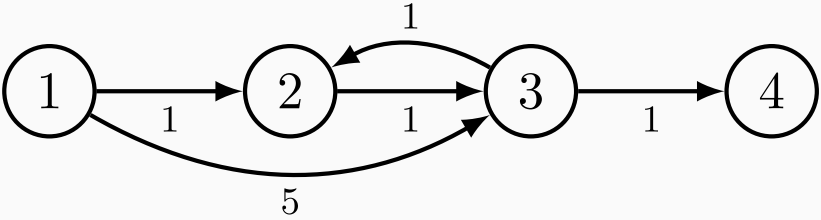
The Shortest Path Tour Problem

Let be $G = (V, A)$ be a directed graph, where

- V is a set of n nodes;
- A is a set of m arcs;
- C: $A \to \mathbb{R}^+ \cup \{0\}$

Let $T_1, \ldots, T_N$ be disjoint subsets of V. The Shortest Path Tour Problem (SPTP) consists in finding a single-origin single-destination shortest path by ensuring that at least one node of each node subset $T_1, \ldots, T_N$ is involved according to the sequence wherewith the subsets are ordered.

The Shortest Path Tour Problem

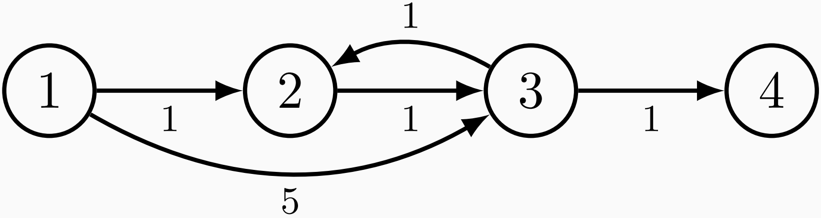


$$T_1 = \{1\}, T_2 = \{3\}, T_3 = \{2\}, T_4 = \{4\}$$

$$P_T = \{1, 2, 3, 2, 3, 4\} \quad c(P_T) = 5$$

The Constrained Shortest Path Tour Problem

Additional constraint: the path does not include repeated arcs.



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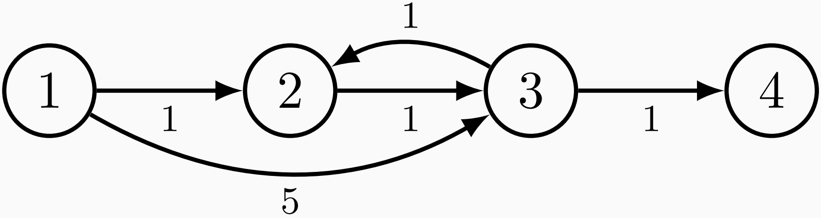

$T_1 = \{1\}, T_2 = \{3\}, T_3 = \{2\}, T_4 = \{4\}$

$P_T = \{1, 3, 2, 3, 4\} \quad c(P_T) = 8$

Complexity

Hardness

Theorem

*The CSPTP is **NP**-hard.*

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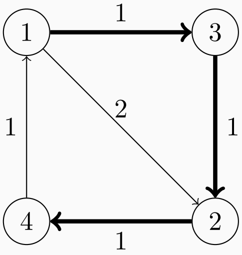

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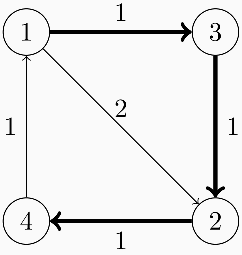

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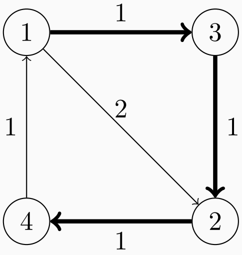

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- for each node $i \in V$,
    - insert in V' nodes i^- and i^+;

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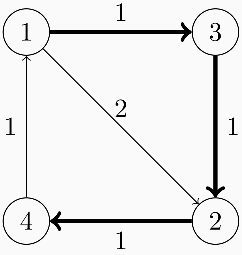

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- for each node $i \in V$,
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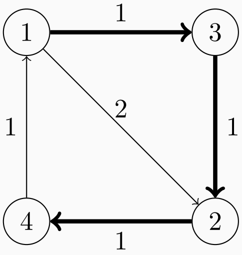

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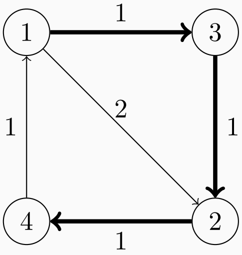

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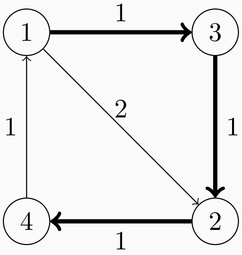

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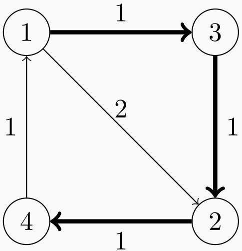

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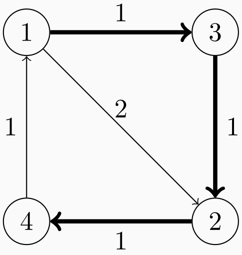

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- for each arc $(i, j) \in A$ and for each $k = 2, \ldots, n$,
  - insert in V' node ij^k;
  - insert in T_k node ij^k;
  - insert in A' arc (i^+, ij^k) with cost c_{ij} and arc (ij^k, j^-) with cost 0;
- set $T_1 = \{s^-\}$ and $T_{n+1} = \{d^+\}$.

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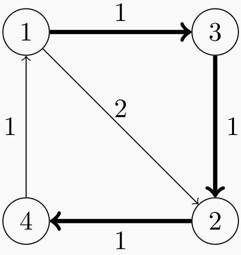

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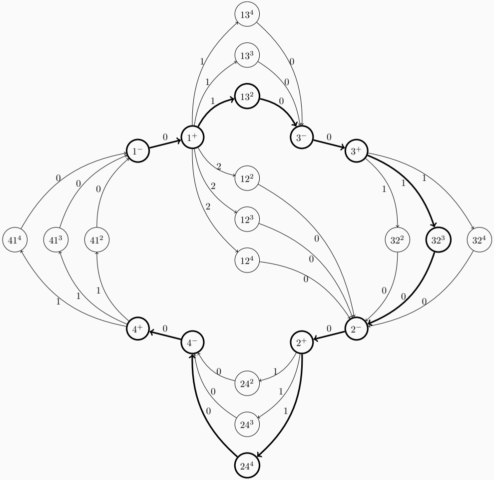

Path existence lemma

Lemma

There exists a path $P = i_1, i_2, \ldots, i_k$, $k \leq n$, *in*

$$\langle G = (V, A, C), s, d \rangle ,$$

if and only if in

$$\left\langle G' = (V', A', C'), s^-, d^+, \{T_h\}_{h=1,\ldots,n+1} \right\rangle$$

there exists a path P' *from* i_1^- *to* i_k^+*, such that*

$$P' = \left\{ \bigoplus_{l=1}^{k-1} \left(i_l^-, i_l^+, {i_l i_{l+1}}^{l+1} \right), i_k^-, i_k^+ \right\}.$$

Path existence lemma (proof)

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- by construction $(i_l^-, i_l^+) \in A'$ for each $l = 1, \ldots, k$;
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$\Leftarrow$ exists P', P is not present.

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- $i_l \notin V$: i_l^- and i_l^+ would not be in V';
- $(i_l, i_{l+1}) \notin A$, then arcs $(i_l^+, {i_l i_{l+1}}^{l+1})$ and $({i_l i_{l+1}}^{l+1}, i_{l+1}^-)$ would not be in A'.

Reduction theorem

Theorem (Ham-Path $<_m^p$ CSPTP)
There exists in G an Hamiltonian path P from s to d with length $L(P)$ if and only if there exists in G' a constrained path tour P' from s^- to d^+ with length $L(P') = L(P)$.

Reduction theorem

$\Rightarrow$ By hypothesis, there exists in G an Hamiltonian path $P = \{ i_1, i_2, \ldots, i_n \}$, where $i_1 = s$ and $i_n = d$.

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$\Leftarrow$ By construction the feasible path tour must be as P', it implies exists a path $P = \{\, i_1, \ldots, i_n \,\}$ in G.

- Suppose P is not Hamiltonian;
- There exist $i_k = i_j$, where $k \neq j$;
- P' must cross twice the arc $(i_k^-, i_k^+) = (i_j^-, i_j^+)$.

Mathematical model

Mathematical model

$$\min \sum_{(i,j)\in A} \sum_{k=1}^{N-1} x_{ij}^k c_{ij}$$

$s.a.$

$$\sum_{j\in FS(i)} x_{ij}^k - \sum_{j\in BS(i)} x_{ji}^k = \begin{cases} y_i & \text{if } i \in T_k, \\ -y_i & \text{if } i \in T_{k+1}, \\ 0, & \text{otherwise.} \end{cases} \quad \forall i \in V, k = 1, \ldots, N-1$$

$$\sum_{i\in T_k} y_i = 1 \qquad \forall k = 1, \ldots, N-1$$

$$\sum_{k=1}^{N-1} x_{ij}^k \leq 1 \qquad \forall (i,j) \in A$$

$$y_s = 1, y_d = 1$$

$$x_{ij}^k \in \{0,1\}, y_i \in \{0,1\}$$

Exact approach

Main ideas

- A path tour is a concatenation of simple paths $T_i \rightsquigarrow T_{i+1}$ for $i = 1, \ldots, N-1$;

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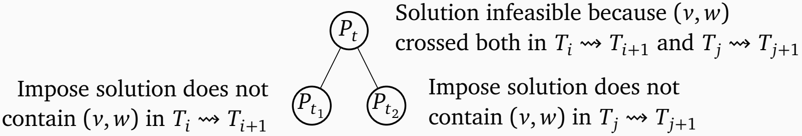


The B&B algorithm

Function B&B($G = \langle V, A, C \rangle, s, d, \{T_i\}_{i=1,\dots,N}$)

  ShortestPaths $\leftarrow$ $\textsc{FloydWarshall}(G)$;

  $x \leftarrow$ DP($V, A, s, \{T_i\}_{i=1,\dots,N},$) ;

  if x *is feasible* **then**

    return $(x, z(x))$

  ...;

The B&B algorithm

Function `B&B(`$G = \langle V, A, C \rangle, s, d, \{T_i\}_{i=1,\dots,N}$`)`

…;

for $i \leftarrow 1$ ***to*** $N - 1$ **do**

  foreach $v \in T_i$ **do**

    foreach $w \in T_{i+1}$ **do**

      Paths[i] $\leftarrow$ Paths[i] $\cup$ {ShortestPaths[v][w]} ;

$Q \leftarrow$ `GenerateNodes`$(x, Paths, [\emptyset]_{i=1}^{N-1})$;

$x^* \leftarrow \mathbf{Nil}; z(x^*) \leftarrow +\infty$;

…;

The B&B algorithm

Function `B&B`($G = \langle V, A, C \rangle$, $s, d, \{T_i\}_{i=1,\dots,N}$)

  …;

  while Q *is not empty* **do**

    $Node \leftarrow$ `Pop`(Q);

    $i \leftarrow Node.index$;

    $A \leftarrow A \setminus Node.costraints[i]$;

    …;

The B&B algorithm

```
Function B&B(G = ⟨V, A, C⟩, s, d, {T_i}_{i=1,...,N})
    ...;
    while Q is not empty do
        ...;
        foreach v ∈ T_i do
            foreach w ∈ T_{i+1} do
                Node.paths[i] ←
                  Node.paths[i] ∪ {Dijkstra(G, v, w)};
        ...;
```

The B&B algorithm

Function `B&B`($G = \langle V, A, C \rangle, s, d, \{T_i\}_{i=1,\dots,N}$)

    …;

    while Q *is not empty* **do**

        …;

        $x \leftarrow$ `DP`$(Node.paths)$;

        $A \leftarrow A \cup Node.costraints[i]$;

        if x *is feasible* **then**

            if $z(x) < z(x^*)$ **then**

                $x^* \leftarrow x, z(x^*) \leftarrow z(x)$;

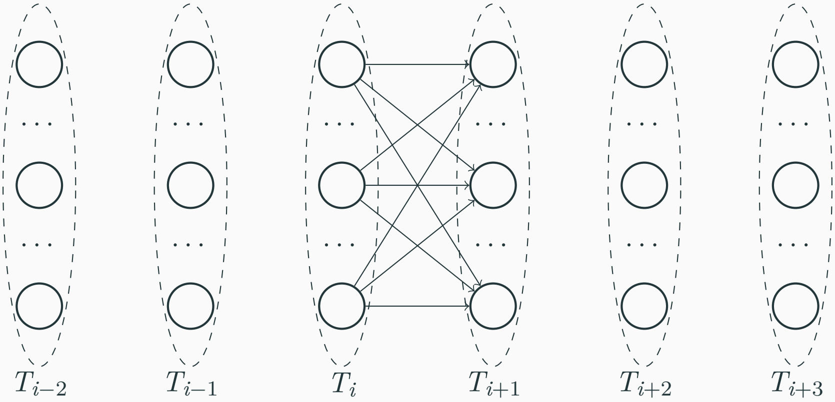
        else if $z(x) < z(x^*)$ **then**

            $Q \leftarrow$
             $Q \cup$`GenerateNodes`$(x, Node.paths, Node.constraints)$;

    return $(x^*, z(x^*))$;

GRASP

Construction phase



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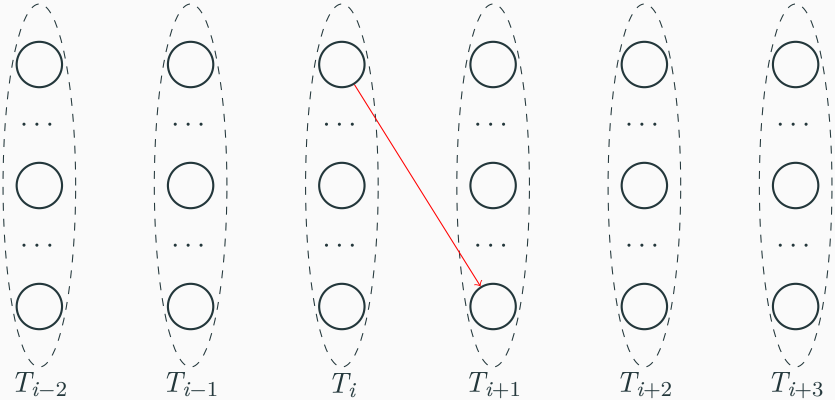

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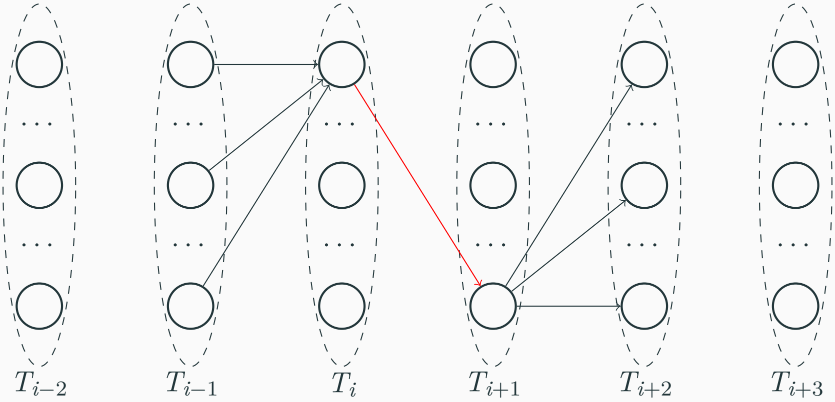

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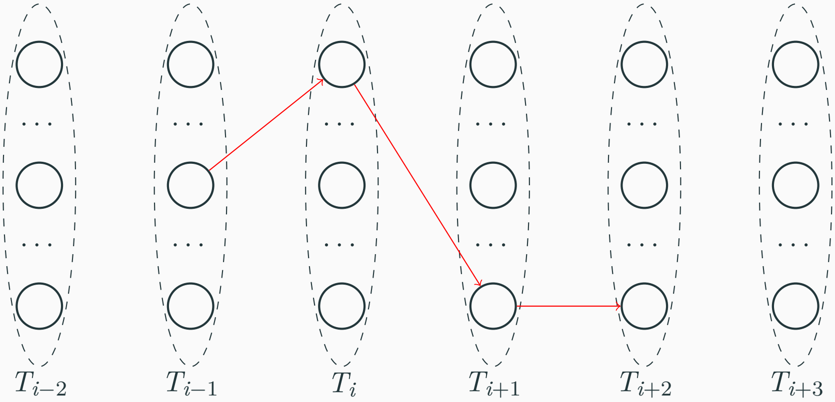

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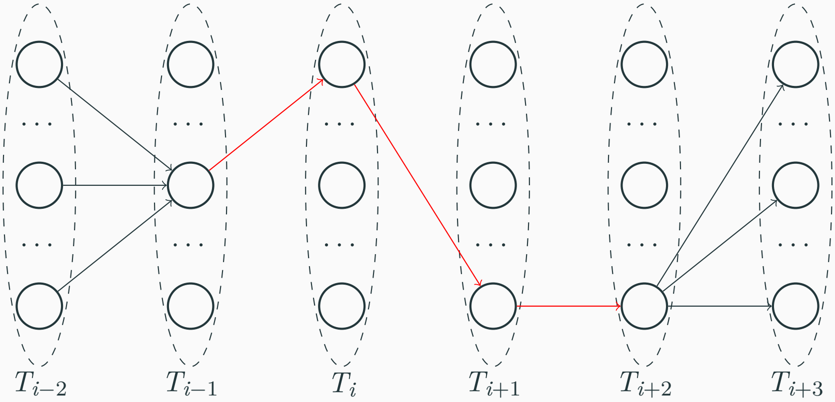
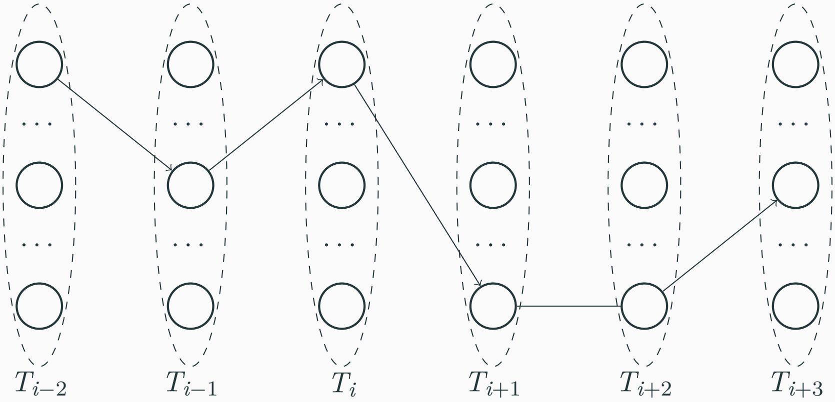


Local search



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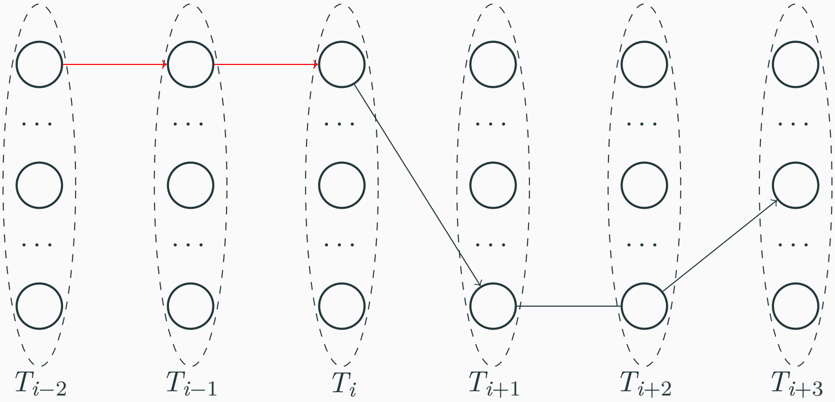

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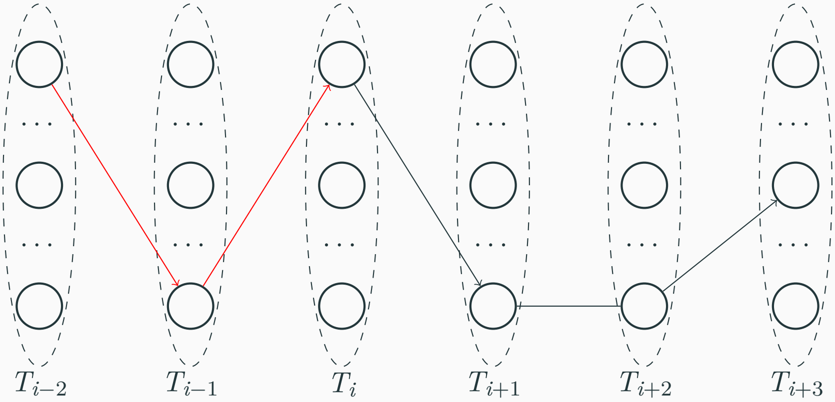

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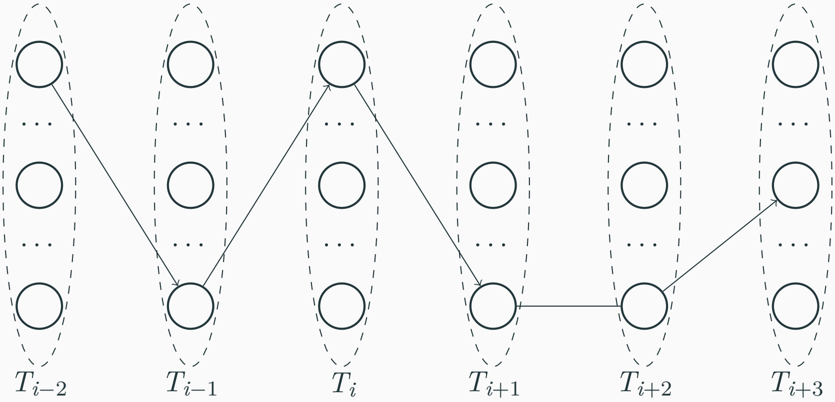

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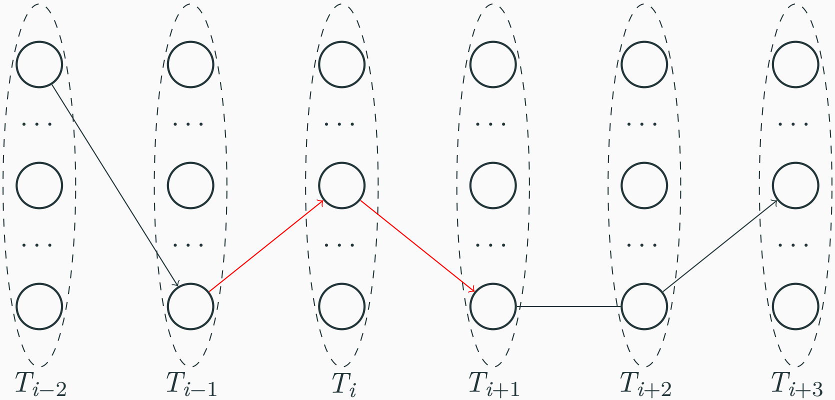

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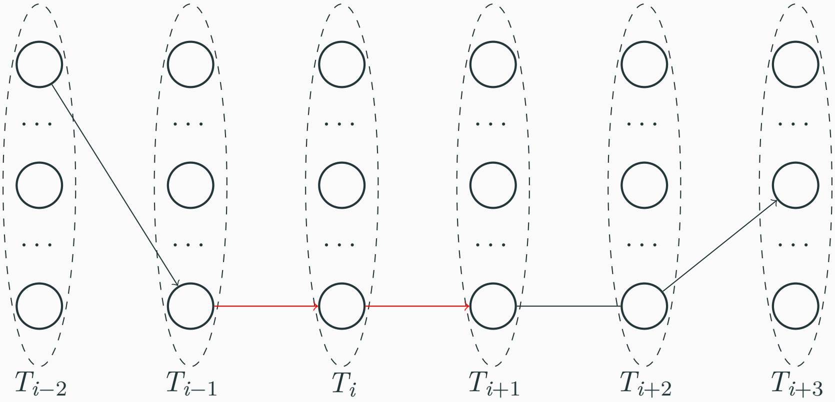


Experimental results

Experimental settings

Implemented in C++, executed on S.Co.P.E., a cluster of nodes, connected by 10 Gigabit Infiniband technology, each of them with two processors Intel Xeon E5-4610v2@2.30 Ghz.

Instances:

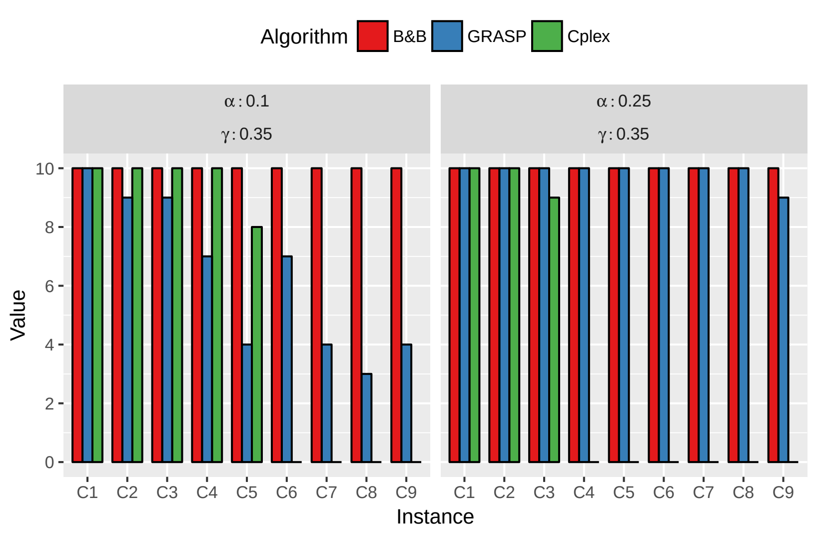
Complete Graphs The number of nodes are $n \in \{ 100, \ldots, 500 \}$ with a step of 50. ($C1, \ldots, C9$)

Random Sparse Graphs $n \in \{250, 500\}$, $m \in \{0.1, 0.2, 0.3, 0.4, 0.5\} \cdot n(n-1)$. (R1, …, R10)

Grid Graphs 5×10, 10×20, 15×30, 5×5, 10×10, 15×15. (G1, …, G6)

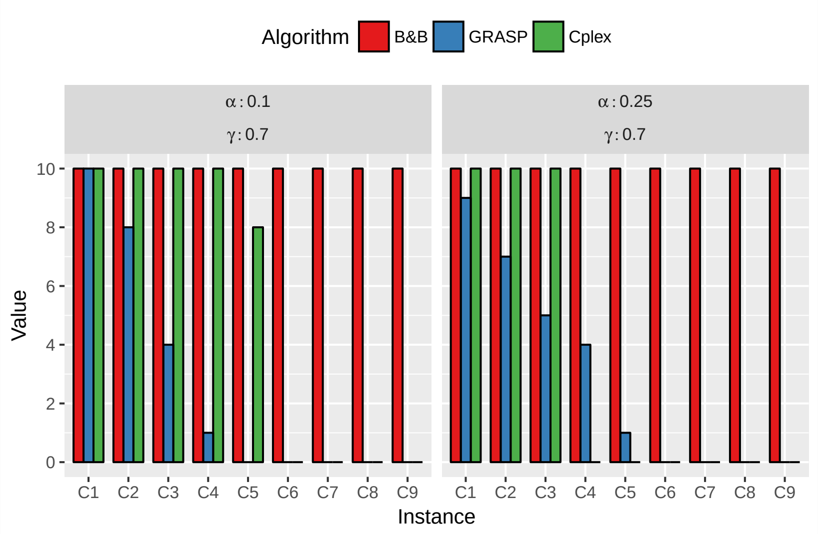
M2 vs B&B vs GRASP on complete graphs

Optimal solutions



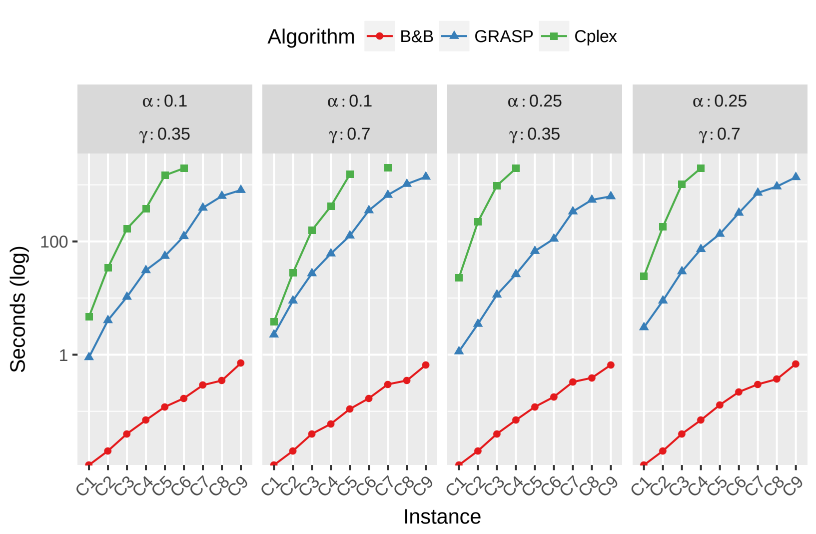
M2 vs B&B vs GRASP on complete graphs

Optimal solutions



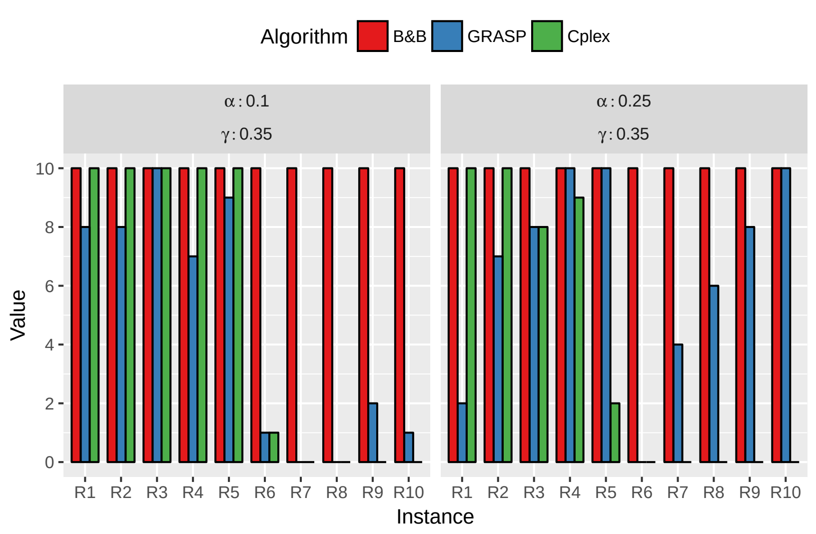
M2 vs B&B vs GRASP on complete graphs

Computational times



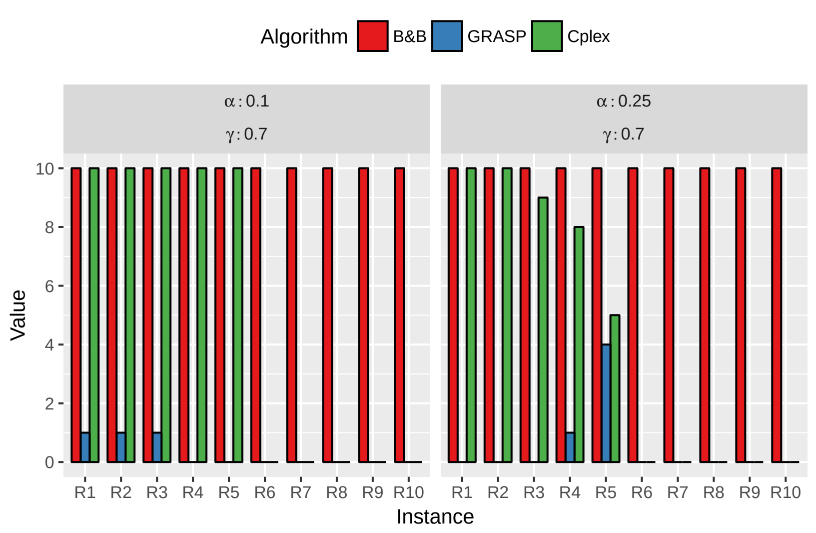
M2 vs B&B vs GRASP on random graphs

Optimal solutions



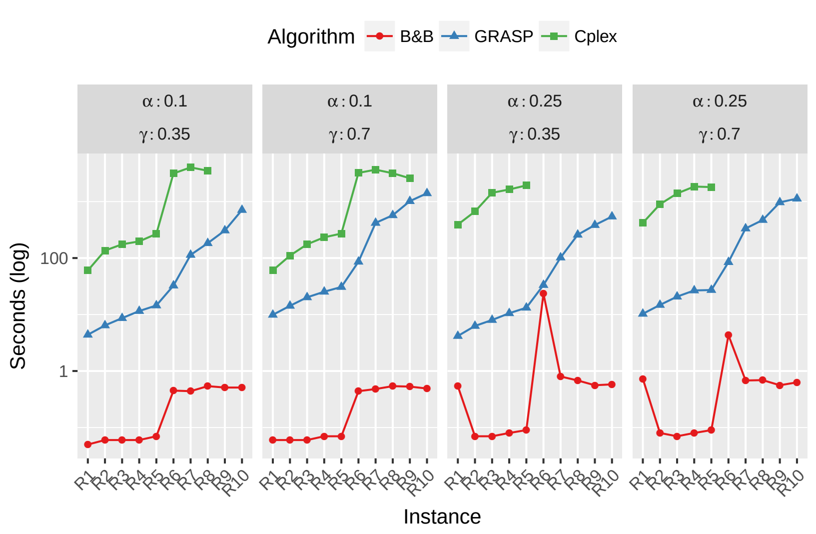
M2 vs B&B vs GRASP on random graphs

Optimal solutions



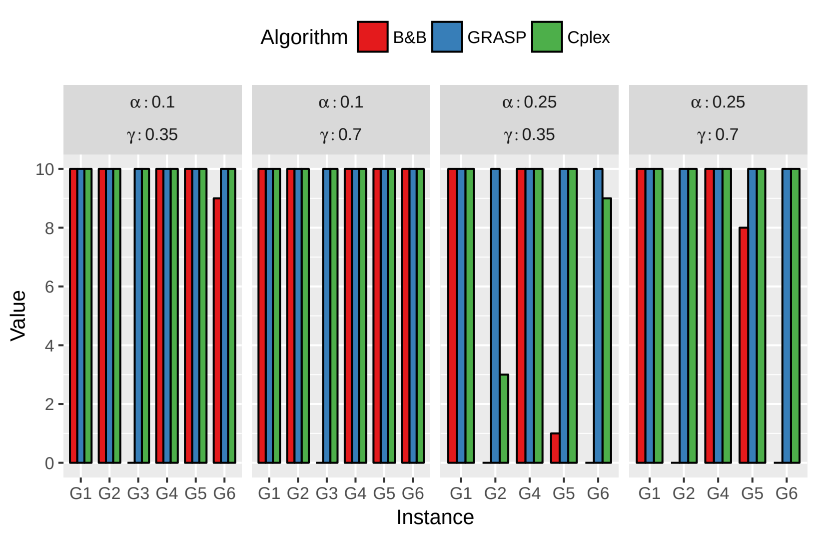
M2 vs B&B vs GRASP on random graphs

Computational times



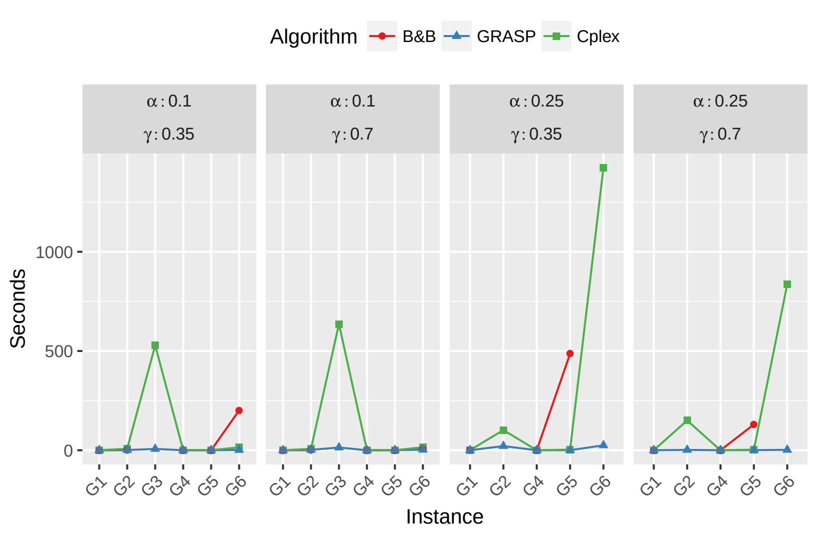
M2 vs B&B vs GRASP on grid graphs

Feasible solutions



M2 vs B&B vs GRASP on grid graphs

Computational times



Conclusions and future work

- B&B has very good performances expecially on dense graphs;

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- B&B has very good performances expecially on dense graphs;
- GRASP is useful when B&B is not able to find feasible solutions;
- as future work, we are investigating further variants of the problem resulting from the introduction of further constraints defined on the arcs and/or on the nodes of the graph.

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(**E** **N** **D**)

Thank you.