UNIVERSITAT DE BARCELONA

>>> IMAGE PROCESSING AND COMPUTATIONAL PHOTOGRAPHY

SESSION 3: FREQUENCY DOMAIN

Oriol Pujol

REVIEW QUESTIONS

1. Write down a 3x3 filter that returns a positive value if the average value of the 4-adjacent neighbors is less than the center and a negative value otherwise

2. Write down a filter that will compute the gradient in the x-direction:

   gradx(y,x) = im(y,x+1)-im(y,x) for each x, y

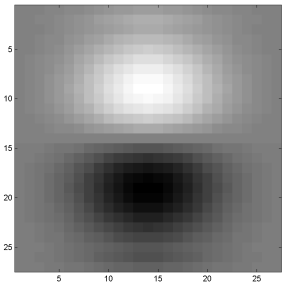
REVIEW QUESTIONS

Filtering Operator

3. Fill in the blanks:

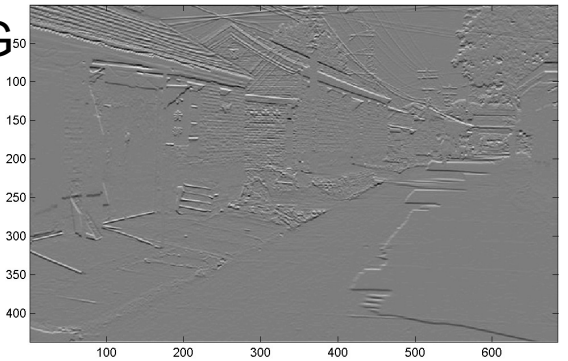
a) _ = D * B
b) A = _ * _
c) F = D * _
d) _ = D * D

A

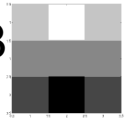


E

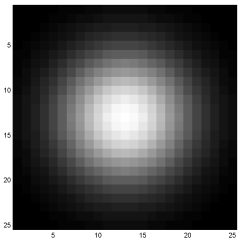
G



B



C



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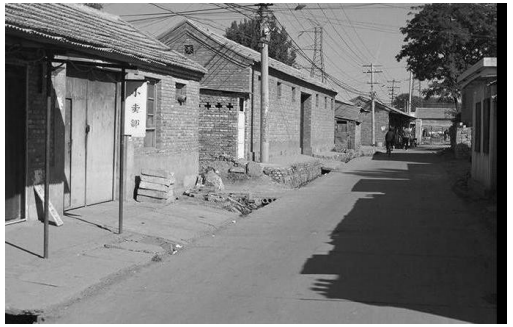
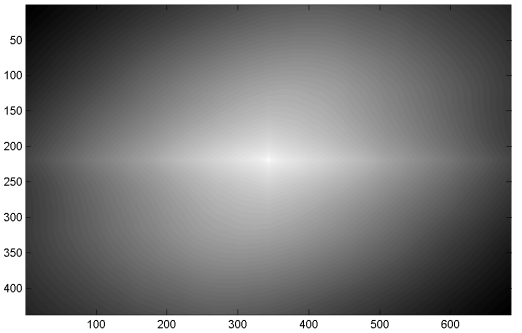
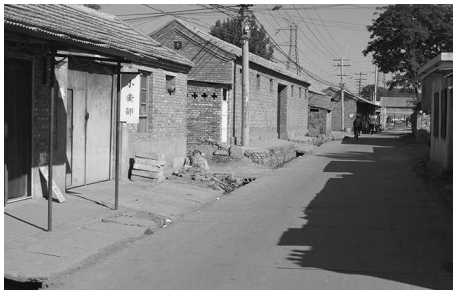


H

I

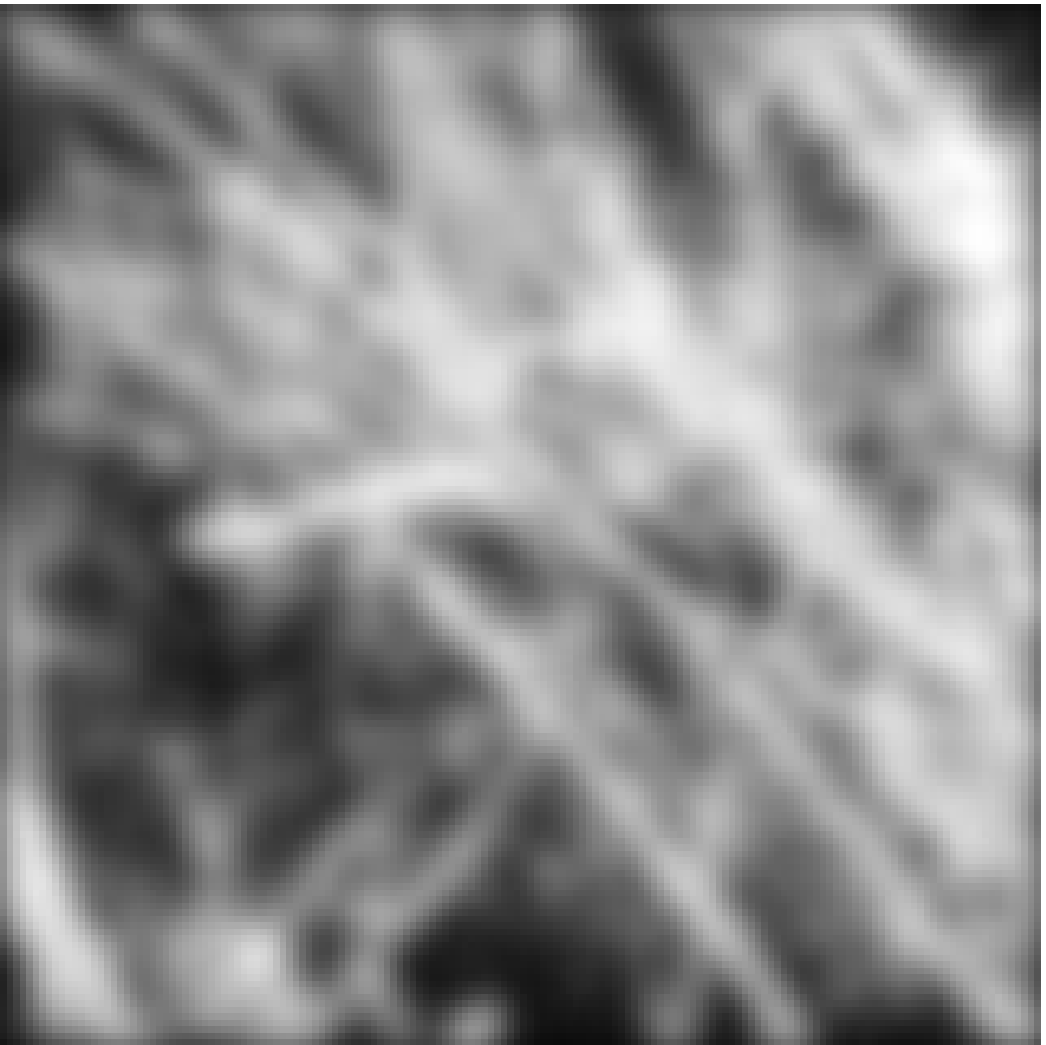


D



WHY DOES BOX FILTER SMOOTHING HAVE SO MANY ARTIFACTS?

Gaussian 



Box filter 

WHY DO WE GET DIFFERENT, DISTANCE-DEPENDENT INTERPRETATIONS OF HYBRID IMAGES?

WHY DOES A LOWER RESOLUTION IMAGE STILL MAKE SENSE TO US? WHAT DO WE LOSE?

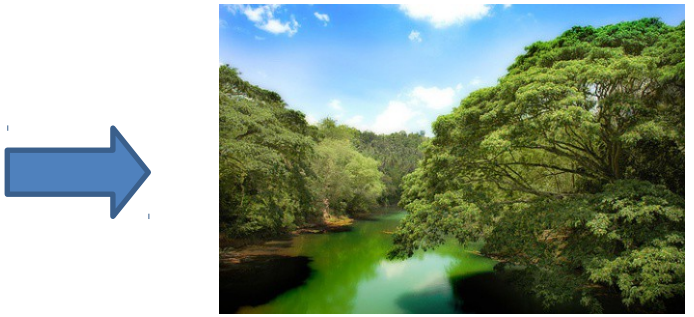


Image: http://www.flickr.com/photos/igorms/136916757/

TODAY'S LECTURE

Fourier transform and frequency domain

- Frequency view of filtering
- Another look at hybrid images
- Sampling

JEAN BAPTISTE JOSEPH FOURIER (1768-1830)

Idea (1807):

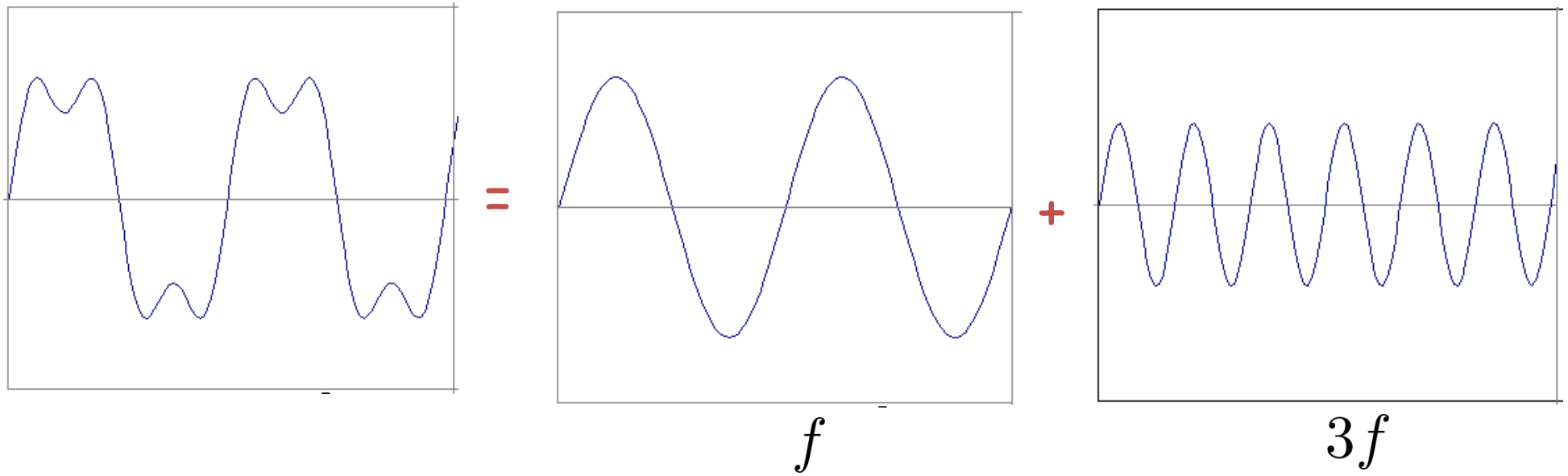
Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Don't believe it?
Neither did Lagrange, Laplace, Poisson, etc.
Not translated into English until 1878!

But it's (mostly) true!
called Fourier Series but there are some subtle restrictions.

EXAMPLE

- example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$



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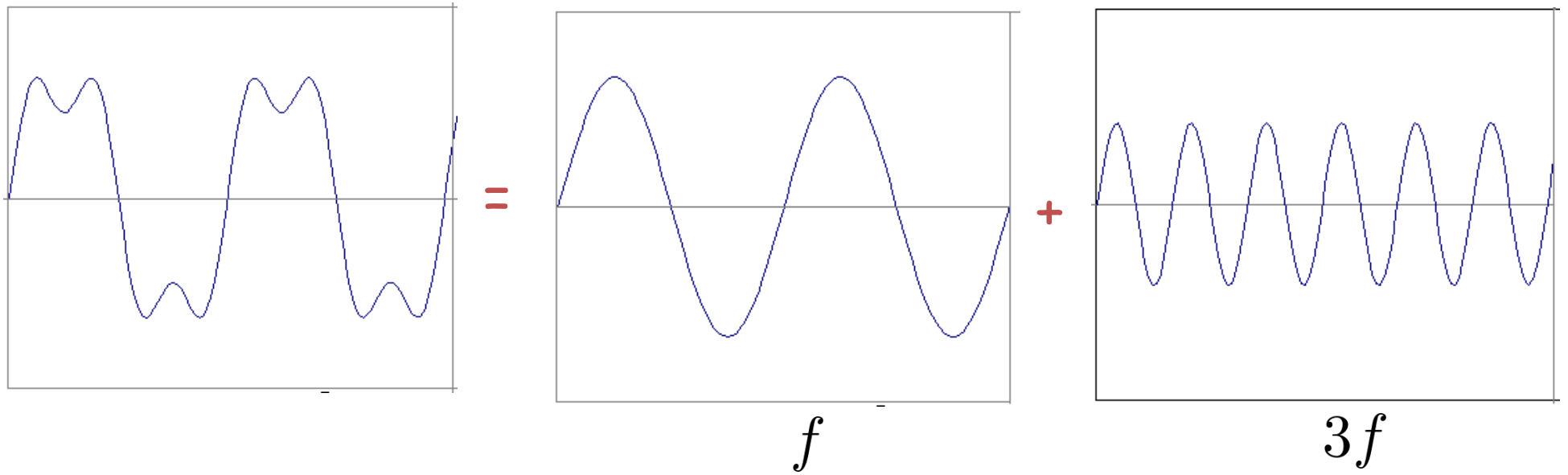
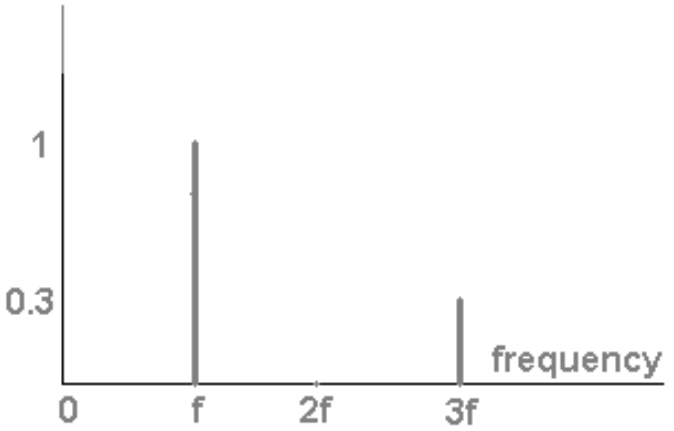
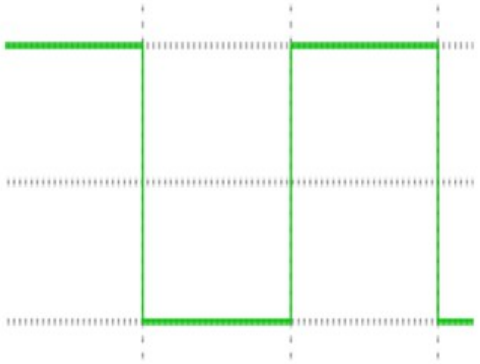


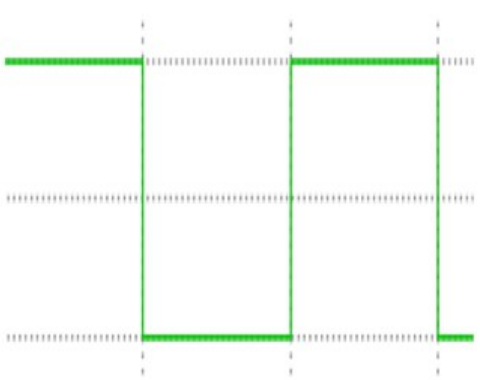


Slides: Efros

EXAMPLE



EXAMPLE



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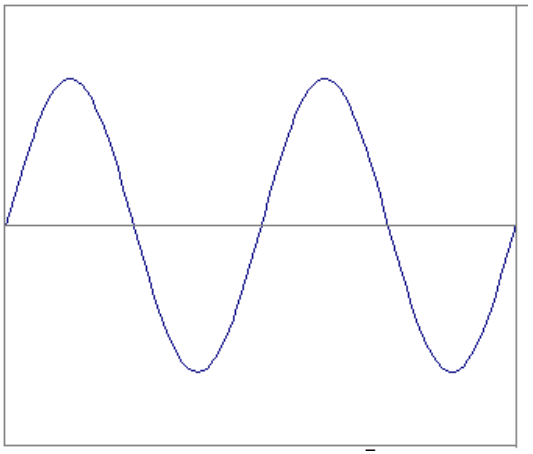

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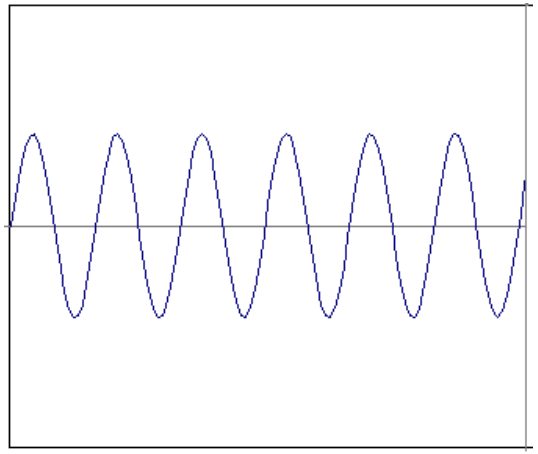

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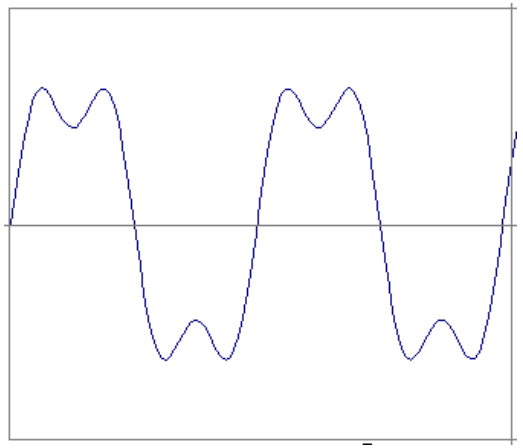
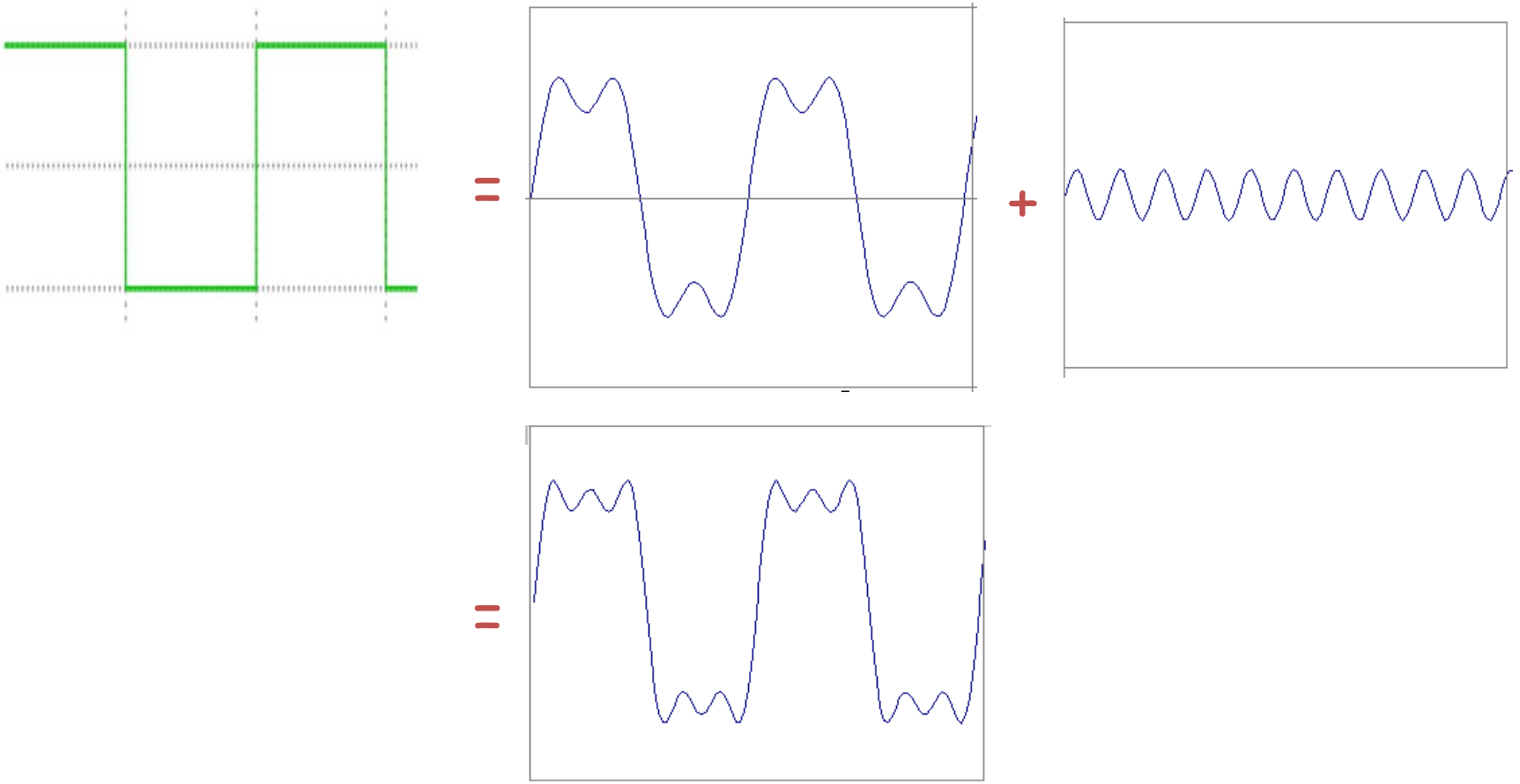
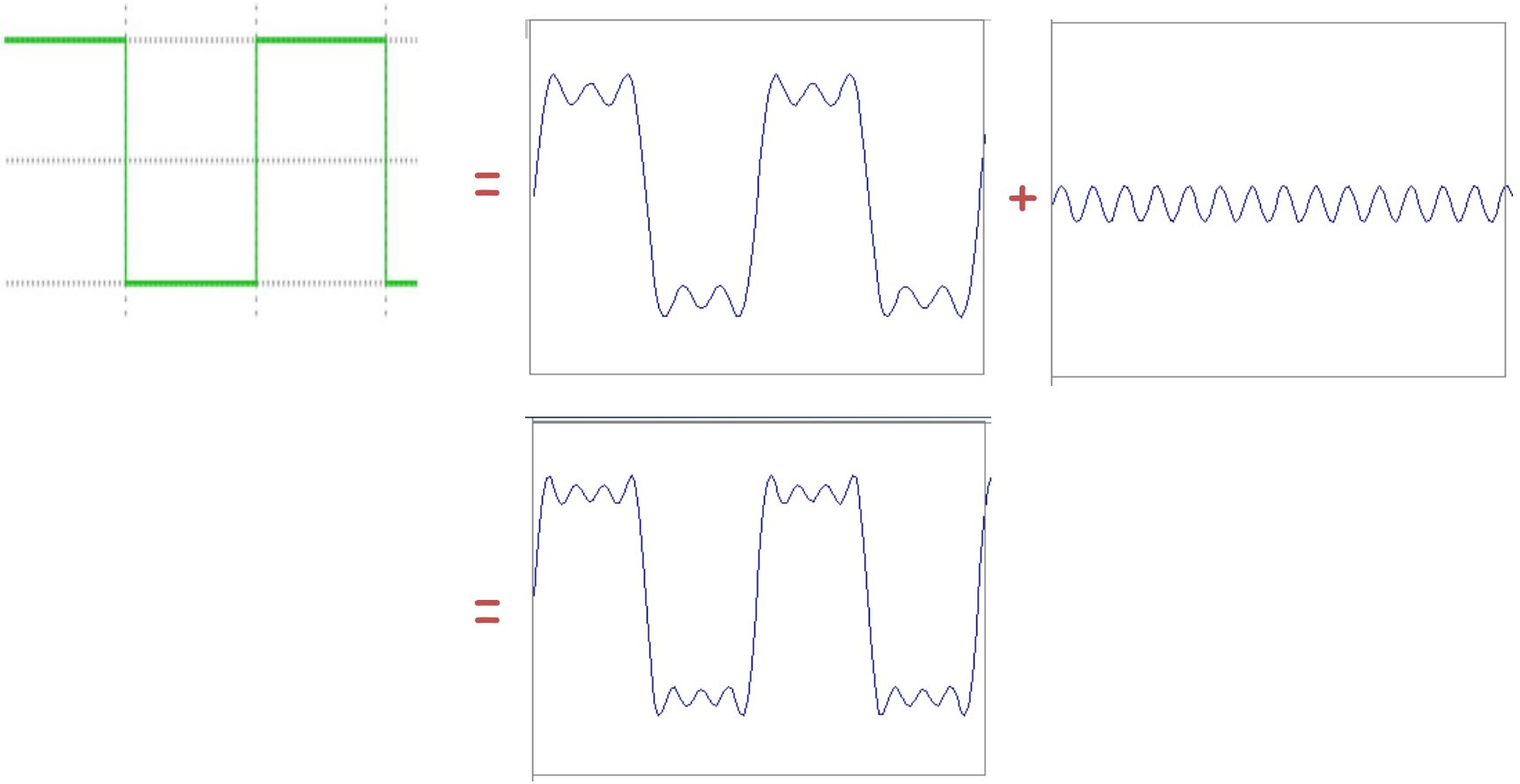


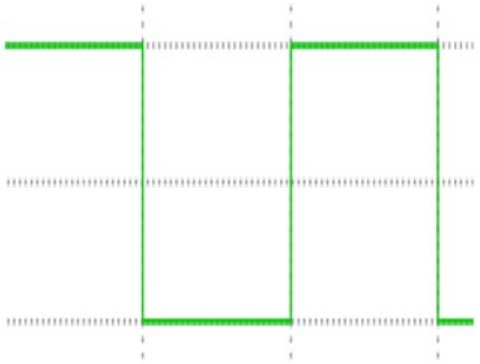
EXAMPLE



EXAMPLE



EXAMPLE



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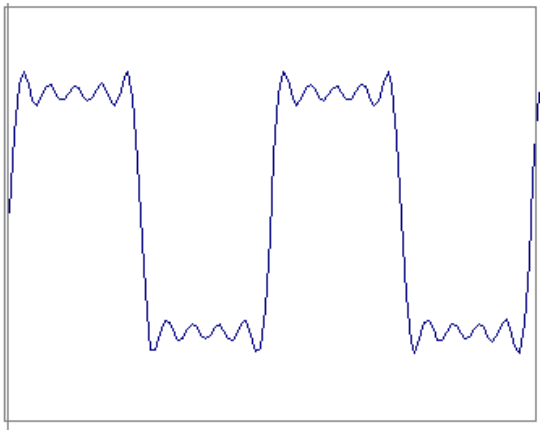

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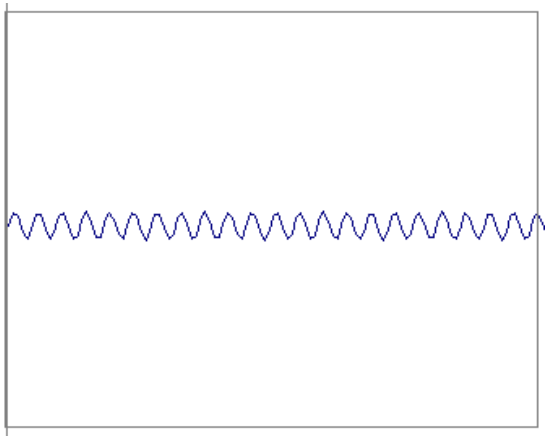

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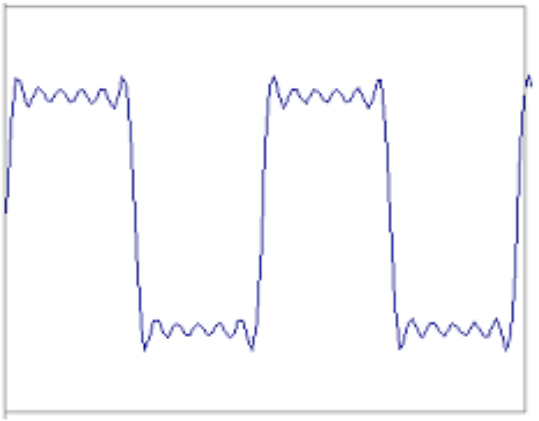
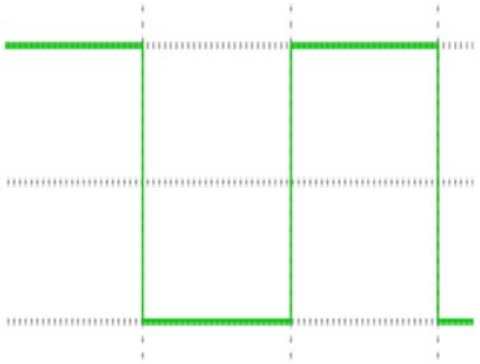


EXAMPLE



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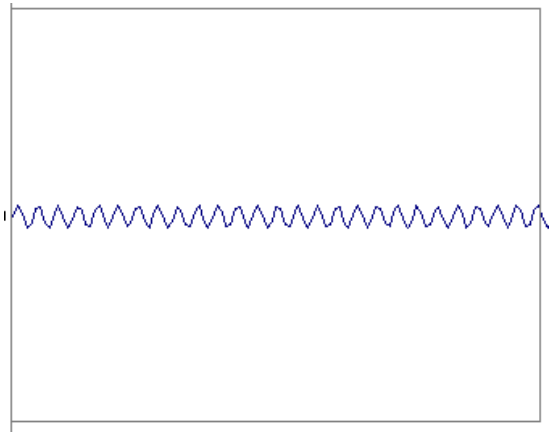

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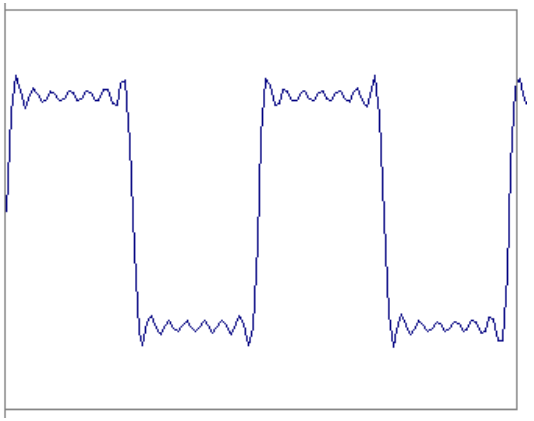
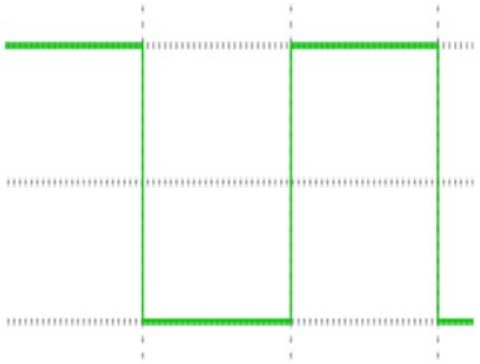
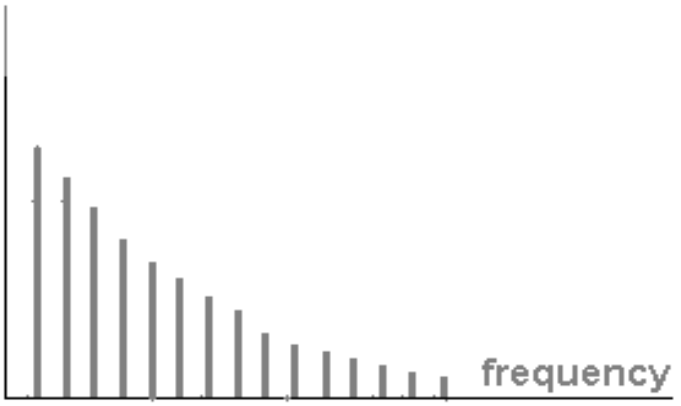


EXAMPLE



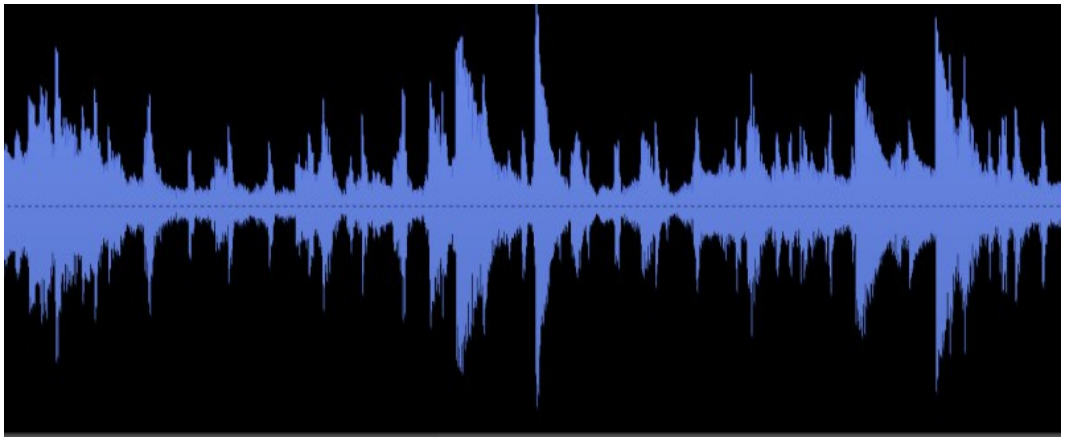
$$= A \sum_{i=1}^{\infty} \frac{1}{k} sin(2\pi kt)$$



EXAMPLE

- We think of music in terms of frequencies at different magnitudes

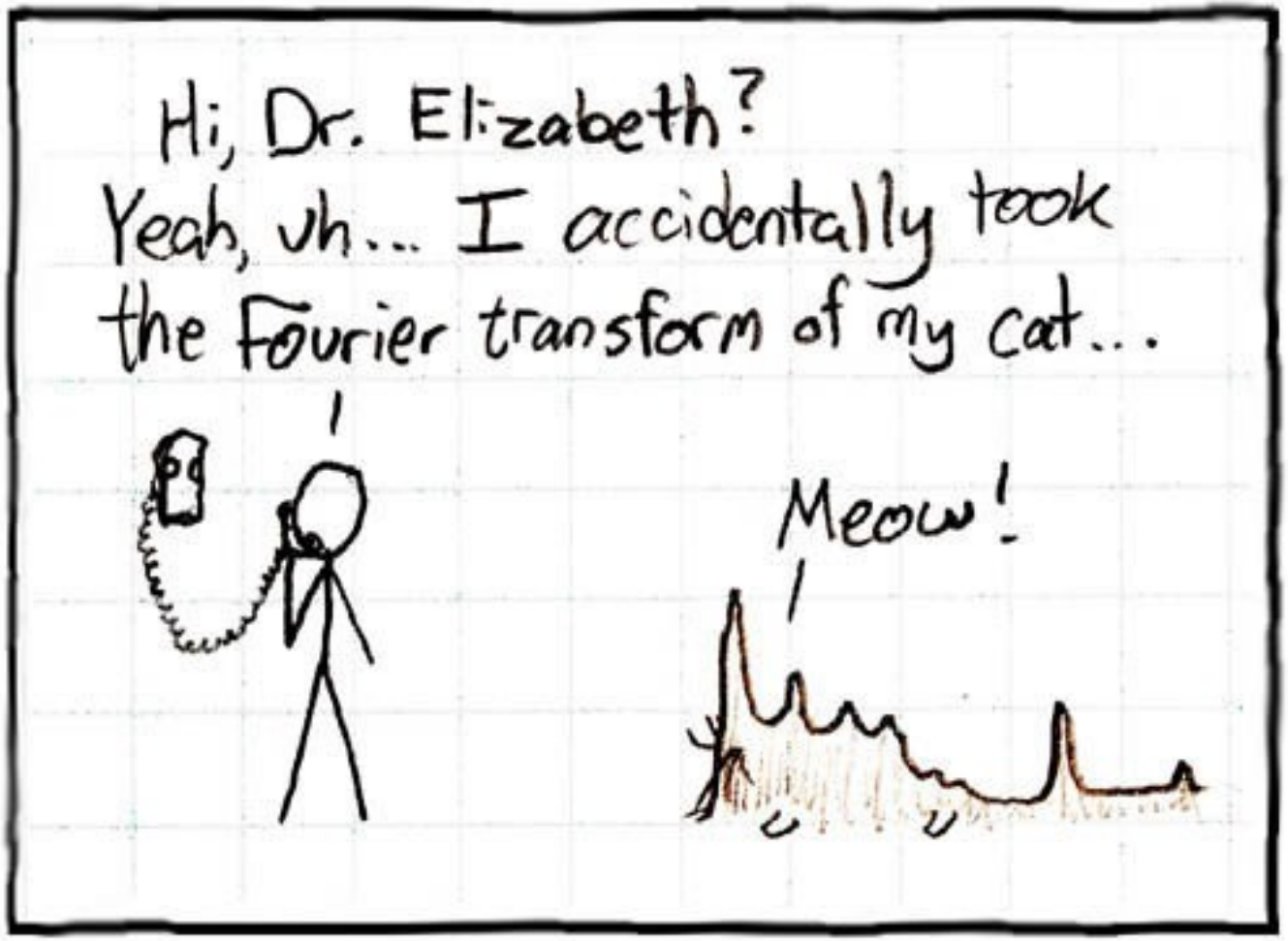
Music signal



Frequency spectrum

OTHER EXAMPLES

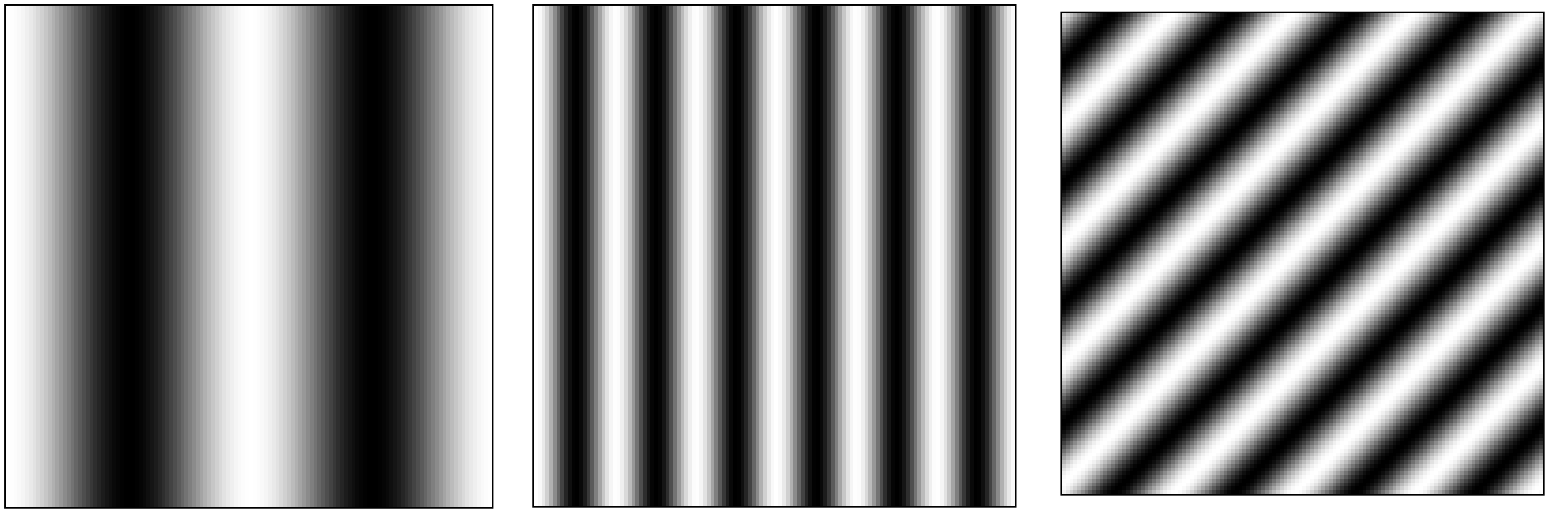
- We can also think of all kinds of other signals the same way



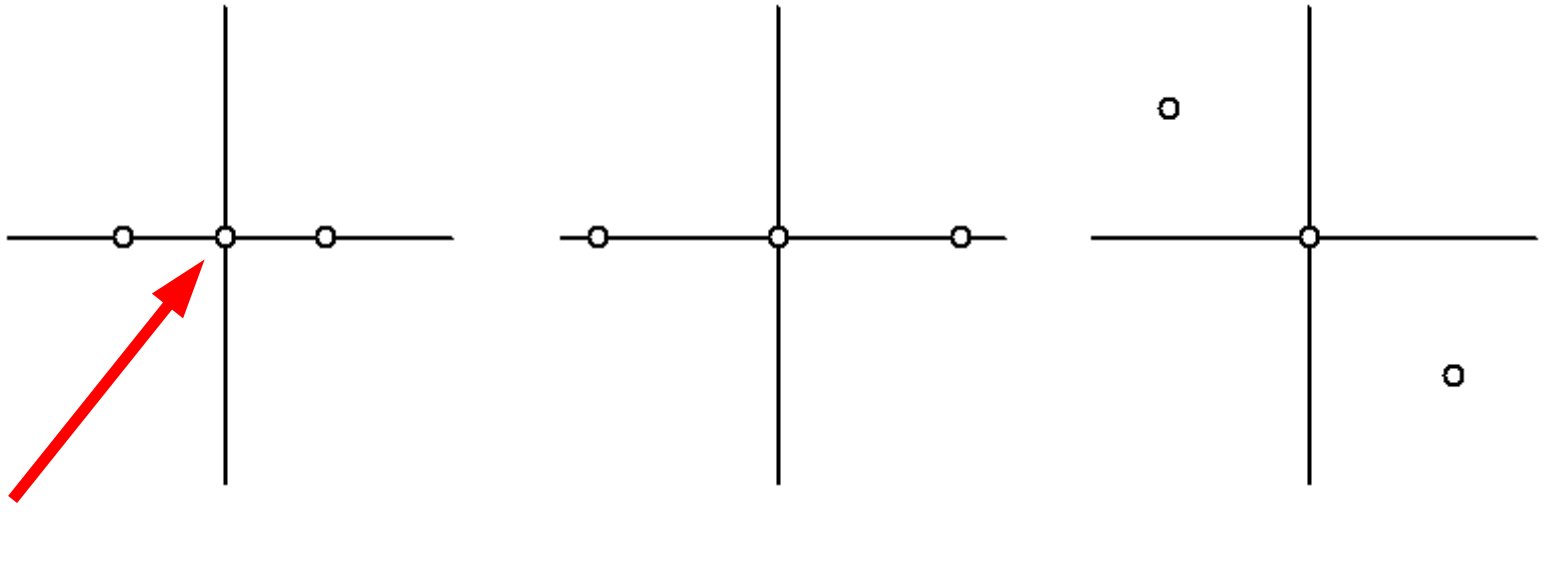
xkcd.com

FOURIER ANALYSIS IN IMAGES

Intensity Image

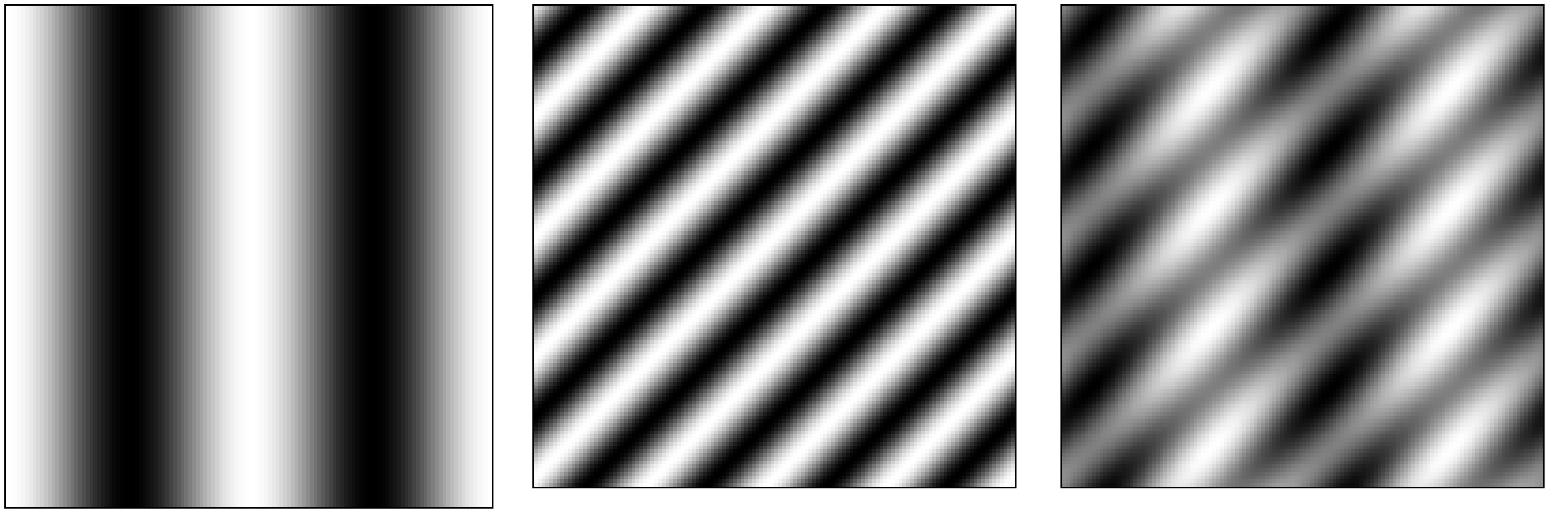


Fourier Image

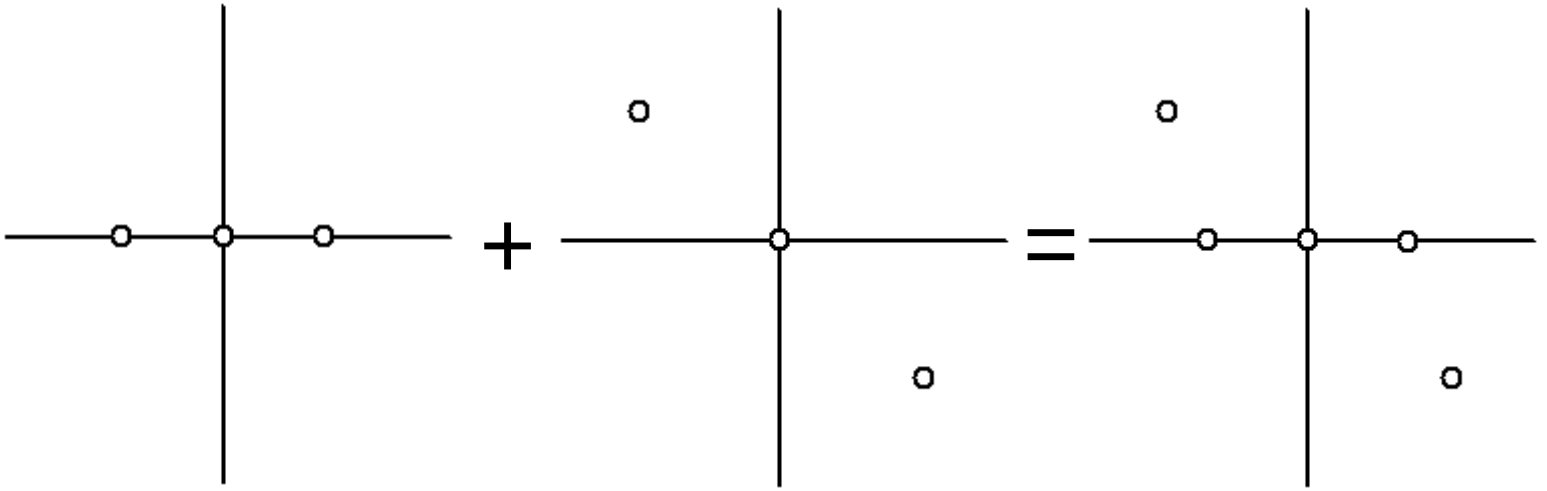


FOURIER ANALYSIS IN IMAGES

Intensity Image



Fourier Image



THE REAL DEAL (SERIOUS MATHS AHEAD)

THE FOURIER SERIES

Fourier series:

Any even function can be decomposed as follows

$$C(x) = a_0 + a_1 \cos x + a_2 \cos 2x$$

Any odd function can be decomposed as follows

$$S(x) = b_1 \sin x + b_2 \sin 2x$$

Any function can be decomposed as follows

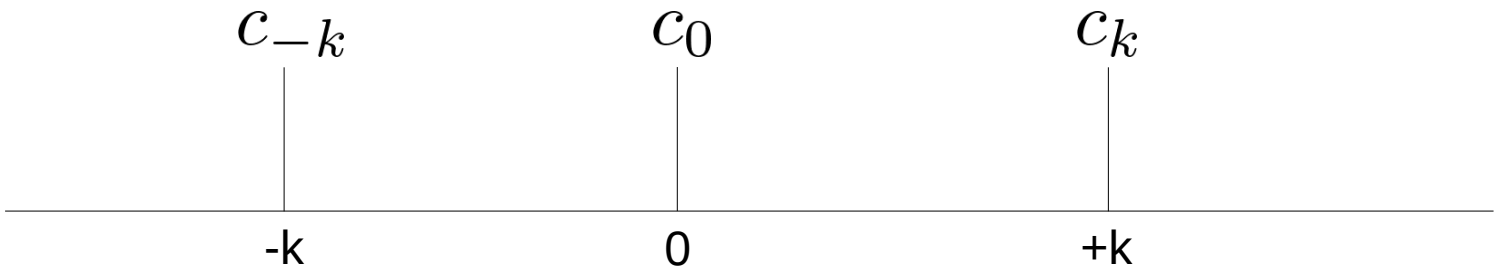
$$F(x) = C(x) + S(x)$$

In compact (cool) notation $e^{ix} = \cos x + i \sin x$

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

THE FOURIER TRANSFORM

Take the coefficients and put them in one axis. We obtain the **“frequency spectrum”**.



THE DISCRETE FOURIER TRANSFORM

A detail: up to this moment we assume functions periodic with period 2π

The discretization of the former periodic signal in N samples take values at locations

$$F(n) = f(x)|_{x=\frac{2\pi}{N}n} \quad n = 0, \ldots N-1$$

THE DISCRETE FOURIER TRANSFORM

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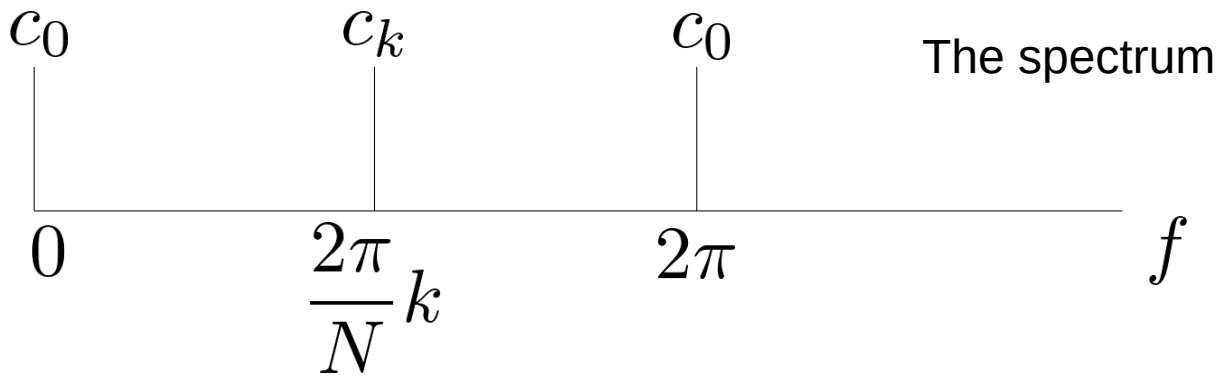
The discretization of the former periodic signal in N samples take values at locations

$$F(n) = f(x)|_{x=\frac{2\pi}{N}n} \quad n = 0, \dots N-1$$

Replacing in the continuous case

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \longrightarrow F(n) = \sum_{k=0}^{N-1} c_k e^{ik\frac{2\pi}{N}n}$$

and the spectrum is defined as



The spectrum is periodic with period 2π

A BRIEF NOTE ABOUT THE COEFFICIENTS

$$c_k = \sum_{n=0}^{N-1} f(\frac{2\pi}{N}n)e^{-ik\frac{2\pi}{N}n}$$

The coefficients are complex numbers. As such, in the Fourier transform we can refer to the magnitude and phase at each frequency:

- Magnitude encodes how much signal there is at a particular frequency (usual value displayed in the spectrum)
- Phase encodes spatial information (indirectly)

$$\mathcal{F}(f(x - x_0)) = F(x)e^{-i\omega x_0}$$

$$|c_k| = \pm\sqrt{R(\omega_k) + I(\omega_k)} \qquad \phi_k = \tan^{-1}\frac{I(w_k)}{R(w_k)}$$

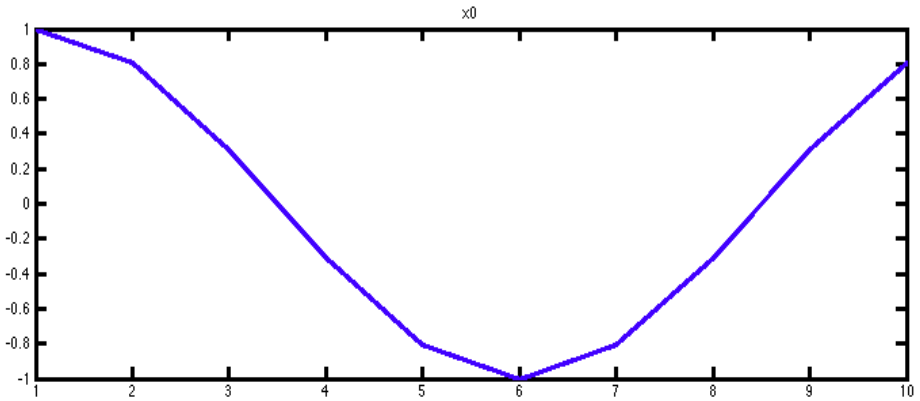
A BRIEF NOTE ABOUT THE COEFFICIENTS

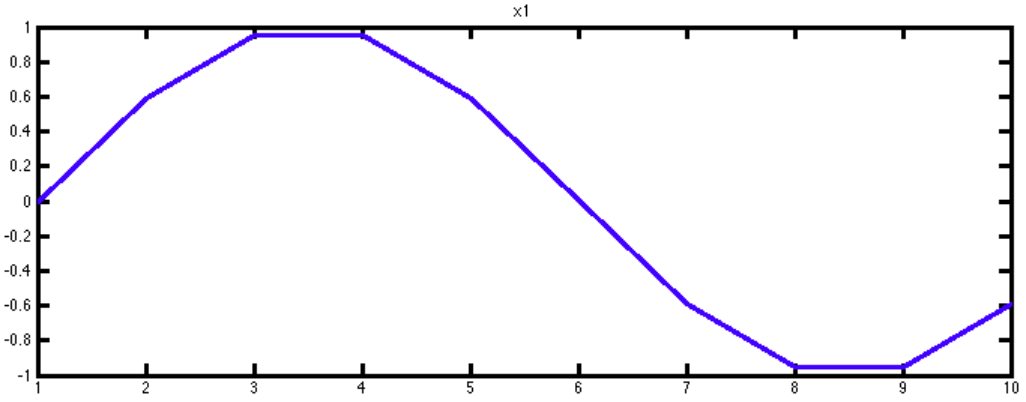
They can be computed in $\mathcal{O}(N \log N)$!!!!
(with some limitations)

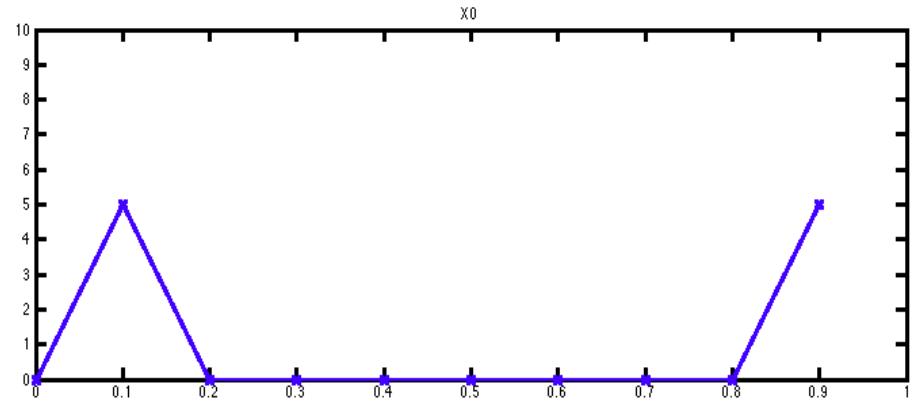
ENTER THE FAST FOURIER TRANSFORM

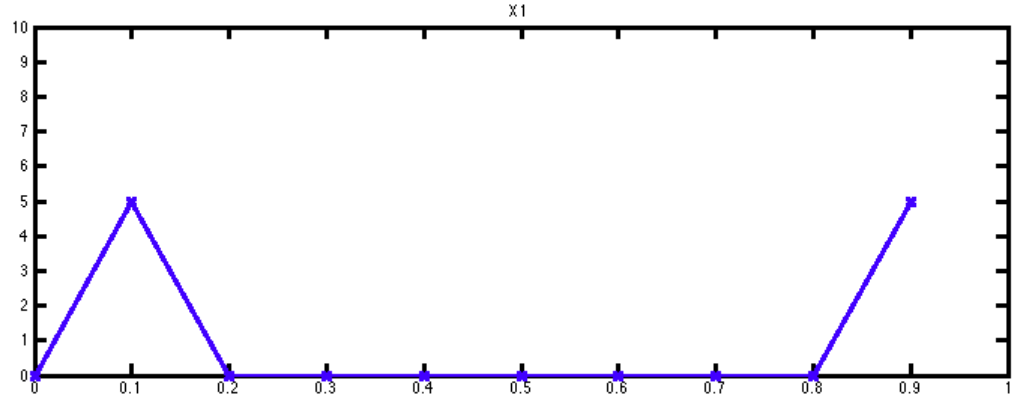
CODING TIME: sine and cosine transforms magnitude

```
n = [0:9];
x0 = cos(2*pi*n/10); % 1 period
N0 = 10;
X0 = fft(x0,N0);
plot(abs(X0))
```



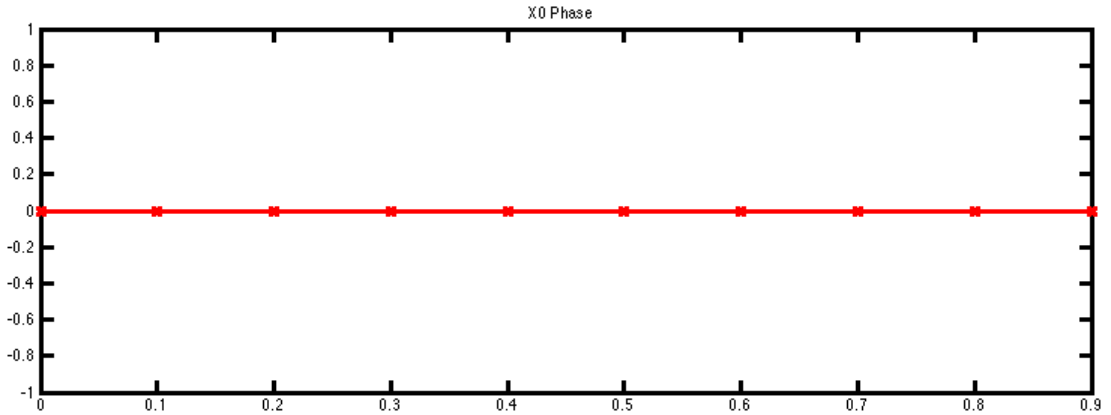




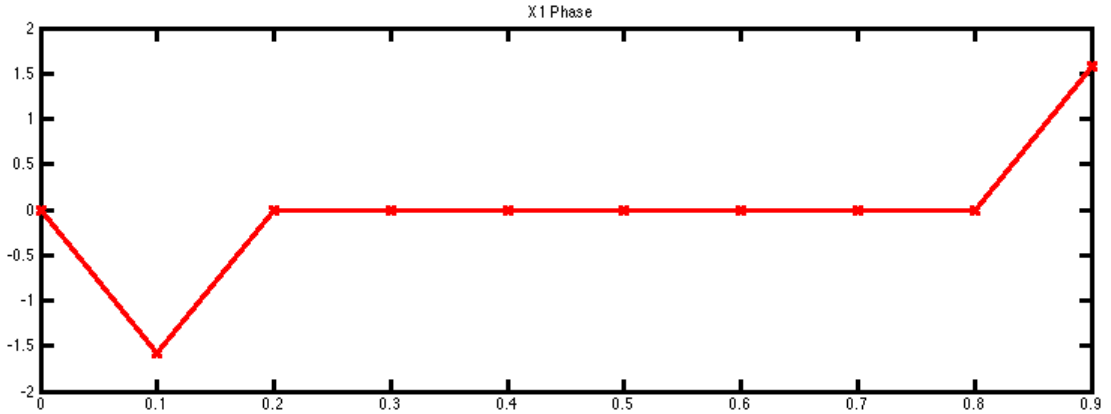


CODING TIME: sine and cosine transforms phases

```
plot(angle(X0));
```



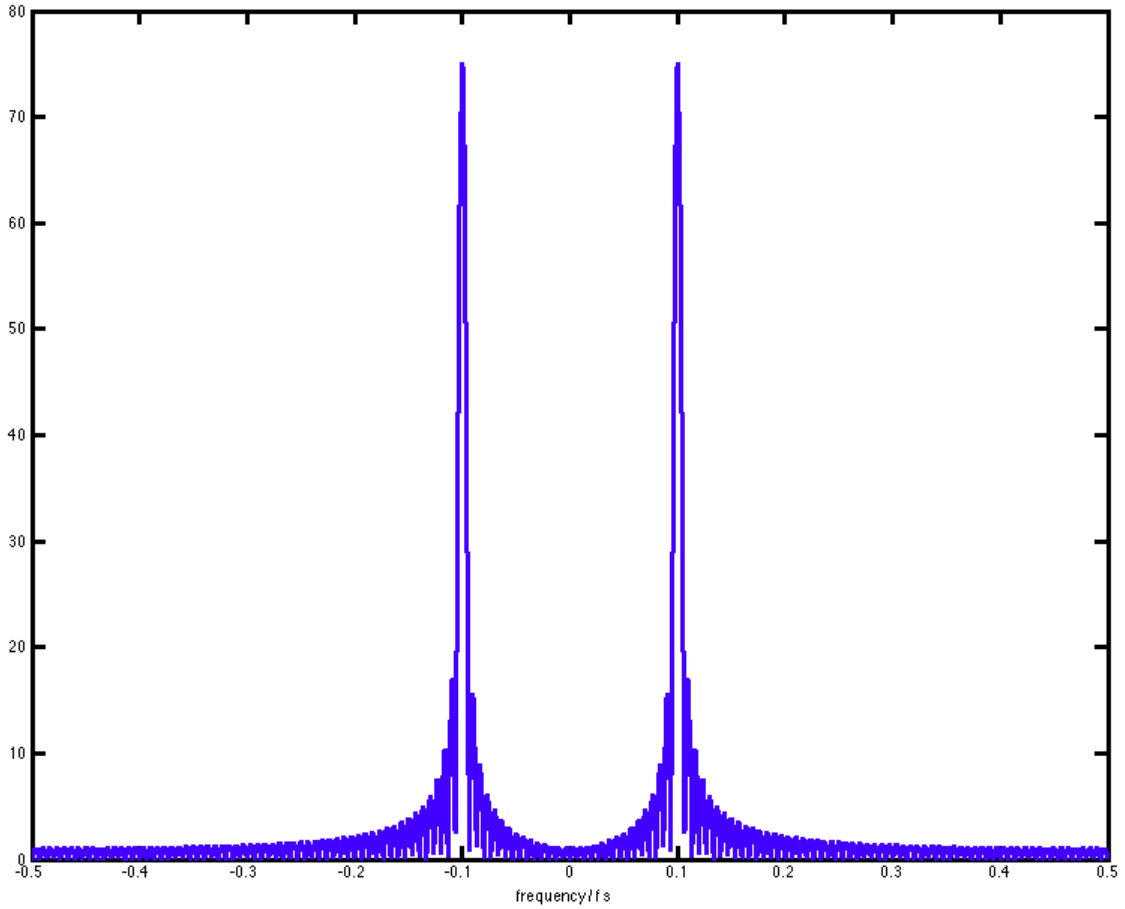
cosine



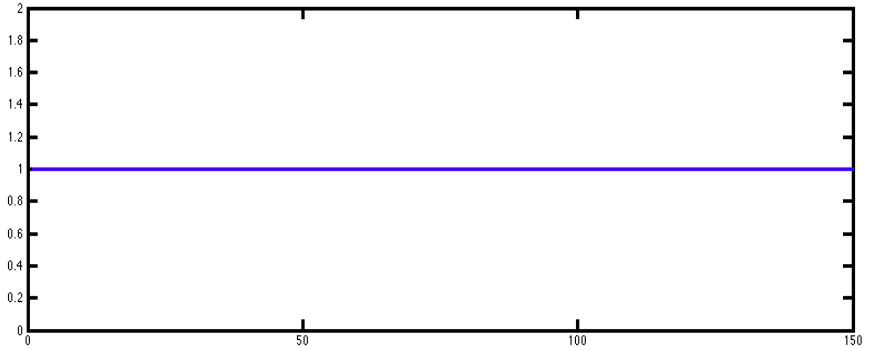
sine

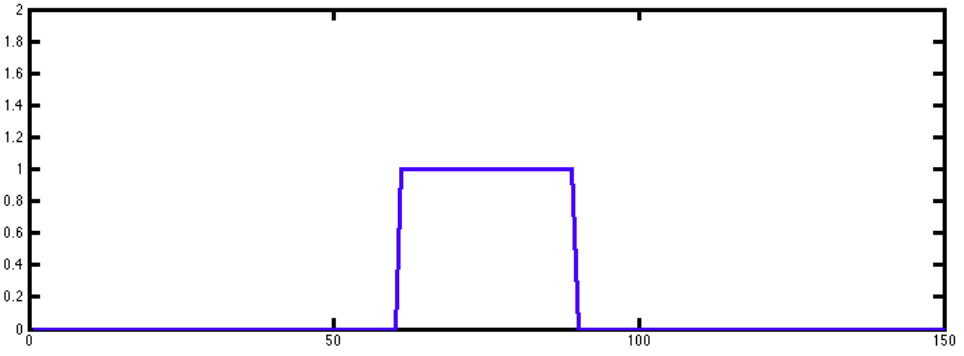
CODING TIME: symmetry and visualization

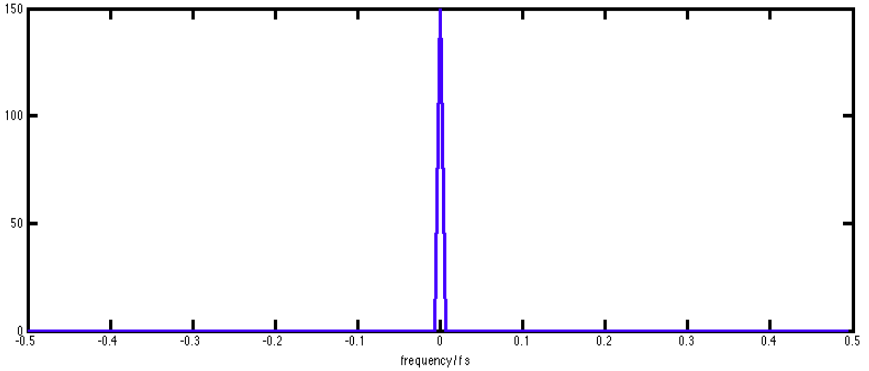
```
n = [0:149];
x = cos(2*pi*n/10);
N = 2048;
X = abs(fft(x,N));
X = fftshift(X);
F = [-N/2:N/2-1]/N;
plot(F,X),
```

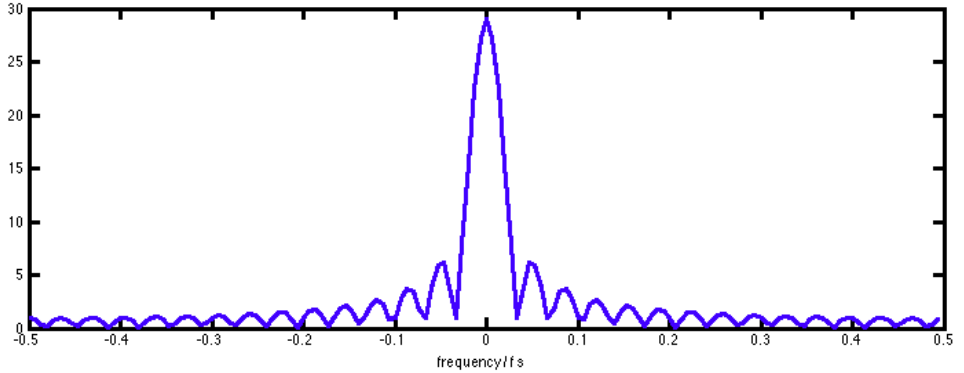


CODING TIME: constant and box transforms

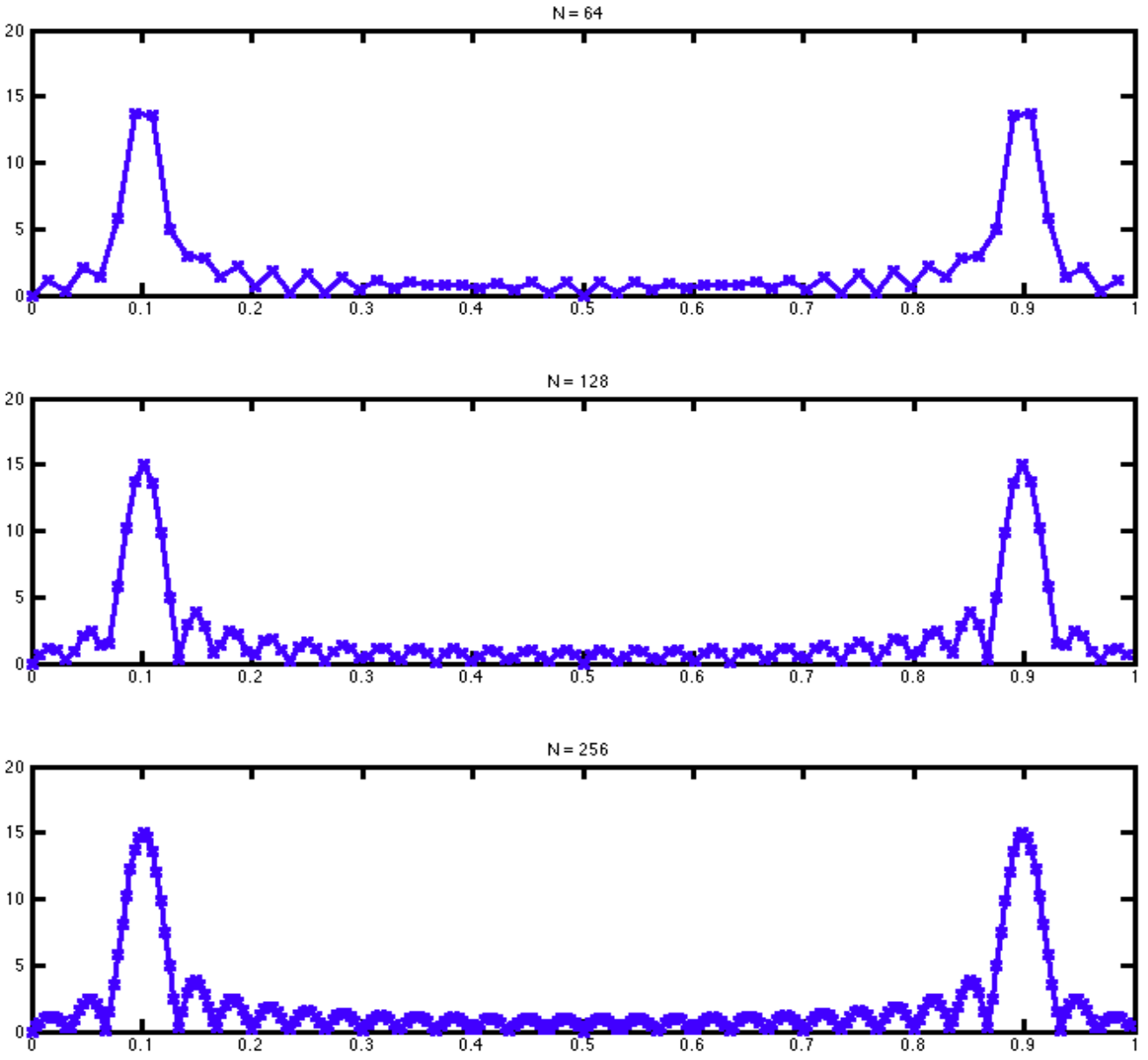




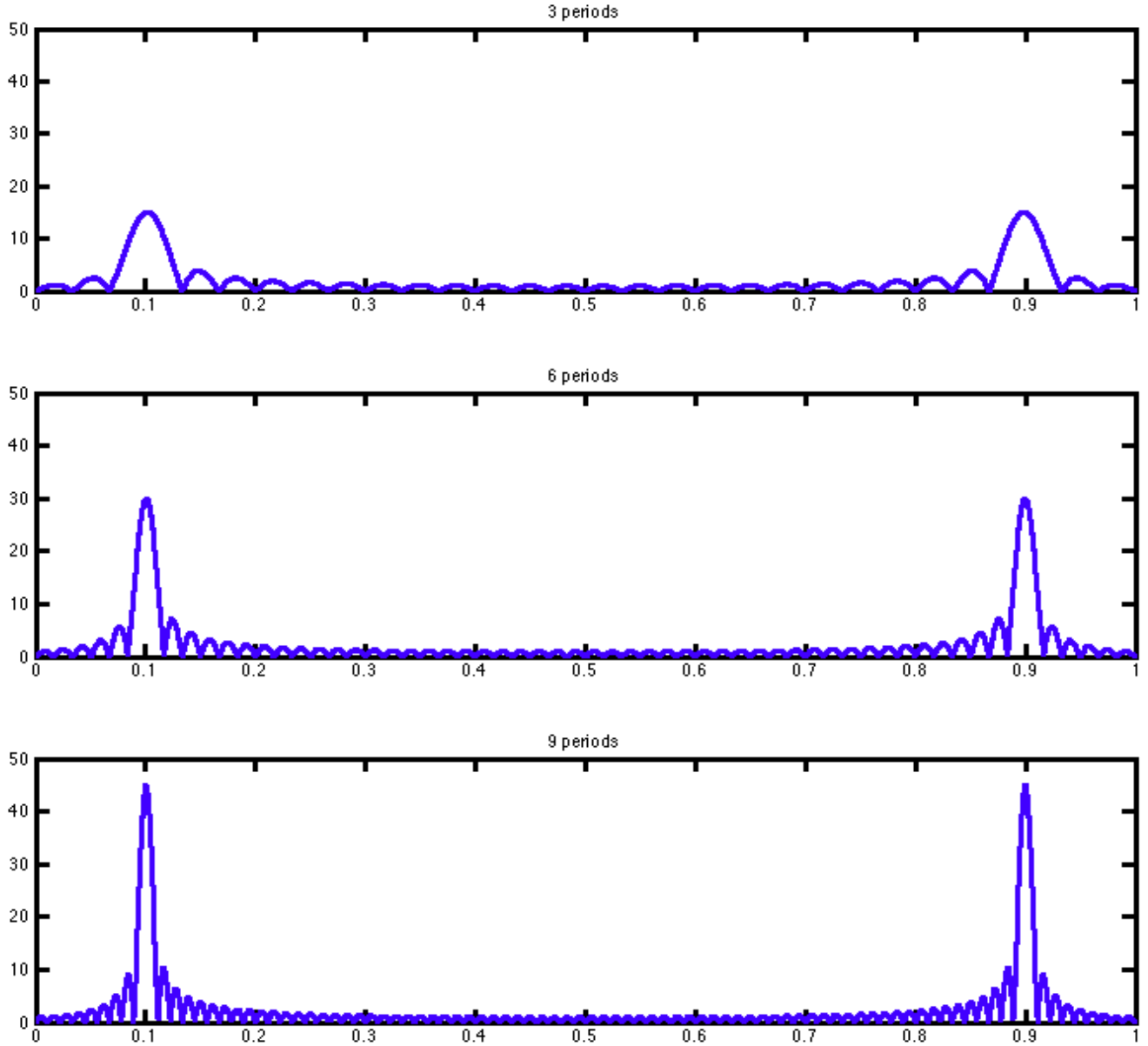




CODING TIME: size effect

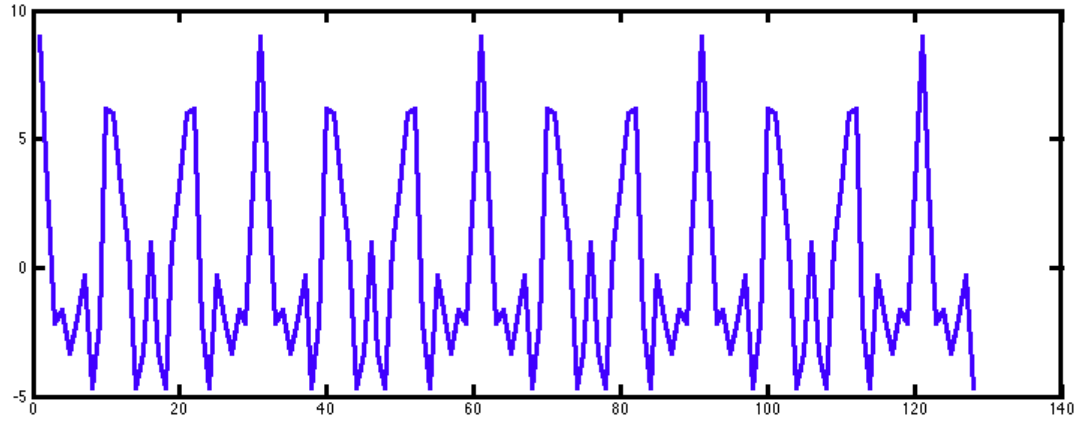


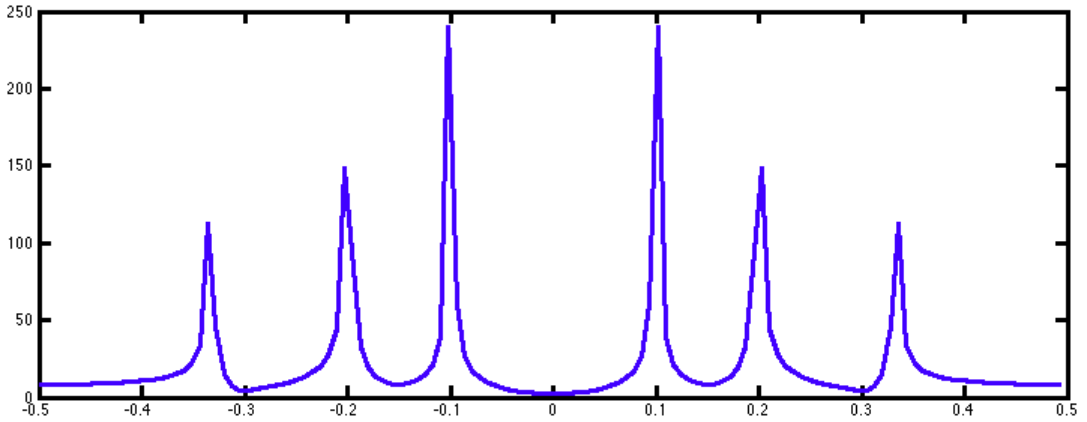
CODING TIME: signal length effect



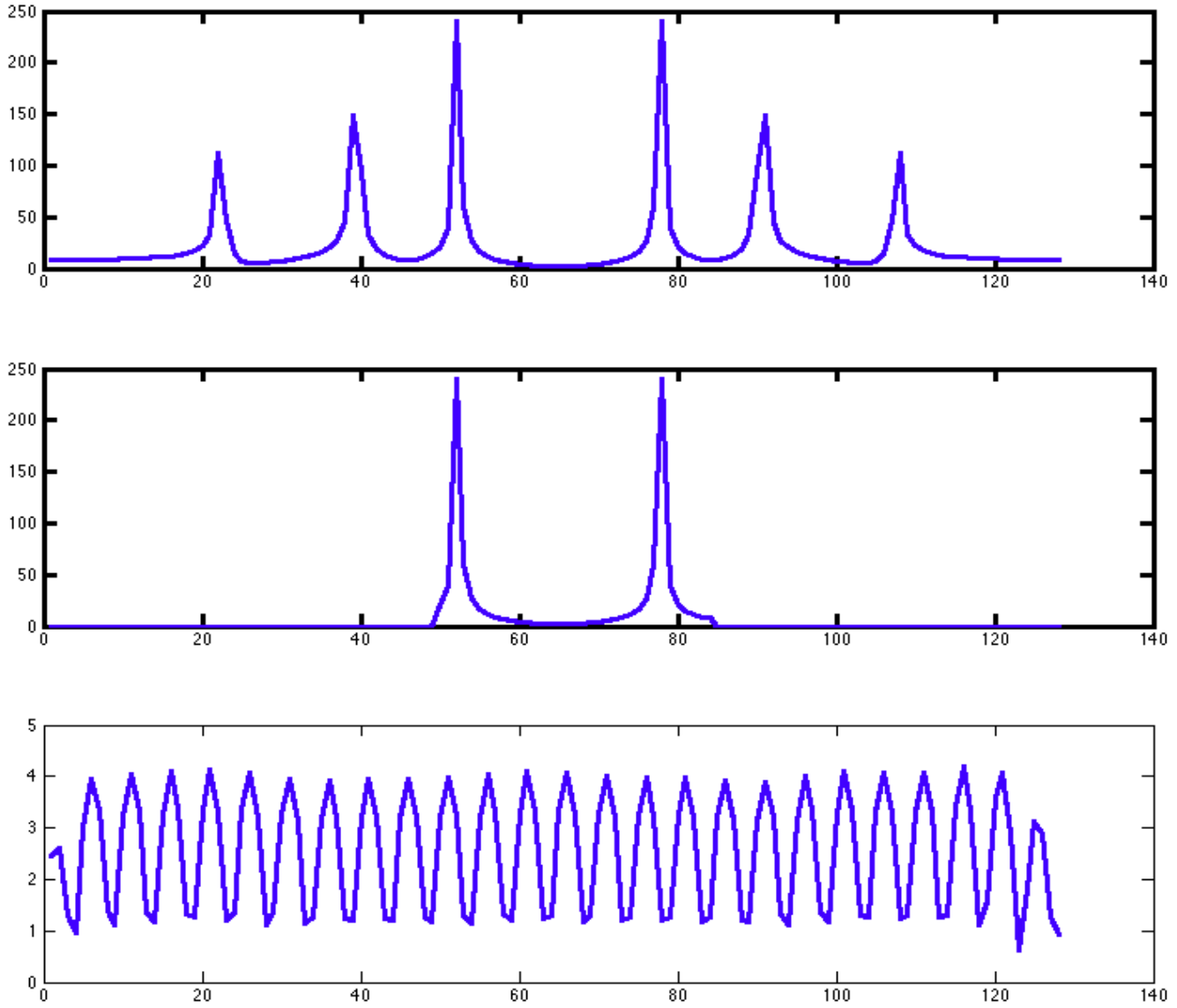
CODING TIME: composed signal

```
n = [0:127];
x1 = 4*cos(2*pi*n/10); % f=1/10
x2 = 3*cos(2*pi*n/5); % f=1/5
x3 = 2*cos(2*pi*n/3); % f=1/3
x= x1+x2+x3;
```

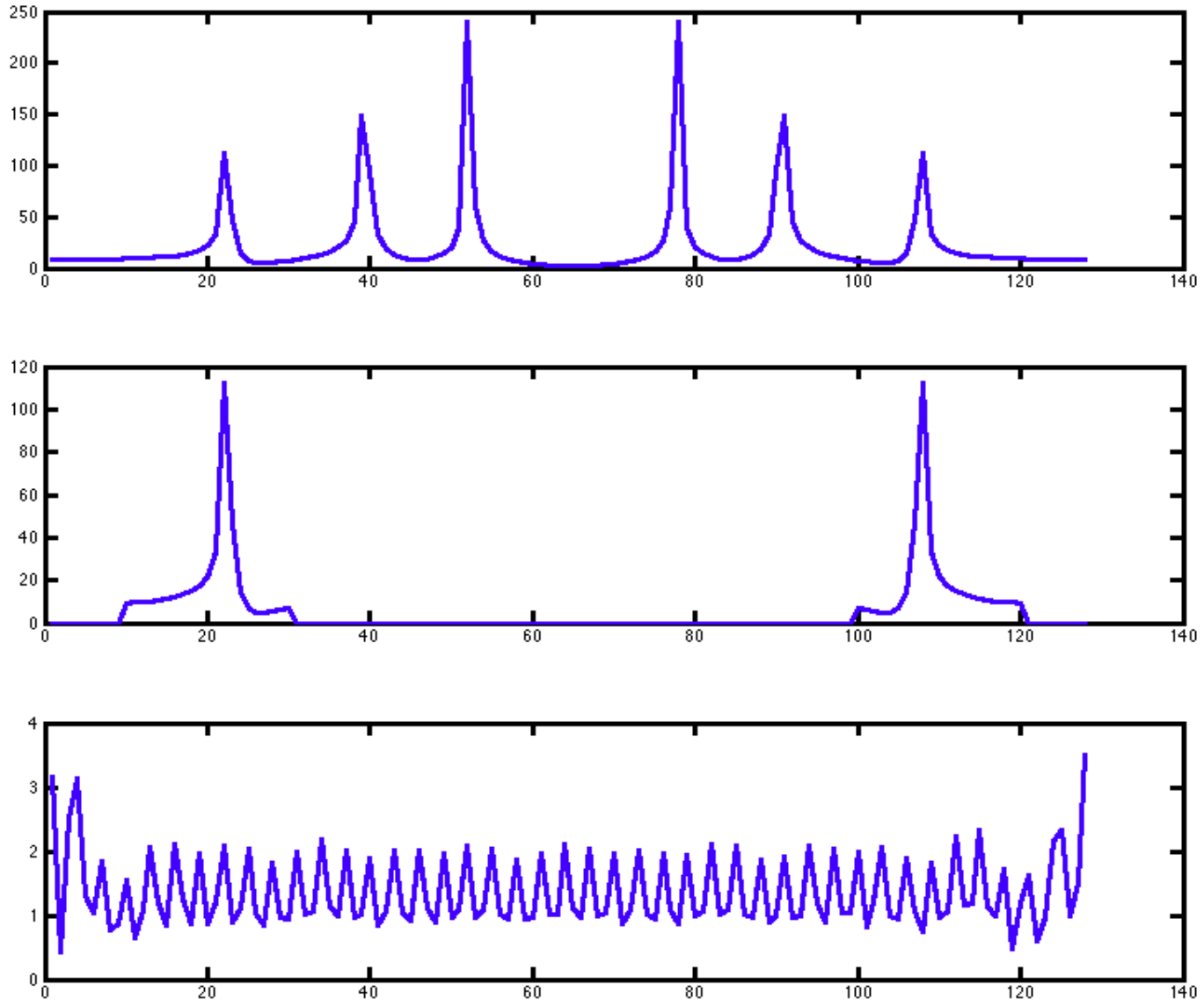




CODING TIME: low frequency selection



CODING TIME: high frequency selection



APPLICATIONS AND PROPERTIES OF THE FOURIER TRANSFORM

COMMUNICATIONS: Because of this your cellphone, TV, radio, satellite communications, etc work.

$$\mathcal{F}(f(x)e^{i2\pi x w_0}) = F(w - w_0)$$

MATHEMATICAL MODELLING: EDP numerical solution.

$$\mathcal{F}(f'(x)) = i\omega F(\omega)$$

SIGNAL PROCESSING: A fast way of applying a convolution.

$$\mathcal{F}(f(x) \otimes g(x)) = F(\omega) \cdot G(\omega)$$

SIGNAL ANALYSIS: Interpretation in term of frequencies.

PROPERTIES OF THE FOURIER TRANSFORM

Parseval's theorem: The energy of the signal is the same as the energy of its Fourier transform

$$\sum_{n=1}^{N-1} |f(n)|^2 = \sum_{k=1}^{N-1} |F(k)|^2$$

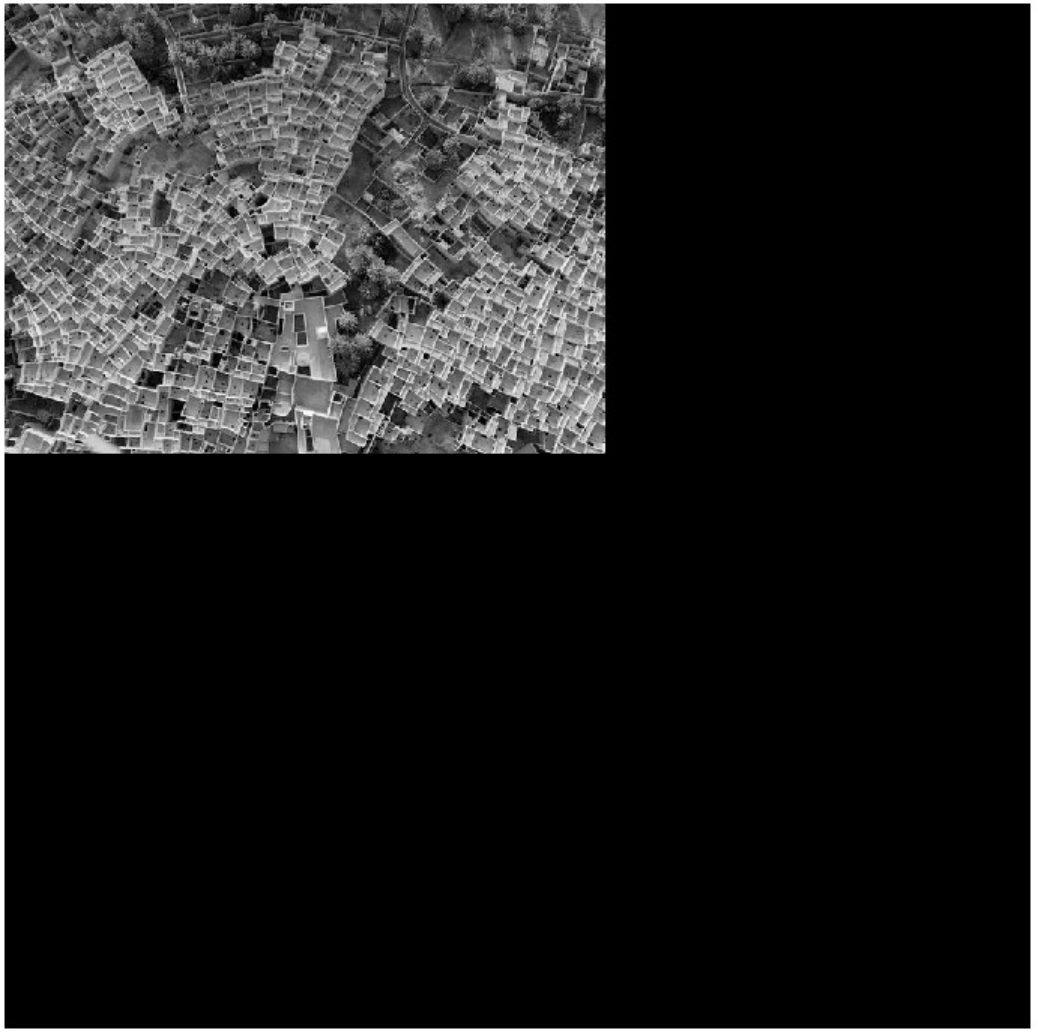
Linearity:

$$\mathcal{F}(af(x) + bg(x)) = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x))$$

The Fourier transform of a real signal is symmetric about the origin.

CODING TIME: Fourier transform in 2D

```
fftsize = 1024; % should be order of 2 (for speed) and include padding
im_fft = fft2(im, fftsize, fftsize);
disp('Observe the inverse fft to see the padding');
im_rec = ifft2(im_fft);
```



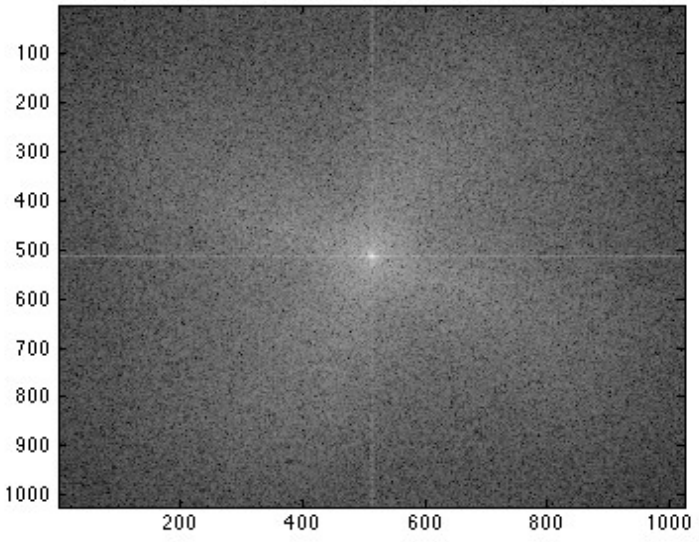
CODING TIME: 5 steps for using Fourier transform

```
Disp('(1) Transform the original image. Fftsize should be order
of 2 and include padding')
fftsize = 1024;
im_fft = fft2(im, fftsize, fftsize);

disp('We create a gaussian filter of half size value of 50')
hs = 50;
fil = fspecial('gaussian', hs*2+1, 10);

disp('(2) fft filter and pad to the same size as the image')
fil_fft = fft2(fil, fftsize, fftsize);

disp('(3) multiply fft images')
im_fil_fft = im_fft .* fil_fft;

disp('(4) inverse fft2')
im_fil = ifft2(im_fil_fft);

disp('(5) remove padding');
im_fil = im_fil(1+hs:size(im,1)+hs, 1+hs:size(im, 2)+hs);
```

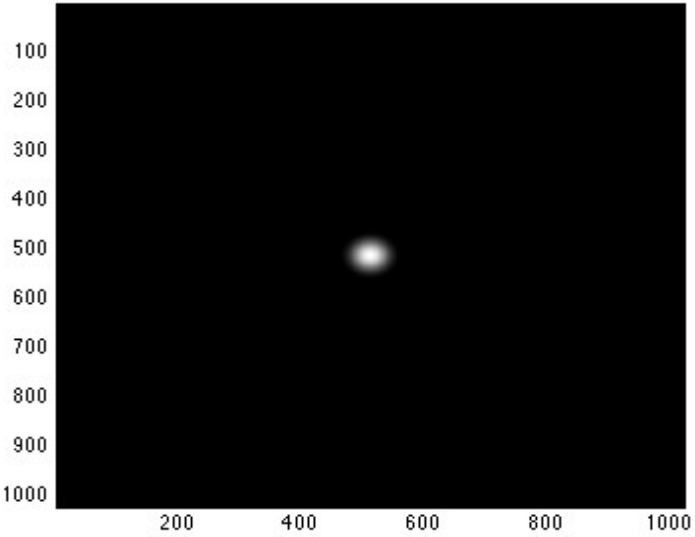
CODING TIME: Fourier transform in 2D

original

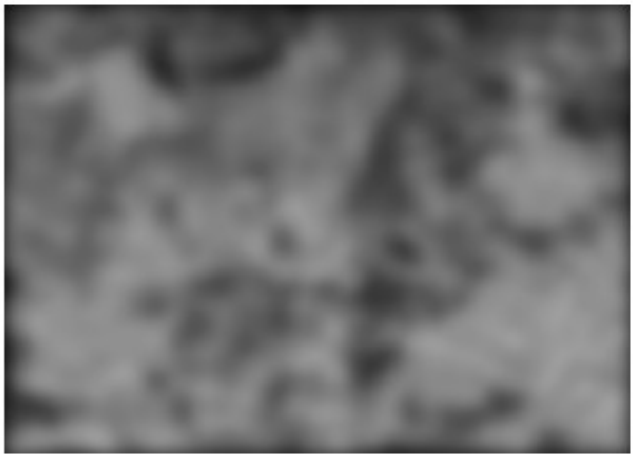
Fourier transform



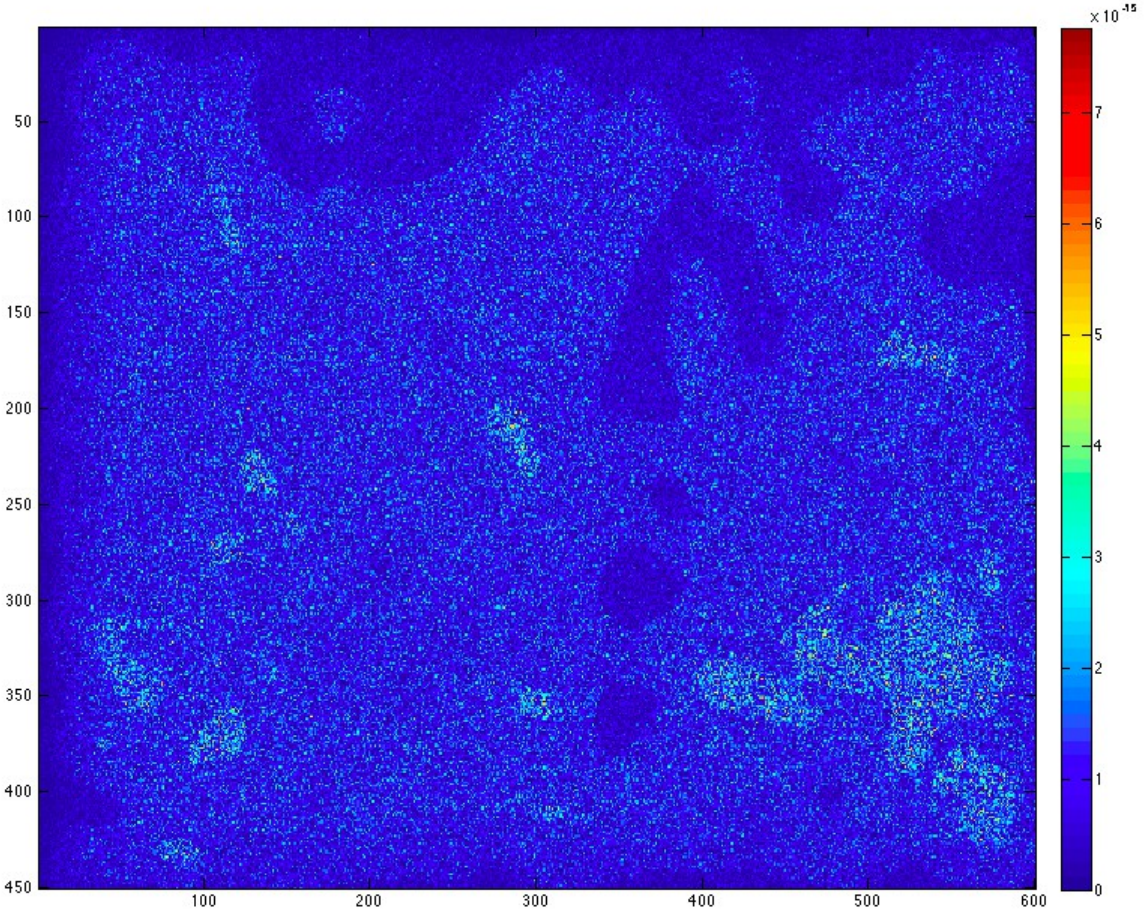
Fourier tranform of the filter



result

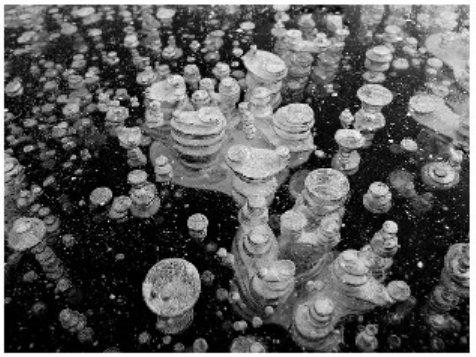


CODING TIME: Absolute difference between convolution and the Fourier counterpart

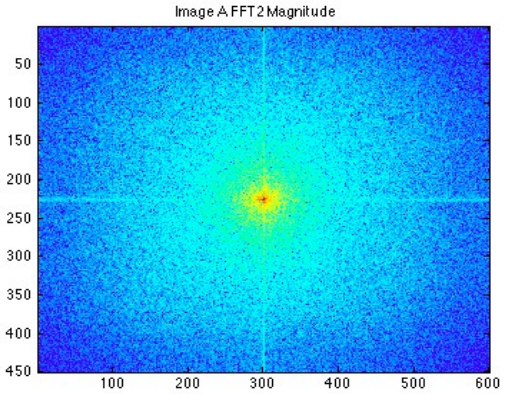


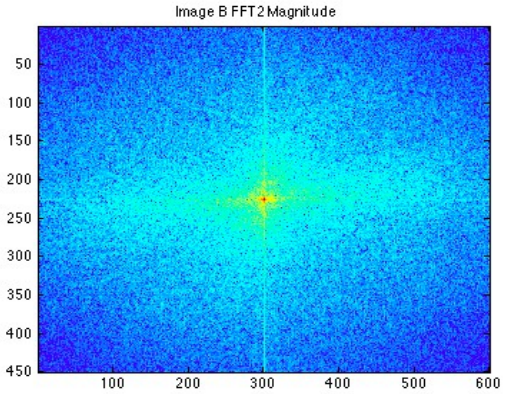
CODING TIME: Magnitude and Phase in Fourier

Imagen original

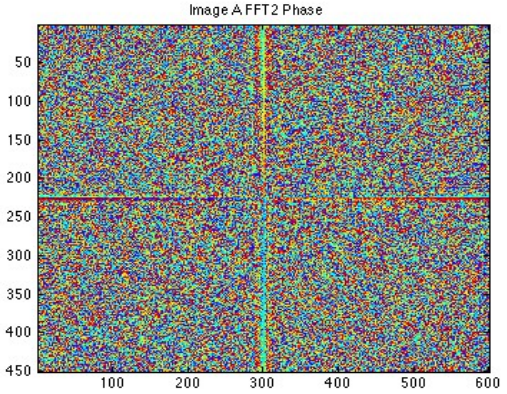


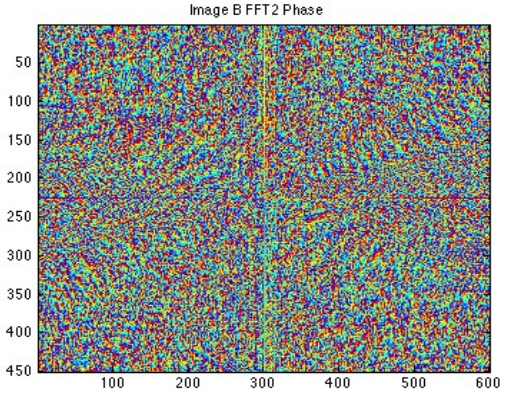
Magnitude





Phase





CODING TIME: Magnitude and Phase in Fourier

Image A

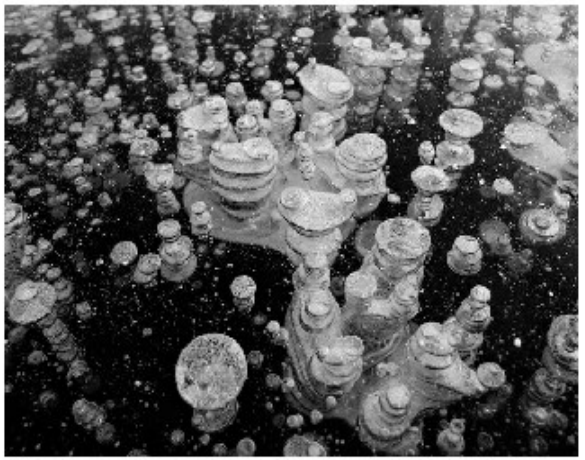
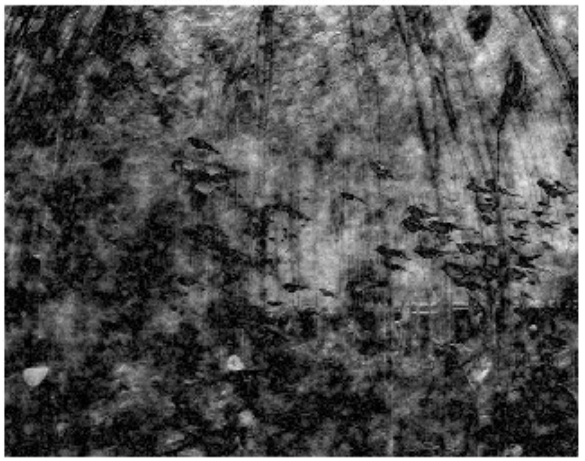
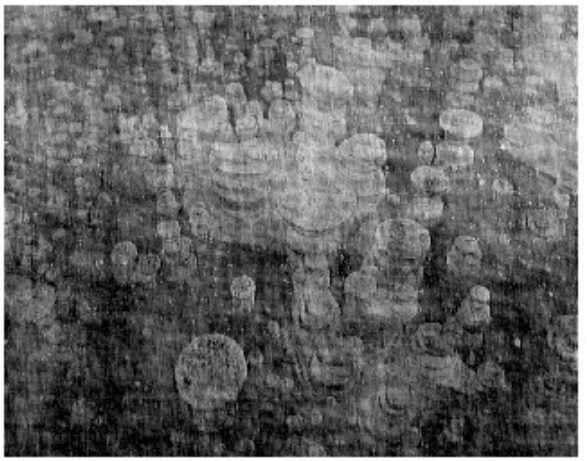


Image B

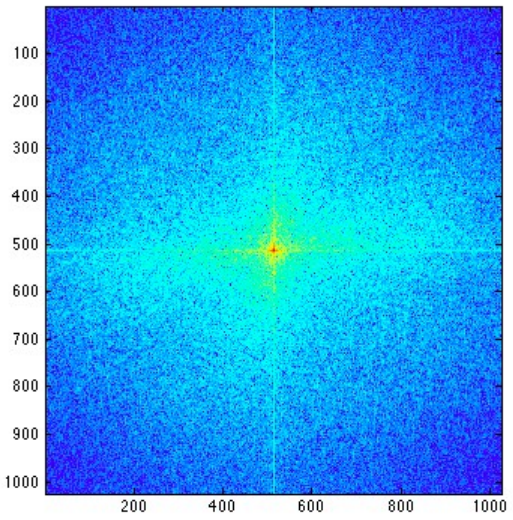
Magnitude from Image A and Phase from Image B

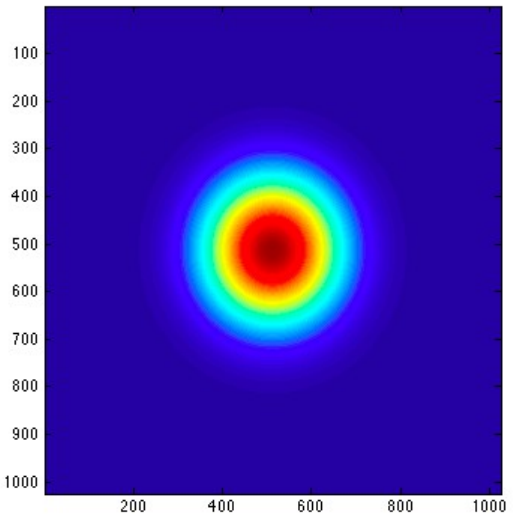


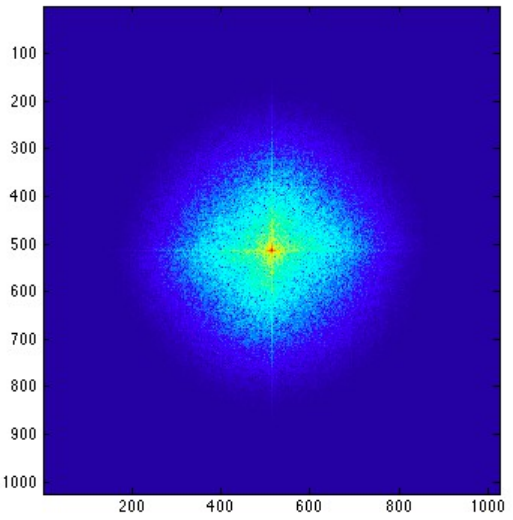
Magnitude from Image B and Phase from Image A



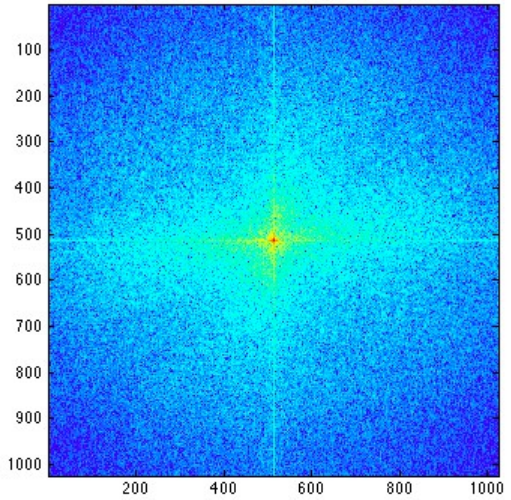
CODING TIME: Basic filtering in Fourier

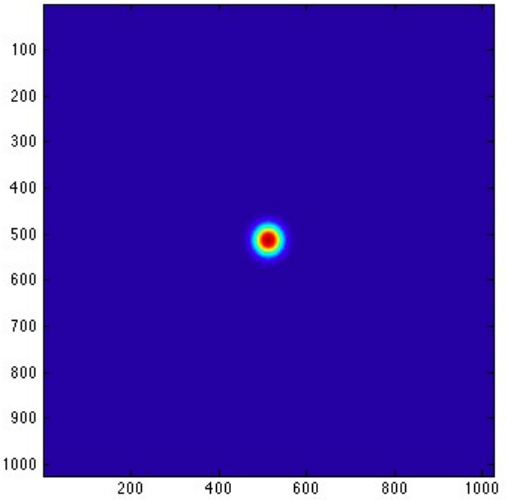


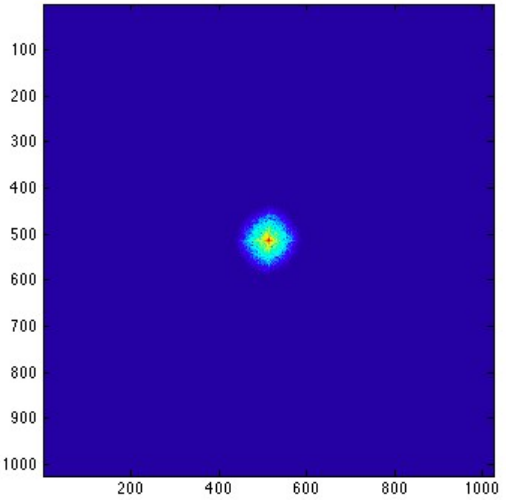




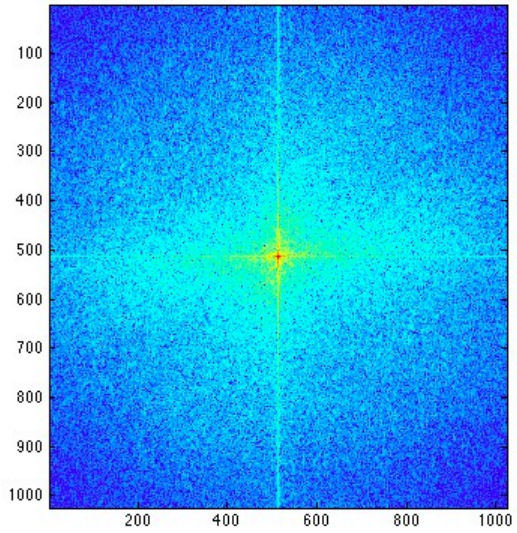
CODING TIME: Basic filtering in Fourier

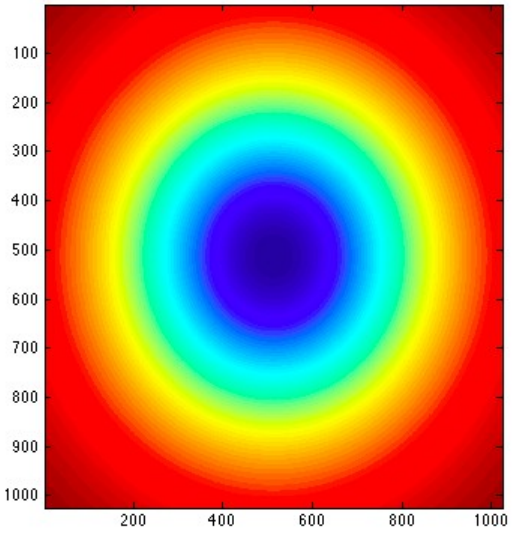


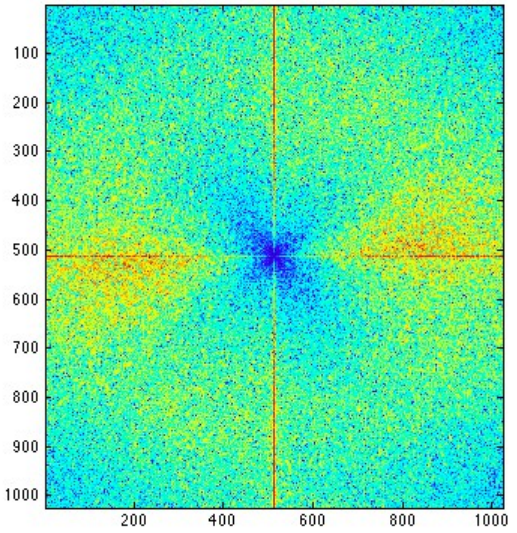




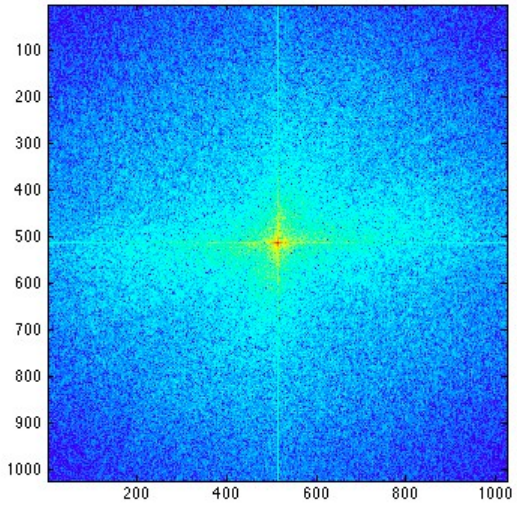
CODING TIME: Basic filtering in Fourier

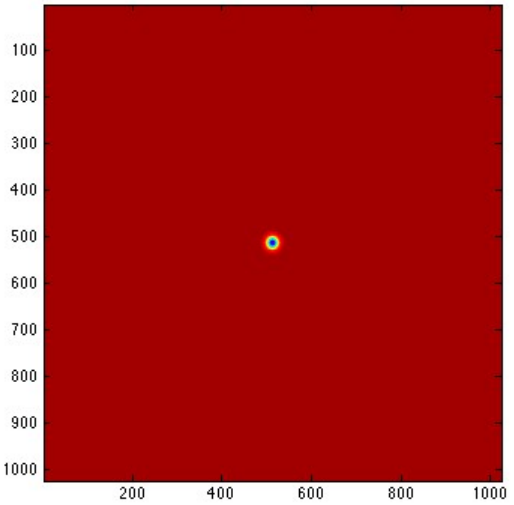


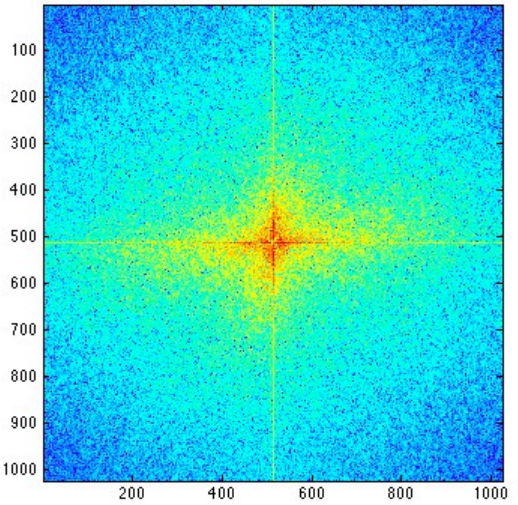




CODING TIME: Basic filtering in Fourier

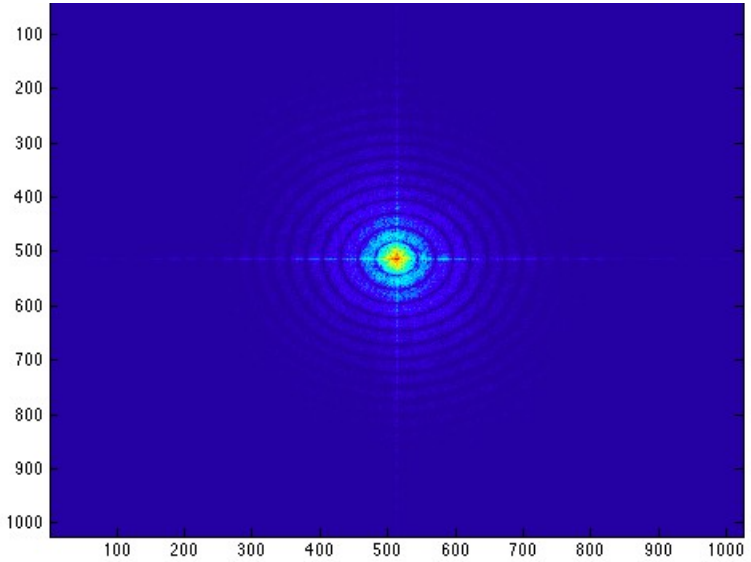






CODING TIME: Inverse filtering (for academic purposes)

$$G(w) = I(w)H(w) \implies I(w) = G(w)/H(w)$$



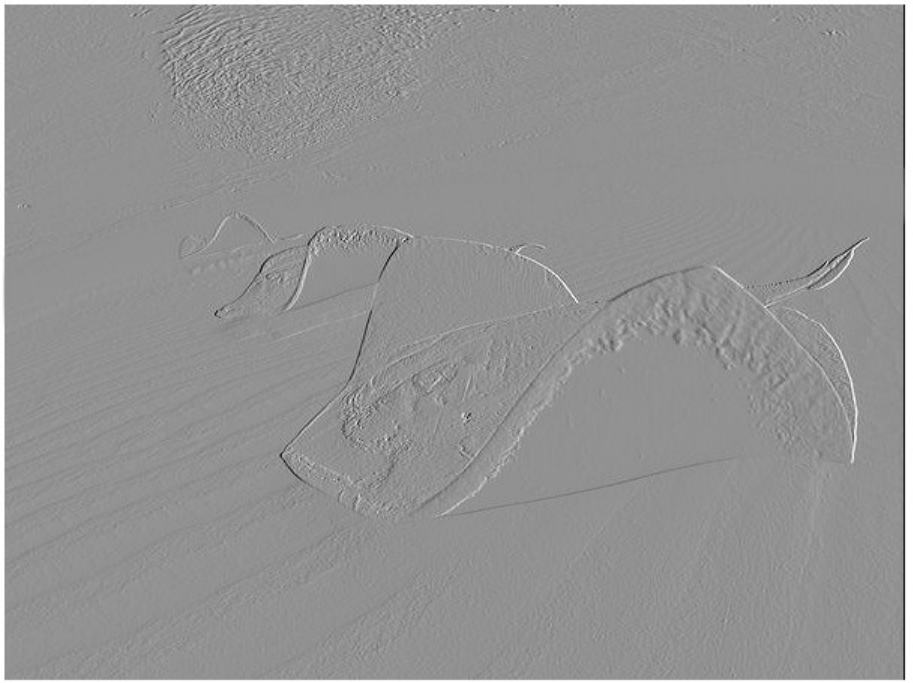
CODING TIME: Inverse filtering (for academic purposes)

$$G(w) = I(w)H(w) \implies I(w) = G(w)/H(w)$$

```
psf=fspecial('disk',20);
filtered=conv2(im,psf);

fft_filtered=fft2(filtered,1024,1024);
fft_psf=fft2(psf,1024,1024);
fft_recons=fft_filtered./fft_psf;

recons=ifft2(fft_recons);
```

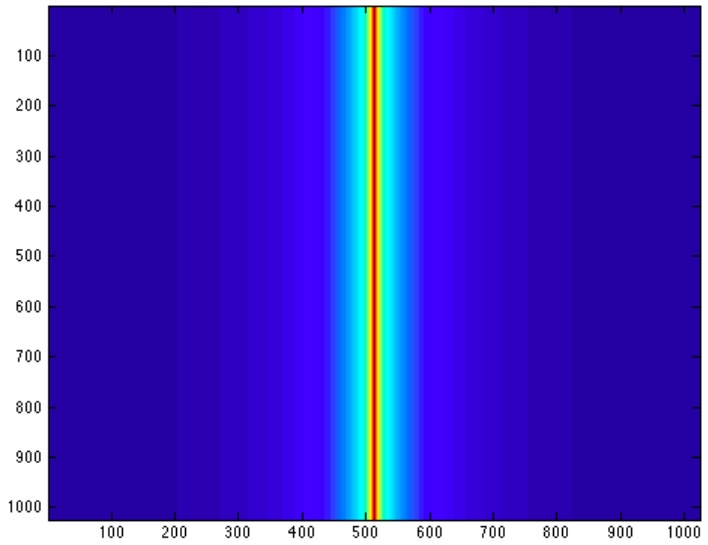
We must know the degradation model H(w)
(aka **Point Spread Function PSF**).

CODING TIME: Inverse filtering (for academic purposes)



input

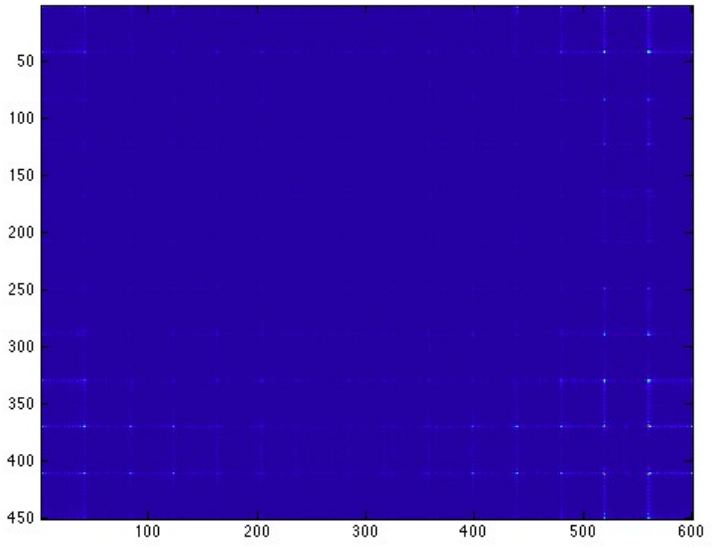
output

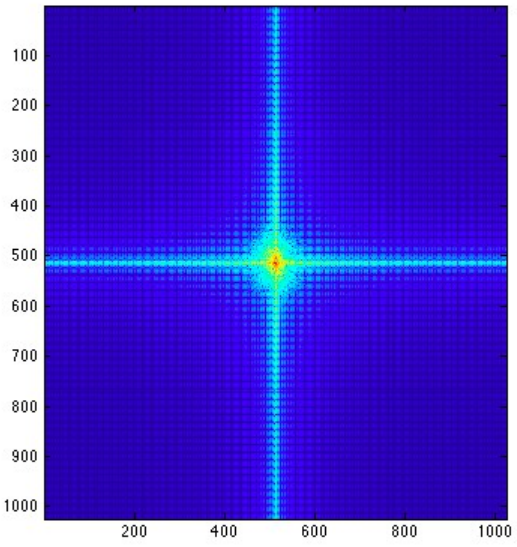


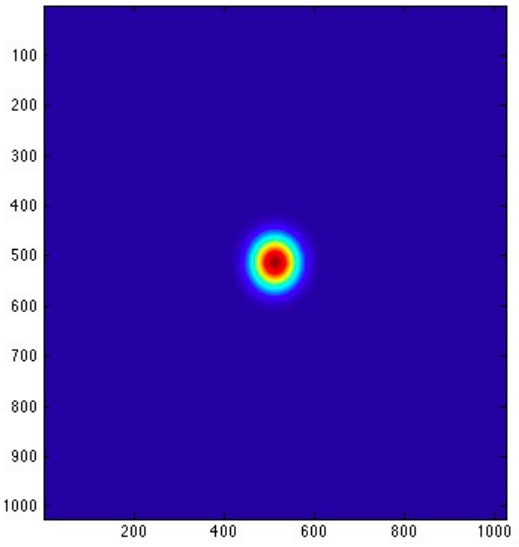
CODING TIME: Inverse filtering (for academic purposes)

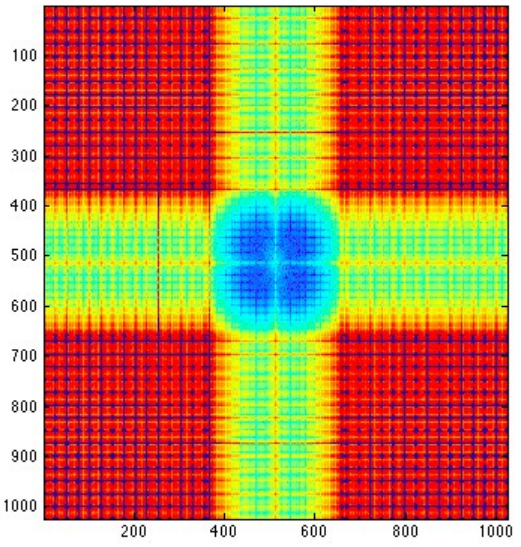
input

output



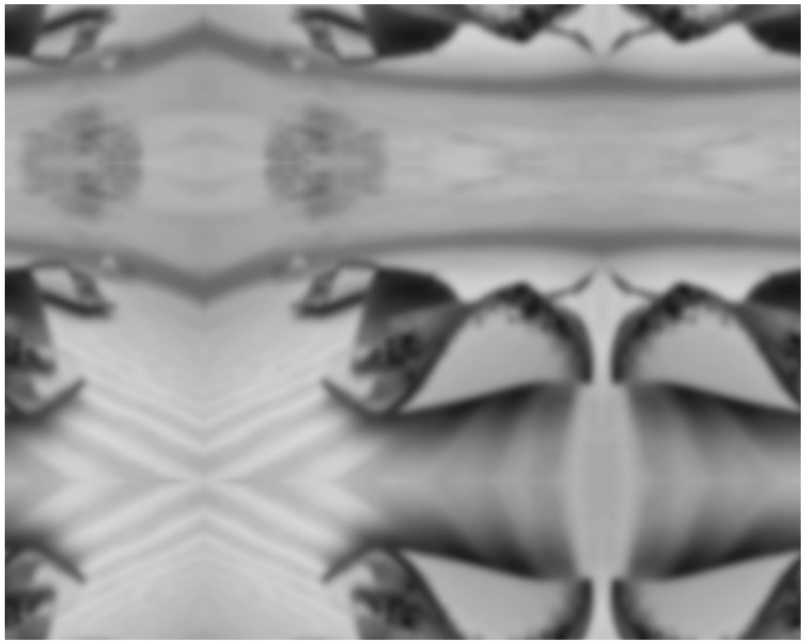




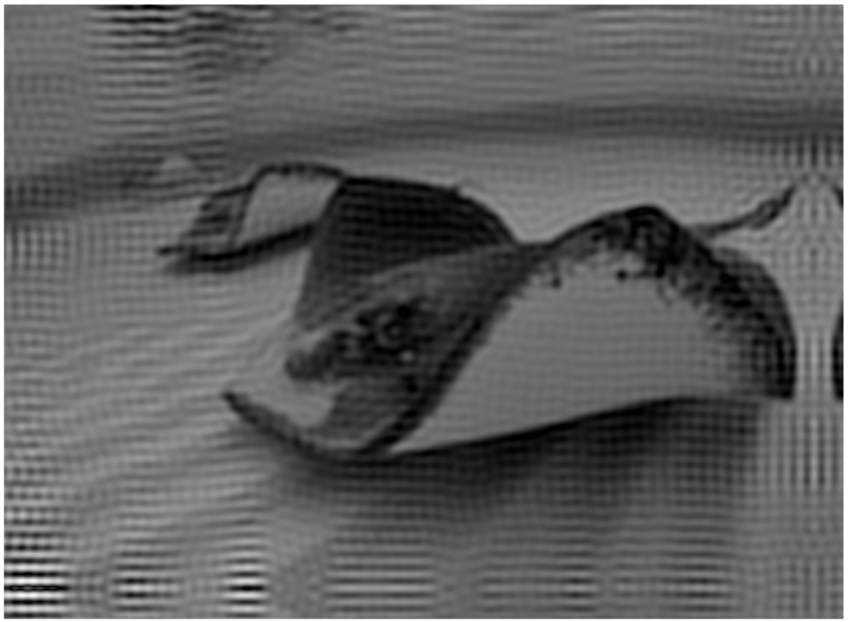


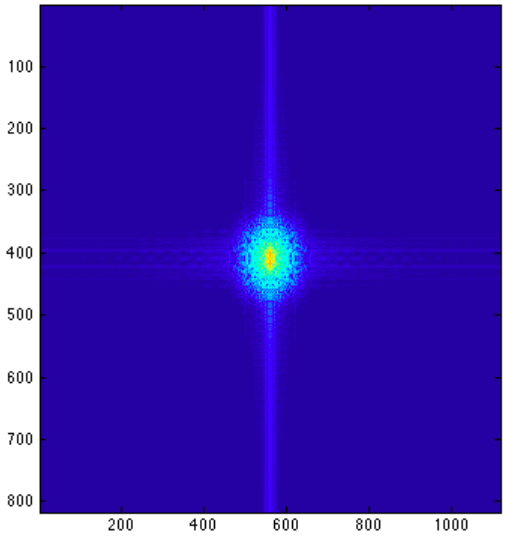
CODING TIME: Inverse filtering and borders

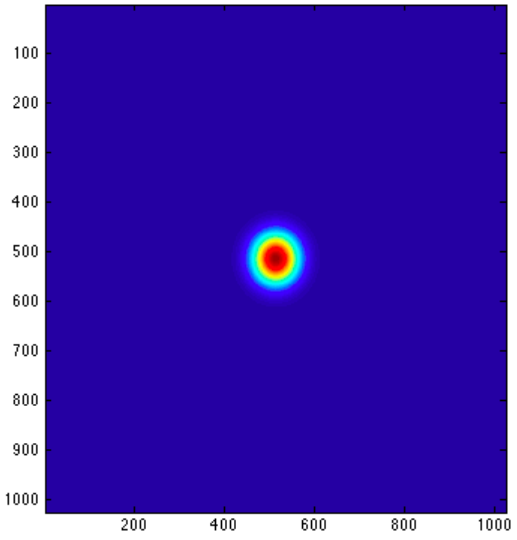
input

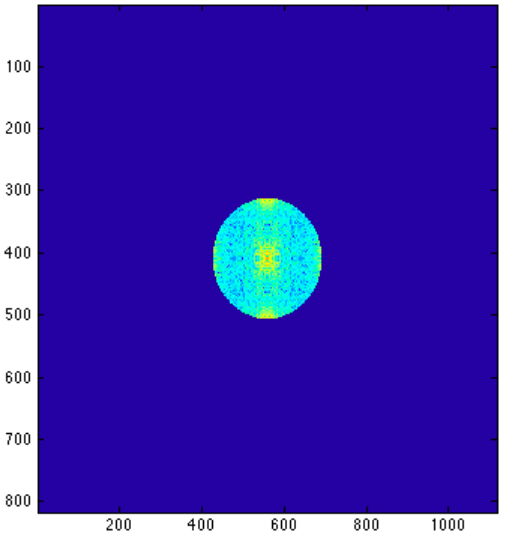


output





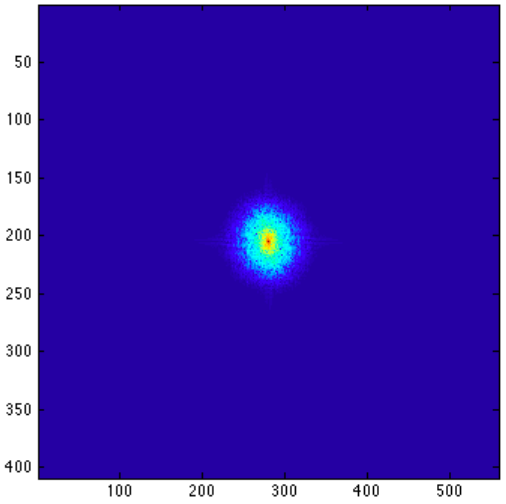


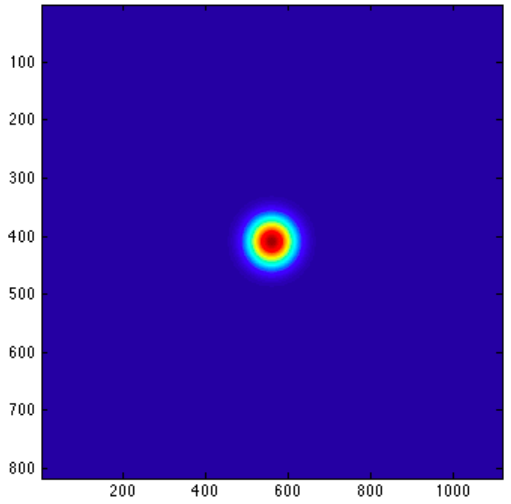


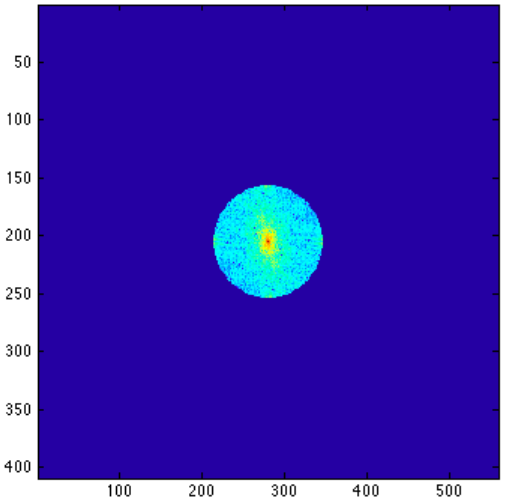
CODING TIME: Inverse filtering and borders

input

output







CODING TIME: Inverse filtering and borders

input

CODING TIME: Inverse filtering and borders

output

DECONVOLUTION

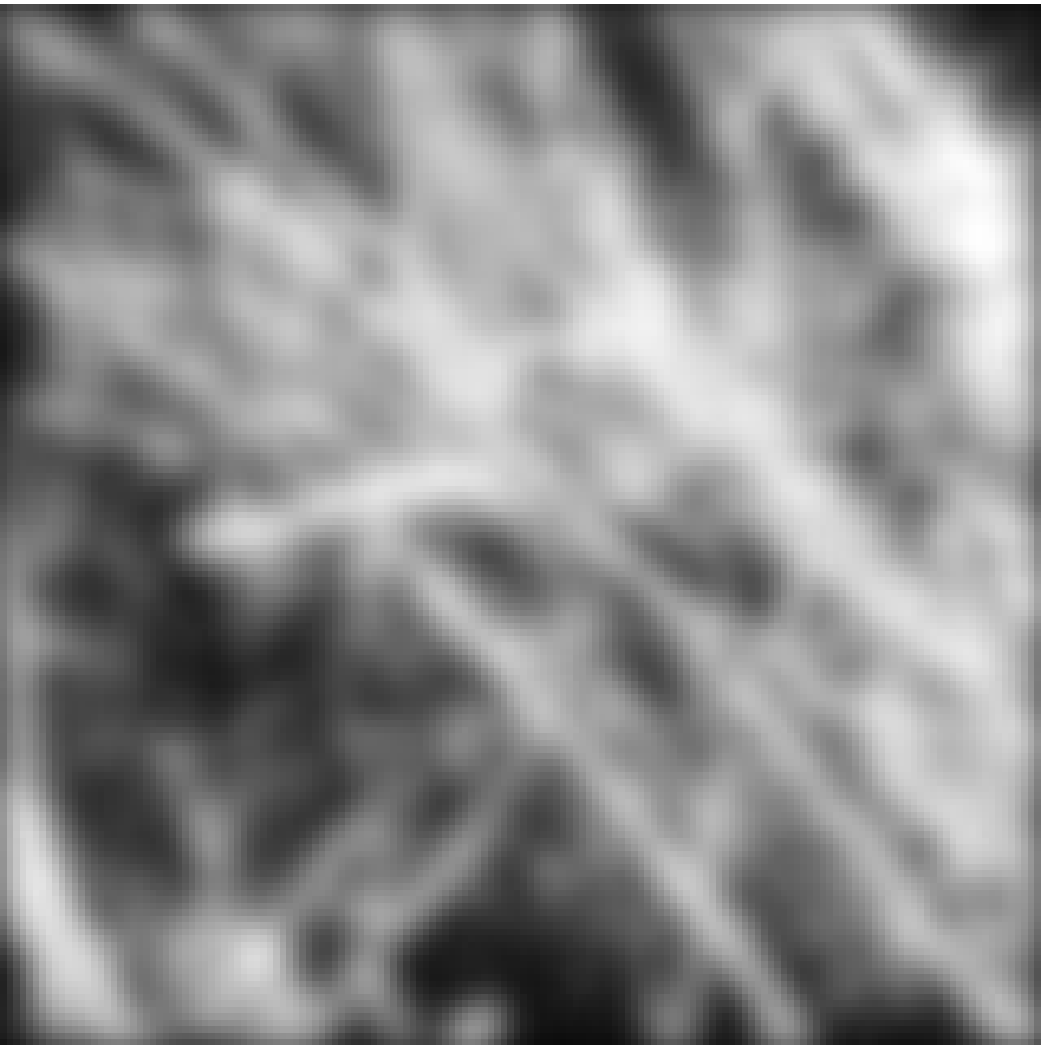
Inverse filtering is the naïve way of deconvolution.
The most well known method for deconvolution is the Wiener filter.

It is worth checking out the following...

- Lucy-Richardson
- Iterative methods in the primal (spatial domain)

WHY DOES BOX FILTER SMOOTHING HAVE SO MANY ARTIFACTS?

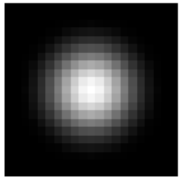
Gaussian 

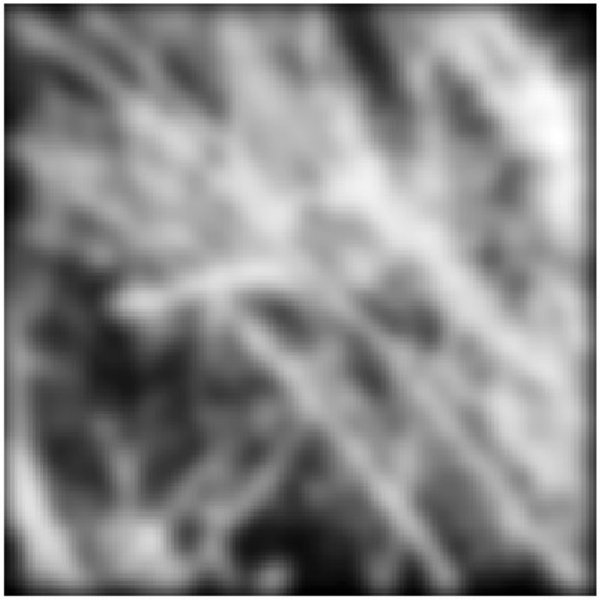


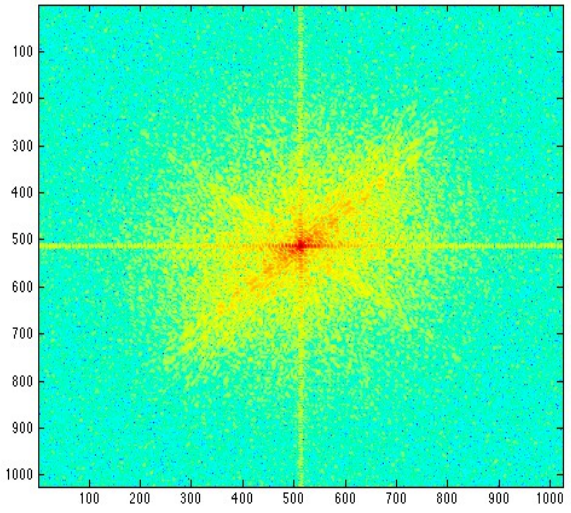
Box filter 

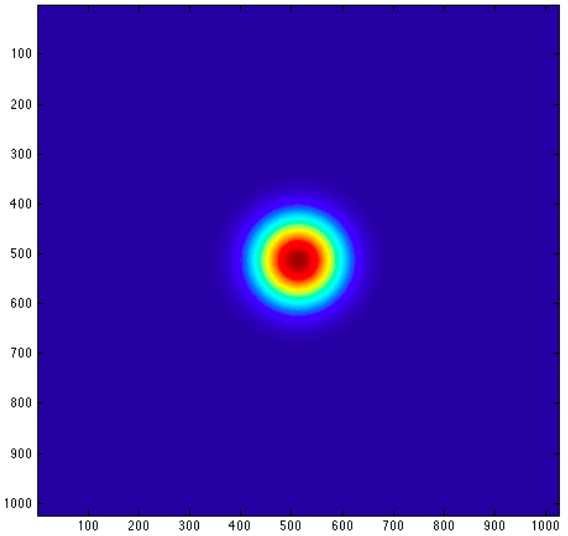
WHY DOES BOX FILTER SMOOTHING HAVE SO MANY ARTIFACTS?

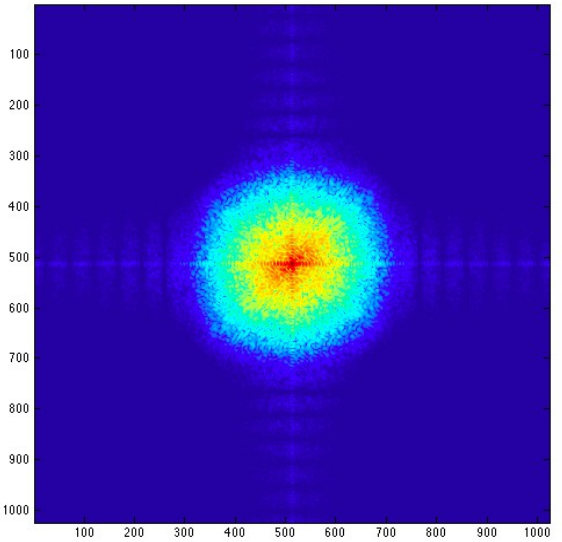
Gaussian Filter





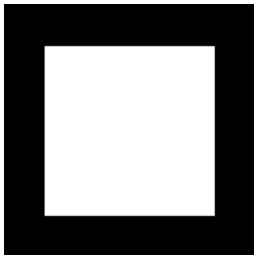


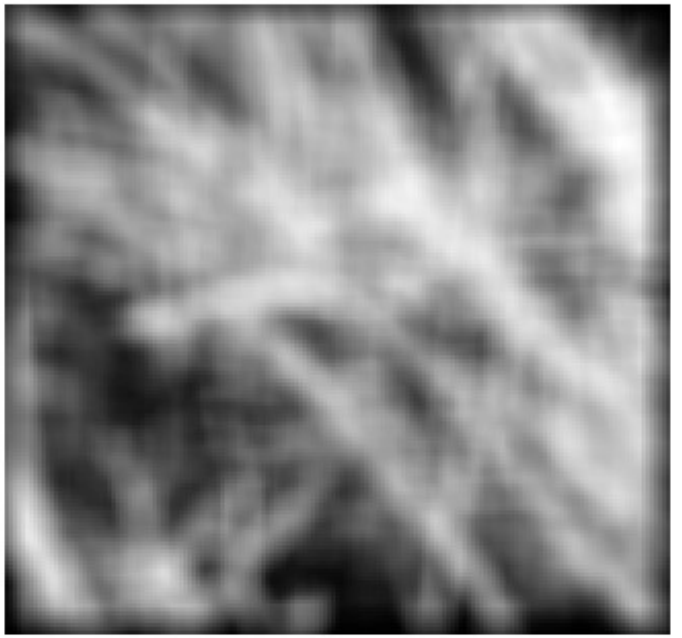


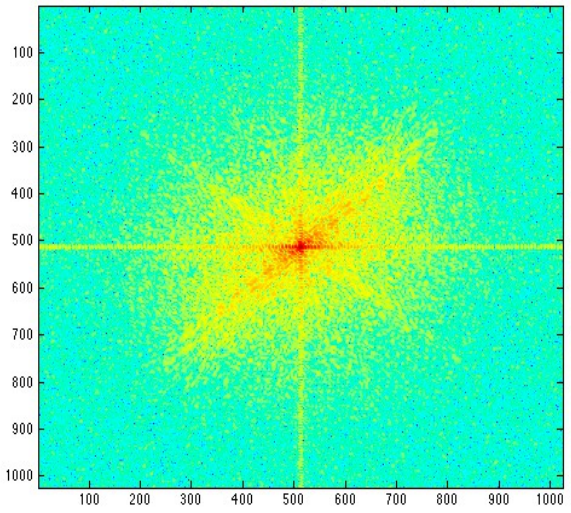


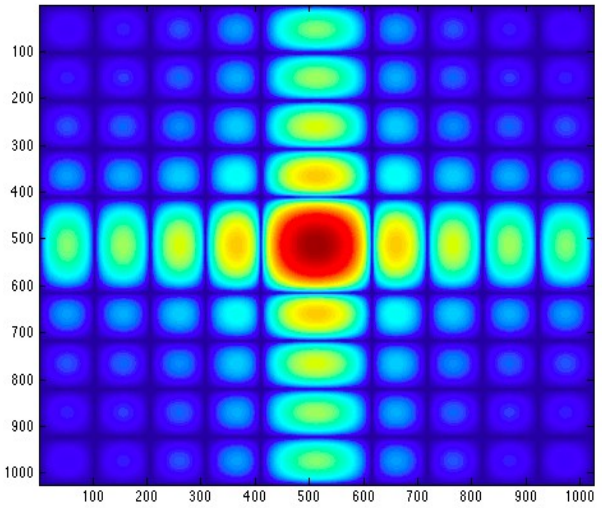
WHY DOES BOX FILTER SMOOTHING HAVE SO MANY ARTIFACTS?

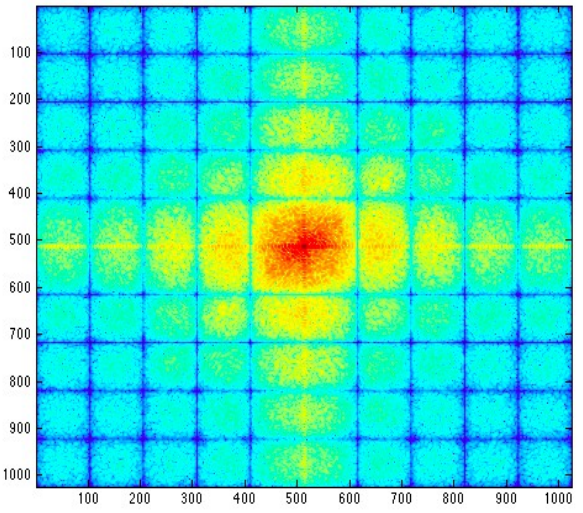
Box Filter











WHY DOES A LOWER RESOLUTION IMAGE STILL MAKE SENSE TO US? WHAT DO WE LOSE?

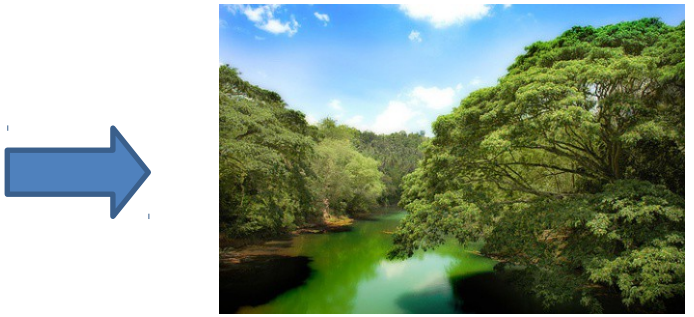
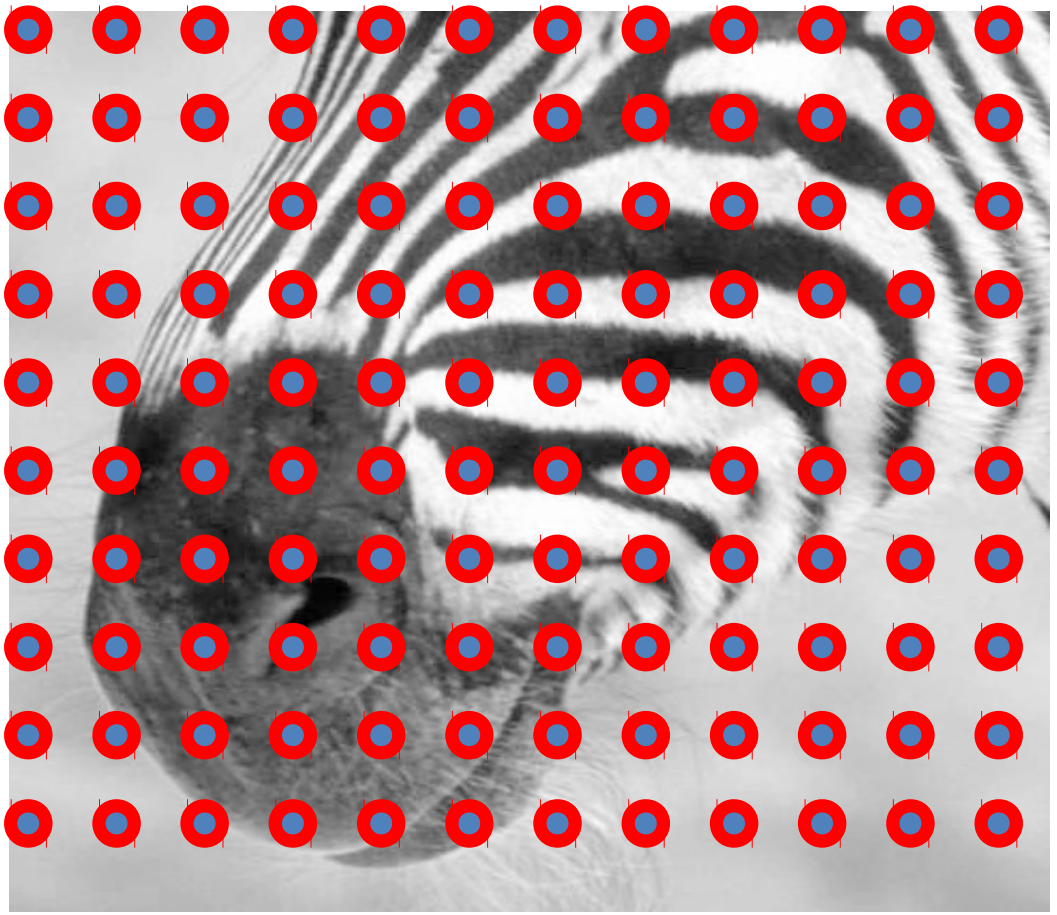


Image: http://www.flickr.com/photos/igorms/136916757/

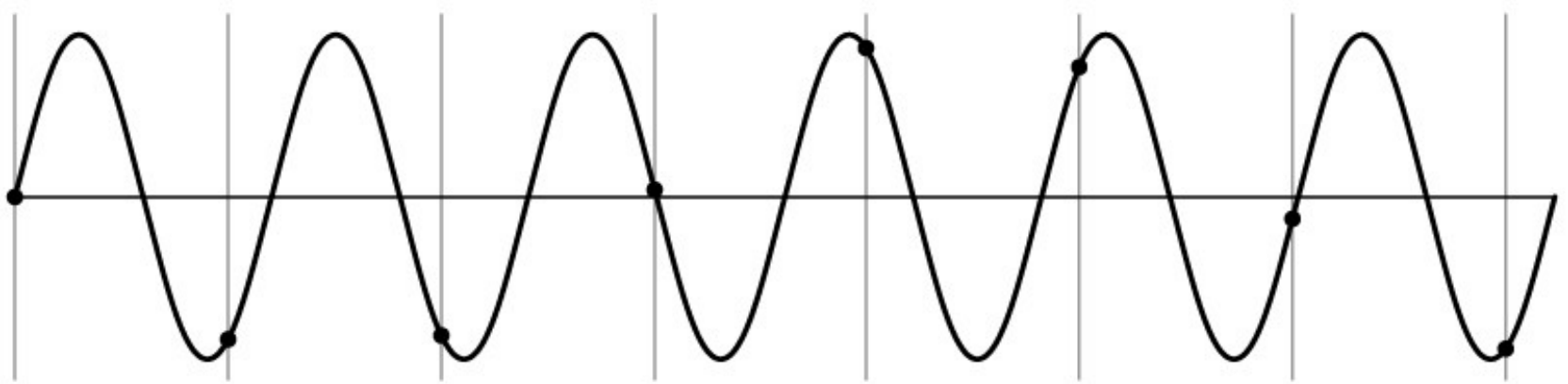
SUBSAMPLING BY A FACTOR OF 2



Throw away every other row and column
to create a 1/2 size image

ALIASING PROBLEM

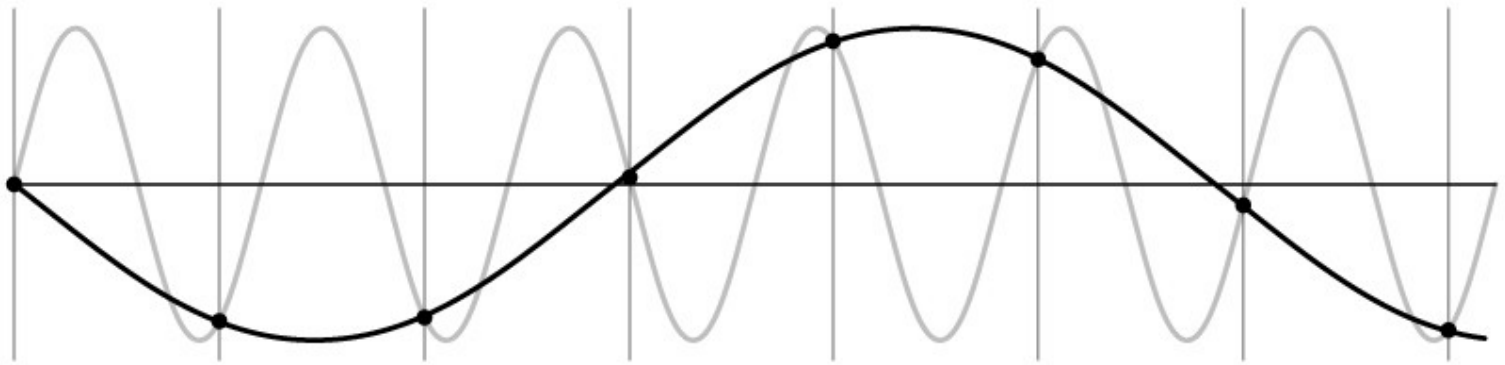
1D example (sinewave):



Source: S. Marschner

ALIASING PROBLEM

1D example (sinewave):

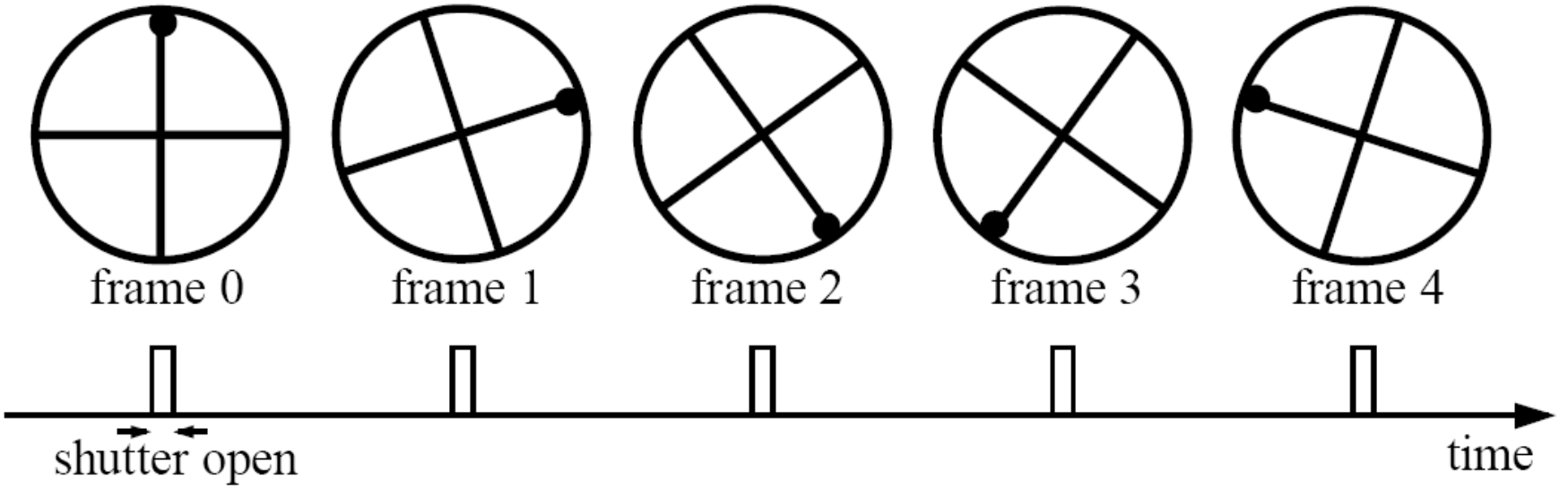


Source: S. Marschner

ALIASING IN VIDEO

Imagine a spoked wheel moving to the right (rotating clockwise). Mark wheel with dot so we can see what's happening.

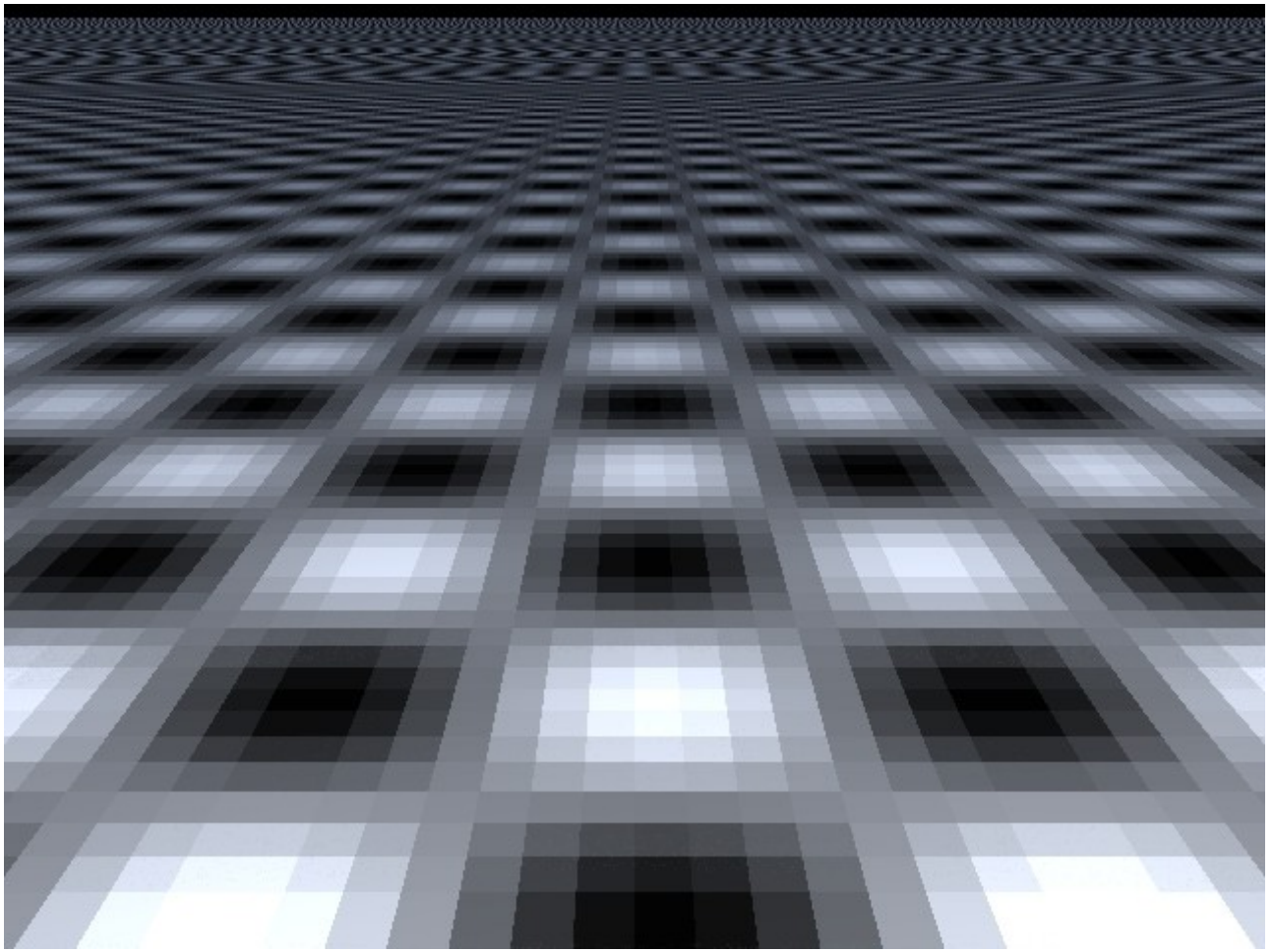
If camera shutter is only open for a fraction of a frame time (frame time = 1/30 sec. for video, 1/24 sec. for film):



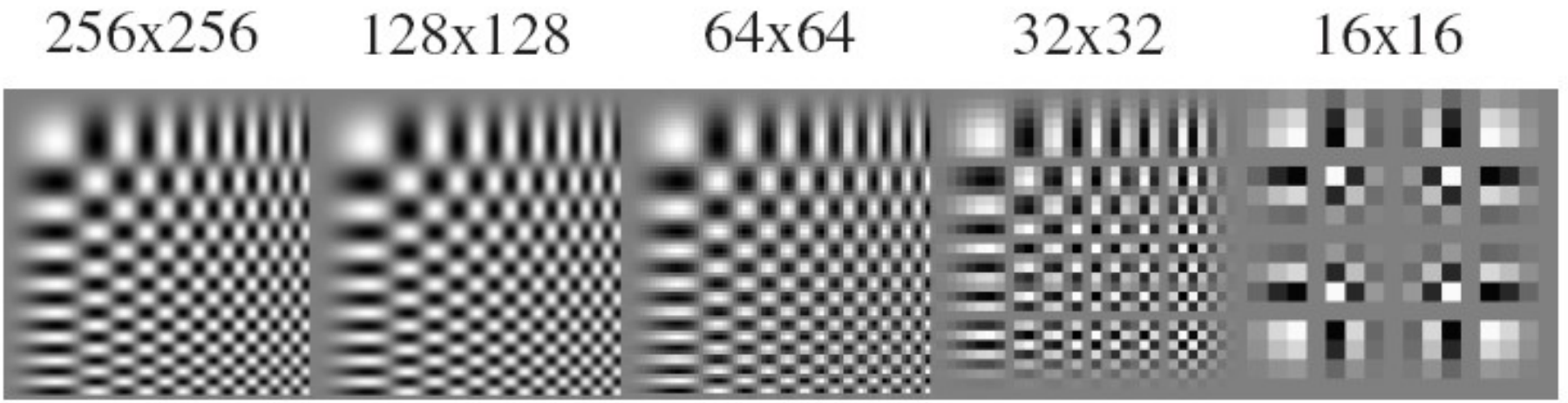
Without dot, wheel appears to be rotating slowly backwards! (counterclockwise)

Slide by Steve Seitz

ALIASING IN CG

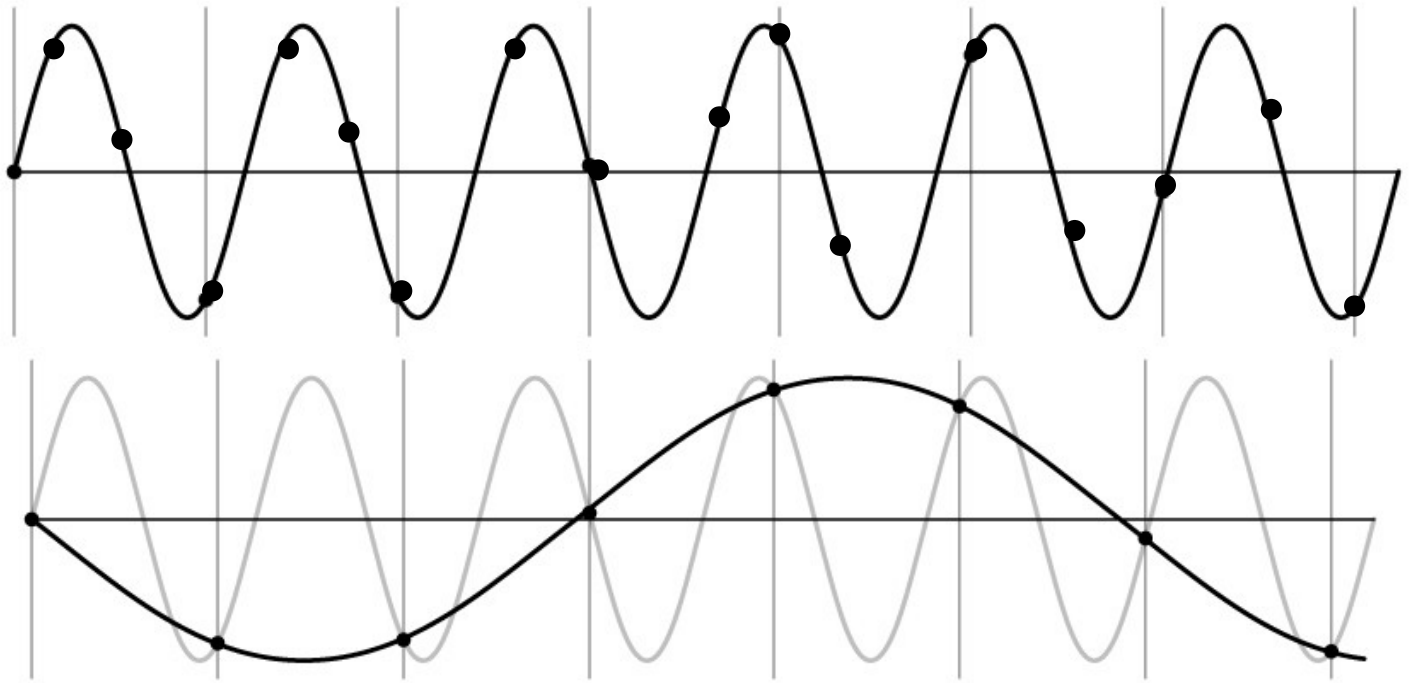


SAMPLING AND ALIASING



NYQUIST-SHANNON SAMPLING THEOREM

- When sampling a signal at discrete intervals, the sampling frequency must be $f_{sampling} \geq 2f_{max}$
- f_{max} = max frequency of the input signal
- This will allows to reconstruct the original perfectly from the sampled version



ANTI-ALIASING

Solutions:

- Sample more often

- Get rid of all frequencies that are greater than half the new sampling frequency
  - Will lose information
  - But it’s better than aliasing
  - Apply a smoothing filter

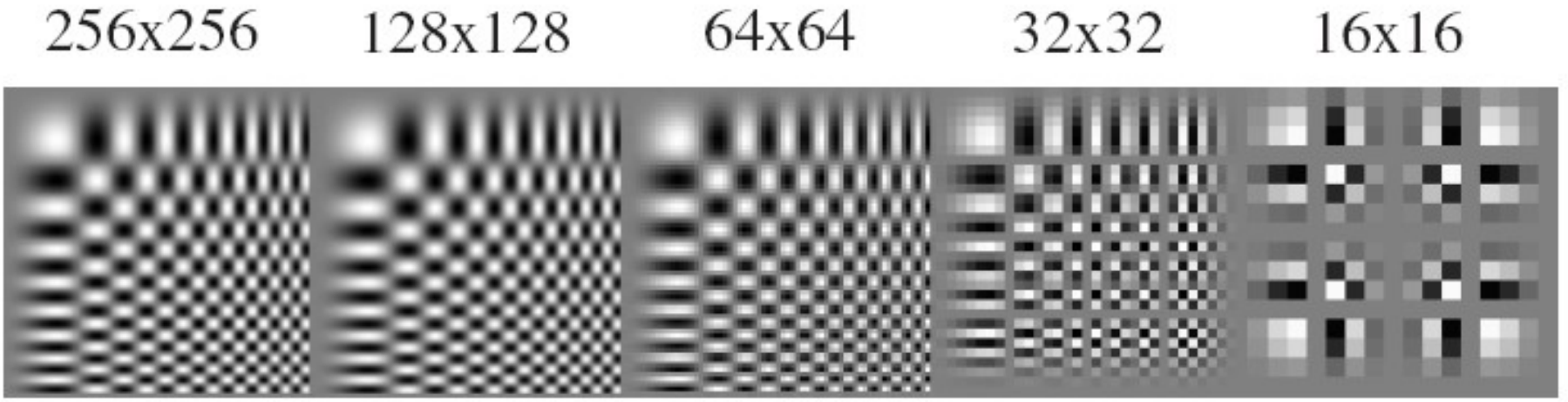
DOWNSAMPLING ALGORITHM

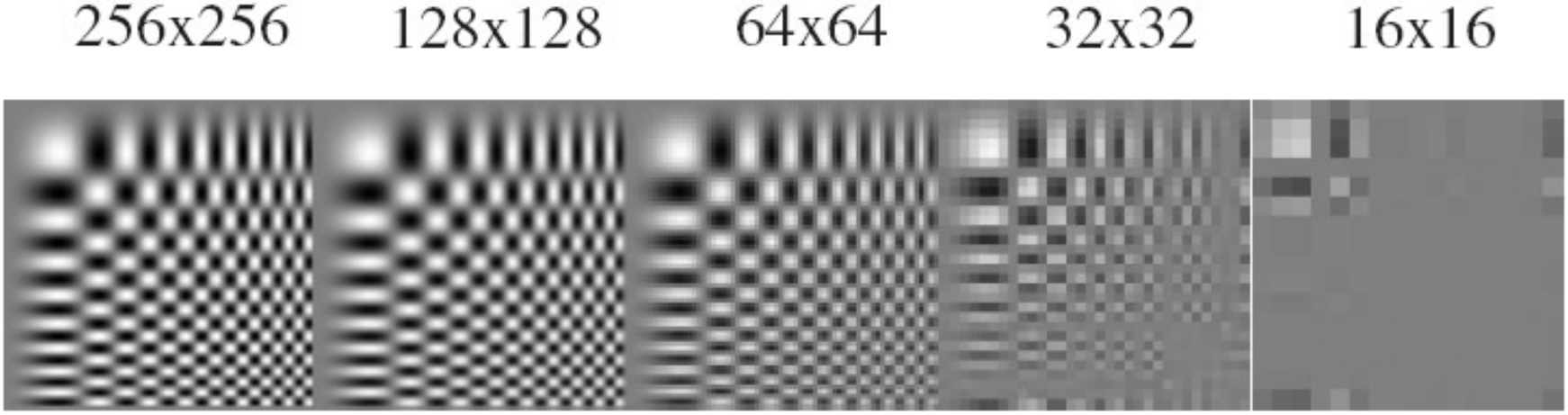
1. Start with image(h, w)
2. Apply low-pass filter

   im_blur = imfilter(image, fspecial(‘gaussian’, 7, 1))
3. Sample every other pixel

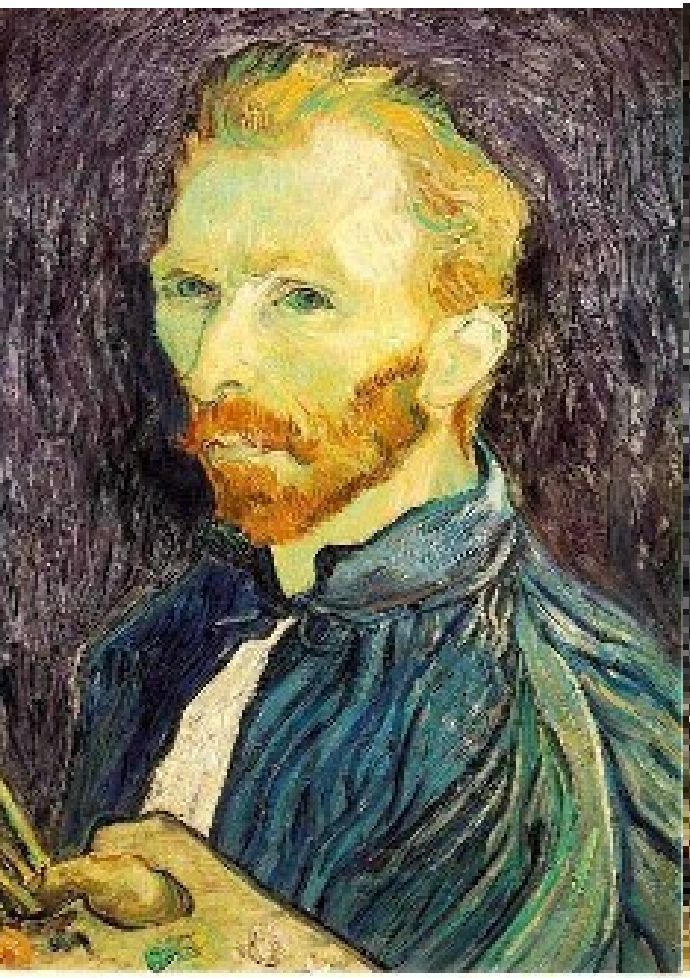
   im_small = im_blur(1:2:end, 1:2:end);

ANTI-ALIASING





SUBSAMPLING WITHOUT PREFILTERING



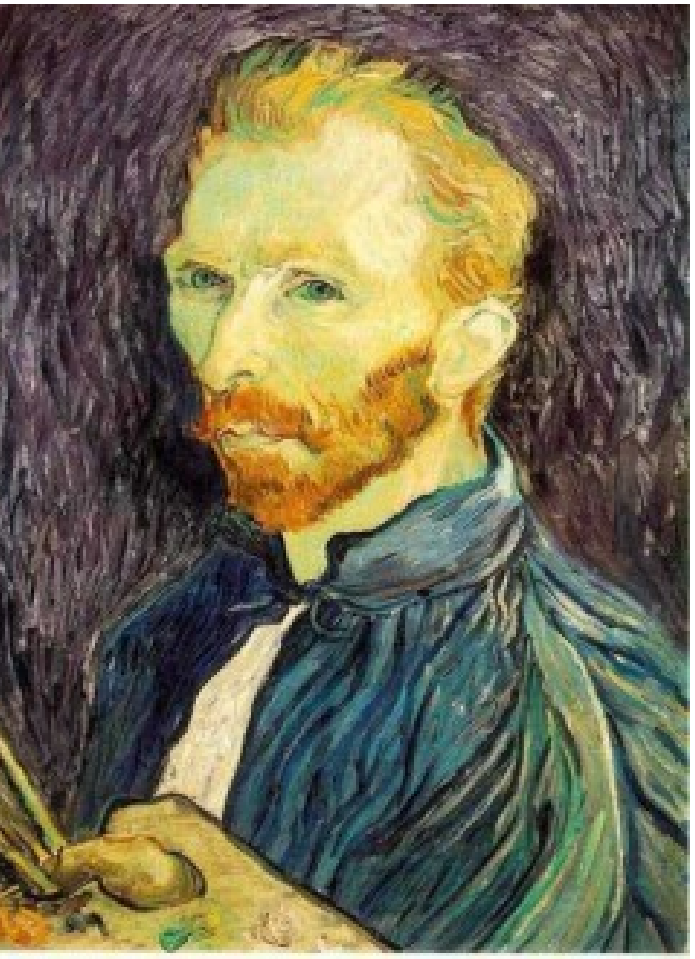
1/2

1/4 (2x zoom)

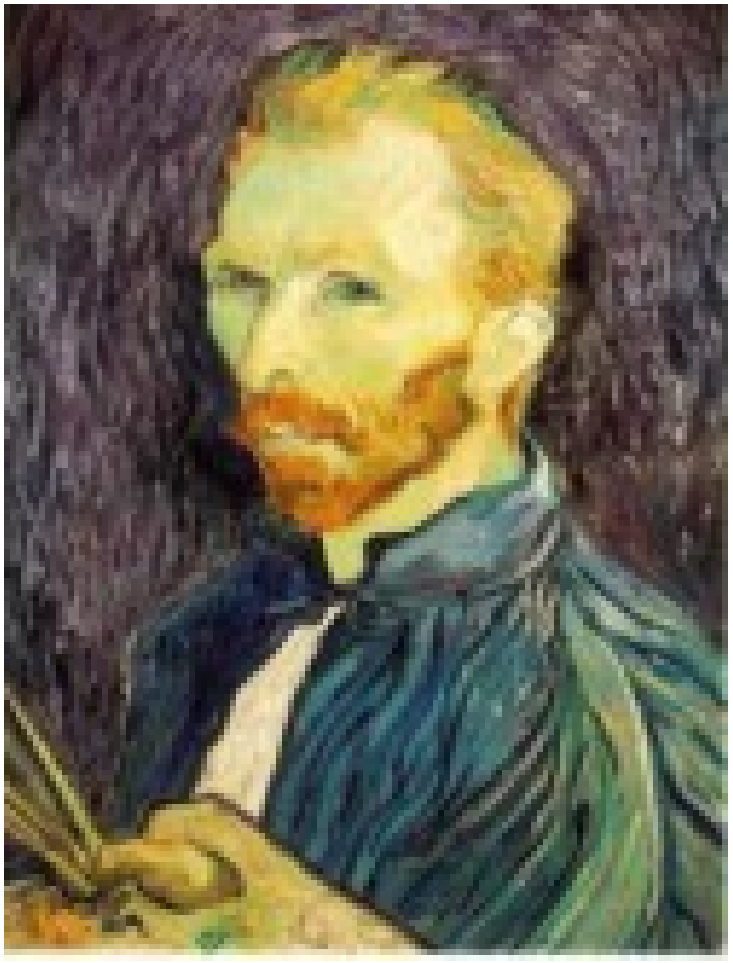
1/8 (4x zoom)

Slide by Steve Seitz

SUBSAMPLING WITH PREFILTERING



Gaussian 1/2



G 1/4

G 1/8

Slide by Steve Seitz

WHY DOES A LOWER RESOLUTION IMAGE STILL MAKE SENSE TO US? WHAT DO WE LOSE?

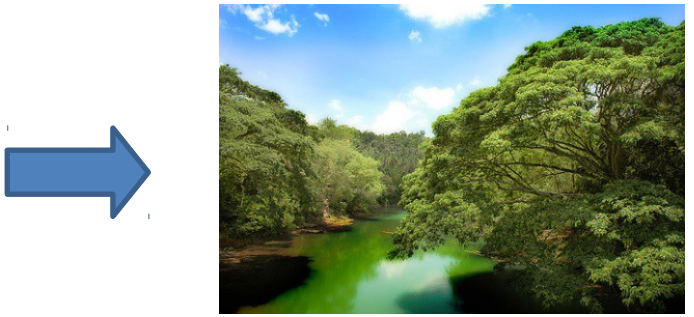
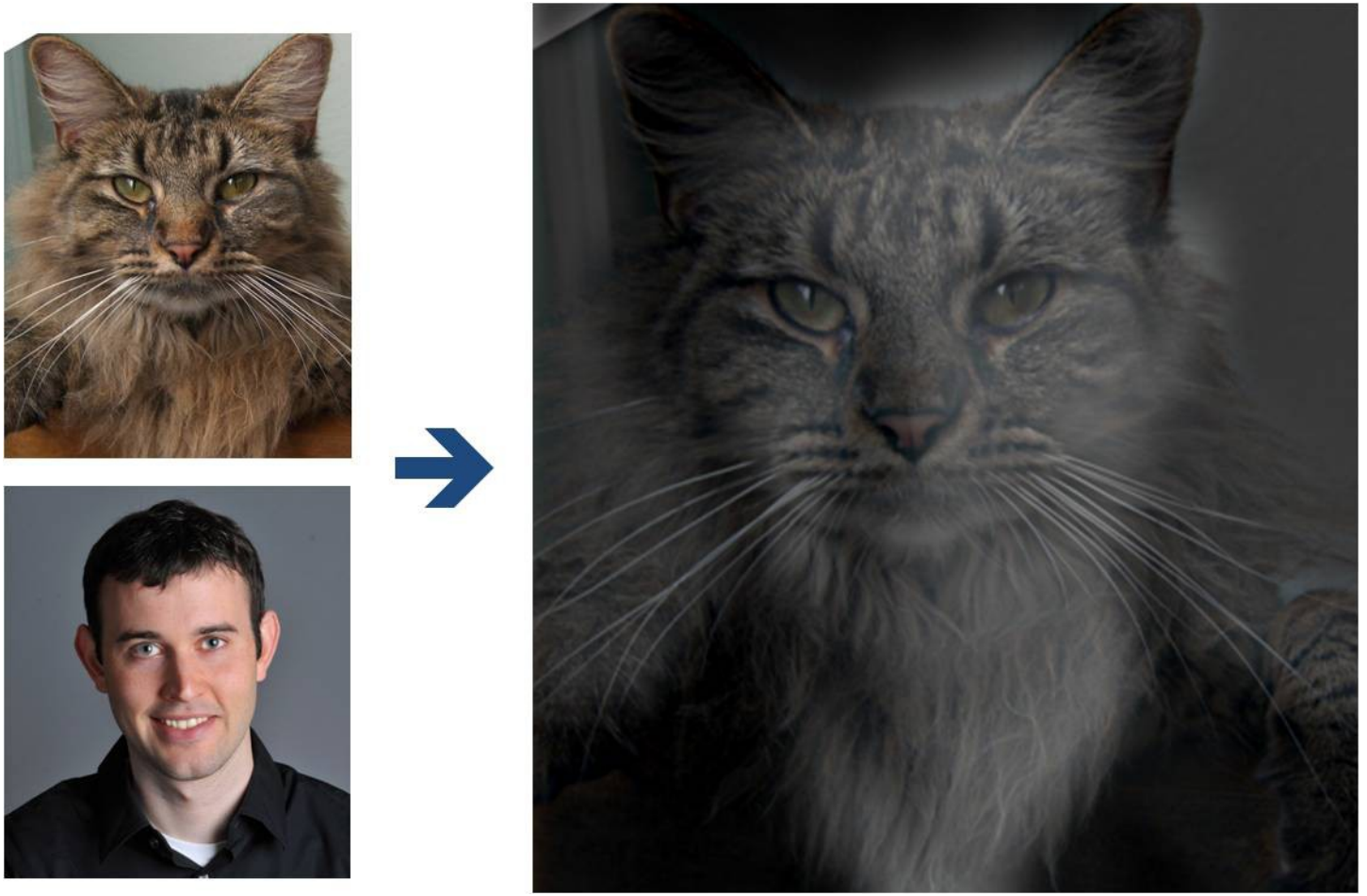


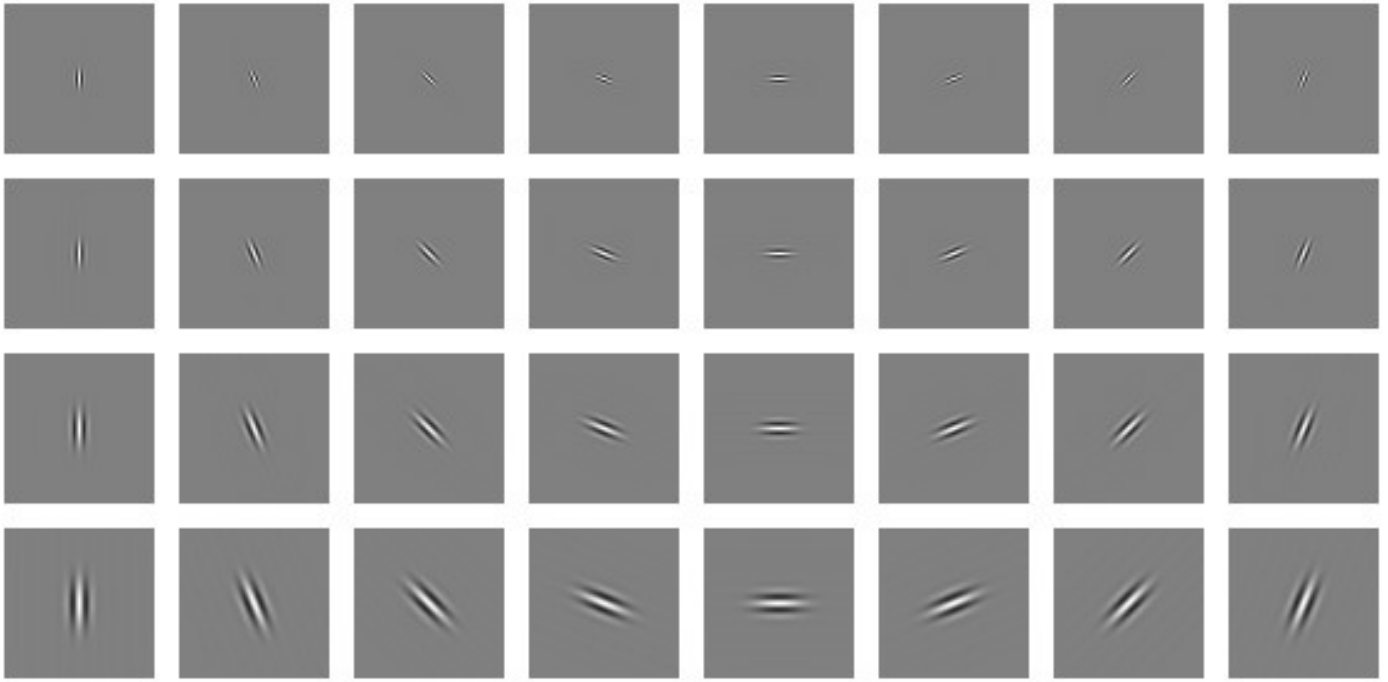
Image: http://www.flickr.com/photos/igorms/136916757/

WHY DO WE GET DIFFERENT, DISTANCE-DEPENDENT INTERPRETATIONS OF HYBRID IMAGES?



CLUES FOR HUMAN PERCEPTION

- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid frequencies dominate perception
- When we see an image from far away, we are effectively subsampling it



Early Visual Processing: Multi-scale edge and blob filters

HIBRID IMAGE IN FFT

YOUR TURN ...

Sharpening revisited

- What does blurring take away?

–

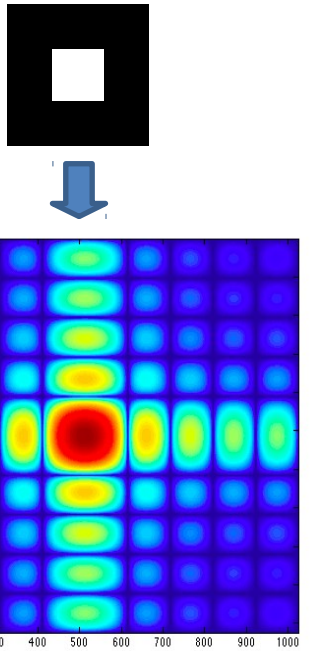
=

Let’s add it back:

+ α

=

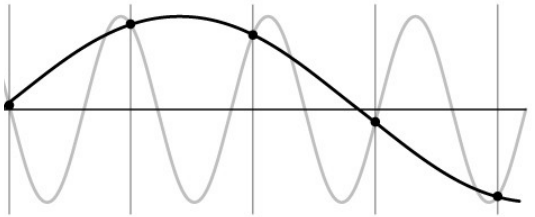
THINGS TO REMEMBER



- Sometimes it makes sense to think of images and filtering in the frequency domain
  - Fourier analysis

- Can be faster to filter using FFT for large images (N logN vs. N2 for auto-correlation)

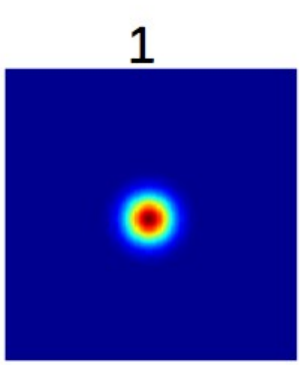
- Images are mostly smooth
  - Basis for compression

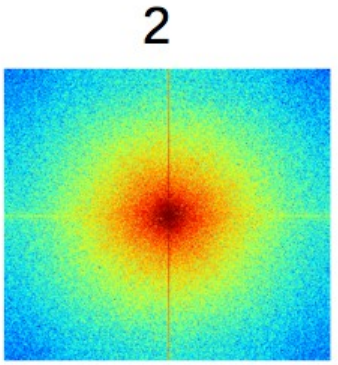


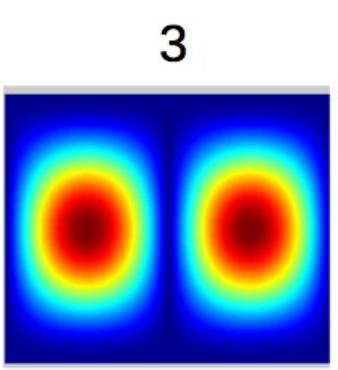
- Remember to low-pass before sampling

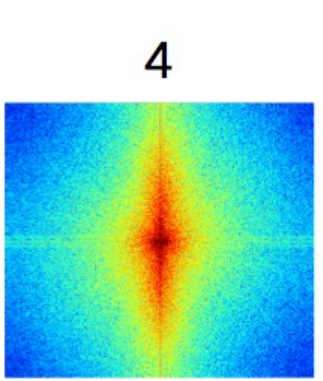
TAKE HOME QUESTIONS

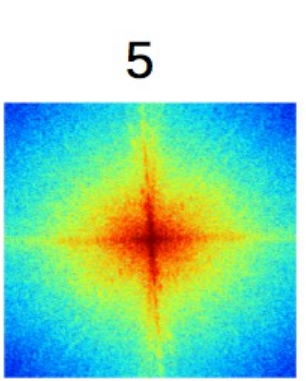
1. Match the spatial domain image to the Fourier magnitude image

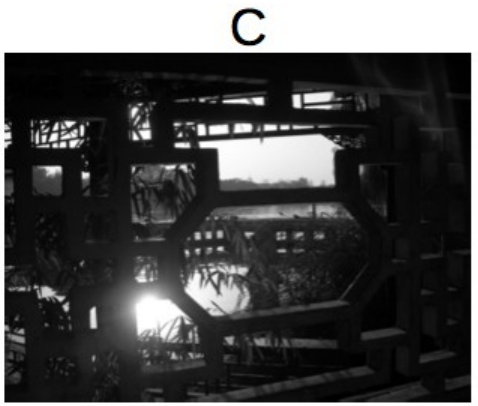


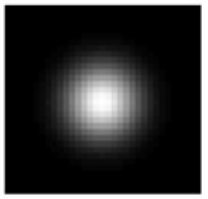












NEXT CLASS

- Denoising
- Template matching
- Image pyramids
- Bilateral filtering