AWtools: a toolbox for designing MIMO anti-windup strategies

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AWtools is a short toolbox for Matlab aimed to design anti-windup (AW) strategies for MIMO plants. The design tools are based on the traditional two-step approaches, in which the AW compensator is designed after a controller is tuned without considering saturation. **AWtools** covers LTI and LPV plants and is based on the comprime factor formulation introduced in [4, 3].

1 Getting started

This toolbox provides the synthesis function

```
[Kaw,Taw,g] = awsyn(G,K,method,nr,opts,F1,F2);
```

to design the anti-windup (AW) compensation scheme depicted in Figure 1 [4], where `Kaw` is the combination of the controller `K` and the AW compensator `Taw`.

The plant `G` and the controller `K` can be LTI or LPV systems. These systems can be modelled with Control System/Robust Control Toolbox objects, LMI Toolbox objects, or LPV objects from **LPVtools**. The input argument `method` is a string defining the synthesis method:

- `'fsec'`: produces a full order AW compensator `Taw` designed according to the method proposed in [3] based on sector boundedness,

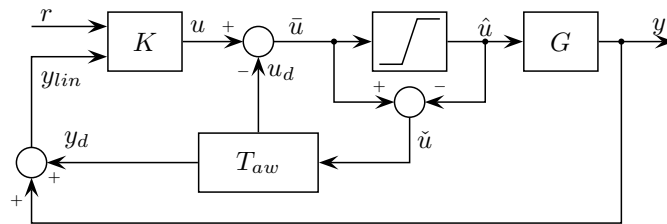- `'fsgt'`: produces a full order AW compensator `Taw` designed using Small Gain Theorem arguments,

Figure 1: Anti-windup compensation scheme

- `'ssec'`: produces a static AW compensator `Taw` designed according to the method proposed in [4] based on sector boundedness,

- `'psec'`: produces a low-order AW compensator `Taw` using the same ideas as `'ssec'` but with the addition of two filters, `F1` and `F2`.

The argument `nr` is the size of the reference r for cases of 2-dof controllers, by default it is assumed zero, *i.e.*, the controller input is the error $e = r - y_{lin}$.

The argument `opts` is a struct with settings for the design algorithm. This will be explained next as it depends on the particular design method.

2 Design methods

Consider an LPV plant of the form

$$G(\rho) : \begin{cases} \dot{x} = A(\rho)x + B(\rho)\hat{u}, \\ y = C(\rho)x + D(\rho)\hat{u}, \end{cases} \tag{1}$$

and the controller

$$K(\rho) : \begin{cases} \dot{x}_c = A_c(\rho)x + B_{cr}(\rho)r + B_{cy}(\rho)y_{lin}, \\ u = C_c(\rho)x + D_{cr}(\rho)r + D_{cy}(\rho)y_{lin}, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^{n_s}$ is the plant state vector, $\hat{u} \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the controlled output, $x_c \in \mathbb{R}^{n_k}$ is the controller state vector, $u \in \mathbb{R}^{n_u}$ is the controller output, and $r \in \mathbb{R}^{n_r}$ is the reference. The parameter ρ takes values in a compact set $\mathcal{P}$. Notice that an LPV model evaluated at a constant parameter value reduces to an LTI model. Hence, the explanations will be focused on LPV models since the LTI ones can be considered as a particular case.

The AW compensation scheme in Figure 1 consists of two compensation terms: one acting on the controller output $u_d \in \mathbb{R}^{n_u}$ and another on the controller input $y_d \in \mathbb{R}^{n_y}$. Defining

$$\begin{bmatrix} u_d \\ y_d \end{bmatrix} = T_{\text{aw}}(\rho)\check{u} = \begin{bmatrix} M(\rho) - I \\ N(\rho) \end{bmatrix} \check{u},$$

where $N = GM$ and $\check{u} = u - u_d - \text{sat}(\bar{u}) = u - u_d - \hat{u}$. It can be proved, after some system manipulations, that the compensation scheme in Figure 1 is equivalent to the scheme in Figure 2. After this reformulation and providing the nominal closed-loop system is stable, it is clear that the design of the AW compensator reduces to ensuring stability of the closed-loop system formed by $M - I$ and the dead-zone operator (nonlinear loop) and minimizing the effect of the y_d on the controlled variable (disturbance filter).

The AW compensation schemes can be:

- *static*:

$$T_{\text{aw}}(\rho) : \begin{cases} u_d = D_{\text{aw},u}(\rho)\check{u}, \\ y_d = D_{\text{aw},y}(\rho)\check{u}, \end{cases}$$
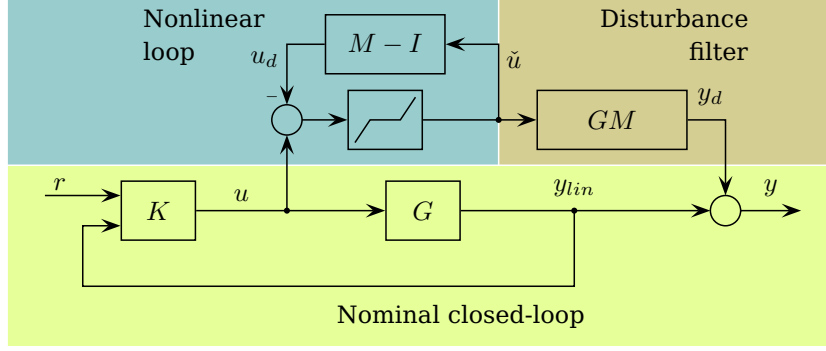
Figure 2: Equivalent representation of the anti-windup compensation scheme in Figure 1

- *low-order*:

$$T_{\mathrm{aw}}(\rho) : \begin{cases} u_d = F_1(s)D_{\mathrm{aw},u}(\rho)\check{u}, \\ y_d = F_2(s)D_{\mathrm{aw},y}(\rho)\check{u}, \end{cases}$$

- *full order* (*i.e.* with the same order than the plant G):

$$T_{\mathrm{aw}}(\rho) : \begin{cases} \dot{x}_e = A_{\mathrm{aw}}(\rho)x_e + B_{\mathrm{aw}}(\rho)\check{u}, \\ u_d = C_{\mathrm{aw},u}(\rho)x_e + D_{\mathrm{aw},u}(\rho)\check{u}, \\ y_d = C_{\mathrm{aw},y}(\rho)x_e + D_{\mathrm{aw},y}(\rho)\check{u}. \end{cases}$$

Therefore, the controller with the AW compensator results:

- *in the static case*

$$K_{\mathrm{aw}}(\rho) : \begin{cases} \dot{x}_c = A_c(\rho)x_c + B_{cr}(\rho)r + B_{cy}(\rho)y + B_{cy}(\rho)D_{\mathrm{aw},y}(\rho)\check{u}, \\ \bar{u} = C_c(\rho)x_c + D_{cr}(\rho)r + D_{cy}(\rho)y + (D_{cy}(\rho)D_{\mathrm{aw},y}(\rho) - D_{\mathrm{aw},u}(\rho))\check{u}, \end{cases}$$

- *in the low-order and full order cases*

$$K_{\mathrm{aw}}(\rho) : \begin{cases} \dot{x}_c = A_c(\rho)x_c + B_{cy}(\rho)C_{\mathrm{aw},y}(\rho)x_e + B_{cr}(\rho)r + B_{cy}(\rho)y + B_{cy}(\rho)D_{\mathrm{aw},y}\check{u}, \\ \dot{x}_e = A_{\mathrm{aw}}(\rho)x_e + B_{\mathrm{aw}}(\rho)\check{u}, \\ \bar{u} = C_c(\theta)x_c + (D_{cy}(\rho)C_{\mathrm{aw},y}(\rho) - C_{\mathrm{aw},u}(\rho))x_e + D_{cr}(\rho)r + D_{cy}(\rho)y \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + (D_{cy}(\rho)D_{\mathrm{aw},y}(\rho) - D_{\mathrm{aw},u}(\rho))\check{u}. \end{cases}$$

3 Full order AW compensators

In case the AW compensator is full order, *i.e.* with the same order than the plant G, and if M and N are the coprime factors of the plant G [6], the design of the AW compensator can be expressed as the design of a state-feedback gain fulfilling an induced $\mathcal{L}_2$ norm condition. More precisely, the design of the AW compensation consists in finding the matrix $H(\rho)$ such that

$$\begin{bmatrix} M(\rho) - I \\ N(\rho) \end{bmatrix} : \begin{cases} \dot{x}_e = (A(\rho) + B(\rho)H(\rho))x_e + B(\rho)\check{u}, \\ u_d = H(\rho)x_e, \\ y_d = (C(\rho) + D(\rho)H(\rho))x_e + D(\rho)\check{u}. \end{cases} \tag{3}$$

is internally stable and the operator $\check{u} \to y_d$ is small in a $\mathcal{L}_2$ sense.

3.1 Design based on sector boundedness (`method='fsec'`)

The dead-zone nonlinearity

$$\text{Dz}(v_j) = v_j - \text{sat}(v_j),$$

satisfies the inequality

$$\text{Dz}(v_j) w_j (v_j - \text{Dz}(v_j)) \geq 0,$$

for each input channel j ($j = 1, \ldots, n_u$) and a positive weight w_j. Then, collecting the inequalities for all j, the nonlinearity satisfies

$$\text{Dz}(v)^T W (v - \text{Dz}(v)) \geq 0 \tag{4}$$

with $W = \text{diag}(w_1, \ldots, w_{n_u}) > 0$.

Skogestad and Postlethwaite [3] combine (4) with the standard Bounded Real Lemma constraint to ensure internal stability and that the $\mathcal{L}_2$-norm of the operator $\check{u} \to y_d$ is lower that $\gamma > 0$. More concretely, substituting $v = u - u_d$ in (4), this inequality results

$$2\check{u}^T W (u - H(\rho)x - \check{u}) \geq 0. \tag{5}$$

The constraints on the $\mathcal{L}_2$-norm of the operator $\check{u} \to y_d$ can be expressed as

$$\dot{V}(x) + y_d^T y_d - \gamma^2 \check{u}^T \check{u} < 0, \tag{6}$$

where the Lyapunov function is defined as $V(x) = x^T \mathbf{Y} x$. Combining these inequalities using the S-procedure, the design of the AW compensator results in finding a positive definite matrix $\mathbf{Y} > 0$, a diagonal matrix $\mathbf{U} = \tau W^{-1}$ ($\tau > 0$) and a general matrix function $\mathbf{L}(\rho)$ such that

$$\begin{bmatrix} (A(\rho)\mathbf{Y} + B(\rho)\mathbf{L}(\rho)) + (\star) & B(\rho)\mathbf{U} - \mathbf{L}(\rho) & 0 & (C(\rho)\mathbf{Y} + D(\rho)\mathbf{L}(\rho))^T \\ \star & -2\mathbf{U} & I & \mathbf{U}D(\rho)^T \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0, \tag{7}$$

for all $\rho \in \mathcal{P}$. Then, the state feedback gain H is given by

$$H(\rho) = \mathbf{L}(\rho)\mathbf{Y}^{-1}.$$

3.2 Design based on Small Gain Theorem (`method='fsgt'`)

The AW compensator (3) can also be found using Small Gain Theorem concepts. In this line, the system (3) is internally stable if the $\mathcal{L}_2$-norm of the operator $\check{u} \to u_d$ is lower than 1. Therefore, the design of the AW compensator can be stated as the following optimization problem:

$$\underset{H}{\text{minimize}} \quad \gamma \tag{8}$$

$$\text{subtject to} \quad ||GM||_{\mathcal{L}_2} < \gamma, \tag{9}$$

$$||M - I||_{\mathcal{L}_2} < 1. \tag{10}$$

Using the Bounded Real Lemma, the state feedback gain H must satisfy the constraint

$$\begin{bmatrix} (A(\rho)\mathbf{Y} + B(\rho)\mathbf{V}(\rho)) + (\star) & B(\rho) & \mathbf{V}(\rho)^T \\ \star & -I & 0 \\ \star & \star & -I \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} (A(\rho)\mathbf{Y} + B(\rho)\mathbf{V}(\rho)) + (\star) & B(\rho) & (C(\rho)\mathbf{Y} + D(\rho)\mathbf{V}(\rho))^T \\ \star & -\gamma I & D(\rho)^T \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{12}$$

for all $\rho \in \mathcal{P}$, with $\mathbf{Y} > 0$ and $H(\rho) = \mathbf{V}(\rho)\mathbf{Y}^{-1}$.

3.3 Computational aspects

Finding the state-feedback satisfying LMIs (7), (11) and (12) involves to solve an infinite dimensional optimization problem with infinity constraints. If the matrices B, C and D in the plant (1) are constants, the matrix A is an affine function of ρ or polytopic, *i.e.*,

$$A(\rho) = \sum_{i=1}^{n_v} \alpha_i A_i$$

where

$$\rho = \sum_{i=1}^{n_v} \alpha_i \rho^i, \quad \alpha_1 + \cdots + \alpha_{n_v} = 1, \quad 0 \le \alpha_i \le 1, \quad \forall\, i = 1, \dots, n_v.$$

with ρ^i the vertices of the polytopic parameter set $\mathcal{P}$, and the matrix functions $\mathbf{L}$ and $\mathbf{V}$, respectively, are also affine or polytopic, LMIs (7), (11) and (12) are satisfied for all $\rho \in \mathcal{P}$ if and only if they are satisfied at the vertices ρ^i of the polytope $\mathcal{P}$ [2].

When the previous conditions are not fulfilled, the feedback gains can be obtained by evaluating LMIs (7), (11) and (12) at a grid of points in the set $\mathcal{P}$. If the grid of points is sufficiently dense, it can be assumed that the computed feedback gains satisfy the constraints at the rest of points in $\mathcal{P}$ [1, 5].

AWtools will analyze the plant matrices and will compute an polytopic H when it is possible.

4 Static and low-order AW compensators

In the case of a static AW compensator (**method='ssec'**),

$$\begin{bmatrix} u_d \\ y_d \end{bmatrix} = T_{aw}\breve{u}, \tag{13}$$

it can be proved [4] that

$$\begin{bmatrix} M(\rho) - I \\ N(\rho) \end{bmatrix} : \begin{cases} \dot{x}_e = \bar{A}(\rho)x_e + (\bar{B}_0(\rho) + \bar{B}_1(\rho)T_{\mathrm{aw}}(\rho))\breve{u}, \\ u_d = \bar{C}_u(\rho)x_e + (\bar{D}_{u,0}(\rho) + \bar{D}_{u,1}(\rho)T_{\mathrm{aw}}(\rho))\breve{u}, , \\ y_d = \bar{C}_y(\rho)x_e + (\bar{D}_{y,0}(\rho) + \bar{D}_{y,1}(\rho)T_{\mathrm{aw}}(\rho))\breve{u}, \end{cases} \tag{14}$$

where

$$\bar{A}(\rho) = \begin{bmatrix} A(\rho) + B(\rho)\tilde{\Delta}(\rho)D_{cy}(\rho)C(\rho) & B(\rho)\tilde{\Delta}(\rho)C_c(\rho) \\ B_{cy}(\rho)\Delta(\rho)C(\rho) & A_c(\rho) + B_{cy}(\rho)\Delta(\rho)D(\rho)C_c(\rho) \end{bmatrix},$$

$$\bar{B}_0(\rho) = \begin{bmatrix} B(\rho)\tilde{\Delta}(\rho) \\ B_{cy}(\rho)\Delta(\rho)D(\rho) \end{bmatrix}, \qquad \bar{B}_1(\rho) = \begin{bmatrix} B(\rho)\tilde{\Delta}(\rho) & -B(\rho)\tilde{\Delta}(\rho)D_{cy}(\rho) \\ B_{cy}(\rho)\Delta(\rho)D(\rho) & -B_{cy}(\rho)\Delta(\rho) \end{bmatrix},$$

$$\bar{C}_u(\rho) = \begin{bmatrix} \tilde{\Delta}(\rho)D_{cy}(\rho)C(\rho) & \tilde{\Delta}(\rho)C_c(\rho) \end{bmatrix}, \qquad \bar{C}_y(\rho) = \begin{bmatrix} \Delta(\rho)C(\rho) & \Delta(\rho)D(\rho)C_c(\rho) \end{bmatrix},$$

$$\bar{D}_{u,0}(\rho) = \tilde{\Delta}(\rho)D_{cy}(\rho)D(\rho), \qquad \bar{D}_{u,1}(\rho) = \begin{bmatrix} \tilde{\Delta}(\rho) & -\tilde{\Delta}(\rho)D_{cy}(\rho) \end{bmatrix},$$

$$\bar{D}_{y,0}(\rho) = \Delta(\rho)D(\rho), \qquad \bar{D}_{y,1}(\rho) = \begin{bmatrix} \Delta(\rho)D(\rho) & -\Delta(\rho)D(\rho)D_{cy}(\rho) \end{bmatrix},$$

$$\Delta(\rho) = (I - D(\rho)D_{cy}(\rho))^{-1}, \qquad \tilde{\Delta}(\rho) = (I - D_{cy}(\rho)D(\rho))^{-1}.$$

In the low-order case (`method='psec'`), the AW compensator is taken as

$$\begin{bmatrix} u_d \\ y_d \end{bmatrix} = \begin{bmatrix} F_1(s) & 0 \\ 0 & F_2(s) \end{bmatrix} T_{aw}\breve{u}, \tag{15}$$

where F_1 and F_2 are LTI filters with realizations

$$F_1(s) = C_1(sI - A_1)^{-1}B_1 + D_1, \qquad F_2(s) = C_2(sI - A_2)^{-1}B_2 + D_2.$$

Then the matrices in (14) are

$$\bar{A} = \begin{bmatrix} A + B\tilde{\Delta}D_{cy}C & B\tilde{\Delta}C_c & B\tilde{\Delta}C_1 & -B\tilde{\Delta}D_{cy}C_2 \\ B_{cy}\Delta C & A_c + B_{cy}\Delta DC_c & B_{cy}\Delta DC_1 & B_{cy}\Delta C_2 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix},$$

$$\bar{B}_0 = \begin{bmatrix} B\tilde{\Delta} \\ B_{cy}\Delta D \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{B}_1 = \begin{bmatrix} B\tilde{\Delta}D_1 & -B\tilde{\Delta}D_{cy}D_2 \\ B_{cy}\Delta DD_1 & -B_{cy}\Delta D_2 \\ B1 & 0 \\ 0 & B_2 \end{bmatrix},$$

$$\bar{C}_u = \begin{bmatrix} \tilde{\Delta}D_{cy}C & \tilde{\Delta}C_c & \tilde{\Delta}C_1 & -\tilde{\Delta}D_{cy}C_2 \end{bmatrix}, \qquad \bar{C}_y = \begin{bmatrix} \Delta C & \Delta DC_c & \Delta DC_1 & \Delta DD_{cy}C_1 \end{bmatrix},$$

$$\bar{D}_{u,0} = \tilde{\Delta}D_{cy}D, \qquad \bar{D}_{u,1} = \begin{bmatrix} \tilde{\Delta}D_1 & -\tilde{\Delta}D_{cy}D_2 \end{bmatrix},$$

$$\bar{D}_{y,0} = \Delta D, \qquad \bar{D}_{y,1} = \begin{bmatrix} \Delta DD_1 & -\Delta DD_{cy}D_2 \end{bmatrix},$$

where the parameter dependence has been dropped for ease the notation.

Similarly to the full order case, Turner and Postlethwaite [4] have proved with sector boundedness arguments that after finding a positive definite matrix $\mathbf{Y} > 0$, a diagonal matrix $\mathbf{U}$ and a general matrix function $\mathbf{L}(\rho)$ such that

$$\begin{bmatrix} \bar{A}(\rho)\mathbf{Y} + (\star) & \bar{B}_0(\rho)\mathbf{U} + \bar{B}_1(\rho)\mathbf{L}(\rho) - \mathbf{Y}\bar{C}_u^T(\rho) & 0 & (\bar{C}_y(\rho)\mathbf{Y})^T \\ \star & -(\mathbf{U} + \bar{D}_{u,0}(\rho)\mathbf{U} + \bar{D}_{u,1}(\rho)\mathbf{L}(\rho)) - (\star) & I & (\bar{D}_{y,0}(\rho)\mathbf{U} + \bar{D}_{y,1}(\rho)\mathbf{L}(\rho))^T \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0,$$

for all $\rho \in \mathcal{P}$, then the AW compensator is given by

$$T_{\text{aw}}(\rho) = \mathbf{L}(\rho)\mathbf{U}^{-1}.$$

		Method			
Field	Default	`'fsec'`	`'fsgt'`	`'ssec'`	`'psec'`
MinDecay	(1)	✓	✓		
MinDamping	(1)	✓	✓		
MaxFreq	(1)	✓	✓		
weigthY	10^8	✓	✓	✓	✓
weigthU	0.1	✓		✓	✓
MinBndU	1			✓	✓
weigthGamma	0.01		✓		

Table 1: Design settings for **awsyn**

4.1 Computational aspects

As discussed in Section 3.3, in order to be able to use the vertex property and obtain a finite dimensional optimization problem, some structural constraints must be imposed to the plant, the controller and the AW compensator. First, B, C and D must be constant matrices. In addition D or D_{cy} should be zero to avoid a fractional parameter dependence in (14). In the case of $D = 0$, the first column of $\bar{B}_1$, $\bar{D}_{u,1}$, and $\bar{D}_{y,1}$ are parameter independent and T_{aw} can be taken as

$$T_{\mathrm{aw}}(\rho) = \begin{bmatrix} D_{\mathrm{aw},u}(\rho) \\ D_{\mathrm{aw},y} \end{bmatrix}.$$

to achieve better performance. This fact is checked in awsyn and when this condition is fulfilled, the produced AW compensator will result an LPV system.

5 Optional arguments

The fifth argument of awsyn is a `struct` variable that allows the user to modify the design algorithm settings. This input argument opts has several fields that depends on the particular design method. The available options are summarizes in Table 1 together with the default values.

In case of full order compensators, the fields MinDecay, MinDamping and MaxFreq set the minimum decay rate, the minimum damping and the maximum frequency, respectively, of the compensator poles. The default values are determined according to the poles of the plant.

The fields weigthY and weigthU penalize the traces of $\mathbf{Y}$ and $\mathbf{U}$, which might help to improve numerical issues. The field MinBndU imposes a lower bound on $\mathbf{U}$ in order to avoid close to singular matrices, which might lead to large T_{aw}, in cases of static and low-order compensators.

Finally, when weigthGamma is set lower than 1, the hard constraints $||M - I||_{\mathcal{L}_2} < 1$ in (10) is substituted by $||M - I||_{\mathcal{L}_2} < \gamma_u$, and the objective function by

$\gamma_u + \texttt{weigthGamma}\,\gamma$.

This relaxation might help to find a suitable solution in some cases. The function `awsyn` will check the value of $||M - I||_{\mathcal{L}_2}$ and warn in case of values greater than 1.

References

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