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Robust identification/invalidation in an LPV framework

Fernando D. Bianchi[1,*] Ricardo S. Sánchez-Peña[1,2]

[1] *SAC-ESAII, Universitat Politècnica de Catalunya, Rb. Sant Nebridi 10, 08222 Terrassa, Barcelona, Spain.*
[2] *Institució Catalana de Recerca y Estudis Avançats (ICREA), Barcelona, Spain*

SUMMARY

A robust LPV identification/invalidation method is presented. Starting from a given initial model, the proposed method modifies it and produces an LPV model consistent with the assumed uncertainty/noise bounds and the experimental information. This procedure may complement existing *nominal* LPV identification algorithms, by adding the uncertainty and noise bounds which produces a set of models consistent with the experimental evidence. Unlike standard invalidation results, the proposed method allows the computation of the necessary changes to the initial model in order to place it within the consistency set. Similar to previous LPV identification procedures, the initial parameter dependency is fixed in advance, but here a methodology to modify this dependency is presented. In addition, all calculations are made on state space matrices which simplifies further controller design computations. The application of the proposed method to the identification of nonlinear systems is also discussed.

KEY WORDS: Linear parameter varying, control-oriented identification, model invalidation, robust control.

1. INTRODUCTION

Since the introduction of new and effective synthesis techniques more than a decade ago, linear parameter varying (LPV) systems have awoken great interest. It fits numerous design and analysis methods used for linear time invariant (LTI) models in robust control (see [1] and references therein). Besides, LPV models are commonly used for "covering" the description of many nonlinear systems. However, the application of these tools in practical cases requires methodologies capable of finding LPV models from experimental data.

Early attempts to identify LPV models which are not based on the knowledge of a nonlinear model use linear regression algorithms, Gray Box LPV identification, plain optimization or

*Correspondence to: F. Bianchi, Email address: fernando.bianchi@upc.edu, Tel.(+34)93 739 8705, Fax.+(34)93 739 8628

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even orthonormal basis functions ([2, 3, 4, 5, 6, 7]). To solve the MIMO problem, an important area of research in the last years is the identification through subspace methods. These methods have the clear advantage that direct state space matrices are identified by means of algebraic computations, hence no optimization or convergence problems need to be addressed. These methodologies have already been applied to LPV models in [8, 9], but its application is limited by the dimension of the data matrices involved. To overcome these difficulties, alternatives have been proposed as the use of periodic scheduling sequences ([10]), piecewise linear (PWL) models ([11, 12]) and more recently in [13]. Furthermore in [12], the problem derived from composing individual SISO models into a MIMO one is also addressed. A very recent overview of this area can be found in [14]. In all these works, a nominal LPV model is obtained, but no uncertainty description is computed nor a worst case criteria has been used.

Further steps have been taken to include model uncertainty, stating the problem in the framework of robust identification: a deterministic, worst case, set–membership approach. Identification through interpolation in this framework has been explored in [15]. It produces an LPV model from experimental and *a priori* information, but with very conservative uncertainty bounds. A subsequent invalidation step is required in order to find less conservative bounds, in terms of uncertainty and external noise bounds ([16]).

The present paper follows the identification guidelines introduced in [15]. Basically, the idea is to propose a general LPV model structure and to compute its system matrices such that the model is consistent with the experimental data and the assumed uncertainty and noise bounds. The main difference with the previous algorithm in [15] is that the nominal model and the uncertainty and noise bounds are found in a single step, using the invalidation procedures presented in [16, 17]. On the other hand, the proposed procedure provides the model directly in state space form avoiding subsequent conversions from input-output descriptions ([12]). As in any identification method where a model structure needs to be proposed, the selection of a model structure is crucial. Hence, we introduce a systematic procedure to select the model structure in order to find the most adequate LPV model. Another topic discussed here is the application of the proposed method to fit a nonlinear system in an LPV model. This is a topic of great practical importance because of the common use of LPV techniques in the control of nonlinear processes, which is explained by means of an example.

The paper is organized as follows. Next section presents a brief background on LPV invalidation. Section 3 introduces the problem formulation and the main results of this work. Practical implementation issues for the basis selection are derived to the Appendix. Two examples are presented in section 4: a simulated LPV system which illustrates in detail this methodology, and a nonlinear system based on a Hi-Fi (high fidelity) helicopter simulator developed in [18]. Final conclusions and future research directions end this presentation in section 5.

2. BACKGROUND MATERIAL

This material has been extracted from [17, 16]. Consider an LPV system

$$G(\rho) : \begin{cases} x_{k+1} = A(\rho_k)x_k + B_1(\rho_k)w_k + B_2(\rho_k)u_k, \\ z_k = C_1(\rho_k)x_k + D_{11}(\rho_k)w_k + D_{12}(\rho_k)u_k \\ y_k = C_2(\rho_k)x_k + D_{21}(\rho_k)w_k + D_{22}(\rho_k)u_k \end{cases} \tag{1}$$

with $x_k \in \mathbb{R}^{n_s}$, $z_k \in \mathbb{R}^{n_z}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$, $u_k \in \mathbb{R}^{n_u}$ and $\rho_k \in \mathbb{R}^{n_\rho}$. The system (1) can also be represented by its convolution kernel $\{g_{k,i}\}$. The latter can be truncated up to time $k = n$: $\{g_0, g_1, \cdots, g_n\}$ and represented by its associated Toeplitz matrix as follows:

$$T_G^n = \begin{bmatrix} g_{0,1} & 0 & \cdots & 0 \\ g_{1,2} & g_{0,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-1,n} & g_{n-2,n} & \cdots & g_{0,n} \end{bmatrix}. \tag{2}$$

Similarly, for a given signal sequence $\{h_k\} \in \ell_2$, its truncated version can also be represented by its associated Toeplitz matrix:

$$T_h^n = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & \cdots & h_0 \end{bmatrix}. \tag{3}$$

With the previous definitions, the uncertain LPV models depicted in Figure 1 can be described in a matrix form as follows (the superscript n will be dropped from here on to simplify the notation):

$$\begin{aligned} T_z &= T_{G_{11}} T_w + T_{G_{12}} T_u, \\ T_y &= T_{G_{21}} T_w + T_{G_{22}} T_u + T_d, \\ T_w &= T_\Delta T_z. \end{aligned} \tag{4}$$

The uncertainty and external signal *a priori* sets are defined as follows:

$$\mathbf{\Delta} \stackrel{\triangle}{=} \{\Delta \in \mathcal{H}_\infty : \|\Delta\|_\infty \le \delta \le 1\}, \tag{5}$$

$$\mathcal{D} \stackrel{\triangle}{=} \{d \in \mathbb{R}^r : \|d\|_2 / m < d_{\max}\} \tag{6}$$

where m is the number of samples in the sequence d. The following theorem allows us to verify if the uncertain model (4), the uncertainty set (5) and the noise set (6) are consistent with the experimental data. Here, vectors $\mathbf{d}, \mathbf{y}, \mathbf{u}$ and $\mathbf{w}$ represent their truncated versions as follows: $\mathbf{x} = \left[x_o^T \ \cdots \ x_{n-1}^T \ x_n^T\right]^T$.

Theorem 2.1. *Given time-domain measurements of the input u, the output y and the time-varying parameter ρ, the LPV model $G(\rho)$ is not invalidated by this experimental information if and only if there exists a vector w, such that*

$$\begin{bmatrix} X(w) & T_w^T \\ T_w & (\delta^{-2} I - T_{G_{11}}^T T_{G_{11}})^{-1} \end{bmatrix} > 0, \quad \begin{bmatrix} d_{\max}^2 & \mathbf{d}^T \\ \mathbf{d} & I \end{bmatrix} > 0 \tag{7}$$

where

$$\begin{aligned} \mathbf{d} &= \mathbf{y} - T_{G_{21}} \mathbf{w} - T_{G_{22}} \mathbf{u} \\ X(w) &= T_u^T T_{G_{12}}^T T_{G_{12}} T_u + T_u^T T_{G_{12}}^T T_{G_{11}} T_w + T_w^T T_{G_{11}}^T T_{G_{12}} T_u \end{aligned}$$

The previous result is used to compute the lowest bounds on the uncertainty and noise, either fixing one bound and minimizing the other, or minimizing a weighted combination of both bounds simultaneously.

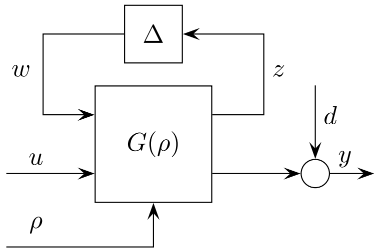

Figure 1. Identification/invalidation setup

3. MAIN RESULTS

3.1. Problem Statement

Consider the robust LPV control-oriented identification setup sketched in Figure 1. The signals u represents the test input, y the output corrupted by the measurement noise d and ρ corresponds to the time-varying parameter.

Our attention will be centered on the most common uncertainty descriptions, *i.e.* additive and multiplicative representations. Therefore, $G_{11} = 0$ and for simplicity $G_{21} = I$ and the matrices associated to the input w in the model (1) become $B_1 = 0$, $D_{11} = 0$ and $D_{21} = I^\dagger$. The *a priori* error sets (uncertainty and disturbance noise) are the ones previously described by equations (5) and (6), with $\Delta \in \mathcal{H}_\infty^{n_z \times n_z}$ and $d_k \in \mathbb{R}^{n_y}$.

It will be assumed that the system matrices in (1) can be expressed as

$$\begin{bmatrix} A(\rho) & 0 & B_2(\rho) \\ C_1(\rho) & 0 & D_{12}(\rho) \\ C_2(\rho) & I & D_{22}(\rho) \end{bmatrix} = S_\mathrm{I}(\rho) + S_\mathrm{C}(\rho) \tag{8}$$

where

$$S_\mathrm{I}(\rho) = \begin{bmatrix} A_0(\rho) & 0 & B_{2,0}(\rho) \\ C_{1,0}(\rho) & 0 & D_{12,0}(\rho) \\ C_{2,0}(\rho) & I & D_{22,0}(\rho) \end{bmatrix}, \tag{9}$$

$$S_\mathrm{C}(\rho) = \sum_{i=1}^{N} f_i(\rho) \begin{bmatrix} A_i & 0 & B_{2,i} \\ \mathbf{C}_{1,i} & 0 & \mathbf{D}_{12,i} \\ \mathbf{C}_{2,i} & 0 & \mathbf{D}_{22,i} \end{bmatrix}. \tag{10}$$

Matrices A_i, $B_{2,i}$ and the set of nonlinear functions $\mathcal{F}^N = \{f_i, i = 1, \cdots, N\}$ are fixed beforehand and $\mathbf{C}_{1,i}$, $\mathbf{D}_{12,i}$, $\mathbf{C}_{2,i}$, $\mathbf{D}_{22,i}$ are matrices to be determined.

The term $S_\mathrm{I}(\rho)$ represents an LPV model which could have been obtained by means of available identification algorithms or according to some *a priori* information based on mathematical expressions describing the dynamic behavior of the physical process‡. On the

†This simplification has been made without loss of generality, but more general situations may also be handled, as the case where uncertainty input and output weights are included, i.e. $B_1 \neq 0$, $G_{21} \neq I$.
‡Here S_I is defined as a LPV model for generality, but it can also be described by a LTI model.

other hand, $S_{\mathrm{C}}(\rho)$ is a term to be determined such that the complete model is consistent with the *a priori* assumptions on the uncertainty and noise bounds and with the *a posteriori* experimental information.

That is, given an initial LPV model S_{I} for which the bounds on the modelling errors have not been stated yet, the objective is to find the term $S_{\mathrm{C}}(\rho)$ that makes the model governed by equation (8) consistent with the *a priori* and *a posteriori* information:

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}, \qquad \boldsymbol{\rho} = \begin{bmatrix} \rho_0 \\ \vdots \\ \rho_n \end{bmatrix}.$$

This approach has two possible interpretations. On one hand, it is a robust LPV identification algorithm in the sense that an LPV model and uncertainty/noise bounds are computed based on *a priori* and *a posteriori* information. The main difference with previous robust LPV identification results ([16]) is that the nominal model and the uncertainty/noise bounds are computed *via* invalidation in only one step. This provides a less conservative set of models in general and avoids subsequent verifications. In addition, the identification is completely addressed in a state-space context, more convenient for MIMO systems and controller design. On the other hand, the approach can be regarded as a tool that complements other available identification algorithms. That is, once a model is found by means of these algorithms, the proposed procedure computes the uncertainty/noise bounds as well as the necessary changes to achieve consistency.

3.2. Consistency and robust LPV model identification

Based on the invalidation Theorem 2.1 and the previous model definition, the following theorem proposes an LPV identification procedure based on an initial model.

Theorem 3.1. *Given time-domain sequences of the input u, the output y and the parameter ρ, an initial model S_{I}, A_i, $B_{2,i}$ and a basis of nonlinear functions $\mathcal{F}^N$, the LPV model (1) is consistent with the experimental data if and only if there exists a sequence w and matrices $\mathbf{C}_{1,i}$, $\mathbf{D}_{12,i}$, $\mathbf{C}_{2,i}$ and $\mathbf{D}_{22,i}$, such that:*

$$\begin{bmatrix} T_u^T X(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) T_u & T_w^T \\ T_w & \delta^2 I \end{bmatrix} > 0, \tag{11}$$

$$\begin{bmatrix} d_{max}^2 & [\mathbf{y} - T_{G_{22}}(\rho, \mathbf{C}_{2,i}, \mathbf{D}_{22,i})\mathbf{u} - \mathbf{w}]^T \\ [\mathbf{y} - T_{G_{22}}(\rho, \mathbf{C}_{2,i}, \mathbf{D}_{22,i})\mathbf{u} - \mathbf{w}] & I \end{bmatrix} > 0 \tag{12}$$

where

$$X(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) = T_{G_{12}}^T(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) T_{G_{12}}(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}), \tag{13}$$

$$T_{G_{12}}(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) = T_{G_{12,0}}(\rho) + \sum_{i=1}^{N} T_{G_{12},i}(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}), \tag{14}$$

$$T_{G_{22}}(\rho, \mathbf{C}_{2,i}, \mathbf{D}_{22,i}) = T_{G_{22,0}}(\rho) + \sum_{i=1}^{N} T_{G_{22},i}(\rho, \mathbf{C}_{2,i}, \mathbf{D}_{22,i}). \tag{15}$$

with $T_{G_{21},i}$ and $T_{G_{22},i}$ depending linearly on the unknowns $\mathbf{C}_{1,i}$, $\mathbf{D}_{12,i}$ and $\mathbf{C}_{2,i}$, $\mathbf{D}_{22,i}$, respectively.

Proof:

Replacing the matrices (8) in the impulsive response, the Toeplitz matrices $T_{G_{12}}$ and $T_{G_{22}}$ become (14)-(15). Then, matrix inequalities (11)-(12) result from applying Theorem 2.1 to model (1) and replacement of the perturbation d as a function of the uncertainty input w, according to equation (12).

□

Note that because of the term

$$T^T_{G_{12,j}}(\rho, \mathbf{C}_{1,j}, \mathbf{D}_{12,j}) T_{G_{12,i}}(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) \tag{16}$$

in $X(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i})$, equation (11) is a bilinear matrix inequality (BMI). These optimization problems may result difficult to solve, but there are several available algorithms which may work ([19]). Alternatively, it is possible to obtain a relaxed version of Theorem 3.1 by eliminating the nonlinear term (16). In fact, if the problem (11)-(12) with $X(\cdot)$ replaced by

$$\sum_{i=1}^{N} \left[T^T_{G_{12,0}}(\rho) T_{G_{12,i}}(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) + T^T_{G_{12,i}}(\rho, \mathbf{C}_{1,i}, \mathbf{D}_{12,i}) T_{G_{12,0}}(\rho) \right] + T^T_{G_{12,0}}(\rho) T_{G_{12,0}}(\rho)$$

is feasible then the original optimization problem will also be feasible.

According to Theorem 3.1, the model obtained is consistent with both, the *a priori* noise and uncertainty sets and the experimental data. This implies that the model is inside the consistency set, and therefore the algorithmic identification procedure is known as *interpolatory.* It is a well known fact that all interpolatory algorithms have a worst case identification error which is bounded by the diameter of information $\mathcal{D}(I)$. Hence, these are convergent in the sense that the worst case error vanishes as the data increases ($n \to \infty$) and the measurement error vanishes ($d_{max} \to 0$) (see details Chap.10 in [20]). A generalization of this concept to LPV models can also be found in [16].

The results in this section require proposing an initial model and a basis of nonlinear functions $\mathcal{F}^N$. As mentioned previously, the initial model can be obtained by means of the available identification algorithms or the mathematical expressions describing the physical phenomena. On the other hand, the basis of functions can be selected based on physical insight or following the procedure proposed in the Appendix. In addition, this procedure can be used to "cover" a nonlinear system by an LPV representation in order to design a gain scheduled controller, as illustrated in the second example.

4. EXAMPLES

Here we apply the method to two different examples. The first one illustrates the case of a pure LPV system. The second example is a nonlinear system based on a Hi-Fi helicopter simulator developed in [18].

4.1. Example 1

In this first example, the proposed methodology is illustrated with simulated data from the following two-input two-output LPV system

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -(0.3+0.01\rho_{1,k}) & -(0.6+0.6\rho_{2,k}) \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_k,$$
$$y_k = \begin{bmatrix} \rho_{2,k}^2 & 0 \\ 0 & 1 \end{bmatrix} x_k. \tag{17}$$

The system has been excited with a pulse signal at the input u and a decreasing time-varying parameter trajectory (Figure 2) to record the "experimental data". The following initial LPV model has been proposed:

$$S_I = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ -0.3 & -0.6 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rho_{1,k} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rho_{2,k}. \tag{18}$$

The uncertainty and noise sets are defined by equations (5)-(6), with $W_\Delta = 0.01$ (*i.e.*, 1% of relative uncertainty). For a clearer result interpretation, measurement noise has not been included in this stage. In this circumstance, by adding enough nonlinear terms in S_C, the algorithm should find the matrices $\mathbf{C}_{1,i}$, $\mathbf{D}_{12,i}$, $\mathbf{C}_{2,i}$, $\mathbf{D}_{22,i}$ that achieve a perfect fit with noise and uncertain bounds close to zero.

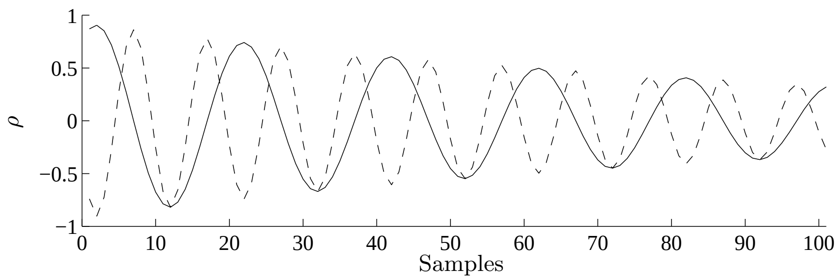

Figure 2. Parameter trajectory corresponding to Example 1

In order to evaluate the effectiveness of the algorithm, several basis of functions have been tested. The results obtained in each case are summarized in Table I. In the second and third columns, the uncertainty and noise bounds are listed, respectively. These results indicate that as more nonlinear terms are included in the functions basis, the bounds decrease to zero and thus the model responses become closer to the "experimental data". In particular, it can be observed that when the basis of functions include all terms present in the real model, the algorithm, in the case of absence of noise, is capable of finding a model that perfectly matches the real one. In fact, the computed nonlinear terms are:

$$S_C = \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{C}_{1,1} & 0 & \mathbf{D}_{12,1} \\ \mathbf{C}_{2,1} & 0 & \mathbf{D}_{22,1} \end{bmatrix} \rho_{1,k}\rho_{2,k} + \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{C}_{1,2} & 0 & \mathbf{D}_{12,2} \\ \mathbf{C}_{2,2} & 0 & \mathbf{D}_{22,2} \end{bmatrix} \rho_{1,k}^2 + \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{C}_{1,3} & 0 & \mathbf{D}_{12,3} \\ \mathbf{C}_{2,3} & 0 & \mathbf{D}_{22,3} \end{bmatrix} \rho_{2,k}^2. \quad (19)$$

with

$$\mathbf{C}_{1,1} = \mathbf{C}_{2,1} = \begin{bmatrix} 0.0000 & 0.0001 \\ 0.0001 & 0.0001 \end{bmatrix} \qquad \mathbf{D}_{12,1} = \mathbf{D}_{22,1} = \begin{bmatrix} -0.0000 & 0.0000 \\ -0.0009 & -0.0041 \end{bmatrix}$$

$$\mathbf{C}_{1,2} = \mathbf{C}_{2,2} = \begin{bmatrix} 0.0000 & 0.0001 \\ 0.0001 & 0.0001 \end{bmatrix} \qquad \mathbf{D}_{12,2} = \mathbf{D}_{22,2} = \begin{bmatrix} -0.0000 & -0.0000 \\ -0.0004 & -0.0053 \end{bmatrix}$$

$$\mathbf{C}_{1,3} = \mathbf{C}_{2,3} = \begin{bmatrix} 1.0000 & 0.0002 \\ 0.0001 & 0.0001 \end{bmatrix} \qquad \mathbf{D}_{12,3} = \mathbf{D}_{22,3} = \begin{bmatrix} -0.0000 & -0.0000 \\ -0.0005 & 0.0122 \end{bmatrix}$$

This shows that the algorithm has found a model really close to the actual system. Similar conclusions can be drawn from Figure 3, where the step responses of each model subject to the parameter trajectory shown in Figure 2 are displayed. It can be seen that the responses of the system and model (iv) are coincident.

Model	δ	$d_{\max}$
(i) Affine LPV	0.217	0.505
(ii) LPV with terms ρ_1, ρ_2, $\rho_1\rho_2$	0.238	0.472
(iii) LPV with terms ρ_1, ρ_2, $\rho_1\rho_2$, ρ_1^2	0.217	0.468
(iv) LPV with terms ρ_1, ρ_2, $\rho_1\rho_2$, ρ_1^2, ρ_2^2	0.000	0.000

Table I. Uncertainty and noise bounds for several models in Example 1 (in absence of noise)

With the aim of checking the algorithm in a less ideal case, the same "experiment" was repeated with the measurement output corrupted by white noise $(0, 0.1)$. The results obtained are listed in Table II. It can be noted that although it is not possible to achieve a perfect fit as previously, the results show a similar trend to the values in Table I. In this case, model (iv) achieves the lowest bounds and results the best representation of the dynamic behavior of the system.

Model	δ	$d_{\max}$
(i) Affine LPV	0.308	0.627
(ii) LPV with terms ρ_1, ρ_2, $\rho_1\rho_2$	0.654	0.583
(iii) LPV with terms ρ_1, ρ_2, $\rho_1\rho_2$, ρ_1^2	0.438	0.565
(iv) LPV with terms ρ_1, ρ_2, $\rho_1\rho_2$, ρ_1^2, ρ_2^2	0.437	0.308

Table II. Noise and uncertainty bounds for several models in Example 1 (with noise)

4.2. Example 2

In the second example, we investigate the application of Theorem 3.1 to nonlinear systems. With this aim, consider a nonlinear system linearized around a trajectory defined by an input

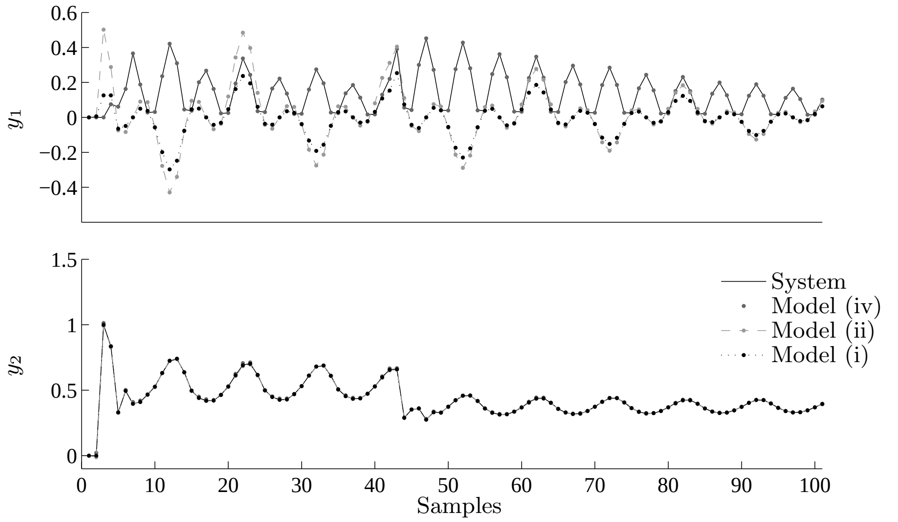

Figure 3. Step response of the several model in Table I

u^* and the corresponding values of the states x^*. Assuming that x^* is a one-to-one function of u^*, we can model the nonlinear system as

$$\begin{aligned} \hat{x}_{k+1} &= A(u_k^*)\hat{x}_k + B_2(u_k^*)\hat{u}_k, \\ z_k &= C_1(u_k^*)x_k + D_{12}(u_k^*)\hat{u}_k \\ y_k &= C_2(u_k^*)\hat{x}_k + w_k + D_{22}(u_k^*)\hat{u}_k + \Phi(u_k^*) \end{aligned} \tag{20}$$

where $\hat{x} = x - x^*$ and $\hat{u} = u - u^*$. Basically, model (20) fits the LPV representation (1) except for the term $\Phi(u_k^*)$. This shows the main difference between LPV models and nonlinear ones, the latter having equilibrium points different from the origin. Therefore, the identification methodology presented previously can be applied to nonlinear systems of the form (20) which generalizes model (1), with a small modification.

In this second example, a Hi-Fi simulation of a 6-DOF model of an autonomous helicopter ([18]) is analyzed. This system is described by complex nonlinear expressions and presents an open-loop unstable behavior. The model has eleven states: Euler angles, angular velocities and accelerations, and the angles describing the orientation of the main rotor. In order to simplify the data recollection, an LQR controller was included to stabilize the system. In this circumstances, the system was excited by the reference angles and only the Euler angles were measured. Therefore, the resulting system has three inputs and three outputs. The model in this case has been parameterized by the input u^*. The nonlinear model information was not used in any circumstance, only the result of the simulated "experiments" performed on the simulation. As in the previous example, the simulated output was corrupted by zero mean white noise and variance 0.5.

The system was excited by u^* with the signal shown in Figure 4 plus a random signal $\hat{u}$ of zero mean value. The initial model has been computed by means of PWL identification ([11]). In the first step, the system is excited by a square wave, which forces the operation to the limits of the parameter space, thus identifying LTI models to construct a first LPV model. The second portion of u^* allows the computation of the nonlinear gain $\Phi(\cdot)$, while the input of the LPV model is zero ($\hat{u} \equiv 0$). Finally, the last section of u^* is used to test the proposed LPV model for invalidation.

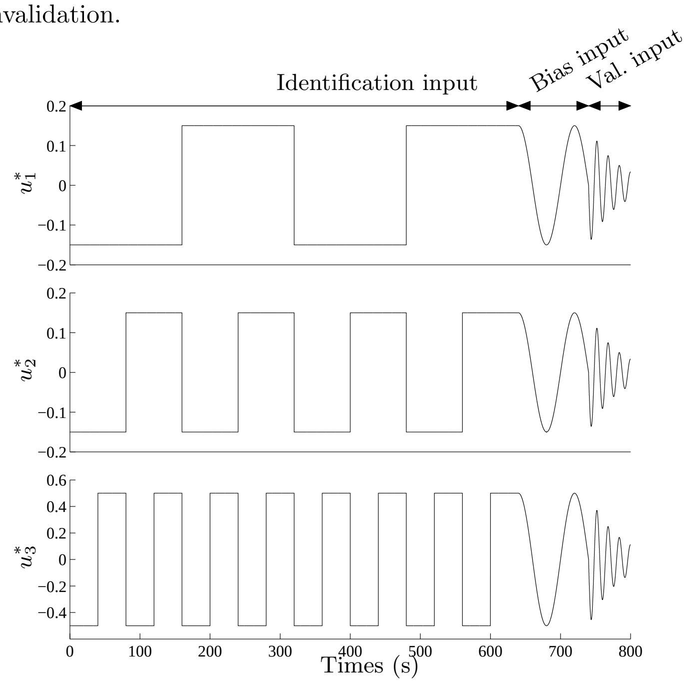

Figure 4. Parameter sequence used to excite the system in Example 2

Since the model is parameterized by u^*, the nonlinear gain $\Phi(\cdot)$ should also be determined before checking the invalidation of any LPV model. This nonlinear gain was computed by means of a polynomial fitting, which results in the following functions:

$$\Phi(u^*) = \begin{bmatrix} 0.167 - 0.881u_2^*u_3^* \\ 0.016 - 0.757u_3^* \\ 0.010 - 0.940u_3^* \end{bmatrix}.$$

In Table III, the uncertainty and noise bounds obtained with several basis of functions are listed. As in the previous example, A_i and $B_{2,i}$ have been set to 0. These results indicate that the inclusion of the multiaffine term $u_1^*u_2^*$ in S_C is useful to reduce the uncertainty and noise bounds, specially the first one. It can also be observed that the inclusion of additional multiaffine terms does not improve the model fit. The first test produces an affine LPV model, which is more practical for further controller design. If a lower disturbance bound is sought, the model (ii) is a better option at the expense of a more complex model. On the other hand, the results in the last row in Table III show that the invalidation results are almost identical when simplifying the model by eliminating the parameter u_3^*. As a conclusion, the results indicate that an affine LPV model is good enough to explain the data, depending on the desired noise bound.

Model	δ	$d_{\max}$
(i) Affine LPV	0.992	0.432
(ii) LPV with terms u_1^*, u_2^*, u_3^*, $u_1^*u_2^*$	0.214	0.370
(iii) LPV with terms u_1^*, u_2^*, u_3^*, $u_1^*u_2^*$, $u_1^*u_3^*$	0.214	0.370
(iv) LPV with terms u_1^*, u_2^*, $u_1^*u_2^*$	0.221	0.392

Table III. Noise and uncertainty bounds for several models in Example 2

Figure 5 shows the responses of the system and the model (ii) to a series of consecutive doublet signals in each Euler angle. Parameter trajectory u^* is the same used in the invalidation stage in Figure 4. In this case, the response of the model does not perfectly match the response of the system, but the observed differences are within the computed uncertainty and noise bounds. Note that the use of norm-2 criteria permits local deviations and then a typical response convergence like those observed in other identification methods is rather unlikely.

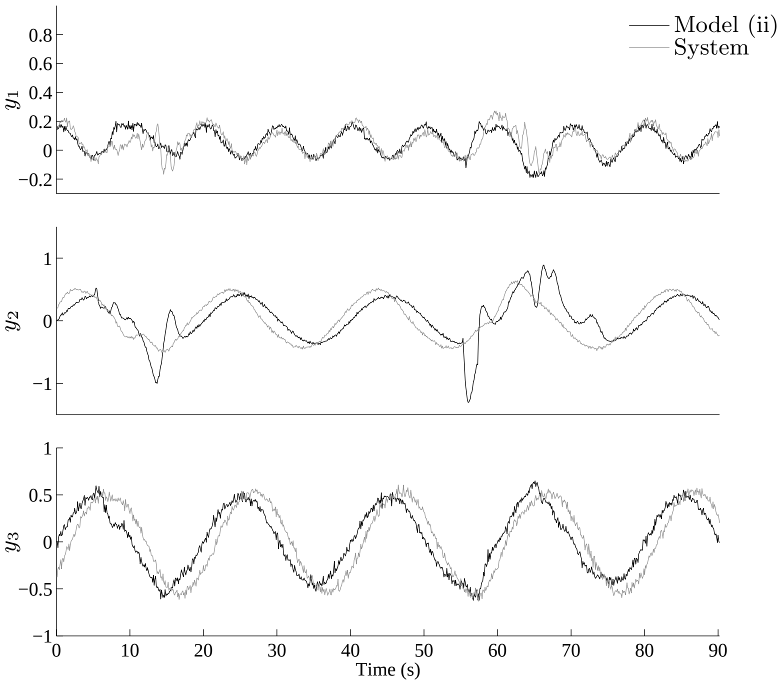

Figure 5. Responses of the real model (black lines) and the model (ii) (gray lines) in Example 2

Note that the resulting model is not invalidated only for trajectories as fast as the signal in Figure 4, which therefore provides a consistent bound on the parameter rate. This last issue can be useful for further controller designs applied to this model that use this information [21].

5. CONCLUSIONS

A robust LPV identification method has been presented, based on invalidation concepts. The methodology is able to compute a model consistent with the uncertainty/noise bounds and a given model structure. A procedure that gradually increase the complexity of the model is discussed in order to help in the selection of the model structure. For fixed uncertainty and noise bounds and depending on the system under test, the LPV model could have constant parameters, i.e. LTI, affine parameter dependency, or more complex parameter dependency, i.e. multilinear, quadratic, etc., in order to make the model consistent with the experiment. Otherwise, increasing the uncertainty bound may simplify the parameter dependency of the model in order to use simpler control design methods, e.g. affine LPV models. Of course the controller synthesis simplification has a cost: the resulting performance of the closed loop system will in general be lower. The application to nonlinear systems with the aim of designing gain scheduling controllers has also been discussed.

REFERENCES

1. Rugh WJ, Shamma J. Research on gain scheduling. *Automatica* 2000; **36**:1401–1425.
2. Giarré L, Bausoa D, Falugib P, Bamieh B. LPV model identification for gain scheduling control: An application to rotating stall and surge control problem. *Control Engineering Practice* 2006; **14**:351–361.
3. Gaspar P, Szabo Z, Bokor J. Gray-box continuous-time parameter identification for LPV models with vehicle dynamics applications. *Mediterranean Conference on Control and Automation*, 2005.
4. Tóth R, Heuberger P, den Hof PV. Optimal pole selection for LPV system identificaiton with OBFs, a clustering approach. *14th IFAC Symposium on System Identification*, Newcastle, Australia, 2006; 356–361.
5. Lee L, Poolla K. Identification of linear parameter-varying systems via LFTs. *Proceedings of the 35th IEEE Decision and Control*, vol. 2, 1996; 1545–1550, doi:10.1109/CDC.1996.572742.
6. Mazzaro MC, Movsichoff B, Sánchez Peña RS. Robust identification of linear parameter varying systems. *Proceedings of the 1999 American Control Conference*, 1999; 2282–2284.
7. Salcedo J, Martínez M. Identificación de modelos LPV para el Control de Sistemas Nolineales. *RIAI (in spanish)* 2006; **3**(3):92–107.
8. Verdult V, Verhaegen M. Subspace identification of multivariable LPV systems. *Automatica* 2002; **38**:805–814.
9. Verdult V, Verhaegen M. Kernel methods for subspace identification of multivariable LPV and bilinear systems. *Automatica* 2005; **41**:1557–1565.
10. Felici F, vanWingerden JW, Verhaegen M. Subspace identification of MIMO LPV systems using a periodic scheduling sequence. *Automatica* 2007; **43**:1684–1697.
11. Verdult V, Verhaegen M. Subspace identification of piecewise linear systems. *Proceedings on the 43rd Conference on Decision and Control*, Bahamas, 2004; 3838–3843.
12. Tóth R, Felici F, Heuberger PSC, Van den Hof PMJ. Discrete time LPV I/O and state-space representations, differences of behavior and pitfalls of interpolation. *Proceedings of the European Control Conference (ECC)*, Kos, Greece, 2007.
13. van Wingerden JW, Verhaegen M. Subspace identification of multivariable LPV systems: a novel approach. *Proceedings of the IEEE Multi-conference on Systems and Control (MSC)*, San Antonio, USA, 2008.
14. Lovera M, Casella F. LPV/LFT modelling and identification: overview, synergies and a case study. *Proceedings of the IEEE Multi-conference on Systems and Control (MSC)*, San Antonio, USA, 2008.
15. Sznaier M, Mazzaro MC, Inanc T. An LMI approach to control oriented identification of LPV systems. *Proceedings of the 2000 American Control Conference*, Chicago, Illinois, 2000; 3682–3686.
16. Sznaier M, Mazzaro MC. An LMI approach to control-oriented identification and model (in)validation of LPV systems. *IEEE Transactions on Automatic Control* 2003; **48**(9):1619–1624.
17. Smith R, Dullerud G, Rangan S, Poolla K. Model validation for dynamically uncertain systems. *Mathematical and Computer Modelling of Dynamical Systems* 1997; **3**(1):43–58.
18. Hald U, Hesselbk M, Holmgaard J, Jensen C, Jakobsen S, Siegumfeldt M. Autonomous helicopter. modelling and control. *Technical Report*, Aalborg University, Department of Control Engineering 2005.

19. VanAntwerp JG, Braatz RD. A tutorial on linear and bilinear matrix inequalities. *Journal of Process Control* 2000; **10**:363–385.
20. Sánchez Peña RS, Sznaier M. *Robust Systems Theory and Applications.* John Wiley & Sons, Inc., 1998.
21. Wu F, Yang XH, Packard A, Becker GS. Induced $\mathcal{L}_2$–norm control for LPV systems with bounded parameter variation rates. *International Journal of Nonlinear and Robust Control* 1996; .

APPENDIX

Practical implementation issues: selection of the basis of functions

A simple methodology to select the basis of functions $\mathcal{F}^N$ in those cases where the use of physical insight is not possible is described next. Basically, the idea is to apply Theorem 3.1 iteratively in order to find an adequate basis of functions. Given the initial nominal model as well as external noise and uncertainty sets, we propose a basis of functions and check consistency with Theorem 3.1. If the nominal LPV model obtained by adding the term $S_{\mathrm{C}}(\rho)$ is not consistent, then new terms are added to the basis and again consistency is tested. The procedure continues until a model that does not invalidate the experimental information is obtained. This iterative methodology allows to increase gradually the complexity of the model only when necessary.

A natural choice is the gradual selection of components from the basis of polynomial functions of $\rho_k = \begin{bmatrix}\rho_{1,k} & \cdots & \rho_{n_\rho,k}\end{bmatrix}^T \in \mathbb{R}^{n_\rho}$. The iterative selection would be as follows:

$$\begin{array}{rclr}
\mathcal{F}^1 & = & \{1\} & \text{(LTI)} \\
\mathcal{F}^2 & = & \{1, \rho_1\} & \text{(affine LPV)} \\
\vdots & = & \vdots & \text{” ”} \\
\mathcal{F}^{n_\rho+1} & = & \{1, \rho_1, \cdots \rho_{n_\rho}\} & \text{(affine LPV)} \\
\mathcal{F}^{n_\rho+2} & = & \{1, \rho_1, \cdots, \rho_{n_\rho}, \rho_1\rho_2\} & \text{(multilinear LPV)} \\
\vdots & = & \vdots & \vdots \\
\mathcal{F}^{n_\rho+n_\rho!} & = & \{1, \rho_1, \cdots, \rho_{n_\rho}, \rho_1\rho_2, \rho_1\rho_3, \cdots, \rho 1 \cdots \rho_{n_\rho}\} & \text{(multilinear LPV)} \\
\vdots & = & \vdots & \vdots
\end{array}$$

Finally, the methodology can be summarized in the following algorithm.

Test the consistency of the initial model $S_{\mathrm{I}}(\rho)$ against the experimental measurements $\{\mathbf{u}, \mathbf{y}\}$ and the assumed uncertainty/noise sets using Theorem 2.1.
if the model is not invalidated **then**
  The dynamic behavior of the system can be covered by the initial model $S_{\mathrm{I}}(\rho_k)$ and the uncertainty/noise bounds, and the procedure ends.
else
  Set $N = 1$ in (10), fix $(A_1, B_{2,1})$ and compute a new LPV model according to equation (8) with the use of Theorem 3.1.
  while the LPV model is not found **do**
    Increase N and define new pairs $(A_N, B_{2,N})$, and apply Theorem 3.1 to find a new LPV model.
  end while

The dynamic behavior of the system can be covered by the model $S_{\mathrm{I}}(\rho) + S_{\mathrm{C}}(\rho)$ and the uncertainty/noise bounds, and the procedure ends.

end if

In order to investigate if there exists a simpler model consistent with the experimental data, we can gradually eliminate parameters in the resulting model and check its consistency.