2 Beam Elements

In the second-year course lumped-parameter models were considered. In these systems mass, stiffness and damping are thought of as lumped at discrete points within the system. In Part A of this course distributed-parameter systems have been investigated. However these systems are often difficult to solve theoretically for all but the simplest problems. For example, we have restricted ourselves to looking at beams with uniform cross-section and stiffness along their length, with additional discrete mass or inertia at either end (but not partway along the length of the beam). Even for these simple cases we have seen that the resulting equations can be transcendental and so need numerical iteration to solve. These restrictions can be relaxed slightly by using energy-based approximate methods such as Rayleigh methods.

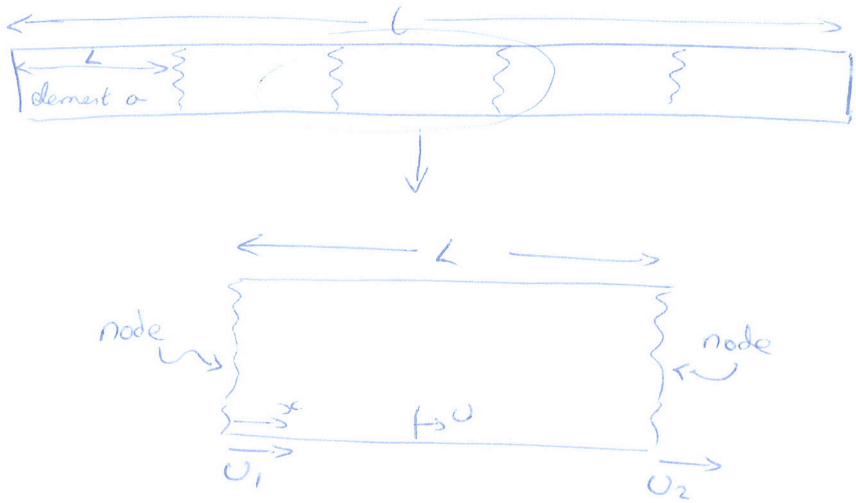
Finite Element Analysis (FEA), or Modelling, is an alternative method for studying distributed-parameter systems. It is potentially more accurate than approximating distributed-parameter systems to lumped-parameter systems and easier to solve than the partial differential equations that arise when considering distributed-parameter systems theoretically. However it is an approximate method. Here we restrict ourselves to considering exclusively beams, although the concepts can be extended to consider plates and 3D objects.

The stages involved in the FEA are:

1. Split the system into multiple simple beam elements
2. Derive the equation of motion for each small (finite) element based on a low order polynomial approximation to the deformed shape.
3. Arrange these local (based on individual elements) solutions in the form of global (representing the whole system) mass and stiffness matrices.
4. Compute natural frequencies and mode shapes using these mass and stiffness matrices.

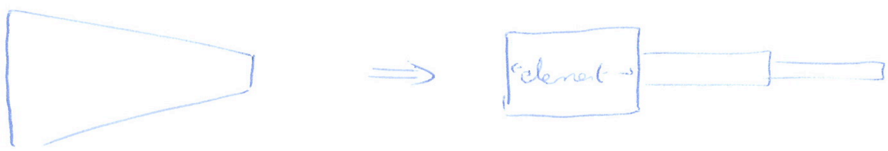
2.1 Axial Vibration - Matrices

A beam may be split into several elements:



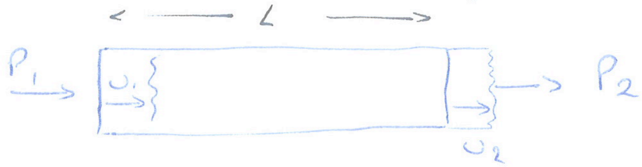
Now the local mass and stiffness matrix for a beam element can be derived. These can be used to generate a global mass and stiffness matrix for the entire beam. These local matrices are in terms of the displacements at either end of the element, which lead to global matrices in terms of the displacements of the joints between beam elements or *nodes*. **Note:** the definition of node here is different to that in modal analysis where node refers to a point of no motion.

The element don't have to be identical, for example a tapered bar could be approximated to a series of elements with reducing cross sections.



Stiffness Matrix

Stiffness is associated with static deflection, therefore the stiffness matrix can be derived by considering the static force-deflection relationship for a bar:



From equilibrium $P_1 = -P_2$. The extension of the bar is $u_2 - u_1$, therefore using Hooke's law we can write:

$$P_2 = EA\frac{\overbrace{u_2 - u_1}^{\epsilon}}{L}$$

since $P = A\sigma = AE\epsilon = AE\Delta/L$ (i.e. the effective spring stiffness is $k = AE/L$). We can now write an equations for the forces at the nodes in terms of the displacements at the nodes:

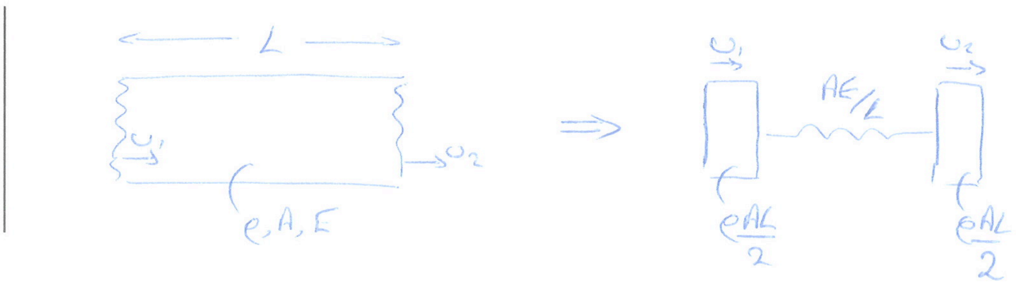
$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underline{\mathbf{K}}\underline{\mathbf{u}}$$

where $\underline{\mathbf{K}}$ is the *stiffness matrix*:

$$\underline{\mathbf{u}}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad , \quad \underline{\mathbf{K}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Discrete Lumped-Mass Matrix

Before considering the continuous system, it is useful to consider a lumped-mass approximation to the element as it is relatively straightforward.



The lumped-mass approach treats the mass in the system to be concentrated at the nodal points. For our beam element, we represent the mass of the element ρAL as a point mass of $\rho AL/2$ at both nodes. So, we can write the mass matrix as:

$$\underline{\mathbf{M}} = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Alternatively, treating the system as a two DOF discrete system we can derive the equation of motion for each of the masses:

$$\frac{\rho AL}{2}\ddot{u}_1 = \frac{AE}{L}(u_2 - u_1)$$
$$\frac{\rho AL}{2}\ddot{u}_2 = -\frac{AE}{L}(u_2 - u_1)$$

This may be rewritten in terms of our mass and stiffness matrices:

$$\frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightsquigarrow \quad \underline{\mathbf{M}}\underline{\ddot{\mathbf{u}}} + \underline{\mathbf{K}}\underline{\mathbf{u}} = \mathbf{0}$$

* state (appendix) $T = \frac{1}{2}\underline{\dot{u}}^T \underline{M}\, \underline{\dot{u}}$, $V = \frac{1}{2}\underline{u}^T \underline{K}\, \underline{u}$ $\left\{ M\ddot{u} + Ku = 0 \right.$

calc T in terms of $\underline{u}$ ie nodal deflections

Continuous Mass Matrix

For the continuous element, we assume that the deflected shape is a first order polynomial with respect to position along the beam, x:

$$u(x,t) = a(t) + b(t)\,x$$

We can now apply the end conditions, at the nodes the displacements are:

$$u(0,t) = u_1(t) \quad , \quad u(L,t) = u_2(t)$$

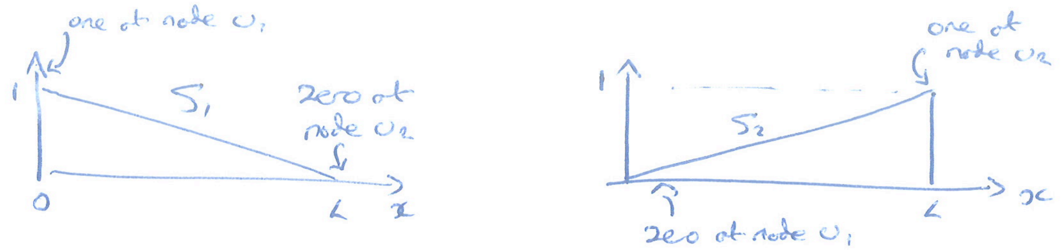
which gives

$$u_1(t) = a(t) \quad \text{and} \quad u_2(t) = a(t) + b(t)L$$

Therefore, using our linear approximation to the deflected shape, we can write an expression for the deflection along the beam in terms of the deflection at the nodes:

$$u(x,t) = \left(1 - \frac{x}{L}\right) u_1(t) + \frac{x}{L} u_2(t) = S_1(x)u_1(t) + S_2(x)u_2(t)$$

where $S_1(x)$ and $S_2(x)$ are called *shape functions.*



The shape functions are so called because their geometric shape corresponds to the motion of one node whilst the other is constrained. The approximation for u is a summation of these shapes multiplied by the nodal displacements.

The kinetic energy in the bar element can be written as *

$$T = \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

By defining a vector of the nodal displacements, $\underline{\mathbf{u}}$, and a shape vector $\underline{\mathbf{S}}$:

$$\underline{\mathbf{u}}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad , \quad \underline{\mathbf{S}} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

note $\underline{u}$ function of t

$\underline{S}$ function of x

The linear approximation to the deflected shape may be written as:

$$u(x,t) = S_1(x)u_1(t) + S_2(x)u_2(t) = \underline{\mathbf{u}}^T\underline{\mathbf{S}} = \underline{\mathbf{S}}^T\underline{\mathbf{u}}$$

Therefore:

$$\frac{\partial u}{\partial t} = \underline{\dot{\mathbf{u}}}^T\underline{\mathbf{S}} = \underline{\mathbf{S}}^T\underline{\dot{\mathbf{u}}}$$

Substituting this expression into the equation for the kinetic energy gives

$$T = \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x = \frac{1}{2}\rho A \int_0^L \underline{\dot{\mathbf{u}}}^T \underline{\mathbf{S}}\underline{\mathbf{S}}^T \underline{\dot{\mathbf{u}}} \mathrm{d}x = \frac{1}{2}\underline{\dot{\mathbf{u}}}^T \ \rho A \int_0^L \underline{\mathbf{S}}\underline{\mathbf{S}}^T \mathrm{d}x \ \ \underline{\dot{\mathbf{u}}}$$

It can be shown that the kinetic energy may be written in terms of the mass matrix and nodal displacements (beyond the scope of the course – see appendix):

if $T = \frac{1}{2}\dot{u}^T \underline{M}\, \dot{\underline{u}}$, $V = \frac{1}{2} u^T K u$ $\Big\}$ $\underline{M}\ddot{u} + \underline{K}\underline{u} = 0$

$$T = \frac{1}{2}\underline{\dot{\mathbf{u}}}^T \underline{\mathbf{M}}\underline{\dot{\mathbf{u}}}$$

~ diagonal mass matrix $T = \frac{1}{2} M_{11} \dot{u}_1^2 + \frac{1}{2} M_{22} \dot{u}_2^2$

Therefore the mass matrix may be written as:

$$\underline{\mathbf{M}} = \rho A \int_0^L \underline{\mathbf{S}}\underline{\mathbf{S}}^T \mathrm{d}x$$

or the $\{i, j\}$ element on the mass matrix may be written as:

$$\underline{\mathbf{M}}(i,j) = \rho A \int_0^L S_i S_j \mathrm{d}x \quad \rightsquigarrow \quad \underline{\mathbf{M}} = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Note that the mass matrix is dependent on the assumed linear deflected shape. (10 min left) Lec 2

We can use a similar approach to derive the stiffness matrix by rewriting the expression for potential energy:

$$V = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

in the form $V = \frac{1}{2}\underline{\mathbf{u}}^T \underline{\mathbf{K}}\underline{\mathbf{u}}$. Taking the double derivative of the assumed shape with respect to position we can write:

$$\frac{\partial u}{\partial x} = \underline{\mathbf{u}}^T \underline{\mathbf{S}}' = \underline{\mathbf{S}}'^T \underline{\mathbf{u}}$$

where $\{*\}'$ indicates the derivative with respect to position. Substituting this expression into the equation for the potential energy gives:

$$V = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x = \frac{1}{2}\underline{\mathbf{u}}^T EA \int_0^L \underline{\mathbf{S}}'\underline{\mathbf{S}}'^T \mathrm{d}x \ \ \underline{\mathbf{u}}$$

Therefore the stiffness matrix may be written as

$$\underline{\mathbf{K}} = EA \int_0^L \underline{\mathbf{S}}'\underline{\mathbf{S}}'^T \mathrm{d}x$$

or the $\{i, j\}$ element on the stiffness matrix may be written as:

$$\underline{\mathbf{K}}(i,j) = EA \int_0^L S_i' S_j' \mathrm{d}x \quad \rightsquigarrow \quad \underline{\mathbf{K}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Note that the stiffness matrix derived here is the same as that derived earlier because the assumed linear deformed shape used here is identical to the statically deformed shape used earlier.

2.2 Natural Frequencies

We have already seen, when considering the lumped-mass model, that we can write the equation of motion in terms of the mass and stiffness matrices

$$\underline{\mathbf{M}}\ddot{\underline{\mathbf{u}}} + \underline{\mathbf{K}}\underline{\mathbf{u}} = 0$$

For the case where the system is continuous, the conversion of the energy expressions to this equation of motion is based on the Lagrange approach and is given in the appendix (beyond the scope of the course). As we have seen in part A of the course, the beam vibrations are sinusoidal with respect to time so we can write a trial solution in the form:

$$\underline{\mathbf{u}} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \mathrm{e}^{\,j\omega t} = \underline{\mathbf{U}}\mathrm{e}^{\,j\omega t}$$

where U_1 and U_2 are the amplitudes of response and ω is the natural frequency. Noting that $\ddot{\mathbf{u}} = -\omega^2\mathbf{u}$, this gives

$$-\omega^2\underline{\mathbf{M}}\underline{\mathbf{U}} + \underline{\mathbf{K}}\underline{\mathbf{U}} = 0$$

Pre-multiplying by the inverse of the mass matrix, we can write this expression in the eigenvector form:

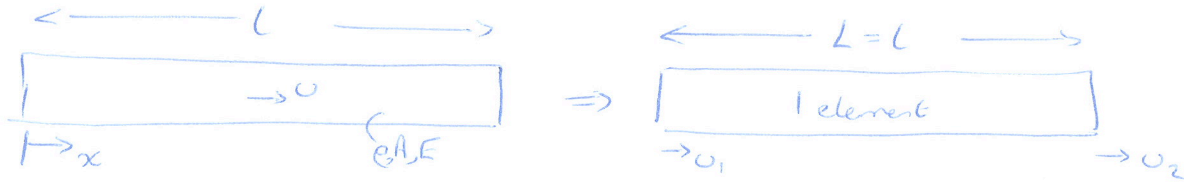
$$-\omega^2\underline{\mathbf{M}}^{-1}\underline{\mathbf{M}}\underline{\mathbf{U}} + \underline{\mathbf{M}}^{-1}\underline{\mathbf{K}}\underline{\mathbf{U}} = 0 \quad \rightsquigarrow \quad \left(-\omega^2\underline{\mathbf{I}} + \underline{\mathbf{M}}^{-1}\underline{\mathbf{K}}\right)\underline{\mathbf{U}} = 0$$

Won't invert M in future examples

This can now be solved as an eigenvalue/eigenvector problem, i.e. find values of ω that set the determinant of $-\omega^2\underline{\mathbf{I}} + \underline{\mathbf{M}}^{-1}\underline{\mathbf{K}}$ to zero. Then the ratio of the values in the $\mathbf{U}$ vector, which gives the mode shape, can be found by substituting the value of ω back into the equation.

$$\underline{A}\underline{x} = \lambda \underline{x}$$
$$(\underline{A} - \lambda \underline{I})\underline{x} = 0$$

Example: A One-Element Axial vibration model of a Free-Free bar



$$\underline{M} = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \underline{K} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{6}{\rho A L}\,\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(-\omega^2\underline{I} + \underline{M}^{-1}\underline{K})\underline{U} = 0 \quad \Rightarrow$$

$$\left(\frac{6EA}{\rho A L^2}\,\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2\underline{I}\right)\underline{U} = 0$$

$$\left(\frac{6E}{\rho L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \begin{bmatrix}\omega^2 & 0\\ 0 & \omega^2\end{bmatrix}\right)\underline{U} = 0 \qquad (a)$$

Det P20 Part A

$$\text{Det} = 0 : \quad \begin{vmatrix}1 - \frac{\rho L^2\omega^2}{6E} & -1\\ -1 & 1 - \frac{\rho L^2 \omega^2}{6E}\end{vmatrix} = 0$$

$$\left(1 - \frac{\rho L^2\omega^2}{6E}\right)^2 - 1 = 0 \quad \Rightarrow \quad \omega^2 = 0, \ \frac{12E}{\rho L^2}$$

modeshape :

$$\omega^2 = \frac{12E}{\rho L^2} \text{ in } (a) : \quad \frac{6E}{\rho L^2}\left(\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)\begin{pmatrix}U_1\\ U_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore U_1 = -U_2$$

$L = l \Rightarrow \omega = \frac{3.46}{l}\sqrt{\frac{E}{\rho}}$ with modeshape

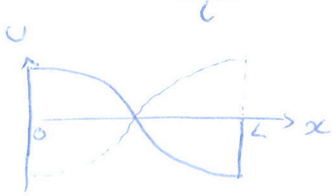
$(\omega = 0,$

$)$

exact :
part A
Example Sheet 1
Question 1

$\omega = \frac{n\pi c}{l} = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}$ modeshape $= \cos\frac{n\pi x}{l}$

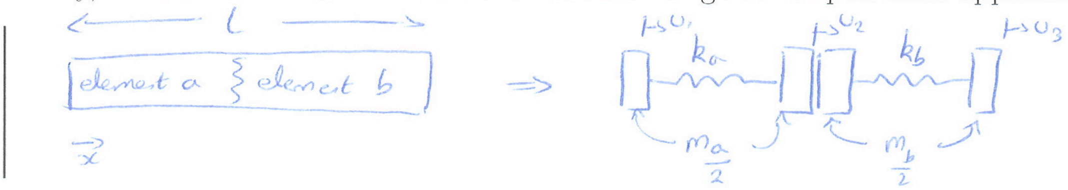
$\therefore$ first mode $\omega = \frac{3.14}{l}\sqrt{\frac{E}{\rho}}$,



A way of improving the estimated frequency and estimated modeshapes is to increase the number of beam elements used. Currently with only one beam element the model is equivalent to using the Rayleigh method with a linear deformed shape approximation. Additional elements allow more complex assumed shapes to exist (in the form of a series of linear relationships between adjacent nodes) which can more closely match the real mode shapes and hence real natural frequencies. Also they allow higher order modes to be approximated - the number of modes matches the number of nodes used (which in the lumped-mass model is the same as the number of DOFs).

2.3 Multiple Elements

Firstly, consider a two-element model of a beam using the lumped-mass approximation.



The resulting system has three DOFs, with the middle mass, corresponding to the u_2 node, being a combination of the half the mass of element a and half the mass of element b. The equations of motion for the three DOFs may now be derived:

$$\begin{aligned}
\frac{m_a}{2}\ddot{u}_1 &= k_a(u_2 - u_1) \\
\frac{m_a + m_b}{2}\ddot{u}_2 &= k_a(u_1 - u_2) + k_b(u_3 - u_2) \\
\frac{m_b}{2}\ddot{u}_3 &= k_b(u_2 - u_3)
\end{aligned}$$

where m_i is the mass of element i ($m_i = \rho_i A_i L_i$) and k_i is the effective stiffness of element i ($k_i = A_i E_i / L_i$). This distinction between the elements has been made since in general A, L, E, or ρ may vary from element to element. These equations of motion may be rewritten in matrix form as:

$$\frac{1}{2}\begin{bmatrix} m_a & 0 & 0 \\ 0 & m_a + m_b & 0 \\ 0 & 0 & m_b \end{bmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{pmatrix} + \begin{bmatrix} k_a & -k_a & 0 \\ -k_a & k_a + k_b & -k_b \\ 0 & -k_b & k_b \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The same result can be derived by considering the local mass and stiffness matrices, which for element i may be expressed as

$$\underline{\mathbf{M_i}} = \frac{m_i}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \underline{\mathbf{K_i}} = k_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Using these local matrices, we can build up global mass and stiffness matrices by writing (for the case where there are two elements):

$$\left(\begin{bmatrix} \multicolumn{2}{c}{(\underline{\mathbf{M_a}})} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{(\underline{\mathbf{M_b}})} \\ 0 & & \end{bmatrix}\right)\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{pmatrix} +$$

$$\left(\begin{bmatrix} \multicolumn{2}{c}{(\underline{\mathbf{K_a}})} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{(\underline{\mathbf{K_b}})} \\ 0 & & \end{bmatrix}\right)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

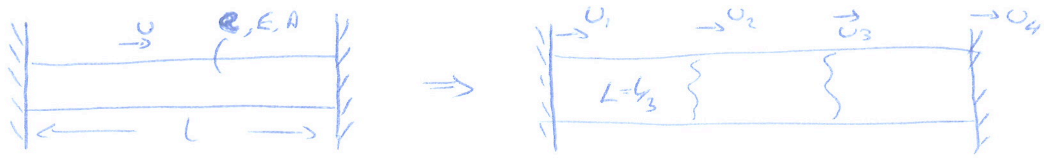
The same procedure may be adopted for larger numbers of degrees of freedom, where we must ensure that the local matrices for each element lines up with the relevant nodes.

2.4 Boundary Conditions

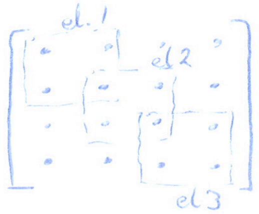
Up to now only the *free* support condition has been considered. To apply a *fixed* support boundary condition the displacement, velocity and acceleration at that node must be set to zero. This may be done by crossing out the rows and columns of the mass and stiffness matrices relating to the fixed node.

explain this

Example: A Three-Element Axial vibration model of a Fixed-Fixed bar



global : $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$



local : $\underline{M} = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\underline{K} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

global : $\underline{M} = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $\underline{K} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

∴ if free-free bar :

$-\omega^2 \underline{M}\underline{u} + \underline{K}\,\underline{u} = 0 \Rightarrow$

here

$$\left(-\omega^2 \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\right)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

apply BCs : $u_1 = 0$ (fixed at $x = 0$)

$u_4 = 0$ (fixed at $x = L$)

$$\Rightarrow \left(-\omega^2 \frac{\rho A L}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$M^{-1} = \frac{6}{\rho A L}\,\frac{1}{15}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

$$\left(-\omega^2 \underline{I} + \frac{6E\cancel{A}}{\rho \cancel{A} L^2 15}\underbrace{\begin{bmatrix}4 & -1\\-1 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}}_{\begin{bmatrix}9 & -6\\-6 & 9\end{bmatrix}}\right)\begin{pmatrix}U_2\\U_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad (a)$$

Det = 0 :
$$\begin{vmatrix} 9 - \frac{15\rho L^2}{6E}\omega^2 & -6 \\ -6 & 9 - \frac{15\rho L^2}{6E}\omega^2 \end{vmatrix} = 0 \quad \Rightarrow \quad \omega^2 = \cancel{3}\left(\frac{6E}{\cancel{15}\rho L^2}\right),\ \cancel{15}\left(\frac{6E}{\cancel{15}\rho L^2}\right)$$

modeshapes :

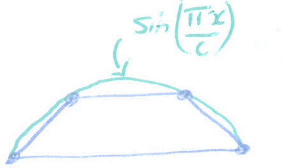
$$\omega_1^2 = 3\left(\frac{6E}{15\rho L^2}\right) \overset{\text{in (a)}}{\Longrightarrow} \begin{pmatrix}9-3 & -6\\ -6 & 9-3\end{pmatrix}\begin{pmatrix}U_2\\U_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \therefore\ U_2 = U_3$$

$$\omega_2^2 = 15\left(\frac{6E}{15\rho L^2}\right) \Rightarrow \begin{pmatrix}9-15 & -6\\ -6 & 9-15\end{pmatrix}\begin{pmatrix}U_2\\U_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \therefore\ -U_2 = U_3$$

$$L = \frac{l}{3}$$

FE

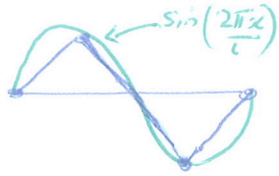
$$\omega_1 = \sqrt{\frac{6\times 3}{15}}\sqrt{\frac{E}{\rho}}\,\frac{3}{l} = \frac{3.286}{l}\sqrt{\frac{E}{\rho}}$$



Exact

$$\omega = \frac{\overset{3.14}{\pi}}{l}\sqrt{\frac{E}{\rho}}$$

$$\omega_2 = \sqrt{\frac{15\times 6}{15}}\sqrt{\frac{E}{\rho}}\,\frac{3}{l} = \frac{7.348}{l}\sqrt{\frac{E}{\rho}}$$



$$\omega = \frac{\overset{6.28}{2\pi}}{l}\sqrt{\frac{E}{\rho}}$$

if 5 elements used

$$\omega_1 = \frac{3.19}{l}\sqrt{\frac{E}{\rho}}$$

$$\omega_2 = \frac{6.70}{l}\sqrt{\frac{E}{\rho}}$$

$$\underline{M} = \frac{\rho A L}{6}\underline{M}^*, \quad \underline{K} = \frac{EA}{L}\underline{K}^*$$

$$\Rightarrow \left(-\omega^2\frac{\rho L^2}{6E} + \underline{M}^{*-1}\underline{K}^*\right)\underline{U} = 0$$

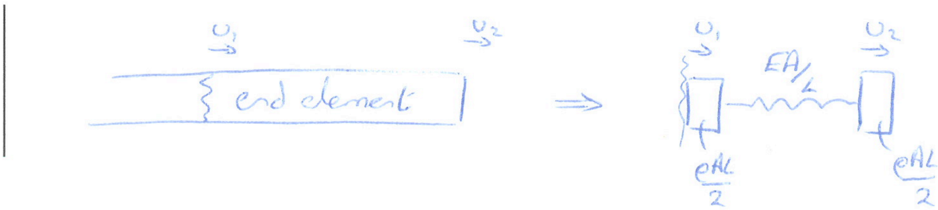
eigenvalues of $\underline{M}^{*-1}\underline{K}^* = \lambda$: $\omega^2 = \frac{6E}{\rho L^2}\lambda$

demo

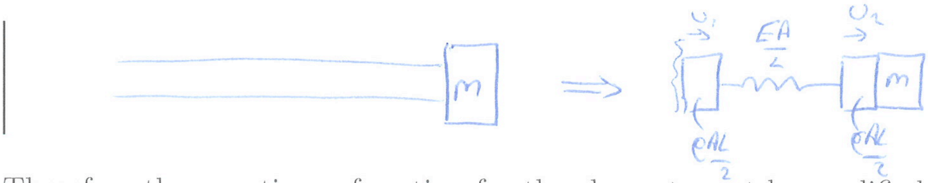
fe-axial.m

Note that the FE model has provided estimates for the first two modes since there are two non-zero nodes. Also note that higher order modes are increasingly inaccurate since the mode shape they are attempting to recreate using linear functions between nodes are more complex.

To include a lumped mass or a flexible support condition the global mass or global stiffness matrix must be modified respectively. It is easiest to see this if we consider the lumped-mass approximation to the continuous element.



If an additional lumped mass is added to the end of the beam (let us consider the right-hand end), the resulting beam elements becomes



Therefore the equations of motion for the element must be modified:

$$\text{from}: \quad \tfrac{\rho AL}{2}\ddot{u}_1 = \tfrac{AE}{L}(u_2 - u_1) \quad , \quad \tfrac{\rho AL}{2}\ddot{u}_2 = \tfrac{AE}{L}(u_1 - u_2)$$
$$\text{to}: \quad \tfrac{\rho AL}{2}\ddot{u}_1 = \tfrac{AE}{L}(u_2 - u_1) \quad , \quad \left(\tfrac{\rho AL}{2} + m\right)\ddot{u}_2 = \tfrac{AE}{L}(u_1 - u_2)$$

just state result

Therefore we must modify the local mass matrix for this element:

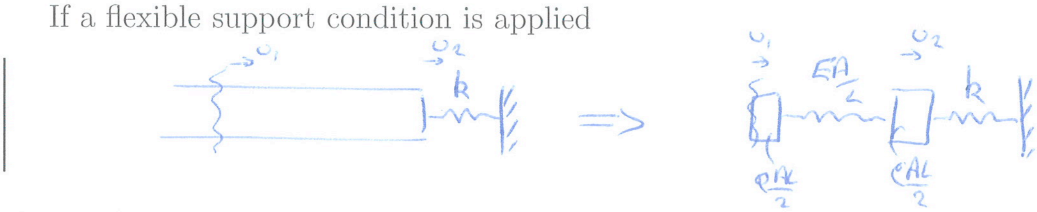
$$\text{from}: \quad \underline{\mathbf{M}} = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{to}: \quad \underline{\mathbf{M}} = \begin{bmatrix} \frac{\rho AL}{2} & 0 \\ 0 & \frac{\rho AL}{2} + m \end{bmatrix}$$

but the stiffness remains unaltered. Likewise for a continuous element, the local mass matrix is modified:

$$\text{from}: \quad \underline{\mathbf{M}} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{to}: \quad \underline{\mathbf{M}} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 + m\frac{6}{\rho AL} \end{bmatrix}$$

These type of modification is valid for a mass added to any node, not just the nodes corresponding to the ends of the bar. If a lumped mass is added to the i^{th} node, then the global mass matrix is modified by adding that mass to the $\underline{\mathbf{M}}(i,i)$ value in the matrix $\underline{\mathbf{M}}$.

If a flexible support condition is applied



the resulting beam equations for the local lumped-mass element become:

$$\tfrac{\rho AL}{2}\ddot{u}_1 = \tfrac{AE}{L}(u_2 - u_1) \quad , \quad \tfrac{\rho AL}{2}\ddot{u}_2 = \tfrac{AE}{L}(u_1 - u_2) - ku_2$$

resulting in the following modification to the local stiffness matrix:

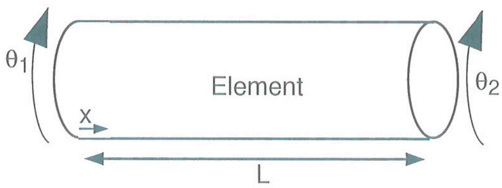
$$\text{from}: \quad \underline{\mathbf{K}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{to}: \quad \underline{\mathbf{K}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 + \frac{L}{EA}k \end{bmatrix}$$

Again this type of modification can be made to any node i by modifying the $\underline{\mathbf{K}}(i,i)$ element of the global stiffness matrix.

I don't lecture!

2.5 Torsional Vibration

Mass (strictly inertia) and stiffness matrices can be derived for torsional beam elements in an analogous way to the axial vibrating beam elements. Assuming the deformed



shape has a linear relationship with distance along the element $\theta(x,t) = a(t) + b(t)x$ and applying the node conditions $\theta(0,t) = \theta_1(t)$ and $\theta(L,t) = \theta_2(t)$ gives the relationship:

$$\theta(x,t) = \left(1 - \frac{x}{L}\right)\theta_1(t) + \frac{x}{L}\theta_2(t) = S_1\theta_1(t) + S_2\theta_2(t)$$

where S_1 and S_2 are the shape functions for torsional rotation (and are identical to those we derived when considering axial vibrations).

Using the kinetic energy equation and substituting in the assumed deflected shape gives:

$$T = \int_0^L \frac{1}{2}\rho I_p \left(\frac{\partial \theta}{\partial t}\right)^2 \mathrm{d}x = \int_0^L \frac{1}{2}\rho I_p \left(S_1(x)\frac{\partial \theta_1}{\partial t} + S_2(x)\frac{\partial \theta_2}{\partial t}\right)^2 \mathrm{d}x$$

The energy may be rewritten in matrix form as

$$T = \frac{1}{2}\underline{\dot{\theta}}^T \underline{\mathbf{M}}\underline{\dot{\theta}}$$

where θ is a vector of the nodal displacements and $\underline{\mathbf{M}}$ is the mass matrix for torsional vibrations (see section 2.1 for details of derivation):

$$\underline{\theta}(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} \quad , \quad \underline{\mathbf{M}} = \frac{\rho I_p L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Likewise by considering the potential energy equation:

$$V = \int_0^L \frac{1}{2} G I_p \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

we can derive the stiffness matrix for torsional vibrations:

$$\underline{\mathbf{K}} = \frac{\cancel{\rho} G I_p}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This analysis is identical in form to the axial vibration with the following analogies:

displacement	$u(x,t)$	$\Leftrightarrow$	$\theta(x,t)$
stiffness	$P = \frac{EA}{L}x = kx$	$\Leftrightarrow$	$T = \frac{GI_p}{L}\theta = k_r\theta$
mass/inertia	$m_{element} = \rho A L$	$\Leftrightarrow$	$J_{element} = \rho I_p L$

In exactly the same way as for the axial vibrations, global matrices can be generated, boundary conditions applied and, the equation of motion for torsional vibration:

$$\underline{\mathbf{M}}\ddot{\underline{\theta}} + \underline{\mathbf{K}}\underline{\theta} = \mathbf{0}$$

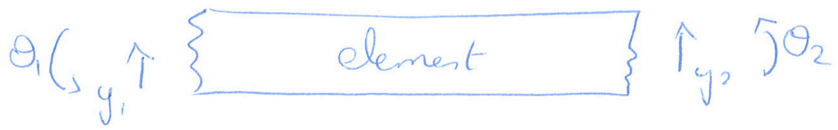
can be solved to find the mode shapes and natural frequencies.

able to do all ~~test~~ ex. Shed.

lecture 4
(able to do all
first example sheet)

2.6 Flexural Vibration

For flexural vibration we need four variables, two displacements and two rotations, to describe the two end conditions of an element:



rather than the two variables needed for axial vibration (just as we needed four boundary conditions rather than the two for axially vibrating beams). The displacement vector for an element may be written as

$$\underline{\mathrm{y}} = \begin{pmatrix} y_1 & \theta_1 & y_2 & \theta_2 \end{pmatrix}^T$$

As we have four end conditions we can use a cubic approximation for the deformed shape of the element (which results in four unknowns corresponding to the four end conditions)

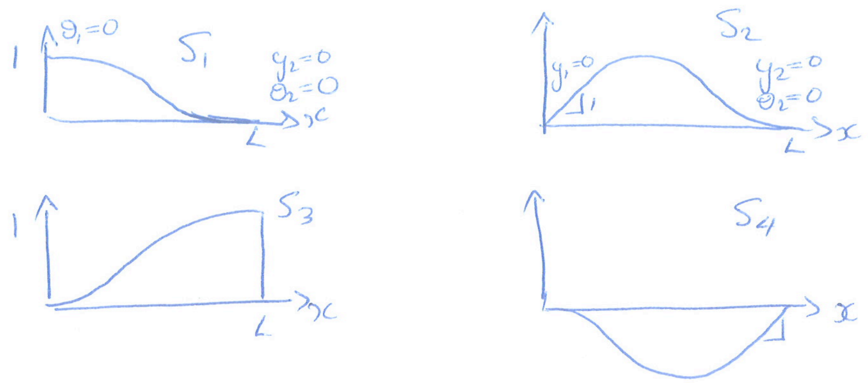
$$y(x,t) = a(t) + b(t)\,x + c(t)\,x^2 + d(t)\,x^3$$

Applying the end conditions gives:

$$\begin{array}{rcll}
\text{at } x=0 & : & y(0,t)=y_1(t) & \rightsquigarrow \quad y_1 = a(t) \\
\text{at } x=0 & : & \frac{\partial y(0,t)}{\partial x}=\theta_1(t) & \rightsquigarrow \quad \theta_1 = b(t) \\
\text{at } x=L & : & y(L,t)=y_2(t) & \rightsquigarrow \quad y_2 = a(t)+b(t)L+c(t)L^2+d(t)L^3 \\
\text{at } x=L & : & \frac{\partial y(L,t)}{\partial x}=\theta_2(t) & \rightsquigarrow \quad \theta_2 = b(t)+2c(t)L+3d(t)L^2
\end{array}$$

Solving for $a(t)$, $b(t)$, $c(t)$ and $d(t)$, substituting back into the equation for the assumed shape and rearranging gives the following expression in terms of shape functions:

$$\begin{aligned}
y = & \left(1-3\tfrac{x^2}{L^2}+2\tfrac{x^3}{L^3}\right)y_1(t) + \left(x-2\tfrac{x^2}{L^2}+\tfrac{x^3}{L^3}\right)\theta_1(t) + \left(3\tfrac{x^2}{L^2}-2\tfrac{x^3}{L^3}\right)y_2(t) + \left(-2\tfrac{x^2}{L^2}+\tfrac{x^3}{L^3}\right)\theta_2(t) \\
= & \; S_1 y_1(t) + S_2\theta_1(t) + S_3 y_2(t) + S_4\theta_2(t)
\end{aligned}$$



The kinetic energy of the element may be expressed as:

$$T = \int_0^L \frac{1}{2}\rho A \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x$$ $\frac{1}{2}\dot{y}^T M \dot{y}$

As with the axial case, to find the mass matrix, we need to rewrite this equation in the form $T = \frac{1}{2}\dot{\mathbf{y}}^T\underline{\mathbf{M}}\dot{\mathbf{y}}$, where $\underline{\mathbf{M}}$ is the mass matrix. We can write the expression for the assumed shape in vector form as:

← function of x
← function of t

$$y(x,t) = S_1 y_1 + S_2\theta_1 + S_3 y_2 + S_4\theta_2 = \mathbf{y}^T\mathbf{S} = \mathbf{S}^T\mathbf{y}$$

where S is a vector of the shape functions $\mathbf{S} = \begin{pmatrix} S_1 & S_2 & S_3 & S_4 \end{pmatrix}^T$. Taking the derivative with respect to time gives

$$\frac{\partial y}{\partial t} = \dot{\mathbf{y}}^T\mathbf{S} = \mathbf{S}^T\dot{\mathbf{y}}$$

where $\{\dot{*}\}$ indicates the derivative with respect to time and noting that the shape functions are functions of position x only. Substituting this expression into the equation for the kinetic energy gives:

$$T = \int_0^L \frac{1}{2}\rho A\dot{\mathbf{y}}^T\mathbf{S}\mathbf{S}^T\dot{\mathbf{y}}\mathrm{d}x = \frac{1}{2}\dot{\mathbf{y}}^T\ \rho A\int_0^L \mathbf{S}\mathbf{S}^T\mathrm{d}x\ \dot{\mathbf{y}}$$

noting that $\dot{\mathbf{y}}$ may be taken outside the integral as it is not a function of x (just time). Therefore the mass matrix may be written as:

$$\underline{\mathbf{M}} = \rho A\int_0^L \mathbf{S}\mathbf{S}^T\mathrm{d}x$$

or the $\{i,j\}$ element on the mass matrix may be written as:

$$\underline{\mathbf{M}}(i,j) = \rho A\int_0^L S_i S_j \mathrm{d}x$$

By substituting in the expressions for the shape functions the following mass matrix may be calculated:

$$\underline{\mathbf{M}} = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

The corresponding stiffness matrix may be found by rewriting the potential energy of the element:

$$V = \int_0^L \frac{1}{2}EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x$$ $\frac{1}{2}y^T K y$

in the form $V = \frac{1}{2}\mathbf{y}^T\underline{\mathbf{K}}\mathbf{y}$. Taking the double derivative of the assumed shape with respect to position we can write:

$$\frac{\partial^2 y}{\partial x^2} = \mathbf{y}^T\mathbf{S}'' = \mathbf{S}''^T\mathbf{y}$$

where $\{*\}'$ indicates the derivative with respect to position. Substituting this expression into the equation for the potential energy gives:

$$V = \int_0^L \frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y} \quad \text{where :} \quad \mathbf{K} = EI\int_0^L \mathbf{S}''\mathbf{S}''^T \mathrm{d}x$$

or the $\{i, j\}$ element on the stiffness matrix may be written as:

$$\mathbf{K}(i,j) = EI\int_0^L S_i'' S_j'' \mathrm{d}x$$

Substitution of the shape functions gives the stiffness matrix:

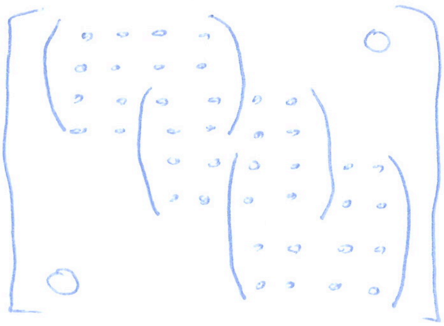
$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

As with the axial case the natural frequencies can be found by solving the equation

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0}$$

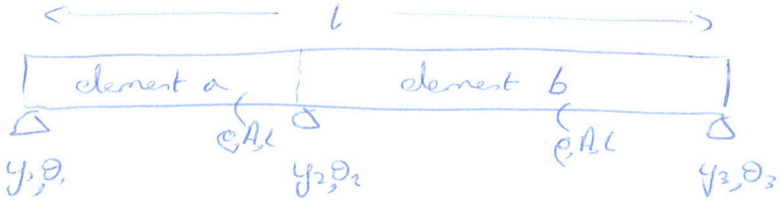
Boundary conditions are applied in the same way as for axial vibrations, by crossing out the relevant rows/columns if the simply-supported or clamped (cantilever). If an added point mass or point inertia is present modifying the relevant diagonal element in the mass matrix (corresponding to the displacement or rotation node respectively). If a flexible spring support is present then the stiffness matrix must be modified.

The process for generating the global mass and stiffness matrices from the local matrices is the same as for the axial case, **however** remember that each node has two degrees of freedom (y and θ) rather than one:



(corrected)

Example: A Two-Element flexural vibration model of a ~~Fixed Fixed~~ Simply Supported bar with a support at the midspan



local
$$\underline{M} = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \qquad \underline{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

2 element global matrices (free/free) : $\underline{M}\ddot{\underline{y}} + \underline{K}\,\underline{y} = 0$

$$\frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L & 0 & 0 \\ 22L & 4L^2 & 13L & -3L^2 & 0 & 0 \\ 54 & 13L & 156+156 & -22L+22L & 54 & -13L \\ -13L & -3L^2 & -22L+22L & 4L^2+4L^2 & 13L & -3L^2 \\ 0 & 0 & 54 & 13L & 156 & -22L \\ 0 & 0 & -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}\begin{pmatrix} \ddot{y}_1 \\ \ddot{\theta}_1 \\ \ddot{y}_2 \\ \ddot{\theta}_2 \\ \ddot{y}_3 \\ \ddot{\theta}_3 \end{pmatrix}$$

$$+ \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ -6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{pmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ y_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Lecture 5

apply BCs $\quad y_1 = \ddot{y}_1 = 0 \quad y_2 = \ddot{y}_2 = 0 \quad y_3 = \ddot{y}_3 = 0$

$\Rightarrow$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} e^{j\omega t} : \left(-\frac{\rho A L}{420}\frac{L^3}{EI}\omega^2 \begin{bmatrix} 4L^2 & -3L^2 & 0 \\ -3L^2 & 8L^2 & -3L^2 \\ 0 & -3L^2 & 4L^2 \end{bmatrix} + \begin{bmatrix} 4L^2 & 2L^2 & 0 \\ 2L^2 & 8L^2 & 2L^2 \\ 0 & 2L^2 & 4L^2 \end{bmatrix} \right)\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = 0 \qquad (1)$$

$$\bar{\omega}^2 = \frac{\rho A L^4}{420 EI}\omega^2$$

$$\text{Det} = 0 : \quad \begin{vmatrix} 4-4\bar{\omega}^2 & 2+3\bar{\omega}^2 & 0 \\ 2+3\bar{\omega}^2 & 8-8\bar{\omega}^2 & 2+3\bar{\omega}^2 \\ 0 & 2+3\bar{\omega}^2 & 4-4\bar{\omega}^2 \end{vmatrix} = 0$$

$$(4-4\bar{\omega}^2)\left[(8-8\bar{\omega}^2)(4-4\bar{\omega}^2)-(2+3\bar{\omega}^2)^2\right] - (2+3\bar{\omega}^2)\left[(2+3\bar{\omega}^2)(4-4\bar{\omega}^2) - 0\right] = 0$$

$$\bar{\omega}^2 = \frac{2}{7}, 1, 6 \quad , \quad \bar{\omega}^2 = \frac{\rho A L^4 \omega^2}{420 EI}$$

$$\therefore \quad \omega_1 = \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}\sqrt{\frac{2}{7}\times 420} = \frac{43.82}{\ell^2}\sqrt{\frac{EI}{\rho A}} \quad \text{noting } L = \frac{\ell}{2}$$

theory:

mode 1

mode 2

$$\omega_n^2 = \frac{EI}{\rho A}\alpha_n^4 \qquad \alpha_n = \frac{n\pi}{\ell}$$

$$\therefore \omega_2 = \frac{4\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}} = \frac{39.48}{\ell^2}\sqrt{\frac{EI}{\rho A}}$$

mode 2 same as mode 1 for 

Mode shapes : nodes $\underline{\Theta}_{1st} = \begin{pmatrix} 0.5872 \\ -0.5571 \\ 0.5872 \end{pmatrix}$ $\Rightarrow$ just rotations at nodes

as in ①

to work out complete shape need to insert into expression $y = S_1 y_1 + S_2\theta_1 + S_3 y_2 + S_4\theta_2$ for each element in turn

demo

fe_flex.m

3 Orthogonality and Modes

Regardless of the type of vibration being considered, the matrix form the equation of motion may be expressed in the form:

$$\underline{\mathbf{M}}\underline{\ddot{\mathbf{y}}} + \underline{\mathbf{K}}\underline{\mathbf{y}} = \mathbf{0}$$

Recapping, assuming simple harmonic motion, $\underline{\mathbf{y}} = \underline{\mathbf{Y}}e^{j\omega t}$, this may be written as

$$\underline{\mathbf{K}}\underline{\mathbf{Y}} = \omega^2 \underline{\mathbf{M}}\underline{\mathbf{Y}}$$

which is the eigenvalue/vector problem defining the natural frequencies and mode shapes.

For the l^{th} natural frequency/mode shapes pair we can write:

$$\underline{\mathbf{K}}\underline{\mathbf{Y}}_l = \omega_l^2 \underline{\mathbf{M}}\underline{\mathbf{Y}}_l$$

Pre-multiplying by the transpose of the m^{th} eigenvector solution gives:

$$\underline{\mathbf{Y}}_m^T \underline{\mathbf{K}}\underline{\mathbf{Y}}_l = \omega_l^2 \underline{\mathbf{Y}}_m^T \underline{\mathbf{M}}\underline{\mathbf{Y}}_l \tag{1}$$

Alternatively we write the equation for the m^{th} solution and pre-multiplying by the transpose of the l^{th} eigenvector solution to give:

$$\underline{\mathbf{Y}}_l^T \underline{\mathbf{K}}\underline{\mathbf{Y}}_m = \omega_m^2 \underline{\mathbf{Y}}_l^T \underline{\mathbf{M}}\underline{\mathbf{Y}}_m$$

Using $\underline{\mathbf{A}}\underline{\mathbf{B}} = (\underline{\mathbf{B}}^T \underline{\mathbf{A}}^T)^T$ and noting that $\underline{\mathbf{K}}$ and $\underline{\mathbf{M}}$ are symmetric and that $\underline{\mathbf{C}}^T = \underline{\mathbf{D}}^T$ is the same as $\underline{\mathbf{C}} = \underline{\mathbf{D}}$, this may be rewritten as

$$\underline{\mathbf{Y}}_m^T \underline{\mathbf{K}}\underline{\mathbf{Y}}_l = \omega_m^2 \underline{\mathbf{Y}}_m^T \underline{\mathbf{M}}\underline{\mathbf{Y}}_l \tag{2}$$

Assuming that the natural frequencies are distinct, subtracting (1) from (2) gives

$$\underline{\mathbf{Y}}_m^T \underline{\mathbf{M}}\underline{\mathbf{Y}}_l = 0 \quad \text{where} \quad l \neq m$$

Mode shapes are **orthogonal** with respect to the mass matrix. Substituting back into (2) gives

$$\underline{\mathbf{Y}}_m^T \underline{\mathbf{K}}\underline{\mathbf{Y}}_l = 0 \quad \text{where} \quad l \neq m$$

Mode shapes are also **orthogonal** with respect to the stiffness matrix.

For the case where $l = m$, equation 2 may be rewritten as

$$\underline{\mathbf{Y}}_l^T \underline{\mathbf{K}}\underline{\mathbf{Y}}_l = \omega_l^2 \underline{\mathbf{Y}}_l^T \underline{\mathbf{M}}\underline{\mathbf{Y}}_l \quad \leadsto \quad \omega_l^2 = \frac{k_l}{m_l}$$

where the scalars k_l and m_l are given by $k_l = \underline{\mathbf{Y}}_l^T \underline{\mathbf{K}}\underline{\mathbf{Y}}_l$ and $m_l = \underline{\mathbf{Y}}_l^T \underline{\mathbf{M}}\underline{\mathbf{Y}}_l$.

Defining the *modal matrix*, $\underline{\mathbf{\Phi}}$, as a collection of all the mode shape column vectors $\underline{\mathbf{Y}}_i$:

$$\underline{\mathbf{\Phi}} = \begin{bmatrix} | & \cdots & | & \cdots \\ \underline{\mathbf{Y}}_1 & \cdots & \underline{\mathbf{Y}}_i & \cdots \\ | & \cdots & | & \cdots \end{bmatrix}$$

allows us to decompose the mass and stiffness matrices into diagonal matrices

$$\underline{\Phi}^T \underline{M}\, \underline{\Phi} = \begin{bmatrix} m_1 & 0 & \dots \\ 0 & m_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} = \underline{M}_{dia} \quad , \quad \underline{\Phi}^T \underline{K}\, \underline{\Phi} = \begin{bmatrix} k_1 & 0 & \dots \\ 0 & k_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} = \underline{K}_{dia}$$

Starting from the equation of motion in matrix form $\underline{M}\ddot{\mathbf{y}} + \underline{K}\mathbf{y} = \mathbf{0}$ and making the substitution

$$\mathbf{y} = \underline{\Phi}\mathbf{q} \quad \text{i.e.} \quad \mathbf{y} = q_1\mathbf{Y}_1 + q_2\mathbf{Y}_2 + \ldots + q_n\mathbf{Y}_n$$

($\mathbf{y}$ is a linear combination of the various modes and q_i are the coefficients indicating how much of each mode is present) gives:

$$\underline{M}\,\underline{\Phi}\ddot{\mathbf{q}} + \underline{K}\,\underline{\Phi}\mathbf{q} = 0 \quad \rightsquigarrow \quad \underline{\Phi}^T\underline{M}\,\underline{\Phi}\ddot{\mathbf{q}} + \underline{\Phi}^T\underline{K}\,\underline{\Phi}\mathbf{q} = 0$$

Substituting in for the mass and stiffness matrices leads to:

$$\underline{M}_{dia}\ddot{\mathbf{q}} + \underline{K}_{dia}\mathbf{q} = 0$$

which is effectively n **single DOF equations!** $m_1\ddot{q}_1 + k_1 q_1 = 0 \ldots$

1. In the original coordinate system, y, each of the degree of freedom were coupled, in the coordinate system q they are uncoupled - we refer to these as the *modal coordinates*.

2. Perturb by x_k results in a response from all DOFs, perturb by $\mathbf{Y}_k$ (corresponding to a non zero q_k with other q_i values being zero) results in a response in q_k only - energy does not move across to the other modes as they are uncoupled. Therefore, using the modal coordinates, we can consider the response at each mode individually and sum them up to find the total response - a technique called *modal superposition*.

3. As we can examine the response of each mode separately we might choose to only look at a limited number of modes to estimate the overall response, in forced vibration problems we might limit ourselves to consider only the modes with frequencies close to the forcing frequencies. modal reduction

4. We can use the Rayleigh approach to estimate the natural frequency from a guessed mode shape since

$$\omega_i^2 = \frac{\mathbf{Y}_i^T\underline{K}\mathbf{Y}_i}{\mathbf{Y}_i^T\underline{M}\mathbf{Y}_i} \quad \rightsquigarrow \omega_{i,estimate}^2 = \frac{\mathbf{Y}_{i,guess}^T\underline{K}\mathbf{Y}_{i,guess}}{\mathbf{Y}_{i,guess}^T\underline{M}\mathbf{Y}_{i,guess}}$$

where $\mathbf{Y}_{i,guess}$ is the guessed mode shape for mode i.

$M\ddot{u} + Ku = 0$

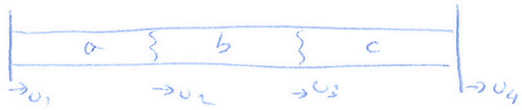
$\Rightarrow M_{diag}\,\ddot{q} + K_{diag}\,q = 0$

$u = \Phi q \qquad M_{diag} = \Phi^T M \Phi$

$\boxed{\text{Lecture 6}}$

Example: A Three-Element Axial vibration model of a Fixed-Fixed bar

from P11 $\quad u = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$



$$\underline{M}_{global} = \frac{\rho A L}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \underline{K}_{global} \quad \frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \omega_1^2 = \frac{6E}{5\rho L^2} \quad \text{modeshape} \quad \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{6E}{\rho L^2} \quad \text{modeshape} \quad \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{\Phi} = \begin{array}{cc} \text{1st mode} & \text{2nd mode} \\ \end{array} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\underline{M}_{diag} = \underline{\Phi}^T \underline{M}\,\underline{\Phi} = \frac{\rho A L}{6}\begin{bmatrix} 10 & 0 \\ 0 & 6 \end{bmatrix}, \quad K_{diag} = \underline{\Phi}^T \underline{K}\,\underline{\Phi} = \frac{EA}{L}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

expect $\quad \omega_i^2 = \dfrac{k_i}{m_i} \quad \therefore \quad \omega_1 = \dfrac{2EA/L}{10\rho AL/6} = \dfrac{6E}{5\rho L^2}$

$\underline{u} = \underline{\Phi}\,\underline{q}$ — modal coordinates

$M_{diag}\,\ddot{q} + K_{diag}\,q = 0$

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 + q_2 \\ q_1 - q_2 \end{pmatrix}$$

q_1, q_2 quantity of each mode

$$M_{diag}\,\ddot{\underset{\sim}{q}} + K_{diag}\,\underset{\sim}{q} = 0 \Rightarrow \left.\begin{array}{l} m_1\ddot{q}_1 + k_1 q_1 = 0 \\ m_2\ddot{q}_2 + k_2 q_2 = 0 \end{array}\right\} \text{1DoF equations}$$

if guess first mode $\underline{u}_{1g} = \begin{pmatrix} 1 \\ 0.9 \end{pmatrix}$:

then estimate for
$$\omega_1^2 = \frac{u_{1g}^T K u_{1g}}{u_{1g}^T M u_{1g}} = \frac{6E}{\rho L^2}\,\frac{(1 \;\; 0.9)\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 0.9 \end{pmatrix}}{(1 \;\; 0.9)\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 0.9 \end{pmatrix}}$$

$$= \frac{6E}{\rho L^2}\,\frac{1.82}{9.04} = 1.208\,\frac{E}{\rho L^2}$$

rather than 1.2.

4 Damping

From the study of one DOF systems we know that the system response is made up of two parts; 1) the transient response (coming from the complementary function) which is governed by the initial conditions and results in oscillates at the natural frequency and 2) the steady-state response (coming from the particular integral) which is due to external forcing and causes oscillations at the same frequency as the external forcing. Damping causes decay of the transient response and a change is steady-state response. However if the damping is light then only the steady-state response near resonance is affected significantly.

Material damping is very poorly understood. If a physical model of the structure or a similar structure exists it is possible to experimentally measure modal damping ratios ζ_i, where i refers to the mode number. A common approach to generating a damping matrix is to **assume** it can be generated from a linear combination of the mass and stiffness matrices, the Rayleigh proportional damping approach:

$$\underline{\mathbf{C}} = \alpha \underline{\mathbf{M}} + \beta \underline{\mathbf{K}}$$

This approach ensures that the modes remain uncoupled:

$$\underline{\Phi}^T \underline{\mathbf{C}}\, \underline{\Phi} = \alpha \underline{\mathbf{M}}_{dia} + \beta \underline{\mathbf{K}}_{dia} = \underline{\mathbf{C}}_{dia} = \begin{bmatrix} c_1 & 0 & \dots \\ 0 & c_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where $c_i = \alpha m_i + \beta k_i$. Therefore the equation of motion:

$$\underline{\mathbf{M}}\ddot{\underline{\mathbf{y}}} + \underline{\mathbf{C}}\dot{\underline{\mathbf{y}}} + \underline{\mathbf{K}}\underline{\mathbf{y}} = \mathbf{0} \quad \leadsto \quad \underline{\mathbf{M}}_{dia}\ddot{\underline{\mathbf{q}}} + \underline{\mathbf{C}}_{dia}\dot{\underline{\mathbf{q}}} + \underline{\mathbf{K}}_{dia}\underline{\mathbf{q}} = \mathbf{0}$$

may be written as a series of one DOF equations:

$$m_i \ddot{q}_i + c_i \dot{q}_k + k_i q_i = 0$$

In standard form this may be written as:

$$\ddot{q}_i + 2\zeta_i \omega_i \dot{q}_k + \omega_i^2 q_i = 0$$

We have already seen that $\omega_i^2 = k_i / m_i$, and now we can write

$$\zeta_i = \frac{c_i}{2 m_i \omega_i} = \frac{\alpha m_i + \beta k_i}{2 m_i \omega_i} = \frac{\alpha + \beta \omega_i^2}{2 \omega_i}$$

We can use this relationship to find the best fit for α and β for experimentally measured ζ_i values.

5 Forced Vibrations

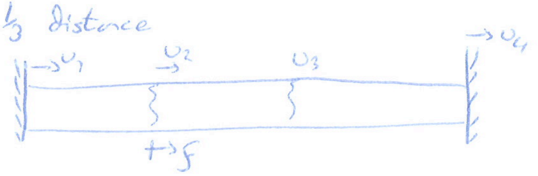
Assuming the forcing function is acting at a node, we can easily modify the equation of motion accordingly:

$$\underset{\sim}{\mathbf{M}}\ddot{\underset{\sim}{\mathbf{y}}} + \underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{y}} = \underset{\sim}{\mathbf{f}} \quad \rightsquigarrow \quad \underline{\mathbf{M}}_{dia}\ddot{\underline{\mathrm{q}}} + \underline{\mathbf{K}}_{dia}\underline{\mathrm{q}} = \underline{\mathbf{\Phi}}^T\underline{\mathbf{f}}$$

(Including distributed forcing functions is more complicated and involves consideration of the shape functions but is beyond the scope of the course.)

Example: A Three-Element Axial vibration model of a Fixed-Fixed bar

Forcing at $\frac{1}{3}$ distance



(P_{11})

global:

$$\underset{\sim}{u} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, \quad \underline{M} = \frac{\rho AL}{6}\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \quad \underline{K} = \frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\underline{M}\,\ddot{\underset{\sim}{u}} + \underline{K}\,\underset{\sim}{u} = \begin{pmatrix} f \\ 0 \end{pmatrix} = \underset{\sim}{f} \quad \longleftarrow \text{acting at node } u_2$$

n.i.f $f = g\sin\omega t$: try $\underset{\sim}{u} = \begin{pmatrix} U_2 \\ U_3 \end{pmatrix}\sin\omega t$, $\ddot{\underset{\sim}{u}} = -\omega^2 \underset{\sim}{u}$ $\quad (PI)_s$ (CF-transients)

$$\Rightarrow \left(-\omega^2 \frac{\rho AL}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)\begin{pmatrix} U_2 \\ U_3 \end{pmatrix}\cancel{\sin\omega t} = \begin{pmatrix} g \\ 0 \end{pmatrix}\cancel{\sin\omega t}$$

let $\bar{\omega}^2 = \omega^2 \frac{\rho L^2}{6E}$, $\bar{g} = \frac{gL}{EA}$ $\Rightarrow \begin{pmatrix} 2-4\bar{\omega}^2 & -1-\bar{\omega}^2 \\ -1-\bar{\omega}^2 & 2-4\bar{\omega}^2 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} \bar{g} \\ 0 \end{pmatrix}$

$$\therefore \quad U_2 = \frac{(2-4\bar{\omega}^2)\,\bar{g}}{(2-4\bar{\omega}^2)^2 - (1+\bar{\omega}^2)^2}, \quad U_3 = \frac{(1+\bar{\omega}^2)\,\bar{g}}{(2-4\bar{\omega}^2)^2 - (1-\bar{\omega}^2)^2}$$

n.i.f $\omega^2 = 1.3\frac{E}{\rho L^2}$ $\Rightarrow$ $\bar{\omega}^2 = 0.2167$ $\Rightarrow$ $U_2 = -5.787\bar{g}$

$U_3 = -6.2128\,\bar{g}$

$\left(\omega_1^2 = 1.2\frac{E}{\rho L^2}, \text{ ms } \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

[illegible] lecture

alternatively :

$\omega_1^2 = 1.2\dfrac{E}{\rho L^2}$

since $\omega \simeq \omega_1$

we could use $\underline{M}_{diag}\,\ddot{\underline{q}} + \underline{K}_{diag}\,\underline{q} = \underline{\Phi}^T\underline{f}$ ①

and assume response to just first mode (ie assume $q_2 = 0$)

$\underline{M}_{diag} = \dfrac{\rho AL}{6}\begin{bmatrix}10 & 0\\ 0 & 6\end{bmatrix}$, $\underline{K}_{diag} = \dfrac{EA}{L}\begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}$, $\underline{\Phi} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ ← don't need to know 2nd mode.

$q = \begin{pmatrix}q_1\\ q_2\end{pmatrix}\sin(\omega t)$ in ①

$\underline{f} = \begin{pmatrix}f\\ 0\end{pmatrix}$

$f = g\sin\omega t$

$$\Rightarrow \left[-\frac{10\rho AL}{6}\omega^2 + 2\frac{EA}{L}\right] q_1 = g$$
$$\left(\left[-\frac{6\rho AL}{6}\omega^2 + 6\frac{EA}{L}\right] q_2 = g\right)$$
$$\Bigg\}\quad q_1 = \frac{\bar{g}}{2 - 10\bar{\omega}^2} = -5.99\bar{g}$$

$$\begin{pmatrix}u_2\\ u_3\end{pmatrix} = \underline{\Phi}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}-5.99\bar{g}\\ 0\end{pmatrix}$$

assume zero

$$= \begin{pmatrix}-5.99\bar{g}\\ -5.99\bar{g}\end{pmatrix}$$

If large matrices inverting M timeconsuming
∴ if found a few modes / freqs near
excitation freq - numerical techniques
modal analysis quicker.

If Bending $\quad \underline{f} = \begin{pmatrix}F_1\\ M_1\\ F_2\\ M_2\\ \vdots\end{pmatrix}$

Appendix - (beyond the scope of the course)

Lagrange states that the equations of motion for a system may be derived from:

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

where q_i is the i^{th} degree of freedom. If the kinetic and potential energies are expressed in the form, $T = \frac{1}{2}\dot{\mathbf{y}}^T\underline{\mathbf{M}}\dot{\underline{\mathbf{y}}}$ and $V = \frac{1}{2}\mathbf{y}^T\underline{\mathbf{K}}\underline{\mathbf{y}}$ respectively, then, assuming there are no external driving forces, we can use Lagrange analysis to show that the equation of motion may be written as $\underline{\mathbf{M}}\ddot{\underline{\mathbf{y}}} + \underline{\mathbf{K}}\underline{\mathbf{y}} = \mathbf{0}$.

The case where a two DOF system and an n DOF system are considered (the equations for each are separated by a $\wr$ symbol). We assume the mass and stiffness matrices are symmetric.

$$\dot{\underline{\mathbf{y}}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} \quad , \quad \underline{\mathbf{M}} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \quad \wr \quad \dot{\underline{\mathbf{y}}} = \begin{pmatrix} \vdots \\ \dot{y}_i \\ \vdots \end{pmatrix} \quad , \quad \underline{\mathbf{M}} = \begin{bmatrix} \ddots & & \\ & m_{ij} & \\ & & \ddots \end{bmatrix}$$

where i and j correspond to the row and column respectively. The kinetic energy may be expanded out from the vector/matrix form by writing

$$T = \frac{1}{2}m_{11}\dot{y}_1^2 + m_{12}\dot{y}_1\dot{y}_2 + \frac{1}{2}m_{22}\dot{y}_2^2 \quad \wr \quad T = \frac{1}{2}\sum_i\sum_j M_{ij}\dot{y}_i\dot{y}_j$$

This gives (for the n DOF system the i^{th} equation is given):

$$\begin{aligned} \frac{\partial T}{\partial \dot{y}_1} &= m_{11}\dot{y}_1 + m_{12}\dot{y}_2 \\ \frac{\partial T}{\partial \dot{y}_2} &= m_{12}\dot{y}_1 + m_{22}\dot{y}_2 \end{aligned} \quad \wr \quad \frac{\partial T}{\partial \dot{y}_i} = \sum_j M_{ij}\dot{y}_j = 2\underline{\mathbf{M}}_i^r\dot{\underline{\mathbf{y}}}$$

where $\underline{\mathbf{M}}_i^r$ is a row vector containing the i^{th} row of $\underline{\mathbf{M}}$, and

$$\begin{aligned} \frac{\partial T}{\partial y_1} &= 0 \\ \frac{\partial T}{\partial y_2} &= 0 \end{aligned} \quad \wr \quad \frac{\partial T}{\partial y_i} = 0$$

Likewise, we can expand out the expression for the potential energy, $V = \frac{1}{2}\mathbf{y}^T\underline{\mathbf{K}}\underline{\mathbf{y}}$:

$$V = \frac{1}{2}k_{11}y_1^2 + k_{12}y_1y_2 + \frac{1}{2}k_{22}y_2^2 \quad \wr \quad V = \frac{1}{2}\sum_i\sum_j K_{ij}y_iy_j$$

where the notation for the stiffness matrix matches that for the mass matrix. The partial derivative with respect to each degree of freedom may be written as:

$$\begin{aligned} \frac{\partial V}{\partial y_1} &= k_{11}y_1 + k_{12}y_2 \\ \frac{\partial V}{\partial y_2} &= k_{12}y_1 + k_{22}y_2 \end{aligned} \quad \wr \quad \frac{\partial V}{\partial y_i} = \sum_j K_{ij}y_j = 2\underline{\mathbf{K}}_i^r\mathbf{y}$$

Substituting into Lagrange's equation gives (in the case of the 2 DOF system the two equations are with respect to y_1 and y_2 respectively, for the n DOF system the equation is for the i^{th} DOF):

$$\begin{aligned} m_{11}\ddot{y}_1 + m_{12}\ddot{y}_2 + k_{11}y_1 + k_{12}y_2 &= 0 \\ m_{12}\ddot{y}_1 + m_{22}\ddot{y}_2 + k_{12}y_1 + k_{22}y_2 &= 0 \end{aligned} \quad \wr \quad \underline{\mathbf{M}}_i^r\ddot{\underline{\mathbf{y}}} + \underline{\mathbf{K}}_i^r\underline{\mathbf{y}} = 0$$

The n equations of motion of this form can be expressed in matrix form as:

$$\underline{\mathbf{M}}\ddot{\underline{\mathbf{y}}} + \underline{\mathbf{K}}\underline{\mathbf{y}} = \mathbf{0}$$