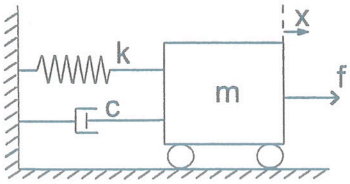
2 Introduction to Continuous Systems

2.1 Summary of Second Year Course

In the second year Vibrations course you considered one and two degree-of-freedom mass-spring-damper systems. These systems were "lumped" mass systems, i.e. there were distinct masses separated by springs and dampers. Practical examples of such systems include car suspension units, aircraft landing gear, washing machines (drum attached to outer casing via springs and dampers) and simple models of buildings (where the floors are represented by masses and the columns, being flexible and light, are springs).



The equation of motion for this one degree-of-freedom system may be written as:

$$m\ddot{x} + c\dot{x} + kx = f$$

For the case where there is no forcing, $f(t) = 0$, and no damping, $c = 0$, we can use the trial solution $x = Ae^{\alpha t}$ to give

$m\ddot{x} + kx = 0 \Rightarrow$

$$m\alpha^2 Ae^{\alpha t} + kAe^{\alpha t} = 0 \quad \rightsquigarrow \quad \alpha = \pm j\omega_n \quad \rightsquigarrow \quad x = B\sin(\omega_n t) + C\cos(\omega_n t)$$

where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency and the constants B and C may be found from the initial conditions. With damping, $c \neq 0$, the trial solution gives

$$m\alpha^2 Ae^{\alpha t} + c\alpha Ae^{\alpha t} + kAe^{\alpha t} = 0 \quad \rightsquigarrow \quad \alpha = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

where $\zeta = c/(2\sqrt{mk})$ is the damping ratio. When the system is over-damped, $\zeta > 1$, or critically damped $\zeta = 1$, α is real and no oscillations occur. However when the system is under-damped, $\zeta < 1$, α is complex resulting in oscillatory behaviour

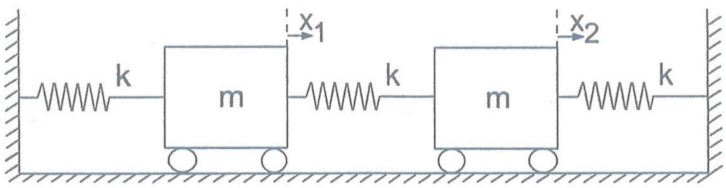
$$\alpha = -\zeta\omega_n \pm j\omega_n\sqrt{\cancel{\zeta^2 - 1}} \quad \rightsquigarrow \quad x = e^{-\zeta\omega_n t}[B\sin(\omega_d t) + C\cos(\omega_d t)]$$

$1 - \zeta^2$

← error

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is the damped natural frequency. If the system is forced sinusoidally, $f = sin(\omega t)$, then a trial solution of $x = Ae^{j\omega t}$ would be used $\cdots$.

$\omega_d \sim \omega_n$ if damping small



The equations of motion for this undamped two degree-of-freedom system may be expressed in vector form as:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since there is no damping we expect the solutions to be purely oscillatory (no exponential decay terms) so choosing trial solutions $x_1 = X_1 e^{j\omega t}$ and $x_2 = X_2 e^{j\omega t}$, we find

$$\begin{bmatrix} 2k/m - \omega^2 & -k/m \\ -k/m & 2k/m - \omega^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad M^{-1}K \qquad (1)$$

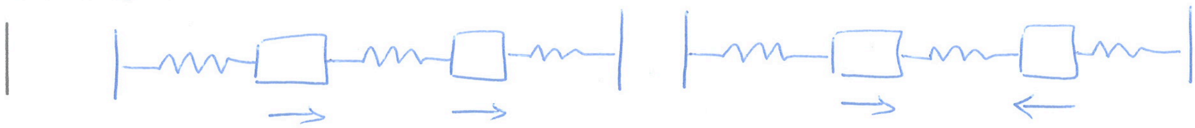
Solving this eigenvalue/eigenvector problem (it is in the form $(\underline{A} - \lambda \underline{I})\vec{X} = 0$, where $\lambda = \omega^2$), by setting the determinant of the matrix to zero, we find that

$$\left(2\frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \quad \rightsquigarrow \quad \omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega = \sqrt{\frac{3k}{m}}$$

We can find the corresponding mode shapes for these two natural frequencies by substituting the natural frequencies back into equation 1

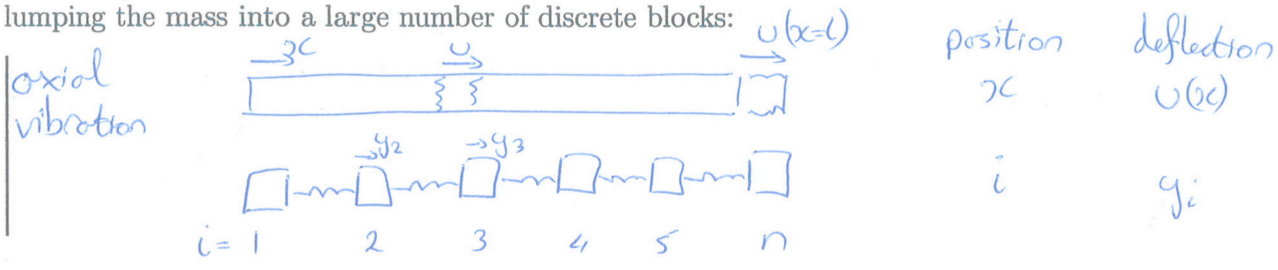
$$\omega = \sqrt{\frac{k}{m}} \quad \text{gives} \quad \frac{X_2}{X_1} = 1 \quad , \quad \omega = \sqrt{\frac{3k}{m}} \quad \text{gives} \quad \frac{X_2}{X_1} = -1$$

If a system has n degrees-of-freedom then it would have n natural frequencies and n mode shapes.



2.2 Continuous Systems

We will use the work *continuous* to denote a system with has mass distributed throughout it, such as flexible bar. The bar could be approximated to a discrete system by lumping the mass into a large number of discrete blocks:



As the number of degrees-of-freedom, n, increases so does the number of modes. In the limit as the number of degrees-of-freedom tends to infinity and the lumped masses infinitesimally small slices of bar the number of modes tends to infinity and the system becomes continuous. Examples of continuous systems include turbine blades, aircraft wings, car propeller shafts and transmitter masts.

In this course, we will study the axial and torsional vibration of bars, flexural vibration of beams and flexural vibration of plates.

1 Why interested?
resonance close to forcing
→ noise, fatigue, discomfort
either - shift freq by altering design or add damping

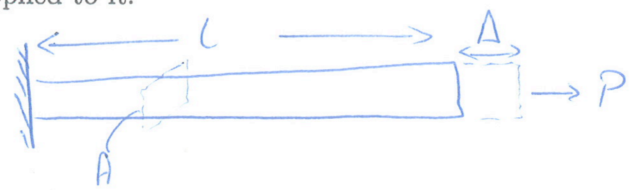
2 avoid nfs near forcing freq

3 slides of example continuous systems

3 Axial and Torsional Vibration of Bars

3.1 Equation of Motion for Axial Vibration

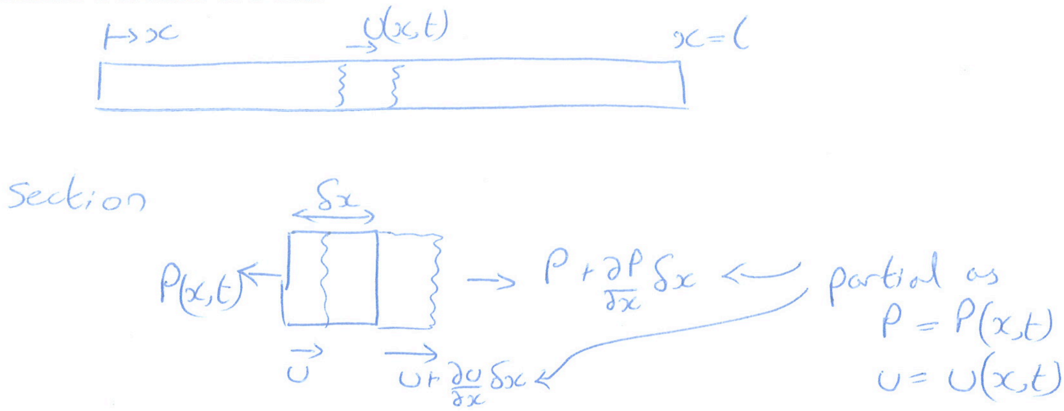
Firstly, consider a bar with a static tensile force P (we define P as positive if the bar is in tension) applied to it:



This force may be written as a stress $\sigma = P/A$ and results in a strain $\epsilon = \Delta/l$, since $E = \sigma/\epsilon$ we can write

$$P = AE\epsilon = \frac{AE\Delta}{l} \tag{2}$$

Now consider a section of a bar:



We define x as the distance along the bar, $u(x,t)$ as the displacement of the bar from its rest position and $P(x,t)$ as the axial force acting on the cross section. Considering a small section of bar we can write the equation of motion as

$ma = F$

$$\rho A \delta x \frac{\partial^2 u}{\partial t^2} = P + \frac{\partial P}{\partial x}\delta x - P = \frac{\partial P}{\partial x}\delta x$$

where ρ is the density. However, we know from equation 2 that a force P corresponds to a stress

$\dfrac{u + \frac{\partial u}{\partial x}\delta x - u}{\delta x}$ $\quad \epsilon = \dfrac{\partial u}{\partial x} = \dfrac{P}{EA} \rightsquigarrow \dfrac{\partial P}{\partial x} = EA\dfrac{\partial^2 u}{\partial x^2}$ extension

Note that $\epsilon = \frac{\partial u}{\partial x}$ is a partial derivative as u is a function of both x and t. Combining these equations gives the equation of motion

$$\rho A \delta x \frac{\partial^2 u}{\partial t^2} = EA\frac{\partial^2 u}{\partial x^2}\delta x \rightsquigarrow \frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

It is this equation (which is sometimes referred to as the 1-D wave equation) that we must solve to understand the axial vibration characteristics of a bar.

Note: In this analysis we have assumed that there is no damping. In reality there will be some damping, however the level of *material damping* will be small so the natural frequency will be very close to that of the undamped system. This is different to the one and two degree-of-freedom systems considered in the second year where you considered systems with mechanical dampers (or dashpots).

3.2 Solving the Equation of Motion

A solution to the wave equation (or equation of motion) may be obtained using a method known as *separation of variables*. Considering the wave equation in its general form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where for the axial vibration of a bar $c^2 = E/\rho$, we can make the trial solution in the form of two separate functions, one in terms of x only and one in terms of t only

separation of variables

$$u(x, t) = F(x)G(t) \tag{3}$$

(or d'Alembert's solution $u(x,t) = \phi(x+ct) + \psi(x-ct)$ travelling wave solution)

We can therefore write

$$\frac{\partial^2 u}{\partial t^2} = F(x)\frac{\mathrm{d}^2 G(t)}{\mathrm{d}t^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\mathrm{d}^2 F(x)}{\mathrm{d}x^2}G(t)$$

Substituting these into the wave equation gives

$$F(x)\frac{d^2 G(t)}{dt^2} = c^2 \frac{d^2 F(x)}{dx^2} G(t)$$

Dividing by $c^2F(x)G(t)$ gives

$$\frac{1}{c^2 G(t)}\frac{d^2 G(t)}{dt^2} = \frac{1}{F(x)}\frac{d^2 F(x)}{dx^2}$$

Now since the left-hand side is only a function of t and the right-hand side is only a function of x both sides must equal a constant, say $-\alpha^2$. Using this we can write two separate ordinary differential equations

$$\frac{\mathrm{d}^2 G(t)}{\mathrm{d}t^2} + \alpha^2 c^2 G(t) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 F(x)}{\mathrm{d}x^2} + \alpha^2 F(x) = 0$$

These equations have the solutions

$$F(x) = A\sin(\alpha x) + B\cos(\alpha x)$$

$$G(t) = C\sin(\alpha ct) + D\cos(\alpha ct) \qquad \text{where} \quad c = \sqrt{\frac{E}{\rho}}$$

Care needed adopted Thomson notation. Here A is not cross-section area but an unknown constant

We are used to seeing oscillatory functions in terms of $\cos(\omega t)$ etc, where ω is the natural frequency (in rads/s, $\omega = 2\pi f_r$ where f_r is the frequency in Hz), therefore substitute $\omega = \alpha c$ and combine the two equations to give a solution for u using equation 3

$$u(x,t) = F(x)G(t) = \left[A\sin\left(\frac{\omega x}{c}\right) + B\cos\left(\frac{\omega x}{c}\right)\right]\left[C\sin(\omega t) + D\cos(\omega t)\right]$$

We still have several unknown constants ω, A, B, C and D. Note that there is one redundant constant - one of A, B, C or D can be set to one without loss of generality. These can be found by applying the boundary conditions and initial conditions. As we are usually interested in the natural frequencies and mode shapes of the system normally we need only apply the boundary conditions (which are sufficient to find ω, A and B).

divide through by A for example

3.3 Boundary Conditions

We apply boundary conditions, ie information about the state of the beam at either ends, $x = 0$ and $x = l$, to find the natural frequencies and mode shapes of the bar. There are several types of boundary (or end) condition, two simple ones are *free* and *clamped*. The conditions for these boundary conditions are:

Type	Condition
Free	$P = 0$
Clamped	$u = 0$

or fixed $\rightsquigarrow$

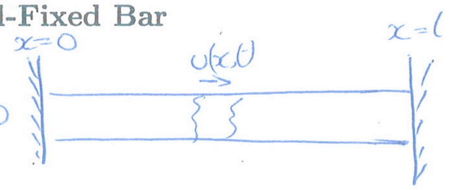
A free end can have no force P acting on it, therefore the stress and hence the strain is zero at the end. At end $x = l$ we can write

$$\overset{P=AE\epsilon}{P = 0} \quad \rightsquigarrow \quad \epsilon = \frac{\partial u}{\partial x} = 0 \quad \rightsquigarrow \quad u'(l,t) = 0$$

where we use $\{*\}'$ as the derivative with respect to x (and $\{\dot{*}\}$ as the derivative with respect to t)

Example: Fixed-Fixed Bar

$u(0,t) = 0$



$u(l,t) = 0$

$$u(x,t) = F(x)G(t) = \underbrace{\left[A\sin\left(\frac{\omega x}{c}\right) + B\cos\left(\frac{\omega x}{c}\right)\right]}_{F(x)}\underbrace{\left[C\sin(\omega t) + D\cos(\omega t)\right]}_{G(t)}$$

$$c = \sqrt{\frac{E}{\rho}}$$

Boundary conditions (for all time):

$u(0,t) = 0 \quad \Rightarrow \quad 0 = B\,G(t)$ $\quad$ (if $G(t) = 0$ for all t, $u = 0$)

$\therefore B = 0$

$u(l,t) = 0 \quad \Rightarrow \quad 0 = A\sin\left(\frac{\omega l}{c}\right)G(t)$

either $G(t) = 0$ all t $\therefore$ no vibrations $\omega = 0$

$A = 0 \longrightarrow$ no vibration

$\sin\left(\frac{\omega l}{c}\right) = 0 \rightarrow$ non-zero solutions

$\frac{\omega l}{c} = 0, \pi, 2\pi \ldots$

$\frac{\omega l}{c} = 0 \rightarrow$ zero solution $\therefore \frac{\omega l}{c} = n\pi \quad n = 1, 2, 3 \ldots$

$$\omega = \frac{c n \pi}{l} = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}$$

infinite number of solutions for ω: $\omega_1, \omega_2, \omega_3 \ldots$

$n = 1$ $\quad u_1(x,t) = A_1 \sin\left(\frac{\omega_1 x}{c}\right)\left[C_1 \sin(\omega_1 t) + D_1 \cos(\omega_1 t)\right]$

absorb A_1 into C_1, D_1

$$u_1(x,t) = \sin\left(\frac{\omega_1 x}{c}\right)\left[C_1 \sin(\omega_1 t) + D_1 \cos(\omega_1 t)\right]$$

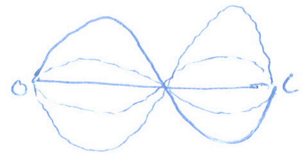
general $\quad u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{\omega_n x}{c}\right)\left[C_n \sin(\omega_n t) + D_n \cos(\omega_n t)\right]$

$$\omega_n = \frac{c n \pi}{l} \qquad c = \sqrt{\frac{E}{\rho}}$$

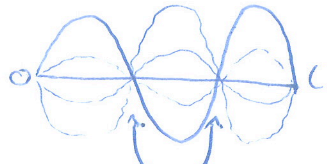
mode $\quad \omega \quad$ modeshape (i.e. snapshot over length at any time)

$n = 1$ $\quad \omega = \frac{c\pi}{l}$ $\quad \sin\frac{\omega_1 x}{c} = \sin\frac{\pi x}{l}$

$n = 2$ $\quad \omega = \frac{2c\pi}{l}$ $\quad \sin\frac{2\pi x}{l}$



$n = 3$ $\quad \omega = \frac{3c\pi}{l}$ $\quad \sin\frac{3\pi x}{l}$



nodes

Note that for the continuous system there are an infinite number of modes.

note u is axial - these are schematic

Lecture 3

3.4 Initial Conditions

For a complete solution, we can find constants C_n and D_n by considering the initial conditions. However this is not normally needed as we are normally interested in the steady-state natural frequencies and mode shapes.

Example: Fixed-Fixed Bar
Continuing the example from the last section, let us consider the simple initial conditions that

$$u(x,0) = 3\sin\left(\frac{\pi x}{l}\right) \quad \text{and} \quad \dot{u}(x,0) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{\omega_n x}{c}\right)\left[C_n \sin(\omega_n t) + D_n \cos(\omega_n t)\right] \quad \text{with} \quad \omega_n = \frac{cn\pi}{l} \quad c = \sqrt{\frac{E}{\rho}}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\left(\frac{\omega_n x}{c}\right)\left[\omega_n C_n \cos(\omega_n t) - \omega_n D_n \sin(\omega_n t)\right]$$

at $t=0$ $\quad \dot{u}(x,0) = 0 \quad \Rightarrow \quad 0 = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c} C_n \omega_n$

$\therefore C_n = 0$ for all n

at $t=0$ $\quad u(x,0) = 3\sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c} D_n \cos(\omega_n t)\Big|_{t=0}$

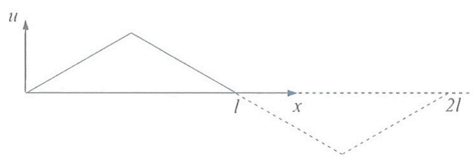
$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} D_n$$

$\therefore D_1 = 3, \; D_2 = D_3 = D_4 = \dots = 0$

$$u(x,t) = 3\cos(\omega_1 t)\sin\left(\frac{\omega_1 x}{c}\right) = 3\cos\left(\sqrt{\frac{E}{\rho}}\frac{\pi}{l}t\right)\sin\left(\frac{\pi x}{l}\right)$$

For more complex initial conditions such as the beam is initial loaded at the midpoint and released:

I don't lecture



$$u(x,0) = \begin{cases} \frac{2ax}{l} & x = 0 \text{ to } x = \frac{l}{2} \\ 2a - \frac{2ax}{l} & x = \frac{l}{2} \text{ to } x = l \end{cases}$$

then $C_n = 0$ for all n and the values of D_n are found from transforming $u(x,0)$ into a Fourier series (over the range $x = 0$ to $x = 2l$ – a complete cycle of the first mode) giving:

$$\int_0^{2l} D_n \sin^2 \frac{n\pi x}{l} \mathrm{d}x = \int_0^{2l} u(x,0) \sin\frac{n\pi x}{l} \mathrm{d}x \quad \rightsquigarrow \quad D_n = \begin{cases} 0 & \text{even } n \\ \frac{8a}{n^2\pi^2}(-1)^{\frac{i-1}{2}} & \text{odd } n \end{cases}$$

section 3.4 not examined

5 min left

Lecture 3

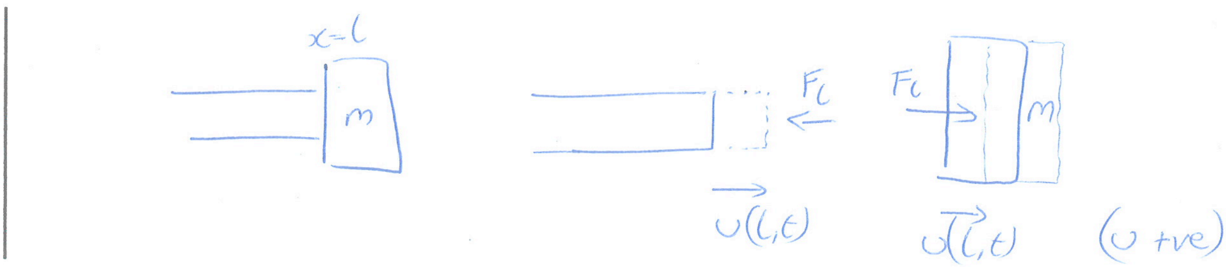
Fixed $u = 0$

Free $P = 0 = AE\epsilon = AE\frac{\partial u}{\partial x}$

3.5 Further Boundary Constraints

3.5.1 Lumped Mass

If there is a lumped mass at the $x = l$ end of the beam:

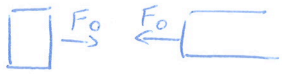


For the acceleration of the mass is positive there must be a force F_l (to the right) applied to it and an equal *but opposite* force applied to the bar. This equal but opposite force is compressive and so must equal $-P$ at $x = l$ since P is defined as positive when the bar is in tension.

$P = EA\epsilon = EA\frac{\partial u}{\partial x}$

$$\text{“F = ma”} \quad \leadsto \quad m\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l} = F_l = -\left.P\right|_{x=l} = -AE\left.\frac{\partial u}{\partial x}\right|_{x=l}$$

On the other hand if the lumped mass is at the $x = 0$ end of the bar and the mass has a positive acceleration. The force F_0 generating this acceleration is acting to the right and so the force on the bar is to the left. Hence the bar will be in tension due to the force F_0, therefore $P = F$.

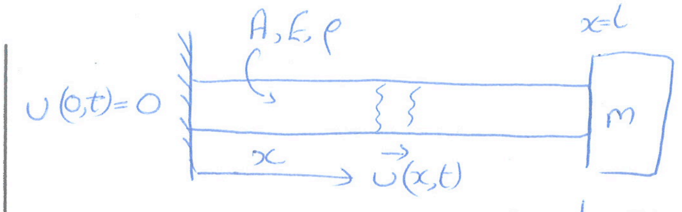


$$\text{“F = ma”} \quad \leadsto \quad m\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0} = F_0 = \left.P\right|_{x=0} = AE\left.\frac{\partial u}{\partial x}\right|_{x=0}$$

$\rightarrow u$ +ve

Using these boundary constraints the resulting equation for ω can not be solved explicitly (it is *transcendental* - it contains functions other than simple polynomials), it has to be solved numerically or graphically. This will be seen in the following example.

Example: Fixed-Lumped Mass Bar



$$AE\left.\frac{\partial u}{\partial x}\right|_{x=l} = -m\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l}$$

← not area!

$$u(x,t) = \underbrace{\left[A\sin\left(\frac{\omega x}{c}\right) + B\cos\left(\frac{\omega x}{c}\right)\right]}_{F(x)}\underbrace{\left[C\sin(\omega t) + D\cos(\omega t)\right]}_{G(t)}$$

RHS $\quad \left\{\frac{\partial^2 u}{\partial t^2} = -\omega^2 u\right\} \quad \Rightarrow \quad AE\left.\frac{\partial u}{\partial x}\right|_{x=l} = m\omega^2 \left.u\right|_{x=l}$

$$U(0,t) = 0 \quad \Rightarrow \quad 0 = B\,G(t) \quad \therefore \quad B = 0$$

$$u(x,t) = \bar{A}\sin\left(\frac{\omega}{c}x\right)G(t)$$

$$AE\left.\frac{\partial u}{\partial x}\right|_{x=l} = m\omega^2 u\Big|_{x=l} \Rightarrow AE\bar{A}\frac{\omega}{c}\cos\left(\frac{\omega}{c}x\right)\Big|_{x=l}G(t) = m\omega^2\bar{A}\sin\left(\frac{\omega}{c}x\right)\Big|_{x=l}G(t)$$

$$AE\frac{\omega}{c}\cos\left(\frac{\omega}{c}l\right) = m\omega^2\sin\left(\frac{\omega}{c}l\right)$$

$c = \sqrt{\frac{E}{\rho}} \quad \therefore \quad E = \rho c^2$ also $M = Al\rho$ total mass of bar

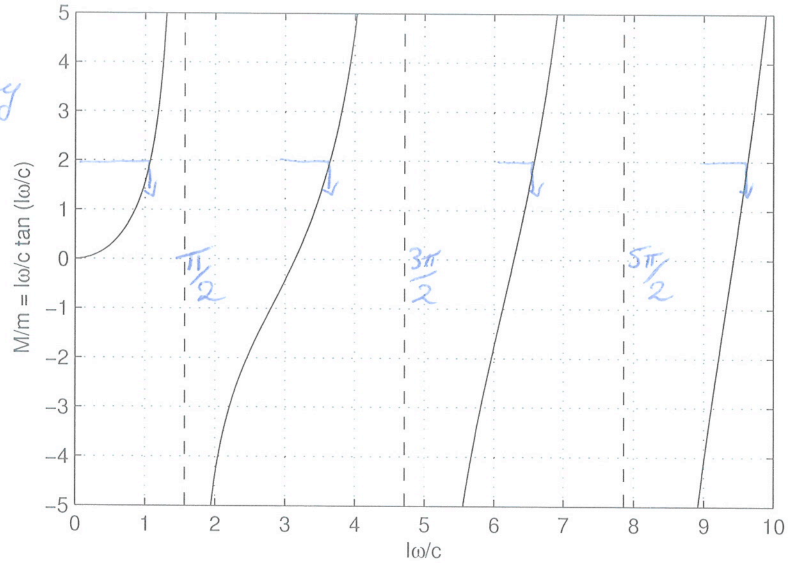
$$\frac{A\rho c^2\omega}{m\omega^2 c} = \tan\left(\frac{\omega}{c}l\right)$$

$$\frac{M}{m}\frac{c}{\omega l} = \tan\left(\frac{\omega}{c}l\right)$$

$$\Rightarrow \quad \left(\frac{\omega l}{c}\right)\tan\left(\frac{\omega}{c}l\right) = \frac{M}{m}$$

1) sketch with reasonable accuracy

2) iterate using calculator



if $\frac{M}{m} = 2$

as $m \to 0$ $\left(\frac{M}{m} \to \infty\right)$ $\quad \frac{L\omega}{c} \to \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

same as fixed/free bar

as $m \to \infty$ $\left(\frac{M}{m} \to 0\right)$ $\quad \frac{L\omega}{c} \to \pi, 2\pi, 3\pi \ldots$

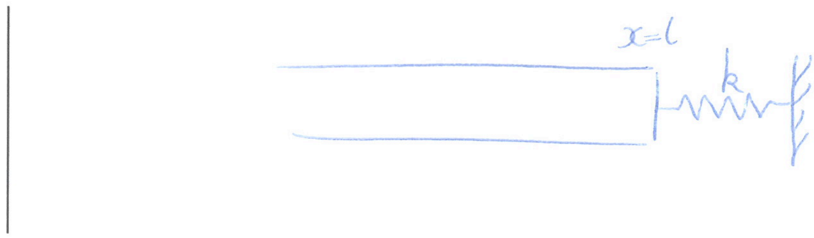
same as fixed/fixed bar

$$u(x,t) = \sum_{n=1}^{\infty} A_n \overbrace{\sin\left(\frac{\omega_n x}{c}\right)}^{\text{modeshapes}} G(t)$$

with ω_n found from graph (depending on $\frac{M}{m}$)

3.5.2 Spring Connection

In practice bars are often not ideally clamped, more realistically the connection with a rigid base is slightly flexible. This flexibility can be modelled using a spring:



For the case where the spring is at the $x = l$ end, if we imagine that u is positive then the spring is in compression so the force exerted on the bar will be resisting the motion, i.e. $P = -ku(l, t)$ where k is the spring stiffness. We can therefore write

defined as tension +ve

$$P = AE \left.\frac{\partial u}{\partial x}\right|_{x=l} = -ku|_{x=l}$$

spring compressed as u +ve to right

$F_l = ku(l,t)$

$F_l = -P(l,t)$

For the case where the spring is at the $x = 0$ end, if we imagine that u is positive then the spring is in tension so the bar around $x = 0$ will be in tension so P will be positive, i.e. $P = ku(0, t)$ where k is the spring stiffness. We can therefore write

$$P = AE \left.\frac{\partial u}{\partial x}\right|_{x=0} = ku|_{x=0}$$

$F_0 = P(0,t)$

These conditions are applied in a similar way to those of the lumped mass boundary conditions.

3.5.3 Summary of Boundary Conditions

Table of Boundary Conditions for Axial vibration of a bar:

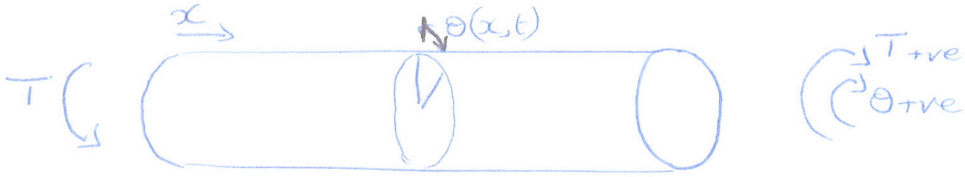
Type	Condition
Free	$P = AE\frac{\partial u}{\partial x} = 0$
Clamped	$u = 0$
Lumped Mass ($x = l$ end)	$\frac{\partial^2 u}{\partial t^2} = -w^2 u = -\frac{AE}{m}\frac{\partial u}{\partial x}$
Spring Connection ($x = l$ end)	$u = -\frac{AE}{k}\frac{\partial u}{\partial x}$

Lecture 4

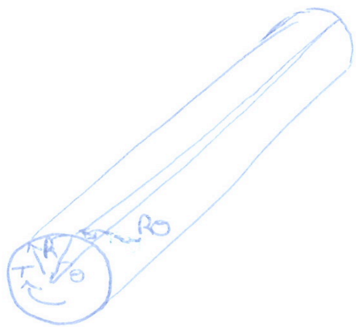
3.6 Torsional Vibration of Rods

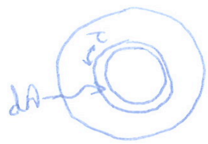
3.6.1 Equation of Motion

As before let x be the length along the shaft and let θ be the angle the shaft has rotated from equilibrium (like u for the axial bar vibration):



Recall from statics that for *circular sections*:





strain $\gamma = \frac{\theta r}{l}$

stress $\tau = G\gamma$

$$\tau = \frac{G\theta r}{l}$$

$$\text{torque} = T = \int_A \tau r \, dA = \int_A \frac{G\theta r^2}{l} dA = \frac{G\theta}{l} I_p$$

I_p = second polar moment of area $= \int_A r^2 dA = \int_0^R r^2 \, 2\pi r \, dr = \frac{\pi R^4}{2}$

$$\boxed{\frac{\tau}{r} = \frac{T}{I_p} = \frac{G\theta}{l}}$$

(J = polar moment of inertia $= \int_m r^2 dm = \int_0^R \rho l 2\pi r \, r^2 dr = \rho l \frac{R^4 \pi}{2}$)

Giving

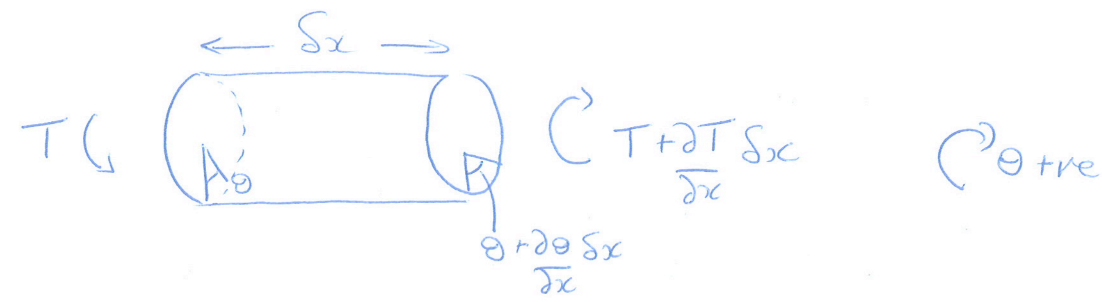
$$\text{statics}: \frac{\theta}{l} = \frac{T}{GI_p} \rightsquigarrow \frac{\partial \theta}{\partial x} = \frac{T}{GI_p} \tag{4}$$

where I_p is the second polar moment of area (J is reserved for polar moment of inertia) and G is the shear modulus of elasticity. Note that $\frac{\partial \theta}{\partial x}$ is a partial derivative because θ is a function of x and t when we consider vibrations.

lecture 4

$$\frac{\partial \theta}{\partial x} = \frac{T}{G I_p}$$

Now consider a small element δx long:



We can write the equation of motion as

polar moment of inertia

$$T + \frac{\partial T}{\partial x}\delta x - T = I_p \rho \delta x \frac{\partial^2 \theta}{\partial t^2}$$

where $I_p \rho \delta x$ is the moment of inertia for the δx long element. Using equation 4 we can write

$$T = G I_p \frac{\partial \theta}{\partial x} \quad \rightsquigarrow \quad \frac{\partial T}{\partial x} = G I_p \frac{\partial^2 \theta}{\partial x^2}$$

Substituting this into our equation of motion gives

$$G I_p \frac{\partial^2 \theta}{\partial x^2}\delta x = I_p \rho \delta x \frac{\partial^2 \theta}{\partial t^2} \quad \rightsquigarrow \quad \frac{G}{\rho}\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$$

Curiously the equation of motion does not include the second polar moment of area, I_p, therefore the natural frequencies will not be effected by the radius of the rod if the rod is clamped or has free boundary conditions (for lumped inertia and spring connection conditions I_p is reintroduced in the boundary conditions - see below) The natural frequencies will however be effected by ρ, G and the length of the rod - which appears when the boundary conditions are applied.

If we write

$$c^2 = \frac{G}{\rho} \qquad \rightsquigarrow \quad \frac{\partial^2 \theta}{\partial t^2} = c^2 \frac{\partial^2 \theta}{\partial x^2}$$

then this equation has the identical form as the equation of motion for a bar when considering axial vibrations (where $c^2 = \frac{E}{\rho}$).

The solution to the torsional rod vibration problem can therefore be solved in an identical way to that already discussed for the axial bar vibrations, through the use of the general solution

$$\theta(x,t) = F(x)G(t) = \left[A \sin\left(\frac{\omega}{c}x\right) + B\cos\left(\frac{\omega}{c}x\right)\right]\left[C\sin(\omega t) + D\cos(\omega t)\right]$$

and the application of boundary conditions.

Don't lecture

3.6.2 Boundary Conditions

Clamped End: There is no rotation, therefore $\theta = 0$

Free End: There is no torque at a free end, therefore the strain is zero

$$\frac{\partial u}{\partial x} = 0$$

Lumped Inertia: If an lumped moment of inertia J is fixed to the $x = l$ end then

$$\text{“T = J}\ddot{\theta}\text{”} \quad \rightsquigarrow \quad T|_{x=l} = -J \left.\frac{\partial^2 \theta}{\partial t^2}\right|_{x=l}$$

using the same arguments regarding signs as for the axial vibration of a bar and making the distinction that J is a moment of inertia and I_p is a second polar moment of area. Since $\frac{\partial^2\theta}{\partial t^2} = -\omega^2\theta$ and $T = GI_p\frac{\partial\theta}{\partial x}$ we can write

$$GI_p \left.\frac{\partial \theta}{\partial x}\right|_{x=l} = J\omega^2 \left.\theta\right|_{x=l}$$

If the lumped inertial is at the $x = 0$ end then the condition is

$$GI_p \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = -J\omega^2 \left.\theta\right|_{x=0}$$

Spring Connection: If the end of the rod is clamped at the $x = l$ end via a rotational spring (of stiffness k_r) then we can write

$$T|_{x=l} = -k \left.\theta\right|_{x=l}$$

Again since $T = GI_p\frac{\partial\theta}{\partial x}$ we can rewrite this as

$$GI_p \left.\frac{\partial \theta}{\partial x}\right|_{x=l} = -k \left.\theta\right|_{x=l}$$

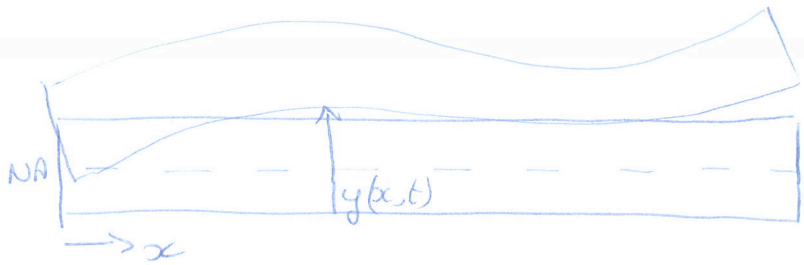
If the spring connection is at the $x = 0$ end then the condition is

$$GI_p \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = k \left.\theta\right|_{x=0}$$

Able to attempt whole of example sheet 1

4 Flexural Vibration of Beams

4.1 Equation of Motion



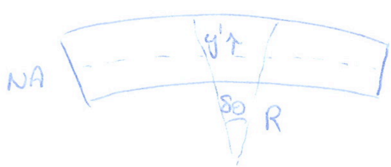
mass = ρA / unit length

In analysing the flexural vibration of a beam we will make several assumptions:

- the beam is slender
- there is no axial load
- plane sections remain plane
- the moment of inertia of a section of beam is negligible
  not area!
- the shear deformation is negligible

These assumptions follow the *Euler-Bernoulli* beam theory. There are more complex beam theory, such as the *Timoshenko* theory which relaxes the assumptions on shear deformation and moment of inertia, but these are beyond the scope of these lectures.

Recall from statics:



$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y'}$$

length before bending $= NA$ length $= R\delta\theta$

length after bending $= (R+y')\delta\theta$

$$\epsilon = \frac{(R+y')\delta\theta - R\delta\theta}{R\delta\theta} = \frac{y'}{R}$$

$$\Rightarrow \frac{\sigma}{y'} = \frac{E}{R}$$

$$M = \int_A y'\sigma \, dA = \int_A \frac{E y'^2}{R} dA = \frac{EI}{R}$$

y' = distance from NA

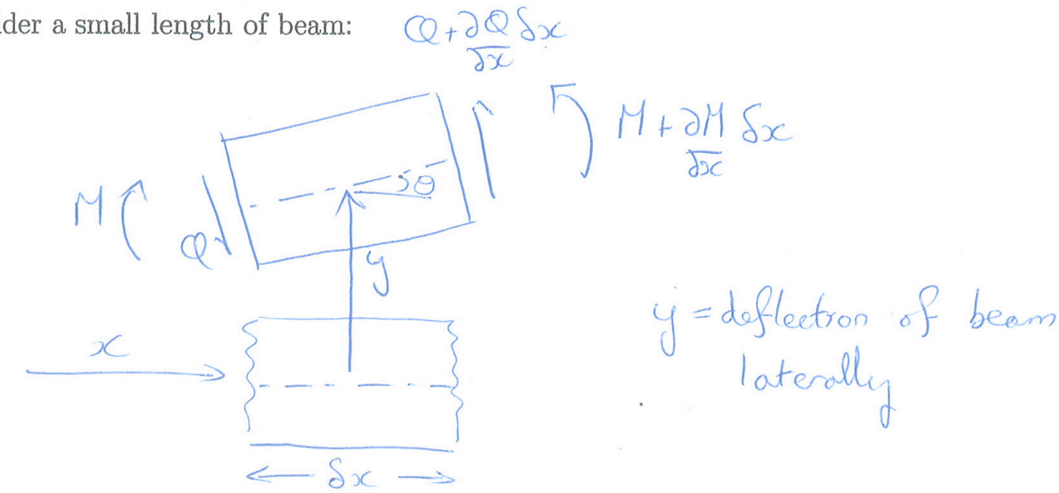
I = second moment of area $= \int_A y'^2 dA$

$$M = \frac{EI}{R} = EI\frac{\partial^2 y}{\partial x^2}$$

partial as $y = y(x,t)$

where I is the second moment of area of the beam cross section.

Consider a small length of beam:



Note the sign convention used in this analysis. Taking moments about the centre gives

$$M + \frac{\partial M}{\partial x}\delta x - M + \frac{1}{2}\delta x Q + \frac{1}{2}\delta x\left(Q + \frac{\partial Q}{\partial x}\delta x\right) = \rho I \frac{\partial^2 \theta}{\partial t^2}\delta x$$

Neglecting the rotary inertia term on the right hand-side and taking the limit as $\delta x \to 0$

$$\frac{\partial M}{\partial x} + Q + \frac{1}{2}\delta x \frac{\partial Q}{\partial x} = 0 \quad \rightsquigarrow \quad \frac{\partial M}{\partial x} = -Q$$

Applying "$F = ma$" vertically gives

$$\frac{\partial Q}{\partial x}\delta x = \rho A \delta x \frac{\partial^2 y}{\partial t^2}$$

Combining the moment and the force equations gives

$$\frac{\partial^2 M}{\partial x^2} = -\rho A \frac{\partial^2 y}{\partial t^2}$$

Substituting in the relationship between moment and deflection gives

$$M = EI\frac{\partial^2 y}{\partial x^2} \quad \rightsquigarrow \quad \frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0$$

which is the general equation for the flexural vibration of a uniform beam.

4.2 Solving the Euler-Bernoulli Equation

We can find a solution to the Euler-Bernoulli equation using separation of variables

$$y(x,t) = F(x)G(t)$$

Substituting this into the equation of motion gives

$$G(t)\frac{\mathrm{d}^4 F(x)}{\mathrm{d}x^4} + \frac{\rho A}{EI}F(x)\frac{\mathrm{d}^2 G(t)}{\mathrm{d}t^2} = 0$$

Separating the variables gives

$$\frac{EI}{\rho A}\frac{1}{F(x)}\frac{\mathrm{d}^4F(x)}{\mathrm{d}x^4} = -\frac{1}{G(t)}\frac{\mathrm{d}^2G(t)}{\mathrm{d}t^2} = \beta^2$$

where β is a constant. This gives the two separate ordinary differential equations:

$$\frac{d^4F(x)}{\partial x^4} - \beta^2\frac{\rho A}{EI}F(x) = 0$$

$$\frac{d^2G(t)}{dt^2} + \beta^2 G(t) = 0$$

The solution to the differential equation in time is

$$G(t) = E\sin(\beta t) + F\cos(\beta t) \quad \rightsquigarrow G(t) = E\sin(\omega t) + F\cos(\omega t)$$

where E and F are unknown constants and β has been replaced by ω to represent angular frequency. For the differential equation in position, using the trial solution $F(x) = a\mathrm{e}^{\gamma x}$ gives

$$\frac{\mathrm{d}^4F(x)}{\mathrm{d}x^4} - \omega^2\frac{\rho A}{EI}F(x) = 0 \quad \rightsquigarrow \quad \gamma^4 - \underbrace{\omega^2\frac{\rho A}{EI}}_{\alpha^4} = 0$$

The solution of which is

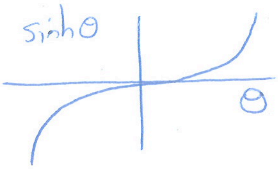
$$\gamma = \pm\alpha\ ,\ \pm j\alpha \quad \text{where} \quad \alpha^4 = \omega^2\frac{\rho A}{EI}$$

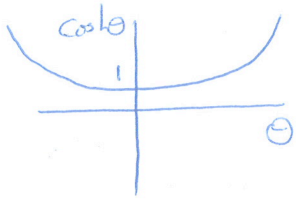
so we can write the solution to the position relationship as

$$F(x) = A\sin(\alpha x) + B\cos(\alpha x) + C\sinh(\alpha x) + D\cosh(\alpha x)$$

where A, B, C, D and α are unknown constants (note that the natural frequency ω is contained within α). Note: using the sinh and cosh functions is more convenient than using real exponential functions

$$\sinh(\theta) = \frac{1}{2}\left(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}\right) \quad \text{and} \quad \cosh(\theta) = \frac{1}{2}\left(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}\right)$$





The complete solution is

$$y(x,t) = [A\sin(\alpha x) + B\cos(\alpha x) + C\sinh(\alpha x) + D\cosh(\alpha x)]\,[E\sin(\omega t) + F\cos(\omega t)]$$

As with the axial vibrations we do not need to worry about the $G(t)$ function for free vibrations, what we are interested in is the natural frequencies and modeshapes which are found from applying the boundary conditions to $F(x)$ (applying initial conditions would allow us to find constants E and F).

lecture 5

4.3 Boundary Conditions

For bending of beams we can write two conditions for each end of the beam. The simple boundary conditions are

Type	Condition	
Free	$M = 0$	$Q = 0$
Clamped	$y = 0$	$\theta = 0$
Simply Supported	$y = 0$	$M = 0$

To apply these we need to remember that

$$\theta = \frac{\partial y}{\partial x} \quad , \quad M = EI\frac{\partial^2 y}{\partial x^2} \quad \text{and} \quad Q = -\frac{\partial M}{\partial x}$$

Example: Simply Supported-Simply Supported Beam

$x \rightarrow$ $\quad x = l$

$M|_{x=0} = 0 = \frac{\partial^2 y}{\partial x^2}\Big|_{x=0}$

$y|_{x=0} = 0$

$M|_{x=l} = 0 = \frac{\partial^2 y}{\partial x^2}\Big|_{x=l}$

$y|_{x=l} = 0$

$$y(x,t) = \left(\bar{A}\sin(\alpha x) + B\cos(\alpha x) + C\sinh(\alpha x) + D\cosh(\alpha x)\right)\left(\bar{E}\sin(\omega t) + F\cos(\omega t)\right)$$

$$\alpha^4 = \omega^2 \frac{\rho A}{EI}$$

$\partial/\partial x$
$\sinh \rightarrow \cosh$
$\cosh \rightarrow \sinh$

$$\left.\begin{aligned} y(0,t) = 0 &\Rightarrow (B + D)G(t) = 0 \\ y''(0,t) = 0 &\Rightarrow (-\alpha^2 B + \alpha^2 D)G(t) = 0 \end{aligned}\right\} \quad B = D = 0$$

$$y(x,t) = G(t)\left(\bar{A}\sin(\alpha x) + C\sinh(\alpha x)\right)$$

$$y(l,t) = 0 \Rightarrow \bar{A}\sin(\alpha l) + C\sinh(\alpha l) = 0$$

$$y''(l,t) = 0 \Rightarrow -\alpha^2\bar{A}\sin(\alpha l) + C\alpha^2\sinh(\alpha l) = 0$$

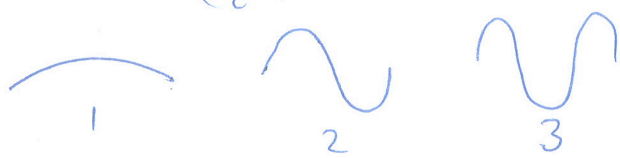
$$\therefore \bar{A}\sin(\alpha l) = C\sinh(\alpha l) = 0$$

$$\sinh(\alpha l) = 0 \text{ only if } \alpha, l = 0 \quad \therefore C = 0$$

$$\sin(\alpha l) = 0 \quad \text{if} \quad \alpha l = \cancel{0}, \pi, 2\pi, \ldots n\pi \ldots \quad n = 1, 2, 3 \ldots$$

$$\alpha^4 = \frac{\rho A \omega^2}{EI} \Rightarrow \omega_n = \sqrt{\frac{EI}{\rho A}}\left(\frac{n\pi}{l}\right)^2$$

$$\text{modeshape} = \sin(\alpha x) = \sin\left(\frac{n\pi x}{l}\right)$$



Example: Clamped-Free Beam

$y(0,t)=0$

$y'(0,t)=0$

$M|_{x=l}=0 \Rightarrow y''(l,t)=0$

$Q|_{x=l}=0 \Rightarrow y'''(l,t)=0$

$$y(x,t)=\left(\bar{A}\sin(\alpha x)+B\cos(\alpha x)+C\sinh(\alpha x)+D\cosh(\alpha x)\right)\left(\bar{E}\sin(\omega t)+F\cos(\omega t)\right)$$

$$\alpha^4=\omega^2\frac{\rho A}{EI}$$

$y(0,t)=0 \Rightarrow B+D=0$ ①

$y'(0,t)=0 \Rightarrow \cancel{\alpha}A+\cancel{\alpha}C=0$ ②

$y''(l,t)=0 \Rightarrow -A\sin(\alpha l)-B\cos(\alpha l)+C\sinh(\alpha l)+D\cosh(\alpha l)=0$ ③

$y'''(l,t)=0 \Rightarrow -A\cos(\alpha l)+B\sin(\alpha l)+C\cosh(\alpha l)+D\sinh(\alpha l)=0$ ④

①,② in ③,④ :
$$(\sin(\alpha l)+\sinh(\alpha l))C+(\cos(\alpha l)+\cosh(\alpha l))D=0$$
$$(\cos(\alpha l)+\cosh(\alpha l))C+(-\sin(\alpha l)+\sinh(\alpha l))D=0$$

$$\begin{pmatrix}\sin(\alpha l)+\sinh(\alpha l) & \cos(\alpha l)+\cosh(\alpha l)\\ \cos(\alpha l)+\cosh(\alpha l) & -\sin(\alpha l)+\sinh(\alpha l)\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Det Handout

$\therefore$ either $C=D=0 \Rightarrow y=0$ or $\text{Det}=0$

$$\text{Det}=-\sin^2(\alpha l)+\sinh^2(\alpha l)-\cos^2(\alpha l)-2\cos(\alpha l)\cosh(\alpha l)-\cosh^2(\alpha l)=0$$

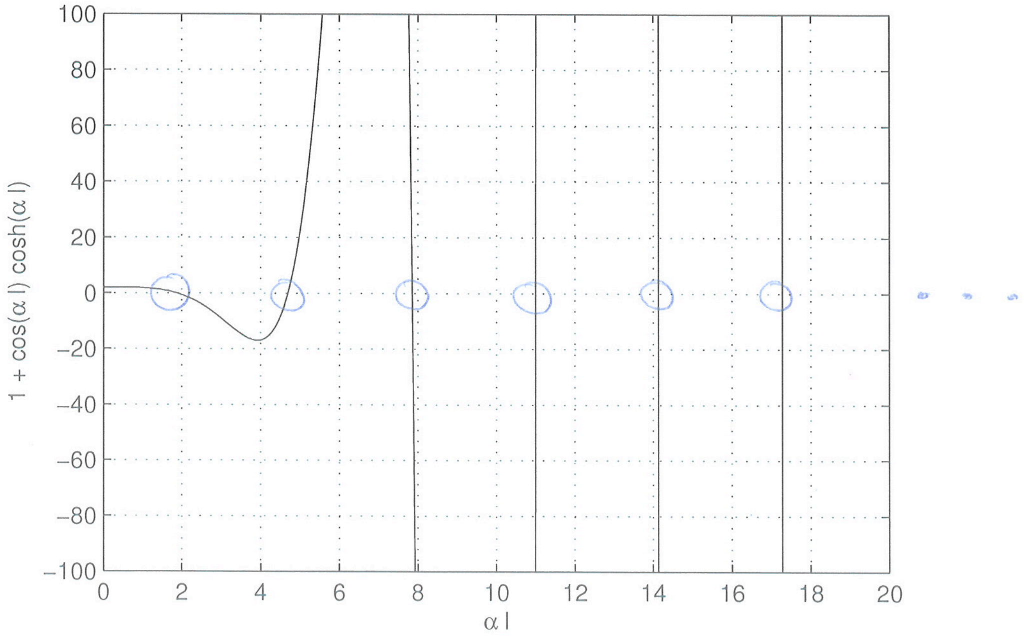
$\cosh^2\theta-\sinh^2\theta=1$

$\cos^2\theta+\sin^2\theta=1$

Det = 0 : results in the two equations being the same so we can't find values for C and D but a relationship between them

$$1+\cos(\alpha l)\cosh(\alpha l)=0$$

This is called the *frequency equation*. It can be solved numerically or graphically.

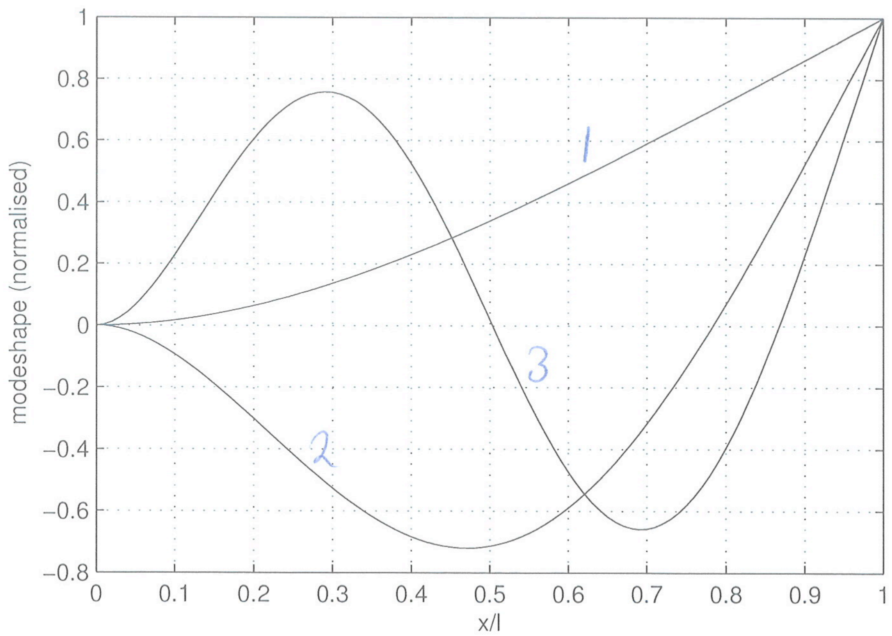


$$\alpha_n l = 1.875,\ 4.694,\ 7.855,\ 10.996$$

$$\omega_n = \alpha_n^2 \sqrt{\frac{EI}{\rho A}} = \frac{1.875^2}{l^2} \sqrt{\frac{EI}{\rho A}},\ \ldots$$

$$C_n = -\bar{A}_n = -\left(\frac{\cos(\alpha_n l) + \cosh(\alpha_n l)}{\sin(\alpha_n l) + \sinh(\alpha_n l)}\right) D_n = \left(\frac{\cos(\alpha_n l) + \cosh(\alpha_n l)}{\sin(\alpha_n l) + \sinh(\alpha_n l)}\right) B_n$$

nth modeshape $= \bar{A}_n \sin(\alpha_n x) + B_n \cos(\alpha_n x) + C_n \sinh(\alpha_n x) + D_n \cosh(\alpha_n x)$



ratio 1st → 2nd 6.27   2nd - 7.5 Hz   2.4 V

2nd → 3rd 2.8   21   3rd   21.2 Hz   1.5 V

21 Hz

3rd → 4th 1.96   4th   40.3 Hz   1.6 V

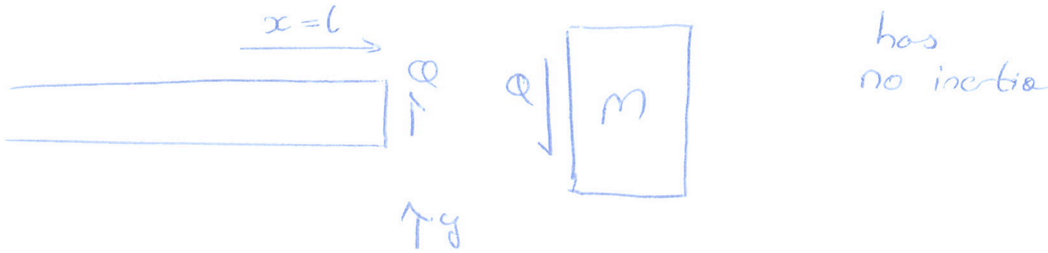
41 Hz

demo

4.4 Mass and Spring End Conditions

As with the axial vibration and torsional vibration of bars, mass or spring end conditions can also be included in the analysis. However, for flexural vibration we need two boundary conditions at each end of the bar.

4.4.1 Lumped Mass



Considering the $x = l$ end, using "F=ma" we can write

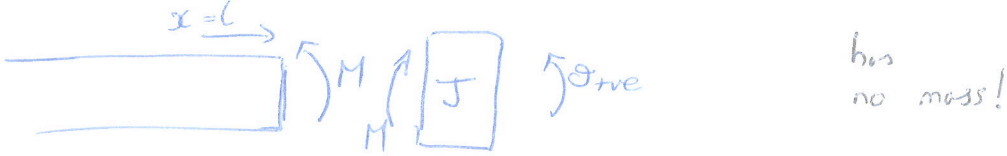
$$m\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=l} = -\left.Q\right|_{x=l} = \left.\frac{\partial M}{\partial x}\right|_{x=l} = EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=l}$$

This can be simplified by noting that $\ddot{y} = -\omega^2 y$ to give

$$x = l: \quad -m\omega^2 \left.y\right|_{x=l} = EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=l} \quad , \quad x = 0: \quad m\omega^2 \left.y\right|_{x=0} = EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=0}$$

Examples of the second end condition are:

4.4.2 Lumped Inertia



Considering the $x = l$ end, using "$T = J\ddot{\theta}$" (where J is the second moment of inertia) we can write

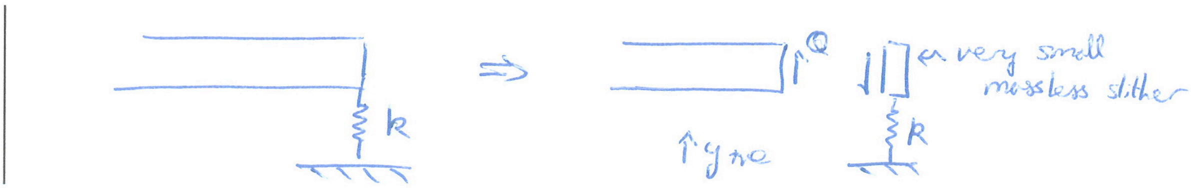
$$\left.M\right|_{x=l} = EI\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=l} = -J\left.\frac{\partial^2 \theta}{\partial t^2}\right|_{x=l} = -J\left.\frac{\partial^3 y}{\partial t^2 \partial x}\right|_{x=l} = J\omega^2\left.\frac{\partial y}{\partial x}\right|_{x=l}$$

Giving

$$x = l: \quad J\omega^2\left.\frac{\partial y}{\partial x}\right|_{x=l} = EI\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=l} \quad , \quad x = 0: \quad -J\omega^2\left.\frac{\partial y}{\partial x}\right|_{x=0} = EI\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0}$$

4.5 Spring Support

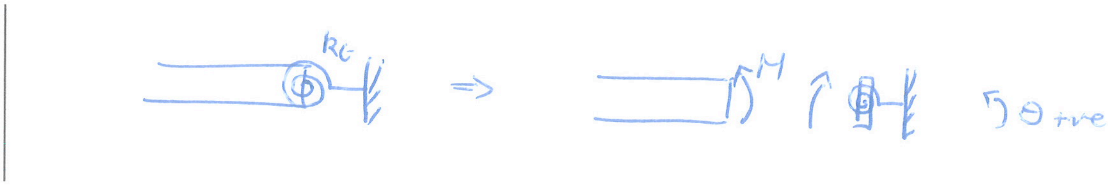
For a lateral spring support condition;



considering the $x = l$ end, we can write

$$Q|_{x=l} = -\left.\frac{\partial M}{\partial x}\right|_{x=l} = -\left.EI\frac{\partial^3 y}{\partial x^3}\right|_{x=l} = -k\, y|_{x=l}$$

As with the lumped mass or lumped inertia this is not a sufficient condition on its own, since two conditions are needed for each end of the beam. A second condition could be that the end is unconstrained in rotation such that $M_{x=l} = 0$ or that in addition to the lateral spring there is a torsional spring acting on the rotation of the beam;

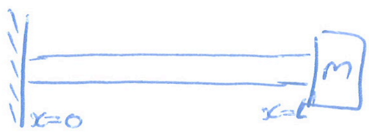


Considering the $x = l$ end, we can write

$$M|_{x=l} = \left.EI\frac{\partial^2 y}{\partial x^2}\right|_{x=l} = -k_t\, \theta|_{x=l} = -k_t \left.\frac{\partial y}{\partial x}\right|_{x=l}$$

subscript
error

Example: Cantilever Beam with a Lumped Mass



$x=0$: $y(0,t)=0$ (1)

$\theta(0,t)=0$ (2)

$x=l$: $M(l,t)=0$ (3)

$-m\omega^2 y(l,t) = EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=l}$ (4)

$$y(x,t) = [A\sin(\alpha x) + B\cos(\alpha x) + C\sinh(\alpha x) + D\cosh(\alpha x)] \times [E\sin(\omega t) + F\cos(\omega t)]$$

$$= [A\sin(\alpha x) + B\cos(\alpha x) + C\sinh(\alpha x) + D\cosh(\alpha x)]\, G(t)$$

with $\alpha^4 = \frac{\omega^2 \rho A}{EI}$

(1) $y(0,t) = 0 \Rightarrow \quad 0 = [B + D]\,G(t) \quad \therefore \quad B = -D$ (a)

$$\theta = \frac{\partial y}{\partial x} = \alpha\left[\bar{A}\cos(\alpha x) - B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x)\right]G(t)$$

(2) $\theta(0,t) = 0 \Rightarrow \quad 0 = \alpha[\bar{A} + C]\,G(t) \quad \therefore \quad \bar{A} = -C$ (b)

(3) $M(l,t) = 0$

$$M = EI\frac{\partial^2 y}{\partial x^2} = EI\alpha^2\left[-\bar{A}\sin(\alpha x) - B\cos(\alpha x) + C\sinh(\alpha x) + D\cosh(\alpha x)\right]G(t)$$

$$M(l,t) = 0 \Rightarrow -\bar{A}\sin(\alpha l) - B\cos(\alpha l) + C\sinh(\alpha l) + D\cosh(\alpha l) = 0$$

sub in (a), (b) : $\quad -\bar{A}\left(\sin(\alpha l) + \sinh(\alpha l)\right) - B\left(\cos(\alpha l) + \cosh(\alpha l)\right) = 0$ (c)

(4) $-m\omega^2 y(l,t) = EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=l}$

$$EI\frac{\partial^3 y}{\partial x^3} = EI\alpha^3\left[-\bar{A}\cos(\alpha x) + B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x)\right]G(t)$$

$$-m\omega^2 y(l,t) = EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=l} \Rightarrow$$

$$-m\omega^2\left[\bar{A}\sin(\alpha l) + B\cos(\alpha l) + C\sinh(\alpha l) + D\cosh(\alpha l)\right]G(t) =$$
$$EI\alpha^3\left[-\bar{A}\cos(\alpha l) + B\sin(\alpha l) + C\cosh(\alpha l) + D\sinh(\alpha l)\right]G(t)$$

sub in (a), (b) :

$$\bar{A}\left[\left(\sin(\alpha l) - \sinh(\alpha l)\right)(-m\omega^2) - EI\alpha^3\left(-\cos(\alpha l) - \cosh(\alpha l)\right)\right]$$
$$+ B\left[\left(\cos(\alpha l) - \cosh(\alpha l)\right)(-m\omega^2) - EI\alpha^3\left(\sin(\alpha l) - \sinh(\alpha l)\right)\right] = 0$$

sub in (c) $\quad \bar{A} = -B\left(\frac{c + ch}{s + sh}\right) \Rightarrow (c + ch)\left[m\omega^2(s - sh) - EI\alpha^3(c + ch)\right]$

(shorthand $ch = \cosh(\alpha l)$ etc, $c = \cos(\alpha l)$) $\quad + (s + sh)\left[-m\omega^2(c - ch) - EI\alpha^3(s - sh)\right] = 0$

$ch^2 - sh^2 = 1$
$c^2 + s^2 = 1$

$$\Rightarrow \quad EI\alpha^3(1 + c\,ch) = m\omega^2(s\,ch - c\,sh)$$

$$\frac{m\omega^2}{EI\alpha^3} = \frac{1 + \cos(\alpha l)\cosh(\alpha l)}{\sin(\alpha l)\cosh(\alpha l) - \cos(\alpha l)\sinh(\alpha l)} = \frac{m}{EI\alpha^3}\left(\frac{EI\alpha^4}{\rho A}\right) = \frac{m\alpha l}{M}$$

$\omega^2 = \frac{EI}{\rho A}\alpha^4 \qquad M = \rho A l =$ mass of beam

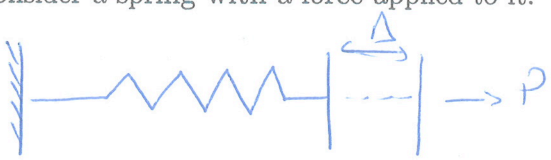
Lecture 7

5 Rayleigh Method

For approximate methods such as Rayleigh Method and finite element analysis, equations for the kinetic and potential (or elastic) energy of the system are required.

5.1 Energy

Stored elastic energy in an element may be calculated from the work done in extending the element. Consider a spring with a force applied to it:



The stored potential energy, V, is equal to the work done extending the element

$$V = \text{Work Done} = \int_0^{\Delta} P \mathrm{d}\Delta$$

Using the relationship $P = k\Delta$ we can write

$$V = \int_0^{\Delta} k\Delta \mathrm{d}\Delta = \frac{1}{2}k\Delta^2$$

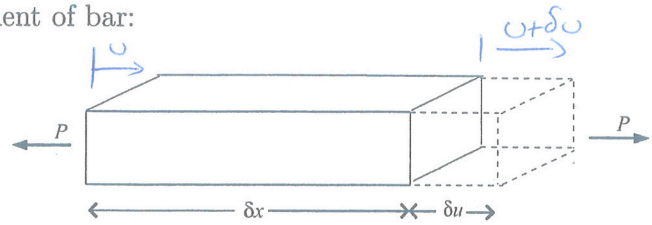
For a torsional spring subjected to a torque T resulting in a rotational deflection ϕ we can write

$$V = \text{Work Done} = \int_0^{\phi} T \mathrm{d}\phi \quad \rightsquigarrow \quad V = \frac{1}{2}k_r\phi^2$$

using the relationship $T = k_r\phi$. We can use these relationships to calculate the stored energy in bars, rods and beams by relating the applied force (or torque) to the resulting deflection using an effective spring stiffness k (or k_r). However we must be careful as the force (or torque) in the system is a function of length along the bar/rod/beam x (and t). We must therefore consider the stored elastic energy in a small element δx long and integrate over the length $x = 0$ to $x = l$.

5.1.1 Axial Vibrations

Consider the element of bar:



For this element, we can relate the force P to extension δu (extension of one end relative to the other) using

$$P = A\sigma = EA\epsilon = EA\frac{\delta u}{\delta x} = k_{axial}\delta u$$

$k_{axial} = \frac{EA}{\delta x}$

where k_{axial} is the effective spring stiffness relating the force to the extension. Therefore the elastic energy stored in this element, δV may be written as

$$\delta V = \frac{1}{2} k_{axial} \delta u^2$$

But we can write

$$\delta u = \frac{\partial u}{\partial x} \delta x$$

Therefore

$$\delta V = \frac{1}{2} k_{axial} \left(\frac{\partial u}{\partial x} \delta x \right)^2 = \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \delta x$$

Integrating over the whole beam gives a total stored elastic potential energy of

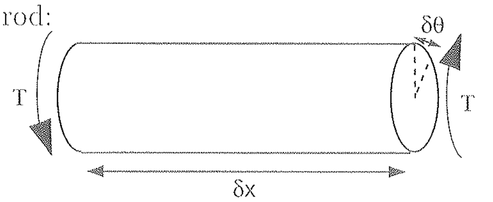
$$V = \int_0^l \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x$$

Again, for the kinetic energy, T, we must consider a small element of length δx and integrate over the whole bar since the displacement u is a function of x (and t):

$$\delta T = \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 \rho A \delta x \quad \rightsquigarrow \quad T = \int_0^l \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 \mathrm{d}x$$

5.1.2 Torsional Vibration

Consider the element of rod:



For this element, we can relate the torque T to rotational deflection $\delta\theta$ using

$$T = \frac{GI_p}{\delta x} \delta\theta = k_{torsion} \delta\theta$$

where $k_{torsion}$ is the effective torsional spring stiffness. Therefore the elastic energy stored in this element, δV may be written as

$$\delta V = \frac{1}{2} k_{torsion} \delta\theta^2$$

Using $\delta\theta = \frac{\partial\theta}{\partial x}\delta x$ we get:

$$\delta V = \frac{1}{2} k_{torsion} \left(\frac{\partial \theta}{\partial x} \delta x \right)^2 = \frac{1}{2} GI_p \left(\frac{\partial \theta}{\partial x} \right)^2 \delta x$$

Integrating over the rod gives:

$$V = \int_0^l \frac{1}{2} G I_p \left(\frac{\partial \theta}{\partial x} \right)^2 \mathrm{d}x$$

The kinetic energy, δT, for a small length of inertial element, δx, is given by:

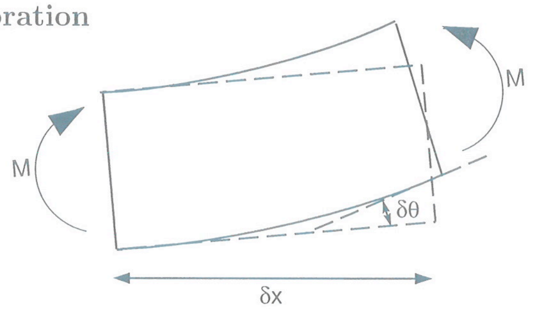
$$\delta T = \frac{1}{2} J \left(\frac{\partial \theta}{\partial t} \right) = \frac{1}{2} I_p \rho \left(\frac{\partial \theta}{\partial t} \right) \delta x$$

recalling that J is the moment of inertia of the element and I_p is the second polar moment of area of the cross-section of the element. Integrating over the length of the bar gives the total kinetic energy T

$$T = \int_0^l \frac{1}{2} \rho I_p \left(\frac{\partial \theta}{\partial t} \right)^2 \mathrm{d}x$$

The standard symbol for kinetic energy is T which is also the usual symbol for torque - care is needed!

5.1.3 Flexural Vibration



The moment applied to a beam can be related to the curvature of the beam and hence the slope of the beam (assuming small deflections):

$$M = \frac{EI}{R} = EI \frac{\delta \theta}{\delta x} = k_{flexural} \delta \theta$$

Therefore the elastic energy stored in this element of beam, δV may be written as

$$\delta V = \frac{1}{2} k_{flexural} \delta \theta^2$$

Using $\delta \theta = \frac{\partial \theta}{\partial x} \delta x$ we get:

$$\delta V = \frac{1}{2} k_{flexural} \left(\frac{\partial \theta}{\partial x} \delta x \right)^2 = \frac{1}{2} EI \left(\frac{\partial \theta}{\partial x} \right)^2 \delta x$$

Integrating over the bar and substituting $\theta = \frac{\partial y}{\partial x}$ gives:

$$V = \int_0^l \frac{1}{2} EI \left(\frac{\partial \theta}{\partial x} \right)^2 \mathrm{d}x = \int_0^l \frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2} \right)^2 \mathrm{d}x$$

Since the amplitude of vibration is assumed small the rotational kinetic energy is assumed to be negligible. The translational kinetic energy may be written as

$$T = \int_0^l \frac{1}{2} \rho A \left(\frac{\partial y}{\partial t} \right)^2 \mathrm{d}x$$

Section 5.1 not examined ; V, T will be given

5.2 Applying the Rayleigh Method

The Rayleigh method allows the estimation of the fundamental frequency of a system. The principle behind the Rayleigh Method is that, over a cycle, the maximum potential energy stored in system is the same as the maximum kinetic energy.

$$V_{max} = T_{max}$$

The method involves the following steps:

- Guess deformed shape (trying to satisfy boundary conditions) ← better guesses → better estimates of n.f.
- Calculate the energies based on the guessed shape
- Equate maximum energies over a cycle to find the frequency

Note that lumped masses and spring supports may be included in the kinetic and potential energy calculations respectively. (The method is also applicable to multi-degree-of-freedom discrete systems.)

Example: Axial vibration of Fixed-Fixed bar

(P8) corresponds to $\omega_1 = \frac{c\pi}{l}$ $\quad c = \sqrt{\frac{E}{\rho}}$

$$\text{Guess 1:} \quad u(x,t) = C\sin\left(\frac{\pi x}{l}\right)\cos(\omega t)$$

(this is the correct mode shape for the fundamental frequency)

Potential Energy $\quad V = \int_0^l \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 dx$

using guessed mode shape

$$= \int_0^l \frac{1}{2} EA \left[\frac{\pi}{l}\cos\left(\frac{\pi x}{l}\right)\cos(\omega t)\right]^2 dx$$

$$= \frac{EA\pi^2}{2l^2}\cos^2(\omega t)\int_0^l \cos^2\left(\frac{\pi x}{l}\right) dx$$

$\therefore V = V(t) \qquad V = V_{max}$ when $\cos(\omega t) = 1$

$$V_{max} = \frac{EA\pi^2}{2l^2}\int_0^l \cos^2\left(\frac{\pi x}{l}\right) dx$$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$$= \frac{EA\pi^2}{4l^2}\int_0^l \cos\left(\frac{2\pi x}{l}\right) + 1 \, dx$$

$$= \frac{EA\pi^2}{4l^2}\left[\frac{l}{2\pi}\sin\left(\frac{2\pi x}{l}\right) + x\right]_0^l = \frac{EA\pi^2}{4l} C^2$$

Kinetic Energy $T = \int_0^l \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t}\right)^2 dx$

$$\frac{\partial u}{\partial t} = -\omega \sin\left(\frac{\pi x}{l}\right) \sin(\omega t)$$ using guessed mode shape

$$T = \int_0^l \frac{1}{2} \rho A \omega^2 \sin^2\left(\frac{\pi x}{l}\right) dx \ \sin^2(\omega t)$$

$T = T(t)$ $\quad T = T_{max}$ when $\sin(\omega t) = 1$

$$T_{max} = \int_0^l \frac{1}{2} \rho A \omega^2 \sin^2\left(\frac{\pi x}{l}\right) dx$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$= \frac{1}{4} \rho A \omega^2 \int_0^l 1 - \cos \frac{2\pi x}{l} \, dx$$

$$= \frac{1}{4} \rho A \omega^2 \left[x - \frac{l}{2\pi} \sin\left(\frac{2\pi x}{l}\right) \right]_0^l$$

$$= \frac{1}{4} \rho A \omega^2 l \ C^2$$

$$T_{max} = V_{max} \Rightarrow \frac{\cancel{C^2} E A \pi^2}{4l} = \frac{1}{4} \rho A \omega^2 l \ \cancel{C^2}$$

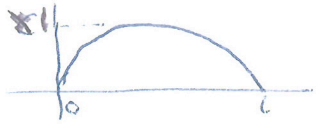
$$\omega^2 = \frac{E \pi^2}{\rho l^2}$$

$$\omega = \frac{\pi}{l} \sqrt{\frac{E}{\rho}}$$

using Rayleigh method with the correct mode shape $\Rightarrow$ correct fundamental frequency

(A8)

Guess 2: Quadratic: $u(x,t) = 4\left(\frac{x}{l} - \frac{x^2}{l^2}\right)\cos(\omega t)$



Potential $V = \int_0^l \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 dx$ guessed mode shape

$= \int_0^l \frac{1}{2} EA \,(4)^2 \left(\frac{1}{l} - \frac{2x}{l^2}\right)^2 dx \; \cos^2(\omega t)$

$\Rightarrow V_{max} = \int_0^l 8EA \left(\frac{1}{l} - \frac{2x}{l^2}\right)^2 dx$

$= \frac{8}{3}\frac{EA}{l}$

Kinetic $T = \int_0^l \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t}\right)^2 dx$

$= \int_0^l \frac{1}{2} \rho A \, 16 \left(\frac{x}{l} - \frac{x^2}{l^2}\right)^2 dx \quad \omega^2 \sin^2 \omega t$

$T_{max} = \omega^2 \int_0^l 8 \rho A \left(\frac{x}{l} - \frac{x^2}{l^2}\right)^2 dx$

$= \frac{4 \rho A \,\omega^2 l}{15}$

$V_{max} = T_{max} \qquad \frac{4 \rho A \,\omega^2 l}{15} = \frac{8}{3}\frac{EA}{l}$

$\omega^2 = 10 \frac{E}{\rho l^2}$

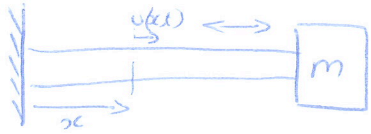
$\omega = \sqrt{\frac{E}{\rho}} \frac{\sqrt{10}}{l} = \frac{3.16}{l}\sqrt{\frac{E}{\rho}}$

not bad! using correct mode shape $\frac{3.142}{l}\sqrt{\frac{E}{\rho}}$

Note: The form of the sinusoidal time varying part ($\sin(\omega t)$, $\cos(\omega t)$ or $\sin(\omega t + \phi)$) of the guess does not matter, it does not affect the estimated frequency of response as we are interested in the maximum energy over a cycle

Lecture 8

Example: Axial vibration of Fixed-Lumped Mass bar



P11 transcendental equation for ω:

$$\left(\frac{\omega l}{c}\right)\tan\left(\frac{\omega l}{c}\right) = \frac{M}{m}$$

$M = \rho A l$

$c = \sqrt{\frac{E}{\rho}}$

Guess 1: $u(x,t) = \bar{C}x\cos(\omega t)$ — linear mode shape, 0 at $x=0$

Potential (Elastic) $V = \int_0^l \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 dx$

$$= \int_0^l \frac{1}{2}EA\bar{C}^2 \, dx \, \cos^2(\omega t)$$

$$V_{max} = \frac{1}{2}EA\bar{C}^2\left[x\right]_0^l = \frac{1}{2}EA\bar{C}^2 l$$

mention the positions + kinetic energy

KE $T = \int_0^l \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}m\left(\left.\frac{\partial u}{\partial t}\right|_{x=l}\right)^2$

beam mass

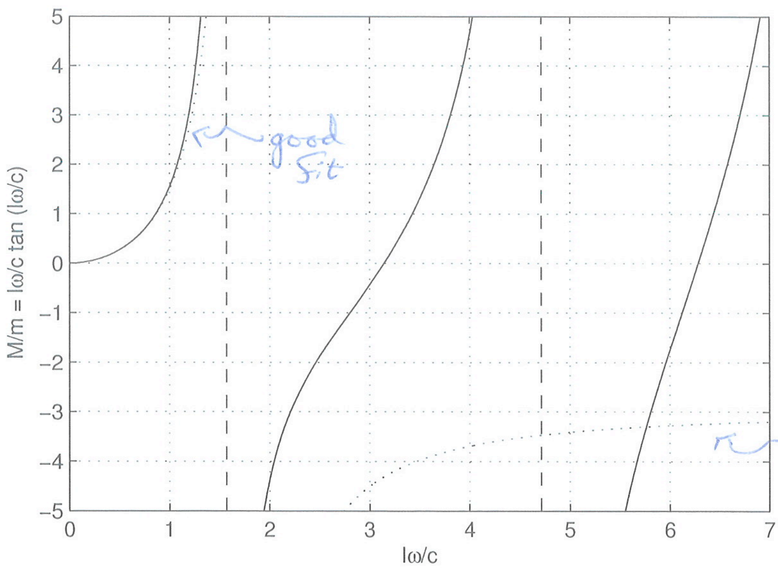
$\frac{\partial u}{\partial t} = -\omega\bar{C}x\sin\omega t$

$$= \left[\int_0^l \frac{1}{2}\rho A(\bar{C}x)^2 dx + \frac{1}{2}m(\bar{C}l)^2\right]\omega^2\sin^2(\omega t)$$

$$T_{max} = \left(\frac{1}{2}\rho A\bar{C}^2\frac{l^3}{3} + \frac{1}{2}m\bar{C}^2 l^2\right)\omega^2$$

$V_{max} = T_{max} \Rightarrow$

$$\omega^2 = \frac{\cancel{\frac{1}{2}}E\cancel{\bar{C}^2}\frac{M}{\rho}}{\cancel{\frac{1}{2}}M\frac{\cancel{\bar{C}^2}l^2}{3} + \cancel{\frac{1}{2}}m\cancel{\bar{C}^2}l^2} = \frac{c^2}{l^2}\frac{1}{\frac{1}{3}+\frac{m}{M}}$$



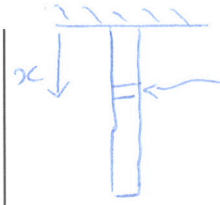
or:

$$\frac{M}{m} = \frac{\left(\frac{\omega l}{c}\right)^2}{1 - \frac{1}{3}\left(\frac{\omega l}{c}\right)^2}$$

$\frac{M}{m}$ −ve

Guess 2: Static Deflected Shape

$m = 0$



Force = weight below $= A\rho g(l-x)$

$$F = A\sigma = AE\epsilon = AE\frac{\partial u}{\partial x}$$

$$\therefore \cancel{A}E\frac{\partial u}{\partial x} = \cancel{A}\rho g(l-x)$$

$$u = \frac{\rho g}{E}\left(lx - \frac{x^2}{2} + C\right)$$

$u = 0$ at $x = 0$ $\therefore$ $C = 0$

$\Rightarrow$ deflected shape $\quad u = \dfrac{\rho g}{E}\left(lx - \dfrac{x^2}{2}\right)$

$\therefore$ guess mode shape $\quad u = \left(lx - \dfrac{x^2}{2}\right)\cos\omega t$

$\Rightarrow$ Potential Energy $\quad V_{max} = \dfrac{EAl^3}{6}$

Kinetic Energy $\quad T_{max} = \dfrac{2}{30}\rho A\omega^2 l^5$

$V_{max} = T_{max}$ $\Rightarrow$ $\quad \omega^2 = \dfrac{EAl^3}{6}\,\dfrac{30}{2\rho A l^5} = \dfrac{5}{2}\,\dfrac{E}{\rho l^2}$

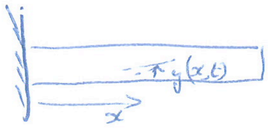
$$\omega = \frac{1.581}{l}\sqrt{\frac{E}{\rho}} = 1.581\,\frac{c}{l}$$

correct solution from P11 $\quad \omega = \dfrac{\pi}{2}\dfrac{c}{l} = 1.57\dfrac{c}{l}$

(if guess of "x" (p31) setting $m = 0$

$$\omega = \frac{c}{l}\sqrt{\frac{1}{\frac{1}{3} + \frac{m}{M}}} = \sqrt{3}\,\frac{c}{l} = 1.73\,\frac{c}{l}$$)

Example: Flexural vibration of a Cantilever Beam



guess a parabola $\quad y = kx^2 \cos(\omega t)$

BC ok at $x=0$
not at $x=l$ $\quad M = EI \frac{\partial^2 y}{\partial x^2} \neq 0$

Potential Energy $\quad V = \int_0^l \frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$

$\therefore \quad V_{max} = \int_0^l \frac{1}{2} EI (2k)^2 \, dx = 2EIk^2 l$

Kinetic Energy $\quad T = \int_0^l \frac{1}{2} \rho A \left(\frac{\partial y}{\partial t}\right)^2 dx$

$\therefore \quad T_{max} = \int_0^l \frac{1}{2} \rho A (kx^2)^2 \omega^2 dx = \frac{\rho A k^2 l^5 \omega^2}{10}$

$\Rightarrow \quad 2EIk^2 l = \frac{\rho A k^2 l^5 \omega^2}{10}$

$\omega^2 = \frac{20 EI}{\rho A l^4} \quad \Rightarrow \omega = 4.47 \sqrt{\frac{EI}{\rho A l^4}}$

exact (P21) $\quad \omega_1 = 3.52 \sqrt{\frac{EI}{\rho A l^4}}$ $\quad$ 27% error in estimate.

if use deflected shape : $\quad y_{static} = \frac{\rho g A}{EI} \left(-\frac{l^2 x^2}{4} + \frac{l x^3}{6} - \frac{x^4}{24} \right)$

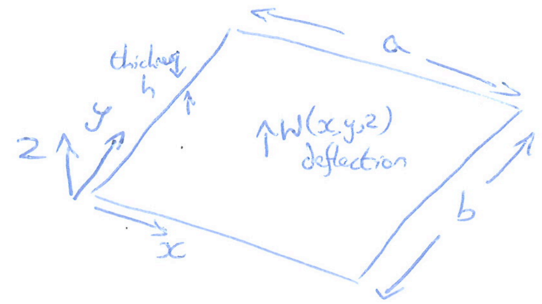
$\Rightarrow$ guess $\quad y = k\left(\frac{l^2 x^2}{4} - \frac{l x^3}{6} + \frac{x^4}{24}\right) \cos(\omega t)$

$\Rightarrow \quad \omega_1 = 3.53 \sqrt{\frac{EI}{\rho A l^4}}$

It can be shown that the estimated frequency is always higher that (or equal to) the actual frequency. – Provided BCs met

Lecture 9

6 Vibration of Plates

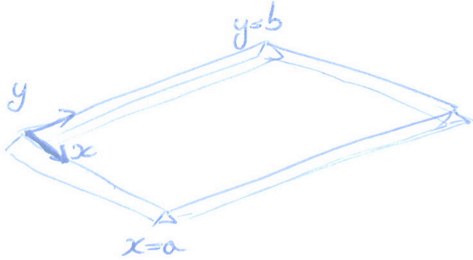


The equation for motion for the flexural vibration of a plate is given by

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{12\rho(1-\nu^2)}{Eh^2}\frac{\partial^2 w}{\partial t^2} = 0$$

nu ν = poisson's ratio

This equation can not be solved explicitly except for the case where the plate is **rectangular** and **simply supported**.



$W(0,y,t) = 0$ $\quad M(0,y,t) = 0$

$W(a,y,t) = 0$

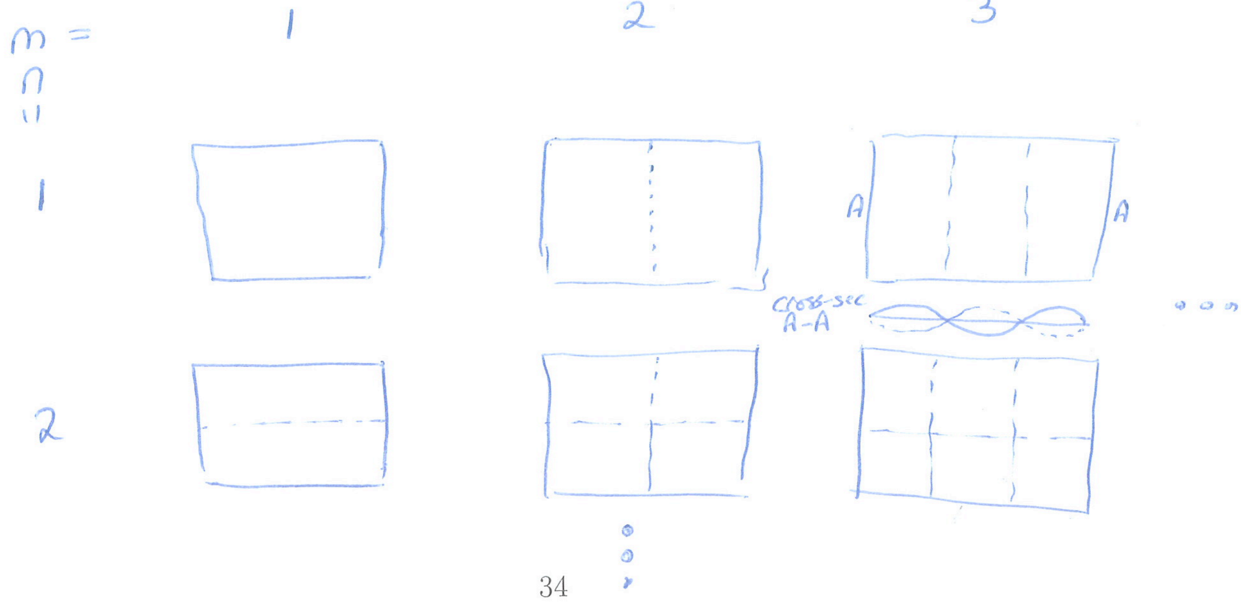
$W(x,0,t) = 0$

$W(x,b,t) = 0$

In this case we can write

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[A_{m,n}\sin(\omega_{m,n}t) + B_{m,n}\cos(\omega_{m,n}t)\right]$$

The nodal patterns can therefore be defined as:



By substituting the expression for w into the equation of motion the natural frequency can be calcluated for the simply-supported rectanglar plate:

$w = \sin \alpha x \sin \beta y \sin \omega t$ form

sub into equation of motion:

$$\alpha^4 + 2\alpha^2\beta^2 + \beta^4 = \frac{12\rho(1-\nu^2)}{Eh^2}\omega^2$$

$$\frac{12\rho(1-\nu^2)}{Eh^2}\omega^2 = (\alpha^2+\beta^2)^2$$

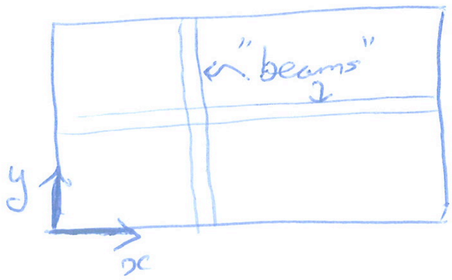
$$\omega = \sqrt{\frac{Eh^2}{12\rho(1-\nu^2)}}(\alpha^2+\beta^2)$$

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b} \quad \therefore \quad \omega_{m,n} = \pi^2\sqrt{\frac{Eh^2}{12\rho(1-\nu^2)}}\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)$$

For other boundary conditions we need functions of x and y that satisfy the boundary conditions and give the deflected shape

$$w(x, y, t) = X(x)Y(y)G(t)$$

where the mode shapes are defined by $X(x)$ and $Y(y)$. One way of obtaining a guessed shape is to base these functions on beam functions:



$$X(x) = A_x \sin(\alpha x) + B_x \cos(\alpha x) + C_x \sinh(\alpha x) + D_x \cosh(\alpha x)$$
$$Y(y) = A_y \sin(\alpha y) + B_y \cos(\alpha y) + C_y \sinh(\alpha y) + D_y \cosh(\alpha y)$$

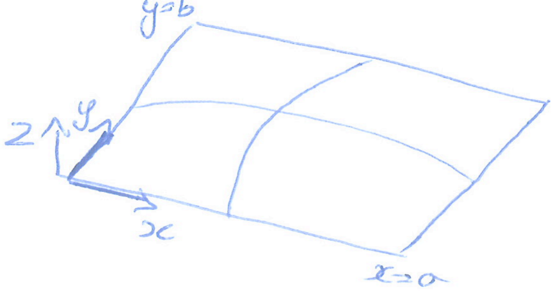
Using the beam functions, as a guess, we then progress by fitting boundary conditions for the individual beam functions and then using **Rayleigh method**. For flexural vibration, the elastic potential energy in a plate is given by

$$V = \int_0^a \int_0^b \frac{Eh^3}{24(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] \mathrm{d}y\mathrm{d}x$$

and the kinetic energy is given by

$$T = \int_0^a \int_0^b \frac{\rho h}{2} \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}y\mathrm{d}x$$

Example: Simply Supported Plate Vibration



Guess: Beam functions with quadratic form

$$X(x) = \left(x - \frac{x^2}{a}\right) \qquad Y(y) = \left(y - \frac{y^2}{b}\right)$$

Rayleigh $\quad W = \left(x - \frac{x^2}{a}\right)\left(y - \frac{y^2}{b}\right)\cos \omega t$

$$\left(\frac{\partial^2 W}{\partial x^2}\right)^2 = \left[\frac{-2}{a}\left(y - \frac{y^2}{b}\right)\cos \omega t\right]^2 = \frac{4}{a^2}\left(y - \frac{y^2}{b}\right)^2 \cos^2 \omega t$$

$$\left(\frac{\partial^2 W}{\partial y^2}\right)^2 = \frac{4}{b^2}\left(x - \frac{x^2}{a}\right)^2 \cos^2 \omega t$$

$$2\nu \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} = \frac{8\nu}{ab}\left(x - \frac{x^2}{a}\right)\left(y - \frac{y^2}{b}\right)\cos^2 \omega t$$

$$2(1-\nu)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 = 2(1-\nu)\left(1 - \frac{2x}{a}\right)^2\left(1 - \frac{2y}{b}\right)^2 \cos^2 \omega t$$

$$\therefore \quad V_{max} = \int_0^a \int_0^b \frac{Eh^3}{24(1-\nu^2)}\left[\frac{4}{a^2}\left(y - \frac{y^2}{b}\right)^2 + \frac{4}{b^2}\left(x - \frac{x^2}{a}\right)^2 + \frac{8\nu}{ab}\left(x - \frac{x^2}{a}\right)\left(y - \frac{y^2}{b}\right) + 2(1-\nu)\left(1 - \frac{2x}{a}\right)^2\left(1 - \frac{2y}{b}\right)^2\right] dy\,dx$$

$$= \frac{Eh^3}{24(1-\nu^2)} \frac{2}{45ab}\left(3a^4 + 10a^2b^2 + 3b^4\right)$$

$$T = \int_0^a \int_0^b \frac{\rho h}{2} \left(\frac{\partial w}{\partial t}\right)^2 dy\,dx$$

$$T_{max} = \int_0^a \int_0^b \frac{\rho h}{2} \left(x - \frac{x^2}{a}\right)^2 \left(y - \frac{y^2}{b}\right)^2 \omega^2 \, dy\,dx$$

$$= \frac{\rho h}{2} \frac{b^3}{30} \frac{a^3}{30} \omega^2$$

equate $T_{max} = V_{max}$:

$$\omega^2 = \frac{Eh^2}{12\rho(1-\nu^2)} \frac{40}{a^4 b^4} \left(3a^4 + 10a^2b^2 + 3b^4\right)$$

if square $a = b$: $\omega = \frac{\sqrt{640}}{a^2} \sqrt{\frac{Eh^2}{12\rho(1-\nu^2)}}$

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using "exact" theory $\omega_{11} = \frac{19.7}{a^2} \sqrt{\frac{Eh^2}{12\rho(1-\nu^2)}}$