or

$$\frac{\overline{PQ}}{\overline{BD}} + \frac{\overline{PQ}}{\overline{AC}} = 1$$

Solving for $\overline{PQ}$, we obtain

$$\overline{PQ} = \frac{1}{\dfrac{1}{\overline{AC}} + \dfrac{1}{\overline{BD}}}$$

So if lengths $\overline{AC}$ and $\overline{BD}$ represent the spring constants k_1 and k_2, respectively, then length $\overline{PQ}$ represents the equivalent spring constant k_{eq}. That is,

$$\overline{PQ} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}} = k_{eq}$$

Problem A–3–8

In Figure 3–23, the simple pendulum shown consists of a sphere of mass m suspended by a string of negligible mass. Neglecting the elongation of the string, find a mathematical model of the pendulum. In addition, find the natural frequency of the system when θ is small. Assume no friction.

Solution The gravitational force mg has the tangential component $mg \sin \theta$ and the normal component $mg \cos \theta$. The torque due to the tangential component is $-mgl \sin \theta$, so the equation of motion is

$$J\ddot{\theta} = -mgl \sin \theta$$

where $J = ml^2$. Therefore,

$$ml^2\ddot{\theta} + mgl \sin \theta = 0$$

For small θ, $\sin \theta \doteqdot \theta$, and the equation of motion simplifies to

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

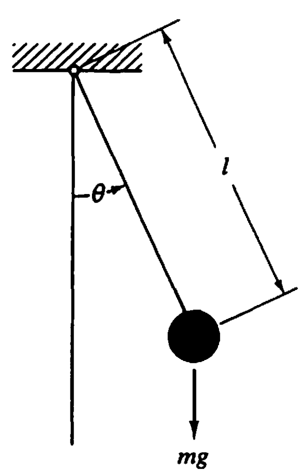


Figure 3–23 Simple pendulum.

This is a mathematical model of the system. The natural frequency is then obtained as

$$\omega_n = \sqrt{\frac{g}{l}}$$

Problem A-3-9

Consider the spring-loaded pendulum system shown in Figure 3-24. Assume that the spring force acting on the pendulum is zero when the pendulum is vertical ($\theta = 0$). Assume also that the friction involved is negligible and the angle of oscillation, θ, is small. Obtain a mathematical model of the system.

Solution Two torques are acting on this system, one due to the gravitational force and the other due to the spring force. Applying Newton's second law, we find that the equation of motion for the system becomes

$$J\ddot{\theta} = -mgl\sin\theta - 2(ka\sin\theta)(a\cos\theta)$$

where $J = ml^2$. Rewriting this last equation, we obtain

$$ml^2\ddot{\theta} + mgl\sin\theta + 2\,ka^2\sin\theta\cos\theta = 0$$

For small θ, we have $\sin\theta = \theta$ and $\cos\theta = 1$. So the equation of motion can be simplified to

$$ml^2\ddot{\theta} + (mgl + 2\,ka^2)\theta = 0$$

or

$$\ddot{\theta} + \left(\frac{g}{l} + 2\frac{ka^2}{ml^2}\right)\theta = 0$$

This is a mathematical model of the system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{g}{l} + 2\frac{ka^2}{ml^2}}$$

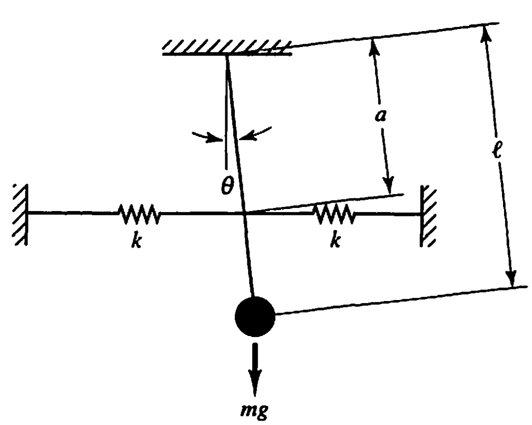


Figure 3-24 Spring-loaded pendulum system.

Problem A–3–10

Consider the rolling motion of the ship shown in Figure 3–25. The force due to buoyancy is $-w$ and that due to gravity is w. These two forces produce a couple that causes rolling motion of the ship. The point where the vertical line through the center of buoyancy, C, intersects the symmetrical line through the center of gravity, which is in the ship's centerline plane, is called the *metacenter* (point M). Define

R = distance of the metacenter to the center of gravity of the ship = $\overline{MG}$

J = moment of inertia of the ship about its longitudinal centroidal axis

Derive the equation of rolling motion of the ship when the rolling angle θ is small.

Solution From Figure 3–25, we obtain

$$J\ddot{\theta} = -wR\sin\theta$$

or

$$J\ddot{\theta} + wR\sin\theta = 0$$

For small θ, we have $\sin\theta \doteqdot \theta$. Hence, the equation of rolling motion of the ship is

$$J\ddot{\theta} + wR\theta = 0$$

The natural frequency of the rolling motion is $\sqrt{wR/J}$. Note that the distance $R(=\overline{MG})$ is considered positive when the couple of weight and buoyancy tends to rotate the ship toward the upright position. That is, R is positive if point M is above point G, and R is negative if point M is below point G.

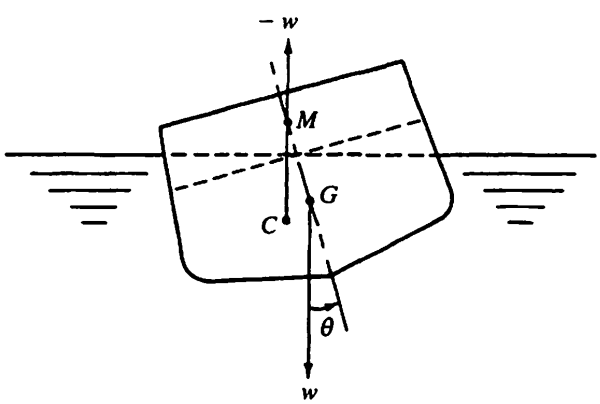


Figure 3–25 Rolling motion of a ship.

Problem A–3–11

In Figure 3–26, a homogeneous disk of radius R and mass m that can rotate about the center of mass of the disk is hung from the ceiling and is spring preloaded. (Two springs are connected by a wire that passes over a pulley as shown.) Each spring is prestretched by an amount x. Assuming that the disk is initially rotated by a small angle θ and then released, obtain both a mathematical model of the system and the natural frequency of the system.

Solution If the disk is rotated by an angle θ as shown in Figure 3–26, then the right spring is stretched by $x + R\theta$ and the left spring is stretched by $x - R\theta$. So, applying Newton's second law to the rotational motion of the disk gives

$$J\ddot{\theta} = -k(x + R\theta)R + k(x - R\theta)R$$

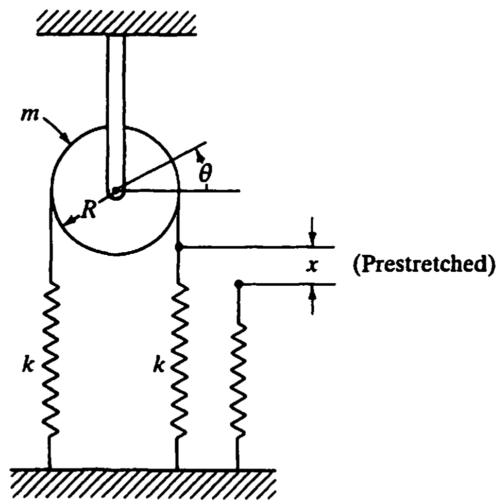


Figure 3–26 Spring–pulley system.

where the moment of inertia J is $\frac{1}{2}mR^2$. Simplifying the equation of motion, we have

$$\ddot{\theta} + \frac{4k}{m}\theta = 0$$

This is a mathematical model of the system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{4k}{m}}$$

Problem A–3–12

For the spring–mass–pulley system of Figure 3–27, the moment of inertia of the pulley about the axis of rotation is J and the radius is R. Assume that the system is initially at equilibrium. The gravitational force of mass m causes a static deflection of the spring such that $k\delta = mg$. Assuming that the displacement x of mass m is measured from the equilibrium position, obtain a mathematical model of the system. In addition, find the natural frequency of the system.

Solution Applying Newton's second law, we obtain, for mass m,

$$m\ddot{x} = -T \tag{3–20}$$

where T is the tension in the wire. (Note that since x is measured from the static equilibrium position the term mg does not enter into the equation.) For the rotational motion of the pulley,

$$J\ddot{\theta} = TR - kxR \tag{3–21}$$

If we eliminate the tension T from Equations (3–20) and (3–21), the result is

$$J\ddot{\theta} = -m\ddot{x}R - kxR \tag{3–22}$$

Noting that $x = R\theta$, we can simplify Equation (3–22) to

$$(J + mR^2)\ddot{\theta} + kR^2\theta = 0$$

or

$$\ddot{\theta} + \frac{kR^2}{J + mR^2}\theta = 0$$

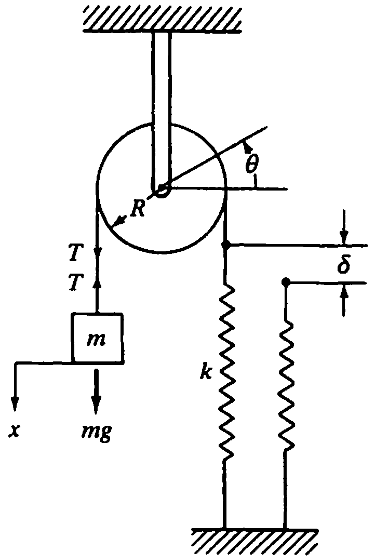


Figure 3–27
Spring-mass-pulley system.

This is a mathematical model of the system. The natural frequency is

$$\omega_n = \sqrt{\frac{kR^2}{J + mR^2}}$$

Problem A-3-13

In the mechanical system of Figure 3–28, one end of the lever is connected to a spring and a damper, and a force F is applied to the other end of the lever. Derive a mathematical model of the system. Assume that the displacement x is small and the lever is rigid and massless.

Solution From Newton's second law, for small displacement x, the rotational motion about pivot P is given by

$$Fl_1 - (b\dot{x} + kx)l_2 = 0$$

or

$$b\dot{x} + kx = \frac{l_1}{l_2}F$$

which is a mathematical model of the system.

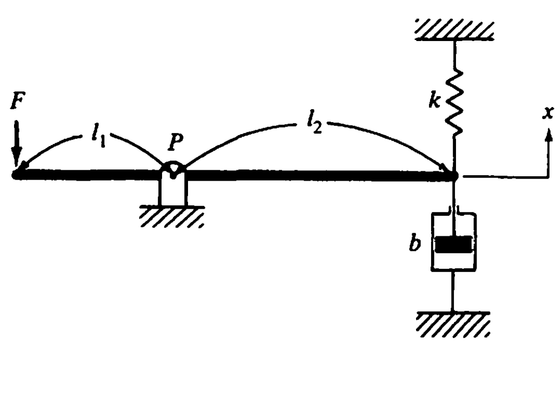


Figure 3–28 Lever system.

Problem A–3–14

Consider the mechanical system shown in Figure 3–29(a). The massless bar AA' is displaced 0.05 m by a constant force of 100 N. Suppose that the system is at rest before the force is abruptly released. The time-response curve when the force is abruptly released at $t = 0$ is shown in Figure 3–29(b). Determine the numerical values of b and k.

Solution Since the system is at rest before the force is abruptly released, the equation of motion is

$$kx = F \qquad t \leq 0$$

Note that the effect of the force F is to give the initial condition

$$x(0) = \frac{F}{k}$$

Since $x(0) = 0.05$ m, we have

$$k = \frac{F}{x(0)} = \frac{100}{0.05} = 2000 \text{ N/m}$$

At $t = 0$, F is abruptly released, so, for $t > 0$, the equation of motion becomes

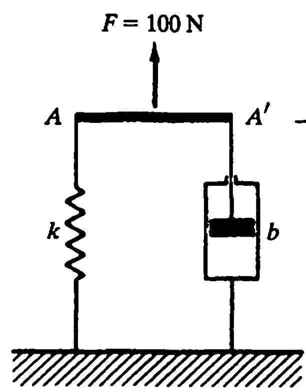
$$b\dot{x} + kx = 0 \qquad t > 0$$

Taking the Laplace transform of this last equation, we have

$$b[sX(s) - x(0)] + kX(s) = 0$$

Substituting $x(0) = 0.05$ and solving the resulting equation for $X(s)$, we get

$$X(s) = \frac{0.05}{s + \dfrac{k}{b}}$$



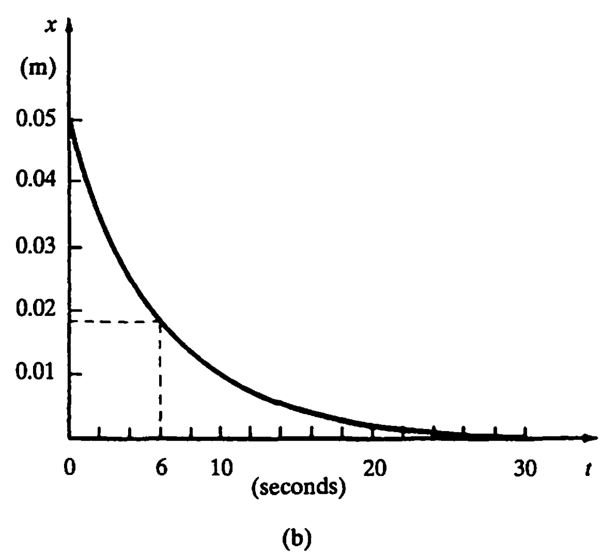


Figure 3–29 (a) Mechanical system; (b) response curve.

The inverse Laplace transform of $X(s)$, using the value of $k = 2000$ just obtained, is

$$x(t) = 0.05e^{-(2000/b)t}$$

Since the solution is an exponential function, at t = time constant $= b/2000$ the response becomes

$$x\left(\frac{b}{2000}\right) = 0.05 \times 0.368 = 0.0184 \text{ m}$$

From Figure 3–29(b), $x = 0.0184$ m occurs at $t = 6$ s. Hence,

$$\frac{b}{2000} = 6$$

from which it follows that

$$b = 12{,}000 \text{ N-s/m}$$

Problem A-3-15

In the rotating system shown in Figure 3–30, assume that the torque T applied to the rotor is of short duration, but large amplitude, so that it can be considered an impulse input. Assume also that initially the angular velocity is zero, or $\omega(0-) = 0$. Given the numerical values

$$J = 10 \text{ kg-m}^2$$

and

$$b = 2 \text{ N-s/m}$$

find the response $\omega(t)$ of the system. Assume that the amplitude of torque T is 300 N-m/s and that the duration of T is 0.1 s; that is, the magnitude of the impulse input is $300 \times 0.1 = 30$ N-m. Show that the effect of an impulse input on a first-order system that is at rest is to generate a nonzero initial condition at $t = 0+$.

Solution The equation of motion for the system is

$$J\dot{\omega} + b\omega = T, \qquad \omega(0-) = 0$$

Let us define the impulsive torque of magnitude 1 N-m as $\delta(t)$. Then, by substituting the given numerical values into this last equation, we obtain

$$10\dot{\omega} + 2\omega = 30\delta(t)$$

Taking the $\mathscr{L}_-$ transform of this last equation, we have

$$10[s\Omega(s) - \omega(0-)] + 2\Omega(s) = 30$$

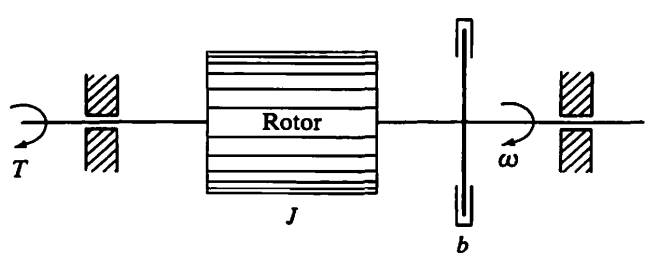


Figure 3–30 Mechanical rotating system.

or

$$\Omega(s) = \frac{30}{10s + 2} = \frac{3}{s + 0.2}$$

The inverse Laplace transform of $\Omega(s)$ is

$$\omega(t) = 3e^{-0.2t} \tag{3-23}$$

Note that $\omega(0+) = 3$ rad/s. The angular velocity of the rotor is thus changed instantaneously from $\omega(0-) = 0$ to $\omega(0+) = 3$ rad/s.

If the system is subjected only to the initial condition $\omega(0) = 3$ rad/s and there is no external torque ($T = 0$), then the equation of motion becomes

$$10\dot{\omega} + 2\omega = 0, \qquad \omega(0) = 3$$

Taking the Laplace transform of this last equation, we obtain

$$10[s\Omega(s) - \omega(0)] + 2\Omega(s) = 0$$

or

$$\Omega(s) = \frac{10\omega(0)}{10s + 2} = \frac{30}{10s + 2} = \frac{3}{s + 0.2}$$

The inverse Laplace transform of $\Omega(s)$ gives

$$\omega(t) = 3e^{-0.2t}$$

which is identical to Equation (3–23).

From the preceding analysis, we see that the response of a first-order system that is initially at rest to an impulse input is identical to the motion from the initial condition at $t = 0+$. That is, the effect of the impulse input on a first-order system that is initially at rest is to generate a nonzero initial condition at $t = 0+$.

Problem A–3–16

A mass $M = 8$ kg is supported by a spring with spring constant $k = 400$ N/m and a damper with $b = 40$ N-s/m, as shown in Figure 3–31. When a mass $m = 2$ kg is gently placed on the top of mass M, the system exhibits vibrations. Assuming that the displacement x of the masses is measured from the equilibrium position before mass m is placed on mass M, determine the response $x(t)$ of the system. Determine also the static deflection δ—the deflection of the spring when the transient response died out. Assume that $x(0) = 0$ and $\dot{x}(0) = 0$.

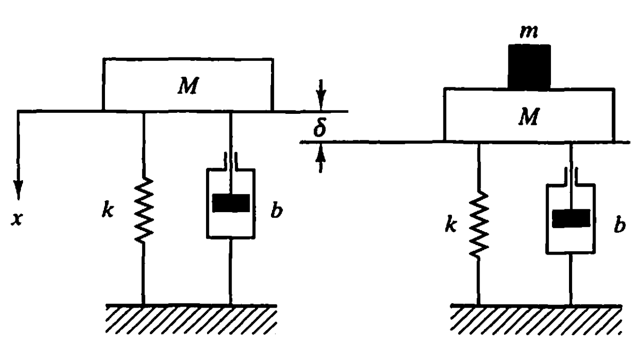


Figure 3–31 Mechanical system.

Notice that the numerical values of M, m, b, and k are given in the SI system of units. If the units are changed to BES units, how does the mathematical model change? How will the solution be changed?

Solution We shall first solve this problem using SI units. The input to the system is a constant force mg that acts as a step input to the system. The system is at rest before $t = 0$, and at $t = 0+$ the masses start to move up and down. A mathematical model, or equation of motion, is

$$(M + m)\ddot{x} + b\dot{x} + kx = mg$$

where $M + m = 10$ kg, $b = 40$ N-s/m, $k = 400$ N/m, and $g = 9.807$ m/s^2.

Substituting the numerical values into the equation of motion, we find that

$$10\ddot{x} + 40\dot{x} + 400x = 2 \times 9.807$$

or

$$\ddot{x} + 4\dot{x} + 40x = 1.9614 \tag{3–24}$$

Equation (3–24) is a mathematical model for the system when the units used are SI units. To obtain the response $x(t)$, we take the Laplace transform of Equation (3–24) and substitute the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ into the Laplace-transformed equation as follows:

$$s^2X(s) + 4sX(s) + 40X(s) = \frac{1.9614}{s}$$

Solving for $X(s)$ yields

$$\begin{aligned} X(s) &= \frac{1.9614}{(s^2 + 4s + 40)s} \\ &= \frac{1.9614}{40}\left(\frac{1}{s} - \frac{s + 4}{s^2 + 4s + 40}\right) \\ &= 0.04904\left[\frac{1}{s} - \frac{2}{6}\frac{6}{(s + 2)^2 + 6^2} - \frac{s + 2}{(s + 2)^2 + 6^2}\right] \end{aligned}$$

The inverse Laplace transform of this last equation gives

$$x(t) = 0.04904\left(1 - \frac{1}{3}e^{-2t}\sin 6t - e^{-2t}\cos 6t\right)\text{ m}$$

This solution gives the up-and-down motion of the total mass $(M + m)$. The static deflection δ is 0.04904 m.

Next, we shall solve the same problem using BES units. If we change the numerical values of M, m, b, and k given in the SI system of units to BES units, we obtain

$$\begin{aligned} M &= 8\text{ kg} = 0.54816\text{ slug} \\ m &= 2\text{ kg} = 0.13704\text{ slug} \\ b &= 40\text{ N-s/m} = 2.74063\text{ lb}_f\text{-s/ft} \\ k &= 400\text{ N/m} = 27.4063\text{ lb}_f\text{/ft} \\ mg &= 0.13704\text{ slug} \times 32.174\text{ ft/s}^2 = 4.4091\text{ slug-ft/s}^2 \\ &= 4.4091\text{ lb}_f \end{aligned}$$

Then the equation of motion for the system becomes

$$0.6852\ddot{x} + 2.74063\dot{x} + 27.4063x = 4.4091$$

which can be simplified to

$$\ddot{x} + 4\dot{x} + 40x = 6.4348 \tag{3–25}$$

Equation (3–25) is a mathematical model for the system. Comparing Equations (3–24) and (3–25), we notice that the left-hand sides of the equations are the same, which means that the characteristic equation remains the same. The solution of Equation (3–25) is

$$x(t) = 0.1609\left(1 - \frac{1}{3}e^{-2t}\sin 6t - e^{-2t}\cos 6t\right)\ \text{ft}$$

The static deflection δ is 0.1609 ft. (Note that 0.1609 ft = 0.04904 m.) Notice that, whenever consistent systems of units are used, the results carry the same information.

Problem A–3–17

Consider the spring-loaded inverted pendulum shown in Figure 3–32. Assume that the spring force acting on the pendulum is zero when the pendulum is vertical ($\theta = 0$). Assume also that the friction involved is negligible. Obtain a mathematical model of the system when the angle θ is small, that is, when $\sin\theta \doteqdot \theta$ and $\cos\theta \doteqdot 1$. Also, obtain the natural frequency ω_n of the system.

Solution Suppose that the inverted pendulum is given an initial angular displacement $\theta(0)$ and released with zero initial angular velocity. Then, from Figure 3–32, for small θ such that $\sin\theta \doteqdot \theta$ and $\cos\theta \doteqdot 1$, the left-hand side spring is stretched by $h\theta$ and the right-hand side spring is compressed by $h\theta$. Hence, the torque acting on the pendulum in the counterclockwise direction is $2kh^2\theta$. The torque due to the gravitational force is $mgl\theta$, which acts in the clockwise direction. The moment of inertia of the pendulum is ml^2. Thus, the equation of motion of the system for small θ is

$$ml^2\ddot{\theta} = mgl\theta - 2kh^2\theta$$

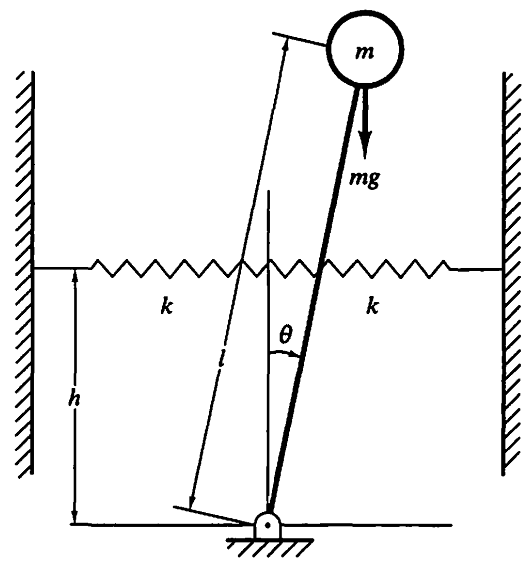


Figure 3–32 Spring-loaded inverted pendulum.

or

$$\ddot{\theta} + \left(\frac{2kh^2}{ml^2} - \frac{g}{l}\right)\theta = 0$$

This is a mathematical model of the system for small θ. If $2kh^2 > mgl$, the torques acting in the system cause it to vibrate. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{2kh^2}{ml^2} - \frac{g}{l}}$$

If, however, $2kh^2 < mgl$, then, starting with a small disturbance, the angle θ increases and the pendulum will fall down or hit the vertical wall and stop. The vibration will not occur.

Problem A-3-18

Consider the spring-mass-pulley system of Figure 3-33(a). If the mass m is pulled downward a short distance and released, it will vibrate. Obtain the natural frequency of the system by applying the law of conservation of energy.

Solution Define x, y, and θ as the displacement of mass m, the displacement of the pulley, and the angle of rotation of the pulley, measured respectively from their corresponding equilibrium positions. Note that $x = 2y$, $R\theta = x - y = y$, and $J = \frac{1}{2}MR^2$.

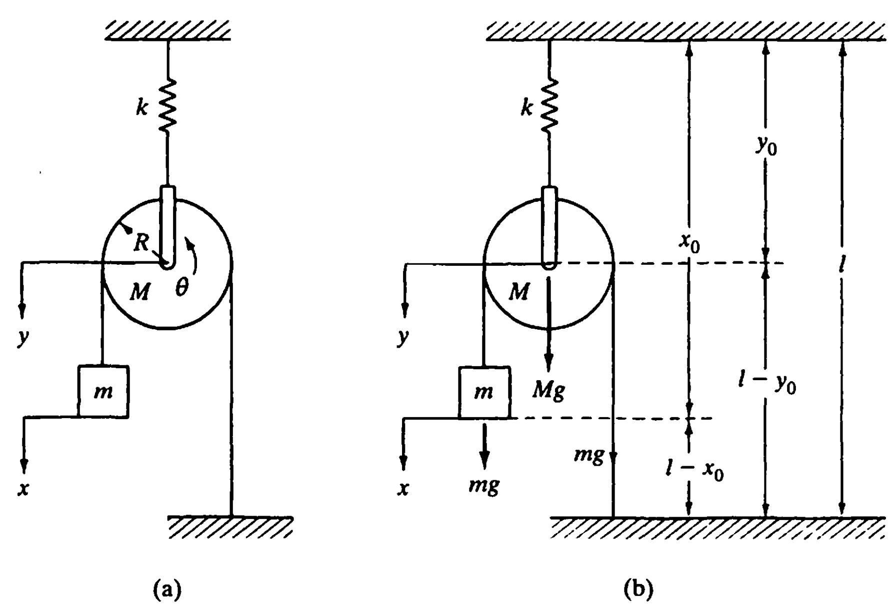


Figure 3-33 (a) Spring-mass-pulley system; (b) diagram for figuring out potential energy of the system.

The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{8}M\dot{x}^2 + \frac{1}{4}MR^2\left(\frac{\dot{y}}{R}\right)^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{3}{16}M\dot{x}^2$$

The potential energy U of the system can be obtained from Figure 3–33(b). At the equilibrium state, the potential energy is

$$U_0 = \frac{1}{2}ky_\delta^2 + Mg(l - y_0) + mg(l - x_0)$$

where y_δ is the static deflection of the spring due to the hanging masses M and m. When masses m and M are displaced by x and y, respectively, the instantaneous potential energy can be obtained as

$$U = \frac{1}{2}k(y_\delta + y)^2 + Mg(l - y_0 - y) + mg(l - x_0 - x)$$
$$= \frac{1}{2}ky_\delta^2 + ky_\delta y + \frac{1}{2}ky^2 + Mg(l - y_0) - Mgy + mg(l - x_0) - mgx$$
$$= U_0 + \frac{1}{2}ky^2 + ky_\delta y - Mgy - mgx$$

Again from Figure 3–33(b), the spring force ky_δ must balance with $Mg + 2mg$, or

$$ky_\delta = Mg + 2mg$$

Therefore,

$$ky_\delta y = Mgy + 2mgy = Mgy + mgx$$

and

$$U = U_0 + \frac{1}{2}ky^2 = U_0 + \frac{1}{8}kx^2$$

where U_0 is the potential energy at the equilibrium state.

Applying the law of conservation of energy to this conservative system gives

$$T + U = \frac{1}{2}m\dot{x}^2 + \frac{3}{16}M\dot{x}^2 + U_0 + \frac{1}{8}kx^2 = \text{constant}$$

and differentiating this last equation with respect to t yields

$$m\dot{x}\ddot{x} + \frac{3}{8}M\dot{x}\ddot{x} + \frac{1}{4}kx\dot{x} = 0$$

or

$$\left[\left(m + \frac{3}{8}M\right)\ddot{x} + \frac{1}{4}kx\right]\dot{x} = 0$$

Since $\dot{x}$ is not always zero, we must have

$$\left(m + \frac{3}{8}M\right)\ddot{x} + \frac{1}{4}kx = 0$$

or

$$\ddot{x} + \frac{2k}{8m + 3M}x = 0$$

The natural frequency of the system, therefore, is

$$\omega_n = \sqrt{\frac{2k}{8m + 3M}}$$

Problem A-3-19

If, for the spring–mass system of Figure 3–34, the mass m_s of the spring is small, but not negligibly small, compared with the suspended mass m, show that the inertia of the spring can be allowed for by adding one-third of its mass m_s to the suspended mass m and then treating the spring as a massless spring.

Solution Consider the free vibration of the system. The displacement x of the mass is measured from the static equilibrium position. In free vibration, the displacement can be written as

$$x = A\cos\omega t$$

Since the mass of the spring is comparatively small, we can assume that the spring is stretched uniformly. Then the displacement of a point in the spring at a distance ξ from the top is given by $(\xi/l)A\cos\omega t$.

In the mean position, where $x = 0$ and the velocity of mass m is maximum, the velocity of the suspended mass is $A\omega$ and that of the spring at the distance ξ from the

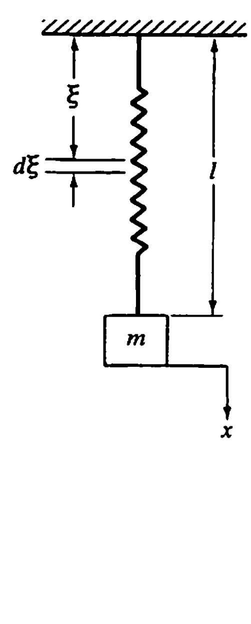


Figure 3–34 Spring–mass system.

top is $(\xi/l)A\omega$. The maximum kinetic energy is

$$T_{\max} = \frac{1}{2}m(A\omega)^2 + \int_0^l \frac{1}{2}\left(\frac{m_s}{l}\right)\left(\frac{\xi}{l}A\omega\right)^2 d\xi$$
$$= \frac{1}{2}mA^2\omega^2 + \frac{1}{2}\left(\frac{m_s}{l}\right)\left(\frac{A^2\omega^2}{l^2}\right)\frac{1}{3}l^3$$
$$= \frac{1}{2}\left(m + \frac{m_s}{3}\right)A^2\omega^2$$

Note that the mass of the spring does not affect the change in the potential energy of the system and that, if the spring were massless, the maximum kinetic energy would have been $\frac{1}{2}mA^2\omega^2$. Therefore, we conclude that the inertia of the spring can be allowed for simply by adding one-third of mass m_s to the suspended mass m and then treating the spring as a massless spring, provided that m_s is small compared with m.

PROBLEMS

Problem B–3–1

A homogeneous disk has a diameter of 1 m and mass of 100 kg. Obtain the moment of inertia of the disk about the axis perpendicular to the disk and passing through its center.

Problem B–3–2

Figure 3–35 shows an experimental setup for measuring the moment of inertia of a rotating body. Suppose that the moment of inertia of a rotating body about axis AA' is known. Describe a method to determine the moment of inertia of any rotating body, using this experimental setup.

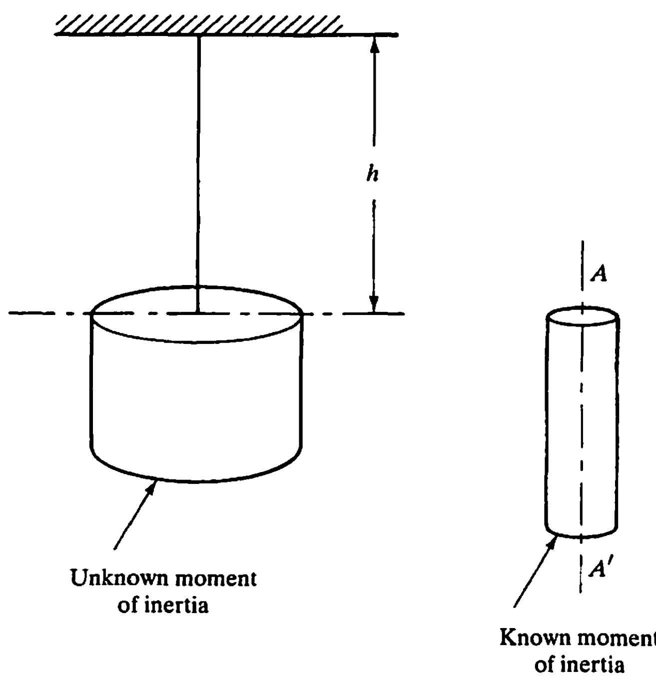


Figure 3–35 Experimental setup for measuring the moment of inertia of a rotating body.