EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–2–1

Obtain the real and imaginary parts of

$$\frac{2 + j1}{3 + j4}$$

Also, obtain the magnitude and angle of this complex quantity.

Solution

$$\frac{2 + j1}{3 + j4} = \frac{(2 + j1)(3 - j4)}{(3 + j4)(3 - j4)} = \frac{6 + j3 - j8 + 4}{9 + 16} = \frac{10 - j5}{25}$$
$$= \frac{2}{5} - j\frac{1}{5}$$

Hence,

$$\text{real part} = \frac{2}{5}, \qquad \text{imaginary part} = -j\frac{1}{5}$$

The magnitude and angle of this complex quantity are obtained as follows:

$$\text{magnitude} = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-1}{5}\right)^2} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}} = 0.447$$
$$\text{angle} = \tan^{-1}\frac{-1/5}{2/5} = \tan^{-1}\frac{-1}{2} = -26.565°$$

Problem A–2–2

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 \qquad && t < 0 \\ &= te^{-3t} && t \geq 0 \end{aligned}$$

Solution Since

$$\mathscr{L}[t] = G(s) = \frac{1}{s^2}$$

referring to Equation (2–2), we obtain

$$F(s) = \mathscr{L}[te^{-3t}] = G(s + 3) = \frac{1}{(s + 3)^2}$$

Problem A–2–3

What is the Laplace transform of

$$\begin{aligned} f(t) &= 0 \qquad && t < 0 \\ &= \sin(\omega t + \theta) && t \geq 0 \end{aligned}$$

where θ is a constant?

Solution Noting that

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta$$

we have

$$\begin{aligned}\mathscr{L}[\sin(\omega t + \theta)] &= \cos\theta\,\mathscr{L}[\sin \omega t] + \sin\theta\,\mathscr{L}[\cos \omega t] \\ &= \cos\theta\frac{\omega}{s^2 + \omega^2} + \sin\theta\frac{s}{s^2 + \omega^2} \\ &= \frac{\omega\cos\theta + s\sin\theta}{s^2 + \omega^2}\end{aligned}$$

Problem A-2-4

Find the Laplace transform $F(s)$ of the function $f(t)$ shown in Figure 2-9. Also, find the limiting value of $F(s)$ as a approaches zero.

Solution The function $f(t)$ can be written

$$f(t) = \frac{1}{a^2}1(t) - \frac{2}{a^2}1(t - a) + \frac{1}{a^2}1(t - 2a)$$

Then

$$\begin{aligned}F(s) &= \mathscr{L}[f(t)] \\ &= \frac{1}{a^2}\mathscr{L}[1(t)] - \frac{2}{a^2}\mathscr{L}[1(t - a)] + \frac{1}{a^2}\mathscr{L}[1(t - 2a)] \\ &= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as} \\ &= \frac{1}{a^2 s}(1 - 2e^{-as} + e^{-2as})\end{aligned}$$

As a approaches zero, we have

$$\begin{aligned}\lim_{a\to 0}F(s) &= \lim_{a\to 0}\frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} = \lim_{a\to 0}\frac{\dfrac{d}{da}(1 - 2e^{-as} + e^{-2as})}{\dfrac{d}{da}(a^2 s)} \\ &= \lim_{a\to 0}\frac{2se^{-as} - 2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as} - e^{-2as}}{a}\end{aligned}$$

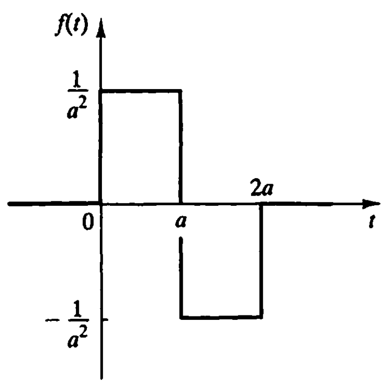

Figure 2-9 Function $f(t)$.

$$= \lim_{a\to 0} \frac{\dfrac{d}{da}(e^{-as} - e^{-2as})}{\dfrac{d}{da}(a)} = \lim_{a\to 0} \frac{-se^{-as} + 2se^{-2as}}{1}$$

$$= -s + 2s = s$$

Problem A–2–5

Obtain the Laplace transform of the function $f(t)$ shown in Figure 2–10.

Solution The given function $f(t)$ can be defined as follows:

$$\begin{aligned} f(t) &= 0 && t \le 0 \\ &= \frac{b}{a}t && 0 < t \le a \\ &= 0 && a < t \end{aligned}$$

Notice that $f(t)$ can be considered a sum of the three functions $f_1(t), f_2(t)$, and $f_3(t)$ shown in Figure 2–11. Hence, $f(t)$ can be written as

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) \\ &= \frac{b}{a}t \cdot 1(t) - \frac{b}{a}(t-a) \cdot 1(t-a) - b \cdot 1(t-a) \end{aligned}$$

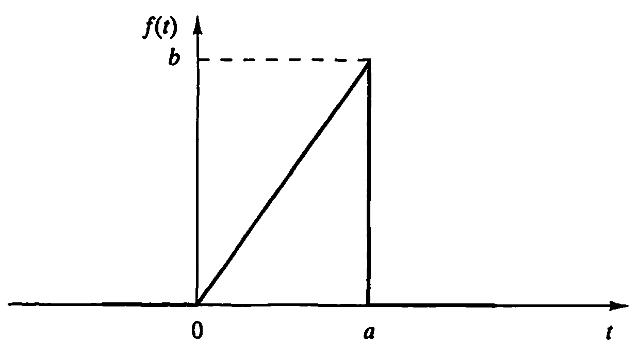

Figure 2–10 Function $f(t)$.

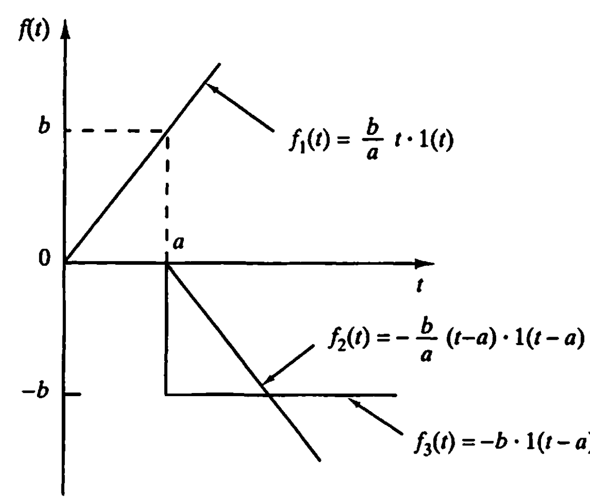

Figure 2–11 Functions $f_1(t), f_2(t)$, and $f_3(t)$.

Then the Laplace transform of $f(t)$ becomes

$$\begin{aligned} F(s) &= \frac{b}{a}\frac{1}{s^2} - \frac{b}{a}\frac{1}{s^2}e^{-as} - b\frac{1}{s}e^{-as} \\ &= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as} \end{aligned}$$

The same $F(s)$ can, of course, be obtained by performing the following Laplace integration:

$$\begin{aligned} \mathscr{L}[f(t)] &= \int_0^a \frac{b}{a}te^{-st}\,dt + \int_a^\infty 0\,e^{-st}\,dt \\ &= \frac{b}{a}t\frac{e^{-st}}{-s}\bigg|_0^a - \int_0^a \frac{b}{a}\frac{e^{-st}}{-s}\,dt \\ &= b\frac{e^{-as}}{-s} + \frac{b}{as}\frac{e^{-st}}{-s}\bigg|_0^a \\ &= b\frac{e^{-as}}{-s} - \frac{b}{as^2}(e^{-as} - 1) \\ &= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as} \end{aligned}$$

Problem A-2-6

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then, except at poles of $F(s)$,

$$\begin{aligned} \mathscr{L}[tf(t)] &= -\frac{d}{ds}F(s) \\ \mathscr{L}[t^2f(t)] &= \frac{d^2}{ds^2}F(s) \end{aligned}$$

and in general,

$$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Solution

$$\begin{aligned} \mathscr{L}[tf(t)] &= \int_0^\infty tf(t)e^{-st}\,dt = -\int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt \\ &= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = -\frac{d}{ds}F(s) \end{aligned}$$

Similarly, by defining $tf(t) = g(t)$, the result is

$$\begin{aligned} \mathscr{L}[t^2f(t)] &= \mathscr{L}[tg(t)] = -\frac{d}{ds}G(s) = -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right] \\ &= (-1)^2\frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}F(s) \end{aligned}$$

Repeating the same process, we obtain

$$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Problem A–2–7

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= t^2 \sin \omega t && t \geq 0 \end{aligned}$$

Solution Since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

referring to **Problem A–2–6**, we have

$$\mathscr{L}[f(t)] = \mathscr{L}[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2 + \omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2 + \omega^2)^3}$$

Problem A–2–8

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as) \qquad a > 0$$

Solution If we define $t/a = \tau$ and $as = s_1$, then

$$\begin{aligned} \mathscr{L}\left[f\left(\frac{t}{a}\right)\right] &= \int_0^\infty f\left(\frac{t}{a}\right)e^{-st}\,dt = \int_0^\infty f(\tau)e^{-as\tau}a\,d\tau \\ &= a\int_0^\infty f(\tau)e^{-s_1\tau}\,d\tau = aF(s_1) = aF(as) \end{aligned}$$

Problem A–2–9

Prove that if $f(t)$ is of exponential order and if $\int_0^\infty f(t)\,dt$ exists [which means that $\int_0^\infty f(t)\,dt$ assumes a definite value], then

$$\int_0^\infty f(t)\,dt = \lim_{s \to 0} F(s)$$

where $F(s) = \mathscr{L}[f(t)]$.

Solution Note that

$$\int_0^\infty f(t)\,dt = \lim_{t \to \infty}\int_0^t f(t)\,dt$$

Referring to Equation (2–5), we have

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Since $\int_0^\infty f(t)\,dt$ exists, by applying the final-value theorem to this case, we obtain

$$\lim_{t \to \infty}\int_0^t f(t)\,dt = \lim_{s \to 0} s\frac{F(s)}{s}$$

or

$$\int_0^\infty f(t)\,dt = \lim_{s \to 0} F(s)$$

Problem A–2–10

The convolution of two time functions is defined by

$$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$$

A commonly used notation for the convolution is $f_1(t)*f_2(t)$, which is defined as

$$f_1(t)*f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$$

Show that if $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then

$$\mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] = F_1(s)F_2(s)$$

where $F_1(s) = \mathscr{L}[f_1(t)]$ and $F_2(s) = \mathscr{L}[f_2(t)]$.

Solution Noting that $1(t-\tau) = 0$ for $t < \tau$, we have

$$\begin{aligned}\mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] &= \mathscr{L}\left[\int_0^\infty f_1(\tau)f_2(t-\tau)1(t-\tau)\,d\tau\right]\\ &= \int_0^\infty e^{-st}\left[\int_0^\infty f_1(\tau)f_2(t-\tau)1(t-\tau)\,d\tau\right]dt\\ &= \int_0^\infty f_1(\tau)\,d\tau\int_0^\infty f_2(t-\tau)1(t-\tau)e^{-st}\,dt\end{aligned}$$

Changing the order of integration is valid here, since $f_1(t)$ and $f_2(t)$ are both Laplace transformable, giving convergent integrals. If we substitute $\lambda = t - \tau$ into this last equation, the result is

$$\begin{aligned}\mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] &= \int_0^\infty f_1(\tau)e^{-s\tau}\,d\tau\int_0^\infty f_2(\lambda)e^{-s\lambda}\,d\lambda\\ &= F_1(s)F_2(s)\end{aligned}$$

or

$$\mathscr{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s)$$

Thus, the Laplace transform of the convolution of two time functions is the product of their Laplace transforms.

Problem A–2–11

Determine the Laplace transform of $f_1(t)*f_2(t)$, where

$$\begin{aligned}f_1(t) &= f_2(t) = 0 && \text{for } t < 0\\ f_1(t) &= t && \text{for } t \geq 0\\ f_2(t) &= 1 - e^{-t} && \text{for } t \geq 0\end{aligned}$$

Solution Note that

$$\mathscr{L}[t] = F_1(s) = \frac{1}{s^2}$$

$$\mathscr{L}[1 - e^{-t}] = F_2(s) = \frac{1}{s} - \frac{1}{s+1}$$

The Laplace transform of the convolution integral is given by

$$\begin{aligned}\mathscr{L}[f_1(t)*f_2(t)] &= F_1(s)F_2(s) = \frac{1}{s^2}\left(\frac{1}{s} - \frac{1}{s+1}\right)\\ &= \frac{1}{s^3} - \frac{1}{s^2(s+1)} = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}\end{aligned}$$

To verify that the expression after the rightmost equal sign is indeed the Laplace transform of the convolution integral, let us first integrate the convolution integral and then take the Laplace transform of the result. We have

$$\begin{aligned}f_1(t)*f_2(t) &= \int_0^t \tau[1 - e^{-(t-\tau)}]\,d\tau\\ &= \int_0^t (t-\tau)(1 - e^{-\tau})\,d\tau\\ &= \int_0^t (t - \tau - te^{-\tau} + \tau e^{-\tau})\,d\tau\end{aligned}$$

Noting that

$$\int_0^t (t-\tau)\,d\tau = \frac{t^2}{2}$$

$$\int_0^t te^{-\tau}\,d\tau = -te^{-t} + t$$

$$\int_0^t \tau e^{-\tau}\,d\tau = -\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau}\,d\tau = -te^{-t} - e^{-t} + 1$$

we have

$$f_1(t)*f_2(t) = \frac{t^2}{2} - t + 1 - e^{-t}$$

Thus,

$$\mathscr{L}\left[\frac{t^2}{2} - t + 1 - e^{-t}\right] = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

Problem A–2–12

Prove that if $f(t)$ is a periodic function with period T, then

$$\mathscr{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}}$$

Solution

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt$$

By changing the independent variable from t to $\tau = t - nT$, we obtain

$$\mathscr{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\int_0^T f(\tau + nT)e^{-s\tau}\,d\tau$$

Since $f(t)$ is a periodic function with period $T, f(\tau + nT) = f(\tau)$. Hence,

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs} \int_0^T f(\tau)e^{-s\tau}\, d\tau$$

Noting that

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-nTs} &= 1 + e^{-Ts} + e^{-2Ts} + \cdots \\ &= 1 + e^{-Ts}(1 + e^{-Ts} + e^{-2Ts} + \cdots) \\ &= 1 + e^{-Ts}\left(\sum_{n=0}^{\infty} e^{-nTs}\right)\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}}$$

It follows that

$$\mathcal{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\, dt}{1 - e^{-Ts}}$$

Problem A-2-13

What is the Laplace transform of the periodic function shown in Figure 2-12?

Solution Note that

$$\begin{aligned}\int_0^T f(t)e^{-st}\, dt &= \int_0^{T/2} e^{-st}\, dt + \int_{T/2}^T (-1)e^{-st}\, dt \\ &= \frac{e^{-st}}{-s}\bigg|_0^{T/2} - \frac{e^{-st}}{-s}\bigg|_{T/2}^T \\ &= \frac{e^{-(1/2)Ts} - 1}{-s} + \frac{e^{-Ts} - e^{-(1/2)Ts}}{s} \\ &= \frac{1}{s}[e^{-Ts} - 2e^{-(1/2)Ts} + 1] \\ &= \frac{1}{s}[1 - e^{-(1/2)Ts}]^2\end{aligned}$$

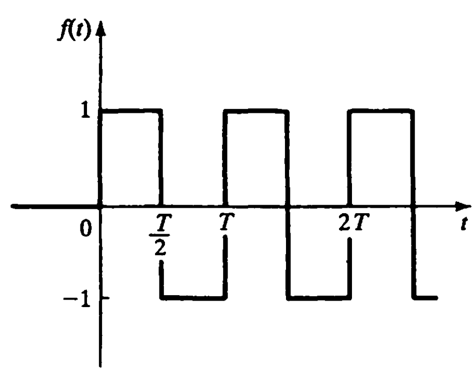

Figure 2-12 Periodic function (square wave).

Consequently,

$$F(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}} = \frac{(1/s)[1 - e^{-(1/2)Ts}]^2}{1 - e^{-Ts}}$$
$$= \frac{1 - e^{-(1/2)Ts}}{s[1 + e^{-(1/2)Ts}]} = \frac{1}{s}\tanh\frac{Ts}{4}$$

Problem A–2–14

Find the initial value of $df(t)/dt$, where the Laplace transform of $f(t)$ is given by

$$F(s) = \mathscr{L}[f(t)] = \frac{2s + 1}{s^2 + s + 1}$$

Solution Using the initial-value theorem, we obtain

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s(2s + 1)}{s^2 + s + 1} = 2$$

Since the $\mathscr{L}_+$ transform of $df(t)/dt = g(t)$ is given by

$$\mathscr{L}_+[g(t)] = sF(s) - f(0+)$$
$$= \frac{s(2s + 1)}{s^2 + s + 1} - 2 = \frac{-s - 2}{s^2 + s + 1}$$

the initial value of $df(t)/dt$ is obtained as

$$\lim_{t\to 0+}\frac{df(t)}{dt} = g(0+) = \lim_{s\to\infty} s[sF(s) - f(0+)]$$
$$= \lim_{s\to\infty}\frac{-s^2 - 2s}{s^2 + s + 1} = -1$$

To verify this result, notice that

$$F(s) = \frac{2(s + 0.5)}{(s + 0.5)^2 + (0.866)^2} = \mathscr{L}[2e^{-0.5t}\cos 0.866t]$$

Hence,

$$f(t) = 2e^{-0.5t}\cos 0.866t$$

and

$$\dot{f}(t) = -e^{-0.5t}\cos 0.866t + 2e^{-0.5t}0.866\sin 0.866t$$

Thus,

$$\dot{f}(0) = -1 + 0 = -1$$

Problem A–2–15

Obtain the inverse Laplace transform of

$$F(s) = \frac{cs + d}{(s^2 + 2as + a^2) + b^2}$$

where a, b, c, and d are real and a is positive.

Solution Since $F(s)$ can be written as

$$F(s) = \frac{c(s+a) + d - ca}{(s+a)^2 + b^2}$$
$$= \frac{c(s+a)}{(s+a)^2 + b^2} + \frac{d - ca}{b} \frac{b}{(s+a)^2 + b^2}$$

we obtain

$$f(t) = ce^{-at} \cos bt + \frac{d - ca}{b} e^{-at} \sin bt$$

Problem A–2–16

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Solution Since

$$s^2 + 2s + 2 = (s + 1 + j1)(s + 1 - j1)$$

it follows that $F(s)$ involves a pair of complex-conjugate poles, so we expand $F(s)$ into the form

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2s + 2}$$

where a_1, a_2, and a_3 are determined from

$$1 = a_1(s^2 + 2s + 2) + (a_2 s + a_3)s$$

By comparing corresponding coefficients of the s^2, s, and s^0 terms on both sides of this last equation respectively, we obtain

$$a_1 + a_2 = 0, \qquad 2a_1 + a_3 = 0, \qquad 2a_1 = 1$$

from which it follows that

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1$$

Therefore,

$$F(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{s^2 + 2s + 2}$$
$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+1)^2 + 1^2} - \frac{1}{2}\frac{s+1}{(s+1)^2 + 1^2}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t} \sin t - \frac{1}{2}e^{-t} \cos t \qquad t \geq 0$$

Problem A-2-17

Derive the inverse Laplace transform of

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}$$

Solution

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)} = \frac{b_2}{s^2} + \frac{b_1}{s} + \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

where

$$a_1 = \left.\frac{5(s+2)}{s^2(s+3)}\right|_{s=-1} = \frac{5}{2}$$

$$a_2 = \left.\frac{5(s+2)}{s^2(s+1)}\right|_{s=-3} = \frac{5}{18}$$

$$b_2 = \left.\frac{5(s+2)}{(s+1)(s+3)}\right|_{s=0} = \frac{10}{3}$$

$$b_1 = \frac{d}{ds}\left[\frac{5(s+2)}{(s+1)(s+3)}\right]_{s=0}$$

$$= \left.\frac{5(s+1)(s+3) - 5(s+2)(2s+4)}{(s+1)^2(s+3)^2}\right|_{s=0} = -\frac{25}{9}$$

Thus,

$$F(s) = \frac{10}{3}\frac{1}{s^2} - \frac{25}{9}\frac{1}{s} + \frac{5}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s+3}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{10}{3}t - \frac{25}{9} + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t} \qquad t \geq 0$$

Problem A-2-18

Find the inverse Laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 5}{s(s+1)}$$

Solution Since the numerator polynomial is of higher degree than the denominator polynomial, by dividing the numerator by the denominator until the remainder is a fraction, we obtain

$$F(s) = s^2 + s + 2 + \frac{2s+5}{s(s+1)} = s^2 + s + 2 + \frac{a_1}{s} + \frac{a_2}{s+1}$$

where

$$a_1 = \left.\frac{2s+5}{s+1}\right|_{s=0} = 5$$

$$a_2 = \left.\frac{2s+5}{s}\right|_{s=-1} = -3$$

It follows that

$$F(s) = s^2 + s + 2 + \frac{5}{s} - \frac{3}{s+1}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{d^2}{dt^2}\delta(t) + \frac{d}{dt}\delta(t) + 2\delta(t) + 5 - 3e^{-t} \qquad t \geq 0-$$

Problem A–2–19

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 4s + 6}{s^2(s^2 + 2s + 10)} \tag{2–10}$$

Solution Since the quadratic term in the denominator involves a pair of complex-conjugate roots, we expand $F(s)$ into the following partial-fraction form:

$$F(s) = \frac{a_1}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10}$$

The coefficient a_1 can be obtained as

$$a_1 = \left.\frac{2s^2 + 4s + 6}{s^2 + 2s + 10}\right|_{s=0} = 0.6$$

Hence, we obtain

$$\begin{aligned} F(s) &= \frac{0.6}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10} \\ &= \frac{(a_2 + b)s^3 + (0.6 + 2a_2 + c)s^2 + (1.2 + 10a_2)s + 6}{s^2(s^2 + 2s + 10)} \end{aligned} \tag{2–11}$$

By equating corresponding coefficients in the numerators of Equations (2–10) and (2–11), respectively, we obtain

$$\begin{aligned} a_2 + b &= 0 \\ 0.6 + 2a_2 + c &= 2 \\ 1.2 + 10a_2 &= 4 \end{aligned}$$

from which we get

$$a_2 = 0.28, \qquad b = -0.28, \qquad c = 0.84$$

Hence,

$$\begin{aligned} F(s) &= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28s + 0.84}{s^2 + 2s + 10} \\ &= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28(s + 1) + (1.12/3) \times 3}{(s + 1)^2 + 3^2} \end{aligned}$$

The inverse Laplace transform of $F(s)$ gives

$$f(t) = 0.6t + 0.28 - 0.28e^{-t}\cos 3t + \frac{1.12}{3}e^{-t}\sin 3t$$

Problem A-2-20

Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + \omega^2)}$$

Solution

$$\begin{aligned} F(s) &= \frac{1}{s(s^2 + \omega^2)} = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega^2}\right) \\ &= \frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

Thus, the inverse Laplace transform of $F(s)$ is obtained as

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{\omega^2}(1 - \cos \omega t) \qquad t \geq 0$$

Problem A-2-21

Obtain the solution of the differential equation

$$\dot{x} + ax = A \sin \omega t, \qquad x(0) = b$$

Solution Laplace transforming both sides of this differential equation, we have

$$[sX(s) - x(0)] + aX(s) = A\frac{\omega}{s^2 + \omega^2}$$

or

$$(s + a)X(s) = \frac{A\omega}{s^2 + \omega^2} + b$$

Solving this last equation for $X(s)$, we obtain

$$\begin{aligned} X(s) &= \frac{A\omega}{(s + a)(s^2 + \omega^2)} + \frac{b}{s + a} \\ &= \frac{A\omega}{a^2 + \omega^2}\left(\frac{1}{s + a} - \frac{s - a}{s^2 + \omega^2}\right) + \frac{b}{s + a} \\ &= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)\frac{1}{s + a} + \frac{Aa}{a^2 + \omega^2}\frac{\omega}{s^2 + \omega^2} - \frac{A\omega}{a^2 + \omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

The inverse Laplace transform of $X(s)$ then gives

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] \\ &= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)e^{-at} + \frac{Aa}{a^2 + \omega^2}\sin \omega t - \frac{A\omega}{a^2 + \omega^2}\cos \omega t \qquad t \geq 0 \end{aligned}$$