FDCL UAV Flight Software

Contents

1 Reference Frames

There are 8 reference frames formulated. Understanding the definition of each frame, and converting between these frames precisely are critical for successful flight experiments.

1.1 Frames Fixed to UAV

UAV body-fixed frame: b This corresponds to the body-fixed frame defined in [2]. The center of rotor is numbered counterclockwise when observed from the top. The origin of the b-frame is located at the mass center of the UAV, and the first axis points toward the first rotor, and the second axis points toward the second rotor. Thus, the third axis points downward.

IMU-fixed frame: i This corresponds to the frame marked on the surface of the IMU.

VICON object markers frame: m When creating an object from the VICON tracker software, a reference frame fixed to the marker is defined. The orientation of this frame is identical to the above v-frame at the time of object creation. Therefore, it is common that the object is aligned in a specific desired way with respect to the v-frame.

ZED fixed frame: z This is a frame fixed to the ZED camera. See the documents at Stereolabs website.[1] The ZED fixed frame is referred to as the camera frame in the above document, and it is right-handed with the positive Y-axis pointing down, X-axis pointing right when viewed from behind, and Z-axis pointing away from the camera.

1.2 Framed Fixed to Ground

Local NED frame: n This frame is common in flight dynamics. Its origin is located at the base station. Three axes points toward the due North, the due East, and downward, respectively.

VICON reference frame: v This corresponds to the *world* frame of the VICON system. It is determined when calibrating the VICON sensors with the marker wand. At SEH2200, there is a black electrical tape marked on the floor. The first axis points toward the right when sitting at the base station, or the 22nd street; the second axis points front when sitting at the base station, or toward the I street; the third axis points upward.

ZED world frame: w This is the global reference frame for the ZED. If no initial parameters are set, the world Frame is located at the place where the ZED first started motion tracking, and is oriented in the direction where the device was looking.

FDCL SEH2200 frame: f This frame is fixed to SEH2200, and its origin is identical to the origin of the v-frame. The first axis points front when sitting at the base station, or toward the I street; the second axis points toward the right when sitting at the base station, or the 22nd street; the third axis points downward.

1.3 Conversion between Frames

Conversion among UAV-fixed frames Conversion among the three UAV-fixed frames do not change over time, as all of them are fixed to the UAV body.

[1] https://docs.stereolabs.com/overview/positional-tracking/coordinate-frames/

It will be convenient if the IMU is aligned to the UAV body so that $R_{bi} \in \mathsf{SO}(3)$, i.e., the transform from the i-frame to the b-frame is identical, or a $\frac{\pi}{2}n$ rotation about a b-frame axis. However, in practice, it is challenging to align the IMU in such way. However, the IMU orientation can be precisely determined when designing the layout of the PCB board. As such, the rotation matrix $R_{bi} \in \mathsf{SO}(3)$ will be visually determined.

As discussed above, the orientation of the marker-fixed m-frame relative to the body-fixed b-frame can be determined when creating the VICON object upon the error in aligning the UAV to the black tapes on the floor.

As both of the IMU and the VICON measures attitudes, the rotation matrix $R_{mi} \in \mathsf{SO}(3)$ from the i-frame to the m-frame can be calibrated as described in Section 2.

Conversion among ground-fixed frames Conversion among the three ground-fixed frames do not change over time, as all of them are fixed to the ground.

The conversion between the v-frame and the f-frame is trivial, as the f-frame is constructed by flipping the v-frame. More explicitly,

$$R_{fv} = [e_2, e_1, -e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_{vf} = R_{fv}^T = R_{fv}. \tag{1}$$

The conversion between the n-frame and the v-frame is not straightforward, as the magnetic field in SEH2200 is neither uniform nor stationary. However, as both of the IMU and the VICON measures attitudes, the rotation matrix $R_{nv} \in \mathsf{SO}(3)$ from the v-frame to the n-frame can be calibrated as described in Section 2.

Conversion between a UAV-fixed frame and a ground-fixed frame In contrast to the two types conversion described above, the conversion between a UAV-fixed frame and a ground-fixed frame depends on the actual attitude of the UAV with respect to the ground.

Two types of measurements are available.

- $R_{ni}(t) \in \mathsf{SO}(3)$, the rotation matrix from the IMU i-frame to the NED n-frame is measured by IMU. There are several ways to convert Euler-angles to the corresponding rotation matrix. The following equations should be used for VN100.

```
YPR(0)=data->ypr.yaw*M_PI/180.;
YPR(1)=data->ypr.pitch*M_PI/180.;
YPR(2)=data->ypr.roll*M_PI/180.;

R_ni(0,0)=cos(YPR(0))*cos(YPR(1));
R_ni(0,1)=cos(YPR(0))*sin(YPR(2))*sin(YPR(1)) - cos(YPR(2))*sin(YPR(0));
R_ni(0,2)=sin(YPR(0))*sin(YPR(2)) + cos(YPR(0))*cos(YPR(2))*sin(YPR(1));
R_ni(1,0)=cos(YPR(1))*sin(YPR(0));
R_ni(1,1)=cos(YPR(0))*cos(YPR(2)) + sin(YPR(0))*sin(YPR(2))*sin(YPR(1));
R_ni(1,2)=cos(YPR(2))*sin(YPR(0))*sin(YPR(1)) - cos(YPR(0))*sin(YPR(2));
R_ni(2,0)=-sin(YPR(1));
R_ni(2,1)=cos(YPR(1))*sin(YPR(2));
R_ni(2,2)=cos(YPR(2))*cos(YPR(1));
```

- $R_{vm}(t) \in \mathsf{SO}(3)$, the rotation matrix from the marker m-frame to the VICON reference v-frame is measured by VICON. There are several ways to convert a quaternion to the corresponding rotation matrix. The following equations should be used for VICON VRPN.

```
R_vm(0,0) = 1-(2*(tdata.quat[1])*(tdata.quat[1]))-(2*(tdata.quat[2])*(tdata.quat[2]));
R_vm(0,1) = (2*tdata.quat[0]*tdata.quat[1])-(2*tdata.quat[3]*tdata.quat[2]);
R_vm(0,2) = (2*tdata.quat[0]*tdata.quat[2])+(2*tdata.quat[3]*tdata.quat[1]);
R_vm(1,0) = (2*tdata.quat[0]*tdata.quat[1])+(2*tdata.quat[3]*tdata.quat[2]);
R_vm(1,1) = 1-(2*(tdata.quat[0])*(tdata.quat[0]))-(2*(tdata.quat[2])*(tdata.quat[2]));
R_vm(1,2) = (2*(tdata.quat[1])*(tdata.quat[2]))-(2*(tdata.quat[3])*(tdata.quat[0]));
R_vm(2,0) = (2*tdata.quat[0]*tdata.quat[2])-(2*tdata.quat[3]*tdata.quat[1]);
R_vm(2,1) = (2*tdata.quat[0]*tdata.quat[3])+(2*tdata.quat[2]*tdata.quat[1]);
R_vm(2,2) = 1-(2*(tdata.quat[0])*(tdata.quat[0]))-(2*(tdata.quat[1])*(tdata.quat[1]));
```

1.4 Reference Frame Selection

In indoor flight experiments at SEH2200, we use the following two reference frames:

- The body-fixed frame, or the b-frame, as the UAV-fixed frame.
- The FDCL SEH2200 frame, or the f-frame, as the ground-fixed frame.

In the implementation of the controller or the generation of tracking commands,

- $x \in \mathbb{R}^3$ is the position of the origin of the b-frame, or the center of gravity of the UAV, resolved in the f-frame.
- $v = \dot{x} \in \mathbb{R}^3$.
- $R \triangleq R_{fb} \in \mathsf{SO}(3)$ is the rotation matrix from the b-frame to the f-frame.
- $\Omega \in \mathbb{R}^3$ is the angular velocity resolved in the b-frame.

IMU The output of the IMU corresponds to $(R_{\mathrm{IMU}}, \Omega_{\mathrm{IMU}}, a_{\mathrm{IMU}}) = (R_{ni}, \Omega_i, a_{\mathrm{IMU}})$, where a_{IMU} is the relative acceleration with respect to the gravitational acceleration resolved in the i-frame. They can be converted into (R, Ω, a) as

$$R(t) = R_{fb}(t) = R_{fv}R_{vn}R_{ni}(t)R_{ib}, \tag{2}$$

$$\dot{R}(t) = R_{fv}R_{vn}R_{ni}(t)\hat{\Omega}_i(t)R_{ib} = R_{fv}R_{vn}R_{ni}(t)R_{ib}\widehat{R_{ib}^T\Omega_i(t)} = R(t)\widehat{R_{ib}^T\Omega_i(t)} = R(t)\hat{\Omega}(t) \tag{3}$$

$$\Omega(t) = \Omega_b(t) = R_{bi}\Omega_i(t), \tag{4}$$

$$a(t) = \ddot{x}(t) = R_{fi}(t)a_{\mathrm{IMU}} + ge_3 = R_{fv}R_{vn}R_{ni}(t)a_{\mathrm{IMU}} + ge_3 = R_{fb}(t)R_{bi}a_{\mathrm{IMU}} + ge_3. \tag{5}$$

VICON The output of the VICON measurements corresponds to (x_v, R_{vm}). They can be converted into (x, R) as

$$x(t) = x_f(t) = R_{fv}x_v(t), \tag{6}$$

$$R(t) = R_{fb}(t) = R_{fv}R_{vm}(t)R_{mb}. \tag{7}$$

ZED The output of the ZED measurements corresponds to (x_w, R_{wz}). They can be converted into (x, R) as

$$x(t) = x_f(t) = R_{fw} x_w(t), \tag{8}$$

$$R(t) = R_{fb}(t) = R_{fw} R_{wz}(t) R_{zb}. \tag{9}$$

2 Sensor Calibration

2.1 IMU and VICON Calibration

The IMU measures $R_{ni} \in \mathsf{SO}(3)$ which represents the linear transformation of a representation of a vector from the i-frame to the n-frame, and the VICON measures $R_{vm} \in \mathsf{SO}(3)$ that is the rotation matrix from the m-frame to the v-frame. They are related as

$$R_{ni}(t) = R_{nv} R_{vm}(t) R_{mi}, \tag{10}$$

where $R_{mi} \in \mathsf{SO}(3)$ is the fixed rotation matrix from the i-frame to the m-frame, and $R_{nv} \in \mathsf{SO}(3)$ is the fixed rotation matrix from the v-frame to the n-frame.

The objective of this section is to compute R_{mi} and R_{nv} for several measurements of $\{(R_{ni}(t), R_{vm}(t)) \in \mathsf{SO}(3) \times \mathsf{SO}(3) \,|\, t \in \{t_1, t_2, \ldots t_N\}\}$ for a positive integer $N \geq 2$. This can be formulated as (11) with

$$R = R_{ni}, \quad Q = R_{vm}, \quad X = R_{nv}, \quad Y = R_{mi}.$$

The procedures to solve (11) is summarized at the end of Section 2.4.

2.2 VICON and ZED Calibration

For indoor flight experiments, it is desirable to calibrate the ZED measuring $R_{wz}(t)$ with the VICON measuring $R_{vm}(t)$. They are related as

$$R_{wz}(t) = R_{wv} R_{vm}(t) R_{mi} R_{ib} R_{bz},$$

which can be formulated as (11) with

$$R = R_{wz}, \quad Q = R_{vm}, \quad R_{wv} = X, \quad Y = R_{mi} R_{ib} R_{bz}.$$

The procedures to solve (11) is summarized at the end of Section 2.4. Once Y is obtained, R_{bz} can be constructed by

$$R_{bz} = R_{ib}^T R_{mi}^T Y.$$

2.3 IMU and ZED Calibration

For outdoor flight experiments, it is desirable to calibrate the ZED measuring $R_{wz}(t)$ with the IMU measuring $R_{ni}(t)$. They are related as

$$R_{wz}(t) = R_{wn} R_{ni}(t) R_{ib} R_{bz},$$

which can be formulated as (11) with

$$R = R_{wz}, \quad Q = R_{ni}, \quad R_{wn} = X, \quad Y = R_{ib} R_{bz}.$$

The procedures to solve (11) is summarized at the end of Section 2.4. Once Y is obtained, R_{bz} can be constructed by

$$R_{bz} = R_{ib}^T Y.$$

2.4 Solution of $R = XQY$

The problem of sensor calibration can be formulated as finding $X, Y \in \mathsf{SO}(3)$ for multiple pairs of measurements, $\{R_i, Q_i\}_{i=1}^N$ so that

$$R_i = XQ_iY, \tag{11}$$

for $i \in \{1, \ldots, N\}$.

For another pair of measurement indexed by $j \neq i$, we have $R_j = XQ_jY$. Since $Y = Q_j^T X^T R_j$, substituting it into the above,

$$R_i = XQ_iQ_j^T X^T R_j,$$

which is rearranged

$$R_iR_j^T = XQ_iQ_j^T X^T, \tag{12}$$

which follows the form of $A = XBX^T$, or equivalently $AX = XB$, that is an equation well-known in sensor calibrations.

We solve it using the matrix exponential. Define $r_{ij}, q_{ij} \in \mathbb{R}^3$ such that

$$R_iR_j^T = \exp \hat{r}_{ij}, \quad Q_iQ_j^T = \exp \hat{q}_{ij}. \tag{13}$$

Using this, (12) is rewritten as $\exp \hat{r}_{ij} = X \exp \hat{q}_{ij} X^T = \exp(\widehat{Xq_{ij}})$, which is equivalent to

$$r_{ij} = Xq_{ij}. \tag{14}$$

The problem of finding $X \in \mathsf{SO}(3)$ for given $\{(r_{ij}, q_{ij}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i, j \in \{1, \ldots, N\}\}$ has been investigated in the field of attitude determination. In particular, the least-square determination approach is recognized as Wahba's problem.

One of the most popular solution to Wahba's problem is based on the singular value decomposition. Define the matrix $Z \in \mathbb{R}^{3\times 3}$ as

$$Z = \sum_{i,j=1}^{N} w_{ij} r_{ij} q_{ij}^T, \tag{15}$$

where $w_{ij} \in \mathbb{R}$ denotes a positive weighting parameters. Let the singular value decomposition of Z be $Z = USV^T$. The least-square solution of (14) minimizing $\mathcal{J} = \sum_{i,j=1}^N w_{ij} \|r_{ij} - Xq_{ij}\|^2$ is given by

$$X = U \,\mathrm{diag}[1, 1, \det[U]\det[V]]\, V^T. \tag{16}$$

Compute X In summary, $X \in \mathsf{SO}(3)$ is determined by

1. Collect $\{(R_i(t), Q_i(t)) \in \mathsf{SO}(3) \times \mathsf{SO}(3) \,|\, t \in \{t_1, t_2, \ldots t_N\}\}$.
2. Define $\{(r_{ij}, q_{ij}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i, j \in \{1, \ldots, N\}\}$ from (13).
3. Define Z as (15), and perform singular value decomposition to obtain $Z = USV^T$.
4. We have $X = U \,\mathrm{diag}[1, 1, \det[U]\det[V]]\, V^T$.

Compute Y The rotation matrix Y can be constructed directly by using the above solution via $Y = Q_j^T X^T R_j$ for any $j \in \{1, \dots, N\}$. In order to compute Y in the lease-square sense, the above procedure can be repeated as follows.

We have $X = R_j Y^T Q_j^T$, which yield $R_i = R_j Y^T Q_j^T Q_i Y$ that is equivalent to

$$R_j^T R_i = Y^T Q_j^T Q_i Y,$$

which is comparable to (12). As such, the procedure to compute X can be repeated by using the following definition of r_{ij}, q_{ij} instead of (13),

$$R_j^T R_i = \exp \hat{r}_{ij}, \quad Q_j^T Q_i = \exp \hat{q}_{ij}. \tag{17}$$

In summary, the rotation matrix $Y \in \mathsf{SO}(3)$ is determined by

1. Collect $\{(R_i(t), Q_i(t)) \in \mathsf{SO}(3) \times \mathsf{SO}(3) \,|\, t \in \{t_1, t_2, \dots t_N\}\}$.
2. Define $\{(r_{ij}, q_{ij}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, i, j \in \{1, \dots, N\}\}$ from (17).
3. Define Z as (15), and perform singular value decomposition to obtain $Z = USV^T$.
4. We have $Y = V \,\mathrm{diag}[1, 1, \det[U]\det[V]]\, U^T$.

3 State Estimation

This estimator estimates position, velocity, attitude, and the vertical accelerometer bias.

3.1 Equations of Motion

The equations of motion for $(x, v, R = R_{fb})$ are given by

$$\dot{x} = v, \tag{18}$$

$$\dot{v} = a = RR_{bi}(a_{\mathrm{IMU}} + w_a + e_3 b_a) + ge_3, \tag{19}$$

$$\dot{R} = R\hat{\Omega} = R(R_{bi}(\Omega_{\mathrm{IMU}} + w_\Omega))^\wedge \tag{20}$$

$$\dot{b}_a = w_{b_a} \tag{21}$$

where the acceleration measurement and the angular velocity measurements, namely $(a_{\mathrm{IMU}}, \Omega_{\mathrm{IMU}})$ are treated as an exogenous time-varying signal. As such, the measurement noise for the acceleration and the angular velocity, namely $w_a, w_\Omega \in \mathbb{R}^3$ are considered as the process noise, instead of the measurement noise in the formulation of the estimator. The IMU measures the acceleration and the angular velocity and this formulation assumes the measurements do not have sensor bias.

Let $(\delta x, \delta v, \delta R = R\hat{\eta}, \delta b_a)$ be the perturbation of the state for $\eta \in \mathbb{R}^3$, from the ideal case of the absence of noise. After ignoring the higher order terms of perturbations and noise, the linearized equations of motion for $\delta x, \delta v$, and δb_a are given by

$$\begin{aligned}
\delta \dot{x} &= \delta v, \\
\delta \dot{v} &= R\hat{\eta} R_{bi} a_{\mathrm{IMU}} + RR_{bi} w_a = -R(R_{bi} a_{\mathrm{IMU}})^\wedge \eta + RR_{bi} e_3 \delta b_a + RR_{bi} w_a, \\
\delta \dot{b}_a &= w_{b_a}.
\end{aligned}$$

Also the perturbation of (20) is written as

$$\begin{aligned}
\frac{d}{dt}(\delta R) = \frac{d}{dt}(R\hat{\eta}) &= R\hat{\Omega}\hat{\eta} + R\dot{\hat{\eta}} = R(R_{bi}\Omega_{\mathrm{IMU}})^\wedge \hat{\eta} + R\dot{\hat{\eta}} \\
\delta(\dot{R}) &= R\hat{\eta}(R_{bi}\Omega_{\mathrm{IMU}})^\wedge + R(R_{bi} w_\Omega)^\wedge,
\end{aligned}$$

which yields

$$\dot{\eta} = -(R_{bi}\Omega_{\mathrm{IMU}})^\wedge \eta + R_{bi} w_\Omega,$$

after ignoring the higher-order terms of the process noise and the perturbation.

The linearized equations of motion are rearranged into a matrix form with $\mathbf{x} = [\delta x; \delta v; \eta; \delta b_a] \in \mathbb{R}^{10}$

and $\mathbf{w} = [w_a; w_\Omega; w_{b_a}] \in \mathbb{R}^7$ as

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & -R(R_{bi}a_{\mathrm{IMU}})^\wedge & RR_{bi}e_3 \\ 0_{3\times3} & 0_{3\times3} & -(R_{bi}\Omega_{\mathrm{IMU}})^\wedge & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ RR_{bi} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & R_{bi} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 1 \end{bmatrix} \mathbf{w} \\
&\triangleq A(t)\mathbf{x} + F(t)\mathbf{w}. \qquad (22)
\end{aligned}$$

3.2 Prediction

Mean Let $h_k = t_k - t_{k-1}$ be the discrete time step. The mean values $(\bar{x}, \bar{v}, \bar{R})$ are updated by discretizing (18)–(20) in the absence of the process noise via the following second order explicit method:

$$\bar{R}_k = \bar{R}_{k-1} \exp\left\{\frac{h_k}{2}(\bar{\Omega}_{k-1} + \bar{\Omega}_k)^\wedge\right\}, \qquad (23)$$

$$\bar{x}_k = \bar{x}_{k-1} + h_k\bar{v}_{k-1} + \frac{h_k^2}{2}\bar{a}_{k-1}, \qquad (24)$$

$$\bar{v}_k = \bar{v}_{k-1} + \frac{h_k}{2}(\bar{a}_{k-1} + \bar{a}_k), \qquad (25)$$

$$\bar{b}_{a_k} = \bar{b}_{a_{k-1}}, \qquad (26)$$

with

$$\bar{\Omega}_k = R_{bi}\Omega_{\mathrm{IMU}_k}, \qquad (27)$$

$$\bar{a}_k = \bar{R}_k R_{bi}(a_{\mathrm{IMU}_k} + b_{a_k}) + ge_3. \qquad (28)$$

Covariance The linearized equation is discretized according to [1, p 330] as

$$\mathbf{x}_k = A_{k-1}\mathbf{x}_{k-1} + F_{k-1}\mathbf{w}_{k-1}, \qquad (29)$$

where

$$\Psi = I_{10\times10} + \frac{h_k}{2}A(t_{k-1})\left(I + \frac{h_k}{3}A(t_{k-1})\left(I + \cdots\left(I + \frac{h_k}{N}A(t_{k-1})\right)\right)\right), \qquad (30)$$

$$A_{k-1} = I_{10\times10} + h_k A(t_{k-1})\Psi, \qquad (31)$$

$$F_{k-1} = h_k\Psi F(t_{k-1}). \qquad (32)$$

For the predictor run in the FDCL-UAV code, we neglect the higher order terms of Ψ and use

$$\Psi = I_{10\times10} + \frac{h_k}{2}A(t_{k-1}). \qquad (33)$$

Let the covariance of $\mathbf{x}_k$ be $P_k \in \mathbb{R}^{10\times10}$ and let the covariance of $\mathbf{w}_k$ be $Q_k \in \mathbb{R}^{7\times7}$. It is updated as

$$P_k = A_{k-1}P_{k-1}A_{k-1}^T + F_{k-1}Q_{k-1}F_{k-1}^T. \qquad (34)$$

3.3 Correction for Indoor Tests

3.3.1 Correction by IMU

In addition to $(\Omega_{\mathrm{IMU}}, a_{\mathrm{IMU}})$, the IMU measures R_{ni}. The measurement equation is given by

$$R_{\mathrm{IMU}} = R_{nv}R_{vf}RR_{bi}\exp\hat{\zeta}.$$

where $\zeta \in \mathbb{R}^3$ denotes the measurement noise for R_{IMU}. The estimation of the IMU measurement is given by

$$\bar{R}_{\mathrm{IMU}} = R_{nv}R_{vf}\bar{R}R_{bi}.$$

The difference between R_{IMU} and $\bar{R}_{\mathrm{IMU}}$ is referred to as the residual error, which is formulated by $\eta, \zeta \in \mathbb{R}^3$ as

$$\begin{aligned} R_{\mathrm{IMU}} &= R_{nv}R_{vf}\bar{R}\exp\hat{\eta}R_{bi}\exp\hat{\zeta} \\ &= R_{nv}R_{vf}\bar{R}R_{bi} + R_{nv}R_{vf}\bar{R}\hat{\eta}R_{bi} + R_{nv}R_{vf}\bar{R}R_{bi}\hat{\zeta}, \end{aligned}$$

for small η and ζ, which is rearranged into

$$\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T = I_{3\times 3} + \hat{\eta} + \widehat{R_{bi}\zeta},$$

which yields

$$(\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T - I_{3\times 3})^\vee = \eta + R_{bi}\zeta.$$

In practice, $\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T - I_{3\times 3}$ may not be skew-symmetric. After projecting it to the space of skew-symmetric matrices,

$$\frac{1}{2}(\bar{R}^T R_{vf}^T R_{nv}^T R_{\mathrm{IMU}} R_{bi}^T - R_{bi}R_{\mathrm{IMU}}^T R_{nv}R_{vf}\bar{R})^\vee = \delta z = \eta + R_{bi}\zeta = H\mathbf{x} + G\mathbf{v}, \tag{35}$$

where $H = [0_{3\times 3}, 0_{3\times 3}, I_{3\times 3}, 0_{3\times 1}] \in \mathbb{R}^{3\times 10}$, $G = R_{bi} \in \mathbb{R}^{3\times 3}$, and $\mathbf{v} = \zeta \in \mathbb{R}^{3\times 3}$.

Correction Define $T \in \mathbb{R}^{10\times 10}$ as

$$T = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 1} \\ I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 1 \end{bmatrix}.$$

Let $\chi = Tx \in \mathbb{R}^{15}$. We have the similarity transform, $\bar{A} = TAT^{-1}$, $\bar{F} = TF$, and $\bar{H} = HT^{-1}$. The first three elements of χ, namely $\chi_o = T_o\chi \in \mathbb{R}^3$ corresponds to the observable subspace, where $T_0 \in \mathbb{R}^{3\times 9}$ is

$$T_o = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \end{bmatrix}.$$

And it is governed by

$$\begin{aligned}\dot{\chi}_o &= (T_o\bar{A}(t)T_o^T)T_o\chi + T_o\bar{F}(t)\mathbf{w}, \\ &\triangleq \bar{A}_o(t)\chi_o + \bar{F}_o(t)\mathbf{w}, \end{aligned} \tag{36}$$

$$\begin{aligned}\delta z &= (\bar{H}T_o^T)T_o\chi \\ &\triangleq \bar{H}_o\chi_o. \end{aligned} \tag{37}$$

Also

$$P_o = \mathrm{E}[(\chi_o-\bar{\chi}_o)(\chi_o-\bar{\chi}_o)^T] = \mathrm{E}[T_o(\chi-\bar{\chi})(\chi-\bar{\chi})T_o^T] = \mathrm{E}[T_oT(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})T^TT_o^T] = T_oTPT^TT_o^T.$$

The above equation is considered as the linearized measurement equation for EKF with the residual or the innovation term δz. Let $V \in \mathbb{R}^{3\times 3}$ be the covariance of the measurement error ζ. The residual (or innovation) covariance $S \in \mathbb{R}^{3\times 3}$ is

$$S = \bar{H}_oP_o\bar{H}_o^T + GVG^T \tag{38}$$

The Kalman gain $\bar{K}_o \in \mathbb{R}^{6\times 3}$ is given by

$$\bar{K}_o = P_o\bar{H}_o^TS^{-1} \tag{39}$$

and the a posteriori value of the perturbed state is given by

$$\mathbf{x}^+ = T^TT_o^T\bar{K}_o\delta z \triangleq K\delta z. \tag{40}$$

The a posteriori covariance is

$$P^+ = (I_{10\times 10} - KH)P^- = (I_{10\times 10} - KH)P^-(I_{10\times 10} - KH)^T + KGVG^TK^T, \tag{41}$$

3.3.2 Correction via VICON

The VICON measures the location in the v-frame and the attitude from the m-frame to the v-frame, namely $(x_{\mathrm{VICON}}, R_{\mathrm{VICON}})$.

$$\begin{aligned}x_{\mathrm{VICON}} &= R_{vf}x + \zeta_x, \\ R_{\mathrm{VICON}} &= R_{vm}\exp\hat{\zeta}_R = R_{vf}RR_{bi}R_{im}\exp\hat{\zeta}_R,\end{aligned}$$

where ζ_x, ζ_R denote the measurement noise for x_{VICON} and R_{VICON}, respectively.

Similar to above, the estimate of R_{VICON} is given by $\bar{R}_{\mathrm{VICON}} = R_{vf}RR_{bm}$, and the difference between R_{VICON} and $\bar{R}_V$ is formulated as

$$\begin{aligned}R_{\mathrm{VICON}} &= R_{vf}\bar{R}\exp\hat{\eta}R_{bi}R_{im}\exp\hat{\zeta}_R \\ &= R_{vf}\bar{R}R_{bi}R_{im} + R_{vf}\bar{R}\hat{\eta}R_{bi}R_{im} + R_{vf}\bar{R}R_{bi}R_{im}\hat{\zeta}_R,\end{aligned}$$

after ignoring the higher order terms of ζ_R and η. This is rearranged into

$$\bar{R}^T R_{vf}^T R_{\text{VICON}} R_{im}^T R_{bi}^T = I_{3\times 3} + \hat{\eta} + \widehat{R_{bm}\zeta_R},$$

or equivalently

$$(\bar{R}^T R_{vf}^T R_{\text{VICON}} R_{im}^T R_{bi}^T - I_{3\times 3})^\vee = \eta + R_{bm}\zeta_R.$$

Projecting the left hand side onto the skew-symmetric matrices,

$$\frac{1}{2}(\bar{R}^T R_{vf}^T R_{\text{VICON}} R_{im}^T R_{bi}^T - R_{bi} R_{im} R_{\text{VICON}}^T R_{vf} \bar{R})^\vee = \eta + R_{bm}\zeta_R.$$

Similarly,

$$x_{\text{VICON}} = R_{vf}\bar{x} + R_{vf}\delta x + \zeta_x.$$

The linearized measurement equation is written as

$$\delta z = \begin{bmatrix} \frac{1}{2}(\bar{R}^T R_{vf}^T R_{\text{VICON}} R_{im}^T R_{bi}^T - R_{bi} R_{im} R_{\text{VICON}}^T R_{vf} \bar{R})^\vee \\ x_{\text{VICON}} - R_{vf}\bar{x} \end{bmatrix} = \begin{bmatrix} \eta \\ R_{vf}\delta x \end{bmatrix} + \begin{bmatrix} R_{bi} R_{im} \zeta_R \\ \zeta_x \end{bmatrix} = H\mathbf{x} + G\mathbf{v},$$

where $\mathbf{v} = [\zeta_R; \zeta_x] \in \mathbb{R}^6$, and the matrices $H \in \mathbb{R}^{6\times 10}$ and $G \in \mathbb{R}^{6\times 6}$ are given by

$$H = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 1} \\ R_{vf} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \end{bmatrix}, \quad G = \begin{bmatrix} R_{bi} R_{im} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix}.$$

Let $V \in \mathbb{R}^{6\times 6}$ be the covariance of $\mathbf{v}$. The residual covariance $S \in \mathbb{R}^{6\times 6}$ is given by

$$S = HPH^T + GVG^T.$$

The Kalman gain $K \in \mathbb{R}^{10\times 6}$ is given by

$$K = PH^T S^{-1}, \tag{42}$$

and the a posteriori value of the perturbed state is given by

$$\mathbf{x}^+ = K\delta z. \tag{43}$$

The a posteriori state is

$$x^+ = x^- + \delta x^+, \quad v^+ = v^- + \delta v^+, \quad \bar{R}^+ = \bar{R}^- \exp(\hat{\eta}^+), \quad \bar{b}_a^+ = \bar{b}_a^- + \delta b_a^+. \tag{44}$$

The a posteriori covariance is

$$P^+ = (I_{10\times 10} - KH)P^- = (I_{10\times 10} - KH)P^-(I_{10\times 10} - KH)^T + KGVG^T K^T, \tag{45}$$

where the latter is known as the Joseph form, which is implemented to preserve the symmetry and the positive-definiteness of P in numerical computation.

3.3.3 Correction via Mini Lidar Sensor

The Mini Lidar sensor measures the distance from the sensor to the closest obstacles. If the sensor is mounted perfectly facing positive b_3 axis and if the only horizontal surface is the ground, the sensor measures the altitude.

The predictions are corrected from the Lidar sensor measurement. Let $d_{\mathrm{LDR}} \in \mathbb{R}^1$ be the measurement and $\zeta_{\mathrm{LDR}} \in \mathbb{R}^1$ be the measurement error for the Lidar sensor. Let $x_{\mathrm{LDR}} \in \mathbb{R}^1$ be the altitude determined by the Lidar sensor.

The Lidar sensor measures the distance from the sensor to the ground which is the distance from the origin of the b-frame to the ground plane, along the b_3 axis. Let p_0 a known point on the ground plane and n a vector normal to the ground plane. The point p is chosen as a point both common to the above line and the landing plane. Considering the line-plane intersection, the distance measured by the Lidar can be derives as follows. The landing plane that goes through both p and p_0 can be expressed as

$$(p - p_0) \cdot n = 0. \tag{46}$$

The line along b_3 that goes through the UAV and p is defined as

$$p = \bar{d}_{\mathrm{LDR}} R e_3 + x \tag{47}$$

where $\bar{d}_{\mathrm{LDR}}$ is the calculated distance measured by the Lidar sensor. Substituting (47) in (46), the relationship between x and the d_{LDR} can be expressed as,

$$\bar{d}_{\mathrm{LDR}} = \frac{(p_o - x) \cdot n}{R e_3 \cdot n}. \tag{48}$$

The error in the Lidar measurement is given by

$$\begin{aligned} \delta z_d &= \begin{bmatrix} 0 & 0 & (\bar{d}_{\mathrm{LDR}} - d_{\mathrm{LDR}}) \end{bmatrix}^T \\ \delta z &= -R \delta z_d \cdot e_3 \\ &= H\mathbf{x} + G\mathbf{v}. \end{aligned}$$

Here $\mathbf{v} \in \mathbb{R}^1 = \zeta_{\mathrm{LDR}}$, and matrices $H \in \mathbb{R}^{1\times 10}$ and $G \in \mathbb{R}^{1\times 1}$ are given by,

$$H = \begin{bmatrix} 0 & 0 & 1 & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \end{bmatrix} \in \mathbb{R}^{3\times 10}, \quad G = \begin{bmatrix} 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The third element of the position (x_3) corresponds to the observable subspace. Let $\chi = T_0 T x \in \mathbb{R}^1$ where

$$T = \begin{bmatrix} M_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 1_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 1_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 1 \end{bmatrix}, \quad T_0 = \begin{bmatrix} 1 & 0 & 0 & 0_{3\times3} & 0_{3\times1} & 0_{3\times3} \end{bmatrix}.$$

Then, with $\bar{A} = TAT^{-1}$, $\bar{F} = TF$, and $\bar{H} = HT^{-1}$, the correction equations are expressed as

$$\dot{\chi} = T_0 \bar{A}(t) T_0^{-1} T_0 \chi + T_0 \bar{F}(t)\mathbf{w} \triangleq \bar{A}_0(t)\chi_0 + \bar{F}_0(t)\mathbf{w}, \tag{49}$$

$$\delta z = (\bar{H} T_0^{-1}) T_0 \chi \triangleq \bar{H}_0 \chi_0, \tag{50}$$

Also

$$P_o = \mathrm{E}[(\chi_o - \bar{\chi}_o)(\chi_o - \bar{\chi}_o)^T] = \mathrm{E}[T_o(\chi - \bar{\chi})(\chi - \bar{\chi})T_o^T] = \mathrm{E}[T_o T(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})T^T T_o^T] = T_o T P T^T T_o^T.$$

Let $V \in \mathbb{R}^1$ be the covariance of $\mathbf{v}$. The residual covariance $S \in \mathbb{R}^1$ is given by

$$S = HPH^T + GVG^T.$$

The Kalman gain $K \in \mathbb{R}^{10\times1}$ is given by

$$K = PH^T S^{-1}, \tag{51}$$

and the a posteriori value of the perturbed state is given by

$$\mathbf{x}^+ = K\delta z. \tag{52}$$

The a posteriori state is

$$x_3^+ = x_3^- + \delta x_3^+. \tag{53}$$

The a posteriori covariance is

$$P^+ = (I_{10\times10} - KH)P^- = (I_{10\times10} - KH)P^-(I_{10\times10} - KH)^T + KGVG^T K^T, \tag{54}$$

3.3.4 Correction via Camera

The camera measures the location in the camera frame, namely x_{CAM}. Since the time difference between two camera frames are known, the velocity (v_{CAM}) can be calculated from the position data.

$$x_{\mathrm{CAM}} = R_{cb} R_{bf} x + \zeta_{x_{\mathrm{CAM}}}$$
$$v_{\mathrm{CAM}} = R_{cb} R_{bf} v + \zeta_{v_{\mathrm{CAM}}}$$

where $\zeta_{x_{\mathrm{CAM}}} \in \mathbb{R}^{3\times1}$ and $\zeta_{v_{\mathrm{CAM}}} \in \mathbb{R}^{3\times1}$ denotes the measurement noise for x_{CAM} and v_{CAM}, respectively. Note that this estimation assumes that the camera is fixed at the center of gravity of the UAV. If it is not the case, the input from the camera must be adjusted so that the measurement is zero at the center of the gravity of the UAV.

Similar to above, the estimate of $(x_{\text{CAM}}, v_{\text{CAM}})$ are given by,

$$
\begin{aligned}
x_{\text{CAM}} &= R_{cb}R_{bf}\bar{x} + R_{cb}R_{bf}\delta x + \zeta_{x_{\text{CAM}}},\\
v_{\text{CAM}} &= R_{cb}R_{bf}\bar{v} + R_{cb}R_{bf}\delta v + \zeta_{v_{\text{CAM}}}.
\end{aligned}
$$

The linearized measurement equation is written as

$$
\delta z = \begin{bmatrix} x_{\text{CAM}} - R_{cb}R_{bf}\bar{x} \\ v_{\text{CAM}} - R_{cb}R_{bf}\bar{v} \end{bmatrix} = \begin{bmatrix} R_{cb}R_{bf}\delta x \\ R_{cb}R_{bf}\delta v \end{bmatrix} + \begin{bmatrix} \zeta_{x_{\text{CAM}}} \\ \zeta_{v_{\text{CAM}}} \end{bmatrix} = H\mathbf{x} + G\mathbf{v},
$$

where $\mathbf{v} = [\zeta_{x_{\text{CAM}}};\ \zeta_{v_{\text{CAM}}}] \in \mathbb{R}^6$, and the matrices $H \in \mathbb{R}^{6\times 10}$ and $G \in \mathbb{R}^{6\times 6}$ are given by

$$
H = \begin{bmatrix} R_{cb}R_{bf} & 0 & 0 & 0 & 0 \\ 0 & R_{cb}R_{bf} & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} I_{6\times 6} \end{bmatrix}.
$$

It is clear that η is not observable from the camera position, x_{CAM}. Define $T \in \mathbb{R}^{10\times 10}$ as

$$
T = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 1 \\ 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 1} \end{bmatrix}.
$$

Let $\chi = Tx \in \mathbb{R}^{10}$. We have the similarity transform, $\bar{A} = TAT^{-1}$, $\bar{F} = TF$, and $\bar{H} = HT^{-1}$,

$$
\begin{aligned}
\dot{\chi} &= TAT^{-1}\chi + TF\mathbf{w} \\
&\triangleq \bar{A}(t)\chi + \bar{F}(t)\mathbf{w}, \qquad (55)\\
\delta z &= HT^{-1}\chi, \\
&\triangleq \bar{H}\chi, \qquad (56)
\end{aligned}
$$

The first seven elements of χ, namely $\chi_o = T_o[\delta x, \delta v, \delta b_a] \in \mathbb{R}^7$ corresponds to the observable subspace, where $T_0 \in \mathbb{R}^{7\times 10}$ is

$$
T_o = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 1} & 0_{3\times 3} \\ 0_{1\times 3} & 0_{1\times 3} & 1 & 0_{1\times 3} \end{bmatrix}.
$$

And it is governed by

$$
\begin{aligned}
\dot{\chi}_o &= (T_o\bar{A}(t)T_o^T)T_o\chi + T_o\bar{F}(t)\mathbf{w}, \\
&\triangleq \bar{A}_o(t)\chi_o + \bar{F}_o(t)\mathbf{w}, \qquad (57)\\
\delta z &= (\bar{H}T_o^T)T_o\chi \\
&\triangleq \bar{H}_o\chi_o. \qquad (58)
\end{aligned}
$$

Also

$$P_o = \mathrm{E}[(\chi_o - \bar{\chi}_o)(\chi_o - \bar{\chi}_o)^T] = \mathrm{E}[T_o(\chi - \bar{\chi})(\chi - \bar{\chi})T_o^T] = \mathrm{E}[T_o T(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})T^T T_o^T] = T_o T P T^T T_o^T.$$

Correction The above equation is considered as the linearized measurement equation for EKF with the residual or the innovation term δz. The residual (or innovation) covariance $S \in \mathbb{R}^{6\times 6}$ is

$$\begin{aligned} S &= \bar{H}_o P_o \bar{H}_o^T + GVG^T = (HT^T T_o^T)(T_o T P T^T T_o^T) T_o T H^T + GVG^T \\ &= HPH^T + GVG^T. \end{aligned} \tag{59}$$

The Kalman gain $\bar{K}_o \in \mathbb{R}^{6\times 3}$ is given by

$$\bar{K}_o = P_o \bar{H}_o^T S^{-1} = (T_o T P T^T T_o^T)(T_o T H^T) S^{-1}, \tag{60}$$

and the a posteriori value of the perturbed state is given by

$$\chi^+ = \bar{K}_o \delta z. \tag{61}$$

Or equivalently,

$$\mathbf{x}^+ = T^T T_o^T \bar{K}_o \delta z \triangleq K \delta z. \tag{62}$$

The a posteriori state is

$$x^+ = x^- + \delta x^+, \quad v^+ = v^- + \delta v^+, \quad b_a^+ = b_a^- + \delta b_a^+ \tag{63}$$

and the a posteriori covariance is

$$P^+ = (I_{15\times 15} - KH)P^- = (I_{15\times 15} - KH)P^-(I_{15\times 15} - KH)^T + KGVG^T K^T, \tag{64}$$

where the latter is known as the Joseph form, which is implemented to preserve the symmetry and the positive-definiteness of P in numerical computation.

3.4 Correction for Outdoor Tests

Lets define the n-frame as the ground fixed frame (f-frame) with the origin fixed at the location of the base.

3.4.1 Correction via RTK GPS for Stationary Base

The predictions are corrected from the RTK GPS measurements for the outdoor tests. The RTK GPS measures the location and velocity in the n-frame, namely x_{GPS} and v_{GPS} respectively.

$$\begin{aligned} x_{\mathrm{GPS}} &= x + \zeta_x, \\ v_{\mathrm{GPS}} &= v + \zeta_v, \end{aligned}$$

where ζ_x and ζ_v denotes the measurement noises for x_{GPS} and v_{GPS}, respectively. Similar to above, the estimate of x_{GPS} is given by $\bar{x}_{\mathrm{GPS}}$, and the difference between x_{GPS} and $\bar{x}_{\mathrm{GPS}}$ is given by

$$x_{\mathrm{GPS}} = \bar{x} + \delta x + \zeta_x.$$

Similarly, the estimate of v_{GPS} is given by

$$v_{\mathrm{GPS}} = \bar{v} + \delta v + \zeta_v.$$

The linearized measurement equation is written as

$$\delta z = \begin{bmatrix} x_{\mathrm{GPS}} - \bar{x} \\ v_{\mathrm{GPS}} - \bar{v} \end{bmatrix} = \begin{bmatrix} \delta x + \zeta_x \\ \delta v + \zeta_v \end{bmatrix} = H\mathbf{x} + G\mathbf{v},$$

where $\mathbf{v} = \begin{bmatrix} \zeta_x & \zeta_v \end{bmatrix}^T \in \mathbb{R}^6$, and the matrices $H \in \mathbb{R}^{6\times 10}$ and $G \in \mathbb{R}^{6\times 6}$ are given by

$$H = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \end{bmatrix}, \quad G = \begin{bmatrix} I_{3\times 3} & 0 \\ 0 & I_{3\times 3} \end{bmatrix}.$$

It is clear that η is not observable from the GPS position and velocity measurements $(x_{\mathrm{GPS}}, v_{\mathrm{GPS}})$. Define $T \in \mathbb{R}^{10\times 10}$ as

$$T = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 1 \\ 0_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & 0_{3\times 1} \end{bmatrix}.$$

Let $\chi = Tx \in \mathbb{R}^{10}$. We have the similarity transform, $\bar{A} = TAT^{-1}$, $\bar{F} = TF$, and $\bar{H} = HT^{-1}$,

$$\begin{aligned} \dot{\chi} &= TAT^{-1}\chi + TF\mathbf{w} \\ &\triangleq \bar{A}(t)\chi + \bar{F}(t)\mathbf{w}, && (65) \\ \delta z &= HT^{-1}\chi, \\ &\triangleq \bar{H}\chi, && (66) \end{aligned}$$

The first seven elements of χ, namely $\chi_o = T_o[\delta x, \delta v, \delta b_a] \in \mathbb{R}^7$ corresponds to the observable subspace, where $T_0 \in \mathbb{R}^{7\times 10}$ is

$$T_o = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} & 0_{3\times 1} & 0_{3\times 3} \\ 0_{1\times 3} & 0_{1\times 3} & 1 & 0_{1\times 3} \end{bmatrix}.$$

And it is governed by

$$
\begin{aligned}
\dot{\chi}_o &= (T_o\bar{A}(t)T_o^T)T_o\chi + T_o\bar{F}(t)\mathbf{w}, \\
&\triangleq \bar{A}_o(t)\chi_o + \bar{F}_o(t)\mathbf{w}, && (67) \\
\delta z &= (\bar{H}T_o^T)T_o\chi \\
&\triangleq \bar{H}_o\chi_o. && (68)
\end{aligned}
$$

Also

$$P_o = \mathrm{E}[(\chi_o - \bar{\chi}_o)(\chi_o - \bar{\chi}_o)^T] = \mathrm{E}[T_o(\chi - \bar{\chi})(\chi - \bar{\chi})T_o^T] = \mathrm{E}[T_oT(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})T^TT_o^T] = T_oTPT^TT_o^T.$$

Correction The above equation is considered as the linearized measurement equation for EKF with the residual or the innovation term δz. The residual (or innovation) covariance $S \in \mathbb{R}^{6\times 6}$ is

$$
\begin{aligned}
S &= \bar{H}_oP_o\bar{H}_o^T + GVG^T = (HT^TT_o^T)(T_oTPT^TT_o^T)T_oTH^T + GVG^T \\
&= HPH^T + GVG^T. && (69)
\end{aligned}
$$

The Kalman gain $\bar{K}_o \in \mathbb{R}^{6\times 3}$ is given by

$$\bar{K}_o = P_o\bar{H}_o^TS^{-1} = (T_oTPT^TT_o^T)(T_oTH^T)S^{-1}, \tag{70}$$

and the a posteriori value of the perturbed state is given by

$$\chi^+ = \bar{K}_o\delta z. \tag{71}$$

Or equivalently,

$$\mathbf{x}^+ = T^TT_o^T\bar{K}_o\delta z \triangleq K\delta z. \tag{72}$$

The a posteriori state is

$$x^+ = x^- + \delta x^+, \quad v^+ = v^- + \delta v^+, \quad b_a^+ = b_a^- + \delta b_a^+ \tag{73}$$

and the a posteriori covariance is

$$P^+ = (I_{15\times 15} - KH)P^- = (I_{15\times 15} - KH)P^-(I_{15\times 15} - KH)^T + KGVG^TK^T, \tag{74}$$

where the latter is known as the Joseph form, which is implemented to preserve the symmetry and the positive-definiteness of P in numerical computation.

3.4.2 Correction by IMU

The correction by IMU for the outdoor tests is same as the correction by IMU for indoor tests, except $R_{nv} = R_{vf} = I_{3\times 3}$.

4 Filters

4.1 Low-Pass Filter for Ω

In the formulation of Kalman filter described in **??**, the angular velocity measurement is considered as a known time-varying signal. Even after correcting it with the gyro bias, $\bar{\Omega}_k$ obtained by (**??**) is subject to high-frequency noise, which causes undesired irregular vibrations in the corresponding control input.

A simple first order low-pass filter is represented by

$$G(s) = \frac{1}{\tau s + 1},$$

where τ represents the time-constant, i.e., the time which the initial condition becomes reduced by the factor of $\frac{1}{e} = 0.368$. For the sinusoidal inputs,

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 \tau^2}}$$

Therefore, the bandwidth, the frequency where $|G(j\omega)|^2 = 0.5$ becomes $\omega = \frac{1}{\tau}$. For example, when the bandwidth is set for $f = 100\,\text{Hz}$, or equivalently $\omega = 2\pi f$, then the time constant is chosen as

$$\tau = \frac{1}{\omega} = \frac{1}{2\pi f} = 0.0016\,\text{sec}.$$

In the continuous time, the filter equation is written as

$$\tau \dot{y} + y = u,$$

which can be approximated by

$$\tau(y_k - y_{k-1}) + h y_k = h u_k,$$

or equivalently

$$y_k = \frac{1}{h + \tau}(\tau y_{k-1} + h u_k) = (1 - \alpha) y_{k-1} + \alpha u_k,$$

where $\alpha = \frac{h}{\tau + h}$.

5 Geometric Control System

5.1 Attitude Control

The equations of motion for the attitude dynamics is given by

$$J\dot{\Omega} + \Omega \times J\Omega = M,$$
$$\dot{R} = R\hat{\Omega}.$$

Suppose $J = \mathrm{diag}[J_1, J_2, J_3]$ and $J_1 = J_2$. The first equation is rearranged into

$$\begin{aligned} J_1\dot{\Omega}_1 + (J_3 - J_1)\Omega_2\Omega_3 &= M_1, \\ J_1\dot{\Omega}_2 - (J_3 - J_1)\Omega_3\Omega_1 &= M_2, \\ J_3\dot{\Omega}_3 &= M_3. \end{aligned}$$

Let $b_i = Re_i \in \mathsf{S}^2$ be the direction of the i-th body-fixed axis resolved in the inertial frame, for $i \in \{1, 2, 3\}$. The kinematics equation for b_3 is given by

$$\begin{aligned} \dot{b}_3 &= R\hat{\Omega}e_3 = \widehat{R\Omega}Re_3 = \widehat{R\Omega}b_3 = (b_1\Omega_1 + b_2\Omega_2 + b_3\Omega_3) \times b_3 \\ &= \omega_{12} \times b_3, \end{aligned} \tag{75}$$

where $\omega_{12} = b_1\Omega_1 + b_2\Omega_2 = R(\Omega - \Omega_3 e_3) \in \mathbb{R}^3$, with $b_3 \cdot \omega_{12} = 0$. Therefore, $\dot{b}_3 \cdot \omega_{12} + b_3 \cdot \dot{\omega}_{12} = b_3 \cdot \dot{\omega}_{12} = 0$.

Taking the time derivative of ω_{12},

$$\begin{aligned} \dot{\omega}_{12} &= (b_2\Omega_2 + b_3\Omega_3) \times b_1\Omega_1 + b_1\dot{\Omega}_1 + (b_1\Omega_1 + b_3\Omega_3) \times b_2\Omega_2 + b_2\dot{\Omega}_2 \\ &= -\Omega_1\Omega_2 b_3 + \Omega_3\Omega_1 b_2 + \Omega_1\Omega_2 b_3 - \Omega_3\Omega_2 b_1 \\ &\quad + b_1((1 - \frac{J_3}{J_1}\Omega_2\Omega_3) + \frac{1}{J_1}M_1) + b_2((\frac{J_3}{J_1} - 1)\Omega_3\Omega_1 + \frac{1}{J_1}M_2) \\ &= b_1(-\frac{J_3}{J_1}\Omega_2\Omega_3 + \frac{1}{J_1}M_1) + b_2(\frac{J_3}{J_1}\Omega_3\Omega_1 + \frac{1}{J_1}M_2). \end{aligned}$$

Define $\tau_1, \tau_2 \in \mathbb{R}$ be

$$\begin{aligned} \tau_1 &= -J_3\Omega_3\Omega_2 + M_1, \\ \tau_2 &= J_3\Omega_3\Omega_1 + M_2. \end{aligned}$$

Also, let $\tau = \tau_1 b_1 + \tau_2 b_2 \in \mathbb{R}^3$. Then,

$$\dot{\omega}_{12} = \frac{1}{J_1}\tau, \tag{76}$$

with $\tau \cdot b_3 = 0$.

Suppose that the desired direction of b_3 is given by $b_{3_d}(t)$ as a function of time. It satisfies

$$\dot{b}_{3_d} = \omega_{12_d} \times b_{3_d} \tag{77}$$

for $\omega_{12_d} \in \mathbb{R}^3$ satisfying $\omega_{12_d} \cdot b_{3_d} = 0$.

Next, we design τ such that b_3 follows $b_{3_d}(t)$ asymptotically. Define the error variables as

$$e_b = b_{3_d} \times b_3, \quad e_\omega = \omega_{12} + \hat{b}_3^2 \omega_{12_d} = \omega_{12} + (b_3 \cdot \omega_{12_d}) b_3 - \omega_{12_d}.$$

It has been shown that the following input guarantees that $(e_b, e_\omega) = (0, 0)$ is exponentially stable.

$$\tau = -k_b e_b - k_\omega e_\omega - J_1 (b_3 \cdot \omega_{12_d}) \dot{b}_3 - J_1 \hat{b}_3^2 \dot{\omega}_{12_d}.$$

The resulting control moment is given by

$$\begin{aligned} M_1 &= \tau \cdot b_1 + J_3 \Omega_3 \Omega_1, \\ M_2 &= \tau \cdot b_2 - J_3 \Omega_3 \Omega_2. \end{aligned}$$

The third element of the moment can be selected such that $\Omega_3 \to 0$,

$$M_3 = -k_{\Omega_3} \Omega_3.$$

(TODO: develop M_3 for yaw control.)

References

[1] W. Brogan. *Modern Control Theory*. Prentice Hall, 1991.

[2] T. Lee, M. Leok, and N. McClamroch. Geometric tracking control of a quadrotor UAV on SE(3). In *Proceedings of the IEEE Conference on Decision and Control*, pages 5420–5425, Atlanta, GA, Dec. 2010.