Product rule for FFT on SO(3)

Let $f, g \in L^2(\mathrm{SO}(3))$, $F^l_{m,n}$, $G^l_{m,n}$ be the corresponding Fourier coefficients for $0 \le l \le B-1$ and $-l \le m, n \le l$. Then we have the approximation

$$\begin{aligned} f(R) &= \sum_{l=0}^{B-1} \sum_{m,n=-l}^{l} (2l+1) F^l_{m,n} D^l_{m,n}(R) \\ g(R) &= \sum_{l=0}^{B-1} \sum_{m,n=-l}^{l} (2l+2) G^l_{m,n} D^l_{m,n}(R). \end{aligned}$$

The product of f and g has approximation

$$\begin{aligned} (fg)(R) &= \sum_{l_1=0}^{B-1} \sum_{l_2=0}^{B-1} \sum_{m_1,n_1=-l_1}^{l_1} \sum_{m_2,n_2=-l_2}^{l_2} (2l_1+1)(2l_2+1) F^{l_1}_{m_1,n_1} G^{l_2}_{m_2,n_2} D^{l_1}_{m_1,n_1}(R) D^{l_2}_{m_2,n_2}(R) \\ &= \sum_{l_1=0}^{B-1} \sum_{l_2=0}^{B-1} (2l_1+1)(2l_2+1) \mathrm{tr}\left(F^{l_1} \otimes G^{l_2} \cdot \left(D^{l_1}(R) \otimes D^{l_2}(R) \right)^T \right) \\ &= \sum_{l_1=0}^{B-1} \sum_{l_2=0}^{B-1} (2l_1+1)(2l_2+1) \mathrm{tr}\left(C^T_{l_1,l_2} F^{l_1} \otimes G^{l_2} C_{l_1,l_2} \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} D^l(R)^T \right), \end{aligned}$$

where C_{l_1,l_2} is the Clebsch-Gordan coefficient. Let

$$A_{l_1,l_2} = (2l_1+1)(2l_2+1) C^T_{l_1,l_2} F^{l_1} \otimes G^{l_2} C_{l_1,l_2},$$

and denote the Fourier coefficients of fg as $H^l_{m,n}$, then

$$H^l = \sum_{\substack{|l_1-l_2| \le l \le l_1+l_2 \\ 0 \le l_1 \le B-1 \\ 0 \le l_2 \le B-1}} = \frac{1}{2l+1} A_{l_1,l_2}(l),$$

where $A_{l_1,l_2}(l)$ is a sub-diagonal block of A_{l_1,l_2}, whose row and column indices range from $l^2 - (l_1-l_2)^2 + 1$ to $l^2 - (l_1-l_2)^2 + 2l + 1$.