Quadrotor with Suspended Payload

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The system under consideration in this article, described below, is very similar to the one in [1], except for the additional thrust force and control moment by the quadrotor. Hence, the derivation remains largely the same as given in [1], along with addition of virtual work because of the thrust force and control moments. The relevant equations are, hence, borrowed directly from [1].

1 Vehicle configuration

We aim to develop continuous-time dynamical equations for a quadrotor with an extensible-cable suspended payload having a reeling mechanism. The following assumptions are set in place:

- The reeling drum is located at the center of mass of the quadrotor.
- Gravity is assumed constant.
- the radius of the reeling drum and the length of the guideway is very small when compared to the length of the cable.
- The part of the cable inside the guideway is inextensible, while the deployed portion is extensible.
- The reeling drum rotates about the second axis of the quadrotor body-fixed axis (the e_y axis).

Please refer to figure 1 for the vehicle configuration. A spatial reference frame is defined as showing in the figure. A quadrotor body-fixed frame is also defined, which has it's origin at the center of mass of the quadrotor. Next, a payload body-fixed frame is defined with it's origin at the point where the cable is attached to the payload. We use the following notation throughout this article.

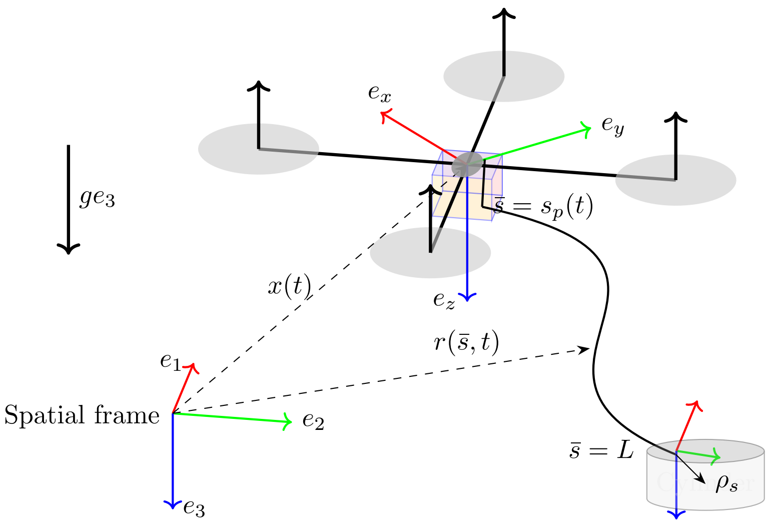

Figure 1: Quadrotor with suspended payload

$m \in \mathbb{R}$	mass of the quadrotor
$J \in \mathbb{R}^{3\times 3}$	inertia matrix of the quadrotor
$J_s \in \mathbb{R}^{3\times 3}$	inertia matrix of the payload
$R \in \mathbb{SO}(3)$	rotation matrix from quadrotor body frame to spatial frame
$R_s \in \mathbb{SO}(3)$	rotation matrix from payload body frame to spatial frame
$x \in \mathbb{R}^3$	location of quadrotor center of mass
$\Omega \in \mathbb{R}^3$	angular velocity of the quadrotor in its body-frame axes
$b \in \mathbb{R}$	radius of the reeling drum
$d \in \mathbb{R}$	length of the guideway
$\rho \in \mathbb{R}^3$	the vector from the center of mass of the quadrotor to the beginning of the guideway, represented in quadrotor body frame; $\rho = [b, 0, d]$
$\rho_s \in \mathbb{R}^3$	the vector from the point of attachment of the cable on the payload to the payload center of mass, represented in the payload body frame axes
$m_r \in \mathbb{R}$	mass of the reeling drum
$J_r \in \mathbb{R}^{3\times 3}$	inertia matrix of the reeling drum, given by $J_r = \kappa_r d^2$
$\bar{\mu} \in \mathbb{R}$	mass of the cable per unit unstretched length of the cable
$L \in \mathbb{R}$	the total unstretched length of the cable
$\bar{s} \in [0, L]$	the unstretched arc length of the cable from the point where it is attached to the reeling drum to a material point P on the cable
$s(\bar{s}, t) \in \mathbb{R}^+$	the stretched cable arc length of a material point at distance $\bar{s}$
$s_p(t) \in [b, L]$	the arc length of the cable between the point where the cable is attached to the reeling drum and the beginning of the guideway
$r(\bar{s}, t) \in \mathbb{R}^3$	the location of the cable material point in the spatial frame; $r(s_p, t) = x(t) + R\rho$
$m_s \in \mathbb{R}$	mass of the payload
$\Omega_s \in \mathbb{R}^3$	angular velocity of the payload in its body fixed frame
$u \in \mathbb{R}$	control moment applied at the reeling drum
$f \in \mathbb{R}$	thrust force generated by the rotors of the quadrotor
$M \in \mathbb{R}^3$	control moment generated by the rotors of the quadrotor

2 Continuous-time dynamical model

We refer to [1] to come up with a continuous time dynamical model that describes the motion of the system, and we use Hamilton's variational principle for the same. We note that the attitude kinematics are given as follows.

$$\dot{R} = R\hat{\Omega} \tag{1}$$

$$\dot{R}_s = R_s\hat{\Omega}_s \tag{2}$$

$$\tag{3}$$

where the *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is such that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$.

2.1 Lagrangian

2.1.1 Kinetic energy

The kinetic energy is composed of the kinetic energy of the quadrotor (T_{b_1}), the kinetic energy of the reeling mechanism (T_{b_2}), the kinetic energy of the deployed tether (T_c) and the kinetic energy of the payload (T_p). We will calculate each of these. The kinetic energy of the quadrotor is given below.

$$T_{b_1} = \frac{1}{2}m\dot{x}\cdot\dot{x} + \frac{1}{2}J\Omega\cdot\Omega \tag{4}$$

The kinetic energy of the reeling mechanism (including the reeling drum and the undeployed tether) is calculated below. the kinetic energy of infinitesimal material points is integrated to obtain the total kinetic energy due to the reeling mechanism. Let $\tilde{\rho}$ be the position vector of an infinitesimal material point in the quadrotor body frame. The position vector of this point in the spatial frame will be given by $x(\rho, t) = x + R\rho$.

$$T_{b_2} = \int_{\mathcal{B}} \frac{1}{2}\|\dot{x} + R\hat{\Omega}\tilde{\rho}\|^2 dm$$

Simplifying the above equation, along with the assumptions stated beforehand, result in the following expression for the kinetic energy.

$$T_{b_2} = \frac{1}{2}(m_r + \bar{\mu}s_p)\dot{x}\cdot\dot{x} + \frac{1}{2}\bar{\mu}s_p\dot{s}_p^2 + \frac{1}{2}\kappa_2\dot{s}_p^2 \tag{5}$$

where $\kappa_2 = e_2 \cdot \kappa_r e_2$. Next, let $\dot{r}(\bar{s}, t)$ denote the partial derivative of $r(\bar{s}, t)$ with respect to t. The kinetic energy of the deployed portion of the tether is given as follows.

$$T_c = \int_{s_p}^{L} \frac{1}{2}\bar{\mu}\dot{r}(\bar{s})\cdot\dot{r}(\bar{s})d\bar{s} \tag{6}$$

Next, to calculate the kinetic energy of the payload, consider an infinitesimal material point, the position vector of which is given by $\tilde{\rho}$ in the payload body-fixed frame. This position vector in the spatial frame would be given by $r(L) + R_s\tilde{\rho}$. Thus, the kinetic energy of the payload is given as follows.

$$T_p = \int_{\mathcal{B}_c} \frac{1}{2}\|\dot{r}(L) + R_s\hat{\Omega}_s\tilde{\rho}\|^2 dm$$

$$\frac{1}{2}m_s\dot{r}(L)\cdot\dot{r}(L) + m_s\dot{r}(L)\cdot R_s\hat{\Omega}_s\left(\int_{\mathcal{B}_c}\tilde{\rho}dm\right) + \frac{1}{2}\Omega_s\cdot\left(\int_{\mathcal{B}_c} -\hat{\rho}^2 dm\right)\Omega_s$$

Since $\rho_s = \int_{\mathcal{B}_c} \tilde{\rho} dm$ and $J_s = \int_{\mathcal{B}_c} -\hat{\rho}^2 dm$, we get the following expression for T_p.

$$T_p = \frac{1}{2} m_s \dot{r}(L) \cdot \dot{r}(L) + m_s \dot{r}(L) \cdot R_s \hat{\Omega}_s \rho_s + \frac{1}{2} \Omega_s \cdot J_s \Omega_s \tag{7}$$

2.1.2 Potential energy

The total potential energy will have contributions from the quadrotor plus the reeling mechanism (V_b), the deployed part of the cable (V_c) and the payload V_p. Now,

$$V_b = -m_b g e_3 \cdot x \tag{8}$$

where $m_b = m + m_r + \bar{\mu} s_p$. Next, the potential energy of the deployed portion will contain the gravitational potential energy as well as the strain energy. We refer to [1] and write the corresponding potential energy as follows.

$$V_c = \frac{1}{2} \int_{s_p}^{L} EA(\|r'(\bar{s})\| - 1)^2 d\bar{s} - \int_{s_p}^{L} \bar{\mu} g r(\bar{s}) \cdot e_3 d\bar{s} \tag{9}$$

where E denotes the Young's modulus, A denotes the cross-sectional area of the cable and $()'$ denotes the partial derivative with respect to $\bar{s}$. Next, we find out the potential energy of the payload. Since the position of the center of mass of the payload is given by $r(L) + R_s \rho_s$, its potential energy is given as follows.

$$V_p = -m_s g e_3 \cdot (r(L) + R_s \rho_s)) \tag{10}$$

Now, from equations 4-10, the Lagrangian is as defined below.

$$L = L_b + L_c + L_p \tag{11}$$

where $L_b = T_{b_1} + T_{b_2} - V_b$, $L_c = T_c - V_c$ and $L_p = T_p - L_p$.

3 Euler-Lagrange equations

Let the action integral be defined as follows.

$$\mathfrak{S} = \int_{t_0}^{t_f} (L_b + L_c + L_p) \, dt = \mathfrak{S}_b + \mathfrak{S}_c + \mathfrak{S}_p \tag{12}$$

Using the Lagrange-d'Alembert principle, the variation of action integral is equal to the negative of virtual work for fixed boundary conditions. Thus, the first course of action is to calculate the variation of the action integral. Since equations 1-10 are the same as those in [1], the expression for $\mathfrak{S}$ is the same as given in [1]. Hence, we directly directly refer to equations 14, 18, 19 in [1] to obtain the variation of $\mathfrak{S}$, denoted by $\delta\mathfrak{S}$. These are as given below.

$$\begin{aligned} \delta\mathfrak{S}_b = \int_{t_0}^{t_f} & \Big\{ - m_b \ddot{x} - \bar{\mu} \dot{s}_p \dot{x} + m_b g e_3 \Big\} \cdot \delta x \\ & + \Big\{ - (\bar{\mu} s_p + \kappa_2) \ddot{s}_p + \frac{1}{2} \bar{\mu} (\dot{x} \cdot \dot{x} - \dot{s}_p^2) + \bar{\mu} g e_3 \cdot x \Big\} \delta s_p \\ & - \big\{ J\dot{\Omega} + \hat{\Omega} J \Omega \big\} \cdot \eta dt \end{aligned} \tag{13}$$

$$
\begin{aligned}
\delta\mathfrak{S}_c = &\int_{t_0}^{t_f}\int_{s_p}^{L}\Big\{-\bar{\mu}\ddot{r}(\bar{s}) + F'(\bar{s}) + \bar{\mu}ge_3\Big\}\cdot\delta r(\bar{s})d\bar{s}dt \\
&+\int_{t_0}^{t_f}\Big\{-\frac{1}{2}\bar{\mu}\dot{r}(s_p^+)\cdot\dot{r}(s_p^+) + \frac{1}{2}EA(\|r'(s_p^+)\|-1)^2 - \bar{\mu}ge_3\cdot r(s_p)\Big\}\delta s_p \\
&+\bar{\mu}\dot{r}(s_p^+)\cdot\delta r(s_p^+)\dot{s}_p - F(L)\cdot\delta r(L) + F(s_p)\cdot\delta r(s_p^+)\,dt
\end{aligned}
\tag{14}
$$

where $F(\bar{s}) = EA\frac{\|r'(\bar{s})\|-1}{\|r'(\bar{s})\|}r'(\bar{s})$.

$$
\begin{aligned}
\delta\mathfrak{S}_p = &\int_{t_0}^{t_f}\Big\{-m_s\ddot{r}(L) - m_sR_s\hat{\Omega}_s^2\rho_s + m_sge_3\Big\}\cdot\delta r(L) \\
&+\Big\{-J_s\dot{\Omega}_s - m_s\hat{\rho}_sR_s^T\ddot{r}(L) - \hat{\Omega}_sJ_s\Omega_s + m_sg\hat{\rho}_sR_s^Te_3\Big\}\cdot\eta_s\ dt
\end{aligned}
\tag{15}
$$

Now, using $\delta\mathfrak{S} = \delta\mathfrak{S}_b + \delta\mathfrak{S}_c + \delta\mathfrak{S}_p$ and equations 13-15, we obtain the expression for $\delta\mathfrak{S}$. Next, we use the Lagrange - d'Alembert principle with discontinuity as given in [1]. This results in the following.

$$
\delta\mathfrak{S} + \int_{t_0}^{t_f}(Q + u/d)\delta s_p - ue_y\cdot\eta + M\cdot\eta - fRe_z\cdot\delta x\,dt = 0 \tag{16}
$$

where the expression for Q (obtained from [1]) is as given below.

$$
Q = -\frac{1}{2}\bar{\mu}(\|r'(s_p^+)\|-1)^2\dot{s}_p^2 - \frac{1}{2}EA(\|r'(s_p^+)\|-1)^2 \tag{17}
$$

3.1 EL equations

Using 16, we obtain the following continuous-time equations which describe the dynamics of the system.

$$
-m_b\ddot{x} + m_bge_3 - \bar{\mu}\dot{s}_p(r'(s_p^+)\dot{s}_p + R\hat{\rho}\Omega) + F(s_p) - fRe_z = 0 \tag{18}
$$

$$
-J\dot{\Omega} - \hat{\Omega}J\Omega + \bar{\mu}\dot{s}_p\hat{\rho}R^T(\dot{x} - r'(s_p^+)\dot{s}_p - R\hat{\rho}\Omega) + \hat{\rho}R^TF(s_p) - ue_y + M = 0 \tag{19}
$$

$$
\begin{aligned}
&-(\bar{\mu}s_p + \kappa_2)\ddot{s}_p - \frac{1}{2}\bar{\mu}(\dot{x} - R\hat{\rho}\Omega)\cdot(\dot{x} - R\hat{\rho}\Omega) + \frac{1}{2}\bar{\mu}\dot{x}\cdot\dot{x} - \bar{\mu}ge_3\cdot r(s_p) \\
&+\bar{\mu}ge_3\cdot x - F(s_p)\cdot r'(s_p^+) + \bar{\mu}\dot{s}_p^2\big(\|r'(s_p^+)\|-1\big) + u/d = 0
\end{aligned}
\tag{20}
$$

$$
-\bar{\mu}\ddot{r}(\bar{s}) + F'(\bar{s}) + \bar{\mu}ge_3 = 0, \quad (\bar{s}\in[s_p, L],\ r(s_p) = x + R\rho) \tag{21}
$$

$$
-m_s\ddot{r}(L) + m_sR_s\hat{\rho}_s\dot{\Omega}_s - m_sR_s\hat{\Omega}_s^2\rho_s + m_sge_3 - F(L) = 0 \tag{22}
$$

$$
-J_s\dot{\Omega}_s - m_s\hat{\rho}_sR_s^T\ddot{r}(L) - \hat{\Omega}_sJ_s\Omega_s + m_sg\hat{\rho}_sR_s^Te_3 = 0 \tag{23}
$$

Equations 18 and 22 represent positional dynamics of the quadrotor and the payload respectively. Similarly, equations 19 and 23 represent the rotational dynamics of quadrotor and payload respectively. Equation 20 represents the variation of s_p with time and equation 21 gives the motion of each material point on the cable. The discretized version of equation 21 is given in the following section.

4 Cable discretisation

To simulate the dynamics, we need to discretize the cable into N number of smaller parts, which is done as follows. Since there are N discrete parts of the cable, there would be $N+1$ *nodes*.

Since we are looking at continuous time dynamics, we discretize the cable in space and not in time. Let the position of the nodes of the k-th discrete element be given by r_{k-1} and r_k respectively. Hence, the position of every point in this discrete element will be given as follows:

$$r_k(\bar{s}, t) = (1 - \zeta_k(\bar{s}))r_{k-1} + \zeta_k(\bar{s})r_k$$

where $\zeta_k(\bar{s}) = \frac{(\bar{s}-s_p)-(k-1)(L-s_p)/N}{(L-s_p)/N}$ and $\bar{s} \in [s_p + (k-1)\frac{L-s_p}{N}, s_p + k\frac{L-s_p}{N}]$.

For the k-th discrete cable element, we now write $r'(\bar{s})$, $F(\bar{s})$ and $F'(\bar{s})$ as follows for all $\bar{s} \in [s_p + (k-1)\frac{L-s_p}{N}, s_p + k\frac{L-s_p}{N}]$.

$$\begin{aligned} r_k'(\bar{s}) &= \frac{r_k - r_{k-1}}{(L-s_p)/N} \\ F_k(\bar{s}) &= EA\frac{\|r_k'(\bar{s})\| - 1}{\|r_k'(\bar{s})\|}r_k'(\bar{s}) \\ F_k'(\bar{s}) &= \frac{F_k(\bar{s}) - F_{k-1}(\bar{s})}{(L-s_p)/N} \end{aligned}$$

With these values for the discrete cable elements, we break down equation 21 for each discrete cable element as shown below.

$$-\bar{\mu}\ddot{r}_k(\bar{s}) + F_k'(\bar{s}) + \bar{\mu}ge_3 = 0, \quad k = 1, 2, ..., N \tag{24}$$

5 Simplified dynamics

We attempt simplified simulations to begin with, with the following assumptions in place:

- $\rho = rho_s 0_{3\times 1}$
- $\dot{s}_p = 0$
- no control forces and moments, i.e. $f = 0$, $M = 0_{3\times 1}$

This results in the simplified dynamical equations as given below.

$$-m_b\ddot{x} + m_bge_3 + F(s_p) = 0 \tag{25}$$

$$-m_s\ddot{r}(L) + m_sge_3 - F(L) = 0 \tag{26}$$

$$-J\Omega - \hat{\Omega}J\Omega - ue_y = 0 \tag{27}$$

$$F(s_p) \cdot r'(s_p^+) = u/d \tag{28}$$

$$-J_s\dot{\Omega}_s - \hat{\Omega}J_s\Omega_s = 0 \tag{29}$$

Since the rotational dynamics and translational dynamics are decoupled, we begin with simulating only the translational motion of the system, given by equations 25 and 26.

Appendix

This section contains the derivation of the equations described in the previous section, and are largely a repetition of the equations provided in [1].

Variation of $\mathfrak{S}_b$: We refer to [1] to find out the variation of the rotation matrix, which is as given below.

$$\delta R = R\hat{\eta} \tag{30}$$

for $\eta \in \mathbb{R}^3$. It is noted that the variation of R lies in $\mathfrak{so}(3)$, i.e. the set of 3×3 skew-symmetric matrices. Next, since the variation of angular velocity is obtained from the attitude kinematics, and is given as below.

$$\delta\hat{\Omega} = (\dot{\eta} + \Omega \times \eta)^{\wedge} \tag{31}$$

Next, we simplify the variation of $\mathcal{S}_b$ as follows.

$$\begin{aligned}
\mathfrak{S}_b &= \int_{t_0}^{t_f} \left\{ \frac{1}{2} m\dot{x} \cdot \dot{x} + \frac{1}{2} J\Omega \cdot \Omega + \frac{1}{2}(m_r + \bar{\mu}s_p)\dot{x} \cdot \dot{x} + \frac{1}{2}\bar{\mu}s_p\dot{s}_p^2 + \frac{1}{2}\kappa_2\dot{s}_p^2 + m_b g e_3 \cdot x \right\} dt \\
&= \int_{t_0}^{t_f} \left\{ \frac{1}{2} m_b\dot{x} \cdot \dot{x} + \frac{1}{2} J\Omega \cdot \Omega + \frac{1}{2}\bar{\mu}s_p\dot{s}_p^2 + \frac{1}{2}\kappa_2\dot{s}_p^2 + m_b g e_3 \cdot x \right\} dt
\end{aligned}$$

Next, using equations 30, 31 and using integration by parts, we obtain the following variation.

$$\begin{aligned}
\delta\mathfrak{S}_b = \int_{t_0}^{t_f} & \Big\{ - m_b\ddot{x} - \bar{\mu}\dot{s}_p\dot{x} + m_b g e_3 \Big\} \cdot \delta x \\
&+ \Big\{ - (\bar{\mu}s_p + \kappa_2)\ddot{s}_p + \frac{1}{2}\bar{\mu}(\dot{x} \cdot \dot{x} - \dot{s}_p^2) + \bar{\mu} g e_3 \cdot x \Big\} \delta s_p \\
&- \big\{ J\dot{\Omega} + \hat{\Omega}J\Omega \big\} \cdot \eta dt
\end{aligned} \tag{32}$$

Variation of $\mathfrak{S}_c$: We write the expression of $\mathfrak{S}_c$ for clarity.

$$\mathfrak{S}_c = \int_{t_0}^{t_f} \int_{s_p}^{L} \frac{1}{2}\bar{\mu}\dot{r}(\bar{s}) \cdot \dot{r}(\bar{s}) - \frac{1}{2}EA(\|r'(\bar{s})\| - 1)^2 + \bar{\mu} g e_3 \cdot r(\bar{s}) \;\; d\bar{s}dt$$

Since s_p which is the limit of the inner integral is a function of time, the variation of $\mathfrak{S}_c$ has to account for the variation of s_p as well. We refer to equation 15 of [1]. Thus, we get the following.

$$\begin{aligned}
\delta\mathfrak{S}_c = & \int_{t_0}^{t_f} \int_{s_p}^{L} \left\{ \bar{\mu}\dot{r}(\bar{s}) \cdot \delta\dot{r}(\bar{s}) - EA\frac{\|r'(\bar{s})\| - 1}{\|r'(\bar{s})\|} r'(\bar{s}) \cdot \delta r'(s) + \bar{\mu} g e_3 \cdot \delta r(\bar{s}) \right\} d\bar{s}dt \\
& - \int_{t_0}^{t_f} \left\{ \frac{1}{2}\bar{\mu}\dot{r}(s_p^+) \cdot \dot{r}(s_p^+) - \frac{1}{2}EA(\|r'(s_p^+)\| - 1)^2 + \bar{\mu} g e_3 \cdot r(s_p) \right\} \delta s_p \;\; dt
\end{aligned} \tag{33}$$

As shown in [1], integration by parts cannot be applied to the first term in equation 33 as $s_p(t)$ is a function of t. Thus, we apply Green's theorem and then, using integration by parts, we obtain the following as shown in [1].

$$\delta\mathfrak{S}_c = \int_{t_0}^{t_f}\int_{s_p}^{L}\Big\{ -\bar{\mu}\ddot{r}(\bar{s}) + F'(\bar{s}) + \bar{\mu}ge_3 \Big\}\cdot\delta r(\bar{s})d\bar{s}dt$$
$$+\int_{t_0}^{t_f}\Big\{ -\frac{1}{2}\bar{\mu}\dot{r}(s_p^+)\cdot\dot{r}(s_p^+) + \frac{1}{2}EA(\|r'(s_p^+)\|-1)^2 - \bar{\mu}ge_3\cdot r(s_p)\Big\}\delta s_p$$
$$+\bar{\mu}\dot{r}(s_p^+)\cdot\delta r(s_p^+)\dot{s}_p - F(L)\cdot\delta r(L) + F(s_p)\cdot\delta r(s_p^+)\,dt \tag{34}$$

where $F(\bar{s}) = EA\frac{\|r'(\bar{s})\|-1}{\|r'(s)\|}r'(\bar{s})$. Further, let the location of the beginning of the guideway be r_{s_p}, which is a function of s_p and t. Next, the terms $\delta r(s_p^+)$ and $\dot{r}(s_p^+)$ in equation 34 are simplified follows. Let the position vector of the guideway entry point be given by $r(s_p(t),t)$. Since $r(s_p(t),t)$ is also equal to $x + r\rho$, we take its variation and obtain the following equation for $\delta r(s_p^+)$:

$$\delta r(s_p(t),t) = r'(s_p)\delta s_p + \delta r(s_p^+) = \delta x - R\hat{\rho}\eta$$
$$\Longrightarrow\ \delta r(s_p^+) = \delta x - r'(s_p)\delta s_p - R\hat{\rho}\eta$$

Similarly,

$$\dot{r}(s_p(t),t) = r'(s_p)\dot{s}_p + \dot{r}(s_p^+) = \dot{x} - R\hat{\rho}\Omega$$
$$\Longrightarrow\ \dot{r}(s_p^+) = \dot{x} - r'(s_p^+)\dot{s}_p - R\hat{\rho}\Omega$$

Substituting these values of $\delta r(s_p^+)$ and $\dot{r}(s_p^+)$ in equation 34, we get the following expression for $\delta\mathfrak{S}_c$.

$$\delta\mathfrak{S}_c = \int_{t_0}^{t_f}\int_{s_p}^{L}\Big\{ -\bar{\mu}\ddot{r}(\bar{s}) + F'(\bar{s}) + \bar{\mu}ge_3 \Big\}\cdot\delta r(\bar{s})d\bar{s}dt$$
$$+\int_{t_0}^{t_f}\Big\{ -\frac{1}{2}\bar{\mu}\Big(\dot{x} - r'(s_p^+)\dot{s}_p - R\hat{\rho}\Omega\Big)\cdot\Big(\dot{x} - r'(s_p^+)\dot{s}_p - R\hat{\rho}\Omega\Big) + \frac{1}{2}EA(\|r'(s_p^+)\|-1)^2 - \bar{\mu}ge_3\cdot r(s_p)\Big\}\delta s_p$$
$$+\bar{\mu}\Big(\dot{x} - r'(s_p^+)\dot{s}_p - R\hat{\rho}\Omega\Big)\cdot\delta r(s_p^+)\dot{s}_p - F(L)\cdot\delta r(L) + F(s_p)\cdot\Big(\delta x - r'(s_p)\delta s_p - R\hat{\rho}\eta\Big)\,dt \tag{35}$$

Variation of $\mathfrak{S}_p$:

$$\mathfrak{S}_p = \int_{t_0}^{t_f}\frac{1}{2}m_s\dot{r}(L)\cdot\dot{r}(L) + m_s\dot{r}(L)\cdot R_s\hat{\Omega}_s\rho_s + \frac{1}{2}\Omega_s\cdot J_s\Omega_s + m_sge_3\cdot(r(L)+R_s\rho_s))\,dt$$

Taking the variation of $\mathfrak{S}_p$ and using integration by parts, we arrive at the following expression.

$$\delta\mathfrak{S}_p = \int_{t_0}^{t_f}\Big\{ -m_s\ddot{r}(L) - m_sR_s\hat{\Omega}_s^2\rho_s + m_sge_3\Big\}\cdot\delta r(L) \tag{36}$$
$$+\Big\{ -J_s\dot{\Omega}_s - m_s\hat{\rho}R_s^T\ddot{r}(L) - \hat{\Omega}_sJ_s\Omega_s + m_sg\hat{\rho}_sR_s^Te_3\Big\}\cdot\eta_s\ dt \tag{37}$$

Accounting for discoutinuity: The cable just outside the guideway is assumed extensible, whereas the cable just inside the guideway is assumed inextensible. This creates a discontinuity, i.e. there is an abrupt change in the speed of the material point of the cable just inside and just outside the guideway. Hence, the variation of action integral is no more equal to the negative of the virtual work done by all forces and moments. An additional term Q, known as Carnot energy loss term, is introduced as shown in [1].

Work is done by the control moment applied on the reeling drum. Since a moment equal to ue_y is applied on the reeling drum and it is an internal moment for the quadrotor+reeling mechanism system, a moment equal to $-ue_y$ is applied on the quadrotor as a reaction in the quadrotor body-fixed frame. Next, it is assumed that the net thrust force due to the four rotors of the quadrotor is $-fRe_z$, and the net control moment due to the rotors is equal to M in the quadrotor body-fixed frame. This results in the following by the Lagrange-d'Alembert principle.

$$\delta\mathfrak{S} + \int_{t_0}^{t_f} (Q + u/d)\delta s_p - ue_y \cdot \eta + M \cdot \eta - fRe_z \cdot \delta x \, dt = 0 \tag{38}$$

where the expression for Q is as given below [1].

$$Q = -\frac{1}{2}\bar{\mu}(\|r'(s_p^+)\| - 1)^2 \dot{s}_p^2 - \frac{1}{2}EA(\|r'(s_p^+)\| - 1)^2 \tag{39}$$

References

[1] Taeyoung Lee, Melvin Leok, and N. Harris McClamroch. High-fidelity numerical simulation of complex dynamics of tethered spacecraft. *Acta Astronautica*, 99:215–230, 2014.