An Approximation Algorithm for the Two-Node-Connected Star Problem with Steiner Nodes

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Abstract

The goal in topological network design is to build a minimum-cost topology meeting specific real-life constraints. There is a cost-robustness trade-off under single and multiple failures.

Previous works in the field suggest that a backbone composed by a two-node-connected toplogy provides savings with respect to elementary cycles. Consequently, we introduce the Two-Node Connected Star Problem with Steiner Nodes (2NCSP-SN). The goal is to design a minimum-cost topology, where the backbone is two-node connected, the access network is connected in a *star* topology or by direct links to the backbone, and optional nodes (called Steiner nodes) could be included in the solution. The 2NCSP-SN belongs to the class of $\mathcal{NP}$-Hard problems. This promotes the development of heuristics and approximation algorithms.

An approximation algorithm of factor 4α for the 2NCSP-SN is introduced, being $\alpha \geq 1/2$ the cost-ratio between backbone and access links. This is a generalization of the well-known factor 2 for the design of minimum-cost two-connected spanning networks (if we fix $\alpha = 1/2$). Finally, an exact Integer Linear Programming (ILP) formulation is proposed in order to highlight the effectiveness of the approximation algorithm. The results confirm a small gap between the globally optimum solution and the topology offered by our approximation algorithm when the ratio α is close to $1/2$.

Keywords: Network Optimization, Approximation Algorithm, Integer Linear Programming

1   Introduction

The minimum-cost toplogy meeting simple connectivity is the Minimum Spanning Tree, or MST. In this basic setting, Greedy algorithm efficiently finds the MST [3]. However, the design of communication systems is more challenging, since the cost-redundancy trade-off should be tackled. In the physical layer, this means that the network must be robust under single failures of any component, or two-node connected.

A natural way to connect terminal nodes in order to fulfill two-connectivity is to consider an elementary cycle. The design of a minimum-cost elementary cycle is a celebrated problem, known as the Travelling Salesman Problem or TSP. Supported by the concept of reducitibility and Karp list, it is well-known that the TSP belongs to the class of $\mathcal{NP}$-Hard problems [2]. Indeed, a natural reduction to Hamiltonian tour is obtained using 0-1 costs in their links.

A foundational work credited by Clyde Monma et. al. confirms that elementary cycles are sub-optimal for the design of minimum-weight 2-node-connected spanning network problem or MW2CSNP [5]. However, the authors show that cost of the cheapest Hamiltonian tour, $C_{\mathcal{H}}$, always respects the inequality $C_{\mathcal{H}} \leq 4/3 OPT$, being OPT the cost of the globally optimum solution. Furthermore, the bound is tight, since they build an asymptotic family of graphs such that the cost-ratio tends to 4/3. The authors find structural properties of the globally optimum solution for the MW2CSNP and, they remark that the optimum 2-edge connected is also 2-node connected. The MW2CSNP is at least as hard as the TSP (using 0-1 costs, OPT is not greater than the number of nodes iff the graph is Hamiltonian). Therefore, several authors address the MW2CSNP using heuristics and approximation algorithms. Christofides provides a 3/2-factor for metric TSP which, together with the 4/3 factor credited by Monma et. al. results in an approximation algorithm with factor 2 for the MW2CSNP [1]. This factor-2 can be extended to several topological network design problems such as Steiner Networks, using either combinatorial analysis or strong duality theorem from linear programming [8,9].

The physical implementation of fiber-optics communication imposed a new challenge, *geographical diversity*. As a consequence, real-life communication networks are hierarchically structured in a backbone and access network, where

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customers in the last-mile have elementary connectivity requirements [7]. Martine Labbé et. al. introduced a hierarquically organized network, called Ring Star Problem, where the backbone is a ring and the access network is a *star*, this is, direct links connected to the ring [4]. A natural topological extension called Two-Node Connected Star Problem (2NCSP) has been introduced by Recoba et. al. [6]. The ring is replaced by an arbitrary two-node connected topology, and the authors confirm savings with respect to optimal solutions for the RSP (which are in turn feasible for the 2NCSP). Here, an extension of the 2NCSP is proposed, by the introduction of optional Steiner nodes in feasible solutions.

Complexity: Being RSP and 2NCSP $\mathcal{NP}$-Hard problems, 2NCSP-SN is $\mathcal{NP}$-Hard too.

The contributions of this article are the following:

- The Two-Node-Connected Star Problem with Steiner Nodes (2NCSP-SN) is here introduced.

- A generalization of the factor-2 result is offered. Specifically, an approximation algorithm with factor 4α is introduced for the 2NCSP, being $\alpha \geq 1/2$ the relation between the link-costs from the backbone/access network.

- An exact Integer Linear Programming (ILP) formulation for the 2NCSP-SN is presented.

- A sensibility analysis is carried out in order to understand the effectiveness of our approximation algorithm.

The paper is organized as follows. Section 2 formally presents the 2NCSP-SN using an ILP formulation. An approximation algorithm with factor 4α is introduced in Section 3. A fair comparison between the globally optimum solution and the approximation algorithm is performed in Section 4. Finally, Section 5 presents concluding remarks and trends for future work.

2   Problem

We are given a simple graph $G = (V, E)$, internal link-costs $IC = \{c_e\}_{e \in E}$, external link-costs $EC = \{d_e\}_{e \in E}$, and $V = S \cup T$, being T the terminal-set and S Steiner nodes with costs $\{a(s)\}_{s \in S}$. The goal in 2NCSP-SN is to build a minimum-cost spanning subgraph $H = (V_H, E_H)$, where $T \subseteq V_H$ and $V_H = I \cup L$, being L the set of leaf-nodes (in the access network), such that $deg_H(v) = 1, \forall v \in L$, and the induced subgraph $H(I)$ is two-node-connected.

Let us develop an ILP for the problem under study. The key idea is to consider connectivity requirement $r_q = 2$ for every pair of terminals q from the backbone,

while $r_q = 1$ otherwise. Let $Q = \{q = (i, j), \forall i \neq j, i, j \in T \subseteq V\}$. Consider the following set of binary variables:

- $z_{ij} = 1$ iff $(i, j) \in E$ is in the backbone;
- $y_{ij} = 1$ iff $(i, j) \in E$ is in the access network;
- x_{ij}^q is the $i - j$ flow for every pair of terminals q;
- $p_i = 1$ iff the i is included in the access network.

An ILP formulation for the 2NCSP-SN can be expressed as follows:

$$\min_{H \subseteq G} c(H) = \sum_{ij \in E} c_{ij}.z_{ij} + \sum_{s \in S} a_s.p_s + \sum_{ij \in E} d_{ij}.y_{ij} \tag{1}$$

$$\text{s.t.} \sum_{j:(j,i) \in E} x_{ji}^q - \sum_{j:(i,j) \in Ed^q} x_{ij}^q = I(i).r_i \ \forall i \in V, \ \forall q = (q_o, q_d) \in Q \tag{2}$$

$$I(i) = 1 \ \forall i \in V \backslash \{q_o\}, \ I(q_o) = -1 \tag{3}$$

$$r_i = 0 \ \forall i \in V \backslash \{q_o, q_d\} \tag{4}$$

$$\max(1, p_{q_o} + p_{q_d}) \leq r_i \leq 1 + \min(p_{q_o}, p_{q_d}), \ \forall i \in \{q_o, q_d\}, \ \forall q \in Q \tag{5}$$

$$x_{ij}^q + x_{ji}^q \leq z_{ij} + y_{ij}, \ \forall ij \in E, \ \forall q \in Q \tag{6}$$

$$\sum_{j \in \delta(i)} y_{ij} \leq 1 + Mp_i, \ \forall i \in T \tag{7}$$

$$\sum_{j \in \delta(i)} (z_{ij} + y_{ij}) \leq Mp_i, \ \forall i \in S \tag{8}$$

$$y_{ij} \leq 2 - p_i - p_j, \ \forall ij \in E, i, j \in V \tag{9}$$

$$z_{ij} \leq \min(p_i, p_j), \ \forall ij \in E, i, j \in V \tag{10}$$

$$z_{ij} + y_{ij} \leq 1 \ \forall ij \in E, i, j \tag{11}$$

$$2p_i \leq \sum_{j \in \delta(i)} z_{ij} \leq Mp_i, \ \forall i \in V \tag{12}$$

Where $\delta(i)$ is neighbor-set for node i, and M is an arbitrarily large integer. The objective function (1) is the contribution of internal/external connections and Steiner nodes. Constraints (2)-(5) ensure connectivity using Kirchhoff equations. Constraints (6) and (7) force one-way flow. By Constraint (8), optional Steiner nodes belong to the backbone, if needed. The definitions of binary variables y_{ij} and z_{ij} are captured by Constraints (9) and (10). Constraints (11) state that either y_{ij} or z_{ij} can be set to 1, but not both. Constraints (12) state that a terminal from the backbone could have multilinks, but nodes from the access network have one link.

3 Methodology

From now on, we assume that the internal/external costs are positive and internal costs satisfy the triangle inequality. Without loss of generality, a complete graph $G = (V, E)$ is considered. Let $\alpha_e = \frac{c_e}{d_e}, \forall e \in E$ be the primary/secondary cost ratio for each arc. In this section we build an approximation algorithm for the the 2NCSP-SN of factor 4α, being α:

$$\alpha = \max_{e \in E} \{\alpha_e\} \tag{13}$$

Recall that Christofides's algorithm is a 3/2-factor for the metric TSP. The key concept of our approximation algorithm is Christofides in order to span the terminal-set with an elementary cycle. Greedy augmentations of the solution including Steiner nodes also takes place, whenever the cost is reduced.

In Line 1, *Christofides* is called in order to build an elementary cycle C that spans the terminal-set T. The corresponding solution is updated in Lines 2-3, where the backbone is C and the access network is empty yet. In the while-loop (Lines 4-11), Steiner nodes are greedily included in the backbone, whenever the cost is reduced (Line 5). If this happens, some terminal node v is included in the access network, and the evidence $s \in S$ is added to the backbone (Lines 6-7). Observe that candidate terminals $t \in T$ are iteratively checked (Line 9), and the condition $|J| \geq 3$ forces to have a cycle in the backbone. The corresponding feasible solution F is finally returned (Line 12).

Require: $G = (T \cup S, E), c(e), d(e) \forall e \in E$,
  1: $C \leftarrow Christofides(G, c)$
  2: $L \leftarrow \emptyset, I \leftarrow T, E' \leftarrow E(C), J \leftarrow T$
  3: $F \leftarrow (L \cup I, E')$
  4: **while** $|J| \geq 3$ **do**
  5:    **if** there are $s \in S, (t, v), (v, w) \in F$: $c(t, v) + c(v, w) > d(v, s) + c(t, s) + c(s, w)$ **then**
  6:      $L \leftarrow L \cup \{v\}, I \leftarrow I \cup \{s\} \setminus \{v\}, J \leftarrow J \backslash \{t, v\}$
  7:      $E' \leftarrow E' \cup \{(t, s), (s, v), (s, w)\} \setminus \{(t, v), (v, w)\}$
  8:    **else**
  9:      $J \leftarrow J \setminus \{t\}$
10:    **end if**
11: **end while**
12: **return** $F = (I \cup L, E')$

Lemma 3.1 *If $L(F) \neq \emptyset$ then $\alpha > 1/2$*

Proof. By the triangle inequality, $c_{(t,v)} \leq c_{(t,s)} + c_{(s,v)}$ and $c_{(v,w)} \leq c_{(v,s)} + c_{(s,w)}$, so $c_{(t,v)} + c_{(v,w)} \leq c_{(t,s)} + 2.c_{(s,v)} + c_{(s,w)}$. If $L \neq \emptyset$, there exists $v \in L$, $s \in S \cap I$, $t, w \in I$ such that $c_{(t,v)} + c_{(v,w)} > d_{(v,s)} + c_{(t,s)} + c_{(s,w)}$,
Therefore $d_{(v,s)} = c_{(v,s)}/\alpha_{v,s} < 2.c_{(s,v)}$, so there exists $e = (v, s)$ such that $\alpha_e > 1/2$, that is $\alpha > 1/2$ with $\alpha = \max_{e \in E} \{\alpha_e\}$.

$\square$

Theorem 3.2 $c(F) \leq \max\{2, 4\alpha\} \times OPT$.

Proof. Let $G^* = (S^* \cup T, E^*)$ be the optimal solution, H the cheapest Hamilton tour spanning T and H^S the cheapest Hamilton tour spanning $T \cup S^*$. Analogously, let us denote TNC (TEC) to the optimal 2-node (resp. 2-edge) connected spanning subgraph for $T \cup S^*$. Recall that F is the output and C the cycle obtained using Christofides algorithm. Combining Monma and Christofides theorems:

$$c(G^*) \leq c(F) \leq c(C) \leq \frac{3}{2}c(H) \leq \left(\frac{3}{2}\right)\left(\frac{4}{3}\right)c(TNC) = 2c(TNC) \qquad (14)$$

Let $G^* = B \cup L$, being B its backbone. Consider an augmentation F' for G^*, doubling edges from L with cost $c_{r,j} = \alpha_{r,j}d_{r,j}$ and adding them. F' is 2-edge connected and

$$c(F') = c(B) + 2\sum_{r \in L} c_{r,j} \leq c(B) + 2\alpha \sum_{r \in L} d_{r,j} = 2\alpha c(G^*) + (1 - 2\alpha)c(B), \qquad (15)$$

If $1 - 2\alpha > 0$, $c(F') \leq 2\alpha c(G^*) + 2(1 - 2\alpha)c(G^*) \leq 2c(G^*)$. In this case a factor 2 is provided, and by Lemma 3.1 F consists of an elementary cycle.

Otherwise, combining (14) and (15) we have that:

$$\begin{aligned} c(G^*) \leq c(F) \leq 2c(TNC) = 2c(TEC) &\leq 2c(F') \\ &\leq 4\alpha c(G^*) + 2(1 - 2\alpha)c(B) \\ &\leq 4\alpha c(G^*). \end{aligned}$$

$\square$

4 Experimental Analysis

In order to highlight the effectiveness of our approximation algorithm, a sensibility analysis with respect to the ratio α is carried out. We consider a single instance from TSPLIB named berlin52.tsp. This is the case of a real-life network with Euclidean costs. In order to find the globally optimum solution, an induced subgraph with 22 nodes is considered (with 10 terminal-nodes and 12 Steiner nodes). The ILP has been executed in CPLEX 12.6.3 MIP solver using an Intel i7 processor, 2.30 GHz, 8GB RAM. Table 1 illustrates the performance $c(F)/OPT$ as a function of α. The cycle C obtained using Christofides algorithm is $c(C) = 3164.8$, while the cheapest Hamiltonian tour H spanning the terminal-set has a cost $c(H) = 2826.5$. Naturally, since F considers greedy augmentations of C, we get that $c(F) \leq c(C)$ in all cases.

Table 1
Sensibility Analysis as a function of α

α	OPT	$c(F)$	$c(F)/OPT$
10	485.78	3117.3	6.42
4	1110.48	3164.8	2.85
2	2115.02	3164.8	1.50
4/3	2611.45	3164.8	1.21
1	2786.80	3164.8	1.14
4/5	2811.63	3164.8	1.13
2/3	2822.88	3164.8	1.12
4/7	2825.60	3164.8	1.12

Our approximation algorithm outperforms Christofides only when $\alpha = 10$. In this case Steiner nodes are included in the solution. The factor is far away from 4α in all cases. Furthermore, when $\alpha = 4/7$ the ratio 1.12 is the lowest. Curiously enough, the results suggest that the performance is consistently better when the ratio α is decreased.

5 Conclusions

The Two-Node Connected Star Problem with Steiner Nodes (2NCSP-SN) is introduced. Its hardness promotes heuristics and approximation algorithms. Here, we introduce an approximation algorithm of factor 4α for the problem, being $\alpha \geq 1/2$ the minimum ratio among internal/external costs. An exact ILP formulation is also proposed. The celebrated factor 2 from Steiner networks is retrieved when $\alpha = 1/2$. Furthermore, a proof of concept suggests that the performance of the algorithm is consistently better when α is close to $1/2$.

Currently, we are working to develop a full GRASP methodology enriched with Variable Neighborhood Descent (VND) to find high-competitive solutions for the 2NCSP-SN.

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References

[1] N. Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, 1976.

[2] R. Karp. Reducibility among combinatorial problems. In R. Miller and J. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, 1972.

[3] Joseph B. Kruskal. On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem. *Proceedings of the American Mathematical Society*, 7(1):48–50, February 1956.

[4] Martine Labbé, Gilbert Laporte, Inmaculada Rodríguez Martín, and Juan Jose Salazar Gonzalez. The ring star problem: Polyhedral analysis and exact algorithm. *Networks*, 43(3):177–189, 2004.

[5] Clyde L Monma, Beth Spellman Munson, and William R Pulleyblank. Minimum-weight two-connected spanning networks. *Mathematical Programming*, 46(1-3):153–171, 1990.

[6] Rodrigo Recoba, Franco Robledo, Pablo Romero, and Omar Viera. Two-node-connected star problem. *International Transactions in Operational Research*, 25(2):523–543, 2018.

[7] F. Robledo. *GRASP heuristics for Wide Area Network design*. PhD thesis, IRISA/INRIA Université de Rennes I, 2005.

[8] Vijay V Vazirani. *Approximation algorithms*. Springer Science & Business Media, 2003.

[9] David P. Williamson and David B. Shmoys. *The Design of Approximation Algorithms*. Cambridge University Press, New York, NY, USA, 1st edition, 2011.