A note about the inverse of a random matrix

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I. AN IDEA

Let A be an invertible $N \times N$ matrix whose elements are

$$A_{ii} = 1 \text{ for all } i = 1, \dots, N \text{ and } A_{ij} \in \mathcal{N}(0, \sigma^2) \text{ for all } i, j = 1, \dots, N,\ i \neq j\,, \tag{1}$$

namely the off-diagonal elements of A are i.i.d. random variables drawn from a normal distribution with 0 mean and σ standard deviation.

Let B be the inverse of A, namely $AB = BA = \mathbb{I}$, we would like to estimate the diagonal and (mean of) off-diagonal elements of B.

Let us make a crude assumption, that is

$$A_{ii} = 1 \text{ for all } i = 1, \dots, N \text{ and } A_{ij} = \sigma \text{ for all } i, j = 1, \dots, N,\ i \neq j\,, \tag{2}$$

namely all the off-diagonal elements of A have the same value σ. We can thus rewrite A as follows

$$A = \mathbb{I} + \sigma(E - \mathbb{I})\,,$$

where E is the matrix such that $E_{ij} = 1$ for all i, j. Namely

$$A = \mathbb{I}(1 - \sigma) + \sigma E\,.$$

We can thus look for $B = \alpha\mathbb{I} + \beta E$ and then

$$\begin{aligned} AB &= (\mathbb{I}(1-\sigma) + \sigma E)(\alpha\mathbb{I} + \beta E) \\ &= (1-\sigma)\alpha\mathbb{I} + [(1-\sigma)\beta + \sigma\alpha]E + \sigma\beta N E\,, \end{aligned}$$

where we used in the last step that E satisfies $E^2 = NE$. The values of α and β that return $AB = \mathbb{I}$ are thus

$$\alpha = \frac{1}{1-\sigma} \text{ and } \beta = -\frac{\sigma}{(1-\sigma)[1+\sigma(N-1)]}\,. \tag{3}$$

The inverse of A is thus

$$B = \frac{1}{1-\sigma}\mathbb{I} - \frac{\sigma}{(1-\sigma)[1+\sigma(N-1)]}E\,, \tag{4}$$

and to emphasize the diagonal and off-diagonal terms we can rewrite it as follows

$$\begin{aligned} B &= \frac{1}{1-\sigma}\left(1 - \frac{\sigma}{1+\sigma(N-1)}\right)\mathbb{I} - \frac{\sigma}{(1-\sigma)[1+\sigma(N-1)]}(E-\mathbb{I}) \\ &= \frac{1+\sigma(N-2)}{(1-\sigma)[1+\sigma(N-1)]}\mathbb{I} - \frac{\sigma}{(1-\sigma)[1+\sigma(N-1)]}(E-\mathbb{I})\,. \end{aligned} \tag{5}$$

We can thus conclude about the diagonal and off-diagonal elements of the inverse

$$B_{ii} = \frac{1+\sigma(N-2)}{(1-\sigma)[1+\sigma(N-1)]} \sim 1 + (N-1)\sigma^2 + \mathcal{O}(\sigma^3) \text{ and } B_{ij} = -\frac{\sigma}{(1-\sigma)[1+\sigma(N-1)]} \sim \sigma - (N-2)\sigma^2 + \mathcal{O}(\sigma^3)\,. \tag{6}$$

To check the goodness of the approximation we build n_{iter} matrices of the form of A, for each matrix we numerically computed its inverse, X, then we computed the average value of the diagonal and off diagonal elements

$$\langle X_{diag}\rangle = \frac{1}{N}\sum_{i=1}^{N} X_{ii} \text{ and } \langle X_{off}\rangle = \frac{1}{N(N-1)}\sum_{i,j=1, i\neq j}^{N} X_{ij}\,,$$

and finally we computed the error with respect to the theoretical prediction (6), i.e.,

$$\varepsilon_{diag} = \langle X_{diag} \rangle - \frac{1+\sigma(N-2)}{(1-\sigma)[1+\sigma(N-1)]} \text{ and } \varepsilon_{off} = \langle X_{off} \rangle + \frac{\sigma}{(1-\sigma)[1+\sigma(N-1)]} .$$

In Fig. 1 we report the distribution of ε_{diag} and ε_{off} in the form of histograms for $n_{iter} = 1000$ matrices. The results agree quite well with the theoretical prediction, however let us observe that the agreement seems to depend on the property of A to be with dominant diagonal, i.e.,

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}| .$$

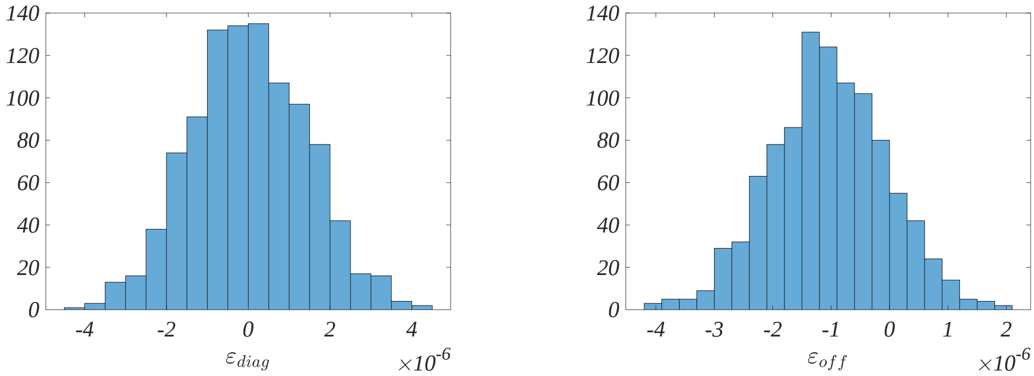

FIG. 1: Distribution of the error with respect to the theory of the average diagonal terms (left panel) and average off-diagonal terms (right panel) of $n_{iter} = 1000$ matrices A defined as in the text.