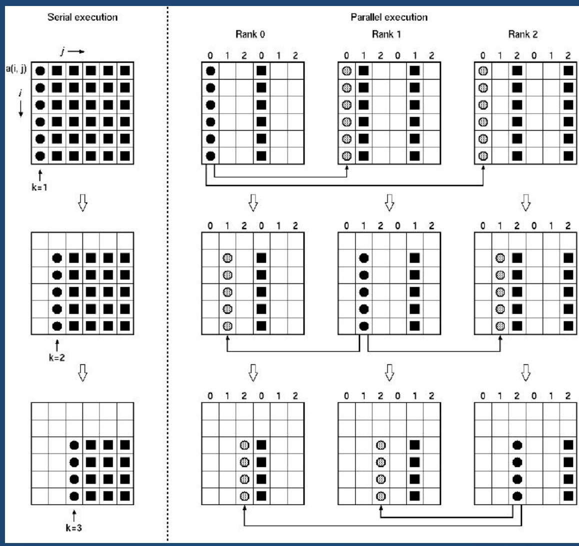


Collective Communications

Some references

- **Parallel Algorithms,** H. Casanova, A. Legrand, Y. Robert
- **Parallel Programming – For Multicore and Cluster System,** T. Rauber, G. Rünger

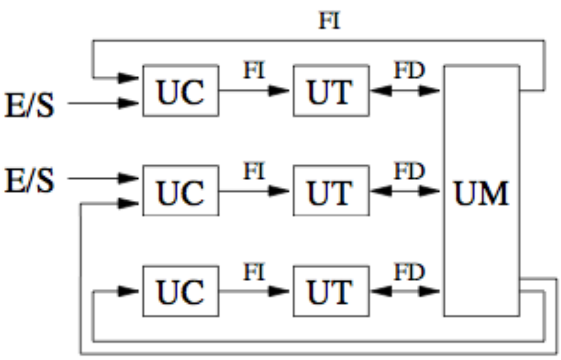
MIMD: Multiple Instructions stream, multiple data stream

Multi-Processor Machines

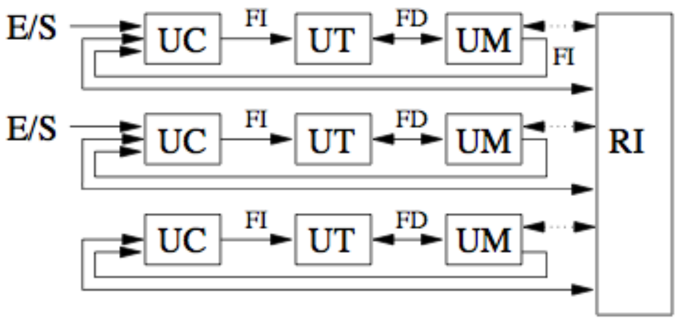
Each processor runs its own code asynchronously and independently

Two sub-classes

Shared memory



Distributed memory

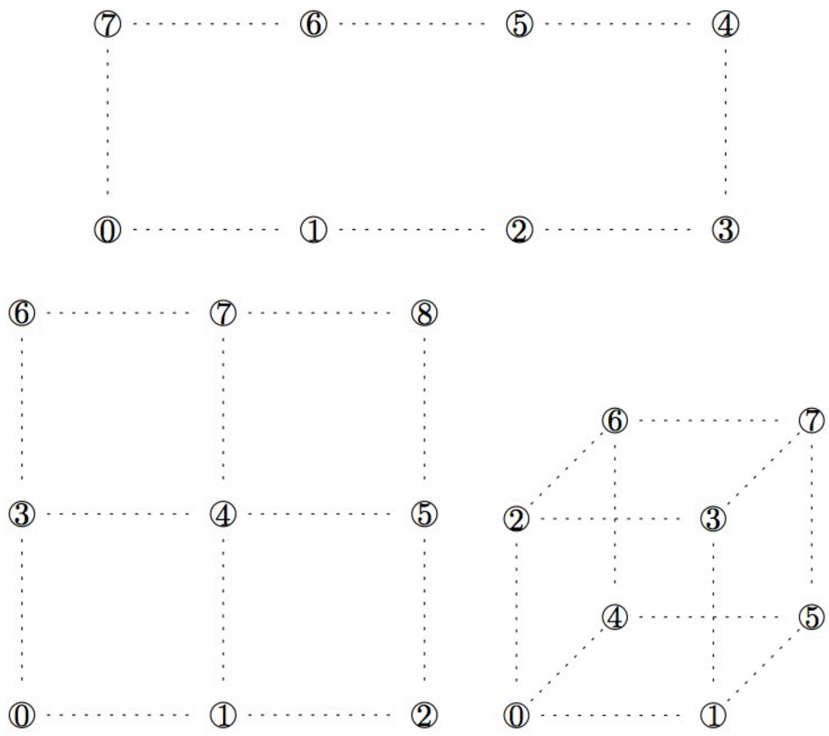


A mix between SIMD and MIMD: SPMD (**Single Program, Multiple Data**)

Collectives communications

- Interactions between parts of a parallel program mapped in a set of processors happen following **well defined schemes** between groups of processors/cores
  - Not only point-to-point communications
- To write parallel algorithms, we need **collectives operations**
  - Broadcast, scatter, gather, all-to-all, ...
  - Used in most parallel applications
- MPI provides many of them
  - They should be designed to use efficiently hardware resources (processors, network, memory interfaces, bus, …)
- Optimizing these operations can
  - Improve global performance of programs
  - Reduce the development cost of applications
  - Improve parallel software quality
- If possible, take the hardware architecture into account
- So why should we take a look at the way they are designed ?

Topologies



Communication costs

Global communications are usually written using point-to-point communications

Difficulty to find accurate models

- MPI implementations have different optimisations depending of the message sizes
- Smart optimizations taking into account special hardware/software features

Here we use a simplified model

- Time = L + m/B (without contentions)
- L: startup (or latency) time
- B: bandwidth (b = 1/B)
- m: message size

- Store-and-forward
  - If we suppose that a message of length m is sent from de P_0 to P_q, then the communication cost is

$$T_c(m) = q(L + m\,b)$$

Suppositions about communications

Several options

- Send() and Recv() are both blocking
  - Called “rendez-vous” mode
- Recv() is blocking, but Send() is not
  - Pretty standard
  - MPI supports it
- Recv() and Send() are both non-blocking
  - Pretty standard too
  - MPI supports it as well

Supposition about concurrency

An important question: can the processor perform several operations at the same time?

Generally we suppose that the processor is able to send, receive, and compute at the same time

- MPI_IRecv()
- MPI_ISend()
- Compute something

We need these three operations to be independent

- We can not send the result of a computation before it is computed
- We can not send what we receive (*forwarding*) unless we pipeline the communication

When we write parallel algorithms (in pseudo-code), we write concurrent activities with the || sign

Virtual topology versus physical topology

- We have chosen that our virtual topology is a ring
  - We suppose that the topology is a ring too
- Maybe an other virtual topology is more adapted to the physical one we have for our cluster

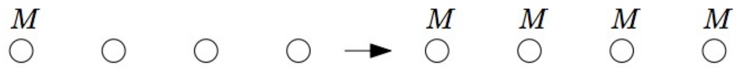
- The ring of processes allows to have simple algorithms
- With quite good performances

- Good candidate for our first approach of parallel algorithmics

Some global operations

- One-to-all broadcast and reduction
- All-to-all broadcast and reduction
- All-Reduce operation and prefix sum
- Scatter and Gather
- Personnalized all-to-all communication
- Circular shift

Broadcast (one-to-all communication)



- **Input**
  - Message M is stored on root processor

- **Output**
  - Message M is stored locally on every processors

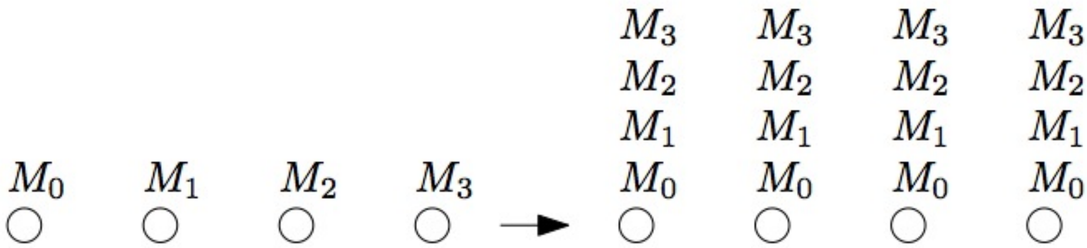
Reduction (all-to-one reduction)

$$M := M_0 \oplus M_1 \oplus M_2 \oplus M_3$$

- **Input**
  - The p messages M_k for k = 0, 1, …, p-1
  - Message M_k is stored locally on processor k
  - An associative operation (+, x, max, min)

- **Output**
  - The “sum” is stored on root processor

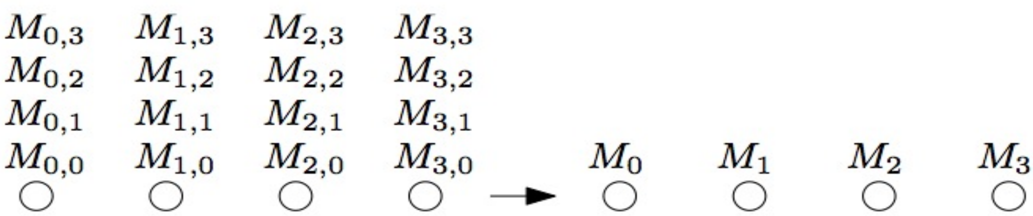
All-to-all broadcast



- **Input**
  - The p messages M_k for k = 0, 1, …, p-1
  - Message M_k is stored locally processor k

- **Output**
  - The p messages M_k for k = 0, 1, …, p-1 are stored locally on every processors

All-to-all reduction



- **Input**
  - The p^2 messages $M_{r,k}$ for r, k = 0, 1, …, p-1
  - Message $M_{r,k}$ is stored locally on processor r
  - An associative operation (+, x, max, min)

- **Output**
  - The "sum" is stored on the root processor

$$M_r := M_{0,r} \oplus M_{1,r} \oplus \cdots \oplus M_{p-1,r}$$

Prefix sum

$$M^{(k)} := M_0 \oplus M_1 \oplus \cdots \oplus M_k$$

- **Input**
  - The p messages M_k for k = 0, 1, …, p-1
  - Message M_k is stored locally on processor k
  - An associative operation (+, x, max, min)

- **Output**
  - The "sum" is stored locally on processor k for all k

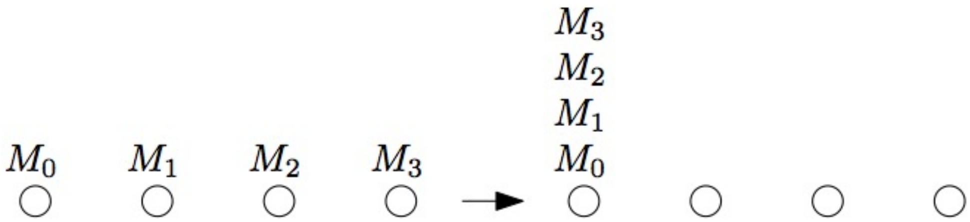
$$M^{(k)} := M_0 \oplus M_1 \oplus \cdots \oplus M_k$$

Scatter

- **Input**
  - The p messages M_k for k = 0, 1, …, p-1 are stored locally on root processor

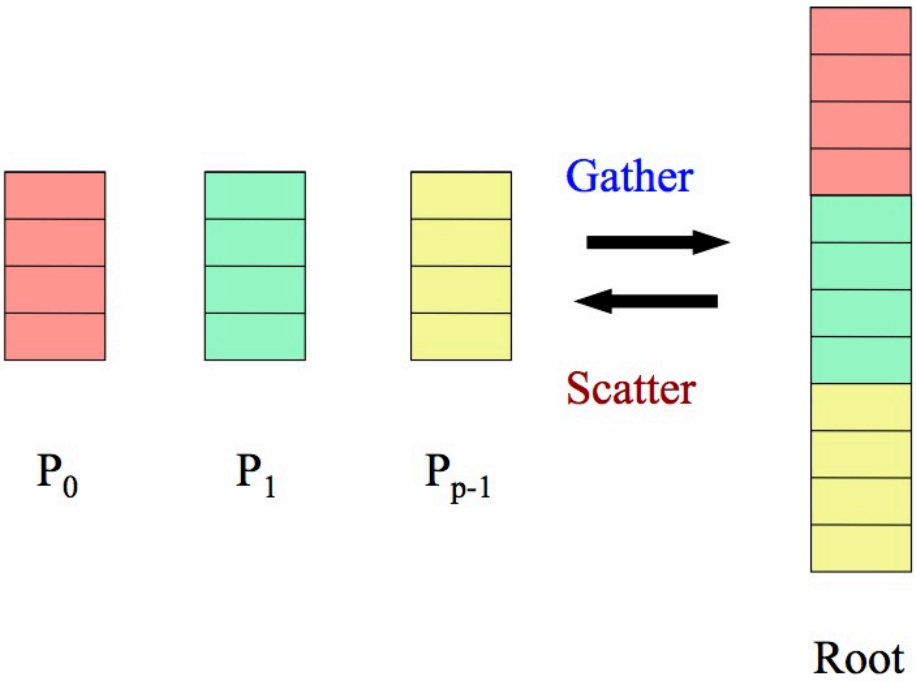
- **Output**
  - Message M_k is stored locally processor k for all k

Gather

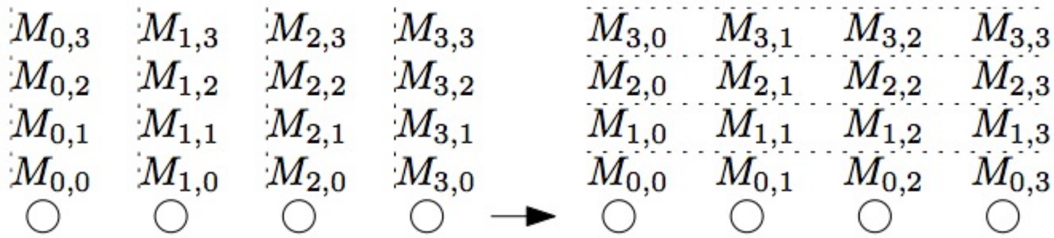


- **Input**
  - The p messages M_k for k = 0, 1, …, p-1
  - Message M_k is stored locally on processor k
- **Output**
  - The p messages M_k are stored locally on root processor

Scatter/Gather



Personnalized All-to-all (transposition)



- **Input**
  - The p^2 messages $M_{r,k}$ for r, k = 0, 1, …, p-1
  - Message $M_{r,k}$ is stored locally on processor r

- **Output**
  - The p messages $M_{r,k}$ are stored locally processor k for all k

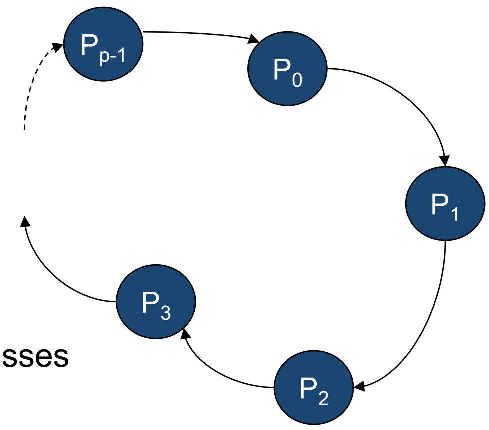
Circular shift

- **Input**
  - The p messages M_k for k = 0, 1, …, p-1 are stored locally on each processor

- **Output**
  - Message $M_{(k-1)\%p}$ is stored locally on k for each k

ALGORITHMS ON A RING OF PROCESSORS

Ring of processors



Each process is identified by his rank

- `MY_NUM()`

We have a way of finding the total number of processes

- `NUM_PROCS()`

Each process can send message to each successor

- `SEND(addr, L)`

And receive a message to its predecessor

- `RECV(addr, L)`

Broadcast

We want to write a program in which P_k sends the same message of length m to all other processors

- `Broadcast (k, addr, m)`

On a ring, the **naive algorithm** consists in sending message to the neighbor processor and so on an so forth, with **no parallel communication**

It should not be written like this if the physical topology is not a ring

- MPI uses some kind of tree

Broadcast

```
Broadcast(k,addr,m)
  q = MY_NUM()
  p = NUM_PROCS()

  if (q == k)
     SEND(addr,m)
  else
     if (q == k-1 mod p)
        RECV(addr,m)
     else
        RECV(addr,m)
        SEND(addr,m)
     endif
  endif
```

- Assumes a blocking receive
- Send can be non-blocking

- The broadcast time is the following
  $(p-1)(L+m\,b)$

Optimized broadcast

- How to improve performance?
- We can split the message in smaller packets
  - r packets where m can be divided by r
- The root process sends r messages

- **The model of the broadcast can be computed like this**
  - Consider the last process to obtain the last packet of the message
  - We need p-1 steps for the first packet to reach its destination, thus
    $(p-1)(L + m\,b / r)$
  - The the next r-1 packets arrive one after an other
    $(r-1)(L + m\,b / r)$
  - Thus a total of
    $(p + r - 2)(L + mb / r)$

Optimized broadcast, contd.

The next question is, what is the value r that that minimizes

$$(p + r - 2)(M + m\,b / r)\ ?$$

- We can see the previous expression as
  $(c+ar)(d+b/r)$, with four constant values a, b, c, d
- The non-constant part of the expression is thus
  $ad.r + cb/r$, that should be minimized
- This value is minimized for
  $\mathrm{sqrt}(cb / ad)$

  thus we have
  $r_{opt} = \mathrm{sqrt}(m(p-2)\,b / L)$

  With the optimal time
  $(\mathrm{sqrt}((p-2)\,L) + \mathrm{sqrt}(m\,b))^2$

  that tends towards mb when m is large (independent of p !)

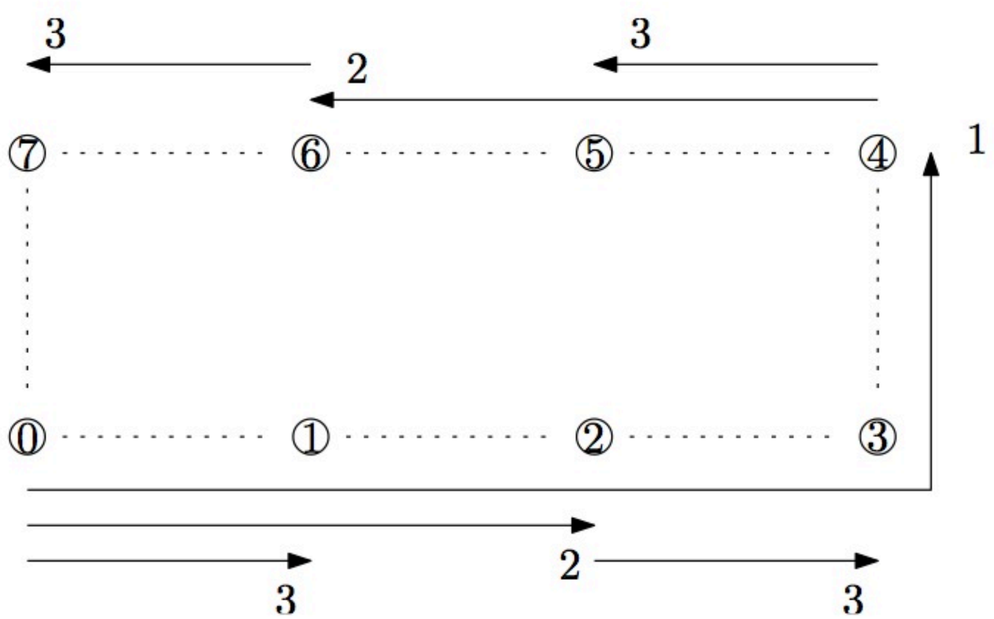
Classical network principle

We have seen that if we cut a (large) message into a large number of (small) messages, then we can send the message through several jumps (in our case p-1) virtually as fast as sending it to just one jump

This is the fundamental principle of IP networks

- Messages are divided into several IP frames
- They are sent on several routers
- But the execution time is limited by the slowest router time

Other solution: Recursive Doubling



Double the number of active processes at each step

Scatter

- Process k sends a different message to all other processes (including it)
  - P_k stores messages for P_q at address `addr[q]`, including a message to `addr[k]`

- At the end of the execution, each processor has the message it received in `msg`

- The principle of the algorithm is just pipelining the communications starting with the message intended for P_{k-1}, the most distant process

Scatter

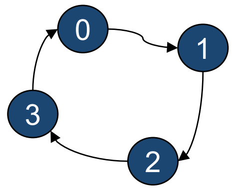
```
Scatter(k,msg,addr,m)
  q = MY_NUM()
  p = NUM_PROCS()
  if (q == k)
     for i = 0 to p-2
        SEND(addr[k+p-1-i mod p],m)
     msg ← addr[k]
  else
     RECV(tempR,L)
     for i = 1 to k-1-q mod p
        tempS ↔ tempR
        SEND(tempS,m) || RECV(tempR,m)
     msg ← tempR
```

Same execution time than broadcast
(p-1)(L + m b)

Exchange of Send Buffer and Receive Buffer (Pointer)

Send and receive in parallel, with a non-blocking send

Scatter



k = 2, p = 4

```
Scatter(k,msg,addr,m)
  q = MY_NUM()
  p = NUM_PROCS()
  if (q == k)
     for i = 0 to p-2
        SEND(addr[k+p-1-i mod p],m)
     msg ← addr[k]
  else
     RECV(tempR,L)
     for i = 1 to k-1-q mod p
        tempS ↔ tempR
        SEND(tempS,m) || RECV(tempR,m)
     msg ← tempR
```

Proc q=2

```
send addr[2+4-1-0 % 4 = 1]
send addr[2+4-1-1 % 4 = 0]
send addr[2+4-1-2 % 4 = 3]
msg = addr[2]
```

Proc q=3

```
recv  (addr[1])
// loop 2-1-3 % 4 = 2 times
send (addr[1]) || recv (addr[0])
send (addr[0]) || recv (addr[3])

msg = addr[3]
```

Proc q=0

```
recv (addr[1])
// loop 2-1-2 % 4 = 1 time
send (addr[1]) || recv (addr[0])

msg = addr[0]
```

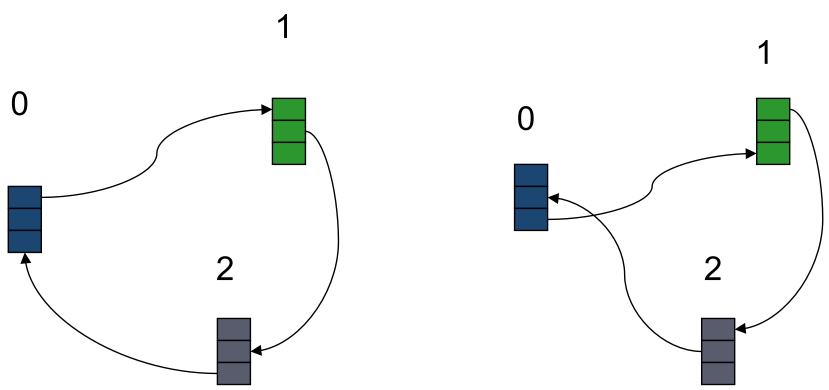
Proc q=1

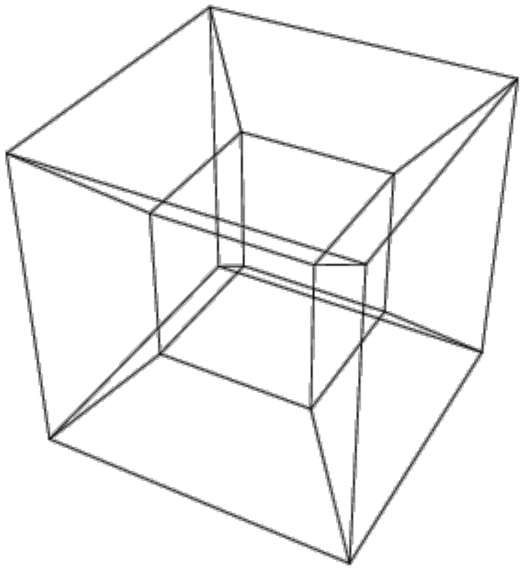
```
 // loop 2-1-1 % 4 = 0 time
recv (addr[1])

msg = addr[1]
```


All-to-all

```
All2All(my_addr, addr, m)
  q = MY_NUM()
  p = NUM_PROCS()
  addr[q] ← my_addr
  for i = 1 to p-1
     SEND(addr[q-i+1 mod p],m) || RECV(addr[q-i mod p],m)
```

Same execution time than scatter
(p-1)(L + m b)

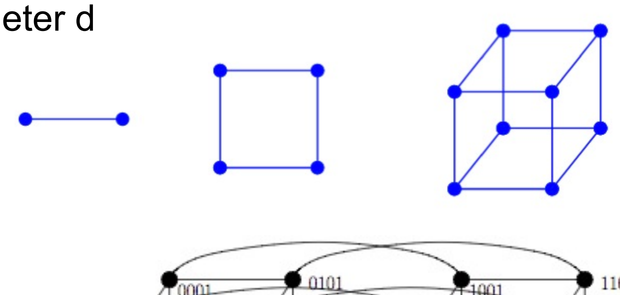


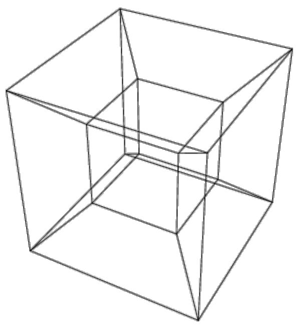


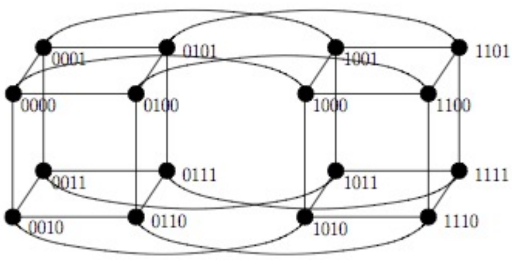
ALGORITHMS ON HYPERCUBE

Reminder on hypercubes

- d dimensional graph
- 2^d nodes with d neighbor each
- A 0-cube is a simple node simple, a 1-cube a row of processors, a 2-cube a mesh, etc
- Log(p) dimensions if p processors
- Diameter d



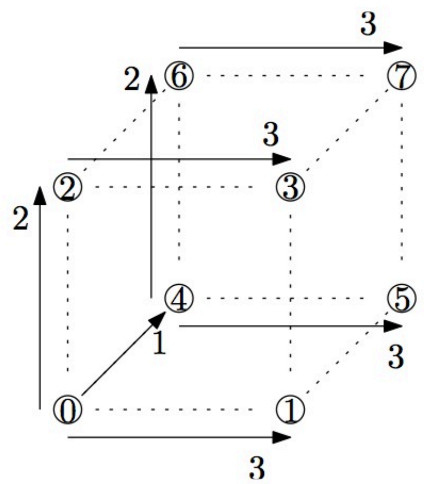




Broadcast in hypercubes

Same algorithm as the ring one but generalized to d dimensions

```
Assume that p = 2^d
mask ← 2^d − 1 (set all bits)
for k = d − 1, d − 2, . . . , 0 do
    mask ← mask XOR 2^k (clear bit k)
    if me AND mask = 0 then
        (lower k bits of me are 0)
        partner ← me XOR 2^k (partner has opposite bit k)
        if me AND 2^k = 0 then
            Send M to partner
        else
            Receive M from partner
        end if
    end if
end for
```



If the root process is not 0

rename processes `me = me XOR root`

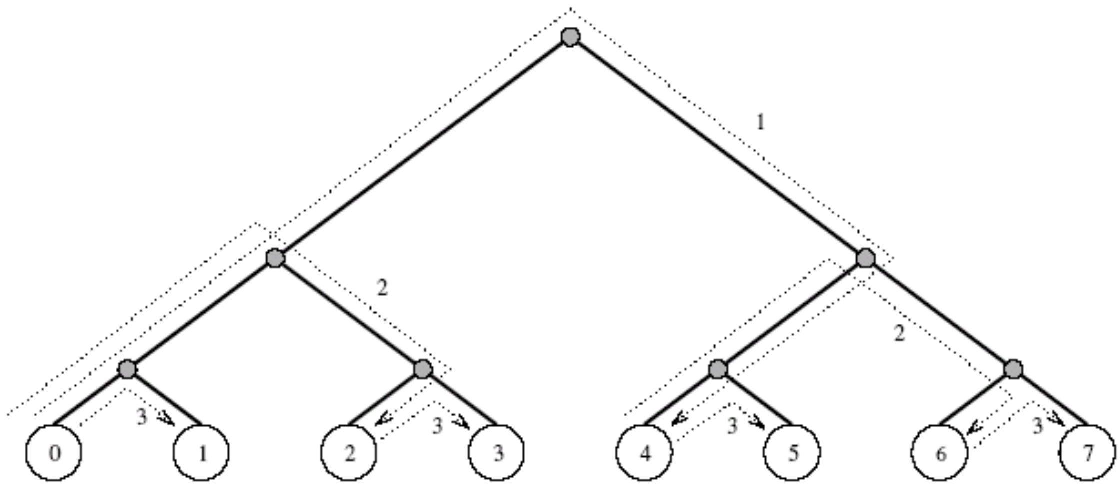
Broadcast cost

- Number of steps: $d = \log_2(p)$
- Cost per step: $L + m/B$
- Total cost: $(L + m/B) \log_2(p)$

The broadcast cost with p^2 processors is only the double of the broadcast cost with p processors

$$\log_2(p^2) = 2 \log_2(p)$$

Broadcast in a binary tree



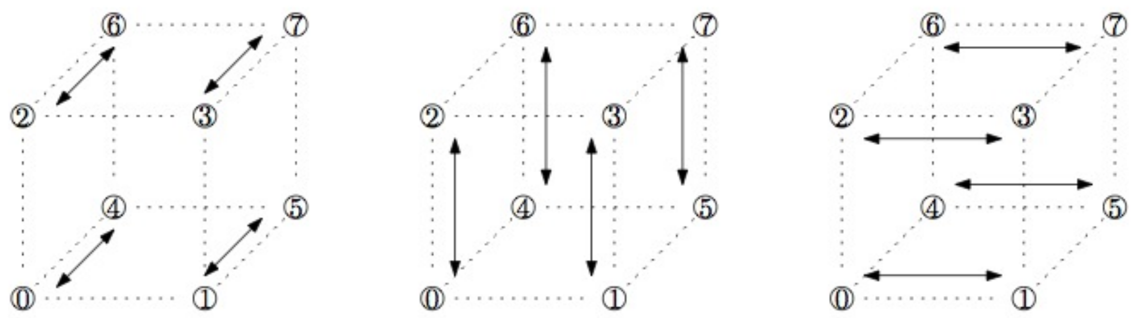
Reduction (all-to-one)

- Same algorithm as broadcast but reversing the communication order and directions
- Same execution time (adding the reduction cost)
- Combining the incoming message with the local data with the operation

All-to-all broadcast in a hypercube

Using the ring algorithm

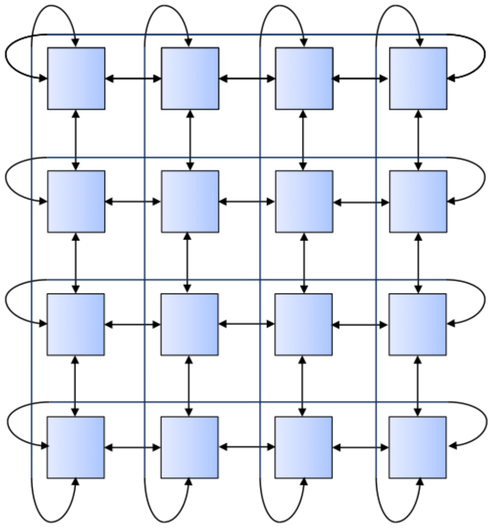
- For each dimension d of the hypercube, apply in sequence the algorithm on a ring on the 2^{d-1} links of the current dimension in parallel



All-to-all broadcast in a hypercube

- **Cost**
  - Number of steps: $d = \log_2(p)$
  - Cost for step k = 0, 1, …, d-1: $L + \frac{m2^k}{B}$
  - Total cost:

$$\sum_{k=0}^{d-1}(L + 2^k \frac{m}{B}) = \log_2(p) * L + (p-1) * \frac{m}{B}$$



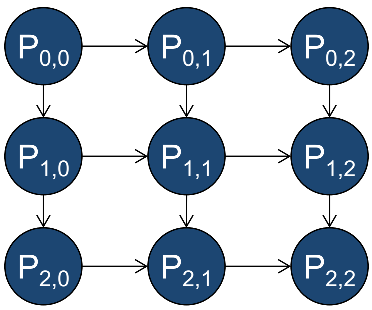
ALGORITHMS ON A GRID OF PROCESSORS

Bi-dimensional grid of processors

Let $p = q^2$ processors

They can be seen as being arranged in the form of a square grid
- One can also have a rectangular grid

Each processor is identified by two indexes
- i: its row
- j: its column

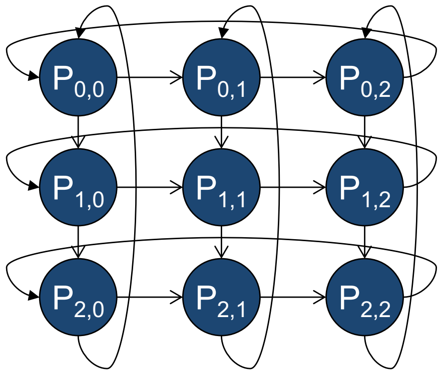


Bi-dimensional torus (2D torus)

We have links which connect each side of the grid

Each processor belongs to two different rings
- Possibility to use algorithms designed for ring topologies

Mono-directional or bi-directional links
- Depends on what we need for our algorithm and/or physical resources



Overlaps

In our performance analyzes, it is often assumed that a processor can perform **three activities in parallel**
- Computation
- Send
- Receive

It is also necessary to know whether the links are bi-directional or not
- **Two models**
  - **Half-duplex:** two messages on the same link going in opposite directions share the link bandwidth
  - **Full-duplex:** it's like having two links between each processor
- To be checked (and to measure and verify sometimes) with the target platform

Multiple concurrent communications?

- We now have four (logical) links on each processor
- You need to know how many concurrent calls can be made at the same time
  - There can be 4 sends and 4 receives in the model with bi-directional links
  - Assuming that the 4 sends and the 4 receives can take place in parallel, one has a **multi-port model**
  - If we assume 1 send and 1 receive in parallel, we have a **1-port model**
  - Other possible variations
    - k-port (bounded multi-port), inputs/outputs

Next

We have several options

- Grid or torus
- Mono- ou bi-directional links
- 1-port or multi-port (or k-port)
- Half- or full-duplex

- We will generally assume a bi-directional and full-duplex torus
- We will examine the 1-port and multi-port assumptions

"Easy" to modify a performance analysis to stick with the physical resources of the target machines studied

Is the grid topology realistic?

Some parallel machines are(were) built with physical networks in the form of grids (2D or 3D)

- Examples: Intel Paragon, IBM's Blue Gene/L

If the platform uses a switch with all-to-all communications, then the grid is assumed to be valid

- On the other hand, the assumptions of full-duplex or multi-port are not necessarily valid

We will see that even if the physical platform is a unique shared medium (such as a non-switched Ethernet network), it is sometimes better to think of it as a grid when developing algorithms!

Communications in a grid

- A process can call two functions to know its position in the grid:

```
My_Proc_Row() and My_Proc_Col()
```

- A process can know how many total processes are in the topology with:

```
Num_Procs()
```

  - Assume that we have a square grid

- There are two point-to-point communications functions:

```
Send(dest, addr, L)
Recv(src, addr, L)
```

- Broadcast functions can be created in rows and columns

```
BroadcastRow(i, j, srcaddr, dstaddr, L)
BroadcastCol(i, j, srcaddr, dstaddr, L)
```

- It is assumed that a call to such a function in a row or column that is not right returns immediately

Row and column broadcast

If we have a torus

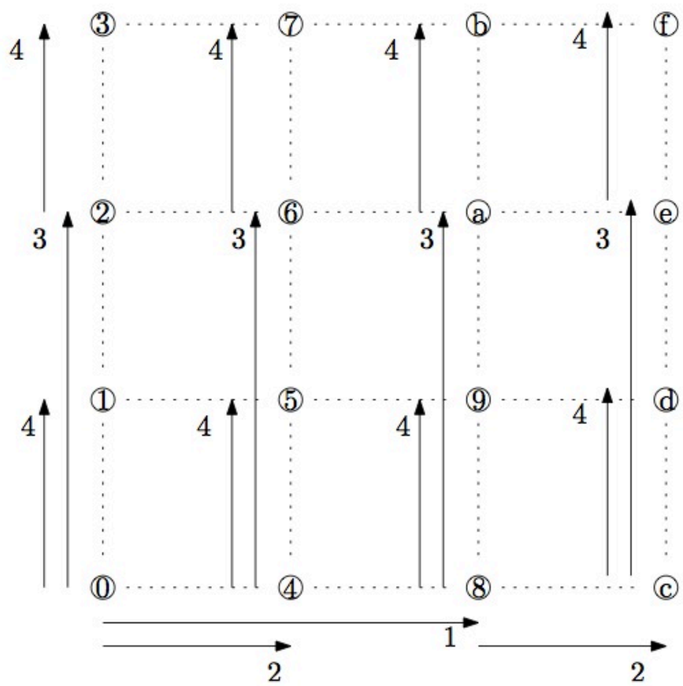
- If one has mono-directional links, one can re-use the broadcast function developed for the rings of processors
- Pipelined or not
- If you have bi-directional links and a multi-port model, you can improve performance by sending data on both sides of the ring
- Asymptotic performances are not changed

If you have a grid

- If the links are bi-directional, then we can send the messages on both sides from the source processor concurrently or not, depending on whether we have a 1-port or multi-port model
- If the links are mono-directional, one can simply not implement the broadcast

Broadcast in a grid

- Use the ring broadcast algorithm on the row where the root is located
- Use the ring broadcast algorithm on all columns in //

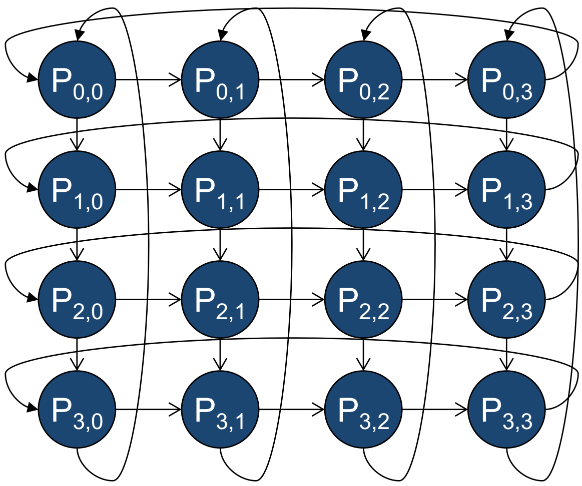


All-to-all in a grid of processors

- **Use the ring broadcast algorithm on each row in //**
  - Cost (we suppose that we have a $\sqrt{p} * \sqrt{p}$ grid of processors)
    - Number of steps: $\sqrt{p} - 1$
    - Time per step: $L + \frac{m}{B}$
    - Total time: $\left(\sqrt{p} - 1\right) * \left(L + \frac{m}{B}\right)$
- **Use the ring broadcast algorithm on each column in //**
  - Cost
    - Number of steps: $\sqrt{p} - 1$
    - Time per step: $L + \sqrt{p}\frac{m}{B}$
    - Total time: $\left(\sqrt{p} - 1\right) * \left(L + \frac{m}{B}\right)$
- Total time:

$$2 * \left(\sqrt{p} - 1\right) * L + \left(p - 1\right) * \frac{m}{B}$$

Bi-dimensional matrix distribution



- Let $a_{i,j}$ be a element of the matrix
- We denote by $A_{i,j}$ (or A_{ij}) the block of matrix A assigned to $P_{i,j}$

C_{00}	C_{01}	C_{02}	C_{03}
C_{10}	C_{11}	C_{12}	C_{13}
C_{20}	C_{21}	C_{22}	C_{23}
C_{30}	C_{31}	C_{32}	C_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}

Cannon matrix product algorithm

Old algorithm
- Designed for systolic architectures (SIMD)
- Adapted to a 2D grid

The algorithm starts with a redistribution of matrices A and B
- Called "*preskewing*"

Then matrices are multiplied together

At the end, the matrices are re-distributed to find their initial distribution
- Called "*postskewing*"

Cannon Preskewing

Matrix A

Each block of matrix A is shifted to the left until the process of the first process column contains a block of the diagonal of the matrix

A_{00}	A_{01}	A_{02}	A_{03}
A_{10}	A_{11}	A_{12}	A_{13}
A_{20}	A_{21}	A_{22}	A_{23}
A_{30}	A_{31}	A_{32}	A_{33}

A_{00}	A_{01}	A_{02}	A_{03}
A_{11}	A_{12}	A_{13}	A_{10}
A_{22}	A_{23}	A_{20}	A_{21}
A_{33}	A_{30}	A_{31}	A_{32}

Cannon Preskewing, contd.

Matrix B

Each block of matrix B is shifted upward until process of the first process line contains a block of the diagonal of the matrix

B_{00}	B_{01}	B_{02}	B_{03}
B_{10}	B_{11}	B_{12}	B_{13}
B_{20}	B_{21}	B_{22}	B_{23}
B_{30}	B_{31}	B_{32}	B_{33}

B_{00}	B_{11}	B_{22}	B_{33}
B_{10}	B_{21}	B_{32}	B_{03}
B_{20}	B_{31}	B_{02}	B_{13}
B_{30}	B_{01}	B_{12}	B_{23}

Cannon algorithm

- The algorithm runs in q steps
- At each step, each processor executes a multiplication of its block of A and its block of B and adds it to its block of C
- Then the blocks of A are shifted to the left and the blocks of B are shifted upwards
- C blocks do not move

```
Participate to the preskewing of A
Participate to the preskewing of B
For k = 1 to q
   Local C = C + A*B
   Horizontal shift of A
   Vertical shift of B
Participate to the postskewing of A
Participate to the postskewing of B
```

Steps of the Cannon algorithm

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{01}	A_{02}	A_{03}	B_{00}	B_{11}	B_{22}	B_{33}
C_{10}	C_{11}	C_{12}	C_{13}	A_{11}	A_{12}	A_{13}	A_{10}	B_{10}	B_{21}	B_{32}	B_{03}
C_{20}	C_{21}	C_{22}	C_{23}	A_{22}	A_{23}	A_{20}	A_{21}	B_{20}	B_{31}	B_{02}	B_{13}
C_{30}	C_{31}	C_{32}	C_{33}	A_{33}	A_{30}	A_{31}	A_{32}	B_{30}	B_{01}	B_{12}	B_{23}

Local computation on processor (0,0)

C_{00}	C_{01}	C_{02}	C_{03}	A_{01}	A_{02}	A_{03}	A_{00}	B_{10}	B_{21}	B_{32}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}	A_{12}	A_{13}	A_{10}	A_{11}	B_{20}	B_{31}	B_{02}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}	A_{23}	A_{20}	A_{21}	A_{22}	B_{30}	B_{01}	B_{12}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{00}	B_{11}	B_{22}	B_{33}

Shifts

C_{00}	C_{01}	C_{02}	C_{03}	A_{01}	A_{02}	A_{03}	A_{00}	B_{10}	B_{21}	B_{32}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}	A_{12}	A_{13}	A_{10}	A_{11}	B_{20}	B_{31}	B_{02}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}	A_{23}	A_{20}	A_{21}	A_{22}	B_{30}	B_{01}	B_{12}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{00}	B_{11}	B_{22}	B_{33}

Local computation on processor (0,0)

Fox algorithm

This algorithm was originally developed to run on a hypercube topology

- But in fact it uses a grid, mapped on a hypercube

• It does not require any pre / post-skewing

• It is based on horizontal broadcast of the diagonals of matrix A and vertical shifts of matrix B

• Sometimes also called the **broadcast-multiply-roll** algorithm

Steps of the Fox algorithm

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{01}	A_{02}	A_{03}	B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}	A_{10}	A_{11}	A_{12}	A_{13}	B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}	A_{20}	A_{21}	A_{22}	A_{23}	B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{30}	B_{31}	B_{32}	B_{33}

Initial state

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{00}	A_{00}	A_{00}	B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}	A_{11}	A_{11}	A_{11}	A_{11}	B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}	A_{22}	A_{22}	A_{22}	A_{22}	B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}	A_{33}	A_{33}	A_{33}	A_{33}	B_{30}	B_{31}	B_{32}	B_{33}

Broadcast of the 1st diagonal of A (stored in a separate buffer)

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{00}	A_{00}	A_{00}	B_{00}	B_{01}	B_{02}	B_{03}
C_{10}	C_{11}	C_{12}	C_{13}	A_{11}	A_{11}	A_{11}	A_{11}	B_{10}	B_{11}	B_{12}	B_{13}
C_{20}	C_{21}	C_{22}	C_{23}	A_{22}	A_{22}	A_{22}	A_{22}	B_{20}	B_{21}	B_{22}	B_{23}
C_{30}	C_{31}	C_{32}	C_{33}	A_{33}	A_{33}	A_{33}	A_{33}	B_{30}	B_{31}	B_{32}	B_{33}

Local computations

Steps of the Fox algorithm, contd.

C_{00}	C_{01}	C_{02}	C_{03}	A_{00}	A_{01}	A_{02}	A_{03}	B_{10}	B_{11}	B_{12}	B_{13}
C_{10}	C_{11}	C_{12}	C_{13}	A_{10}	A_{11}	A_{12}	A_{13}	B_{20}	B_{21}	B_{22}	B_{23}
C_{20}	C_{21}	C_{22}	C_{23}	A_{20}	A_{21}	A_{22}	A_{23}	B_{30}	B_{31}	B_{32}	B_{33}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{31}	A_{32}	A_{33}	B_{00}	B_{01}	B_{02}	B_{03}

Shift of B

C_{00}	C_{01}	C_{02}	C_{03}	A_{01}	A_{01}	A_{01}	A_{01}	B_{10}	B_{11}	B_{12}	B_{13}
C_{10}	C_{11}	C_{12}	C_{13}	A_{12}	A_{12}	A_{12}	A_{12}	B_{20}	B_{21}	B_{22}	B_{23}
C_{20}	C_{21}	C_{22}	C_{23}	A_{23}	A_{23}	A_{23}	A_{23}	B_{30}	B_{31}	B_{32}	B_{33}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{30}	A_{30}	A_{30}	B_{00}	B_{01}	B_{02}	B_{03}

Broadcast of the 2nd diagonal of A (stored in a separate buffer)

C_{00}	C_{01}	C_{02}	C_{03}	A_{01}	A_{01}	A_{01}	A_{01}	B_{10}	B_{11}	B_{12}	B_{13}
C_{10}	C_{11}	C_{12}	C_{13}	A_{12}	A_{12}	A_{12}	A_{12}	B_{20}	B_{21}	B_{22}	B_{23}
C_{20}	C_{21}	C_{22}	C_{23}	A_{23}	A_{23}	A_{23}	A_{23}	B_{30}	B_{31}	B_{32}	B_{33}
C_{30}	C_{31}	C_{32}	C_{33}	A_{30}	A_{30}	A_{30}	A_{30}	B_{00}	B_{01}	B_{02}	B_{03}

Local computations

Fox algorithm

```
// No initial move
for k = 1 to q  in parallel
  Broadcast of the k-th diagonal of A
  Local computation C = C + A*B
  Vertical shift of B
// No final move
```

- We need an additional array to store the diagonal blocks that are received on processes
- This is the array used for multiplication A * B