

Decomposition methods for nonlinear optimization and data mining

By

BRANDON E. DUTRA

B.S. (University of California, Davis) 2012

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Applied Mathematics

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Committee Member 1

Committee Member 2

Committee Member 3

Committee in Charge

2016

Contents

Acknowledgments	iv
Chapter 1. Introduction	1
Chapter 2. Background	2
2.1. Working with generating functions: an example	2
2.2. Indicator functions	3
2.3. Generating functions of simple cones	4
2.4. Generating function for non-simple cones	5
Chapter 3. Long Title of Second Chapter	7
Appendix A. Long Title of Appendix A	8
Bibliography	9

This Abstract Found

Acknowledgments

I'd like to thank the little people.

CHAPTER 1

Introduction

Allow me to introduce you to my dissertation.

CHAPTER 2

Background

In this chapter, we first review some polyhedral decompositions and how these relate to generating functions for the lattice points of a polyhedra.

2.1. Working with generating functions: an example

Let us start with an easy example. Consider the one dimensional polyhedra in \mathbb{R} given by $P = [0, n]$. We encode the lattice points of $P \cap \mathbb{Z}$ by placing each integer point as the power of a monomial, thereby obtaining the polynomial $S(P; z) := z^0 + z + z^2 + z^3 + \dots + z^n$. The polynomial $S(P; z)$ is called the *generating function* of P . Notice that counting $P \cap \mathbb{Z}$ is equivalent to evaluating $S(P, 1)$.

In terms of the computational complexity, listing each monomial in the polynomial $S(P, z)$ results in a polynomial with exponential length in the bit length of n . However, we can rewrite the summation with one term:

$$S(P, z) = z^0 + z^1 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Counting the number of point in $|P \cap \mathbb{Z}|$ is no longer as simple as evaluating $\frac{1 - z^{n+1}}{1 - z}$ at $z = 1$ because this is a singularity. However, this singularity is removable. One could perform long-polynomial division, but this would result in a exponentially long polynomial in the bit length of n . Another option that yields a polynomial time algorithm would be to apply L'Hospital's rule:

$$\lim_{z \rightarrow 1} S(P, z) = \lim_{z \rightarrow 1} \frac{-(n+1)z^n}{1} = n + 1.$$

Notice that $S(P, z)$ can be written in two ways:

$$S(P, z) = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z} = \frac{1}{1 - z} + \frac{z^n}{1 - z^{-1}}.$$

The first two rational expressions have a nice description in terms of their series expansion:

$$1 + z + \dots + z^n = (z^0 + z^1 + \dots) - (z^{n+1} + z^{n+2} + \dots).$$

For the second two rational functions, we have to be careful about the domain of convergence when computing the series expansion. Notice that in the series expansion,

$$\begin{aligned} \frac{1}{1 - z} &= \begin{cases} z^0 + z^1 + z^2 \dots & \text{if } |z| < 1 \\ -z^{-1} - z^{-2} - z^{-3} - \dots & \text{if } |z| > 1 \end{cases} \\ \frac{z^n}{1 - z^{-1}} &= \begin{cases} -z^{n+1} - z^{n+2} - z^{n+3} - \dots & \text{if } |z| < 1 \\ z^n + z^{n-1} + z^{n-2} + \dots & \text{if } |z| > 1 \end{cases} \end{aligned}$$

adding the terms when $|z| < 1$ or $|z| > 1$ results in the desired polynomial: $z^0 + z^1 + \dots + z^n$. But we can also get the correct polynomial by adding the series that corresponds to different domains of convergence. However, to do this we must now add the series $\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots$ which corresponds to the polyhedra that is the entire real line:

$$\begin{aligned} 1 + z + \dots + z^n &= (1 + z + z^2 + \dots) \\ &\quad + (z^n + z^{n-1} + \dots) \\ &\quad - (\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots) \end{aligned}$$

and

$$\begin{aligned} 1 + z + \dots + z^n &= (-z^{-1} - z^{-2} - z^{-3} - \dots) \\ &\quad + (-z^{n+1} - z^{n+2} - z^{n+3} - \dots) \\ &\quad + (\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots) \end{aligned}$$

Hence by including the series $\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots$, we can perform the series expansion of $\frac{1}{1-z} + \frac{z^n}{1-z^{-1}}$ by computing the series expansion of each term on potentially different domains of convergence.

In the next sections, we will develop rigorous justification for adding the series $\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots$.

2.2. Indicator functions

DEFINITION 2.2.1. The indicator function, $[A] : \mathbb{R}^d \rightarrow \mathbb{R}$, of a set $A \subseteq \mathbb{R}^d$ takes two values: $[A](x) = 1$ if $x \in A$ and $[A](x) = 0$ otherwise.

The set of indicator functions on \mathbb{R}^d forms a vectorspace with pointwise additions and scalar multiplication. Notice that $[A] \cdot [B] = [A \cap B]$, and $[A] + [B] = [A \cup B] + [A \cap B]$.

DEFINITION 2.2.2. The *cone* of a set $A \subseteq \mathbb{R}^d$ is all conic combinations of the points from A :

$$\text{Cone}(A) := \left\{ \sum_i \alpha_i a_i \mid a_i \in A, \alpha_i \in \mathbb{R}_{\geq 0} \right\}$$

DEFINITION 2.2.3. Let P be a polyhedron and $x \in P$. Then the *tangent cone*, of P at x is the polyhedral cone

$$\text{TCone}(P, x) := x + \text{Cone}(P - x)$$

Note that if x is a vertex of P , and P is given by an inequality description, then the tangent cone $\text{TCone}(P, x)$ is the intersection of inequalities that are tight at x . Also, $\text{TCone}(P, x)$ includes the affine hull of the face that x is in, so the tangent cone is pointed only if x is a vertex.

When F is a face of P , we will also use the notation $\text{TCone}(P, F)$ to denote $\text{TCone}(P, x)$ where x is any interior point of F .

THEOREM 2.2.4 ([Bri37], [Gra71]). Let P be a polyhedron, then

$$[P] = \sum_F (-1)^{\dim(F)} [\text{TCone}(P, F)]$$

where the sum ranges over all faces F of P including $F = P$ but excluding $F = \emptyset$

This theorem is saying that if the generating function of a polytope is desired, it is sufficient to just find the generating function for every face of P . The next corollary takes this a step further and says it is sufficient to just construct the generating functions associated at each vertex. This is because, as we will see, the generating functions for non-pointed polyhedra can be ignored.

COROLLARY 2.2.5. *Let P be a polyhedron, then*

$$[P] \equiv \sum_{v \in V} [\text{TCone}(P, v)] \pmod{\text{indicator functions of non-pointed polyhedra}},$$

where V is the vertex set of P .

2.3. Generating functions of simple cones

In this section, we quickly review the generating function for summation and integration when the polyhedra is a cone.

The next Proposition serves as a basis for all the summation algorithms we will discuss.

PROPOSITION 2.3.1 (Theorem 13.8 in [Bar08]). *There exists a unique valuation $S(\cdot, \ell)$ which associates to every polyhedron $P \subset \mathbb{R}^d$ a meromorphic function in ℓ so that the following properties hold*

- (1) *If $\ell \in \mathbb{R}^d$ such that $e^{\langle \ell, x \rangle}$ is summable over the lattice points of P , then*

$$S(P, \ell) = \sum_{P \cap \mathbb{Z}^d} e^{\langle \ell, x \rangle}.$$

- (2) *For every point $s \in \mathbb{Z}^d$, one has*

$$S(s + P, \ell) = e^{\langle \ell, s \rangle} S(P, \ell).$$

- (3) *If P contains a straight line, then $S(P, \ell) = 0$.*

A consequence of the valuation property is the following fundamental theorem. It follows from the Brion–Lasserre–Lawrence–Varchenko decomposition theory of a polyhedron into the supporting cones at its vertices [BHS09, Bri88, Bar08, Las83].

LEMMA 2.3.2. *Let P be a polyhedron with set of vertices $V(P)$. For each vertex s , let $C_s(P)$ be the cone of feasible directions at vertex s . Then*

$$S(P, \ell) = \sum_{s \in V(P)} S(s + C_s(P), \ell).$$

This last lemma can be identified as the natural result of combining Corollary 2.2.5 and Proposition 2.3.1 part (3). A non-pointed polyhedra is another characterization of a polyhedra that contain a line.

Note that the cone $C_s(P)$ in Lemma 2.3.2 may not be simplicial, but for simplicial cones there are explicit rational function formulas. As we will see in Proposition 2.3.3, one can derive an explicit formula for the rational function $S(s + C_s(P), \ell)$ in terms of the geometry of the cones.

PROPOSITION 2.3.3. *For a simplicial full-dimensional pointed cone C generated by rays u_1, u_2, \dots, u_d (with vertex 0) and for any point s*

$$S(s + C, \ell) = \sum_{a \in (s + \Pi_C) \cap \mathbb{Z}^d} e^{\langle \ell, a \rangle} \prod_{i=1}^d \frac{1}{1 - e^{\langle \ell, u_i \rangle}}$$

where $\Pi_c := \{\sum_{i=1}^d \alpha_i u_i \mid 0 \leq \alpha_i < 1\}$. These identities holds as a meromorphic function of ℓ and pointwise for every ℓ such that $\langle \ell, u_i \rangle \neq 0$ for all u_i .

The continuous generating function for P almost mirrors the discrete case.

PROPOSITION 2.3.4 (Theorem 8.4 in [Bar08]). *There exists a unique valuation $I(\cdot, \ell)$ which associates to every polyhedron $P \subset \mathbb{R}^d$ a meromorphic function so that the following properties hold*

- (1) *If ℓ is a linear form such that $e^{\langle \ell, x \rangle}$ is integrable over P with the standard Lebesgue measure on \mathbb{R}^d , then*

$$I(P, \ell) = \int_P e^{\langle \ell, x \rangle} dx$$

- (2) *For every point $s \in \mathbb{R}^d$, one has*

$$I(s + P, \ell) = e^{\langle \ell, s \rangle} I(P, \ell).$$

- (3) *If P contains a line, then $I(P, \ell) = 0$.*

LEMMA 2.3.5. *Let P be a polyhedron with set of vertices $V(P)$. For each vertex s , let $C_s(P)$ be the cone of feasible directions at vertex s . Then*

$$I(P, \ell) = \sum_{s \in V(P)} I(s + C_s(P), \ell).$$

Again, this last lemma can be identified as the natural result of combining Corollary 2.2.5 and Proposition 2.3.4 part (3).

PROPOSITION 2.3.6. *For a simplicial full-dimensional pointed cone C generated by rays u_1, u_2, \dots, u_d (with vertex 0) and for any point s*

$$I(s + C, \ell) = \text{vol}(\Pi_C) e^{\langle \ell, s \rangle} \prod_{i=1}^d \frac{1}{-\langle \ell, u_i \rangle}.$$

These identities holds as a meromorphic function of ℓ and pointwise for every ℓ such that $\langle \ell, u_i \rangle \neq 0$ for all u_i .

2.4. Generating function for non-simple cones

In the last section, we reviewed results that reduced the problem of finding the generating function of $\sum_{x \in P \cap \mathbb{Z}^d} e^{\langle \ell, x \rangle}$ and $\int_P e^{\langle \ell, x \rangle} dx$ of a general polytope P to finding the same generating functions of a simplicial cone C . This works perfectly if the polytope P is simplicial, meaning at every vertex v , the cone $\text{TCone}(P, v)$ is simplicial. When $\text{TCone}(P, v)$ is not simplicial, the solution is to triangulate it.

DEFINITION 2.4.1. A triangulation of a cone C is the set Γ of simplicial cones C_i of the same dimension as the affine hull of C such that

- (1) the union of all the simplicial cones in Γ is C ,
- (2) the intersection of any pair of simplicial cones in Γ is a common face of both simplicial cones,
- (3) and every ray of every simplicial cone is also a ray of C .

Let C be a full-dimensional pointed polyhedral cone, and $\Gamma = \{C_i \mid i \in I_1\}$ be a triangulation into simplicial cones C_i where I_1 is a finite index set. It is true that $C = \bigcup_{i \in I_1} C_i$, but $[C] = \sum_{i \in I_1} [C_i]$ is false as points on the boundary of two adjacent simplicial cones are counted multiple times. The correct approach is to use the inclusion-exclusion formula:

$$[C] = \sum_{\emptyset \neq J \subseteq I_1} (-1)^{|J|-1} [\cap_{j \in J} C_j]$$

Also, note that this still holds true when C (and the C_i) is shifted by a point s . When $|J| \geq 2$, $I(\cap_{j \in J} C_j, \ell) = 0$ as $\cap_{j \in J} C_j$ is not full-dimensional, and the integration is done with respect to the Lebesgue measure on \mathbb{R}^d . This leads us to the next lemma.

LEMMA 2.4.2. *For any triangulation D_s of the feasible cone C_s at each of the vertices s of the polytope P we have $I(P, \ell) = \sum_{s \in V(P)} \sum_{C \in D_s} I(s + C, \ell)$*

Lemma 2.4.2 states that we can triangulate a polytope's feasible cones and apply the integration formulas on each simplicial cone without worrying about shared boundaries among the cones. Note that there is no restriction on how the triangulation is performed.

More care is needed for the discrete case as $S(\cap_{j \in J} C_j, \ell) \neq 0$ when $|J| \geq 2$. We want to avoid using the inclusion-exclusion formula as it contains exponentially many terms (in size of $|I_1|$).

In the general case for computing $S(s + C, \ell)$, the cone of feasible directions at a vertex $C_s(P)$ may not be simplicial. Moreover, the number of lattice points in $\Pi_C \cap \mathbb{Z}^d$ may be exponential in the encoding of C . A standard solution to these problems is to compute a triangulation for each cone C_s , and then decompose each simplicial cone into unimodular cones via Algorithm ??.

- (1) ex of a line via polynomial
- (2) 2d tile case
- (3) Brion, tangent cones
- (4) Barvinok

FIGURE 2.1. A picture of a gull.



CHAPTER 3

Long Title of Second Chapter

Rain is wet. The conclusions are immediate and self-evident. We leave them as an exercise for the reader.

APPENDIX A

Long Title of Appendix A

Observations of non-wet rain have recently appeared in the literature. In this Appendix, we briefly consider the implications of these observations for the analysis offered in this dissertation.

Bibliography

- [Bar08] A. I. Barvinok, *Integer points in polyhedra*, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, Switzerland, 2008.
- [BHS09] M. Beck, C. Haase, and F. Sottile, (*formulas of brion, lawrence, and varchenko on rational generating functions for cones*), The Mathematical Intelligencer **31** (2009), no. 1, 9–17 (English).
- [Bri37] C. Brianchon, *Théorème nouveau sur les polyèdres*, École Polytechnique **15** (1837), 317–319.
- [Bri88] M. Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. École Norm. Sup. **21** (1988), no. 4, 653–663.
- [Gra71] J. P. Gram., *Om rumvinklerne i et polyeder*, Tidsskrift for Math **3(4)** (1871), 161–163.
- [Las83] J. B. Lasserre, *An analytical expression and an algorithm for the volume of a convex polyhedron in \mathbb{R}^n* , Journal of Optimization Theory and Applications **39** (1983), no. 3, 363–377 (English).