Decomposition methods for nonlinear optimization and data mining

By

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Acknowledgments

I'd like to thank the little people.

CHAPTER 1

Introduction

Allow me to introduce you to my dissertation.

CHAPTER 2

Background

In this chapter, we first review some polyhedral decompositions and how these relate to generating functions for the lattice points of a polyhedra.

2.1. Working with generating functions: an example

Let us start with an easy example. Consider the one dimensional polyhedra in $\mathbb{R}$ given by $P = [0, n]$. We encode the latticd points of $P \cap \mathbb{Z}$ by placing each integer point as the power of a monomial, there by obtaining the polynomial $S(P; z) := z^0 + z + z^2 + z^3 + \cdots + z^n$. The polynomial $S(P; z)$ is called the *generating function of* P. Notice that counting $P \cap \mathbb{Z}$ is equivalat to evaluating $S(P, 1)$.

In terms of the computational complexity, listing each monomial in the polynomial $S(P, z)$ results in a polynomial with exponential length in the bit length of n. However, we can rewrite the summation with one term:

$$S(P, z) = z^0 + z^1 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Counting the number of point in $|P \cap \mathbb{Z}|$ is no longer as simple as evaluating $\frac{1-z^{n+1}}{1-z}$ at $z = 1$ because this is a singularity. However, this singularity is removable. One could perform long-polynomial division, but this would result in a expoentially long polynomial in the bit length of n. Another option that yeilds a polynomail time algorithm would be to apply L'Hospital's rule:

$$\lim_{z \to 1} S(P, z) = \lim_{z \to 1} \frac{-(n + 1)z^n}{1} = n + 1.$$

Notice that $S(P, z)$ can be written in two ways:

$$S(P, z) = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z} = \frac{1}{1 - z} + \frac{z^n}{1 - z^{-1}}.$$

The first two rational expressions have a nice description in terms of their series expansion:

$$1 + z + \cdots + z^n = (z^0 + z^1 + \cdots) - (z^{n+1} + z^{n+2} + \cdots).$$

For the secont two rational functions, we have to be careful about the domain of convergence when computing the series expansion. Notice that in the series expansion,

$$\frac{1}{1 - z} = \begin{cases} z^0 + z^1 + z^2 \cdots & \text{if } |z| < 1 \\ -z^{-1} - z^{-2} - z^{-3} - \cdots & \text{if } |z| > 1 \end{cases}$$

$$\frac{z^n}{1 - z^{-1}} = \begin{cases} -z^{n+1} - z^{n+2} - z^{n+3} - \cdots & \text{if } |z| < 1 \\ z^n + z^{n-1} + z^{n-2} + \cdots & \text{if } |z| > 1 \end{cases}$$

adding the terms when $|z| < 1$ or $|z| > 1$ results in the desired polynomial: $z^0 + z^1 + \cdots + z^n$. But we can also get the correct polynomial by adding the series that corresponds to different domains of convergence. However, to do this we must now add the series $\cdots + z^{-2} + z^{-1} + 0 + z + z^2 + \cdots$ which corresponds to the polyhedra that is the entire real line:

$$\begin{aligned} 1 + z + \cdots z^n = &(1 + z + z^2 + \cdots) \\ &+ (z^n + z^{n-1} + \cdots) \\ &- (\cdots + z^{-2} + z^{-1} + 0 + z + z^2 + \cdots) \end{aligned}$$

and

$$\begin{aligned} 1 + z + \cdots z^n = &(-z^{-1} - z^{-2} - z^{-3} - \cdots) \\ &+ (-z^{n+1} - z^{n+2} - z^{n+3} - \cdots) \\ &+ (\cdots + z^{-2} + z^{-1} + 0 + z + z^2 + \cdots) \end{aligned}$$

Hence by including the series $\cdots + z^{-2} + z^{-1} + 0 + z + z^2 + \cdots$, we can perform the series expansion of $\frac{1}{1-z} + \frac{z^n}{1-z^{-1}}$ by computing the series expansion of each term on potentially different domains of convergence.

In the next sections, we will develop rigorous justification for adding the series $\cdots + z^{-2} + z^{-1} + 0 + z + z^2 + \cdots$.

2.2. Indicator functions

Definition 2.2.1. The indicator function, $[A] : \mathbb{R}^d \to \mathbb{R}$, of a set $A \subseteq \mathbb{R}^d$ takes two values: $[A](x) = 1$ if $x \in A$ and $[A](x) = 0$ otherwise.

The set of indicator functions on $\mathbb{R}^d$ froms a vectorspace with pointwise additions and scalar multiplication. Notice that $[A] \cdot [B] = [A \cap B]$, and $[A] + [B] = [A \cup B] + [A \cap B]$.

Definition 2.2.2. The *cone* of a set $A \subseteq \mathbb{R}^d$ is all conic combinations of the points from A:

$$\text{Cone}(A) := \left\{ \sum_i \alpha_i a_i \mid a_i \in A, \alpha_i \in \mathbb{R}_{\geq 0} \right\}$$

Definition 2.2.3. Let P be a polyhedron and $x \in P$. Then the *tangent cone*, of P at x is the polyhedral cone

$$\text{TCone}(P, x) := x + \text{Cone}(P - x)$$

Note that if x is a vertex of P, and P is given by an inequality description, then the tangent cone $\text{TCone}(P, x)$ is the intersection of inequalities that are tight at x. Also, $\text{TCone}(P, x)$ includes the affine hull of the face that x is in, so the tangent cone is pointed only if x is a vertex.

When F is a face of P, we will also use the notation $\text{TCone}(P, F)$ to denote $\text{TCone}(P, x)$ where x is any interior point of F.

Theorem 2.2.4 ([**Bri37**], [**Gra71**]). *Let P be a polyhedron, then*

$$[P] = \sum_F (-1)^{\dim(F)} [\text{TCone}(P, F)]$$

where the sum ranges over all faces F of P including $F = P$ but excluding $F = \emptyset$

(1) ex of a line via polynomial
(2) 2d tile case

(3) Brion, tangent cones
(4) Barvinok

FIGURE 2.1. A picture of a gull.

CHAPTER 3

Long Title of Second Chapter

Rain is wet. The conclusions are immediate and self-evident. We leave them as an exercise for the reader.

APPENDIX A

Long Title of Appendix A

Observations of non-wet rain have recently appeared in the literature. In this Appendix, we briefly consider the implications of these observations for the analysis offered in this dissertation.

Bibliography

[Bri37] C. Brianchon, *Théorème nouveau sur les polyèdres*, École Polytechnique **15** (1837), 317–319.
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