

Rain is Wet

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B.S. (University of California, Davis) 2001

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Meteorology

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

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2006

Contents

Acknowledgments	iv
Chapter 1. Introduction	1
Chapter 2. Background	2
2.1. Working with generating functions: an example	2
2.2. Indicator functions	3
Chapter 3. Long Title of Second Chapter	5
Appendix A. Long Title of Appendix A	6
Bibliography	7

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Acknowledgments

I'd like to thank the little people.

CHAPTER 1

Introduction

Allow me to introduce you to my dissertation.

CHAPTER 2

Background

In this chapter, we first review some polyhedral decompositions and how these relate to generating functions for the lattice points of a polyhedra.

2.1. Working with generating functions: an example

Let us start with an easy example. Consider the one dimensional polyhedra in \mathbb{R} given by $P = [0, n]$. We encode the lattice points of $P \cap \mathbb{Z}$ by placing each integer point as the power of a monomial, thereby obtaining the polynomial $S(P; z) := z^0 + z + z^2 + z^3 + \dots + z^n$. The polynomial $S(P; z)$ is called the *generating function* of P . Notice that counting $P \cap \mathbb{Z}$ is equivalent to evaluating $S(P, 1)$.

In terms of the computational complexity, listing each monomial in the polynomial $S(P, z)$ results in a polynomial with exponential length in the bit length of n . However, we can rewrite the summation with one term:

$$S(P, z) = z^0 + z^1 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Counting the number of point in $|P \cap \mathbb{Z}|$ is no longer as simple as evaluating $\frac{1 - z^{n+1}}{1 - z}$ at $z = 1$ because this is a singularity. However, this singularity is removable. One could perform long-polynomial division, but this would result in a exponentially long polynomial in the bit length of n . Another option that yields a polynomial time algorithm would be to apply L'Hospital's rule:

$$\lim_{z \rightarrow 1} S(P, z) = \lim_{z \rightarrow 1} \frac{-(n+1)z^n}{1} = n + 1.$$

Notice that $S(P, z)$ can be written in two ways:

$$S(P, z) = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z} = \frac{1}{1 - z} + \frac{z^n}{1 - z^{-1}}.$$

The first two rational expressions have a nice description in terms of their series expansion:

$$1 + z + \dots + z^n = (z^0 + z^1 + \dots) - (z^{n+1} + z^{n+2} + \dots).$$

For the second two rational functions, we have to be careful about the domain of convergence when computing the series expansion. Notice that in the series expansion,

$$\begin{aligned} \frac{1}{1 - z} &= \begin{cases} z^0 + z^1 + z^2 \dots & \text{if } |z| < 1 \\ -z^{-1} - z^{-2} - z^{-3} - \dots & \text{if } |z| > 1 \end{cases} \\ \frac{z^n}{1 - z^{-1}} &= \begin{cases} -z^{n+1} - z^{n+2} - z^{n+3} - \dots & \text{if } |z| < 1 \\ z^n + z^{n-1} + z^{n-2} + \dots & \text{if } |z| > 1 \end{cases} \end{aligned}$$

adding the terms when $|z| < 1$ or $|z| > 1$ results in the desired polynomial: $z^0 + z^1 + \dots + z^n$. But we can also get the correct polynomial by adding the series that corresponds to different domains of convergence. However, to do this we must now add the series $\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots$ which corresponds to the polyhedra that is the entire real line:

$$\begin{aligned} 1 + z + \dots + z^n &= (1 + z + z^2 + \dots) \\ &\quad + (z^n + z^{n-1} + \dots) \\ &\quad - (\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots) \end{aligned}$$

and

$$\begin{aligned} 1 + z + \dots + z^n &= (-z^{-1} - z^{-2} - z^{-3} - \dots) \\ &\quad + (-z^{n+1} - z^{n+2} - z^{n+3} - \dots) \\ &\quad + (\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots) \end{aligned}$$

Hence by including the series $\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots$, we can perform the series expansion of $\frac{1}{1-z} + \frac{z^n}{1-z^{-1}}$ by computing the series expansion of each term on potentially different domains of convergence.

In the next sections, we will develop rigorous justification for adding the series $\dots + z^{-2} + z^{-1} + 0 + z + z^2 + \dots$.

2.2. Indicator functions

DEFINITION 2.2.1. The indicator function, $[A] : \mathbb{R}^d \rightarrow \mathbb{R}$, of a set $A \subseteq \mathbb{R}^d$ takes two values: $[A](x) = 1$ if $x \in A$ and $[A](x) = 0$ otherwise.

The set of indicator functions on \mathbb{R}^d forms a vectorspace with pointwise additions and scalar multiplication. Notice that $[A] \cdot [B] = [A \cap B]$, and $[A] + [B] = [A \cup B] + [A \cap B]$.

DEFINITION 2.2.2. The *cone* of a set $A \subseteq \mathbb{R}^d$ is all conic combinations of the points from A :

$$\text{Cone}(A) := \left\{ \sum_i \alpha_i a_i \mid a_i \in A, \alpha_i \in \mathbb{R}_{\geq 0} \right\}$$

DEFINITION 2.2.3. Let P be a polyhedron and $x \in P$. Then the *tangent cone*, of P at x is the polyhedral cone

$$\text{TCone}(P, x) := x + \text{Cone}(P - x)$$

Note that if x is a vertex of P , and P is given by an inequality description, then the tangent cone $\text{TCone}(P, x)$ is the intersection of inequalities that are tight at x . Also, $\text{TCone}(P, x)$ includes the affine hull of the face that x is in, so the tangent cone is pointed only if x is a vertex.

When F is a face of P , we will also use the notation $\text{TCone}(P, F)$ to denote $\text{TCone}(P, x)$ where x is any interior point of F .

THEOREM 2.2.4 ([Bri37], [Gra71]). *Let P be a polyhedron, then*

$$[P] = \sum_F (-1)^{\dim(F)} [\text{TCone}(P, F)]$$

where the sum ranges over all faces F of P including $F = P$ but excluding $F = \emptyset$

- (1) ex of a line via polynomial
- (2) 2d tile case

- (3) Brion, tangent cones
- (4) Barvinok

FIGURE 2.1. A picture of a gull.



CHAPTER 3

Long Title of Second Chapter

Rain is wet. The conclusions are immediate and self-evident. We leave them as an exercise for the reader.

APPENDIX A

Long Title of Appendix A

Observations of non-wet rain have recently appeared in the literature. In this Appendix, we briefly consider the implications of these observations for the analysis offered in this dissertation.

Bibliography

- [Bri37] C. Brianchon, *Théorème nouveau sur les polyèdres*, École Polytechnique **15** (1837), 317–319.
- [Gra71] J. P. Gram., *Om rumvinklerne i et polyeder*, Tidsskrift for Math **3(4)** (1871), 161–163.