Math 231br Final Paper: the Hirzebruch Riemann Roch theorem

Fengning (David) Ding

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This paper discusses the Atiyah-Hirzebruch formulation of the Riemann Roch theorem for differentiable manifolds, which was first proven in [1]. This formulation, which is closely related to the Grothendieck-Riemann-Roch formula for smooth algebraic varieties ([1]), relates the Gysin maps from K theory to the Gysin maps of ordinary cohomology via the Chern character. From the formula, we can deduce many divisibility properties of various characteristic classes. We mostly follow the exposition in [3].

1 Preliminary Definitions

Recall the definition of a G-bundle, where G is some topological group.

Definition 1. A G-principle bundle of a space X is a fiber bundle $\pi : P \to X$ with a properly continuous right action $P \times G \to P$ that preserves the fibers of P and acts freely and transitively.

Any topological group G admits a classifying space BG, so that any G-principal bundle over a paracompact manifold X is isomorphic to a pullback of the principal bundle $EG \to BG$.

If V is a representation of G and P a G-bundle over X, then V can be regarded as a G-vector bundle $V \times P/\{(v, gp) \sim (gv, p)\} \to X$. For instance, complex vector bundles are $U(n)$-bundles, real vector bundles are $O(n)$-bundles, and orientable real vector bundles are $SO(n)$-bundles. Now, recall that $\mathrm{Spin}(2n)$ is the double cover of $SO(2n)$. If a vector bundle is a $\mathrm{Spin}(2n)$-bundle, we say that the bundle has spin structure; a bundle ξ has spin structure iff $w_2(\xi) = 0$ (see [1] and [2]). More generally, we define $\mathrm{Spin}^c(2n) = \mathrm{Spin}(n) \times U(1)/(p, z) \sim (-p, -z)$, and we can consider bundles with c-spin structures. The homomorphism $\mathrm{Spin}^c(2n) \to U(1)$ given by $(p, z) \mapsto z^2$ induces a map $H^1(X; \mathrm{Spin}^c(2n)) \to H^1(X; U(1)) \cong \mathrm{Vect}^1_{\mathbb{C}}(X)$. Hence, we can associate a complex line bundle $L(W)$ to the real bundle W. We cite the following proposition from Karoubi without proof.

Proposition 1 (see [3] p. 218)**.** *If W is a c-spin bundle, then $W \oplus W$ is a c-spin bundle* and *a complex bundle.*

2 The Gysin map

Let X and Y be compact manifolds, and let $f : X \to Y$ be a continuous map. We know that f induces ring homomorphisms $f_K^* : K(Y) \to K(X)$ and $f_H^* : H^*(Y) \to H^*(X)$, since cohomology and K-theory are contravariant functors. It turns out that if $\dim X \equiv \dim Y \bmod 2$, we are able to construct a map going in the "wrong" direction, $f_{K*} : K(X) \to K(Y)$ and $f_{H*} : H^*(Y) \to H^*(X)$. This map is called the Gysin map, and to do this, we use the Thom isomorphism theorem:

Theorem 1 (Thom isomorphism theorem, see [3] p. 185)**.** *Let V be a complex vector bundle over X. Then, $K(B(V), S(V))$ is a free $K(X)$ module (with the module structure given by the map $p^* : K(X) \to K(B(V), S(V))$ induced by the projection $B(V)/S(V) \to X$) with generator $U_V \in K(B(V), S(V))$, known as the Thom class.*

Remark 1. The theorem is also true in cohomology, and we will use T_V to denote the Thom class of the bundle V in cohomology. Moreover, if W is a 2n-dimensional real vector bundle with a c-spin structure, then there is a Thom isomorphism $K(X) \to K(B(W), S(W))$ (see [3], p. 227).

We state without proof some properties of the Thom class (this proposition is proved in Karoubi):

Proposition 2.

1. *If V and W are vector bundles over X, $U_{V \oplus W} = U_V \cdot U_W \in K(B(V \oplus W), S(V \oplus W))$, where the product is performed after pulling U_V and U_W back to $K(B(V \oplus W), S(V \oplus W))$ along the projection $V \oplus W \to V$ and $V \oplus W \to W$ (see [3], p. 185).*

2. *If V is a bundle over X and $s : X \to V$ is the inclusion of X into V as the zero bundle, $s^*(U_V) = \chi_K(V)$, where $\chi_K(V)$ is the Euler class of V. If V is a line bundle, $\chi_K(V) = 1 - V$ (see [3], p. 187).*

3. *Let W is a $2n$-dimensional real-bundle with c-spin structure, $U'_{W \oplus W}$ be the Thom class for the Thom isomorphism when regarding $W \oplus W$ as a real bundle, and $U_{W \otimes \mathbb{C}}$ the Thom class for the Thom isomorphsm when regarding $W \oplus W \cong W \otimes \mathbb{C}$ as a complex bundle. Then, $U'_{W \oplus W} = U_{W \otimes \mathbb{C}} L(W)$ (see [3], p. 224).*

Remark 2. Statements 1 and 2 in the above proposition is also true in cohomology, and $\chi_H(V)$ is the "normal" cohomology Euler class.

The Thom isomorphism is the main ingredient in the construction of the Gysin map. First, consider the case when $f : X \to Y$ is a proper embedding. Let N be normal bundle of X in Y, $B(N)$ be the ball bundle, and $S(N)$ the sphere bundle. Then, $K(B(N), S(N))$ is a $K(X)$ module, with module structure given by the map $p^* : K(X) \to K(B(N), S(N))$ induced by the projection $B(N)/S(N) \to X$. The Thom isomorphism theorem then implies that $K(B(N), S(N))$ is a free $K(X)$ module generated by $U_N \in K(B(N), S(N))$, the Thom class. Hence, we have a map $\phi_K : K^*(X) \to K(B(N), S(N))$. Now, let A be a tubular neighborhood of X in Y with boundary $S(N)$. Then, $K(B(N), S(N)) \cong K(A, S(N))$. By excision, $K(A, S(N)) \cong K(Y, Y - A')$ (where A' is a slightly smaller tubular neighborhood of Y, so that the closure of $Y - A$ is in the interior of $Y - A'$), and finally, we have a map $K(Y, Y - A') \to K(Y)$. We define the Gysin map to be the composition of all these maps:

$$f_{K*} : K(X) \xrightarrow{\phi_K} K(B(N), S(N)) \to K(A, S(N)) \xrightarrow{\text{excision}} K(Y, Y - A') \to K(Y).$$

Now, we consider the general case (when f is not necessarily an embedding). Suppose $\dim Y \equiv \dim X \bmod 2$. We can factor a general differentiable map $f : X \to Y$ into an embedding $i : X \to X_+ \to (Y \times \mathbb{R}^{2n})_+$ for large enough n (where X_+ is the one-point compactification), followed by the projection $p : (Y \times S^{2n}) + \to Y$. Note that $(Y \times \mathbb{R}^{2n})_+ \cong Y_+ \wedge S^{2n}$. Now, we already have a map $i_{K*} : K(X) \cong \tilde{K}(X_+) \to \tilde{K}(Y_+ \wedge S^{2n})$ since i is an embedding. We now note that $\alpha : K(Y) \cong \tilde{K}(Y_+) \to \tilde{K}(Y_+ \wedge S^{2n})$ is the Bott-isomorphism, given by external multiplication by an element of $\tilde{K}(S^{2n})$. We can then define $f_{K*} = \alpha^{-1} \circ i_{K*}$:

$$f_{K*} : K(X) \xrightarrow{\cong} \tilde{K}(X_+) \xrightarrow{i_{K*}} \tilde{K}(Y_+ \wedge S^{2n}) \xrightarrow{\alpha} \tilde{K}(Y_+) \xrightarrow{\cong} K(Y).$$

We list some useful properties of the Gysin map below.

Proposition 3 (see [3], p. 233). 1. *The Gysin homomorphism*

$$f_{K*} : K(X) \to K(Y)$$

does not depend on the choice of the embedding i but only on the homotopy class of f.

2. *If $f : X \to Y$ and $g : Y \to Z$ are differentiable maps, $(g \circ f)_{K*} = g_{K*} \circ f_{K*}$.*

3. *For all $x \in K(X)$ and $y \in K(Y)$,*

$$f_*(xf^*(y)) = f_*(x)y.$$

We can analogously define the Gysin map for ordinary cohomology since we have a Thom isomorphism theorem for cohomology also. We replace the Bott-periodicity in the last step with the isomorphism

$$H^*(Y_+, *; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(S^{2n}, *; \mathbb{Q}) \to H^*(Y_+ \times S^{2n}, S^{2n} \cup Y_+; \mathbb{Q}) \cong H^*(Y_+ \wedge S^{2n}, *; \mathbb{Q})$$

from the cohomology cross product with a generator in $H^{2n}(S^{2n}, *; \mathbb{Q})$, we choose the image under the Chern character of the generator of $\tilde{K}(S^{2n})$. It turns out that the Gysin map for cohomology is just the Poincare dual of the natural homology map ([1]):

$$\begin{array}{ccc} H^*(X) & \xrightarrow{f_{H*}} & H^{\dim Y - \dim X + *}(Y) \\ \big\downarrow & & \big\uparrow \\ H_{\dim X - *}(X) & \xrightarrow{f^*} & H_{\dim X - *}(Y). \end{array}$$

Moreover, this construction also can be used to give a Gysin map $f_{KO*} : KO(X) \to KO(Y)$ from the real K-theory of X to that of Y. Namely, if W is a real bundle with spin structure of dimension $8n$ over X, we have the Thom isomorphism $\phi_{KO} : KO(X) \to KO(W)$. Take W to be the normal bundle in $Y_+ \wedge S^{8n}$. Then, we can compose with the real Bott-peridocity $\tilde{KO}(Y_+ \wedge S^{8n}) \to \tilde{KO}(Y_+)$ to get the Gysin map $f_{KO*} : KO(X) \to KO(Y)$.

3 Baby Atiyah-Hirzebruch for complex bundles

Let $f(x) = 1 + a_1 x_1 + \cdots$ be a formal power series with coefficients in $\mathbb{Q}$. Then, the product $f(x_1)f(x_2)\ldots f(x_n)$ is a formal power series that is symmetric in $x_1, \ldots, x_n$. Hence, we can write

$$f(x) = \sum_{k=0}^{\infty} R_f^k(\sigma_1, \ldots, \sigma_n)$$

where σ_i is the i-th elementary symmetric function on x_j and R_f^k is of degree k in x_j. Then, if V is a vector bundle over X, $T_f^k(V) = R_f^k(c_1(V), c_2(V), \ldots, c_n(V))$ is a well-defined element of $H^{2k}(X; \mathbb{Q})$.

The following properties then follow immediately from properties of the Chern class:

Proposition 4. *Define $T_f(V) = \sum_{k=0}^{\infty} T_f^k(V)$.*

1. *$T_f^0(V) = 1$.*
2. *T_f is natural, i.e., if $\phi : X \to Y$ is a map of spaces and ξ a bundle over Y, $T_f(\phi^*\xi) = \phi^* T_f(\xi)$.*
3. *$T_f(V_1 \oplus V_2) = T_f(V_1)T_f(V_2)$.*
4. *If V is of dimension one, $T_f(V) = f(\chi_H(V))$, where $\chi_H(V) = c_1(V)$ is the Euler class of V.*

Example 1.

1. When $f(x) = 1 + x$, $T_f(V)$ is the total Chern class $c(V) = 1 + c_1(V) + \cdots + c_n(V)$.
2. When $f(x) = \frac{1-e^{-x}}{x}$, $T_f(V)$ is the *Todd class* of V and denoted by $\tau(V)$.

Theorem 2. *Let V be a complex vector bundle of rank n, and let $\phi_K : K(X) \to K(B(\bar{V}), S(\bar{V}))$ and $\phi_H : H^*(X;\mathbb{Q}) \to H^*(B(V), S(V);\mathbb{Q})$ be the Thom isomorphisms in K theory and ordinary cohomology. Then,*

$$\mathrm{ch}(\phi_K(x)) = \phi_H(\mathrm{ch}(x)\tau(V)).$$

Remark 3. If $\tau(V) = 1$ (for example, if V was the trivial bundle), the theorem shows that the Thom isomorphism commutes with the Chern character:

$$\begin{array}{ccc} K(B(\bar{V}), S(\bar{V})) & \xrightarrow{\mathrm{ch}} & H^*(B(V), S(V)) \\ \phi_K \big\downarrow & & \phi_H \big\downarrow \\ K(X) & \xrightarrow{\mathrm{ch}} & H^*(X). \end{array}$$

Hence, in some sense, $\tau(V)$ measures the difference between the Thom isomorphism for K theory and for cohomology.

Proof. We first show that $\phi_H(\tau(V)) = \mathrm{ch}(\phi_K(1))$. Denote $\phi_H^{-1}(\mathrm{ch}(\phi_K(1)))$ by $\tilde{\tau}(V)$ (which we want to equal $\tau(V)$). By definition, $\phi_K(1)$ is the Thom class $U_{\bar{V}}$. Recall that $U_{V_1 \oplus V_2} = U_{V_1} U_{V_2} \subseteq K(B(V_1 \oplus V_2), S(V_1 \oplus V_2))$, and similarly, $T_{V_1 \oplus V_2} = T_{V_1} T_{V_2} \subseteq H^*(B(V_1 \oplus V_2), S(V_1 \oplus V_2);\mathbb{Q})$. Hence,

$$\begin{aligned} \phi_H(\tilde{\tau}(V_1 \oplus V_2)) &= \mathrm{ch}(\phi_K(1)) T_{V_1 \oplus V_2} = \mathrm{ch}(U_{\bar{V_1} \oplus \bar{V_2}}) T_{V_1} T_{V_2} = \mathrm{ch}(U_{\bar{V_1}}) T_{V_1} \mathrm{ch}(U_{\bar{V_2}}) T_{V_2} \\ &= \phi_H(\tilde{\tau}(V_1)) \phi_H(\tilde{\tau}(V_2)), \end{aligned}$$

so if we show that $\tilde{\tau}(L) = \tau(L)$ for all line bundles L, we can use the splitting principle to conclude that $\tilde{\tau}(V) = \tau(V)$ for all vector bundles V.

Let $s : X \to B(L)/S(L)$ be the inclusion of X as the zero section. This induces maps $s_K^* : K(B(\bar{L}), S(\bar{L})) \to K(X)$ and $s_H^* : H^*(B(L), S(L);\mathbb{Q}) \to H^*(X;\mathbb{Q})$. Now, recall that $s_K^*(U_L) = \chi_K(L) = 1 - L$ and $s_H^*(T_L) = \chi_H(L)$, where χ is the Euler class of V in K-theory/cohomology. Since $\phi_H(\tilde{\tau}(L)) = \tilde{\tau}(L) \smile T_L$, $s_H^*(\phi_H(\tilde{\tau}(L))) = \tilde{\tau}(L) s_H^*(T_L) = \tilde{\tau}(L)\chi_H(L)$, so

$$\begin{aligned} \tilde{\tau}(L)\chi_H(L) &= s_H^*(\phi_H(\tilde{\tau}(L))) = s_H^*(\mathrm{ch}(\phi_K(1))) = \mathrm{ch}(s_K^*(\phi_K(1))) \\ &= \mathrm{ch}(1 - \bar{L}) = 1 - \exp(-c_1(L)). \end{aligned}$$

Since $\chi_H(L) = c_1(L)$, we see that $\tilde{\tau}$ is generated by the power series $\frac{1-e^{-x}}{x}$, the same series as τ, so $\tilde{\tau}(V) = \tau(V)$ for all bundles V.

We conclude that

$$\tau(V)\mathrm{ch}(x) = \phi_H^{-1}(\mathrm{ch}(\phi_K(1)))\mathrm{ch}(x) = \phi_H^{-1}(\mathrm{ch}(x\phi_K(1))) = \phi_H^{-1}(\mathrm{ch}(\phi_K(x))).$$

□

4 Baby Atiyah-Hirzebruch for real bundles

We now prove an analogous formula for real bundles W over X by constructing a class $A(W)$ that will play the role of the Todd class $\tau(V)$. Recall that the *Pontrjagin classes* of W are defined to be $p_i(W) = (-1)^i c_{2i}(W \otimes \mathbb{C})$. If the rank of W is even and W has a spin structure, then there is a Thom isomorphism $K(X) \to K(B(W), S(W))$ (note that we use complex K groups). In the previous section, we saw that the class $\tau(V) = \phi_H^{-1}(\mathrm{ch}(\phi_K(1)))$ was useful in arriving at a relationship between the Chern character and Thom isomorphism; hence, let us define for real bundles an analogous class, $A(W) = \phi_H^{-1}(\mathrm{ch}(\phi_K(1)))$ (where ϕ_K is now the Thom isomorphism to the real vector bundle W). Then, the exact same argument of the previous section will show the following theorem:

Theorem 3 (Atiyah-Hirzebruch)**.** *If W is a real $2n$-dimensional real vector bundle with c-spin structure,*

$$\mathrm{ch}(\phi_K(x)) = \phi_H(A(W)\mathrm{ch}(x)).$$

Now, we construct $A(W)$ in terms of the Pontryagin classes of W. Define $d(W) = c_1(L(W))$ (where $L(W)$ is the complex line bundle associated with W; see Section 1).

Proposition 5. *Let $V = W \otimes \mathbb{C}$. Then, $A(W) = e^{d/2}\sqrt{\tau(V)}$.*

Proof. We note that $A(W_1 \oplus W_2) = A(W_1)A(W_2)$ and $A(W) = 1$ for trivial bundles (the same proof for $\tilde{\tau}$ can be applied). Now, recall that $W \oplus W$ can be regarded has having a complex structure, and the Thom class when regarding it as a complex bundle, $U_{W\oplus W}$, is related to the Thom class when regarding it as a real bundle, $U'_{W\oplus W}$ by the formula $U_{W\oplus W}L(W) = U'_{W\oplus W}$. Hence,

$$\begin{aligned} A(W)^2 &= A(W \oplus W) = \phi_H^{-1}(\mathrm{ch}(U'_{W\oplus W})) = \phi_H^{-1}(\mathrm{ch}(U_{W\oplus W})\mathrm{ch}(L(W))) \\ &= e^{d(W)}\tau(W \otimes \mathbb{C}), \end{aligned}$$

so $A(W) = e^{d(W)}\sqrt{\tau(W \otimes \mathbb{C})}$ as desired. ☐

Now, let us write $c(V) = \prod_{i=1}^n (1 - x_i)(1 + x_i)$, so that the Pontryagin classes are the elementary symmetric polynomials of x_i^2. Then,

$$\begin{aligned} A(W) &= e^{d(W)/2}\prod_{i=1}^n \sqrt{\frac{1 - e^{x_i}}{-x_i} \times \frac{1 - e^{-x_i}}{x_i}} \\ &= e^{d(W)/2}\prod_{i=1}^n \sqrt{\frac{e^{x_i/2}(e^{-x_i/2} - e^{x_i/2})}{-x_i} \times \frac{e^{-x_i/2}(e^{x_i/2} - e^{-x_i})}{x_i}} \\ &= e^{d(W)/2}\prod_{i=1}^n \frac{\sinh(x_i/2)}{x_i/2}, \end{aligned}$$

which we regard as a function of the Pontrjagin classes $p_i(x_1, \ldots) = \sigma_i(x_1^2, \ldots)$. We write

$$\hat{A}(W) = \prod_{i=1}^{n} \frac{\sinh(x_i/2)}{x_i/2}.$$

5 The Atiyah-Hirzebruch-Riemann-Roch Theorem

In this section, we study the interaction of the Chern character with the Gysin maps $f_{K*} : K(X) \to K(Y)$ and $f_{H*} : H^*(X) \to H^*(Y)$. The previous two sections can be regarded as the special cases where Y is the Thom space of some bundle over X.

Theorem 4 (Atiyah-Hirzebruch-Riemann-Roch)**.** *Let X and Y be compact differentiable manifolds such that $\dim Y - \dim X \equiv 0 \bmod 2$, and let $f : X \to Y$ be a continuous map such that $v_f = f^*(TY) - TX$ has a stable c-spin structure. Then,*

$$\mathrm{ch}(f_{K*}(x)) = f_{H*}(e^{d(v_f)/2}\hat{A}(v_f)\mathrm{ch}(x)).$$

Proof. We first consider the case where f is an embedding. Then, $v_f = N$ in $K(X)$, where N is the normal bundle of X when embedded into Y. Note that N is an even-dimensional bundle over X, as in the hypothesis of Theorem 3. Let A be a tubular neighborhood of X in Y whose boundary is $S(N)$. Then, we have a commutative diagram

$$\begin{array}{ccc} K(B(N), S(N)) & \xrightarrow{u} & K(Y) \\ {\scriptstyle \mathrm{ch}}\downarrow & & {\scriptstyle \mathrm{ch}}\downarrow \\ H^*(B(N), S(N); \mathbb{Q}) & \xrightarrow{v} & H^*(Y; \mathbb{Q}), \end{array}$$

where u (resp. v) is the map $K(B(N), S(N)) \to K(Y, Y - B(N)) \to K(Y)$ (resp. $H^*(B(N), S(N); \mathbb{Q}) \to H^*(Y, Y - B(N); \mathbb{Q}) \to H^*(Y)$) induced by excision followed by the natural map of relative groups (as in the definition of the Gysin map in Section 2).

Now, by definition, the Gysin map f_{K*} is the composition $u \circ \phi_K$, and similarly, $f_{H*} = v \circ \phi_K$. By Theorem 3, we know that $\mathrm{ch}(\phi_K(x)) = \phi_H(e^{d(v_f)/2}\hat{A}(v_f)\mathrm{ch}(x))$. Hence,

$$\begin{aligned} \mathrm{ch}(f_{K*}(x)) = \mathrm{ch}(u(\phi_K(x))) &= v(\mathrm{ch}(\phi_K(x))) = v(\phi_H(e^{d(v_f)/2}\hat{A}(v_f)\mathrm{ch}(x))) \\ &= f_{H*}(e^{d(v_f)/2}\hat{A}(v_f)\mathrm{ch}(x)) \end{aligned}$$

as desired.

To show the general case, we write $f = p \circ i$, where i is the embedding $i : X \to Y_+ \wedge S^{2n}$, and p is the projection $Y_+ \wedge S^{2n} \to Y$ (this is the same factorization used in the definition of the Gysin map). Then, $f_{K*} =$

$p_{K*} \circ i_{K*}$ and $f_{H*} = p_{H*} \circ i_{H*}$. By definition, p_{K*} is the inverse of the Bott periodicity isomorphism $K(Y) \to \tilde{K}(Y_+ \wedge S^{2n})$ given by $y \mapsto yh$ for some h that generates $\tilde{K}(S^{2n})$. Moreover, p_{H*} is defined to be the inverse of the isomorphism $H^*(Y;\mathbb{Q}) \to H^*(Y_+ \wedge S^{2n}, *; \mathbb{Q})$ given by $y \mapsto y\mathrm{ch}(h)$. Hence, the following diagram commutes:

$$\begin{array}{ccc} K(Y_+ \wedge S^{2n}) & \xrightarrow{p_{K*}} & K(Y) \\ \mathrm{ch}\big\downarrow & & \mathrm{ch}\big\downarrow \\ H^*(Y_+ \wedge S^{2n}; \mathbb{Q}) & \xrightarrow{p_{H*}} & H^*(Y;\mathbb{Q}). \end{array}$$

Now

$$\begin{aligned} v_f = f^*(TY) - TX &= i^*p^*(TY) - i^*(T(Y \times S^{2n})) + i^*(T(Y \times S^{2n})) - TX \\ &= i^*(p^*(Y) - T(Y \times S^{2n})) + v_i \\ &= i^*(TY - TY - TS^{2n}) + v_i, \end{aligned}$$

so v_f and v_i are stably isomorphic. Hence, $d(v_f) = d(v_i)$. We conclude that

$$\begin{aligned} \mathrm{ch}(f_{K*}(x)) &= \mathrm{ch}(p_{K*} \circ i_{K*}(x)) = p_{H*}(\mathrm{ch}(i_{K*}(x))) \\ &= p_{H*}i_{H*}(e^{d(v_f)/2}\hat{A}(v_f)\mathrm{ch}(x)) = f_{H*}(e^{d(v_f)/2}\hat{A}(v_f)\mathrm{ch}(x)). \end{aligned}$$

□

Corollary 1. *Suppose v_f has a stable complex structure. Then,*

$$\mathrm{ch}(f_{K*}(x)) = f_{H*}(\tau(v_f)\mathrm{ch}(x)).$$

Proof. This is an immediate consequence of the above theorem, since $e^{d(v_f)/2}\hat{A}(v_f) = \tau(v_f)$ by definition of $\hat{A}$. □

5.1 Atiyah-Hirzebruch for real K theory

We briefly consider an extension of Atiyah-Hirzeburch to real K-theory. Suppose that $\dim X \equiv \dim Y \bmod 8$. Recall that we can define a Gysin map $f_{KO*} : KO(X) \to KO(Y)$. Now, we define the Pontrjagin character $\mathrm{ch}O : KO(Y) \to H^*(Y;\mathbb{Q})$ by the composition

$$\mathrm{ch}O : KO(Y) \to K(Y) \xrightarrow{\mathrm{ch}} H^*(Y;\mathbb{Q}),$$

where the first map is the complexification homomorphism. Then, the Atiyah-Hirzebruch formula becomes

Theorem 5. *Let W be a real vector bundle with spin structure over X of rank $8n$. Then,*

$$\mathrm{ch}O(f_{KO*}(x)) = f_{H*}(\hat{A}(v_f)\mathrm{ch}O(x)).$$

6 Consequences of the Atiyah-Hirzebruch formula

In this section, we will consider various divisibility results that follow from the Atiyah-Hirzebruch Riemann Roch formula. The main idea is to take Y to be a particularly simple space, such as a point. Then, the formula will tell us properties of $\mathrm{ch}(x)$.

Corollary 2. *Let X be a compact differentiable manifold of even dimension, such that TX is a stable c-spin bundle. Then,*

$$e^{-d(TX)/2}/\hat{A}(TX)$$

evaluated on the fundamental class of X is an integer.

Proof. Take Y to be a point. Then, $f_{H*} : H^p(X;\mathbb{Q}) \to H^{p-\dim X}(*;\mathbb{Q})$ is nonzero iff $p = \dim X$, and in that case, $f_{H*} : H^{\dim X}(X;\mathbb{Q}) \to H^0(*,\mathbb{Q}) \cong \mathbb{Q}$ is an isomorphism mapping $x \in H^{\dim X}(X;\mathbb{Q})$ to x evaluated on the fundamental class of X (this is easy to see using the Poincare-duality formulation of the Gysin map). Since $\mathrm{ch}(K(*)) = \mathbb{Z} \subset \mathbb{Q}$ and $v_f = -TX$, we conclude that $e^{-d(TX)/2}\hat{A}(TX)^{-1}\mathrm{ch}(x)$ evaluated on the fundamental class of X is an integer. $\square$

Corollary 3. *If X is a compact differentiable manifold of even dimension and TX a complex-bundle,* $\mathrm{ch}(x)\tau(TX)^{-1}$ *is an integer when evaluated on the fundamental class of X.*

Example 2. Let $\tau'(V) = \tau(V)^{-1}$ and $\hat{A}'(W) = \hat{A}(W)^{-1}$. Then, according to computations in [3] p. 286 and 287,

$$\begin{aligned}
\tau_1'(V) &= \frac{1}{2}c_1,\\
\tau_2'(V) &= \frac{1}{12}(c_2 + c_1^2),\\
\tau_3'(V) &= \frac{1}{24}c_2c_1,\\
\tau_4'(V) &= \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1),
\end{aligned}$$

and

$$\begin{aligned}
\hat{A}_1'(V) &= -\frac{1}{24}p_1,\\
\hat{A}_2'(V) &= \frac{1}{5760}p_1^2 - \frac{1}{1440}p_2.
\end{aligned}$$

Now, let $p_i = p_i(TX)$, $d = d(TX)$, and $c_i = c_i(TX)$. If we set $x = 1$ in Corollary 2, we see that

$$\left(1 + \frac{d}{2} + \frac{d^2}{8} + \frac{d^3}{48} + \cdots\right)\left(\frac{1}{24}p_1 + \frac{1}{5760}p_1^2 - \frac{1}{1440}p_2\right)$$

must have integer coefficient at degree $\dim X$. We thus get the following divisibility results:

- if $\dim X = 4$ (and TX a c-spin bundle), $p_1 - 3d^2$ is divisible by 24;
- if $\dim X = 6$, $d^3 - dp_1$ is divisible by 48;
- if $\dim X = 8$, $15d^4 + 30p_1d^2 + 7p_1^2 - 4p_2$ is divisible by 5760.

Analogously, if we assume TX to be a complex bundle, we get the following divisibility results from Corollary 3:

- if $\dim X = 2$, c_1 is divisible by 2;
- if $\dim X = 4$, $c_2 + c_1^2$ is divisible by 12;
- if $\dim X = 6$, c_2c_1 is divisible by 24;

and so on.

In similar spirit, we can extract divisibility results from Theorem 5.

Corollary 4. *Let X be a compact manifold with* $\dim X \equiv 4 \bmod 8$*, such that TX has spin structure. Then,* $\mathrm{ch}O(x)\hat{A}(TX)^{-1}$ *evaluated on the fundamental class of X is an* even *integer.*

Proof. Let $f : X \to S^4$ be the constant map. Now, recall that ch is an isomorphism from $K(S^4)$ to $H^4(S^4;\mathbb{Z})$. Since the complexification homomorphism $KO(S^4) \to K(S^4)$ is multiplication by 2, we see that $\mathrm{ch}O$ is an isomorphism between $KO(S^4)$ and $2\mathbb{Z}$. Now, f can be factored into the maps $\alpha : X \to *$ and $\beta : * \to S^4$, and since β_{H*} is an isomorphism between $H^0(*)$ and $H^4(S^4)$, we see that $f_{H*}(\mathrm{ch}O(x)\hat{A}(TX)^{-1}) \in 2\mathbb{Z}$ is just the evaluation of $\mathrm{ch}O(x)\hat{A}(TX)^{-1}$ on the fundamental cycle of X. □

Example 3. Recall that TX has spin-structure iff $w_2(TX) = 0$. Hence, if $\dim X = 4$ and $w_2(TX) = 0$, $p_1(TX)/24$ is an even integer, i.e., $p_1(TX)$ is divisible by 48. This is Rohlin's Theorem ([1], [4]).

As a final application, we consider $p_1(TX)$ for a manifold X. It is known that $p_1(TX)$ is not a homotopy invariant of X ([3]). However:

Corollary 5. *If X is a compact manifold, $p_1(TX)$, considered as an element of $H^4(X;\mathbb{Z}/\Gamma^4(X)$ (where Γ^4 is the torsion subgroup) is a homotopy invariant modulo 24.*

Proof. Let $f : X \to Y$ be a homotopy equivalence. Then, by Theorem 5, $f_{H*}(\hat{A}(v_f)) \in \mathrm{ch}O(KO(Y))$. Since f_{H*} is an isomorphism, we see that $\hat{A}(v_f) \in \mathrm{ch}O(KO(X))$, so $\hat{A}(TY)/\hat{A}(TX) = \mathrm{ch}O(\xi)$ for some bundle O. Expanding and equating the degree 4 terms, we get

$$\frac{1}{24}(p_1(TY) - p_1(TX)) = p_1(\xi),$$

so $p_1(TY) - p_1(TX) \equiv 0 \bmod 24$ as desired. □

Remark 4. If $H^2(X;\mathbb{Z}/2) = 0$, then $p_1(TX)$ is actually invariant modulo 48, since $p_1(\xi) \equiv w_2(\xi)^2 \bmod 2$ ([1]). If $H^2(X;\mathbb{Z}/2) = 0$, $p_1(\xi)$ is even, so $p_1(TY) - p_1(TX) \equiv 0 \bmod 48$.

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