Mis-Classified, Binary, Endogenous Regressors: Identification and Inference

Francis J. DiTraglia

Camilo García-Jimeno

University of Pennsylvania

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What is the effect of T^*?

$$y = c(\mathbf{x}) + \beta(\mathbf{x})T^* + \varepsilon$$

- y – Outcome of interest
- T^* – Unobserved, endogenous binary regressor
- T – Observed, mis-measured binary surrogate for T^*
- $\mathbf{x}$ – Exogenous covariates
- z – Discrete (typically binary) instrumental variable

(Additively Separable ε and binary $T^* \Rightarrow$ linear model given $\mathbf{x}$)

Using a discrete IV to learn about $\beta(\mathbf{x})$

$$y = c(\mathbf{x}) + \beta(\mathbf{x})\,T^* + \varepsilon$$

Constributions of This Paper

1. Show that only existing point identification result for mis-classified, endogenous T^* is incorrect.
2. Derive sharp identified set for $\beta(\mathbf{x})$ under standard assumptions.
3. Prove point identification of $\beta(\mathbf{x})$ under slightly stronger assumptions.
4. Point out problem of weak identification in mis-classification models, develop identification-robust inference for $\beta(\mathbf{x})$.

Example: Smoking and Birthweight (SNAP Trial)

Coleman et al. (N Engl J Med, 2012)

RCT with pregnant smokers in England: half given nicotine patches, the rest given placebo patches. Some given nicotine fail to quit; some given placebo quit.

- y – Birthweight
- T^* – True smoking behavior
- T – Self-reported smoking behavior
- $\mathbf{x}$ – Mother characteristics
- z – Indicator of nicotine patch

Example: Schooling and Test Scores

Burde & Linden (2013, AEJ Applied)

RCT in Afghanistan: schools built in randomly selected villages. In treatment villages only some girls attend school; in control villages some girls attend school elsewhere.

- y – Girl's score on math and language test
- T^* – Girl's true school attendance
- T – Parent's report of child's school attendance
- $\mathbf{x}$ – Child and household characteristics
- z – School built in village

Related Literature

Continuous Regressor

Lewbel (1997, 2012), Schennach (2004, 2007), Chen et al. (2005), Hu & Schennach (2008), Song (2015), Hu et al. (2015)...

Binary/Discrete, "Exogenous"

Aigner (1973), Bollinger (1996), Kane et al. (1999), Black et al. (2000), Frazis & Loewenstein (2003), Mahajan (2006), Lewbel (2007), Hu (2008), Molinari (2008)

Binary, Endogenous Regressor

Mahajan (2006),

Shiu (2015), Denteh et al. (2016), Ura (2016), Calvi et al. (2017)

"Baseline" Assumptions I – Model & Instrument

Additively Separable Model

$y = c(\mathbf{x}) + \beta(\mathbf{x})T^* + \varepsilon, \quad \mathbb{E}[\varepsilon] = 0$

Valid & Relevant Instrument: $z \in \{0, 1\}$

- $\mathbb{P}(T^* = 1|\mathbf{x}, z = 1) \neq \mathbb{P}(T^* = 1|\mathbf{x}, z = 0)$
- $\mathbb{E}[\varepsilon|\mathbf{x}, z] = 0$
- $0 < \mathbb{P}(z = 1|\mathbf{x}) < 1$

If T^* were observed, these conditions would identify β.

"Baseline" Assumptions II – Measurement Error

Notation: Mis-classification Rates

"↑" $\alpha_0(\mathbf{x}, z) \equiv \mathbb{P}\left(T = 1 \middle| T^* = 0, \mathbf{x}, z\right)$

"↓" $\alpha_1(\mathbf{x}, z) \equiv \mathbb{P}\left(T = 0 \middle| T^* = 1, \mathbf{x}, z\right)$

Mis-classification unaffected by z

$$\alpha_0(\mathbf{x}, z) = \alpha_0(\mathbf{x}), \quad \alpha_1(\mathbf{x}, z) = \alpha_1(\mathbf{x})$$

Extent of Mis-classification

$\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1$ (T is positively correlated with T^*)

Non-differential Mis-classification

$$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, z, T^*]$$

Identification Results from the Literature

Mahajan (2006) Theorem 1, Frazis & Loewenstein (2003)

$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*] = 0$, plus “Baseline” $\implies$ $\beta(\mathbf{x})$ identified

Requires (T^*, z) jointly exogenous.

Mahajan (2006) A.2

$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*]$, plus “Baseline” $\implies$ $\beta(\mathbf{x})$ identified

Allows T^* endogenous, but we prove this claim is false.

Open Question

Can we identify $\beta(\mathbf{x})$ when T^* is endogenous? If so, how?

First-stage Probabilities & Mis-classification Bounds

Unobserved	Observed
$p_k^*(\mathbf{x}) \equiv \mathbb{P}(T^* = 1 \mid \mathbf{x}, z = k)$	$p_k(\mathbf{x}) \equiv \mathbb{P}(T = 1 \mid \mathbf{x}, z = k)$

Relationship

$$p_k^*(\mathbf{x}) = \frac{p_k(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}, \quad k = 0, 1$$

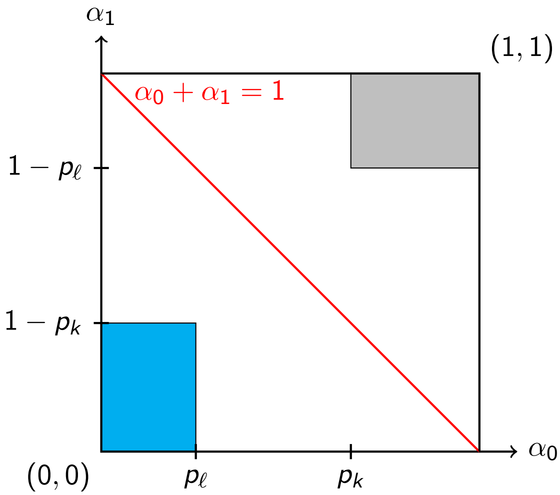
z does not affect (α_0, α_1); denominator $\neq 0$

Bounds for Mis-classification

$$\alpha_0(\mathbf{x}) \leq p_k(\mathbf{x}) \leq 1 - \alpha_1(\mathbf{x}), \quad k = 0, 1$$

$\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1$

$\alpha_0 \leq \min_k \{p_k\}\,, \quad \alpha_1 \leq \min_k \{1 - p_k\}$



What does IV estimate under mis-classification?

Unobserved

$$\beta(\mathbf{x}) = \frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x})}$$

Wald (Observed)

$$\frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1(\mathbf{x}) - p_0(\mathbf{x})} = \beta(\mathbf{x})\left[\frac{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x})}{p_1(\mathbf{x}) - p_0(\mathbf{x})}\right] = \frac{\beta(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}$$

$$\boxed{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x}) = \frac{p_1(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0 - \alpha_1(\mathbf{x})} - \frac{p_0(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0 - \alpha_1(\mathbf{x})} = \frac{p_1(\mathbf{x}) - p_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}}$$

Partial Identification Bounds for $\beta(\mathbf{x})$

Known Result

- $\beta(\mathbf{x})$ is between Wald and Reduced form; same sign as Wald.
- Doesn't rely on non-differential assumption or additive sep.
- Frazis & Loewenstein (2003), Ura (2016), ...

Non-differential Assumption

- $\mathbb{E}[\varepsilon|\mathbf{x}, T^*, T, z] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*, z]$
- Used in literature to identify $\beta(\mathbf{x})$ when T^* is exogenous.
- Does it restrict the identified set when T^* is endogenous?

Restrictions from Non-differential Mis-classification?

(Suppress **x** for simplicity)

Notation

- $r_{tk} \equiv \mathbb{P}(T^* = 1 | T = t, z = k)$, function of (α_0, α_1) and observables only
- z_k is shorthand for $z = k$

Iterated Expectations over T^*

$$\mathbb{E}(y|T = 0, z_k) = (1 - r_{0k})\mathbb{E}(y|T^* = 0, T = 0, z_k) + r_{0k}\mathbb{E}(y|T^* = 1, T = 0, z_k)$$

$$\mathbb{E}(y|T = 1, z_k) = (1 - r_{1k})\mathbb{E}(y|T^* = 0, T = 1, z_k) + r_{1k}\mathbb{E}(y|T^* = 1, T = 1, z_k)$$

Restrictions from Non-differential Mis-classification?

(Suppress $\mathbf{x}$ for simplicity)

Notation

- $r_{tk} \equiv \mathbb{P}(T^* = 1 | T = t, z = k)$, function of (α_0, α_1) and observables only
- z_k is shorthand for $z = k$

Adding Non-differential Assumption

$$\mathbb{E}(y|T = 0, z_k) = (1 - r_{0k})\mathbb{E}(y|T^* = 0, z_k) \qquad + r_{0k}\mathbb{E}(y|T^* = 1, z_k)$$
$$\mathbb{E}(y|T = 1, z_k) = (1 - r_{1k})\mathbb{E}(y|T^* = 0, z_k) \qquad + r_{1k}\mathbb{E}(y|T^* = 1, z_k)$$

$\boxed{\text{2 equations in 2 unknowns} \Rightarrow \text{solve for } \mathbb{E}(y|T^* = t^*, z = k) \text{ given } (\alpha_0, \alpha_1).}$

Restrictions from Non-differential Mis-classification?

Law of Total Probability

$$\boxed{F_{tk} = (1 - r_{tk})F_{tk}^0 + r_{tk}F_{tk}^1}$$

$F_{tk} \equiv$ Observed CDF: $y|(T = t, z = k)$

$F_{tk}^{t^*} \equiv$ Unobserved CDF: $y|(T^* = t^*, T = t, z = k)$

Previous Slide

- r_{tk} observable given (α_0, α_1)
- $\mathbb{E}(T^*, T, z) = \mathbb{E}(T^*, z)$ observable given (α_0, α_1)

Key Question

Given (α_0, α_1) can we always find (F_{tk}^0, F_{tk}^1) to satisfy the mixture model?

Restrictions from Non-differential Mis-classification?

Equivalent Problem

Given a specified CDF F, probability p and mean μ, do there exist valid CDFs (G, H) with $F = (1-p)G + pH$ and $\mu = \text{mean}(H)$?

Valid CDFS

$$\begin{aligned} &0 \leq H \leq 1 \\ &0 \leq G \leq 1 \quad \iff \quad [F - (1-p)]/p \leq H \leq F/p \end{aligned}$$

$$\boxed{\max\left\{0,\ \frac{F(x)}{p} - \frac{1-p}{p}\right\} \leq H(x) \leq \min\left\{1, \frac{F(x)}{p}\right\}}$$

Restrictions from Non-differential Mis-classification?

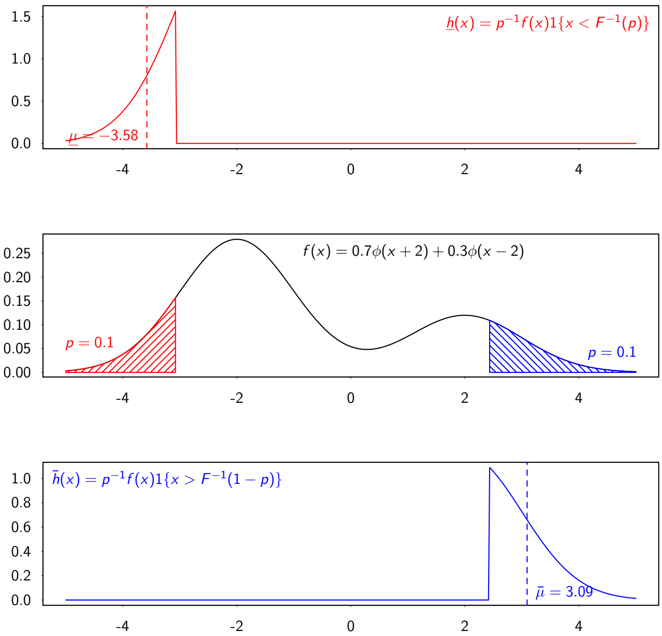
Notation

$$\overline{H} \equiv \max\left\{0,\ \frac{F(x)}{p} - \frac{1-p}{p}\right\}, \quad \underline{H} \equiv \min\left\{1, \frac{F(x)}{p}\right\}$$

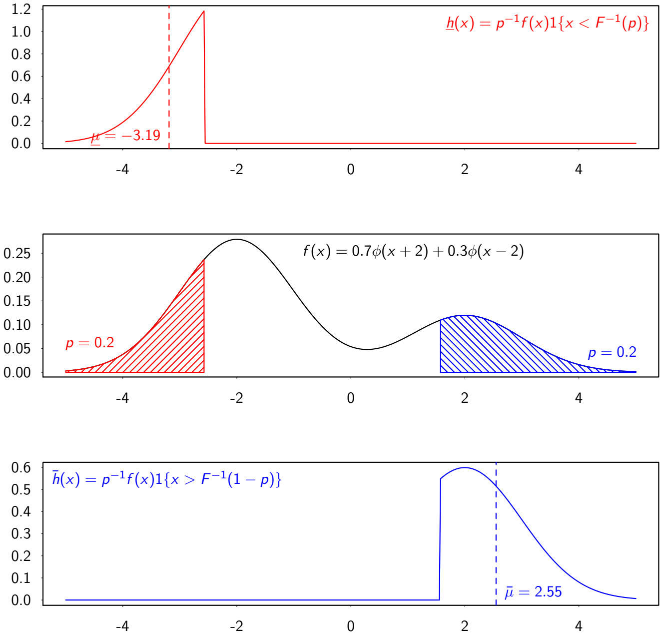
1st Order Stochastic Dominance

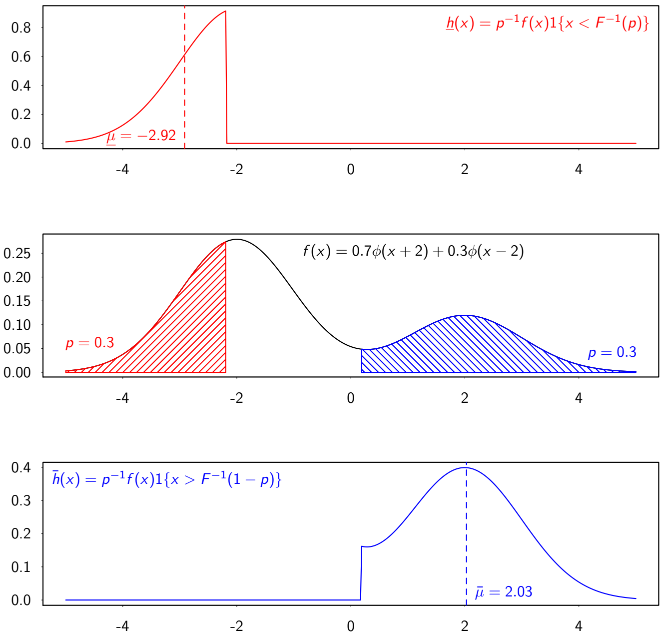
$$\overline{H}(x) \leq H(x) \leq \underline{H}(x) \quad \text{for all } x$$

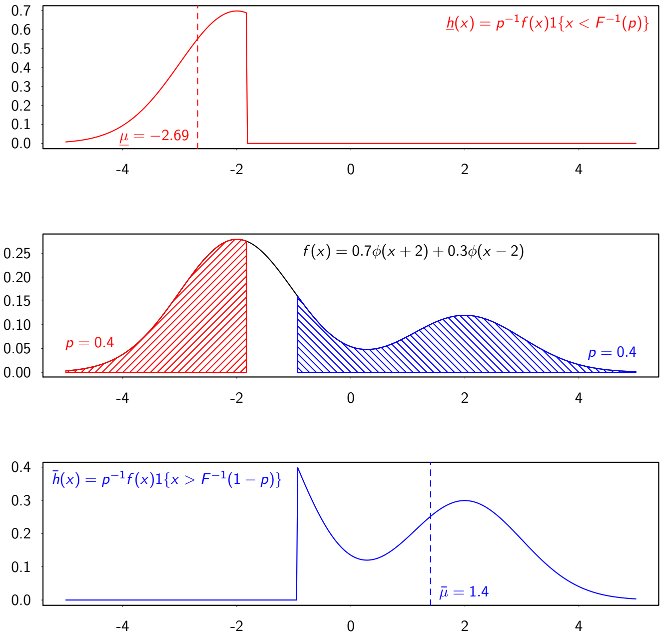
$$\implies \underbrace{\int_{\mathbb{R}} x\, \underline{H}(dx)}_{\underline{\mu}(p,F)} \leq \underbrace{\int_{\mathbb{R}} x\, H(dx)}_{\mu} \leq \underbrace{\int_{\mathbb{R}} x\, \overline{H}(dx)}_{\overline{\mu}(p,F)}$$



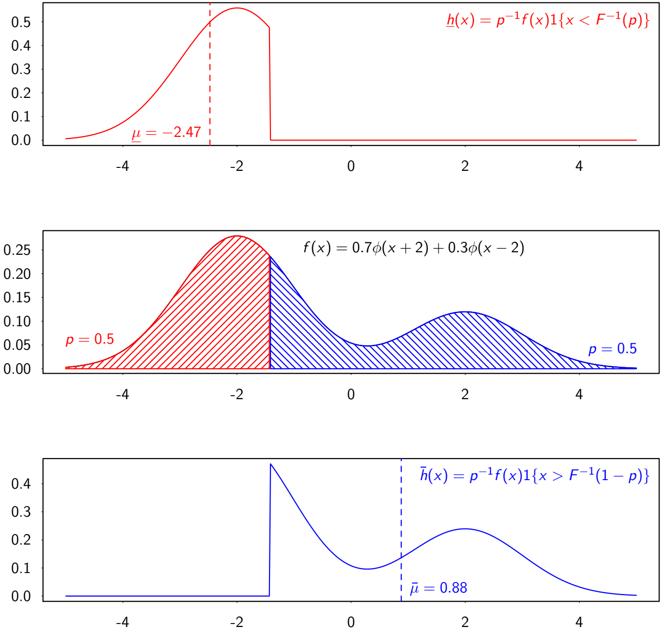
$\underline{h}(x) = p^{-1}f(x)1\{x < F^{-1}(p)\}$
$\underline{\mu} = -3.58$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.1$
$p = 0.1$
$\bar{h}(x) = p^{-1}f(x)1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 3.09$


$\underline{h}(x) = p^{-1}f(x)1\{x < F^{-1}(p)\}$
$\underline{\mu} = -3.19$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.2$
$p = 0.2$
$\bar{h}(x) = p^{-1}f(x)1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 2.55$

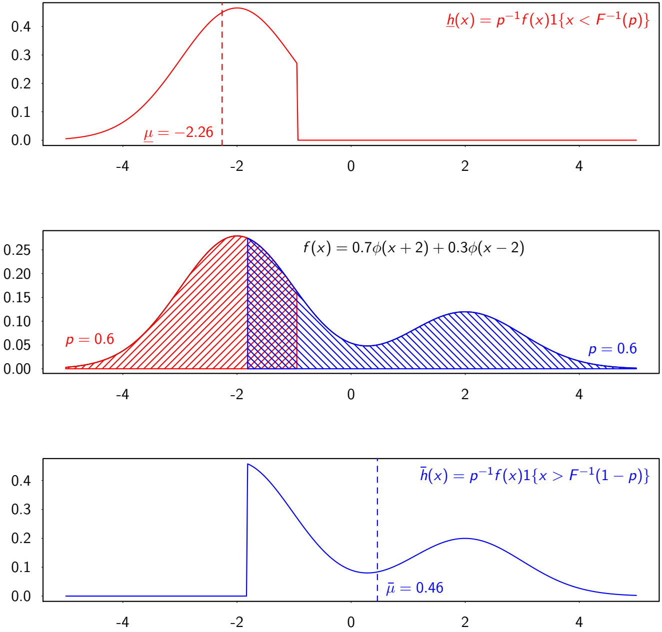

$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.92$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.3$
$p = 0.3$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 2.03$

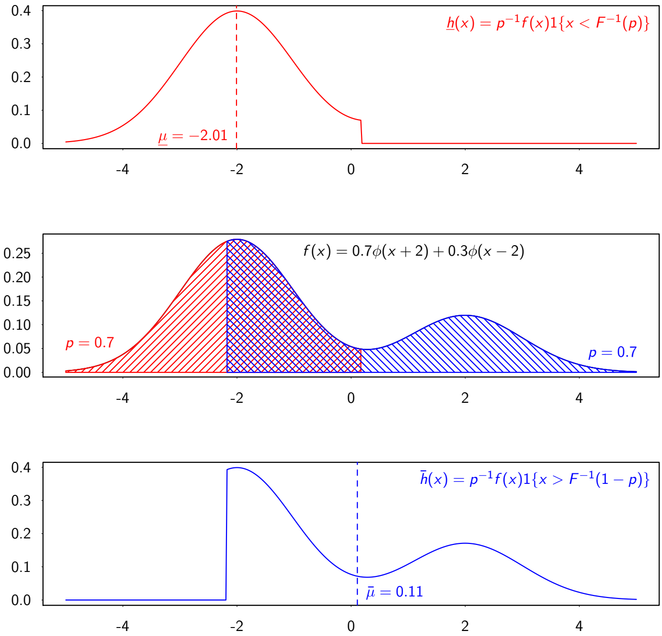


$\underline{h}(x) = p^{-1}f(x)1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.69$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.4$
$p = 0.4$
$\bar{h}(x) = p^{-1}f(x)1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 1.4$

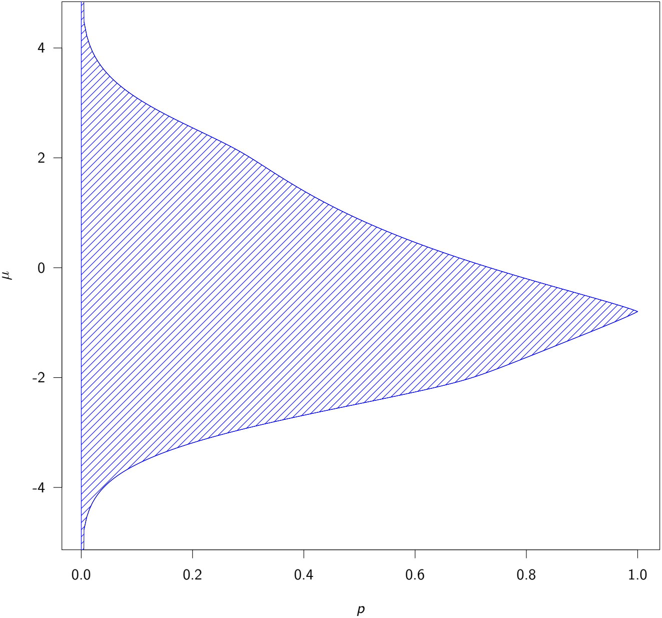


$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.47$
0.5
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.5$
$p = 0.5$
0.25
0.20
0.15
0.10
0.05
0.00
-4
-2
0
2
4
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 0.88$
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4


$\underline{h}(x) = p^{-1}f(x)1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.26$
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.6$
$p = 0.6$
0.25
0.20
0.15
0.10
0.05
0.00
$\bar{h}(x) = p^{-1}f(x)1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 0.46$



$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.01$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.7$
$p = 0.7$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 0.11$



4
2
0
-2
-4
μ
0.0
0.2
0.4
0.6
0.8
1.0
p

Restrictions from Non-differential Mis-classification

Equivalent Problem

Given a specified CDF F, probability p and mean μ, do there exist valid CDFs (G, H) with $F = (1-p)G + pH$ and $\mu = \text{mean}(H)$?

Necessary and Sufficient Condition if F is Continuous

$$\int_{-\infty}^{F^{-1}(p)} \frac{x}{p}\, f(x)\, dx \leq \ \mu \ \leq \int_{F^{-1}(1-p)}^{+\infty} \frac{x}{p}\, f(x)\, dx$$

Sharp Identified Set

Includes only those (α_0, α_1) at which the preceding condition is satisfied jointly for the mixtures $F_{tk} = (1 - r_{tk})F_{tk}^0 + r_{tk}F_{tk}^1$.

Sharp Identified Set under Baseline Assumptions

Theorem

Under baseline assumptions, sharp identified set for $\beta(\mathbf{x})$ is never a singleton, regardless of how many (discrete) values z takes on.

Point identification from stronger assumptions?

Point Identification: 1st Ingredient

Reparameterization

$$\theta_1(\mathbf{x}) = \beta(\mathbf{x}) / \left[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]$$

$$\theta_2(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^2 \left[1 + \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]$$

$$\theta_3(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^3 \left[\left\{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right\}^2 + 6\alpha_0(\mathbf{x}) \left\{1 - \alpha_1(\mathbf{x})\right\}\right]$$

$$\boxed{\beta(\mathbf{x}) = 0 \iff \theta_1(\mathbf{x}) = \theta_2(\mathbf{x}) = \theta_3(\mathbf{x}) = 0}$$

Lemma

Baseline Assumptions $\implies$ $\text{Cov}(y, z|\mathbf{x}) = \theta_1(\mathbf{x})\text{Cov}(z, T|\mathbf{x})$.

Point Identification: 2nd Ingredient

Assumption (II)

$\mathbb{E}[\varepsilon^2|\mathbf{x}, z] = \mathbb{E}[\varepsilon^2|\mathbf{x}]$

Lemma

(Baseline) + (II) $\implies$

$$\mathrm{Cov}(y^2, z|\mathbf{x}) = 2\mathrm{Cov}(yT, z|\mathbf{x})\theta_1(\mathbf{x}) - \mathrm{Cov}(T, z|\mathbf{x})\theta_2(\mathbf{x})$$

Corollary

(Baseline) + (II) + $[\beta(\mathbf{x}) \neq 0] \implies [\alpha_1(\mathbf{x}) - \alpha_0(\mathbf{x})]$ is identified.

Hence, $\beta(\mathbf{x})$ is identified if mis-classification is one-sided.

Point Identification: 3rd Ingredient

Assumption (III)

(i) $\mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*]$

(ii) $\mathbb{E}[\varepsilon^3|\mathbf{x}, z] = \mathbb{E}[\varepsilon^3|\mathbf{x}]$

Lemma

(Baseline) + (II) + (III) $\implies$

$$\text{Cov}(y^3, z|\mathbf{x}) = 3\text{Cov}(y^2 T, z|\mathbf{x})\theta_1(\mathbf{x}) - 3\text{Cov}(yT, z|\mathbf{x})\theta_2(\mathbf{x}) + \text{Cov}(T, z|\mathbf{x})\theta_3(\mathbf{x})$$

Point Identification Result

Theorem

(Baseline) + (II) + (III) $\implies \beta(\mathbf{x})$ is point identified. If $\beta(\mathbf{x}) \neq 0$, then $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are likewise point identified.

Proof Sketch

1. $\beta(\mathbf{x}) = 0 \iff \theta_1(\mathbf{x}) = 0$ so suppose this is not the case.
2. Lemmas: full-rank linear system in $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})$ & observables.
3. Non-linear eqs. relating $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})$ to $\beta(\mathbf{x})$ and $\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x})$. Show that solution exists and is unique.

Sufficient Conditions for (II) and (III)

(i) T is conditionally independent of (ε, z) given $(T^*, \mathbf{x})$

(ii) z is conditionally independent of ε given $\mathbf{x}$

MENTION THE ANGRIST ARGUMENT...

Just-Identified System of Moment Equalities

Suppress dependence on $\mathbf{x}$ to simplify the notation from here on...

Collect Lemmas from Above:

$$\begin{aligned}
\mathrm{Cov}(y,z) - \mathrm{Cov}(T,z)\theta_1 &= 0 \\
\mathrm{Cov}(y^2,z) - 2\mathrm{Cov}(yT,z)\theta_1 + \mathrm{Cov}(T,z)\theta_2 &= 0 \\
\mathrm{Cov}(y^3,z) - 3\mathrm{Cov}(y^2T,z)\theta_1 + 3\mathrm{Cov}(yT,z)\theta_2 - \mathrm{Cov}(T,z)\theta_3 &= 0
\end{aligned}$$

Notation: Observed Data Vector

$$\mathbf{w}_i' = (T_i, y_i, y_i T_i, y_i^2, y_i^2 T_i, y_i^3)$$

Just-Identified System of Moment Equalities

$$\boxed{\mathbb{E}\left[\left(\boldsymbol{\Psi}'(\boldsymbol{\theta})\mathbf{w}_i - \boldsymbol{\kappa}\right) \otimes \begin{pmatrix} 1 \\ z_i \end{pmatrix}\right] = \mathbf{0}}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1 & \boldsymbol{\psi_2} & \boldsymbol{\psi_3} \end{bmatrix} \qquad \boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)' \equiv \text{ “Intercepts”}$$

$$\psi_1' = \begin{bmatrix} -\theta_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \theta_1 = \beta/(1 - \alpha_0 - \alpha_1)$$

$$\psi_2' = \begin{bmatrix} \theta_2 & 0 & -2\theta_1 & 1 & 0 & 0 \end{bmatrix} \qquad \theta_2 = \theta_1^2\left[1 + \alpha_0 - \alpha_1\right]$$

$$\psi_3' = \begin{bmatrix} -\theta_3 & 0 & 3\theta_2 & 0 & -3\theta_1 & 1 \end{bmatrix} \qquad \theta_3 = \theta_1^3\left[(1 - \alpha_0 - \alpha_1)^2 + 6\alpha_0(1 - \alpha_1)\right]$$

Weak Identification Problem

Moment conditions are uninformative about (α_0, α_1) when β is small.

Moreover, (α_0, α_1) could be on the boundary of the parameter space. LINK TO SIMULATION RESULTS!!!

Non-standard Inference Problem

- β small $\Rightarrow$ moment equalities uninformative about (α_0, α_1)
- (α_0, α_1) could be on the boundary of the parameter space
- Partial identification bounds remain informative even if β is small or zero
- Same problem for other estimators from the literature but hasn't been pointed out...

Our Approach

Identification-robust inference combining equality and inequality moment conditions based on generalized moment selection (GMS)

SHOW FULL CONTINUUM OF MOMENTS? (TO ILLUSTRATE THAT IT'S NOT JUST A PROBLEM WITH THE MOMENTS WE HAPPEN TO USE)

Inference With Moment Equalities and Inequalities

Moment Conditions

$$\mathbb{E}\left[m_j(\mathbf{w}_i, \vartheta_0)\right] \geq 0, \quad j = 1, \cdots, J$$
$$\mathbb{E}\left[m_j(\mathbf{w}_i, \vartheta_0)\right] = 0, \quad j = J+1, \cdots, J+K$$

Test Statistic

$$T_n(\vartheta) = \sum_{j=1}^{J} \left[\frac{\sqrt{n}\ \bar{m}_{n,j}(\vartheta)}{\widehat{\sigma}_{n,j}(\vartheta)}\right]_-^2 + \sum_{j=J+1}^{J+K} \left[\frac{\sqrt{n}\ \bar{m}_{n,j}(\vartheta)}{\widehat{\sigma}_{n,j}(\vartheta)}\right]^2$$

$$[x]_- = \min\{x, 0\}$$

$$\bar{m}_{n,j}(\vartheta) = n^{-1} \sum_{i=1}^{n} m_j(\mathbf{w}_i, \vartheta)$$

$$\widehat{\sigma}^2_{n,j}(\vartheta) = \text{consistent est. of AVAR} \left[\sqrt{n}\ \bar{m}_{n,j}(\vartheta)\right]$$

Moment Inequalities: Part I

$\alpha_0(\mathbf{x}) \leq p_k \leq 1 - \alpha_1$ becomes $\mathbb{E}\left[m^I_{1k}(\mathbf{w}_i, \boldsymbol{\vartheta})\right] \geq \mathbf{0}$ for all k where

$$m^I_{1k}(\mathbf{w}_i, \boldsymbol{\vartheta}) \equiv \begin{bmatrix} \mathbf{1}(z_i = k)(T - \alpha_0) \\ \mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \end{bmatrix}$$

Moment Inequalities: Part II

For all k, we have $\mathbb{E}[m^l_{2k}\left(\mathbf{w}_i, \vartheta, \mathbf{q}_k\right)] \geq 0$ where

$$m^l_{2k}\left(\mathbf{w}_i, \vartheta, \mathbf{q}_k\right) \equiv \left[\begin{array}{r} y_i \mathbf{1}\left(z_i = k\right)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i \leq \underline{q}_{0k})(1 - T_i)\left(\frac{1-\alpha_0-\alpha_1}{\alpha_1}\right)\right\} \\ -y_i \mathbf{1}(z_i = k)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i > \overline{q}_{0k})(1 - T_i)\left(\frac{1-\alpha_0-\alpha_1}{\alpha_1}\right)\right\} \\ y_i \mathbf{1}\left(z_i = k\right)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i \leq \underline{q}_{1k}) T_i \left(\frac{1-\alpha_0-\alpha_1}{1-\alpha_1}\right)\right\} \\ -y_i \mathbf{1}(z_i = k)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i > \overline{q}_{1k}) T_i \left(\frac{1-\alpha_0-\alpha_1}{1-\alpha_1}\right)\right\} \end{array} \right]$$

and $\mathbf{q}_k \equiv (\underline{q}_{0k}, \overline{q}_{0k}, \underline{q}_{1k}, \overline{q}_{1k})'$ defined by $\mathbb{E}[h^l_k(\mathbf{w}_i, \vartheta, \mathbf{q}_k)] = 0$ with

$$h^l_k(\mathbf{w}_i, \vartheta, \mathbf{q}_k) = \left[\begin{array}{l} \mathbf{1}(y_i \leq \underline{q}_{0k})\mathbf{1}(z_i = k)(1 - T_i) - \left(\frac{\alpha_1}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(T_i - \alpha_0) \\ \mathbf{1}(y_i \leq \overline{q}_{0k})\mathbf{1}(z_i = k)(1 - T_i) - \left(\frac{1-\alpha_0}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \\ \mathbf{1}(y_i \leq \underline{q}_{1k})\mathbf{1}(z_i = k) T_i - \left(\frac{1-\alpha_1}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(T_i - \alpha_0) \\ \mathbf{1}(y_i \leq \overline{q}_{1k})\mathbf{1}(z_i = k) T_i - \left(\frac{\alpha_0}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \end{array} \right]$$

Inference via Generalized Moment Selection

Andrews & Soares (2010)

Moment Selection Step

If $\dfrac{\sqrt{n}\,\bar{m}_{n,j}(\vartheta_0)}{\widehat{\sigma}_{n,j}(\vartheta_0)} > \sqrt{\log n}$ then drop inequality j

Critical Value

- $\sqrt{n}\,\bar{m}_n(\vartheta_0) \to_d$ normal limit with covariance matrix $\Sigma(\vartheta_0)$
- Use this to bootstrap the limit distribution of the test statistic.

Theoretical Guarantees

Uniformly valid test of $H_0\colon \vartheta = \vartheta_0$ regardless of whether ϑ_0 is identified.

Not asymptotically conservative.

Drawback

Joint test for the whole parameter vector but we're only interested in β

Bonferroni-Based Inference Procedure

Leverage Special Structure of Model

- β only enters MCs through $\theta_1 = \beta/(1 - \alpha_0 - \alpha_1)$
- Inference for θ_1 is standard if z is a strong IV.
- $(\boldsymbol{\kappa}, \mathbf{q})$ strongly identified under null for (α_0, α_1)

Procedure

1. Concentrate out $(\theta_1, \boldsymbol{\kappa}, \boldsymbol{q}) \implies$ joint GMS test for (α_0, α_1)
2. Invert $\implies (1 - \delta_1) \times 100\%$ confidence set for (α_0, α_1)
3. Project $\implies$ CI for $(1 - \alpha_0 - \alpha_1)$
4. Construct standard $(1 - \delta_2) \times 100\%$ IV CI for θ_1
5. Bonferroni $\implies (1 - \delta - \delta_2) \times 100\%$ CI for β

Short empirical illustration using Burde & Linden, including picture of joint confidence region for (α_0, α_1) etc.

Conclusion

1. Identification and inference for effect of binary, mis-classified, endogenous regressor.
2. Show that only existing point identification result is incorrect.
3. Derive sharp identified set for $\beta(\mathbf{x})$ under standard assumptions.
4. Prove point identification of $\beta(\mathbf{x})$ under slightly stronger assumptions.
5. Point out problem of weak identification in mis-classification models, develop identification-robust inference for $\beta(\mathbf{x})$.

Related Past and Current Research

Talk about how this paper fits into a research agenda concerning measurement error: the beliefs paper, this paper, returns to lying (with Arthur), and biased measurements of displacement in the paper with Camilo.

Simulation DGP: $y = \beta T^* + \varepsilon$

Sample Size = 1000; Simulation Replications = 2000

Errors

$(\varepsilon, \eta) \sim$ jointly normal, mean 0, variance 1, correlation 0.5.

First-Stage

- Half of observations have $z = 1$, the rest have $z = 0$.
- $T^* = \mathbf{1}\{\gamma_0 + \gamma_1 z + \eta > 0\}$
- $\mathbb{P}(T^* = 0|z = 1) = \mathbb{P}(T^* = 1|z = 0) = 0.15$

Mis-classification

- $T|T^* = 0 \sim \text{Bernoulli}(\alpha_0)$
- $T|T^* = 1 \sim \text{Bernoulli}(1 - \alpha_1)$

α_0	α_1	β 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	27	33	30	14	1	0	0	0
	0.1	27	32	29	13	2	0	0	0
	0.2	26	33	32	15	4	0	0	0
	0.3	26	34	30	17	5	0	0	0
0.1	0.0	26	32	31	14	2	0	0	0
	0.1	26	36	32	16	4	0	0	0
	0.2	27	35	31	18	8	0	0	0
	0.3	25	35	32	21	11	1	0	0
0.2	0.0	26	33	30	15	3	0	0	0
	0.1	26	33	30	19	6	0	0	0
	0.2	26	35	33	22	12	1	0	0
	0.3	26	35	33	26	15	3	0	0
0.3	0.0	26	32	32	16	6	0	0	0
	0.1	24	35	33	21	11	1	0	0
	0.2	26	32	35	27	15	4	0	0
	0.3	26	35	35	28	21	7	2	0

Table: Percentage of simulation replications for which the standard GMM CI fails to exist.

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	72	62	62	80	92	95	94	95
	0.1	72	62	63	79	92	95	96	95
	0.2	73	61	61	77	90	96	96	96
	0.3	73	59	62	76	88	95	96	95
0.1	0.0	73	63	60	78	91	95	96	96
	0.1	73	58	59	77	90	95	95	94
	0.2	73	59	61	75	86	95	95	94
	0.3	74	59	58	71	82	94	96	96
0.2	0.0	74	62	60	78	91	95	96	96
	0.1	73	60	61	74	87	95	96	94
	0.2	73	58	57	70	81	93	95	95
	0.3	73	58	56	66	78	92	95	96
0.3	0.0	74	62	60	76	89	95	96	96
	0.1	75	59	58	71	82	93	96	95
	0.2	74	61	56	65	78	90	96	96
	0.3	73	58	55	64	71	88	93	96

Table: Coverge of nominal 95% GMM CI, conditional on existence.

α_0	α_1	β 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	19.07	3.44	1.86	1.32	0.87	0.47	0.37	0.35
	0.1	17.52	3.47	1.92	1.41	1	0.61	0.51	0.46
	0.2	17.41	3.51	1.9	1.45	1.1	0.76	0.65	0.58
	0.3	18.23	3.34	1.92	1.48	1.24	0.91	0.79	0.7
0.1	0.0	17.13	3.51	1.86	1.38	0.97	0.61	0.51	0.46
	0.1	17.88	3.33	1.85	1.45	1.13	0.78	0.67	0.6
	0.2	17.37	3.36	1.95	1.54	1.24	0.97	0.85	0.75
	0.3	18.07	3.33	1.98	1.63	1.41	1.17	1.04	0.92
0.2	0.0	17.79	3.39	1.92	1.45	1.11	0.75	0.65	0.58
	0.1	18.98	3.43	1.96	1.54	1.26	0.97	0.84	0.75
	0.2	18.25	3.26	1.92	1.64	1.45	1.2	1.06	0.95
	0.3	19.03	3.31	2.02	1.75	1.66	1.49	1.33	1.19
0.3	0.0	18.27	3.48	1.87	1.5	1.25	0.9	0.79	0.7
	0.1	19.4	3.41	1.96	1.63	1.43	1.18	1.04	0.92
	0.2	18.22	3.56	1.96	1.74	1.67	1.49	1.35	1.19
	0.3	17.56	3.55	2.13	1.96	1.86	1.86	1.74	1.55

Table: Median width of nominal 95% GMM CI, conditional on existence.

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	97.7	97.7	97.6	97.7	98.0	98.0	97.4	97.9
	0.1	98.0	98.7	98.8	99.1	98.8	98.4	97.1	96.4
	0.2	98.4	98.5	98.9	98.9	98.8	98.6	98.0	97.0
	0.3	98.5	98.8	98.8	99.0	98.7	98.4	97.8	97.5
0.1	0.0	98.1	98.5	98.3	98.8	98.8	98.4	96.8	95.7
	0.1	98.6	99.1	99.5	99.6	99.6	98.8	97.7	95.2
	0.2	99.0	99.3	99.7	99.8	99.7	98.9	97.5	95.7
	0.3	99.4	99.7	99.8	99.8	99.6	99.0	98.2	96.7
0.2	0.0	98.6	98.5	98.6	98.9	98.7	98.2	97.7	97.0
	0.1	99.0	99.5	99.7	99.7	99.4	99.0	98.1	96.5
	0.2	99.5	99.7	99.8	99.7	99.4	99.0	97.8	96.8
	0.3	99.7	99.8	99.8	99.8	99.5	99.0	98.7	97.7
0.3	0.0	98.7	98.7	98.8	98.7	98.7	98.2	98.1	97.6
	0.1	99.4	99.6	99.6	99.7	99.4	98.9	98.3	96.8
	0.2	99.8	99.8	99.7	99.8	99.5	99.1	98.5	97.8
	0.3	100.0	99.9	99.9	99.8	99.6	99.5	99.1	98.8

Table: Coverage (1 - size) of nominal 97.5% GMS joint test for (α_0, α_1).

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	96	97	97	96	97	97	95	96
	0.1	97	99	99	99	99	100	100	99
	0.2	98	99	99	100	100	100	100	100
	0.3	97	100	100	100	100	100	100	100
0.1	0.0	97	99	99	99	100	100	100	98
	0.1	98	100	100	100	100	100	100	100
	0.2	98	100	100	100	100	100	100	100
	0.3	97	100	100	100	100	100	100	100
0.2	0.0	97	99	99	100	100	100	100	100
	0.1	98	100	100	100	100	100	100	100
	0.2	98	100	100	100	100	100	100	100
	0.3	98	100	100	100	100	100	100	100
0.3	0.0	97	99	100	100	100	100	100	100
	0.1	97	100	100	100	100	100	100	100
	0.2	98	100	100	100	100	100	100	100
	0.3	98	100	100	100	100	100	100	100

Table: Coverage of nominal $> 95\%$ Bonferroni CI for β

α_0	α_1	β 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	0.4	0.41	0.43	0.43	0.43	0.42	0.41	0.41
	0.1	0.45	0.47	0.54	0.59	0.63	0.7	0.75	0.86
	0.2	0.51	0.54	0.65	0.76	0.85	0.95	1.01	1.17
	0.3	0.58	0.62	0.79	0.95	1.07	1.17	1.24	1.48
0.1	0.0	0.45	0.47	0.54	0.59	0.63	0.7	0.76	0.88
	0.1	0.51	0.54	0.66	0.77	0.86	1.03	1.18	1.46
	0.2	0.58	0.63	0.8	0.98	1.12	1.38	1.55	1.88
	0.3	0.67	0.75	1	1.25	1.46	1.74	1.94	2.4
0.2	0.0	0.51	0.54	0.65	0.76	0.86	0.96	1.02	1.19
	0.1	0.58	0.63	0.81	0.99	1.14	1.42	1.64	2.08
	0.2	0.67	0.75	1.01	1.29	1.54	1.97	2.33	2.9
	0.3	0.81	0.91	1.3	1.7	2.09	2.73	3.13	3.9
0.3	0.0	0.58	0.62	0.8	0.95	1.09	1.18	1.25	1.5
	0.1	0.68	0.74	1.01	1.26	1.49	1.84	2.13	2.78
	0.2	0.81	0.91	1.3	1.7	2.11	2.8	3.4	4.48
	0.3	1.01	1.16	1.74	2.35	2.93	4.17	5.2	6.85

Table: Median width of nominal $> 95\%$ Bonferroni CI for β.

α_0	α_1	β = 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	0.4	0.41	0.43	0.43	0.43	0.42	0.41	0.41
	0.1	0.45	0.47	0.54	0.59	0.63	0.7	0.75	0.86
	0.2	0.51	0.54	0.65	0.76	0.85	0.95	1.01	1.17
	0.3	0.58	0.62	0.79	0.95	1.07	1.17	1.24	1.48
0.1	0.0	0.45	0.47	0.54	0.59	0.63	0.7	0.76	0.88
	0.1	0.51	0.54	0.66	0.77	0.86	1.03	1.18	1.46
	0.2	0.58	0.63	0.8	0.98	1.12	1.38	1.55	1.88
	0.3	0.67	0.75	1	1.25	1.46	1.74	1.94	2.4
0.2	0.0	0.51	0.54	0.65	0.76	0.86	0.96	1.02	1.19
	0.1	0.58	0.63	0.81	0.99	1.14	1.42	1.64	2.08
	0.2	0.67	0.75	1.01	1.29	1.54	1.97	2.33	2.9
	0.3	0.81	0.91	1.3	1.7	2.09	2.73	3.13	3.9
0.3	0.0	0.58	0.62	0.8	0.95	1.09	1.18	1.25	1.5
	0.1	0.68	0.74	1.01	1.26	1.49	1.84	2.13	2.78
	0.2	0.81	0.91	1.3	1.7	2.11	2.8	3.4	4.48
	0.3	1.01	1.16	1.74	2.35	2.93	4.17	5.2	6.85

Table: Median width of nominal $> 95\%$ Bonferroni CI for β.

α_0	α_1	β = 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	19.07	3.44	1.86	1.32	0.87	0.47	0.37	0.35
	0.1	17.52	3.47	1.92	1.41	1	0.61	0.51	0.46
	0.2	17.41	3.51	1.9	1.45	1.1	0.76	0.65	0.58
	0.3	18.23	3.34	1.92	1.48	1.24	0.91	0.79	0.7
0.1	0.0	17.13	3.51	1.86	1.38	0.97	0.61	0.51	0.46
	0.1	17.88	3.33	1.85	1.45	1.13	0.78	0.67	0.6
	0.2	17.37	3.36	1.95	1.54	1.24	0.97	0.85	0.75
	0.3	18.07	3.33	1.98	1.63	1.41	1.17	1.04	0.92
0.2	0.0	17.79	3.39	1.92	1.45	1.11	0.75	0.65	0.58
	0.1	18.98	3.43	1.96	1.54	1.26	0.97	0.84	0.75
	0.2	18.25	3.26	1.92	1.64	1.45	1.2	1.06	0.95
	0.3	19.03	3.31	2.02	1.75	1.66	1.49	1.33	1.19
0.3	0.0	18.27	3.48	1.87	1.5	1.25	0.9	0.79	0.7
	0.1	19.4	3.41	1.96	1.63	1.43	1.18	1.04	0.92
	0.2	18.22	3.56	1.96	1.74	1.67	1.49	1.35	1.19
	0.3	17.56	3.55	2.13	1.96	1.86	1.86	1.74	1.55

Table: Median width of nominal 95% GMM CI, conditional on existence.

α_0	α_1	β: 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	96	97	97	96	97	97	95	93
	0.1	97	99	99	99	99	98	96	95
	0.2	98	99	99	100	100	97	96	96
	0.3	97	100	100	100	99	96	96	96
0.1	0.0	97	99	99	99	100	98	97	95
	0.1	98	100	100	100	100	96	96	96
	0.2	98	100	100	100	99	96	96	95
	0.3	97	100	100	100	97	95	96	96
0.2	0.0	97	99	99	100	100	96	96	96
	0.1	98	100	100	100	99	96	96	96
	0.2	98	100	100	100	96	95	95	96
	0.3	98	100	100	98	95	95	95	96
0.3	0.0	97	99	100	100	100	95	96	97
	0.1	97	100	100	100	97	94	96	96
	0.2	98	100	100	98	94	94	96	96
	0.3	98	100	99	96	92	94	95	96

Table: Coverage of hybrid CI constructed from nominal 95% GMM and $>$ 95% Bonferroni intervals.

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	0.4	0.41	0.43	0.43	0.43	0.42	0.4	0.35
	0.1	0.45	0.47	0.54	0.59	0.63	0.67	0.52	0.46
	0.2	0.51	0.54	0.65	0.76	0.84	0.82	0.65	0.58
	0.3	0.58	0.62	0.79	0.95	1.05	0.96	0.79	0.7
0.1	0.0	0.45	0.47	0.54	0.59	0.63	0.67	0.51	0.46
	0.1	0.51	0.54	0.66	0.77	0.86	0.92	0.69	0.61
	0.2	0.58	0.63	0.8	0.97	1.11	1.17	0.87	0.75
	0.3	0.67	0.75	1	1.25	1.4	1.4	1.06	0.92
0.2	0.0	0.51	0.54	0.65	0.76	0.85	0.83	0.65	0.58
	0.1	0.58	0.63	0.81	0.99	1.12	1.18	0.86	0.75
	0.2	0.67	0.75	1.01	1.29	1.48	1.56	1.08	0.95
	0.3	0.81	0.91	1.3	1.67	1.95	1.77	1.35	1.2
0.3	0.0	0.58	0.62	0.8	0.95	1.07	0.95	0.8	0.7
	0.1	0.68	0.74	1.01	1.26	1.43	1.48	1.06	0.93
	0.2	0.81	0.91	1.3	1.66	1.98	1.94	1.37	1.19
	0.3	1.01	1.16	1.73	2.24	2.71	2.33	1.78	1.55

Table: Median width of hybrid CI constructed from nominal 95% GMM and $>$ 95% Bonferroni intervals.

Figure: Coverage of hybrid vs. $> 95\%$ Bonferroni CIs: $\beta = 1$

Figure: Coverage of hybrid vs. $> 95\%$ Bonferroni CIs: $\beta = 2$

Figure: Coverage of hybrid vs. $> 95\%$ Bonferroni CIs: $\beta = 3$