Estimating the Effect of a Mis-measured, Endogenous, Binary Treatment

Francis J. DiTraglia

Camilo Garcia-Jimeno

University of Pennsylvania

November 19th, 2015

What is the causal effect of T^*?

$$y_i = h(T_i^*, \mathbf{x}_i) + \varepsilon_i$$

- y – Outcome of interest
- h – Unknown function that *does not depend on i*
- T^* – Unobserved, endogenous binary treatment
- T – Observed, mis-measured binary surrogate for T^*
- $\mathbf{x}$ – Exogenous covariates
- ε – Mean-zero error term
- z – Discrete instrumental variable

Example 1: Smoking and Birthweight (SNAP Trial)

Coleman et al. (N Engl J Med, 2012)

RCT with 1050 pregnant smokers in England: 521 given nicotine patches, the rest given placebo patches.

- y – Birthweight
- T^* – True smoking behavior
- T – Self-reported smoking behavior
- $\mathbf{x}$ – Mother characteristics
- z – Indicator of nicotine patch

Example 2: Schooling and Test Scores

Burde & Linden (2013, AEJ Applied)

RCT in Afghanistan: 32 villages divided into 11 clusters. Randomly choose 6 and build a school in each village of these clusters.

- y – Score on math and language test
- T^* – True school attendance
- T – Self-reported school attendance
- $\mathbf{x}$ – Household characteristics
- z – School built in village

Example 3: Job Training Partnership Act (JPTA)

Heckman et al. (2000, QJE)

Randomized offer of job training, but about 30% of those *not* offered also obtain training and about 40% of those offered training don't attend. Estimate causal effect of *training* rather than *offer* of training.

- y – Log wage
- T^* – True training attendence
- T – Self-reported training attendance
- $\mathbf{x}$ – Individual characteristics
- z – Offer of job training

Related Literature

Continuous Treatment

Lewbel (1997, 2012), Schennach (2004, 2007), Chen et al. (2005), Hu & Schennach (2008)...

Binary, Exogenous Treatment

Aigner (1973), Bollinger (1996), Kane et al. (1999), Black et al. (2000), Frazis & Loewenstein (2003), Mahajan (2006), Lewbel (2007)

Binary, Endogenous Treatment

Only existing result in Mahajan (2006)

Model: $y = h(T^*, \mathbf{x}) + \varepsilon$

ATE Function

$\tau(\mathbf{x}) = h(1, \mathbf{x}) - h(0, \mathbf{x})$

First-stage

$p_k^*(\mathbf{x}) \equiv \mathbb{P}(T^* = 1 | z = z_k, \mathbf{x}) \neq \mathbb{P}(T^* = 1 | z = z_\ell) \equiv p_\ell^*(\mathbf{x})$, $k \neq \ell$

Measurement Error

Non-differential, $\mathbb{E}[\varepsilon | T^*, T, z, \mathbf{x}] = \mathbb{E}[\varepsilon | T^*, z, \mathbf{x}]$, and does not depend on z:

$$
\begin{aligned}
\alpha_0(\mathbf{x}) &= \mathbb{P}(T = 1 | T^* = 0, z, \mathbf{x}) \\
\alpha_1(\mathbf{x}) &= \mathbb{P}(T = 0 | T^* = 1, z, \mathbf{x})
\end{aligned}
$$

Notation

- Treat exog. covariates $\mathbf{x}$ non-parametrically: hold fixed at $\mathbf{x}_a$ throughout:

$$\begin{aligned} y &= \beta T^* + u \\ u &= \varepsilon + c \end{aligned}$$

where $\beta = \tau(\mathbf{x}_a)$ and $c = h(0, \mathbf{x}_a)$.

- Similarly:

$$\begin{aligned} \alpha_0 &= \mathbb{P}(T = 1 | T^* = 0) \\ \alpha_1 &= \mathbb{P}(T = 0 | T^* = 1) \\ p_k^* &= \mathbb{P}(T^* = 1 | z = z_k) \end{aligned}$$

Observable Moments: $y = \beta T^* + u$

	$z = 1$	$z = 1$	$\dots$	$z = K$
$T = 0$	$\bar{y}_{01}$ p_{01}	$\bar{y}_{02}$ p_{02}	$\dots$	$\bar{y}_{0K}$ p_{0K}
$T = 1$	$\bar{y}_{11}$ p_{11}	$\bar{y}_{12}$ p_{12}	$\dots$	$\bar{y}_{1K}$ p_{1K}

$$\bar{y}_{tk} = \mathbb{E}[y | T = t, z = z_k], \quad p_{tk} = q_k p_k$$

$$q_k = \mathbb{P}(z = z_k), \quad p_k = \mathbb{P}(T = 1 | z = z_k)$$

Unobservable Moments: $y = \beta T^* + u$

	$z = 1$	$z = 1$	$\dots$	$z = K$
$T^* = 0$	m^*_{01} / p^*_{01}	m^*_{02} / p^*_{02}	$\dots$	m^*_{0K} / p^*_{0K}
$T^* = 1$	m^*_{11} / p^*_{11}	m^*_{12} / p^*_{12}	$\dots$	m^*_{1K} / p^*_{1K}

$$m^*_{tk} = \mathbb{E}[u | T^* = t, z = z_k], \quad p^*_{tk} = q_k p^*_k$$

$$p^*_k = \mathbb{P}(T^* = 1 | z = z_k) = \frac{p_k - \alpha_0}{1 - \alpha_0 - \alpha_1}$$

Possible Assumptions On m^*_{tk}

Joint Exogeneity: $\mathbb{E}[\varepsilon | T^*, z] = 0$

$$\implies m^*_{tk} = c \quad \text{for all } t, k$$

Exogenous Treatment: $\mathbb{E}[\varepsilon | T^*] = 0$

$$\implies \frac{1}{\mathbb{P}(T^* = t)} \sum_k p^*_{tk} m^*_{tk} = c \quad \text{for all } t$$

Exogenous Instrument: $\mathbb{E}[\varepsilon | z] = 0$

$$\implies (1 - p^*_k) m^*_{0k} + p^*_k m^*_{1k} = c \quad \text{for all } k$$

Mahajan (2006, Econometrica)

Regression Model

$y = \mathbb{E}[y|T^*] + \nu$

$\mathbb{E}[\nu|T^*] = 0$ by construction

Causal Model

$y = c + \beta T^* + \varepsilon$

$\mathbb{E}[\varepsilon|T^*] \neq 0$

Main Result (Correct) – Exogenous Treatment

Relevant binary instrument z ($p_1^* \neq p_2^*$) identifies α_0, α_1 and $\mathbb{E}[y|T^*]$ provided that $\mathbb{E}[\nu|T^*, T, z] = 0$.

Mahajan (2006, Econometrica)

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$\mathbb{E}[\varepsilon|T^*] \neq 0$

Additional Result (Incorrect) – Endogenous Treatment

$\mathbb{E}[\varepsilon|z] = 0$, $p_1^* \neq p_2^*$, $\mathbb{E}[\varepsilon|T, T^*, z] = \mathbb{E}[\varepsilon|T^*] \implies \beta$ identified.

Mahajan's Argument

Regression Model

$y = \mathbb{E}[y|T^*] + \nu$

$\mathbb{E}[\nu|T^*] = 0$ by construction

Causal Model

$y = c + \beta T^* + \varepsilon$

$\mathbb{E}[\varepsilon|T^*] \neq 0$

Ingredients

1. If $p_1^* \neq p_2^*$, $\mathbb{E}[\varepsilon|z] = 0$ then, since $\beta_{IV} = \beta/(1 - \alpha_0 - \alpha_1)$, knowledge of α_0, α_1 is sufficient to recover β. (Correct)
2. If $p_1^* \neq p_2^*$, $\mathbb{E}[\nu|T^*, T, z] = 0$, α_0, α_1 are identified. (Correct)

   How to satisfy both 1 and 2 while allowing $\mathbb{E}[\varepsilon|T^*] \neq 0$?

3. Assume that $\mathbb{E}[\varepsilon|T^*, T, z] = \mathbb{E}[\varepsilon|T^*]$

   (i.e. $m_{01}^* = m_{02}^*$ and $m_{11}^* = m_{12}^*$)

Mahajan's Argument

Proposition

If $\mathbb{E}[\varepsilon|T^*] \neq 0$ then $\mathbb{E}[\varepsilon|T^*, T, z] = \mathbb{E}[\varepsilon|T^*]$ combined with $\mathbb{E}[\varepsilon|z] = 0$ implies $p_1^* = p_2^*$, i.e. z is irrelevant for T^*.

Proof

Recall that $\mathbb{E}[\varepsilon|z] = 0$ implies

$$(1 - p_1^*)m_{01}^* + p_1^* m_{11}^* = c$$

$$(1 - p_2^*)m_{02}^* + p_2^* m_{12}^* = c$$

while Mahajan's assumption implies $m_{01}^* = m_{02}^*$ and $m_{11}^* = m_{12}^*$. Therefore either $m_{01}^* = m_{02}^* = m_{11}^* = m_{12}^* = c$, which is ruled out by $E[\varepsilon|T^*] = 0$, or $p_1^* = p_2^*$.

What about increasing the support of z?

$\boxed{\mathbb{E}[\varepsilon|z] = 0 \implies \textit{pair} \text{ of equations for each } k = 1, \ldots, K}$

$$\hat{y}_{0k} = \alpha_1(p_k - \alpha_0)\left(\frac{\beta}{1-\alpha_0-\alpha_1}\right) + (1-\alpha_0)c - (p_k - \alpha_0)m^*_{1k}$$

$$\hat{y}_{1k} = (1-\alpha_1)(p_k - \alpha_0)\left(\frac{\beta}{1-\alpha_0-\alpha_1}\right) + \alpha_0 c + (p_k - \alpha_0)m^*_{1k}$$

where $\hat{y}_{0k} = (1-p_k)\bar{y}_{0k}$ and $\hat{y}_{0k} = p_k\bar{y}_{1k}$

$\boxed{2K \text{ Equations in } K+4 \text{ Unknowns}}$

Theorem: β is undentified regardless of K.

(For general case, see paper.)

Proof of special case: $\alpha_0 = 0$

1. System of equations simplifies to

$$\hat{y}_{0k} = c + p_k \beta \left(\frac{\alpha_1}{1 - \alpha_1} \right) - p_k m^*_{1k}$$

$$\hat{y}_{1k} = p_k \beta + p_k m^*_{1k}$$

2. $\beta / (1 - \alpha_1) \equiv \mathcal{W}$ is identified and imposing this, algebra gives $\beta \alpha_1 / (1 - \alpha_1) = \mathcal{W} - \beta$.

Theorem: β is undentified regardless of K.

(For general case, see paper.)

Proof of special case: $\alpha_0 = 0$ continued...

3. Substituting:

$$(c + p_k \mathcal{W} - \hat{y}_{0k})/p_k = \beta + m^*_{1k}$$
$$\hat{y}_{1k}/p_k = \beta + m^*_{1k}$$

4. Linear system in (β, m^*_{1k}) – no solution or ∞ of solutions.
5. Sum original pair of equations $\implies c + p_k \mathcal{W} - \hat{y}_{0k} = \hat{y}_{1k}$ thus ∞ of solutions. The model is unidentified.

Conditional *Second* Moment Independence.

New Assumption

Homoskedastic errors w.r.t. the *instrument*: $E[\varepsilon^2|z] = E[\varepsilon^2]$

Not Crazy!

Holds in an RCT or a *true* natural experiment.

New Moment Conditions

Defining $\mu^*_{k\ell} = (p_k - \alpha_0)m^*_{1k} - (p_\ell - \alpha_0)m^*_{k\ell}$,

$$\mathbb{E}(y^2|z_k) - \mathbb{E}(y^2|z_\ell) \equiv \Delta\overline{y^2} = \beta\mathcal{W}(p_k - p_\ell) + 2\mathcal{W}\mu^*_{k\ell}$$
$$\mathbb{E}(yT|z_k) - \mathbb{E}(yT|z_\ell) \equiv \Delta\overline{yT} = (1 - \alpha_1)\mathcal{W}(p_k - p_\ell) + \mu^*_{k\ell}$$

Theorem: $(\alpha_1 - \alpha_0)$ is Identified.

(Requires only binary z)

Proof

$$\begin{aligned} \Delta\overline{y^2} &= \beta\mathcal{W}(p_k - p_\ell) + 2\mathcal{W}\mu^*_{k\ell} \\ \Delta\overline{yT} &= (1-\alpha_1)\mathcal{W}(p_k - p_\ell) + \mu^*_{k\ell} \end{aligned}$$

Solve for $\mu^*_{k\ell}$, substitute and rearrange:

$$\mathcal{R} \equiv \beta - 2(1-\alpha_1)\mathcal{W} = \frac{\Delta\overline{y^2} - 2\mathcal{W}\Delta\overline{yT}}{\mathcal{W}(p_k - p_\ell)}.$$

Rearrange and substitute $\beta = \mathcal{W}(1 - \alpha_0 - \alpha_1)$ to find

$$\alpha_1 - \alpha_0 = 1 + \mathcal{R}/\mathcal{W}.$$

What Good is $(\alpha_1 - \alpha_0)$?

- Test a necessary condition for *no mis-classification*: $\alpha_0 = \alpha_1$
- Simple, tighter partial identification bounds for β
- In some settings, one of the mis-classification probabilities is known to be zero $\implies$ β point identified

Conditional *Third* Moment Independence

New Assumption

Third moment independence w.r.t instrument: $E[\varepsilon^3|z] = E[\varepsilon^3]$

New Moment Conditions

Define $\lambda^*_{k\ell} = (p_k - \alpha_0)v^*_{1k} - (p_\ell - \alpha_0)v^*_{1\ell}$
where $v^*_{tk} = \mathbb{E}(u^2|\, T^* = t, z_k)$. Then

$$
\begin{aligned}
\mathbb{E}(y^3|z_k) \;&-\; \mathbb{E}(y^3|z_\ell) \equiv \\
\Delta\overline{y^3} \;&=\; \beta^2\mathcal{W}(p_k - p_\ell) + 3\beta\mathcal{W}\mu^*_{k\ell} + 3\mathcal{W}\lambda^*_{k\ell} \\
\mathbb{E}(y^2T|z_k) \;&-\; \mathbb{E}(y^2T|z_\ell) \equiv \\
\Delta\overline{y^2T} \;&=\; \beta(1-\alpha_1)\mathcal{W}(p_k - p_\ell) + 2(1-\alpha_1)\mathcal{W}\mu^*_{k\ell} + \lambda^*_{k\ell}
\end{aligned}
$$

Theorem: β, α_0 and α_1 are identified.

Requires $\alpha_0 + \alpha_1 < 1$, but z need only be binary.

Proof

$$\begin{aligned} \Delta\overline{y^3} &= \beta^2 \mathcal{W}(p_k - p_\ell) + 3\beta\mathcal{W}\mu^*_{k\ell} + 3\mathcal{W}\lambda^*_{k\ell} \\ \Delta\overline{y^2 T} &= \beta(1-\alpha_1)\mathcal{W}(p_k - p_\ell) + 2(1-\alpha_1)\mathcal{W}\mu^*_{k\ell} + \lambda^*_{k\ell} \end{aligned}$$

Solve for $\lambda^*_{k\ell}$, substitute and rearrange:

$$\mathcal{S} \equiv \beta^2 - 3\mathcal{W}(1-\alpha_1)(\beta + \mathcal{R}) = \frac{\Delta\overline{y^3} - 3\mathcal{W}\left[\Delta\overline{y^2 T} + \mathcal{R}\Delta\overline{yT}\right]}{\mathcal{W}(p_k - p_\ell)}.$$

Theorem: β, α_0 and α_1 are identified.

Requires $\alpha_0 + \alpha_1 < 1$, but z need only be binary.

Proof continued…

$$\mathcal{S} \equiv \beta^2 - 3\mathcal{W}(1-\alpha_1)(\beta + \mathcal{R}) = \frac{\Delta\overline{y^3} - 3\mathcal{W}\left[\Delta\overline{y^2 T} + \mathcal{R}\Delta\overline{yT}\right]}{\mathcal{W}(p_k - p_\ell)}$$

Use the fact that $\mathcal{R} = \beta - 2(1-\alpha_1)\mathcal{W}$ to eliminate β from $\mathcal{S}$:

$$2\mathcal{W}^2(1-\alpha_1)^2 + 2\mathcal{R}\mathcal{W}(1-\alpha_1) + (\mathcal{S} - \mathcal{R}^2) = 0$$

which is a quadratic in $(1-\alpha_1)$ and observables only! Can show that there are always two real roots: one is $(1-\alpha_1)$ and the other is α_0. To tell which is which, need $\alpha_0 + \alpha_1 < 1$.

Recap of Results

1. Using first-moment information alone, β is unidentified regardless of how many values the instrument takes on.
2. Using second moment information $\alpha_1 - \alpha_0$ is identified
   - Partial identification bound for β
   - Identifies β if α_0 is known (e.g. smoking/birthweight example)
3. Using third moment information β, α_0 and α_1 are identified so long as $\alpha_0 + \alpha_1 < 1$.

Simulation Study

$y = \beta T^* + \varepsilon$

$T^* = \mathbf{1}\left\{\gamma_0 + \gamma_1 z + \eta > 0\right\}$

$\gamma_0 = \Phi^{-1}(\delta)$, $\gamma_1 = \Phi^{-1}(1-\delta) - \Phi(\delta)$ so that δ

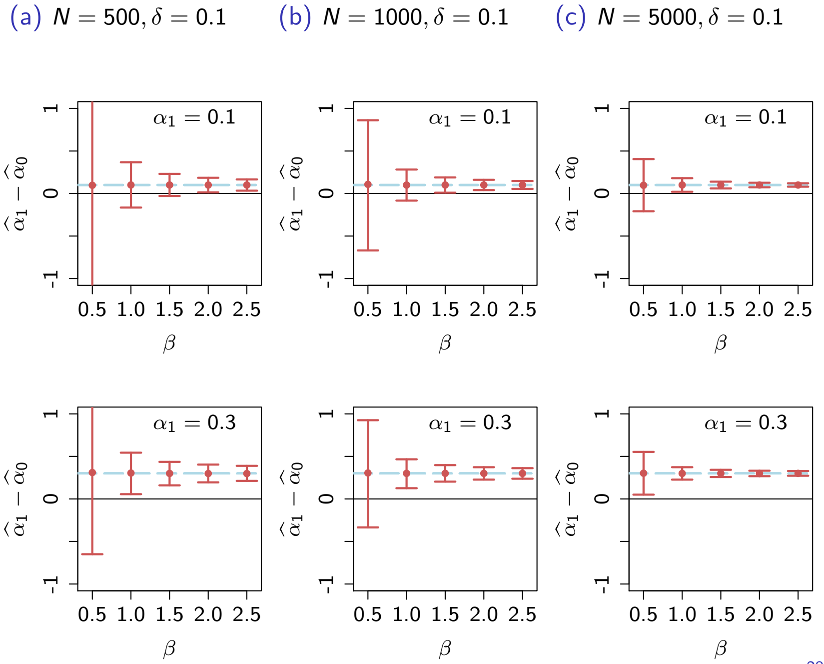
E.g. if $\delta = 0.1$ then 10% of those *not* offered treatment get it anyway, and 10% of those offered treatment don't take it up.

If $T^* = 0$ then $T = 0$ (E.g. Birthweight and smoking)

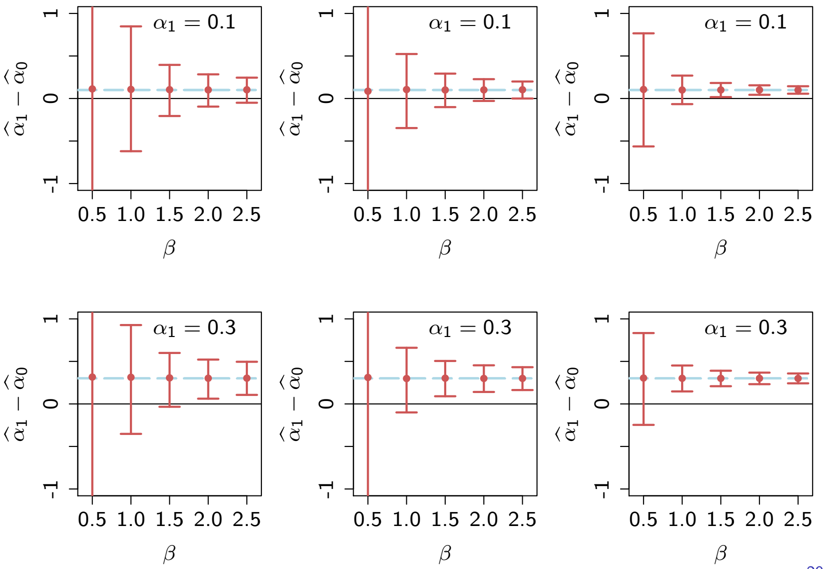
$T | T^* = 1 \sim$ Bernoulli(?)

$$\begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}\right)$$

$$\boxed{\text{Sampling Distribution of } \widehat{\alpha}_1 - \widehat{\alpha}_0}$$

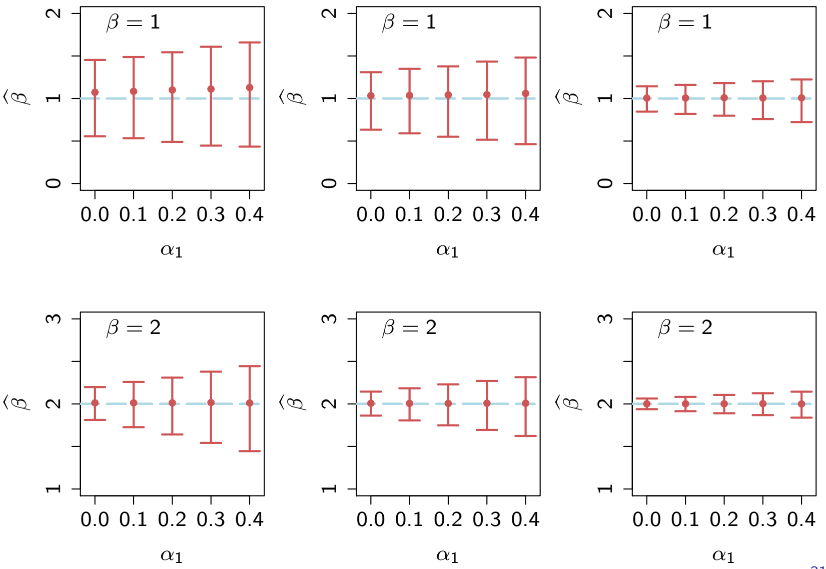

(a) $N = 500, \delta = 0.1$
(b) $N = 1000, \delta = 0.1$
(c) $N = 5000, \delta = 0.1$
$\alpha_1 = 0.1$
$\alpha_1 = 0.3$
$\hat{\alpha}_1 - \hat{\alpha}_0$
β
0.5 1.0 1.5 2.0 2.5
-1 0 1

(a) $N = 500, \delta = 0.3$ (b) $N = 1000, \delta = 0.3$ (c) $N = 5000, \delta = 0.3$

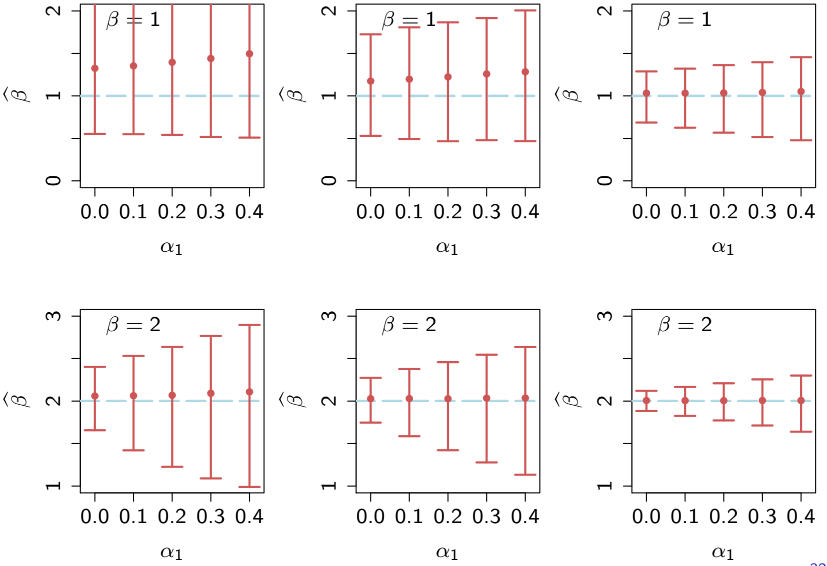


$$\boxed{\text{Sampling Distribution of } \widehat{\beta} = (1 - \widehat{\alpha}_0 - \widehat{\alpha}_1)\widehat{\beta}_{IV}}$$

(a) $N = 500, \delta = 0.1$ (b) $N = 1000, \delta = 0.1$ (c) $N = 5000, \delta = 0.1$



(a) $N = 500, \delta = 0.3$ (b) $N = 1000, \delta = 0.3$ (c) $N = 5000, \delta = 0.3$



Empirical Illustration: Schooling and Test Scores