# Mis-Classified, Binary, Endogenous Regressors: Identification and Inference


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# Additively Separable Model

$$y = h(T^*, \mathbf{x}) + \varepsilon$$

- $y$ – Outcome of interest
- $h$ – Known or unknown function
- $T^*$ – Unobserved, endogenous binary regressor
- $T$ – Observed, mis-measured binary surrogate for $T^*$
- $\mathbf{x}$ – Exogenous covariates
- $\varepsilon$ – Mean-zero error term

# What is the Effect of $T^*$?

## Re-write the Model

$$\begin{aligned} y &= c(\mathbf{x}) + \beta(\mathbf{x})T^* + \varepsilon \\ \beta(\mathbf{x}) &= h(1,\mathbf{x}) - h(0,\mathbf{x}) \\ c(\mathbf{x}) &= h(0,\mathbf{x}) \end{aligned}$$

## This Paper:

- Does a discrete instrument $z$ (typically binary) identify $\beta(\mathbf{x})$?
- What assumptions are required for $z$ and the surrogate $T$?
- How to carry out inference for a mis-classified regressor?

# Example: Job Training Partnership Act (JTPA)

Heckman et al. (2000, QJE)

Randomized offer of job training, but about 30% of those *not* offered also obtain training and about 40% of those offered training don't attend. Estimate causal effect of *training* rather than *offer* of training.

- $y$ – Log wage
- $T^*$ – True training attendence
- $T$ – Self-reported training attendance
- $\mathbf{x}$ – Individual characteristics
- $z$ – Offer of job training

# Related Literature

### Continuous Regressor

Lewbel (1997, 2012), Schennach (2004, 2007), Chen et al. (2005), Hu & Schennach (2008), Song (2015), Hu et al. (2015)...

### Binary, Exogenous Regressor

Aigner (1973), Bollinger (1996), Kane et al. (1999), Black et al. (2000), Frazis & Loewenstein (2003), Mahajan (2006), Lewbel (2007), Hu (2008)

### Binary, Endogenous Regressor

Mahajan (2006), Shiu (2015), Ura (2015), Denteh et al. (2016)

## "Baseline" Assumptions I – Model & Instrument

Additively Separable Model

$$y = c(\mathbf{x}) + \beta(\mathbf{x}) T^* + \varepsilon, \quad \mathbb{E}[\varepsilon] = 0$$

Valid & Relevant Instrument: $z \in \{0, 1\}$

- $\mathbb{P}(T^* = 1 | \mathbf{x}, z = 1) \neq \mathbb{P}(T^* = 1 | \mathbf{x}, z = 0)$
- $\mathbb{E}[\varepsilon | \mathbf{x}, z] = 0$
- $0 < \mathbb{P}(z = 1 | \mathbf{x}) < 1$

If $T^*$ were observed, these conditions would identify $\beta$.

# "Baseline" Assumptions II – Measurement Error

Notation: Mis-classification Rates

"↑" $\alpha_0(\mathbf{x}, z) \equiv \mathbb{P}(T = 1 | T^* = 0, \mathbf{x}, z)$

"↓" $\alpha_1(\mathbf{x}, z) \equiv \mathbb{P}(T = 0 | T^* = 1, \mathbf{x}, z)$

Mis-classification unaffected by $z$

$$\alpha_0(\mathbf{x}, z) = \alpha_0(\mathbf{x}), \quad \alpha_1(\mathbf{x}, z) = \alpha_1(\mathbf{x})$$

Extent of Mis-classification

$\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1 \quad$ ($T$ is positively correlated with $T^*$)

Non-differential Mis-classification

$$\mathbb{E}[\varepsilon | \mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon | \mathbf{x}, z, T^*]$$

# Identification Results from the Literature

## Mahajan (2006) Theorem 1, Frazis & Loewenstein (2003)

$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*] = 0$, plus "Baseline" $\implies \beta(\mathbf{x})$ identified

Requires $(T^*, z)$ jointly exogenous.

## Mahajan (2006) A.2

$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*]$, plus "Baseline" $\implies \beta(\mathbf{x})$ identified

Allows $T^*$ endogenous, but we prove this claim is false.

## Open Question

Do the baseline assumptions identify $\beta(\mathbf{x})$ when $T^*$ is endogenous?

# First-stage Probabilities & Mis-classification Bounds

| Unobserved | Observed |
|---|---|
| $p_k^*(\mathbf{x}) \equiv \mathbb{P}(T^* = 1 \vert \mathbf{x}, z = k)$ | $p_k(\mathbf{x}) \equiv \mathbb{P}(T = 1 \vert \mathbf{x}, z = k)$ |

## Relationship

$$p_k^*(\mathbf{x}) = \frac{p_k(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}, \quad k = 0, 1$$

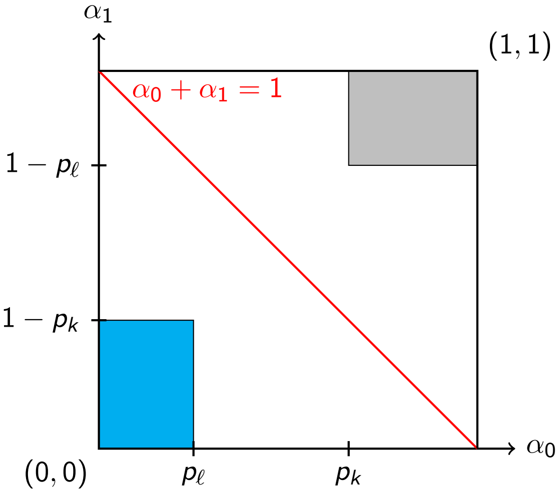
$z$ does not affect $(\alpha_0, \alpha_1)$; denominator $\neq 0$

## Bounds for Mis-classification

$$\alpha_0(\mathbf{x}) \leq p_k(\mathbf{x}) \leq 1 - \alpha_1(\mathbf{x}), \quad k = 0, 1$$

$\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1$

$\alpha_0 \leq \min_k \{p_k\}, \quad \alpha_1 \leq \min_k \{1 - p_k\}$



# Instrumental Variable Estimands

## Unobserved

$$\beta(\mathbf{x}) = \frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x})}$$

## Wald (Observed)

$$\frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1(\mathbf{x}) - p_0(\mathbf{x})} = \beta(\mathbf{x}) \left[\frac{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x})}{p_1(\mathbf{x}) - p_0(\mathbf{x})}\right] = \frac{\beta(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}$$

$$\boxed{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x}) = \frac{p_1(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0 - \alpha_1(\mathbf{x})} - \frac{p_0(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0 - \alpha_1(\mathbf{x})} = \frac{p_1(\mathbf{x}) - p_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}}$$

# Partial Identification Bounds for $\beta(\mathbf{x})$

$$\boxed{\beta(\mathbf{x}) = [1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})]\left[\frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1(\mathbf{x}) - p_0(\mathbf{x})}\right]}$$

$$\boxed{0 \leq \alpha_0 \leq \min_k\{p_k(\mathbf{x})\}, \quad 0 \leq \alpha_1 \leq \min_k\{1 - p_k(\mathbf{x})\}}$$

## No Mis-classification

$$\alpha_0(\mathbf{x}) = \alpha_1(\mathbf{x}) = 0 \implies \beta(\mathbf{x}) = \text{Wald}$$

## Maximum Mis-classification

$$\alpha_0(\mathbf{x}) = p_{\min}(\mathbf{x}),\ \alpha_1(\mathbf{x}) = 1 - p_{\max}(\mathbf{x})$$

$$\implies 1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x}) = p_{\max}(\mathbf{x}) - p_{\min}(\mathbf{x}) = |p_1(\mathbf{x}) - p_0(\mathbf{x})|$$

$$\implies \beta(\mathbf{x}) = \text{sign}\left\{p_1(\mathbf{x}) - p_0(\mathbf{x})\right\} \times (\text{Reduced Form})$$

# Partial Identification Bounds for $\beta(\mathbf{x})$

## No Mis-classification

$\beta(\mathbf{x}) = \text{Wald}$

## Maximum Mis-classification

$$\begin{aligned} \beta(\mathbf{x}) &= \text{sign}\left\{p_1(\mathbf{x}) - p_0(\mathbf{x})\right\} \times (\text{Reduced Form}) \\ &= \text{sign}\left\{\text{Wald}\right\} \times \left|\text{Reduced Form}\right| \end{aligned}$$

$\text{Wald} > 0 \iff \text{sign}\left\{p_1(\mathbf{x}) - p_0(\mathbf{x})\right\} = \text{sign}\left\{\text{Reduced Form}\right\}$

$\text{Wald} < 0 \iff \text{sign}\left\{p_1(\mathbf{x}) - p_0(\mathbf{x})\right\} \neq \text{sign}\left\{\text{Reduced Form}\right\}$

$\beta(\mathbf{x})$ has the same sign as the Wald and its magnitude is between that of Wald and Reduced Form.

# Sharp Bounds?

- Bounds from the preceding slide are known in the literature.
- We prove that they are *not* sharp under the baseline assumptions from above.
- $\mathbb{E}[\varepsilon|\mathbf{x}, T^*, T, z] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*, z]$ restricts $(\alpha_0, \alpha_1)$ hence $\beta$.
- Description of the sharp set is somewhat complicated...
- Corollary: $\beta$ is not point identified regardless of how many (discrete) values $z$ takes on.

Point identification from slightly stronger assumptions?

# Point Identification: 1st Ingredient

## Reparameterization

$$\theta_1(\mathbf{x}) = \beta(\mathbf{x}) / \left[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]$$

$$\theta_2(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^2 \left[1 + \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]$$

$$\theta_3(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^3 \left[\left\{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right\}^2 + 6\alpha_0(\mathbf{x})\left\{1 - \alpha_1(\mathbf{x})\right\}\right]$$

$$\boxed{\beta(\mathbf{x}) = 0 \iff \theta_1(\mathbf{x}) = \theta_2(\mathbf{x}) = \theta_3(\mathbf{x}) = 0}$$

## Lemma

Baseline Assumptions $\implies$ $\text{Cov}(y, z|\mathbf{x}) = \theta_1(\mathbf{x})\text{Cov}(z, T|\mathbf{x})$.

# Point Identification: 2nd Ingredient

Assumption (♠)

$\mathbb{E}[\varepsilon^2|\mathbf{x}, z] = \mathbb{E}[\varepsilon^2|\mathbf{x}]$

Lemma

(Baseline) + (♠) $\implies$

$$\mathrm{Cov}(y^2, z|\mathbf{x}) = 2\mathrm{Cov}(yT, z|\mathbf{x})\theta_1(\mathbf{x}) - \mathrm{Cov}(T, z|\mathbf{x})\theta_2(\mathbf{x})$$

Corollary

(Baseline) + (♠) + $[\beta(\mathbf{x}) \neq 0] \implies [\alpha_1(\mathbf{x}) - \alpha_0(\mathbf{x})]$ is identified.

Hence, $\beta(\mathbf{x})$ is identified if mis-classification is one-sided.

# Point Identification: 1st Ingredient

Assumption (♣)

(i) $\mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*]$

(ii) $\mathbb{E}[\varepsilon^3|\mathbf{x}, z] = \mathbb{E}[\varepsilon^3|\mathbf{x}]$

Lemma

(Baseline) + (♠) + (♣) $\implies$

$$\mathrm{Cov}(y^3, z|\mathbf{x}) = 3\mathrm{Cov}(y^2 T, z|\mathbf{x})\theta_1(\mathbf{x}) - 3\mathrm{Cov}(yT, z|\mathbf{x})\theta_2(\mathbf{x}) + \mathrm{Cov}(T, z|\mathbf{x})\theta_3(\mathbf{x})$$

# Point Identification Result

## Theorem

(Baseline) + ($\spadesuit$) + ($\clubsuit$) $\implies$ $\beta(\mathbf{x})$ is point identified. If $\beta(\mathbf{x}) \neq 0$, then $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are likewise point identified.

## Proof Sketch

1. $\beta(\mathbf{x}) = 0 \iff \theta_1(\mathbf{x}) = 0$ so suppose this is not the case.
2. Lemmas: full-rank linear system in $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})$ & observables.
3. Non-linear eqs. relating $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})$ to $\beta(\mathbf{x})$ and $\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x})$. Show that solution exists and is unique.

## Sufficient Conditions for ($\spadesuit$) and ($\clubsuit$)

(i) $T$ is conditionally independent of $(\varepsilon, z)$ given $(T^*, \mathbf{x})$

(ii) $z$ is conditionally independent of $\varepsilon$ given $\mathbf{x}$

# Just-Identified System of Moment Equalities

Suppress dependence on $\mathbf{x}$ to simplify the notation from here on...

## Collect Lemmas from Above:

$$
\begin{aligned}
\mathrm{Cov}(y,z) - \mathrm{Cov}(T,z)\theta_1 &= 0\\
\mathrm{Cov}(y^2,z) - 2\mathrm{Cov}(yT,z)\theta_1 + \mathrm{Cov}(T,z)\theta_2 &= 0\\
\mathrm{Cov}(y^3,z) - 3\mathrm{Cov}(y^2T,z)\theta_1 + 3\mathrm{Cov}(yT,z)\theta_2 - \mathrm{Cov}(T,z)\theta_3 &= 0
\end{aligned}
$$

## Notation: Observed Data Vector

$$
\mathbf{w}_i' = (T_i, y_i, y_iT_i, y_i^2, y_i^2T_i, y_i^3)
$$

# Just-Identified System of Moment Equalities

$$\boxed{\mathbb{E}\left[\left(\mathbf{\Psi}'(\boldsymbol{\theta})\mathbf{w}_i - \boldsymbol{\kappa}\right) \otimes \begin{pmatrix} 1 \\ z_i \end{pmatrix}\right] = \mathbf{0}}$$

$$\mathbf{\Psi} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \boldsymbol{\psi}_3 \end{bmatrix} \qquad \boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)' \equiv \text{ "Intercepts"}$$

$$\boldsymbol{\psi}_1' = \begin{bmatrix} -\theta_1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \theta_1 = \beta/(1-\alpha_0-\alpha_1)$$

$$\boldsymbol{\psi}_2' = \begin{bmatrix} \theta_2 & 0 & -2\theta_1 & 1 & 0 & 0 \end{bmatrix} \qquad \theta_2 = \theta_1^2\left[1+\alpha_0-\alpha_1\right]$$

$$\boldsymbol{\psi}_3' = \begin{bmatrix} -\theta_3 & 0 & 3\theta_2 & 0 & -3\theta_1 & 1 \end{bmatrix} \qquad \theta_3 = \theta_1^3\left[(1-\alpha_0-\alpha_1)^2 + 6\alpha_0(1-\alpha_1)\right]$$

## Weak Identification Problem

Moment conditions are uninformative about $(\alpha_0, \alpha_1)$ when $\beta$ is small.

Moreover, $(\alpha_0, \alpha_1)$ could be on the boundary of the parameter space.

# Simulation DGP: $y = \beta T^* + \varepsilon$

Sample Size = 1000; Simulation Replications = 2000

## Errors

$(\varepsilon, \eta) \sim$ jointly normal, mean 0, variance 1, correlation 0.5.

## First-Stage

- Half of individuals have $z = 1$, the rest have $z = 0$.
- $T^* = \mathbf{1}\{\gamma_0 + \gamma_1 z + \eta > 0\}$
- $\mathbb{P}(T^* = 0|z = 1) = \mathbb{P}(T^* = 1|z = 0) = 0.15$

## Mis-classification

- $T|T^* = 0 \sim \text{Bernoulli}(\alpha_0)$
- $T|T^* = 1 \sim \text{Bernoulli}(1 - \alpha_1)$

| | | $\beta$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha_0$ | $\alpha_1$ | 0 | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 3 |
| 0.0 | 0.0 | 27 | 33 | 30 | 14 | 1 | 0 | 0 | 0 |
| | 0.1 | 27 | 32 | 29 | 13 | 2 | 0 | 0 | 0 |
| | 0.2 | 26 | 33 | 32 | 15 | 4 | 0 | 0 | 0 |
| | 0.3 | 26 | 34 | 30 | 17 | 5 | 0 | 0 | 0 |
| 0.1 | 0.0 | 26 | 32 | 31 | 14 | 2 | 0 | 0 | 0 |
| | 0.1 | 26 | 36 | 32 | 16 | 4 | 0 | 0 | 0 |
| | 0.2 | 27 | 35 | 31 | 18 | 8 | 0 | 0 | 0 |
| | 0.3 | 25 | 35 | 32 | 21 | 11 | 1 | 0 | 0 |
| 0.2 | 0.0 | 26 | 33 | 30 | 15 | 3 | 0 | 0 | 0 |
| | 0.1 | 26 | 33 | 30 | 19 | 6 | 0 | 0 | 0 |
| | 0.2 | 26 | 35 | 33 | 22 | 12 | 1 | 0 | 0 |
| | 0.3 | 26 | 35 | 33 | 26 | 15 | 3 | 0 | 0 |
| 0.3 | 0.0 | 26 | 32 | 32 | 16 | 6 | 0 | 0 | 0 |
| | 0.1 | 24 | 35 | 33 | 21 | 11 | 1 | 0 | 0 |
| | 0.2 | 26 | 32 | 35 | 27 | 15 | 4 | 0 | 0 |
| | 0.3 | 26 | 35 | 35 | 28 | 21 | 7 | 2 | 0 |

Table: Percentage of simulation replications for which the standard GMM CI fails to exist.

| $\alpha_0$ | $\alpha_1$ | $\beta$ 0 | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 72 | 62 | 62 | 80 | 92 | 95 | 94 | 95 |
| | 0.1 | 72 | 62 | 63 | 79 | 92 | 95 | 96 | 95 |
| | 0.2 | 73 | 61 | 61 | 77 | 90 | 96 | 96 | 96 |
| | 0.3 | 73 | 59 | 62 | 76 | 88 | 95 | 96 | 95 |
| 0.1 | 0.0 | 73 | 63 | 60 | 78 | 91 | 95 | 96 | 96 |
| | 0.1 | 73 | 58 | 59 | 77 | 90 | 95 | 95 | 94 |
| | 0.2 | 73 | 59 | 61 | 75 | 86 | 95 | 95 | 94 |
| | 0.3 | 74 | 59 | 58 | 71 | 82 | 94 | 96 | 96 |
| 0.2 | 0.0 | 74 | 62 | 60 | 78 | 91 | 95 | 96 | 96 |
| | 0.1 | 73 | 60 | 61 | 74 | 87 | 95 | 96 | 94 |
| | 0.2 | 73 | 58 | 57 | 70 | 81 | 93 | 95 | 95 |
| | 0.3 | 73 | 58 | 56 | 66 | 78 | 92 | 95 | 96 |
| 0.3 | 0.0 | 74 | 62 | 60 | 76 | 89 | 95 | 96 | 96 |
| | 0.1 | 75 | 59 | 58 | 71 | 82 | 93 | 96 | 95 |
| | 0.2 | 74 | 61 | 56 | 65 | 78 | 90 | 96 | 96 |
| | 0.3 | 73 | 58 | 55 | 64 | 71 | 88 | 93 | 96 |

Table: Coverge of nominal 95% GMM CI, conditional on existence.

# Weak Identification Problem

- $\beta$ small $\Rightarrow$ moment equalities uninformative about $\alpha_1$
- Same problem for other estimators from the literature but hasn't been pointed out.
- Identification robust inference: GMM Anderson-Rubin statistic
- But we can do better...
- Bounds for $(\alpha_0, \alpha_1)$ immune to weak identification problem: remain informative if $\beta$ is small or zero.
- Inference using Generalized Moment Selection (Andrews & Soares, 2010)

# What we do

- Explain our multi-step procedure
- Key ingredient is the preliminary estimation of strongly identified parameters
- Details of the steps involved along with the inequalities we use and why it's ok to eliminate some parameters but not others $(\alpha_0, \alpha_1)$
- Then explain about Bonferroni and projection etc
- Projection to get inference for $\beta$, but can be conservative

# Moment Inequalities: Part I

$\alpha_0(\mathbf{x}) \leq p_k \leq 1 - \alpha_1$ becomes $\mathbb{E}\left[m^I_{1k}(\mathbf{w}_i, \vartheta)\right] \geq \mathbf{0}$ for all $k$ where

$$m^I_{1k}(\mathbf{w}_i, \vartheta) \equiv \begin{bmatrix} \mathbf{1}(z_i = k)(T - \alpha_0) \\ \mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \end{bmatrix}$$

# Moment Inequalities: Part II

For all $k$, we have $\mathbb{E}[m^l_{2k}\left(\mathbf{w}_i, \vartheta, \mathbf{q}_k\right)] \geq 0$ where

$$m^l_{2k}\left(\mathbf{w}_i, \vartheta, \mathbf{q}_k\right) \equiv \left[\begin{array}{r} y_i \mathbf{1}\left(z_i = k\right)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i \leq \underline{q}_{0k})(1 - T_i)\left(\frac{1-\alpha_0-\alpha_1}{\alpha_1}\right)\right\} \\ -y_i \mathbf{1}(z_i = k)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i > \overline{q}_{0k})(1 - T_i)\left(\frac{1-\alpha_0-\alpha_1}{\alpha_1}\right)\right\} \\ y_i \mathbf{1}\left(z_i = k\right)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i \leq \underline{q}_{1k}) T_i\left(\frac{1-\alpha_0-\alpha_1}{1-\alpha_1}\right)\right\} \\ -y_i \mathbf{1}(z_i = k)\left\{(T_i - \alpha_0) - \mathbf{1}(y_i > \overline{q}_{1k}) T_i\left(\frac{1-\alpha_0-\alpha_1}{1-\alpha_1}\right)\right\} \end{array}\right]$$

and $\mathbf{q}_k \equiv (\underline{q}_{0k}, \overline{q}_{0k}, \underline{q}_{1k}, \overline{q}_{1k})'$ defined by $\mathbb{E}[h^l_k(\mathbf{w}_i, \vartheta, \mathbf{q}_k)] = 0$ with

$$h^l_k(\mathbf{w}_i, \vartheta, \mathbf{q}_k) = \left[\begin{array}{l} \mathbf{1}(y_i \leq \underline{q}_{0k})\mathbf{1}(z_i = k)(1 - T_i) - \left(\frac{\alpha_1}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(T_i - \alpha_0) \\ \mathbf{1}(y_i \leq \overline{q}_{0k})\mathbf{1}(z_i = k)(1 - T_i) - \left(\frac{1-\alpha_0}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \\ \mathbf{1}(y_i \leq \underline{q}_{1k})\mathbf{1}(z_i = k) T_i - \left(\frac{1-\alpha_1}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(T_i - \alpha_0) \\ \mathbf{1}(y_i \leq \overline{q}_{1k})\mathbf{1}(z_i = k) T_i - \left(\frac{\alpha_0}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \end{array}\right]$$

# Moment Equalities

Let $\boldsymbol{\vartheta} = (\alpha_0, \alpha_1)$ and $\boldsymbol{\gamma} = (\boldsymbol{\kappa}, \theta_1)$

$$\mathbb{E}[m^I(\mathbf{w}_i, \boldsymbol{\vartheta}_0, \mathbf{q}_0)] \geq \mathbf{0}, \quad \mathbb{E}[m^E(\mathbf{w}_i, \boldsymbol{\vartheta}_0, \boldsymbol{\gamma}_0)] = \mathbf{0} \tag{1}$$

where $m^I = (m_1^{I\prime}, m_2^{I\prime})'$ and

$$m^E(\mathbf{w}_i, \boldsymbol{\vartheta}_0, \boldsymbol{\gamma}_0) = \begin{bmatrix} \left\{\boldsymbol{\psi}_2'(\theta_1, \alpha_0, \alpha_1)\mathbf{w}_i - \kappa_2\right\} z_i \\ \left\{\boldsymbol{\psi}_3'(\theta_1, \alpha_0, \alpha_1)\mathbf{w}_i - \kappa_3\right\} z_i \end{bmatrix}. \tag{2}$$

$$h^E(\mathbf{w}_i, \boldsymbol{\vartheta}, \boldsymbol{\gamma}) = \begin{bmatrix} \boldsymbol{\Psi}'(\theta_1, \alpha_0, \alpha_1)\mathbf{w}_i - \boldsymbol{\kappa} \\ \left\{\boldsymbol{\psi}_1'(\theta_1)\mathbf{w}_i - \kappa_1\right\} z_i \end{bmatrix}. \tag{3}$$

# Inference With Moment Equalities and Inequalities

## Moment Conditions

$\mathbb{E}\left[m_j(\mathbf{w}_i, \vartheta_0)\right] \geq 0, \quad j = 1, \cdots, p$

$\mathbb{E}\left[m_j(\mathbf{w}_i, \vartheta_0)\right] = 0, \quad j = p+1, \cdots, p+v$

## Test Statistic

$$T_n(\vartheta) = \sum_{j=1}^{p} \left[ \frac{\sqrt{n}\ \bar{m}_{n,j}(\vartheta)}{\widehat{\sigma}_{n,j}(\vartheta)} \right]_-^2 + \sum_{j=p+1}^{p+v} \left[ \frac{\sqrt{n}\ \bar{m}_{n,j}(\vartheta)}{\widehat{\sigma}_{n,j}(\vartheta)} \right]^2$$

$$[x]_- = \min\{x, 0\}$$

$$\bar{m}_{n,j}(\vartheta) = n^{-1} \sum_{i=1}^{n} m_j(\mathbf{w}_i, \vartheta)$$

$$\widehat{\sigma}^2_{n,j}(\vartheta) = \text{consistent est. of AVAR} \left[\sqrt{n}\ \bar{m}_{n,j}(\vartheta)\right]$$

# Inference via Generalized Moment Selection

Andrews & Soares (2010)

### Moment Selection Step

If $\dfrac{\sqrt{n}\,\bar{m}_{n,j}(\vartheta_0)}{\widehat{\sigma}_{n,j}(\vartheta_0)} > \sqrt{\log n}$ then drop inequality $j$

### Critical Value

- $\sqrt{n}\,\bar{m}_n(\vartheta_0) \to_d$ normal limit with covariance matrix $\Sigma(\vartheta_0)$
- Use this to bootstrap the limit distribution of the test statistic.

### Theoretical Guarantees

Uniformly valid test of $H_0\colon \vartheta = \vartheta_0$ regardless of whether $\vartheta_0$ is identified. Not asymptotically conservative.

| $\alpha_0$ | $\alpha_1$ | $\beta$ 0 | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 97.7 | 97.7 | 97.6 | 97.7 | 98.0 | 98.0 | 97.4 | 97.9 |
| | 0.1 | 98.0 | 98.7 | 98.8 | 99.1 | 98.8 | 98.4 | 97.1 | 96.4 |
| | 0.2 | 98.4 | 98.5 | 98.9 | 98.9 | 98.8 | 98.6 | 98.0 | 97.0 |
| | 0.3 | 98.5 | 98.8 | 98.8 | 99.0 | 98.7 | 98.4 | 97.8 | 97.5 |
| 0.1 | 0.0 | 98.1 | 98.5 | 98.3 | 98.8 | 98.8 | 98.4 | 96.8 | 95.7 |
| | 0.1 | 98.6 | 99.1 | 99.5 | 99.6 | 99.6 | 98.8 | 97.7 | 95.2 |
| | 0.2 | 99.0 | 99.3 | 99.7 | 99.8 | 99.7 | 98.9 | 97.5 | 95.7 |
| | 0.3 | 99.4 | 99.7 | 99.8 | 99.8 | 99.6 | 99.0 | 98.2 | 96.7 |
| 0.2 | 0.0 | 98.6 | 98.5 | 98.6 | 98.9 | 98.7 | 98.2 | 97.7 | 97.0 |
| | 0.1 | 99.0 | 99.5 | 99.7 | 99.7 | 99.4 | 99.0 | 98.1 | 96.5 |
| | 0.2 | 99.5 | 99.7 | 99.8 | 99.7 | 99.4 | 99.0 | 97.8 | 96.8 |
| | 0.3 | 99.7 | 99.8 | 99.8 | 99.8 | 99.5 | 99.0 | 98.7 | 97.7 |
| 0.3 | 0.0 | 98.7 | 98.7 | 98.8 | 98.7 | 98.7 | 98.2 | 98.1 | 97.6 |
| | 0.1 | 99.4 | 99.6 | 99.6 | 99.7 | 99.4 | 98.9 | 98.3 | 96.8 |
| | 0.2 | 99.8 | 99.8 | 99.7 | 99.8 | 99.5 | 99.1 | 98.5 | 97.8 |
| | 0.3 | 100.0 | 99.9 | 99.9 | 99.8 | 99.6 | 99.5 | 99.1 | 98.8 |

Table: Coverage (1 - size) of nominal 97.5% GMS joint test for $(\alpha_0, \alpha_1)$.

| $\alpha_0$ | $\alpha_1$ | $\beta$ = 0 | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0 | 96 | 97 | 97 | 96 | 97 | 97 | 95 | 96 |
| | 0.1 | 97 | 99 | 99 | 99 | 99 | 100 | 100 | 99 |
| | 0.2 | 98 | 99 | 99 | 100 | 100 | 100 | 100 | 100 |
| | 0.3 | 97 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 0.1 | 0.0 | 97 | 99 | 99 | 99 | 100 | 100 | 100 | 98 |
| | 0.1 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.2 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.3 | 97 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 0.2 | 0.0 | 97 | 99 | 99 | 100 | 100 | 100 | 100 | 100 |
| | 0.1 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.2 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.3 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 0.3 | 0.0 | 97 | 99 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.1 | 97 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.2 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| | 0.3 | 98 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

Table: Coverage of nominal $> 95\%$ Bonferroni CI for $\beta$

| | | $\beta$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha_0$ | $\alpha_1$ | 0 | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 3 |
| 0.0 | 0.0 | 0.4 | 0.41 | 0.43 | 0.43 | 0.43 | 0.42 | 0.41 | 0.41 |
| | 0.1 | 0.45 | 0.47 | 0.54 | 0.59 | 0.63 | 0.7 | 0.75 | 0.86 |
| | 0.2 | 0.51 | 0.54 | 0.65 | 0.76 | 0.85 | 0.95 | 1.01 | 1.17 |
| | 0.3 | 0.58 | 0.62 | 0.79 | 0.95 | 1.07 | 1.17 | 1.24 | 1.48 |
| 0.1 | 0.0 | 0.45 | 0.47 | 0.54 | 0.59 | 0.63 | 0.7 | 0.76 | 0.88 |
| | 0.1 | 0.51 | 0.54 | 0.66 | 0.77 | 0.86 | 1.03 | 1.18 | 1.46 |
| | 0.2 | 0.58 | 0.63 | 0.8 | 0.98 | 1.12 | 1.38 | 1.55 | 1.88 |
| | 0.3 | 0.67 | 0.75 | 1 | 1.25 | 1.46 | 1.74 | 1.94 | 2.4 |
| 0.2 | 0.0 | 0.51 | 0.54 | 0.65 | 0.76 | 0.86 | 0.96 | 1.02 | 1.19 |
| | 0.1 | 0.58 | 0.63 | 0.81 | 0.99 | 1.14 | 1.42 | 1.64 | 2.08 |
| | 0.2 | 0.67 | 0.75 | 1.01 | 1.29 | 1.54 | 1.97 | 2.33 | 2.9 |
| | 0.3 | 0.81 | 0.91 | 1.3 | 1.7 | 2.09 | 2.73 | 3.13 | 3.9 |
| 0.3 | 0.0 | 0.58 | 0.62 | 0.8 | 0.95 | 1.09 | 1.18 | 1.25 | 1.5 |
| | 0.1 | 0.68 | 0.74 | 1.01 | 1.26 | 1.49 | 1.84 | 2.13 | 2.78 |
| | 0.2 | 0.81 | 0.91 | 1.3 | 1.7 | 2.11 | 2.8 | 3.4 | 4.48 |
| | 0.3 | 1.01 | 1.16 | 1.74 | 2.35 | 2.93 | 4.17 | 5.2 | 6.85 |

Table: Median width of nominal $> 95\%$ Bonferroni CI for $\beta$.

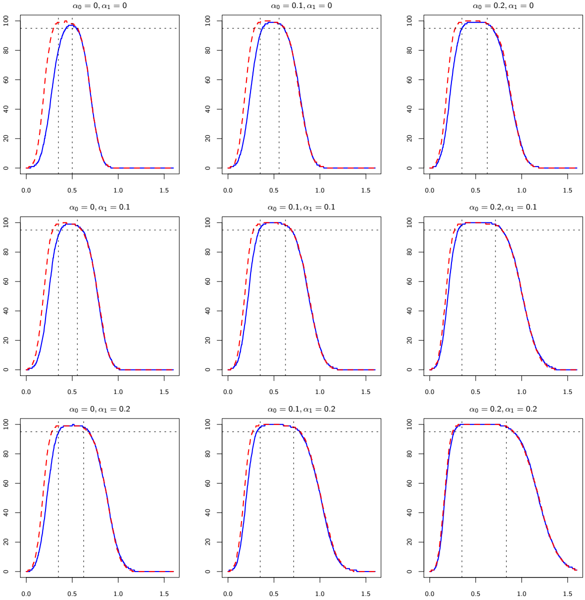



Figure: Coverage of nominal $> 95\%$ Bonferroni CI for $\beta$: with and without non-differential assumption, $\beta = 0.5$

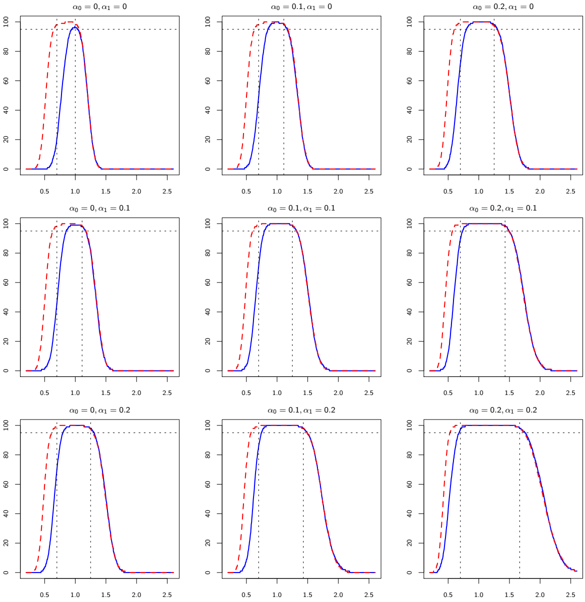



Figure: Coverage of nominal $> 95\%$ Bonferroni CI for $\beta$: with and without non-differential assumption, $\beta = 1$

# Conclusion

- Endogenous, mis-measured binary treatment.
- Important in applied work but no solution in the literature.
- Usual (1st moment) IV assumption fails to identify $\beta$
- Higher moment / independence restrictions identify $\beta$
- Identification-Robust Inference incorportating additional inequality moment conditions.