Mis-Classified, Binary, Endogenous Regressors: Identification and Inference

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What is the effect of T^*?

$$y = c(\mathbf{x}) + \beta(\mathbf{x}) T^* + \varepsilon$$

- y – Outcome of interest
- T^* – Unobserved, endogenous binary regressor
- T – Observed, mis-measured binary surrogate for T^*
- $\mathbf{x}$ – Exogenous covariates
- z – Discrete (typically binary) instrumental variable

(Additively Separable ε and binary T^* $\Rightarrow$ linear model given $\mathbf{x}$)

Using a discrete IV to learn about $\beta(\mathbf{x})$

$$y = c(\mathbf{x}) + \beta(\mathbf{x}) T^* + \varepsilon$$

Constributions of This Paper

1. Show that only existing point identification result for mis-classified, endogenous T^* is incorrect.
2. Derive sharp identified set for $\beta(\mathbf{x})$ under standard assumptions.
3. Prove point identification of $\beta(\mathbf{x})$ under slightly stronger assumptions.
4. Point out problem of weak identification in mis-classification models, develop identification-robust inference for $\beta(\mathbf{x})$.

Example: Smoking and Birthweight (SNAP Trial)

Coleman et al. (N Engl J Med, 2012)

RCT with pregnant smokers in England: half given nicotine patches, the rest given placebo patches. Some given nicotine fail to quit; some given placebo quit.

- y – Birthweight
- T^* – True smoking behavior
- T – Self-reported smoking behavior
- $\mathbf{x}$ – Mother characteristics
- z – Indicator of nicotine patch

Related Literature

Continuous Regressor

Lewbel (1997, 2012), Schennach (2004, 2007), Chen et al. (2005), Hu & Schennach (2008), Song (2015), Hu et al. (2015)...

Binary/Discrete, "Exogenous"

Aigner (1973), Bollinger (1996), Kane et al. (1999), Black et al. (2000), Frazis & Loewenstein (2003), Mahajan (2006), Lewbel (2007), Hu (2008), Molinari (2008)

Binary, Endogenous Regressor

Mahajan (2006),

Shiu (2015), Denteh et al. (2016), Ura (2016), Calvi et al. (2017)

“Baseline” Assumptions I – Model & Instrument

Additively Separable Model

$$y = c(\mathbf{x}) + \beta(\mathbf{x}) T^* + \varepsilon, \quad \mathbb{E}[\varepsilon] = 0$$

Valid & Relevant Instrument: $z \in \{0, 1\}$

- $\mathbb{P}(T^* = 1|\mathbf{x}, z = 1) \neq \mathbb{P}(T^* = 1|\mathbf{x}, z = 0)$
- $\mathbb{E}[\varepsilon|\mathbf{x}, z] = 0$
- $0 < \mathbb{P}(z = 1|\mathbf{x}) < 1$

If T^* were observed, these conditions would identify β.

“Baseline” Assumptions II – Measurement Error

Notation: Mis-classification Rates

“$\uparrow$” $\alpha_0(\mathbf{x}, z) \equiv \mathbb{P}\left(T = 1 \middle| T^* = 0, \mathbf{x}, z\right)$

“$\downarrow$” $\alpha_1(\mathbf{x}, z) \equiv \mathbb{P}\left(T = 0 \middle| T^* = 1, \mathbf{x}, z\right)$

Mis-classification unaffected by z

$$\alpha_0(\mathbf{x}, z) = \alpha_0(\mathbf{x}), \quad \alpha_1(\mathbf{x}, z) = \alpha_1(\mathbf{x})$$

Extent of Mis-classification

$\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1$ (T is positively correlated with T^*)

Non-differential Mis-classification

$$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, z, T^*]$$

Existing Result for Endogenous T^* is Incorrect

Mahajan (2006; Ecta) A.2

$\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*]$, plus "Baseline" $\Rightarrow \beta(\mathbf{x})$ point identified when T^* is endogenous.

This is incorrect...

We prove that, under Mahajan's assumptions, the instrument must be uncorrelated with T^* unless T^* is in fact *exogenous*.

Simple Bounds for Mis-classification from First-stage

Unobserved	Observed
$p_k^*(\mathbf{x}) \equiv \mathbb{P}(T^* = 1 \mid \mathbf{x}, z = k)$	$p_k(\mathbf{x}) \equiv \mathbb{P}(T = 1 \mid \mathbf{x}, z = k)$

Relationship

$$p_k^*(\mathbf{x}) = \frac{p_k(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}, \quad k = 0, 1$$

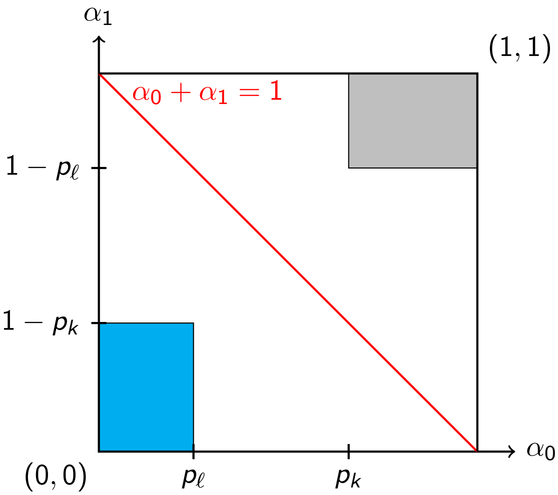
z does not affect (α_0, α_1); denominator $\neq 0$

Bounds for Mis-classification

$$\alpha_0(\mathbf{x}) \leq p_k(\mathbf{x}) \leq 1 - \alpha_1(\mathbf{x}), \quad k = 0, 1$$

$\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1$

$$\alpha_0 \leq \min_k \{p_k\}, \quad \alpha_1 \leq \min_k \{1 - p_k\}$$



What does IV estimate under mis-classification?

Unobserved

$$\beta(\mathbf{x}) = \frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x})}$$

Wald (Observed)

$$\frac{\mathbb{E}[y|\mathbf{x}, z=1] - \mathbb{E}[y|\mathbf{x}, z=0]}{p_1(\mathbf{x}) - p_0(\mathbf{x})} = \beta(\mathbf{x}) \left[\frac{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x})}{p_1(\mathbf{x}) - p_0(\mathbf{x})}\right] = \frac{\beta(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}$$

$$\boxed{p_1^*(\mathbf{x}) - p_0^*(\mathbf{x}) = \frac{p_1(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0 - \alpha_1(\mathbf{x})} - \frac{p_0(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0 - \alpha_1(\mathbf{x})} = \frac{p_1(\mathbf{x}) - p_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}}$$

Partial Identification Bounds for $\beta(\mathbf{x})$

Known Result

- $\beta(\mathbf{x})$ is between Wald and Reduced form; same sign as Wald.
- Doesn't rely on non-differential assumption or additive sep.
- Frazis & Loewenstein (2003), Ura (2016), ...

Non-differential Assumption

- $\mathbb{E}[\varepsilon|\mathbf{x}, T^*, T, z] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*, z]$
- Used in literature to identify $\beta(\mathbf{x})$ when T^* is exogenous.
- Does it restrict the identified set when T^* is endogenous?

Restrictions from Non-differential Mis-classification?

(Suppress **x** for simplicity)

Notation

- $r_{tk} \equiv \mathbb{P}(T^* = 1 | T = t, z = k)$, function of (α_0, α_1) and observables only
- z_k is shorthand for $z = k$

Iterated Expectations over T^*

$$\mathbb{E}(y|T = 0, z_k) = (1 - r_{0k})\mathbb{E}(y|T^* = 0, T = 0, z_k) + r_{0k}\mathbb{E}(y|T^* = 1, T = 0, z_k)$$
$$\mathbb{E}(y|T = 1, z_k) = (1 - r_{1k})\mathbb{E}(y|T^* = 0, T = 1, z_k) + r_{1k}\mathbb{E}(y|T^* = 1, T = 1, z_k)$$

Restrictions from Non-differential Mis-classification?

(Suppress **x** for simplicity)

Notation

- $r_{tk} \equiv \mathbb{P}(T^* = 1 | T = t, z = k)$, function of (α_0, α_1) and observables only
- z_k is shorthand for $z = k$

Adding Non-differential Assumption

$$\mathbb{E}(y|T=0,z_k) = (1-r_{0k})\mathbb{E}(y|T^*=0,z_k) + r_{0k}\mathbb{E}(y|T^*=1,z_k)$$
$$\mathbb{E}(y|T=1,z_k) = (1-r_{1k})\mathbb{E}(y|T^*=0,z_k) + r_{1k}\mathbb{E}(y|T^*=1,z_k)$$

2 equations in 2 unknowns $\Rightarrow$ solve for $\mathbb{E}(y|T^*=t^*, z=k)$ given (α_0, α_1).

Restrictions from Non-differential Mis-classification?

Law of Total Probability

$$\boxed{F_{tk} = (1 - r_{tk})F_{tk}^0 + r_{tk}F_{tk}^1}$$

$F_{tk} \equiv$ Observed CDF: $y|(T = t, z = k)$

$F_{tk}^{t^*} \equiv$ Unobserved CDF: $y|(T^* = t^*, T = t, z = k)$

Previous Slide

- r_{tk} observable given (α_0, α_1)
- $\mathbb{E}(y|T^*, T, z) = \mathbb{E}(y|T^*, z)$ observable given (α_0, α_1)

Key Question

Given (α_0, α_1) can we always find (F_{tk}^0, F_{tk}^1) to satisfy the mixture model?

Restrictions from Non-differential Mis-classification?

Equivalent Problem

Given a specified CDF F, for what values of p and μ do there exist valid CDFs (G, H) with $F = (1-p)G + pH$ and $\mu = \text{mean}(H)$?

Valid CDFS

$$\begin{aligned} &0 \leq H \leq 1 \\ &0 \leq G \leq 1 \quad \iff \quad [F - (1-p)]/p \leq H \leq F/p \end{aligned}$$

$$\boxed{\max\left\{0,\ \frac{F(x)}{p} - \frac{1-p}{p}\right\} \leq H(x) \leq \min\left\{1, \frac{F(x)}{p}\right\}}$$

Restrictions from Non-differential Mis-classification?

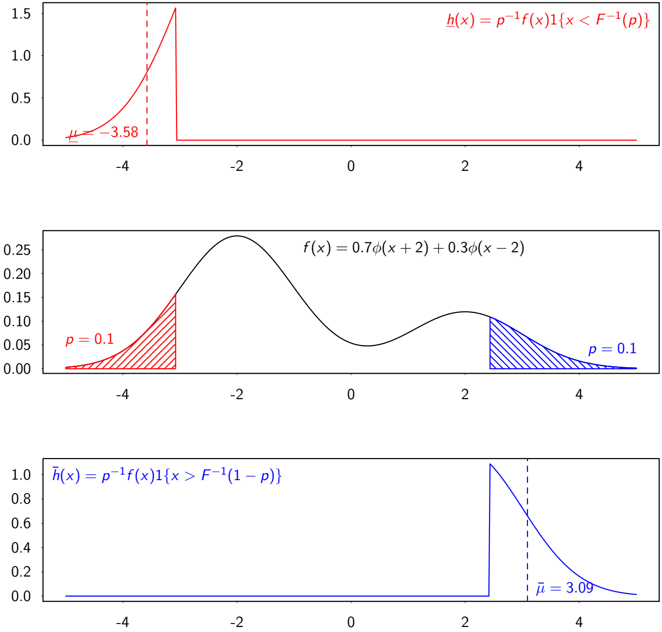
Notation

$$\overline{H} \equiv \max\left\{0,\ \frac{F(x)}{p} - \frac{1-p}{p}\right\}, \quad \underline{H} \equiv \min\left\{1, \frac{F(x)}{p}\right\}$$

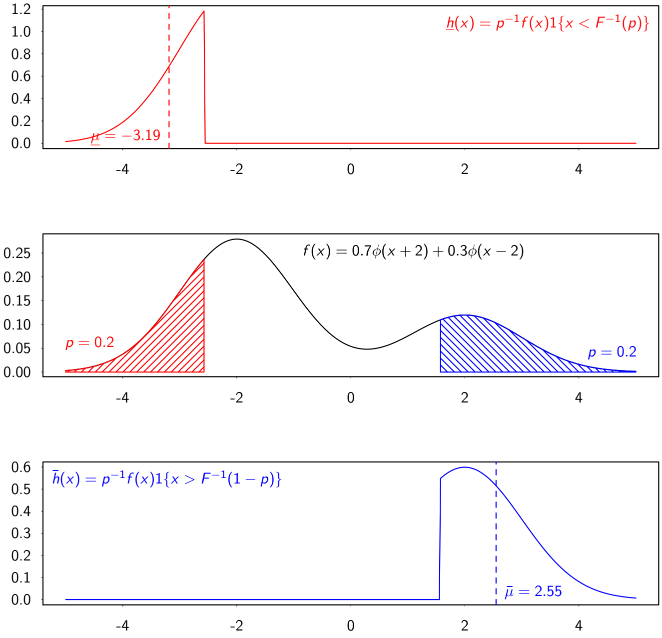
1st Order Stochastic Dominance

$$\overline{H}(x) \leq H(x) \leq \underline{H}(x) \quad \text{for all } x$$

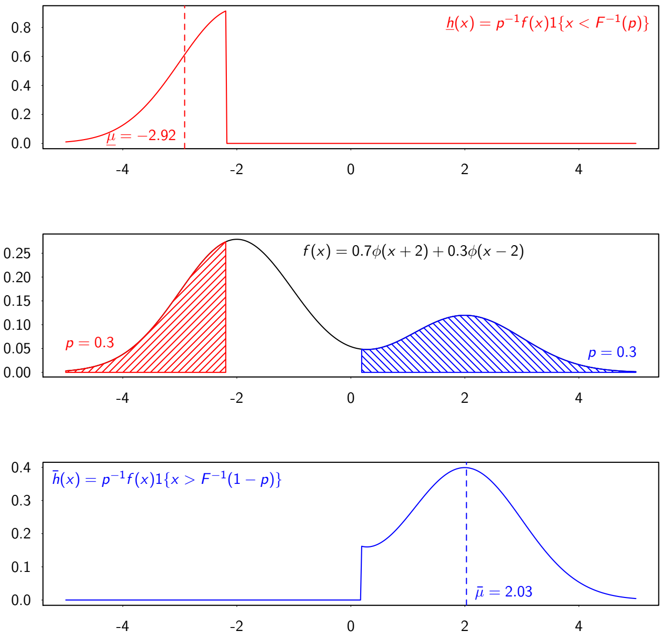
$$\implies \underbrace{\int_{\mathbb{R}} x\,\underline{H}(dx)}_{\underline{\mu}(p,F)} \leq \underbrace{\int_{\mathbb{R}} x\,H(dx)}_{\mu} \leq \underbrace{\int_{\mathbb{R}} x\,\overline{H}(dx)}_{\overline{\mu}(p,F)}$$



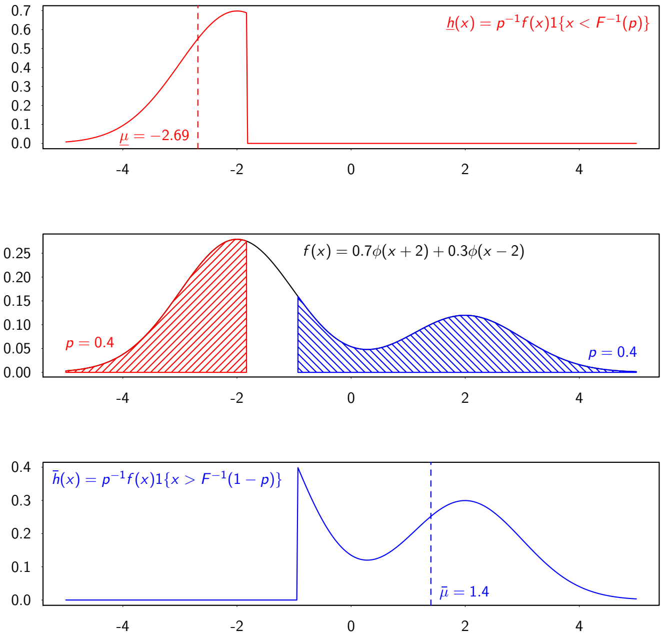
$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -3.58$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.1$
$p = 0.1$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 3.09$



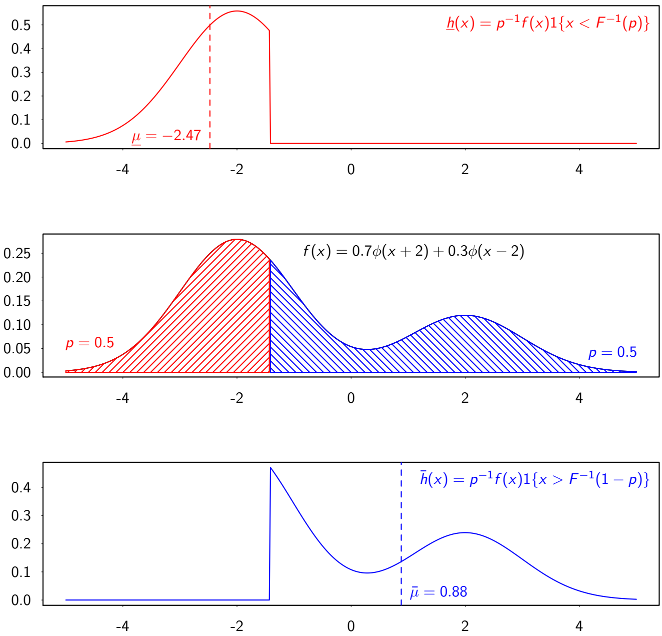
$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -3.19$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.2$
$p = 0.2$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 2.55$

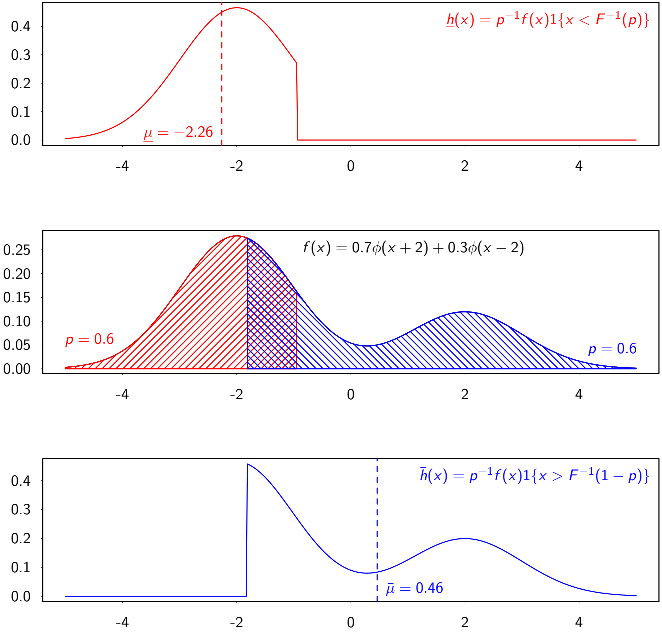


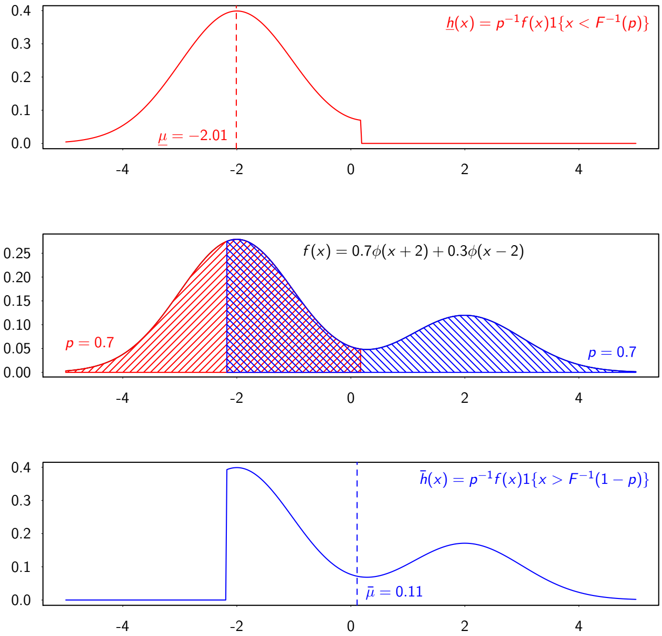
$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.92$
0.8
0.6
0.4
0.2
0.0
-4
-2
0
2
4
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.3$
$p = 0.3$
0.25
0.20
0.15
0.10
0.05
0.00
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 2.03$
0.4
0.3
0.2
0.1
0.0

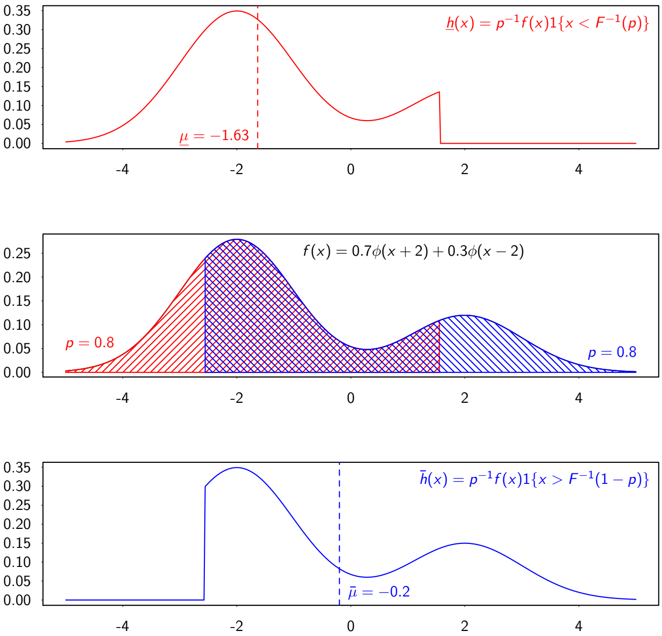


$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.69$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.4$
$p = 0.4$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 1.4$

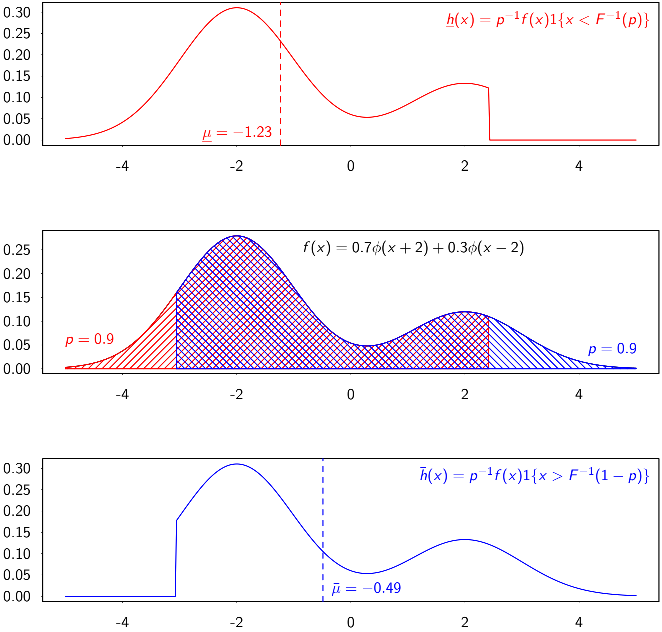

$\underline{h}(x) = p^{-1}f(x)1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.47$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.5$
$p = 0.5$
$\bar{h}(x) = p^{-1}f(x)1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 0.88$

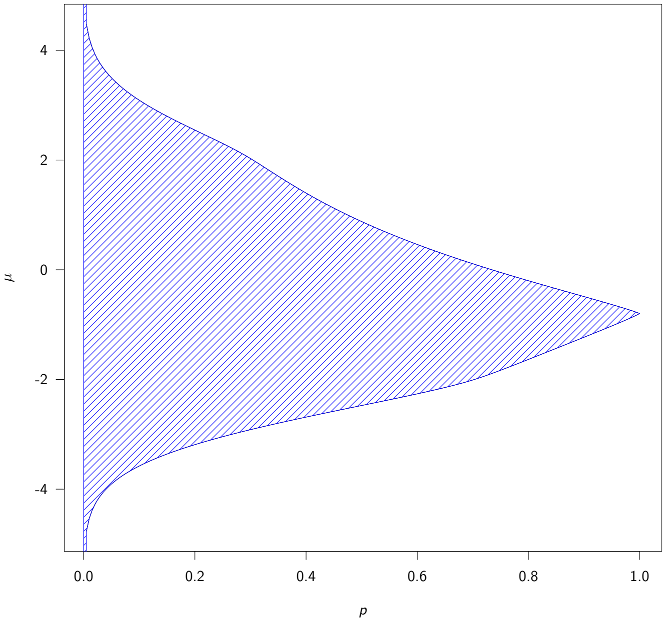

$\underline{h}(x) = p^{-1}f(x)1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.26$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.6$
$p = 0.6$
$\bar{h}(x) = p^{-1}f(x)1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 0.46$


$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -2.01$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.7$
$p = 0.7$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = 0.11$
0.4
0.3
0.2
0.1
0.0
0.25
0.20
0.15
0.10
0.05
0.00
-4
-2
0
2
4



$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -1.63$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.8$
$p = 0.8$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = -0.2$


$\underline{h}(x) = p^{-1} f(x) 1\{x < F^{-1}(p)\}$
$\underline{\mu} = -1.23$
$f(x) = 0.7\phi(x+2) + 0.3\phi(x-2)$
$p = 0.9$
$p = 0.9$
$\bar{h}(x) = p^{-1} f(x) 1\{x > F^{-1}(1-p)\}$
$\bar{\mu} = -0.49$


4
2
0
-2
-4
μ
0.0
0.2
0.4
0.6
0.8
1.0
p

Restrictions from Non-differential Mis-classification

Necessary and Sufficient Condition if F is Continuous

$$\int_{-\infty}^{F^{-1}(p)} \frac{x}{p}\, f(x)\, dx \leq \ \mu \ \leq \int_{F^{-1}(1-p)}^{+\infty} \frac{x}{p}\, f(x)\, dx$$

Back to Our Original Problem

- Observe F_{tk} for all (t, k)
- r_{tk} pinned down by (α_0, α_1)
- Can we find $F_{tk}^{t^*}$ so that $F_{tk} = (1 - r_{tk})F_{tk}^0 + r_{tk}F_{tk}^1$?
- Non-diff. assumption $\Rightarrow$ mean of F_{tk}^1 pinned down by (α_0, α_1).
- Implies joint restrictions on (α_0, α_1), hence β.

Sharp Identified Set under Baseline Assumptions

Theorem

Under baseline assumptions, sharp identified set for $\beta(\mathbf{x})$ is never a singleton, regardless of how many (discrete) values z takes on.

Intuition

No mis-classification $\Rightarrow r_{tk} = 0$ or 1 and we can always form a valid mixture in this case. Show that Wald estimand always lies within the sharp identified set for β.

Point Identification: 1st Ingredient

Reparameterization

$$\theta_1(\mathbf{x}) = \beta(\mathbf{x}) / \left[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]$$

$$\theta_2(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^2 \left[1 + \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]$$

$$\theta_3(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^3 \left[\left\{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right\}^2 + 6\alpha_0(\mathbf{x})\left\{1 - \alpha_1(\mathbf{x})\right\}\right]$$

$$\boxed{\beta(\mathbf{x}) = 0 \iff \theta_1(\mathbf{x}) = \theta_2(\mathbf{x}) = \theta_3(\mathbf{x}) = 0}$$

Lemma

Baseline Assumptions $\implies$ $\text{Cov}(y, z|\mathbf{x}) = \theta_1(\mathbf{x})\text{Cov}(z, T|\mathbf{x})$.

Point Identification: 2nd Ingredient

Assumption (II)

$\mathbb{E}[\varepsilon^2|\mathbf{x}, z] = \mathbb{E}[\varepsilon^2|\mathbf{x}]$

Lemma

(Baseline) + (II) $\implies$

$$\text{Cov}(y^2, z|\mathbf{x}) = 2\text{Cov}(yT, z|\mathbf{x})\theta_1(\mathbf{x}) - \text{Cov}(T, z|\mathbf{x})\theta_2(\mathbf{x})$$

Corollary

(Baseline) + (II) + $[\beta(\mathbf{x}) \neq 0] \implies [\alpha_1(\mathbf{x}) - \alpha_0(\mathbf{x})]$ is identified.

Hence, $\beta(\mathbf{x})$ is identified if mis-classification is one-sided.

Point Identification: 3rd Ingredient

Assumption (III)

(i) $\mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*]$

(ii) $\mathbb{E}[\varepsilon^3|\mathbf{x}, z] = \mathbb{E}[\varepsilon^3|\mathbf{x}]$

Lemma

(Baseline) + (II) + (III) $\implies$

$$\text{Cov}(y^3, z|\mathbf{x}) = 3\text{Cov}(y^2 T, z|\mathbf{x})\theta_1(\mathbf{x}) - 3\text{Cov}(yT, z|\mathbf{x})\theta_2(\mathbf{x}) + \text{Cov}(T, z|\mathbf{x})\theta_3(\mathbf{x})$$

Point Identification Result

Theorem

(Baseline) + (II) + (III) $\implies \beta(\mathbf{x})$ is point identified. If $\beta(\mathbf{x}) \neq 0$, then $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are likewise point identified.

Proof Sketch

1. $\beta(\mathbf{x}) = 0 \iff \theta_1(\mathbf{x}) = 0$ so suppose this is not the case.
2. Lemmas: full-rank linear system in $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})$ & observables.
3. Non-linear eqs. relating $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})$ to $\beta(\mathbf{x})$ and $\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x})$. Show that solution exists and is unique.

Sufficient Conditions for (II) and (III)

(i) T is conditionally independent of (ε, z) given $(T^*, \mathbf{x})$

(ii) z is conditionally independent of ε given $\mathbf{x}$

Just-Identified System of Moment Equalities

Suppress dependence on **x** to simplify the notation from here on. . .

Collect Lemmas from Above:

$$
\begin{aligned}
\mathrm{Cov}(y,z) - \mathrm{Cov}(T,z)\theta_1 &= 0 \\
\mathrm{Cov}(y^2,z) - 2\mathrm{Cov}(yT,z)\theta_1 + \mathrm{Cov}(T,z)\theta_2 &= 0 \\
\mathrm{Cov}(y^3,z) - 3\mathrm{Cov}(y^2T,z)\theta_1 + 3\mathrm{Cov}(yT,z)\theta_2 - \mathrm{Cov}(T,z)\theta_3 &= 0
\end{aligned}
$$

Notation: Observed Data Vector

$$
\mathbf{w}_i' = (T_i, y_i, y_i T_i, y_i^2, y_i^2 T_i, y_i^3)
$$

Just-Identified System of Moment Equalities

$$\boxed{\mathbb{E}\left[\left(\boldsymbol{\Psi}'(\boldsymbol{\theta})\mathbf{w}_i - \boldsymbol{\kappa}\right) \otimes \begin{pmatrix} 1 \\ z_i \end{pmatrix}\right] = \mathbf{0}}$$

$$\boldsymbol{\Psi} = \left[\begin{array}{ccc} \psi_1 & \psi_2 & \psi_3 \end{array}\right] \qquad \boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)' \equiv \text{ ``Intercepts''}$$

$$\psi_1' = \left[\begin{array}{cccccc} -\theta_1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \theta_1 = \beta/(1-\alpha_0-\alpha_1)$$

$$\psi_2' = \left[\begin{array}{cccccc} \theta_2 & 0 & -2\theta_1 & 1 & 0 & 0 \end{array}\right] \qquad \theta_2 = \theta_1^2\left[1+\alpha_0-\alpha_1\right]$$

$$\psi_3' = \left[\begin{array}{cccccc} -\theta_3 & 0 & 3\theta_2 & 0 & -3\theta_1 & 1 \end{array}\right] \qquad \theta_3 = \theta_1^3\left[(1-\alpha_0-\alpha_1)^2 + 6\alpha_0(1-\alpha_1)\right]$$

Weak Identification Problem

Moment are conditions uninformative about (α_0, α_1) when β is small: GMM performs very badly. sims

Inference for a Mis-classified Regressor

The Problem

- β small $\Rightarrow$ moment equalities uninformative about (α_0, α_1) more
- (α_0, α_1) could be on the boundary of the parameter space
- Also true of existing estimators that assume T^* exogenous

Our Solution

- Sharp identified set result from above remains informative even if β is small or zero.
- Implies a number of *inequality* moment conditions
- Identification-robust inference combining equality and inequality moment conditions based on generalized moment selection (GMS)

Moment Inequalities I – First-stage Probabilities

$\alpha_0 \leq p_k \leq 1 - \alpha_1$ becomes $\mathbb{E}\left[m^I_{1k}(\mathbf{w}_i, \vartheta)\right] \geq \mathbf{0}$ for all k where

$$m^I_{1k}(\mathbf{w}_i, \vartheta) \equiv \begin{bmatrix} \mathbf{1}(z_i = k)(T - \alpha_0) \\ \mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \end{bmatrix}$$

Moment Inequalities II – Non-differential Assumption

For all k, we have $\mathbb{E}[m^l_{2k}\left(\mathbf{w}_i, \vartheta, \mathbf{q}_k\right)] \geq 0$ where

$$m^l_{2k}\left(\mathbf{w}_i, \vartheta, \mathbf{q}_k\right) \equiv \left[\begin{array}{r} y_i \mathbf{1}\left(z_i = k\right)\left\{\left(T_i - \alpha_0\right) - \mathbf{1}(y_i \leq \underline{q}_{0k})(1 - T_i)\left(\frac{1-\alpha_0-\alpha_1}{\alpha_1}\right)\right\} \\ -y_i \mathbf{1}(z_i = k)\left\{\left(T_i - \alpha_0\right) - \mathbf{1}(y_i > \overline{q}_{0k})(1 - T_i)\left(\frac{1-\alpha_0-\alpha_1}{\alpha_1}\right)\right\} \\ y_i \mathbf{1}\left(z_i = k\right)\left\{\left(T_i - \alpha_0\right) - \mathbf{1}(y_i \leq \underline{q}_{1k}) T_i \left(\frac{1-\alpha_0-\alpha_1}{1-\alpha_1}\right)\right\} \\ -y_i \mathbf{1}(z_i = k)\left\{\left(T_i - \alpha_0\right) - \mathbf{1}(y_i > \overline{q}_{1k}) T_i \left(\frac{1-\alpha_0-\alpha_1}{1-\alpha_1}\right)\right\} \end{array}\right]$$

and $\mathbf{q}_k \equiv (\underline{q}_{0k}, \overline{q}_{0k}, \underline{q}_{1k}, \overline{q}_{1k})'$ defined by $\mathbb{E}[h^l_k(\mathbf{w}_i, \vartheta, \mathbf{q}_k)] = 0$ with

$$h^l_k(\mathbf{w}_i, \vartheta, \mathbf{q}_k) = \left[\begin{array}{l} \mathbf{1}(y_i \leq \underline{q}_{0k})\mathbf{1}(z_i = k)(1 - T_i) - \left(\frac{\alpha_1}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(T_i - \alpha_0) \\ \mathbf{1}(y_i \leq \overline{q}_{0k})\mathbf{1}(z_i = k)(1 - T_i) - \left(\frac{1-\alpha_0}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \\ \mathbf{1}(y_i \leq \underline{q}_{1k})\mathbf{1}(z_i = k) T_i - \left(\frac{1-\alpha_1}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(T_i - \alpha_0) \\ \mathbf{1}(y_i \leq \overline{q}_{1k})\mathbf{1}(z_i = k) T_i - \left(\frac{\alpha_0}{1-\alpha_0-\alpha_1}\right)\mathbf{1}(z_i = k)(1 - T_i - \alpha_1) \end{array}\right]$$

Inference with Moment Equalities and Inequalities

Moment Conditions

$\mathbb{E}\left[m_j(\mathbf{w}_i, \vartheta_0)\right] \geq 0, \quad j = 1, \cdots, J$

$\mathbb{E}\left[m_j(\mathbf{w}_i, \vartheta_0)\right] = 0, \quad j = J+1, \cdots, J+K$

Test Statistic

$$T_n(\vartheta) = \sum_{j=1}^{J} \left[\frac{\sqrt{n}\ \bar{m}_{n,j}(\vartheta)}{\widehat{\sigma}_{n,j}(\vartheta)}\right]_{-}^{2} + \sum_{j=J+1}^{J+K} \left[\frac{\sqrt{n}\ \bar{m}_{n,j}(\vartheta)}{\widehat{\sigma}_{n,j}(\vartheta)}\right]^{2}$$

Critical Value

- $\sqrt{n}\,\bar{m}_n(\vartheta_0) \to_d$ normal limit with covariance matrix $\Sigma(\vartheta_0)$
- Use this to bootstrap the limit dist. of $T_n(\vartheta)$ under $H_0\colon \vartheta = \vartheta_0$

Inference with Moment Equalities and Inequalities

Generalized Moment Selection – Andrews & Soares (2010)

- Inequalities that don't bind reduce power of test, so eliminate those that are "far from binding" before calculating critical value.
- Drop inequality j if $\dfrac{\sqrt{n}\,\bar{m}_{n,j}(\vartheta_0)}{\widehat{\sigma}_{n,j}(\vartheta_0)} > \sqrt{\log n}$
- Uniformly valid test of $H_0\colon \vartheta = \vartheta_0$ even if ϑ_0 is not point identified.
- Not asymptotically conservative.

Problem

Joint test for the whole parameter vector but we're only interested in β.

Projection is conservative and computationally intensive.

Our Solution: Bonferroni-Based Inference

Leverage Special Structure of Model

- β only enters MCs through $\theta_1 = \beta/(1-\alpha_0-\alpha_1)$
- If z is a strong instrument, inference for θ_1 is standard.
- $(\boldsymbol{\kappa}, \mathbf{q})$ strongly identified under null for (α_0, α_1)

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Procedure

1. Concentrate out $(\theta_1, \boldsymbol{\kappa}, \boldsymbol{q}) \Rightarrow$ joint GMS test for (α_0, α_1)

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Procedure

1. Concentrate out $(\theta_1, \boldsymbol{\kappa}, \boldsymbol{q}) \Rightarrow$ joint GMS test for (α_0, α_1)
2. Invert test $\Rightarrow (1 - \delta_1) \times 100\%$ confidence set for (α_0, α_1)

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3. Project $\Rightarrow$ CI for $(1 - \alpha_0 - \alpha_1)$
4. Construct standard $(1 - \delta_2) \times 100\%$ IV CI for θ_1

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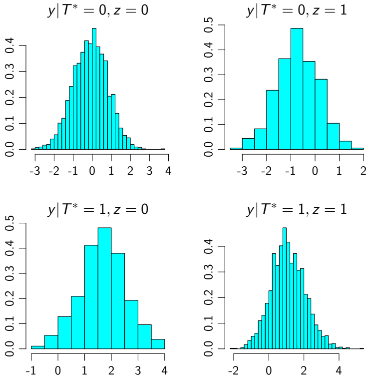
2. Invert test $\Rightarrow (1 - \delta_1) \times 100\%$ confidence set for (α_0, α_1)
3. Project $\Rightarrow$ CI for $(1 - \alpha_0 - \alpha_1)$
4. Construct standard $(1 - \delta_2) \times 100\%$ IV CI for θ_1
5. Bonferroni $\Rightarrow (1 - \delta - \delta_2) \times 100\%$ CI for β

Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

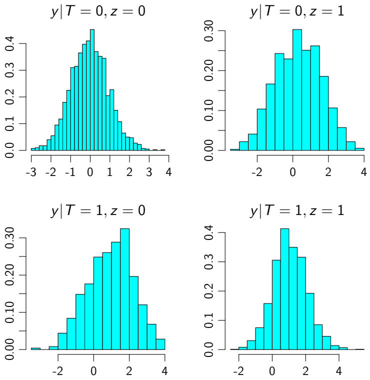
Results if T^* were observed



$$\widehat{\beta}_{IV} = 0.96, \quad 95\% \text{ CI } = (0.88, 1.04)$$

Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

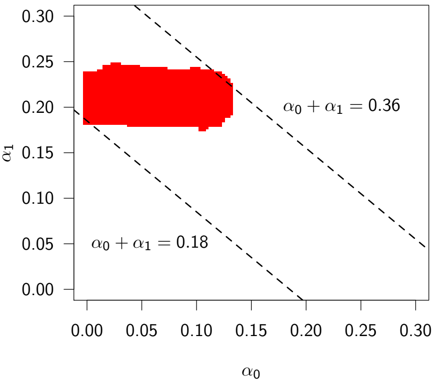
Results using T instead of T^*



$$\widehat{\beta}_{IV} = 1.34, \quad 95\% \text{ CI } = (1.22, 1.45)$$

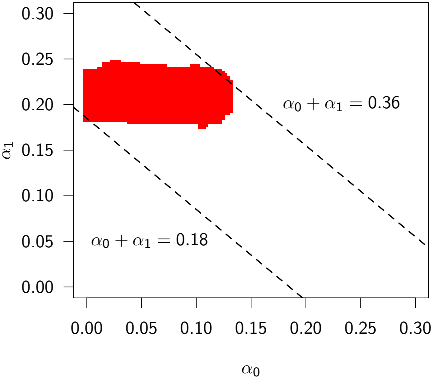
Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

97.5% GMS Confidence Region for (α_0, α_1)



Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

97.5% GMS Confidence Region for (α_0, α_1)

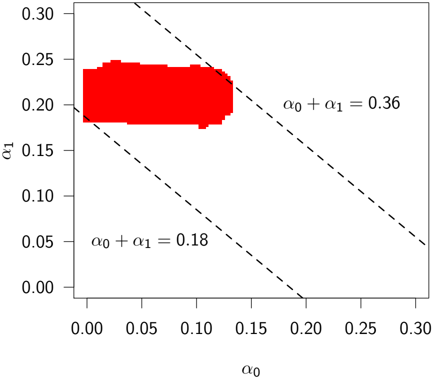


Bonferroni Interval

1. 97.5% CI for $(1 - \alpha_0 - \alpha_1) = (0.64, 0.82)$

Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

97.5% GMS Confidence Region for (α_0, α_1)

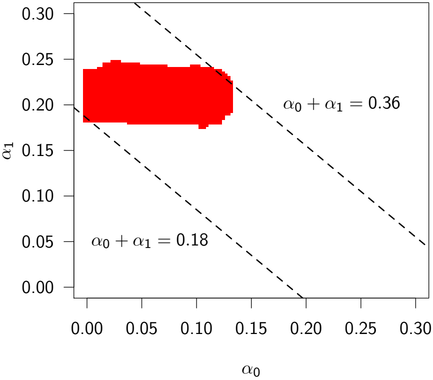


Bonferroni Interval

1. 97.5% CI for $(1 - \alpha_0 - \alpha_1) = (0.64, 0.82)$
2. 97.5% CI for $\theta_1 = (1.20, 1.47)$

Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

97.5% GMS Confidence Region for (α_0, α_1)

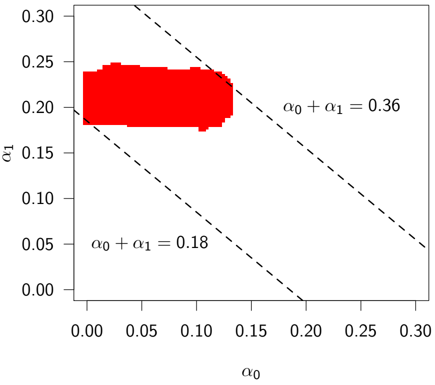


Bonferroni Interval

1. 97.5% CI for $(1 - \alpha_0 - \alpha_1) = (0.64, 0.82)$
2. 97.5% CI for $\theta_1 = (1.20, 1.47)$
3. $>$ 95% CI for β:
   $(0.64 \times 1.20, 0.82 \times 1.47) = (0.77, 1.21)$

Simulation Example $(\beta = 1, \alpha_0 = 0.1, \alpha_1 = 0.2, n = 5000)$

97.5% GMS Confidence Region for (α_0, α_1)



Bonferroni Interval

1. 97.5% CI for $(1 - \alpha_0 - \alpha_1) = (0.64, 0.82)$
2. 97.5% CI for $\theta_1 = (1.20, 1.47)$
3. $>$ 95% CI for β: $(0.64 \times 1.20, 0.82 \times 1.47) = (0.77, 1.21)$

Comparisons

- (0.88, 1.04) for IV if T^* were observed
- (1.22,1.45) for naive IV interval using T

Conclusion

1. Identification and inference for effect of binary, mis-classified, endogenous regressor.
2. Show that only existing point identification result is incorrect.
3. Derive sharp identified set for $\beta(\mathbf{x})$ under standard assumptions.
4. Prove point identification of $\beta(\mathbf{x})$ under slightly stronger assumptions.
5. Point out problem of weak identification in mis-classification models, develop identification-robust inference for $\beta(\mathbf{x})$.

Related Past and Current Research on Measurement Error

DiTraglia & Garcia–Jimeno (2017b)

What if z is invalid? Bayesian sensitivity analysis using joint restrictions on measurement error, regressor endogeneity, and instrument invalidity.

DiTraglia & Lewbel (in progress)

What if the surrogate T can directly effect y? E.g. higher wage by *claiming* to have college degree. Estimate the "returns to lying."

DiTraglia & Garcia-Jimeno (in progress)

Structural model of forced migration and de facto land reform during the Columbian civil conflict (1990s–2000s). How to exploit multiple biased measures of migration?

Full Independence $\implies$ Continuum of Moment Equalities

Suppose that

(i) T is conditionally independent of (ε, z) given $(T^*, \mathbf{x})$

(ii) z is conditionally independent of ε given $\mathbf{x}$

Then for all τ we have

$$\widetilde{\Delta}_1(\tau+\beta) - \widetilde{\Delta}_1(\tau) = \alpha_0 \Delta(\tau+\beta) - (1-\alpha_1)\Delta(\tau)$$

$$\Delta(\tau) = F_k(\tau) - F_\ell(\tau)$$

$$\widetilde{\Delta}_1(\tau) = p_k F_{1k}(\tau) - p_\ell F_{1\ell}(\tau)$$

But if $\beta = 0$ this reduces to $F_k(\tau) - F_\ell(\tau) = 0$

back

Simulation DGP: $y = \beta T^* + \varepsilon$

Sample Size = 1000; Simulation Replications = 2000

Errors

$(\varepsilon, \eta) \sim$ jointly normal, mean 0, variance 1, correlation 0.5.

First-Stage

- Half of observations have $z = 1$, the rest have $z = 0$.
- $T^* = \mathbf{1}\{\gamma_0 + \gamma_1 z + \eta > 0\}$
- $\mathbb{P}(T^* = 0|z = 1) = \mathbb{P}(T^* = 1|z = 0) = 0.15$

Mis-classification

- $T|T^* = 0 \sim \text{Bernoulli}(\alpha_0)$
- $T|T^* = 1 \sim \text{Bernoulli}(1 - \alpha_1)$

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	27	33	30	14	1	0	0	0
	0.1	27	32	29	13	2	0	0	0
	0.2	26	33	32	15	4	0	0	0
	0.3	26	34	30	17	5	0	0	0
0.1	0.0	26	32	31	14	2	0	0	0
	0.1	26	36	32	16	4	0	0	0
	0.2	27	35	31	18	8	0	0	0
	0.3	25	35	32	21	11	1	0	0
0.2	0.0	26	33	30	15	3	0	0	0
	0.1	26	33	30	19	6	0	0	0
	0.2	26	35	33	22	12	1	0	0
	0.3	26	35	33	26	15	3	0	0
0.3	0.0	26	32	32	16	6	0	0	0
	0.1	24	35	33	21	11	1	0	0
	0.2	26	32	35	27	15	4	0	0
	0.3	26	35	35	28	21	7	2	0

Table: Percentage of simulation replications for which the standard GMM CI fails to exist.

α_0	α_1	β = 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	72	62	62	80	92	95	94	95
	0.1	72	62	63	79	92	95	96	95
	0.2	73	61	61	77	90	96	96	96
	0.3	73	59	62	76	88	95	96	95
0.1	0.0	73	63	60	78	91	95	96	96
	0.1	73	58	59	77	90	95	95	94
	0.2	73	59	61	75	86	95	95	94
	0.3	74	59	58	71	82	94	96	96
0.2	0.0	74	62	60	78	91	95	96	96
	0.1	73	60	61	74	87	95	96	94
	0.2	73	58	57	70	81	93	95	95
	0.3	73	58	56	66	78	92	95	96
0.3	0.0	74	62	60	76	89	95	96	96
	0.1	75	59	58	71	82	93	96	95
	0.2	74	61	56	65	78	90	96	96
	0.3	73	58	55	64	71	88	93	96

Table: Coverge of nominal 95% GMM CI, conditional on existence.

α_0	α_1	β = 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	19.07	3.44	1.86	1.32	0.87	0.47	0.37	0.35
	0.1	17.52	3.47	1.92	1.41	1	0.61	0.51	0.46
	0.2	17.41	3.51	1.9	1.45	1.1	0.76	0.65	0.58
	0.3	18.23	3.34	1.92	1.48	1.24	0.91	0.79	0.7
0.1	0.0	17.13	3.51	1.86	1.38	0.97	0.61	0.51	0.46
	0.1	17.88	3.33	1.85	1.45	1.13	0.78	0.67	0.6
	0.2	17.37	3.36	1.95	1.54	1.24	0.97	0.85	0.75
	0.3	18.07	3.33	1.98	1.63	1.41	1.17	1.04	0.92
0.2	0.0	17.79	3.39	1.92	1.45	1.11	0.75	0.65	0.58
	0.1	18.98	3.43	1.96	1.54	1.26	0.97	0.84	0.75
	0.2	18.25	3.26	1.92	1.64	1.45	1.2	1.06	0.95
	0.3	19.03	3.31	2.02	1.75	1.66	1.49	1.33	1.19
0.3	0.0	18.27	3.48	1.87	1.5	1.25	0.9	0.79	0.7
	0.1	19.4	3.41	1.96	1.63	1.43	1.18	1.04	0.92
	0.2	18.22	3.56	1.96	1.74	1.67	1.49	1.35	1.19
	0.3	17.56	3.55	2.13	1.96	1.86	1.86	1.74	1.55

Table: Median width of nominal 95% GMM CI, conditional on existence.

back

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	97.7	97.7	97.6	97.7	98.0	98.0	97.4	97.9
	0.1	98.0	98.7	98.8	99.1	98.8	98.4	97.1	96.4
	0.2	98.4	98.5	98.9	98.9	98.8	98.6	98.0	97.0
	0.3	98.5	98.8	98.8	99.0	98.7	98.4	97.8	97.5
0.1	0.0	98.1	98.5	98.3	98.8	98.8	98.4	96.8	95.7
	0.1	98.6	99.1	99.5	99.6	99.6	98.8	97.7	95.2
	0.2	99.0	99.3	99.7	99.8	99.7	98.9	97.5	95.7
	0.3	99.4	99.7	99.8	99.8	99.6	99.0	98.2	96.7
0.2	0.0	98.6	98.5	98.6	98.9	98.7	98.2	97.7	97.0
	0.1	99.0	99.5	99.7	99.7	99.4	99.0	98.1	96.5
	0.2	99.5	99.7	99.8	99.7	99.4	99.0	97.8	96.8
	0.3	99.7	99.8	99.8	99.8	99.5	99.0	98.7	97.7
0.3	0.0	98.7	98.7	98.8	98.7	98.7	98.2	98.1	97.6
	0.1	99.4	99.6	99.6	99.7	99.4	98.9	98.3	96.8
	0.2	99.8	99.8	99.7	99.8	99.5	99.1	98.5	97.8
	0.3	100.0	99.9	99.9	99.8	99.6	99.5	99.1	98.8

Table: Coverage (1 - size) of nominal 97.5% GMS joint test for (α_0, α_1).

α_0	α_1	β = 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	96	97	97	96	97	97	95	96
	0.1	97	99	99	99	99	100	100	99
	0.2	98	99	99	100	100	100	100	100
	0.3	97	100	100	100	100	100	100	100
0.1	0.0	97	99	99	99	100	100	100	98
	0.1	98	100	100	100	100	100	100	100
	0.2	98	100	100	100	100	100	100	100
	0.3	97	100	100	100	100	100	100	100
0.2	0.0	97	99	99	100	100	100	100	100
	0.1	98	100	100	100	100	100	100	100
	0.2	98	100	100	100	100	100	100	100
	0.3	98	100	100	100	100	100	100	100
0.3	0.0	97	99	100	100	100	100	100	100
	0.1	97	100	100	100	100	100	100	100
	0.2	98	100	100	100	100	100	100	100
	0.3	98	100	100	100	100	100	100	100

Table: Coverage of nominal $> 95\%$ Bonferroni CI for β

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	0.4	0.41	0.43	0.43	0.43	0.42	0.41	0.41
	0.1	0.45	0.47	0.54	0.59	0.63	0.7	0.75	0.86
	0.2	0.51	0.54	0.65	0.76	0.85	0.95	1.01	1.17
	0.3	0.58	0.62	0.79	0.95	1.07	1.17	1.24	1.48
0.1	0.0	0.45	0.47	0.54	0.59	0.63	0.7	0.76	0.88
	0.1	0.51	0.54	0.66	0.77	0.86	1.03	1.18	1.46
	0.2	0.58	0.63	0.8	0.98	1.12	1.38	1.55	1.88
	0.3	0.67	0.75	1	1.25	1.46	1.74	1.94	2.4
0.2	0.0	0.51	0.54	0.65	0.76	0.86	0.96	1.02	1.19
	0.1	0.58	0.63	0.81	0.99	1.14	1.42	1.64	2.08
	0.2	0.67	0.75	1.01	1.29	1.54	1.97	2.33	2.9
	0.3	0.81	0.91	1.3	1.7	2.09	2.73	3.13	3.9
0.3	0.0	0.58	0.62	0.8	0.95	1.09	1.18	1.25	1.5
	0.1	0.68	0.74	1.01	1.26	1.49	1.84	2.13	2.78
	0.2	0.81	0.91	1.3	1.7	2.11	2.8	3.4	4.48
	0.3	1.01	1.16	1.74	2.35	2.93	4.17	5.2	6.85

Table: Median width of nominal $>95\%$ Bonferroni CI for β.

α_0	α_1	β = 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	0.4	0.41	0.43	0.43	0.43	0.42	0.41	0.41
	0.1	0.45	0.47	0.54	0.59	0.63	0.7	0.75	0.86
	0.2	0.51	0.54	0.65	0.76	0.85	0.95	1.01	1.17
	0.3	0.58	0.62	0.79	0.95	1.07	1.17	1.24	1.48
0.1	0.0	0.45	0.47	0.54	0.59	0.63	0.7	0.76	0.88
	0.1	0.51	0.54	0.66	0.77	0.86	1.03	1.18	1.46
	0.2	0.58	0.63	0.8	0.98	1.12	1.38	1.55	1.88
	0.3	0.67	0.75	1	1.25	1.46	1.74	1.94	2.4
0.2	0.0	0.51	0.54	0.65	0.76	0.86	0.96	1.02	1.19
	0.1	0.58	0.63	0.81	0.99	1.14	1.42	1.64	2.08
	0.2	0.67	0.75	1.01	1.29	1.54	1.97	2.33	2.9
	0.3	0.81	0.91	1.3	1.7	2.09	2.73	3.13	3.9
0.3	0.0	0.58	0.62	0.8	0.95	1.09	1.18	1.25	1.5
	0.1	0.68	0.74	1.01	1.26	1.49	1.84	2.13	2.78
	0.2	0.81	0.91	1.3	1.7	2.11	2.8	3.4	4.48
	0.3	1.01	1.16	1.74	2.35	2.93	4.17	5.2	6.85

Table: Median width of nominal $> 95\%$ Bonferroni CI for β.

α_0	α_1	β 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	19.07	3.44	1.86	1.32	0.87	0.47	0.37	0.35
	0.1	17.52	3.47	1.92	1.41	1	0.61	0.51	0.46
	0.2	17.41	3.51	1.9	1.45	1.1	0.76	0.65	0.58
	0.3	18.23	3.34	1.92	1.48	1.24	0.91	0.79	0.7
0.1	0.0	17.13	3.51	1.86	1.38	0.97	0.61	0.51	0.46
	0.1	17.88	3.33	1.85	1.45	1.13	0.78	0.67	0.6
	0.2	17.37	3.36	1.95	1.54	1.24	0.97	0.85	0.75
	0.3	18.07	3.33	1.98	1.63	1.41	1.17	1.04	0.92
0.2	0.0	17.79	3.39	1.92	1.45	1.11	0.75	0.65	0.58
	0.1	18.98	3.43	1.96	1.54	1.26	0.97	0.84	0.75
	0.2	18.25	3.26	1.92	1.64	1.45	1.2	1.06	0.95
	0.3	19.03	3.31	2.02	1.75	1.66	1.49	1.33	1.19
0.3	0.0	18.27	3.48	1.87	1.5	1.25	0.9	0.79	0.7
	0.1	19.4	3.41	1.96	1.63	1.43	1.18	1.04	0.92
	0.2	18.22	3.56	1.96	1.74	1.67	1.49	1.35	1.19
	0.3	17.56	3.55	2.13	1.96	1.86	1.86	1.74	1.55

Table: Median width of nominal 95% GMM CI, conditional on existence.

		β							
α_0	α_1	0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	96	97	97	96	97	97	95	93
	0.1	97	99	99	99	99	98	96	95
	0.2	98	99	99	100	100	97	96	96
	0.3	97	100	100	100	99	96	96	96
0.1	0.0	97	99	99	99	100	98	97	95
	0.1	98	100	100	100	100	96	96	96
	0.2	98	100	100	100	99	96	96	95
	0.3	97	100	100	100	97	95	96	96
0.2	0.0	97	99	99	100	100	96	96	96
	0.1	98	100	100	100	99	96	96	96
	0.2	98	100	100	100	96	95	95	96
	0.3	98	100	100	98	95	95	95	96
0.3	0.0	97	99	100	100	100	95	96	97
	0.1	97	100	100	100	97	94	96	96
	0.2	98	100	100	98	94	94	96	96
	0.3	98	100	99	96	92	94	95	96

Table: Coverage of hybrid CI constructed from nominal 95% GMM and $>95\%$ Bonferroni intervals.

α_0	α_1	β 0	0.25	0.5	0.75	1	1.5	2	3
0.0	0.0	0.4	0.41	0.43	0.43	0.43	0.42	0.4	0.35
	0.1	0.45	0.47	0.54	0.59	0.63	0.67	0.52	0.46
	0.2	0.51	0.54	0.65	0.76	0.84	0.82	0.65	0.58
	0.3	0.58	0.62	0.79	0.95	1.05	0.96	0.79	0.7
0.1	0.0	0.45	0.47	0.54	0.59	0.63	0.67	0.51	0.46
	0.1	0.51	0.54	0.66	0.77	0.86	0.92	0.69	0.61
	0.2	0.58	0.63	0.8	0.97	1.11	1.17	0.87	0.75
	0.3	0.67	0.75	1	1.25	1.4	1.4	1.06	0.92
0.2	0.0	0.51	0.54	0.65	0.76	0.85	0.83	0.65	0.58
	0.1	0.58	0.63	0.81	0.99	1.12	1.18	0.86	0.75
	0.2	0.67	0.75	1.01	1.29	1.48	1.56	1.08	0.95
	0.3	0.81	0.91	1.3	1.67	1.95	1.77	1.35	1.2
0.3	0.0	0.58	0.62	0.8	0.95	1.07	0.95	0.8	0.7
	0.1	0.68	0.74	1.01	1.26	1.43	1.48	1.06	0.93
	0.2	0.81	0.91	1.3	1.66	1.98	1.94	1.37	1.19
	0.3	1.01	1.16	1.73	2.24	2.71	2.33	1.78	1.55

Table: Median width of hybrid CI constructed from nominal 95% GMM and $>$ 95% Bonferroni intervals.

Figure: Coverage of hybrid vs. $> 95\%$ Bonferroni CIs: $\beta = 1$

Figure: Coverage of hybrid vs. $> 95\%$ Bonferroni CIs: $\beta = 2$

Figure: Coverage of hybrid vs. $> 95\%$ Bonferroni CIs: $\beta = 3$