Notes for Paper on Mis-measured, Binary, Endogenous Regressors

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September 21, 2016

1 Model and Notation

Probabilities

$$
\begin{aligned}
p^*_{tk} &= P(T^* = t, Z = k) \\
p_{tk} &= P(T = t, Z = k) \\
p^*_k &= P(T^* = 1|Z = k) \\
p_k &= P(T = 1|Z = k) \\
q &= P(Z = 1)
\end{aligned}
$$

$$
\begin{aligned}
p^*_{00} &= P(T^* = 0|Z = 0)P(Z = 0) = (1 - p^*_0)(1 - q) = \left(\frac{1 - p_0 - \alpha_1}{1 - \alpha_0 - \alpha_1}\right)(1 - q) \\
p^*_{10} &= P(T^* = 1|Z = 0)P(Z = 0) = p^*_0(1 - q) = \left(\frac{p_0 - \alpha_0}{1 - \alpha_0 - \alpha_1}\right)(1 - q) \\
p^*_{01} &= P(T^* = 0|Z = 1)P(Z = 1) = (1 - p^*_1)q = \left(\frac{1 - p_1 - \alpha_1}{1 - \alpha_0 - \alpha_1}\right)q \\
p^*_{11} &= P(T^* = 1|Z = 1)P(Z = 1) = p^*_1(1 - q) = \left(\frac{p_1 - \alpha_0}{1 - \alpha_0 - \alpha_1}\right)q
\end{aligned}
$$

CDFs For $t, Z \in \{0, 1\}$ define

$$
\begin{aligned}
F^*_{tk}(\tau) &= P(Y \leq \tau|T^* = t, Z = k) \\
F_{tk}(\tau) &= P(Y \leq \tau|T = t, Z = k) \\
F_k(\tau) &= P(Y \leq \tau|Z = k)
\end{aligned}
$$

Note that the second two are observed for all t, k while the first is never observed since it depends on the unobserved RV T^*.

2 Weakest Bounds on α_0, α_1

Assume that $\alpha_0 + \alpha_1 < 1$ that T is independent of Z conditional on T^*. These standard assumptions turn out to yield informative bounds on α_0 and α_1 without *any further restrictions of any kind*. In particular, we assume nothing about the validity of the instrument Z and nothing about the relationship between the mis-classification error and the outcome Y: we impose only that the mis-classification error rates do not depend on z and that the mis-classification is not so bad that $1 - T$ is a better measure of T^* than T.

By the Law of Total Probability and the assumption that T is conditionally independent of Z given T^*,

$$\begin{aligned}
p_k &= P(T=1|Z=k, T^*=0)(1-p_k^*) + P(T=1|Z=k, T^*=1)p_k^* \\
&= P(T=1|T^*=0)(1-p_k^*) + P(T=1|T^*=1)p_k^* \\
&= \alpha_0(1-p_k^*) + (1-\alpha_1)p_k^* \\
&= \alpha_0 + (1-\alpha_0-\alpha_1)p_k^*
\end{aligned}$$

and similarly

$$\begin{aligned}
1-p_k &= P(T=0|Z=k, T^*=0)(1-p_k^*) + P(T=0|Z=k, T^*=1)p_k^* \\
&= P(T=0|T^*=0)(1-p_k^*) + P(T=0|T^*=1)p_k^* \\
&= (1-\alpha_0)(1-p_k^*) + \alpha_1 p_k^* \\
&= \alpha_1 + (1-p_k^*)(1-\alpha_0-\alpha_1)
\end{aligned}$$

and hence

$$\begin{aligned}
p_k - \alpha_0 &= (1-\alpha_0-\alpha_1)p_k^* \\
(1-p_k) - \alpha_1 &= (1-\alpha_0-\alpha_1)(1-p_k^*)
\end{aligned}$$

Now, since p_k^* and $(1-p_k^*)$ are probabilities they are between zero and one which means that the sign of $p_k - \alpha_0$ as well as that of $(1-p_k) - \alpha_1$ are both determined by that of $1-\alpha_0-\alpha_1$.

Accordingly, provided that $1 - \alpha_0 - \alpha_1 < 1$, we have

$$\begin{aligned} \alpha_0 &< p_k \\ \alpha_1 &< (1 - p_k) \end{aligned}$$

so long as p_k^* does not equal zero or one, which is not a realistic case for any example that we consider. Since these bounds hold for all k, we can take the tightest bound over all values of Z.

3 Stronger Bounds for α_0, α_1

Now suppose we add the assumption that T is conditionally independent of Y given T^*. This is essentially the non-differential measurement error assumption although it is slightly stronger than the version used by Mahajan (2006) who assumes only conditional mean independence. This assumption allows us to considerably strengthen the bounds from the preceding section by exploiting information contained in the conditional distribution of Y given T and Z. The key ingredient is a relationship that we can derive between the unobservable distributions F_{tk}^* and the observable distributions F_{tk} using this new conditional independence assumption. To begin, note that by Bayes' rule we have

$$\begin{aligned} P(T^* = 1|T = 1, Z = k) &= P(T = 1|T^* = 1)\left(\frac{p_k^*}{p_k}\right) = (1 - \alpha_1)\left(\frac{p_k^*}{p_k}\right) \\ P(T^* = 1|T = 0, Z = k) &= P(T = 0|T^* = 1)\left(\frac{p_k^*}{1 - p_k}\right) = \alpha_1\left(\frac{p_k^*}{1 - p_k}\right) \\ P(T^* = 0|T = 1, Z = k) &= P(T = 1|T^* = 0)\left(\frac{1 - p_k^*}{p_k}\right) = \alpha_0\left(\frac{1 - p_k^*}{p_k}\right) \\ P(T^* = 0|T = 0, Z = k) &= P(T = 0|T^* = 0)\left(\frac{1 - p_k^*}{1 - p_k}\right) = (1 - \alpha_0)\left(\frac{1 - p_k^*}{1 - p_k}\right) \end{aligned}$$

Now, by the conditional independence assumption

$$\begin{aligned} P(Y \leq \tau|T^* = 0, T = t, Z = k) = P(Y \leq \tau|T^* = 0, Z = k) = F_{0k}^*(\tau) \\ P(Y \leq \tau|T^* = 1, T = t, Z = k) = P(Y \leq \tau|T^* = 1, Z = k) = F_{1k}^*(\tau) \end{aligned}$$

Finally, putting everything together using the Law of Total Probability, we find that

$$\begin{aligned} (1 - p_k)F_{0k}(\tau) &= (1 - \alpha_0)(1 - p_k^*)F_{0k}^*(\tau) + \alpha_1 p_k^* F_{1k}^*(\tau) \\ p_k F_{1k}(\tau) &= \alpha_0(1 - p_k^*)F_{0k}^*(\tau) + (1 - \alpha_1)p_k^* F_{1k}^*(\tau) \end{aligned}$$

for all k. Defining the shorthand

$$\begin{aligned}\widetilde{F}_{0k}(\tau) &\equiv (1-p_k)F_{0k}(\tau)\\ \widetilde{F}_{1k}(\tau) &\equiv p_k F_{1k}(\tau)\end{aligned}$$

this becomes

$$\widetilde{F}_{0k}(\tau) = (1-\alpha_0)(1-p_k^*)F_{0k}^*(\tau) + \alpha_1 p_k^* F_{1k}^*(\tau) \tag{1}$$

$$\widetilde{F}_{1k}(\tau) = \alpha_0(1-p_k^*)F_{0k}^*(\tau) + (1-\alpha_1) p_k^* F_{1k}^*(\tau) \tag{2}$$

Now, solving Equation 1 for $p_k^* F_{1k}^*(\tau)$ we have

$$p_k^* F_{1k}^*(\tau) = \frac{1}{\alpha_1}\left[\widetilde{F}_{0k}(\tau) - (1-\alpha_0)(1-p_k^*)F_{0k}^*(\tau)\right]$$

Substituting this into Equation 2,

$$\begin{aligned}\widetilde{F}_{1k}(\tau) &= \alpha_0(1-p_k^*)F_{0k}^*(\tau) + \frac{1-\alpha_1}{\alpha_1}\left[\widetilde{F}_{0k}(\tau) - (1-\alpha_0)(1-p_k^*)F_{0k}^*(\tau)\right]\\ &= \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) + \left[\alpha_0 - \frac{(1-\alpha_1)(1-\alpha_0)}{\alpha_1}\right](1-p_k^*)F_{0k}^*(\tau)\\ &= \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) + \left[\frac{\alpha_0\alpha_1 - (1-\alpha_1)(1-\alpha_0)}{\alpha_1}\right](1-p_k^*)F_{0k}^*(\tau)\\ &= \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) - \left[\frac{(1-\alpha_1)(1-\alpha_0) - \alpha_0\alpha_1}{\alpha_1}\right](1-p_k^*)F_{0k}^*(\tau)\\ &= \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) - \left[\frac{1-\alpha_1-\alpha_0}{\alpha_1}\right]\left(\frac{1-p_k-\alpha_1}{1-\alpha_0-\alpha_1}\right)F_{0k}^*(\tau)\end{aligned}$$

and therefore

$$\widetilde{F}_{1k}(\tau) = \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) - \frac{1-p_k-\alpha_1}{\alpha_1}F_{0k}^*(\tau) \tag{3}$$

Equation 3 relates the observable $\widetilde{F}_{1k}(\tau)$ to the mis-classification error rate α_1 and the unobservable CDF $F_{0k}^*(\tau)$. Since $F_{0k}^*(\tau)$ is a CDF, however, it lies in the interval $[0,1]$. Accordingly, substituting 0 in place of $F_{0k}^*(\tau)$ gives

$$\widetilde{F}_{1k}(\tau) \leq \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) \tag{4}$$

while substituting 1 gives

$$\widetilde{F}_{1k}(\tau) \geq \frac{1-\alpha_1}{\alpha_1}\widetilde{F}_{0k}(\tau) - \frac{1-p_k-\alpha_1}{\alpha_1} \tag{5}$$

Rearranging Equation 4

$$\begin{aligned}
\alpha_1 \widetilde{F}_{1k}(\tau) &\leq (1-\alpha_1)\widetilde{F}_{0k}(\tau) \\
\alpha_1 \widetilde{F}_{1k}(\tau) &\leq \widetilde{F}_{0k}(\tau) - \alpha_1 \widetilde{F}_{0k}(\tau) \\
\alpha_1 \left[\widetilde{F}_{0k}(\tau) + \widetilde{F}_{1k}(\tau)\right] &\leq \widetilde{F}_{0k}(\tau)
\end{aligned}$$

since $\alpha_1 \in [0,1]$ and therefore

$$\alpha_1 \leq \frac{\widetilde{F}_{0k}(\tau)}{\widetilde{F}_{0k}(\tau) + \widetilde{F}_{1k}(\tau)} = (1-p_k)\left[\frac{F_{0k}(\tau)}{F_k(\tau)}\right] \tag{6}$$

since $\widetilde{F}_{1k}(\tau) + \widetilde{F}_{1k}(\tau) \geq 0$. Proceeding similarly for Equation 5,

$$\begin{aligned}
\alpha_1 \widetilde{F}_{1k}(\tau) &\geq (1-\alpha_1)\widetilde{F}_{0k}(\tau) - (1-p_k-\alpha_1) \\
\alpha_1 \left[\widetilde{F}_{1k}(\tau) + \widetilde{F}_{0k}(\tau) - 1\right] &\geq \widetilde{F}_{0k}(\tau) - (1-p_k) \\
-\alpha_1 \left[1 - \widetilde{F}_{1k}(\tau) - \widetilde{F}_{0k}(\tau)\right] &\geq -\left[1 - \widetilde{F}_{0k}(\tau) - p_k\right] \\
\alpha_1 \left[1 - \widetilde{F}_{1k}(\tau) - \widetilde{F}_{0k}(\tau)\right] &\leq 1 - \widetilde{F}_{0k}(\tau) - p_k
\end{aligned}$$

Now since $\widetilde{F}_{1k}(\tau) = p_k F_{1k}(\tau) \leq p_k$ and $\widetilde{F}_{0k}(\tau) = (1-p_k)F_{0k}(\tau) \leq (1-p_k)$ it follows that $1 - \widetilde{F}_{1k}(\tau) - \widetilde{F}_{0k}(\tau) \geq 0$ and hence

$$\alpha_1 \leq \frac{1 - \widetilde{F}_{0k}(\tau) - p_k}{1 - \widetilde{F}_{1k}(\tau) - \widetilde{F}_{0k}(\tau)} = (1-p_k)\left[\frac{1 - F_{0k}(\tau)}{1 - F_k(\tau)}\right] \tag{7}$$

The bounds given in Equations 6 and 7 relate α_1 to observable quantities *only* and hold for all values of τ for which their respective denominators are non-zero. Moreover, these bounds hold for any value k that the instrument takes on.

We can proceed similarly for α_0. First solve Equation 1 for $(1-p_k^*)F_{0k}^*(\tau)$:

$$(1-p_k^*)F_{0k}^*(\tau) = \frac{1}{1-\alpha_0}\left[\widetilde{F}_{0k}(\tau) - \alpha_1 p_k^* F_{1k}^*(\tau)\right]$$

and then substitute into Equation 2:

$$
\begin{aligned}
\widetilde{F}_{1k}(\tau) &= \frac{\alpha_0}{1-\alpha_0}\left[\widetilde{F}_{0k}(\tau) - \alpha_1 p_k^* F_{1k}^*(\tau)\right] + (1-\alpha_1)p_k^* F_{1k}^*(\tau) \\
&= \frac{\alpha_0}{1-\alpha_0}\widetilde{F}_{0k}(\tau) + \left[(1-\alpha_1) - \frac{\alpha_0\alpha_1}{1-\alpha_0}\right] p_k^* F_{1k}^*(\tau) \\
&= \frac{\alpha_0}{1-\alpha_0}\widetilde{F}_{0k}(\tau) + \left[\frac{(1-\alpha_1)(1-\alpha_0) - \alpha_0\alpha_1}{1-\alpha_0}\right] p_k^* F_{1k}^*(\tau) \\
&= \frac{\alpha_0}{1-\alpha_0}\widetilde{F}_{0k}(\tau) + \left[\frac{1-\alpha_0-\alpha_1}{1-\alpha_0}\right] \frac{p_k - \alpha_0}{1-\alpha_0-\alpha_1} F_{1k}^*(\tau)
\end{aligned}
$$

and therefore

$$
\widetilde{F}_{1k}(\tau) = \frac{\alpha_0}{1-\alpha_0}\widetilde{F}_{0k}(\tau) + \frac{p_k - \alpha_0}{1-\alpha_0} F_{1k}^*(\tau) \tag{8}
$$

Now we can again obtain two bounds by substituting the smallest and largest possible values of $F_{1k}^*(\tau)$. Substituting zero gives

$$
\widetilde{F}_{1k}(\tau) \geq \frac{\alpha_0}{1-\alpha_0}\widetilde{F}_{0k}(\tau) \tag{9}
$$

while substituting one gives

$$
\widetilde{F}_{1k}(\tau) \leq \frac{\alpha_0}{1-\alpha_0}\widetilde{F}_{0k}(\tau) + \frac{p_k - \alpha_0}{1-\alpha_0} \tag{10}
$$

Now, rearranging Equation 9,

$$
\begin{aligned}
(1-\alpha_0)\widetilde{F}_{1k}(\tau) &\geq \alpha_0 \widetilde{F}_{0k}(\tau) \\
\widetilde{F}_{1k}(\tau) &\geq \alpha_0 \left[\widetilde{F}_{0k}(\tau) + \widetilde{F}_{1k}(\tau)\right]
\end{aligned}
$$

since $1 - \alpha_0 \geq 0$. Therefore,

$$
\alpha_0 \leq \frac{\widetilde{F}_{1k}(\tau)}{\widetilde{F}_{0k}(\tau) + \widetilde{F}_{1k}(\tau)} = p_k \left[\frac{F_{1k}(\tau)}{F_k(\tau)}\right] \tag{11}
$$

since $\left[\widetilde{F}_{0k}(\tau)+\widetilde{F}_{1k}(\tau)\right] \geq 0$. Similarly, rearranging Equation 10

$$\begin{aligned}
(1-\alpha_0)\widetilde{F}_{1k}(\tau) &\leq \alpha_0\widetilde{F}_{0k}(\tau)+p_k-\alpha_0\\
\widetilde{F}_{1k}(\tau)-p_k &\leq \alpha_0\left[\widetilde{F}_{0k}(\tau)+\widetilde{F}_{1k}(\tau)-1\right]\\
-\left[1-\widetilde{F}_{1k}(\tau)-(1-p_k)\right] &\leq -\alpha_0\left[1-\widetilde{F}_{0k}(\tau)-\widetilde{F}_{1k}(\tau)\right]\\
\left[1-\widetilde{F}_{1k}(\tau)-(1-p_k)\right] &\geq \alpha_0\left[1-\widetilde{F}_{0k}(\tau)-\widetilde{F}_{1k}(\tau)\right]
\end{aligned}$$

Therefore

$$\alpha_0 \leq \frac{1-\widetilde{F}_{1k}(\tau)-(1-p_k)}{1-\widetilde{F}_{0k}(\tau)-\widetilde{F}_{1k}(\tau)} = p_k\left[\frac{1-F_{1k}(\tau)}{1-F_k(\tau)}\right] \tag{12}$$

Putting Everything Together For all k we have

$$\alpha_0 \leq p_k \min_\tau\left\{\left[\frac{F_{1k}(\tau)}{F_k(\tau)}\right]\wedge\left[\frac{1-F_{1k}(\tau)}{1-F_k(\tau)}\right]\right\} \leq p_k \tag{13}$$

$$\alpha_1 \leq (1-p_k) \min_\tau\left\{\left[\frac{F_{0k}(\tau)}{F_k(\tau)}\right]\wedge\left[\frac{1-F_{0k}(\tau)}{1-F_k(\tau)}\right]\right\} \leq (1-p_k) \tag{14}$$

Note that these bounds can only improve upon those derived in the previous section since the ratio of CDFs tends to one as $\tau\to\infty$. To derive these tighter bounds we have made no assumption regarding the relationship between Z and the error term ε. These bounds use only the assumption that $\alpha_0+\alpha_1<1$, and the assumption that T is conditionally independent of Z, Y given T^*. Notice that that the bounds are related. In particular,

$$p_k\left[\frac{F_{1k}(\tau)}{F_k(\tau)}\right] = 1-(1-p_k)\left[\frac{F_{0k}(\tau)}{F_k(\tau)}\right]$$

and

$$p_k\left[\frac{1-F_{1k}(\tau)}{1-F_k(\tau)}\right] = 1-(1-p_k)\left[\frac{1-F_{0k}(\tau)}{1-F_k(\tau)}\right]$$

4 Even Stronger Bounds on α_0, α_1

Try applying the stochastic dominance conditions from our simulation study.

5 Independent Instrument

Assume that $Z \perp U$. The model is $Y = \beta T^* + U$ and

$$F_U(\tau) = P(U \leq \tau) = P(Y - \beta T^* \leq \tau)$$

but if Z is independent of U then it follows that

$$\begin{aligned}
F_U(\tau) &= F_{U|Z=k}(\tau) = P(U \leq \tau | Z = k) = P(Y - \beta T^* \leq \tau | Z = k) \\
&= P(Y \leq \tau | T^* = 0, Z = k)(1 - p_k^*) + P(Y \leq \tau + \beta | T^* = 1, Z = k) p_k^* \\
&= (1 - p_k^*) F_{0k}^*(\tau) + p_k^* F_{1k}^*(\tau + \beta)
\end{aligned}$$

for all k by the Law of Total Probability. Similarly,

$$F_k(\tau) = (1 - p_k^*) F_{0k}^*(\tau) + p_k^* F_{1k}^*(\tau)$$

and rearranging

$$(1 - p_k^*) F_{0k}^*(\tau) = F_k(\tau) - p_k^* F_{1k}^*(\tau)$$

Substituting this expression into the equation for $F_U(\tau)$ from above, we have

$$F_U(\tau) = F_k(\tau) + p_k^* \left[F_{1k}^*(\tau + \beta) - F_{1k}^*(\tau) \right]$$

for all k and all τ. Evaluating at two values k and ℓ in the support of Z and equating

$$F_k(\tau) + p_k^* \left[F_{1k}^*(\tau + \beta) - F_{1k}^*(\tau) \right] = F_\ell(\tau) + p_\ell^* \left[F_{1\ell}^*(\tau + \beta) - F_{1\ell}^*(\tau) \right]$$

or equivalently

$$F_k(\tau) - F_\ell(\tau) = p_\ell^* \left[F_{1\ell}^*(\tau + \beta) - F_{1\ell}^*(\tau) \right] - p_k^* \left[F_{1k}^*(\tau + \beta) - F_{1k}^*(\tau) \right] \tag{15}$$

for all τ. Now we simply need to re-express all of the "star" quantities, namely p_k^*, p_ℓ^* and $F_{1k}^*, F_{1\ell}^*$ in terms of α_0, α_1 and the *observable* probability distributions F_{1k} and $F_{1\ell}$ and observable probabilities p_k, p_ℓ. To do this, we use the fact that

$$\begin{aligned}
F_{0k}(\tau) &= \frac{1 - \alpha_0}{1 - p_k}(1 - p_k^*) F_{0k}^*(\tau) + \frac{\alpha_1}{1 - p_k} p_k^* F_{1k}^*(\tau) \\
F_{1k}(\tau) &= \frac{\alpha_0}{p_k}(1 - p_k^*) F_{0k}^*(\tau) + \frac{1 - \alpha_1}{p_k} p_k^* F_{1k}^*(\tau)
\end{aligned}$$

for all k by Bayes' rule. Solving these equations,

$$p_k^* F_{1k}^*(\tau) = \frac{1-\alpha_0}{1-\alpha_0-\alpha_1} p_k F_{1k}(\tau) - \frac{\alpha_0}{1-\alpha_0-\alpha_1}(1-p_k)F_{0k}(\tau)$$

for all k. Combining this with Equation 15, we find that

$$\begin{aligned}(1-\alpha_0-\alpha_1)\left[F_k(\tau)-F_\ell(\tau)\right] = \alpha_0 \left\{(1-p_k)\left[F_{0k}(\tau+\beta)-F_{0k}(\tau)\right]-(1-p_\ell)\left[F_{0\ell}(\tau+\beta)-F_{0\ell}(\tau)\right]\right\}\\ -(1-\alpha_0)\left\{p_k\left[F_{1k}(\tau+\beta)-F_{1k}(\tau)\right]-p_\ell\left[F_{1\ell}(\tau+\beta)-F_{1\ell}(\tau)\right]\right\}\end{aligned}$$

Now, define

$$\Delta_{tk}^\tau(\beta) = F_{tk}(\tau+\beta)-F_{tk}(\tau) = E\left[\frac{\mathbf{1}\{T=t, Z=k\}}{p_{tk}}\left(\mathbf{1}\{Y \le \tau+\beta\} - \mathbf{1}\{Y \le \tau\}\right)\right]$$

and note that we can express $F_k(\tau)-F_\ell(\tau)$ similarly as

$$F_k(\tau)-F_\ell(\tau) = E\left[\mathbf{1}\{Y \le \tau\}\left(\frac{\mathbf{1}\{Z=k\}}{q_k} - \frac{\mathbf{1}\{Z=\ell\}}{q_\ell}\right)\right]$$

Using this notation, we can write the preceding as

$$(1-\alpha_0-\alpha_1)\left[F_k(\tau)-F_\ell(\tau)\right] = \alpha_0\left[(1-p_k)\Delta_{0k}^\tau(\beta)-(1-p_\ell)\Delta_{0\ell}^\tau(\beta)\right] - (1-\alpha_0)\left[p_k\Delta_{1k}^\tau(\beta)-p_\ell\Delta_{1\ell}^\tau(\beta)\right]$$

or in moment-condition form

$$\begin{aligned}E\Bigg[&(1-\alpha_0-\alpha_1)\mathbf{1}\{Y\le\tau\}\left(\frac{\mathbf{1}\{Z=k\}}{q_k}-\frac{\mathbf{1}\{Z=\ell\}}{q_\ell}\right) - \left(\mathbf{1}\{Y\le\tau+\beta\}-\mathbf{1}\{Y\le\tau\}\right)\Bigg\{\\ &\alpha_0\left((1-p_k)\frac{\mathbf{1}\{T=0,Z=k\}}{p_{0k}} - (1-p_\ell)\frac{\mathbf{1}\{T=0,Z=\ell\}}{p_{0\ell}}\right)\\ &-(1-\alpha_0)\left(p_k\frac{\mathbf{1}\{T=1,Z=k\}}{p_{1k}} - p_\ell\frac{\mathbf{1}\{T=1,Z=\ell\}}{p_{1\ell}}\right)\Bigg\}\Bigg] = 0\end{aligned}$$

Each value of τ yields a moment condition.

6 Special Case: $\alpha_0 = 0$

In this case the expressions from above simplify to

$$(1-\alpha_1)\left[F_k(\tau)-F_\ell(\tau)\right] = \left[p_\ell F_{1\ell}(\tau+\beta) - p_k F_{1k}(\tau+\beta) - p_\ell F_{1\ell}(\tau) + p_k F_{1k}(\tau)\right] \qquad (16)$$

for all τ. Now, provided that all of the CDFs are differentiable we have[1]

$$e^{i\omega\tau}(1-\alpha_1)\left[f_k(\tau)-f_\ell(\tau)\right] = e^{i\omega\tau}\left[p_\ell f_{1\ell}(\tau+\beta) - p_k f_{1k}(\tau+\beta) - p_\ell f_{1\ell}(\tau) + p_k f_{1k}(\tau)\right]$$

where we have pre-multiplied both sides by $e^{i\omega\tau}$. Finally, integrating both sides with respect to τ over $(-\infty,\infty)$, we have

$$(1-\alpha_1)\left[\varphi_k(\omega)-\varphi_\ell(\omega)\right] = \left\{\int_{-\infty}^{\infty} e^{i\omega\tau}\left[p_\ell f_{1\ell}(\tau+\beta) - p_k f_{1k}(\tau+\beta)\right]\, d\tau - p_\ell\varphi_{1\ell}(\omega) + p_k\varphi_{1k}(\omega)\right\}$$

where φ_k is the conditional characteristic function of Y given $Z=k$ and φ_{1k} is the conditional characteristic function of Y given $T=1, Z=k$. Finally,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\omega\tau} p_\ell f_{1\ell}(\tau+\beta)\, d\tau &= e^{i\omega\beta} p_\ell \int_{u=-\infty+\beta}^{u=\infty+\beta} e^{i\omega u} f_{1\ell}(u)\, du \\ &= e^{-i\omega\beta} p_\ell \varphi_{1\ell}(\omega)\end{aligned}$$

using the substitution $u=\tau+\beta$. Changing subscripts, the same holds for k and thus

$$(1-\alpha_1)\left[\varphi_k(\omega)-\varphi_\ell(\omega)\right] = e^{-i\omega\beta}\left[p_\ell\varphi_{1\ell}(\omega) - p_k\varphi_{1k}(\omega)\right] + \left[p_k\varphi_{1k}(\omega) - p_\ell\varphi_{1\ell}(\omega)\right]$$

which, after collecting terms, simplifies to

$$(1-\alpha_1)\left[\varphi_k(\omega)-\varphi_\ell(\omega)\right] = \left(e^{-i\omega\beta}-1\right)\left[p_\ell\varphi_{1\ell}(\omega) - p_k\varphi_{1k}(\omega)\right] \tag{17}$$

for all ω. Equation 17 contains exactly the same information as Equation 16 but gives us a more convenient way to prove identification since β enters in a simpler way. Leibniz's formula for the rth derivative of a product of two functions f and g is:

$$(fg)^{(r)} = \sum_{s=0}^{r}\binom{r}{s} f^{(s)} g^{(r-s)}$$

[1]There must be a way to generalize this using Lebesgue.

where $f^{(r)}$ denotes the rth derivative of the function f and $g^{(r-s)}$ denotes the $(r-s)$th derivative of the function g. Applying this to the RHS, $R(\omega)$ of Equation 17 gives

$$\begin{aligned}\frac{d}{d\omega^r}R(\omega) &= \sum_{s=0}^{r}\binom{r}{s}\frac{d}{d\omega^s}\left(e^{-i\omega\beta}-1\right)\frac{d}{d\omega^{r-s}}\left[p_\ell\varphi_{1\ell}(\omega)-p_k\varphi_{1k}(\omega)\right] \\ &= \left(e^{-i\omega\beta}-1\right)\left[p_\ell\varphi_{1\ell}^{(r)}(\omega)-p_k\varphi_{1k}^{r}(\varphi)\right]+e^{-i\omega\beta}\sum_{s=1}^{r}\binom{r}{s}(-i\beta)^s\left[p_\ell\varphi_{1\ell}^{(r-s)}(\omega)-p_k\varphi_{1k}^{(r-s)}(\omega)\right]\end{aligned}$$

where we split off the $s=0$ term because our generic expression for the sth derivative of $(e^{-i\omega\beta}-1)$ only applies for $s\geq 1$. Evaluating at zero:

$$\frac{d}{d\omega^r}R(0)=\sum_{s=1}^{r}\binom{r}{s}(-i\beta)^s\left[p_\ell\varphi_{1\ell}^{(r-s)}(0)-p_k\varphi_{1k}^{(r-s)}(0)\right]$$

Combining this with the LHS of Equation 17, also differentiated r times and evaluated at zero, we have

$$(1-\alpha_1)\left[\varphi_k^{(r)}(0)-\varphi_\ell^{(r)}(0)\right]=\sum_{s=1}^{r}\binom{r}{s}(-i\beta)^s\left[p_\ell\varphi_{1\ell}^{(r-s)}(0)-p_k\varphi_{1k}^{(r-s)}(0)\right]$$

Now, recall that if $\varphi(\omega)$ is the characteristic function of Y then $\varphi^{(r)}(0)=i^rE[Y^r]$ provided that the expectation exists where $\varphi^{(r)}$ denotes the rth derivative of φ. The same applies for the conditional characteristic functions we consider here. Hence, provided that the rth moments exist,

$$i^r(1-\alpha_1)\left\{E[Y^r|Z=k]-E[Y^r|Z=\ell]\right\}=\sum_{s=1}^{r}\binom{r}{s}(-i\beta)^s i^{r-s}\left(p_\ell E\left[Y^{r-s}|T=1,Z=\ell\right]-p_kE\left[Y^{r-s}|T=1,Z=k\right]\right)$$

After simplifying the terms involving i and cancelling them from both sides,

$$(1-\alpha_1)\left(E[Y^r|Z=k]-E[Y^r|Z=\ell]\right)=\sum_{s=1}^{r}\binom{r}{s}(-\beta)^s\left(p_\ell E\left[Y^{r-s}|T=1,Z=\ell\right]-p_kE\left[Y^{r-s}|T=1,Z=k\right]\right)$$

again provided that the moments exist. Abbreviating the conditional expectations according to $E[Y^r|Z=k]=E_k[Y^r]$ and $E[Y^r|T=t,Z=k]=E_{tk}[Y^r]$, this becomes

$$(1-\alpha_1)\left(E_k[Y^r]-E_\ell[Y^r]\right)=\sum_{s=1}^{r}\binom{r}{s}(-\beta)^s\left(p_\ell E_{1\ell}\left[Y^{r-s}\right]-p_kE_{1k}\left[Y^{r-s}\right]\right) \tag{18}$$

Equation 18 can be used to generate moment equations that are implied by the Equation 17 and the equivalent representation in terms of CDFs: Equation 16. Assuming that the conditional first moments exist, we can evaluate Equation 18 at $r = 1$, yielding

$$\begin{aligned}(1-\alpha_1)\left(E_k[Y]-E_\ell[Y]\right) &= \sum_{s=1}^{1}\binom{1}{s}(-\beta)^s\left(p_\ell E_{1\ell}\left[Y^{1-s}\right]-p_k E_{1k}\left[Y^{1-s}\right]\right)\\ &= -\beta\left(p_\ell - p_k\right)\end{aligned}$$

Rearranging, this gives us the expression for the probability limit of the Wald estimator

$$\mathcal{W} \equiv \frac{E_k[Y]-E_\ell[Y]}{p_k-p_\ell} = \frac{\beta}{1-\alpha_1} \tag{19}$$

Evaluating Equation 18 at $r = 2$, we have

$$\begin{aligned}(1-\alpha_1)\left(E_k[Y^2]-E_\ell[Y^2]\right) &= \sum_{s=1}^{2}\binom{2}{s}(-\beta)^s\left(p_\ell E_{1\ell}\left[Y^{2-s}\right]-p_k E_{1k}\left[Y^{2-s}\right]\right)\\ &= 2\beta\left(p_k E_{1k}[Y]-p_\ell E_{1\ell}[Y]\right)-\beta^2\left(p_k-p_\ell\right)\end{aligned}$$

Rearranging, we have

$$E_k[Y^2]-E_\ell[Y^2] = \frac{\beta}{1-\alpha_1}\left[2\left(p_k E_{1k}[Y]-p_\ell E_{1\ell}[Y]\right)-\beta(p_k-p_\ell)\right] \tag{20}$$

Substituting Equation 19, we can replace $\beta/(1-\alpha_1)$ with a function of observables only, namely $\mathcal{W}$. Solving, we find that

$$\beta = \frac{2\left(p_k E_{1k}[Y]-p_\ell E_{1\ell}[Y]\right)}{p_k-p_\ell} - \frac{E_k[Y^2]-E_\ell[Y^2]}{E_k[Y]-E_\ell[Y]} \tag{21}$$

This allows us to state low-level sufficient conditions for identification:

(a) $\alpha_1 < 1$

(b) $p_k \neq p_\ell$

(c) $E_k[Y] \neq E_\ell[Y]$

(d) $E_{1k}[|Y|], E_{1\ell}[|Y|], E_k[|Y^2|], E_\ell[|Y^2|] < \infty$.

Note that, although $\beta = 0$ is always a solution of Equation 16 this solution is ruled out by the assumption that $E_k[Y] \neq E_\ell[Y]$ via Equation 19. The mis-classification error rate α_1 is likewise uniquely identified under these assumptions. Substituting $\beta/\mathcal{W} = 1 - \alpha_1$ into

Equation 21

$$
\begin{aligned}
(1-\alpha_1) &= \left\{\frac{p_k - p_\ell}{E_k[Y]-E_\ell[Y]}\right\}\left\{\frac{2\left(p_k E_{1k}[Y] - p_\ell E_{1\ell}[Y]\right)}{p_k - p_\ell} - \frac{E_k[Y^2]-E_\ell[Y^2]}{E_k[Y]-E_\ell[Y]}\right\}\\
&= \frac{2\left(p_k E_{1k}[Y] - p_\ell E_{1\ell}[Y]\right)}{E_k[Y]-E_\ell[Y]} - (p_k - p_\ell)\left\{\frac{E_k[Y^2]-E_\ell[Y^2]}{\left(E_k[Y]-E_\ell[Y]\right)^2}\right\}
\end{aligned}
$$

and thus

$$
\alpha_1 = 1 + (p_k - p_\ell)\left\{\frac{E_k[Y^2]-E_\ell[Y^2]}{\left(E_k[Y]-E_\ell[Y]\right)^2}\right\} - \frac{2\left(p_k E_{1k}[Y] - p_\ell E_{1\ell}[Y]\right)}{E_k[Y]-E_\ell[Y]}
$$

7 Identification in the General Case

8 Characteristic Functions

Recall from above that in the general case an independent instrument combined with non-differential measurement error implies that

$$
\begin{aligned}
(1-\alpha_0-\alpha_1)\left[F_k(\tau)-F_\ell(\tau)\right] = \alpha_0 &\left\{(1-p_k)\left[F_{0k}(\tau+\beta)-F_{0k}(\tau)\right] - (1-p_\ell)\left[F_{0\ell}(\tau+\beta)-F_{0\ell}(\tau)\right]\right\}\\
&- (1-\alpha_0)\left\{p_k\left[F_{1k}(\tau+\beta)-F_{1k}(\tau)\right] - p_\ell\left[F_{1\ell}(\tau+\beta)-F_{1\ell}(\tau)\right]\right\}
\end{aligned}
$$

Using the same steps as in the preceding section, we can convert this expression into characteristic function form by differentiating each side, multiplying by $e^{i\omega\tau}$ and then integrating with respect to τ, yielding

$$
\begin{aligned}
(1-\alpha_0-\alpha_1)\left[\varphi_k(\omega)-\varphi_\ell(\omega)\right] = \alpha_0 &\left\{(1-p_k)\left(e^{-i\omega\beta}-1\right)\varphi_{0k}(\omega) - (1-p_\ell)\left(e^{-i\omega\beta}-1\right)\varphi_{0\ell}(\omega)\right\}\\
&- (1-\alpha_0)\left\{p_k\left(e^{-i\omega\beta}-1\right)\varphi_{1k}(\omega) - p_\ell\left(e^{-i\omega\beta}-1\right)\varphi_{1\ell}(\omega)\right\}
\end{aligned}
$$

which simplifies to

$$
\varphi_k(\omega)-\varphi_\ell(\omega) = \left(e^{-i\omega\beta}-1\right)\left(\frac{\alpha_0\left[(1-p_k)\varphi_{0k}(\omega)-(1-p_\ell)\varphi_{0\ell}(\omega)\right] - (1-\alpha_0)\left[p_k\varphi_{1k}(\omega)-p_\ell\varphi_{1\ell}(\omega)\right]}{1-\alpha_0-\alpha_1}\right)
$$

As above, we will differentiate both sides of this expression r times and evaluate at $\omega = 0$. Steps nearly identical to those given above yield

$$
\begin{aligned}
(1-\alpha_0-\alpha_1)\left(E_k[Y^r]-E_\ell[Y^r]\right) = \alpha_0 \sum_{s=1}^{r} \binom{r}{s} (-\beta)^s \left\{(1-p_k)E_{0k}[Y^{r-s}] - (1-p_\ell)E_{0\ell}[Y^{r-s}]\right\} \\
- (1-\alpha_0) \sum_{s=1}^{r} \binom{r}{s} (-\beta)^s \left\{p_k E_{1k}[Y^{r-s}] - p_\ell E_{1\ell}[Y^{r-s}]\right\}
\end{aligned}
$$

First Moments Taking $r = 1$ gives

$$
(1-\alpha_0-\alpha_1)\left(E_k[Y]-E_\ell[Y]\right) = \beta(p_k - p_\ell)
$$

Simplifying,

$$
\mathcal{W} \equiv \frac{E_k[Y]-E_\ell[Y]}{p_k-p_\ell} = \frac{\beta}{1-\alpha_0-\alpha_1} \tag{22}
$$

Second Moments Now, taking $r = 2$ gives

$$
\begin{aligned}
(1-\alpha_0-\alpha_1)\left(E_k[Y^2]-E_\ell[Y^2]\right) &= \alpha_0 \left\{[(1-p_k)E_{0k}[Y] - (1-p_\ell)E_{0\ell}] - \beta^2\left(p_k-p_\ell\right)\right\} \\
&\quad - (1-\alpha_0)\left\{-2\beta\left(p_k E_{1k}[Y] - p_\ell E_{1\ell}[Y]\right) + \beta^2\left(p_k-p_\ell\right)\right\} \\
&= -2\beta\alpha_0 \left\{(1-p_k)E_{0k}[Y] - (1-p_\ell)E_{0\ell}[Y] p_k E_{1k}[Y] + p_\ell E_{1\ell}[Y]\right\} \\
&\quad + 2\beta\left(p_k E_{1k}[Y] - p_\ell E_{1\ell}[Y]\right) - (p_k-p_\ell)\beta^2\left(\alpha_0 + 1 - \alpha_0\right) \\
&= -2\beta\left\{\alpha_0\left(E_k[Y]-E_\ell[Y]\right) - \left(p_k E_{1k}[Y] - p_\ell E_{1\ell}[Y]\right)\right\} - \beta^2(p_k-p_\ell)
\end{aligned}
$$

Now, simplifying

$$
(1-\alpha_0-\alpha_1)\left(\frac{E_k[Y^2]-E_\ell[Y^2]}{p_k-p_\ell}\right) = -2\beta\alpha_0\left(\frac{E_k[Y]-E_k[Y]}{p_k-p_\ell}\right) + 2\beta\left(\frac{p_{1k}E_{1k}[Y]-p_\ell E_{1\ell}[Y]}{p_k-p_\ell}\right) - \beta^2
$$

and substituting Equation 22 to eliminate β, this becomes

$$
\begin{aligned}
(1-\alpha_0-\alpha_1)\left(\frac{E_k[Y^2]-E_\ell[Y^2]}{p_k-p_\ell}\right) &= -2\alpha_0(1-\alpha_0-\alpha_1)\mathcal{W}^2 + 2\mathcal{W}(1-\alpha_0-\alpha_1)\left(\frac{p_{1k}E_{1k}[Y]-p_\ell E_{1\ell}[Y]}{p_k-p_\ell}\right) \\
&\quad - (1-\alpha_0-\alpha_1)^2\mathcal{W}^2 \\
\left(\frac{E_k[Y^2]-E_\ell[Y^2]}{p_k-p_\ell}\right) &= -2\alpha_0\mathcal{W}^2 + 2\mathcal{W}\left(\frac{p_{1k}E_{1k}[Y]-p_\ell E_{1\ell}[Y]}{p_k-p_\ell}\right) - (1-\alpha_0-\alpha_1)\mathcal{W}^2
\end{aligned}
$$

And thus, simplifying

$$
\begin{aligned}
-2\alpha_0\mathcal{W}^2 - (1-\alpha_0-\alpha_1)\mathcal{W}^2 &= \left(\frac{E_k[Y^2]-E_\ell[Y^2]}{p_k-p_\ell}\right) - 2\mathcal{W}\left(\frac{p_{1k}E_{1k}[Y]-p_\ell E_{1\ell}[Y]}{p_k-p_\ell}\right)\\
\alpha_1-\alpha_0 &= 1+\left[\frac{E_k[Y^2]-E_\ell[Y^2]}{\mathcal{W}^2(p_k-p_\ell)}\right] - 2\left[\frac{p_{1k}E_{1k}[Y]-p_\ell E_{1\ell}[Y]}{\mathcal{W}(p_k-p_\ell)}\right]
\end{aligned}
$$

and therefore

$$
\alpha_1-\alpha_0 = 1+(p_k-p_\ell)\left[\frac{E_k[Y^2]-E_\ell[Y^2]}{\left(E_k[Y]-E_\ell[Y]\right)^2}\right] - 2\left[\frac{p_{1k}E_{1k}[Y]-p_\ell E_{1\ell}[Y]}{E_k[Y]-E_\ell[Y]}\right] \tag{23}
$$

"Product" Moments Recall that in our initial draft of the paper we worked with moments such as $E[TY|Z=k], E[TY|Z=\ell]$ and $E[TY^2|Z=k], E[TY^2|Z=\ell]$. In the notation of this document, we can express these quantities as follows:

$$
\begin{aligned}
E[TY^r|z=k] &= E[TY^r|T=1,z=k]p_k + E[TY^r|T=0,z=k](1-p_k)\\
&= p_k E[Y^r|T=1,z=k]+0\\
&= p_k E_{1k}[Y^r]
\end{aligned}
$$

for any r. We will use this relationship to motivate some shorthand notation below.

Some Shorthand The notation above is becoming very cumbersome and we haven't even looked at the third moments yet! To make life easier, define the following:

$$
\begin{aligned}
\widetilde{y^r_{1k}} &= p_k E_{1k}[Y^r]\\
\widetilde{y^r_{0k}} &= (1-p_k)E_{1k}[Y^r]\\
\Delta\overline{y^r} &= E_k[Y^r]-E_\ell[Y^r]\\
\Delta\overline{Ty^r} &= p_k E_{1k}[Y^r] - p_\ell E_{1\ell}[Y^r] = \widetilde{y^r_{1k}} - \widetilde{y^r_{1k}}\\
\mathcal{W} &= (E_k[Y]-E_\ell[Y])/(p_k-p_\ell)
\end{aligned}
$$

for all r. When no r superscript is given this means $r=1$. Note, moreover, that when $r=0$ we have $\widetilde{y^0_{1k}} = p_k$ and $\widetilde{y^0_{0k}} = (1-p_k)$. Thus $\Delta\overline{Ty^0} = p_k - p_\ell$. In contrast, $\Delta y^0 = 0$.

Among other things, this notation will make it easier for us to link the derivations here to our earlier derivations from the first draft of the paper that used slightly different notation and did not work explicitly with the independence of the instrument.

Simplifying the Moment Equalities Using the final two pieces of notation defined in the preceding section, we can re-rewrite the collection of moment equalities arising from the characteristic function equations as

$$(1-\alpha_0-\alpha_1)\Delta\overline{y^r} = \sum_{s=1}^{r}\binom{r}{s}(-\beta)^s\left[\alpha_0\left(\widetilde{y_{0k}^{r-s}}-\widetilde{y_{0\ell}^{r-s}}\right)-(1-\alpha_0)\left(\widetilde{y_{1k}^{r-s}}-\widetilde{y_{1\ell}^{r-s}}\right)\right]$$

Now, simplifying the terms in the square brackets,

$$\begin{aligned}
\alpha_0\left(\widetilde{y_{0k}^{r-s}}-\widetilde{y_{0\ell}^{r-s}}\right)-(1-\alpha_0)\left(\widetilde{y_{1k}^{r-s}}-\widetilde{y_{1\ell}^{r-s}}\right) &= \alpha_0\left[\left(\widetilde{y_{0k}^{r-s}}+\widetilde{y_{1k}^{r-s}}\right)-\left(\widetilde{y_{0\ell}^{r-s}}+\widetilde{y_{1\ell}^{r-s}}\right)\right]-\left(\widetilde{y_{1k}^{r-s}}-\widetilde{y_{1\ell}^{r-s}}\right)\\
&= \alpha_0\left(E_k[Y^{r-s}]-E_\ell[Y^{r-s}]\right)-\Delta\overline{Ty^{r-s}}\\
&= \alpha_0\Delta\overline{y^{r-s}}-\Delta\overline{Ty^{r-s}}
\end{aligned}$$

and hence

$$(1-\alpha_0-\alpha_1)\Delta\overline{y^r} = \sum_{s=1}^{r}\binom{r}{s}(-\beta)^s\left(\alpha_0\Delta\overline{y^{r-s}}-\Delta\overline{Ty^{r-s}}\right) \tag{24}$$

Third Moments Evaluating Equation 24 at $r=3$

$$\begin{aligned}
(1-\alpha_0-\alpha_1)\Delta\overline{y^3} &= \sum_{s=1}^{3}\binom{3}{s}(-\beta)^s\left(\alpha_0\Delta\overline{y^{3-s}}-\Delta\overline{Ty^{3-s}}\right)\\
&= -3\beta\left(\alpha_0\Delta\overline{y^2}-\Delta\overline{Ty^2}\right)+3\beta^2\left(\alpha_0\Delta\overline{y}-\Delta\overline{Ty}\right)+\beta^3(p_k-p_\ell)
\end{aligned}$$

Solving the System Using $\mathcal{W}=\beta/(1-\alpha_0-\alpha_1)$ we can re-write the third moment expression as follows

$$\begin{aligned}
\Delta\overline{y^3} &= -3\mathcal{W}\left(\alpha_0\Delta\overline{y^2}-\Delta\overline{Ty^2}\right)+3\beta\mathcal{W}\left(\alpha_0\Delta\overline{y}-\Delta\overline{Ty}\right)+\beta^2\mathcal{W}(p_k-p_\ell)\\
\frac{\Delta\overline{y^3}}{\mathcal{W}(p_k-p_\ell)} &= \beta^2+3\beta\left(\frac{\alpha_0\Delta\overline{y}-\Delta\overline{Ty}}{p_k-p_\ell}\right)-3\left(\frac{\alpha_0\Delta\overline{y^2}-\Delta\overline{Ty^2}}{p_k-p_\ell}\right)\\
\frac{\Delta\overline{y^3}-3\mathcal{W}\Delta\overline{y^2T}}{\mathcal{W}(p_k-p_\ell)} &= \beta^2+3\beta\left(\frac{\alpha_0\Delta\overline{y}-\Delta\overline{Ty}}{p_k-p_\ell}\right)-3\left(\frac{\alpha_0\Delta\overline{y^2}}{p_k-p_\ell}\right)
\end{aligned}$$

Now, translating the second moment equation into the shorthand notation defined above, we have

9 New Results from September 2016

9.1 Is the Treatment Effect Identified when Y is Discrete?

Suppose that Y is discrete with arbitrary support. This allows for the possibility that Y is binary. Using our standard argument,

$$
\begin{aligned}
P(Y|T^* = 0, z_k) &= P(Y|T = 0, z_k) + \left(\frac{\alpha_1 p_k}{1 - p_k - \alpha_1}\right)[P(Y|T = 0, z_k) - P(Y|T = 1, z_k)] \\
P(Y|T^* = 1, z_k) &= P(Y|T = 1, z_k) + \left(\frac{\alpha_0(1 - p_k)}{p_k - \alpha_0}\right)[P(Y|T = 1, z_k) - P(Y|T = 0, z_k)]
\end{aligned}
$$

under non-differential measurement error. Now, if we assume that Z is conditionally independent of Y given T^*, then we have $P(Y, Z|T^*) = P(Z|T^*)P(Y|T^*)$ or equivalently $P(Y|T^*, Z) = P(Y|T^*)$. I originally derived this in terms of the first condition, but here I'll present both ways.

9.1.1 Derivation Using $P(Y, Z|T^*) = P(Y|T^*)P(Z|T^*)$

We have

$$
\begin{aligned}
P(Z|T^* = 1) &= P(Z)\left[\frac{P(T = 1|Z) - \alpha_0}{P(T = 1) - \alpha_0}\right] \\
P(Z|T^* = 0) &= P(Z)\left[\frac{P(T = 0|Z) - \alpha_1}{P(T = 0) - \alpha_1}\right]
\end{aligned}
$$

Now, by our usual argument

$$
\begin{aligned}
P(Y, Z|T^* = 0) &= P(Y, Z|T = 0) + \left(\frac{\alpha_1 p}{1 - p - \alpha_1}\right)[P(Y, Z|T = 0) - P(Y, Z|T = 1)] \\
P(Y, Z|T^* = 1) &= P(Y, Z|T = 1) + \left(\frac{\alpha_0(1 - p)}{p - \alpha_0}\right)[P(Y, Z|T = 1) - P(Y, Z|T = 0)]
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
P(Y, Z|T^* = 1) &= \frac{P(Y, Z, T = 1) - \alpha_0 P(Y, Z)}{P(T = 1) - \alpha_0} \\
P(Y, Z|T^* = 0) &= \frac{P(Y, Z, T = 0) - \alpha_1 P(Y, Z)}{P(T = 0) - \alpha_1}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P(Y|T^*=1) &= \left(\frac{1-\alpha_0}{p-\alpha_0}\right) pP(Y|T=1) - \left(\frac{\alpha_0}{p-\alpha_0}\right)(1-p)P(Y|T=0)\\
P(Y|T^*=0) &= \left(\frac{-\alpha_1}{1-p-\alpha_1}\right) pP(Y|T=1) - \left(\frac{1-\alpha_1}{1-p-\alpha_1}\right)(1-p)P(Y|T=0)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
P(Y|T^*=1) &= \frac{P(Y,T=1)-\alpha_0 P(Y)}{P(T=1)-\alpha_0}\\
P(Y|T^*=0) &= \frac{P(Y,T=0)-\alpha_1 P(Y)}{P(T=0)-\alpha_1}
\end{aligned}
$$

Thus, imposing $P(Y,Z|T^*) = P(Z|T^*)P(Y|T^*)$ we have two restrictions for every pair (y,z) in the joint support of (Y,Z): one corresponding to $T^*=0$ and the other to $T^*=1$, namely

$$
\begin{aligned}
\left[P(T=1)-\alpha_0\right]\left[P(Y,Z,T=1)-\alpha_0 P(Y,Z)\right] &= P(Z)\left[P(T=1|Z)-\alpha_0\right]\left[P(Y,T=1)-\alpha_0 P(Y)\right]\\
\left[P(T=0)-\alpha_1\right]\left[P(Y,Z,T=0)-\alpha_1 P(Y,Z)\right] &= P(Z)\left[P(T=0|Z)-\alpha_1\right]\left[P(Y,T=0)-\alpha_1 P(Y)\right]
\end{aligned}
$$

The first of these gives a quadratic in α_0 and observables only while the second gives a quadratic in α_1 and observables only. Both quadratices have the same coefficient on the squared variable: $P(Y,Z) - P(Y)P(Z)$. This means that the quadratic term vanishes when Y and Z are independent.

The first equation can be re-written in polynomial form:

$$
\begin{aligned}
\alpha_0^2\left[P(Y)-P(Y|Z)\right] - \alpha_0\left[P(Y,T=1) - P(Y,T=1|Z) + P(T=1|Z)P(Y) - P(T=1)P(Y|Z)\right]&\\
+\left[P(T=1|Z)P(Y,T=1) - P(T=1)P(Y,T=1|Z)\right] &= 0
\end{aligned}
$$

The second equation can be re-written in polynomial form:

$$
\begin{aligned}
\alpha_1^2\left[P(Y)-P(Y|Z)\right] - \alpha_1\left[P(Y,T=0) - P(Y,T=0|Z) + P(T=0|Z)P(Y) - P(T=0)P(Y|Z)\right]&\\
+\left[P(T=0|Z)P(Y,T=0) - P(T=0)P(Y,T=0|Z)\right] &= 0
\end{aligned}
$$

In both cases we make use of the fact that

$$
\frac{P(Y,Z,T)}{P(Z)} = P(Y,T|Z)
$$

After some algebra, the discriminant ($D = b^2 - 4ac$) for the α_0 roots can be written as:

$$
\begin{aligned}
&(P(Y,T=1) - P(Y)P(T=1|Z))^2 + (P(Y,T=1|Z) - P(T=1)P(Y|Z))^2 \\
&+4P(Y|Z)P(T=1|Z)P(Y,T=1) - 2P(Y|Z)P(T=1|Z)P(Y)P(T=1) \\
&+4P(Y,T=1|Z)P(Y)P(T=1) - 2P(Y,T=1|Z)P(Y)P(T=1|Z) \\
&-2P(Y,T=1)P(Y,T=1|Z) - 2P(Y,T=1)P(Y|Z)P(T=1)
\end{aligned}
$$

The question is whether the last six terms can be factored in a way that shows that the whole discriminant is always positive.

9.1.2 Derivation Using $P(Y|T^*, Z) = P(Y|T^*)$

Let's write this in a way that allows for Y to follow an arbitrary distribution. Let

$$
\begin{aligned}
F^*_{tk}(\tau) &= P(Y \leq \tau | T^* = t, z_k) \\
F_{tk}(\tau) &= P(Y \leq \tau | T = t, z_k)
\end{aligned}
$$

Now, by the assumption of non-differential measurement error,

$$
\begin{aligned}
p_k F_{1k}(\tau) &= (1-\alpha_1)p^*_k F^*_{1k}(\tau) + \alpha_0(1-p^*_k)F^*_{0k}(\tau) \\
(1-p_k)F_{0k}(\tau) &= \alpha_1 p^*_k F^*_{1k}(\tau) + (1-\alpha_0)(1-p^*_k)F^*_{0k}(\tau)
\end{aligned}
$$

Solving the linear system as above, we find that

$$
\begin{aligned}
F^*_{0k}(\tau) &= F_{0k}(\tau) + \left(\frac{\alpha_1 p_k}{1-p_k-\alpha_1}\right)[F_{0k}(\tau) - F_{1k}(\tau)] \\
F^*_{1k}(\tau) &= F_{1k}(\tau) + \left(\frac{\alpha_0(1-p_k)}{p_k-\alpha_0}\right)[F_{1k}(\tau) - F_{0k}(\tau)]
\end{aligned}
$$

The conditional independence assumption is equivalent to

$$
\begin{aligned}
F^*_{0k}(\tau) &= F^*_{0\ell}(\tau) \\
F^*_{1k}(\tau) &= F^*_{1\ell}(\tau)
\end{aligned}
$$

for all τ. (If Y is discrete, it's simpler to work with the analogous statement for probability mass functions.) Imposing this on the solutions for F^*_{0k} and F^*_{1k} gives,

$$F_{0k}(\tau) + \left(\frac{\alpha_1 p_k}{1 - p_k - \alpha_1}\right)[F_{0k}(\tau) - F_{1k}(\tau)] = F_{0\ell}(\tau) + \left(\frac{\alpha_1 p_\ell}{1 - p_\ell - \alpha_1}\right)[F_{0\ell}(\tau) - F_{1\ell}(\tau)]$$
$$F_{1k}(\tau) + \left(\frac{\alpha_0(1 - p_k)}{p_k - \alpha_0}\right)[F_{1k}(\tau) - F_{0k}(\tau)] = F_{1\ell}(\tau) + \left(\frac{\alpha_0(1 - p_\ell)}{p_\ell - \alpha_0}\right)[F_{1\ell}(\tau) - F_{0\ell}(\tau)]$$

Simplifying these should give quadratics in α_0 and α_1 with the same solutions as those from above derived under the alternative version of the independence condition.

9.2 Can we relax the measurement error assumptions?

Suppose that we continue to assume that $P(Y|T^*, T, z) = P(Y|T^*, z)$ but relax the assumption that $P(T|T^*, z) = P(T|T^*)$. Define:

$$\alpha_{0k} = P\left(T = 1|T^* = 1, z_k\right)$$
$$\alpha_{1k} = P\left(T = 1|T^* = 0, z_k\right)$$

As before, the Wald estimator converges in probability to

$$\mathcal{W} = \frac{E[Y|z_k] - E[Y|z_\ell]}{p_k - p_\ell}$$

but the relationship between $p_1 - p_0$ and the unobserved $p^*_1 - p^*_0$ changes. By the law of total probability

$$\begin{aligned}p_k = P(T = 1|z_k) &= P(T = 1|T^* = 1, z_k)P(T^* = 1|z_k) + P(T = 1|T^* = 0, z_k)P(T^* = 0|z_k)\\ &= (1 - \alpha_{1k})p^*_k + \alpha_{0k}(1 - p^*_k) = p^*_k(1 - \alpha_{0k} - \alpha_{1k}) + \alpha_{0k}\end{aligned}$$

and thus

$$p^*_k = \frac{p_k - \alpha_{0k}}{1 - \alpha_{0k} - \alpha_{1k}}, \quad 1 - p^*_k = \frac{1 - p_k - \alpha_{1k}}{1 - \alpha_{0k} - \alpha_{1k}}.$$

Thus, we have

$$\begin{aligned}p^*_k - p^*_\ell &= \left(\frac{p_k - \alpha_{0k}}{1 - \alpha_{0k} - \alpha_{1k}}\right) - \left(\frac{p_0 - \alpha_{0\ell}}{1 - \alpha_{0\ell} - \alpha_{1\ell}}\right)\\ &= \frac{(p_k - \alpha_{0k})(1 - \alpha_{0\ell} - \alpha_{1\ell}) - (p_0 - \alpha_{0\ell})(1 - \alpha_{0k} - \alpha_{1k})}{(1 - \alpha_{0k} - \alpha_{1k})(1 - \alpha_{0\ell} - \alpha_{1\ell})}\end{aligned}$$

9.3 Is there a LATE interpretation of our results?

Let $J \in \{a, c, d, n\}$ index an individual's *type*: always-taker, complier, defier, or never-taker. Let $\pi_a, \pi_c, \pi_d, \pi_n$ denote the population proportions of always-takers, compliers, defiers, and never-takers. The unconfounded type assumption is $P(J = j|z = 1) = P(J = j|z = 0)$. Combined with the law of total probability, this gives

$$\begin{aligned}
p_1^* &= P(T^* = 1|z = 1) = \pi_a + \pi_c \\
1 - p_1^* &= P(T^* = 0|z = 1) = \pi_d + \pi_n \\
p_0^* &= P(T^* = 1|z = 0) = \pi_d + \pi_a \\
1 - p_0^* &= P(T^* = 0|z = 0) = \pi_n + \pi_c
\end{aligned}$$

Imposing no-defiers, $\pi_d = 0$, these expressions simplify to

$$\begin{aligned}
p_1^* &= \pi_a + \pi_c \\
1 - p_1^* &= \pi_n \\
p_0^* &= \pi_a \\
1 - p_0^* &= \pi_n + \pi_c
\end{aligned}$$

Solving for π_c, we see that

$$\begin{aligned}
\pi_c &= p_1^* - p_0^* \\
\pi_a &= p_0^* \\
\pi_n &= 1 - p_1^*
\end{aligned}$$

Now, let $Y(1)$ indicate the potential outcome when $T^* = 1$ and $Y(0)$ indicate the potential outcome when $T^* = 0$. The standard LATE assumptions (no defiers, mean exclusion, unconfounded type) imply

$$\begin{aligned}
\mathbb{E}\left(Y|T^* = 1, z = 1\right) &= \left(\frac{p_0^*}{p_1^*}\right)\mathbb{E}\left[Y(1)|J = a\right] + \left(\frac{p_1^* - p_0^*}{p_1^*}\right)\mathbb{E}\left[Y(1)|J = c\right] \\
\mathbb{E}\left(Y|T^* = 0, z = 0\right) &= \left(\frac{p_1^* - p_0^*}{1 - p_0^*}\right)\mathbb{E}\left[Y(0)|J = c\right] + \left(\frac{1 - p_1^*}{1 - p_0^*}\right)\mathbb{E}\left[Y(0)|J = n\right] \\
\mathbb{E}\left(Y|T^* = 1, z = 0\right) &= \mathbb{E}\left[Y(1)|J = a\right] \\
\mathbb{E}\left(Y|T^* = 0, z = 1\right) &= \mathbb{E}\left[Y(0)|J = n\right]
\end{aligned}$$

9.3.1 LATE Version of Theorem 2 from the Draft

$$
\begin{aligned}
\Delta\overline{yT} &= \mathbb{E}\left(yT|z=1\right) - \mathbb{E}\left(yT|z=0\right) \\
&= (1-\alpha_1)\left[p_1^*\mathbb{E}\left(y|T^*=1,z=1\right) - p_0^*\mathbb{E}\left(y|T^*=1,z=0\right)\right] \\
&\qquad + \alpha_0\left[(1-p_1^*)\mathbb{E}\left(y|T^*=0,z=1\right) - (1-p_0^*)\mathbb{E}\left(y|T^*,z=0\right)\right]
\end{aligned}
$$

So we find that

$$
\begin{aligned}
\Delta\overline{yT} &= (p_1^*-p_0^*)\left\{(1-\alpha_1)\mathbb{E}\left[Y(1)|J=c\right] - \alpha_0\mathbb{E}\left[Y(0)|J=c\right]\right\} \\
&= (1-\alpha_1)\left\{\frac{\mathbb{E}\left[Y(1)-Y(0)|J=c\right]}{1-\alpha_0-\alpha_1}(p_1-p_0)\right\} + (p_1-p_0)\mathbb{E}\left[Y(0)|J=c\right]
\end{aligned}
$$

Recall that the analogous expression in the homogeneous treatment effect case is

$$
\begin{aligned}
\Delta\overline{yT} &= (1-\alpha_1)\mathcal{W}(p_1-p_0) + \mu_{10}^* \\
&= (1-\alpha_1)\left(\frac{\beta}{1-\alpha_0-\alpha_1}\right)(p_1-p_0) + (p_1-\alpha_0)m_{11}^* - (p_0-\alpha_0)m_{10}^*
\end{aligned}
$$

while the expression for the difference of variances is

$$
\Delta\overline{y^2} = \beta\mathcal{W}(p_1-p_0) + 2\mathcal{W}\mu_{10}^*
$$

From above we see that the analogue of μ_{10}^* in the heterogeneous treatment effects setting is $(p_1-p_0)E\left[Y(0)|J=c\right]$ and since the LATE is $\mathbb{E}\left[Y(1)-Y(0)|J=c\right]$, the analogue of $\mathcal{W}$ is

$$
\frac{\mathbb{E}\left[Y(1)-Y(0)|J=c\right]}{1-\alpha_0-\alpha_1}
$$

so *if* we could establish that

$$
\Delta\overline{y^2} = \left(\frac{p_1-p_0}{1-\alpha_0-\alpha_1}\right)\mathbb{E}\left[Y(1)-Y(0)|J=c\right]\cdot\mathbb{E}\left[Y(1)+Y(0)|J=c\right]
$$

in the heterogeneous treatment effects case, the proof of Theorem 2 would go through immediately. Now, if we assume an exclusion restriction on the *second* moment of y an argument almost identical to the standard LATE derivation gives

$$
\Delta\overline{y^2} = \frac{\mathbb{E}\left[Y^2(1)-Y^2(0)|J=c\right]}{p_1^*-p_0^*} = \left(\frac{p_1-p_0}{1-\alpha_0-\alpha_1}\right)\mathbb{E}\left[Y^2(1)-Y^2(0)|J=c\right]
$$

so we see that the necessary and sufficient condition for our proof to go through is

$$\mathbb{E}\left[Y^2(1) - Y^2(0)|J = c\right] = \mathbb{E}\left[Y(1) - Y(0)|J = c\right] \cdot \mathbb{E}\left[Y(1) + Y(0)|J = c\right]$$

Rearranging, this in turn is equivalent to

$$\text{Var}\left[Y(1)|J = c\right] = \text{Var}\left[Y(0)|J = c\right]$$

9.3.2 Independence Restriction

If we strengthen the exclusion restriction from $E[Y(T^*, z)] = E[Y(T^*)]$ to $Y(T^*, z) = Y(T^*)$,

$$\begin{aligned}
\mathbb{P}\left(Y|T^* = 1, z = 1\right) &= \left(\frac{p_0^*}{p_1^*}\right)\mathbb{P}\left[Y(1)|J = a\right] + \left(\frac{p_1^* - p_0^*}{p_1^*}\right)\mathbb{P}\left[Y(1)|J = c\right] \\
\mathbb{P}\left(Y|T^* = 0, z = 0\right) &= \left(\frac{p_1^* - p_0^*}{1 - p_0^*}\right)\mathbb{P}\left[Y(0)|J = c\right] + \left(\frac{1 - p_1^*}{1 - p_0^*}\right)\mathbb{P}\left[Y(0)|J = n\right] \\
\mathbb{P}\left(Y|T^* = 1, z = 0\right) &= \mathbb{P}\left[Y(1)|J = a\right] \\
\mathbb{P}\left(Y|T^* = 0, z = 1\right) &= \mathbb{P}\left[Y(0)|J = n\right].
\end{aligned}$$

Now suppose impose $P(Y|T^*, z) = P(Y|T^*)$ to identify α_0 and α_1. What would this require us to assume about the potential outcomes? Evaluating at each combination of (T^*, z), the assumption is equivalent to

$$\begin{aligned}
P(Y|T^* = 1, z = 1) = P(Y|T^* = 1, z = 0) = P(Y|T^* = 1) \\
P(Y|T^* = 0, z = 1) = P(Y|T^* = 0, z = 0) = P(Y|T^* = 0)
\end{aligned}$$

Since

$$P(Y|T^*) = P(Y|T^*, z = 1)P(z = 1|T^*) + P(Y|T^*, z = 0)\left[1 - P(z = 1|T^*)\right]$$

it suffices to establish that

$$\begin{aligned}
P(Y|T^* = 1, z = 0) = P(Y|T^* = 1, z = 1) \\
P(Y|T^* = 0, z = 1) = P(Y|T^* = 0, z = 0)
\end{aligned}$$

From the expressions given at the beginning of this section, these are equivalent to

$$\begin{aligned}
\mathbb{P}\left[Y(1)|J=a\right] &= \left(\frac{p_0^*}{p_1^*}\right)\mathbb{P}\left[Y(1)|J=a\right] + \left(\frac{p_1^*-p_0^*}{p_1^*}\right)\mathbb{P}\left[Y(1)|J=c\right] \\
\mathbb{P}\left[Y(0)|J=n\right] &= \left(\frac{p_1^*-p_0^*}{1-p_0^*}\right)\mathbb{P}\left[Y(0)|J=c\right] + \left(\frac{1-p_1^*}{1-p_0^*}\right)\mathbb{P}\left[Y(0)|J=n\right]
\end{aligned}$$

which are in turn equivalent to

$$\begin{aligned}
\mathbb{P}\left[Y(1)|J=a\right] &= \mathbb{P}\left[Y(1)|J=c\right] \\
\mathbb{P}\left[Y(0)|J=n\right] &= \mathbb{P}\left[Y(0)|J=c\right]
\end{aligned}$$

Thus, under the LATE assumptions from Kitagawa etc. $P(Y|T^*,z) = P(Y|T^*)$ is equivalent to assuming that the distribution of $Y(1)$ is the same for always-takers and compliers, and that the distribution of $Y(0)$ is the same for never-takers and compliers. If we are prepared to assume this, then α_0 and α_1 are identified and can be used to identify a LATE.

9.4 Partial Identification Under Weak Independence Assumption

Suppose we only make the LATE independence assumption $Y(T^*,z) = Y(T^*)$ rather than the conditional independence assumption $P(Y<\tau|T^*,z_k) = P(Y<\tau|T^*,z_\ell)$. Then we still obtain

$$\begin{aligned}
\mathbb{P}\left(Y|T^*=1,z=1\right) &= \left(\frac{p_0^*}{p_1^*}\right)\mathbb{P}\left[Y(1)|J=a\right] + \left(\frac{p_1^*-p_0^*}{p_1^*}\right)\mathbb{P}\left[Y(1)|J=c\right] \\
\mathbb{P}\left(Y|T^*=0,z=0\right) &= \left(\frac{p_1^*-p_0^*}{1-p_0^*}\right)\mathbb{P}\left[Y(0)|J=c\right] + \left(\frac{1-p_1^*}{1-p_0^*}\right)\mathbb{P}\left[Y(0)|J=n\right] \\
\mathbb{P}\left(Y|T^*=1,z=0\right) &= \mathbb{P}\left[Y(1)|J=a\right] \\
\mathbb{P}\left(Y|T^*=0,z=1\right) &= \mathbb{P}\left[Y(0)|J=n\right]
\end{aligned}$$

From above, we also know that

$$\begin{aligned}
P(Y|T^*=0,z_k) &= P(Y|T=0,z_k) + \left(\frac{\alpha_1 p_k}{1-p_k-\alpha_1}\right)\left[P(Y|T=0,z_k) - P(Y|T=1,z_k)\right] \\
P(Y|T^*=1,z_k) &= P(Y|T=1,z_k) + \left(\frac{\alpha_0(1-p_k)}{p_k-\alpha_0}\right)\left[P(Y|T=1,z_k) - P(Y|T=0,z_k)\right]
\end{aligned}$$

The notation is getting a bit unwieldy so let $\pi^*_{tk}(y) = P(Y = y|T^* = t, z_k)$ and similarly define $\pi_{tk}(y) = P(Y = y|T = t, z_k)$. Using this new notation, we have

$$\begin{aligned}
(1 - p_k - \alpha_1)\pi^*_{0k}(y) &= (1 - p_k - \alpha_1)\pi_{0k}(y) + \alpha_1 p_k \left[\pi_{0k}(y) - \pi_{1k}(y)\right] \\
(p_k - \alpha_0)\pi^*_{1k}(y) &= (p_k - \alpha_0)\pi_{1k}(y) + \alpha_0(1 - p_k) \left[\pi_{1k}(y) - \pi_{0k}(y)\right]
\end{aligned}$$

Writing these out for all values of k,

$$\begin{aligned}
(p_1 - \alpha_0)\pi^*_{11}(y) &= (p_1 - \alpha_0)\pi_{11}(y) + \alpha_0(1 - p_1) \left[\pi_{11}(y) - \pi_{01}(y)\right] \\
(1 - p_0 - \alpha_1)\pi^*_{00}(y) &= (1 - p_0 - \alpha_1)\pi_{00}(y) + \alpha_1 p_0 \left[\pi_{00}(y) - \pi_{10}(y)\right] \\
(p_0 - \alpha_0)\pi^*_{10}(y) &= (p_0 - \alpha_0)\pi_{10}(y) + \alpha_0(1 - p_0) \left[\pi_{10}(y) - \pi_{00}(y)\right] \\
(1 - p_1 - \alpha_1)\pi^*_{01}(y) &= (1 - p_1 - \alpha_1)\pi_{01}(y) + \alpha_1 p_1 \left[\pi_{01}(y) - \pi_{11}(y)\right]
\end{aligned}$$

Similarly, using the fact that $p^*_k = (p_k - \alpha_0)/(1 - \alpha_0 - \alpha_1)$,

$$\begin{aligned}
\pi^*_{11}(y) &= \left(\frac{p_0 - \alpha_0}{p_1 - \alpha_0}\right) P\left[Y(1)|J = a\right] + \left(\frac{p_1 - p_0}{p_1 - \alpha_0}\right) P\left[Y(1)|J = c\right] \\
\pi^*_{00}(y) &= \left(\frac{p_1 - p_0}{1 - p_0 - \alpha_1}\right) P\left[Y(0)|J = c\right] + \left(\frac{1 - p_1 - \alpha_1}{1 - p_0 - \alpha_1}\right) P\left[Y(0)|J = n\right] \\
\pi^*_{10}(y) &= P\left[Y(1)|J = a\right] \\
\pi^*_{01}(y) &= P\left[Y(0)|J = n\right]
\end{aligned}$$

or equivalently,

$$\begin{aligned}
(p_1 - \alpha_0)\pi^*_{11}(y) &= (p_0 - \alpha_0)\, P\left[Y(1)|J = a\right] + (p_1 - p_0)\, P\left[Y(1)|J = c\right] \\
(1 - p_0 - \alpha_1)\pi^*_{00}(y) &= (p_1 - p_0)\, P\left[Y(0)|J = c\right] + (1 - p_1 - \alpha_1)\, P\left[Y(0)|J = n\right] \\
(p_0 - \alpha_0)\pi^*_{10}(y) &= (p_0 - \alpha_0) P\left[Y(1)|J = a\right] \\
(1 - p_1 - \alpha_1)\pi^*_{01}(y) &= (1 - p_1 - \alpha_1) P\left[Y(0)|J = n\right]
\end{aligned}$$

Equating,

$$\begin{aligned}
(p_0 - \alpha_0)\, P\left[Y(1)|J = a\right] + (p_1 - p_0)\, P\left[Y(1)|J = c\right] &= (p_1 - \alpha_0)\pi_{11}(y) + \alpha_0(1 - p_1) \left[\pi_{11}(y) - \pi_{01}(y)\right] \\
(p_1 - p_0)\, P\left[Y(0)|J = c\right] + (1 - p_1 - \alpha_1)\, P\left[Y(0)|J = n\right] &= (1 - p_0 - \alpha_1)\pi_{00}(y) + \alpha_1 p_0 \left[\pi_{00}(y) - \pi_{10}(y)\right] \\
(p_0 - \alpha_0) P\left[Y(1)|J = a\right] &= (p_0 - \alpha_0)\pi_{10}(y) + \alpha_0(1 - p_0) \left[\pi_{10}(y) - \pi_{00}(y)\right] \\
(1 - p_1 - \alpha_1) P\left[Y(0)|J = n\right] &= (1 - p_1 - \alpha_1)\pi_{01}(y) + \alpha_1 p_1 \left[\pi_{01}(y) - \pi_{11}(y)\right]
\end{aligned}$$

and substituting the third and fourth equalities into the first and second we obtain

$$(p_0-\alpha_0)\pi_{10}(y)+\alpha_0(1-p_0)\left[\pi_{10}(y)-\pi_{00}(y)\right]+(p_1-p_0)\,P\left[Y(1)|J=c\right]=(p_1-\alpha_0)\pi_{11}(y)+\alpha_0(1-p_1)\left[\pi_{11}(y)-\pi_{01}(y)\right]$$
$$(p_1-p_0)\,P\left[Y(0)|J=c\right]+(1-p_1-\alpha_1)\pi_{01}(y)+\alpha_1p_1\left[\pi_{01}(y)-\pi_{11}(y)\right]=(1-p_0-\alpha_1)\pi_{00}(y)+\alpha_1p_0\left[\pi_{00}(y)-\pi_{10}(y)\right]$$
$$(p_0-\alpha_0)P\left[Y(1)|J=a\right]=(p_0-\alpha_0)\pi_{10}(y)+\alpha_0(1-p_0)\left[\pi_{10}(y)-\pi_{00}(y)\right]$$
$$(1-p_1-\alpha_1)P\left[Y(0)|J=n\right]=(1-p_1-\alpha_1)\pi_{01}(y)+\alpha_1p_1\left[\pi_{01}(y)-\pi_{11}(y)\right]$$

Simplifying and re-arranging,

$$P\left[Y(1)=y|J=c\right]=\left[\frac{p_1\pi_{11}(y)-p_0\pi_{10}(y)}{p_1-p_0}\right]-\alpha_0\left[\frac{p_1\pi_{11}(y)-p_0\pi_{10}(y)+(1-p_1)\pi_{01}(y)-(1-p_0)\pi_{00}(y)}{p_1-p_0}\right]$$
$$P\left[Y(0)=y|J=c\right]=\left[\frac{(1-p_0)\pi_{00}(y)-(1-p_1)\pi_{01}(y)}{p_1-p_0}\right]-\alpha_1\left[\frac{(1-p_0)\pi_{00}(y)-(1-p_1)\pi_{01}(y)+p_0\pi_{10}(y)-p_1\pi_{11}(y)}{p_1-p_0}\right]$$
$$P\left[Y(1)=y|J=a\right]=\pi_{10}(y)+\left[\frac{\alpha_0(1-p_0)}{p_0-\alpha_0}\right]\left[\pi_{10}(y)-\pi_{00}(y)\right]$$
$$P\left[Y(0)=y|J=n\right]=\pi_{01}(y)+\left[\frac{\alpha_1p_1}{1-p_1-\alpha_1}\right]\left[\pi_{01}(y)-\pi_{11}(y)\right]$$

Notice that the first two equations can be simplified as follows

$$P\left[Y(1)=y|J=c\right]=\left[\frac{P(Y=y,T=1|z=1)-P(Y,T=1|z=0)}{p_1-p_0}\right]-\alpha_0\left[\frac{P(Y=y|z=1)-P(Y=y|z=0)}{p_1-p_0}\right]$$
$$P\left[Y(0)=y|J=c\right]=\left[\frac{P(Y=y,T=0|z=1)-P(Y=y,T=0|z=1)}{p_1-p_0}\right]-\alpha_1\left[\frac{P(Y=y|z=0)-P(Y=y|z=1)}{p_1-p_0}\right]$$

Now, since probabilities must be between zero and one, we obtain the bounds

$$0\leq\left[\frac{P(Y=y,T=1|z=1)-P(Y=y,T=1|z=0)}{p_1-p_0}\right]-\alpha_0\left[\frac{P(Y=y|z=1)-P(Y=y|z=0)}{p_1-p_0}\right]\leq 1$$
$$0\leq\left[\frac{P(Y=y,T=0|z=1)-P(Y=y,T=0|z=1)}{p_1-p_0}\right]-\alpha_1\left[\frac{P(Y=y|z=0)-P(Y=y|z=1)}{p_1-p_0}\right]\leq 1$$

which we abbreviate

$$0\leq\left[\frac{\Delta P(Y=y,T=1)}{p_1-p_0}\right]-\alpha_0\left[\frac{\Delta P(Y=y)}{p_1-p_0}\right]\leq 1$$
$$0\leq\alpha_1\left[\frac{\Delta P(Y=y)}{p_1-p_0}\right]-\left[\frac{\Delta P(Y=y,T=0)}{p_1-p_0}\right]\leq 1$$

where

$$\Delta P(Y=y)=P(Y=y|z=1)-P(Y=y|z=0)$$
$$\Delta P(Y=y,T=t)=P(Y=y,T=t|z=1)-P(Y=y,T=t|z=0).$$

To manipulate these bounds, we need to know the sign of $R = \Delta P(Y = y)/(p_1 - p_0)$. Presumably this will be positive for most values of y, but it could be negative.

Case I: R is positive.

$$\frac{\Delta P(Y=y, T=1) - (p_1 - p_0)}{\Delta P(Y=y)} \leq \alpha_0 \leq \frac{\Delta P(Y=y, T=1)}{\Delta P(Y=y)}$$
$$\frac{\Delta P(Y=y, T=0)}{\Delta P(Y=y)} \leq \alpha_1 \leq \frac{\Delta P(Y=y, T=0) + (p_1 - p_0)}{\Delta P(Y=y)}$$

Case II: R is negative.

$$\frac{\Delta P(Y=y, T=1)}{\Delta P(Y=y)} \leq \alpha_0 \leq \frac{\Delta P(Y=y, T=1) - (p_1 - p_0)}{\Delta P(Y=y)}$$
$$\frac{\Delta P(Y=y, T=0) + (p_1 - p_0)}{\Delta P(Y=y)} \leq \alpha_1 \leq \frac{\Delta P(Y=y, T=0)}{\Delta P(Y=y)}$$

Note that we *two-sided* bounds for the misclassification probabilities. These may be trivial in some cases, but I don't think it's obvious that they always will be.

Do these bounds have anything to do with the testability of the LATE assumptions? That is, do we get a lower bound for measurement error *precisely when* we would otherwise violate a testable LATE assumption?

Note that we also obtain bounds from the potential outcome distributions of always-takers and never-takers, namely

$$0 \leq \pi_{10}(y) + \left[\frac{\alpha_0(1-p_0)}{p_0 - \alpha_0}\right][\pi_{10}(y) - \pi_{00}(y)] \leq 1$$
$$0 \leq \pi_{01}(y) + \left[\frac{\alpha_1 p_1}{1 - p_1 - \alpha_1}\right][\pi_{01}(y) - \pi_{11}(y)] \leq 1$$

These appear to be redundant: I think they're the same as the "CDF bounds" from above that don't require a valid instrument.