Estimating the Effect of a Mis-measured, Endogenous, Binary Treatment

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December 6th, 2016

What is the causal effect of T^*?

$$y_i = h(T_i^*, \mathbf{x}_i) + \varepsilon_i$$

- y – Outcome of interest
- h – Unknown function that *does not depend on i*
- T^* – Unobserved, endogenous binary treatment
- T – Observed, mis-measured binary surrogate for T^*
- $\mathbf{x}$ – Exogenous covariates
- ε – Mean-zero error term
- z – Discrete (typically binary) instrumental variable

Target of Inference:

ATE function: $\tau(\mathbf{x}) = h(1, \mathbf{x}) - h(0, \mathbf{x})$

Example: Smoking and Birthweight (SNAP Trial)

Coleman et al. (N Engl J Med, 2012)

RCT with 1050 pregnant smokers in England: 521 given nicotine patches, the rest given placebo patches.

- y – Birthweight
- T^* – True smoking behavior
- T – Self-reported smoking behavior
- $\mathbf{x}$ – Mother characteristics
- z – Indicator of nicotine patch

Example: Schooling and Test Scores

Burde & Linden (2013, AEJ Applied)

RCT in Afghanistan: 32 villages divided into 11 clusters. Randomly choose 6 and set up school in each village of these clusters.

- y – Child's score on math and language test
- T^* – Child's true school attendance
- T – Parent's report of child's school attendance
- $\mathbf{x}$ – Child and household characteristics
- z – School built in village

Example: Returns to Schooling

Oreopoulos (2006, AER)

Fuzzy RD: minimum school-leaving age in UK increased from 14 to 15 in 1947 but some already stayed until 15 before the law and others failed to comply after it.

- y – Log wage
- T^* – School attendance at age 15
- T – Self-report of school attendance at age 15
- $\mathbf{x}$ – Individual characteristics
- z – Indicator: born in or after 1933

Related Literature

Continuous Treatment

Lewbel (1997, 2012), Schennach (2004, 2007), Chen et al. (2005), Hu & Schennach (2008), Song (2015), Hu et al. (2015)...

Binary, Exogenous Treatment

Aigner (1973), Bollinger (1996), Kane et al. (1999), Black et al. (2000), Frazis & Loewenstein (2003), Mahajan (2006), Lewbel (2007)

Binary, Endogenous Treatment

Mahajan (2006), Shiu (2015), Ura (2015)

▸ Mahajan Details

Model: $y = c + \beta T^* + \varepsilon$

Valid Instrument

$\mathbb{E}[\varepsilon|z] = 0.$

First-stage

$p_k^* \equiv \mathbb{P}(T^* = 1|z = z_k) \neq \mathbb{P}(T^* = 1|z = z_\ell) \equiv p_\ell^*,\ k \neq \ell$

Non-differential Measurement Error

- $\mathbb{E}[\varepsilon|T^*, T, z] = \mathbb{E}[\varepsilon|T^*, z]$
- $\alpha_0 = \mathbb{P}(T = 1|T^* = 0, z)$
- $\alpha_1 = \mathbb{P}(T = 0|T^* = 1, z)$
- $\alpha_0 + \alpha_1 < 1$

Observable Moments: $y = c + \beta T^* + \varepsilon$

	$z = 1$	$z = 2$	$\dots$	$z = K$
$T = 0$	$\bar{y}_{01}$ p_{01}	$\bar{y}_{02}$ p_{02}	$\dots$	$\bar{y}_{0K}$ p_{0K}
$T = 1$	$\bar{y}_{11}$ p_{11}	$\bar{y}_{12}$ p_{12}	$\dots$	$\bar{y}_{1K}$ p_{1K}

$$\bar{y}_{tk} = \mathbb{E}[y|T = t, z = z_k], \quad p_{tk} = q_k p_k$$

$$q_k = \mathbb{P}(z = z_k), \quad p_k = \mathbb{P}(T = 1|z = z_k)$$

Unobservable Moments: $y = \beta T^* + u$

Define error term that absorbs constant: $u = c + \varepsilon$

	$z = 1$	$z = 2$	$\ldots$	$z = K$
$T^* = 0$	m^*_{01} p^*_{01}	m^*_{02} p^*_{02}	$\ldots$	m^*_{0K} p^*_{0K}
$T^* = 1$	m^*_{11} p^*_{11}	m^*_{12} p^*_{12}	$\ldots$	m^*_{1K} p^*_{1K}

$$m^*_{tk} = \mathbb{E}[u|T^* = t, z = z_k], \quad p^*_{tk} = q_k p^*_k$$

$$q_k = \mathbb{P}(z = z_k), \quad p^*_k = \mathbb{P}(T^* = 1|z = z_k)$$

Possible Restrictions On m^*_{tk}

Joint Exogeneity: $\mathbb{E}[\varepsilon | T^*, z] = 0$

$$\implies m^*_{tk} = c \quad \text{for all } t, k$$

Exogenous Treatment: $\mathbb{E}[\varepsilon | T^*] = 0$

$$\implies \frac{1}{\mathbb{P}(T^* = t)} \sum_k p^*_{tk} m^*_{tk} = c \quad \text{for all } t$$

Exogenous Instrument: $\mathbb{E}[\varepsilon | z] = 0$

$$\implies (1 - p^*_k) m^*_{0k} + p^*_k m^*_{1k} = c \quad \text{for all } k$$

System of Equations given $E[\varepsilon|z] = 0$

$$\boxed{\mathbb{E}[\varepsilon|z] = 0 \implies \textit{pair} \text{ of equations for each } k = 1, \ldots, K}$$

$$\hat{y}_{0k} = \alpha_1(p_k - \alpha_0)\left(\frac{\beta}{1 - \alpha_0 - \alpha_1}\right) + (1 - \alpha_0)c - (p_k - \alpha_0)m^*_{1k}$$

$$\hat{y}_{1k} = (1 - \alpha_1)(p_k - \alpha_0)\left(\frac{\beta}{1 - \alpha_0 - \alpha_1}\right) + \alpha_0 c + (p_k - \alpha_0)m^*_{1k}$$

where $\hat{y}_{0k} = (1 - p_k)\bar{y}_{0k}$ and $\hat{y}_{0k} = p_k \bar{y}_{1k}$

$$\boxed{2K \text{ Equations in } K + 4 \text{ Unknowns}}$$

β is undentified regardless of K.

Proof of special case: $\alpha_0 = 0$

1. System of equations:

$$\widetilde{y}_{0k} = c + p_k \left(\frac{\beta\alpha_1}{1-\alpha_1} \right) - p_k m^*_{1k}$$

$$\widetilde{y}_{1k} = p_k\beta + p_k m^*_{1k}$$

2. $\beta/(1-\alpha_1) \equiv \beta_{IV}$ identified, $\beta\alpha_1/(1-\alpha_1) = \beta_{IV} - \beta \implies$

$$(c + p_k\beta_{IV} - \widetilde{y}_{0k})/p_k = \beta + m^*_{1k}$$

$$\widetilde{y}_{1k}/p_k = \beta + m^*_{1k}$$

3. Sum equations from 1. $\implies (c + p_k\beta_{IV} - \widetilde{y}_{0k}) = \widetilde{y}_{1k}$

Bounds for Mis-classification Probabilities

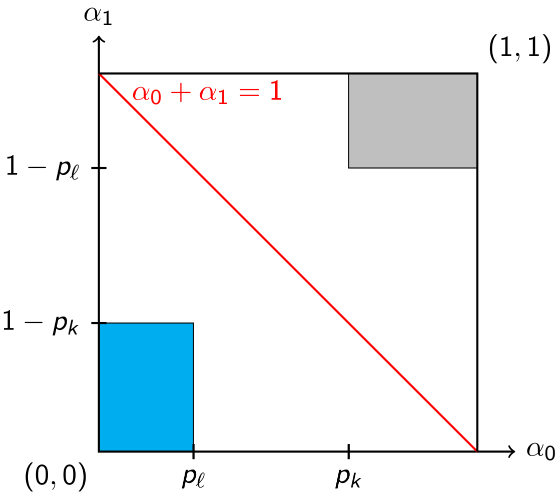
$\alpha_0(z) = \alpha_0,\ \alpha_1(z) = \alpha_1$

$$\Longrightarrow \quad p_k^* = \frac{p_k - \alpha_0}{1 - \alpha_0 - \alpha_1}, \quad 1 - p_k^* = \frac{1 - p_k - \alpha_1}{1 - \alpha_0 - \alpha_1}$$

$\alpha_0 + \alpha_1 < 1 \iff \mathrm{Cor}(T, T^*) > 0 \iff (1 - \alpha_0 - \alpha_1) > 0$

$\alpha_0 < \min_k\{p_k\},\ \alpha_1 < \min_k\{1 - p_k\}$

$\alpha_0 \leq \min_k \{p_k\}, \quad \alpha_1 \leq \min_k \{1 - p_k\}$



Bounds for β

$\mathbb{E}[\varepsilon|z] = 0$

$\implies \beta_{RF} = \mathbb{E}[y|z_k] - \mathbb{E}[y|z_\ell] = \beta(p_k^* - p_\ell^*)$

Mis-classification

$\implies p_k^* - p_\ell^* = (p_k - p_\ell)/(1 - \alpha_0 - \alpha_1)$

Combining: $\beta_{IV} = \beta/(1 - \alpha_0 - \alpha_1)$

$\alpha_0 + \alpha_1 < 1 \implies$

- β is between β_{RF} and β_{IV}
- β_{IV} *inflated* but has correct sign
- β_{RF} bound equivalent to substituting α_0, α_1 bounds

Strengthening the Measurement Error Assumptions

- $\alpha_0 = \mathbb{P}(T = 1 | T^* = 0, z)$
- $\alpha_1 = \mathbb{P}(T = 0 | T^* = 1, z)$
- $\alpha_0 + \alpha_1 < 1$
- $\mathbb{E}[\varepsilon | T^*, T, z] = \mathbb{E}[\varepsilon | T^*, z]$

Additional Assumption

$\mathbb{E}[\varepsilon^2 | T^*, T, z] = \mathbb{E}[\varepsilon^2 | T^*, z]$

Improve bounds for α_0, α_1 to tighten lower bound for β...

Tighter Bounds for α_0, α_1 from Conditional Variances

Assume

$\mathbb{E}[\varepsilon^2 | T^*, T, z] = \mathbb{E}[\varepsilon^2 | T^*, z]$

Observables

$\sigma^2_{tk} = \text{Var}(y | T = t, z = k)$

Constrain Unobservables

$s^{*2}_{tk} = Var(u | T^* = t, z_k) > 0$

$$
\begin{aligned}
(p_k - \alpha_0)\left[(1-\alpha_0)p_k\sigma^2_{1k} - \alpha_0(1-p_k)\sigma^2_{0k}\right] &> \alpha_0(1-\alpha_0)p_k(1-p_k)(\bar{y}_{1k} - \bar{y}_{0k})^2 \\
(1 - p_k - \alpha_1)\left[(1-\alpha_1)(1-p_k)\sigma^2_{0k} - \alpha_1 p_k\sigma^2_{1k}\right] &> \alpha_1(1-\alpha_1)p_k(1-p_k)(\bar{y}_{1k} - \bar{y}_{0k})^2
\end{aligned}
$$

Schooling and Test Scores – Afghan RCT

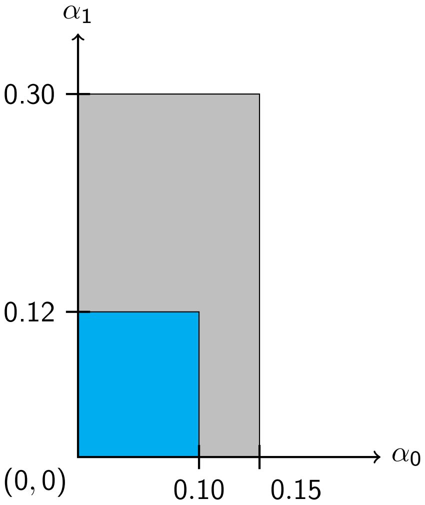
Burde & Linden (2013, AEJ Applied)

"Weak" Bounds

$\beta \in [0.65 \times \beta_{IV},\ \beta_{IV}]$

Add 2nd Moments

$\beta \in [0.78 \times \beta_{IV},\ \beta_{IV}]$



Independence Assumption: $\varepsilon \perp T | (T^*, z)$

Define $F_{tk}(\tau) = \mathbb{P}(Y \leq \tau | T = t, z_k)$ and $F_k(\tau) = \mathbb{P}(Y \leq \tau | z_k)$

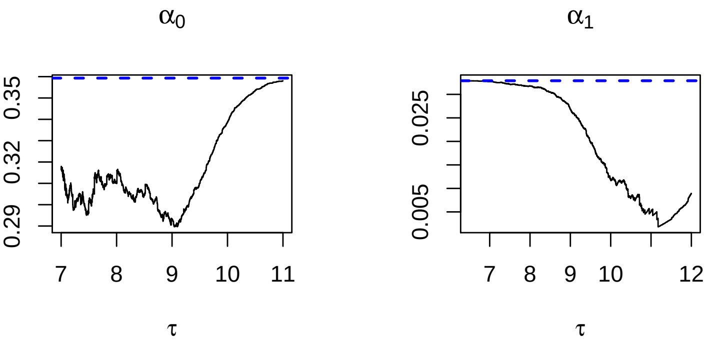
$$\alpha_0 \leq p_k \inf_\tau \left\{ \left[\frac{F_{1k}(\tau)}{F_k(\tau)} \right] \wedge \left[\frac{1 - F_{1k}(\tau)}{1 - F_k(\tau)} \right] \right\} \leq p_k$$

$$\alpha_1 \leq (1 - p_k) \inf_\tau \left\{ \left[\frac{F_{0k}(\tau)}{F_k(\tau)} \right] \wedge \left[\frac{1 - F_{0k}(\tau)}{1 - F_k(\tau)} \right] \right\} \leq (1 - p_k)$$

Bounds for (α_0, α_1) do *not* require z to be a valid instrument!

Upper Bounds for Mis-Classification Rates

Returns to Schooling Example: Oreopoulos (2006)



Sufficient Conditions To Identify α_0, α_1, and β

Baseline Assumptions

- $\mathbb{E}[\varepsilon|z] = 0$
- $\mathbb{E}[\varepsilon|T^*, T, z] = \mathbb{E}[\varepsilon|T^*, z]$
- $\alpha_0 = \mathbb{P}(T = 1|T^* = 0, z),\ \alpha_1 = \mathbb{P}(T = 0|T^* = 1, z),\ \alpha_0 + \alpha_1 < 1$

Strengthen IV Assumption

- $\mathbb{E}[\varepsilon^2|z] = \mathbb{E}[\varepsilon^2]$
- $\mathbb{E}[\varepsilon^3|z] = \mathbb{E}[\varepsilon^3]$

Strengthen Measurement Error Assumption

- $\mathbb{E}[\varepsilon^2|T^*, T, z] = \mathbb{E}[\varepsilon^2|T^*, z]$
- $\mathbb{E}[\varepsilon^3|T^*, T, z] = \mathbb{E}[\varepsilon^3|T^*, z]$

First Moment Condition

Assumptions

- $\mathbb{E}[\varepsilon|z] = 0$
- $\mathbb{E}[\varepsilon|T^*, T, z] = \mathbb{E}[\varepsilon|T^*, z]$
- $\alpha_0 = \mathbb{P}(T = 1|T^* = 0, z)$
- $\alpha_1 = \mathbb{P}(T = 0|T^* = 1, z)$

Moment Condition

$$\mathrm{Cov}(y, z) - \left(\frac{\beta}{1 - \alpha_0 - \alpha_1}\right) \mathrm{Cov}(T, z) = 0$$

MC # 1 identifies $\beta/(1 - \alpha_0 - \alpha_1)$

Second Moment Condition

Additional Assumptions

- $\mathbb{E}[\varepsilon^2|z] = \mathbb{E}[\varepsilon^2]$
- $\mathbb{E}[\varepsilon^2|T^*, T, z] = \mathbb{E}[\varepsilon|T^*, z]$

Moment Condition

$$\mathrm{Cov}(y^2, z) - \frac{\beta}{1-\alpha_0-\alpha_1}\left\{2\mathrm{Cov}(yT, z) - \beta\mathrm{Cov}(T, z)\left(\frac{1+\alpha_0-\alpha_1}{1-\alpha_0-\alpha_1}\right)\right\} = 0$$

Given MC #1, MC #2 identifies $(\alpha_1 - \alpha_0)$

Third Moment Condition

Additional Assumptions

- $\mathbb{E}[\varepsilon^3|z] = \mathbb{E}[\varepsilon^2]$
- $\mathbb{E}[\varepsilon^3|T^*, T, z] = \mathbb{E}[\varepsilon|T^*, z]$

Moment Condition

$$\mathrm{Cov}(y^3, z) - \left(\frac{\beta}{1-\alpha_0-\alpha_1}\right)\Bigg\{ \beta^2\left[1 + \frac{6\alpha_0(1-\alpha_1)}{(1-\alpha_0-\alpha_1)^2}\right]\mathrm{Cov}(T, z) \\ -3\beta\left[\frac{1-(\alpha_1-\alpha_0)}{1-\alpha_0-\alpha_1}\right]\mathrm{Cov}(yT, z) + 3\mathrm{Cov}(y^2T, z)\Bigg\} = 0$$

Sketch of Identification Argument

Very tedious algebra...

1. Use 1st MC to eliminate $\beta/(1-\alpha_0-\alpha_1)$ from others
2. Use 2nd MC to solve for α_1 in terms of α_0
3. 3rd MC becomes a quadratic in $(1-\alpha_1)$ and observables only.
4. The quadratic always has two real roots: $(1-\alpha_1)$ and α_0.
5. To tell which root is which, use $\alpha_0+\alpha_1<1$.
6. Calculate $\alpha_0+\alpha_1$ and substitute into 1st MC to obtain β.

Unfortunately, identification of α_0, α_1 fails if $\beta = 0$...

Simple Special Case: $\alpha_0 = 0$

$$\text{Cov}(y,z) - \left(\frac{\beta}{1-\alpha_0}\right)\text{Cov}(T,z) = 0$$

$$\text{Cov}(y^2,z) - \frac{\beta}{1-\alpha_0}\left\{2\text{Cov}(yT,z) - \beta\text{Cov}(T,z)\right\} = 0$$

$$\beta = \frac{2\text{Cov}(yT,z)}{\text{Cov}(T,z)} - \frac{\text{Cov}(y^2,z)}{\text{Cov}(y,z)}$$

Simulation Example: $y = \beta T^* + \varepsilon$

Errors

$(\varepsilon, \eta) \sim$ jointly normal, mean 0, variance 1, correlation 0.5.

First-Stage

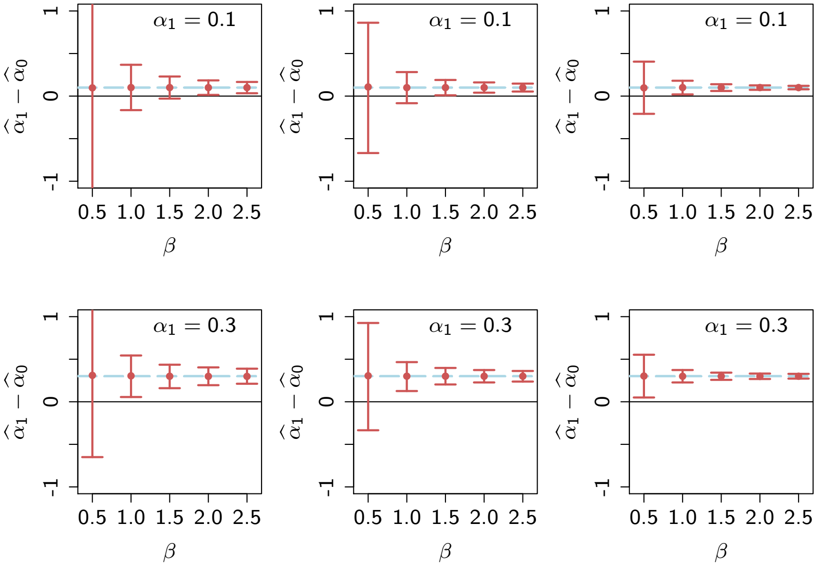
- Half of subjects have $z = 1$, the rest have $z = 0$.
- $T^* = \mathbf{1}\{\gamma_0 + \gamma_1 z + \eta > 0\}$
- $\delta = \mathbb{P}(T^* = 0|z = 1) = \mathbb{P}(T^* = 1|z = 0)$

Mis-classification

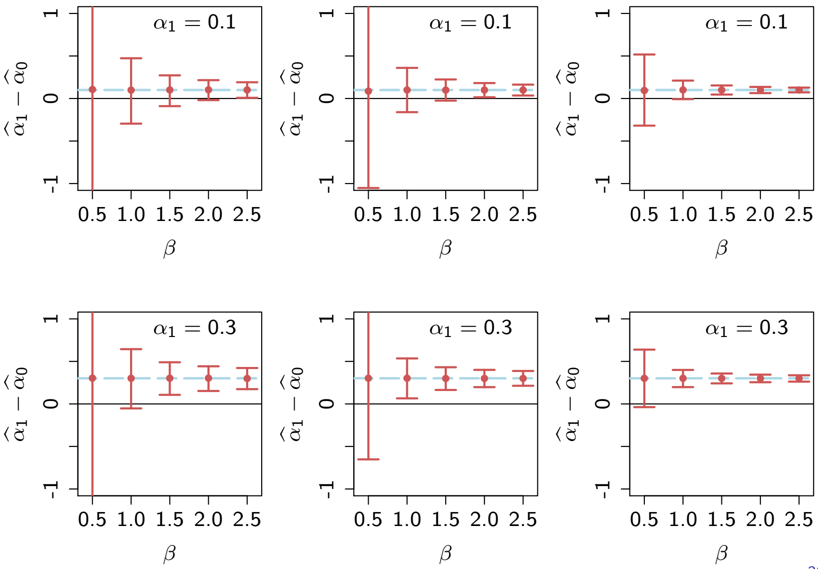
- Set $\alpha_0 = 0$ so $T^* = 0 \implies T = 0$
- $T|T^* = 1 \sim \text{Bernoulli}(1 - \alpha_1)$

$$\boxed{\text{Sampling Distribution of } \widehat{\alpha}_1 - \widehat{\alpha}_0}$$

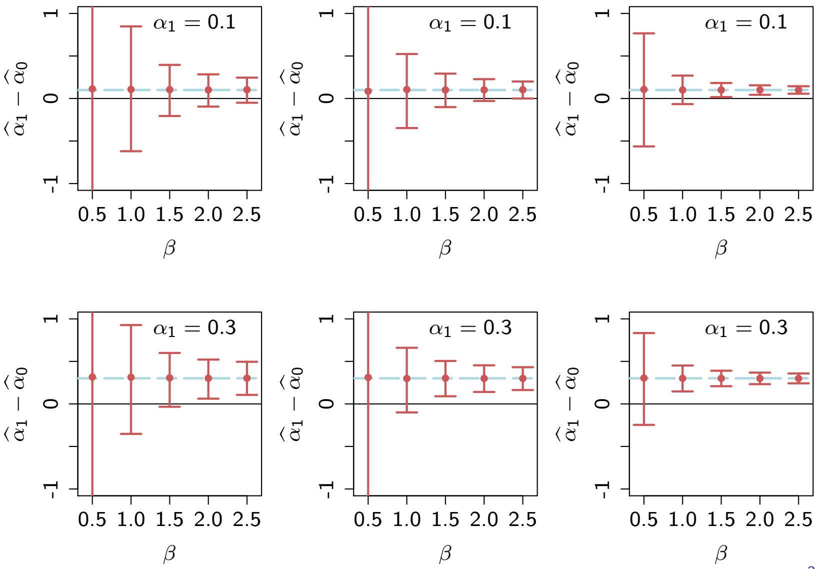
(a) $N = 500, \delta = 0.1$ (b) $N = 1000, \delta = 0.1$ (c) $N = 5000, \delta = 0.1$



(a) $N = 500, \delta = 0.2$ (b) $N = 1000, \delta = 0.2$ (c) $N = 5000, \delta = 0.2$

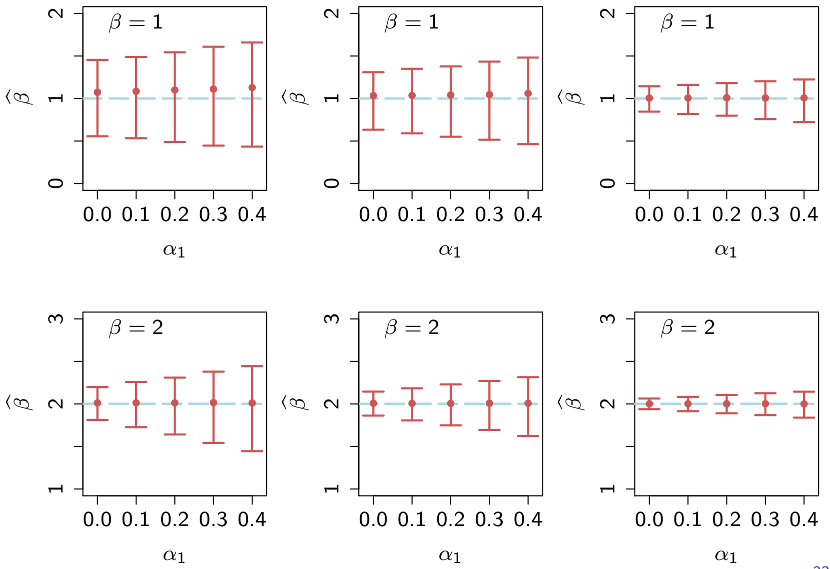


(a) $N = 500, \delta = 0.3$ (b) $N = 1000, \delta = 0.3$ (c) $N = 5000, \delta = 0.3$



$$\boxed{\text{Sampling Distribution of } \widehat{\beta} = (1 - \widehat{\alpha}_0 - \widehat{\alpha}_1)\widehat{\beta}_{IV}}$$

(a) $N = 500, \delta = 0.1$ (b) $N = 1000, \delta = 0.1$ (c) $N = 5000, \delta = 0.1$



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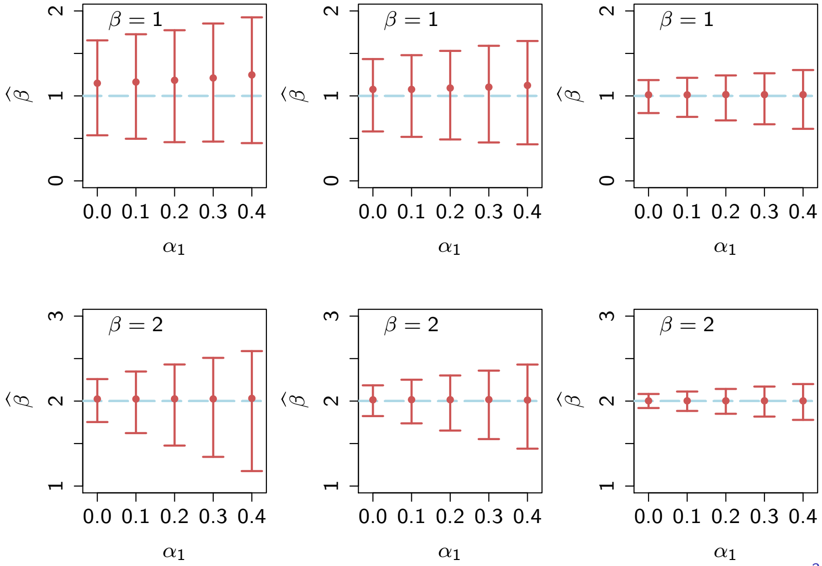

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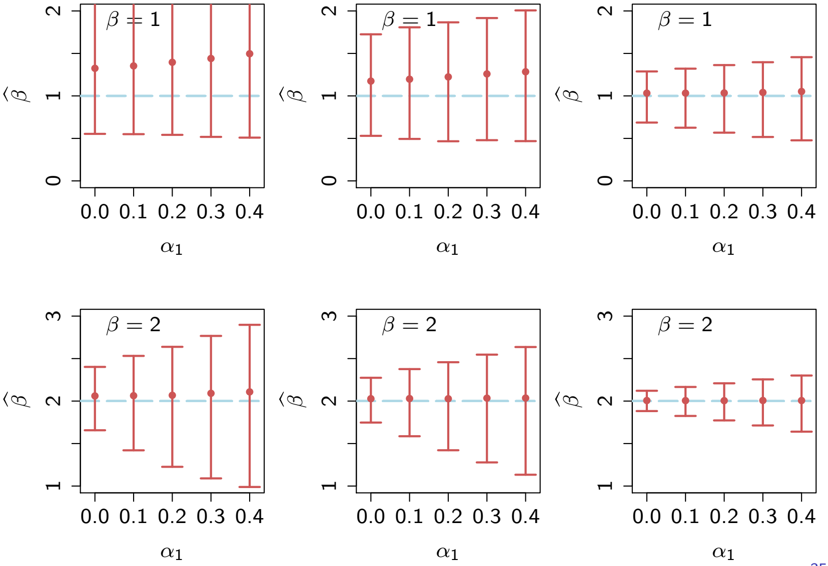


$(z \perp \varepsilon)$ and $(T \perp \varepsilon | T^*, z) \Rightarrow$ Continuum of MCs

Characteristic Functions

$$e^{i\omega\beta}\left[(1-\alpha_1) - \xi(\omega)\right] = \alpha_0 - \xi(\omega)$$

$$\xi(\omega) \equiv \frac{\varphi_k(\omega) - \varphi_\ell(\omega)}{p_k\varphi_{1k}(\omega) - p_\ell\varphi_{1\ell}(\omega)}$$

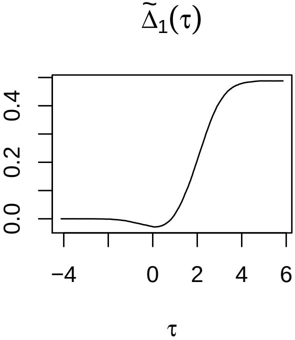
Distribution Functions

$$\widetilde{\Delta}_1(\tau+\beta) - \widetilde{\Delta}_1(\tau) = \alpha_0\Delta(\tau+\beta) - (1-\alpha_1)\Delta(\tau)$$

$$\Delta(\tau) = F_k(\tau) - F_\ell(\tau)$$

$$\widetilde{\Delta}_1(\tau) = p_k F_{1k}(\tau) - p_\ell F_{1\ell}(\tau)$$

CDF Conditions for Simulation DGP



$$\widetilde{\Delta}_1(\tau+\beta)-\widetilde{\Delta}_1(\tau)=\alpha_0\Delta(\tau+\beta)-(1-\alpha_1)\Delta(\tau)$$

Conclusion

Summary

- Endogenous, mis-measured binary treatment.
- Important in applied work but no solution in the literature.
- Usual (1st moment) IV assumption fails to identify β
- Bounds for mis-classification probabilities and β.
- Higher moment / independence restrictions identify β

Extensions / Work in Progress

- Efficient estimation w/ continuum of MCs
- Inference / Specification Testing
- LATE interpretation

Mahajan (2006, ECTA)

Regression Model

$y = \mathbb{E}[y|T^*] + \nu$

$\mathbb{E}[\nu|T^*] = 0$ by construction

Causal Model

$y = c + \beta T^* + \varepsilon$

$\mathbb{E}[\varepsilon|T^*] \neq 0$

Main Result (Correct) – Exogenous Treatment

Relevant binary instrument z ($p_k^* \neq p_\ell^*$) identifies α_0, α_1 and $\mathbb{E}[y|T^*]$ provided that $\mathbb{E}[\nu|T^*, T, z] = 0$ and $\alpha_0 + \alpha_1 < 1$.

Extension (Incorrect) – Endogenous Treatment

$\mathbb{E}[\varepsilon|z] = 0$, $p_k^* \neq p_\ell^*$, $\mathbb{E}[\varepsilon|T, T^*, z] = \mathbb{E}[\varepsilon|T^*] \implies \beta$ identified.

Mahajan (2006, ECTA)

Regression Model

$y = \mathbb{E}[y|T^*] + \nu$

$\mathbb{E}[\nu|T^*] = 0$ by construction

Causal Model

$y = c + \beta T^* + \varepsilon$

$\mathbb{E}[\varepsilon|T^*] \neq 0$

Ingredients

1. If $p_k^* \neq p_\ell^*$, $\mathbb{E}[\varepsilon|z] = 0$ then, since $\beta_{IV} = \beta/(1-\alpha_0-\alpha_1)$, knowledge of α_0, α_1 is sufficient to recover β. (Correct)
2. If $p_k^* \neq p_\ell^*$, $\mathbb{E}[\nu|T^*, T, z] = 0$, α_0, α_1 are identified. (Correct)

   How to satisfy both 1 and 2 while allowing $\mathbb{E}[\varepsilon|T^*] \neq 0$?
3. Assume that $\mathbb{E}[\varepsilon|T^*, T, z] = \mathbb{E}[\varepsilon|T^*]$

   (i.e. $m_{0k}^* = m_{0\ell}^*$ and $m_{1k}^* = m_{1\ell}^*$)

Flaw in the Argument

Proposition

If $\mathbb{E}[\varepsilon | T^*] \neq 0$ then $\mathbb{E}[\varepsilon | T^*, T, z] = \mathbb{E}[\varepsilon | T^*]$ combined with $\mathbb{E}[\varepsilon | z] = 0$ implies $p_k^* = p_\ell^*$, i.e. z is irrelevant for T^*.

Proof

$\mathbb{E}[\varepsilon | z] = 0$ implies

$$(1 - p_1^*) m_{0k}^* + p_1^* m_{1k}^* = c$$

$$(1 - p_2^*) m_{0\ell}^* + p_2^* m_{1\ell}^* = c$$

while Mahajan's assumption implies $m_{0k}^* = m_{0\ell}^*$ and $m_{1k}^* = m_{1\ell}^*$. Therefore either $m_{0k}^* = m_{0\ell}^* = m_{1k}^* = m_{1\ell}^* = c$, which is ruled out by $E[\varepsilon | T^*] = 0$, or $p_k^* = p_\ell^*$.

Back