Identifying the Effect of a Mis-classified, Binary, Endogenous Regressor*

Francis J. DiTraglia[1] and Camilo García-Jimeno[2,3]

[1]Department of Economics, University of Pennsylvania
[2]Department of Quantitative Social Science, Emory University
[3]NBER

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Abstract

This paper studies identification and inference for the effect of a mis-classified, binary, endogenous regressor when a discrete-valued instrumental variable is available. We begin by showing that the only existing point identification result for this model is incorrect. We go on to derive the sharp identified set under mean independence assumptions for the instrument and measurement error. The resulting bounds are novel and informative, but fail to point identify the effect of interest. This motivates us to consider alternative and slightly stronger assumptions: we show that adding second and third moment independence assumptions suffices to identify the model.

Keywords: Instrumental variables, Measurement error, Endogeneity

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1 Introduction

Measurement error and endogeneity are pervasive features of economic data. Conveniently, a valid instrumental variable corrects for both problems when the measurement error is classical, i.e. uncorrelated with the true value of the regressor. Many regressors of interest in applied work, however, are binary and thus cannot be subject to classical measurement error.[1] When faced with non-classical measurement error, the instrumental variables estimator can be severely biased. In this paper, we study an additively separable model of the form

$$y = c(\mathbf{x}) + \beta(\mathbf{x})T^* + \varepsilon \tag{1}$$

where ε is a mean-zero error term, T^* is a binary, potentially endogenous regressor of interest, and $\mathbf{x}$ is a vector of exogenous controls.[2] We ask whether, and if so under what conditions, a discrete instrumental variable z suffices to non-parametrically identify the causal effect $\beta(\mathbf{x})$ of T^*, when we observe not T^* but a mis-classified binary surrogate T.

We proceed under the assumption of non-differential measurement error. This condition has been widely used in the existing literature and imposes that T provides no additional information beyond that contained in $(T^*, \mathbf{x})$. Even in this fairly standard setting, identification remains an open question: we begin by showing that the only existing identification result for this model is incorrect. We then go on to derive the sharp identified set under the standard first-moment assumptions from the related literature. We show that regardless of the number of values that z takes on, the model is not point identified. This motivates us to consider alternative, and slightly stronger assumptions. We show that, given a binary instrument, the addition of a second moment independence assumption suffices to identify a model with one-sided mis-classification. Adding a second moment restriction on the measurement error along with a third moment independence assumption for the instrument suffices to identify the model in general. This result likewise requires only a binary z.

Our work relates to a large literature studying departures from the textbook linear, classical measurement error setting. One strand of this literature considers relaxing the assumption of linearity while maintaining that of classical measurement error. Schennach (2004), for example, uses repeated measures of each mis-measured regressor to obtain identification, while Schennach (2007) uses an instrumental variable. More recently, Song et al. (2015) rely on a repeated measure of the mis-measured regressor and the existence of a set of additional regressors, conditional upon which the regressor of interest is unrelated to the unobservables,

[1] The only way to mis-classify a true one is downwards, as a zero, while the only way to mis-classify a true zero is upwards, as a one. This creates negative dependence between the truth and measurement error.

[2] Because T^* is binary, there is no loss of generality from writing the model in this form rather than the more familiar $y = h(T^*, \mathbf{x}) + \varepsilon$. Simply define $\beta(\mathbf{x}) = h(1, \mathbf{x}) - h(0, \mathbf{x})$ and $c(\mathbf{x}) = h(0, \mathbf{x})$.

to obtain identification. For comprehensive reviews of the challenges of addressing measurement error in non-linear models, see Chen et al. (2011) and Schennach (2013). Another strand of the literature considers relaxing the assumption of classical measurement error, by allowing the measurement error to be related to the true value of the unobserved regressor. Chen et al. (2005) obtain identification in a general class of moment condition models with mis-measured data by relying on the existence of an auxiliary dataset from which they can estimate the measurement error process. In contrast, Hu and Shennach (2008) and Song (2015) rely on an instrumental variable and an additional conditional location assumption on the measurement error distribution. More recently, Hu et al. (2015) use a continuous instrument to identify the ratio of partial effects of two continuous regressors, one measured with error, in a linear single index model. Unfortunately, these approaches cannot be applied to the case of a mis-measured binary regressor.

A number of papers have studied models with an exogenous binary regressor subject to non-differential measurement error. One group of papers asks what can be learned without recourse to an instrumental variable. An early contribution by Aigner (1973) characterizes the asymptotic bias of OLS in this setting, and proposes a correction using outside information on the mis-classification process. Related work by Bollinger (1996) provides partial identification bounds. More recently, Chen et al. (2008a) use higher moment assumptions to obtain identification in a linear model, and Chen et al. (2008b) extend these results to the non-parametric setting. van Hasselt and Bollinger (2012) and Bollinger and van Hasselt (2015) provide additional partial identification results. For results on the partial identification of discrete probability distributions under mis-classification, see Molinari (2008).

Continuing under the assumption of exogeneity and non-differential measurement error, another group of papers relies on the availability of either an instrumental variable or a second measure of T^*. Black et al. (2000) and Kane et al. (1999) consider a linear model and show that when *two* alternative measures T_1 and T_2 of T^* are available, a non-linear GMM estimator can be used to recover the effect of interest. Subsequently, Frazis and Loewenstein (2003) note that an instrumental variable can take the place of one of the measures. Mahajan (2006) extends the results of Black et al. (2000) and Kane et al. (1999) to a more general setting using a binary instrument in place of one of the treatment measures, establishing non-parametric identification of the conditional mean function. When T^* is in fact exogenous, this coincides with the causal effect. Hu (2008) derives related results when the mis-classified discrete regressor may take on more than two values. Lewbel (2007a) provides an identification result for the same model as Mahajan (2006) under different assumptions. In particular, his "instrument-like variable" need not satisfy the usual exclusion restriction so long as it does not interact with T^* and takes on three or more values.

Much less is known about the case in which a binary, or discrete, regressor is not only mis-classified but endogenous. The first paper to provide a formal result for this case is Mahajan (2006). He extends his main result to the case of an endogenous treatment, providing an explicit proof of identification under the usual IV assumption in a model with additively separable errors. As we show below, however, this result is false.[3] Several more recent papers also consider the case of a mis-classified, endogenous, binary regressor. Kreider et al. (2012), partially identify the effects of food stamps on health outcomes of children under weak measurement error assumptions by relying on auxiliary data. Similarly, Battistin et al. (2014) study the returns to schooling in a setting with multiple mis-reported measures of educational qualifications. Unlike these two papers, our approach does not depend on the availability of auxiliary data. In a different vein, Shiu (2016) uses an exclusion restriction for the participation equation and an additional valid instrument to identify the effect of a discrete, mis-classified endogenous regressor in a semi-parametric selection model. Similarly, Nguimkeu et al. (2016) use exclusion restrictions for both the participation equation and measurement error equation to identify a parametric model with endogenous participation and one-sided endogenous mis-reporting. Unlike those of the preceding two papers, our results rely neither on parametric assumptions nor additional exclusion restrictions. Other than Mahajan (2006), the paper most closely related to our own is that of Ura (Forthcoming), who derives partial identification results for a local average treatment effect without the non-differential assumption. In contrast, we study an additively separable model under non-differential measurement error and derive both partial and point identification results.

The remainder of the paper is organized as follows. Section 2.1 describes our model and assumptions, Section 2.2 relates our results to existing work, and Sections 2.3–2.4 present our identification results. Section 3 provides a brief discussion of how to carry out inference using our identification results, and Section 4 concludes. Proofs appear in Appendix A, and we give a detailed explanation of the error in Mahajan (2006) in Appendix B.

2 Identification Results

2.1 Baseline Assumptions

As defined in the preceding section, our model is $y = c(\mathbf{x}) + \beta(\mathbf{x})T^* + \varepsilon$, where ε is a mean-zero error term, and the parameter of interest is $\beta(\mathbf{x})$ – the effect of an unobserved, binary, endogenous regressor T^*. Suppose we observe a valid and relevant binary instrument z. In the discussion following Corollary 2.3 below, we explain how these results generalize to the

[3] Appendix B provides a detailed explanation of the error in Mahajan's proof.

case of an arbitrary discrete-valued instrument. We assume that the model and instrument satisfy the following conditions:

Assumption 2.1.

(i) $y = c(\mathbf{x}) + \beta(\mathbf{x})T^* + \varepsilon$ *where* $T^* \in \{0, 1\}$ *and* $\mathbb{E}[\varepsilon] = 0$*;*

(ii) $z \in \{0, 1\}$*, where* $0 < \mathbb{P}(z = 1|\mathbf{x}) < 1$*, and* $\mathbb{P}(T^* = 1|\mathbf{x}, z = 1) \neq \mathbb{P}(T^* = 1|\mathbf{x}, z = 0)$*;*

(iii) $\mathbb{E}[\varepsilon|\mathbf{x}, z] = 0$*.*

Assumptions 2.1(ii) and (iii) are the standard instrument relevance and mean independence assumptions.[4] If T^* were observed, Assumption 2.1 would suffice to identify $\beta(\mathbf{x})$. Unfortunately we observe not T^* but a mis-classified binary surrogate T. Define the following mis-classification probabilities:

$$\alpha_0(\mathbf{x}, z) = \mathbb{P}\left(T = 1|T^* = 0, \mathbf{x}, z\right), \quad \alpha_1(\mathbf{x}, z) = \mathbb{P}\left(T = 0|T^* = 1, \mathbf{x}, z\right). \tag{2}$$

Following the existing literature for the case of an exogenous regressor (Black et al., 2000; Frazis and Loewenstein, 2003; Kane et al., 1999; Lewbel, 2007a; Mahajan, 2006), we impose the following conditions on the mis-classification process.

Assumption 2.2.

(i) $\alpha_0(\mathbf{x}, z) = \alpha_0(\mathbf{x})$*,* $\alpha_1(\mathbf{x}, z) = \alpha_1(\mathbf{x})$

(ii) $\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) < 1$

(iii) $\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, z, T^*]$

Assumption 2.2 (i) states that the mis-classification probabilities do not depend on z. As we maintain this assumption throughout, we drop the dependence of α_0 and α_1 on z and write $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$. Assumption 2.2 (ii) restricts the extent of mis-classification and is equivalent to requiring that T and T^* be positively correlated. Assumption 2.2 (iii) is often referred to as "non-differential measurement error." Intuitively, it maintains that T provides no additional information about ε, and hence y, given knowledge of $(T^*, z, \mathbf{x})$.

[4] Assumption 2.1 (ii) states that z is a relevant instrument for the unobserved regressor T^*. Under Assumption 2.2, however, this is equivalent to assuming that z is a relevant instrument for the observed regressor T by Lemma 2.1 below.

2.2 Point Identification Results from the Literature

Existing results from the literature – see for example Frazis and Loewenstein (2003) and Mahajan (2006) – establish that $\beta(\mathbf{x})$ is point identified if Assumptions 2.1–2.2 are augmented to include the following condition:

Assumption 2.3 (Joint Exogeneity)**.** $\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*] = 0$.

Assumption 2.3 strengthens the mean independence condition from Assumption 2.1 (iii) to hold *jointly* for T^* and z. By iterated expectations, this implies that T^* is exogenous, i.e. $\mathbb{E}[\varepsilon|\mathbf{x}, T^*] = 0$. If T^* is endogenous, Assumption 2.3 clearly fails. Mahajan (2006) argues, however, that the following restriction, along with our Assumptions 2.1–2.2, suffices to identify $\beta(\mathbf{x})$ when T^* may be endogenous:

Assumption 2.4 (Mahajan (2006) Equation 11)**.** $\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon|\mathbf{x}, T^*]$.

Assumption 2.4 does not require $\mathbb{E}[\varepsilon|\mathbf{x}, T^*]$ to be zero, but maintains that it does not vary with z. We show in Appendix B, however, that under Assumptions 2.1–2.2, Assumption 2.4 can only hold if T^* is exogenous. If z is a valid instrument and T^* is endogenous, then Assumption 2.4 implies that there is no first-stage relationship between z and T^*. As such, identification in the case where T^* is endogenous is an open question.

2.3 Partial Identification

In this section we derive the sharp identified set under Assumptions 2.1–2.2 and show that $\beta(\mathbf{x})$ is not point identified. To simplify the notation, define the following shorthand for the unobserved and observed first stage probabilities

$$p_k^*(\mathbf{x}) = \mathbb{P}(T^* = 1|\mathbf{x}, z = k) \tag{3}$$

$$p_k(\mathbf{x}) = \mathbb{P}(T = 1|\mathbf{x}, z = k). \tag{4}$$

We first state two lemmas that have appeared in various guises throughout the literature. These will be used repeatedly below.

Lemma 2.1. *Under Assumption 2.2 (i),*

$$[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})]\, p_k^*(\mathbf{x}) = p_k(\mathbf{x}) - \alpha_0(\mathbf{x})$$

$$[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})]\, [1 - p_k^*(\mathbf{x})] = 1 - p_k(\mathbf{x}) - \alpha_1(\mathbf{x})$$

where the first-stage probabilities $p_k^*(\mathbf{x})$ and $p_k(\mathbf{x})$ are as defined in Equations 3–4.

Lemma 2.2. *Under Assumptions 2.1 and 2.2 (i)–(ii),*

$$\beta(\mathbf{x})\,Cov(z,T|\mathbf{x}) = [1-\alpha_0(\mathbf{x})-\alpha_1(\mathbf{x})]\,Cov(y,z|\mathbf{x})$$

Lemma 2.1 relates the observed first-stage probabilities $p_k(\mathbf{x})$ to their unobserved counterparts $p_k^*(\mathbf{x})$ in terms of the mis-classification probabilities $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$. By Assumption 2.2 (ii), $1-\alpha_0(\mathbf{x})-\alpha_1(\mathbf{x})>0$ so that Lemma 2.1 provides non-trivial bounds for $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ in terms of the observed first-stage probabilities. Lemma 2.2 relates the instrumental variables (IV) estimand, $\text{Cov}(y,z|\mathbf{x})/\text{Cov}(z,T|\mathbf{x})$, to the mis-classification probabilities. Since $1-\alpha_0(\mathbf{x})-\alpha_1(\mathbf{x})>0$, IV is biased *upwards* in the presence of mis-classification. Combining the two lemmas yields a well-known bound, namely that $\beta(\mathbf{x})$ lies between the reduced form and IV estimators. Our first result shows that *without* Assumption 2.2 (non-differential measurement error) these bounds are sharp.

Theorem 2.1. *Under Assumptions 2.1 and 2.2 (i)–(ii), the sharp identified set is characterized by*

$$\mathbb{E}[y|\mathbf{x},z=k] = c(\mathbf{x}) + \beta(\mathbf{x})\left[\frac{p_k(\mathbf{x})-\alpha_0(\mathbf{x})}{1-\alpha_0(\mathbf{x})-\alpha_1(\mathbf{x})}\right] \tag{5}$$

and $\alpha_0(\mathbf{x}) \leq p_k(\mathbf{x}) \leq 1-\alpha_1(\mathbf{x})$ for $k=0,1$ where $p_k(\mathbf{x})$ is defined in Equation 4.

Corollary 2.1. *Under the conditions of Theorem 2.1, the sharp identified set for $\beta(\mathbf{x})$ is the closed interval between the reduced form estimand $Cov(y,z|\mathbf{x})/Var(z|\mathbf{x})$ and the IV estimand $Cov(y,z|\mathbf{x})/Cov(z,T|\mathbf{x})$.*

Corollary 2.1 follows by taking differences of the expression for $\mathbb{E}[y|\mathbf{x},z=k]$ across $k=1$ and $k=0$, and substituting the maximum and minimum value for $\alpha_0(\mathbf{x})+\alpha_1(\mathbf{x})$ consistent with the observed first-stage probabilities. When the mis-classification probabilities are known *a priori* to satisfy additional restrictions, these bounds can be tightened.[5] The following corollary collects results for two common cases: one-sided misclassification (either $\alpha_0(\mathbf{x})$ or $\alpha_1(\mathbf{x})$ equals zero), and symmetric mis-classification ($\alpha_0(\mathbf{x})=\alpha_1(\mathbf{x})$).

Corollary 2.2. *Under the conditions of Theorem 2.1, the following restrictions on the mis-classification probabilities $\alpha_0(\mathbf{x}),\alpha_1(\mathbf{x})$ shrink the sharp identified set for $\beta(\mathbf{x})$ to the closed interval between $\Delta\times[Cov(y,z|\mathbf{x})/Cov(z,T|\mathbf{x})]$ and $Cov(y,z|\mathbf{x})/Cov(z,T|\mathbf{x})$.*

(i) If $\alpha_0(\mathbf{x})=0$ then $\Delta=\max_k\, p_k(\mathbf{x})$.

(ii) If $\alpha_1(\mathbf{x})=0$ then $\Delta=1-\min_k\, p_k(\mathbf{x})$.

[5] Frazis and Loewenstein (2003) consider a model in which α_0 and α_1 do not depend on the exogenous covariates $\mathbf{x}$. In this case $\alpha_0 \leq \mathbb{P}(T=1|\mathbf{x},z) \leq 1-\alpha_1$ and they suggest minimizing the bounds over $\mathbf{x}$.

(iii) If $\alpha_0(\mathbf{x}) = \alpha_1(\mathbf{x})$ then $\Delta = 1 - 2\min\{\min_k p_k(\mathbf{x}), 1 - \max_k p_k(\mathbf{x})\}$.

Theorem 2.1 and Corollaries 2.1–2.2 do not impose Assumption 2.2 (iii) – non-differential measurement error. We now show that this assumption yields further restrictions on the misclassification probabilities $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$. While these restrictions are more complicated to describe than those from Theorem 2.1, they are straightforward to implement in practice and can be extremely informative, as we will show in our simulation exercises below. To the best of our knowledge, the sharp bounds that we derive by adding Assumption 2.2 (iii) are new to the literature. Our result uses two additional conditions to simplify the proof of sharpness. First, we assume that y is continuously distributed. This is natural in an additively separable model and holds in our simulation examples below. Without this assumption, the bounds that we derive are still valid, but may not be sharp. Nevertheless, the reasoning from our proof can be generalized to cases in which y does not have a continuous support set. We also impose $\mathbb{E}[y|\mathbf{x}, T = 0, z = k] \neq \mathbb{E}[y|\mathbf{x}, T = 1, z = k]$ for any k. This holds generically and is not essential to the proof: it merely simplifies the description of the identified set.

Theorem 2.2. *Suppose that the conditional distribution of y given $(\mathbf{x}, T, z)$ is continuous for any values of the conditioning variables and $\mathbb{E}[y|\mathbf{x}, T = 0, z = k] \neq \mathbb{E}[y|\mathbf{x}, T = 1, z = k]$ for all k. Then, under Assumptions 2.1 and 2.2, the sharp identified set is characterized by Equation 5 from Theorem 2.1 along with $\alpha_0(\mathbf{x}) < p_k(\mathbf{x}) < 1 - \alpha_1(\mathbf{x})$ for $k = 0, 1$ and*

$$\underline{\mu}_{tk}\left(\underline{q}_{tk}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}),\ \mathbf{x}\right) \leq \mu_k(\alpha_0(\mathbf{x}), \mathbf{x}) \leq \overline{\mu}_{tk}\left(\overline{q}_{tk}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}),\ \mathbf{x}\right)$$

for all pairs (t, k) where

$$\underline{\mu}_{tk}(q, \mathbf{x}) = \mathbb{E}\left[y \mid y \leq q, \mathbf{x}, T = t, z = k\right], \qquad \overline{\mu}_{tk}(q, \mathbf{x}) = \mathbb{E}\left[y \mid y > q, \mathbf{x}, T = t, z = k\right]$$

$$\mu_k(\alpha_0(\mathbf{x}), \mathbf{x}) = \frac{p_k(\mathbf{x})\mathbb{E}[y|\mathbf{x}, z = k, T = 1] - \alpha_0(\mathbf{x})\mathbb{E}[y|\mathbf{x}, z = k]}{p_k(\mathbf{x}) - \alpha_0(\mathbf{x})}$$

and we define

$$\begin{aligned}\underline{q}_{tk}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}) &= F_{tk}^{-1}\left(r_{tk}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}) \,\middle|\, \mathbf{x}\right)\\ \overline{q}_{tk}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}) &= F_{tk}^{-1}\left(1 - r_{tk}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}) \,\middle|\, \mathbf{x}\right)\end{aligned}$$

where $F_{tk}^{-1}(\cdot|\mathbf{x})$ *is the conditional quantile function of* y *given* $(\mathbf{x}, T = t, z = k)$,

$$r_{0k}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}) = \frac{\alpha_1(\mathbf{x})}{1 - p_k(\mathbf{x})}\left[\frac{p_k(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}\right]$$
$$r_{1k}(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \mathbf{x}) = \frac{1 - \alpha_1(\mathbf{x})}{p_k(\mathbf{x})}\left[\frac{p_k(\mathbf{x}) - \alpha_0(\mathbf{x})}{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})}\right]$$

and $p_k(\mathbf{x})$ *is defined in Equation 4.*

The intuition for Theorem 2.2 is as follows. For simplicity, suppress dependence on $\mathbf{x}$. Now, fix $(T = t, z = k)$ and (α_0, α_1). The observed distribution of y given $(T = t, z = k)$, call it F_{tk}, is a mixture of two unobserved distributions: the distribution of y given $(T = 1, z = k, T^* = 1)$, call it F_{tk}^1, and the distribution of y given $(T = t, z = k, T^* = 0)$, call it F_{tk}^0. The mixing probabilities are r_{tk} and $1 - r_{tk}$ from the statement of Theorem 2.2 and are fully determined by (α_0, α_1) and p_k. Assumptions 2.1 (i) and 2.2 (ii) imply that the unobserved means $\mathbb{E}[y|T^*, T, z]$ are fully determined by (α_0, α_1) given the observed means $\mathbb{E}[y|T, z]$. The question is whether it is possible, given the observed distribution F_{tk}, to construct F_{tk}^1 and F_{tk}^0 with the required values for $\mathbb{E}[y|T^*, T, z]$ such that $F_{tk} = r_{tk}F_{tk}^1 + (1 - r_{tk})F_{tk}^0$ for all combinations (t, k). If not, then (α_0, α_1) does not belong to the identified set. Our proof provides necessary and sufficient conditions for such a mixture to exist at a given point (α_0, α_1). We can then appeal to the reasoning from Theorem 2.1 to complete the argument. By ruling out values for α_0 and α_1, Theorem 2.2 restricts β via Lemma 2.2. While these restrictions can be very informative in practice, they do not yield point identification.

Corollary 2.3. *Under Assumptions 2.1 and 2.2 the identified set for* $\beta(\mathbf{x})$ *contains both the IV estimand* $Cov(y, z|\mathbf{x})/Cov(z, T|\mathbf{x})$ *and the true coefficient* $\beta(\mathbf{x})$.

Corollary 2.3 follows by Lemma 2.2 because $\alpha_0(\mathbf{x}) = \alpha_1(\mathbf{x}) = 0$ always belongs to the sharp identified set from Theorem 2.2. Non-differential measurement error cannot exclude the possibility that there is no mis-classification because in this case it is trivial to construct the required mixtures. Although we focus throughout this paper on the case of a binary instrument, one might wonder whether point identification can be achieved by increasing the support of z, perhaps along the lines of Lewbel (2007a). The answer turns out to be no. Suppose that we were to modify Assumptions 2.1 and 2.2 to hold for all values of z in some discrete support set. By Lemma 2.2, a binary instrument identifies $\beta(\mathbf{x})$ up to knowledge of the mis-classification probabilities $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$. It follows that *any* pair of values (k, ℓ) in the support set of z identifies the same object. Accordingly, to identify $\beta(\mathbf{x})$ it is necessary and sufficient to identify the mis-classification probabilities. A binary instrument fails to identify these probabilities because we can never exclude the possibility of zero mis-classification. The same is true of a discrete K-valued instrument. Increasing the support of z does, however, shrink the identified set by increasing the number of restrictions available: in this case Theorems 2.1–2.2 continue to apply replacing "$k = 0, 1$" with "for all k."

2.4 Point Identification

The results of the preceding section establish that $\beta(\mathbf{x})$ is not point identified under Assumptions 2.1 and 2.2. In light of this, there are two possible ways to proceed: either one can report partial identification bounds based on our characterization of the sharp identified set from Theorem 2.2, or one can attempt to impose stronger assumptions to obtain point identification. In this section we consider the second possibility. We begin by defining the following functions of the model parameters:

$$\theta_1(\mathbf{x}) = \beta(\mathbf{x})\left[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right]^{-1} \tag{6}$$

$$\theta_2(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^2\left[1 + \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right] \tag{7}$$

$$\theta_3(\mathbf{x}) = \left[\theta_1(\mathbf{x})\right]^3\left[\left\{1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})\right\}^2 + 6\alpha_0(\mathbf{x})\left\{1 - \alpha_1(\mathbf{x})\right\}\right] \tag{8}$$

Now consider the following additional assumption:

Assumption 2.5. $\mathbb{E}[\varepsilon^2|\mathbf{x}, z] = \mathbb{E}[\varepsilon^2|\mathbf{x}]$

Assumption 2.5 is a *second moment* version of the standard mean exclusion restriction for the instrument z – Assumption 2.1 (iii). It requires that the conditional variance of the error term given the covariates $\mathbf{x}$ does not depend on z, but does *not* require homoskedasticity with respect to $\mathbf{x}, T^*$ or T. Assumption 2.5 allows us to derive the following lemma:

Lemma 2.3. *Under Assumptions 2.1, 2.2 and 2.5,*

$$Cov(y^2, z|\mathbf{x}) = 2\,Cov(yT, z|\mathbf{x})\theta_1(\mathbf{x}) - Cov(T, z|\mathbf{x})\theta_2(\mathbf{x})$$

where $\theta_1(\mathbf{x})$ and $\theta_2(\mathbf{x})$ are defined in Equations 6–7.

Lemma 2.2 identifies $\theta_1(\mathbf{x})$. Since $\text{Cov}(z, T|\mathbf{x}) \neq 0$ by Assumption 2.1 (ii), we can solve for $\theta_2(\mathbf{x})$ in terms of observables only, using Lemma 2.3. Given knowledge of $\theta_1(\mathbf{x})$, we can solve Equation 7 for the difference of mis-classification rates so long as $\beta(\mathbf{x}) \neq 0$.

Corollary 2.4. *Under Assumptions 2.1–2.2 and 2.5, $\alpha_1(\mathbf{x}) - \alpha_0(\mathbf{x})$ is identified so long as $\beta(\mathbf{x}) \neq 0$.*

Corollary 2.4 identifies the difference of mis-classification error rates. Hence, under one-sided mis-classification, $\alpha_0(\mathbf{x}) = 0$ or $\alpha_1(\mathbf{x}) = 0$, augmenting our baseline Assumptions 2.1–2.2 with Assumption 2.5 suffices to identify $\beta(\mathbf{x})$. Notice that $\beta(\mathbf{x}) = 0$ if and only if $\theta_1(\mathbf{x}) = 0$. Thus, $\beta(\mathbf{x})$ is still identified in the case where Corollary 2.4 fails to apply.

Assumption 2.5 does not suffice to identify $\beta(\mathbf{x})$ without *a priori* restrictions on the mis-classification error rates. To achieve identification in the general case, we impose the following additional conditions:

Assumption 2.6.

(i) $\mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*, T] = \mathbb{E}[\varepsilon^2|\mathbf{x}, z, T^*]$

(ii) $\mathbb{E}[\varepsilon^3|\mathbf{x}, z] = \mathbb{E}[\varepsilon^3|\mathbf{x}]$

Assumption 2.6 (i) is a second moment version of the non-differential measurement error assumption, Assumption 2.2 (iii). It requires that, given knowledge of $(\mathbf{x}, T^*, z)$, T provides no additional information about the variance of the error term. Note that Assumption 2.6 (i) does not require homoskedasticity of ε with respect to $\mathbf{x}$ or T^*. Assumption 2.6 (ii) is a third moment version of Assumption 2.5. It requires that the conditional third moment of the error term given $\mathbf{x}$ does not depend on z. This condition neither requires nor excludes skewness in the error term conditional on covariates: it merely states that the skewness is unaffected by the instrument.

While Assumptions 2.5 and 2.6 may appear unfamiliar, we consider them to be fairly natural in the context of an additively separable model in which one has already assumed that $\mathbb{E}[\varepsilon|z] = 0$ and $\mathbb{E}[\varepsilon|\mathbf{x}, z, T^*, T] = \mathbb{E}[\mathbf{x}, z, T^*]$ – Assumptions 2.1 (iii) and 2.2 (iii) from above.[6] For example, if an applied researcher reports results both for an outcome in logs and levels, she has implicitly assumed *independence* rather than first moment exclusion. Assumptions 2.1 (iii), 2.5 and 2.6 (ii) are of course implied by $\varepsilon \perp z|\mathbf{x}$ while Assumptions 2.2 (iii) and 2.6 (i) are implied by $\varepsilon \perp T|(\mathbf{x}, T^*, z)$. Achieving identification via Assumptions 2.5–2.6 involves using information beyond first moments and as such does places higher demands on the data. Assumption 2.6 allows us to derive the following Lemma which, combined with Lemma 2.3, leads to point identification:

Lemma 2.4. *Under Assumptions 2.1–2.2 and 2.5–2.6,*

$$Cov(y^3, z|\mathbf{x}) = 3Cov(y^2T, z|\mathbf{x})\theta_1(\mathbf{x}) - 3Cov(yT, z|\mathbf{x})\theta_2(\mathbf{x}) + Cov(T, z|\mathbf{x})\theta_3(\mathbf{x})$$

where $\theta_1(\mathbf{x}), \theta_2(\mathbf{x})$ *and* $\theta_3(\mathbf{x})$ *are defined in Equations 6–7.*

[6] If one wishes to weaken our Assumption 2.1 (i) to allow for some form of unobserved heterogeneity, our higher moment assumptions may impose additional restrictions.

Theorem 2.3. *Under Assumptions 2.1–2.2 and 2.5–2.6, $\beta(\mathbf{x})$ is identified. If $\beta(\mathbf{x}) \neq 0$, then $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are likewise identified.*

Lemmas 2.2–2.4 yield a linear system of three equations in $\theta_1(\mathbf{x}), \theta_2(\mathbf{x})$ and $\theta_3(\mathbf{x})$. Under Assumption 2.1 (ii), the system has a unique solution so $\theta_1(\mathbf{x}), \theta_2(\mathbf{x})$ and $\theta_3(\mathbf{x})$ are identified. The proof of Theorem 2.3 shows that, so long as $\beta(\mathbf{x}) \neq 0$, Equations 6–8 can be solved for $\beta(\mathbf{x})$, $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$. If we relax Assumption 2.2 (ii) and assume $\alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x}) \neq 1$ only, $\beta(\mathbf{x})$ is only identified up to sign.

3 Estimation and Inference

We now briefly outline how the identification results from Section 2 can be used to estimate and carry out statistical inference for the structural parameters of interest: $\big(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \beta(\mathbf{x})\big)$. Lemmas 2.2–2.4 yield a system of linear moment equations in the reduced form parameters $\boldsymbol{\theta}'(\mathbf{x}) = \big(\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \theta_3(\mathbf{x})\big)$. Defining a vector of intercepts $\boldsymbol{\kappa}'(\mathbf{x}) = \big(\kappa_1(\mathbf{x}), \kappa_2(\mathbf{x}), \kappa_3(\mathbf{x})\big)$, and a vector of observables $\mathbf{w}' = (T, y, yT, y^2, y^2T, y^3)$, we can write this system as

$$\mathbb{E}\left[\left\{\boldsymbol{\Psi}\big(\boldsymbol{\theta}(\mathbf{x})\big)\mathbf{w}_i - \boldsymbol{\kappa}(\mathbf{x})\right\} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} \middle| \mathbf{x} = \boldsymbol{x}\right] = \mathbf{0} \tag{9}$$

where the matrix of functions $\boldsymbol{\Psi}\big(\boldsymbol{\theta}(\mathbf{x})\big)$ is given by

$$\boldsymbol{\Psi}\big(\boldsymbol{\theta}(\mathbf{x})\big) = \begin{bmatrix} -\theta_1(\mathbf{x}) & 1 & 0 & 0 & 0 & 0 \\ \theta_2(\mathbf{x}) & 0 & -2\theta_1(\mathbf{x}) & 1 & 0 & 0 \\ -\theta_3(\mathbf{x}) & 0 & 3\theta_2(\mathbf{x}) & 0 & -3\theta_1(\mathbf{x}) & 1 \end{bmatrix}.$$

Using Equations 6–8, we can re-write $\boldsymbol{\Psi}$ as a function of $\big(\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x}), \beta(\mathbf{x})\big)$, leaving us with a just-identified, non-parametric conditional moment problem. Because the conditioning variables in Equation 9 are the same as the arguments of the unknown functions $(\alpha_0, \alpha_1, \beta)$, this problem fits within the framework of Lewbel (2007b), permitting straightforward estimation and inference via a local GMM procedure. If $\beta(\mathbf{x})$ is close to zero, however, this procedure can perform poorly; in this case the moment conditions from Equations 9, are only weakly informative about $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$. For more discussion of this problem, along with proposed solution combining the partial identification bounds from Section 2.3 with tools from the moment inequality literature, see DiTraglia and García-Jimeno (2017).

4 Conclusion

This paper has studied identification and inference for a mis-classified, binary, endogenous regressor in an additively separable model using a discrete instrumental variable. We have shown that the only existing identification result for this model is incorrect, and gone on to derive the sharp identified set under standard first-moment assumptions from the literature. Strengthening these assumptions to hold for second and third moments, we have established point identification for the effect of interest. A interesting extension of the results presented above would be to consider the case of discrete regressors that take on more than two values.

A Proofs

Throughout the following arguments, we suppress dependence on $\mathbf{x}$ for simplicity.

A.1 Partial Identification Results

Proof of Lemma 2.1. Follows from a simple calculation using the law of total probability. $\square$

Proof of Lemma 2.2. Immediate since $\text{Cov}(z,T) = (1-\alpha_0-\alpha_1)\text{Cov}(z,T^*)$ by Lemma 2.1. $\square$

Proof of Theorem 2.1. We first show that so long as $\alpha_0 \leq p_k \leq 1-\alpha_1$ then we can construct a valid joint probability distribution for (T^*, T, z) that satisfies our assumptions. First decompose the joint probability mass function as

$$p(T^*, T, z) = p(T|T^*, z)p(T^*|z)p(z).$$

By Assumption 2.2 (ii), $p(T|T^*, z) = p(T|T^*)$ and thus α_0 and α_1 fully determine $p(T|T^*, z)$. Under the proposed bounds, α_0 and α_1 are clearly valid probabilities. Since $p(z)$ is observed, it thus suffices to ensure that $p(T^*|z)$ is a valid probability mass function. By Lemma 2.1, $p_k^* = (p_k - \alpha_0)/(1-\alpha_0-\alpha_1)$ and hence $0 \leq p_k^* \leq 1$ if and only if $\alpha_0 \leq p_k \leq 1-\alpha_1$. Since $(p_k - p_\ell) = (p_k^* - p_\ell^*)(1-\alpha_0-\alpha_1)$, we have $p_k^* \neq p_\ell^*$ provided that $p_k - p_\ell \neq 0$

We now show how to construct a valid conditional distribution for y given (T^*, T, z) that satisfies our assumptions if $\beta(p_k - \alpha_0) = (1-\alpha_0-\alpha_1)[\mathbb{E}(y|z=k) - c]$ for all k. Define

$$\begin{aligned} r_{tk} &\equiv \mathbb{P}(T^*=1|T=t, z=k) & F_t(\tau) &\equiv \mathbb{P}(y \leq \tau|z=k) \\ F_{tk}(\tau) &\equiv \mathbb{P}(y \leq \tau|T=t, z=k) & F_{tk}^{t^*}(\tau) &\equiv \mathbb{P}(y \leq \tau|T^*=t^*, T=t, z=k) \\ G_k(\tau) &\equiv \mathbb{P}(\varepsilon \leq \tau|z=k) & G_{tk}^{t^*}(\tau) &\equiv \mathbb{P}(\varepsilon \leq \tau|T^*=t^*, T=t, z=k). \end{aligned}$$

Assumption 2.1 (i) implies a relationship between $G_{tk}^{t^*}$ and $F_{tk}^{t^*}$ for each t^*, namely

$$G_{tk}^0(\tau) = F_{tk}^0(\tau + c), \quad G_{tk}^1(\tau) = F_{tk}^1(\tau + c + \beta) \tag{A.1}$$

and thus we see that

$$\begin{aligned} G_k(\tau) = {} & r_{1k}p_kF_{1k}^1(\tau + c + \beta) + r_{0k}(1-p_k)F_{0k}^1(\tau + c + \beta) \\ & + (1-r_{1k})p_kF_{1k}^0(\tau + c) + (1-r_{0k})(1-p_k)F_{0k}^0(\tau + c) \end{aligned} \tag{A.2}$$

applying the law of total probability and Bayes' rule. Moreover, again applying the law of total probability,

$$F_{tk}(\tau) = r_{tk}F^1_{tk}(\tau) + (1-r_{tk})F^0_{tk}(\tau) \tag{A.3}$$

for all $t, k \in \{0,1\}$, and by Bayes' rule,

$$r_{1k} = \frac{(1-\alpha_1)p^*_k}{p_k}, \quad r_{0k} = \frac{\alpha_1 p^*_k}{1-p_k}. \tag{A.4}$$

There are four cases, corresponding to different possibilities for the r_{tk}.

Case I: $r_{1k} = 0, r_{0k} \neq 0$ By Equation A.4, this requires $\alpha_1 = 1$ which is ruled out by Assumption 2.2 (ii).

Case II: $r_{0k} = r_{1k} = 0$ By Equation A.4, this requires $p^*_k = 0$ which in turn requires $p_k = \alpha_0$. Moreover, by Equation A.3 we have $F^0_{tk} = F_{tk}$, while F^1_{tk} is undefined. Substituting into Equation A.2,

$$G_k(\tau) = p_k F_{1k}(\tau + c) + (1-p_k)F_{0k}(\tau + c) = F_k(\tau + c)$$

Now, since $F_k(\tau + c)$ is the conditional CDF of $y - c$ given that $z = k$, and G_k is the conditional CDF of ε given $z = k$, we see that Assumption 2.1 (i) is satisfied if and only if $\mathbb{E}(y|z=k) = c$. But since $p_k = \alpha_0$ in this case, $c = c + \beta(p_k - \alpha_0)/(1 - \alpha_0 - \alpha_1)$.

Case III: $r_{1k} \neq 0, r_{0k} = 0$ By Equation A.4 this requires $\alpha_1 = 0$ and $p^*_k \neq 0$. By Equation A.3 we have $F^0_{0k} = F_{0k}$ and since $r_{1k} \neq 1$, we can solve to obtain

$$F^1_{1k}(\tau) = \frac{1}{r_{1k}}\left[F_{1k}(\tau) - (1-r_{1k})F^0_{1k}(\tau)\right]$$

Substituting into Equation A.2, we obtain

$$\begin{aligned} G_k(\tau) = &\left[(1-p_k)F_{0k}(\tau + c) + p_k F_{1k}(\tau + c + \beta)\right] \\ &+ p_k(1-r_{1k})\left[F^0_{1k}(\tau + c) - F^0_{1k}(\tau + c + \beta)\right] \end{aligned}$$

Now, $F_{0k}(\tau + c)$ is the conditional CDF of $(y - c)$ given $(T = 0, z = k)$ while $F_{1k}(\tau + c + \beta)$ is the conditional CDF of $(y - c - \beta)$ given $(T = 1, z = k)$. Similarly, $F^0_{1k}(\tau + c)$ is the conditional CDF of ε given $(T^* = 0, T = 1, z = k)$ while $F^0_{1k}(\tau + c + \beta)$ is the conditional CDF of $(\varepsilon - \beta)$ given $(T^* = 0, T = 1, z = k)$. Since $G_k(\tau)$ is the conditional CDF of ε given $z = k$, we see that Assumption 2.1 (iii) is satisfied if and only if

$$\begin{aligned} 0 = &(1-p_k)\mathbb{E}(y - c|T = 0, z = k) + p_k\mathbb{E}(y - c - \beta|T = 1, z = k) \\ &+ p_k(1-r_{1k})\left[\mathbb{E}(\varepsilon|T^* = 0, T = 1, z = k) - \mathbb{E}(\varepsilon - \beta|T^* = 0, T = 1, z = k)\right] \end{aligned}$$

Rearranging, this is equivalent to

$$\mathbb{E}(y|z=k) = c + (1-\alpha_1)\beta\left(\frac{p_k - \alpha_0}{1-\alpha_0-\alpha_1}\right) = c + \beta\left(\frac{p_k - \alpha_0}{1-\alpha_0-\alpha_1}\right)$$

since $\alpha_1 = 0$ in this case. As explained above, $F^0_{0k} = F_{0k}$ in the present case while F^1_{0k} is undefined. We are free to choose any distributions for F^0_{1k} and F^1_{1k} that satisfy Equation A.3, for example

$F^0_{1k} = F^1_{1k} = F_{1k}$.

Case IV: $r_{1k} \neq 0, r_{0k} \neq 0$ In this case, we can solve Equation A.3 to obtain

$$F^1_{tk}(\tau) = \frac{1}{r_{tk}}\left[F_{tk}(\tau) - (1 - r_{tk})F^0_{tk}(\tau)\right]$$

Substituting this into Equation A.2, we have

$$\begin{aligned} G_k(\tau) = F_k(\tau + c + \beta) + p_k(1 - r_{1k})\left[F^0_{1k}(\tau + c) - F^0_{1k}(\tau + c + \beta)\right] \\ + (1 - p_k)(1 - r_{0k})\left[F^0_{0k}(\tau + c) - F^0_{0k}(\tau + c + \beta)\right] \end{aligned}$$

using the fact that $F_k(\tau) = p_k F_{1k}(\tau) + (1 - p_k)F_{0k}(\tau)$. Now, $F_k(\tau + c + \beta)$ is the conditional CDF of $(y - c - \beta)$ given $z = k$, while $F^0_{tk}(\tau + c)$ is the conditional CDF of ε given $(T = t, z = k)$ and $F^0_{tk}(\tau + c + \beta)$ is the conditional CDF of $(\varepsilon - \beta)$ given $(T = t, z = k)$. Since $G_k(\tau)$ is the conditional CDF of ε given $z = k$, we see that Assumption 2.1 (iii) is satisfied if and only if

$$\begin{aligned} 0 &= \mathbb{E}[y - c - \beta|z = k] + p_k(1 - r_{1k})\left[\mathbb{E}(\varepsilon|T^* = 0, T = 1, z = k) - \mathbb{E}(\varepsilon - \beta|T^* = 0, T = 1, z = k)\right] \\ &\quad + (1 - p_k)(1 - r_{0k})\left[\mathbb{E}(\varepsilon|T^* = 0, T = 0, z = k) - \mathbb{E}(\varepsilon - \beta|T^* = 0, T = 0, z = k)\right] \\ 0 &= \mathbb{E}[y - c - \beta|z = k] + \beta\left[p_k(1 - r_{1k}) + (1 - p_k)(1 - r_{0k})\right] \end{aligned}$$

But since $[p_k(1 - r_{1k}) + (1 - p_k)(1 - r_{0k})] = (1 - p^*_k)$ and $p^*_k = (p_k - \alpha_0)/(1 - \alpha_0 - \alpha_1)$, this becomes

$$\mathbb{E}[y|z = k] = c + \beta\left[(p_k - \alpha_0)(1 - \alpha_0 - \alpha_1)\right].$$

Thus, in this case we are free to choose *any* distributions for F^0_{tk} and F^1_{tk} that satisfy Equation A.3. For example we could take $F^0_{tk} = F^1_{tk} = F_{tk}$. □

Proof of Corollary 2.1. Follows by plugging in the largest and smallest possible values for $\alpha_0 + \alpha_1$ and taking the difference of the expressions for $\mathbb{E}[y|z = k]$ □

Proof of Theorem 2.2. Under Assumption 2.1 (i) and Assumption 2.2 (iii), we obtain $\mathbb{E}(y|T^*, T, z) = \mathbb{E}(y|T^*, z)$. Hence, by iterated expectations

$$\begin{aligned} \mathbb{E}(y|T = 0, z = k) = (1 - r_{0k})\mathbb{E}(y|T^* = 0, z = k) + r_{0k}\mathbb{E}(y|T^* = 1, z = k) \\ \mathbb{E}(y|T = 1, z = k) = (1 - r_{1k})\mathbb{E}(y|T^* = 0, z = k) + r_{1k}\mathbb{E}(y|T^* = 1, z = k) \end{aligned}$$

where r_{tk} is defined as in the proof of Theorem 2.1. This is system of two linear equations in two unknowns: $\mathbb{E}(y|T^* = 0, z = k)$ and $\mathbb{E}(y|T^* = 1, z = k)$. After some algebra, we find that the determinant is

$$r_{1k} - r_{0k} = \left[\frac{p_k - \alpha_0}{1 - \alpha_0 - \alpha_1}\right]\left[\frac{1 - p_k - \alpha_1}{p_k(1 - p_k)}\right]$$

and thus a unique solution exists provided that $\alpha_0 \neq p_k$ and $\alpha_1 \neq 1 - p_k$. By our assumption that $\mathbb{E}[y|T = 0, z = k] \neq \mathbb{E}[y|T = 1, z = k]$, the system has no solution when the determinant condition fails. Thus, Assumption 2.2 (iii) rules out $\alpha_0 = p_k$ and $\alpha_1 = 1 - p_k$. Solving,

$$\mu_k^0 \equiv \mathbb{E}(y|T^* = 0, z = k) = \left(\frac{1}{1 - p_k - \alpha_1}\right)[(1 - p_k)\mathbb{E}(y|T = 0, z = k) - \alpha_1 \mathbb{E}(y|z = k)]$$
$$\mu_k^1 \equiv \mathbb{E}(y|T^* = 1, z = k) = \left(\frac{1}{p_k - \alpha_0}\right)[p_k \mathbb{E}(y|T = 1, z = k) - \alpha_0 \mathbb{E}(y|z = k)]$$

Given (α_0, α_1), we see that r_{tk}, μ_k^0, and μ_k^1 are fixed. The question is whether, for a given pair (α_0, α_1) and observed CDFs F_{tk}, we can construct valid CDFs F_{tk}^0, F_{tk}^1 such that

$$\int_{\mathbb{R}} \tau F_{tk}^0(d\tau) = \mu_k^0, \quad \int_{\mathbb{R}} \tau F_{tk}^1(d\tau) = \mu_k^1, \quad F_{tk}(\tau) = r_{tk} F_{tk}^1(\tau) + (1 - r_{tk}) F_{tk}^0(\tau)$$

where F_{tk} and $F_{tk}^{t^*}$ are as defined in the proof of Theorem 2.2. For a given pair (t, k), there are two cases: $0 < r_{tk} < 1$ and $r_{tk} \in \{0, 1\}$.

Case I: $r_{tk} \in \{0, 1\}$ Suppose that $r_{tk} = 1$. Then, $\mu_k^1 = \mathbb{E}[y|T = t, z = k]$ so we can simply set $F_{tk}^1 = F_{tk}$. In this case F_{tk}^0 is undefined. If instead $r_{tk} = 0$, then $\mu_k^0 = \mathbb{E}[y|T = t, z = k]$ so we can simply set $F_{tk}^0 = F_{tk}$. In this case F_{tk}^1 is undefined.

Case II: $0 < r_{tk} < 1$ Define

$$\mu_{tk}(\xi) = \mathbb{E}[y|y \in I_{tk}(\xi), T = t, z = k]$$
$$I_{tk}(\xi) = \left[F_{tk}^{-1}(1 - \xi - r_{tk}), F_{tk}^{-1}(1 - \xi)\right]$$

for $t, k = 0, 1$ where $0 \leq \xi \leq 1 - r_{tk}$ and F_{tk}^{-1} is the quantile function of y given $(T = t, z = k)$. We see that μ_{tk} is a decreasing function of ξ that attains its maximum at $\xi = 0$ and minimum at $\xi = 1 - r_{tk}$. Define these extrema as $\underline{\mu}_{tk} = \mu_{tk}(1 - r_{tk})$ and $\overline{\mu}_{tk} = \mu_{tk}(0)$.

Suppose first that μ_k^1 does *not* lie in the interval $[\underline{\mu}_{tk}, \overline{\mu}_{tk}]$. We show that it is impossible to construct valid CDFs F_{tk}^0 and F_{tk}^1 that satisfy $F_{tk}(\tau) = r_{tk} F_{tk}^1(\tau) + (1 - r_{tk}) F_{tk}^0(\tau)$ where F_{tk} and $F_{tk}^{t^*}$ are as defined in the proof of Theorem 2.2. Since $r_{tk} \neq 1$, we can solve the expression for F_{tk} to yield $F_{tk}^0(\tau) = \left[F_{tk}(\tau) - r_{tk} F_{tk}^1(\tau)\right]/(1 - r_{tk})$. Hence, since $r_{tk} \neq 0$, the requirement that $0 \leq F_{tk}^0(\tau) \leq 1$ implies

$$\frac{F_{tk}(\tau) - (1 - r_{tk})}{r_{tk}} \leq F_{tk}^1(\tau) \leq \frac{F_{tk}(\tau)}{r_{tk}} \tag{A.5}$$

Now define

$$\underline{F}_{tk}^1(\tau) = \min\{1, F_{tk}(\tau)/r_{tk}\}$$
$$\overline{F}_{tk}^1(\tau) = \max\{0, F_{tk}(\tau)/r_{tk} - (1 - r_{tk})/r_{tk}\}$$

Combining Equation A.5 with the requirement that $0 \leq F_{tk}^1(\tau) \leq 1$, we see that

$$\overline{F}_{tk}^1(\tau) \leq F_{tk}^1(\tau) \leq \underline{F}_{tk}^1(\tau)$$

Hence $\overline{F}_{tk}^1$ first-order stochastically dominates F_{tk}^1 which in turn first-order stochastically dominates $\underline{F}_{tk}^1$. It follows that

$$\int \tau \underline{F}_{tk}^1(d\tau) \leq \int \tau F_{tk}^1(d\tau) \leq \int \tau \overline{F}_{tk}^1(d\tau)$$

But notice that

$$\underline{\mu}_{tk} = \int \tau \underline{F}^1_{tk}(d\tau), \quad \mu^1_k = \int \tau F^1_{tk}(d\tau), \quad \overline{\mu}_{tk} = \int \tau \overline{F}^1_{tk}(d\tau)$$

so we have $\underline{\mu}_{tk} \leq \mu^1_k \leq \overline{\mu}_{tk}$ which contradicts $\mu^1_k \notin [\underline{\mu}_{tk}, \overline{\mu}_{tk}]$.

Now suppose that $\mu^1_k \in \left[\underline{\mu}_{tk}, \overline{\mu}_{tk}\right]$. Since y is assumed to follow a continuous distribution conditional on (T, z), μ_{tk} is continuous on its domain and takes on all values in $\left[\underline{\mu}_{tk}, \overline{\mu}_{tk}\right]$ by the intermediate value theorem. Thus, there exists a ξ^* such that $\mu_{tk}(\xi^*) = \mu^1_k$. Now let $f_{tk}(\tau) = dF_{tk}(\tau)/d\tau$ which is non-negative by the assumption that y is continuously distributed. Define the densities

$$f^1_{tk}(\tau) = \frac{f_{tk}(\tau) \times \mathbf{1}\{\tau \in I_{tk}(\xi^*)\}}{r_{tk}}, \quad f^0_{tk}(\tau) = \frac{f_{tk}(\tau) \times \mathbf{1}\{\tau \in I_{tk}(\xi^*)\}}{1 - r_{tk}}.$$

Clearly $f^1_{tk} \geq 0$ and $f^0_{tk} \geq 0$. Integrating,

$$\int_{\mathbb{R}} f^1_{tk}(\tau)\, d\tau = \frac{1}{r_{tk}} \int_{I_{tk}(\xi^*)} f_{tk}(\tau)\, d\tau = 1$$
$$\int_{\mathbb{R}} f^0_{tk}(\tau)\, d\tau = \frac{1}{1 - r_{tk}} \int_{I^C_{tk}(\xi^*)} f_{tk}(\tau)\, d\tau = 1$$

where I^C_{tk} is the complement of I_{tk}. And, by construction

$$r_{tk} \int_A f^1_{tk}(\tau)\, d\tau + (1 - r_{tk}) \int_A f^0_{tk}(\tau)\, d\tau = \int_A f_{tk}(\tau)\, d\tau$$

for any set A. Finally,

$$\int_{\mathbb{R}} \tau f^1_{tk}(\tau)\, d\tau = \frac{1}{r_{tk}} \int_{I_{tk}(\xi^*)} \tau f_{tk}(\tau)\, d\tau = \mu_{tk}(\xi^*) = \mu^1_k.$$

The result now follows by appealing to the proof of Theorem 2.1.

□

A.2 Point Identification Results

In the proofs of Lemma 2.3, Lemma 2.4, and Theorem 2.3, we use the shorthand

$$\pi \equiv \text{Cov}(T, z), \quad \eta_j \equiv \text{Cov}(y^j, z), \quad \tau_j \equiv \text{Cov}(Ty^j, z)$$

for $j = 1, 2, 3$. Using this notation, Lemma 2.2 becomes $\eta_1 = \pi\theta_1$, while Lemma 2.3 becomes $\eta_2 = 2\tau_1\theta_1 - \pi\theta_2$, and Lemma 2.4 becomes $\eta_3 = 3\tau_2\theta_1 - 3\tau_1\theta_2 + \pi\theta_3$.

Proof of Lemma 2.3. By Assumption 2.1 (i) and the basic properties of covariance,

$$\eta_2 = \beta^2 \text{Cov}(T^*, z) + 2\beta\left[c\,\text{Cov}(T^*, z) + \text{Cov}(T^*\varepsilon, z)\right] + 2c\,\text{Cov}(\varepsilon, z) + \text{Cov}(\varepsilon^2, z)$$
$$\tau_1 = c\pi + \text{Cov}(T\varepsilon, z) + \beta\text{Cov}(TT^*, z)$$

using the fact that T^* is binary. Now, by Assumptions 2.1 (iii) and 2.5 we have $\text{Cov}(\varepsilon, z) = \text{Cov}(\varepsilon^2, z) = 0$. And, using Assumptions 2.2 (i) and (ii), one can show that $\text{Cov}(TT^*, z) = (1 -$

$\alpha_1)\text{Cov}(T^*, z)$ and $\text{Cov}(T^*, z) = \pi/(1 - \alpha_0 - \alpha_1)$. Hence,

$$\eta_2 = \theta_1 (\beta + 2c) \pi + 2\beta \text{Cov}(T^*\varepsilon, z)$$
$$2\tau_1\theta_1 - \pi\theta_2 = \left[2\theta_1 c + 2\theta_1^2(1 - \alpha_1) - \theta_2\right] \pi + 2\theta_1 \text{Cov}(T\varepsilon, z)$$

but since $\theta_2 = \theta_1^2 [(1 - \alpha_1) + \alpha_0]$, we see that $[2\theta_1^2(1 - \alpha_1) - \theta_2] = \theta_1\beta$. Thus, it suffices to show that $\beta\text{Cov}(T^*\varepsilon, z) = \theta_1\text{Cov}(T\varepsilon, z)$. This equality is trivially satisfied when $\beta = 0$, so suppose that $\beta \neq 0$. In this case it suffices to show that $(1 - \alpha_0 - \alpha_1)\text{Cov}(T^*\varepsilon, z) = \text{Cov}(T\varepsilon, z)$. Define $m^*_{tk} = \mathbb{E}[\varepsilon | T^* = t, z = k]$ and $p^*_k = \mathbb{P}(T^* = 1 | z = k)$. Then, by iterated expectations, Bayes' rule, and Assumption 2.2 (iii)

$$\text{Cov}(T^*\varepsilon, z) = q(1 - q) \left(p^*_1 m^*_{11} - p^*_0 m^*_{10}\right)$$
$$\text{Cov}(T\varepsilon, z) = q(1 - q) \left\{(1 - \alpha_1) \left[p^*_1 m^*_{11} - p^*_0 m^*_{10}\right] + \alpha_0 \left[(1 - p^*_1)m^*_{01} - (1 - p^*_0)m^*_{00}\right]\right\}$$

But by Assumption 2.1 (iii), $\mathbb{E}[\varepsilon | z = k] = m^*_{1k}p^*_k + m^*_{0k}(1 - p^*_k) = 0$ and thus we obtain $m^*_{0k}(1 - p^*_k) = -m^*_{1k}p^*_k$. Therefore $(1 - \alpha_0 - \alpha_1)\text{Cov}(T^*\varepsilon, z) = \text{Cov}(T\varepsilon, z)$ as required. □

Proof of Lemma 2.4. Since T^* is binary, if follows from the basic properties of covariance that,

$$\eta_3 = \text{Cov}\left[(c + \varepsilon)^3, z\right] + 3\beta\text{Cov}[(c + \varepsilon)^2 T^*, z] + 3\beta^2\text{Cov}[(c + \varepsilon)T^*, z] + \beta^3\text{Cov}(T^*, z)$$
$$\tau_2 = \text{Cov}\left[(c + \varepsilon)^2 T, z\right] + 2\beta\text{Cov}\left[(c + \varepsilon)TT^*, z\right] + \beta^2\text{Cov}(TT^*, z)$$

By Assumptions 2.1 (iii), 2.5, and 2.6 (ii) , $\text{Cov}\left[(c + \varepsilon)^3, z\right] = 0$. Expanding,

$$\eta_3 = 3\beta\text{Cov}(T^*\varepsilon^2, z) + \left(3\beta^2 + 6c\beta\right)\text{Cov}(T^*\varepsilon, z) + \left(\beta^3 + 3c\beta^2 + 3c^2\beta\right)\text{Cov}(T^*, z)$$
$$\tau_2 = c^2\text{Cov}(T, z) + \beta(\beta + 2c)\text{Cov}(TT^*, z) + \text{Cov}(T\varepsilon^2, z) + 2c\text{Cov}(T\varepsilon, z) + 2\beta\,\text{Cov}(TT^*\varepsilon, z)$$

Now, define $s^*_{tk} = \mathbb{E}[\varepsilon^2 | T^* = t, z = k]$ and $p^*_k = \mathbb{P}(T^* = 1 | z = k)$. By iterated expectations, Bayes' rule, and Assumption 2.6 (i),

$$\text{Cov}(T^*\varepsilon^2, z) = q(1 - q)(p^*_1 s^*_{11} - p^*_0 s^*_{10})$$
$$\text{Cov}(T\varepsilon^2, z) = q(1 - q) \left\{(1 - \alpha_1) \left[p^*_1 s^*_{11} - p^*_0 s^*_{10}\right] + \alpha_0 \left[(1 - p^*_1)s^*_{01} - (1 - p^*_0)s^*_{00}\right]\right\}$$

By Assumption 2.5, $\mathbb{E}[\varepsilon^2 | z = 1] = \mathbb{E}[\varepsilon^2 | z = 0]$ and thus, by iterated expectations we have $p^*_1 s^*_{11} - p^*_0 s^*_{10} = -\left[(1 - p^*_1)s^*_{01} - (1 - p^*_0)s^*_{00}\right]$ which implies

$$\text{Cov}(T\varepsilon^2, z) = (1 - \alpha_0 - \alpha_1)\text{Cov}(T^*\varepsilon^2, z). \tag{A.6}$$

Similarly by iterated expectations and Assumptions 2.2 (i)–(ii)

$$\text{Cov}(TT^*\varepsilon, z) = q(1 - q)(1 - \alpha_1)(p^*_1 m^*_{1k} - p^*_0 m^*_{10}) = (1 - \alpha_1)\text{Cov}(T^*\varepsilon, z) \tag{A.7}$$

where m^*_{tk} is defined as in the proof of Lemma 2.3. As shown in the proof of Lemma 2.3,

$$\text{Cov}(TT^*, z) = (1 - \alpha_1)\text{Cov}(T^*, z)$$
$$\text{Cov}(T^*, z) = \pi/(1 - \alpha_0 - \alpha_1)$$
$$\text{Cov}(T^*\varepsilon, z) = \text{Cov}(T\varepsilon, z)/(1 - \alpha_0 - \alpha_1)$$

and combining these equalities with Equations A.6 and A.7, it follows that

$$
\begin{aligned}
\tau_2 &= 2\left[(1-\alpha_1)(c+\beta) - c\alpha_0\right]\text{Cov}(T^*\varepsilon, z) + \left[(1-\alpha_1)(c+\beta)^2 - c^2\alpha_0\right]\text{Cov}(T^*, z)\\
&\quad + (1-\alpha_0-\alpha_1)\text{Cov}(T^*\varepsilon^2, z)\\
\tau_1 &= (1-\alpha_0-\alpha_1)\text{Cov}(T^*\varepsilon, z) + \left[(1-\alpha_1)(c+\beta) - c\alpha_0\right]\text{Cov}(T^*, z)
\end{aligned}
$$

using $\tau_1 = c\pi + \text{Cov}(T\varepsilon, z) + \beta\text{Cov}(TT^*, z)$ as shown in the proof of Lemma 2.3. Thus,

$$
3\tau_2\theta_1 - 3\tau_1\theta_2 + \pi\theta_3 = K_1\text{Cov}(T^*\varepsilon^2, z) + K_2\text{Cov}(T^*\varepsilon, z) + K_3\text{Cov}(T^*, z)
$$

where $K_1 \equiv 3\theta_1(1-\alpha_0-\alpha_1) = 3\beta$ and

$$
\begin{aligned}
K_2 &\equiv 6\theta_1\left[(1-\alpha_1)(c+\beta) - c\alpha_0\right] - 3\theta_2(1-\alpha_0-\alpha_1)\\
K_3 &\equiv 3\theta_1\left[(1-\alpha_1)(c+\beta)^2 - c^2\alpha_0\right] - 3\theta_2\left[(1-\alpha_1)(c+\beta) - c\alpha_0\right] + \theta_3(1-\alpha_0-\alpha_1)
\end{aligned}
$$

Substituting the definitions of θ_1, θ_2, and θ_3 from Equations 6–8, tedious but straightforward algebra shows that $K_2 = 3\beta^2 + 6c\beta$ and $K_3 = \beta^3 + 3c\beta^2 + 3c^2\beta$. Therefore the coefficients of η_3 equal those of $3\tau_2 - 3\tau_1\theta_2 + \pi\theta_3$ and the result follows. $\square$

Proof of Theorem 2.3. Collecting the results of Lemmas 2.2–2.4, we have

$$
\eta_1 = \pi\theta_1, \quad \eta_2 = 2\tau_1\theta_1 - \pi\theta_2, \quad \eta_3 = 3\tau_2\theta_1 - 3\tau_1\theta_2 + \pi\theta_3
$$

which is a linear system in $\theta_1, \theta_2, \theta_3$ with determinant $-\pi^3$. Since $\pi \neq 0$ by assumption 2.1 (ii), θ_1, θ_2 and θ_3 are identified. Now, so long as $\beta \neq 0$, we can rearrange Equations 7 and 8 to obtain

$$
A = \theta_2/\theta_1^2 = 1 + (\alpha_0 - \alpha_1) \tag{A.8}
$$

$$
B = \theta_3/\theta_1^3 = (1-\alpha_0-\alpha_1)^2 + 6\alpha_0(1-\alpha_1) \tag{A.9}
$$

Equation A.8 gives $(1-\alpha_1) = A - \alpha_0$. Hence $(1-\alpha_0-\alpha_1) = A - 2\alpha_0$ and $\alpha_0(1-\alpha_1) = \alpha_0(A-\alpha_0)$. Substituting into Equation A.9 and simplifying, $(A^2 - B) + 2A\alpha_0 - 2\alpha_0^2 = 0$. Substituting for α_0 analogously yields a quadratic in $(1-\alpha_1)$ with *identical* coefficients. It follows that one root of $(A^2 - B) + 2Ar - 2r^2 = 0$ is α_0 and the other is $1 - \alpha_1$. Solving,

$$
r = \frac{A}{2} \pm \sqrt{3A^2 - 2B} = \frac{1}{\theta_1^2}\left(\frac{\theta_2}{2} \pm \sqrt{3\theta_2^2 - 2\theta_1\theta_3}\right). \tag{A.10}
$$

By Equations 7 and 8,

$$
\begin{aligned}
3\theta_2^2 - 2\theta_1\theta_3 &= 3\left[\theta_1^2(1+\alpha_0-\alpha_1)\right]^2 - 2\theta_1\left\{\theta_1^3\left[(1-\alpha_0-\alpha_1)^2 + 6\alpha_0(1-\alpha_1)\right]\right\}\\
&= \theta_1^4\left\{3(1+\alpha_0-\alpha_1)^2 - 2\left[(1-\alpha_0-\alpha_1)^2 + 6\alpha_0(1-\alpha_1)\right]\right\}.
\end{aligned}
$$

Expanding the first term we find that

$$
\begin{aligned}
3(1+\alpha_0-\alpha_1)^2 &= 3\left[1 + 2(\alpha_0-\alpha_1) + (\alpha_0-\alpha_1)^2\right]\\
&= 3 + 6\alpha_0 - 6\alpha_1 + 3\alpha_0^2 + 3\alpha_1^2 - 6\alpha_0\alpha_1
\end{aligned}
$$

and expanding the second

$$\begin{aligned} 2\left[(1-\alpha_0-\alpha_1)^2+6\alpha_0(1-\alpha_1)\right] &= 2\left[1-2(\alpha_0+\alpha_1)+(\alpha_0+\alpha_1)^2+6\alpha_0-6\alpha_0\alpha_1\right] \\ &= 2+8\alpha_0-4\alpha_1+2\alpha_0^2+2\alpha_1^2-8\alpha_0\alpha_1. \end{aligned}$$

Therefore

$$\begin{aligned} 3\theta_2^2-2\theta_1\theta_3 &= \theta_1^4\left\{1-2\alpha_0-2\alpha_1+\alpha_0^2-\alpha_1^2+2\alpha_0\alpha_1\right\} \\ &= \theta_1^4\left[(1-\alpha_0-\alpha_1)^2\right] \end{aligned}$$

which is strictly greater than zero since $\theta_1 \neq 0$ and $\alpha_0+\alpha_1 \neq 0$. It follows that both roots of the quadratic are real. Moreover, $3\theta_2^2/\theta_1^4 - 2\theta_3/\theta_1^3$ identifies $(1-\alpha_0-\alpha_1)^2$. Substituting into Equation 6, it follows that β is identified up to sign. If $\alpha_0+\alpha_1<1$ then $\text{sign}(\beta)=\text{sign}(\theta_1)$ so that both the sign and magnitude of β are identified. If $\alpha_0+\alpha_1<1$ then $1-\alpha_1>\alpha_0$ so $(1-\alpha_1)$ is the larger root of $(A^2-B)+2Ar-2r^2=0$ and α_0 is the smaller root. □

B Comment on Mahajan (2006) A.2

Expanding on our discussion from Section 2.2 above, we now show that Mahajan's identification argument for an endogenous regressor in an additively separable model (A.2) is incorrect. Unless otherwise indicated, all notation used below is as defined in Section 2.

The first step of Mahajan (2006) A.2 argues (correctly) that under Assumptions 2.1 and 2.2 (i)–(ii), knowledge of $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ is sufficient to identify $\beta(\mathbf{x})$. This step is equivalent to our Lemma 2.2 above. The second step appeals to Mahajan (2006) Theorem 1 to argue that $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are indeed point identified. To understand the logic of this second step, we first re-state Mahajan (2006) Theorem 1 in our notation. As in Section 2 above, T^* denotes an unobserved binary random variable, z is a instrument, T an observed binary surrogate for T^*, y an outcome of interest, and $\mathbf{x}$ a vector covariates.

Assumption B.1 (Mahajan (2006) Theorem 1)**.** *Define $g(T^*,\mathbf{x}) \equiv \mathbb{E}[y|\mathbf{x},T^*]$ and $v \equiv y-g(T^*,\mathbf{x})$. Suppose that knowledge of $(y,T^*,\mathbf{x})$ is sufficient to identify g and that:*

(i) $\mathbb{P}(T^*=1|\mathbf{x},z=0) \neq \mathbb{P}(T^*=1|\mathbf{x},z=1)$.

(ii) T is conditionally independent of z given $(\mathbf{x},T^*)$.

(iii) $\alpha_0(\mathbf{x})+\alpha_1(\mathbf{x})<1$

(iv) $\mathbb{E}[v|\mathbf{x},z,T^*,T]=0$

(v) $g(1,\mathbf{x}) \neq g(0,\mathbf{x})$

Theorem B.1 (Mahajan (2006) Theorem 1)**.** *Under Assumption B.1, $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are point identified, as is $g(T^*,\mathbf{x})$.*

Assumption B.1 (i) is equivalent to our Assumption 2.1 (ii), while Assumptions B.1 (ii)–(iii) are equivalent to our Assumptions 2.2 (i)–(ii). Assumption B.1 (v) serves the same purpose as $\beta(\mathbf{x}) \neq 0$ in our Theorem 2.3: unless T^* affects y, we cannot identify the mis-classification probabilities. The key difference between Theorem B.1 and the setting we consider in Section 2 comes from Assumption

B.1 (iv). This is essentially a stronger version of our Assumptions 2.1 (iii) and 2.2 (iii) but applies to the *projection error* v, defined in Assumption B.1 rather than the structural error ε, defined in Assumption 2.1 (i). Accordingly, Theorem B.1 identifies the conditional mean function g rather than the causal effect $\beta(\mathbf{x})$.

Although the meaning of the error term changes when we move from a structural to a reduced form model, the meaning of the mis-classification error rates does not: $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ are simply conditional probabilities for T given $(T^*, \mathbf{x})$. Step 2 of Mahajan (2006) A.2 relies on this insight. The idea is to find a way to satisfy Assumption B.1 (iv) simultaneously with Assumptions 2.1 (iii) and 2.2 (iii), while allowing T^* to be endogenous. If this can be achieved, $\alpha_0(\mathbf{x}), \alpha_1(\mathbf{x})$ will be identified via Theorem B.1, and identification of $\beta(\mathbf{x})$ will follow from step 1 of A.2 (our Lemma 2.2). To this end, Mahajan (2006) invokes the condition

$$\mathbb{E}(y|\mathbf{x}, z, T^*, T) = \mathbb{E}(y|\mathbf{x}, T^*). \tag{B.1}$$

Because Mahajan (2006) A.2 assumes an additively separable model – our Assumption 2.1 (i) – we see that

$$\mathbb{E}(y|\mathbf{x}, z, T^*, T) = c(\mathbf{x}) + \beta(\mathbf{x})T^* + \mathbb{E}(\varepsilon|\mathbf{x}, z, T^*, T)$$

so Equation B.1 is equivalent to $\mathbb{E}(\varepsilon|\mathbf{x}, z, T^*, T) = \mathbb{E}(\varepsilon|\mathbf{x}, T^*)$. Note that this allows T^* to be endogenous, as it does not require $\mathbb{E}(\varepsilon|\mathbf{x}, T^*) = 0$. Now, applying Equation B.1 to the definition of v from Assumption B.1, we have

$$\mathbb{E}(v|\mathbf{x}, z, T^*, T) = \mathbb{E}\left[y - \mathbb{E}(y|\mathbf{x}, T^*) \mid \mathbf{x}, z, T^*, T\right] = 0$$

which satisfies Assumption B.1 (iv) as required. Based on this reasoning, Mahajan (2006) claims that Equation B.1 along with Assumptions B.1 (iv), 2.1, and 2.2 (i)–(ii) suffice to identify the effect $\beta(\mathbf{x})$ of an endogenous T^*, so long as $g(1, \mathbf{x}) \neq g(0, \mathbf{x})$. As we now show, however, these Assumptions are contradictory unless T^* is exogenous.

By Equation B.1 and Assumption 2.1 (i), $\mathbb{E}(\varepsilon|\mathbf{x}, z, T^*, T) = \mathbb{E}(\varepsilon|\mathbf{x}, T^*)$ and thus by iterated expectations, we obtain

$$\mathbb{E}(\varepsilon|\mathbf{x}, T^*, z) = \mathbb{E}_{T|\mathbf{x}, T^*, z}\left[\mathbb{E}(\varepsilon|\mathbf{x}, T^*, T, z)\right] = \mathbb{E}_{T|\mathbf{x}, T^*, z}\left[\mathbb{E}(\varepsilon|\mathbf{x}, T^*)\right] = \mathbb{E}(\varepsilon|\mathbf{x}, T^*). \tag{B.2}$$

Now, let $m^*_{tk}(\mathbf{x}) = \mathbb{E}(\varepsilon|\mathbf{x}, T^* = t, z = k)$. Using this notation, Equation B.2 is equivalent to $m^*_{t0}(\mathbf{x}) = m^*_{t1}(\mathbf{x})$ for $t = 0, 1$. Combining iterated expectations with Assumption 2.1 (iii),

$$\mathbb{E}(\varepsilon|\mathbf{x}, z = k) = [1 - p^*_k(\mathbf{x})]m^*_{0k}(\mathbf{x}) + p^*_k(\mathbf{x})m^*_{1k}(\mathbf{x}) = 0 \tag{B.3}$$

for $k = 0, 1$ where $p^*_k(\mathbf{x}) \equiv \mathbb{P}(T^* = 1|\mathbf{x}, z = k)$. But substituting $m^*_{t0}(\mathbf{x}) = m^*_{t1}(\mathbf{x})$ into Equation B.3 for $k = 0, 1$, we obtain

$$\begin{aligned}
[1 - p^*_0(\mathbf{x})]m^*_{00}(\mathbf{x}) + p^*_0(\mathbf{x})m^*_{10}(\mathbf{x}) = 0 \\
[1 - p^*_1(\mathbf{x})]m^*_{00}(\mathbf{x}) + p^*_1(\mathbf{x})m^*_{10}(\mathbf{x}) = 0
\end{aligned}$$

The preceding two equalities are convex combinations of m^*_{00} and m^*_{10}. The only way that both can equal zero simultaneously is if either $p^*_0(\mathbf{x}) = p^*_1(\mathbf{x})$, contradicting Assumption 2.1 (ii), or if $m^*_{tk}(\mathbf{x}) = 0$ for all (t, k), which implies that T^* is exogenous. Hence Mahajan (2006) A.2 fails: given the assumption that z is a valid instrument for ε, Equation B.1 implies that either there is no first-stage relationship between z and T^* or that T^* is exogenous.

The root of the problem with A.2 is the attempt to use *one* instrument to satisfy both the assumptions of Theorem B.1 and Lemma 2.2. If one had access to a second instrument w, or equivalently a second mis-measured surrogate for T^*, that satisfied Assumptions B.1, one could use w to recover $\alpha_0(\mathbf{x})$ and $\alpha_1(\mathbf{x})$ via Theorem B.1 and z to recover the IV estimand $\beta(\mathbf{x})/[1 - \alpha_0(\mathbf{x}) - \alpha_1(\mathbf{x})]$ via Lemma 2.2. This is effectively the approach used by Battistin et al. (2014) to evaluate the returns to schooling in a setting with multiple misreported measures of educational qualifications.

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