MPhil Econometrics – Limited Dependent Variables and Selection

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Housekeeping

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Course Materials: `https://economictricks.com`

References

- **Wooldridge (2010) – *Econometric Analysis of Cross Section & Panel Data***
- Cameron & Trivedi (2005) – *Microeconometrics: Methods and Applications*
- Train (2009) – *Discrete Choice Methods with Simulation*

Lecture #1 – Maximum Likelihood Estimation Under Mis-specification

Review: the Poisson Distribution

The Kullback-Leibler Divergence

Example: Consistency of Poisson MLE

Asymptotic Theory for MLE Under Mis-specification

Example: Asymptotic Variance Calculations for Poisson MLE

Appendix: The Information Matrix Equality

"All models are wrong; some are useful."

Question

What happens if we carry out maximum likelihood estimation, but our model is *wrong*?

This Lecture

Examine a simple example in excruciating detail; present the general theory.

Next Lecture

Apply what we've learned to study Poisson Regression, a model for count data.

Suppose that $y \sim \text{Poisson}(\theta)$

Support Set: $\{0, 1, 2, \ldots\}$

A Poisson Random Variable is a *count*.

$$\sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} = e^{-\theta} \sum_{y=0}^{\infty} \frac{\theta^y}{y!} = e^{-\theta}\left(e^{\theta}\right) = 1$$

Probability Mass Function

$$f(y;\theta) = \frac{e^{-\theta}\theta^y}{y!}$$

Expected Value: $\mathbb{E}(y) = \theta$

Poisson parameter θ equals the mean of y.

$$\begin{aligned}
\mathbb{E}(y) &= \sum_{y=0}^{\infty} y \frac{e^{-\theta}\theta^y}{y!} = \sum_{y=1}^{\infty} y \frac{e^{-\theta}\theta^y}{y!} \\
&= \theta \sum_{y=1}^{\infty} \frac{e^{-\theta}\theta^{y-1}}{(y-1)!} = \theta \sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} = \theta
\end{aligned}$$

Variance: $\text{Var}(y) = \theta$

You will show this on the problem set.

MLE for θ where $y_1, y_2, \ldots, y_N \sim$ iid Poisson(θ).

The Likelihood (iid data)

$L_N(\theta) \equiv \prod_{i=1}^N \frac{e^{-\theta}\theta^{y_i}}{y_i!}$

The Log-Likelihood

$\ell_N(\theta) = \sum_{i=1}^N [y_i \log(\theta) - \theta - \log(y_i!)]$

Maximum Likelihood Estimator

$\widehat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max}\, \ell_N(\theta) = \bar{y}$

$$\frac{d}{d\theta}\ell_N(\theta) = \sum_{i=1}^N \left[\frac{y_i}{\theta} - 1\right]$$

$$\frac{d}{d\theta}\ell_N(\widehat{\theta}) = 0$$

$$\sum_{i=1}^N \left[y_i/\widehat{\theta} - 1\right] = 0$$

$$\left(\sum_{i=1}^N y_i\right)/\widehat{\theta} = N$$

$$\frac{1}{N}\sum_{i=1}^N y_i = \bar{y} = \widehat{\theta}$$

The Kullback-Leibler (KL) Divergence

Motivation

How well does a parametric model $f(\mathbf{y};\boldsymbol{\theta})$ approximate a *true* density/pmf $p_o(\mathbf{y})$?

Definition

$$KL(p_o; f_{\boldsymbol{\theta}}) \equiv \mathbb{E}\left[\log\left\{\frac{p_o(\mathbf{y})}{f(\mathbf{y};\boldsymbol{\theta})}\right\}\right]$$

KL Properties

1. *Asymmetric*: $KL(p_o; f_{\boldsymbol{\theta}}) \neq KL(f_{\boldsymbol{\theta}}; p_o)$
2. $KL(p_o; f_{\boldsymbol{\theta}}) \geq 0$; zero iff $p_o = f_{\boldsymbol{\theta}}$
3. Min KL iff max expected log-likelihood

Alternative Expression

$$\boxed{\mathbb{E}\left[\log\left\{\frac{p_o(\mathbf{y})}{f(\mathbf{y};\boldsymbol{\theta})}\right\}\right] = \underbrace{\mathbb{E}\left[\log p_o(\mathbf{y})\right]}_{\text{Constant wrt } \boldsymbol{\theta}} - \underbrace{\mathbb{E}\left[\log f(\mathbf{y};\boldsymbol{\theta})\right]}_{\text{Expected Log-like.}}}$$

All expectations are wrt p_o

$p_o(\mathbf{y})$ and $f(\mathbf{y};\boldsymbol{\theta})$ are merely *functions* of the RV $\mathbf{y}$

$$\mathbb{E}[\log p_o(\mathbf{y})] = \int \log p_o(\mathbf{y}) p_o(\mathbf{y})\, d\mathbf{y}$$

$$\mathbb{E}[\log f(\mathbf{y};\boldsymbol{\theta})] = \int \log f(\mathbf{y};\boldsymbol{\theta}) p_o(\mathbf{y})\, d\mathbf{y}$$

Watch Out!

$KL = \infty$ if $\exists \mathbf{y}$ with $f(\mathbf{y};\boldsymbol{\theta}) = 0$ & $p_o(\mathbf{y}) \neq 0$

$\text{KL}(p_o; f) \geq 0$ with equality iff $p_o = f$

Jensen's Inequality

If φ is convex then $\varphi(\mathbb{E}[y]) \leq \mathbb{E}[\varphi(y)]$, with equality iff φ is linear or y is constant.

log is concave so $(-\log)$ is convex

$$
\begin{aligned}
\mathbb{E}\left[\log\left\{\frac{p_o(y)}{f(y)}\right\}\right] = \mathbb{E}\left[-\log\left\{\frac{f(y)}{p_o(y)}\right\}\right] &\geq -\log\left\{\mathbb{E}\left[\frac{f(y)}{p_o(y)}\right]\right\} \\
&= -\log\left\{\int_{-\infty}^{\infty} \frac{f(y)}{p_o(y)} \cdot p_o(y)\, dy\right\} \\
&= -\log\left\{\int_{-\infty}^{\infty} f(y)\, dy\right\} \\
&= -\log(1) = 0
\end{aligned}
$$

A Simple Example: Calculating the KL Divergence

Remember: all expectations are calculated using p_o.

True Distribution p_o

$y_1, \ldots, y_N \sim$ iid p_o where:

$p_o(0) = \frac{2}{5}, p_o(1) = \frac{1}{5}, p_o(2) = \frac{2}{5}$.

Mis-specified Model f_θ

$y_1, \ldots, y_N \sim$ iid Poisson(θ)

KL Divergence

$KL(p_o; f_\theta) = \theta - \log\theta + (\text{Constant})$

$$\boxed{KL(p_o; f_\theta) = \mathbb{E}[\log p_o(y)] - \mathbb{E}[\log f(y;\theta)]}$$

$$\begin{aligned}
\mathbb{E}[\log p_o(y)] &= \sum_{\text{all } y} \log\left[p_o(y)\right] p_o(y) \\
&= \log\left(\frac{2}{5}\right)\cdot\frac{2}{5} + \log\left(\frac{1}{5}\right)\cdot\frac{1}{5} + \log\left(\frac{2}{5}\right)\cdot\frac{2}{5}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\log f(y;\theta)] &= \sum_{\text{all } y} \log\left[\frac{e^{-\theta}\theta^y}{y!}\right] p_o(y) \\
&= \log\left(e^{-\theta}\right)\times\frac{2}{5} + \log\left(e^{-\theta}\theta\right)\times\frac{1}{5} + \log\left(\frac{e^{-\theta}\theta^2}{2}\right)\times\frac{2}{5} \\
&= -\left[\theta - \log(\theta) + \log(2)\times\frac{2}{5}\right]
\end{aligned}$$

A Simple Example Continued: Minimizing the KL Divergence

Model $=$ Poisson(θ); True Dist. $p_o(0) = p_o(2) = \frac{2}{5}$ and $p_o(1) = \frac{1}{5}$

Best Approximation

What parameter value θ_o makes the Poisson(θ) model *as close as possible* to the true distribution p_o, where we measure "closeness" using the KL-divergence?

Using the previous slide

$$KL(p_o; f_\theta) = \theta - \log\theta + (\text{Const.})$$

$$\text{FOC:} \quad 0 = 1 - \frac{1}{\theta} \implies \boxed{\theta = 1}$$

A more direct approach

Min KL $\iff$ Max Expected Log-like.

$$\begin{aligned}
\frac{d}{d\theta}\mathbb{E}[\log f(y;\theta)] &= \frac{d}{d\theta}\mathbb{E}\left[-\theta + y\log(\theta) - \log(y!)\right] \\
&= \frac{d}{d\theta}\left\{-\theta + \mathbb{E}[y]\log(\theta) - \mathbb{E}[\log(y!)]\right\} \\
&= -1 + \mathbb{E}[y]/\theta = 0 \\
&\implies \boxed{\theta = \mathbb{E}[y]}
\end{aligned}$$

A Simple Example Continued: Minimizing the KL Divergence

Model = Poisson(θ); True Dist. $p_o(0) = p_o(2) = \frac{2}{5}$ and $p_o(1) = \frac{1}{5}$

Best Approximation

What parameter value θ_o makes the Poisson(θ) model *as close as possible* to the true distribution p_o, where we measure "closeness" using the KL-divergence?

First approach: $\theta_o = 1$

Second approach: $\theta_o = \mathbb{E}[y]$

Both Methods Agree

- For the specified p_o we have: $\mathbb{E}[y] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 2 \cdot \frac{2}{5} = 1.$
- The "Direct approach" is general: works for *any* p_o.

Is this just a coincidence?

We have shown that:

1. Under an iid Poisson(θ) model for $y_1, \ldots, y_N$, the MLE for θ is $\widehat{\theta} = \bar{y}$
2. For *any* (reasonable) p_o, setting $\theta_o = \mathbb{E}[y_i]$ minimizes $KL(p_o; f_\theta)$.

Law of Large Numbers & Central Limit Theorem:

$\widehat{\theta} = \bar{y}$ is a consistent, asymptotically normal estimator of $\mathbb{E}[y_i]$ as $N \to \infty$.

So at least in this example...

The maximum likelihood estimator $\widehat{\theta}$ is a consistent estimator of θ_o, the minimizer the KL divergence from the true distribution p_o to the Poisson(θ) model $f(y; \theta)$.

Maximum Likelihood Estimation Under Mis-specification

Note: expectations and variances are calculated using p_o

Theorem

Suppose that $\mathbf{y}_1, \ldots, \mathbf{y}_N \sim$ iid p_o and let $\widehat{\boldsymbol{\theta}}$ denote the MLE for $\boldsymbol{\theta}$ under the possibly mis-specified model $f(\mathbf{y}; \boldsymbol{\theta})$. Then, under mild regularity conditions:

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for the pseudo-true parameter value $\boldsymbol{\theta}_o$, defined as the minimizer of $KL(p_o, f_{\boldsymbol{\theta}})$ over the parameter space Θ.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$

where we define $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right]$ and $\mathbf{K} \equiv \text{Var}\left[\dfrac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right]$.

Why is this result such a big deal?

1. Provides an interpretation of MLE when we acknowledge that our models are only an *approximation* or reality: MLE recovers the pseudo-true parameter $\boldsymbol{\theta}_o$.
2. Yields a formula for standard errors that is robust to mis-specification of our model: compare to Heteroskedasticity consistent SEs for regression.
3. If the model is correctly specified, we recover the "classical" MLE result.

Maximum Likelihood Estimation Under Correct Specification

"Classical" large-sample theory for MLE

Theorem

Suppose that $\mathbf{y}_1, \ldots, \mathbf{y}_N \sim$ iid $f(\mathbf{y}; \boldsymbol{\theta}_o)$. Then, under mild regularity conditions:

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for $\boldsymbol{\theta}_o$.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$ where $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right]$.

Why? If $p_o(\mathbf{y}) = f(\mathbf{y}; \boldsymbol{\theta}_o)$, then:

1. $KL(p_o; f_{\boldsymbol{\theta}})$ equals *zero* at $\boldsymbol{\theta} = \boldsymbol{\theta}_o$.
2. The *information matrix equality* gives $\mathbf{K} = \mathbf{J}$ which implies $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1} = \mathbf{J}^{-1}$.

A Consistent Asymptotic Variance Matrix Estimator: $\widehat{\mathbf{J}}^{-1}\widehat{\mathbf{K}}\widehat{\mathbf{J}}^{-1}$

$\widehat{\boldsymbol{\theta}} \to_p \boldsymbol{\theta}_o$ plus Uniform Weak Law of Large Numbers: $\boxed{\text{Newey \& McFadden (1994)}}$

$$\boldsymbol{\theta}_o \equiv \underset{\boldsymbol{\theta}\in\Theta}{\arg\max}\ \mathbb{E}\left[\log f(\mathbf{y}_i;\boldsymbol{\theta})\right] \qquad \widehat{\theta} \equiv \underset{\boldsymbol{\theta}\in\Theta}{\arg\max}\ \frac{1}{N}\sum_{i=1}^{N}\log f(\mathbf{y};\boldsymbol{\theta})$$

$$\sqrt{N}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0},\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1}) \qquad \widehat{\boldsymbol{\theta}} \approx \mathcal{N}(\boldsymbol{\theta}_o,\widehat{\mathbf{J}}^{-1}\widehat{\mathbf{K}}\widehat{\mathbf{J}}^{-1}/N)$$

$$\mathbf{J} \equiv -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}_i;\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] \qquad \widehat{\mathbf{J}} \equiv -\frac{1}{N}\sum_{i=1}^{N}\frac{\partial^2 \log f(\mathbf{y}_i;\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}$$

$$\mathbf{K} \equiv \mathrm{Var}\left[\frac{\partial \log f(\mathbf{y}_i;\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right] \qquad \widehat{\mathbf{K}} \equiv \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log f(\mathbf{y}_i;\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]\left[\frac{\partial \log f(\mathbf{y}_i;\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]'$$

Some Notes on the Preceding Slide

What happened to the KL divergence?

$\mathbb{E}[\log p_o(\mathbf{y})]$ does not involve $\boldsymbol{\theta}$. Hence, $\underset{\boldsymbol{\theta}\in\Theta}{\arg\max}\ \mathbb{E}\left[\log f(\mathbf{y}_i;\boldsymbol{\theta})\right] = \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\ KL(p_o, f_{\boldsymbol{\theta}})$.

Isn't $\widehat{\mathbf{K}}$ missing a term?

The sample variance of $\mathbf{x}$ is given by $\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i\mathbf{x}_i'\right) - (\bar{\mathbf{x}}\bar{\mathbf{x}}')$ where $\bar{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$. In our formula for $\widehat{\mathbf{K}}$, the "$\bar{\mathbf{x}}\bar{\mathbf{x}}'$" term appears to be missing, but it is in fact equal to zero, since $\widehat{\boldsymbol{\theta}}$ is the solution to the MLE first-order condition.

Some Terminology

I will call $\widehat{\mathbf{J}}^{-1}\widehat{\mathbf{K}}\widehat{\mathbf{J}}^{-1}$ the robust asymptotic variance matrix estimator, since it is correct regardless of whether the model is correctly specified.

A Simple Example Continued Again: Asymptotic Variance Calculations

Poisson(θ) model, possibly mis-specified.

Ingredients

$$\boxed{\begin{aligned} \log f(y;\theta) &= -\theta + y\log(\theta) - \log(y!) \\ \frac{d}{d\theta}\log f(y;\theta) &= -1 + y/\theta \\ \frac{d^2}{d\theta^2}\log f(y;\theta) &= -y/\theta^2 \\ \theta_o = \mathbb{E}[y], &\quad \widehat{\theta} = \bar{y} \end{aligned}}$$

$$J = -\mathbb{E}\left[\frac{d^2}{d\theta^2}\log f(y;\theta_o)\right] = 1/\mathbb{E}[y]$$

$$\widehat{J} = -\frac{1}{N}\sum_{i=1}^{N}\frac{d^2}{d\theta^2}\log f(y_i;\widehat{\theta}) = 1/\bar{y}$$

$$K = \text{Var}\left[\frac{d}{d\theta}\log f(y;\theta_o)\right] = \text{Var}(y)/\mathbb{E}[y]^2$$

$$\widehat{K} = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{d}{d\theta}\log f(y_i;\widehat{\theta})\right]^2 = s_y^2/(\bar{y})^2$$

where $s_y^2 \equiv \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2$ and $\bar{y} \equiv \frac{1}{N}\sum_{i=1}^{n} y_i$

A Simple Example Continued Again: Asymptotic Variance Calculations

From Previous Slide

$$\theta_0 = \mathbb{E}[y], \quad J = 1/\mathbb{E}[y], \quad \widehat{J} = 1/\bar{y}, \quad K = \text{Var}(y)/\mathbb{E}[y]^2, \quad \widehat{K} = s_y^2/(\bar{y})^2$$

Correct Specification

$$\boxed{y_1, \ldots, y_N \sim \text{ iid Poisson}(\theta_o)} \implies \boxed{J = K = 1/\theta_o} \implies \boxed{J^{-1}KJ^{-1} = \theta_o = \mathbb{E}[y]}$$

Potential Mis-specification

$$\boxed{y_1, \ldots, y_N \sim \text{ iid}} \implies \boxed{J = 1/\mathbb{E}[y], \quad K = \text{Var}(y)/\mathbb{E}[y]^2} \implies \boxed{J^{-1}KJ^{-1} = \text{Var}(y)}$$

A Simple Example Continued Again: Asymptotic Variance Calculations

Comparison of Asymptotic Distributions

$$\boxed{y_1, \ldots, y_N \sim \text{ iid Poisson}(\theta_o)} \implies \sqrt{N}(\widehat{\theta} - \theta_o) = \sqrt{N}(\bar{y} - \mathbb{E}[y]) \to_d \mathcal{N}(0, \mathbb{E}[y])$$

$$\boxed{y_1, \ldots, y_N \sim \text{ iid}} \implies \sqrt{N}(\widehat{\theta} - \theta_o) = \sqrt{N}(\bar{y} - \mathbb{E}[y]) \to_d \mathcal{N}(0, \text{Var}[y])$$

Comparison of Asymptotic 95% CIs

$$\boxed{y_1, \ldots, y_N \sim \text{ iid Poisson}(\theta_o)} \implies \bar{y} \pm 1.96 \times \sqrt{\bar{y}/N}$$

$$\boxed{y_1, \ldots, y_N \sim \text{ iid}} \implies \bar{y} \pm 1.96 \times s_y/\sqrt{N}$$

Punch Line

Unless $\text{Var}(y) = \mathbb{E}[y]$, CIs/tests that assume the Poisson model is true are wrong!

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y}; \boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\mathbf{J} \equiv -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right], \quad \mathbf{K} \equiv \mathrm{Var}\left[\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right]}$$

Step 1: Alternative Expression for $\mathbf{K}$

$$\mathrm{Var}\left[\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right\}'\right] - \mathbb{E}\left[\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right]\mathbb{E}\left[\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right]'$$

but since $\boldsymbol{\theta}_o$ maximizes $\mathbb{E}\left[\log f(\mathbf{y}; \boldsymbol{\theta})\right]$,

$$\mathbb{E}\left[\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right] = \frac{\partial}{\partial \boldsymbol{\theta}}\mathbb{E}\left[\log f(\mathbf{y}; \boldsymbol{\theta}_o)\right] = \mathbf{0}$$

so it suffices to show that

$$-\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y}; \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}}\right\}'\right]$$

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y};\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\text{suffices to show} \quad -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}'\right]}$$

Step 2: Chain Rule & Product Rule

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta}) = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta})\right] = \frac{\partial}{\partial\theta_i}\left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})\right]$$

$$= \left[-\frac{1}{f^2(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_i}f(\mathbf{y};\boldsymbol{\theta})\right]\left[\frac{\partial}{\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})\right] + \frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})$$

$$= -\left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_i}f(\mathbf{y};\boldsymbol{\theta})\right]\left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})\right] + \frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})$$

$$= -\frac{\partial}{\partial\theta_i}\log f(\mathbf{y};\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta}) + \frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})$$

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y};\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\text{suffices to show} \quad -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}'\right]}$$

Step 3: Multiply by -1, Evaluate at $\boldsymbol{\theta}_o$, and Take Expectations

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta}) = -\frac{\partial}{\partial\theta_i}\log f(\mathbf{y};\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta}) + \frac{1}{f(\mathbf{y};\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta})$$

$$-\mathbb{E}\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta}_o)\right] = \mathbb{E}\left[\frac{\partial}{\partial\theta_i}\log f(\mathbf{y};\boldsymbol{\theta}_o)\frac{\partial}{\partial\theta_j}\log f(\mathbf{y};\boldsymbol{\theta}_o)\right] - \underbrace{\mathbb{E}\left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta}_o)}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y};\boldsymbol{\theta}_o)\right]}_{\text{suffices to show this is zero!}}$$

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y};\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\text{suffices to show } \mathbb{E}\left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y};\boldsymbol{\theta}_o)\right] = 0}$$

Step 4: Use $p_o(\mathbf{y}) = f(\mathbf{y};\boldsymbol{\theta}_o)$

$$\mathbb{E}\left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y};\boldsymbol{\theta}_o)\right] \equiv \int \left[\frac{1}{f(\mathbf{y};\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y};\boldsymbol{\theta}_o)\right] p_o(\mathbf{y})\, d\mathbf{y}$$

$$= \int \left[\frac{1}{\color{red}{f(\mathbf{y};\boldsymbol{\theta}_o)}} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y};\boldsymbol{\theta}_o)\right] \color{red}{f(\mathbf{y};\boldsymbol{\theta}_o)}\, d\mathbf{y} = \int \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y};\boldsymbol{\theta}_o)\, d\mathbf{y}$$

$$= \frac{\partial^2}{\partial\theta_i\partial\theta_j} \int f(\mathbf{y};\boldsymbol{\theta}_o)\, d\mathbf{y} = \frac{\partial^2}{\partial\theta_i\partial\theta_j}(1) = 0$$

Lecture #2 – Poisson Regression

Review: Minimum MSE Predictor / Minimum MSE Linear Predictor

What's special about count data?

Conditional Maximum Likelihood Estimation

Poisson Regression: A Robust Model for Count Data

Asymptotic Variance Calculations for Poisson Regression

Review: Minimum MSE Predictor / Minimum MSE Linear Predictor

Suppose we want to predict y using $\mathbf{x}$

Minimum MSE Predictor

$\mu(\mathbf{x}) \equiv \mathbb{E}(y|\mathbf{x})$ minimizes $\mathbb{E}\left[\{y - \varphi(\mathbf{x})\}^2\right]$ over all possible predictors $\varphi(\cdot)$.

Minimum MSE Linear Predictor

$\boldsymbol{\beta} \equiv \mathbb{E}\left[\mathbf{x}\mathbf{x}'\right]^{-1}\mathbb{E}[\mathbf{x}y]$ minimizes $\mathbb{E}\left[(y - \mathbf{x}'\boldsymbol{\theta})^2\right]$ over all linear predictors $\mathbf{x}'\boldsymbol{\theta}$.

Proof: $\mathbb{E}(y|\mathbf{x})$ is the minimum MSE predictor

Step 1: add and subtract $\mu(\mathbf{x}) \equiv \mathbb{E}(y|\mathbf{x})$

$$\begin{aligned}
\mathbb{E}\left[\{y-\varphi(\mathbf{x})\}^2\right] &= \mathbb{E}\left[\{(y-\mu(\mathbf{x})) - (\varphi(\mathbf{x})-\mu(\mathbf{x}))\}^2\right] \\
&= \mathbb{E}\left[\{y-\mu(\mathbf{x})\}^2\right] - 2\mathbb{E}\left[\{y-\mu(\mathbf{x})\}\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}\right] + \mathbb{E}\left[\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}^2\right]
\end{aligned}$$

Step 2: iterated expectations

$$\begin{aligned}
\mathbb{E}\left[\{y-\mu(\mathbf{x})\}\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}\right] &= \mathbb{E}\Big(\mathbb{E}\left[\{y-\mu(\mathbf{x})\}\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}\,|\mathbf{x}\right]\Big) \\
&= \mathbb{E}\Big([\varphi(\mathbf{x})-\mu(\mathbf{x})]\,[\mathbb{E}(y|\mathbf{x})-\mu(\mathbf{x})]\Big) = 0
\end{aligned}$$

Step 3: combine steps 1 & 2

$$\mathbb{E}\left[\{y-\varphi(\mathbf{x})\}^2\right] = \underbrace{\mathbb{E}\left[\{y-\mu(\mathbf{x})\}^2\right]}_{\text{constant wrt } \varphi} + \underbrace{\mathbb{E}\left[\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}^2\right]}_{\text{cannot be negative; zero if } \varphi=\mu}$$

Proof: OLS is the Minimum MSE Linear Predictor

Objective Function

$$\mathbb{E}\left[\left(y - \mathbf{x}'\boldsymbol{\theta}\right)^2\right] = \mathbb{E}[y^2] - 2\mathbb{E}[y\mathbf{x}']\boldsymbol{\theta} + \boldsymbol{\theta}'\mathbb{E}\left[\mathbf{x}\mathbf{x}'\right]\boldsymbol{\theta}$$

Recall: Matrix Differentiation

$$\frac{\partial(\mathbf{a}'\mathbf{z})}{\partial \mathbf{z}} = \mathbf{a}, \quad \frac{\partial(\mathbf{z}'\mathbf{A}\mathbf{z})}{\partial \mathbf{z}} = (\mathbf{A} + \mathbf{A}')\mathbf{z}$$

First-Order Condition

$$-2\mathbb{E}\left[\mathbf{x}y\right] + 2\mathbb{E}[\mathbf{x}\mathbf{x}']\boldsymbol{\beta} = 0 \implies \boldsymbol{\beta} = \mathbb{E}\left[\mathbf{x}\mathbf{x}'\right]^{-1}\mathbb{E}\left[\mathbf{x}y\right]$$

How to predict a count variable?

Example

Suppose we want to predict y using $\mathbf{x}$, where:

- $y \equiv \#$ of children a woman has: a count variable, i.e. $y \in \{0, 1, 2, \ldots\}$
- $\mathbf{x} \equiv$ {years of schooling, age, married, etc.}

Problems with linear-in-parameters models for count data

Best predictor is $\mathbb{E}(y|\mathbf{x})$ but how can we estimate this?

Plain-vanilla OLS?

- If $\mathbb{E}(y|\mathbf{x}) \approx \mathbf{x}'\boldsymbol{\beta}$, OLS is a reasonable approach.
- Problem: y is a count so it *can't* be negative, but OLS prediction $\mathbf{x}'\boldsymbol{\beta}$ could be.

OLS for $\log(y)$?

- Log-linear model $\log(y) = \mathbf{x}'\beta + \varepsilon$
- Solves the problem of negative predictions: $\log(y)$ *can* be negative.
- Problem: if y is a count it could equal zero but $\log(0) = -\infty$!

A realistic model for count data *must* be nonlinear in parameters.

General Approach

- Assume that $\mathbb{E}(y|\mathbf{x}) = m(\mathbf{x};\boldsymbol{\beta})$ where m is a known parametric function.
- Choose m so that it is always positive, regardless of $\mathbf{x}$ and $\boldsymbol{\beta}$.
- This means m *cannot* be linear.

This Lecture: $m(\mathbf{x};\boldsymbol{\beta}) = \exp(\mathbf{x}'\boldsymbol{\beta})$

- Always strictly positive
- Common choice in practice
- Everything I'll discuss works with other choices of m, making appropriate changes.

How to estimate β_o?

Assumption: $\mathbb{E}(y|\mathbf{x}) = \exp\left(\mathbf{x}'\boldsymbol{\beta}_o\right)$

Using our argument from above, $\boldsymbol{\beta}_o$ minimizes $\mathbb{E}\left[\{y_i - \exp(\mathbf{x}_i'\boldsymbol{\beta})\}^2\right]$ over all $\boldsymbol{\beta}$.

Nonlinear Least Squares (NLLS)

$\widehat{\beta}_{NLLS}$ is the minimizer of $\sum_{i=1}^{N}\{y_i - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\}^2$

Poisson Regression (MLE)

$\widehat{\beta}_{MLE}$ is the MLE for β_o under the model $y_i|\mathbf{x}_i \sim$ indep. Poisson$\Big(\exp(\mathbf{x}_i'\boldsymbol{\beta}_o)\Big)$

Conditional versus Unconditional MLE

Last Lecture: Unconditional MLE

Model *unconditional* dist. of a random vector $\mathbf{y}$: $f(\mathbf{y};\boldsymbol{\theta})$.

This Lecture: Conditional MLE

Model *conditional* dist. of a random variable y *given* a random vector $\mathbf{x}$: $f(\mathbf{y}|\mathbf{x};\boldsymbol{\theta})$.

Why Conditional MLE?

- Unconditional MLE requires joint distribution: $f(y,\mathbf{x};\boldsymbol{\theta}) = f(y|\mathbf{x};\boldsymbol{\theta})f(\mathbf{x};\boldsymbol{\theta})$
- $\mathbb{E}(y|\mathbf{x})$ only depends on $f(y|\mathbf{x};\boldsymbol{\theta})$ not $f(\mathbf{x};\boldsymbol{\theta})$.
- Not interested in $f(\mathbf{x};\boldsymbol{\theta})$; coming up with a good model for it is challenging.

The Conditional Maximum Likelihood Estimator

Assuming iid data.

Sample

$$\widehat{\boldsymbol{\theta}} \equiv \underset{\boldsymbol{\theta} \in \Theta}{\arg\max} \frac{1}{N} \sum_{i=1}^{N} \log f(y_i | \mathbf{x}_i; \boldsymbol{\theta})$$

Population

$$\boldsymbol{\theta}_o \equiv \underset{\boldsymbol{\theta} \in \Theta}{\arg\max}\, \mathbb{E}\left[\log f(y_i | \mathbf{x}_i; \boldsymbol{\theta})\right]$$

Important

- We only model the conditional distribution $y|\mathbf{x}$, but...
- ...the expectation $\mathbb{E}[\log f(y_i|\mathbf{x}_i; \boldsymbol{\theta})]$ is taken over the *joint distribution* of $(y, \mathbf{x})$.
- $f(y_i|\mathbf{x}_i; \boldsymbol{\theta})$ is merely a *function* of the RVs $(y_i, \mathbf{x}_i)$.

Conditional MLE Under Mis-specification

Theorem

Suppose that $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^N \sim$ iid p_o and let $\widehat{\boldsymbol{\theta}}$ denote the Conditional MLE for $\boldsymbol{\theta}$ under the possibly mis-specified model $f(\mathbf{y}|\mathbf{x};\boldsymbol{\theta})$. Then, under regularity conditions:

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for the pseudo-true parameter value $\boldsymbol{\theta}_o$, defined as the *maximizer* of the expected log likelihood $\mathbb{E}\left[\log f(y|\mathbf{x};\boldsymbol{\theta})\right]$ over the parameter space Θ.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$

where we define $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}|\mathbf{x};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$ and $\mathbf{K} \equiv \text{Var}\left[\dfrac{\partial \log f(\mathbf{y}|\mathbf{x};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right]$ and all expectations are taken with respect to p_o, the true joint distribution of $(\mathbf{y}, \mathbf{x})$.

Conditional MLE Under Correct Specification

Corollary

Suppose that $f(\mathbf{y}|\mathbf{x};\boldsymbol{\theta}_o)$ is the true conditional distribution of $\mathbf{y}_i|\mathbf{x}_i$. Then, under the conditions of the preceding theorem,

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for $\boldsymbol{\theta}_o$

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0},\mathbf{J}^{-1})$ where $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}|\mathbf{x};\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$

Poisson Regression as a Conditional MLE

Model: $y_i|\mathbf{x}_i \sim \text{Poisson}\big(\exp\{\mathbf{x}_i'\boldsymbol{\beta}\}\big)$

$$\ell_i(\boldsymbol{\beta}) \equiv \log f(y_i|\mathbf{x}_i;\boldsymbol{\beta}) = y_i\mathbf{x}_i'\boldsymbol{\beta} - \exp(\mathbf{x}_i'\boldsymbol{\beta}) - \log(y_i!)$$

$$\underbrace{\mathbf{s}_i(\boldsymbol{\beta})}_{\text{score vector}} \equiv \frac{\partial \ell_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{x}_i\left[y_i - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right]$$

$$\widehat{\boldsymbol{\beta}} \text{ solves } \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i\underbrace{\left[y_i - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right]}_{\text{residual: } u_i} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i u_i(\boldsymbol{\beta}) = \mathbf{0}$$

What value of $\boldsymbol{\beta}$ maximizes $\mathbb{E}\left[\ell_i(\boldsymbol{\beta})\right]$ for Poisson Regression?

Iterated Expectations

$$\mathbb{E}[\ell_i(\boldsymbol{\beta})] = \mathbb{E}\left\{\mathbb{E}\left[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i\right]\right\} = \mathbb{E}\left\{\mathbb{E}\left[y_i\mathbf{x}_i'\boldsymbol{\beta} - \exp(\mathbf{x}_i'\boldsymbol{\beta}) - \log\left(y_i!\right)|\mathbf{x}_i\right]\right\}$$

Simplify Inner Expectation

$$\mathbb{E}\left[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i\right] = \mathbf{x}_i'\boldsymbol{\beta}\mathbb{E}\left[y_i|\mathbf{x}_i\right] - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right) - \underbrace{\mathbb{E}\left[\log\left(y_i!\right)|\mathbf{x}_i\right]}_{\text{constant wrt } \boldsymbol{\beta}}$$

FOC for Inner Expectation

$$\frac{\partial}{\partial\boldsymbol{\beta}}\mathbb{E}\left[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i\right] = \left\{\mathbb{E}\left[y_i|\mathbf{x}_i\right] - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right\}\mathbf{x}_i = \mathbf{0}$$

What value of $\boldsymbol{\beta}$ maximizes $\mathbb{E}\left[\ell_i(\boldsymbol{\beta})\right]$?

$$\boxed{\frac{\partial}{\partial \boldsymbol{\beta}} \mathbb{E}\left[\ell_i(\boldsymbol{\beta}) | \mathbf{x}_i\right] = \left\{\mathbb{E}\left[y_i | \mathbf{x}_i\right] - \exp\left(\mathbf{x}_i' \boldsymbol{\beta}\right)\right\} \mathbf{x}_i = \mathbf{0}}$$

What does this mean?

Since $\mathbb{E}\left[y_i | \mathbf{x}_i\right] = \exp\left(\mathbf{x}_i' \boldsymbol{\beta}_o\right)$, setting $\boldsymbol{\beta} = \boldsymbol{\beta}_o$ solves the FOC for the inner expectation!

In other words:

For any realization of $\mathbf{x}_i$ and any $\boldsymbol{\beta}$,

$$\mathbb{E}[\ell_i(\boldsymbol{\beta}) | \mathbf{x}_i] \leq \mathbb{E}[\ell_i(\boldsymbol{\beta}_o) | \mathbf{x}_i]$$

so taking expectations of both sides:

$$\mathbb{E}\left[\ell_i(\boldsymbol{\beta})\right] = \mathbb{E}\left\{\mathbb{E}[\ell_i(\boldsymbol{\beta}) | \mathbf{x}_i]\right\} \leq \mathbb{E}\left\{\mathbb{E}[\ell_i(\boldsymbol{\beta}_o) | \mathbf{x}_i]\right\} = \mathbb{E}\left[\ell_i(\boldsymbol{\beta}_o)\right]$$

Poisson Regression is consistent if $\mathbb{E}(y|\mathbf{x})$ is correctly specified.

We showed this for a particular choice of $m(\mathbf{x};\boldsymbol{\beta})$ but the result is general.

Result

Provided that we have correctly specified $\mathbb{E}(y_i|\mathbf{x}_i)$, it *doesn't matter* if $y_i|\mathbf{x}_i$ actually follows a Poisson distribution: Poisson regression is *still consistent* for $\boldsymbol{\beta}_o$.

Compare

This is very similar to our result for the Poisson(θ) model from last lecture.

Caveat

Strictly speaking we need to show that $\boldsymbol{\beta}_o$ is the *unique* maximizer of the expected log likelihood. *Multiple solutions* if $\mathbf{x}_i$ perfectly co-linear (compare to OLS regression).

Average Partial Effects

Partial Effects

For continuous x_j, we call $\frac{\partial}{\partial x_j}\mathbb{E}(y|\mathbf{x})$ the partial effect of x_j. For discrete x_j the partial effect is the difference of $\mathbb{E}(y|\mathbf{x})$ at two different values of x_j

Average Partial Effects (APE)

In nonlinear models, partial effects typically vary with $\mathbf{x}$. The average partial effect is the expectation of the partial effect over the distribution of $\mathbf{x}$.

Average Partial Effects for Poisson Regression

Partial Effect

$$\frac{\partial}{\partial x_j}\mathbb{E}(y|\mathbf{x}) = \frac{\partial}{\partial x_j}\exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right) = \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\beta_j$$

Average Partial Effect

$$\mathbb{E}\left[\frac{\partial}{\partial x_j}\exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right] = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right]\beta_j$$

Estimated Partial Effect

$$\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\widehat{\beta}_j$$

Estimated Average Partial Effect

$$\left[\frac{1}{N}\sum_{i=1}^N \exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\right]\widehat{\beta}_j$$

Relative Effects

The *ratio* of partial effects does not depend on $\mathbf{x}$: relative effects are constant.

Problem Set

Poisson regression: APE$=\bar{y}\widehat{\beta}_j$. Multiply by $\bar{y}$ to put coefficients on the scale of OLS.

Asymptotic Variance Calculations for Poisson Regression

$$\underbrace{\mathbf{s}_i(\boldsymbol{\beta})}_{\text{score vector}} \equiv \frac{\partial \ell_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{x}_i \left[y_i - \exp\left(\mathbf{x}_i' \boldsymbol{\beta} \right) \right] = \mathbf{x}_i u_i(\boldsymbol{\beta})$$

$$\underbrace{\mathbf{H}_i(\boldsymbol{\beta})}_{\text{Hessian matrix}} \equiv \frac{\partial \mathbf{s}_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'} = -\exp\left(\mathbf{x}_i' \boldsymbol{\beta} \right) \mathbf{x}_i \mathbf{x}_i'$$

$$\mathbf{J} \equiv -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o) \right] = \mathbb{E}\left[\exp\left(\mathbf{x}_i' \boldsymbol{\beta}_o \right) \mathbf{x}_i \mathbf{x}_i' \right]$$

$$\mathbf{K} \equiv \mathrm{Var}\left[\mathbf{s}_i(\boldsymbol{\beta}_o) \right] = \mathbb{E}\left[\mathbf{s}_i(\boldsymbol{\beta}_o) \mathbf{s}_i(\boldsymbol{\beta}_o)' \right] = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i' \right]$$

Asymptotic Variance Calculations for Poisson Regression

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)\mathbf{x}_i\mathbf{x}_i'\right]}$$

Notice

$\mathbf{J}$ does not depend on y but $\mathbf{K}$ does:

$$\mathbf{K} = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)\mathbf{x}_i\mathbf{x}_i'\right] = \mathbb{E}\left\{\mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)|\mathbf{x}_i\right]\mathbf{x}_i\mathbf{x}_i'\right\} = \mathbb{E}\left(\mathbb{E}\left[\{y_i - \mathbb{E}(y_i|\mathbf{x}_i)\}^2|\mathbf{x}_i\right]\mathbf{x}_i\mathbf{x}_i'\right)$$
$$= \mathbb{E}\left[\mathrm{Var}(y_i|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'\right]$$

Assumptions about $\mathrm{Var}(y|\mathbf{x})$ affect the asymptotic variance through $\mathbf{K}$.

Possible Assumptions for $\text{Var}(y|\mathbf{x})$: Strongest to Weakest

1. Poisson Assumption: $\text{Var}(y|\mathbf{x}) = \mathbb{E}(y|\mathbf{x})$
   - holds if Poisson model is correct.
2. Quasi-Poisson Assumption: $\text{Var}(y|\mathbf{x}) = \sigma^2\mathbb{E}(y|\mathbf{x})$
   - Allows for possibility that $y|\mathbf{x}$ is *not* Poisson
   - Overdispersion: $\sigma^2 > 1 \implies \text{Var}(y|\mathbf{x}) > \mathbb{E}(y|\mathbf{x})$
   - Underdispersion $\sigma^2 < 1 \implies \text{Var}(y|\mathbf{x}) < \mathbb{E}(y|\mathbf{x})$
   - If $\sigma^2 = 1$ we're back to the Poisson Assumption.
3. No Assumption: $\text{Var}(y|\mathbf{x})$ unspecified

Asymptotic Variance Under Poisson Assumption

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[\text{Var}(y_i|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'\right]}$$

Assumption: $\text{Var}(y_i|\mathbf{x}_i) = \mathbb{E}(y_i|\mathbf{x}_i) = \exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)$

- Implies $\mathbf{K} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right]$
- Hence $\mathbf{K} = \mathbf{J}$ (Information Matrix Equality)
- Therefore: $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$
- Consistent Estimator: $\widehat{\mathbf{J}}^{-1} = \left[\frac{1}{N}\sum_{i=1}^{N}\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\mathbf{x}_i\mathbf{x}_i'\right]^{-1}$

Asymptotic Variance Under Quasi-Poisson Assumption

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[\mathrm{Var}(y_i|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'\right]}$$

Assumption: $\mathrm{Var}(y_i|\mathbf{x}_i) = \sigma^2\mathbb{E}(y_i|\mathbf{x}_i) = \sigma^2\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)$

- Implies $\mathbf{K} = \sigma^2\mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right] = \sigma^2\mathbf{J}$
- Hence $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1} = \sigma^2\mathbf{J}^{-1}$
- Therefore: $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{J}^{-1})$
- Consistent estimator of $\mathbf{J}^{-1}$ on prev. slide but how can we estimate σ^2?

How to estimate σ^2 under the Quasi-Poisson Assumption?

$$\begin{aligned}
\text{Var}(y|\mathbf{x}) &= \sigma^2\mathbb{E}(y|\mathbf{x}) \\
\sigma^2 &= \text{Var}(y|\mathbf{x})/\mathbb{E}(y|\mathbf{x}) \\
\sigma^2 &= \mathbb{E}\left[\{y-\mathbb{E}(y|\mathbf{x})\}^2\,|\mathbf{x}\right]/\mathbb{E}(y|\mathbf{x}) \\
\sigma^2 &= \mathbb{E}\left[\left.\frac{\{y-\mathbb{E}(y|\mathbf{x})\}^2}{\mathbb{E}(y|\mathbf{x})}\right|\mathbf{x}\right] \\
\sigma^2 &= \mathbb{E}\left[\left.\frac{\{y-\exp(\mathbf{x}'\boldsymbol{\beta}_o)\}^2}{\exp(\mathbf{x}'\boldsymbol{\beta}_o)}\right|\mathbf{x}\right] \\
\mathbb{E}[\sigma^2] &= \mathbb{E}\left(\mathbb{E}\left[\left.\frac{\{y-\exp(\mathbf{x}'\boldsymbol{\beta}_o)\}^2}{\exp(\mathbf{x}'\boldsymbol{\beta}_o)}\right|\mathbf{x}\right]\right) \\
\sigma^2 &= \mathbb{E}\left[\frac{\{y-\exp(\mathbf{x}'\boldsymbol{\beta}_o)\}^2}{\exp(\mathbf{x}'\boldsymbol{\beta}_o)}\right] \\
\sigma^2 &= \mathbb{E}\left[u^2(\boldsymbol{\beta}_o)/\exp(\mathbf{x}'\boldsymbol{\beta}_o)\right]
\end{aligned}$$

Consistent Estimator of σ^2

$$\widehat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}\frac{[y_i-\exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})]^2}{\exp(\mathbf{x}_i\widehat{\boldsymbol{\beta}})} = \frac{1}{N}\sum_{i=1}^{N}\frac{\widehat{u}_i^2}{\exp(\mathbf{x}_i\widehat{\boldsymbol{\beta}})}$$

Robust Asymptotic Variance Matrix

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)\mathbf{x}_i\mathbf{x}_i'\right]}$$

No Assumption on Var$(y_i|\mathbf{x}_i)$

- $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$
- Consistent Estimator: $\widehat{\mathbf{J}}^{-1} = \left[\frac{1}{N}\sum_{i=1}^{N}\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\mathbf{x}_i\mathbf{x}_i'\right]^{-1}$
- Consistent Estimator: $\widehat{\mathbf{K}} = \frac{1}{N}\sum_{i=1}^{N}\left[y_i - \exp(\mathbf{x}_i\widehat{\boldsymbol{\beta}})\right]^2\mathbf{x}_i\mathbf{x}_i' = \frac{1}{N}\sum_{i=1}^{N}\widehat{u}_i^2\mathbf{x}_i\mathbf{x}_i'$

Why Poisson Regression rather than NLLS?

Assume that $\mathbb{E}(y|\mathbf{x}) = \exp(\mathbf{x}'\beta_o)$

Both Poisson Reg. & NLLS are consistent if the conditional mean is correctly specified.

Count data are typically heteroskedastic.

If $\text{Var}(y|\mathbf{x})$ varies with $\mathbf{x}$, NLLS will be relatively inefficient.

Efficiency of Poisson Regression

- Correct model $\implies$ lowest variance among all estimators that leave the distribution of $\mathbf{x}$ unspecified.
- $\text{Var}(y|\mathbf{x}) = \sigma^2\mathbb{E}(y|\mathbf{x}) \implies$ Poisson regression is more efficient than NLLS and various other count data models.

Lecture #3 – Models for Binary Outcomes

Properties of Binary Outcome Models

Linear Probability Model

Index Models (e.g. Logit & Probit)

Partial Effects

Conditional MLE for Index Models

Pseudo R-squared

Models for Binary Outcomes

Example

- Outcome: $y = 1$ if employed, 0 otherwise
- Predictors/Regressors: $\mathbf{x} = \{\text{age, sex, education, experience, ...}\}$

Question

How does x_j affect our prediction of y holding the other regressors constant?

We'll consider three models:

1. Linear Probability Model (LPM)
2. Logistic Regression (Logit)
3. Probit Regression (Probit)

Properties of Binary Outcome Models: $y \in \{0, 1\}$

Notation

$p(\mathbf{x}) \equiv \mathbb{P}(y = 1|\mathbf{x})$

Conditional Mean

$\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x})$

Conditional Variance

$\text{Var}(y|\mathbf{x}) = p(\mathbf{x})\,[1 - p(\mathbf{x})]$

$$\begin{aligned}
\mathbb{E}(y|\mathbf{x}) &= 0 \times \mathbb{P}(y = 0|\mathbf{x}) + 1 \times \mathbb{P}(y = 1|\mathbf{x}) \\
&= \mathbb{P}(y = 1|\mathbf{x}) \equiv p(\mathbf{x})
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(y^2|\mathbf{x}) &= \left\{0^2 \times [1 - p(\mathbf{x})] + 1^2 \times p(\mathbf{x})\right\} \\
&= p(\mathbf{x})
\end{aligned}$$

$$\begin{aligned}
\text{Var}(y|\mathbf{x}) &= \mathbb{E}(y^2|\mathbf{x}) - \mathbb{E}(y|\mathbf{x})^2 \\
&= \left\{0^2 \times [1 - p(\mathbf{x})] + 1^2 \times p(\mathbf{x})\right\} - p(\mathbf{x})^2 \\
&= p(\mathbf{x})\,[1 - p(\mathbf{x})]
\end{aligned}$$

The Linear Probability Model: Assume $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$

Conditional Mean & Variance

- $\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$
- $\text{Var}(y|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}\,(1 - \mathbf{x}'\boldsymbol{\beta})$

$$
\begin{aligned}
\mathbb{E}(u|\mathbf{x}) &= \mathbb{E}(y - \mathbf{x}'\boldsymbol{\beta}|\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) - \mathbf{x}'\boldsymbol{\beta} \\
&= \mathbf{x}'\boldsymbol{\beta} - \mathbf{x}'\boldsymbol{\beta} = 0
\end{aligned}
$$

This is just Linear Regression!

$$y = \mathbf{x}'\boldsymbol{\beta} + u, \quad \mathbb{E}(u|\mathbf{x}) = 0$$

But u is Heteroskedastic

$$\text{Var}(u|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}(1 - \mathbf{x}'\boldsymbol{\beta})$$

$$
\begin{aligned}
\text{Var}(u|\mathbf{x}) &= \mathbb{E}\left[\{u - \mathbb{E}(u|\mathbf{x})\}^2 |\mathbf{x}\right] = \mathbb{E}\left[u^2|\mathbf{x}\right] \\
&= \mathbb{E}\left[(y - \mathbf{x}'\boldsymbol{\beta})^2 |\mathbf{x}\right] \\
&= \mathbb{E}\left(y^2|\mathbf{x}\right) - 2\mathbb{E}\left(y|\mathbf{x}\right)\mathbf{x}'\boldsymbol{\beta} + \left(\mathbf{x}'\boldsymbol{\beta}\right)^2 \\
&= p(\mathbf{x}) - 2p(\mathbf{x})p(\mathbf{x}) + p(\mathbf{x})^2 \\
&= p(\mathbf{x})\left[1 - p(\mathbf{x})\right]
\end{aligned}
$$

The Linear Probability Model: Assume $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$

Estimation

Since $\mathbb{E}(u|\mathbf{x}) = 0$ OLS estimation of $y = \mathbf{x}'\boldsymbol{\beta} + u$ is unbiased and consistent.

Inference

Since u is heteroskedastic, tests and CIs should use robust standard errors.

Is the LPM actually reasonable?

- Assumes $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta} \implies$ changing x_j by Δ changes $p(\mathbf{x})$ by $\beta_j\Delta$
- If $\mathbf{x}$ contains regressors without upper/lower bounds, $p(\mathbf{x})$ could be > 1 or < 0!
- LPM could be a reasonable approximation but cannot be *literally* true.

Index Models: Constrain $p(\mathbf{x})$ to lie in $[0, 1]$

Index Model: $p(\mathbf{x}) = G(\mathbf{x}'\boldsymbol{\beta})$

Assume $\mathbf{x}$ includes a constant, $0 \leq G(\cdot) \leq 1$, G is differentiable and strictly increasing, $\lim_{z\to\infty} G(z) = 1$, and $\lim_{z\to-\infty} G(z) = 0$.

Terminology

We call $\mathbf{x}'\boldsymbol{\beta}$ the linear index and G the index function.

Partial Effects

Let $g(z) \equiv \frac{d}{dz}G(z)$. Then $\frac{\partial}{\partial x_j}p(\mathbf{x}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j$. Hence:

- The partial effect of x_j depends on the value of $\mathbf{x}$ at which we evaluate g.
- G strictly increasing $\implies g(\cdot) > 0 \implies$ sign of partial effect determined by β_j.

Possible Choices of Index Function

CDFs as Index Functions

G satisfies the index model assumptions (prev. slide) iff it is a continuous CDF.

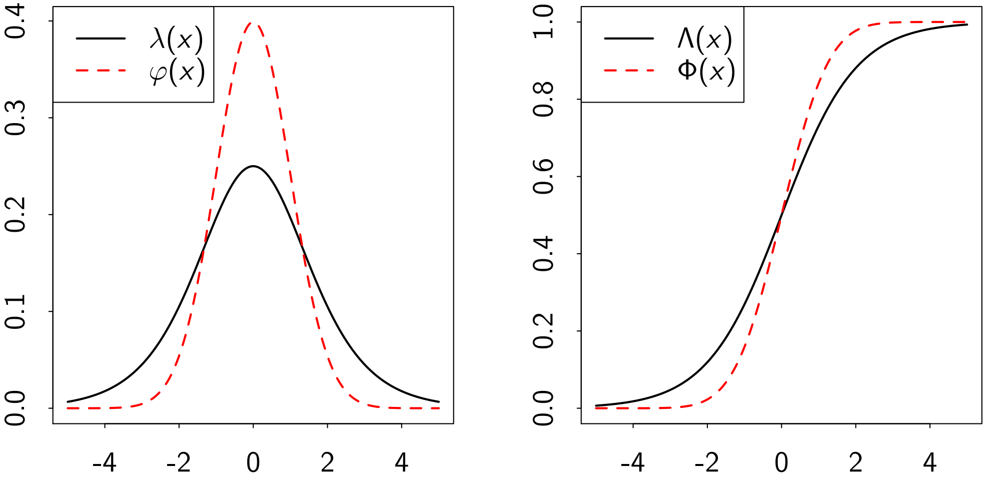
We focus on two examples:

1. Logit: $G(z) = \Lambda(z) \equiv \exp(z)/\left[1 + \exp(z)\right]$
2. Probit: $G(z) = \Phi(z) \equiv \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-t^2/2\right)\, dt$

Notation:

- Λ is the CDF of a "standard logistic" RV and Φ of a standard normal RV.
- λ is the density of a "standard logistic" RV and φ of a standard normal
- To treat Logit and Probit simultaneously, we'll write G as a placeholder.

Standard $\boxed{\text{Logistic}}$ and $\boxed{\text{Normal}}$ Densities and CDFs



Partial Effects: $\partial p(\mathbf{x})/\partial x_j$

LPM

$$\frac{\partial}{\partial x_j}\mathbf{x}'\boldsymbol{\beta} = \beta_j$$

Logit

$$\frac{\partial}{\partial x_j}\Lambda(\mathbf{x}'\boldsymbol{\beta}) = \frac{\beta_j \exp(\mathbf{x}'\boldsymbol{\beta})}{[1+\exp(\mathbf{x}'\boldsymbol{\beta})]^2}$$

Probit

$$\frac{\partial}{\partial x_j}\Phi(\mathbf{x}'\beta) = \frac{\beta_j \exp\{-(\mathbf{x}'\boldsymbol{\beta})^2/2\}}{\sqrt{2\pi}}$$

$$\frac{\partial}{\partial x_j}G(\mathbf{x}'\boldsymbol{\beta}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j$$

$$\begin{aligned}\frac{d}{dz}\Lambda(z) \equiv \lambda(z) = \frac{d}{dz}\left(\frac{e^z}{1+e^z}\right) &= \frac{e^z(1+e^z) - e^z e^z}{(1+e^z)^2} \\ &= \frac{e^z}{(1+e^z)^2}\end{aligned}$$

$$\frac{d}{dz}\Phi(z) = \varphi(z) = \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}}$$

Comparing Logit, Probit, and LPM Partial Effects

$$\boxed{\frac{\partial}{\partial x_j} G(\mathbf{x}'\boldsymbol{\beta}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j, \quad \frac{d}{dz}\Lambda(z) \equiv \lambda(z) = \frac{e^z}{(1+e^z)^2}, \quad \frac{d}{dz}\Phi(z) \equiv \varphi(z) = \frac{\exp\left\{-z^2/2\right\}}{\sqrt{2\pi}}}$$

Maximum Partial Effects

- λ and φ are unimodal with mode at 0

  Logit $\lambda(0) = 0.25$

  Probit $\varphi(0) = (2\pi)^{-1/2} \approx 0.4$

- *Maximum* partial effect when $\mathbf{x}'\boldsymbol{\beta} = 0$

  Logit $\beta_j \lambda(0) = 0.25\beta_j$

  Probit $\beta_j \varphi(0) \approx 0.4\beta_j$

- LPM has constant partial effects β_j

Relative Effects

$$\frac{\frac{\partial}{\partial x_j} p(\mathbf{x})}{\frac{\partial}{\partial x_h} p(\mathbf{x})} = \frac{\beta_j g(\mathbf{x}'\boldsymbol{\beta})}{\beta_h g(\mathbf{x}'\boldsymbol{\beta})} = \frac{\beta_j}{\beta_h}$$

Relative effects do not depend on $\mathbf{x}$.

Average Partial Effects for Index Models

Partial Effect

$$\frac{\partial}{\partial x_j} G(\mathbf{x}_i'\boldsymbol{\beta}) = g(\mathbf{x}_i'\boldsymbol{\beta})\beta_j$$

Average Partial Effect

$$\mathbb{E}\left[\frac{\partial}{\partial x_j} G(\mathbf{x}_i'\boldsymbol{\beta})\right] = \mathbb{E}[g(\mathbf{x}_i'\boldsymbol{\beta})]\beta_j$$

Estimated Partial Effect

$$\frac{\partial}{\partial x_j} G(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}) = g(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\widehat{\beta}_j$$

Estimated Average Partial Effect

$$\left[\frac{1}{N}\sum_{i=1}^{N} g(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\right]\widehat{\beta}_j$$

Warning:

APE $\neq$ partial effect evaluated at the average value of $\mathbf{x}$ since $\mathbb{E}[f(Z)] \neq f(\mathbb{E}[Z])$.

Conditional MLE for Index Models: iid Observations

Conditional Likelihood

$$f(y_i|\mathbf{x}_i,\boldsymbol{\beta}) = \begin{cases} 1 - G(\mathbf{x}_i'\boldsymbol{\beta}) & \text{if } y_i = 0 \\ G(\mathbf{x}_i'\boldsymbol{\beta}) & \text{if } y_i = 1 \end{cases} \iff f(y_i|\mathbf{x}_i,\boldsymbol{\beta}) = G(\mathbf{x}_i'\boldsymbol{\beta})^{y_i}\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]^{1-y_i}$$

Conditional Log-Likelihood

$$\ell_i(\boldsymbol{\beta}) \equiv \log f(y_i|\mathbf{x}_i,\boldsymbol{\beta}) = y_i \log\left[G(\mathbf{x}_i'\boldsymbol{\beta})\right] + (1-y_i)\log\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]$$

Sample

$$\widehat{\boldsymbol{\beta}} \equiv \underset{\boldsymbol{\beta}\in\Theta}{\arg\max}\,\frac{1}{N}\sum_{i=1}^{N}\ell_i(\boldsymbol{\beta})$$

Population

$$\boldsymbol{\beta}_o \equiv \underset{\boldsymbol{\beta}\in\Theta}{\arg\max}\,\mathbb{E}\left[\ell(\boldsymbol{\beta})\right]$$

$\boxed{\text{Correct specification: } \mathbb{E}(y|\mathbf{x}) = p(\mathbf{x}) = G(\mathbf{x}'\boldsymbol{\beta}_o). \text{ Otherwise } \boldsymbol{\beta}_o = \text{KL-minimizer.}}$

Asymptotic Variance Calculations for Index Models

Recall from last lecture.

Possibly Mis-specified Model

$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$ where $\mathbf{J} = -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)\right]$ and $\mathbf{K} = \mathbb{E}\left[\mathbf{s}_i(\boldsymbol{\beta}_o)\mathbf{s}_i(\boldsymbol{\beta}_o)'\right]$

Correct Specification

$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$ where $\mathbf{J} = -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)\right]$

Asymptotic variance calculations for index models are complicated, but there's a clever trick for computing $\mathbf{J}$ under correct specification.

Asymptotic Variance Calculation for Correctly Specified Index Models

$$\boxed{\ell_i(\boldsymbol{\beta}) = y_i \log\left\{G(\mathbf{x}_i'\boldsymbol{\beta})\right\} + (1-y_i)\log\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}$$

Step 1: Calculate The Score Vector

$$\mathbf{s}_i \equiv \frac{\partial}{\partial \boldsymbol{\beta}}\ell_i(\boldsymbol{\beta}) = \frac{y_i g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})} - \frac{(1-y_i)g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{1 - G(\mathbf{x}_i'\boldsymbol{\beta})}$$

$$= \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}\left\{\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]y_i - G(\mathbf{x}_i'\boldsymbol{\beta})(1-y_i)\right\}$$

$$= \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i\left[y_i - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}{G(\mathbf{x}_i'\boldsymbol{\beta})\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}$$

Asymptotic Variance Calculation for Correctly Specified Index Models

$$\boxed{\mathbf{s}_i = \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i\left\{y_i - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}}$$

Step 2: Start Calculating the Hessian but give up because it's a nightmare.

$$\mathbf{H}_i(\boldsymbol{\beta}) \equiv \frac{\partial \mathbf{s}_i}{\partial \boldsymbol{\beta}'} = \frac{\partial}{\partial \boldsymbol{\beta}'}\left([y_i - G(\mathbf{x}_i'\boldsymbol{\beta})]\left[\frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}\right]\right)$$

$$= \frac{-g(\mathbf{x}_i'\boldsymbol{\beta})^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}} + [y_i - G(\mathbf{x}_i'\boldsymbol{\beta})]\underbrace{\frac{\partial}{\partial \boldsymbol{\beta}'}\left\{\frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}\right\}}_{\text{a nasty awful mess: call it } \mathbf{M}(\mathbf{x}_i,\boldsymbol{\beta})}$$

Asymptotic Variance Calculation for Correctly Specified Index Models

$$\boxed{\mathbf{H}_i(\boldsymbol{\beta}) = \frac{-g(\mathbf{x}_i'\boldsymbol{\beta})^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}} + \left[y_i - G(\mathbf{x}_i'\boldsymbol{\beta})\right]\mathbf{M}(\mathbf{x}_i, \boldsymbol{\beta})}$$

Step 3: Calculate the *Conditional Expectation* at $\boldsymbol{\beta}_o$ instead...

$$\begin{aligned}
\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i\right] &= \frac{-g(\mathbf{x}_i'\boldsymbol{\beta}_o)^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta}_o)\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta}_o)\right\}} + \underbrace{\mathbb{E}\left[y_i - G(\mathbf{x}_i'\boldsymbol{\beta}_o)|\mathbf{x}_i\right]}_{\text{equals zero under correct spec.}}\ \mathbf{M}(\mathbf{x}_i, \boldsymbol{\beta}_o) \\
&= \frac{-g(\mathbf{x}_i'\boldsymbol{\beta}_o)^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta}_o)\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta}_o)\right\}}
\end{aligned}$$

Step 4: Iterated Expectations

$$\mathbf{J} = -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)\right] = -\mathbb{E}\left\{\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i\right]\right\} = \mathbb{E}\left\{\frac{g(\mathbf{x}_i'\boldsymbol{\beta}_o)^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta}_o)\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta}_o)\right\}}\right\}$$

Asymptotic Variance Calculation for Correctly Specified Index Models

Asymptotic Distribution

$$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}\left(\mathbf{0}, \mathbf{J}^{-1}\right), \quad \mathbf{J}^{-1} = \mathbb{E}\left\{\frac{g(\mathbf{x}_i'\boldsymbol{\beta_o})^2 \mathbf{x}_i \mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta_o})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta_o})\right\}}\right\}^{-1}$$

Consistent Estimator

$$\widehat{\mathbf{J}}^{-1} \equiv \left\{\frac{1}{N}\sum_{i=1}^{N} \frac{g(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})^2 \mathbf{x}_i \mathbf{x}_i'}{G(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\left[1 - G(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\right]}\right\}^{-1}$$

Notes

- Assumes correct specification, i.e. $p(\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) = G(\mathbf{x}'\boldsymbol{\beta}_o)$
- In contrast, *robust* variance matrix $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1}$ is complicated, but R can do it.

McFadden (1974) – "Pseudo R-squared"

Model with Intercept Only

$L(\bar{y}) \equiv$ maximized sample Likelihood

$\ell(\bar{y}) \equiv$ maximized sample log-likelihood

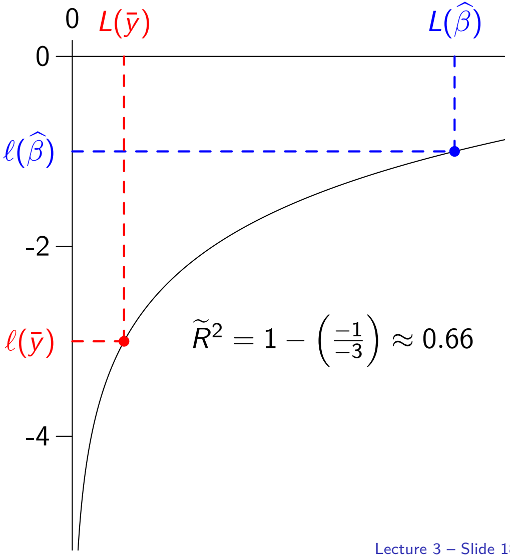
Full Model

$L(\widehat{\beta}) \equiv$ maximized sample Likelihood

$\ell(\widehat{\beta}) \equiv$ maximized sample log-likelihood

Pseudo R-squared

$\widetilde{R}^2 \equiv 1 - \ell(\widehat{\beta})/\ell(\bar{y})$



Problem set: $\widetilde{R}^2 \in [0, 1]$

Lecture #4 – Random Utility Models

Overview of Random Utility Models

Identification of Choice Models

Index Models as Special Cases (e.g. Logit & Probit)

The Logit Family of Choice Models

The Independence of Irrelevant Alternatives (IIA)

Discrete Choice – Basic Terminology

Decision-maker

Household, person, firm, etc.

Alternatives

Products, courses of action, etc.

Choice Set

The collection of all alternatives available to the decision-maker.

Restrictions on the Choice Set

We assume that:

1. Choices are mutually exclusive: choose only *one* alternative.
2. Choice set is *exhaustive*: contains every alternative (always choose something)
3. The number of alternatives is finite.

We can always redefine the choice set to satisfy 1 and 2

$$\underbrace{\{\text{Beer}, \text{Pizza}\}}_{\text{not mutually exclusive}} \quad \rightarrow \quad \underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}\}}_{\text{mutually exclusive}}$$

$$\underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}\}}_{\text{not exhaustive}} \quad \rightarrow \quad \underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}, \text{Something Else}\}}_{\text{exhaustive}}$$

Random Utility Models (RUMs)

Following Train (2009), use n to index individuals!

Notation

- N decision-makers $n = 1, \ldots, N$
- J alternatives $j = 1, \ldots, J$.

Utility and Decision Rule

- Decision-maker n obtains utility U_{nj} from choosing alternative j
- Maximize utility: decision-maker n chooses alternative i iff $U_{ni} > U_{nj}$ for any $j \neq i$

Random Utility Models

Researcher Observes

- Attributes x_{nj} of each alternative (e.g. product characteristics)
- Attributes s_n of the decision-maker (e.g. demographics)
- Choices but not utilities

Representative Utility V_{nj}

Assume researcher can specify a function $V_{nj}(x_{nj}, s_n)$ relating attributes x_{nj} of each alternative j and attributes s_n of each decision-maker n to her utilities U_{nj}.

Error Terms ε_{nj}

$\varepsilon_{nj} \equiv U_{nj} - V_{nj}$ is the difference between *true* utility U_{nj} and representative utility V_{nj}

Random Utility Models (RUMs)

What are the error terms?

ε_{nj} for $j = 1, \ldots, J$ represent unobserved factors that affect choices but are not captured by representative utilities (i.e. our model)

Treat Errors as Random

Let $\boldsymbol{\varepsilon}' \equiv [\ \varepsilon_{n1} \quad \ldots \quad \varepsilon_{nJ}\ ]$ have density function $f(\boldsymbol{\varepsilon}_n)$. Characterizes unobserved heterogeneity across decision-makers.

Choice Probabilities

$$P_{ni} \equiv \mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \int_{\mathbb{R}^J} \mathbb{1}\left\{\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i\right\} f(\boldsymbol{\varepsilon}_n)\, d\boldsymbol{\varepsilon}_n$$

This all sounds a bit abstract...

Basic Idea

1. Write down a parametric model for $V_{nj}(x_{nj}, s_n)$ with unknown parameters $\boldsymbol{\theta}$.
2. Choose a distribution f for the errors (heterogeneity) ε_n.
3. Back out choice probabilities as a function of parameters $\boldsymbol{\theta}$.
4. Use observed choices and attributes to find the MLE $\widehat{\boldsymbol{\theta}}$.

Looking Back; Looking Ahead

- Logit and Probit are special cases of RUMs: choice between two alternatives.
- RUMs provide a framework to estimate more complicated discrete choice models.

A Very Simple Example

Transport Decision

- Exactly two ways to get to work: by car or by bus.
- Observe two attributes: cost in time T and money M of each mode of transport.

Econometrician's Model: (β, γ) unknown

$$V_{\text{car}} = \beta T_{\text{car}} + \gamma M_{\text{car}} \qquad U_{\text{car}} = V_{\text{car}} + \varepsilon_{\text{car}}$$

$$V_{\text{bus}} = \beta T_{\text{bus}} + \gamma M_{\text{bus}} \qquad U_{\text{bus}} = V_{\text{bus}} + \varepsilon_{\text{bus}}$$

Choice Probabilities

$$P_{\text{car}} = \mathbb{P}(\varepsilon_{\text{bus}} - \varepsilon_{\text{car}} < V_{\text{car}} - V_{\text{bus}})$$

$$P_{\text{bus}} = \mathbb{P}(\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} < V_{\text{bus}} - V_{\text{car}}) = 1 - P_{\text{car}}$$

A Very Simple Example: Who drives to work?

What is common?

Parameters: (β, γ). Our goal is to estimate these.

Observed Heterogeneity

- Alice lives next to the bus stop: her T_{bus} is low.
- Bob is 70 and gets a discount on public transport: his M_{bus} is low.
- Clara and her roommates work at the same office and can carpool: her M_{car} is low.

Unobserved Heterogeneity

James hates to drive ($\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} < 0$) but Steve loves driving ($\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} > 0$).

The Likelihood for Random Utility Models

Notation

- $y_n \in \{1, \ldots, J\} \equiv n$'s choice.
- $\mathbf{z}_n$ vector of all regressors for n
- $\boldsymbol{\theta}$ vector of all unknown parameters
- Choice Probs. $P_{ni} \equiv \mathbb{P}(y_n = i | \mathbf{z}_n, \boldsymbol{\theta})$

Note

Likelihood is easy, but choice probabilities are usually hard (logit is an exception).

Likelihood

$f(y_n | \mathbf{z}_n, \boldsymbol{\theta}) = \prod_{j=1}^{J} P_{nj}^{\mathbb{1}\{y_n = j\}}$

Log Likelihood

$\ell_N(\boldsymbol{\theta}) = \sum_{n=1}^{N} \sum_{j=1}^{J} \mathbb{1}\{y_n = j\} \log P_{nj}$

Example: Logit Choice Probabilities

$P_{ni} = \exp(V_{ni}) / \sum_{j=1}^{J} \exp(V_{nj})$

Identification – What can we learn from data?

Identification

A parameter is identified if it could be uniquely determined by observing the whole population of data from which our sample was drawn.

E.g. Car versus Bus

Are (β, γ) from $V_{nj} = \beta T_{nj} + \gamma M_{nj}$ identified?

Recall from Microeconomics

1. Only differences in utility matter for choices.
2. The scale of utility is irrelevant.

Only Differences in Utility Matter

All that matters for choices is how much better/worse an alternative is than the others:

$$\mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \mathbb{P}\left(U_{ni} - U_{nj} > 0 \quad \forall j \neq i\right)$$

Consequences

1. Only differences in errors matter.
2. We cannot identify a different intercept for each alternative.
3. We can only identify differences of effects for decision-maker attributes.

Only Differences in Errors Matter

Notation

- $\widetilde{\varepsilon}_{njk} \equiv \varepsilon_{nj} - \varepsilon_{nk}$ be the *difference* of errors ε_{nj} and ε_{nk}.
- $\widetilde{\boldsymbol{\varepsilon}}_{ni} \equiv$ vector of all unique differences, taking ε_{ni} as the "base case"
  - E.g. $\boldsymbol{\varepsilon}'_n = (\varepsilon_{n1}, \varepsilon_{n2}, \varepsilon_{n3}) \implies \widetilde{\boldsymbol{\varepsilon}}'_{n1} = (\varepsilon_{n2} - \varepsilon_{n1}, \varepsilon_{n3} - \varepsilon_{n1})$
  - Note: J errors $\Rightarrow (J-1)$ unique *differences*
- Let g be the joint density of $\widetilde{\boldsymbol{\varepsilon}}_{ni}$.

Choice Probabilities

$$
\begin{aligned}
P_{ni} &\equiv \mathbb{P}\left(U_{ni} > U_{nj} \quad \forall j \neq i\right) = \mathbb{P}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i) \\
&= \mathbb{P}(\widetilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \quad \forall j \neq i) = \int_{\mathbb{R}^{J-1}} \mathbb{1}\left\{\widetilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \quad \forall j \neq i\right\} g(\widetilde{\boldsymbol{\varepsilon}}_{ni})\, d\widetilde{\boldsymbol{\varepsilon}}_{ni}
\end{aligned}
$$

If there are J alternatives, we can identify only $(J-1)$ intercepts.

Equivalently: normalize one intercept to zero.

Intercept $\Rightarrow \mathbb{E}\left[\varepsilon_{nj}\right]=0$

- Suppose $U_{nj}=\mathbf{x}_{nj}'\boldsymbol{\beta}+\varepsilon_{nj}^*$ where $\mathbf{x}_{nj}$ *excludes* a constant and $\mathbb{E}[\varepsilon_{nj}^*]\neq 0$.
- Equivalent model: $U_{nj}=\alpha_j+\mathbf{x}_{nj}'\boldsymbol{\beta}+\varepsilon_{nj}$ where $\mathbb{E}[\varepsilon_{nj}]=0$ by construction.

Why not a different intercept for each alternative?

$$U_{\text{car}}=\alpha_{\text{car}}+\beta T_{\text{car}}+\gamma M_{\text{car}}+\varepsilon_{\text{car}}$$

$$U_{\text{bus}}=\alpha_{\text{bus}}+\beta T_{\text{bus}}+\gamma M_{\text{bus}}+\varepsilon_{\text{bus}}$$

$$U_{\text{bus}}-U_{\text{car}}=\left(\alpha_{\text{bus}}-\alpha_{\text{car}}\right)+\beta\left(T_{\text{bus}}-T_{\text{car}}\right)+\gamma\left(M_{\text{bus}}-M_{\text{car}}\right)+\left(\varepsilon_{\text{bus}}-\varepsilon_{\text{car}}\right)$$

Only differences of effects for decision-maker attributes are identified.

Can we identify the effects of income Y separately for Bus and Car?

$$U_{\text{car}} = \theta_{\text{car}} Y + \beta T_{\text{car}} + \gamma M_{\text{car}} + \varepsilon_{\text{car}}$$

$$U_{\text{bus}} = \theta_{\text{bus}} Y + \beta T_{\text{bus}} + \gamma M_{\text{bus}} + \varepsilon_{\text{bus}}$$

$$U_{\text{bus}} - U_{\text{car}} = (\theta_{\text{bus}} - \theta_{\text{car}})\, Y + \beta\,(T_{\text{bus}} - T_{\text{car}}) + \gamma\,(M_{\text{bus}} - M_{\text{car}}) + (\varepsilon_{\text{bus}} - \varepsilon_{\text{car}})$$

Equivalent to normalizing one of the θs to zero.

More on Identification – The Scale of Utility is Irrelevant

Why?

- Let λ be an arbitrary positive constant.
- Rational Choice: select i if and only if $U_{ni} > U_{nj}$ for all $j \neq i$
- Equivalently: select i if and only if $\lambda U_{ni} > \lambda U_{nj}$ for all $j \neq i$

$\text{Var}(\varepsilon_{nj})$ determines the scale of $\boldsymbol{\beta}$

- $\boxed{U_{nj} = \mathbf{x}_{nj}'\boldsymbol{\beta} + \varepsilon_{nj},\ \text{Var}(\varepsilon_{nj}) = \sigma^2} \iff \boxed{U_{nj}^* = \mathbf{x}_{nj}'(\boldsymbol{\beta}/\sigma) + \varepsilon_{nj}^*,\ \text{Var}(\varepsilon_{nj}^*) = 1}$
- Can't directly compare coefs. across models with different normalizations for ε_{nj}.
- Recall: we had to re-scale Logit and Probit coefs. to compare them.

How to obtain the index models from last lecture? (E.g. Probit and Logit)

1. Two alternatives, e.g. Bus or Something Else
2. Let $y_n = 1$ if decision-maker n chooses alternative 1; zero otherwise.
3. $V_{nj} = \mathbf{s}_n'\gamma_j$ (representative utility depends only on attributes of decision-maker)
4. $(\varepsilon_{n2} - \varepsilon_{n1}) \sim G$ independently of $\mathbf{s}_n$.

$$\begin{aligned} U_{n1} - U_{n2} &= (\mathbf{s}_n'\gamma_1 - \mathbf{s}_n'\gamma_2) + (\varepsilon_{n1} - \varepsilon_{n2}) = \mathbf{s}_n'(\gamma_1 - \gamma_2) + (\varepsilon_{n1} - \varepsilon_{n2}) \\ &= \mathbf{s}_n'\gamma + (\varepsilon_{n1} - \varepsilon_{n2}) \end{aligned}$$

$$\mathbb{P}(y_n = 1|\mathbf{s}_n) = \mathbb{P}(U_{n1} - U_{n2} > 0|\mathbf{s}_n) = \mathbb{P}(\varepsilon_{n2} - \varepsilon_{n1} < \mathbf{s}_n'\gamma|\mathbf{s}_n) = G(\mathbf{s}_n'\gamma)$$

The Logit Family of Choice Models

Theorem

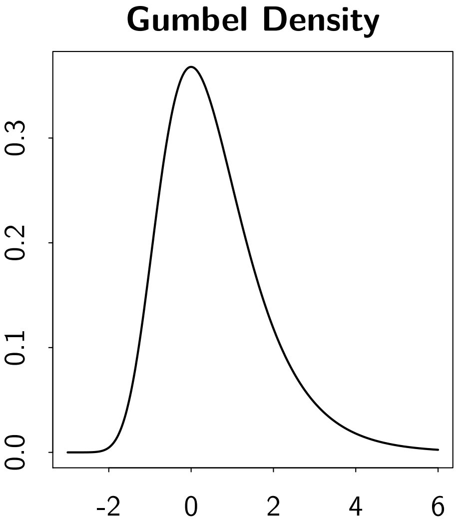
Suppose that $\varepsilon_{n1}, \ldots \varepsilon_{nJ} \sim$ iid F where $F(z) = \exp\{-\exp(-z)\}$. Then,

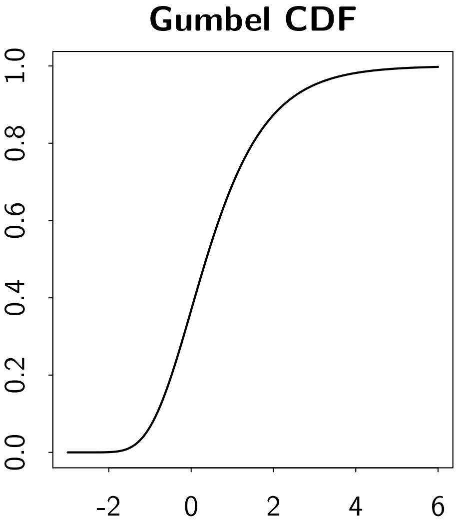
$$P_{ni} = \mathbb{P}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i) = \frac{\exp(V_{ni})}{\sum_{j=1}^{J} \exp(V_{nj})}$$

Notes

- This is a special case where the choice probabilities have a closed-form solution!
- $F(z) = \exp\{-\exp(-z)\}$ is the Gumbel aka Type I Extreme Value CDF
- Corollary: the *difference* of independent Gumbel RVs is a standard Logistic RV

The Gumbel Distribution (aka Type I Extreme Value)





Different specifications of V_{nj} yield different models.

Multinomial Logit

- $V_{nj} = \mathbf{s}_n' \gamma_j \quad \leftarrow$ only attributes that are fixed across alternatives (e.g. n's income)
- Can only identify differences $(\gamma_j - \gamma_i)$. Typical to normalize $\boldsymbol{\gamma}_1 = \mathbf{0}$.

Conditional Logit

- $V_{nj} = \mathbf{x}_{nj}' \boldsymbol{\beta} \quad \leftarrow$ only attributes that vary across alternatives (e.g. price)
- Note that $\boldsymbol{\beta}$ is fixed across alternatives.

Mixed Logit

- $V_{nj} = \mathbf{s}_n' \gamma_j + \mathbf{x}_{nj}' \boldsymbol{\beta} \quad \leftarrow$ a combination of the two

Interpreting Multinomial Logit Coefficients

- Partial effects tricky to derive and interpret.
- Better approach: partial effects for relative risk
- Normalizing $\gamma_1 = \mathbf{0}$, we have $\exp(\mathbf{s}_n \gamma_1) = \exp(0) = 1$. Hence,

$$\frac{P_{ni}}{P_{n1}} = \frac{\exp\left(\mathbf{s}_n \gamma_i\right)}{\sum_{j=1}^{J} \exp\left(\mathbf{s}_n \gamma_j\right)} \times \frac{\sum_{j=1}^{J} \exp\left(\mathbf{s}_n \gamma_j\right)}{\exp\left(\mathbf{s}_n \gamma_1\right)} = \frac{\exp(\mathbf{s}_n \gamma_i)}{\exp(\mathbf{s}_n \gamma_1)} = \exp(\mathbf{s}_n \gamma_i)$$

- Taking logs: $\log\left(P_{ni}/P_{n1}\right) = \log\left[\exp(\mathbf{s}_n \gamma_i)\right] = \mathbf{s}_n' \gamma_i$.

Punchline

$\gamma_i^{(k)}$ is the marginal effect of $s_n^{(k)}$ on the relative probability that $y = i$ compared to $y = 1$ measured on the log scale – e.g. taking the bus relative to driving.

Interpreting Conditional Logit Coefficients

You'll derive these on the problem set!

Partial Effects

- The attributes $\mathbf{x}_{nj}$ are *specific* to a particular alternative j.
- Hence: partial effects are much simpler for conditional logit than multinomial.

Own Attribute

$$\frac{\partial P_{nj}}{\partial \mathbf{x}_{nj}} = P_{nj}(1 - P_{nj})\boldsymbol{\beta}$$

Cross-Attribute ($j \neq i$)

$$\frac{\partial P_{nj}}{\partial \mathbf{x}_{ni}} = -P_{nj}P_{ni}\boldsymbol{\beta}$$

If increasing $\mathbf{x}_{nj}^{(k)}$ makes $y = j$ *more likely*, it must make $y = i$ *less likely*

The Independence of Irrelevant Alternatives (IIA)

Or why people don't like logit models...

Logit Choice Probabilities

$$P_{ni} = \frac{\exp(V_{ni})}{\sum_{j=1}^{J} \exp(V_{nj})} \implies \frac{P_{ni}}{P_{nj}} = \exp(V_{ni} - V_{nj})$$

In Words

The relative probability of choosing i versus j only depends on the representative utilties for i and j. This is called the independence of irrelevant alternatives (IIA).

Why is this a problem

IIA arises in logit models because $\varepsilon_{n1}, \ldots, \varepsilon_{nJ}$ are *independent*. In reality "some alternatives are more similar than others," i.e. errors may be correlated.

An Example where IIA is Unreasonable – Choosing Presidential Candidates

Model

- $V_{nj} = (\text{Demographics}_n)'\gamma_j + (\text{Ideology}_{nj})'\beta$
- $(\text{Ideology}_{nj}) =$ similarity between voter n's ideology and candidate j's.
- $\text{Candidates} = \{\text{Trump}, \text{Sanders}, \text{Warren}\}$

Consider a group of voters who all have the *same* demographics and ideology

E.g. white, centrist, female, mid-westerners between the age of 45 and 50 with an average household income between \$50 and \$55 thousand USD.

Same regressors $\Rightarrow$ same V_{nj}

V_{nj} doesn't vary over n within the group: $\{V_{\text{Trump}}, V_{\text{Sanders}}, V_{\text{Warren}}\}$

An Example where IIA is Unreasonable – Choosing Presidential Candidates

Two-way Race

Suppose 2/3 of this group of voters chooses Sanders over Trump: $P_{\text{Sanders}}/P_{\text{Trump}} = 2$

Assumption

Sanders and Warren are ideologically similar $\implies V_{\text{Warren}} \approx V_{\text{Sanders}}$

Implications of Logit

- Relative choice probabilities are the *same* in a two-way race or a three-way race.
- $P_{\text{Warren}}/P_{\text{Sanders}} = \exp\left(V_{\text{Warren}} - V_{\text{Sanders}}\right) \approx 1$

An Example where IIA is Unreasonable – Choosing Presidential Candidates

Logit Implication for Three-way Race

$$P_{\text{Sanders}} = 2P_{\text{Trump}}, \quad P_{\text{Sanders}} \approx P_{\text{Warren}}, \quad P_{\text{Trump}} + P_{\text{Sanders}} + P_{\text{Warren}} = 1$$

$$\begin{aligned} \implies P_{\text{Trump}} + 2P_{\text{Trump}} + 2P_{\text{Trump}} &= 1 \\ P_{\text{Trump}} &= 1/5 \\ P_{\text{Warren}} &= P_{\text{Sanders}} = 2/5 \end{aligned}$$

What we'd actually expect in a Three-way Race

1/3 Trump, 1/3 Sanders and 1/3 Warren – i.e. Warren "splits" the Sanders vote.

What's going wrong?

Logit assumes $\varepsilon_{\text{Warren}}$ and $\varepsilon_{\text{Sanders}}$ are independent but in reality they're not.

Lecture #5 – Sample Selection

Examples of Sample Selection

The Heckman Selection Model

Proof of First Lemma

Proof of Second Lemma

The Expectation of a Truncated Normal

What is sample selection?

Question

Thus far we have always assumed that $(y_1, \mathbf{x}_1), \ldots, (y_N, \mathbf{x}_N)$ are a random sample from the population of interest. What if they aren't?

Example 1: MPhil Admissions

- Suppose we want to improve admissions decisions at Oxford.
- $y \equiv$ overall marks in 1st year of Oxford Economics MPhil
- $\mathbf{x} \equiv \{\text{undergrad grades}, \text{letters of reference}, \ldots\}$
- What we observe: $\mathbf{x}$ for all applicants; y for applicants who were admitted.
- What we want: $\mathbb{E}(y|\mathbf{x})$ for all applicants.

Example 2: A Model of Wage Offers

Gronau (1974; JPE)

Question

How do wage offers offers w_i^o vary with $\mathbf{x}_i$ for all people in the population.

Problem

Only observe w_i^o for people who *accept* their offer, i.e. those who are employed.

Mathematically

$\mathbb{E}(w_i^o|\mathbf{x}_i) \neq \mathbb{E}(w_i^o|\mathbf{x}_i, \text{Employed})$

The Heckman Selection Model — Is β_1 identified?

Outcome Equation

$$y_1 = \mathbf{x}_1'\beta_1 + u_1$$

Participation Equation

$$y_2 = \mathbb{1}\left\{\mathbf{x}'\boldsymbol{\delta}_2 + v_2 > 0\right\}$$

Assumptions

(a) Observe $y_2, \mathbf{x}' = (\mathbf{x}_1', \mathbf{x}_2')$; only observe y_1 if $y_2 = 1$.

(b) (u_1, v_2) are mean zero and jointly independent of $\mathbf{x}$.

(c) $v_2 \sim \text{Normal}(0, 1)$

(d) $\mathbb{E}(u_1 | v_2) = \gamma_1 v_2$ where γ_1 is an unknown constant.

Notes

- $\mathbb{E}(u_1) = \mathbb{E}(v_2) = 0$ is not restrictive: just include intercepts in both equations.
- Assumption (d) would be *implied* by assuming that (u_1, v_2) are jointly normal.
- These assumptions are strong. They can be weakened somewhat.

Two Lemmas $\implies \beta_1$ Identified from Two Simple Regressions

Lemma 1: $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$

- Shorthand: $h(\mathbf{x}) \equiv \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$
- $(\boldsymbol{\beta}_1, \gamma_1)$ identified from regression of y_1 on $[\mathbf{x}_1, h(\mathbf{x})]$ for selected population.

Lemma 2: $\mathbb{E}(v_2|\mathbf{x}, y_2 = 1) = \varphi(\mathbf{x}'\boldsymbol{\delta}_2)/\Phi(\mathbf{x}'\boldsymbol{\delta}_2)$

- $h(\mathbf{x}) = \lambda(\mathbf{x}'\boldsymbol{\delta}_2)$ where $\lambda(c) \equiv \varphi(c)/\Phi(c)$ is called the inverse Mills ratio

Probit Identifies $\boldsymbol{\delta}_2$

- $(y_2, \mathbf{x})$ observed for full sample and $y_2 = \mathbb{1}\{\mathbf{x}'\boldsymbol{\delta}_2 + v_2 > 0\}$ where $v_2 \sim N(0,1)$

The Heckman Two-step Estimator aka "Heckit"

Observables

Observe $(y_{2i}, \mathbf{x}_i)$ for a random sample of size N; only observe y_{1i} for those with $y_{2i} = 1$.

First Step – Estimate δ_2 from Full Sample

- Run Probit on the Participation Eq. $\mathbb{P}(y_{2i} = 1|\mathbf{x}_i) = \Phi(\mathbf{x}_i'\boldsymbol{\delta}_2)$ for the full sample.
- Define $\widehat{\lambda}_i \equiv \lambda(\mathbf{x}_i'\widehat{\boldsymbol{\delta}}_2)$ where $\widehat{\boldsymbol{\delta}}_2$ is the MLE for $\boldsymbol{\delta}_2$.

Second Step – Estimate (β_1, γ_1) from Selected Sample

Using the observations for which y_{i1} is observed, regress y_{i1} on $(\mathbf{x}_{1i}, \widehat{\lambda}_i)$ by OLS to obtain estimates $(\widehat{\boldsymbol{\beta}}_1, \widehat{\gamma}_1)$.

The Big Picture: How does Heckit solve the selection problem?

- If we regress y_{1i} on $\mathbf{x}_{1i}$ for the selected sample, there is an omitted variable.
- Under the Heckit assumptions, the omitted variable is precisely $\lambda(\mathbf{x}_i'\boldsymbol{\delta}_2)$.
- Hence: a regression of y_{1i} on $\mathbf{x}_{1i}$ and $\lambda(\mathbf{x}_i'\boldsymbol{\delta}_2)$ is correctly specified.

Why is the second step regression identified?

Second Step Regression

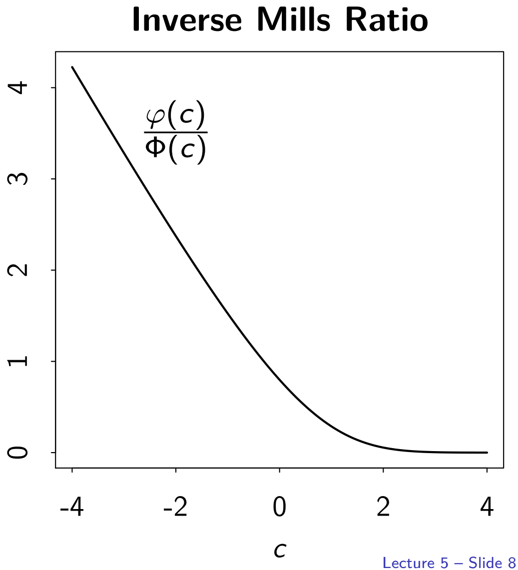
y_{1i} on $[\mathbf{x}_{1i}, \lambda(\mathbf{x}_i'\widehat{\boldsymbol{\delta}}_2)]$ for selected sample

Exclusion Restriction

$\mathbf{x}_i$ contains some variables *not* in $\mathbf{x}_{1i}$

No Exclusion Restriction

- $\lambda(c) \equiv \varphi(c)/\Phi(c)$ is nonlinear
- $\lambda(\mathbf{x}_{1i}'\boldsymbol{\delta}_2)$ and $\mathbf{x}_{1i}$ are not co-linear
- Identification is less credible
- λ *close* to linear: noisy estimates



Asymptotics for "Heckit"

Theorem

Under our assumptions and some regularity conditions, the "Heckit" estimators satisfy

$$\begin{bmatrix} \widehat{\boldsymbol{\delta}}_2 \\ \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\gamma}_1 \end{bmatrix} \to_p \begin{bmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\beta}_1 \\ \gamma_1 \end{bmatrix} \quad \text{and} \quad \sqrt{N} \begin{bmatrix} \widehat{\boldsymbol{\delta}}_2 - \boldsymbol{\delta}_2 \\ \widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1 \\ \widehat{\gamma}_1 - \gamma_1 \end{bmatrix} \to_d \text{Normal}(\mathbf{0}, \Omega) \quad \text{as } N \to \infty.$$

Standard Errors

The asymptotic variance matrix Ω is complicated: the usual OLS standard errors from step two are incorrect as they do not account for the estimation of $\boldsymbol{\delta}_2$ in step one.

Proof of First Lemma

Lemma 1: $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$

Steps in the Proof

1. u_1 is conditionally independent of $\mathbf{x}$ given v_2
2. $\mathbb{E}(y_1|\mathbf{x}, v_2) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 v_2$
3. Relate unobserved $\mathbb{E}(y_1|\mathbf{x}, v_2)$ to observed $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1)$.

Step 1: u_1 and $\mathbf{x}$ are conditionally independent given v_2.

Assumption (b)

(u_1, v_2) are jointly independent of $\mathbf{x}$.

Equivalently

$$f_{1,2|\mathbf{x}}(u_1, v_2|\mathbf{x}) = f_{1,2}(u_1, v_2), \quad \text{and} \quad f_{1|\mathbf{x}}(u_1|\mathbf{x}) = f_1(u_1), \quad \text{and} \quad f_{2|\mathbf{x}}(v_2|\mathbf{x}) = f_2(v_2)$$

Therefore

$$f_{1|2,\mathbf{x}}(u_1|v_2, \mathbf{x}) = \frac{f_{1,2|\mathbf{x}}(u_1, v_2|\mathbf{x})}{f_{2|\mathbf{x}}(v_2|\mathbf{x})} = \frac{f_{1,2}(u_1, v_2)}{f_2(v_2)} = f_{1|2}(u_1|v_2)$$

In Words

Conditioning on $(v_2, \mathbf{x})$ gives the same information about u_1 as conditioning on v_2 only.

Step 2: $\mathbb{E}(y_1|\mathbf{x}, v_2) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 v_2$

$$
\begin{aligned}
\mathbb{E}(y_1|\mathbf{x}, v_2) &= \mathbb{E}(\mathbf{x}_1'\boldsymbol{\beta}_1 + u_1|\mathbf{x}, v_2) && \text{(Substitute Outcome Eq.)} \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \mathbb{E}(u_1|\mathbf{x}, v_2) && (\mathbf{x}_1 \text{ is a subset of } \mathbf{x}) \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \mathbb{E}(u_1|v_2) && \text{(apply result of Step 1)} \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 v_2 && \text{(apply Assumption (d))}
\end{aligned}
$$

Step 3: Relate unobserved $\mathbb{E}(y_1|\mathbf{x}, v_2)$ to observed $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1)$.

$$
\begin{aligned}
\mathbb{E}(y_1|\mathbf{x}, y_2) &= \mathbb{E}_{v_2|(\mathbf{x},y_2)}\left[\mathbb{E}\left(y_1|\mathbf{x}, y_2, v_2\right)\right] && \text{(Law of Iterated Expectations)} \\
&= \mathbb{E}_{v_2|(\mathbf{x},y_2)}\left[\mathbb{E}\left(y_1|\mathbf{x}, v_2\right)\right] && \text{(Participation Eq: } y_2 = g(\mathbf{x}, v_2)\text{)} \\
&= \mathbb{E}\left[\mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 v_2|\mathbf{x}, y_2\right] && \text{(apply result of Step 2)} \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}\left(v_2|\mathbf{x}, y_2\right) && \text{(}\mathbf{x}_1\text{ is a subset of }\mathbf{x}\text{)}
\end{aligned}
$$

Therefore

$$\mathbb{E}\left(y_1|\mathbf{x}, y_2 = 1\right) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2|\mathbf{x}, y_2 = 1) \ \checkmark$$

Note: Selection Bias Enters Through γ_1

Assumption (d)

$\mathbb{E}(u_1|v_2) = \gamma_1 v_2$ allows *dependence* between errors in participation and outcome eqs.

Step 3

$\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$

Therefore

If $\gamma_1 = 0$ there is no selection bias: in this case $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}$ so regressing y_1 on $\mathbf{x}_1$ for the subset of individuals with $y_2 = 1$ identifies $\boldsymbol{\beta}_1$.

Proof of Second Lemma

Lemma 2: $\mathbb{E}(v_2|\mathbf{x}, y_2 = 1) = \varphi(\mathbf{x}'\boldsymbol{\delta}_2)/\Phi(\mathbf{x}'\boldsymbol{\delta}_2)$

Steps in the Proof

1. Determine the distribution of v_2 given $(\mathbf{x}, y_2 = 1)$
2. Apply a result for truncated normal distributions.

Step 1: Determine the distribution of v_2 given $(\mathbf{x}, y_2 = 1)$.

$$\mathbb{P}(v_2 \leq t|\mathbf{x}, y_2 = 1) = \mathbb{P}(v_2 \leq t|\mathbf{x}, v_2 > -\mathbf{x}'\boldsymbol{\delta}_2) \qquad \text{(participation eq.)}$$

$$= \frac{\mathbb{P}\left(\{v_2 \leq t\} \cap \{v_2 > -\mathbf{x}'\boldsymbol{\delta}_2\} |\mathbf{x}\right)}{\mathbb{P}(v_2 > -\mathbf{x}'\boldsymbol{\delta}_2|\mathbf{x})} \qquad \text{(defn. of cond. prob.)}$$

$$= \frac{\mathbb{P}\left\{v_2 \in (-\mathbf{x}'\boldsymbol{\delta}_2, t]\right\}}{\mathbb{P}(v_2 > -\mathbf{x}'\boldsymbol{\delta}_2)} \qquad (v_2 \text{ and } \mathbf{x} \text{ are indep.})$$

$$= \frac{\mathbb{P}\left\{v_2 \in (c, t]\right\}}{\mathbb{P}(v_2 > c)} \qquad (\text{shorthand: } c \equiv -\mathbf{x}'\boldsymbol{\delta}_2)$$

$$= \mathbb{P}(v_2 \leq t|v_2 > c) \qquad \text{(defn. of cond. prob.)}$$

Step 2: Apply a result for truncated normal distributions.

Result of Step 1

$\mathbb{P}(v_2 \leq t|\mathbf{x}, y_2 = 1) = \mathbb{P}(v_2 \leq t|v_2 > c)$ where $c \equiv -\mathbf{x}'\boldsymbol{\delta}_2$.

Assumption (c)

v_2 is a standard normal random variable

Combining

$$\mathbb{E}(v_2|\mathbf{x}, y_2 = 1) = \mathbb{E}(v_2|v_2 > c) = \frac{\varphi(c)}{1 - \Phi(c)} \qquad (\mathbb{E}[\text{truncated normal}])$$

$$= \frac{\varphi(-\mathbf{x}'\boldsymbol{\delta}_2)}{1 - \Phi(-\mathbf{x}'\boldsymbol{\delta}_2)} = \frac{\varphi(\mathbf{x}'\boldsymbol{\delta}_2)}{\Phi(\mathbf{x}'\boldsymbol{\delta}_2)} \quad (\varphi(-c) = \varphi(c),\ 1 - \Phi(c) = \Phi(-c))$$

The Expectation of a Truncated Normal

Lemma

If $z \sim \mathcal{N}(0,1)$ then for any constant c we have $\mathbb{E}[z|z > c] = \dfrac{\varphi(c)}{1 - \Phi(c)}$.

CDF

$$\mathbb{P}(z \leq t | z > c) = \frac{\mathbb{P}\left\{z \in (c, t]\right\}}{\mathbb{P}(z > c)} = \mathbb{1}\left\{c \leq t\right\} \left[\frac{\Phi(t) - \Phi(c)}{1 - \Phi(c)}\right]$$

Density

$$f(t|z > c) = \frac{d}{dt}\mathbb{P}(z \leq t|z > c) = \begin{cases} 0, & t \leq c \\ \varphi(t)/\left[1 - \Phi(c)\right], & t > c \end{cases}$$

The Expectation of a Truncated Normal

$$\mathbb{E}(z|z>c) = \int_{-\infty}^{\infty} tf(t|z>c)\,dt = \frac{1}{1-\Phi(c)}\int_{c}^{\infty} t\varphi(t)\,dt$$

$$= \left[\frac{1}{1-\Phi(c)}\right]\left(\frac{1}{\sqrt{2\pi}}\right)\int_{c}^{\infty} t\exp\left\{-t^2/2\right\}\,dt$$

$$= \left[\frac{1}{1-\Phi(c)}\right]\left(\frac{1}{\sqrt{2\pi}}\right)\left[-\exp\left\{-t^2/2\right\}\right]_{c}^{\infty}$$

$$= \left[\frac{1}{1-\Phi(c)}\right]\left(\frac{\exp\left\{-c^2/2\right\}}{\sqrt{2\pi}}\right) = \frac{\varphi(c)}{1-\Phi(c)}$$