

# MPhil Econometrics – Limited Dependent Variables and Selection

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# Housekeeping

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**Course Materials:** <https://economictricks.com>

## References

- ▶ **Wooldridge (2010) – *Econometric Analysis of Cross Section & Panel Data***
- ▶ Cameron & Trivedi (2005) – *Microeconometrics: Methods and Applications*
- ▶ Train (2009) – *Discrete Choice Methods with Simulation*

# Lecture #1 – Maximum Likelihood Estimation Under Mis-specification

Review: the Poisson Distribution

The Kullback-Leibler Divergence

Example: Consistency of Poisson MLE

Asymptotic Theory for MLE Under Mis-specification

The Information Matrix Equality

Example: Asymptotic Variance Calculations for Poisson MLE

“All models are wrong; some are useful.”

### Question

What happens if we carry out maximum likelihood estimation, but our model is *wrong*?

### This Lecture

Examine a simple example in excruciating detail; present the general theory.

### Next Lecture

Apply what we've learned to study **Poisson Regression**, a model for count data.

Suppose that  $y \sim \text{Poisson}(\theta)$

Support Set:  $\{0, 1, 2, \dots\}$

A Poisson Random Variable is a *count*.

Probability Mass Function

$$f(y|\theta) = \frac{e^{-\theta} \theta^y}{y!}$$

Expected Value:  $\mathbb{E}(y) = \theta$

Poisson parameter  $\theta$  equals the mean of  $y$ .

Variance:  $\text{Var}(y) = \theta$

You will show this on the problem set.

$$\sum_{y=0}^{\infty} \frac{e^{-\theta} \theta^y}{y!} = e^{-\theta} \sum_{y=0}^{\infty} \frac{\theta^y}{y!} = e^{-\theta} (e^{\theta}) = 1$$

$$\begin{aligned} \mathbb{E}(y) &= \sum_{y=0}^{\infty} y \frac{e^{-\theta} \theta^y}{y!} = \sum_{y=1}^{\infty} y \frac{e^{-\theta} \theta^y}{y!} \\ &= \theta \sum_{y=1}^{\infty} \frac{e^{-\theta} \theta^{y-1}}{(y-1)!} = \theta \sum_{y=0}^{\infty} \frac{e^{-\theta} \theta^y}{y!} = \theta \end{aligned}$$

MLE for  $\theta$  where  $y_1, y_2, \dots, y_N \sim \text{iid Poisson}(\theta)$ .

The Likelihood (iid data)

$$L_N(\theta) \equiv \prod_{i=1}^N \frac{e^{-\theta} \theta^{y_i}}{y_i!}$$

The Log-Likelihood

$$\ell_N(\theta) = \sum_{i=1}^N [y_i \log(\theta) - \theta - \log(y_i!)]$$

Maximum Likelihood Estimator

$$\hat{\theta} \equiv \arg \max_{\theta \in \Theta} \ell_N(\theta) = \bar{y}$$

$$\frac{d}{d\theta} \ell_N(\theta) = \sum_{i=1}^N \left[ \frac{y_i}{\theta} - 1 \right]$$

$$\frac{d}{d\theta} \ell_N(\hat{\theta}) = 0$$

$$\sum_{i=1}^N \left[ y_i / \hat{\theta} - 1 \right] = 0$$

$$\left( \sum_{i=1}^N y_i \right) / \hat{\theta} = N$$

$$\frac{1}{N} \sum_{i=1}^N y_i = \bar{y} = \hat{\theta}$$

# The Kullback-Leibler (KL) Divergence

## Motivation

How well does a parametric model  $f(\mathbf{y}|\boldsymbol{\theta})$  approximate a *true* density/pmf  $p_o(\mathbf{y})$ ?

## Definition

$$KL(p_o; f_{\boldsymbol{\theta}}) \equiv \mathbb{E} \left[ \log \left\{ \frac{p_o(\mathbf{y})}{f(\mathbf{y}|\boldsymbol{\theta})} \right\} \right]$$

## KL Properties

1. *Asymmetric*:  $KL(p_o; f_{\boldsymbol{\theta}}) \neq KL(f_{\boldsymbol{\theta}}; p_o)$
2.  $KL(p_o; f_{\boldsymbol{\theta}}) \geq 0$ ; zero iff  $p_o = f_{\boldsymbol{\theta}}$
3. Min KL iff max expected log-likelihood

## Alternative Expression

$$\mathbb{E} \left[ \log \left\{ \frac{p_o(\mathbf{y})}{f(\mathbf{y}|\boldsymbol{\theta})} \right\} \right] = \underbrace{\mathbb{E} [\log p_o(\mathbf{y})]}_{\text{Constant wrt } \boldsymbol{\theta}} - \underbrace{\mathbb{E} [\log f(\mathbf{y}|\boldsymbol{\theta})]}_{\text{Expected Log-like.}}$$

## All expectations are wrt $p_o$

$p_o(\mathbf{y})$  and  $f(\mathbf{y}|\boldsymbol{\theta})$  are merely *functions* of the RV  $\mathbf{y}$

$$\mathbb{E}[\log p_o(\mathbf{y})] = \int \log p_o(\mathbf{y}) p_o(\mathbf{y}) d\mathbf{y}$$

$$\mathbb{E}[\log f(\mathbf{y}|\boldsymbol{\theta})] = \int \log f(\mathbf{y}|\boldsymbol{\theta}) p_o(\mathbf{y}) d\mathbf{y}$$

## Watch Out!

$KL = \infty$  if  $\exists \mathbf{y}$  with  $f(\mathbf{y}|\boldsymbol{\theta}) = 0$  &  $p_o(\mathbf{y}) \neq 0$

$\text{KL}(p_o; f) \geq 0$  with equality iff  $p_o = f$

### Jensen's Inequality

If  $\varphi$  is convex then  $\varphi(\mathbb{E}[y]) \leq \mathbb{E}[\varphi(y)]$ , with equality iff  $\varphi$  is linear or  $y$  is constant.

$\log$  is concave so  $(-\log)$  is convex

$$\begin{aligned}\mathbb{E} \left[ \log \left\{ \frac{p_o(y)}{f(y)} \right\} \right] &= \mathbb{E} \left[ -\log \left\{ \frac{f(y)}{p_o(y)} \right\} \right] \geq -\log \left\{ \mathbb{E} \left[ \frac{f(y)}{p_o(y)} \right] \right\} \\ &= -\log \left\{ \int_{-\infty}^{\infty} \frac{f(y)}{p_o(y)} \cdot p_o(y) dy \right\} \\ &= -\log \left\{ \int_{-\infty}^{\infty} f(y) dy \right\} \\ &= -\log(1) = 0\end{aligned}$$



# A Simple Example: Calculating the KL Divergence

Remember: all expectations are calculated using  $p_o$ .

True Distribution  $p_o$

$y_1, \dots, y_N \sim \text{iid } p_o$  where:

$$p_o(0) = \frac{2}{5}, p_o(1) = \frac{1}{5}, p_o(2) = \frac{2}{5}.$$

Mis-specified Model  $f_\theta$

$y_1, \dots, y_N \sim \text{iid Poisson}(\theta)$

KL Divergence

$$KL(p_o; f_\theta) = \theta - \log \theta + (\text{Constant})$$

$$KL(p_o; f_\theta) = \mathbb{E}[\log p_o(y)] - \mathbb{E}[\log f(y|\theta)]$$

$$\begin{aligned}\mathbb{E}[\log p_o(y)] &= \sum_{\text{all } y} \log [p_o(y)] p_o(y) \\ &= \log \left( \frac{2}{5} \right) \times \frac{2}{5} + \log \left( \frac{1}{5} \right) \times \frac{1}{5} + \log \left( \frac{2}{5} \right) \times \frac{2}{5}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\log f(y|\theta)] &= \sum_{\text{all } y} \log \left[ \frac{e^{-\theta} \theta^y}{y!} \right] p_o(y) \\ &= \log \left( e^{-\theta} \right) \times \frac{2}{5} + \log \left( e^{-\theta} \theta \right) \times \frac{1}{5} + \log \left( \frac{e^{-\theta} \theta^2}{2} \right) \times \frac{2}{5} \\ &= - \left[ \theta - \log(\theta) + \log(2) \times \frac{2}{5} \right]\end{aligned}$$

## A Simple Example Continued: Minimizing the KL Divergence

Model = Poisson( $\theta$ ); True Dist.  $p_o(0) = p_o(2) = \frac{2}{5}$  and  $p_o(1) = \frac{1}{5}$

### Best Approximation

What parameter value  $\theta_o$  makes the Poisson( $\theta$ ) model *as close as possible* to the true distribution  $p_o$ , where we measure “closeness” using the KL-divergence?

### Using the previous slide

$$KL(p_o; f_\theta) = \theta - \log \theta + (\text{Const.})$$

$$\text{FOC: } 0 = 1 - \frac{1}{\theta} \implies \boxed{\theta = 1}$$

### A more direct approach

Min KL  $\iff$  Max Expected Log-like.

$$\begin{aligned} \frac{d}{d\theta} \mathbb{E}[\log f(y|\theta)] &= \mathbb{E} \left[ \frac{d}{d\theta} \{-\theta + y \log(\theta) - \log(y!)\} \right] \\ &= \mathbb{E}[-1 + y/\theta] = \mathbb{E}[y]/\theta - 1 = 0 \\ &\implies \boxed{\theta = \mathbb{E}[y]} \end{aligned}$$

## A Simple Example Continued: Minimizing the KL Divergence

Model = Poisson( $\theta$ ); True Dist.  $p_o(0) = p_o(2) = \frac{2}{5}$  and  $p_o(1) = \frac{1}{5}$

### Best Approximation

What parameter value  $\theta_o$  makes the Poisson( $\theta$ ) model *as close as possible* to the true distribution  $p_o$ , where we measure “closeness” using the KL-divergence?

Using the previous slide:  $\theta_o = 1$

A more direct approach:  $\theta_o = \mathbb{E}[y]$

### Both Methods Agree

- ▶ For the specified  $p_o$  we have:  $\mathbb{E}[y] = 0 \times \frac{1}{5} + 1 \times \frac{2}{5} + 2 \times \frac{2}{5} = 1$ .
- ▶ The “Direct approach” is general: works for *any*  $p_o$  (under regularity conditions)

## Is this just a coincidence?

We have shown that:

1. Under an iid  $\text{Poisson}(\theta)$  model for  $y_1, \dots, y_N$ , the MLE for  $\theta$  is  $\hat{\theta} = \bar{y}$
2. For *any* (reasonable)  $p_o$ , setting  $\theta_o = \mathbb{E}[y_i]$  minimizes  $KL(p_o; f_\theta)$ .

By the (weak) law of large numbers:

If  $y_1, \dots, y_N \sim \text{iid}$ , then  $\bar{y}$  is a consistent estimator of  $\mathbb{E}[y_i]$  as  $N$  approaches infinity.

So at least in this example...

The maximum likelihood estimator  $\hat{\theta}$  is a consistent estimator of  $\theta_o$ , the minimizer the KL divergence from the true distribution  $p_o$  to the  $\text{Poisson}(\theta)$  model  $f(y|\theta)$ .

# Maximum Likelihood Estimation Under Mis-specification

Note: expectations and variances are calculated using  $p_o$

## Theorem

Suppose that  $\mathbf{y}_1, \dots, \mathbf{y}_N \sim \text{iid } p_o$  and let  $\hat{\boldsymbol{\theta}}$  denote the MLE for  $\boldsymbol{\theta}$  under the possibly mis-specified model  $f(\mathbf{y}|\boldsymbol{\theta})$ . Then, under mild regularity conditions:

(i)  $\hat{\boldsymbol{\theta}}$  is consistent for the **pseudo-true** parameter value  $\boldsymbol{\theta}_o$ , defined as the minimizer of  $KL(p_o, f_{\boldsymbol{\theta}})$  over the parameter space  $\Theta$ .

(ii)  $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$

where we define  $\mathbf{J} \equiv -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]$  and  $\mathbf{K} \equiv \text{Var} \left[ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right]$ .

## Why is this result such a big deal?

1. Provides an interpretation of MLE when we acknowledge that our models are only an *approximation* or reality: MLE recovers the pseudo-true parameter  $\theta_o$ .
2. Yields a formula for standard errors that is **robust** to mis-specification of our model: compare to Heteroskedasticity consistent SEs for regression.
3. If the model is correctly specified, we recover the “classical” MLE result.

# A Consistent Asymptotic Variance Matrix Estimator: $\hat{\mathbf{J}}^{-1}\hat{\mathbf{K}}\hat{\mathbf{J}}^{-1}$

$\hat{\boldsymbol{\theta}} \rightarrow_p \boldsymbol{\theta}_o$  plus Uniform Weak Law of Large Numbers: Newey & McFadden (1994)

$$\boldsymbol{\theta}_o \equiv \arg \max_{\boldsymbol{\theta} \in \Theta} \mathbb{E} [\log f(\mathbf{y}_i | \boldsymbol{\theta})] \quad \hat{\boldsymbol{\theta}} \equiv \arg \max_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{i=1}^N \log f(\mathbf{y}_i | \boldsymbol{\theta})$$

$$\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1}) \quad \hat{\boldsymbol{\theta}} \approx \mathcal{N}(\boldsymbol{\theta}_o, \hat{\mathbf{J}}^{-1}\hat{\mathbf{K}}\hat{\mathbf{J}}^{-1}/N)$$

$$\mathbf{J} \equiv -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}_i | \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right] \quad \hat{\mathbf{J}} \equiv -\frac{1}{N} \sum_{i=1}^N \frac{\partial^2 \log f(\mathbf{y}_i | \hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}$$

$$\mathbf{K} \equiv \text{Var} \left[ \frac{\partial \log f(\mathbf{y}_i | \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right] \quad \hat{\mathbf{K}} \equiv \frac{1}{N} \sum_{i=1}^N \left[ \frac{\partial \log f(\mathbf{y}_i | \hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} \right] \left[ \frac{\partial \log f(\mathbf{y}_i | \hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} \right]'$$

## Some Notes on the Preceding Slide

What happened to the KL divergence?

$\mathbb{E}[\log p_o(\mathbf{y})]$  does not involve  $\boldsymbol{\theta}$ . Hence,  $\arg \max_{\boldsymbol{\theta} \in \Theta} \mathbb{E}[\log f(\mathbf{y}_i|\boldsymbol{\theta})] = \arg \min_{\boldsymbol{\theta} \in \Theta} KL(p_o, f_{\boldsymbol{\theta}})$ .

Isn't  $\hat{\mathbf{K}}$  missing a term?

The sample variance of  $\mathbf{x}$  is given by  $\left(\frac{1}{N} \sum_{i=1}^N \mathbf{x}_i \mathbf{x}_i'\right) - (\bar{\mathbf{x}} \bar{\mathbf{x}}')$  where  $\bar{\mathbf{x}} = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i$ . In our formula for  $\hat{\mathbf{K}}$ , the “ $\bar{\mathbf{x}} \bar{\mathbf{x}}'$ ” term appears to be missing, but it is in fact equal to zero, since  $\hat{\boldsymbol{\theta}}$  is the solution to the MLE first-order condition.

### Some Terminology

I will call  $\hat{\mathbf{J}}^{-1} \hat{\mathbf{K}} \hat{\mathbf{J}}^{-1}$  the **robust** asymptotic variance matrix estimator, since it is correct regardless of whether the model is correctly specified.



# Maximum Likelihood Estimation Under Correct Specification

“Classical” large-sample theory for MLE

## Theorem

Suppose that  $\mathbf{y}_1, \dots, \mathbf{y}_N \sim \text{iid } f(\mathbf{y}|\boldsymbol{\theta}_o)$ . Then, under mild regularity conditions:

- (i)  $\hat{\boldsymbol{\theta}}$  is consistent for  $\boldsymbol{\theta}_o$ .
- (ii)  $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$  where  $\mathbf{J} \equiv -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]$ .

Why? If  $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$ , then:

1.  $KL(p_o; f_{\boldsymbol{\theta}})$  equals zero at  $\boldsymbol{\theta} = \boldsymbol{\theta}_o$ .
2. The *information matrix equality* gives  $\mathbf{K} = \mathbf{J}$  which implies  $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1} = \mathbf{J}^{-1}$ .

The Information Matrix Equality: if  $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$ , then  $\mathbf{K} = \mathbf{J}$ .

$$\mathbf{J} \equiv -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right], \quad \mathbf{K} \equiv \text{Var} \left[ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right]$$

Step 1: Alternative Expression for  $\mathbf{K}$

$$\text{Var} \left[ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right] = \mathbb{E} \left[ \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\} \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\}' \right] - \mathbb{E} \left[ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right] \mathbb{E} \left[ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right]'$$

but since  $\boldsymbol{\theta}_o$  minimizes  $\mathbb{E} [\log f(\mathbf{y}|\boldsymbol{\theta})]$ ,

$$\mathbb{E} \left[ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right] = \frac{\partial}{\partial \boldsymbol{\theta}} \mathbb{E} [\log f(\mathbf{y}|\boldsymbol{\theta}_o)] = \mathbf{0}$$

so it suffices to show that

$$-\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right] = \mathbb{E} \left[ \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\} \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\}' \right]$$

The Information Matrix Equality: if  $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$ , then  $\mathbf{K} = \mathbf{J}$ .

$$\text{suffices to show } -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right] = \mathbb{E} \left[ \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\} \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\}' \right]$$

Step 2: Chain Rule & Product Rule

$$\begin{aligned} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_i} \left[ \frac{\partial}{\partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}) \right] = \frac{\partial}{\partial \theta_i} \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial}{\partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}) \right] \\ &= \left[ -\frac{1}{f^2(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial}{\partial \theta_i} f(\mathbf{y}|\boldsymbol{\theta}) \right] \left[ \frac{\partial}{\partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}) \right] + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}) \\ &= -\left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial}{\partial \theta_i} f(\mathbf{y}|\boldsymbol{\theta}) \right] \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial}{\partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}) \right] + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}) \\ &= -\frac{\partial}{\partial \theta_i} \log f(\mathbf{y}|\boldsymbol{\theta}) \frac{\partial}{\partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}) + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}) \end{aligned}$$

The Information Matrix Equality: if  $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$ , then  $\mathbf{K} = \mathbf{J}$ .

$$\text{suffices to show } -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right] = \mathbb{E} \left[ \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\} \left\{ \frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right\}' \right]$$

Step 3: Multiply by  $-1$ , Evaluate at  $\boldsymbol{\theta}_o$ , and Take Expectations

$$\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}) = -\frac{\partial}{\partial \theta_i} \log f(\mathbf{y}|\boldsymbol{\theta}) \frac{\partial}{\partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}) + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta})$$

$$-\mathbb{E} \left[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}_o) \right] = \mathbb{E} \left[ \frac{\partial}{\partial \theta_i} \log f(\mathbf{y}|\boldsymbol{\theta}_o) \frac{\partial}{\partial \theta_j} \log f(\mathbf{y}|\boldsymbol{\theta}_o) \right] - \underbrace{\mathbb{E} \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o) \right]}_{\text{suffices to show this is zero!}}$$

The Information Matrix Equality: if  $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$ , then  $\mathbf{K} = \mathbf{J}$ .

$$\text{suffices to show } \mathbb{E} \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o) \right] = 0$$

Step 4: Use  $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o) \right] &\equiv \int \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o) \right] p_o(\mathbf{y}) d\mathbf{y} \\ &= \int \left[ \frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o) \right] f(\mathbf{y}|\boldsymbol{\theta}_o) d\mathbf{y} = \int \frac{\partial^2}{\partial \theta_i \partial \theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o) d\mathbf{y} \\ &= \frac{\partial^2}{\partial \theta_i \partial \theta_j} \int f(\mathbf{y}|\boldsymbol{\theta}_o) d\mathbf{y} = \frac{\partial^2}{\partial \theta_i \partial \theta_j} (1) = 0 \end{aligned}$$

# A Simple Example Continued Again: Asymptotic Variance Calculations

Poisson( $\theta$ ) model, possibly mis-specified.

## Ingredients

$$\begin{aligned}\log f(y|\theta) &= -\theta + y \log(\theta) - \log(y!) \\ \frac{d}{d\theta} \log f(y|\theta) &= -1 + y/\theta \\ \frac{d^2}{d\theta^2} \log f(y|\theta) &= -y/\theta^2 \\ \theta_o &= \mathbb{E}[y], \quad \hat{\theta} = \bar{y}\end{aligned}$$

$$J = -\mathbb{E} \left[ \frac{d^2}{d\theta^2} \log f(y|\theta_o) \right] = 1/\mathbb{E}[y]$$

$$\hat{J} = -\frac{1}{N} \sum_{i=1}^N \frac{d^2}{d\theta^2} \log f(y_i|\hat{\theta}) = 1/\bar{y}$$

$$K = \text{Var} \left[ \frac{d}{d\theta} \log f(y|\theta_o) \right] = \text{Var}(y)/\mathbb{E}[y]^2$$

$$\hat{K} = \frac{1}{N} \sum_{i=1}^N \left[ \frac{d}{d\theta} \log f(y_i|\hat{\theta}) \right]^2 = s_y^2/(\bar{y})^2$$

where  $s_y^2 \equiv \frac{1}{N} \sum_{i=1}^N (y_i - \bar{y})^2$  and  $\bar{y} \equiv \frac{1}{N} \sum_{i=1}^N y_i$

# A Simple Example Continued Again: Asymptotic Variance Calculations

From Previous Slide

$$\theta_0 = \mathbb{E}[y], \quad J = 1/\mathbb{E}[y], \quad \hat{J} = 1/\bar{y}, \quad K = \text{Var}(y)/\mathbb{E}[y]^2, \quad \hat{K} = s_y^2/(\bar{y})^2$$

Correct Specification

$$\boxed{y_1, \dots, y_N \sim \text{iid Poisson}(\theta_o)} \implies \boxed{J = K = 1/\theta_o} \implies \boxed{J^{-1} K J^{-1} = \theta_o = \mathbb{E}[y]}$$

Potential Mis-specification

$$\boxed{y_1, \dots, y_N \sim \text{iid}} \implies \boxed{J = 1/\mathbb{E}[y], \quad K = \text{Var}(y)/\mathbb{E}[y]^2} \implies \boxed{J^{-1} K J^{-1} = \text{Var}(y)}$$

# A Simple Example Continued Again: Asymptotic Variance Calculations

## Comparison of Asymptotic Distributions

$$\boxed{y_1, \dots, y_N \sim \text{iid Poisson}(\theta_o)} \implies \sqrt{N}(\hat{\theta} - \theta_o) = \sqrt{N}(\bar{y} - \mathbb{E}[y]) \rightarrow_d \mathcal{N}(0, \mathbb{E}[y])$$

$$\boxed{y_1, \dots, y_N \sim \text{iid}} \implies \sqrt{N}(\hat{\theta} - \theta_o) = \sqrt{N}(\bar{y} - \mathbb{E}[y]) \rightarrow_d \mathcal{N}(0, \text{Var}[y])$$

## Comparison of Asymptotic 95% CIs

$$\boxed{y_1, \dots, y_N \sim \text{iid Poisson}(\theta_o)} \implies \bar{y} \pm 1.96 \times \sqrt{\bar{y}/N}$$

$$\boxed{y_1, \dots, y_N \sim \text{iid}} \implies \bar{y} \pm 1.96 \times s_y / \sqrt{N}$$

## Punch Line

Unless  $\text{Var}(y) = \mathbb{E}[y]$ , CIs/tests that assume the Poisson model is true are wrong!



# Lecture #2 – Poisson Regression

Review: Minimum MSE Predictor / Minimum MSE Linear Predictor

Why not just use OLS?

Conditional Maximum Likelihood Estimation

Poisson Regression: A Robust Model for Count Data

Asymptotic Variance Calculations for Poisson Regression

# How to predict a count variable?

## Example

Suppose we want to predict  $y$  using  $\mathbf{x}$ , where:

- ▶  $y \equiv \#$  of children a woman has: a **count variable**, i.e.  $y \in \{0, 1, 2, \dots\}$
- ▶  $\mathbf{x} \equiv \{\text{years of schooling, age, married, etc.}\}$

## Minimum MSE Predictor

$\mu(\mathbf{x}) \equiv \mathbb{E}(y|\mathbf{x})$  minimizes  $\mathbb{E} \left[ \{y - \varphi(\mathbf{x})\}^2 \right]$  over all possible predictors  $\varphi(\cdot)$ .

## Minimum MSE Linear Predictor

$\beta \equiv \mathbb{E}[\mathbf{x}\mathbf{x}']^{-1} \mathbb{E}[\mathbf{x}y]$  minimizes  $\mathbb{E} \left[ (y - \mathbf{x}'\theta)^2 \right]$  over all linear predictors  $\mathbf{x}'\theta$ .

## Proof: $\mathbb{E}(y|\mathbf{x})$ is the minimum MSE predictor

Step 1: add and subtract  $\mu(\mathbf{x}) \equiv \mathbb{E}(y|\mathbf{x})$

$$\begin{aligned}\mathbb{E} \left[ \{y - \varphi(\mathbf{x})\}^2 \right] &= \mathbb{E} \left[ \{ (y - \mu(\mathbf{x})) - (\varphi(\mathbf{x}) - \mu(\mathbf{x})) \}^2 \right] \\ &= \mathbb{E} \left[ \{y - \mu(\mathbf{x})\}^2 \right] - 2\mathbb{E} [\{y - \mu(\mathbf{x})\} \{\varphi(\mathbf{x}) - \mu(\mathbf{x})\}] + \mathbb{E} \left[ \{\varphi(\mathbf{x}) - \mu(\mathbf{x})\}^2 \right]\end{aligned}$$

Step 2: iterated expectations

$$\begin{aligned}\mathbb{E} [\{y - \mu(\mathbf{x})\} \{\varphi(\mathbf{x}) - \mu(\mathbf{x})\}] &= \mathbb{E} \left( \mathbb{E} [\{y - \mu(\mathbf{x})\} \{\varphi(\mathbf{x}) - \mu(\mathbf{x})\} | \mathbf{x}] \right) \\ &= \mathbb{E} \left( [\varphi(\mathbf{x}) - \mu(\mathbf{x})] [\mathbb{E}(y|\mathbf{x}) - \mu(\mathbf{x})] \right) = 0\end{aligned}$$

Step 3: combine steps 1 & 2

$$\mathbb{E} \left[ \{y - \varphi(\mathbf{x})\}^2 \right] = \underbrace{\mathbb{E} \left[ \{y - \mu(\mathbf{x})\}^2 \right]}_{\text{constant wrt } \varphi} + \underbrace{\mathbb{E} \left[ \{\varphi(\mathbf{x}) - \mu(\mathbf{x})\}^2 \right]}_{\text{cannot be negative; zero if } \varphi = \mu}$$

# Proof: OLS is the Minimum MSE Linear Predictor

## Objective Function

$$\mathbb{E} \left[ (y - \mathbf{x}'\boldsymbol{\theta})^2 \right] = \mathbb{E}[y^2] - 2\mathbb{E}[y\mathbf{x}']\boldsymbol{\theta} + \boldsymbol{\theta}'\mathbb{E}[\mathbf{x}\mathbf{x}']\boldsymbol{\theta}$$

## Recall: Matrix Differentiation

$$\frac{\partial(\mathbf{a}'\mathbf{z})}{\partial\mathbf{z}} = \mathbf{a}, \quad \frac{\partial(\mathbf{z}'\mathbf{A}\mathbf{z})}{\partial\mathbf{z}} = (\mathbf{A} + \mathbf{A}')\mathbf{z}$$

## First-Order Condition

$$-2\mathbb{E}[\mathbf{x}y] + 2\mathbb{E}[\mathbf{x}\mathbf{x}']\boldsymbol{\beta} = 0 \implies \boldsymbol{\beta} = \mathbb{E}[\mathbf{x}\mathbf{x}']^{-1} \mathbb{E}[\mathbf{x}y]$$

# Problems with linear-in-parameters models for count data

Best predictor is  $\mathbb{E}(y|\mathbf{x})$  but how can we estimate this?

Plain-vanilla OLS?

- ▶ If  $\mathbb{E}(y|\mathbf{x}) \approx \mathbf{x}'\beta$ , OLS is a reasonable approach.
- ▶ **Problem:**  $y$  is a count so it *can't* be negative, but OLS prediction  $\mathbf{x}'\beta$  could be.

OLS for  $\log(y)$ ?

- ▶ Log-linear model  $\log(y) = \mathbf{x}'\beta + \varepsilon$
- ▶ Solves the problem of negative predictions:  $\log(y)$  *can* be negative.
- ▶ **Problem:** if  $y$  is a count it could equal zero but  $\log(0) = -\infty$ !

A realistic model for count data *must* be nonlinear in parameters.

## General Approach

- ▶ Assume that  $\mathbb{E}(y|\mathbf{x}) = m(\mathbf{x}; \beta)$  where  $m$  is a known parametric function.
- ▶ Choose  $m$  so that it is always positive, regardless of  $\mathbf{x}$  and  $\beta$ .
- ▶ This means  $m$  *cannot* be linear.

This Lecture:  $m(\mathbf{x}; \beta) = \exp(\mathbf{x}'\beta)$

- ▶ Always strictly positive
- ▶ Common choice in practice
- ▶ Everything I'll discuss works with other choices of  $m$ , making appropriate changes.

## How to estimate $\beta_o$ ?

Assumption:  $\mathbb{E}(y|\mathbf{x}) = \exp(\mathbf{x}'\beta_o)$

Using our argument from above,  $\beta_o$  minimizes  $\mathbb{E} \left[ \{y_i - \exp(\mathbf{x}'_i\beta)\}^2 \right]$  over all  $\beta$ .

Nonlinear Least Squares (NLLS)

$\hat{\beta}_{NLLS}$  is the minimizer of  $\sum_{i=1}^N \{y_i - \exp(\mathbf{x}'_i\beta)\}^2$

Poisson Regression (MLE)

$\hat{\beta}_{MLE}$  is the MLE for  $\beta_o$  under the model  $y_i|\mathbf{x}_i \sim \text{indep. Poisson}(\exp(\mathbf{x}'_i\beta_o))$

# Conditional versus Unconditional MLE

## Last Lecture: Unconditional MLE

Model *unconditional* dist. of a random vector  $\mathbf{y}$ :  $f(\mathbf{y}|\boldsymbol{\theta})$ .

## This Lecture: Conditional MLE

Model *conditional* dist. of a random variable  $y$  *given* a random vector  $\mathbf{x}$ :  $f(y|\mathbf{x}, \boldsymbol{\theta})$ .

## Why Conditional MLE?

- ▶ Unconditional MLE requires joint distribution:  $f(y, \mathbf{x}|\boldsymbol{\theta}) = f(y|\mathbf{x}, \boldsymbol{\theta})f(\mathbf{x}|\boldsymbol{\theta})$
- ▶  $\mathbb{E}(y|\mathbf{x})$  only depends on  $f(y|\mathbf{x}, \boldsymbol{\theta})$  not  $f(\mathbf{x}|\boldsymbol{\theta})$ .
- ▶ Not interested in  $f(\mathbf{x}|\boldsymbol{\theta})$ ; coming up with a good model for it is challenging.
- ▶ Caveat: unconditional MLE is more efficient provided the model for  $\mathbf{x}$  is correct.



# The Conditional Maximum Likelihood Estimator

Assuming iid data.

Sample

$$\hat{\theta} \equiv \arg \max_{\theta \in \Theta} \frac{1}{N} \sum_{i=1}^N \log f(y_i | \mathbf{x}_i, \theta)$$

Population

$$\theta_o \equiv \arg \max_{\theta \in \Theta} \mathbb{E} [\log f(y_i | \mathbf{x}_i, \theta)]$$

## Important

- ▶ We only model the conditional distribution  $y|\mathbf{x}$ , but...
- ▶ ...the expectation  $\mathbb{E}[\log f(y_i|\mathbf{x}_i, \theta)]$  is taken over the *joint distribution* of  $(y, \mathbf{x})$ .
- ▶  $f(y_i|\mathbf{x}_i, \theta)$  is merely a *function* of the RVs  $(y_i, \mathbf{x}_i)$ .

# Poisson Regression as a Conditional MLE

Model:  $y_i | \mathbf{x}_i \sim \text{Poisson}(\exp\{\mathbf{x}_i' \boldsymbol{\beta}\})$

$$\ell_i(\boldsymbol{\beta}) \equiv \log f(y_i | \mathbf{x}_i, \boldsymbol{\beta}) = y_i \mathbf{x}_i' \boldsymbol{\beta} - \exp(\mathbf{x}_i' \boldsymbol{\beta}) - \log(y_i!)$$

$$\underbrace{\mathbf{s}_i(\boldsymbol{\beta})}_{\text{score vector}} \equiv \frac{\partial \ell_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{x}_i [y_i - \exp(\mathbf{x}_i' \boldsymbol{\beta})]$$

$$\hat{\boldsymbol{\beta}} \text{ solves } \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i \underbrace{[y_i - \exp(\mathbf{x}_i' \boldsymbol{\beta})]}_{\text{residual: } u_i} = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i u_i(\boldsymbol{\beta}) = \mathbf{0}$$

# Average Partial Effects

## Partial Effects

For continuous  $x_j$ , we call  $\frac{\partial}{\partial x_j} \mathbb{E}(y|\mathbf{x})$  the **partial effect** of  $x_j$ . For discrete  $x_j$  the partial effect is the difference of  $\mathbb{E}(y|\mathbf{x})$  at two different values of  $x_j$

## Average Partial Effects (APE)

In nonlinear models, partial effects typically vary with  $\mathbf{x}$ . The **average partial effect** is the expectation of the partial effect over the distribution of  $\mathbf{x}$ .

# Average Partial Effects for Poisson Regression

## Partial Effect

$$\frac{\partial}{\partial x_j} \mathbb{E}(y|\mathbf{x}) = \frac{\partial}{\partial x_j} \exp(\mathbf{x}'_i \boldsymbol{\beta}) = \exp(\mathbf{x}'_i \boldsymbol{\beta}) \beta_j$$

## Estimated Partial Effect

$$\exp(\mathbf{x}'_i \hat{\boldsymbol{\beta}}) \hat{\beta}_j$$

## Average Partial Effect

$$\mathbb{E} \left[ \frac{\partial}{\partial x_j} \exp(\mathbf{x}'_i \boldsymbol{\beta}) \right] = \mathbb{E} [\exp(\mathbf{x}'_i \boldsymbol{\beta})] \beta_j$$

## Estimated Average Partial Effect

$$\left[ \frac{1}{N} \sum_{i=1}^N \exp(\mathbf{x}'_i \hat{\boldsymbol{\beta}}) \right] \hat{\beta}_j$$

## Relative Effects

The *ratio* of partial effects does not depend on  $\mathbf{x}$ : relative effects are constant.

## Problem Set

Poisson regression:  $\text{APE} = \bar{y} \hat{\beta}_j$ . Multiply by  $\bar{y}$  to put coefficients on the scale of OLS.

# Conditional MLE Under Mis-specification

Basically identical to the unconditional version.

## Theorem

Suppose that  $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^N \sim \text{iid } p_o$  and let  $\hat{\boldsymbol{\theta}}$  denote the Conditional MLE for  $\boldsymbol{\theta}$  under the possibly mis-specified model  $f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta})$ . Then, under mild regularity conditions:

- (i)  $\hat{\boldsymbol{\theta}}$  is consistent for the **pseudo-true** parameter value  $\boldsymbol{\theta}_o$ , defined as the *maximizer* of the expected log likelihood  $\mathbb{E} [\log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta})]$  over the parameter space  $\Theta$ .
- (ii)  $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1} \mathbf{K} \mathbf{J}^{-1})$

where we define  $\mathbf{J} \equiv -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]$  and  $\mathbf{K} \equiv \text{Var} \left[ \frac{\partial \log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta}} \right]$ .

# Conditional MLE Under Correct Specification

Basically identical to the unconditional version.

## Theorem

Suppose that  $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^N \sim \text{iid}$  where the conditional distribution of  $y_i|\mathbf{x}_i$  is given by  $f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)$ . Then, under mild regularity conditions,

(i)  $\hat{\boldsymbol{\theta}}$  is consistent for  $\boldsymbol{\theta}_o$

(ii)  $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$  where  $\mathbf{J} \equiv -\mathbb{E} \left[ \frac{\partial^2 \log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]$

What value of  $\beta$  maximizes  $\mathbb{E} [\ell_i(\beta)]$ ?

Iterated Expectations

$$\mathbb{E}[\ell_i(\beta)] = \mathbb{E} \{ \mathbb{E} [\ell_i(\beta) | \mathbf{x}_i] \} = \mathbb{E} \{ \mathbb{E} [y_i \mathbf{x}_i' \beta - \exp(\mathbf{x}_i' \beta) - \log(y_i!) | \mathbf{x}_i] \}$$

Simplify Inner Expectation

$$\mathbb{E} [\ell_i(\beta) | \mathbf{x}_i] = \mathbf{x}_i' \beta \mathbb{E} [y_i | \mathbf{x}_i] - \exp(\mathbf{x}_i' \beta) - \underbrace{\mathbb{E} [\log(y_i!) | \mathbf{x}_i]}_{\text{constant wrt } \mathbf{x}_i}$$

FOC for Inner Expectation

$$\frac{\partial}{\partial \beta} \mathbb{E} [\ell_i(\beta) | \mathbf{x}_i] = \{ \mathbb{E} [y_i | \mathbf{x}_i] - \exp(\mathbf{x}_i' \beta) \} \mathbf{x}_i = \mathbf{0}$$

What value of  $\beta$  maximizes  $\mathbb{E} [\ell_i(\beta)]$ ?

$$\frac{\partial}{\partial \beta} \mathbb{E} [\ell_i(\beta) | \mathbf{x}_i] = \{ \mathbb{E} [y_i | \mathbf{x}_i] - \exp(\mathbf{x}_i' \beta) \} \mathbf{x}_i = \mathbf{0}$$

What does this mean?

Since  $\mathbb{E} [y_i | \mathbf{x}_i] = \exp(\mathbf{x}_i' \beta_o)$ , setting  $\beta = \beta_o$  solves the FOC for the inner expectation!

In other words:

For any realization of  $\mathbf{x}_i$  and any  $\beta$ ,

$$\mathbb{E}[\ell_i(\beta) | \mathbf{x}_i] \leq \mathbb{E}[\ell_i(\beta_o) | \mathbf{x}_i]$$

so taking expectations of both sides:

$$\mathbb{E} [\ell_i(\beta)] = \mathbb{E} \{ \mathbb{E}[\ell_i(\beta) | \mathbf{x}_i] \} \leq \mathbb{E} \{ \mathbb{E}[\ell_i(\beta_o) | \mathbf{x}_i] \} = \mathbb{E} [\ell_i(\beta_o)]$$



Poisson Regression is consistent if  $\mathbb{E}(y|\mathbf{x})$  is correctly specified.

We showed this for a particular choice of  $m(\mathbf{x};\beta)$  but the result is general.

## Result

Provided that we have correctly specified  $\mathbb{E}(y_i|\mathbf{x}_i)$ , it *doesn't matter* if  $y_i|\mathbf{x}_i$  actually follows a Poisson distribution: Poisson regression is *still consistent* for  $\beta_o$ .

## Compare

This is very similar to our result for the  $\text{Poisson}(\theta)$  model from last lecture.

## Caveat

Strictly speaking we need to show that  $\beta_o$  is the *unique* maximizer of the expected log likelihood. *Multiple solutions* if  $\mathbf{x}_i$  perfectly co-linear (compare to OLS regression).

## Asymptotic Variance Calculations for Poisson Regression

$$\underbrace{\mathbf{s}_i(\boldsymbol{\beta})}_{\text{score vector}} \equiv \frac{\partial \ell_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{x}_i [y_i - \exp(\mathbf{x}_i' \boldsymbol{\beta})] = \mathbf{x}_i u_i(\boldsymbol{\beta})$$

$$\underbrace{\mathbf{H}_i(\boldsymbol{\beta})}_{\text{Hessian matrix}} \equiv \frac{\partial \mathbf{s}_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'} = -\exp(\mathbf{x}_i' \boldsymbol{\beta}) \mathbf{x}_i \mathbf{x}_i'$$

$$\mathbf{J} \equiv -\mathbb{E} [\mathbf{H}_i(\boldsymbol{\beta}_o)] = \mathbb{E} [\exp(\mathbf{x}_i' \boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i']$$

$$\mathbf{K} \equiv \text{Var} [\mathbf{s}_i(\boldsymbol{\beta}_o)] = \mathbb{E} [\mathbf{s}_i(\boldsymbol{\beta}_o) \mathbf{s}_i(\boldsymbol{\beta}_o)'] = \mathbb{E} [u_i^2(\boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i']$$

# Asymptotic Variance Calculations for Poisson Regression

$$\mathbf{J} = \mathbb{E} \left[ \exp(\mathbf{x}_i' \boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i' \right], \quad \mathbf{K} = \mathbb{E} \left[ u_i^2(\boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i' \right]$$

## Notice

$\mathbf{J}$  does not depend on  $y$  but  $\mathbf{K}$  does:

$$\begin{aligned} \mathbf{K} &= \mathbb{E} \left[ u_i^2(\boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i' \right] = \mathbb{E} \left\{ \mathbb{E} \left[ u_i^2(\boldsymbol{\beta}_o) | \mathbf{x}_i \right] \mathbf{x}_i \mathbf{x}_i' \right\} = \mathbb{E} \left( \mathbb{E} \left[ \{y_i - \mathbb{E}(y_i | \mathbf{x}_i)\}^2 | \mathbf{x}_i \right] \mathbf{x}_i \mathbf{x}_i' \right) \\ &= \mathbb{E} \left[ \text{Var}(y_i | \mathbf{x}_i) \mathbf{x}_i \mathbf{x}_i' \right] \end{aligned}$$

Assumptions about  $\text{Var}(y|\mathbf{x})$  affect the asymptotic variance through  $\mathbf{K}$ .

## Possible Assumptions for $\text{Var}(y|\mathbf{x})$ : Strongest to Weakest

1. Poisson Assumption:  $\text{Var}(y|\mathbf{x}) = \mathbb{E}(y|\mathbf{x})$ 
  - ▶ holds if Poisson model is correct.
2. Quasi-Poisson Assumption:  $\text{Var}(y|\mathbf{x}) = \sigma^2 \mathbb{E}(y|\mathbf{x})$ 
  - ▶ Allows for possibility that  $y|\mathbf{x}$  is *not* Poisson
  - ▶ Overdispersion:  $\sigma^2 > 1 \implies \text{Var}(y|\mathbf{x}) > \mathbb{E}(y|\mathbf{x})$
  - ▶ Underdispersion  $\sigma^2 < 1 \implies \text{Var}(y|\mathbf{x}) < \mathbb{E}(y|\mathbf{x})$
  - ▶ If  $\sigma^2 = 1$  we're back to the Poisson Assumption.
3. No Assumption:  $\text{Var}(y|\mathbf{x})$  unspecified

## Asymptotic Variance Under Poisson Assumption

$$\mathbf{J} = \mathbb{E} [\exp(\mathbf{x}_i' \boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i'] , \quad \mathbf{K} = \mathbb{E} [\text{Var}(y_i | \mathbf{x}_i) \mathbf{x}_i \mathbf{x}_i']$$

Assumption:  $\text{Var}(y_i | \mathbf{x}_i) = \mathbb{E}(y_i | \mathbf{x}_i) = \exp(\mathbf{x}_i' \boldsymbol{\beta}_o)$

- ▶ Implies  $\mathbf{K} = \mathbb{E} [\exp(\mathbf{x}_i' \boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i']$
- ▶ Hence  $\mathbf{K} = \mathbf{J}$  (Information Matrix Equality)
- ▶ Therefore:  $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$
- ▶ Consistent Estimator:  $\hat{\mathbf{J}}^{-1} = \left[ \frac{1}{N} \sum_{i=1}^N \exp(\mathbf{x}_i' \hat{\boldsymbol{\beta}}) \mathbf{x}_i \mathbf{x}_i' \right]^{-1}$

## Asymptotic Variance Under Quasi-Poisson Assumption

$$\mathbf{J} = \mathbb{E} [\exp(\mathbf{x}_i' \boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i'] , \quad \mathbf{K} = \mathbb{E} [\text{Var}(y_i | \mathbf{x}_i) \mathbf{x}_i \mathbf{x}_i']$$

Assumption:  $\text{Var}(y_i | \mathbf{x}_i) = \sigma^2 \mathbb{E}(y_i | \mathbf{x}_i) = \sigma^2 \exp(\mathbf{x}_i' \boldsymbol{\beta}_o)$

- ▶ Implies  $\mathbf{K} = \sigma^2 \mathbb{E} [\exp(\mathbf{x}_i' \boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i'] = \sigma^2 \mathbf{J}$
- ▶ Hence  $\mathbf{J}^{-1} \mathbf{K} \mathbf{J}^{-1} = \sigma^2 \mathbf{J}^{-1}$
- ▶ Therefore:  $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{J}^{-1})$
- ▶ Consistent estimator of  $\mathbf{J}^{-1}$  on prev. slide but how can we estimate  $\sigma^2$ ?

## How to estimate $\sigma^2$ under the Quasi-Poisson Assumption?

$$\text{Var}(y|\mathbf{x}) = \sigma^2 \mathbb{E}(y|\mathbf{x})$$

$$\sigma^2 = \text{Var}(y|\mathbf{x}) / \mathbb{E}(y|\mathbf{x})$$

$$\sigma^2 = \mathbb{E} \left[ \{y - \mathbb{E}(y|\mathbf{x})\}^2 \middle| \mathbf{x} \right] / \mathbb{E}(y|\mathbf{x})$$

$$\sigma^2 = \mathbb{E} \left[ \frac{\{y - \mathbb{E}(y|\mathbf{x})\}^2}{\mathbb{E}(y|\mathbf{x})} \middle| \mathbf{x} \right]$$

$$\sigma^2 = \mathbb{E} \left[ \frac{\{y - \exp(\mathbf{x}'\beta_o)\}^2}{\exp(\mathbf{x}'\beta)} \middle| \mathbf{x} \right]$$

$$\mathbb{E}[\sigma^2] = \mathbb{E} \left( \mathbb{E} \left[ \frac{\{y - \exp(\mathbf{x}'\beta_o)\}^2}{\exp(\mathbf{x}'\beta)} \middle| \mathbf{x} \right] \right)$$

$$\sigma^2 = \mathbb{E} \left[ \frac{\{y - \exp(\mathbf{x}'\beta_o)\}^2}{\exp(\mathbf{x}'\beta)} \right]$$

$$\sigma^2 = \mathbb{E} \left[ u^2(\beta_o) / \exp(\mathbf{x}'\beta_o) \right]$$

### Consistent Estimator of $\sigma^2$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N \frac{[y_i - \exp(\mathbf{x}_i' \hat{\beta})]^2}{\exp(\mathbf{x}_i \hat{\beta})} = \frac{1}{N} \sum_{i=1}^N \frac{\hat{u}_i^2}{\exp(\mathbf{x}_i \hat{\beta})}$$

# Robust Asymptotic Variance Matrix

$$\mathbf{J} = \mathbb{E} \left[ \exp(\mathbf{x}'_i \beta_o) \mathbf{x}_i \mathbf{x}'_i \right], \quad \mathbf{K} = \mathbb{E} \left[ u_i^2(\beta_o) \mathbf{x}_i \mathbf{x}'_i \right]$$

No Assumption on  $\text{Var}(y_i | \mathbf{x}_i)$

- ▶  $\sqrt{N}(\hat{\beta} - \beta_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1} \mathbf{K} \mathbf{J}^{-1})$
- ▶ Consistent Estimator:  $\hat{\mathbf{J}}^{-1} = \left[ \frac{1}{N} \sum_{i=1}^N \exp(\mathbf{x}'_i \hat{\beta}) \mathbf{x}_i \mathbf{x}'_i \right]^{-1}$
- ▶ Consistent Estimator:  $\hat{\mathbf{K}} = \frac{1}{N} \sum_{i=1}^N \left[ y_i - \exp(\mathbf{x}_i \hat{\beta}) \right]^2 \mathbf{x}_i \mathbf{x}'_i = \frac{1}{N} \sum_{i=1}^N \hat{u}_i^2 \mathbf{x}_i \mathbf{x}'_i$



# Why Poisson Regression rather than NLLS?

Assume that  $\mathbb{E}(y|\mathbf{x}) = \exp(\mathbf{x}'\beta_o)$

Both Poisson Reg. & NLLS are consistent if the conditional mean is correctly specified.

Count data are typically heteroskedastic.

If  $\text{Var}(y|\mathbf{x})$  varies with  $\mathbf{x}$ , NLLS will be relatively inefficient.

## Efficiency of Poisson Regression

- ▶ Correct model  $\implies$  lowest variance among all estimators that leave the distribution of  $\mathbf{x}$  unspecified.
- ▶  $\text{Var}(y|\mathbf{x}) = \sigma^2 \mathbb{E}(y|\mathbf{x}) \implies$  Poisson regression is more efficient than NLLS and various other count data models.

# Lecture #3 – Models for Binary Outcomes

Properties of Binary Outcome Models

Linear Probability Model

Index Models (e.g. Logit & Probit)

Partial Effects

Conditional MLE for Index Models

Pseudo R-squared

# Models for Binary Outcomes

## Example

- ▶ Outcome:  $y = 1$  if employed, 0 otherwise
- ▶ Predictors/Regressors:  $\mathbf{x} = \{\text{age, sex, education, experience, ...}\}$

## Question

How does  $x_j$  affect our prediction of  $y$  holding the other regressors constant?

We'll consider three models:

1. Linear Probability Model (LPM)
2. Logistic Regression (Logit)
3. Probit Regression (Probit)

# Properties of Binary Outcome Models: $y \in \{0, 1\}$

## Notation

$$p(\mathbf{x}) \equiv \mathbb{P}(y = 1|\mathbf{x})$$

## Conditional Mean

$$\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x})$$

## Conditional Variance

$$\text{Var}(y|\mathbf{x}) = p(\mathbf{x}) [1 - p(\mathbf{x})]$$

$$\begin{aligned}\mathbb{E}(y|\mathbf{x}) &= 0 \times \mathbb{P}(y = 0|\mathbf{x}) + 1 \times \mathbb{P}(y = 1|\mathbf{x}) \\ &= \mathbb{P}(y = 1|\mathbf{x}) \equiv p(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\mathbb{E}(y^2|\mathbf{x}) &= \{0^2 \times [1 - p(\mathbf{x})] + 1^2 \times p(\mathbf{x})\} \\ &= p(\mathbf{x})\end{aligned}$$

$$\begin{aligned}\text{Var}(y|\mathbf{x}) &= \mathbb{E}(y^2|\mathbf{x}) - \mathbb{E}(y|\mathbf{x})^2 \\ &= \{0^2 \times [1 - p(\mathbf{x})] + 1^2 \times p(\mathbf{x})\} - p(\mathbf{x})^2 \\ &= p(\mathbf{x}) [1 - p(\mathbf{x})]\end{aligned}$$

## The Linear Probability Model: Assume $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$

### Conditional Mean & Variance

- ▶  $\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$
- ▶  $\text{Var}(y|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}(1 - \mathbf{x}'\boldsymbol{\beta})$

### This is just Linear Regression!

$$y = \mathbf{x}'\boldsymbol{\beta} + u, \quad \mathbb{E}(u|\mathbf{x}) = 0$$

### But $u$ is Heteroskedastic

$$\text{Var}(u|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}(1 - \mathbf{x}'\boldsymbol{\beta})$$

$$\begin{aligned}\mathbb{E}(u|\mathbf{x}) &= \mathbb{E}(y - \mathbf{x}'\boldsymbol{\beta}|\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) - \mathbf{x}'\boldsymbol{\beta} \\ &= \mathbf{x}'\boldsymbol{\beta} - \mathbf{x}'\boldsymbol{\beta} = 0\end{aligned}$$

$$\begin{aligned}\text{Var}(u|\mathbf{x}) &= \mathbb{E} \left[ \{u - \mathbb{E}(u|\mathbf{x})\}^2 | \mathbf{x} \right] = \mathbb{E} [u^2 | \mathbf{x}] \\ &= \mathbb{E} \left[ (y - \mathbf{x}'\boldsymbol{\beta})^2 | \mathbf{x} \right] \\ &= \mathbb{E} (y^2 | \mathbf{x}) - 2\mathbb{E}(y|\mathbf{x})\mathbf{x}'\boldsymbol{\beta} + (\mathbf{x}'\boldsymbol{\beta})^2 \\ &= p(\mathbf{x}) - 2p(\mathbf{x})p(\mathbf{x}) + p(\mathbf{x})^2 \\ &= p(\mathbf{x}) [1 - p(\mathbf{x})]\end{aligned}$$

# The Linear Probability Model: Assume $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$

## Estimation

Since  $\mathbb{E}(u|\mathbf{x}) = 0$  OLS estimation of  $y = \mathbf{x}'\boldsymbol{\beta} + u$  is unbiased and consistent.

## Inference

Since  $u$  is heteroskedastic, tests and CIs should use robust standard errors.

## Is the LPM actually reasonable?

- ▶ Assumes  $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta} \implies$  changing  $x_j$  by  $\Delta$  changes  $p(\mathbf{x})$  by  $\beta_j\Delta$
- ▶ If  $\mathbf{x}$  contains regressors without upper/lower bounds,  $p(\mathbf{x})$  could be  $> 1$  or  $< 0$ !
- ▶ LPM could be a reasonable approximation but cannot be *literally* true.

## Index Models: Constrain $p(\mathbf{x})$ to lie in $[0, 1]$

Index Model:  $p(\mathbf{x}) = G(\mathbf{x}'\beta)$

Assume  $\mathbf{x}$  includes a constant,  $0 \leq G(\cdot) \leq 1$ ,  $G$  is differentiable and strictly increasing,  $\lim_{z \rightarrow \infty} G(z) = 1$ , and  $\lim_{z \rightarrow -\infty} G(z) = 0$ .

### Terminology

We call  $\mathbf{x}'\beta$  the **linear index** and  $G$  the **index function**.

### Partial Effects

Let  $g(z) \equiv \frac{d}{dz} G(z)$ . Then  $\frac{\partial}{\partial x_j} p(\mathbf{x}) = g(\mathbf{x}'\beta)\beta_j$ . Hence:

- ▶ The partial effect of  $x_j$  depends on the value of  $\mathbf{x}$  at which we evaluate  $g$ .
- ▶  $G$  strictly increasing  $\implies g(\cdot) > 0 \implies$  sign of partial effect determined by  $\beta_j$ .

# Possible Choices of Index Function

## CDFs as Index Functions

$G$  satisfies the index model assumptions (prev. slide) iff it is a continuous CDF.

We focus on two examples:

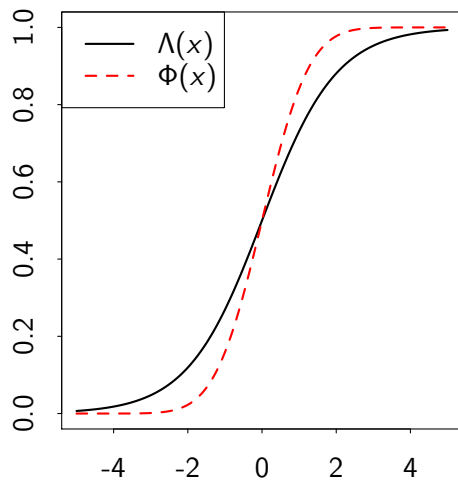
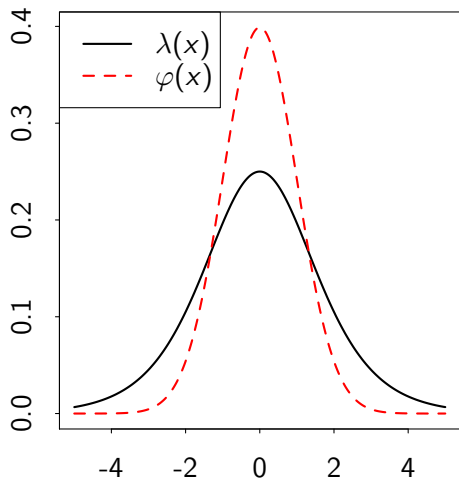
1. Logit:  $G(z) = \Lambda(z) \equiv \exp(z) / [1 + \exp(z)]$
2. Probit:  $G(z) = \Phi(z) \equiv \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) dt$

Notation:

- ▶  $\Lambda$  is the CDF of a “standard logistic” RV and  $\Phi$  of a standard normal RV.
- ▶  $\lambda$  is the density of a “standard logistic” RV and  $\varphi$  of a standard normal
- ▶ To treat Logit and Probit simultaneously, we'll write  $G$  as a placeholder.



## Standard Logistic and Normal Densities and CDFs



## Partial Effects: $\partial p(\mathbf{x})/\partial x_j$

### LPM

$$\frac{\partial}{\partial x_j} \mathbf{x}'\boldsymbol{\beta} = \beta_j$$

### Logit

$$\frac{\partial}{\partial x_j} \Lambda(\mathbf{x}'\boldsymbol{\beta}) = \frac{\beta_j \exp(\mathbf{x}'\boldsymbol{\beta})}{[1 + \exp(\mathbf{x}'\boldsymbol{\beta})]^2}$$

### Probit

$$\frac{\partial}{\partial x_j} \Phi(\mathbf{x}'\boldsymbol{\beta}) = \frac{\beta_j \exp\{-(\mathbf{x}'\boldsymbol{\beta})^2/2\}}{\sqrt{2\pi}}$$

$$\frac{\partial}{\partial x_j} G(\mathbf{x}'\boldsymbol{\beta}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j$$

$$\begin{aligned} \frac{d}{dz} \Lambda(z) &\equiv \lambda(z) = \frac{d}{dz} \left( \frac{e^z}{1 + e^z} \right) = \frac{e^z(1 + e^z) - e^z e^z}{(1 + e^z)^2} \\ &= \frac{e^z}{(1 + e^z)^2} \end{aligned}$$

$$\frac{d}{dz} \Phi(z) = \varphi(z) = \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}}$$

# Comparing Logit, Probit, and LPM Partial Effects

$$\frac{\partial}{\partial x_j} G(\mathbf{x}'\beta) = g(\mathbf{x}'\beta)\beta_j, \quad \frac{d}{dz}\Lambda(z) \equiv \lambda(z) = \frac{e^z}{(1+e^z)^2}, \quad \frac{d}{dz}\Phi(z) \equiv \varphi(z) = \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}}$$

## Maximum Partial Effects

- ▶  $\lambda$  and  $\varphi$  are unimodal with mode at 0

Logit  $\lambda(0) = 0.25$

Probit  $\varphi(0) = (2\pi)^{-1/2} \approx 0.4$

- ▶ *Maximum* partial effect when  $\mathbf{x}'\beta = 0$

Logit  $\beta_j\lambda(0) = 0.25\beta_j$

Probit  $\beta_j\varphi(0) \approx 0.4\beta_j$

- ▶ LPM has constant partial effects  $\beta_j$

## Relative Effects

$$\frac{\frac{\partial}{\partial x_j} p(\mathbf{x})}{\frac{\partial}{\partial x_h} p(\mathbf{x})} = \frac{\beta_j g(\mathbf{x}'\beta)}{\beta_h g(\mathbf{x}'\beta)} = \frac{\beta_j}{\beta_h}$$

Relative effects do not depend on  $\mathbf{x}$ .

# Average Partial Effects for Index Models

Partial Effect

$$\frac{\partial}{\partial x_j} G(\mathbf{x}'_i \boldsymbol{\beta}) = g(\mathbf{x}'_i \boldsymbol{\beta}) \beta_j$$

Average Partial Effect

$$\mathbb{E} \left[ \frac{\partial}{\partial x_j} G(\mathbf{x}'_i \boldsymbol{\beta}) \right] = \mathbb{E}[g(\mathbf{x}'_i \boldsymbol{\beta})] \beta_j$$

Estimated Partial Effect

$$\frac{\partial}{\partial x_j} G(\mathbf{x}'_i \hat{\boldsymbol{\beta}}) = g(\mathbf{x}'_i \hat{\boldsymbol{\beta}}) \hat{\beta}_j$$

Estimated Average Partial Effect

$$\left[ \frac{1}{N} \sum_{i=1}^N g(\mathbf{x}'_i \hat{\boldsymbol{\beta}}) \right] \hat{\beta}_j$$

Warning:

APE  $\neq$  partial effect evaluated at the average value of  $\mathbf{x}$  since  $\mathbb{E}[f(Z)] \neq f(\mathbb{E}[Z])$ .

# Conditional MLE for Index Models: iid Observations

## Conditional Likelihood

$$f(y_i|\mathbf{x}_i, \beta) = \begin{cases} 1 - G(\mathbf{x}'_i\beta) & \text{if } y_i = 0 \\ G(\mathbf{x}'_i\beta) & \text{if } y_i = 1 \end{cases} \iff f(y_i|\mathbf{x}_i, \beta) = G(\mathbf{x}'_i\beta)^{y_i} [1 - G(\mathbf{x}'_i\beta)]^{1-y_i}$$

## Conditional Log-Likelihood

$$\ell_i(\beta) \equiv \log f(y_i|\mathbf{x}_i, \beta) = y_i \log [G(\mathbf{x}'_i\beta)] + (1 - y_i) \log [1 - G(\mathbf{x}'_i\beta)]$$

## Sample

$$\hat{\beta} \equiv \arg \max_{\beta \in \Theta} \frac{1}{N} \sum_{i=1}^N \ell_i(\beta)$$

## Population

$$\beta_o \equiv \arg \max_{\beta \in \Theta} \mathbb{E} [\ell(\beta)]$$

Correct specification:  $\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x}) = G(\mathbf{x}'\beta_o)$ . Otherwise  $\beta_o = \text{KL-minimizer}$ .

# Asymptotic Variance Calculations for Index Models

Recall from last lecture.

## Possibly Mis-specified Model

$\sqrt{N}(\hat{\beta} - \beta_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$  where  $\mathbf{J} = -\mathbb{E} [\mathbf{H}_i(\beta_o)]$  and  $\mathbf{K} = \mathbb{E} [\mathbf{s}_i(\beta_o)\mathbf{s}_i(\beta_o)']$

## Correct Specification

$\sqrt{N}(\hat{\beta} - \beta_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$  where  $\mathbf{J} = -\mathbb{E} [\mathbf{H}_i(\beta_o)]$

Asymptotic variance calculations for index models are complicated, but there's a clever trick for computing  $\mathbf{J}$  under correct specification.

# Asymptotic Variance Calculation for Correctly Specified Index Models

$$\ell_i(\beta) = y_i \log \{ G(\mathbf{x}'_i \beta) \} + (1 - y_i) \log \{ 1 - G(\mathbf{x}'_i \beta) \}$$

## Step 1: Calculate The Score Vector

$$\begin{aligned} \mathbf{s}_i &\equiv \frac{\partial}{\partial \beta} \ell_i(\beta) = \frac{y_i g(\mathbf{x}'_i \beta) \mathbf{x}_i}{G(\mathbf{x}'_i \beta)} - \frac{(1 - y_i) g(\mathbf{x}'_i \beta) \mathbf{x}_i}{1 - G(\mathbf{x}'_i \beta)} \\ &= \frac{g(\mathbf{x}'_i \beta) \mathbf{x}_i}{G(\mathbf{x}'_i \beta) [1 - G(\mathbf{x}'_i \beta)]} \{ [1 - G(\mathbf{x}'_i \beta)] y_i - G(\mathbf{x}'_i \beta) (1 - y_i) \} \\ &= \frac{g(\mathbf{x}'_i \beta) \mathbf{x}_i [y_i - G(\mathbf{x}'_i \beta)]}{G(\mathbf{x}'_i \beta) [1 - G(\mathbf{x}'_i \beta)]} \end{aligned}$$

## Asymptotic Variance Calculation for Correctly Specified Index Models

$$\mathbf{s}_i = \frac{g(\mathbf{x}'_i\boldsymbol{\beta})\mathbf{x}_i \{y_i - G(\mathbf{x}'_i\boldsymbol{\beta})\}}{G(\mathbf{x}'_i\boldsymbol{\beta}) \{1 - G(\mathbf{x}'_i\boldsymbol{\beta})\}}$$

Step 2: Start Calculating the Hessian but give up because it's a nightmare.

$$\begin{aligned}\mathbf{H}_i(\boldsymbol{\beta}) &\equiv \frac{\partial \mathbf{s}_i}{\partial \boldsymbol{\beta}'} = \frac{\partial}{\partial \boldsymbol{\beta}'} \left( [y_i - G(\mathbf{x}'_i\boldsymbol{\beta})] \left[ \frac{g(\mathbf{x}'_i\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}'_i\boldsymbol{\beta}) \{1 - G(\mathbf{x}'_i\boldsymbol{\beta})\}} \right] \right) \\ &= \frac{-g(\mathbf{x}'_i\boldsymbol{\beta})^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i\boldsymbol{\beta}) \{1 - G(\mathbf{x}'_i\boldsymbol{\beta})\}} + [y_i - G(\mathbf{x}'_i\boldsymbol{\beta})] \underbrace{\frac{\partial}{\partial \boldsymbol{\beta}'} \left\{ \frac{g(\mathbf{x}'_i\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}'_i\boldsymbol{\beta}) [1 - G(\mathbf{x}'_i\boldsymbol{\beta})]} \right\}}_{\text{a nasty awful mess: call it } \mathbf{M}(\mathbf{x}_i, \boldsymbol{\beta})}\end{aligned}$$



# Asymptotic Variance Calculation for Correctly Specified Index Models

$$\mathbf{H}_i(\beta) = \frac{-g(\mathbf{x}'_i\beta)^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i\beta) \{1 - G(\mathbf{x}'_i\beta)\}} + [y_i - G(\mathbf{x}'_i\beta)] \mathbf{M}(\mathbf{x}_i, \beta)$$

Step 3: Calculate the *Conditional Expectation* instead...

$$\begin{aligned} -\mathbb{E} [\mathbf{H}_i(\beta) | \mathbf{x}_i] &= \frac{g(\mathbf{x}'_i\beta)^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i\beta) \{1 - G(\mathbf{x}'_i\beta)\}} + \underbrace{\mathbb{E} [y_i - G(\mathbf{x}'_i\beta) | \mathbf{x}_i]}_{\text{equals zero under correct spec.}} \mathbf{M}(\mathbf{x}_i, \beta) \\ &= \frac{g(\mathbf{x}'_i\beta)^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i\beta) \{1 - G(\mathbf{x}'_i\beta)\}} \end{aligned}$$

Step 4: Iterated Expectations

$$\mathbf{J} = -\mathbb{E} [\mathbf{H}_i(\beta_o)] = \mathbb{E} \{ \mathbb{E} [\mathbf{H}_i(\beta_o) | \mathbf{x}_i] \} = \mathbb{E} \left\{ \frac{g(\mathbf{x}'_i\beta_o)^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i\beta_o) \{1 - G(\mathbf{x}'_i\beta_o)\}} \right\}$$

# Asymptotic Variance Calculation for Correctly Specified Index Models

## Asymptotic Distribution

$$\sqrt{N}(\hat{\beta} - \beta_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}), \quad \mathbf{J}^{-1} = \mathbb{E} \left\{ \frac{g(\mathbf{x}'_i \beta_o)^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i \beta_o) \{1 - G(\mathbf{x}'_i \beta_o)\}} \right\}^{-1}$$

## Consistent Estimator

$$\hat{\mathbf{J}}^{-1} \equiv \left\{ \frac{1}{N} \sum_{i=1}^N \frac{g(\mathbf{x}'_i \hat{\beta})^2 \mathbf{x}_i \mathbf{x}'_i}{G(\mathbf{x}'_i \hat{\beta}) [1 - G(\mathbf{x}'_i \hat{\beta})]} \right\}^{-1}$$

## Notes

- ▶ Assumes correct specification, i.e.  $p(\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) = G(\mathbf{x}'\beta_o)$
- ▶ In contrast, *robust* variance matrix  $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1}$  is complicated, but R can do it.

# McFadden (1974) – “Pseudo R-squared”

## Model with Intercept Only

$L(\bar{y}) \equiv$  maximized sample Likelihood

$\ell(\bar{y}) \equiv$  maximized sample log-likelihood

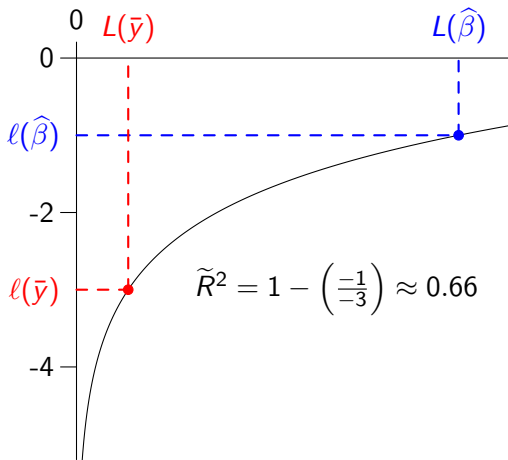
## Full Model

$L(\hat{\beta}) \equiv$  maximized sample Likelihood

$\ell(\hat{\beta}) \equiv$  maximized sample log-likelihood

## Pseudo R-squared

$$\tilde{R}^2 \equiv 1 - \ell(\hat{\beta})/\ell(\bar{y})$$



## McFadden (1974) – “Pseudo R-squared”

Pseudo R-squared

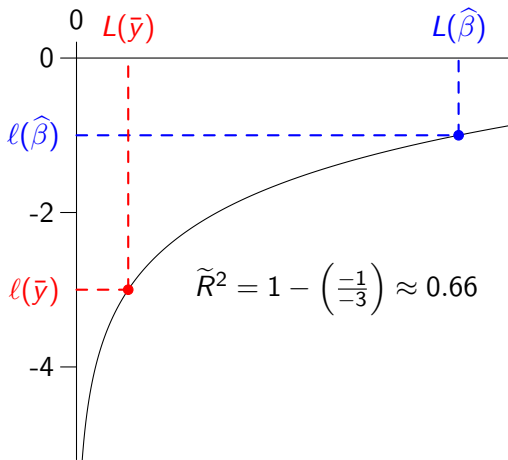
$$\tilde{R}^2 \equiv 1 - \ell(\hat{\beta}) / \ell(\bar{y})$$

Always between 0 and 1

Show this on the problem set!

**Health Warning**

I don't recommend using pseudo- $R^2$ : it's arbitrary and can be misleading. Other people use it so I'm telling you what it is.



# Lecture #4 – Random Utility Models

Overview of Random Utility Models

Identification of Choice Models

Index Models as Special Cases (e.g. Logit & Probit)

Multinomial and Conditional Logit

# Discrete Choice – Basic Terminology

## Decision-maker

Household, person, firm, etc.

## Alternatives

Products, courses of action, etc.

## Choice Set

The collection of all alternatives available to the decision-maker.

# Restrictions on the Choice Set

We assume that:

1. Choices are mutually exclusive: choose only *one* alternative.
2. Choice set is *exhaustive*: contains every alternative (always choose something)
3. The number of alternatives is finite.

We can always redefine the choice set to satisfy 1 and 2

$$\underbrace{\{\text{Beer}, \text{Pizza}\}}_{\text{not mutually exclusive}} \rightarrow \underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}\}}_{\text{mutually exclusive}}$$

$$\underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}\}}_{\text{not exhaustive}} \rightarrow \underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}, \text{Something Else}\}}_{\text{exhaustive}}$$

# Random Utility Models (RUMs)

Following Train (2009), use  $n$  to index individuals!

## Notation

- ▶  $N$  decision-makers  $n = 1, \dots, N$
- ▶  $J$  alternatives  $j = 1, \dots, J$ .

## Utility and Decision Rule

- ▶ Decision-maker  $n$  obtains utility  $U_{nj}$  from choosing alternative  $j$
- ▶ Maximize utility: decision-maker  $n$  chooses alternative  $i$  iff  $U_{ni} > U_{nj}$  for any  $j \neq i$



# Random Utility Models

## Researcher Observes

- ▶ Attributes  $x_{nj}$  of each alternative (e.g. product characteristics)
- ▶ Attributes  $s_n$  of the decision-maker (e.g. demographics)
- ▶ Choices but **not utilities**

## Representative Utility $V_{nj}$

Assume researcher can specify a function  $V_{nj}(x_{nj}, s_n)$  relating attributes  $x_{nj}$  of each alternative  $j$  and attributes  $s_n$  of each decision-maker  $n$  to her utilities  $U_{nj}$ .

## Error Terms $\varepsilon_{nj}$

$\varepsilon_{nj} \equiv U_{nj} - V_{nj}$  is the difference between *true* utility  $U_{nj}$  and representative utility  $V_{nj}$

# Random Utility Models (RUMs)

What are the error terms?

$\varepsilon_{nj}$  for  $j = 1, \dots, J$  represent unobserved factors that affect choices but are not captured by representative utilities (i.e. our model)

Treat Errors as Random

Let  $\varepsilon' \equiv [\varepsilon_{n1} \dots \varepsilon_{nJ}]$  have density function  $f(\varepsilon_n)$ . Characterizes unobserved heterogeneity across decision-makers.

Choice Probabilities

$$P_{ni} \equiv \mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \int_{\mathbb{R}^J} \mathbb{1} \{ \varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i \} f(\varepsilon_n) d\varepsilon_n$$

This all sounds a bit abstract...

### Basic Idea

1. Write down a parametric model for  $V_{nj}(x_{nj}, s_n)$  with unknown parameters  $\theta$ .
2. Choose a distribution  $f$  for the errors (heterogeneity)  $\varepsilon_n$ .
3. Back out choice probabilities as a function of parameters  $\theta$ .
4. Use observed choices and attributes to find the MLE  $\hat{\theta}$ .

### Looking Back; Looking Ahead

- ▶ Logit and Probit are special cases of RUMs: choice between two alternatives.
- ▶ RUMs provide a framework to estimate more complicated discrete choice models.

# Some Complications

## Computation

- ▶ Integral linking choice probabilities to parameters  $\theta$  rarely has a closed form.
- ▶ Logit-type models are a well-known *exception*.
- ▶ More generally: use *Monte Carlo Simulation* to approximate the integral.

## Identification

- ▶ Roughly speaking, we say that a parameter is *identified* if it could be uniquely determined by observing the *whole population*.
- ▶ What parameters of RUMs are identified from choices and attributes?

# A Very Simple Example

## Transport Decision

- ▶ Exactly two ways to get to work: by **car** or by **bus**.
- ▶ Observe two attributes: cost in **time**  $T$  and **money**  $M$  of each mode of transport.

## Econometrician's Model: $(\beta, \gamma)$ unknown

$$V_{\text{car}} = \beta T_{\text{car}} + \gamma M_{\text{car}}$$

$$U_{\text{car}} = V_{\text{car}} + \varepsilon_{\text{car}}$$

$$V_{\text{bus}} = \beta T_{\text{bus}} + \gamma M_{\text{bus}}$$

$$U_{\text{bus}} = V_{\text{bus}} + \varepsilon_{\text{bus}}$$

## Choice Probabilities

$$P_{\text{car}} = \mathbb{P}(\varepsilon_{\text{bus}} - \varepsilon_{\text{car}} < V_{\text{car}} - V_{\text{bus}})$$

$$P_{\text{bus}} = \mathbb{P}(\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} < V_{\text{bus}} - V_{\text{car}}) = 1 - P_{\text{car}}$$

# A Very Simple Example: Who drives to work?

What is common?

Parameters:  $(\beta, \gamma)$ . Our goal is to identify and estimate these.

## Observed Heterogeneity

- ▶ Alice lives next to the bus stop: her  $T_{\text{bus}}$  is low.
- ▶ Bob is 70 and gets a discount on public transport: his  $M_{\text{bus}}$  is low.
- ▶ Clara and her roommates work at the same office and can carpool: her  $M_{\text{car}}$  is low.

## Unobserved Heterogeneity

James hates to drive ( $\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} < 0$ ) but Steve loves driving ( $\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} > 0$ ).

# Identification – What can we learn from data?

## Only differences in utility matter

- ▶  $\mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \mathbb{P}(U_{ni} - U_{nj} > 0 \quad \forall j \neq i)$
- ▶ All that matters is how much better or worse a given alternative is than the others.

## Consequences

1. We cannot identify a different intercept for each alternative.
2. We can only identify differences of effects for decision-maker attributes.

If there are  $J$  alternatives, we can identify only  $(J - 1)$  intercepts.

Equivalently: normalize one intercept to zero.

$$\text{Intercept} \Rightarrow \mathbb{E}[\varepsilon_{nj}] = 0$$

- ▶ Suppose  $U_{nj} = \mathbf{x}'_{nj}\boldsymbol{\beta} + \varepsilon_{nj}^*$  where  $\mathbf{x}_{nj}$  *excludes* a constant and  $\mathbb{E}[\varepsilon_{nj}^*] \neq 0$ .
- ▶ Equivalent model:  $U_{nj} = \alpha + \mathbf{x}'_{nj}\boldsymbol{\beta} + \varepsilon_{nj}$  where  $\mathbb{E}[\varepsilon_{nj}] = 0$  by construction.

Why not a different intercept for each alternative?

$$U_{\text{car}} = \alpha_{\text{car}} + \beta T_{\text{car}} + \gamma M_{\text{car}} + \varepsilon_{\text{car}}$$

$$U_{\text{bus}} = \alpha_{\text{bus}} + \beta T_{\text{bus}} + \gamma M_{\text{bus}} + \varepsilon_{\text{bus}}$$

$$U_{\text{bus}} - U_{\text{car}} = (\alpha_{\text{bus}} - \alpha_{\text{car}}) + \beta (T_{\text{bus}} - T_{\text{car}}) + \gamma (M_{\text{bus}} - M_{\text{car}}) + (\varepsilon_{\text{bus}} - \varepsilon_{\text{car}})$$



Only differences of effects for decision-maker attributes are identified.

Can we identify the effects of income  $Y$  separately for Bus and Car?

$$U_{\text{car}} = \theta_{\text{car}} Y + \beta T_{\text{car}} + \gamma M_{\text{car}} + \varepsilon_{\text{car}}$$

$$U_{\text{bus}} = \theta_{\text{bus}} Y + \beta T_{\text{bus}} + \gamma M_{\text{bus}} + \varepsilon_{\text{bus}}$$

$$U_{\text{bus}} - U_{\text{car}} = (\theta_{\text{bus}} - \theta_{\text{car}}) Y + \beta (T_{\text{bus}} - T_{\text{car}}) + \gamma (M_{\text{bus}} - M_{\text{car}}) + (\varepsilon_{\text{bus}} - \varepsilon_{\text{car}})$$

Equivalent to normalizing one of the  $\theta$ s to zero.

# More on Identification – What can we learn from data?

The scale of utility is irrelevant

- ▶ Let  $\lambda$  be an arbitrary positive constant.
- ▶ Original Model:  $U_{nj} = V_{nj} + \varepsilon_{nj}$ ,  $\text{Var}(\varepsilon_{nj}) = \sigma^2$
- ▶ Re-scaled Model:  $\lambda U_{nj} = \lambda V_{nj} + \lambda \varepsilon_{nj} \iff U_{nj}^* = V_{nj}^* + \varepsilon_{nj}^*$ ,  $\text{Var}(\varepsilon_{nj}^*) = \lambda^2 \sigma^2$

$\text{Var}(\varepsilon_{nj})$  determines the scale of  $\beta$

- ▶  $U_{nj} = \mathbf{x}_{nj}'\beta + \varepsilon_{nj}$ ,  $\text{Var}(\varepsilon_{nj}) = \sigma^2 \iff U_{nj}^* = \mathbf{x}_{nj}'(\beta/\sigma) + \varepsilon_{nj}^*$ ,  $\text{Var}(\varepsilon_{nj}^*) = 1$
- ▶ Can't directly compare coefs. across models with different normalizations for  $\varepsilon_{nj}$ .
- ▶ Recall: we had to re-scale Logit and Probit coefs. to compare them.

Only differences in utility matter  $\implies$  only differences in errors matter.

## Notation

- ▶  $\tilde{\varepsilon}_{nj} \equiv \varepsilon_{nj} - \varepsilon_{ni}$  be the *difference* of errors  $\varepsilon_{nj}$  and  $\varepsilon_{ni}$ .
- ▶  $\tilde{\varepsilon}_{ni} \equiv$  vector of all unique differences, taking  $\varepsilon_{ni}$  as the “base case”
  - ▶ E.g.  $\varepsilon'_n = (\varepsilon_{n1}, \varepsilon_{n2}, \varepsilon_{n3}) \implies \tilde{\varepsilon}'_{n1} = (\varepsilon_{n2} - \varepsilon_{n1}, \varepsilon_{n3} - \varepsilon_{n1})$
  - ▶ Note:  $J$  errors  $\Rightarrow (J - 1)$  unique *differences*
- ▶ Let  $g$  be the joint density of  $\tilde{\varepsilon}_{ni}$ .

## Choice Probabilities

$$\begin{aligned} P_{ni} &\equiv \mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \mathbb{P}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i) \\ &= \mathbb{P}(\tilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \quad \forall j \neq i) = \int_{\mathbb{R}^{J-1}} \mathbb{1}\{\tilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \quad \forall j \neq i\} g(\tilde{\varepsilon}_{ni}) d\tilde{\varepsilon}_{ni} \end{aligned}$$

## How to obtain the index models from last lecture? (E.g. Probit and Logit)

1. Two alternatives, e.g. Bus or Something Else
2. Let  $y_n = 1$  if decision-maker  $n$  chooses alternative 1; zero otherwise.
3.  $V_{nj} = \mathbf{s}'_n \gamma_j$  (representative utility depends only on attributes of decision-maker)
4.  $(\varepsilon_{n2} - \varepsilon_{n1}) \sim G$  independently of  $\mathbf{s}_n$ .

$$\begin{aligned} U_{n1} - U_{n2} &= (\mathbf{s}'_n \gamma_1 - \mathbf{s}'_n \gamma_2) + (\varepsilon_{n1} - \varepsilon_{n2}) = \mathbf{s}'_n (\gamma_1 - \gamma_2) + (\varepsilon_{n1} - \varepsilon_{n2}) \\ &= \mathbf{s}'_n \gamma + (\varepsilon_{n1} - \varepsilon_{n2}) \end{aligned}$$

$$\mathbb{P}(y_n = 1 | \mathbf{s}_n) = \mathbb{P}(U_{n1} - U_{n2} > 0 | \mathbf{s}_n) = \mathbb{P}(\varepsilon_{n2} - \varepsilon_{n1} < \mathbf{s}'_n \gamma | \mathbf{s}_n) = G(\mathbf{s}'_n \gamma)$$

# The Logit Family of Choice Models

## Theorem

Suppose that  $\varepsilon_{n1}, \dots, \varepsilon_{nJ} \sim \text{iid } F$  where  $F(z) = \exp\{-\exp(-z)\}$ . Then,

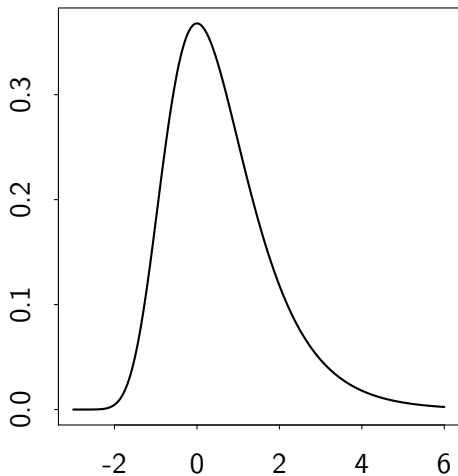
$$P_{ni} = \mathbb{P}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i) = \frac{\exp(V_{ni})}{\sum_{j=1}^J \exp(V_{nj})}$$

## Notes

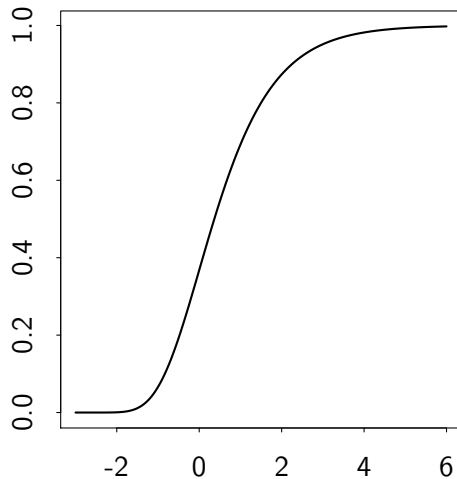
- ▶ This is a special case where the choice probabilities have a closed-form solution!
- ▶  $F(z) = \exp\{-\exp(-z)\}$  is the Gumbel aka Type I Extreme Value CDF
- ▶ Corollary: the *difference* of independent Gumbel RVs is a standard Logistic RV

## The Gumbel Distribution (aka Type I Extreme Value)

**Gumbel Density**



**Gumbel CDF**



## Different specifications of $V_{nj}$ yield different models.

### Multinomial Logit

- ▶  $V_{nj} = \mathbf{s}'_n \gamma_j$  ← only attributes that are fixed across alternatives (e.g.  $n$ 's income)
- ▶ Can only identify differences  $(\gamma_j - \gamma_i)$ . Typical to normalize  $\gamma_1 = \mathbf{0}$ .

### Conditional Logit

- ▶  $V_{nj} = \mathbf{x}'_{nj} \beta$  ← only attributes that vary across alternatives (e.g. price)
- ▶ Note that  $\beta$  is fixed across alternatives.

### Mixed Logit

- ▶  $V_{nj} = \mathbf{s}'_n \gamma_j + \mathbf{x}'_{nj} \beta$  ← a combination of the two

# The Likelihood for Random Utility Models

## Notation

- ▶  $y_n \in \{1, \dots, J\} \equiv n$ 's choice.
- ▶  $\mathbf{z}_n$  vector of all regressors for  $n$
- ▶  $\boldsymbol{\theta}$  vector of all unknown parameters
- ▶ Choice Probs.  $P_{ni} \equiv \mathbb{P}(y_n = i | \mathbf{z}_n, \boldsymbol{\theta})$

## Note

Likelihood is easy, but choice probabilities are usually hard (logit is an exception).

## Likelihood

$$f(y_n | \mathbf{z}_n, \boldsymbol{\theta}) = \prod_{j=1}^J P_{nj}^{\mathbb{1}\{y_n=j\}}$$

## Log Likelihood

$$\ell_N(\boldsymbol{\theta}) = \sum_{n=1}^N \sum_{j=1}^J \mathbb{1}\{y_n = j\} \log P_{nj}$$

## Logit Choice Probabilities

$$P_{ni} = \exp(V_{ni}) / \sum_{j=1}^J \exp(V_{nj})$$



## Interpreting Multinomial Logit Coefficients

- ▶ Partial effects tricky to derive and interpret.
- ▶ Better approach: examine **log-odds ratios**
- ▶ Normalizing  $\gamma_1 = \mathbf{0}$ , we have  $\exp(\mathbf{s}_n \gamma_1) = \exp(0) = 1$ . Hence,

$$\frac{P_{ni}}{P_{n1}} = \frac{\exp(\mathbf{s}_n \gamma_i)}{\sum_{j=1}^J \exp(\mathbf{s}_n \gamma_j)} \times \frac{\sum_{j=1}^J \exp(\mathbf{s}_n \gamma_j)}{\exp(\mathbf{s}_n \gamma_1)} = \frac{\exp(\mathbf{s}_n \gamma_i)}{\exp(\mathbf{s}_n \gamma_1)} = \exp(\mathbf{s}_n \gamma_i)$$

- ▶ Taking logs:  $\log(P_{ni}/P_{n1}) = \log[\exp(\mathbf{s}_n \gamma_i)] = \mathbf{s}'_n \gamma_i$ .

### Punchline

$\gamma_i^{(k)}$  is the marginal effect of  $s_n^{(k)}$  on the **relative probability** that  $y = i$  compared to  $y = 1$  **measured on the log scale** – e.g. taking the bus relative to driving.

# Interpreting Conditional Logit Coefficients

You'll derive these on the problem set!

## Partial Effects

- ▶ The attributes  $\mathbf{x}_{nj}$  are *specific* to a particular alternative  $j$ .
- ▶ Hence: partial effects are much simpler for conditional logit than multinomial.

### Own Attribute

$$\frac{\partial P_{nj}}{\partial \mathbf{x}_{nj}} = P_{nj}(1 - P_{nj})\beta$$

### Cross-Attribute ( $j \neq i$ )

$$\frac{\partial P_{nj}}{\partial \mathbf{x}_{ni}} = -P_{nj}P_{ni}\beta$$

If increasing  $\mathbf{x}_{nj}^{(k)}$  makes  $y = j$  *more likely*, it must make  $y = i$  *less likely*

# The Independence of Irrelevant Alternatives (IIA)

Or why people don't like logit models...

## Logit Choice Probabilities

$$P_{ni} = \frac{\exp(V_{ni})}{\sum_{j=1}^J \exp(V_{nj})} \implies \frac{P_{ni}}{P_{nj}} = \exp(V_{ni} - V_{nj})$$

## In Words

The relative probability of choosing  $i$  versus  $j$  only depends on the representative utilities for  $i$  and  $j$ . This is called the **independence of irrelevant alternatives (IIA)**.

## Why is this a problem

IIA arises in logit models because  $\varepsilon_{n1}, \dots, \varepsilon_{nJ}$  are *independent*. In reality “some alternatives are more similar than others,” i.e. errors may be correlated.

# An Example where IIA is Unreasonable – Choosing Presidential Candidates

## Model

- ▶  $V_{nj} = (\text{Demographics}_n)' \gamma_j + (\text{Ideology}_{nj})' \beta$
- ▶  $(\text{Ideology}_{nj})$  = similarity between voter  $n$ 's ideology and candidate  $j$ 's.
- ▶ Candidates = {Trump, Sanders, Warren}

Consider a group of voters who all have the *same* demographics and ideology

E.g. white, centrist, female, mid-westerners between the age of 45 and 50 with an average household income between \$50 and \$55 thousand USD.

Same regressors  $\Rightarrow$  same  $V_{nj}$

$V_{nj}$  doesn't vary over  $n$  within the group:  $\{V_{\text{Trump}}, V_{\text{Sanders}}, V_{\text{Warren}}\}$

# An Example where IIA is Unreasonable – Choosing Presidential Candidates

## Two-way Race

Suppose 2/3 of this group of voters chooses Sanders over Trump:  $P_{\text{Sanders}}/P_{\text{Trump}} = 2$

## Assumption

Sanders and Warren are ideologically similar  $\implies V_{\text{Warren}} \approx V_{\text{Sanders}}$

## Implications of Logit

- ▶ Relative choice probabilities are the *same* in a two-way race or a three-way race.
- ▶  $P_{\text{Warren}}/P_{\text{Sanders}} = \exp(V_{\text{Warren}} - V_{\text{Sanders}}) \approx 1$

# An Example where IIA is Unreasonable – Choosing Presidential Candidates

## Logit Implication for Three-way Race

$$P_{\text{Sanders}} = 2P_{\text{Trump}}, \quad P_{\text{Sanders}} \approx P_{\text{Warren}}, \quad P_{\text{Trump}} + P_{\text{Sanders}} + P_{\text{Warren}} = 1$$

$$\implies P_{\text{Trump}} + 2P_{\text{Trump}} + 2P_{\text{Trump}} = 1$$

$$P_{\text{Trump}} = 1/5$$

$$P_{\text{Warren}} = P_{\text{Sanders}} = 2/5$$

## What we'd actually expect in a Three-way Race

1/3 Trump, 1/3 Sanders and 1/3 Warren – i.e. Warren “splits” the Sanders vote.

## What's going wrong?

Logit assumes  $\varepsilon_{\text{Warren}}$  and  $\varepsilon_{\text{Sanders}}$  are independent but in reality they're not.

# Lecture #5 – Sample Selection

Examples of Sample Selection

The Heckman Selection Model

# What is sample selection?

## Question

Thus far we have always assumed that  $(y_1, \mathbf{x}_1), \dots, (y_N, \mathbf{x}_N)$  are a random sample from the population of interest. What if they aren't?

## Example 1: MPhil Admissions

- ▶ Suppose we want to improve admissions decisions at Oxford.
- ▶  $y \equiv$  overall marks in 1st year of Oxford Economics MPhil
- ▶  $\mathbf{x} \equiv \{\text{undergrad grades, letters of reference, } \dots\}$
- ▶ What we observe:  $\mathbf{x}$  for all applicants;  $y$  for applicants who were **admitted**.
- ▶ What we want:  $\mathbb{E}(y|\mathbf{x})$  for **all applicants**.



## Example 2: A Model of Wage Offers

Gronau (1974; JPE)

### Question

How do wage offers  $w_i^o$  vary with  $\mathbf{x}_i$  for all people in the population.

### Problem

Only observe  $w_i^o$  for people who *accept* their offer, i.e. those who are employed.

### Mathematically

$$\mathbb{E}(w_i^o | \mathbf{x}_i) \neq \mathbb{E}(w_i^o | \mathbf{x}_i, \text{Employed})$$

# The Heckman Selection Model (Heckit) — Is $\beta_1$ identified?

## Outcome Equation

$$y_1 = \mathbf{x}'_1 \beta_1 + u_1$$

## Assumptions

- (a) Observe  $y_2, \mathbf{x}' = (\mathbf{x}'_1, \mathbf{x}'_2)$ ; only observe  $y_1$  if  $y_2 = 1$ .
- (b)  $(u_1, v_2)$  are mean zero and jointly independent of  $\mathbf{x}$ .
- (c)  $v_2 \sim \text{Normal}(0, 1)$
- (d)  $\mathbb{E}(u_1 | v_2) = \gamma_1 v_2$  where  $\gamma_1$  is an unknown constant.

## Participation Equation

$$y_2 = \mathbb{1} \{ \mathbf{x}' \delta_2 + v_2 > 0 \}$$

## Notes

- ▶  $\mathbb{E}(u_1) = \mathbb{E}(v_2) = 0$  is not restrictive: just include intercepts in both equations.
- ▶ Assumption (d) would be *implied* by assuming that  $(u_1, v_2)$  are jointly normal.
- ▶ These assumptions are strong. They can be weakened a bit, but not too much.

Step 1: Show that  $u_1$  and  $\mathbf{x}$  are conditionally independent given  $v_2$ .

Assumption (b)

$(u_1, v_2)$  are jointly independent of  $\mathbf{x}$ .

Equivalently

$$f_{1,2|\mathbf{x}}(u_1, v_2|\mathbf{x}) = f_{1,2}(u_1, v_2), \quad \text{and} \quad f_{1|\mathbf{x}}(u_1|\mathbf{x}) = f_1(u_1), \quad \text{and} \quad f_{2|\mathbf{x}}(v_2|\mathbf{x}) = f_2(v_2)$$

Therefore

$$f_{1|2,\mathbf{x}}(u_1|v_2, \mathbf{x}) = \frac{f_{1,2|\mathbf{x}}(u_1, v_2|\mathbf{x})}{f_{2|\mathbf{x}}(v_2|\mathbf{x})} = \frac{f_{1,2}(u_1, v_2)}{f_2(v_2)} = f_{1|2}(u_1|v_2)$$

In Words

Conditioning on  $(v_2, \mathbf{x})$  gives the same information about  $u_1$  as conditioning on  $v_2$  only.

Step 2: Calculate  $\mathbb{E}(y_1|\mathbf{x}, v_2)$ ; show that if  $v_2$  were observed we'd be done.

$$\begin{aligned}\mathbb{E}(y_1|\mathbf{x}, v_2) &= \mathbb{E}(\mathbf{x}'_1\beta_1 + u_1|\mathbf{x}, v_2) && \text{(Substitute Outcome Eq.)} \\ &= \mathbf{x}'_1\beta_1 + \mathbb{E}(u_1|\mathbf{x}, v_2) && (\mathbf{x}_1 \text{ is a subset of } \mathbf{x}) \\ &= \mathbf{x}'_1\beta_1 + \mathbb{E}(u_1|v_2) && \text{(apply result of Step 1)} \\ &= \mathbf{x}'_1\beta_1 + \gamma_1 v_2 && \text{(apply Assumption (d))}\end{aligned}$$

Step 3: Relate  $v_2$  (unobserved) to  $\mathbf{x}$  and  $y_2$  (both observed).

$$\begin{aligned}\mathbb{E}(y_1|\mathbf{x}, y_2) &= \mathbb{E}_{v_2|(\mathbf{x}, y_2)} [\mathbb{E}(y_1|\mathbf{x}, y_2, v_2)] && \text{(Law of Iterated Expectations)} \\ &= \mathbb{E}_{v_2|(\mathbf{x}, y_2)} [\mathbb{E}(y_1|\mathbf{x}, v_2)] && \text{(Participation Eq: } y_2 = g(\mathbf{x}, v_2)) \\ &= \mathbb{E} [\mathbf{x}'_1 \boldsymbol{\beta}_1 + \gamma_1 v_2 | \mathbf{x}, y_2] && \text{(apply result of Step 2)} \\ &= \mathbf{x}'_1 \boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2 | \mathbf{x}, y_2) && (\mathbf{x}_1 \text{ is a subset of } \mathbf{x})\end{aligned}$$

Therefore

$$\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}'_1 \boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2 | \mathbf{x}, y_2 = 1).$$

## What is the significance of Step 3?

- ▶ Define  $h(\mathbf{x}) \equiv \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$ . Then:  $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}'_1\beta_1 + \gamma_1 h(\mathbf{x})$
- ▶ Note that  $h(\mathbf{x})$  is a random variable: a function of  $\mathbf{x}$ .
- ▶ Step 3 shows that a linear regression of  $y_1$  on  $\mathbf{x}_1$  and  $h(\mathbf{x})$  for the *selected* sample, those with  $y_2 = 1$ , identifies  $\beta_1$  and  $\gamma_1$ !
- ▶ All that remains is to figure out what function  $h$  is...

## Note: Selection Bias Enters Through $\gamma_1$

### Assumption (d)

$\mathbb{E}(u_1|v_2) = \gamma_1 v_2$  allows *dependence* between errors in participation and outcome eqs.

### Step 3

$$\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}'_1 \beta_1 + \gamma_1 \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$$

### Therefore

If  $\gamma_1 = 0$  there is no selection bias: in this case  $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}'_1 \beta$  so regressing  $y_1$  on  $\mathbf{x}_1$  for the subset of individuals with  $y_2 = 1$  identifies  $\beta_1$ .

Step 4: Determine the distribution of  $v_2$  given  $(\mathbf{x}, y_2 = 1)$ .

$$\mathbb{P}(v_2 \leq t | \mathbf{x}, y_2 = 1) = \mathbb{P}(v_2 \leq t | \mathbf{x}, v_2 > -\mathbf{x}'\delta_2) \quad (\text{participation eq.})$$

$$= \frac{\mathbb{P}(\{v_2 \leq t\} \cap \{v_2 > -\mathbf{x}'\delta_2\} | \mathbf{x})}{\mathbb{P}(v_2 > -\mathbf{x}'\delta_2 | \mathbf{x})} \quad (\text{defn. of cond. prob.})$$

$$= \frac{\mathbb{P}\{v_2 \in (-\mathbf{x}'\delta_2, t]\}}{\mathbb{P}(v_2 > -\mathbf{x}'\delta_2)} \quad (v_2 \text{ and } \mathbf{x} \text{ are indep.})$$

$$= \frac{\mathbb{P}\{z \in (c, t]\}}{\mathbb{P}(z > c)} \quad (v_2 \text{ is standard normal})$$

where  $z \sim \text{Normal}(0, 1)$  and we define the shorthand  $c \equiv -\mathbf{x}'\delta_2$ .



## Step 5: Calculate the Expectation of a Truncated Normal

Recall:  $z \sim \text{Normal}(0, 1)$  and  $c \equiv -\mathbf{x}'\boldsymbol{\delta}_2$

CDF

$$\mathbb{P}(z \leq t | z > c) = \frac{\mathbb{P}\{z \in (c, t]\}}{\mathbb{P}(z > c)} = \mathbb{1}\{c \leq t\} \left[ \frac{\Phi(t) - \Phi(c)}{1 - \Phi(c)} \right]$$

Density

$$f(t | z > c) = \frac{d}{dt} \mathbb{P}(z \leq t | z > c) = \begin{cases} 0, & t \leq c \\ \varphi(t) / [1 - \Phi(c)], & t > c \end{cases}$$

## Step 5: Calculate the Expectation of a Truncated Normal

Recall:  $z \sim \text{Normal}(0, 1)$  and  $c \equiv -\mathbf{x}'\boldsymbol{\delta}_2$

$$\begin{aligned}\mathbb{E}(z|z > c) &= \int_{-\infty}^{\infty} tf(t|z > c) dt = \frac{1}{1 - \Phi(c)} \int_c^{\infty} t\varphi(t) dt \\&= \left[ \frac{1}{1 - \Phi(c)} \right] \left( \frac{1}{\sqrt{2\pi}} \right) \int_c^{\infty} t \exp \{ -t^2/2 \} dt \\&= \left[ \frac{1}{1 - \Phi(c)} \right] \left( \frac{1}{\sqrt{2\pi}} \right) \left[ -\exp \{ -t^2/2 \} \right]_c^{\infty} \\&= \left[ \frac{1}{1 - \Phi(c)} \right] \left( \frac{\exp \{ -c^2/2 \}}{\sqrt{2\pi}} \right) = \frac{\varphi(c)}{1 - \Phi(c)}\end{aligned}$$

## Step 6: Put everything together.

Recall: Step 3

$$y_1 = \mathbf{x}'_1 \beta_1 + \gamma_1 h(\mathbf{x}) + \eta, \quad \mathbb{E}[\eta | \mathbf{x}_1, h(\mathbf{x})] = 0, \quad h(\mathbf{x}) \equiv \mathbb{E}(v_2 | \mathbf{x}, y_2 = 1)$$

Using Steps 4–5

$$h(\mathbf{x}) = \frac{\varphi(-\mathbf{x}'\delta_2)}{1 - \Phi(-\mathbf{x}'\delta_2)} = \frac{\varphi(\mathbf{x}'\delta_2)}{\Phi(\mathbf{x}'\delta_2)} \quad \text{since } \varphi(-c) = \varphi(c) \text{ and } 1 - \Phi(c) = \Phi(-c).$$

Inverse Mills Ratio

$\varphi(c)/\Phi(c)$  is the inverse Mills Ratio, traditionally denoted by  $\lambda \implies h(\mathbf{x}) = \lambda(\mathbf{x}'\delta_2)$ .

Careful!

In an earlier lecture  $\lambda$  denoted the standard logistic density. Here it's something else!

# The Heckman Two-step Estimator aka “Heckit”

## Observables

Observe  $(y_{2i}, \mathbf{x}_i)$  for a random sample of size  $N$ ; only observe  $y_{1i}$  for those with  $y_{2i} = 1$ .

## First Step – Estimate $\delta_2$ from Full Sample

- ▶ Run Probit on the Participation Eq.  $\mathbb{P}(y_{2i} = 1|\mathbf{x}_i) = \Phi(\mathbf{x}_i'\delta_2)$  for the full sample.
- ▶ Define  $\hat{\lambda}_i \equiv \lambda(\mathbf{x}_i'\hat{\delta}_2)$  where  $\hat{\delta}_2$  is the MLE for  $\delta_2$ .

## Second Step – Estimate $(\beta_1, \gamma_1)$ from Selected Sample

Using the observations for which  $y_{1i}$  is observed, regress  $y_{1i}$  on  $(\mathbf{x}_{1i}, \hat{\lambda}_i)$  by OLS to obtain estimates  $(\hat{\beta}_1, \hat{\gamma}_1)$ .

# The Heckman Two-step Estimator aka “Heckit”

## Theorem

Under the assumptions from above, the 2-step “Heckit” estimators satisfy

$$\begin{bmatrix} \hat{\delta}_2 \\ \hat{\beta}_1 \\ \hat{\gamma}_1 \end{bmatrix} \rightarrow_p \begin{bmatrix} \delta_2 \\ \beta_1 \\ \gamma_1 \end{bmatrix} \quad \text{and} \quad \sqrt{N} \begin{bmatrix} \hat{\delta}_2 - \delta_2 \\ \hat{\beta}_1 - \beta_1 \\ \hat{\gamma}_1 - \gamma_1 \end{bmatrix} \rightarrow_d \text{Normal}(\mathbf{0}, \mathbf{\Omega}) \quad \text{as } N \rightarrow \infty.$$

## Standard Errors

The asymptotic variance matrix  $\mathbf{\Omega}$  is complicated: the usual OLS standard errors from step two are incorrect as they do not account for the estimation of  $\delta_2$  in step one.

## The Big Picture: How does Heckit solve the selection problem?

- ▶ If we regress  $y_{1i}$  on  $\mathbf{x}_{1i}$  for the selected sample, there is an omitted variable.
- ▶ Under the Heckit assumptions, the omitted variable is precisely  $\lambda(\mathbf{x}_i'\boldsymbol{\delta}_2)$ .
- ▶ Hence: a regression of  $y_{1i}$  on  $\mathbf{x}_{1i}$  and  $\lambda(\mathbf{x}_i'\boldsymbol{\delta}_2)$  is correctly specified.

## Why is the second step regression identified?

- ▶ If  $\mathbf{x}_i$  contains some variables that are *not* in  $\mathbf{x}_{1i}$ , we have an **exclusion restriction**: i.e. there are variables that affect participation but not outcomes.
- ▶ Even if there are no exclusion restrictions,  $\lambda$  is nonlinear so  $\lambda(\mathbf{x}'_{1i}\boldsymbol{\delta}_2)$  will not be perfectly co-linear with  $\mathbf{x}_{1i}$ .
- ▶ Without exclusion restrictions identification comes *solely* from nonlinearity in  $\lambda$ .
- ▶ Depending on the values where it is evaluated,  $\lambda$  can be *close* to linear, leading to very imprecise estimates unless you have an exclusion restriction.