Practice Problems: Treatment Effects

Advanced Econometrics 1
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This document contains some practice problems to help you prepare for the upcoming exam. I will discuss these in my revision lecture and post full solutions on the course website `https://treatment-effects.com/syllabus`.

1. Let Y be an outcome, D be a binary treatment, and (Y_0, Y_1) be the associated potential outcomes. For an observed vector of covariates $\boldsymbol{X}$, define the propensity score $p(\boldsymbol{X}) \equiv \mathbb{P}(D = 1|\boldsymbol{X})$ and regression functions $m_d(\boldsymbol{X}) \equiv \mathbb{E}(Y|D = d, \boldsymbol{X})$ for $d = 0, 1$. Suppose that $\boldsymbol{X}$ satisfies both the *selection on observables* and *overlap* assumptions.

   (a) Show that $\mathbb{E}(Y_0) = \mathbb{E}\left[\frac{(1-D)Y}{1-p(\boldsymbol{X})}\right]$ and $\mathbb{E}(Y_1) = \mathbb{E}\left[\frac{DY}{p(\boldsymbol{X})}\right]$. Use this result to propose a way of identifying the ATE of D on Y.

   > **Solution:** This part is taken directly from the lecture notes. See Section 3.5: *Identification by Propensity Score Weighting.*

   (b) Show that $\mathbb{E}(Y_0) = \mathbb{E}[m_0(\boldsymbol{X})]$ and $\mathbb{E}(Y_1) = \mathbb{E}[m_1(\boldsymbol{X})]$. Use this result to propose a way of identifying the ATE of D on Y.

   > **Solution:** This part is taken directly from the lecture notes. See Section 3.4: *Identification by Regression Adjustment.*

   (c) Show that $\mathbb{E}(Y_0) = \mathbb{E}\left[\frac{(1-D)m_0(\boldsymbol{X})}{1-p(\mathbf{X})}\right]$ and $\mathbb{E}(Y_1) = \mathbb{E}\left[\frac{Dm_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right]$. While this result could be used to identify the ATE of D on Y, doing so is much less convenient that using one of the methods from the preceding two parts. Explain why.

   > **Solution:** By iterated expectations and the result of part (b),
   >
   > $$\begin{aligned}\mathbb{E}\left[\frac{(1-D)m_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right] &= \mathbb{E}\left\{\mathbb{E}\left[\left.\frac{(1-D)m_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right|\boldsymbol{X}\right]\right\} = \mathbb{E}\left[\mathbb{E}\left(1-D|\,\boldsymbol{X}\right)\frac{m_0(\boldsymbol{X})}{1-p(\mathbf{X})}\right] \\ &= \mathbb{E}\left[\{1-p(\boldsymbol{X})\}\frac{m_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right] = \mathbb{E}\left[m_0(\mathbf{X})\right] = \mathbb{E}(Y_0).\end{aligned}$$

Analogously,

$$\mathbb{E}\left[\frac{Dm_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right] = \mathbb{E}\left\{\mathbb{E}\left[\left.\frac{Dm_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right|\boldsymbol{X}\right]\right\} = \mathbb{E}\left[\mathbb{E}\left(D|\,\boldsymbol{X}\right)\frac{m_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right]$$
$$= \mathbb{E}\left[p(\boldsymbol{X})\frac{m_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right] = \mathbb{E}[m_1(\boldsymbol{X})] = \mathbb{E}(Y_1).$$

Using this result to identify the ATE of D on Y would require us to specify the correct model *both* the propensity score function $p(\cdot)$ *and* the regression functions $m_0(\cdot), m_1(\cdot)$. This may be quite challenging in practice. In contrast, the approaches from the preceding two parts would only require us to specify *either* the propensity score function *or* the regression functions. If we think we have a good model for $p(\cdot)$, we can apply propensity score weighting without having to specify a model for $m_0(\cdot)$ and $m_1(\cdot)$. Similarly, if we think we have a good model for $m_0(\cdot)$ and $m_1(\cdot)$, we can apply regression adjustment without having to specify a model for $p(\cdot)$.

(d) Let $\pi(\cdot)$ be a function that satisfies $0 < \pi(\boldsymbol{X}) < 1$ but *may or may not* equal the propensity score function $p(\cdot)$ defined above. Show that

$$\text{ATE} = \mathbb{E}\left[\frac{DY}{\pi(\boldsymbol{X})} + \left\{1 - \frac{D}{\pi(\boldsymbol{X})}\right\}m_1(\boldsymbol{X}) - \frac{(1-D)Y}{1-\pi(\boldsymbol{X})} - \left\{1 - \frac{1-D}{1-\pi(\boldsymbol{X})}\right\}m_0(\boldsymbol{X})\right].$$

Solution: Define the shorthand Δ for the expression that we are asked to show equals the ATE. Rearranging, we obtain

$$\begin{aligned}\Delta &= \mathbb{E}\left[m_1(\boldsymbol{X}) - m_0(\boldsymbol{X})\right] \\ &\quad + \mathbb{E}\left[\frac{D\left\{Y - m_1(\boldsymbol{X})\right\}}{\pi(\boldsymbol{X})} - \frac{(1-D)\left\{Y - m_0(\boldsymbol{X})\right\}}{1-\pi(\boldsymbol{X})}\right]\end{aligned}$$

by the linearity of expectation. By part (b), the first expectation on the RHS of the expression for Δ equals $\mathbb{E}(Y_1 - Y_0)$, the ATE. Hence, it suffices to show that the second expectation equals zero. This second expectation is itself a difference, so we can split it into two further expectations by linearity. If we show that both are equal, then the difference is zero, giving our desired result. By iterated expectations, and the definition of the propensity score function $p(\cdot)$, we have

$$\begin{aligned}\mathbb{E}\left[\frac{D\left\{Y - m_1(\boldsymbol{X})\right\}}{\pi(\boldsymbol{X})}\right] &= \mathbb{E}\left(\mathbb{E}\left[\left.\frac{D\left\{Y - m_1(\boldsymbol{X})\right\}}{\pi(\boldsymbol{X})}\right|\boldsymbol{X}\right]\right) \\ &= \mathbb{E}\left\{\left[\mathbb{E}(DY|\boldsymbol{X}) - \mathbb{E}(D|\boldsymbol{X})m_1(\boldsymbol{X})\right]\frac{1}{\pi(\boldsymbol{X})}\right\} \\ &= \mathbb{E}\left\{\left[\mathbb{E}(DY_1|\boldsymbol{X}) - p(\boldsymbol{X})m_1(\boldsymbol{X})\right]\frac{1}{\pi(\boldsymbol{X})}\right\}\end{aligned}$$

since $DY = D(1-D)Y_0 + D^2 Y_1 = DY_1$. But by the *selection on observables* assumption, we have

$$\mathbb{E}(DY_1|\boldsymbol{X}) = \mathbb{E}(Y_1|\boldsymbol{X}, D=1)\mathbb{P}(D=1|\boldsymbol{X}) = \mathbb{E}(Y_1|\boldsymbol{X})p(\boldsymbol{X})$$

applying iterated expectations a second time. Thus,

$$\begin{aligned}
\mathbb{E}\left[\frac{D\{Y - m_1(\boldsymbol{X})\}}{\pi(\boldsymbol{X})}\right] &= \mathbb{E}\left\{[\mathbb{E}(Y_1|\boldsymbol{X})p(\boldsymbol{X}) - p(\boldsymbol{X})m_1(\boldsymbol{X})]\frac{1}{\pi(\boldsymbol{X})}\right\} \\
&= \mathbb{E}\left\{[\mathbb{E}(Y_1|\boldsymbol{X}) - m_1(\boldsymbol{X})]\frac{p(\boldsymbol{X})}{\pi(\boldsymbol{X})}\right\} = 0
\end{aligned}$$

since $m_1(\boldsymbol{X}) = \mathbb{E}(Y_1|\boldsymbol{X})$ by part (b). Analogously,

$$\begin{aligned}
\mathbb{E}\left[\frac{(1-D)\{Y - m_1(\boldsymbol{X})\}}{1-\pi(\boldsymbol{X})}\right] &= \mathbb{E}\left(\mathbb{E}\left[\left.\frac{(1-D)\{Y - m_1(\boldsymbol{X})\}}{1-\pi(\boldsymbol{X})}\right|\boldsymbol{X}\right]\right) \\
&= \mathbb{E}\left\{\mathbb{E}\left[(1-D)\{Y - m_1(\boldsymbol{X})\}|\,\boldsymbol{X}\right]\frac{1}{1-\pi(\boldsymbol{X})}\right\} \\
&= \mathbb{E}\left\{[\mathbb{E}(Y_0|\boldsymbol{X}) - m_0(\boldsymbol{X})]\frac{1-p(\boldsymbol{X})}{1-\pi(\boldsymbol{X})}\right\} = 0.
\end{aligned}$$

(e) Let $\mu_0(\cdot)$ and $\mu_1(\cdot)$ be two functions of $\boldsymbol{X}$ that *may or may not* equal the regression functions $m_0(\cdot)$ and $m_1(\cdot)$ defined above. Show that

$$\text{ATE} = \mathbb{E}\left[\frac{DY}{p(\boldsymbol{X})} + \left\{1 - \frac{D}{p(\boldsymbol{X})}\right\}\mu_1(\boldsymbol{X}) - \frac{(1-D)Y}{1-p(\boldsymbol{X})} - \left\{1 - \frac{1-D}{1-p(\boldsymbol{X})}\right\}\mu_0(\boldsymbol{X})\right].$$

Solution: Define the shorthand Δ for the expression that we are asked to show equals the ATE. Rearranging, we obtain

$$\begin{aligned}
\Delta = \mathbb{E}&\left[\frac{DY}{p(\boldsymbol{X})} - \frac{(1-D)Y}{1-p(\boldsymbol{X})}\right] \\
&+ \mathbb{E}\left[\frac{\{p(\boldsymbol{X}) - D\}\mu_1(\boldsymbol{X})}{p(\boldsymbol{X})} - \frac{\{D - p(\boldsymbol{X})\}\mu_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right]
\end{aligned}$$

by the linearity of expectation. By part (a), the first expectation on the RHS of the expression for Δ equals $\mathbb{E}(Y_1 - Y_0)$, the ATE. Hence it suffices to show that the second expectation equals zero. This second expectation is itself a difference, so we can split it into two further expectations by linearity. If we show that these two are equal, then the difference is zero, giving our desired result. By iterated expectations,

$$\begin{aligned}\mathbb{E}\left[\frac{\{p(\boldsymbol{X})-D\}\,\mu_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right] &= \mathbb{E}\left(\mathbb{E}\left[\left.\frac{\{p(\boldsymbol{X})-D\}\,\mu_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right|\boldsymbol{X}\right]\right)\\ &= \mathbb{E}\left\{\mathbb{E}\left[p(\boldsymbol{X})-D|\boldsymbol{X}\right]\frac{\mu_1(\boldsymbol{X})}{p(\boldsymbol{X})}\right\} = 0\end{aligned}$$

since $\mathbb{E}[D|\boldsymbol{X}] = p(X)$. Analogously,

$$\begin{aligned}\mathbb{E}\left[\frac{\{D-p(\boldsymbol{X})\}\,\mu_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right] &= \mathbb{E}\left(\mathbb{E}\left[\left.\frac{\{D-p(\boldsymbol{X})\}\,\mu_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right|\boldsymbol{X}\right]\right)\\ &= \mathbb{E}\left\{\mathbb{E}\left[D-p(\boldsymbol{X})|\boldsymbol{X}\right]\frac{\mu_0(\boldsymbol{X})}{1-p(\boldsymbol{X})}\right\} = 0.\end{aligned}$$

(f) Using the expressions given in the preceding two parts, propose a method for identifying the ATE of D on Y that allows the propensity score to be mis-specified as long as the regression functions are correctly specified, and vice-versa. (FYI: this property is called *double robustness*.)

Solution: Suppose we specify a model $\pi(\cdot)$ for the unknown true propensity score function $p(\cdot)$ and a model $\mu_0(\cdot), \mu_1(\cdot)$ for the unknown true regression functions $m_0(\cdot), m_1(\cdot)$. If the regression models are correctly specified, then $m_0 = \mu_0$ and $m_1 = \mu_1$. Appealing to the result from part (d), we can identify the ATE *regardless* of whether π coincides with the true propensity score function p. If the propensity score model is correctly specified, then $\pi = p$. Appealing to the result from part (e), we can identify the ATE *regardless* of whether μ_0, μ_1 coincide with the true regression functions m_0, m_1. Notice that the expressions from (d) and (e) take *exactly the same form.* If we use this expression to identify the ATE, we'll get the right answer as long as *either* our propensity score model is correct *or* or our regression models are correct. This is true regardless of whether we know in advance which is correct and which is not. If we're extremely lucky and both are correct, we'll still get the right answer.

2. Consider an experiment in which unemployed workers are randomly offered the chance to participate in a job training program. Let $Z = 1$ if a worker is offered training and $Z = 0$ otherwise Let $D = 1$ if a worker *actually attends* job training and $D = 0$ otherwise. Finally, let $Y = 1$ if a worker is employed 18 months after the experiment and $Y = 0$ otherwise. Suppose that only workers who are offered job training can attend the program, so that $Z = 0$ implies $D = 0$. Further suppose that the *unconfounded type* and *mean exclusion* restrictions hold.

   (a) Does $\mathbb{E}[Y|D = 1] - \mathbb{E}[Y|D = 0]$ identify the ATE of job training on later employment? If not, what does it identify? Explain briefly.

Solution: Although Z is randomly assigned, workers *choose* whether or not to attend job training. This choice could be related to their potential outcomes, in which case a comparison of employment rates between those who attend training and those who do not will not in general identify the ATE. In the special case of *perfect compliance*, i.e. if $D = Z$, we have $\mathbb{E}[Y_d|D] = \mathbb{E}[Y_d|Z] = \mathbb{E}[Y_d]$ for $d = 0, 1$. In this case ATE $= \mathbb{E}[Y|D = 1] - \mathbb{E}[Y|D = 0]$ since Z was randomly assigned. More formally, since $Y = (1 - D)Y_0 + DY_1$

$$\begin{aligned}\mathbb{E}[Y|D = 1] - \mathbb{E}[Y|D = 0] &= \mathbb{E}[Y_1|D = 1] - \mathbb{E}[Y_0|D = 0] \\ &= \mathbb{E}(Y_1 - Y_0|D = 1) + [\mathbb{E}(Y_0|D = 1) - \mathbb{E}(Y_0|D = 0)] \,.\end{aligned}$$

The first term in this expression is the average causal effect of the *treatment on the treated* (TOT). This measures the average effect of job training for the kind of person who would be willing to participate in the program. The TOT does not in general equal the ATE: $\mathbb{E}(Y_1 - Y_0)$. In this example, perhaps low-skilled workers would benefit more from job training than high-skilled workers. If so, low skilled workers could be more likely to take up treatment when offered, in which case the TOT would likely be *higher* than the ATE. This phenomenon is called "selection on gains." The second term is sometimes called "selection bias." In the course notes I call it the "difference in outside options." It quantifies the difference of employment prospects for people who are willing to participate, $D = 1$, compared to those who are not, $D = 0$, *in the absence* of job training. In general, this term will not equal zero. Suppose that low-skilled workers are indeed more likely to take up an offer of job training. If low-skilled workers are also less likely to be employed than high-skilled workers in the absence of treatment, the second term will be *negative*.

(b) Does $\mathbb{E}[Y|Z = 1] - \mathbb{E}[Y|Z = 0]$ identify the ATE of job training on later employment? If not, what does it identify? Explain briefly.

Solution: Because Z is randomly assigned, $\mathbb{E}[Y|Z = 1] - \mathbb{E}[Y|Z = 0]$ identifies the causal effect of *offering* someone job training on later employment. This is called the *intent to treat* (ITT) effect. Under perfect compliance, i.e. $Z = D$, the ITT equals the ATE. More generally, however, the two effects will not be equal. This is because $\mathbb{E}[Y|Z = 1]$ contains a mixture of treated and untreated individuals, i.e. those with $D = 1$ and $D = 0$.

(c) The question statement failed to mention the *no defiers* assumption. As it turns out, this assumption holds automatically. Explain how we know that there are neither defiers nor always-takers in this example.

Solution: A defier is someone who takes treatment if she *isn't offered* and *refuses* the treatment if she is offered: $D(Z) = (1 - Z)$. In this example, however, only those who are offered job training can attend the program, so it is impossible to have $D = 1$ when $Z = 0$. For this reason, the no defiers assumption holds automatically. An always-taker is someone who takes the treatment *regardless* of the treatment offer that she receives, i.e. $D(Z) = 1$. Since it is impossible to have $D = 1$ when $Z = 0$, we know that there are no always-takers.

For the remainder of this question, let $T \in \{n, c\}$ indicate a person's *compliance type*, where n denotes never-taker and c denotes complier. As explained in the preceding part, there are no always-takers in this example. Further let (Y_0, Y_1) denote the potential outcomes from *attending* job training.

(d) Show that $\mathbb{E}[Y|Z = 1] = \mathbb{P}(T = n)\mathbb{E}(Y_0|T = n) + \mathbb{P}(T = c)\mathbb{E}(Y_1|T = c)$.

Solution: Defining $p_1 \equiv \mathbb{P}(D = 1|Z = 1)$, we have

$$\begin{aligned}
\mathbb{E}[Y|Z = 1] &= \mathbb{E}_{D|Z=1}\left[\mathbb{E}(Y|Z = 1, D)\right] \\
&= p_1\mathbb{E}(Y|Z = 1, D = 1) + (1 - p_1)\mathbb{E}(Y|Z = 1, D = 0) \\
&= p_1\mathbb{E}(Y_1|Z = 1, D = 1) + (1 - p_1)\mathbb{E}(Y_0|Z = 1, D = 0)
\end{aligned}$$

by iterated expectations, since $Y = (1 - D)Y_0 + DY_1$. From the preceding part, we know that there are neither never-takers nor always-takers in this example. Hence, anyone with $(Z = 1, D = 1)$ is a complier, and anyone with $(Z = 1, D = 0)$ is a never-taker, and we obtain

$$\begin{aligned}
\mathbb{E}[Y|Z = 1] &= p_1\mathbb{E}(Y_1|Z = 1, D = 1) + (1 - p_1)\mathbb{E}(Y_0|Z = 1, D = 0) \\
&= p_1\mathbb{E}(Y_1|Z = 1, T = c) + (1 - p_1)\mathbb{E}(Y_0|Z = 1, T = n) \\
&= p_1\mathbb{E}(Y_1|T = c) + (1 - p_1)\mathbb{E}(Y_0|T = n)
\end{aligned}$$

where the third equality follows from *mean exclusion.* It remains to show that $p_1 = \mathbb{P}(T = c)$ and $1 - p_1 = \mathbb{P}(T = n)$. By the law of total probability,

$$\begin{aligned}
p_1 \equiv \mathbb{P}(D = 1|Z = 1) &= \sum_{t \in \{c,n\}} \mathbb{P}(D = 1|Z = 1, T = t)\mathbb{P}(T = t) \\
&= \mathbb{P}(D = 1|Z = 1, T = c)\mathbb{P}(T = c) + \mathbb{P}(D = 1|Z = 1, T = n)\mathbb{P}(T = n) \\
&= 1 \times \mathbb{P}(T = c) + 0 \times \mathbb{P}(T = n) = \mathbb{P}(T = c).
\end{aligned}$$

Because the only two compliance types in this example are c and n, we have $\mathbb{P}(T = n) = 1 - \mathbb{P}(T = c)$, so the desired result follows.

(e) Show that $\mathbb{E}(Y|Z = 0) = \mathbb{P}(T = n)\mathbb{E}(Y_0|T = n) + \mathbb{P}(T = c)\mathbb{E}(Y_0|T = c)$

Solution: Defining $p_0 \equiv \mathbb{P}(D = 1|Z = 0)$, we have

$$\begin{aligned}
\mathbb{E}[Y|Z = 0] &= \mathbb{E}_{D|Z=0}[\mathbb{E}(Y|Z = 1, D)] \\
&= (1 - p_0)\mathbb{E}(Y|Z = 0, D = 1) + p_0\mathbb{E}(Y|Z = 0, D = 0) \\
&= (1 - p_0)\mathbb{E}(Y_1|Z = 0, D = 1) + p_0\mathbb{E}(Y_0|Z = 0, D = 0).
\end{aligned}$$

by iterated expectations and $Y = (1-D)Y_0 + DY_1$. But since $Z = 0$ implies $D = 0$, we have $p_0 = 1$. By iterated expectations

$$\begin{aligned}
\mathbb{E}[Y|Z = 0] &= (1 - p_0)\mathbb{E}(Y_1|Z = 0, D = 1) + p_0\mathbb{E}(Y_0|Z = 0, D = 0) \\
&= \mathbb{E}(Y_0|Z = 0, D = 0) \\
&= \mathbb{E}_{T|Z=0,D=0}\left[\mathbb{E}(Y_0|T, Z = 0, D = 0)\right] \\
&= \sum_{t\in\{c,n\}} \mathbb{E}(Y_0|T = t, Z = 0, D = 0)\mathbb{P}(T = t|Z = 0, D = 0) \\
&= \sum_{t\in\{c,n\}} \mathbb{E}(Y_0|T = t, Z = 0)\mathbb{P}(T = t|Z = 0, D = 0) \\
&= \sum_{t\in\{c,n\}} \mathbb{E}(Y_0|T = t)\mathbb{P}(T = t|Z = 0, D = 0)
\end{aligned}$$

where the second-to-last equality uses the fact that conditioning on $(T = t, Z = 0, D = 0)$ is equivalent to conditioning on $(T = t, Z = 0)$ given that $Z = 0$ implies $D = 0$ in this example, and the final equality relies on the *mean exclusion* assumption. Now,

$$\begin{aligned}
\mathbb{P}(T = t|Z = 0, D = 0) &= \frac{\mathbb{P}(D = 0|T = t, Z = 0)\mathbb{P}(T = t|Z = 0)}{\mathbb{P}(D = 0|Z = 0)} \\
&= \mathbb{P}(T = t|Z = 0) \\
&= \mathbb{P}(T = t)
\end{aligned}$$

where the first equality follows by Bayes' rule, the second from the fact that $Z = 0$ implies $D = 0$, and the third follows from the *unconfounded type* assumption. Substituting into the expression for $\mathbb{E}(Y|Z = 0)$ from above,

$$\begin{aligned}
\mathbb{E}[Y|Z = 0] &= \sum_{t\in\{c,n\}} \mathbb{E}(Y_0|T = t)\mathbb{P}(T = t) \\
&= \mathbb{P}(T = n)\mathbb{E}(Y_0|T = n) + \mathbb{P}(T = c)\mathbb{E}(Y_0|T = c).
\end{aligned}$$

(f) Suppose that $\mathbb{P}(T = c) > 0$. Based on the results of the preceding two parts, what causal effect does the Wald estimand identify in this example? Explain the economic meaning of this effect in the present context.

Solution: The Wald estimand is given by

$$\mathcal{W} \equiv \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[D|Z=1] - \mathbb{E}[D|Z=0]} = \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[D|Z=1]}$$

since in this example $\mathbb{E}[D|Z=0] = \mathbb{P}(D=1|Z=0) = 0$. The numerator is the *intent to treat effect* (ITT). Using the results from above,

$$\begin{aligned} \text{ITT} &= [\mathbb{P}(T=n)\mathbb{E}(Y_0|T=n) + \mathbb{P}(T=c)\mathbb{E}(Y_1|T=c)] \\ &\quad - [\mathbb{P}(T=n)\mathbb{E}(Y_0|T=n) + \mathbb{P}(T=c)\mathbb{E}(Y_0|T=c)] \\ &= \mathbb{P}(T=c)\left[\mathbb{E}(Y_1|T=c) - \mathbb{E}(Y_0|T=c)\right]. \end{aligned}$$

And since $\mathbb{E}(D|Z=1) = \mathbb{P}(D=1|Z=1) = \mathbb{P}(T=c)$ as shown in the solution to part (d), we obtain

$$\mathcal{W} = \frac{\text{ITT}}{\mathbb{P}(T=c)} = \mathbb{E}(Y_1 - Y_0|T=c).$$

So $\mathcal{W}$ identifies the average treatment effect *compliers*, i.e. the local average treatment effect (LATE). Given that $Z=0$ implies $D=0$, however, it is possible to say more in this example. By iterated expectations,

$$\begin{aligned} \mathbb{E}(Y_d|D=1) &= \mathbb{E}_{(T,Z)|D=1}\left[\mathbb{E}(Y_d|D=1,Z,T)\right] \\ &= \sum_{z\in\{0,1\}} \sum_{t\in\{c,n\}} \mathbb{E}(Y_d|D=1,Z=z,T=t)\mathbb{P}(T=t,Z=z|D=1) \end{aligned}$$

and by Bayes' rule

$$\mathbb{P}(T=t,Z=z|D=1) = \frac{\mathbb{P}(D=1|T=t,Z=z)\mathbb{P}(T=t,Z=z)}{\mathbb{P}(D=1)}.$$

Since $\mathbb{P}(D=1|T=t,Z=0)=0$, it follows that $\mathbb{P}(T=t,Z=0|D=1)=0$ and thus

$$\mathbb{E}(Y_d|D=1) = \sum_{t\in\{c,n\}} \mathbb{E}(Y_d|D=1,Z=1,T=t)\mathbb{P}(T=t,Z=1|D=1).$$

Now, by the multiplication rule for conditional probabilities

$$\begin{aligned} \mathbb{P}(T=t,Z=1|D=1) &= \mathbb{P}(T=t|Z=1,D=1)\mathbb{P}(Z=1|D=1) \\ &= \mathbb{P}(T=t|Z=1,D=1) \times 1 \\ &= \begin{cases} 1, & T=c \\ 0, & T=n \end{cases} \end{aligned}$$

since $D=1$ implies $Z=1$, and anyone with $(Z=1,D=1)$ must be a complier in this example. Substituting into our expression for $\mathbb{E}(Y_d|D=1)$,

$$\begin{aligned} \mathbb{E}(Y_d|D=1) &= \mathbb{E}(Y_d|D=1,Z=1,T=c) \\ &= \mathbb{E}(Y_d|Z=1,T=c) = \mathbb{E}(Y_d|T=c) \end{aligned}$$

by the *mean exclusion* assumption. Therefore,

$$\mathbb{E}(Y_1 - Y_0|T = c) = \mathbb{E}(Y_1 - Y_0|D = 1).$$

In other words, in this example the LATE equals the TOT. Thus, the Wald estimand identifies the causal effect of job training on the subpopulation of unemployed workers who are *willing* to participate in the training program. This result is general: whenever $Z = 0$ implies $D = 0$, a situation called *one-sided non-compliance*, the LATE and TOT coincide. That is because the compliers and the treated are drawn from the *same* subpopulation in this case: anyone who is treated must be a complier, although not all compliers are treated.