Periodic solutions of Euler-Lagrange equations with "sublinear nonlinearity" in an Orlicz-Sobolev space setting

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Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$, of hamiltonian systems with a potential function F satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t)$, with $b_1, b_2 \in L^1$ and for certain N-functions Φ_0.

1 Introduction

This paper deal of system equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t \in (0,T) \\ u(0)-u(T) = u'(0)-u'(T) = 0 \end{cases}, \tag{1}$$

where $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$, and the the unknown function $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous. In other words, we are interesting in to find *periodic weak*

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solutions of *Euler-Lagrange system of ordinary equations*. This topic was addressed deeply for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) = \frac{|y|^p}{p} + F(t,x), \tag{2}$$

for $1 < p < \infty$. For example, the classical book [1] approach mainly with problem (1) through various methods: direct, dual action, minimax, etc. In this context it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respecto to x for a.e. $t \in [0,T]$ and the following conditions

(C) F and its gradient, with respect to $x \in \mathbb{R}^d$, ∇F are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{3}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

The Lagrange function (2) for arbitrary $1 < p < infty$ was considered in several papers. For example in [2], this problem, whose results extend in the present paper.

$$\begin{cases} \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{4}$$

where $T > 0$, $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous and Φ is a differentiable N-function (see section Preliminaries for definitions). Furthermore, the *potential* $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

We will call the differential operator

$$L_\Phi[u] = \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right)$$

the Φ-*laplacian operator*. If $\Phi(x) = |x|^p/p$, $1 < p < \infty$, L_Φ is the well known p-laplacian operator. In this case, we have the *Dirichlet problem* for the p-laplacian

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{5}$$

The problem (4) comes from a variational one, that is, the equation in (4) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \Phi(|u'(t)|) + F(t,u(t))\,dt. \tag{6}$$

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The main result of this article is Theorem 3.9 which establishes conditions to guarantee existence of solutions of the problem (4) by minimization of functional (43). We point out that the hypothesis of Theorem 3.9 are generalizations of those given in [3, 4, 5, 6] about the sublinearity.

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [7, 8, 9]. For Orlicz spaces of vector valued functions, see [10] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is convex and satisfies that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0} \frac{\Phi(t)}{t} = 0$$

In addition, in this paper we assume that Φ is differentiable and we call φ the derivative of Φ. On these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism whose inverse is ψ. We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then, Ψ is an N-function which is called the *complementary function* of Φ.

There exist several order and equivalence relations between N-functions (see [9, Sec. 2.2]). Following [9, Def. 1, p. 15-16] we say that the N-function Φ_2 is *stronger* than the N-function Φ_1, $\Phi_1 \prec \Phi_2$ in symbols, if there exists $a > 0$ and $x_0 \geqslant 0$ such that

$$\Phi_1(x) \leqslant \Phi_2(ax), \quad x \geqslant x_0. \tag{7}$$

The N-functions Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) when $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$. We said that Φ_2 is *essentially stronger* than Φ_1 ($\Phi_1 \ll \Phi_2$) if and only if for every $a > 0$ there exists $x_0 = x_0(a) \geqslant 0$ such that (7) holds. Finally we said that Φ_2 is *completely stronger* than Φ_1 ($\Phi_1 \lessdot \Phi_2$) if and only if for every $a > 0$ there exists $K = K(a) > 0$ and $x_0 = x_0(a) \geqslant 0$ such that

$$\Phi_1(x) \leqslant K\Phi_2(ax), \quad x \geqslant x_0. \tag{8}$$

We also say that a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the *Δ_2-condition*, denoted by $\eta \in \Delta_2$, if there exist constants $K > 0$ and $x_0 \geqslant 0$ such that

$$\eta(2x) \leqslant K\eta(x), \tag{9}$$

for every $x \geqslant x_0$. We note that η is Δ_2 if and only if $\eta \lessdot \eta$. If $x_0 = 0$, the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the *Δ_2-condition globally* ($\eta \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ is defined by

$$C^\Phi_d := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{10}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{11}$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup\left\{\int_0^T u \cdot v \, dt \middle| \rho_\Psi(v) \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [8, Thm. 10.5] and [11]). For every $u \in L^\Phi$,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k}\{1 + \rho_\Phi(ku)\}. \tag{12}$$

In particular

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k}\{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \tag{13}$$

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty_d([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [8, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant \|u\|_{L^\Phi}\|v\|_{L^\Psi}. \tag{14}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset [L^\Phi]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \tag{15}$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi = [L^\Phi]^*$ will not hold. In general, it is true that $[E^\Phi]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* W^1L^Φ (see [7]) by

$$W^1L^\Phi := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi\}.$$

W^1L^Φ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{16}$$

Moreover, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{17}$$

For a function $u \in L^1_d([0,T])$, we write $u = \overline{u} + \widetilde{u}$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty_d \hookrightarrow L^\Phi \hookrightarrow L^1_d$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $u : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t-s|). \tag{18}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (18) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [12, 13, 14, 15, 16]. The next simple lemma, whose proof can be found in [17], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T], \mathbb{R}^d)$ and for every $u \in W^1L^\Phi$*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|) \qquad \textit{(Morrey's inequality)}, \tag{19}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality)}. \tag{20}$$

2. *For every $u \in W^1L^\Phi$ we have $\widetilde{u} \in L^\infty_d$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \qquad \textit{(Sobolev-Wirtinger's inequality)}. \tag{21}$$

3 Lagrangians satisfying sublinear nonlinearity type conditions

Lemma 3.1. *E^Φ is weak* closed in L^Φ.*

Proof. From [9, Thm. 7, p. 110] we have that $L^\Phi = \left[E^\Psi\right]^*$. Then, L^Φ is a dual and therefore we are allowed to speak about the weak* topology of L^Φ. Besides, E^Φ is separable (see [9, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi \,|\, \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [9, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because if $u_n \in S$ and $u_n \xrightarrow{*} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \xrightarrow{*} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ is separable and $L^\Phi = \left[E^\Psi\right]^*$, the ball of L^Φ $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ is weak* metrizable (see [18, Thm. 5.1, p. 138]). Thus, S is closed with respect to the weak* topology. Now, by Krein-Smulian theorem, [18, Cor. 12.6, p. 165] implies that E^Φ is weak* closed. □

The following result is analogous to some lemmata in $W^{1,p}$, see [19].

Lemma 3.2. *If $\|u\|_{W^1L^\Phi} \to \infty$, then $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u} + \tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{22}$$

It is known that $L^\infty_d \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L^\infty_d$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1 \|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2 \|u'\|_{L^\Phi}. \tag{23}$$

Therefore, from (22), (23) and (16), we get

$$\|u\|_{W^1L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. □

Lemma 3.3. *Let Φ a N-function and φ the right continuous derivative of Φ. Then $\Phi \in \Delta_2$ ($\Phi \in \Delta_2$ globally) iff $\varphi \in \Delta_2$ ($\varphi \in \Delta_2$ globally).*

Proof. It is consequence of $[20, Th.11.7]$ and [20, Remark 5, p. 87].

□

Lemma 3.4. *Let Ψ be a N-function satisfying the Δ_2 condition. Then there exists a N-function Ψ^* such that Ψ^* is Δ_2-globally, $\Psi \leqslant \Psi^*$ and for every $a > 1$ there exists $x_0 = x_0(a) \geqslant 0$ such that $\Psi^*(x) \leqslant a\Psi(x)$, for every $x \geqslant x_0$. In particular every Δ_2 N-function is equivalent to a Δ_2-globally N-function.*

Proof. We can assume that the Δ_2 condition for Ψ fails near to 0. Consequently, from Lemma 3.3, we have that the right continuous derivative ψ of Ψ is not Δ_2 near to 0. Therefore, we obtain a sequence x_n, $n = 1, 2, \ldots$ of positive numbers with $x_n \to 0$, and

$$2x_{n+1} < x_n < 2x_n \quad \text{and} \quad \psi(2x_n) > 2\psi(x_n). \tag{24}$$

We define ψ^* inductively on n on the interval $[2x_n, +\infty)$ of the following way. We define $\psi^*(x) = \psi(x)$ when $x \in [2x_1, +\infty)$. Suppose ψ^* defined in $[2x_n, +\infty)$. We define ψ^* in $[2x_{n+1}, 2x_n)$ by

$$\psi^*(x) = \begin{cases} \max\left\{\psi(x), \frac{\psi^*(2x_n)}{2x_n}(x - x_n) + \frac{\psi^*(2x_n)}{2}\right\}, & \text{if } x_n \leqslant x < 2x_n \\ \frac{\psi^*(2x_n)}{2} & \text{if } 2x_{n+1} \leqslant x < x_n \end{cases}$$

Moreover we define $\psi^*(0) = 0$. Next, we will use induction again for to prove that

1. $\psi^*(x_n) = \frac{1}{2}\psi^*(2x_n)$,
2. ψ^* is non-decreasing $[2x_n, +\infty)$,
3. $\psi \leqslant \psi^*$ in $[2x_n, +\infty)$

We suppose $n = 1$. Then items 2 and 3 are obvious. From (24) we have

$$\psi(x_1) < \frac{1}{2}\psi(2x_1) = \frac{1}{2}\psi^*(2x_1).$$

This inequlity implies 1.

Clearly ψ^* is non-decreasing on each interval $[2x_{n+1}, x_n)$ and $[x_n, 2x_n)$. We note that since ψ is right continuous, ψ^* is contiuous at x_n. Therefore ψ^* is non-decreasing on $[2x_{n+1}, 2x_n)$. Suppose $x \in [2x_{n+1}, 2x_n)$ and $y \geqslant 2x_n$. From the definition of ψ^*, inductive hypothesis item 3 and item 2 we obtain

$$\psi^*(x) \leqslant \max\{\psi(2x_n), \psi^*(2x_n)\} = \psi^*(2x_n) \leqslant \psi^*(y).$$

This proves item 2 in the interval $[2x_{n+1}, +\infty)$. Inequality in item 3 holds by inductive hypothesis in $[2x_n, +\infty)$ and is obvious for $x \in [x_n, 2x_n)$. If $x \in [2x_{n+1}, x_n)$, then $\psi(x) \leqslant \psi(x_n) \leqslant \psi^*(x_n) = \psi^*(x)$. This proves 3 in the interval $[2x_{n+1}, +\infty)$

Now, using (24) and the alreadyr proved items 3 for $n + 1$ we deduce

$$\psi(x_{n+1}) < \frac{1}{2}\psi(2x_{n+1}) \leqslant \frac{1}{2}\psi^*(2x_{n+1})$$

Then

$$\psi^*(x_{n+1}) = \max\left\{\psi(x_{n+1}), \frac{1}{2}\psi^*(2x_{n+1})\right\} = \frac{1}{2}\psi^*(2x_{n+1}),$$

i.e. we have proved item 1.

We note that

$$\psi^*(x_{n+1}) = \frac{1}{2}\psi^*(2x_{n+1}) \leqslant \psi^*(x_n).$$

Consequently $\psi(x) \to 0$ when $x \to 0$. Therefore ψ^* is right continuous at 0 and, in fact, right continuous on $[0, +\infty)$. Moreover, since $\psi(x) = \psi^*(x)$ for $x \geqslant 2x_1$ being ψ the right continuous derivative of a N-function, $\psi^*(x) \to +\infty$ when $x \to +\infty$. In this way

$$\Psi^*(x) := \int_0^x \psi^*(t)dt.$$

define a N function.

Let see that ψ^* is Δ_2 globally. For it is sufficient to prove that ψ^* satisfies the Δ_2 conditions near of 0. For this end, suppose $x \leqslant x_1$ and take $n \in \mathbb{N}$ such that $x_{n+1} \leqslant x \leqslant x_n$. Then

$$\psi^*(2x) \leqslant \psi^*(2x_n) = 2\psi^*(2x_{n+1}) = 4\psi^*(x_{n+1}) \leqslant 4\psi^*(x).$$

Consequently Ψ^* is δ_2 globally and $\Psi \leqslant \psi^*$.

It remains to show the ineaulity $\Psi^*(x) \leqslant a\Psi(x)$, for every $a > 1$ and sufficiently large x. We take x_0 sufficiently large for that

$$\frac{1}{a-1}\int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x_0).$$

Therefore, if $x > \max\{x_0, 2x_1\}$ then

$$\Psi^*(x) = \Psi(x) + \int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x) + (a-1)\Psi(x) = a\Psi(x).$$

The last affirmation of the lemma is consequence of that $\Psi(ax) > a\Psi(x)$ when $a > 1$.

□

The following lemma is essentially known, basically it is consequence of the fact that $\Psi \in \Delta_2$ if and only if $\Psi \prec \Psi$, of [9, Prop. 4, p. 20] and [9, Cor. 10, p. 30]. However we prefer included an alternative proof, because we do not see clear that the previous results of [9] contemplate the case of N-functions satisfying the Δ_2 condition globally.

Lemma 3.5. *Let Φ, Ψ be complementary functions. The next statements are equivalent:*

1. $\Psi \in \Delta_2$ *(Δ_2 globally).*
2. *There exists an N-function Φ^* such that*

$$\Phi(rs) \geqslant \Phi^*(r)\Phi(s) \text{ for every } r \geqslant 1, s \geqslant 1\ (s \geqslant 0). \tag{25}$$

Proof. 1)⇒2) By virtue of the Δ_2-condition on Ψ, [20, Thm. 11.7] and [20, Cor. 11.6] (see also [21, Eq. (2.8)]), we get constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geqslant Kr^\nu\Phi(s), \tag{26}$$

for any $1 < \nu < \alpha_\Phi$, $s \geqslant 0$ and $r > 1$. This proves (25) with $\Phi^*(r) = kr^\nu$, which is an N-function.

2)⇒1) Next, we follow [9, p. 32, Prop. 13] and [9, p. 29, Prop. 9]. Assume that

$$\Phi^*(r)\Phi(s) \leqslant \Phi(rs) \quad r > 1,\ s \geqslant 0.$$

Let $u = \Phi^*(r) \geqslant \Phi^*(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [9, p. 13, Prop. 1] and (25), we have for $u \geqslant \Phi^*(1)$ and $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant {\Phi^*}^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{{\Psi^*}^{-1}(u)\Psi^{-1}(v)},$$

then

$${\Psi^*}^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi^*(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi^*(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, {\Psi^*}^{-1}(\Phi^*(1))\}$ we get that $\Psi \in \Delta_2$ globally. $\square$

Remark 1. We note that if Φ^* satisfies (25) then $\Phi^* \prec \Phi$.

Remark 2. The N-function $\Phi(x) = |x|^p$, $p > 1$ has a complementary which is Δ_2.

The following lemma generalizes [17, Lemma 5.2].

Lemma 3.6. *Let Φ, Ψ be complementary N-functions with $\Psi \in \Delta_2$. Then there exists a N-function Φ^*, with $\Phi^* \prec \Phi$, such that for every $\Phi_0 \ll \Phi^*$ and $k > 0$*

$$\lim_{\|u\|_{L^\Phi} \to \infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} = \infty, \tag{27}$$

Reciprocally, if (27) *holds for some N-function Φ_0, then $\Psi \in \Delta_2$.*

Proof. At first we assume that Φ is Δ_2-globally. Let Φ^* be a N function satisfying (25). By the inequality (13), for $r > 1$ we have

$$\int_0^T \Phi(|u|)\,dt \geqslant \Phi^*(r)\int_0^T \Phi(r^{-1}|u|)\,dt \geqslant \Phi^*(r)\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$, as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$. From [9, Thm. 2 (b)(v), p. 16] we get

$$\lim_{\|u\|_{L^\Phi} \to \infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi} \to \infty} \frac{\Phi^*\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(k\|u\|_{L^\Phi})} = \infty.$$

If Ψ is Δ_2, but it is not Δ_2-globally, we use Lemma 3.4. Then there exists a Δ_2-globally N-function Ψ_1, with $\Psi_1 \sim \Psi \leqslant \Psi_1$. Let Φ_1 the complementary function of Ψ_1. Then $\Phi \sim \Phi_1 \leqslant \Phi$ (see [8, Th. 3.1]) and $\|\cdot\|_{L^\Phi}$ and $\|\cdot\|_{L^{\Phi_1}}$ are equivalents norms (see [8, Th. 13.2 and Th. 13.3]). By the previously proved, there exists Φ_0 satisfying (27) with $\Phi = \Phi_1$. Let $C > 0$ be such that $\|\cdot\|_{L^\Phi} \leqslant C\|\cdot\|_{L^{\Phi_1}}$. Then

$$\frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} \geqslant \frac{\int_0^T \Phi_1(|u|)dt}{\Phi_0(kC\|u\|_{L^{\Phi_1}})} = +\infty.$$

Finally, if Φ_0 is an N-function, then $\Phi_0(x) \geqslant \alpha|x|$ for α small enough and $|x| > 1$. Therefore (27) holds for $\Phi_0(x) = |x|$, then [17, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞. $\square$

Remark 3. We point out that this lemma can be applied to more cases than [17, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [8, p. 16] and [8, Sec. 7] for the definition and properties of principal part), then (27) holds for Φ_0. However, $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

Definition 3.7. *We define the functionals* $J_{C,\varphi} : L^\Phi \to (-\infty, +\infty]$ *and* $H_{C,\varphi} : \mathbb{R}^n \to \mathbb{R}$*, where* $C > 0$ *and* $\varphi : [0, +\infty) \to [0, +\infty)$ *is an by*

$$J_{C,\varphi}(u) := \rho_\Phi(u) - C\varphi(\|u\|_{L^\Phi}), \tag{28}$$

and

$$H_{C,\varphi}(x) := \int_0^T F(t,x)dt - C\varphi(2|x|), \tag{29}$$

respectively.

In [22] and [6] the authors considered, for the p-laplacian case, potentials F satisfying the inequality

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^\alpha + b_2(t), \tag{30}$$

where $b_1, b_2 \in L_1^1$ and $\alpha < p$. Thus, they called F a sublinear nonlinearity. In this paper, we will consider bounds on ∇F of a more general type.

Definition 3.8. *Let* Φ_0 *be a differentiable* N*-function. We say that* $G : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ *satisfies a* Φ_0-grow condition *if*

$$|G(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t), \tag{31}$$

with $b_1, b_2 \in L^1([0,T], \mathbb{R})$.

Theorem 3.9. *Let* Φ *be an* N*-function whose complementary function* Ψ *satisfies the* Δ_2 *condition globally. Assume that the* N*-function* Φ^* *satisfies the Lemma 3.6,* F *satisfies (C) and (A), and* ∇F *satisfies a* Φ_0*-grow condition for some* N*-function* Φ_0 *such that* $\Phi_0 \ll \Phi^*$*. Furthermore, we suppose that*

$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Psi_2(\Phi_0'(2|x|))} = +\infty. \tag{32}$$

for some N*-function* Ψ_2 *whose complementary function* Φ_2 *satisfies* $\Phi_0 \ll \Phi_2 \ll \Phi^*$*. Then, the problem* (4) *has at least a solution which minimizes the action integral* I *on* $W^1E_T^\Phi$*.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (31), we have

$$\begin{aligned}
&\left|\int_0^T F(t,u) - F(t,\overline{u})\,dt\right| = \left|\int_0^T \int_0^1 \nabla F(t, \overline{u} + s\tilde{u}(t)) \cdot \tilde{u}(t)\,ds\,dt\right| \\
&\leqslant \int_0^T \int_0^1 b_1(t)\Phi_0'(|\overline{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt + \int_0^T \int_0^1 b_2(t)|\tilde{u}(t)|\,ds\,dt \\
&=: I_1 + I_2.
\end{aligned} \tag{33}$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty} \leqslant C_1\|u'\|_{L^\Phi}, \tag{34}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

We note that, since Φ_0' is increasing function and $\Phi_0'(x) \geqslant 0$ for $x \geqslant 0$, then $\Phi_0'(a+b) \leqslant \Phi_0'(2a) + \Phi_0'(2b)$ for every $a, b \geqslant 0$. In this way, we have

$$\Phi_0'(|\overline{u} + s\tilde{u}(t)|) \leqslant \Phi_0'(2|\overline{u}|) + \Phi_0'(2\|\tilde{u}\|_{L^\infty}), \tag{35}$$

for every $s \in [0,1]$. Now, inequality (35), Hölder's and Sobolev-Wirtinger's inequalities imply that

$$\begin{aligned} I_1 &\leqslant \Phi_0'(2|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} + \Phi_0'(2\|\tilde{u}\|_{L^\infty})\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} \\ &\leqslant C_2\left\{\Phi_0'(2|\overline{u}|)\|u'\|_{L^\Phi} + \Phi_0'(C_3\|u'\|_{L^\Phi})\|u'\|_{L^\Phi}\right\}, \end{aligned} \tag{36}$$

where $C_2 = C_2(T, \|b_1\|_{L^1})$ and $C_3 = C_3(T)$. Next, by Young's inequality with complementary functions Φ_2 and Ψ_2

$$\Phi_0'(2|\overline{u}|)\|u'\|_{L^\Phi} \leqslant \Psi_2(\Phi_0'(2|\overline{u}|)) + \Phi_2(\|u'\|_{L^\Phi}). \tag{37}$$

We have that any N-function Φ_0 satisfies the inequality $x\Phi_0'(x) \leqslant \Phi_0(2x)$ (see [9, p. 17]). Moreover, since $\Phi_0 \lll \Phi_2$ there exists $x_0 = x_0(\Phi_0, \Phi_2, T) \geqslant 0$ such that $\Phi_0(2C_3x) \leqslant \Phi_2(x)$, for every $x \geqslant x_0$. Therefore, $\Phi_0(2C_3x) \leqslant \Phi_2(x) + C_4$, with $C_4 = \Phi_0(2x_0)$. The previous observations imply

$$\Phi_0'(C_3\|u'\|_{L^\Phi})\|u'\|_{L^\Phi} \leqslant C_3^{-1}(\Phi_2(\|u'\|_{L^\Phi}) + C_4). \tag{38}$$

From (36), (37), (38) and (34), we have

$$I_1 + I_2 \leqslant C_5\left\{\Psi_2(\Phi_0'(2|\overline{u}|)) + \Phi_2(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi} + 1\right\} \tag{39}$$

with C_5 depending on $\Phi_0, \Phi_2, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$.

In the subsequent estimates, we use (33), (39), we get

$$\begin{aligned} I(u) &= \rho_\Phi(u') + \int_0^T F(t,u)\,dt \\ &= \rho_\Phi(u') + \int_0^T [F(t,u) - F(t,\overline{u})]\,dt + \int_0^T F(t,\overline{u})\,dt \\ &\geqslant \rho_\Phi(u') - C_5\Phi_2(\|u'\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_5\Psi_2(\Phi_0'(2|\overline{u}|)) - C_5 \\ &\geqslant \rho_\Phi(u') - C_5\Phi_2(\|u'\|_{L^\Phi}) + H_{C_5,\Psi_2\circ\Phi_0}(\overline{u}) - C_5 \\ &= J_{C_5,\Phi_0}(u') + H_{C_5,\Psi_2\circ\Phi_0}(\overline{u}) - C_5. \end{aligned} \tag{40}$$

Let u_n be a sequence in W^1L^Φ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n, $I(u_n)$ is upper bounded, i.e. there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 3.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence is necessary, still denoted u_n, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.6 implies that the functional $J_{C_5,\Phi_0}(u')$ is coercive; and, by (32), the functional $H_{C_5,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_5,\Phi_0}(u_n') \to \infty$ or $H_{C_5,\Phi_0}(\overline{u}_n) \to \infty$. From the condition (A) on F, we have that on a bounded set the functional $H_{C_5,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_5,\Phi_0}(u_n') \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u)|u \in W^1E_T^\Phi\}$. Since $I(u_n)$, $n = 1, 2, \ldots$, is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in W^1E^Φ. Hence, by virtue of [17, Cor. 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T-periodic continuous (therefore in E_T^Φ) function u. As $u_n' \in E^\Phi$ is a norm bounded sequence in L^Φ, there exists a subsequence (again denoted by u_n') such that u_n' converges to a function $v \in L^\Phi$ in the weak* topology of L^Φ. Since E^Φ is weak* closed, by Lemma 3.1, $v \in E^\Phi$. From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \xi' \cdot u\,dt = \lim_{n\to\infty} \int_0^T \xi' \cdot u_n\,dt = -\lim_{n\to\infty} \int_0^T \xi \cdot u_n'\,dt = -\int_0^T \xi \cdot v\,dt$$

for every T-periodic function $\xi \in C^\infty([0,T], \mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [1, p. 6]) and $u \in W^1E_T^\Phi$.

Now, taking into account the relations $\left[L^1\right]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that u_n' converges to u' in the weak topology of L^1. Consequently, from the semicontinuity of I (see [17, Lemma 6.1]) we get

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{v \in W^1E_T^\Phi} I(v).$$

Hence $u \in W^1E_T^\Phi$ is a minimun and, since I is Gâteaux differentiable on W^1E^Φ (see [17, Thm. 3.2]), therefore $I'(u) \in (W^1E_T^\Phi)^\perp$. Thus,

$$\int_0^T \frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t) \cdot v'(t)dt = -\int_0^T \nabla F(t, u(t)) \cdot v(t)dt,$$

for every $v \in W^1E_T^\Phi$.

From [17, Lemma 2.4] we have $u'(t)\Phi'(|u'(t)|)/|u'(t)| \in L^\Psi([0,T], \mathbb{R}^n) \hookrightarrow L^1([0,T], \mathbb{R}^n)$; and, from condition (A) and the fact that $u \in L^\infty$, it follows that $\nabla F(t, u(t)) \in L^1([0,T], \mathbb{R}^n)$. Consequently, from [1, p. 6] we obtain that the differential equations in (4) are verified and $u'(0)\Phi'(|u'(0)|)/|u'(0)| = u'(T)\Phi'(|u'(T)|)/|u'(T)|$ holds. Thus $u'(0) = u'(T)$. □

In the article [17] we have considered more general lagrangian functions $\mathcal{L} : [0,T]\times$

$\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the conditions

$$|\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right) \qquad \text{(A1)},$$

$$|D_x\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right) \qquad \text{(A2)},$$

$$|D_y\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(c(t) + \varphi\left(\frac{|y|}{\lambda} + f(t)\right)\right) \qquad \text{(A3)},$$

where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function, φ is the right continuous derivative of Φ, $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$.

In [17, Thm. 6.2] we obtained existence results of solutions of the problem $\inf\{I(u) : u \in W^1E^\Phi\}$ where the action integral was given by $I(u) = \int_0^T \mathcal{L}(t, u(t), u'(t))\, dt$.

Unfortunately, we made a mistake in the proof of [17, Thm. 4.1], because the minimum of the functional might be out of the domain of differentiability.

Now, we can fix the aforementioned mistake by the process of minimization developed in the last part of the proof of Theorem 3.9.

Furthermore, based on Theorem 3.9, we can get another existence result under different hypothesis than those assumed in [17, Thm. 6.2], as follows.

Corollary 3.10. *Let Φ, Ψ be complementary N-functions with $\Psi \in \Delta_2$ globally. Suppose that $\mathcal{L}(t,x,y)$ is a differentiable Carathéodory function such that (A1),(A2) and (A3) hold. Assume also that $\mathcal{L}(t,x,y)$ is strictly convex at y and*

$$\mathcal{L}(t,x,y) \geqslant \Phi(|y|) + F(t,x), \tag{41}$$

where the function $F(t,x)$ satisfies conditions (A) and (C). Then, the problem

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{42}$$

has at least one solution $u : [0,T] \to \mathbb{R}^d$ absolutely continuous, which minimizes the action integral

$$I(u) = \int_0^T \mathcal{L}(t,u(t),u'(t))\, dt. \tag{43}$$

Proof. In Theorem 3.9 we have just seen that the action integral $\int_0^T \Phi(|u'(t)|) + F(t,u(t))\, dt$ is coercive, then the functional $\mathcal{L}$ does so.

Let $\{u_n\} \subset W^1E^\Phi_T$ be a minimizing sequence for the problem $\inf\{I(u) | u \in W^1E^\Phi_T\}$. Now, $\{u_n\}$ is bounded due to the coercivity of $\mathcal{L}$.

By [17, Thm. 3.2, Lemma 6.1] we have that I is differentiable on W^1E^Φ and lower semi-continuous, respectively.

Next, we find the minimum of I by means of a similar argument to the one developed on the last part of the proof of Theorem 3.9, changing $u'(t)\Phi'(|u'(t)|)/|u'(t)|$ by $D_y\mathcal{L}(t,u(t),u'(t))$ which also belongs to L^Ψ (see [17, Eq. (26)]).

Finally, the strict convexity of $\mathcal{L}(t,x,y)$ at y and the injectivity of the function $D_y\mathcal{L}(T, u(T), \cdot)$ imply that $u'(0) = u'(T)$. $\square$

4 Examples

In this section we developed some applications of our main results so that the reader can appreciate the innovations that bring.

One of the main novelties of our work is that we obtain existence of solutions for lagrangian functions $\mathcal{L}(t,x,y)$ that not satisfy a power like grow condition in y. is not necessarily a power function.

Example 1.We can applied Theorem 3.9 to Lagrangians $\mathcal{L} = \mathcal{L}(t,x,y)$ with an exponential grow in y. For example, suppose

$$\mathcal{L}(t,x,y) = f(y) + F(t,x),$$

with $f(y) \geqslant e^{|y|} - |y| - 1$. The complementary function $\Psi(x)$ of the N-function $\Phi(x) = e^x - x - 1$ is Δ_2 globally (see [8, p. 28]). We suppose that Φ_1 satisfies (25). Taking $s \to 0$ in (25) we obtain that $\Phi_1(r) \leqslant r^2$. On the other hand, is not hard to prove that $\Phi_1(r) = r^2$ satisfies (25). Hence, if $1 < \alpha < \beta < 2$ and $\Phi_0(x) = |x|^\alpha$ and $\Phi_2(x) = |x|^\beta$ then $\Phi_0 \lll \Phi_2 \lll \Phi_1$, we obtain existence of periodic solutions when F satisfies

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^{\alpha-1} + b_2(t) \quad \text{and} \quad \lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{|x|^\gamma} = +\infty,$$

with $\alpha \in (1,2)$ and $\gamma \in (2\alpha - 2, \alpha)$.

Example 2. We want to emphasize that, even in the case of p-laplacian operator (5), our results extend previous ones (see [22, 6]), because we get bounds that may be sharper than those in [22, 6]. For example, in [6, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions of (5) under the assumption (30) for any $\alpha \in (0, p-1)$. Meanwhile, our Theorem 3.9 implies existence for the potential

$$F_0(t,x) = |x|^p / \ln(2 + |x|)^2.$$

We note that this F does not satisfy (30) for any $\alpha < p - 1$. Next we will show an N-function Φ_0 satisfying the hypothesis of Theorem 3.9 for this potential F_0.

We define

$$\Phi_0(u) = \begin{cases} \frac{p-1}{p} u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function Φ_0.

Theorem 4.1. *If $p \geqslant \frac{1+\sqrt{2}}{2}$, then Φ_0 is a differentiable N-function. The N-function Φ_0 satisfies that for every $\varepsilon > 0$, there exists a positive constant $C = C(p,\varepsilon)$ such that*

$$C^{-1} t^{p-\varepsilon} \Phi_0(u) \leqslant \Phi_0(tu) \leqslant C t^p \Phi_0(u) \quad t \geqslant 1, u > 0, \tag{44}$$

Proof. We have

$$\varphi(u) = \Phi_0'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that Φ_0' is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that φ_1 is increasing on $[0, e]$ and φ_2 is increasing on $[e, \infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p > 1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an increasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Moreover $\varphi_2(u) \to \infty$ and $\varphi_1(u) \to 0$ as $u \to \infty$ and $u \to 0$ respectively, provided that $p > 1$. Hence, Φ_0 is an N-function.

Next we will prove (44). If $u \leqslant tu \leqslant e$, then $\Phi_0(tu) = t^p \Phi_0(u)$ and (44) holds with $C = 1$. If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geqslant 1$, we have $\Phi_0(tu) \leqslant t^p u^p = \frac{p}{p-1} t^p \Phi_0(u)$. Thus, the second inequality of (44) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$, then $f((tu)^p) \geqslant f(e^p) = e^p/p$. Now,

$$
\begin{aligned}
\Phi_0(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\
&= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\
&\geqslant \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} \\
&\geqslant \frac{p-1}{p} \frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p.
\end{aligned}
$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t + 1}$ on the interval $[1, +\infty)$ then

$$
\Phi_0(tu) \geqslant \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,
$$

which is the first inequality of (44) with $C = \frac{p}{p-1} \varepsilon^{-1} e^{-1+\varepsilon}$.

If $e \leqslant u \leqslant tu$, then

$$
\Phi_0(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{p t^p v}{\log v}, \tag{45}
$$

where $v := u^p$ and $v \geqslant e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on (α, ∞) and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p, \infty)$. Therefore, we have

$$
\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}
$$

for every $v \geqslant e^p$. In this way, from (45), we have

$$
\Phi_0(tu) \leqslant \frac{p t^p}{p-1} \left(\frac{pv}{\log v} - \frac{e^p}{p} \right) = \frac{p t^p}{p-1} \left(\frac{u^p}{\log u} - \frac{e^p}{p} \right)
$$

and the second inequality of (44) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi_0(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, the function f attains a minimum on $[1, \infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$, f is increasing on $[1, \infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geqslant \varepsilon$ in any case, therefore

$$\Phi_0(tu) \geqslant \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geqslant \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi_0(u).$$

Finally, (44) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. □

Remark 4. The inequality

$$\Phi_0(tu) \geqslant Ct^p\Phi_0(u)$$

is false for every C because for every $u \geqslant e$ we have

$$\lim_{t\to\infty} \frac{\Phi_0(tu)}{t^p\Phi_0(u)} = 0$$

We note that Φ_0 and F_0 satisfy (32). For the p-laplacian operator we have that $\Phi(|u|) = |u|^p/p$. Then we can take $\Phi_1 = \Phi$ in (25). Clearly $\Phi_0 \lll \Phi_1$.

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References

[1] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989.

[2] Y. Tian, W. Ge, Periodic solutions of non-autonomous second-order systems with a p-Laplacian, Nonlinear Anal. 66 (1) (2007) 192–203.

[3] X.-P. Wu, C.-L. Tang, Periodic solutions of a class of non-autonomous second-order systems, J. Math. Anal. Appl. 236 (2) (1999) 227–235.

[4] F. Zhao, X. Wu, Periodic solutions for a class of non-autonomous second order systems, J. Math. Anal. Appl. 296 (2) (2004) 422–434.

[5] F. Zhao, X. Wu, Existence and multiplicity of periodic solution for non-autonomous second-order systems with linear nonlinearity, Nonlinear Analysis: Theory, Methods & Applications 60 (2) (2005) 325–335.

[6] X. Tang, X. Zhang, Periodic solutions for second-order Hamiltonian systems with a p-Laplacian, Ann. Univ. Mariae Curie-Skłodowska Sect. A 64 (1) (2010) 93–113.

[7] R. Adams, J. Fournier, Sobolev spaces, Elsevier/Academic Press, Amsterdam, 2003.

[8] M. A. Krasnosel′skiĭ, J. B. Rutickiĭ, Convex functions and Orlicz spaces, P. Noordhoff Ltd., Groningen, 1961.

[9] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Vol. 146, Marcel Dekker, Inc., New York, 1991.

[10] G. Schappacher, A notion of Orlicz spaces for vector valued functions, Appl. Math. 50 (4) (2005) 355–386.

[11] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general, Indag. Math. (N.S.) 11 (4) (2000) 573–585.

[12] A. Cianchi, A fully anisotropic Sobolev inequality, Pacific J. Math. 196 (2) (2000) 283–295.

[13] A. Cianchi, Some results in the theory of Orlicz spaces and applications to variational problems, in: Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), Acad. Sci. Czech Repub., Prague, 1999, pp. 50–92.

[14] N. Clavero, Optimal Sobolev embeddings and Function Spaces, `http://www.maia.ub.edu/~soria/sobolev1.pdf`, last accessed: 2014-12-22. (2011).

[15] D. Edmunds, R. Kerman, L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, J. Funct. Anal. 170 (2) (2000) 307–355.

[16] R. Kerman, L. Pick, Optimal Sobolev imbeddings, Forum Math. 18 (4) (2006) 535–570.

[17] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, E. Schwindt, Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, Nonlinear Analysis, TMA. 125 (2015) 681 – 698.

[18] J. B. Conway, A Course in Functional Analysis, Springer, USA, 1977.

[19] B. Xu, C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems, J. Math. Anal. Appl. 333 (2) (2007) 1228–1236.

[20] L. Maligranda, Orlicz spaces and interpolation, Vol. 5 of Seminários de Matemática [Seminars in Mathematics], Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[21] S. Acinas, G. Giubergia, F. Mazzone, E. Schwindt, On estimates for the period of solutions of equations involving the ϕ-Laplace operator, J. Abstr. Differ. Equ. Appl. 5 (1) (2014) 21–34.

[22] C.-L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, Proc. Amer. Math. Soc. 126 (11) (1998) 3263–3270.