Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \qquad (\boldsymbol{P})$$

where the function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$ (called the *Lagrange function* or *lagrangian*) satisfying that it is measurable in t for each $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x,y) for almost every $t \in [0,T]$. The unknown function $u : [0,T] \to \mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the articles [14] for definitions and main results on anisotropic Orlicz spaces, see also [2]. These spaces allow us to unify and extend previous results on existences of solutions for systems like $(\boldsymbol{P})$.

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Through this article we say that a function $\Phi : \mathbb{R}^d \to [0, +\infty)$ is of N_∞ class if Φ is convex, $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$ and $\Phi(-y) = \Phi(y)$, and

$$\lim_{|y|\to\infty} \frac{\Phi(y)}{|y|} = +\infty. \tag{1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From [5, Cor. 2.35] a N_∞ function is continuous.

Associated to Φ we have the *complementary function* Ψ which is defined in $\xi \in \mathbb{R}^d$ as

$$\Psi(\xi) = \sup_{y\in\mathbb{R}^d} y \cdot \xi - \Phi(y) \tag{2}$$

then, from the continuity of Φ and (1), we have that $\Psi : \mathbb{R}^d \to [0, \infty)$. Moreover, it is easy to see that Ψ is a convex function such that $\Psi(0) = 0$, $\Psi(-\xi) = \Psi(\xi)$ [9, Chapter 2]. Moreover Ψ satisfies (1) (see [14, Th. 2.2]). i.e. Ψ is N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\Phi_p(y) := |y|^p/p$, for $1 < p < \infty$. In this case $\Psi(\xi) = |\xi|^q/q$, $q = p/(p-1)$.

Example 1.2. If $\Phi : \mathbb{R} \to [0, +\infty)$ is a N_∞ function on $\mathbb{R}$ then $\overline{\Phi}(y) = \Phi(|y|)$ is a N_∞ function on $\mathbb{R}^d$. In this example, as in the previous one, the function Φ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\Phi(y)$ depends on the direction of y. For example, if $1 < p_1, p_2 < \infty$, we define $\Phi_{p_1,p_2} : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ by

$$\Phi_{p_1,p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then Φ_{p_1,p_2} is a N_∞ function. In this case the complementary function is the function Φ_{q_1,q_2} with $q_i = p_i/(p_i - 1)$.

More generally, if $\Phi_k : \mathbb{R}^d \to [0, +\infty)$, $k = 1, \dots, n$, are N_∞ functions, then $\Phi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to [0, +\infty)$ defined by $\Phi(y_1, \dots, y_n) = \Phi_1(y_1) + \cdots + \Phi_n(y_n)$ is a N_∞ function. These functions are truly anisotropic, i.e. $|x| = |y|$ does not imply that $\Phi(x) = \Phi(y)$.

Example 1.4. If $\Phi : \mathbb{R} \to [0, +\infty)$ is a N_∞ function and $O \in GL(d, \mathbb{R})$, then $\Phi(y) = \Phi(Oy)$ is a N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by powers if it does not satisfy the Δ_2 condition (see xxxxx). For example $\Phi : \mathbb{R}^d :\to [0, +\infty)$ defined by $\Phi(y) = \exp(|y|) - 1$ is N_∞ function.

The occurrence of Orlicz Spaces in this paper obeys to we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi(\nabla_y \mathcal{L}) \leqslant a(x) \left\{ b(t) + \Phi\left(\frac{y}{\lambda}\right) \right\}, \tag{$\boldsymbol{S}$}$$

for a.e. $t \in [0, T]$, where $a \in C\left(\mathbb{R}^d, [0, +\infty)\right)$, $b \in L^1\left([0, T], [0, +\infty)\right)$.

Our condition ($\boldsymbol{S}$) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition ($\boldsymbol{S}$) is equivalent to the structure condition in [9, Th. 1.4]. If Φ is a radial N_∞ function such that Ψ satisfies that Δ_2 function then ($\boldsymbol{S}$) is essentially equivalent????? to conditions [1, Eq. (2)-(4)] (see xxxx mas abajo). If Φ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ then inequality ($\boldsymbol{S}$) is related to estructure conditions like [20, Lemma 3.1, Eq. (3.1)]. As can be seen, condition ($\boldsymbol{S}$) is a more compact expression than [20, Lemma 3.1, Eq. (3.1)] and moreover weaker, because ($\boldsymbol{S}$) does not imply a control of $|D_{y_1} L|$ independent of y_2. We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi,F}(t,x,y) := \Phi(y) + F(t,x). \tag{3}$$

Here the function $F(t,x)$, which is often referred to potential, be differentiable with respect to x for a.e. $t \in [0,T]$. Moreover F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla_x F(t,x)| \leqslant a(x) b(t). \tag{4}$$

where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$ and $0 \leqslant b \in L^1([0,T], \mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi,F}$ satisfies condition ($\boldsymbol{S}$). In order to prove this, the only non trivial fact that we should to establishis is that $\Psi(\nabla_y \mathcal{L}) \leqslant a(x)\left\{b(t) + \Phi(y/\lambda)\right\}$. But, from inequality xxxx below, $\Psi(\nabla_y \mathcal{L}) = \Psi\left(\nabla\Phi(y)\right) \leqslant \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi,F}$ leads to the system

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{$\boldsymbol{P_\Phi}$}$$

Problem ($\boldsymbol{P_\Phi}$) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [9] deals mainly with problem ($\boldsymbol{P}$), for the lagrangian $\mathcal{L}_{\Phi,F}$, with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [9] were extended and improved in several articles, see [18, 16, 22, 17, 25] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [20, 19], among other papers, and in this case ($\boldsymbol{P_\Phi}$) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{$\boldsymbol{P_p}$}$$

If Φ is as in Example 1.3 and $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, then the equations ($\boldsymbol{P_\Phi}$) become

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) & \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}, \tag{$\boldsymbol{P_{p_1,p_2}}$}$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as (p_1, p_2)-Laplacian system, see [24, 13, 23, 10, 11, 12, 7].

In conclusion, the problem ($\boldsymbol{P}$) with conditions ($\boldsymbol{S}$) contains several problems that have been considered by many authors in the past. Moreover, our results still improve some results on (p_1, p_2)-lamplacian since our structure conditions are less restrictive even in that particular case.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^n \to [0, +\infty)$. References for these topics are [6, 14, 15, 3, 4, 2, 21].

If Φ is a N_∞ function then from convexity and $\Phi(0) = 0$ we obtain that

$$\Phi(\lambda x) \leqslant \lambda \Phi(x), \quad \lambda \in [0,1], x \in \mathbb{R}^d. \tag{5}$$

One of the greatest difficulties when dealing with anisotropic Orlicz spaces is the lack of monotony with respect to the Euclidean norm, i.e. $|x| \leqslant |y|$ does not imply $\Phi(x) \leqslant \Phi(y)$. This problem is avoided if we consider functions whose values on a sphere are comparable (see[15]). However, from (5), we see that N_∞ functions have the following form of radial monotony: $|x| \leqslant |y|$ and $y = \lambda x$ imply $\Phi(x) \leqslant \Phi(y)$.

We say that $\Phi : \mathbb{R}^d \to [0, +\infty)$ satisfies the Δ_2^∞-*condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant K\Phi(x), \tag{6}$$

for every $|x| \geqslant M$. If Φ es a Δ_2 function then Φ is bounded by powers functions (see [6, Proof Lemma 2.4] and [4, Prop. 1]), i.e. there exists $1 < p < \infty$, $C > 0$ and $r \geqslant 0$ such that

$$\Phi(x) \leqslant C|x|^p, \quad |x| \geqslant r_0.$$

We consider that one of the most important aspects in considering N_∞ functions is that it accounts for the Lagrange functions that present faster growth than powers, for example an exponential growth. Hence we consider it important to avoid imposing hypothesis that Φ to be Δ_2. For some results we will need that Ψ to be Δ_2.

Let Φ_1 and Φ_2 be N_∞ functions. Following to [21] we wrtite $\Phi_1 \dashv \Phi_2$ if there exists non negative numbers k and r_0 such that

$$\Phi_1(x) \leqslant \Phi_2(kx), \quad |x| \geqslant r_0. \tag{7}$$

For example if Φ is Δ_2 then there exist $p \in (1, +\infty)$ such that $\Phi \dashv |x|^p$. If for every $k > 0$ there exists $r_0 = r_0(k) > 0$ such that (7) holds we write $\Phi_1 \lll \Phi_2$.

We denote by $\mathcal{M} := \mathcal{M}\left([0,T], \mathbb{R}^d\right)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Now, we introduce the *Orlicz class* $C^{\Phi} = C^{\Phi}\left([0,T],\mathbb{R}^d\right)$ by setting

$$C^{\Phi} := \{u \in \mathcal{M} | \rho_{\Phi}(u) < \infty\}. \tag{8}$$

The *Orlicz space* $L^{\Phi} = L^{\Phi}\left([0,T],\mathbb{R}^d\right)$ is the linear hull of C^{Φ}; equivalently,

$$L^{\Phi} := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_{\Phi}(\lambda u) < \infty\}. \tag{9}$$

The Orlicz space L^{Φ} equipped with the *Luxemburg norm*

$$\|u\|_{L^{\Phi}} := \inf\left\{\lambda \middle| \rho_{\Phi}\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space.

The subspace $E^{\Phi} = E^{\Phi}\left([0,T],\mathbb{R}^d\right)$ is defined as the closure in L^{Φ} of the subspace $L^{\infty}\left([0,T],\mathbb{R}^d\right)$ of all $\mathbb{R}^d$-valued essentially bounded functions. The equality $L^{\Phi} = E^{\Phi}$ is true if and only if $\Phi \in \Delta_2^{\infty}$ (see [14, Cor. 5.1]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [14, Thm. 7.2]). Namely, if $u \in L^{\Phi}$ and $v \in L^{\Psi}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^{\Phi}} \|v\|_{L^{\Psi}}. \tag{10}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

We consider the subset $\Pi(E^{\Phi}, r)$ of L^{Φ} given by

$$\Pi(E^{\Phi}, r) := \{u \in L^{\Phi} | d(u, E^{\Phi}) < r\}.$$

This set is related to the Orlicz class C^{Φ} by the following inclusions

$$\Pi(E^{\Phi}, r) \subset rC^{\Phi} \subset \overline{\Pi(E^{\Phi}, r)} \tag{11}$$

for any positive r. This relation is a trivial generalization of [14, Thm. 5.6]. If $\Phi \in \Delta_2^{\infty}$, then the sets L^{Φ}, E^{Φ}, $\Pi(E^{\Phi}, r)$ and C^{Φ} are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi} \hookrightarrow \left[L^{\Psi}\right]^*$, where a function $v \in L^{\Phi}$ is associated to $\xi_v \in \left[L^{\Psi}\right]^*$ being

$$\langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \tag{12}$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [14].

Proposition 2.1. *If Ψ satisfies the Δ_2^{∞}-condition then $L^{\Phi}\left([0,T],\mathbb{R}^d\right) = \left[L^{\Psi}\left([0,T],\mathbb{R}^d\right)\right]^*$.*

Consequently if Ψ satisfies the Δ_2^∞-condition then $L^\Phi\left([0,T],\mathbb{R}^d\right)$ can be equipped with the weak* topology.

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi\left([0,T],\mathbb{R}^d\right) := \left\{u|u\in AC\left([0,T],\mathbb{R}^d\right) \text{ and } u'\in L^\Phi\left([0,T],\mathbb{R}^d\right)\right\},$$

where $AC\left([0,T],\mathbb{R}^d\right)$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $[0,T]$. The space $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{13}$$

We introduce the following subspaces of W^1L^Φ

$$W^1E^\Phi = \{u\in W^1L^\Phi|u'\in E^\Phi\},\quad W^1E^\Phi_T = \{u\in W^1E^\Phi|u(0)=u(T)\}. \tag{14}$$

In order to find a modulus of continuity for functios in W^1L^Φ, and from there, to obtain compact embedding of W^1L^Φ, we define the function $A_\Phi:\mathbb{R}^+\to\mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x)\,\middle|\,|x|=s\right\}, \tag{15}$$

Let us establish some elementary properties of A_Φ.

Proposition 2.2. *The function A_Φ has the following properties:*

1. *A_Φ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *$A_\Phi(|x|)$ is the* greatest radial minorant *of $\Phi(x)$,*
4. *Φ is N_∞ if and only if $\lim_{s\to+\infty}A_\Phi(s)/s=+\infty$.*

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [5]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0<r<s$ and $x\in\mathbb{R}^d$ with $A_\Phi(s)=\Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

Example 2.1. Let $\Phi=\Phi_{p_1,p_2}$ be the function given in Example (1.3). We show that

$$K_1\min\left\{\frac{r^{p_1}}{p_1},\frac{r^{p_2}}{p_2}\right\} \leqslant A_\Phi(r) \leqslant \min\left\{\frac{r^{p_1}}{p_1},\frac{r^{p_2}}{p_2}\right\}$$

for some $K > 0$, for every $1 < p_1, p_2 < \infty$. The second inequality follows directly from definition of A_Φ. For the first inequality, we note that $|(y_1, y_2)| = r$ implies that $|y_1| \geqslant r/2$ or $|y_2| \geqslant r/2$. Then

$$\Phi_{p_1,p_2}(y_1, y_2) \geqslant K \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\}, \tag{16}$$

with $K = \min\{2^{-p1}, 2^{-p_2}\}$.

Let us in a little digression to show that

$$A_\Phi(r) = \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\},$$

when $1 < p_1, p_2 \leqslant 2$. We apply the method of Lagrange multipliers (see [8, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1,p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2}y_1 + \lambda y_1 &= 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 &= 0 \\ |y_1|^2 + |y_2|^2 &= r^2 \end{cases} \tag{17}$$

These equations are solved, among others, by the following two sets of citical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2.

If (y_1, y_2) solve (17) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1, y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let L be the Lagrangian associated to the constrained problem: $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) + \lambda H(y_1, y_2)$ being $H = 0$ the constraint. We must analize the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2 H$ on the subspace M. By elementary computations we have for $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace M. We can assume that $p_1 < 2$ or $p_2 < 2$, otherwise the statement we intend to prove would be trivial. Under this assumption, we note that $(-y_2, y_1) \in M$ and $(-y_2, y_1)^t \mathcal{H}(-y_2, y_1) < 0$. Then, by second order necessary conditions [8, p.333], there cannot be a minimum at (y_1, y_2). Therefore follows (16).

As is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0, T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (15)*. Then*

1. *For every $s, t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s-t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty\left([0,T], \mathbb{R}^d\right)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If Φ is N_∞ then the space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is compactly embedded in the space of continuous functions $C\left([0,T], \mathbb{R}^d\right)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) \leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr\right)$$
$$\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leqslant \frac{1}{|s-t|}.$$

By Proposition 2.2(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned} |u(t) - \overline{u}| &= \left|\frac{1}{T}\int_0^T u(t) - u(s)\, ds\right| \\ &\leqslant \frac{1}{T}\int_0^T |u(t) - u(s)|\, ds \\ &\leqslant \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right) \end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.2(3)

$$|\overline{u}| \leqslant A_{\Phi}^{-1}\left(\frac{1}{T}\right)\|u\|_{L^{\Phi}}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}
\|u\|_{L^{\infty}} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^{\infty}} \\
&\leqslant A_{\Phi}^{-1}\left(\frac{1}{T}\right)\|u\|_{L^{\Phi}} + TA_{\Phi}^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^{\Phi}} \\
&\leqslant A_{\Phi}^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^{\Phi}}
\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^{\Phi}\left([0,T],\mathbb{R}^d\right)$. Since Φ is N_{∞}, from Proposition 2.2(4) we obtain $sA_{\Phi}^{-1}(1/s) \to 0$ when $s \to 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (**??**) implies that u_n is bounded in $C\left([0,T],\mathbb{R}^d\right)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C\left([0,T],\mathbb{R}^d\right)$ with $u_{n_k} \to u$ in $C\left([0,T],\mathbb{R}^d\right)$. □

Lemma 2.4. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^{\Phi},1)$ converging to $u \in \Pi(E^{\Phi},1)$ in the L^{Φ}-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0,T],\mathbb{R})$ such that $u_{n_k} \to u \quad$ a.e. and $\Phi(u_{n_k}) \leqslant h \quad$ a.e.*

Proof. Since $d(u,E^{\Phi}) < 1$ and u_n converges to u, there exists $u_0 \in E^{\Phi}$, a subsequence of u_n (again denoted u_n) and $0 < r < 1$ such that $d(u_n,u_0) < r$. Let $\lambda_0 \in (r,1)$. By extracting more subsequences, if necessary, we can assume that $u_n \to u$ a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^{\Phi}} < \frac{1-\lambda_0}{2^n}, \quad \text{for } n \geqslant 1.$$

We can assume $\lambda_n > 0$ for every $n = 0,\ldots$.

Let $\lambda := 1 - \sum_{n=0}^{\infty}\lambda_n$ and define $h : [0,T] \to \mathbb{R}$ by

$$h(x) = \lambda\Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^{\infty}\lambda_n\Phi\left(\frac{u_{n+1}-u_n}{\lambda_n}\right). \tag{18}$$

Note that $\sum_{n=0}^{\infty}\lambda_n + \lambda = 1$, therefore for any $n = 1,\ldots$

$$\begin{aligned}
\Phi(u_n) &= \Phi\left(\lambda\frac{u_0}{\lambda} + \sum_{j=0}^{n-1}\lambda_j\frac{u_{j+1}-u_j}{\lambda_j}\right) \\
&\leqslant \lambda\Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1}\lambda_j\Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right) \leqslant h
\end{aligned}$$

Since $u_0 \in E^{\Phi} \subset C^{\Phi}$ and E^{Φ} is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0,T],\mathbb{R})$. On the other hand $\|u_{n+1}-u_n\|_{L^{\Phi}} \leqslant \lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right)dt \leqslant 1.$$

Then $h \in L^1([0,T],\mathbb{R})$.

□

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

Given a continuous function $a \in C(\mathbb{R}^n,\mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

Lemma 3.1. *If $a \in C(\mathbb{R}^d,\mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

Proof. Let $A \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\|\leqslant s, x\in\mathbb{R}^d} |a(x)|$. If $u \in W^1L^\Phi_d$ then, by Sobolev's inequality, for a.e. $t \in [0,T]$

$$a(u(t)) \leqslant \alpha(\|u\|_{L^\infty}) \leqslant \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\right) =: A(\|u\|_{W^1L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t,u(t),\dot{u}(t))\,dt. \tag{19}$$

Theorem 3.2. *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying ($\boldsymbol{S}$). Then the following statements hold:*

1. *The action integral given by (19) is finitely defined on $\mathcal{E}^\Phi := W^1L^\Phi \cap \{u|\dot{u} \in \Pi(E^\Phi,1)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$. Moreover, I' is given by the following expression*
$$\langle I'(u),v\rangle = \int_0^T \left\{D_x\mathcal{L}\big(t,u,\dot{u}\big)\cdot v + D_y\mathcal{L}\big(t,u,\dot{u}\big)\cdot\dot{v}\right\}\,dt. \tag{20}$$
3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^\Phi$. As

$$\dot{u} \in \Pi(E^\Phi,1) \subset C_1^\Phi \tag{21}$$

and (11), then $\Phi(\dot{u}(t)) \in L^1$. Now,

$$|\mathcal{L}(\cdot,u,\dot{u})| + |D_x\mathcal{L}(\cdot,u,\dot{u})| + \Psi(D_y\mathcal{L}(\cdot,u,\dot{u})) \leqslant A(\|u\|_{W^1L^\Phi})(b+\Phi(\dot{u})) \in L^1, \quad (22)$$

by ($\boldsymbol{S}$) and Lemma 3.1. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto D_x\mathcal{L}(t,u,\dot{u})$ *is continuous from* $\mathcal{E}^\Phi$ *into* $L^1([0,T])$ *with the strong topology on both sets.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^\Phi$ and let $u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in W^1L^Φ. By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leqslant TA_\Phi^{-1}\left(\frac{1}{T}\right)\|u_n - u\|_{L^\Phi}$$

then $u_n \to u$ uniformly. As $\dot{u}_n \to \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.4, there exist a subsequence of $\dot{u}_{n_k}$ (again denoted $\dot{u}_{n_k}$) and a function $h \in L^1([0,T],\mathbb{R})$ such that $\dot{u}_{n_k} \to \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leqslant h$ a.e.

Since u_{n_k}, $k = 1,2,\ldots$, is a strong convergent sequence in W^1L^Φ, it is a bounded sequence in W^1L^Φ. According to item (3) of Lemma 2.3, there exists $M > 0$ such that $\|\boldsymbol{a}(u_{n_k})\|_{L^\infty} \leqslant M$, $k = 1,2,\ldots$. From the previous facts and (22), we get

$$|D_x\mathcal{L}(\cdot,u_{n_k},\dot{u}_{n_k})| \leqslant a(|u_{n_k}|)(b+\Phi(\dot{u}_{n_k})) \leqslant M(b+h) \in L^1.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$D_x\mathcal{L}(t,u_{n_k}(t),\dot{u}_{n_k}(t)) \to D_x\mathcal{L}(t,u(t),\dot{u}(t)) \quad \text{for a.e. } t \in [0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto D_y\mathcal{L}(t,u,\dot{u})$ *is continuous from* $\mathcal{E}^\Phi$ *with the strong topology into* $\left[L^\Phi\right]^*$ *with the weak** *topology.*

Let $u \in \mathcal{E}^\Phi$. From (22) it follows that

$$D_y\mathcal{L}(\cdot,u,\dot{u}) \in C^\Psi. \quad (23)$$

Así? o conviene poner la cota de $\Psi(D_y)$ explícitamente???

Note that (22), (23) and the imbeddings $W^1L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$ imply that the second member of (20) defines an element of $\left[W^1L^\Phi\right]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in the norm of W^1L^Φ. We must prove that $D_y\mathcal{L}(\cdot,u_n,\dot{u}_n) \overset{w^*}{\rightharpoonup} D_y\mathcal{L}(\cdot,u,\dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle D_y\mathcal{L}(\cdot,u_n,\dot{u}_n),v\rangle - \langle D_y\mathcal{L}(\cdot,u,\dot{u}),v\rangle| \geqslant \epsilon. \quad (24)$$

We have $u_n \to u$ in L^Φ and $\dot{u}_n \to \dot{u}$ in L^Φ. By Lemma 2.4, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T],\mathbb{R})$ such that $u_n \to u$ uniformly, $\dot{u}_n \to \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leqslant h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.1 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (22) with u_n instead of u, we have

$$\Psi(D_y\mathcal{L}(\cdot, u_n, \dot{u}_n)) \leqslant M(b+h) \in L^1. \tag{25}$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (25), we have

$$\begin{aligned}
&\lambda D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\
&\leqslant \lambda \left[\Psi(D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right)\right] \\
&\leqslant \lambda M(b+h) + \lambda\Phi\left(\frac{v}{\lambda}\right) \in L^1
\end{aligned} \tag{26}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y\mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v\, dt \to \int_0^T D_y\mathcal{L}(t, u, \dot{u}) \cdot v\, dt \tag{27}$$

which contradicts the inequality (24). This completes the proof of step 2.

Step 3. We will prove (20). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1L^\Phi$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leqslant s_0 := \min\{\left(1 - d(\dot{u}, E^\Phi)\right)/\|v\|_{W^1L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d\left(\dot{u} + s\dot{v}, E^\Phi\right) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.2 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$.

We also have $\|u + sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 3.1, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leqslant M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leqslant s_0\lambda^{-1}$, from the convexity and the parity of Φ, we get

$$\begin{aligned}
\Phi(\dot{u} + s\dot{v}) &= \Phi\left((1 - s_0)\frac{\dot{u}}{1 - s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right) \\
&\leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1
\end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1 - s_0}, E^\Phi\right) = \frac{1}{1 - s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1 - s_0} \in C^\Phi$.

Now, applying (22), (26), the fact that $v \in L^\infty$ and $\dot v \in L^\Phi$, we get

$$\begin{aligned} |D_s H(s,t)| &= \left| D_x\mathcal{L}(t,u+sv,\dot u + s\dot v)\cdot v + \lambda D_y\mathcal{L}(t,u+sv,\dot u+s\dot v)\cdot \frac{\dot v}{\lambda}\right| \\ &\leqslant M\left[b(t)+\Phi(\dot u + s\dot v)\right]|v| \\ &\quad + \lambda\left[\Psi(D_y\mathcal{L}(t,u+sv,\dot u+s\dot v)) + \Phi\left(\frac{\dot v}{\lambda}\right)\right] \\ &\leqslant M\left\{\left[b(t)+\Phi(\dot u+s\dot v)\right]|v|\right\} + \lambda M[b(t)+\Phi(\dot u+s\dot v)] + \lambda\Phi\left(\frac{\dot v}{\lambda}\right) \\ &= M[b(t)+\Phi(\dot u+s\dot v)](|v|+\lambda) + \lambda\Phi\left(\frac{\dot v}{\lambda}\right) \in L^1. \end{aligned} \tag{28}$$

Consequently, I has a directional derivative and

$$\langle I'(u),v\rangle = \frac{d}{ds}I(u+sv)\Big|_{s=0} = \int_0^T \left\{D_x\mathcal{L}(t,u,\dot u)\cdot v + D_y\mathcal{L}(t,u,\dot u)\cdot \dot v\right\}\, dt.$$

Moreover, from the previous formula, (22), (23), and Lemma 2.3, we obtain

$$|\langle I'(u),v\rangle| \leqslant \|D_x\mathcal{L}\|_{L^1}\|v\|_{L^\infty} + \|D_y\mathcal{L}\|_{L^\Psi}\|\dot v\|_{L^\Phi} \leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant C.

This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : \mathcal{E}^\Phi \to \left[W^1L^\Phi_d\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto D_x\mathcal{L}(t,u,\dot u)$ and $u \mapsto D_y\mathcal{L}(t,u,\dot u)$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \to u$ in the norm of W^1L^Φ and $v \in W^1L^\Phi$, then

$$\begin{aligned} \langle I'(u_n),v\rangle &= \int_0^T \left\{D_x\mathcal{L}\left(t,u_n,\dot u_n\right)\cdot v + D_y\mathcal{L}\left(t,u_n,\dot u_n\right)\cdot \dot v\right\}\, dt \\ &\to \int_0^T \left\{D_x\mathcal{L}\left(t,u,\dot u\right)\cdot v + D_y\mathcal{L}\left(t,u,\dot u\right)\cdot \dot v\right\}\, dt \\ &= \langle I'(u),v\rangle. \end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto D_x\mathcal{L}(t,u,\dot u)$ and $u \mapsto D_y\mathcal{L}(t,u,\dot u)$ are norm continuous from $\mathcal{E}^\Phi$ into L^1 and L^Ψ, respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de D_x aquí!!!

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$.

Applying Lemma 2.4 to $\dot u_n$, there exists a subsequence (denoted $\dot u_n$ for simplicity) such that $\dot u_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot u_n) \leqslant h$ and $\dot u_n \to \dot u$ a.e.

Then, by (26) we have $\Psi(v_n) \leqslant m(t) \in L^1$ being $v_n := D_y\mathcal{L}(\cdot,u_n,\dot u_n)$ and $m(t) := M(b+h)$. In addition, from the continuous differentiability of $\mathcal{L}$, we have that $v_n \to v$ a.e. where $D_y\mathcal{L}(\cdot,u,\dot u)$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \to ???$ such that $\Psi(\lambda x) \leqslant c(|\lambda|)\Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})\Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n-v}{\lambda}) \to 0$ a.e. as $n \to \infty$ and $\Psi(\frac{v_n-v}{\lambda}) \leqslant c(|\lambda|^{-1})K\Psi(v_n) + \Psi(v)) \leqslant c(|\lambda|^{-1})K[m(t) + \Psi(v)]) \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda})\, dt \to 0$ for every $\lambda > 0$. Thus, $v_n \to v$ in L^Ψ.

The continuity of I' follows from the continuity of $D_x\mathcal{L}$ and $D_y\mathcal{L}$ using the formula (20). $\square$

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