

# Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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## Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space  $W^1 L^\Phi([0, T])$ , of hamiltonian systems with a potential function  $F$  satisfying the inequality  $|\nabla F(t, x)| \leq b_1(t)\Phi'_0(|x|) + b_2(t)$ , with  $b_1, b_2 \in L^1$  and for certain  $N$ -functions  $\Phi_0$ .

## 1 Introduction

The purpose of this paper is to study the existence of periodic solution for the following non-autonomous second-order systems:

$$\begin{cases} \frac{d}{dt} \left( u'(t) \frac{\Phi'(|u'|)}{|u'|} \right) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \quad (1)$$

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where  $T > 0$ ,  $u : [0, T] \rightarrow \mathbb{R}^d$  is absolutely continuous and  $\Phi$  is a differentiable  $N$ -function (see section Preliminaries for definitions). Furthermore, the *potential*  $F : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies the following conditions:

(C)  $F$  and its gradient  $\nabla F$  are Carathéodory functions, i.e. they are measurable functions with respect to  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ , and continuous functions with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

(A) For a.e.  $t \in [0, T]$ , it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (2)$$

In this inequality we assume that the function  $a : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and nondecreasing and  $0 \leq b \in L^1([0, T], \mathbb{R})$ .

We will call the differential operator

$$L_\Phi[u] = \frac{d}{dt} \left( u'(t) \frac{\Phi'(|u'|)}{|u'|} \right)$$

the  $\Phi$ -laplacian operator. If  $\Phi(x) = |x|^p$ ,  $1 < p < \infty$ ,  $L_\Phi$  is the well known  $p$ -laplacian operator.

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \Phi(|u'(t)|) + F(t, u(t)) dt. \quad (3)$$

## 2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [1, 2, 3]. For Orlicz spaces of vector valued functions, see [4] and the references therein.

Hereafter we denote by  $\mathbb{R}^+$  the set of all non negative real numbers. A function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called an *N-function* if  $\Phi$  is convex and satisfies that

$$\lim_{t \rightarrow +\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$$

In addition, in this paper we assume that  $\Phi$  is differentiable and we call  $\varphi$  the derivative of  $\Phi$ . On these assumptions,  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a homeomorphism whose inverse is  $\psi$ . We denote by  $\Psi$  the primitive of  $\psi$  that satisfies  $\Psi(0) = 0$ . Then,  $\Psi$  is an  $N$ -function which is called the *complementary function* of  $\Phi$ .

There exist several order relations between  $N$ -functions (see [3, Sec. 2.2]). Following [3, Def. 1, p. 15] we say that the  $N$ -function  $\Phi_2$  is *essentially stronger* than the  $N$ -function  $\Phi_1$  ( $\Phi_1 \ll \Phi_2$ ) if and only if there exists  $x_0 \geq 0$  such that  $\Phi_1(x) \leq \Phi_2(ax)$ , for every  $a > 0$  and  $x \geq x_0$ .

We also say that a function  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies the  $\Delta_2$ -condition, denoted by  $\eta \in \Delta_2$ , if there exist constants  $K > 0$  and  $t_0 \geq 0$  such that

$$\eta(2t) \leq K\eta(t), \quad (4)$$

for every  $t \geq t_0$ . If  $t_0 = 0$ , a function  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is said to satisfy the  $\Delta_2$ -condition globally ( $\eta \in \Delta_2$  globally).

Let  $d$  be a positive integer. We denote by  $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$  the set of all measurable functions defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $u = (u_1, \dots, u_d)$  for  $u \in \mathcal{M}$ .

Given an  $N$ -function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|) dt.$$

Here  $|\cdot|$  is the euclidean norm of  $\mathbb{R}^d$ . The *Orlicz class*  $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$  is defined by

$$C_d^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (5)$$

The *Orlicz space*  $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$  is the linear hull of  $C^\Phi$ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (6)$$

The Orlicz space  $L^\Phi$  equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup \left\{ \int_0^T u \cdot v dt \mid \rho_\Psi(v) \leq 1 \right\},$$

is a Banach space. By  $u \cdot v$  we denote the usual dot product in  $\mathbb{R}^d$  between  $u$  and  $v$ . The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [2, Thm. 10.5] and [5]). For every  $u \in L^\Phi$ ,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(ku)\}. \quad (7)$$

In particular

$$\|u\|_{L^\Phi} \leq \frac{1}{k} \{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \quad (8)$$

The subspace  $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$  is defined as the closure in  $L^\Phi$  of the subspace  $L_d^\infty([0, T], \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that  $E^\Phi$  is the only one maximal subspace contained in the Orlicz class  $C^\Phi$ , i.e.  $u \in E^\Phi$  if and only if  $\rho_\Phi(\lambda u) < \infty$  for any  $\lambda > 0$ .

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [2, Thm. 9.3]). Namely, if  $u \in L^\Phi$  and  $v \in L^\Psi$  then  $u \cdot v \in L^1$  and

$$\int_0^T v \cdot u dt \leq \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \quad (9)$$

If  $X$  and  $Y$  are Banach spaces such that  $Y \subset X^*$ , we denote by  $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{R}$  the bilinear pairing map given by  $\langle x^*, x \rangle = x^*(x)$ . Hölder's inequality shows that  $L^\Psi \subset [L^\Phi]^*$ , where the pairing  $\langle v, u \rangle$  is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \quad (10)$$

with  $u \in L^\Phi$  and  $v \in L^\Psi$ . Unless  $\Phi \in \Delta_2$ , the relation  $L^\Psi = [L^\Phi]^*$  will not hold. In general, it is true that  $[E^\Phi]^* = L^\Psi$ .

We define the *Sobolev-Orlicz space*  $W^1 L^\Phi$  (see [1]) by

$$W^1 L^\Phi := \{u \mid u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi\}.$$

$W^1 L^\Phi$  is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (11)$$

Moreover, we introduce the following subspaces of  $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi \mid u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi \mid u(0) = u(T)\}. \end{aligned} \quad (12)$$

For a function  $u \in L_d^1([0, T])$ , we write  $u = \bar{u} + \tilde{u}$  where  $\bar{u} = \frac{1}{T} \int_0^T u(t) \, dt$  and  $\tilde{u} = u - \bar{u}$ .

As usual, if  $(X, \|\cdot\|_X)$  is a Banach space and  $(Y, \|\cdot\|_Y)$  is a subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when the restricted identity map  $i_Y : Y \rightarrow X$  is bounded. That is, there exists  $C > 0$  such that for any  $y \in Y$  we have  $\|y\|_X \leq C\|y\|_Y$ . With this notation, Hölder's inequality states that  $L^\Psi \hookrightarrow [L^\Phi]^*$ ; and, it is easy to see that for every  $N$ -function  $\Phi$  we have that  $L_d^\infty \hookrightarrow L^\Phi \hookrightarrow L_d^1$ .

Recall that a function  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called a *modulus of continuity* if  $w$  is a continuous increasing function which satisfies  $w(0) = 0$ . For example, it can be easily shown that  $w(s) = s\Phi^{-1}(1/s)$  is a modulus of continuity for every  $N$ -function  $\Phi$ . We say that  $u : [0, T] \rightarrow \mathbb{R}^d$  has modulus of continuity  $w$  when there exists a constant  $C > 0$  such that

$$|u(t) - u(s)| \leq Cw(|t - s|). \quad (13)$$

We denote by  $C^w([0, T], \mathbb{R}^d)$  the space of  $w$ -Hölder continuous functions. This is the space of all functions satisfying (13) for some  $C > 0$  and it is a Banach space with norm

$$\|u\|_{C^w([0, T], \mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [6, 7, 8, 9, 10]. The next simple lemma, whose proof can be found in [11], will be used systematically.

**Lemma 2.1.** *Let  $w(s) := s\Phi^{-1}(1/s)$ . Then, the following statements hold:*

1.  $W^1 L^\Phi \hookrightarrow C^w([0, T], \mathbb{R}^d)$  and for every  $u \in W^1 L^\Phi$

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} w(|t - s|) \quad (\text{Morrey's inequality}), \quad (14)$$

$$\|u\|_{L^\infty} \leq \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality}). \quad (15)$$

2. For every  $u \in W^1 L^\Phi$  we have  $\tilde{u} \in L_d^\infty$  and

$$\|\tilde{u}\|_{L^\infty} \leq T \Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality}). \quad (16)$$

### 3 Lagrangians satisfying sublinear nonlinearity type conditions

**Lemma 3.1.**  $E^\Phi$  is weak\* closed in  $L^\Phi$ .

*Proof.* From [3, Thm. 7, p. 110] we have that  $L^\Phi = [E^\Psi]^*$ . Then,  $L^\Phi$  is a dual and therefore we are allowed to speak about the weak\* topology of  $L^\Phi$ . Besides,  $E^\Phi$  is separable (see [3, Thm. 1, p. 87]). Let  $S = E^\Phi \cap \{u \in L^\Phi \mid \|u\|_{L^\Phi} \leq 1\}$ , then  $S$  is closed in the norm  $\|\cdot\|_{L^\Phi}$ . Now, according to [3, Cor. 5, p. 148]  $S$  is weak\* sequentially compact. Thus,  $S$  is weak\* sequentially closed because if  $u_n \in S$  and  $u_n \xrightarrow{*} u \in L^\Phi$  then the weak\* sequential compactness implies the existence of  $v \in S$  and a subsequence  $u_{n_k}$  such that  $u_{n_k} \xrightarrow{*} v$ . Finally, by the uniqueness of the limit, we get  $u = v \in S$ . As  $E^\Psi$  is separable and  $L^\Phi = [E^\Psi]^*$ , the ball of  $L^\Phi$   $\{u \in L^\Phi \mid \|u\|_{L^\Phi} \leq 1\}$  is weak\* metrizable (see [12, Thm. 5.1, p. 138]). Thus,  $S$  is closed with respect to the weak\* topology. Now, by Krein-Smulian theorem, [12, Cor. 12.6, p. 165] implies that  $E^\Phi$  is weak\* closed.  $\square$

The following result is analogous to some lemmata in  $W^{1,p}$ , see [13].

**Lemma 3.2.** If  $\|u\|_{W^1 L^\Phi} \rightarrow \infty$ , then  $(|\bar{u}| + \|u'\|_{L^\Phi}) \rightarrow \infty$ .

*Proof.* By the decomposition  $u = \bar{u} + \tilde{u}$  and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\bar{u} + \tilde{u}\|_{L^\Phi} \leq \|\bar{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\bar{u}| \|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \quad (17)$$

It is known that  $L_d^\infty \hookrightarrow L^\Phi$ , i.e. there exists  $C_1 = C_1(T) > 0$  such that for any  $\tilde{u} \in L_d^\infty$  we have

$$\|\tilde{u}\|_{L^\Phi} \leq C_1 \|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists  $C_2 = C_2(T) > 0$  such that

$$\|\tilde{u}\|_{L^\Phi} \leq C_2 \|u'\|_{L^\Phi}. \quad (18)$$

Therefore, from (17), (18) and (11), we get

$$\|u\|_{W^1 L^\Phi} \leq C_3 (|\bar{u}| + \|u'\|_{L^\Phi})$$

where  $C_3 = C_3(T)$ . Finally, as  $\|u\|_{W^1 L^\Phi} \rightarrow \infty$  we conclude that  $(|\bar{u}| + \|u'\|_{L^\Phi}) \rightarrow \infty$ .  $\square$

**Lemma 3.3.** *Let  $\Phi, \Psi$  be complementary functions. The next statements are equivalent:*

1.  $\Psi \in \Delta_2$  globally.
2. There exists an  $N$ -function  $\Phi_1$  such that

$$\Phi(rs) \geq \Phi_1(r)\Phi(s) \text{ for every } r \geq 1, s \geq 0. \quad (19)$$

*Proof.* 1) $\Rightarrow$ 2) By virtue of the  $\Delta_2$ -condition on  $\Psi$ , [14, Thm. 11.7] and [14, Cor. 11.6] (see also [15, Eq. (2.8)]), we get constants  $K > 0$  and  $\alpha_\Phi > 1$  such that

$$\Phi(rs) \geq Kr^\nu \Phi(s), \quad (20)$$

for any  $1 < \nu < \alpha_\Phi$ ,  $s \geq 0$  and  $r > 1$ . This proves (19) with  $\Phi_1(r) = kr^\nu$ , which is an  $N$ -function.

2) $\Rightarrow$ 1) Next, we follow [3, p. 32, Prop. 13] and [3, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leq \Phi(rs) \text{ } r > 1, s \geq 0.$$

Let  $u = \Phi_1(r) \geq \Phi_1(1)$  and  $v = \Phi(s) \geq 0$ . By a well known inequality [3, p. 13, Prop. 1] and (19), we have for  $u \geq \Phi_1(1)$  and  $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leq \Phi^{-1}(uv) \leq \Phi_1^{-1}(u)\Phi^{-1}(v) \leq \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leq 4\Psi^{-1}(uv).$$

If we take  $x = \Psi_1^{-1}(u) \geq \Psi_1^{-1}(\Phi_1(1))$  and  $y = \Psi^{-1}(v) \geq 0$ , then

$$\Psi\left(\frac{xy}{4}\right) \leq \Psi_1(x)\Psi(y).$$

Now, taking  $x \geq \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$  we get that  $\Psi \in \Delta_2$  globally.  $\square$

The following lemma generalizes [11, Lemma 5.2].

**Lemma 3.4.** *Let  $\Phi, \Psi$  be complementary  $N$ -functions with  $\Psi \in \Delta_2$  globally. Let  $\Phi_1$  be any  $N$ -function satisfying (19). Then*

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \quad (21)$$

for every  $\Phi_0$  with  $\Phi_0 \ll \Phi_1$ .

If (21) holds for some  $N$ -function  $\Phi_0$ , then  $\Psi \in \Delta_2$  (at  $\infty$ ).

*Proof.* By the assumptions on  $\Phi$  and  $\Phi_1$  and inequality (8), for  $r > 1$  we have

$$\frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \Phi_1(r) \frac{\int_0^T \Phi(r^{-1}|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})} \{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose  $r = \frac{\|u\|_{L^\Phi}}{2}$  and as  $\|u\|_{L^\Phi} \rightarrow \infty$  we can assume  $r > 1$  and by [3, Thm. 2 (b), p. 16].

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

Finally, if  $\Phi_0$  is an  $N$ -function, then  $\Phi_0(u) \geq k|u|$  for  $k$  small enough and  $|u| > 1$ . Therefore (21) holds for  $\Phi_0(u) = |u|$ , then [11, Lemma 5.2] implies  $\Psi \in \Delta_2$  at  $\infty$ .  $\square$

*Remark 1.* We point out that this lemma can be applied to more cases than [11, Lemma 5.2]. For example, if  $\Phi(u) = u^2$ ,  $\Phi_1$  and  $\Phi_0$  are  $N$ -functions with principal parts equal to  $u^2/\log u$  and  $u^2/(\log u)^2$  respectively (see [2, p. 16] and [2, Sec. 7] for the definition and properties of principal part), then (21) holds for  $\Phi_0$ . However,  $\Phi_0(u)$  is not dominated for any power function  $|u|^\alpha$  for every  $\alpha < 2$ .

**Definition 3.5.** We define the functionals  $J_{C,\Phi_0} : L^\Phi \rightarrow (-\infty, +\infty]$  and  $H_{C,\Phi_0} : \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $C > 0$  and  $\Phi_0$  is an  $N$ -function, by

$$J_{C,\Phi_0}(u) := \rho_\Phi(u) - C\Phi_0(\|u\|_{L^\Phi}), \quad (22)$$

and

$$H_{C,\Phi_0}(x) := \int_0^T F(t, x) dt - C\Phi_0(|x|), \quad (23)$$

respectively.

In [16] and [17] the authors considered, for the  $p$ -laplacian case, potentials  $F$  satisfying the inequality

$$|\nabla F(t, x)| \leq b_1(t)|x|^\alpha + b_2(t),$$

where  $b_1, b_2 \in L^1_1$  and  $\alpha < p$ . Thus, they called  $F$  a sublinear nonlinearity. In this paper, we will consider bounds on  $\nabla F$  of a more general type.

**Definition 3.6.** Let  $\Phi_0$  be a differentiable  $N$ -function. We say that  $G : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies a  $\Phi_0$ -grow condition if

$$|G(t, x)| \leq b_1(t)\Phi'_0(|x|) + b_2(t), \quad (24)$$

with  $b_1, b_2 \in L^1([0, T], \mathbb{R})$ .

**Theorem 3.7.** Let  $\Phi$  be an  $N$ -function whose complementary function  $\Psi$  satisfies the  $\Delta_2$  condition globally. Assume that the  $N$ -function  $\Phi_1$  satisfies (19),  $F$  satisfies (C) and (A), and  $\nabla F$  satisfies a  $\Phi_0$ -grow condition for some  $\Delta_2$ -globally  $N$ -function  $\Phi_0$  such that  $\Phi_0 \ll \Phi_1$ . Furthermore, we suppose that

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\Phi_0(|x|)} = +\infty. \quad (25)$$

Then, the problem (1) has at least a solution which minimizes the action integral  $I$  on  $W^1 E_T^\Phi$ .

*Proof.* By the decomposition  $u = \bar{u} + \tilde{u}$ , Cauchy-Schwarz's inequality and (24), we have

$$\begin{aligned} \left| \int_0^T F(t, u) - F(t, \bar{u}) dt \right| &= \left| \int_0^T \int_0^1 \nabla F(t, \bar{u} + s\tilde{u}(t)) \cdot \tilde{u}(t) ds dt \right| \\ &\leq \int_0^T \int_0^1 b_1(t) \Phi'_0(|\bar{u} + s\tilde{u}(t)|) |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 b_2(t) |\tilde{u}(t)| ds dt \\ &=: I_1 + I_2. \end{aligned} \quad (26)$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate  $I_2$  as follows

$$I_2 \leq \|b_2\|_{L^1} \|\tilde{u}\|_{L^\infty} \leq C_1 \|u'\|_{L^\Phi}, \quad (27)$$

where  $C_1 = C_1(\|b_2\|_{L^1}, T)$ .

On the other hand, since  $\Phi_0 \in \Delta_2$  globally, then  $\Phi'_0 \in \Delta_2$  globally and consequently  $\Phi'_0$  is a quasi-subadditive function, i.e. there exists  $C(\Phi_0) > 0$  such that  $\Phi'_0(a + b) \leq C(\Phi_0)(\Phi'_0(a) + \Phi'_0(b))$  for every  $a, b \geq 0$ . In this way, we have

$$\Phi'_0(|\bar{u} + s\tilde{u}(t)|) \leq C(\Phi_0)[\Phi'_0(|\bar{u}|) + \Phi'_0(\|\tilde{u}\|_{L^\infty})], \quad (28)$$

for every  $s \in [0, 1]$ .

Now, inequality (28), Hölder's and Sobolev-Wirtinger's inequalities, the monotonicity, the subadditivity and the  $\Delta_2$ -condition on  $\Phi'_0$ , imply that

$$\begin{aligned} I_1 &\leq C(\Phi'_0) \left\{ \Phi'_0(|\bar{u}|) \|b_1\|_{L^1} \|\tilde{u}\|_{L^\infty} + \|b_1\|_{L^1} \Phi'_0(\|\tilde{u}\|_{L^\infty}) \|\tilde{u}\|_{L^\infty} \right\} \\ &\leq C_2 \left\{ \Phi'_0(|\bar{u}|) \|u'\|_{L^\Phi} + \Phi'_0(\|u'\|_{L^\Phi}) \|u'\|_{L^\Phi} \right\}, \end{aligned} \quad (29)$$

where  $C_2 = C_2(\Phi'_0, T, \|b_1\|_{L^1})$ .

Next, by Young's inequality with complementary functions  $\Phi_0$  and  $\Psi_0$  and the fact that  $\Phi_0 \in \Delta_2$  globally, Young's equality [2, Eq. 2.7-2.8] and [3, Thm. 3-(ii), p. 23], we get

$$\begin{aligned} \Phi'_0(|\bar{u}|) \|u'\|_{L^\Phi} &\leq \Psi_0(\Phi'_0(|\bar{u}|)) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leq |\bar{u}| \Phi'_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leq C(\Phi_0) \Phi_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \end{aligned} \quad (30)$$

and

$$\Phi'_0(\|u'\|_{L^\Phi}) \|u'\|_{L^\Phi} \leq C(\Phi_0) \Phi_0(\|u'\|_{L^\Phi}), \quad (31)$$

with  $C(\Phi_0)$  the constant that comes from the  $\Delta_2$ -condition on  $\Phi_0$ .

From (29), (30), (31) and (27), we have

$$\begin{aligned} I_1 + I_2 &\leq C_3 \left\{ \Phi_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi} \right\} \\ &\leq C_4 \left\{ \Phi_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + 1 \right\}, \end{aligned} \quad (32)$$



with  $C_3$  and  $C_4$  depending on  $\Phi_0, T, \|b_1\|_{L^1}$  and  $\|b_2\|_{L^1}$ . The last inequality follows from the fact that  $\Phi_0$  is an  $N$ -function, then there exists  $C > 0$  such that  $\Phi_0(x) \geq Cx$  for every  $x \geq 1$ . Thus  $x \leq C\Phi_0(x) + 1$  for every  $x \geq 0$ .

In the subsequent estimates, we use (26), (32), the fact that  $\Phi_0 \in \Delta_2$  and we get

$$\begin{aligned}
 I(u) &= \rho_\Phi(u') + \int_0^T F(t, u) dt \\
 &= \rho_\Phi(u') + \int_0^T [F(t, u) - F(t, \bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\
 &\geq \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + \int_0^T F(t, \bar{u}) dt - C_4\Phi_0(|\bar{u}|) - C_4 \\
 &\geq \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + H_{C_4, \Phi_0}(\bar{u}) - C_4 \\
 &= J_{C_4, \Phi_0}(u') + H_{C_4, \Phi_0}(\bar{u}) - C_4.
 \end{aligned} \tag{33}$$

Let  $u_n$  be a sequence in  $\mathcal{E}_d^\Phi(\lambda)$  with  $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$  and we have to prove that  $I(u_n) \rightarrow \infty$ . On the contrary, suppose that for a subsequence, still denoted by  $u_n$ ,  $I(u_n)$  is upper bounded, i.e. there exists  $M > 0$  such that  $|I(u_n)| \leq M$ . As  $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$ , from Lemma 3.2, we have  $|\bar{u}_n| + \|u'_n\|_{L^\Phi} \rightarrow \infty$ . Passing to a subsequence is necessary, still denoted  $u_n$ , we can assume that  $|\bar{u}_n| \rightarrow \infty$  or  $\|u'_n\|_{L^\Phi} \rightarrow \infty$ . Now, Lemma 3.4 implies that the functional  $J_{C_4, \Phi_0}(u')$  is coercive; and, by (25), the functional  $H_{C_4, \Phi_0}(\bar{u})$  is also coercive, then  $J_{C_4, \Phi_0}(u'_n) \rightarrow \infty$  or  $H_{C_4, \Phi_0}(\bar{u}_n) \rightarrow \infty$ . From the condition (A) on  $F$ , we have that on a bounded set the functional  $H_{C_4, \Phi_0}(\bar{u}_n)$  is lower bounded and also  $J_{C_4, \Phi_0}(u'_n) \geq 0$ . Therefore,  $I(u_n) \rightarrow \infty$  as  $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$  which contradicts the initial assumption on the behavior of  $I(u_n)$ .

Let  $\{u_n\} \subset W^1 E_T^\Phi$  be a minimizing sequence for the problem  $\inf\{I(u) | u \in W^1 E_T^\Phi\}$ . Since  $I(u_n)$ ,  $n = 1, 2, \dots$ , is upper bounded, the previous part of the proof shows that  $\{u_n\}$  is norm bounded in  $W^1 E^\Phi$ . Hence, by virtue of [11, Cor. 2.2], we can assume, taking a subsequence if necessary, that  $u_n$  converges uniformly to a  $T$ -periodic continuous (therefore in  $E_T^\Phi$ ) function  $u$ . As  $u'_n \in E^\Phi$  is a norm bounded sequence in  $L^\Phi$ , there exists a subsequence (again denoted by  $u'_n$ ) such that  $u'_n$  converges to a function  $v \in L^\Phi$  in the weak\* topology of  $L^\Phi$ . Since  $E^\Phi$  is weak\* closed, by Lemma 3.1,  $v \in E^\Phi$ . From this fact and the uniform convergence of  $u_n$  to  $u$ , we obtain that

$$\int_0^T \xi' \cdot u dt = \lim_{n \rightarrow \infty} \int_0^T \xi' \cdot u_n dt = - \lim_{n \rightarrow \infty} \int_0^T \xi \cdot u'_n dt = - \int_0^T \xi \cdot v dt$$

for every  $T$ -periodic function  $\xi \in C^\infty([0, T], \mathbb{R}^d) \subset E^\Psi$ . Thus  $v = u'$  a.e.  $t \in [0, T]$  (see [18, p. 6]) and  $u \in W^1 E_T^\Phi$ .

Now, taking into account the relations  $[L^1]^* = L^\infty \subset E^\Psi$  and  $L^\Phi \subset L^1$ , we have that  $u'_n$  converges to  $u'$  in the weak topology of  $L^1$ . Consequently, from the semicontinuity of  $I$  (see [11, Lemma 6.1]) we get

$$I(u) \leq \liminf_{n \rightarrow \infty} I(u_n) = \inf_{v \in W^1 E_T^\Phi} I(v).$$

Hence  $u \in W^1 E_T^\Phi$  is a minimum and, since  $I$  is Gâteaux differentiable on  $W^1 E^\Phi$  (see [11, Thm. 3.2]), therefore  $I'(u) \in (W^1 E_T^\Phi)^\perp$ . Thus,

$$\int_0^T \frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t) \cdot v'(t) dt = - \int_0^T \nabla F(t, u(t)) \cdot v(t) dt,$$

for every  $v \in W^1 E_T^\Phi$ .

From [11, Lemma 2.4] we have  $u'(t)\Phi'(|u'(t)|)/|u'(t)| \in L^\Psi([0, T], \mathbb{R}^n) \hookrightarrow L^1([0, T], \mathbb{R}^n)$ ; and, from condition (A) and the fact that  $u \in L^\infty$ , it follows that  $\nabla F(t, u(t)) \in L^1([0, T], \mathbb{R}^n)$ . Consequently, from [18, p. 6] we obtain that the differential equations in (1) are verified and  $u'(0)\Phi'(|u'(0)|)/|u'(0)| = u'(T)\Phi'(|u'(T)|)/|u'(T)|$  holds. Thus  $u'(0) = u'(T)$ .  $\square$

### 3.1 More general action integrals

## 4 Examples

The employment of  $N$ -functions instead of power functions in inequalities like (24) will allow us to extend some results of [16] and [17], not only to the  $\Phi$ -laplacian operator, but even in the case of  $p$ -laplacian operator we get bounds that may be more sharp than those in [16, 17]. More precisely, in [17, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions for the  $p$ -laplacian operator under the assumption

$$|\nabla F| \leq b_1(t)|x|^\alpha + b_2(t).$$

for any  $\alpha \in (0, p-1)$

Assuming  $\|b_1\|_{L^1}$  small enough, in [19, 17] coercivity was obtained even for the limit value  $\mu = p$  in inequality (24).

**OJO que  $\mu$  no aparece en (24)!!!!. Quizás debería decir  $\Phi'_0(x) = x^p$ . O, mencionarse la ecuación anterior donde aparece  $\alpha < p$ , no  $\mu$ .**

This result leans on the fact that

$$\|u\|_{L^\Phi}^{\alpha_\Phi} = O\left(\int_0^T \Phi(|u|) dt\right) \quad \text{for } \|u\|_{L^\Phi} \rightarrow \infty, \quad (34)$$

when  $\Phi(u) = |u|^p$ . Nevertheless, it is no longer the case for any  $N$ -function  $\Phi$  as the following example shows.

In this section, from now on we will suppose that

$$\Phi(u) = \begin{cases} \frac{p-1}{p} u^p & u \leq e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with  $p > 1$ . Next, we will establish some properties of this function  $\Phi$ .

**Theorem 4.1.** *If  $p \geq \frac{1+\sqrt{2}}{2}$ , then  $\Phi$  is an  $N$ -function.*

*Proof.* We have

$$\varphi(u) = \Phi'(u) = \begin{cases} (p-1)u^{p-1} & := \varphi_1(u) \quad \text{if } u \leq e \\ \frac{u^{p-1}}{\log u} \left(p - \frac{1}{\log u}\right) & := \varphi_2(u) \quad \text{if } u \geq e \end{cases}$$

First let us see that  $\Phi'$  is increasing when  $p \geq \frac{1+\sqrt{2}}{2}$ . For this purpose, since  $\varphi_1(e) = \varphi_2(e)$ , it is enough to see that  $\varphi_1$  is increasing on  $[0, e]$  and  $\varphi_2$  is increasing on  $[e, \infty)$  for every  $p \geq \frac{1+\sqrt{2}}{2}$ . Clearly  $\varphi_1$  is an increasing function for  $p > 1$ . On the other hand, an elementary analysis of the function shows that  $\varphi_2'(u) > 0$  on  $[e, \infty)$  if and only if  $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$ . Therefore  $\varphi_2$  is an increasing function when  $p \geq \frac{1+\sqrt{2}}{2}$ .

Besides  $\varphi_2(u) \rightarrow \infty$  and  $\varphi_1(u) \rightarrow 0$  as  $u \rightarrow \infty$  and  $u \rightarrow 0$  respectively, provided that  $p > 1$ . Hence,  $\Phi$  is an  $N$ -function.  $\square$

**Theorem 4.2.** *For every  $\varepsilon > 0$ , there exists a positive constant  $C = C(p, \varepsilon)$  such that*

$$C^{-1}t^{p-\varepsilon}\Phi(u) \leq \Phi(tu) \leq Ct^p\Phi(u) \quad t \geq 1, u > 0, \quad (35)$$

*Proof.* If  $u \leq tu \leq e$ , then  $\Phi(tu) = t^p\Phi(u)$  and (35) holds with  $C = 1$ .

If  $u \leq e \leq tu$ , as  $\frac{e^p}{p} > 0$  and  $\log(tu) \geq 1$ , we have  $\Phi(tu) \leq t^p u^p = \frac{p}{p-1} t^p \Phi(u)$ . Thus, the second inequality of (35) holds with  $C = \frac{p}{p-1}$ . On the other hand, as  $f(t) = \frac{t}{\log t}$  is increasing on  $[e, \infty)$ , then  $f((tu)^p) \geq f(e^p) = e^p/p$ . Now,

$$\begin{aligned} \Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} \\ &\geq \frac{p-1}{p} \frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p. \end{aligned}$$

Since  $\varepsilon e^{1-\varepsilon}$  is the minimum value of  $t \mapsto \frac{t^\varepsilon}{\log t + 1}$  on the interval  $[1, +\infty)$  then

$$\Phi(tu) \geq \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,$$

which is the first inequality of (35) with  $C = \frac{p}{p-1} \varepsilon^{-1} e^{-1+\varepsilon}$ .

If  $e \leq u \leq tu$ , then

$$\Phi(tu) \leq \frac{t^p u^p}{\log(tu)} \leq \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v}, \quad (36)$$

where  $v := u^p$  and  $v \geq e^p$ . If  $\alpha > 0$ , the function  $x \mapsto \frac{x}{x-\alpha}$  is decreasing on  $(\alpha, \infty)$  and the function  $v \mapsto \frac{pv}{\log v}$  is increasing on  $[e^p, \infty)$ . Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leq \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every  $v \geq e^p$ . In this way, from (36), we have

$$\Phi(tu) \leq \frac{pt^p}{p-1} \left( \frac{pv}{\log v} - \frac{e^p}{p} \right) = \frac{pt^p}{p-1} \left( \frac{u^p}{\log u} - \frac{e^p}{p} \right)$$

and the second inequality of (35) holds with  $C = \frac{p}{p-1}$ . For the first inequality we have, as it was proved previously,

$$\Phi(tu) \geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} = \frac{p-1}{p} \frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)} \frac{t^{p-\varepsilon} u^p}{\log u}$$

Let  $f(s) = \frac{sA}{\log s + A}$  with  $s \geq 1$  and  $A \geq \varepsilon$ . If  $A \leq 1$ , the function  $f$  attains a minimum on  $[1, \infty)$  at  $s = e^{1-A}$  and the minimum value is  $f(e^{1-A}) = Ae^{1-A} \geq \varepsilon$ . If  $A > 1$ ,  $f$  is increasing on  $[1, \infty)$  and its minimum value is  $f(1) = 1$ . Then,  $f(s) \geq \varepsilon$  in any case, therefore

$$\Phi(tu) \geq \frac{p-1}{p} \varepsilon \frac{t^{p-\varepsilon} u^p}{\log u} \geq \frac{p-1}{p} \varepsilon t^{p-\varepsilon} \Phi(u).$$

Therefore, (35) holds with  $C = \frac{p}{\varepsilon(p-1)}$ , because this  $C$  is the biggest constant that we have obtained in each case under consideration.  $\square$

*Remark 2.* The inequality

$$\Phi(tu) \geq Ct^p \Phi(u)$$

is false for every  $C$  because for every  $u \geq e$  we have

$$\lim_{t \rightarrow \infty} \frac{\Phi(tu)}{t^p \Phi(u)} = 0$$

**Theorem 4.3.**  $\alpha_\Phi = \beta_\Phi = p$

*Proof.* From (??) and (35), we get

$$\beta_\Phi = \lim_{t \rightarrow \infty} \frac{\log \left[ \sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} \leq \lim_{t \rightarrow \infty} \frac{\log C + p \log t}{\log t} = p.$$

On the other hand, employing (??) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t \rightarrow 0^+} \frac{\log \left[ \sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} = \lim_{s \rightarrow \infty} \frac{\log \left[ \sup_{v>0} \frac{\Phi(v)}{\Phi(sv)} \right]^{-1}}{\log s} = \lim_{s \rightarrow \infty} \frac{\log \left[ \inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s}$$

where  $v := tu$  and  $s := \frac{1}{t}$ . Then, using (35), for every  $\varepsilon > 0$  we have

$$\alpha_\Phi = \lim_{s \rightarrow \infty} \frac{\log \left[ \inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s} \geq \lim_{s \rightarrow \infty} \frac{\log C + (p - \varepsilon) \log s}{\log s} \geq p - \varepsilon,$$

therefore  $\alpha_\Phi \geq p$ .

Finally, as  $\alpha_\Phi \leq \beta_\Phi \leq p$ , we get  $\alpha_\Phi = \beta_\Phi = p$ .  $\square$

Now, we are able to see that

$$\rho_{\Phi}(u) = \int_0^T \Phi(|u|) dx \geq C \|u\|_{L^{\Phi}}^{\alpha_{\Phi}} = C \|u\|_{L^{\Phi}}^p$$

is false.

In fact, if we take  $u \equiv t > 0$ , then  $\|u\|_{L^{\Phi}}^p = C_1 t^p$  where  $C_1 = \|1\|_{L^{\Phi}}$  and  $\int_0^T \Phi(|u|) dx = C_2 \Phi(t)$  with  $C_2 = T$ . Then, if  $\rho_{\Phi}(u) \geq C \|u\|_{L^{\Phi}}^p$  were true, then  $\Phi(t) \geq C t^p$  would also be true; however, this last inequality is false.

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