Euler-Lagragian equations in an Orlicz-Sobolev space setting

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Abstract

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1 Introduction

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. We refer to [2, 12, 19] for additional details and proofs. In the first two references scalar valued function are considered, however the generalization of the results enumerated below to vector valued functions is direct. Last one reference consider vector valued functions.

Hereafter we denote by $\mathbb{R}^+$ to the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an N*-function* if it has the form

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \text{for } u \geq 0,$$

where $\varphi : \mathbb{R}^+ \to: \mathbb{R}^+$ is a right continuous nondecreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Given a function φ as above, we also consider the so-called right inverse function ψ of φ which is defined $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties that function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ is a *function of the* Δ_2 *class* when there exists a constant $k > 0$ and a $t_0 \geq 0$ such that $\Phi(2t) \leqslant K\Phi(t)$, for every $t \geq t_0$. If $t_0 = 0$ we said that Φ is Δ_2 *global*.

In this paper we adopt the convention of to use bold symbols for denote points in $\mathbb{R}^n$ and plain symbols for scalar ones.

For n positive integer we denote by $M_n := M_n([0, T])$ the set of all measurable functions defined in $[0, T]$ with values in $\mathbb{R}^n$. Given a N-function Φ we define the *modular function* $\rho_\Phi : M_n \to: \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_{[0,T]} \Phi(|\boldsymbol{u}|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^n$. The *Orlicz class* $C_n^\Phi = C_n^\Phi([0, T])$ is defined by

$$C_n^\Phi := \{\boldsymbol{u} \in M_n | \rho_\Phi(\boldsymbol{u}) < \infty\}. \tag{1}$$

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The *Orlicz space* $L^\Phi_n = L^\Phi_n([0,T])$ is the linear hull of C^Φ_n. Equivalently

$$L^\Phi_n := \{\boldsymbol{u} \in M_n | \exists \lambda > 0 : \rho_\Phi(\lambda \boldsymbol{u}) < \infty\}. \tag{2}$$

The Orlicz space L^Φ_n equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup\left\{ \int_0^T \boldsymbol{u}\cdot\boldsymbol{v}dt \middle| \rho_\Psi(\boldsymbol{v}) \leqslant 1 \right\},$$

is a Banach space. By $\boldsymbol{u}\cdot\boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^n$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. Sometimes the following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [12, Th. 10.5] and [9]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k}\left\{1 + \rho_\Phi(k\boldsymbol{u})\right\}. \tag{3}$$

The subspace $E^\Phi_n = E^\Phi_n([0,T])$ is defined as the closure in L^Φ_n of the subspace L^∞_n of all the $\mathbb{R}^n$-valued essentially bounded functions. It is showed that E^Φ_n is the only one maximal subspace contained in the Orlicz class C^Φ, that is $\boldsymbol{u} \in E^\Phi_n$ if and only if for any $\lambda > 0$ we have $\rho_\Phi(\lambda\boldsymbol{u}) < \infty$.

A generalizated version of *Hölder inequality* holds in the setting of Orlicz spaces (ver [12, Th 9.3]). Namely, if $\boldsymbol{u} \in L^\Phi_n$ and $\boldsymbol{v} \in L^\Psi_n$ then $\boldsymbol{u}\cdot\boldsymbol{v} \in L^1_1$ and

$$\int_0^T \boldsymbol{v}\cdot\boldsymbol{u}dt \leqslant \|\boldsymbol{u}\|_{L^\Phi}\|\boldsymbol{v}\|_{L^\Psi}. \tag{4}$$

If X and Y are Banach spaces, with $Y \subset X^*$ we denote by $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$ to the bilinear pairing map given by $\langle x^*, x\rangle = x^*(x)$. Hölder inequality shows that $L^\Psi_n \subset \left[L^\Phi_n\right]^*$, where the pairing $\langle\boldsymbol{u},\boldsymbol{v}\rangle$, $\boldsymbol{u} \in L^\Phi_n$ and $\boldsymbol{v} \in L^\Psi_n$, is defined by

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle = \int_0^T \boldsymbol{u}\cdot\boldsymbol{v}dt. \tag{5}$$

Unless Φ be a Δ_2 function, the relation $L^\Psi_n = \left[L^\Phi_n\right]^*$ does not holds. It is true in general that $\left[E^\Phi_n\right]^* = L^\Psi_n$.

Likes in [12], we will consider the subset $\Pi(E^\Phi_n, r)$ of L^Φ_n defined by

$$\Pi(E^\Phi_n, r) := \{\boldsymbol{u} \in L^\Phi_n | d(\boldsymbol{u}, E^\Phi_n) < r\}.$$

This set is related to the Orlicz class C^Φ_n by means of inclusions

$$\Pi(E^\Phi_n, 1) \subset C^\Phi_n \subset \overline{\Pi(E^\Phi_n, 1)}. \tag{6}$$

The proof of this fact, and similar ones, is given by real valued function in [12], the extension to $\mathbb{R}^n$-valued functions does not involve any difficulty. When the function Φ is of the Δ_2 class then the four sets L^Φ_n, E^Φ_n, $\Pi(E^\Phi_n, 1)$ and C^Φ_n are equal.

We will use the following elementary fact frequently

$$\boldsymbol{u} \in \Pi(E^\Phi_n, \lambda) \implies \frac{\boldsymbol{u}}{\lambda} \in \Pi(E^\Phi_n, 1) \subset C^\Phi_n. \tag{7}$$

We define the *Sobolev-Orlicz space* $W^1L^\Phi_n$ (see [2]) by

$$W^1L^\Phi_n := \{\boldsymbol{u}|\boldsymbol{u} \text{ is absolutely continuous and } \boldsymbol{u}, \dot{\boldsymbol{u}} \in L^\Phi_n\}.$$

This space is a Banach space equipped with the norm

$$\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}.$$

For a function $\boldsymbol{u} \in L^1_n([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$, where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t)\,dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the setting of Orlicz-Sobolev spaces, see for example [4, 5, 6, 7, 11]. For this reason the following simple Lemma, which we will use systematically, it is well known. We include a brief proof for sake of completeness.

Lemma 2.1. *Let* $\boldsymbol{u} \in W^1L^\Phi_n$. *Then* $\boldsymbol{u} \in L^\infty_n([0,T])$ *and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Psi^{-1}\left(\frac{1}{T}\right)\|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Wirtinger's inequality)} \tag{8}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Psi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality)} \tag{9}$$

Proof. Since $\boldsymbol{u}$ is contiuous, from the mean value theorem there exists τ such that $\boldsymbol{u}(\tau) = \overline{\boldsymbol{u}}$, thus

$$|\boldsymbol{u}(t) - \overline{\boldsymbol{u}}| \leqslant \int_\tau^t |\dot{\boldsymbol{u}}(s)|ds \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi}\|1\|_{L^\Psi} \leqslant T\Psi^{-1}\left(\frac{1}{T}\right)\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{10}$$

Here we have used Hölder inequality and the formula for the norm of a characteristic function (ver [12, Eq. 9.11]). Inequality (10) proves Wirtinger's inequality (8).

On the other hand, again by Hölder inequality and [12, Eq. 9.11], we obtain

$$|\overline{\boldsymbol{u}}| \leqslant \frac{1}{T}\int_0^T |\boldsymbol{u}(s)|ds \leqslant \Psi^{-1}\left(\frac{1}{T}\right)\|\boldsymbol{u}\|_{L^\Phi}. \tag{11}$$

From (10),(11) and since $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ we obtain (9). $\square$

If $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subespace of X, as is usual we write $Y \hookrightarrow X$ and we say that Y is *embeeded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That means that there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, the Lemma 2.1 states $W^1L^\Phi_n \hookrightarrow L^\infty_n$ and Hölder inequality states that $L^\Psi_n \hookrightarrow \left[L^\Phi_n\right]^*$.

Given a contiuous function $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, we define the composition operator $\boldsymbol{a} : M_n \to M_n$ by $\boldsymbol{a}(\boldsymbol{u})(t) = a(|\boldsymbol{u}(t)|)$. We will use repeatedly the following elementary consequence of the previous lemma.

Corollary 2.2. *If $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi_n \to L^\infty_1([0,T])$ is bounded. More concretely there exists a non decreasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(\boldsymbol{u})\|_{L^\infty([0,T])} \leqslant c(\|\boldsymbol{u}\|_{W^1L^\Phi})$.*

Proof. Let α be a non-decreasing mayorant of a, for example $\alpha(s) := \sup_{0\leqslant t\leqslant s} a(t)$. If $\boldsymbol{u} \in W^1L^\Phi_n$ then by Lemma 2.1

$$a(|\boldsymbol{u}(t)|) \leqslant \alpha(\|\boldsymbol{u}\|_{L^\infty}) \leqslant a\left(\Psi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi}\right) =: c(\|\boldsymbol{u}\|_{W^1L^\Phi}).$$

□

The following lemma is an inmediate consequence of principles related to operators of Nemitskii type, see [12, §17].

Lemma 2.3. *The composition operator $\boldsymbol{\varphi}$ acts from $\Pi(E^\Phi_n,1)$ into C^Ψ_1.*

Proof. As consequence of [12, Lemma 9.1] we have that $\boldsymbol{\varphi}\left(B_{L^\Phi}(0,1)\right) \subset C^\Psi_1$, where $B_X(\boldsymbol{u}_0,r)$ is the open ball with center $\boldsymbol{u}_0$ and radius $r>0$ in the space X. Therefore, applying [12, Lemma 17.1] we deduce that $\boldsymbol{\varphi}$ acts from $\Pi(E^\Phi_n,1)$ into C^Ψ_1. □

We need also the following technical lemma.

Lemma 2.4. *Let $\lambda > 0$ and $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi_n,\lambda)$ converging to $\boldsymbol{u} \in \Pi(E^\Phi_n,\lambda)$ in the L^Φ-norm. Then there exist a subsequence $\boldsymbol{u}_{n_k}$ and a real valued function $h \in \Pi\left(E^\Phi_1([0,T]),\lambda\right)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e. and $|\boldsymbol{u}_{n_k}| \leqslant h$ a.e..*

Proof. Let $r := d(\boldsymbol{u},E^\Phi_n)$, $r<\lambda$. Because $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$, there exists a subsequence (n_k) such that

$$\|\boldsymbol{u}_{n_k}-\boldsymbol{u}\|_{L^\Phi} < \frac{\lambda-r}{2} \quad \text{and} \quad \|\boldsymbol{u}_{n_k}-\boldsymbol{u}_{n_{k+1}}\|_{L^\Phi} < 2^{-(k+1)}(\lambda-r)$$

Let $h:[0,T]\to\mathbb{R}$ defined by

$$h(x) = |\boldsymbol{u}_{n_1}(x)| + \sum_{k=2}^\infty |\boldsymbol{u}_{n_k}(x)-\boldsymbol{u}_{n_{k-1}}(x)|. \tag{12}$$

As a consequence of [12, Lemma 10.1] we have that, for any $\boldsymbol{v}\in L^\Phi_n$, $d(\boldsymbol{v},E^\Phi_n) = d(|\boldsymbol{v}|,E^\Phi_1)$. Therefore

$$d(|\boldsymbol{u}_{n_1}|,E^\Phi_1) = d(\boldsymbol{u}_{n_1},E^\Phi_n) \leqslant d(\boldsymbol{u}_{n_1},\boldsymbol{u}) + d(\boldsymbol{u},E^\Phi_n) < \frac{\lambda+r}{2}.$$

Then

$$d(h,E^\Phi_1) \leqslant d(h,|\boldsymbol{u}_{n_1}|) + d(|\boldsymbol{u}_{n_1}|,E^\Phi_1) < \lambda.$$

Therefore, $h\in\Pi(E^\Phi_1,\lambda)$. In particular, $|h|<\infty$ a.e. We conclude that the series $\boldsymbol{u}_{n_1}(x)+\sum_{k=2}^\infty(\boldsymbol{u}_{n_k}(x)-\boldsymbol{u}_{n_{k-1}}(x))$ is absolutely convergent a.e. This imply that $\boldsymbol{u}_{n_k}\to\boldsymbol{u}$ a.e.. The inequality $|\boldsymbol{u}_{n_k}|\leqslant h$ is clear from the definition of h. □

A common obstacle with Orlicz spaces, that distinguishes it from L^p spaces, is that a sequence $\boldsymbol{u}_n \in L^\Phi_n$ which is uniformly bounded by $h \in L^\Phi_1$ and a.e. convergent to $\boldsymbol{u}$ is not necessarily norm convergent. Fortunately the subspace E^Φ_n has that property.

Lemma 2.5. *Suppose that $\boldsymbol{u_n} \in L^\Phi_n$ is a sequence such that $\boldsymbol{u_n} \to \boldsymbol{u}$ a.e. and suppose that there exist $h \in E^\Phi_1$ with $|\boldsymbol{u_n}| \leqslant h$ a.e. then $\|\boldsymbol{u_n} - \boldsymbol{u}\|_{L^\Phi} \to 0$.*

Proof. [15, p.84] and [12, Th. 10.3]) □

We recall the definition of Gateâux derivative, see [3] for details. Given a function $I : U \to \mathbb{R}$ where U is an open set of a Banach space X, we say that I has a Gâteaux derivative en $\boldsymbol{u} \in U$ if there exists $\boldsymbol{u}^* \in X^*$ such that for every $\boldsymbol{v} \in X$

$$\lim_{s\to 0} \frac{I(\boldsymbol{u} + s\boldsymbol{v}) - I(\boldsymbol{u})}{s} = \langle \boldsymbol{u}^*, \boldsymbol{u}\rangle.$$

We recall the following definition.

Definition 2.6 (see [10]). *Let X a Banach space and $D \subset X$. A non linear operator $T : D \to X^*$ is called* demicontinuous *if it is continuous when X is equipped with the strong topology and X^* with tha weak* topology.*

3 Differetiability of action integrals on Orlicz spaces

Definition 3.1. *We said that a function $\mathcal{L} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory function if for fixed $(\boldsymbol{x}, \boldsymbol{y})$ the map $t \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is measurable and for fixed t the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is continuously differentiable for almost everywhere $t \in [0, T]$.*

In this paper we will consider Lagrangian functions satisfying the following structure conditions. We assume that there exists $\lambda > 0$ and non negative functions $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1_1([0, T])$, $c \in L^\Psi_1([0, T])$ and $d \in E^\Phi_1$ such that

$$\begin{aligned} |\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| &\leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + d(t)\right)\right), && (13)\\ |D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| &\leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + d(t)\right)\right), && (14)\\ |D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| &\leqslant a(|\boldsymbol{x}|)\left(c(t) + \varphi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + d(t)\right)\right). && (15) \end{aligned}$$

Remark 1. These conditions are a generalization of the frequently considered condition (A) (see [17, 20, 18, 21]). In fact, conditions (13),(14), (15) are equivalent to condition (A) when $\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{y}|^p/p + F(t, \boldsymbol{x})$, $\Phi_p(s) = s^p/p$, and $d = 0$.

Remark 2. As a direct consequence of convexity, we can bound the term $\Phi(|\boldsymbol{y}|/\lambda + d(t))$, $d \in E^\Phi_1$, by the expression $\frac{1}{2}\Phi(|\boldsymbol{y}|/\Lambda) + b(t)$ where $b(t) := \frac{1}{2}\Phi(2d(t)) \in L^1_1$, and $\Lambda = \lambda/2$. That is, we can assume $d = 0$ at the price of making smaller the value of λ.

Remark 3. Let us note that if $\Phi \in \Delta_2$ then we can asssume $d = 0$ keeping the same value of λ. This is consequence of that a non decreasing Δ_2 function $G : \mathbb{R}_+ \to \mathbb{R}_+$ is quasi-subadditive. In fact, we suppose $y \leqslant x$, then

$$G(x+y) \leqslant G(2x) \leqslant KG(x) \leqslant K\left(G(x) + G(y)\right).$$

Moreover, if Φ is Δ_2 then φ is also Δ_2, as the following simple argument shows

$$2x\varphi(2x) \leqslant \alpha\Phi(2x) \leqslant K\Phi(x) \leqslant Kx\varphi(x)$$

Here we have used [12, Th. 4.1], the Δ_2 condition for Φ and the inequality $\Phi(x) \leqslant x\varphi(x)$ valid for any N-function. Therefore if Φ is Δ_2 we have that

$$b(t) + \Phi\left(\frac{|\boldsymbol{y}|}{\lambda} + d(t)\right) \leqslant b(t) + K\Phi\left(d(t)\right) + K\Phi\left(\frac{|\boldsymbol{y}|}{\lambda}\right) = b_1(t) + K\Phi\left(\frac{|\boldsymbol{y}|}{\lambda}\right),$$

where $b_1(t) = b(t) + K\Phi\left(d(t)\right) \in L^1_1([0,T])$. A similar fact holds with φ instead Φ namely

$$c(t) + \varphi\left(\frac{|\boldsymbol{y}|}{\lambda} + d(t)\right) \leqslant c_1(t) + \varphi\left(\frac{|\boldsymbol{y}|}{\lambda}\right),$$

where, as consequence of Lemma 2.3 and the Δ_2 condition for Φ, we have $c_1(t) := c(t) + K\varphi\left(d(t)\right) \in L^\Psi_1$.

The following example shows that the condition Δ_2 is essential in the previous discussion. Let $\Phi(s) = e^s - s - 1$ and let d be a function in $L^\Phi_1 \setminus L^\infty_1$. We suppose that there exists $K > 0$ and $b \in L^1_1$ such that $\Phi(s + d(t)) \leqslant K\Phi(s) + b(t)$. This inequality implies for sufficient large s that $\frac{1}{2}e^s e^{d(t)} \leqslant Ke^s + b(t)$. As $d \notin L^\infty_1$ we can fix t with $e^{b(t)} > 4K$. Hence, we obtain $2Ke^s \leqslant Ke^s + b(t)$, which is evidently false as the limit $s \to +\infty$ shows.

Theorem 3.2. *Let $\mathcal{L}$ be a Caratheodory function satisfying* (13),(14)*,* (15)*. Then the following statements hold*

1. *The* action integral
$$I(\boldsymbol{u}) := \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt \tag{16}$$
is finitely defined in $\mathcal{E}^\Phi_n(\lambda) := W^1L^\Phi \cap \{\boldsymbol{u} | \dot{\boldsymbol{u}} \in \Pi(E^\Phi_n, \lambda)\}$.
2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi_n(\lambda)$ and its derivative I' is demi-continuous from $\mathcal{E}^\Phi_n(\lambda)$ into $\left[W^1L^\Phi\right]^*$. Moreover I' is given by the following expression*
$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big)\cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big)\cdot \dot{\boldsymbol{v}}\right\}\, dt. \tag{17}$$
3. *If Ψ is Δ_2 then I' is continuous from $\mathcal{E}^\Phi_n(\lambda)$ into $\left[W^1L^\Phi\right]^*$ when both spaces are equipped with the strong topology.*

Proof. From (7) we have $\dot{\boldsymbol{u}}/\lambda \in \Pi(E_n^\Phi, 1)$. Thus, as $d \in E_1^\Phi$ and attending to (6), we get

$$|\dot{\boldsymbol{u}}|/\lambda + d \in \Pi(E_1^\Phi, 1) \subset C_1^\Phi. \tag{18}$$

From Corollary 2.2 we get a constant $c = c(\|\boldsymbol{u}\|_{W^1L^\Phi})$ such that $a(|\boldsymbol{u}(t)|) \leqslant c$, $t \in [0,T]$. Thus,

$$|\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})| \leqslant c\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}(t)|}{\lambda} + d(t)\right)\right) \in L_1^1.$$

This fact proves item 1.

We split the proof of 2 in three steps.

Step 1. The non linear operator $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ is continuous from $\mathcal{E}_n^\Phi(\lambda)$ into $L_n^1([0,T])$ whith the strong topology on both sets.

We take $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ a sequence of functions in $\mathcal{E}_n^\Phi(\lambda)$, and $\boldsymbol{u} \in \mathcal{E}_n^\Phi(\lambda)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $W^1L_n^\Phi$. Then $\boldsymbol{u}_n \to \boldsymbol{u}$ in L_n^Φ and $\dot{\boldsymbol{u}}_n \to \dot{\boldsymbol{u}}$ in L_n^Φ. By Lemma 2.4 there exist a subsequence $\boldsymbol{u}_{n_k}$ and $h \in \Pi(E_1^\Phi, \lambda))$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k} \to \dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}| \leqslant h$ a.e.. Since $\boldsymbol{u}_{n_k}$, $k = 1, 2, \ldots$ is a strong convergent sequence in $W^1L_n^\Phi$, it is a bounded sequence in $W^1L_n^\Phi$. According to Lemmas 2.1 and Corollary 2.2 there exists $M > 0$ such that $\|\boldsymbol{a}(\boldsymbol{u}_{n_k})\|_{L^\infty} \leqslant M$, $k = 1, 2, \ldots$. From the previous facts, (14) and (18) we get

$$|D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}_{n_k}(t), \dot{\boldsymbol{u}}_{n_k}(t))| \leqslant M\left(b(t) + \Phi\left(\frac{|h|}{\lambda} + d(t)\right)\right) \in L_1^1. \tag{19}$$

By the Caratheodory condition

$$D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}_{n_k}(t), \dot{\boldsymbol{u}}_{n_k}(t)) \to D_x\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad \text{for a.e } t \in [0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $\boldsymbol{u} \mapsto D_y\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ is continuous from $\mathcal{E}_n^\Phi(\lambda)$ with the strong topology into $\left[L^\Phi\right]^*$ with the weak* topology.

Let $\boldsymbol{u} \in \mathcal{E}_n^\Phi(\lambda)$. It follows from (18), Lemma 2.3 and Corollary 2.2 that

$$\varphi\left(\frac{|\boldsymbol{u}|}{\lambda} + d\right) \in C_1^\Psi \tag{20}$$

and $\boldsymbol{a}(|\boldsymbol{u}|) \in L_1^\infty$. Therefore, in virtue of (15) we get

$$|D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))| \leqslant c(\|\boldsymbol{u}\|_{W^1L^\Phi})\left(c + \boldsymbol{\varphi}\left(\frac{|\boldsymbol{u}|}{\lambda} + d\right)\right) \in L_1^\Psi. \tag{21}$$

We note that (19), (21) , the imbedding $W^1L_n^\Phi \hookrightarrow L_n^\infty$ and $L_n^\Psi \hookrightarrow \left[L_n^\Phi\right]^*$ imply that the second member (17) defines an element in $\left[W^1L_n^\Phi\right]^*$.

Now, let us to prove the continuity of the map $\boldsymbol{u} \mapsto D_y\mathcal{L}(\cdot, \boldsymbol{u}, \dot{\boldsymbol{u}})$. We take $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}_n^\Phi(\lambda)$ with $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of $W^1L_n^\Phi$. We must prove that $D_y\mathcal{L}(\cdot, \boldsymbol{u}_n, \dot{\boldsymbol{u}_n}) \overset{w^*}{\rightharpoonup}$

$D_y\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}})$. Suppose, on the contrary, that there exists $\boldsymbol{v}\in L_n^\Phi$, $\epsilon>0$ and a subsequence of $\{\boldsymbol{u}_n\}$ (again denoted for simplicity $\{\boldsymbol{u}_n\}$) such that

$$|\langle D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_{\boldsymbol{n}}),\boldsymbol{v}\rangle-\langle D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}}),\boldsymbol{v}\rangle|\geq\epsilon. \tag{22}$$

We have $\boldsymbol{u}_n\to\boldsymbol{u}$ in L_n^Φ and $\dot{\boldsymbol{u}}_n\to\dot{\boldsymbol{u}}$ in L_n^Φ. By Lemma 2.4, there exist a subsequence $\boldsymbol{u}_{n_k}$ and $h\in\Pi(E^\Phi,\lambda)$ such that $\boldsymbol{u}_{n_k}\to\boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k}\to\dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}|\leqslant h$ a.e.. As in the previous step, since $\boldsymbol{u}_n$ is a convergent sequence, the Corrollary 2.2 implies that $a(|\boldsymbol{u}_n(t)|)$ is uniformly bounded by certain constant C. Therefore, from (15), (20), the fact that $c\in L_1^\Phi$, Hölder inequality we obtain

$$|D_y\mathcal{L}(\cdot,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n)\cdot\boldsymbol{v}|\leqslant C\left(c|\boldsymbol{v}|+\varphi\left(\frac{h}{\lambda}+d\right)\right)|\boldsymbol{v}|\in L_1^1.$$

From the Lebesgue dominated convergence theorem we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k})\cdot\boldsymbol{v}dt\to\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\boldsymbol{v}dt \tag{23}$$

which contradict the inequality (22). This completes the proof of step 2.

Step 3. Finally we prove 2. The proof follows similar lines that [14, Theorem 1.4]. For $\boldsymbol{u}\in\mathcal{E}_n^\Phi(\lambda)$ and $\boldsymbol{v}\in W^1L_n^\Phi$ we define the function

$$f(s,t):=\mathcal{L}(t,\boldsymbol{u}(t)+s\boldsymbol{v}(t),\dot{\boldsymbol{u}}(t)+s\dot{\boldsymbol{v}}(t)).$$

From [12, Th. 10.1] we obtain that if $|\boldsymbol{u}|\leqslant|\boldsymbol{v}|$ then $d(\boldsymbol{u},E_n^\Phi)\leqslant d(\boldsymbol{v},E_n^\Phi)$. Therefore, for $|s|\leqslant s_0:=\left(\lambda-d(\dot{\boldsymbol{u}},E_n^\Phi)\right)/\|\boldsymbol{v}\|_{W^1L^\Phi}$ we have

$$d\left(\dot{\boldsymbol{u}}+s_0\dot{\boldsymbol{v}},E_n^\Phi\right)\leqslant d\left(|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|,E_1^\Phi\right)\leqslant d\left(|\dot{\boldsymbol{u}}|,E_1^\Phi\right)+s_0\|\dot{\boldsymbol{v}}\|_{L^\Phi}<\lambda.$$

As a consequence $\dot{\boldsymbol{u}}+s_0\dot{\boldsymbol{v}}\in\Pi(E_n^\Phi,\lambda)$ and $|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|\in\Pi(E_1^\Phi,\lambda)$. These facts imply, in virtue of Theorem 3.2(1) that $I(\boldsymbol{u}+s\boldsymbol{v})$ is well defined and it is finite for $|s|\leqslant s_0$. Using Corollary 2.2 we see that

$$\|a(|\boldsymbol{u}+s\boldsymbol{v}|)\|_{L^\infty}\leqslant c(\|\boldsymbol{u}+s\boldsymbol{v}\|_{W^1L^\Phi})\leqslant c(\|\boldsymbol{u}\|_{W^1L^\Phi}+s_0\|\boldsymbol{v}\|_{W^1L^\Phi}).$$

Consequently, applying chain rule, inequalities (14)-(15), the previous inequality and using that φ and Φ are non decreasing, we obtain

$$\begin{aligned}|D_sf(s,t)|&=|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}+s\boldsymbol{v},\dot{\boldsymbol{u}}+s\dot{\boldsymbol{v}})\cdot\boldsymbol{v}+D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}+s\boldsymbol{v},\dot{\boldsymbol{u}}+s\dot{\boldsymbol{v}})\cdot\dot{\boldsymbol{v}}|\\&\leqslant c\left[\left(b(t)+\Phi\left(\frac{|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|}{\lambda}\right)\right)|\boldsymbol{v}|\right.\\&\qquad\left.+\left(c(t)+\varphi\left(\frac{|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|}{\lambda}\right)\right)|\dot{\boldsymbol{v}}|\right]\end{aligned} \tag{24}$$

Invoking (19), (21) with $|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|$ instead $\boldsymbol{u}$ and taking account of $\dot{\boldsymbol{v}}\in L^\infty$ and $\boldsymbol{v}\in L^\Phi$ we show that there exists a function $g\in L_1^1([0,T],\mathbb{R}^+)$ such that $|D_sf(s,t)|\leqslant g(t)$. Consequently, I has a directional derivative and

$$\langle v,I'(\boldsymbol{u})\rangle=\frac{d}{ds}I(\boldsymbol{u}+sv)\big|_{s=0}=\int_0^T D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\boldsymbol{v}+D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\dot{\boldsymbol{v}}dt.$$

Moreover, from (19), (21), Lemma 2.1 and previous formula

$$|\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle| \leqslant c\|v\|_{L^\infty} + c\|\dot{\boldsymbol{v}}\|_{L^\Phi} \leqslant c\|\boldsymbol{v}\|_{W^1L^\Phi}.$$

This complete the proof of the Gâteaux differentiability of I. Finally, the demicontinuity of $I' : \mathcal{E}_n^\Phi(\lambda) \to \left[W^1L^\Phi\right]^*$ is a consequence of the continuity of the mappings $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$. Indeed, we set $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}_n^\Phi(\lambda)$ with $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of W^1L^Φ and $\boldsymbol{v} \in W^1L^\Phi$, then

$$\begin{aligned}
\langle I'(\boldsymbol{u}_n), \boldsymbol{v}\rangle &= \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\left(t, \boldsymbol{u}_n, \dot{\boldsymbol{u}}_n\right) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t, \boldsymbol{u}_n, \dot{\boldsymbol{u}}_n\right) \cdot \dot{\boldsymbol{v}}\right\}\, dt \\
&\to \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\left(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\right) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\right) \cdot \dot{\boldsymbol{v}}\right\}\, dt \\
&= \left\langle I'(\boldsymbol{u}), \boldsymbol{v}\right\rangle .
\end{aligned}$$

In order to prove 3, let us see that the maps $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(\cdot, \boldsymbol{u}(\cdot), \dot{\boldsymbol{u}}(\cdot))$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(\cdot, \boldsymbol{u}(\cdot), \dot{\boldsymbol{u}}(\cdot))$ are norm continuous from $\mathcal{E}_n^\Phi(\lambda)$ into L^1 and L^Ψ respectively. The continuity of the first map has already been proved in step 1. We will prove the continuity of the second map. We repeat an argument similar to the one given in step 2. We consider $\boldsymbol{u}_n$ and $\boldsymbol{u}$ in $\mathcal{E}_n^\Phi(\lambda)$ with $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{W^1L^\Phi} \to 0$. By Lemma 2.4, there exist a subsequence $\boldsymbol{u}_{n_k}$ and $h \in \mathcal{E}_n^\Phi(\lambda)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k} \to \dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}| \leqslant h$ a.e.. Then since $\mathcal{L}$ is a Caratheodory function we have $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}_{n_k}(t), \dot{\boldsymbol{u}}_{n_k}(t)) \to D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ a.e. $t \in [0, T]$. Using (15) and that Ψ is of the Δ_2 class, we obtain

$$\begin{aligned}
|D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}_{n_k}(t), \dot{\boldsymbol{u}}_{n_k}(t))| &\leqslant a(|\boldsymbol{u}_{n_k}(t)|)\left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}_{n_k}(t)|}{\lambda} + d(t)\right)\right) \\
&\leqslant C\left(c(t) + \varphi\left(\frac{|h(t)|}{\lambda} + d(t)\right)\right) \in L^\Psi = E^\Psi
\end{aligned}$$

Therefore, invoking Lemma 2.5, we have proved that from any sequence $\boldsymbol{u}_n$ which converge to $\boldsymbol{u}$ in W^1L^Φ we can extract a subsequence with $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ in the strong topology. The desired result follows from a standard argument.

The continuity of I' follows of the previously established continuity for $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ by using the representation (17). □

4 Critical points and Euler-Lagrange equations

In this section we derive the Euler-Lagrange equations associated to critical points of action integrals. We denote by $W^1L_T^\Phi$ the subspace of W^1L^Φ of all T-periodic functions. Similarly we consider the subspaces E_T^Φ, L_T^Φ. As is usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, tghat means the subspace consistent of all the bounded linear functions which are identically zero on Y.

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) < \frac{1}{2}\left(f(\boldsymbol{x}) + f(\boldsymbol{y})\right)$ for $\boldsymbol{x} \neq \boldsymbol{y}$. It is a well known that if f is a strictly convex and differentiable functions then $D_{\boldsymbol{x}} f : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one map (see, for instance [16, Theorem 12.17]).

Theorem 4.1. *Let $\boldsymbol{u} \in \mathcal{E}_n^{\Phi}(\lambda)$. The following statements are equivalent*

1. $I'(\boldsymbol{u}) \in \left(W^1 L_T^{\Phi}\right)^{\perp}$
2. $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and* $\boldsymbol{u}$ *solve the following boundary value problem*

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t \in (0,T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0. \end{cases} \tag{25}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t, x, y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0$ is equivalent to $\dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}(T)$.

Proof. The condition $I'(\boldsymbol{u}) \in \left(W^1 L_T^{\Phi}\right)^{\perp}$ and (17) imply

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \cdot \dot{\boldsymbol{v}}(t) dt = -\int_0^T D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \cdot \boldsymbol{v}(t) dt$$

Using [14, pag. 6] we obtain that $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ is absolutely continuous and T-periodic, therefore it is differentiable a.e.on $[0,T]$ and the first equality of (25) holds true. This complete the proof 1. implies 2. The proof of 2.implies 1. is still easier and so we will omit it.

The last part of the Corollary is a consequence of that $D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) = D_{\boldsymbol{y}}\mathcal{L}(T, u(T), \dot{\boldsymbol{u}}(0))$ and the injectivity of $D_{\boldsymbol{y}}\mathcal{L}(T, u(T), \cdot)$. □

5 Coercivity discussion

We recall the following usual definition in the context of calculus of variations.

Definition 5.1. *Let X be a Banach space and let D be an unbounded subset of X. Suppose $J : D \subset X \to \mathbb{R}$. We said that J is* coercive *if $J(u) \to +\infty$ when $\|\boldsymbol{u}\| \to +\infty$.*

It is well known that coercitivity is an ingredient useful in order to establish existence of minima. Therefore we are interestent in finding conditions which insure the coercitivityof the action integral I acting on $\mathcal{E}_n^{\Phi}(\lambda)$. For this purpose we need to introduce the following extra condition on Lagrange function $\mathcal{L}$

$$\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y}) \geq \alpha_0 \Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t, \boldsymbol{x}), \tag{26}$$

where $\alpha_0, \Lambda > 0$ and the function $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable respect to t for every fix $\boldsymbol{x} \in \mathbb{R}^n$ and continuous in $\boldsymbol{x}$ for a.e. $t \in [0, T]$.

As we shall see in Theorem 5.3, when $\mathcal{L}$ satisfies (13), (14), (15) and (26), the coercitivity of the action integral I is related to the coercitivity of the functional

$$J_{C,\nu}(\boldsymbol{u}) := \int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right) dt - C\|\boldsymbol{u}\|_{L^\Phi}^\nu, \tag{27}$$

for $C, \nu > 0$. If $\Phi(x) = |x|^p/p$ then $J_{C,\nu}$ is clearly coercitive for $\nu < p$. For more general Φ the situation is more interesting as will be shown by the following lemma.

Lemma 5.2. *Let Φ and Ψ complementary N-functions. Then*

1. *If $C\Lambda < 1$ then $J_{C,1}$ is coercitive.*
2. *If Ψ is Δ_2 global, then there exist a constant $\alpha_\Phi > 1$ such that for any $0 < \mu < \alpha_\Phi$,*
$$\lim_{\|\boldsymbol{u}\|_{L^\Phi} \to \infty} \frac{\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} = +\infty. \tag{28}$$
In particular, the functional $J_{C,\mu}$ is coercitive for every $C > 0$ and $0 < \mu < a_\Phi$. The constant α_Φ is one of the so called Matuszewska-Orlicz indices *(see [13, Ch. 11]).*
3. *If $J_{C,1}$ is coercitive with $C\Lambda > 1$, then Ψ is Δ_2 at infinity.*

Proof. Using (3) we obtain

$$(1 - C\Lambda)\|\boldsymbol{u}\|_{L^\Phi} + C\Lambda\|\boldsymbol{u}\|_{L^\Phi} \leqslant \Lambda + \Lambda\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right).$$

Then

$$\frac{(1 - C\Lambda)}{\Lambda}\|\boldsymbol{u}\|_{L^\Phi} - 1 \leqslant \rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^\Phi} = J_{C,1}(\boldsymbol{u}).$$

This show that $J_{C,1}$ is coercitive, and therefore it proves item 1.

In virtue of [1, Eq. (2.8)], the Δ_2 condition for Ψ, [13, Th 11.7] and [13, Cor. 11.6] we obtain a constant $K > 0$ and $\alpha_\Phi > 1$ such that for any $0 < \nu < \alpha_\Phi$, $s \geq 0$ and $r > 1$

$$\Phi(rs) \geq K r^\nu \Phi(s). \tag{29}$$

Let $1 < \mu < \alpha_\Phi$ and we consider a constant $r > \Lambda$ that later will be specify. Then, from (29) and (3), we get

$$\begin{aligned}
\frac{\int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right) dt}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} &\geq K\left(\frac{r}{\Lambda}\right)^\nu \frac{\int_0^T \Phi(r^{-1}|\boldsymbol{u}|)dt}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} \\
&\geq K\left(\frac{r}{\Lambda}\right)^\nu \frac{r^{-1}\|\boldsymbol{u}\|_{L^\Phi} - 1}{\|\boldsymbol{u}\|_{L^\Phi}^\mu}
\end{aligned}$$

We choose $r = \|\boldsymbol{u}\|_{L^\Phi}/2$. Since $\|\boldsymbol{u}\|_{L^\Phi} \to +\infty$ we can assume $\|\boldsymbol{u}\|_{L^\Phi} > 2$. Thus $r > 1$ and

$$\frac{\int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right) dt}{\|\boldsymbol{u}\|^{\mu}_{L^\Phi}} \geq \frac{K}{2^\nu \Lambda^\nu} \|\boldsymbol{u}\|^{\nu-\mu}_{L^\Phi} \to +\infty \quad \text{for } \|\boldsymbol{u}\|_{L^\Phi} \to +\infty,$$

because $\nu > \mu$.

In order to prove the last item, we assume that $\Psi \notin \Delta_2$ at infinity. By [12, Th. 4.2], there exists a sequence of real numbers r_n such that $r_n \to \infty$ and

$$\lim_{n\to\infty} \frac{r_n \psi(r_n)}{\Psi(r_n)} = +\infty. \tag{30}$$

Now, we choose $\boldsymbol{u}_n$, such that $|\boldsymbol{u}_n| = \Lambda \psi(r_n) \chi_{[0, \frac{1}{\Psi(r_n)}]}$, then by [12, Eq. (9.11)], we get

$$\|\boldsymbol{u}_n\|_{L^\Phi} = \Lambda \frac{\psi(r_n)}{\Psi(r_n)} \Psi^{-1}(\Psi(r_n)) = \Lambda \frac{r_n \psi(r_n)}{\Psi(r_n)} \to \infty, \quad \text{as} \quad n \to \infty.$$

Next, using Young's equality (see [12, Eq. (2.7)]), we have

$$\begin{aligned} J_{C,1}(\boldsymbol{u}_n) &= \int_0^T \Phi\left(\frac{|\boldsymbol{u}_n|}{\Lambda}\right) - C\|\boldsymbol{u}_n\|_{L^\Phi} \\ &= \frac{1}{\Psi(r_n)} \left[\Phi(\psi(r_n)) - C\Lambda r_n \psi(r_n)\right) \\ &= \frac{1}{\Psi(r_n)} \left[r_n \psi(r_n) - \Psi(r_n) - C\Lambda r_n \psi(r_n)\right] \\ &= \frac{(1 - C\Lambda) r_n \psi(r_n)}{\Psi(r_n)} - 1. \end{aligned}$$

From (30) and the condition $C\Lambda > 1$, we obtain $J_{C,1}(\boldsymbol{u}_n) \to +\infty$, which is a contradiction. □

Theorem 5.3. *Let $\mathcal{L}$ be a Lagrangian function satisfying* (13)*,* (14)*,* (15) *and* (26)*. We assume following conditions*

1. *There exists a non negative function $b_1 \in L^1_1$ and a constant $\mu > 0$ such that for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^n$ and a.e. $t \in [0, T]$*

$$|F(t, \boldsymbol{x_2}) - F(t, \boldsymbol{x_1})| \leqslant b_1(t)(1 + |\boldsymbol{x_2} - \boldsymbol{x_1}|^\mu). \tag{31}$$

We suppose that the constant μ satisfies that $\mu < \alpha_\Phi$, with α_Φ as in Lemma 5.2, when $\Psi \in \Delta_2$ and that $\mu = 1$ if Ψ is an arbitrary N-function.

2. $$\int_0^T F(t, \boldsymbol{x})\, dt \to \infty \quad \textit{as} \quad |\boldsymbol{x}| \to \infty, \tag{32}$$

In addition we suppose

3. Ψ *is* Δ_2 *or*

$$C := a_0^{-1} T \Psi^{-1}\left(\frac{1}{T}\right) \|b_1\|_{L^1} < 1. \tag{33}$$

Then the action integral I *is coercive.*

Proof. In the following estimates we will use the descomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, Hölder and the Wirtinger inequality (8).

$$\begin{aligned} I(\boldsymbol{u}) &= \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt \\ &\geq a_0 \int_0^T \Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\Lambda}\right) + F(t, \boldsymbol{u}) dt \\ &= a_0 \rho_\Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\Lambda}\right) + \int_0^T F(t, \boldsymbol{u}) - F(t, \overline{\boldsymbol{u}}) dt + \int_0^T F(t, \overline{\boldsymbol{u}}) dt \\ &\geq a_0 \rho_\Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\Lambda}\right) - \int_0^T b_1(t)(1 + |\tilde{\boldsymbol{u}}(t)|^\mu) dt + \int_0^T F(t, \overline{\boldsymbol{u}}) dt \\ &\geq a_0 \rho_\Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\Lambda}\right) - \|b_1\|_{L^1}(1 + \|\tilde{\boldsymbol{u}}\|_{L^\infty}^\mu) + \int_0^T F(t, \overline{\boldsymbol{u}}) dt \\ &\geq a_0 \rho_\Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\Lambda}\right) - \|b_1\|_{L^1}\left(1 + \left[T \Psi^{-1}\left(\frac{1}{T}\right)\right]^\mu \|\dot{\boldsymbol{u}}\|_{L^\Phi}^\mu\right) \\ &\quad + \int_0^T F(t, \overline{\boldsymbol{u}}) dt \\ &= a_0 J_{C,\mu}(\dot{\boldsymbol{u}}) - \|b_1\|_{L^1} + \int_0^T F(t, \overline{\boldsymbol{u}}) dt, \end{aligned} \tag{34}$$

where $C = a_0^{-1}\left[T\Psi^{-1}(1/T)\right]^\mu \|b_1\|_{L^1}$. Suppose $\boldsymbol{u}_n$ a sequence in $\mathcal{E}_n^\Phi(\lambda)$ such that i) the sequence $\overline{\boldsymbol{u}}_n$ is bounded in $\mathbb{R}^n$ and ii) $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$. Then the Wirtinger inequality (8) implies that $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Therefore one of the following affirmation holds true $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$ or $|\overline{\boldsymbol{u}}_n| \to \infty$. On the other hand, (31) and (32) imply that the integral $\int_0^T F(t, \overline{\boldsymbol{u}}_n) dt$ is bounded from below. These observations, the lower bound (34) of I, assumption 3 in Theorem 5.3 and Lemma 5.2 imply the desidered result. □

6 Weak lower semicontinuity of actions integrals

Lemma 6.1. *If the sequence* $\{\boldsymbol{u}_k\}_{k\geq 1}$ *converges weakly to* $\boldsymbol{u}$ *in* W^1L^Φ*, then* $\{\boldsymbol{u}_k\}_{k\geq 1}$ *converges uniformly to* $\boldsymbol{u}$ *on* $[0, T]$.

Proof. By Lemma 2.1, the injection of W^1L^Φ in L^∞ is continuous. Since $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in W^1L^Φ it follows that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $C(0, T; \mathbb{R}^n)$. Since $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in W^1L^Φ, we know that $\{\boldsymbol{u}_k\}_{k\geq 1}$ is bounded in W^1L^Φ and, hence by (**??**) in $C(0, T; \mathbb{R}^n)$. Moreover, the sequence $\{\boldsymbol{u}_k\}_{k\geq 1}$ is equi-uniformly continuous since, for $0 \leqslant s \leqslant t \leqslant T$, we have

$$\begin{aligned}|\boldsymbol{u}_k(t) - \boldsymbol{u}_k(s)| &\leqslant \int_s^t |\dot{\boldsymbol{u}}_k(\tau)| \quad d\tau \leqslant \|t-s\|_{L^\Psi}\|\dot{\boldsymbol{u}}_k\|_{L^\Phi} \\ &\leqslant \|t-s\|_{L^\Psi}\|\boldsymbol{u}_k\|_{W^1L^\Phi} \leqslant C\|t-s\|_{L^\Psi}.\end{aligned}$$

By Arzela-Ascoli theorem, $\{\boldsymbol{u}_k\}_{k\geq 1}$ is relatively compact in $C(0,T;\mathbb{R}^n)$. By the uniqueness of the weak limit in $C(0,T;\mathbb{R}^n)$, every uniformly convergent subsequence of $\{\boldsymbol{u}_k\}_{k\geq 1}$ converges to $\boldsymbol{u}$. Thus, $\{\boldsymbol{u}_k\}_{k\geq 1}$ converges uniformly on $[0,T]$.

□

Theorem 6.2. *We suppose that $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ is a Charateodory functions satisfying* (13)-(15). *Moreover we assume $\mathcal{L}(t,\boldsymbol{x},\cdot)$ is convex for each $t,\boldsymbol{x}$. We suppose that Φ,Ψ are Δ_2 functions. Then the functional* (16) *is weakly lower semicontinuous (w.l.s.c.).*

Proof. We fix any $\boldsymbol{u}\in W^1L^\Phi$. What we must prove that for any sequence $\{\boldsymbol{u}_n\}$ with $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in W^1L^Φ we have that $I(\boldsymbol{u}) \leqslant \liminf_n I(\boldsymbol{u}_n)$. We write

$$\begin{aligned}I(\boldsymbol{v}) &= \int_0^T \mathcal{L}(t,\boldsymbol{v}(t),\dot{\boldsymbol{v}}(t))dt \\ &= \int_0^T \mathcal{L}(t,\boldsymbol{v}(t),\dot{\boldsymbol{v}}(t)) - \mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{v}}(t))dt + \int_0^T \mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{v}}(t))dt \\ &=: J(\boldsymbol{v}) + H(\boldsymbol{v}).\end{aligned}$$

As $\{\boldsymbol{u}_n\}$ is a weakly convergent sequence, by the Lemma 6.1 we have that $\boldsymbol{u}_n \to \boldsymbol{u}$ in L^∞. By the mean value theorem for derivatives, we obtain a function $\boldsymbol{\xi_n}(t)$, with $\boldsymbol{\xi_n}(t)$ belonging to line segment joining $\boldsymbol{u}_n(t)$ and $\boldsymbol{u}(t)$, such that

$$\begin{aligned}&|\mathcal{L}(t,\boldsymbol{u}_n(t),\dot{\boldsymbol{u}}_n(t)) - \mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}_n(t))| \\ &\leqslant |D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{\xi_n}(t),\dot{\boldsymbol{u}}_n(t))||\boldsymbol{u_n}(t)-\boldsymbol{u}(t)|.\end{aligned} \tag{35}$$

The functions $\boldsymbol{u_n}$, and therefore the functions $\boldsymbol{\xi_n}$, are uniformly bounded in L^∞. Thus, there exists $C>0$ such that $a(|\boldsymbol{\xi_n}(t)|)\leqslant C$. Then, using (14) we get

$$|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{\xi_n}(t),\dot{\boldsymbol{u}}_n(t))| \leqslant C\left(b(t)+\Phi(|\dot{\boldsymbol{u}}_n(t))|)\right) \tag{36}$$

Since Φ is a function of the Δ_2 class, we have that the operator $\boldsymbol{v}\mapsto\Phi(|\boldsymbol{v}|)$ acts from L^Φ in L^1. Therefore, by [12, Lemma 17.4] we have that $\{\Phi(|\boldsymbol{v}|) : \|\boldsymbol{v}\|_{L^\Phi}\leqslant r\}$ is bounded in L^1 for any $r>0$. Hence there exists a constant $C>0$ such that $\|\Phi(|\dot{\boldsymbol{u}}_n(t))|)\|_{L^1}\leqslant C$. Then, from (35), (36), Hölder inequality and since $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^\infty}\to 0$ and $b\in L^1$ we get $J(\boldsymbol{u}_n)\to 0$.

Now we will prove that $H(\boldsymbol{v})$ is w.l.s.c. Since $H(\boldsymbol{v})$ is convex it is sufficient to prove that H is l.s.c (see [8, Proposition 4.26]). We suppose that $\|\boldsymbol{v_n}-\boldsymbol{v}\|_{W^1L^\Phi}\to 0$.

There exists $s=s_{n,t}\in[0,1]$ such that

$$|\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{v}}_n(t)) - \mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{v}}(t))| \leqslant |D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),(1-s)\dot{\boldsymbol{v}}_n(t)+s\dot{\boldsymbol{v}}(t))||\dot{\boldsymbol{v}}_n-\dot{\boldsymbol{v}}|.$$

Let $\mathfrak{G}_n$ be the set $\{|\dot{\boldsymbol{v}}_n(t)| \geq |\dot{\boldsymbol{v}}(t)|\}$. Then

$$|(1-s)\dot{\boldsymbol{v}}_n(t) + s\dot{\boldsymbol{v}}(t)| \leqslant \max\{|\dot{\boldsymbol{v}}_n(t)|, |\dot{\boldsymbol{v}}(t)|\} = \chi_{\mathfrak{G}_n}(t)|\dot{\boldsymbol{v}}_n(t)| + \chi_{\mathfrak{G}_n^c}(t)|\dot{\boldsymbol{v}}(t)|$$

Therefore, using (15) and taking account that $a(|\boldsymbol{u}(t)|) \in L^\infty$ we get

$$\begin{aligned}|\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{v}}_n(t)) - \mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{v}}(t))| &\leqslant C\left(c(t) + \varphi(\chi_{\mathfrak{G}}|\dot{\boldsymbol{v}}_n(t)| + \chi_{\mathfrak{G}^c}|\dot{\boldsymbol{v}}(t)|)\right)|\dot{\boldsymbol{v}}_n - \dot{\boldsymbol{v}}| \\ &= C\left(c(t) + \varphi(\chi_{\mathfrak{G}}|\dot{\boldsymbol{v}}_n(t)|) + \varphi(\chi_{\mathfrak{G}^c}|\dot{\boldsymbol{v}}(t)|)\right)|\dot{\boldsymbol{v}}_n - \dot{\boldsymbol{v}}| \\ &\leqslant C\left(c(t) + \varphi(|\dot{\boldsymbol{v}}_n(t)|) + \varphi(|\dot{\boldsymbol{v}}(t)|)\right)|\dot{\boldsymbol{v}}_n - \dot{\boldsymbol{v}}|.\end{aligned}$$

Now, in virtue of [12, Lemma 9.1], [12, Lemma 17.1], [12, Theorem 17.4] and the uniform boundedness of $\dot{\boldsymbol{u}}_n$ in L^Φ we have

$$|H(\boldsymbol{v}_n) - H(\boldsymbol{v})| \leqslant C\|\dot{\boldsymbol{v}}_n - \dot{\boldsymbol{v}}\|_{L^\Phi} \to 0.$$

Which completes the proof. □

We consider the problem (introduced in (25)):

$$\frac{d}{dt}D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t \in (0,T). \tag{37}$$

In the sequel, we will discuss the conditions that guarantee the coercivity of the functional $u \to \int_0^T \Phi(|u|)\,dx$ in L^Φ.

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