

Clarke dual method for Hamiltonian systems with non standard grow

(In alphabetical order)

Sonia Acinas *

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
`sonia.acinas@gmail.com`

Jakub Maksymiuk

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
`fmazzone@exa.unrc.edu.ar`

Abstract

In this paper we consider the problem of finding periodic solutions of
certain Hamiltonian systemsblablabla

1 Main problem

Let $H : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$. We are looking for periodic solutions of the
Hamiltonian system

$$\begin{cases} \dot{q}(t) &= D_p H(t, q(t), p(t)) \\ \dot{p}(t) &= -D_q H(t, q(t), p(t)) \\ p(0) &= p(T), q(0) = q(T) \end{cases} \quad (1)$$

for $t \in [0, T]$. I think that, like in [?], is better to present the Hamiltonian
problem as the main problem

*SECyT-UNRC, FCEyN-UNLPam and UNSL

†SECyT-UNRC, FCEyN-UNLPam and CONICET

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Points.

An alternative writing of (1) using the combined variable $u = (q, p)$ and the canonical symplectic matrix

$$J = \begin{pmatrix} 0 & I_{d \times d} \\ -I_{d \times d} & 0 \end{pmatrix}$$

is the following

$$\dot{u} = J \nabla H(t, u(t)) \quad (2)$$

or equivalently

$$J \dot{u} = -\nabla H(t, u(t)) \quad (3)$$

where ∇H is the gradient of H with respect to the combined variable.

2 Preliminaries

We will use some basic concepts of convex analysis that we list below.

Let $\Gamma_0(\mathbb{R}^d) = \{F : \mathbb{R}^d \rightarrow (-\infty, +\infty]\}$
convex, lower semicontinuous functions with non-empty effective domain.}

The Fenchel conjugate of F is given by

$$F^*(p) = \sup_{q \in \mathbb{R}^d} \langle p, q \rangle - F(q)$$

The Fenchel conjugate satisfies the following properties:

1. $F^* \in \Gamma_0(\mathbb{R}^d)$
2. If $F \leq G$, then $G^* \leq F^*$.
3. If $G(q) = \alpha F(\beta q) + \sigma$ with $\alpha, \beta, \sigma > 0$ then $G^*(p) = \alpha F^*\left(\frac{p}{\beta \alpha}\right) - \sigma$

Let $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ be a differentiable, convex function such that $\Phi(0) = 0$, $\Phi(q) > 0$ if $q \neq 0$, $\Phi(-q) = \Phi(q)$, and

$$\lim_{|q| \rightarrow \infty} \frac{\Phi(q)}{|q|} = +\infty, \quad (4)$$

where $|\cdot|$ denotes the euclidean norm on \mathbb{R}^d . From now on, we say that Φ is an G -function if Φ satisfies the previous properties.

We write Φ^* for the Fenchel conjugate of Φ .

We do not assume that Φ and Φ' satisfy the Δ_2 -condition.

We denote by $\partial F(q)$ the subdifferential of F in the sense of convex analysis (see [?, 7])

The next result is a generalization of [14, Prop. 2.2, p.34]

Proposition 2.1. *Let $F \in \Gamma_0(\mathbb{R}^d)$. Suppose that there exist an anisotropic function Φ and non negative constants β, γ such that*

$$-\beta \leq F(q) \leq \Phi(q) + \gamma, \text{ for all } q \in \mathbb{R}^d. \quad (5)$$

Now, if $p \in \partial F(q)$ then

$$\Phi^*(p) \leq \Phi(2q) + 2(\beta + \gamma). \quad (6)$$

Proof. If $p \in \partial F(q)$, from [14, Thm. 2.2, p.33],

$$F^*(p) = \langle p, q \rangle - F(q) \quad (7)$$

Conjugating (5), we have

$$F^*(p) \geq \Phi^*(p) - \gamma. \quad (8)$$

From Young's inequality, we get

$$\langle p, q \rangle = \frac{1}{2} \langle p, 2q \rangle \leq \frac{1}{2} \Phi^*(p) + \frac{1}{2} \Phi(2q) \quad (9)$$

By eqs. (5) and (7) to (9), we get

$$\Phi^*(p) \leq \frac{1}{2} \Phi^*(p) + \frac{1}{2} \Phi(2q) + \beta + \gamma$$

which implies (6) □

Remark 1. Inequality (6) is a few better than the corresponding in [14, Prop. 2.2] because the the case of power function we obtain $(\beta + \gamma)^{1/p}$, meanwhile in [14] appears $(\beta + \gamma)^{1/(p-1)}$.

3 Optimal bounds for a symplectic bilinear form

We consider the Euclidean space \mathbb{R}^{2d} equipped with the standard symplectic structure given by bilinear canonical symplectic 2-form

$$\Omega(u, v) := \langle Ju, v \rangle.$$

As Jakub observed we can not consider any G -function on the symplectic manifold \mathbb{R}^{2n} . I think that the following can be the appropriate form of the G -function defined on the symplectic manifold \mathbb{R}^{2n}

Definition 3.1. Let Φ a G -function defined in the symplectic manifold \mathbb{R}^{2n} . We say that Φ is a symplectic G -function if

$$\Phi(Ju) = \Phi^*(u). \quad (10)$$

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4 Optimal constant in Poincaré-Wirtinger inequality

5 L^p case

Let $u \in H_T^{1,p}([0, T], \mathbb{R}^d)$. Then

$$\int_0^T |u - \bar{u}|^{p'} dt \leq C_p T^p \int_0^T |u'|^p dt \quad (11)$$

where the optimal constant satisfies

$$C_p^{-1} := \inf \left\{ T^p \frac{\int_0^T |u'|^p dt}{\int_0^T |u - \bar{u}|^p dt} \mid u \in H_T^{1,p} \right\} = \inf \left\{ T^p \frac{\int_0^T |u'|^p dt}{\int_0^T |u|^p dt} \mid u \in H_T^{1,p}, \int_0^T u dt = 0 \right\} \quad (12)$$

Lemma 5.1. C_p given by (18) is independent of T .

Proof. Let $T \neq T'$. If u is a function such that

$$C_p^{-1}(T) + \epsilon > T^p \frac{\int_0^T |u'|^p dt}{\int_0^T |u|^p dt} \quad (13)$$

Performing the change of variable $s = \frac{T'}{T}t$ and calling $r = \frac{T}{T'}$, we have

$$T^p \frac{\int_0^T |u'(t)|^p dt}{\int_0^T |u(t)|^p dt} = T^p \frac{\int_0^T |u'(rs)|^p ds}{\int_0^T |u(rs)|^p ds} = (T')^p \frac{\int_0^T |v'(s)|^p ds}{\int_0^T |v(s)|^p ds} \geq C_p^{-1}(T') \quad (14)$$

where $v(s) = u(rs)$. Therefore, $C_p^{-1}(T') \leq C_p^{-1}(T)$ and consequently $C_p(T') = C_p(T)$. \square

Lemma 5.2.

$$C_p^{-1} = \inf \left\{ T^p \int_0^T |u'|^p dt \mid u \in H_T^{1,p}, \int_0^T u dt = 0, \int_0^T |u|^p dt = 1 \right\} \quad (15)$$

Proof. The existence of a minimum follows as usual by means of a minimizing sequence.

More details....?

We employ the method of Lagrange multipliers to solve an optimization problem with constraints. We will look for critical points of

$$I = \int_0^T |u'|^p dt - \lambda \int_0^T |u|^p dt + \mu \cdot \int_0^T u dt, \quad u \in H_T^{1,p} \quad (16)$$

The Gâteaux derivative of the functional is given by

$$\begin{aligned} \langle I'(u), v \rangle &= \int_0^T p|u'|^{p-2} u' \cdot v' dt - p\lambda \int_0^T |u|^{p-2} u \cdot v dt + \mu \int_0^T v dt = \\ &= \int_0^T \left\{ \frac{d}{dt} (p|u'|^{p-2} u') - p\lambda |u|^{p-2} u + \mu \right\} \cdot v dt + p|u'|^{p-2} u' \cdot v|_0^T = 0 \end{aligned} \quad (17)$$

Since v is an arbitrary function, we choose v such that $v(0) = v(T) = 0$ and we obtain

$$\frac{d}{dt} (p|u'|^{p-2} u') - p\lambda |u|^{p-2} u + \mu = 0 \quad a.e. \quad (18)$$

This fact implies that $p|u'|^{p-2}u' \cdot v|_0^T = 0 \forall v \in H_T^{1,p}$, that is

$$[p|u'(T)|^{p-2}u'(T) - p|u'(0)|^{p-2}u'(0)] \cdot v(0) = 0. \quad (19)$$

Then $u'(T) = u'(0)$. Now, integrating (24), we get

$$p\lambda \int_0^T |u|^{p-2}u \, dt + \mu T = 0.$$

If $p = 2$ then $\mu = 0$ and

$$\begin{cases} u'' + \lambda u &= 0 \\ u(0) &= u(T) \\ \int_0^T u \, dt &= 0 \end{cases}$$

The normalized solution is $u(t) = \cos(\sqrt{\lambda}t)u_0 + \sin(\sqrt{\lambda}t)u_1$ with $u_0, u_1 \in \mathbb{R}^d$.

As $u(0) = u(T)$ and $u'(0) = u'(T)$, the function $u(t)$ has minimal period $\frac{2\pi}{\sqrt{\lambda}}$ and it solves the second order ODE $u'' + \lambda u = 0$

Then $u(0) = u(T)$, $u'(0) = u'(T)$ imply that the function u has period T .

Ésto

As $u \neq 0$, we have $k\frac{2\pi}{\sqrt{\lambda}} = T$ with $k = 1, 3, \dots$. Then $\lambda = k^2\frac{4\pi^2}{T^2}$.

Now, if $u_k(t) = \cos(\frac{2k\pi}{T}t)u_0 + \sin(\frac{2k\pi}{T}t)u_1$, then

$$\begin{aligned} 1 &= \int_0^T |u_k|^2 \, dt \\ &= \int_0^T \left[\cos\left(\frac{2k\pi}{T}t\right) \right]^2 dt |u_0|^2 + \int_0^T \left[\sin\left(\frac{2k\pi}{T}t\right) \right]^2 dt |u_1|^2 \\ &\quad + \int_0^T \cos\left(\frac{2k\pi}{T}t\right) \sin\left(\frac{2k\pi}{T}t\right) dt u_0 \cdot u_1 \\ &= \frac{T}{2}(|u_0|^2 + |u_1|^2) \end{aligned}$$

and

$$\begin{aligned} T^2 \int_0^T |u'_k|^2 \, dt &= T^2 \left(\frac{2k\pi}{T} \right)^2 \left\{ \int_0^T \left[\sin\left(\frac{2k\pi}{T}t\right) \right]^2 |u_0|^2 + \int_0^T \left[\cos\left(\frac{2k\pi}{T}t\right) \right]^2 |u_1|^2 + 0 \right\} \\ &= \left(\frac{2k\pi}{T} \right)^2 \frac{T}{2}(|u_0|^2 + |u_1|^2) \\ &= 4k^2\pi^2 \end{aligned}$$

The minimum occurs when $k = 1$ and we get $C_2^{-1} = 4\pi^2$. Then, $\int_0^T |u|^2 \, dt \leq \frac{T^2}{4\pi^2} \int_0^T |u'|^2 \, dt$

Or...

From $u'' + \lambda u = 0$, we have $u''u + \lambda u^2 = 0$ and integrating over $[0, T]$ we obtain $0 = \int_0^T u''u + \lambda \int_0^T u^2 = -\int_0^T (u')^2 + \lambda \int_0^T u^2 + u'u|_0^T = -\int_0^T (u')^2 + \lambda \int_0^T u^2 + u'(T)u(T) - u'(0)u(0) = -\int_0^T (u')^2 + \lambda \int_0^T u^2$, then $\frac{4\pi^2 k}{T^2} = \lambda = \frac{\int_0^T (u')^2}{\int_0^T u^2} = \frac{1}{C_2}$. The minimum value is attained at $k = 1$ and therefore $C_2 = \frac{T^2}{4\pi^2}$. \square

6 L^Φ case where $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is an anisotropic function

Now, we are looking for the optimal constant $C(\Phi, T)$ such that

$$\int_0^T \Phi(u - \bar{u}_\Phi) dt \leq C(\Phi, T) \int_0^T \Phi(u') dt \quad u \in W^1 L_T^\Phi \quad (20)$$

Then,

$$\int_0^T \Phi(u - \bar{u}_\Phi) dt \leq C(\Phi, T) \int_0^T \Phi(u - a) dt \quad \forall a \in \mathbb{R}^d \quad (21)$$

where $a = \bar{u}_\Phi$ is the unique vector of \mathbb{R}^d such that

$$\int_0^T \nabla \Phi(u - a) dt = 0. \quad (22)$$

Thus, $C^{-1} = \inf \left\{ \frac{\int_0^T \Phi(u') dt}{\int_0^T \Phi(u - \bar{u}_\Phi) dt} \mid u \in W^1 L_T^\Phi \right\}$. Let $v = u - \bar{u}_\Phi$, $v' = u'$ and $\bar{v}_\Phi = 0$ then

$$\lambda := C^{-1} = \inf \left\{ \frac{\int_0^T \Phi(u') dt}{\int_0^T \Phi(u) dt} \mid u \in W^1 L_T^\Phi, \int_0^T \nabla \Phi(u) dt = 0 \right\} \quad (23)$$

Let

$$L(u, u') = \frac{\int_0^T \Phi(u') dt}{\int_0^T \Phi(u) dt} - \mu \cdot \int_0^T \nabla \Phi(u) dt \quad (24)$$

with $\mu \in \mathbb{R}^d$. By Gâteaux derivative we have

$$0 = \frac{\int_0^T \Phi(u) dt \int_0^T \nabla \Phi(u') \cdot v' dt - \int_0^T \Phi(u') dt \int_0^T \nabla \Phi(u) \cdot v dt}{(\int_0^T \Phi(u) dt)^2} - \mu \int_0^T D^2 \Phi(u) \cdot v dt \quad (25)$$

then

$$\begin{aligned}
 0 &= \int_0^T \nabla \Phi(u') \cdot v' dt - \lambda \int_0^T \nabla \Phi(u) \cdot v dt - \int_0^T \Phi(u) dt \mu \cdot \int_0^T D^2 \Phi(u) \cdot v dt = \\
 &= \int_0^T \left\{ -\frac{d}{dt} \nabla \Phi(u') - \lambda \nabla \Phi(u) \right\} \cdot v dt + \nabla \Phi(u') \cdot v|_0^T - \int_0^T \Phi(u) dt \mu \cdot \int_0^T D^2 \Phi(u) \cdot v dt = \\
 &= \int_0^T \left\{ -\frac{d}{dt} \nabla \Phi(u') - \lambda \nabla \Phi(u) - \int_0^T \Phi(u) dt \mu \cdot D^2 \Phi(u) \right\} \cdot v dt + \nabla \Phi(u') \cdot v|_0^T
 \end{aligned} \tag{26}$$

$\forall v \in W^1 L_T^\Phi$.

Now, we consider any $v \in W_0^1 L^\Phi$ and we get

$$-\frac{d}{dt} \nabla \Phi(u') - \lambda \nabla \Phi(u) - \int_0^T \Phi(u) dt \mu \cdot D^2 \Phi(u) = 0 \tag{27}$$

Then,

$$\nabla \Phi(u') v|_0^T = 0 \quad \forall v \in W^1 L_T^\Phi,$$

that is

$$\{\nabla \Phi(u'(T)) - \nabla \Phi(u'(0))\} \cdot v(0) = 0$$

for any $v \in W^1 L_T^\Phi$. Thus, $\nabla \Phi(u'(T)) = \nabla \Phi(u'(0))$

As Φ is strictly convex, then $\nabla \Phi$ is injective and consequently $u'(T) = u'(0)$.

Post-multiplying (33) by μ^t and integrating over $[0, T]$, we get

$$0 = \int_0^T -\frac{d}{dt} \nabla \Phi(u') dt \cdot \mu^t - \lambda \int_0^T \nabla \Phi(u) dt \cdot \mu^t - \int_0^T \Phi(u) dt \int_0^T \mu \cdot D^2 \Phi(u) dt \cdot \mu^t \tag{28}$$

with $\nabla \Phi(u'(T)) = \nabla \Phi(u'(0))$.

We know that $\mu \cdot D^2 \Phi(u) \cdot \mu^t = 0$ iff $\mu = 0$. And, as $\int_0^T \Phi(u) dt \neq 0$, then $\int_0^T \Phi(u) dt \int_0^T \mu \cdot D^2 \Phi(u) \cdot \mu^t dt = 0$ implies that $\mu = 0$.

Therefore,

$$\begin{cases} \frac{d}{dt} \nabla \Phi(u') + \lambda \nabla \Phi(u) = 0 \\ u(0) = u(T), u'(0) = u'(T), \int_0^T \nabla \Phi(u) dt = 0 \end{cases} \tag{29}$$

If $\Phi(u) = \frac{|u|^p}{p}$ and $u \in \mathbb{R}$. We know that $T = \frac{4\pi(p-1)^{\frac{1}{p}}}{p \sin(\frac{\pi}{p}) \lambda^{\frac{1}{p}}} k$ for $k = +1, +2, \dots$

Cuestiones a resolver:

1. Qué dejar? Caso p o caso Φ ?
2. En el caso Φ ? Analizamos la existencia en detalle?
3. Consideramos el caso $\Phi(x_1, \dots, x_d) = \Phi(x_1) + \dots + \Phi(x_d)$?

$$\Phi(u) = \Phi(q, p) := \Phi(q) + \Phi^*(p).$$

We note that if Φ is symplectic then

As a consequence of (??), the matrix J induce a isometry between the spaces $L^\Phi([0, T], \mathbb{R}^{2d})$ and $L^{\Phi^*}([0, T], \mathbb{R}^{2d})$. Therefore we can define a bilinear form $\bar{\Omega}$ on $L^\Phi([0, T], \mathbb{R}^{2d})$ of the following way

$$\Theta(u) := \overline{\Omega}(u, \dot{u}).$$

$$\theta(u) \geq -C \int_0^T \Phi(u) dt, \quad (31)$$

If $\Phi(q) = |q|^2/2$ was proved in [14, Prop. 3.2] (??) holds with $C = T/\pi$. Below we prove that this is the optimal constant satisfying (??). Meanwhile in [26, Lem. 3.3] was proved that $C_\Phi = 2T$ satisfies (??) when $\Phi(q) = |q|^\alpha/\alpha$, $1 < \alpha < \infty$. Since this constant is not equal to T/π when $\alpha = 2$, it is not optimal.

$$\bar{u} = \frac{1}{T} \int_0^T u(t) dt.$$
$$\int_0^T \Phi(\tilde{u}) dt \leq \int_0^T \Phi(T\dot{u}) dt.$$

Then by Young's inequality and using (??)

$$\begin{aligned} \int_0^T \Omega(\dot{u}, u) dt &= T \int_0^T \langle J\dot{u}, T^{-1}\tilde{u} \rangle dt \\ &\geq -T \left\{ \int_0^T \Phi^\star(J\dot{u}) dt + \int_0^T \Phi(T^{-1}\tilde{u}) dt \right\} \\ &\geq -2T \left\{ \int_0^T \Phi(\dot{u}) dt \right\} \end{aligned}$$

□

Clearly the constant $2/T$ is far from optimal. A possible way to improve C is to consider other average \bar{u} . The mean value that it was used is the standard one considered in the literature. But this value is appropriate for the Hilbert setting $\Phi(q) = |q|^2/2$. In this case, the value of \bar{u} is the nearest (in the L^2 -norm) constant vector to u . For an arbitrary G function, it seems more reasonable to consider the nearest constant vector to u with respect to the Φ -integral, i.e.

$$\int_0^T \Phi(u - \bar{u}) dt \leq \int_0^T \Phi(u - u_0) dt, \quad \text{for every } u_0 \in \mathbb{R}^{2n}$$

Equivalently \bar{u} is characterized by

$$\int_0^T \nabla \Phi(u - \bar{u}) dt = 0.$$

There is not an explicit formula as in the Hilbert setting.

PROBLEM 1. We can get a better constant taking this \bar{u} ???

We call to the best constant in (??) C_Φ , i.e.

$$C_\Phi = - \inf \left\{ \frac{\int_0^T \langle J\dot{u}, u \rangle dt}{\int_0^T \Phi(\dot{u}) dt} \mid u \in W^1 L^\Phi([0, T], \mathbb{R}^{2d}) \right\} \quad (32)$$

Proposition 6.2. *The relation $C_\Phi = C_{\Phi^\star}$ holds for every symplectic Φ .*

Proof. Since Φ is symplectic if $u = Jv$

$$\frac{\int_0^T \langle J\dot{u}, u \rangle dt}{\int_0^T \Phi(\dot{u}) dt} = \frac{\int_0^T \langle -\dot{v}, Jv \rangle dt}{\int_0^T \Phi(J\dot{v}) dt} = \frac{\int_0^T \langle J\dot{v}, v \rangle dt}{\int_0^T \Phi^\star(\dot{v}) dt}.$$

Using that $u \mapsto Ju$ is invertible from $W^1 L^{\Phi^\star}([0, T], \mathbb{R}^{2d})$ into $W^1 L^\Phi([0, T], \mathbb{R}^{2d})$ the statement follows taking infimum in previous equality.

□

For the following result we need the theory of indices of G -functions, see [?, ?] for a complete treatment in the case of N -functions defined on \mathbb{R} . The

results are easily extended to the anisotropic setting. We denote by α_Φ and β_Φ the so called *Matuszewska-Orlicz indices* of the function Φ , which are defined next

$$\alpha_\Phi := \lim_{t \rightarrow 0^+} \frac{\log \left(\sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right)}{\log(t)}, \quad \beta_\Phi := \lim_{t \rightarrow +\infty} \frac{\log \left(\sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right)}{\log(t)}. \quad (33)$$

We have that $1 \leq \alpha_\Phi \leq \beta_\Phi \leq +\infty$. The relation $\beta_\Phi < \infty$ holds true if and only if Φ is a Δ_2 -function. The indices satisfy the following relation

$$\frac{1}{\alpha_\Phi} + \frac{1}{\beta_{\Phi^*}} = 1. \quad (34)$$

Therefore if Φ^* is a Δ_2 -function (**I mean Δ_2 as globally Δ_2**) then $\alpha_\Phi > 1$.

We observe that if Φ is symplectic then $\Phi \in \Delta_2$ implies $\Phi^* \in \Delta_2$. It is well known that if Φ and Φ^* are Δ_2 -function, then Φ is controlled by above and below by power functions. More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\Phi, \epsilon)$ and p_0, p_1 with $1 < \alpha_\Phi - \epsilon < p_1 \leq p_2 < \beta_\Phi + \epsilon < \infty$ such that, for every $t, u \geq 0$,

$$K^{-1} \min \{t^{p_2}, t^{p_1}\} \Phi(u) \leq \Phi(tu) \leq K \max \{t^{p_2}, t^{p_1}\} \Phi(u). \quad (35)$$

We recall the following result of [1].

Lemma 6.3. *Let Φ be a G-functions. If $\Phi^* \in \Delta_2$ globally, then for any $0 < \mu < \alpha_\Phi$,*

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi\left(\frac{u}{\Lambda}\right) dt}{\|u\|_{L^\Phi}^\mu} = +\infty. \quad (36)$$

Theorem 6.4. *Suppose that $u \in W^1 L_T^\Phi([0, T], \mathbb{R}^{2d})$ attains the minimum in (??), then $\lambda = 2/C_\Phi$ is the first eigenvalue and u the corresponding eigenfunction of the following problem.*

$$\begin{cases} \frac{d}{dt} \nabla \Phi^*(\dot{u}) + \lambda \nabla \Phi^*(\lambda u) = 0 \\ u(0) = u(T), \int_0^T \nabla \Phi^*(\lambda u) dt = 0 \end{cases} \quad (\text{Eig})$$

Proof. □

7 Differentiability of Hamiltonian dual action

Theorem 7.1. *Suppose that $\Phi : \mathbb{R}^{2d} \rightarrow [0, +\infty)$ is a differentiable G-function, not necessarily symplectic. Additionally*

1. $H : [0, T] \times \mathbb{R}^{2d} \rightarrow \mathbb{R}$ is measurable in t , continuously differentiable with respect to u .
2. there exist $\beta, \gamma \in L^1([0, T], \mathbb{R})$, $\Lambda > \lambda > 0$ such that

$$\Phi^*\left(\frac{u}{\Lambda}\right) - \beta(t) \leq H(t, u) \leq \Phi^*\left(\frac{u}{\lambda}\right) + \gamma(t) \quad (37)$$

Then there exists Λ_0 such that the dual action

$$\chi(v) = \int_0^T \frac{1}{2} \langle J\dot{v}, v \rangle + H^*(t, \dot{v}) dt \quad (38)$$

is continuously differentiable in $W^1 L_T^\Phi([0, T], \mathbb{R}^{2d}) \cap \{u | d(\dot{u}, L^\infty) < \Lambda_0\}$.

If v is a critical point of χ with $d(\dot{v}, L^\infty) < \Lambda_0$, the function defined by $u(t) = \nabla H^*(t, \dot{v})$ solves

$$\begin{cases} \dot{u} &= J \nabla H(t, u) \\ u(T) &= u(T) \end{cases}$$

Proof. Conjugating 2 we obtain

$$\Phi(\lambda u) - \gamma(t) \leq H^*(t, v) \leq \Phi(\Lambda v) + \beta(t) \quad (39)$$

Since H^* is smooth, we have $\partial_v H^*(t, v) = \{\nabla_v H^*(t, v)\}$. Applying Proposition 1.1 with $F = H^*$, $\Phi(\Lambda v)$ instead of $\Phi(u)$ and $u = \nabla H^*(t, v) \in \partial_v H(t, v)$, inequality (10) becomes

$$\Phi^* \left(\frac{\nabla H^*(t, v)}{\Lambda} \right) \leq \Phi(2\Lambda v) + 2(\beta + \gamma). \quad (40)$$

which will be the main inequality in the proof.

We are planning to obtain the structure condition (S) of [15] which guarantees differentiability.

We consider the Lagrangian

$$\mathcal{L}(t, v, \xi) = \frac{1}{2} \langle J\xi, v \rangle + H^*(t, \xi) \quad (41)$$

and we have to prove that there exist $\Lambda_0 > \lambda_0 > 0$ such that

$$|\mathcal{L}| + |\nabla_v \mathcal{L}| + \Phi^* \left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0} \right) \leq a(v) \left(b(t) + \Phi \left(\frac{\xi}{\Lambda_0} \right) \right) \quad (42)$$

We start with $|\mathcal{L}|$. From (11),

$$|\mathcal{L}| \leq \frac{1}{2} |\langle J\xi, v \rangle| + H^*(t, \xi) \leq \frac{1}{2} |\xi| |v| + \Phi(\Lambda\xi) + \beta(t).$$

Since $\frac{\Phi(x)}{|x|} \rightarrow \infty$ as $|x| \rightarrow \infty$, there exists $C > 0$ such that $|x| \leq \Phi(x) + C$ for all $x \in \mathbb{R}^d$. Then,

$$|\mathcal{L}| \leq \frac{1}{2} \frac{|v|}{\Lambda} (\Phi(\Lambda\xi) + C) + \Phi(\Lambda\xi) + \beta(t) \leq \max \left\{ \Lambda, \frac{|v|}{2\Lambda} \right\} [\Phi(\Lambda\xi) + C + \beta(t)]$$

which is an estimate like the second member of (14).

Now, we treat $|\nabla_v \mathcal{L}|$ and we get

$$|\nabla_v \mathcal{L}| = \frac{1}{2} |J\xi| \leq |\xi| \leq \frac{1}{2\Lambda} (\Phi(\Lambda\xi) + C). \quad (43)$$

which is also an estimate of the desired type.

Finally, we deal with $\Phi(\nabla_\xi \mathcal{L} \lambda_0)$. As Φ^\star is a convex, even function, we have

$$\Phi^\star \left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0} \right) = \Phi^\star \left(\frac{-\frac{1}{2} Jv}{\lambda_0} + \frac{\nabla H^\star(t, \xi)}{\lambda_0} \right) \leq \frac{1}{2} \Phi^\star \left(\frac{Jv}{\lambda_0} \right) + \frac{1}{2} \Phi^\star \left(\frac{2\nabla H^\star(t, \xi)}{\lambda_0} \right).$$

We choose $\frac{2}{\lambda_0} = \frac{1}{\Lambda}$ with Λ as in (12) and we finally have

$$\begin{aligned} \Phi^\star \left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0} \right) &\leq \Phi^\star \left(\frac{Jv}{2\Lambda} \right) + \Phi(2\Lambda\xi) + 2(\beta + \gamma) = \\ &\max \left\{ \Phi^\star \left(\frac{Jv}{2\Lambda} \right), 1 \right\} [\Phi(2\Lambda\xi) + 2(\beta + \gamma)] \end{aligned} \quad (44)$$

which is a bound like the second member of (14).

Therefore, from (14), (15), (16) and choosing the worst functions a and b , we obtain condition (S).

Next, [15, Thm. 4.5] implies differentiability of χ in a set like $W^1 L_T^\Phi([0, T], \mathbb{R}^d) \cap \{u | d(\dot{u}, L^\infty) < \lambda_0\}$.

If $v \in W^1 L_T^\Phi([0, T], \mathbb{R}^d)$ is a critical point of χ with $d(\dot{v}, L^\infty) < \lambda_0$ then, from equations (21) of [15] we obtain

$$0 = \int_0^T \frac{1}{2} \langle J\dot{v}, h \rangle - \frac{1}{2} \langle \dot{h}, Jv \rangle + \langle \nabla H^\star(t, \dot{v}), \dot{h} \rangle.$$

The rest of the proof follows as in [14]. □

8 Existence periodic solutions Hamiltonian system

The following theorem extend to a quite general function Φ the result in [14, Th. 3.1] formulated for $\Phi_2(u) = |u|^2/2$. Even more, our result improves a little bit [14, Th. 3.1] in the sense that we obtain existence for Φ_2 when the functions, introduced below, l and γ are in L^2 and L^1 respectively instead that L^4 and L^2 which is assumed in [14, Th. 3.1]. This little improvement is due to the observation in Remark ??.

Theorem 8.1. *Suppose that Φ is a symplectic G-function and*

H1) Exists $\xi \in L^{\Phi^\star}([0, T], \mathbb{R}^{2d})$ such that

$$H(t, u) \geq \langle \xi(t), u \rangle.$$

H2) There exists $\alpha \in (0, C_\Phi)$ (C_Φ is defined in (??)) such that for every $(t, u) \in [0, T] \times \mathbb{R}^{2d}$

$$H(t, u) \leq \frac{1}{\alpha} \Phi(\alpha u) + \gamma(t).$$

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