# Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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**Abstract**

## 1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}, \tag{1}$$

where $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) := \frac{|y|^p}{p} + F(t,x), \tag{2}$$

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for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (1) is reduced to the $p$-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call $F$ a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to $x$ for a.e. $t \in [0,T]$ and the following conditions are verified:

**(C)** $F$ and its gradient $\nabla F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

**(A)** For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{4}$$

In this inequality we assume that the function $a : [0, +\infty) \to [0, +\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0,T], \mathbb{R})$.

In [Acinas et al., 2015] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t, x, y) = \Phi(|y|) + F(t, x), \tag{5}$$

where $\Phi$ is an $N$-function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on $F$ (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (1).

## 2   Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces). References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001]. The choice of anisotropic sobolev spaces is justified since they allow to simplify the writing, and they provide the natural frame of statements of the type [Tian and Ge, 2007, Lemma 3.1].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if $\Phi$ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$. Additionally we assume that the Young's functions which we deal with, satisfy that $\Phi(x) > 0$ when $x \neq 0$.

Following [Schappacher, 2005] we say that $\Phi$ is *coercive* if

$$\lim_{|x|\to\infty}\frac{\Phi(x)}{|x|}=+\infty.$$

Given a Young's function $\Phi$, we define the function $\hat{\Phi}:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\hat{\Phi}(s)=\sup\{|x|:\Phi(x)\leqslant s\},\tag{6}$$

We point out that the function $G(|x|)$ has the following properties, as reader can easily check,

(G1)  $G$ is monotonous increasing, continuous and $G(s)\to\infty$ as $s\to\infty$.

(G2)  $G$ is the *greatest radial minorant* of $\Phi(x)$, i.e. $G(|x|)\leqslant\Phi(x)$ and $G(|x|)$ is the biggest radial function with this property.

(G3)  There exists $G^{-1}$ and $G^{-1}(\Phi(x))\geqslant|x|$.

(G4)  As $\Phi(\alpha x)/\alpha$ is increasing with respect to $\alpha$ for evey $x>0$, $G(\alpha s)/\alpha$ is also increasing with respect to $\alpha$ for every $s>0$. Alternatively $\beta G^{-1}(t/\beta)$ is an increasing function with respect to $\beta$ for every $t>0$.

(G5)  In the event that $\Phi$ is coervive, then $G$ is also coercive. Alternatively $G^{-1}(s)/s\to 0$ when $s\to+\infty$.

We also say that $\Phi:\mathbb{R}^d\to\mathbb{R}^+$ satisfies the $\Delta_2^\infty$-*condition*, denoted by $\Phi\in\Delta_2^\infty$, if there exist constants $K>0$ and $M\geqslant 0$ such that

$$\Phi(2x)\leqslant KH(x),\tag{7}$$

for every $|x|\geqslant M$.

If $\Phi$ is a Young's function we define its *Fenchel conjugate* $\Phi^*:\mathbb{R}^d\to\mathbb{R}^+$ by:

$$\Phi^*(y)=\sup_{x\in\mathbb{R}^d}x\cdot y-\Phi(x)\tag{8}$$

We denote by $\mathcal{M}:=\mathcal{M}([0,T],\mathbb{R}^d)$, with $d\geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u=(u_1,\dots,u_d)$ for $u\in\mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d=1$.

Given an $N$-function $\Phi$ we define the *modular function* $\rho_\Phi:\mathcal{M}\to\mathbb{R}^+\cup\{+\infty\}$ by

$$\rho_\Phi(u):=\int_0^T\Phi(u)\,dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi=C^\Phi([0,T],\mathbb{R}^d)$ by setting

$$C^\Phi:=\{u\in\mathcal{M}|\rho_\Phi(u)<\infty\}.\tag{9}$$

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The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of $C^\Phi$; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{10}$$

The Orlicz space $L^\Phi$ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \bigg| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between $u$ and $v$.

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in $L^\Phi$ of the subspace $L^\infty([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of $E^\Phi$, which is particularly useful for us, is that $u \in E^\Phi$ if and only if $u$ has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \ldots$ then $\|\chi_{E_n}u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^{\Phi^*}}. \tag{11}$$

Like in [Krasnosel′skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of $L^\Phi$ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class $C^\Phi$ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{12}$$

for any positive $r$ (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets $L^\Phi$, $E^\Phi$, $\Pi(E^\Phi, r)$ and $C^\Phi$ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

**Definition 2.1.** *Let $u_n, u \in L^\Phi([0,T],\mathbb{R}^d)$. We say that $u_n$ converges monotonically to $u$ if there exists $\alpha_n \in L^\infty([0,T],\mathbb{R})$, $n = 1, 2, \ldots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t)u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of $X$, we write $Y \hookrightarrow X$ and we say that $Y$ is *embedded* in $X$ when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow [L^\Phi]^*$, where a function $v \in L^{\Phi^*}$ is associated to $\xi_v \in [L^\Phi]^*$ being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \tag{13}$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $[L^\Phi]^*$ which can be identified with $L^{\Phi^*}$.

**Proposition 2.2.** *Let* $F \in \left[ L^{\Phi}([0,T], \mathbb{R}^d) \right]^*$. *Then the following statements are equivalent*

1. $\xi \in L^{\Phi^*}([0,T], \mathbb{R}^d)$

2. $\xi$ *satisfies the* monotone convergence property, *which is if* $u_n$ *converges monotonically to* $u$ *then* $\langle \xi, u_n \rangle \to \langle \xi, u \rangle$.

If $\Phi \in \Delta_2^{\infty}$ and $\Phi$ is coercive then $L^{\Phi^*}([0,T], \mathbb{R}^d) = \left[ L^{\Phi}([0,T], \mathbb{R}^d) \right]^*$ (see [Desch and Grimmer, 2001, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* $W^1 L^{\Phi}$ by

$$W^1 L^{\Phi}([0,T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^{\Phi}([0,T], \mathbb{R}^d)\}.$$

$W^1 L^{\Phi}([0,T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^{\Phi}} = \|u\|_{L^{\Phi}} + \|u'\|_{L^{\Phi}}. \tag{14}$$

And, we introduce the following subspaces of $W^1 L^{\Phi}$

$$
\begin{aligned}
W^1 E^{\Phi} &= \{u \in W^1 L^{\Phi} | u' \in E^{\Phi}\}, \\
W^1 E_T^{\Phi} &= \{u \in W^1 E^{\Phi} | u(0) = u(T)\}.
\end{aligned}
\tag{15}
$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T} \int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

**Lemma 2.3.** *Let* $\Phi : \mathbb{R}^d \to [0, +\infty)$ *be a Young's function and let* $u \in W^1 L^{\Phi}([0,T], \mathbb{R}^d)$. *Let* $G : \mathbb{R}^+ \to \mathbb{R}^+$ *be the function defined by* (6). *Then*

1. *For every* $s, t \in [0,T]$, $s \neq t$,

$$|u(t) - u(s)| \leqslant \|u'\|_{L^{\Phi}} |s - t| G^{-1}\left( \frac{1}{|s-t|} \right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^{\infty}} \leqslant G^{-1}\left( \frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^{\Phi}} \qquad \text{(Sobolev's inequality)}$$

2. *We have* $\widetilde{u} \in L^{\infty}([0,T], \mathbb{R}^d)$ *and*

$$\|\widetilde{u}\|_{L^{\infty}} \leqslant T G^{-1}\left( \frac{1}{T} \right) \|u'\|_{L^{\Phi}} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If* $\Phi$ *is coercive then the space* $W^1 L^{\Phi}([0,T], \mathbb{R}^d)$ *is compactly embedded in the space of continuous functions* $C([0,T], \mathbb{R}^d)$.

*Proof.* By the absolutely continuity of $u$, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) \leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right)$$
$$\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}.$$

By (G1) and (G3) we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant G^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha G^{-1}(1/\alpha)$ is an increasing function with respect to $\alpha \in (0,\infty)$ we have

$$|u(t)-\overline{u}| \leqslant \|u'\|_{L^\Phi}TG^{-1}\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right)ds \leqslant \frac{1}{T}$$

Then by (G1) and (G3)

$$|\overline{u}| \leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\|u\|_{L^\infty} \leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}$$
$$\leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}$$
$$\leqslant G^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}$$

In order to prove item 3, we take a bounded sequence $u_n$ in $W^1L^\Phi([0,T],\mathbb{R}^d)$. From (Morrey's inequality) and (G5) we infer that $u_n$ are equicontinuous. Furthermore (Sobolev's inequality) implies that $u_n$ is bounded in $C([0,T],\mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence $n_k$ and $u \in C([0,T],\mathbb{R}^d)$ with $u_{n_k} \to u$ in $C([0,T],\mathbb{R}^d)$.

$\square$

# 3 Superposition operators in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anistropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined in Sobolev-Orlicz spaces.

Henceforth we assume that $f$ is a *Carathéodory function*,

(C) $f$ is measurable with respect to $t \in [0, T]$ for every $x \in \mathbb{R}^d$, and $f$ is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

**Definition 3.1.** *For $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0, T] \to \mathbb{R}^d$ by*

$$\boldsymbol{f} u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators definied in anisotropic Orlicz spaces of vectorial functions. For the proofs of these results and additional discussions see [Płuciennik, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

**Theorem 3.2.** *We assume that $f$ satisfies condition ((C)). Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps masurable function into measurable functions*

2. Extensibility.? *If*

3. Continuity.? *If*

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