Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

Sonia Acinas *
Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases} \qquad (P_1)$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) := \frac{|y|^p}{p} + F(t,x), \qquad (1)$$

* SECyT-UNRC and FCEyN-UNLPam
† SECyT-UNRC, FCEyN-UNLPam and CONICET

2010 AMS Subject Classification. Primary: . Secondary: .
Keywords and phrases. .

for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (P_1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (1) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (P_1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P_2}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{2}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [Acinas et al., 2015] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{3}$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (P_1).

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions $\Phi : \mathbb{R}^n \to \mathbb{R}_+$, i.e. functions such that $\Phi(x)$ depends on the direction of x, unlike the radial case where $\Phi(x) = \Phi(|x|)$. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of p-laplacian equations as the following example shows.

Example 1. Let $1 < p_1, p_2 < \infty$. We define $\Phi_{p_1,p_2} : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$\Phi_{p_1,p_2}(y_1,y_2) := \frac{|y_1|}{p_1} + \frac{|y_2|}{p_2}.$$

Suppose the following Lagrange function

$$\mathcal{L}(t,x,y) = \Phi_{p_1,p_2}(y) + F(t,x).$$

Then the equations (P_1) becomes

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) & \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} , \tag{P_3}$$

On the other hand, anisotropic Orlicz-Sobolev spaces allow to simplify the writing, and they provide the natural frame of statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question was what motivated us to use these spaces.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$. Additionally, we assume that the Young's functions which we deal with, satisfy that $\Phi(x) > 0$ when $x \neq 0$. Following [Schappacher, 2005] we say that Φ is *coercive* if

$$\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function Φ, we define function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, \|x\| = s\right\}, \tag{4}$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.1. *The function A_Φ has the following properties:*

1. *A_Φ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *A_Φ is the* greatest radial minorant *of $\Phi(x)$,*
4. *Φ is coercive if and only if A_Φ is.*

Proof. It is well known that finite and convex functions defined in finite dimensional vectorial spaces are locally Lipschitz functions (see [Clarke, 2013]). This fact imply item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

We also say that $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ satisfies the Δ_2^∞*-condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant KH(x), \tag{5}$$

for every $|x| \geqslant M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \to \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x\in\mathbb{R}^d} x \cdot y - \Phi(x) \tag{6}$$

We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{7}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{8}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of E^Φ, which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \dots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant 2\|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \tag{9}$$

Like in [Krasnosel′skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^{Φ} by means of inclusions, namely,

$$\Pi(E^{\Phi}, r) \subset rC^{\Phi} \subset \overline{\Pi(E^{\Phi}, r)} \tag{10}$$

for any positive r (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^{\infty}$, then the sets L^{Φ}, E^{Φ}, $\Pi(E^{\Phi}, r)$ and C^{Φ} are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

Definition 2.2. *Let $u_n, u \in L^{\Phi}([0,T], \mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^{\infty}([0,T], \mathbb{R})$, $n = 1, 2, \dots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t)u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow \left[L^{\Phi}\right]^*$, where a function $v \in L^{\Phi^*}$ is associated to $\xi_v \in \left[L^{\Phi}\right]^*$ being

$$\xi_v(u) = \langle \xi_v, u\rangle = \int_0^T v \cdot u \, dt, \tag{11}$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $\left[L^{\Phi}\right]^*$ which can be identified with L^{Φ^*}.

Proposition 2.3. *Let $F \in \left[L^{\Phi}([0,T], \mathbb{R}^d)\right]^*$. Then the following statements are equivalent*

1. $\xi \in L^{\Phi^*}([0,T], \mathbb{R}^d)$
2. *ξ satisfies the* monotone convergence property*, which is if u_n converges monotonically to u then $\langle \xi, u_n\rangle \to \langle \xi, u\rangle$.*

If $\Phi \in \Delta_2^{\infty}$ and Φ is coercive then $L^{\Phi^*}([0,T], \mathbb{R}^d) = \left[L^{\Phi}([0,T], \mathbb{R}^d)\right]^*$ (see [Desch and Grimmer, 2001, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* W^1L^{Φ} by

$$W^1L^{\Phi}([0,T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^{\Phi}([0,T], \mathbb{R}^d)\}.$$

$W^1L^{\Phi}([0,T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^{\Phi}} = \|u\|_{L^{\Phi}} + \|u'\|_{L^{\Phi}}. \tag{12}$$

And, we introduce the following subspaces of W^1L^{Φ}

$$\begin{aligned} W^1E^{\Phi} &= \{u \in W^1L^{\Phi} | u' \in E^{\Phi}\}, \\ W^1E^{\Phi}_T &= \{u \in W^1E^{\Phi} | u(0) = u(T)\}. \end{aligned} \tag{13}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.4. *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (4). *Then*

1. *For every $s, t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi}|s-t|G^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant G^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If Φ is coercive then the space $W^1L^\Phi([0,T],\mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0,T],\mathbb{R}^d)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) \leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right)$$
$$\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}.$$

By **??** and **??** we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant G^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha G^{-1}(1/\alpha)$ is an increasing function with respect to $\alpha \in (0,\infty)$ we have

$$|u(t) - \overline{u}| \leqslant \|u'\|_{L^\Phi}TG^{-1}\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right)ds \leqslant \frac{1}{T}$$

Then by **??** and **??**

$$|\overline{u}| \leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \\ &\leqslant G^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^\Phi([0,T],\mathbb{R}^d)$. From (Morrey's inequality) and **??** we infer that u_n are equicontinuous. Furthermore (Sobolev's inequality) implies that u_n is bounded in $C([0,T],\mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0,T],\mathbb{R}^d)$ with $u_{n_k} \to u$ in $C([0,T],\mathbb{R}^d)$.

□

3 Superposition operators in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anistropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined in Sobolev-Orlicz spaces.

Henceforth we assume that f is a *Carathéodory function*,

(C) f is measurable with respect to $t \in [0,T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

Definition 3.1. *For $f:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u:[0,T]\to\mathbb{R}^d$ by*

$$\boldsymbol{f}u(t) = f(t,u(t))$$

In the following Theorem we enumerate some known properties for superposition operators definied in anisotropic Orlicz spaces of vectorial functions. For the proofs of these results and additional discussions see [Płuciennik, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

Theorem 3.2. *We assume that f satisfies condition* ((C)). *Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps masurable function into measurable functions*
2. Extensibility.? *If*
3. Continuity.? *If*

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

References

[Acinas et al., 2015] Acinas, S., Buri, L., Giubergia, G., Mazzone, F., and Schwindt, E. (2015). Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698.

[Clarke, 2013] Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Graduate Texts in Mathematics.

[Desch and Grimmer, 2001] Desch, W. and Grimmer, R. (2001). On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc*, (353):5095–5120.

[Krasnosel′skiĭ and Rutickiĭ, 1961] Krasnosel′skiĭ, M. A. and Rutickiĭ, J. B. (1961). *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen.

[Mawhin and Willem, 1989] Mawhin, J. and Willem, M. (1989). *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York.

[Płuciennik, 1985a] Płuciennik, R. (1985a). Boundedness of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Bull. Pol. Acad. Sci., Math.*, 33:531â540.

[Płuciennik, 1985b] Płuciennik, R. (1985b). On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 25:321â337.

[Płuciennik, 1987] Płuciennik, R. (1987). The superposition operator in Musielak-Orlicz spaces of vector-valfued functions. Abstract analysis, Proc. 14th Winter Sch., Srní/Czech. 1986, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14, 411-417 (1987).

[Schappacher, 2005] Schappacher, G. (2005). A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386.

[Skaff, 1969] Skaff, M. S. (1969). Vector valued orlicz spaces. ii. *Pacific J. Math.*, 28(2):413–430.

[Tang, 1995] Tang, C.-L. (1995). Periodic solutions of non-autonomous second-order systems with γ-quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675.

[Tang, 1998] Tang, C.-L. (1998). Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270.

[Tang and Wu, 2001] Tang, C. L. and Wu, X.-P. (2001). Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397.

[Tang and Zhang, 2010] Tang, X. and Zhang, X. (2010). Periodic solutions for second-order Hamiltonian systems with a p-Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113.

[Tian and Ge, 2007] Tian, Y. and Ge, W. (2007). Periodic solutions of non-autonomous second-order systems with a p-Laplacian. *Nonlinear Anal.*, 66(1):192–203.

[Wu and Tang, 1999] Wu, X.-P. and Tang, C.-L. (1999). Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235.

[Zhao and Wu, 2004] Zhao, F. and Wu, X. (2004). Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434.