Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

In this paper we consider the problem of finding periodic solutions of certain Euler-Lagrange equations which include, among others, equations involving the p-Laplace operator and, more generally, the (p,q)-Laplace operator. We employ the direct method of the calculus of variations in the framework of anisotropic Orlicz-Sobolev spaces. These spaces appear to be useful in formulating a unified theory of existence of solutions for such a problem.

1 Introduction

Let $\Phi:\mathbb{R}^d\to[0,+\infty)$ be a differentiable, convex function such that $\Phi(0)=0$, $\Phi(y)>0$ if $y\neq 0$, $\Phi(-y)=\Phi(y)$, and

$$\lim_{|y|\to\infty}\frac{\Phi(y)}{|y|}=+\infty, \tag{1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From now on, we say that Φ is an N_∞ function if Φ satisfies the previous properties.

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For $T > 0$, we assume that $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ ($F = F(t,x)$) is a differentiable function with respect to x for a.e. $t \in [0,T]$ and $F(t,x)$ is measurable with respect to t for every $x \in \mathbb{R}$. Moreover, suppose that F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla_x F(t,x)| \leqslant a(x)b(t), \tag{2}$$

where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$ and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

The goal of this paper is to obtain existence of solutions for the following problem:

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)), & \text{for a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P_Φ}$$

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the article [19] for definitions and main results on anisotropic Orlicz spaces. These spaces allow us to unify and extend previous results on existence of solutions for systems like (P_Φ). We will find solutions of (P_Φ) by finding extreme points of the *action integral*

$$I(u) := \int_0^T \Phi(u'(t)) + F(t,u(t))\, dt. \tag{IA}$$

In what follows, we shall denote by $\mathcal{L} = \mathcal{L}_{\Phi,F}$ the function $\Phi(y) + F(t,x)$, and we will call it *Lagrangian*.

Problem (P_Φ) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [14] deals mainly with problem (P_Φ) with $\Phi(x) = \Phi_2(x) := |x|^2/2$, through various methods: direct, dual, saddle points, minimax, etc. The results in [14] were extended and improved in several articles, see [22, 23, 21, 27, 30] to cite some examples. The case $\Phi(x) = \Phi_p(y) := |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [24, 25], among other papers. In this case, (P_Φ) is reduced to the *p-laplacian system*. If $\Phi_{p_1,p_2} : \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ is defined by

$$\Phi_{p_1,p_2}(y_1,y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}, \tag{3}$$

then (P_Φ) becomes (p_1,p_2)-laplacian system, see [13, 15, 16, 17, 18, 28, 29]. In a previous paper (see [1]), we consider similar results in an isotropic Orlicz framework.

Hence (P_Φ) contains several problems that have been considered by many authors in the past. Our results still improve some results on (p_1, p_2)-laplacian systems since we obtain existence of solutions for they under less restrictive conditions. For all this, we believe that anisotropic Sobolev-Orlicz spaces can provide a suitable framework to unify many known results. On the other hand, we point out that one of the most important aspects in our work is the possibility of dealing with functions Φ that grow faster than power functions.

Example 1.1. As an example, we obtain existence of solutions for

$$\begin{cases} \frac{d}{dt}\left[u_1(t)e^{(u_1'(t))^2+(u_2'(t))^2}\right] = F_{x_1}(t,u(t)), & \text{for a.e. } t \in (0,T), \\ \frac{d}{dt}\left[u_2(t)e^{(u_1'(t))^2+(u_2'(t))^2}\right] = F_{x_2}(t,u(t)), & \text{for a.e. } t \in (0,T), \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases} \tag{4}$$

where $p > 1$ and F is any function satisfying conditions (C), (A), (B) and (32) below. For example $F(t,x) = P(t)Q(x)$, with P and Q polynomials satisfying those hypothesis (see Remark 7 below).

The paper is organized as follows. In Section 2, we summarize some known results about Orlicz and Orlicz-Sobolev spaces. In order to obtain existence of minimizers of action integrals it is necessary that the functional I be coercive. In the past, several conditions on F have been useful to obtain coercivity of I for the functions Φ_p and Φ_{p_1,p_2} . In this paper we investigate the condition that in the literature was called sublinearity (see [22, 27, 30] for the laplacian, [12, 24] for the p-laplacian and [13, 15, 16, 29] for (p_1, p_2)-laplacian). In Section 3, we contextualize the sublinearity within our framework (see (B) below) and we establish results of existence of minimizers of (IA) in Theorem 3.2. In Section 4, we establish conditions under which a minimum of (IA) is a solution of (P_Φ).

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^d \to [0,+\infty)$. References for these topics are [4, 5, 6, 9, 10, 19, 20, 26]. For the theory of convex functions in general we suggest [7]. Note that, unlike in [10], we do not require that N_∞ functions be superlinear near from 0, i.e. $\Phi(x)/|x| \to 0$ when $|x| \to 0$. However, most of the results proved in [10] do not depend on this property.

If $\Phi(y)$ is an N_∞ function which depends on $|y|$ ($\Phi(y) = \overline{\Phi}(|y|)$), then we say that Φ is *radial*.

We can use the following example to obtain new N_∞ functions from given N_∞ ones.

Example 2.1. Let $(d_1, \ldots, d_k) \in \mathbb{Z}_+^k$. Suppose that $\Phi_j : \mathbb{R}^{d_j} \to [0,+\infty)$, $j = 1, \ldots k$, are N_∞ functions and $O_j \in L(\mathbb{R}^d, \mathbb{R}^{d_j})$ are bounded linear functions satisfying $\bigcap_{j=1}^k \ker O_j = \{0\}$. Then

$$\Phi(y) := \sum_{j=1}^{k} \Phi_j(O_j y)$$

is an N_∞ function.

Let us briefly show that Φ satisfies (1). Suppose that $|y_n| \to \infty$ and $\Phi(y_n)/|y_n|$ is bounded. If for some $j = 1, \ldots, k$ there exist $\epsilon > 0$ and a subsequence n_s such that $|O_j(y_{n_s})| \geqslant \epsilon |y_{n_s}|$, then $\Phi_j(O_j y_{n_s})/|y_{n_s}| \to \infty$, contrary to our assumption. Hence $O_j(y_n)/|y_n| \to 0$ when $n \to \infty$. Passing to a subsequence, we can assume that there exists $y \in \mathbb{R}^d$ such that $y_n/|y_n| \to y$. Then $y \in \ker O_j$ and $y \neq 0$, which is a contradiction.

As a consequence the function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ defined by

$$\Phi(y_1, y_2) = e^{|y_1 - y_2|} - 1 + |y_2|^p,$$

with $1 < p < \infty$, is an N_∞ function.

Associated to Φ we have the *complementary function* $\Phi^\star$ which is defined on $\zeta \in \mathbb{R}^d$ as

$$\Phi^\star(\zeta) = \sup_{y \in \mathbb{R}^d} y \cdot \zeta - \Phi(y). \tag{5}$$

From the continuity of Φ and (1), we have that $\Phi^\star : \mathbb{R}^d \to [0, \infty)$. The complementary function $\Phi^\star$ is an N_∞ function (see [14, Ch. 2] and [19, Thm. 2.2]). Now, Moreau Theorem (see [7, Thm. 4.21]) implies that $\Phi^{\star\star} = \Phi$.

Some useful properties which are satisfied by N_∞ functions are:

(P1) $\Phi(\lambda x) \leqslant \lambda \Phi(x)$, for every $\lambda \in [0, 1], x \in \mathbb{R}^d$;

(P2) if $0 < |\lambda_1| \leqslant |\lambda_2|$, then $\Phi(\lambda_1 x) \leqslant \Phi(\lambda_2 x)$;

(P3) $x \cdot y \leqslant \Phi(x) + \Phi^\star(y)$;

(P4) $x \cdot \nabla\Phi(x) = \Phi(x) + \Phi^\star(\nabla\Phi(x))$.

We say that $\Phi : \mathbb{R}^d \to [0, +\infty)$ satisfies the Δ_2-*condition* and we denote $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that

$$\Phi(2x) \leqslant C\Phi(x) + 1, \quad x \in \mathbb{R}^d. \tag{6}$$

Throughout this article, we denote by $C = C(\lambda_1, \ldots, \lambda_n)$ a positive constant that may depend on T, Φ (or another N_∞ functions) and the parameters $\lambda_1, \ldots, \lambda_n$. We assume that the value that C represents may change in different occurrences in the same chain of inequalities.

If Φ satisfies the Δ_2-condition, then Φ satisfies the following properties:

(P5) There exists $C > 0$ such that for every $x, y \in \mathbb{R}^d$, $\Phi(x + y) \leqslant C(\Phi(x) + \Phi(y)) + 1$.

(P6) For any $\lambda > 1$ there exists $C(\lambda) > 0$ such that $\Phi(\lambda x) \leqslant C(\lambda)\Phi(x) + 1$.

(P7) There exist $1 < p < \infty$ and $C > 0$ such that $\Phi(x) \leqslant C|x|^p + 1$.

Let Φ_1 and Φ_2 be N_∞ functions. Following to [26] we write $\Phi_1 \dashv \Phi_2$ if there exist $k, C > 0$ such that

$$\Phi_1(x) \leqslant C + \Phi_2(kx), \quad x \in \mathbb{R}^d. \tag{7}$$

For example, if $\Phi \in \Delta_2$ then there exists $p \in (1, +\infty)$ such that $\Phi \dashv |x|^p$. If for every $k > 0$ there exists $C = C(k) > 0$ such that (7) holds we write $\Phi_1 \lll \Phi_2$. We observe that $\Phi_1 \dashv \Phi_2$ implies that $\Phi_2^\star \dashv \Phi_1^\star$. A similar assertion holds for relation $\lll$.

If $\Phi^\star \in \Delta_2$ then Φ satisfies the ∇_2-*condition*, i.e. for every $0 < r < 1$ there exist $l = l(r) > 0$ and $C' = C'(r) > 0$ such that

$$\Phi(x) \leqslant \frac{r}{l}\Phi(lx) + C', \quad x \in \mathbb{R}^d. \tag{8}$$

We denote by $\mathcal{M} := \mathcal{M}\left([0,T], \mathbb{R}^d\right)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\ dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi\left([0,T], \mathbb{R}^d\right)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{9}$$

The *Orlicz space* $L^\Phi = L^\Phi\left([0,T], \mathbb{R}^d\right)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{10}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi\left([0,T], \mathbb{R}^d\right)$ is defined as the closure in L^Φ of the subspace $L^\infty\left([0,T], \mathbb{R}^d\right)$ of all $\mathbb{R}^d$-valued essentially bounded functions. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2$ (see [19, Cor. 5.1]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [19, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^\star}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\ dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^{\Phi^\star}}. \tag{11}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

We consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by the following inclusions

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{12}$$

for any positive r. This relation is a trivial generalization of [19, Thm. 5.6]. If $\Phi \in \Delta_2$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow \left[L^{\Phi^\star}\right]^\star$, where a function $v \in L^\Phi$ is associated to $\xi_v \in \left[L^{\Phi^\star}\right]^\star$ given by

$$\langle \xi_v, u\rangle = \int_0^T v \cdot u \; dt. \tag{13}$$

It is easy to prove that $L^\infty \hookrightarrow L^\Phi \hookrightarrow L^1$ for any N_∞ function Φ.

Suppose $u \in L^\Phi([0,T], \mathbb{R}^d)$ and consider $K := \rho_\Phi(u) + 1 \geqslant 1$. Then, from (P1) we have $\rho_\Phi(K^{-1}u) \leqslant K^{-1}\rho_\Phi(u) \leqslant 1$. Therefore, we conclude

$$\|u\|_{L^\Phi} \leqslant \rho_\Phi(u) + 1. \tag{14}$$

We highlight the following result (see [10, Thm. 3.3]).

Proposition 2.1. $L^\Phi\left([0,T], \mathbb{R}^d\right) = \left[E^{\Phi^\star}\left([0,T], \mathbb{R}^d\right)\right]^\star$.

As a consequence of previous proposition, $L^\Phi\left([0,T], \mathbb{R}^d\right)$ can be equipped with the weak$\star$ topology induced by $E^{\Phi^\star}\left([0,T], \mathbb{R}^d\right)$.

We define the *Sobolev-Orlicz space* $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ by

$$W^1L^\Phi\left([0,T], \mathbb{R}^d\right) := \left\{u | u \in AC\left([0,T], \mathbb{R}^d\right) \text{ and } u' \in L^\Phi\left([0,T], \mathbb{R}^d\right)\right\},$$

where $AC\left([0,T], \mathbb{R}^d\right)$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $[0,T]$. The space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{15}$$

We define the function $A_\Phi : \mathbb{R}^d \to [0, +\infty)$ as the greatest convex radial minorant of Φ, i.e.

$$A_\Phi(x) = \sup\{\Psi(x)\}, \tag{16}$$

where the supremum is taken over all the convex, non negative and radial functions Ψ with $\Psi(x) \leqslant \Phi(x)$.

Proposition 2.2. *A_Φ is a radial and N_∞ function.*

Proof. The convexity and radiality of A_Φ is a consequence of the fact that the supremum preserves these properties. It is only necessary to show that $A_\Phi(x) > 0$, when $x \neq 0$ and $A_\Phi(x)/|x| \to \infty$, when $|x| \to \infty$. We write, for $r \in \mathbb{R}$, $r^+ = \max\{r, 0\}$. Since Φ is an N_∞ function, for every $k > 0$ there exists $r_0 > 0$ such that $\Phi(x) \geqslant k(|x| - r_0)^+$, for $|x| > r_0$. As $k(|x| - r_0)^+$ is a non negative, radial, convex function, it follows that $A_\Phi(x) \geqslant k(|x| - r_0)^+$. Therefore $\liminf_{|x|\to\infty} A_\Phi(s)/|x| \geqslant k$ and consequently $\lim_{|x|\to\infty} A_\Phi(x)/|x| = \infty$.

As Φ is an N_∞ continuous function, for every $r > 0$ there exists $k(r) > 0$ such that $\Phi(x) \geqslant k(r)|x| \geqslant k(r)(|x| - r)^+$, when $|x| \geqslant r$. This fact implies that $A_\Phi(x) > 0$ for $x \neq 0$. □

By abuse of notation, we sometimes identify A_Φ with a function defined on $[0, +\infty)$. This function is invertible.

Corollary 2.3. $L^\Phi([0,T], \mathbb{R}^d) \hookrightarrow L^{A_\Phi}([0,T], \mathbb{R}^d)$.

As is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

Lemma 2.4. *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be an N_∞ function and let $u \in W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$. Let $A_\Phi : \mathbb{R}^d \to [0, +\infty)$ be the isotropic function defined by* (16)*. Then:*

1. Morrey's inequality. *For every $s, t \in [0,T]$, $s \neq t$*
$$|u(t) - u(s)| \leqslant |s - t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \|u'\|_{L^\Phi}. \tag{M.I}$$
2. Sobolev's inequality.
$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1L^\Phi}. \tag{S.I}$$
3. Poincaré-Wirtinger's inequality. *We have $\widetilde{u} \in L^\infty\left([0,T], \mathbb{R}^d\right)$ and*
$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi}. \tag{P-W.I}$$
4. *If Φ is an N_∞ function, then the space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is compactly embedded in the space of continuous functions $C([0,T], \mathbb{R}^d)$.*

Proof. It is an immediate consequence of Corollary 2.3 and [1, Lemma 2.1, Cor. 2.2]. □

Lemma 2.4 gives us estimates for isotropic norms of u. In these type of inequalities some information is lost. The following result gives us an estimate that takes into account the anisotropic nature of the space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$. The proof is similar to that of [4, Thm. 4.5].

Lemma 2.5 (Anisotropic Poincaré-Wirtinger's inequality)**.** *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be an N_∞ function and let $u \in W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$. Then*

$$\Phi\left(\tilde{u}(t)\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(Tu'(r)\right) dr. \tag{A.P-W.I}$$

Proof. Applying Jensen's inequality twice, we get

$$\begin{aligned}
\Phi(\tilde{u}(t)) &= \Phi\left(\frac{1}{T}\int_0^T (u(t)-u(s))\, ds\right)\\
&\leqslant \frac{1}{T}\int_0^T \Phi(u(t)-u(s))ds\\
&\leqslant \frac{1}{T}\int_0^T \Phi\left(\int_s^t |t-s|u'(r)\frac{dr}{|t-s|}\right) ds\\
&\leqslant \frac{1}{T}\int_0^T \frac{1}{|t-s|}\int_s^t \Phi\left(|t-s|u'(r)\right) drds.
\end{aligned}$$

From (P1) we have that $\Phi(rx)/r$ is increasing with respecto to $r > 0$ for a fixed $x \in \mathbb{R}^d$. Therefore, previous inequality implies (A.P-W.I). □

Remark 1. As a consequence of Lemma 2.4, we obtain that

$$\|u\|'_{W^1L^\Phi} = |\overline{u}| + \|u'\|_{L^\Phi},$$

defines an equivalent norm to $\|\cdot\|_{W^1L^\Phi}$ on $W^1L^\Phi([0,T], \mathbb{R}^d)$.

Another immediate consequence of Lemma 2.4 is the following result.

Corollary 2.6. *Every bounded sequence $\{u_n\}$ in $W^1L^\Phi([0,T], \mathbb{R}^d)$ has an uniformly convergent subsequence.*

3 Existence of minimizers

It is well known that an important ingredient in the direct method of the calculus of variations is the coercivity of action integrals. In order to obtain coercivity for the action integral I, defined in (IA), it is necessary to impose more restrictions on the potential F.

There are several restrictions that were explored in the past. The one we will study in this article is based on what is known in the literature as sublinearity (see [22, 27, 30] for the Laplacian, [24, 12] for the p-Laplacian and [29, 13, 15, 16] for (p, q)-Laplacian). In the current article we will use another denomination for this property.

Definition 3.1. *Let $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfying (C) and (A). We say that F satisfies condition (B) if there exist an N_∞ function Φ_0, with $\Phi_0 \lll \Phi$ and a function $d \in L^1([0,T], \mathbb{R})$, with $d \geqslant 1$, such that*

$$\Phi^\star(d^{-1}(t)\nabla_x F) \leqslant \Phi_0(x) + 1. \tag{B}$$

The condition (B) encompasses the sublinearity condition as it was introduced in the context of p-laplacian or (p_1, p_2)-laplacian systems. For example, in [13, Thm. 1.1.] Li, Ou and Tang considered a potential $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying (C) and (A) and the following condition (we recall that $p' = p/(p-1)$).

(H) There exist $f_i, g_i, h_i \in L^1([0,T], \mathbb{R}_+)$, $\alpha_i \in [0, p_i/p_i')$, $i = 1,2$, $\beta_1 \in [0, p_2/p_1')$ and $\beta_2 \in [0, p_1/p_2')$ such that

$$|\nabla_{x_i} F(t, x_1, x_2)| \leqslant f_i(t)|x_1|^{\alpha_i} + g_i(t)|x_2|^{\beta_i} + h_i(t), \quad i = 1,2.$$

It is easy to prove that (H) implies (B), with $\Phi = \Phi_{p_1,p_2}$, $\Phi_0 = \Phi_{\overline{p}_1,\overline{p}_2}$, where $\overline{p}_i$, $i = 1,2$, are taken so that $\max\{\alpha_1 p_1', \beta_2 p_2'\} \leqslant \overline{p}_1 < p_1$ and $\max\{\alpha_2 p_2', \beta_1 p_1'\} \leqslant \overline{p}_2 < p_2$ and $d = C(1 + \sum_i \{f_i + g_i + h_i\}) \in L^1$, with $C > 0$ chosen large enough.

Theorem 3.2. *Let Φ be an N_∞-function whose complementary function $\Phi^\star$ satisfies the Δ_2-condition. Let F be a potential that satisfies (C), (A),(B) and the following condition*

$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\ dt}{\Phi_0(2x)} = +\infty. \tag{17}$$

Let M be a weak$\star$ closed subspace of L^Φ and let $V \subset C([0,T], \mathbb{R}^d)$ be closed in the $C([0,T], \mathbb{R}^d)$-strong topology. Then I attains a minimum on $H = \{u \in W^1 L^\Phi | u \in V$ and $u' \in M\}$.

Proof. Step 1. The action integral is coercive.

Let λ be any positive number with $\lambda > 2\max\{T, 1\}$. Since $\Phi_0 \lll \Phi$ there exists $C(\lambda) > 0$ such that

$$\Phi_0(x) \leqslant \Phi\left(\frac{x}{2\lambda}\right) + C(\lambda), \quad x \in \mathbb{R}^d. \tag{18}$$

By the decomposition $u = \overline{u} + \tilde{u}$, the absolutely continuity of $F(t, x + sy)$ with respect to $s \in \mathbb{R}$, Young's inequality, (B), the convexity of Φ_0, (P2), (18) and (A.P-W.I) we obtain

$$\begin{aligned}
J &:= \left|\int_0^T F(t,u) - F(t,\overline{u})dt\right| \\
&\leqslant \int_0^T \int_0^1 |\nabla_x F(t, \overline{u} + s\tilde{u})\tilde{u}| ds dt \\
&\leqslant \lambda \int_0^T d(t) \int_0^1 \left[\Phi^\star\left(d^{-1}(t)\nabla_x F(t, \overline{u} + s\tilde{u})\right) + \Phi\left(\frac{\tilde{u}}{\lambda}\right)\right] ds dt \\
&\leqslant \lambda \int_0^T d(t) \int_0^1 \left[\frac{1}{2}\Phi_0(2\overline{u}) + \frac{1}{2}\Phi_0(2\tilde{u}) ds + \Phi\left(\frac{\tilde{u}}{\lambda}\right) + 1\right] ds dt \\
&\leqslant \lambda \int_0^T d(t) \int_0^1 \left[\Phi_0(2\overline{u}) + 2\Phi\left(\frac{\tilde{u}}{\lambda}\right) + C(\lambda)\right] ds dt \\
&\leqslant C_1 \Phi_0(2\overline{u}) + \lambda C_2 \int_0^T \Phi\left(\frac{T u'(s)}{\lambda}\right) ds + C_1
\end{aligned}$$

where $C_2 = C_2(\|d\|_{L^1})$ and $C_1 = C_1(\|d\|_{L^1}, \lambda)$. Since $\Phi^\star \in \Delta_2$ we can choose λ large enough so that $l = \lambda T^{-1}$ satisfies (8) for $r = \frac{1}{2}\min\{(C_2T)^{-1}, 1\}$. Thus, we have

$$J \leqslant C_1\Phi_0(2\overline{u}) + \frac{1}{2}\int_0^T \Phi\left(u'(s)\right) ds + C_1.$$

Then

$$\begin{aligned} I(u) &= \int_0^T \Phi(u') + F(t,u)dt \\ &= \int_0^T \{\Phi(u') + [F(t,u) - F(t,\overline{u})] + F(t,\overline{u})\}dt \qquad (19) \\ &\geqslant \frac{1}{2}\int_0^T \Phi(u')dt - C_1\Phi_0(2\overline{u}) + \int_0^T F(t,\overline{u})dt - C_1 \end{aligned}$$

We take $u_n \in W^1L^\Phi$ with $\|u_n\|_{W^1L^\Phi} \to \infty$. From Remark 1, we can suppose that $\|u_n'\|_{L^\Phi} \to \infty$ or $|\overline{u}_n| \to \infty$. In the first case, we have from (14) that $\rho_\Phi(u_n) \to \infty$ and hence $I(u_n) \to \infty$. In the second case, $I(u_n) \to \infty$ as consequence of (17).

Step 2. Suppose that $u_n \to u$ uniformly and $u_n' \overset{\star}{\rightharpoonup} u'$ in $L^\Phi([0,T],\mathbb{R}^d)$ then $I(u) \leqslant \liminf_{n\to\infty} I(u_n)$.

Without loss of generality, passing to subsequences, we may assume that the lim inf is really a lim. The embedding $L^\Phi([0,T],\mathbb{R}^d) \hookrightarrow L^1([0,T],\mathbb{R}^d)$ implies that $u_n' \rightharpoonup u'$ in $L^1([0,T],\mathbb{R}^d)$. Now, applying [3, Thm. 3.6] we obtain $I(u) \leqslant \lim_{n\to\infty} I(u_n)$.

Step 3. Final step. The proof of the theorem is concluded with a usual argument. We take a minimizing sequence $u_n \in H$ of I. From the coercivity of I we have that u_n is bounded on $W^1L^\Phi([0,T],\mathbb{R}^d)$. By Corollary 2.6 (passing to subsequences) we can suppose that u_n converges uniformly to a function $u \in V$. On the other hand, u_n' is bounded on $L^\Phi = \left[E^{\Phi^\star}\right]^\star$. Thus, since $E^{\Phi^\star}$ is separable (see [19, Thm. 6.3]), it follows from [2, Cor. 3.30] there exist a subsequence of u_n' (we denote it u_n' again) and $v \in M$ such that $u_n' \overset{\star}{\rightharpoonup} v$. From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \varphi' \cdot u \; dt = \lim_{n\to\infty}\int_0^T \varphi' \cdot u_n \; dt = -\lim_{n\to\infty}\int_0^T \varphi \cdot u_n' \; dt = -\int_0^T \varphi \cdot v \; dt,$$

for every function $\varphi \in C^\infty([0,T],\mathbb{R}^d) \subset E^{\Phi^\star}$ with $\varphi(0) = \varphi(T) = 0$. Thus u has a derivative in the weak sense in L^Φ. Taking account of $L^\Phi \hookrightarrow L^1$ and [3, Thms. 2.3 and 2.17], we obtain $u \in W^1L^\Phi$ and $v = u'$ a.e. $t \in [0,T]$. Hence, $u \in H$.

Finally, the semicontinuity of I, established in step 2, implies that u is a minimum of I. □

Remark 2. The results of this section extend without difficulty to any Lagrangian $\mathcal{L}$ with $\mathcal{L} \geqslant \mathcal{L}_{\Phi,F}$ (see [1]).

4 Differentiability of solutions and Euler-Lagrange equations

In this section, we will address the question of when minimizers of I are solutions of (P_Φ). It is a classic result that minimizers, on an appropriate set, of action integral satisfying an a priori smothness condition (e.g. Lipschitz continuity) are solutions of (P_Φ) (see [3]). In Theorem 4.1 we obtain a better a priori condition for the action integral under consideration.

We denote by $\mathrm{Lip}([0,T],\mathbb{R}^d)$ the set of $\mathbb{R}^d$-valued Lipschitz continuous functions defined on $[0,T]$. If $X \subset L^\Phi([0,T],\mathbb{R}^d)$ and $u \in L^\Phi([0,T],\mathbb{R}^d)$, we denote by $d(u,X)$ the distance from u to X computed with respect to the Luxemburg norm. We recall that $u \in \mathrm{Lip}([0,T],\mathbb{R}^d)$ implies $d(u', L^\infty([0,T],\mathbb{R}^d)) = 0$. The following is the main result of this section.

Theorem 4.1. *Assume F as in Theorem 3.2 and Φ strictly convex. If u is a minimum of I on the set $H = \{u \in W^1L^\Phi([0,T],\mathbb{R}^d) | u(0) = u(T)\}$ and $d(u', L^\infty([0,T],\mathbb{R}^d)) < 1$ then u is solution of (P_Φ).*

The proof of the previous theorem depends on the Gâteaux differentiability of the action integral on the space $W^1L^\Phi([0,T],\mathbb{R}^d)$. We will deal with a more general lagrangian function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, which is assumed measurable in t for each $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable at (x,y) for almost every $t \in [0,T]$. We consider

$$I(u) = I_{\mathcal{L}}(u) = \int_0^T \mathcal{L}(t, u(t), u'(t))dt \tag{IG}$$

the action integral associated to $\mathcal{L}$. In order to obtain differentiability of I, it is necessary to impose some constraints on $\mathcal{L}$. In the paper [4], Chmara and Maksymiuk obtained differentiability for I on W^1L^Φ assuming a similar condition to Definition 4.2 and additionally $\Phi \in \Delta_2 \cap \nabla_2$. For our purpose, the condition $\Phi \in \Delta_2$ is a very serious limitation since it leaves out of consideration functions that grow faster than power ones. According to our criterion, that is one of the greatest achievements of considering anisotropic Orlicz spaces. For this reason, we present a proof of the results obtained in [4] without the assumption $\Phi \in \Delta_2$. When $\Phi \notin \Delta_2$, the differentiability of I is somewhat more delicate since the effective domain of I is not the whole space W^1L^Φ.

Definition 4.2. *We say that a Lagrangian $\mathcal{L}$ satisfies the condition (S) if*

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Phi^\star\left(\frac{\nabla_y \mathcal{L}}{\lambda}\right) \leqslant a(x)\left[b(t) + \Phi\left(\frac{y}{\Lambda}\right)\right], \tag{S}$$

for a.e. $t \in [0,T]$, where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$, $b \in L^1\left([0,T],[0,+\infty)\right)$ and $\Lambda, \lambda > 0$.

Condition (S) includes structure conditions that have been previously considered in the literature in the case of p-laplacian and p_1, p_2-laplacian systems.

For example, it is easy to see that, when $\Phi(x) = \Phi_p(x) = |x|^p/p$, then condition (S) is equivalent to the structure condition in [14, Thm. 1.4]. If Φ is a radial N_∞ function such that $\Phi^\star$ satisfies the Δ_2-condition, then (S) is related to conditions [1, Eq. (2)-(4)]. If $\Phi = \Phi_{p_1,p_2}$ is as in Equation (3) and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a Lagrangian with $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, then inequality (S) is related to structure conditions like [25, Lemma 3.1, Eq. (3.1)]. As can be seen, condition (S) is a more compact expression than [25, Lemma 3.1, Eq. (3.1)] and moreover weaker, because (S) does not imply a control of $|D_{y_1}L|$ independent of y_2.

Remark 3. We leave to the reader the proof of the fact that if a Lagrange function $\mathcal{L}$ satisfies structure condition (S) and $\Phi \dashv \Phi_0$, then $\mathcal{L}$ satisfies (S) with Φ_0 instead of Φ and possibly with other functions b, a and constants Λ and λ.

Remark 4. The Lagrangian $\mathcal{L} = \mathcal{L}_{\Phi,F} = \Phi(y) + F(t,x)$ satisfies condition (S), for every $\Lambda < 1$. In order to prove this, the only non trivial fact that we should establish is that $\Phi^\star(\nabla_y \mathcal{L}) \leqslant a(x)\,\{b(t) + \Phi\,(y/\Lambda)\}$. From (P4) and the fact that $(d/dt)\Phi(tx) = \nabla\Phi(tx)\cdot x$ is an non decreasing function of t, we obtain

$$\Phi^\star(\nabla\Phi(x)) \leqslant x\cdot\nabla\Phi(x) \leqslant \frac{1}{\Lambda^{-1}-1}\int_1^{\Lambda^{-1}} \frac{d}{dt}\Phi(tx)dt \leqslant \frac{1}{\Lambda^{-1}-1}\Phi(\Lambda^{-1}x).$$

Therefore $\Phi^\star(\nabla_y\mathcal{L}) = \Phi^\star\left(\nabla\Phi(y)\right) \leqslant \Lambda(1-\Lambda)^{-1}\Phi(y/\Lambda)$, for every $\Lambda < 1$.

Given a function $a : \mathbb{R}^d \to \mathbb{R}$, we define the composition operator $\boldsymbol{a} : \mathcal{M} \to \mathcal{M}$ by $\boldsymbol{a}(u)(x) = a(u(x))$. We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

Lemma 4.3. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : [0,+\infty) \to [0,+\infty)$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

The following lemma will be applied repeatedly. We adapted the proof of [1, Lemma 2.5] to the anisotropic case. For an alternative approach, we suggest [4].

Lemma 4.4. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions converging to $u \in \Pi(E^\Phi, \lambda)$ in the L^Φ-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0,T], \mathbb{R})$ such that $u_{n_k} \to u$ a.e. and $\Phi(u_{n_k}/\lambda) \leqslant h$ a.e.*

Proof. Since $d(u, E^\Phi) < \lambda$ and u_n converges to u, there exists a subsequence of u_n (again denoted u_n), $\overline{\lambda} \in (0,\lambda)$ and $u_0 \in E^\Phi$ such that $d(u_n, u_0) < \overline{\lambda}$, $n = 1, \ldots$. Since $L^\Phi\left([0,T], \mathbb{R}^d\right) \hookrightarrow L^1\left([0,T], \mathbb{R}^d\right)$, the sequence u_n converges in measure to u. Therefore, we can extract a subsequence (denoted again u_n) such that $u_n \to u$ a.e. and

$$\lambda_n := \|u_n - u_{n-1}\|_{L^\Phi} < \frac{\lambda - \overline{\lambda}}{2^{n-1}}, \quad \text{for } n \geqslant 2.$$

We can assume $\lambda_n > 0$ for every $n = 1, \ldots$. We write $\lambda_1 := \|u_1 - u_0\|_{L^\Phi}$ and $\lambda_0 := \lambda - \sum_{n=1}^{\infty} \lambda_n$, and we define $h : [0,T] \to \mathbb{R}$ by

$$h(t) = \frac{\lambda_0}{\lambda} \Phi\left(\frac{u_0}{\lambda_0}\right) + \sum_{j=0}^{\infty} \frac{\lambda_{j+1}}{\lambda} \Phi\left(\frac{u_{j+1} - u_j}{\lambda_{j+1}}\right). \tag{20}$$

As $\Phi(0) = 0$ and Φ is a convex function, we have for any $n = 1, \ldots$

$$\begin{aligned}\Phi\left(\frac{u_n}{\lambda}\right) &= \Phi\left(\frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \frac{u_{j+1} - u_j}{\lambda}\right) \\ &\leqslant \frac{\lambda_0}{\lambda} \Phi\left(\frac{u_0}{\lambda_0}\right) + \sum_{j=0}^{n-1} \frac{\lambda_{j+1}}{\lambda} \Phi\left(\frac{u_{j+1} - u_j}{\lambda_{j+1}}\right) \leqslant h\end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and E^Φ is a subspace, we have that $\Phi(u_0/\lambda_0) \in L^1([0,T], \mathbb{R})$. On the other hand, $\|u_{j+1} - u_j\|_{L^\Phi} = \lambda_{j+1}$ and therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_{j+1}}\right) dt \leqslant 1.$$

Then $h \in L^1([0,T], \mathbb{R})$. □

The proof of the following theorem follows the same lines as [1, Thm. 3.2] but with some modifications due to the lack of monotonicity of Φ with respect to the euclidean norm and the fact that the notion of absolutely continuous norm (used intensely in [1, Thm. 3.2]) does not work very well in the framework of anisotropic Orlicz spaces when $\Phi \notin \Delta_2$.

Theorem 4.5. *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying (S). Then the following statements hold:*

1. *The action integral given by (IG) is finitely defined on the set $\mathcal{E}^\Phi_\Lambda := W^1L^\Phi \cap \{u | u' \in \Pi(E^\Phi, \Lambda)\}$.*

2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi_\Lambda$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi_\Lambda$ into $\left[W^1L^\Phi\right]^\star$, i.e. I' is continuous when $\mathcal{E}^\Phi_\Lambda$ is equipped with the strong topology and $\left[W^1L^\Phi\right]^\star$ with the weak$\star$ topology. Moreover, I' is given by the following expression*

$$\langle I'(u), v\rangle = \int_0^T \left[\nabla_x \mathcal{L}\big(t, u, u'\big) \cdot v + \nabla_y \mathcal{L}\big(t, u, u'\big) \cdot v'\right] dt. \tag{21}$$

3. *If $\Phi^\star \in \Delta_2$ then I' is continuous from $\mathcal{E}^\Phi_\Lambda$ into $\left[W^1L^\Phi\right]^\star$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^{\Phi}_{\Lambda}$. From (12) we obtain that $\Phi(u'(t)/\Lambda) \in L^1$. Now, from (S) and Lemma 4.3, we have

$$\begin{aligned}|\mathcal{L}(t,u(t),u'(t))| + |\nabla_x \mathcal{L}(t,u(t),u'(t))| + \Phi^\star\left(\frac{\nabla_y \mathcal{L}(t,u,u')}{\lambda}\right) \\ \leqslant A(\|u\|_{W^1L^\Phi})\left[b(t) + \Phi\left(\frac{u'(t)}{\Lambda}\right)\right] \in L^1.\end{aligned} \tag{22}$$

Thus, by integrating this inequality item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto \nabla_x \mathcal{L}(\cdot,u,u')$ is continuous from $\mathcal{E}^{\Phi}_{\Lambda}$ into $L^1([0,T])$ with the strong topology on both sets.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^{\Phi}_{\Lambda}$ and let $u \in \mathcal{E}^{\Phi}_{\Lambda}$ such that $u_n \to u$ in W^1L^Φ. By (S.I), $u_n \to u$ uniformly. As $u_n' \to u' \in \mathcal{E}^{\Phi}_{\Lambda}$, by Lemma 4.4, there exist a subsequence of u_n' (again denoted u_n') and a function $h \in L^1([0,T],\mathbb{R})$ such that $u_n' \to u'$ a.e. and $\Phi(u_n'/\Lambda) \leqslant h$ a.e.

Since u_n, $n = 1,2,\ldots$, is a bounded sequence in W^1L^Φ, according to Lemma 4.3, there exists $M > 0$ such that $\|\boldsymbol{a}(u_n)\|_{L^\infty} \leqslant M$, $n = 1,2,\ldots$. From the previous facts and (22), we get

$$|\nabla_x \mathcal{L}(\cdot,u_n,u_n')| \leqslant a(u_n)\left[b + \Phi\left(\frac{u_n'}{\Lambda}\right)\right] \leqslant M(b+h) \in L^1.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$\nabla_x \mathcal{L}(t,u_{n_k}(t),u_{n_k}'(t)) \to \nabla_x \mathcal{L}(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T].$$

Applying Lebesgue Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto \nabla_y \mathcal{L}(\cdot,u,u')$ is continuous from $\mathcal{E}^{\Phi}_{\Lambda}$ with the strong topology into $\left[L^\Phi\right]^\star$ with the weak$\star$ topology.

Let $u \in \mathcal{E}^{\Phi}_{\Lambda}$. From (22), it follows that

$$\nabla_y \mathcal{L}(\cdot,u,u') \in \lambda C^{\Phi^\star}\left([0,T],\mathbb{R}^d\right) \subset L^{\Phi^\star}\left([0,T],\mathbb{R}^d\right) \subset \left[L^\Phi\left([0,T],\mathbb{R}^d\right)\right]^\star. \tag{23}$$

Let $u_n, u \in \mathcal{E}^{\Phi}_{\Lambda}$ such that $u_n \to u$ in the norm of W^1L^Φ. We must prove that $\nabla_y \mathcal{L}(\cdot,u_n,u_n') \xrightarrow{w^\star} \nabla_y \mathcal{L}(\cdot,u,u')$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$\left|\langle \nabla_y \mathcal{L}(\cdot,u_n,u_n'), v\rangle - \langle \nabla_y \mathcal{L}(\cdot,u,u'), v\rangle\right| \geqslant \epsilon. \tag{24}$$

We have $u_n \to u$ in L^Φ and $u_n' \to u'$ with $u' \in \Pi(E^\Phi,\Lambda)$. By Lemmas 2.6 and 4.4, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T],\mathbb{R})$ such that $u_n \to u$ uniformly, $u_n' \to u'$ a.e. and $\Phi(u_n'/\Lambda) \leqslant h$ a.e. As in the previous step, Lemma 4.3 implies that $a(u_n(t))$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (22) with u_n instead of u, we have

$$\Phi^\star\left(\frac{\nabla_y \mathcal{L}(\cdot,u_n,u_n')}{\lambda}\right) \leqslant M(b+h) =: h_1 \in L^1. \tag{25}$$

As $v \in L^\Phi$ there exists $\lambda_v > 0$ such that $\Phi(v/\lambda_v) \in L^1$. Now, by Young's inequality and (25), we have

$$\begin{aligned}\nabla_y \mathcal{L}(\cdot, u_n, u_n') \cdot v(t) &\leqslant \lambda\lambda_v \left[\Phi^\star\left(\frac{\nabla_y \mathcal{L}(\cdot, u_n, u_n')}{\lambda}\right) + \Phi\left(\frac{v}{\lambda_v}\right)\right] \\ &\leqslant \lambda\lambda_v M(b+h) + \lambda\lambda_v \Phi\left(\frac{v}{\lambda_v}\right) \in L^1.\end{aligned} \tag{26}$$

Finally, from Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T \nabla_y \mathcal{L}(t, u_n, u_n') \cdot v dt \to \int_0^T \nabla_y \mathcal{L}(t, u, u') \cdot v\, dt, \tag{27}$$

which contradicts the inequality (24). This completes the proof of step 2.

Step 3. We will prove (21). Note that (22), (23) and the imbeddings $W^1L^\Phi \hookrightarrow L^\infty$ and $L^{\Phi^\star} \hookrightarrow \left[L^\Phi\right]^\star$ imply that the second member of (21) defines an element of $\left[W^1L^\Phi\right]^\star$.

The proof follows similar lines as [14, Thm. 1.4]. For $u \in \mathcal{E}^\Phi_\Lambda$ and $0 \neq v \in W^1L^\Phi$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), u'(t) + sv'(t)).$$

For $|s| \leqslant s_0 := \left(\Lambda - d(u', E^\Phi)\right)/\|v\|_{W^1L^\Phi}$ we have that $u' + sv' \in \Pi(E^\Phi, \Lambda)$. This fact implies, in virtue of Theorem 4.5 item 1, that $I(u+sv)$ is well defined and finite for $|s| \leqslant s_0$.

We write $s_1 := \min\{s_0, 1 - d(u', E^\Phi)/\Lambda\}$. Let $\lambda_v > 0$ such that $\Phi(v'/\lambda_v) \in L^1$. As $u' \in \Pi(E^\Phi, \Lambda)$, then

$$d\left(\frac{u'}{(1-s_1)\Lambda}, E^\Phi\right) = \frac{1}{(1-s_1)\Lambda} d(u', E^\Phi) < 1,$$

and consequently $(1-s_1)^{-1}\Lambda^{-1}u' \in C^\Phi$. Hence, if $v' \in L^\Phi$ and $|s| \leqslant s_1\Lambda\lambda_v^{-1}$, from the convexity of Φ and (P2), we get

$$\begin{aligned}\Phi\left(\frac{u' + sv'}{\Lambda}\right) &\leqslant (1-s_1)\Phi\left(\frac{u'}{(1-s_1)\Lambda}\right) + s_1\Phi\left(\frac{s}{s_1\Lambda}v'\right) \\ &\leqslant (1-s_1)\Phi\left(\frac{u'}{(1-s_1)\Lambda}\right) + s_1\Phi\left(\frac{v'}{\lambda_v}\right) \\ &=: h(t) \in L^1.\end{aligned} \tag{28}$$

We also have $\|u + sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 4.3, there exists $M > 0$ independent of s, such that $\|a(u+sv)\|_{L^\infty} \leqslant M$. Now, applying Young's Inequality, (22), the fact that $v \in L^\infty$, (28) and $\Phi(v'/\lambda_v) \in L_1$, we get

$$\begin{aligned}|D_sH(s,t)| &= \left|\nabla_x\mathcal{L}(t,u+sv,u'+sv')\cdot v+\nabla_y\mathcal{L}(t,u+sv,u'+sv')\cdot v'\right|\\ &\leqslant M\left[b(t)+\Phi\left(\frac{u'+sv'}{\Lambda}\right)\right]|v|\\ &\quad+\lambda\lambda_v\left[\Phi^\star\left(\frac{\nabla_y\mathcal{L}(t,u+sv,u'+sv')}{\lambda}\right)+\Phi\left(\frac{v'}{\lambda_v}\right)\right]\\ &\leqslant M\left[b(t)+\Phi\left(\frac{u'+sv'}{\Lambda}\right)\right](|v|+\lambda\lambda_v)+\lambda\lambda_v\Phi\left(\frac{v'}{\lambda_v}\right)\\ &\leqslant M\left(b(t)+h(t)\right)(|v|+\lambda\lambda_v)+\lambda\lambda_v\Phi\left(\frac{v'}{\lambda_v}\right)\in L^1.\end{aligned} \tag{29}$$

Consequently, I has a directional derivative and

$$\langle I'(u),v\rangle=\frac{d}{ds}I(u+sv)\big|_{s=0}=\int_0^T\left[\nabla_x\mathcal{L}(t,u,u')\cdot v+\nabla_y\mathcal{L}(t,u,u')\cdot v'\right]\,dt.$$

Moreover, from the previous formula, (22), (23) and Lemma 2.4, we obtain

$$|\langle I'(u),v\rangle|\leqslant\|\nabla_x\mathcal{L}\|_{L^1}\|v\|_{L^\infty}+\|\nabla_y\mathcal{L}\|_{L^{\Phi^\star}}\|v'\|_{L^\Phi}\leqslant C\|v\|_{W^1L^\Phi},$$

with an appropriate constant C. This completes the proof of the Gâteaux differentiability of I. The previous steps imply the demicontinuity of the operator $I':\mathcal{E}^\Phi_\Lambda\to\left[W^1L^\Phi_d\right]^\star$.

In order to prove item 3, it is necessary to see that the maps $u\mapsto\nabla_x\mathcal{L}(t,u,u')$ and $u\mapsto\nabla_y\mathcal{L}(t,u,u')$ are norm continuous from $\mathcal{E}^\Phi_\Lambda$ into L^1 and $L^{\Phi^\star}$, respectively. It remains to prove the continuity of the second map. To this purpose, we take $u_n,u\in\mathcal{E}^\Phi_\Lambda$, $n=1,2,\dots$, with $\|u_n-u\|_{W^1L^\Phi}\to0$. As before, we can deduce the existence of a subsequence (denoted u_n' for simplicity) and $h_1\in L^1$ such that (25) holds and $u_n\to u$ a.e. Since $\Phi^\star\in\Delta_2$, we have

$$\Phi^\star(\nabla_y\mathcal{L}(\cdot,u_n,u_n'))\leqslant c(\lambda)\Phi^\star\left(\frac{\nabla_y\mathcal{L}(\cdot,u_n,u_n')}{\lambda}\right)+1\leqslant c(\lambda)h_1+1=:h_2\in L^1. \tag{30}$$

Then, from (P5), we get

$$\Phi^\star\left(\nabla_y\mathcal{L}(\cdot,u_n,u_n')-\nabla_y\mathcal{L}(\cdot,u,u')\right)\leqslant K(h_2+\Phi^\star(\nabla_y\mathcal{L}(\cdot,u,u')))+1.$$

Now, by Lebesgue Dominated Convergence Theorem, we obtain $\nabla_y\mathcal{L}(\cdot,u_n,u_n')$ is $\rho_{\Phi^\star}$ modular convergent to $\nabla_y\mathcal{L}\left(\cdot,u,u'\right))$, i.e. $\rho_{\Phi^\star}(u_n-u)\to0$. Since $\Phi^\star\in\Delta_2$, modular convergence implies norm convergence (see [20]). $\square$

Proof Theorem 4.1. We observe that $H=\{u\in W^1L^\Phi|u\in V\text{ and }u'\in M\}$ where

$$V:=\{u\in C([0,T],\mathbb{R}^d)|u(0)=u(T)\},\quad M:=L^\Phi([0,T],\mathbb{R}^d),$$

and V is $C([0,T],\mathbb{R}^d)$-closed. Therefore, the functional I given by (IA), has a minimum u on H. Suppose that $d(u',L^\infty) < 1$. Since $d(u',E^\Phi) = d(u',L^\infty)$, according to Remark 4 and Theorem 4.5, I is Gâteaux differentiable at u. By Fermat's rule (see [7, Prop. 4.12]), we have $\langle I'(u),v\rangle = 0$ for every $v \in H$. Therefore

$$\int_0^T \nabla\Phi(u'(t))\cdot v'(t)dt = -\int_0^T \nabla_x F(t,u(t))\cdot v(t)dt. \tag{31}$$

From Theorem 4.5, $\nabla_x F(t,u(t)) \in L^1([0,T],\mathbb{R}^d)$ and $\nabla\Phi\,(u'(t)) \in L^{\Phi^\star}([0,T],\mathbb{R}) \hookrightarrow L^1([0,T],\mathbb{R})$. Identity (31) holds for every $v \in C^\infty([0,T],\mathbb{R}^d)$ with $v(0) = v(T)$. Using [14, Fundamental Lemma, p. 6], we get that $\nabla\Phi(u'(t))$ is absolutely continuous and $(d/dt)\,(\nabla\Phi(u'(t))) = \nabla_x F(t,u(t))$ a.e. on $[0,T]$. Moreover, $\nabla\Phi(u'(0)) = \nabla\Phi(u'(T))$. Since Φ is *strictly convex*, then $\nabla\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is a one-to-one map (see, e.g. [7, Ex. 4.17, p. 67]). Hence, we conclude that $u'(0) = u'(T)$. Finally, Theorem 4.1 is proven. □

Remark 5. If u is a minimum of I on H then $d(u',E^\Phi) \leqslant 1$. This follows from $\rho_\Phi(u') < \infty$ and (12). Then, the possible minima of I that do not satisfy the hypotheses of Theorem 4.1 lie on a nowhere dense set of the domain of I.

Remark 6. The condition $d(u',L^\infty) < 1$ is trivially satisfied when $\Phi \in \Delta_2$ because, in this case, L^∞ is dense in $L^\Phi([0,T],\mathbb{R}^d)$. In particular, our Theorem 4.1 implies existence of solutions, among others, for the (p_1,p_2)-laplacian system.

It is possible to use the regularity theory in order to get that minimizers u of I satisfy $u' \in L^\infty$. To cite one example, we have the following result.

Corollary 4.6. *Let Φ, F and H be as in Theorem 4.1. Additionally, suppose that $F(t,x)$ is differentiable with respect to (t,x) and that*

$$\left|\frac{\partial}{\partial t}F(t,x)\right| \leqslant a(x)b(t), \tag{32}$$

with a and b as in (2)*. If u is a minimum of I on the set H then u is solution of* (P_Φ).

Proof. We note that u is a minimum of I on the set defined by a Dirichlet boundary condition

$$\{v \in W^1L^\Phi([0,T],\mathbb{R}^d) | v(0) = u(0), v(T) = u(T)\}.$$

Therefore, we can apply Proposition 3.1 in [8] (see also the following remark) and we obtain $u' \in L^\infty$. □

Remark 7. Returning to the system (4) of Example 1.1, we note that the N_∞ function $\Phi(y_1,y_2) = \exp(y_1^2 + y_2^2)$ has a complementary function which satisfies the Δ_2-condition (see [11, p. 28]). In addition, for every $p > 1$ we have $|(y_1,y_2)|^p \ll \Phi(y_1,y_2)$. Therefore $\Phi^\star(y_1,y_2) \ll |(y_1,y_2)|^q$ for $q = p/(p-1)$. Consequently, if $F(t,x_1,x_2) = P(t)Q(x_1,x_2)$ with P and Q polynomials, and

$d(t) := C \max\{1, |P(t)|\}$ then $\Phi^\star(d^{-1}(t)\nabla_x F) \leqslant |(x_1, x_2)|^q + 1$, where p and C are chosen large enough. Hence Φ and F satisfy (B) with $\Phi_0(y_1, y_2) = |(y_1, y_2)|^p$. The conditions (C), (A) and (32) are proven in a direct way. These facts prove the assessment of Example 1.1.

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