# Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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**Abstract**

## 1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases} \tag{1}$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary equations*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y)=\frac{|y|^p}{p}+F(t,x), \tag{2}$$

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for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (1) is reduced to the $p$-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call $F$ a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to $x$ for a.e. $t \in [0,T]$ and the following conditions are verified:

**(C)** $F$ and its gradient $\nabla F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

**(A)** For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{4}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [Acinas et al., 2015] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{5}$$

where $\Phi$ is an $N$-function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on $F$ (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (1).

## 2 Preliminaries

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces) and other brief introduction to superposition operators between these spaces. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001] and [Płuciennik, 1987, Nguen Hong Thai, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if $\Phi$ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$.

Following [Schappacher, 2005] we say that $\Phi$ is *coercive* if

$$\lim_{|x|\to\infty}\frac{\Phi(x)}{|x|}=+\infty.$$

We define the function the $G$ by

$$G(s)=\min\{\Phi(x):|x|=s\}, \tag{6}$$

where $\Phi$ is a Young's function.

The function $F$ defined above is the greatest radial minorant. That is .....????

A simple consequence of the previous fact is the next inequality $G(|x|)\leqslant\Phi(x)$.

The function $G$ is monotonous, continuous and $G(s)\to\infty$ as $s\to\infty$. Pruebas de lo anterior o cómo sale....????

Then, there exists its inverse function $G^{-1}$. And, from (**??**) it is easy to see that $G^{-1}(\Phi(x))\geqslant|x|$.

As $\frac{\Phi(\alpha x)}{\alpha}$ is increasing with respect to $\alpha$, we get that $\frac{G(\alpha s)}{\alpha}$ is also increasing with respect to $\alpha$?????? In fact, if $0<\alpha\leqslant\beta$, we have

$$\begin{aligned}\frac{G(\alpha s)}{\alpha}=\frac{\min\{\Phi(x):|x|=\alpha s\}}{\alpha}=\min\left\{\frac{\Phi(x)}{\alpha}:\frac{|x|}{\alpha}=s\right\}=\\ \min\left\{\frac{\Phi(\alpha y)}{\alpha}:|y|=s\right\}\leqslant\min\left\{\frac{\Phi(\beta y)}{\beta}:|y|=s\right\}=\\ \frac{\min\{\Phi(\beta y):|y|=s\}}{\beta}=\frac{\min\left\{\Phi(x):\frac{|x|}{\beta}=s\right\}}{\beta}=\frac{\min\{\Phi(x):|x|=\beta s\}}{\beta}=\\ \frac{G(\beta s)}{\beta}.\end{aligned}$$

And, in the event that $\Phi$ is an N-function, we obtain that $\frac{G(\alpha s)}{\alpha}\to\infty$ as $\alpha\to\infty$.

Performing some change of variables on....., we can see that $\alpha G^{-1}(\frac{s}{\alpha})$ is an increasing function too.

We also say that a non decreasing function $\eta:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the $\Delta_2^\infty$*-condition*, denoted by $\eta\in\Delta_2^\infty$, if there exist constants $K>0$ and $M\geqslant 0$ such that

$$\eta(2x)\leqslant K\eta(x), \tag{7}$$

for every $|x|\geqslant M$.

If $\Phi$ is a Young function we define its *Fenchel conjugate* $\Phi^*:\mathbb{R}^d\to\mathbb{R}^+$ by:

$$\Phi^*(y)=\sup_{x\in\mathbb{R}^d}x\cdot y-\Phi(x) \tag{8}$$

Let $d$ be a positive integer. We denote by $\mathcal{M}:=\mathcal{M}([0,T],\mathbb{R}^d)$ the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u=(u_1,\ldots,u_d)$ for $u\in\mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d=1$.

Given an $N$-function $\Phi$ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T],\mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{9}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of $C^\Phi$; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{10}$$

The Orlicz space $L^\Phi$ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between $u$ and $v$.

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in $L^\Phi$ of the subspace $L^\infty([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of $E^\Phi$, which is particularly useful for us, is that $u \in E^\Phi$ if and only if $u$ has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \ldots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^{\Phi^*}}. \tag{11}$$

Like in [Krasnosel′skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of $L^\Phi$ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class $C^\Phi$ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{12}$$

for any positive $r$ (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets $L^\Phi$, $E^\Phi$, $\Pi(E^\Phi, r)$ and $C^\Phi$ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

**Definition 2.1.** *Let $u_n, uL^\Phi([0,T],\mathbb{R}^d)$. We say that $u_n$ converges monotonically to $u$ if there exists $\alpha_n \in \|_{L^\infty}([0,T],\mathbb{R}^d)$, $n = 1, 2, \ldots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t)u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of $X$, we write $Y \hookrightarrow X$ and we say that $Y$ is *embedded* in $X$ when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow \left[L^\Phi\right]^*$, where a function $v \in L^{\Phi^*}$ is associated to $F_v \in \left[L^\Phi\right]^*$ where

$$F_v(u) := \langle v, u\rangle = \int_0^T v \cdot u \, dt, \tag{13}$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $\left[L^\Phi\right]^*$ which is identified with $L^{\Phi^*}$. Namely $L^{\Phi^*} = P^{\Phi^*}([0,T],\mathbb{R}^d)$ where $F \in P^{\Phi^*}([0,T],\mathbb{R}^d)$ if and only if $F \in \left[L^\Phi\right]^*$ and satisfying the *monotone convergence property*, which is if $u_n$ converges monotonically to $u$ then $F(u_n) \to F(u)$.

If $\Phi \in \Delta_2^\infty$ and $\Phi$ is coercive then $L^{\Phi^*}([0,T],\mathbb{R}^d) = \left[L^\Phi([0,T],\mathbb{R}^d)\right]^*$ is satisfied (see [Desch and Grimmer, 2001, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* $W^1L^\Phi$ by

$$W^1L^\Phi([0,T],\mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi([0,T],\mathbb{R}^d)\}.$$

$W^1L^\Phi([0,T],\mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{14}$$

And, we introduce the following subspaces of $W^1L^\Phi$

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{15}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

**Lemma 2.2.** *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (6). *Then*

1. *For every $s, t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s-t| G^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant G^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T G^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If $\Phi$ is coercive then the space $W^1L^\Phi([0,T],\mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0,T],\mathbb{R}^d)$.*

*Proof.* By the absolutely continuity of $u$, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned}\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right)\\ &\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}.\end{aligned}$$

By (6)????? we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant G^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha G^{-1}(1/\alpha)$ is an increasing ???? function with respect to $\alpha \in [0,\infty)$ we have

$$|u(t)-\overline{u}| \leqslant \|u'\|_{L^\Phi} T G^{-1}\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right)ds \leqslant \frac{1}{T}$$

Then

$$|\overline{u}| \leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}\\ &\leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}\\ &\leqslant G^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

We take a bounded sequence $u_n$ in $W^1L^\Phi([0,T],\mathbb{R}^d)$ and suppose that $u_n$ has not convergent subsequence. From (Morrey's inequality) and ??????? we infer that $u_n$ are equicontinuous. Furthermore (Sobolev's inequality) implies that $u_n$ is bounded in $C([0,T],\mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence $n_k$ and $u \in C([0,T],\mathbb{R}^d)$ with $u_{n_k} \to u$ in $C([0,T],\mathbb{R}^d)$.

□

# 3 Superposition operators in anisotropic Orlicz spaces

For $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ we denote by $\mathfrak{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0,T] \to \mathbb{R}^d$ by

$$\mathfrak{f}u(t) = f(t, u(t))$$

Referencias y alguna propiedad interesante medibles en medibles? [Krasnosel'skii et al., 2011, Krasnosel′skiĭ and Rutickiĭ, 1961]

**Theorem 3.1.** *Let $\Phi_1, \Phi_2, \dots, \Phi_n$ be $N$-functions. Assume that $M$ is another $N$-functions that satisfy the $\Delta_2$-condition. We write $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ with $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}^d$. Let $f(t, x_1, \dots, x_n, y_1, \dots, y_n)$ be a function Charathéodory? with $f : [0,T] \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathbb{R}^{d'}$.*

*Suppose that $a : (\mathbb{R}^d)^n \to [0, +\infty)$ is a bounded function on bounded sets and $b \in L^M([0,T])$, for a.e. $t \in [0,T]$ such that*

$$|f| \leqslant a(x)[b(t) + \sum_{i=1}^{n} M^{-1}(\Phi_i(|y_i|))], \tag{16}$$

*then*

$$\mathfrak{f} : \left( \prod_{i=1}^{n} L^\infty([0,T], \mathbb{R}^d) \right) \times \left( \prod_{i=1}^{n} \Pi(E^{\Phi_i}([0,T], \mathbb{R}^d), \lambda = 1) \right) \to L^M.$$

*Proof.* If $(u,v) \in \left( \prod_{i=1}^{n} L^\infty([0,T], \mathbb{R}^d) \right) \times \left( \prod_{i=1}^{n} \Pi(E_d^{\Phi_i}, \lambda = 1) \right)$. By [Krasnosel′skiĭ and Rutickiĭ, 1961, Thm. 17.6] (y otras cosas), we get

$$|\mathfrak{f}u(t)| = |f(t, u(t), v(t))| \leqslant M_a[b_j(t) + \sum_{i=1}^{n} M_j^{-1}(\Phi_i(|v_i(t)|))] \in L_1^{M_j}.$$

□

We define the space $X$ by $X = \{v = (v_1, v_2) : v_1 \in W^1 L_T^{\Phi_1}, v_2 \in W^1 L_T^{\Phi_2}\}$ and $X^* = \{v = (v_1, v_2) : v_1 \in (W^1 L_T^{\Phi_1})^*, v_2 \in (W^1 L_T^{\Phi_2})^*\}$ where $(W^1 L_T^{\Phi_i})^*$ stands for the conjugate space of $W^1 L_T^{\Phi_i}$ for $i = 1, 2$.

**Corollary 3.2.** *We will consider the Lagrange function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \to \mathcal{L}(t, x_1, x_2, y_1, y_2)$ which is measurable in $t$ for each $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in $(x_1, x_2, y_1, y_2)$ for almost every $t \in [0,T]$.*

*Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^d$ and let*

$$I(x) = \int_0^T \mathcal{L}(t, x, y)\, dt \tag{17}$$

*If there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2$, $b \in L_1^1([0,T])$, $j = 1, \dots, d'$ for a.e. $t \in [0,T]$ and every $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying the structure conditions*

*The nonlinear operator* $(x_1, x_2) \mapsto D_x\mathcal{L}(t, x_1, y_1, y_2)$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)\times\cdots\times\mathcal{E}_d^{\Phi_n}(\lambda)$ *with the strong topology into* $L^1([0,T])$ *with the strong topology on both sets.*

*The nonlinear operator* $(x_1, x_2) \mapsto D_y\mathcal{L}(t, x_1, y_1, y_2)$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)\times\cdots\times\mathcal{E}_d^{\Phi_n}(\lambda)$ *with the strong topology into* $X$ *with the weak* topology.*

*The function* $I$ *is Gâteaux differentiable on* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ *and its derivative* $I'$ *is demicontinuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ *into* $X^*$*. Moreover,* $I'$ *is given by the following expression*

$$\begin{aligned}\langle I'(x), w\rangle = \int_0^T [&(D_{x_1}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1(t)) + \\ &(D_{x_2}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2(t)) + \\ &(D_{y_1}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1'(t)) + \\ &(D_{y_2}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2'(t))]\,dt\end{aligned} \tag{18}$$

*If* $\Phi^* \in \Delta_2$ *then* $I'$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ *into* $X^*$ *when both spaces are equipped with the strong topology.*

We denote by $\mathfrak{A}(a, b, c, \lambda, f, \Phi)$ the set of all Lagrange functions satisfying (**??**), (**??**) and (**??**).

*Proof.* **OJO!!!! Es algo que teníamos del trabajo anterior!!! con algunas adaptaciones a 2 variables sin controlar y a lo bruto!!!!!**

Let $u \in \mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$.

*Step 1. The non linear operator* $(x_1, x_2) \mapsto (D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2), D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2))$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ *into* $L_d^1([0,T])\times L_d^1([0,T])$ *with the strong topology on both sets.*

If $u \in \mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$, from (**??**) and (**??**), we obtain Let $\{x_n = (x_{1n}, x_{2n})\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ and let $x = (x_1, x_2) \in \mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ such that $x_n \to x$ in $X$. From $x_{in} \to x_i$ in $L^{\Phi_i}$, there exists a subsequence $x_{in_k}$ such that $x_{in_k} \to x_i$ a.e.; and, as $x_{in} \to x_i \in \mathcal{E}_d^{\Phi}(\lambda)$, by Lemma **??**, there exist a subsequence of $x_{in_k}$ (again denoted $x_{in_k}$) and a function $h_i \in \Pi(E_1^{\Phi}, \lambda))$ such that $x_{in_k} \to u_i$ a.e. and $|x_{in_k}| \leqslant h_i$ a.e. Since $x_{in_k}$, $k = 1, 2, \ldots$, is a strong convergent sequence in $W^1L_d^{\Phi_i}$, it is a bounded sequence in $W^1L_d^{\Phi_i}$. According to Lemma 2.2 and Corollary **??**, there exist $M_i > 0$ such that $\|\boldsymbol{a}(x_{in_k})\|_{L^\infty} \leqslant M_i$, $k = 1, 2, \ldots$. From the previous facts and (**??**), we get

$$|D_{x_i}\mathcal{L}(\cdot, x_{1n_k}, x_{2n_k}, y_{1n_k}, y_{2n_k})| \leqslant M_i(b + \Phi_i(|h_i|)) \in L_1^1 \;\; i = 1, 2.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$D_{x_i}\mathcal{L}(t, x_{in_k}(t), y_{in_k}(t)) \to D_{x_i}\mathcal{L}(t, x_i(t), y_i(t)) \quad \text{for a.e. } t \in [0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

*Step 2. The non linear operator* $(x_1, x_2) \mapsto (D_{y_1}\mathcal{L}(t, x_1, y_1, D_{y_2}\mathcal{L}(t, x_2, y_2)$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ *with the strong topology into* $X$ *with the weak* topology.*

Note that (**??**), (**??**) and the imbeddings $W^1L^\Phi_d \hookrightarrow L^\infty_d$ and $L^{\Phi^*}_d \hookrightarrow \left[L^\Phi\right]^*$ imply that the second member of (18) defines an element in $\left[W^1L^\Phi_d\right]^*$.

Let $(x_{1n}, x_{2n}) \in \mathcal{E}^\Phi_d(\lambda)$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$ in the norm of $X$. We must prove that $D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}) \overset{w^*}{\rightharpoonup} D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2)$ para $i = 1, 2$. On the contrary, there exist $v = (v_1, v_2) \in L^{\Phi_1} \times L^{\Phi_2}$, $\epsilon > 0$ and a subsequence of $\{x_n\}$ (denoted $\{x_n\}$ for simplicity) such that

$$|\langle D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}, y_{1n}, y_{2n}),\ v\rangle - \langle D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2, v\rangle| \geqslant \epsilon. \tag{19}$$

We have $x_n \to x$ in $X$ and $y_n \to y$ in $X$. By Lemma **??**, there exist a subsequence $x_{n_k}$ and a function $h \in \Pi(E^{\Phi_1}_1, \lambda) \times \Pi(E^{\Phi_2}_1, \lambda)$ such that $x_{n_k} \to x$ a.e., $y_{n_k} \to y$ a.e. and $|y_{n_k}| \leqslant h$ a.e. As in the previous step, since $x_n$ is a convergent sequence, the Corollary **??** implies that $a(|y_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, with $x_{n_k}$ instead of $x$, inequality (**??**) becomes Consequently, as $v \in L^\Phi$ and employing Hölder's inequality, we obtain that

$$\sup_k |D_{\boldsymbol{y}}\mathcal{L}(\cdot, u_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot v| \in L^1_1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, u_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot \boldsymbol{v}\, dt \to \int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, u, \dot{\boldsymbol{u}}) \cdot \boldsymbol{v}\, dt \tag{20}$$

which contradicts the inequality (19). This completes the proof of step 2.

*Step 3.* We will prove (18). The proof follows similar lines as [Mawhin and Willem, 1989, Thm. 1.4]. For $u \in \mathcal{E}^\Phi_d(\lambda)$ and $\mathbf{0} \neq v \in W^1L^\Phi_d$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), \dot{\boldsymbol{u}}(t) + s\dot{\boldsymbol{v}}(t)).$$

From [Krasnosel′skiĭ and Rutickiĭ, 1961, Lemma 10.1] (or [Schappacher, 2005, Thm. 5.5] ) we obtain that if $|u| \leqslant |v|$ then $d(u, E^\Phi) \leqslant d(v, E^\Phi)$. Therefore, for $|s| \leqslant s_0 := \left(\lambda - d(\dot{\boldsymbol{u}}, E^\Phi)\right)/\|v\|_{W^1L^\Phi}$ we have

$$d\left(\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}, E^\Phi\right) \leqslant d\left(|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}|, E^\Phi_1\right) \leqslant d\left(|\dot{\boldsymbol{u}}|, E^\Phi_1\right) + s\|\dot{\boldsymbol{v}}\|_{L^\Phi} < \lambda.$$

Thus $\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}} \in \Pi(E^\Phi, \lambda)$ and $|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}| \in \Pi(E^\Phi_1, \lambda)$. These facts imply, in virtue of Theorem **??** item **??**, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$. And, using Corollary **??**, we also see that

$$\|a(|u + sv|)\|_{L^\infty} \leqslant A(\|u + sv\|_{W^1L^\Phi}) \leqslant A(\|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}) =: M$$

Now, applying Chain Rule, (**??**), (**??**) the monotonicity of $\varphi$ and $\Phi$, the fact that $v \in L^\infty_d$ and $\dot{\boldsymbol{v}} \in L^\Phi$ and Hölder's inequality, we get

$$\begin{aligned} |D_sH(s,t)| &= |D_{\boldsymbol{x}}\mathcal{L}(t, u + sv, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot v + D_{\boldsymbol{y}}\mathcal{L}(t, u + sv, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \dot{\boldsymbol{v}}| \\ &\leqslant M\left[\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|v|\right. \\ &\qquad \left. + \left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|\dot{\boldsymbol{v}}|\right] \in L^1_1. \end{aligned} \tag{21}$$

Consequently, $I$ has a directional derivative and

$$\langle I'(u),v\rangle = \frac{d}{ds}I(u+sv)\Big|_{s=0} = \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})\cdot v + D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})\cdot\dot{\boldsymbol{v}}\}\ dt.$$

Moreover, from (**??**), (**??**), Lemma 2.2 and the previous formula, we obtain

$$|\langle I'(u),v\rangle| \leqslant \|D_{\boldsymbol{x}}\mathcal{L}\|_{L^1}\|v\|_{L^\infty} + \|D_{\boldsymbol{y}}\mathcal{L}\|_{L^{\Phi^*}}\|\dot{\boldsymbol{v}}\|_{L^\Phi} \leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant $C$. This completes the proof of the Gâteaux differentiability of $I$.

*Step 4. The operator $I' : \mathcal{E}_d^\Phi(\lambda) \to \left[W^1L_d^\Phi\right]^*$ is demicontinuous.* This is a consequence of the continuity of the mappings $u \mapsto D_{\boldsymbol{x}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ and $u \mapsto D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$. Indeed, if $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$ with $u_n \to u$ in the norm of $W^1L_d^\Phi$ and $v \in W^1L_d^\Phi$, then

$$\begin{aligned}\langle I'(u_n),v\rangle &= \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t,u_n,\dot{\boldsymbol{u}}_n\right)\cdot v + D_{\boldsymbol{y}}\mathcal{L}\left(t,u_n,\dot{\boldsymbol{u}}_n\right)\cdot\dot{\boldsymbol{v}}\}\ dt\\ &\to \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t,u,\dot{\boldsymbol{u}}\right)\cdot v + D_{\boldsymbol{y}}\mathcal{L}\left(t,u,\dot{\boldsymbol{u}}\right)\cdot\dot{\boldsymbol{v}}\}\ dt\\ &= \langle I'(u),v\rangle.\end{aligned}$$

In order to prove item **??**, it is necessary to see that the maps $u \mapsto D_{\boldsymbol{x}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ and $u \mapsto D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ are norm continuous from $\mathcal{E}_d^\Phi(\lambda)$ into $L_d^1$ and $L_d^{\Phi^*}$ respectively. The continuity of the first map has already been proved in step 1. Let $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$. Therefore, there exist a subsequence $u_{n_k} \in \mathcal{E}_d^\Phi(\lambda)$ and a function $h \in \Pi(E_1^\Phi,\lambda)$ such that (**??**) holds true. And, as $\Phi^* \in \Delta_2$ then the right hand side of (**??**) belongs to $E_1^{\Phi^*}$. Now, invoking Lemma **??**, we prove that from any sequence $u_n$ which converges to $u$ in $W^1L_d^\Phi$ we can extract a subsequence such that $D_{\boldsymbol{y}}\mathcal{L}(t,u_{n_k},\dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of $I'$ follows from the continuity of $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ using the formula (18). □

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