Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}, \tag{1}$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary*

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equations. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) = \frac{|y|^p}{p} + F(t,x), \tag{2}$$

for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{4}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [Acinas et al., 2015] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{5}$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (1).

2 Preliminaries

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces) and other brief introduction to superposition operators between these spaces. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001] and [Płuciennik, 1987, Nguen Hong Thai, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$.

Following [Schappacher, 2005] we say tha Φ is *coercive* if

$$\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = +\infty.$$

We also say that a non decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Δ_2^∞*-condition*, denoted by $\eta \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\eta(2x) \leqslant K\eta(x), \tag{6}$$

for every $\|x\| \geqslant M$.

If Φ is a Young function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \to \mathbb{R}_+$by:

$$\Phi^*(y) = \sup_{x\in\mathbb{R}^d} x \cdot y - \Phi(x) \tag{7}$$

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{8}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{9}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Th. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Th. 5.2]). Another alternative characterization of E^Φ, which is particularly usefull for us, is that

$u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \ldots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \tag{10}$$

Like in [Krasnosel′skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{11}$$

for any positive r (see [Schappacher, 2005, Th. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

Definition 2.1. *Let $u_n, uL^\Phi([0,T], \mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in \|_{L^\infty}([0,T], \mathbb{R}^d)$, $n = 1, 2, \ldots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t)u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow \left[L^\Phi\right]^*$, where a function $v \in L^{\Phi^*}$ is associated to $F_v \in \left[L^\Phi\right]^*$ where

$$F_v(u) := \langle v, u\rangle = \int_0^T v \cdot u \, dt, \tag{12}$$

In [Desch and Grimmer, 2001, Theorem 2.9] it was characterized a subspace of $\left[L^\Phi\right]^*$ which is identified with L^{Φ^*}. Namely $L^{\Phi^*} = P^{\Phi^*}([0,T], \mathbb{R}^d)$ where $F \in P^{\Phi^*}([0,T], \mathbb{R}^d)$ if and only if $F \in \left[L^\Phi\right]^*$ and satisfying the *monotone convergence property*, which is if u_n converges monotonically to u then $F(u_n) \to F(u)$.

If $\Phi \in \Delta_2^\infty$ and Φ is coercive then $L^{\Phi^*}([0,T], \mathbb{R}^d) = \left[L^\Phi([0,T], \mathbb{R}^d)\right]^*$ is satisfied (see [Desch and Grimmer, 2001, Th. 2.9 , 2.10]).

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi([0,T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi([0,T], \mathbb{R}^d)\}.$$

$W^1L^\Phi([0,T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{13}$$

And, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E_T^\Phi &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{14}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anistropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.2. *Let $\Phi : \mathbb{R}^d \to [0,+\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Then*

1. *For every $s,t \in [0,T]$, $s \neq t$,*

$$\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) \leqslant \frac{1}{|s-t|} \qquad \text{(Morrey's inequality)}$$

$$\left\|\Phi\left(\frac{u}{2\max\{1,T\}\|u\|_{W^1L^\Phi}}\right)\right\|_{L^\infty} \leqslant \frac{1}{T} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ and*

$$\left\|\Phi\left(\frac{\widetilde{u}}{\|u\|_{L^\Phi}T}\right)\right\|_{L^\infty} \leqslant \frac{1}{T} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *The space $W^1L^\Phi([0,T],\mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0,T],\mathbb{R}^d)$.*

Proof. By the absolutely continuity of u and Jensen's inequality we have

$$\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) \leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right) \leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|},$$

which proves 1.

Morrey's inequality implies Sobolev-Wirtinger's inequality by the following argument. Taking account that $\alpha\Phi(x/\alpha)$ is a non increasing function with respect to $\alpha \in [0,\infty)$ for every $x \in \mathbb{R}^d$ we have

$$\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}T}\right) \leqslant \frac{1}{T}.$$

Dividing by T this inequality, integrating respect to s and using Jensen inequality again

$$\Phi\left(\int_0^T \frac{u(t)-u(s)}{\|u'\|_{L^\Phi}T^2}ds\right) \leqslant \frac{1}{T}.$$

which implies

$$\Phi\left(\frac{u(t)-\overline{u}}{\|u'\|_{L^\Phi}T}\right) \leqslant \frac{1}{T}.$$

i.e. Sobolev-Wirtinger inequality.

For the Sobolev inequality we note that (we write $T^* = \max\{1, T\}$)

$$\begin{aligned}\Phi\left(\frac{u}{2T^*\|u\|_{W^1L^\Phi}}\right) &\leqslant \frac{1}{2}\Phi\left(\frac{\tilde{u}}{T^*\|u\|_{W^1L^\Phi}}\right)+\frac{1}{2}\Phi\left(\frac{\overline{u}}{T^*\|u\|_{W^1L^\Phi}}\right)\\ &\leqslant \frac{1}{2}\Phi\left(\frac{\tilde{u}}{T^*\|u'\|_{L^\Phi}}\right)+\frac{1}{2}\Phi\left(\frac{\overline{u}}{T^*\|u\|_{L^\Phi}}\right)\\ &=: I_1 + I_2\end{aligned}$$

Using Sobolev-Wirtinger's inequality, the inequality $T^* \geqslant T$ and that Φ is increasing function we get

$$2I_1 \leqslant \Phi\left(\frac{\tilde{u}}{T\|u'\|_{L^\Phi}}\right) \leqslant \frac{1}{T}$$

Using Jensen inequality and that $T^* \geqslant 1$ we have

$$2I_2 = \Phi\left(\frac{1}{T}\int_0^T \frac{u(s)}{T^*\|u\|_{L^\Phi}} ds\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then $I_1 + I_2 \leqslant 1/T$, which completes the proof of Sobolev's inequality.

Next we prove part 3 of the lemma. First we prove that there exist a non decreasing function $F : (0, +\infty) \to (0, +\infty)$ sucht that $\|u\|_{L^\infty} \leqslant F(\|\Phi(u)\|_{L^\infty})$. In fact, since $\Phi(x) \to +\infty$ when $\|x\| \to +\infty$, for every $K > 0$ there exist $G(K) > 0$ such that $|x| \geqslant G(K)$ then $\Phi(x) \geqslant K$. Suppose that, for certain u, $\|u\|_{L^\infty} > G(\|\Phi(u)\|_{L^\infty})$. Then there exists a set $A \subset [0, T]$ with positive measure such $|u(t)| > G(\|\Phi(u)\|_{L^\infty})$, when $t \in A$. Then $\Phi(u(t)) > \|\Phi(u)\|_{L^\infty}$, for $t \in A$, which is a contradiction. Now we take $F(K) := \sup\{G(s) | 0 < s \leqslant k\}$.

We take a bounded sequence u_n in $W^1L^\Phi([0, T], \mathbb{R}^d)$ and suppose that u_n has not convergent subsequence.

□

3 Superposition operators in anisotropic Orlicz spaces

Vamos escribiendo lo que queremos...(de acuerdo a mis apuntes y sin ver las hojitas de la semana pasada)

For $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ we denote by $\mathfrak{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0, T] \to \mathbb{R}^d$ by

$$\mathfrak{f}u(t) = f(t, u(t))$$

Referencias y alguna propiedad interesante medibles en medibles? [Krasnosel'skii et al., 2011, Krasnosel′skiĭ and Rutickiĭ, 1961]

Theorem 3.1. *Let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be N-functions. Assume that M is another N-functions that satisfy the Δ_2-condition. We write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$*

with $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}^d$. Let $f(t, x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a function Chatratheodory? with $f : [0,T] \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathbb{R}^{d'}$.

Suppose that $a : (\mathbb{R}^d)^n \to [0, +\infty)$ is a bounded function on bounded sets and $b \in L^M([0,T])$, for a.e. $t \in [0,T]$ such that

$$|f| \leqslant a(x)[b(t) + \sum_{i=1}^{n} M^{-1}(\Phi_i(|y_i|))], \tag{15}$$

then

$$\mathfrak{f} : \left(\prod_{i=1}^{n} L^\infty([0,T], \mathbb{R}^d)\right) \times \left(\prod_{i=1}^{n} \Pi(E^{\Phi_i}([0,T], \mathbb{R}^d), \lambda = 1)\right) \to L^M.$$

Proof. If $(u,v) \in \left(\prod_{i=1}^{n} L^\infty([0,T], \mathbb{R}^d)\right) \times \left(\prod_{i=1}^{n} \Pi(E_d^{\Phi_i}, \lambda = 1)\right)$. By [Krasnosel′skiĭ and Rutickiĭ, 1961, Thm. 17.6] (y otras cosas), we get

$$|\mathfrak{f}u(t)| = |f(t, u(t), v(t))| \leqslant M_a[b_j(t) + \sum_{i=1}^{n} M_j^{-1}(\Phi_i(|v_i(t)|))] \in L_1^{M_j}.$$

□

We define the space X by $X = \{v = (v_1, v_2) : v_1 \in W^1 L_T^{\Phi_1}, v_2 \in W^1 L_T^{\Phi_2}\}$ and $X^* = \{v = (v_1, v_2) : v_1 \in (W^1 L_T^{\Phi_1})^*, v_2 \in (W^1 L_T^{\Phi_2})^*\}$ where $(W^1 L_T^{\Phi_i})^*$ stands for the conjugate space of $W^1 L_T^{\Phi_i}$ for $i = 1, 2$.

Corollary 3.2. *We will consider the Lagrange function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \to \mathcal{L}(t, x_1, x_2, y_1, y_2)$ which is measurable in t for each $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x_1, x_2, y_1, y_2) for almost every $t \in [0,T]$.*

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^d$ and let

$$I(x) = \int_0^T \mathcal{L}(t, x, y)\, dt \tag{16}$$

If there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2$, $b \in L_1^1([0,T])$, $j = 1, \ldots, d'$ for a.e. $t \in [0,T]$ and every $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying the structure conditions

The nonlinear operator $(x_1, x_2) \mapsto D_x \mathcal{L}(t, x_1, y_1, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \cdots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into $L^1([0,T])$ with the strong topology on both sets.

The nonlinear operator $(x_1, x_2) \mapsto D_y \mathcal{L}(t, x_1, y_1, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \cdots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into X with the weak* topology.

The function I is Gâteaux differentiable on $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^*. Moreover, I' is given by the following expression

$$\begin{aligned} \langle I'(x), w \rangle = \int_0^T [&(D_{x_1} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1(t)) + \\ &(D_{x_2} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2(t)) + \\ &(D_{y_1} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1'(t)) + \\ &(D_{y_2} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2'(t))]\, dt \end{aligned} \tag{17}$$

If $\Phi^* \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^* when both spaces are equipped with the strong topology.

We denote by $\mathfrak{A}(a, b, c, \lambda, f, \Phi)$ the set of all Lagrange functions satisfying (**??**), (**??**) and (**??**).

Proof. **OJO!!!! Es algo que teníamos del trabajo anterior!!! con algunas adaptaciones a 2 variables sin controlar y a lo bruto!!!!!**

Let $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$.

Step 1. The non linear operator $(x_1, x_2) \mapsto (D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2), D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2))$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ into $L^1_d([0,T])\times L^1_d([0,T])$ with the strong topology on both sets.

If $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$, from (**??**) and (**??**), we obtain Let $\{x_n = (x_{1n}, x_{2n})\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ and let $x = (x_1, x_2) \in \mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ such that $x_n \to x$ in X. From $x_{in} \to x_i$ in L^{Φ_i}, there exists a subsequence x_{in_k} such that $x_{in_k} \to x_i$ a.e.; and, as $x_{in} \to x_i \in \mathcal{E}_d^{\Phi}(\lambda)$, by Lemma **??**, there exist a subsequence of x_{in_k} (again denoted x_{in_k}) and a function $h_i \in \Pi(E_1^{\Phi}, \lambda))$ such that $x_{in_k} \to u_i$ a.e. and $|x_{in_k}| \leqslant h_i$ a.e. Since x_{in_k}, $k = 1, 2, \dots$, is a strong convergent sequence in $W^1L_d^{\Phi_i}$, it is a bounded sequence in $W^1L_d^{\Phi_i}$. According to Lemma 2.2 and Corollary **??**, there exist $M_i > 0$ such that $\|\boldsymbol{a}(x_{in_k})\|_{L^\infty} \leqslant M_i$, $k = 1, 2, \dots$. From the previous facts and (**??**), we get

$$|D_{x_i}\mathcal{L}(\cdot, x_{1n_k}, x_{2n_k}, y_{1n_k}, y_{2n_k})| \leqslant M_i(b + \Phi_i(|h_i|)) \in L^1_1 \quad i = 1, 2.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$D_{x_i}\mathcal{L}(t, x_{in_k}(t), y_{in_k}(t)) \to D_{x_i}\mathcal{L}(t, x_i(t), y_i(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $(x_1, x_2) \mapsto (D_{y_1}\mathcal{L}(t, x_1, y_1, D_{y_2}\mathcal{L}(t, x_2, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ with the strong topology into X with the weak* topology.

Note that (**??**), (**??**) and the imbeddings $W^1L_d^{\Phi} \hookrightarrow L_d^\infty$ and $L_d^{\Phi^*} \hookrightarrow [L^\Phi]^*$ imply that the second member of (17) defines an element in $[W^1L_d^\Phi]^*$.

Let $(x_{1n}, x_{2n}) \in \mathcal{E}_d^{\Phi}(\lambda)$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$ in the norm of X. We must prove that $D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}) \overset{w^*}{\rightharpoonup} D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2)$ para $i = 1, 2$. On the contrary, there exist $v = (v_1, v_2) \in L^{\Phi_1} \times L^{\Phi_2}$, $\epsilon > 0$ and a subsequence of $\{x_n\}$ (denoted $\{x_n\}$ for simplicity) such that

$$|\langle D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}, y_{1n}, y_{2n}),\ v\rangle - \langle D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2, v\rangle| \geqslant \epsilon. \tag{18}$$

We have $x_n \to x$ in X and $y_n \to y$ in X. By Lemma **??**, there exist a subsequence x_{n_k} and a function $h \in \Pi(E_1^{\Phi_1}, \lambda) \times \Pi(E_1^{\Phi_2}, \lambda)$ such that $x_{n_k} \to x$ a.e., $y_{n_k} \to y$ a.e. and $|y_{n_k}| \leqslant h$ a.e. As in the previous step, since x_n is a convergent sequence, the Corollary **??** implies that $a(|y_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, with x_{n_k} instead of x, inequality (**??**) becomes Consequently, as $v \in L^\Phi$ and employing Hölder's inequality, we obtain that

$$\sup_k |D_{\boldsymbol{y}}\mathcal{L}(\cdot, u_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot v| \in L^1_1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,u_{n_k},\dot{\boldsymbol{u}}_{n_k})\cdot\boldsymbol{v}\,dt \to \int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})\cdot\boldsymbol{v}\,dt \tag{19}$$

which contradicts the inequality (18). This completes the proof of step 2.

Step 3. We will prove (17). The proof follows similar lines as [Mawhin and Willem, 1989, Thm. 1.4]. For $u\in\mathcal{E}^\Phi_d(\lambda)$ and $\mathbf{0}\neq v\in W^1L^\Phi_d$, we define the function

$$H(s,t):=\mathcal{L}(t,u(t)+sv(t),\dot{\boldsymbol{u}}(t)+s\dot{\boldsymbol{v}}(t)).$$

From [Krasnosel′skiĭ and Rutickiĭ, 1961, Lemma 10.1] (or [Schappacher, 2005, Thm. 5.5]) we obtain that if $|u|\leqslant|v|$ then $d(u,E^\Phi)\leqslant d(v,E^\Phi)$. Therefore, for $|s|\leqslant s_0:=\left(\lambda-d(\dot{\boldsymbol{u}},E^\Phi)\right)/\|v\|_{W^1L^\Phi}$ we have

$$d\left(\dot{\boldsymbol{u}}+s\dot{\boldsymbol{v}},E^\Phi\right)\leqslant d\left(|\dot{\boldsymbol{u}}|+s|\dot{\boldsymbol{v}}|,E^\Phi_1\right)\leqslant d\left(|\dot{\boldsymbol{u}}|,E^\Phi_1\right)+s\|\dot{\boldsymbol{v}}\|_{L^\Phi}<\lambda.$$

Thus $\dot{\boldsymbol{u}}+s\dot{\boldsymbol{v}}\in\Pi(E^\Phi,\lambda)$ and $|\dot{\boldsymbol{u}}|+s|\dot{\boldsymbol{v}}|\in\Pi(E^\Phi_1,\lambda)$. These facts imply, in virtue of Theorem **??** item **??**, that $I(u+sv)$ is well defined and finite for $|s|\leqslant s_0$. And, using Corollary **??**, we also see that

$$\|a(|u+sv|)\|_{L^\infty}\leqslant A(\|u+sv\|_{W^1L^\Phi})\leqslant A(\|u\|_{W^1L^\Phi}+s_0\|v\|_{W^1L^\Phi})=:M$$

Now, applying Chain Rule, (**??**), (**??**) the monotonicity of φ and Φ, the fact that $v\in L^\infty_d$ and $\dot{\boldsymbol{v}}\in L^\Phi$ and Hölder's inequality, we get

$$\begin{aligned}|D_sH(s,t)|&=|D_{\boldsymbol{x}}\mathcal{L}(t,u+sv,\dot{\boldsymbol{u}}+s\dot{\boldsymbol{v}})\cdot v+D_{\boldsymbol{y}}\mathcal{L}(t,u+sv,\dot{\boldsymbol{u}}+s\dot{\boldsymbol{v}})\cdot\dot{\boldsymbol{v}}|\\&\leqslant M\left[\left(b(t)+\Phi\left(\frac{|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|}{\lambda}+f(t)\right)\right)|v|\right.\\&\qquad\left.+\left(c(t)+\varphi\left(\frac{|\dot{\boldsymbol{u}}|+s_0|\dot{\boldsymbol{v}}|}{\lambda}+f(t)\right)\right)|\dot{\boldsymbol{v}}|\right]\in L^1_1.\end{aligned}\tag{20}$$

Consequently, I has a directional derivative and

$$\langle I'(u),v\rangle=\frac{d}{ds}I(u+sv)\Big|_{s=0}=\int_0^T\left\{D_{\boldsymbol{x}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})\cdot v+D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})\cdot\dot{\boldsymbol{v}}\right\}\,dt.$$

Moreover, from (**??**), (**??**), Lemma 2.2 and the previous formula, we obtain

$$|\langle I'(u),v\rangle|\leqslant\|D_{\boldsymbol{x}}\mathcal{L}\|_{L^1}\|v\|_{L^\infty}+\|D_{\boldsymbol{y}}\mathcal{L}\|_{L^{\Phi^*}}\|\dot{\boldsymbol{v}}\|_{L^\Phi}\leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant C. This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I':\mathcal{E}^\Phi_d(\lambda)\to\left[W^1L^\Phi_d\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u\mapsto D_{\boldsymbol{x}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ and $u\mapsto D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$. Indeed, if $u_n,u\in\mathcal{E}^\Phi_d(\lambda)$ with $u_n\to u$ in the norm of $W^1L^\Phi_d$ and $v\in W^1L^\Phi_d$, then

$$\begin{aligned}\langle I'(u_n),v\rangle&=\int_0^T\left\{D_{\boldsymbol{x}}\mathcal{L}\left(t,u_n,\dot{\boldsymbol{u}}_n\right)\cdot v+D_{\boldsymbol{y}}\mathcal{L}\left(t,u_n,\dot{\boldsymbol{u}}_n\right)\cdot\dot{\boldsymbol{v}}\right\}\,dt\\&\to\int_0^T\left\{D_{\boldsymbol{x}}\mathcal{L}\left(t,u,\dot{\boldsymbol{u}}\right)\cdot v+D_{\boldsymbol{y}}\mathcal{L}\left(t,u,\dot{\boldsymbol{u}}\right)\cdot\dot{\boldsymbol{v}}\right\}\,dt\\&=\langle I'(u),v\rangle\,.\end{aligned}$$

In order to prove item **??**, it is necessary to see that the maps $u \mapsto D_{\boldsymbol{x}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ and $u \mapsto D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ are norm continuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into L_d^1 and $L_d^{\Phi^*}$ respectively. The continuity of the first map has already been proved in step 1. Let $u_n, u \in \mathcal{E}_d^{\Phi}(\lambda)$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$. Therefore, there exist a subsequence $u_{n_k} \in \mathcal{E}_d^{\Phi}(\lambda)$ and a function $h \in \Pi(E_1^{\Phi}, \lambda)$ such that (**??**) holds true. And, as $\Phi^* \in \Delta_2$ then the right hand side of (**??**) belongs to $E_1^{\Phi^*}$. Now, invoking Lemma **??**, we prove that from any sequence u_n which converges to u in $W^1L_d^{\Phi}$ we can extract a subsequence such that $D_{\boldsymbol{y}}\mathcal{L}(t,u_{n_k},\dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t,u,\dot{\boldsymbol{u}})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of I' follows from the continuity of $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ using the formula (17). □

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