

# Periodic solutions for a generalized Sitnikov problem with primaries in rigid motion

Gastón Beltritti \*

Dpto. de Matemática, Facultad de Ciencias Exactas Físico-Químicas y Naturales  
Universidad Nacional de Río Cuarto (5800) Río Cuarto, Córdoba, Argentina,  
`gbeltritti@exa.unrc.edu.ar`

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales  
Universidad Nacional de Río Cuarto  
(5800) Río Cuarto, Córdoba, Argentina,  
`fmazzone@exa.unrc.edu.ar`

Martina G. Oviedo ‡

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales  
Universidad Nacional de Río Cuarto  
(5800) Río Cuarto, Córdoba, Argentina,  
`martinagoviedo@gmail.com`

## Abstract

## 1 Introduction

In this paper we study the following restricted Newtonian  $n + 1$ -body problem  $P$  (see figure 2):

- $P_1$  We have  $n$  primary bodies of masses  $m_1, \dots, m_n$  and an additional massless particle.
- $P_2$  The primary bodies are in a homographic motion (see [13, Section 2.9]). This motion is carried out in a plane  $\Pi$ .
- $P_3$  The massless particle is moving on the perpendicular line to  $\Pi$  passing through the center of masses.

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\*SECyT-UNRC and CONICET

†SECyT-UNRC, FCEyN-UNLPam

‡SECyT-UNRC, CIN

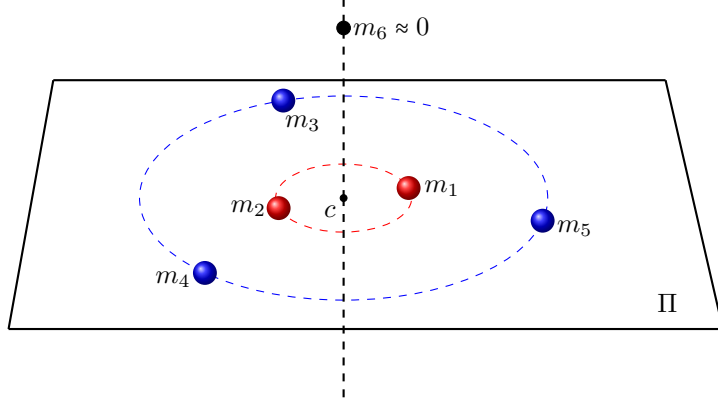


Figure 1: Six-body problem

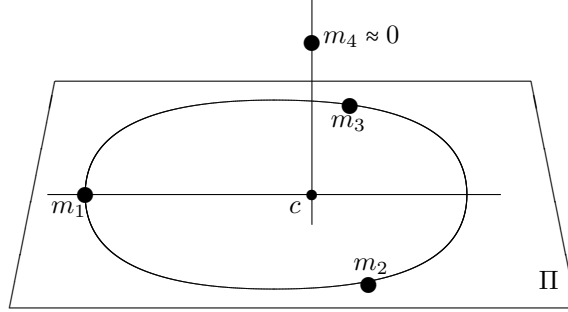


Figure 2: Four-body problem with three primaries

Problems like the one presented have been extensively discussed in the literature. In [27] K. Sitnikov considered the problem of two body in a Keplerian elliptic motion, for small enough eccentricity  $e > 0$ , and a massless particle moving in the perpendicular line to the orbital plane passing through the center of masses. Sitnikov obtained deep results about existence of solutions with a chaotic behavior (see [19, III(5)]). Periodic solutions for a Sitnikov configuration were considered in [25, 5, 6, 14].

Generalized circular Sitnikov problems ( $n \geq 3$ ) were addressed more recently. In [28] Soulis, Papadakis and Bountis studied existence, linear stability and bifurcations for a problem similar to  $P$ . They considered a Lagrangian equilateral triangle configuration for the primaries bodies, which were supposed to have the same mass  $m_1 = m_2 = m_3$ . In [4] Papadakis and Bountis extend of results of [28] to  $n$  primaries ( $n \geq 3$ ) in a polygonal equal mass configuration. Later in [23], Pandey and Ahmad extend the analysis started in [28] to the case when the primaries are oblate (not mass points). In [12] Li, Zhang and Zhao studied

a special type of restricted circular  $n + 1$ -body problem with equal masses for the primaries in a regular polygon configuration. In the previous papers the primary bodies are in the vertices of a regular polygon. In [16] Marchesin and Vidal studied the problem  $P$  for a rigid motion of primaries in a rhomboidal configuration. Periodic solutions for generalized Sitnikov problems with primaries performing no rigid motions were studied in [26, 25]. In [3] Bakker and Simmons studied scape regions for the massless particle in a problem similar to  $P$  for a certain type of periodic orbits for primaries.

In the present paper, after introducing preliminaries facts in Section 2, we obtain in Section 3 necessary and sufficient conditions on configuration of primaries bodies in order that the  $z$ -axis is invariant for the flow associated to the motion equations of the massless particle. For this type of configurations, that we call *admissible*, the Sitnikov problem has sense. In Section 4 we will find all admissible configurations for  $n \leq 4$  primaries. The Section 5 is devoted to describe all possible motion of the massless particle when the primaries are in a relative equilibrium (or rigid) motion. In this direction we observe that only are possible scape (both parabolic and hiperbolic) and periodic motions. Moreover, we will give a formula expressing the period of solutions by means of integrals. We prove in Corollary 5.4 that the complete  $n + 1$ -body system has infinite quantity of periodic solutios. In Section 6 we discuss the situation when the entire system has a solution with the same period that the rigid motion of primaries. Surprisingly this fact is related to the existence of certain pyramidal central configurations (see [7, 8, 21] for the definition of this concept). Finally, in the last section, we study certain non admissible configurations which allows some particular solutions of problem  $P$ .

## 2 Preliminaries

We start considering  $n$  mass points,  $n > 2$ , of masses  $m_1, \dots, m_n$  moving in a Euclidean 3-dimensional space according to Newton's laws of motion. We assume that  $x_1(t), \dots, x_n(t)$  are the coordinates (column vectors) of the bodies in some inertial Cartesian coordinate system. We denote by  $r_{ij} = |x_i - x_j|$  the Euclidean distance between  $x_i$  and  $x_j$ . We can suppose, without any loss of generality, that the center of mass  $c := \sum_j m_j x_j / M$  ( $M := \sum_j m_j$ ) is fixed at the origin ( $c = 0$ ).

Initially we assume that the bodies are in a *planar homographic motion* on the plane  $\Pi$  (see [13]). That means, assuming that  $\Pi$  is the plane determined by the first two coordinates axes, that

$$x_j(t) = r(t)Q(\theta(t))q_j, \quad (1)$$

where

$$Q(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and  $q_j \in \Pi$ ,  $j = 1, \dots, n$  are vectors in a planar *central configuration* (CC) in  $\mathbb{R}^3$ , i.e. there exists  $\lambda \in \mathbb{R}$  such that

$$\nabla_j U(q_1, \dots, q_n) + \lambda m_j q_j = 0, \quad j = 1, \dots, n. \quad (2)$$

where the *potential function*  $U$  is defined by:

$$U(x) = \sum_{i < j} \frac{m_i m_j}{r_{ij}}, \quad (3)$$

and  $\nabla_j$  denotes the 3-dimensional partial gradient with respect to  $x_j$ . For simplicity we denote  $q := (q_1, \dots, q_n)$ .

According to [13, Eq. (2.16)] the functions  $r(t)$  and  $\theta(t)$  solves the two-dimensional Kepler problems in polar coordinates, i.e.

$$\begin{aligned} \ddot{r}(t) - r(t)\dot{\theta}(t)^2 &= -\frac{\lambda}{r(t)^2} \\ \frac{d}{dt} [r(t)^2 \dot{\theta}(t)] &= 0. \end{aligned} \quad (4)$$

In the particular case of *rigid motion*, we have  $r(t) \equiv 1$  and  $\theta(t) = \sqrt{\lambda}t + \theta(0)$ . Therefore the primaries bodies perform a periodic motion with period  $T := 2\pi/\nu$ , where  $\nu^2 = \lambda$ .

Let  $x_0(t)$  denotes the position of the massless particle. According to the Newtonian equations of motion  $q$  satisfies

$$\ddot{x}_0 = \sum_{i=1}^n \frac{m_i(x_i - x_0)}{|x_i - x_0|^3} =: f(t, x_0). \quad (5)$$

### 3 Admissible configurations

Henceforth we denote by  $L$  the coordinate  $z$  axis. It is well know that a necessary and sufficient condition for that  $L$  be invariant under the flow associated to the non autonomous system (5), is that  $f(t, L) \subset L$  for all  $t$ , i.e.  $L$  is *f-invariant* for every  $t$ .

**Definition 3.1.** *We say that a central configuration  $q$  is admissible if and only if satisfies that for any  $r > 0$ , such that the set*

$$F_r := \{i : |q_i| = r\}$$

*is non empty, then*

$$\sum_{i \in F_r} m_i q_i = 0. \quad (6)$$

*i.e. every maximal set of bodies which are equidistant from origin has center of mass equal to 0.*

**Theorem 3.2.**  *$L$  is f-invariant for every  $t$  if and only  $q$  is admissible.*

For the proof of the previous theorem we need the following result

**Lemma 3.3.** For  $c > 0$  we define the function  $y_c(t) := (c + t)^{-3/2}$ . If  $0 < t_1 < t_2 < \dots < t_k$  then the functions  $y_j(t) := y_{t_j}(t)$  are linearly independent on each open interval  $\mathcal{I} \subset \mathbb{R}^+$ .

*Proof.* It is sufficient to prove that Wronskian

$$W := W(y_1, \dots, y_k)(t) = \det \begin{pmatrix} y_1 & \dots & y_k \\ \frac{dy_1}{dt} & \dots & \frac{dy_k}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1}{dt^{k-1}} & \dots & \frac{d^{k-1}y_k}{dt^{k-1}} \end{pmatrix}$$

is not null on  $\mathcal{I}$ .

Using induction is easy to show that

$$\frac{d^i y_c}{dt^i} = \beta_i y_c^{\frac{2i+3}{3}}, \quad \text{for some } \beta_i \neq 0, \text{ and for all } i = 1, \dots \quad (7)$$

Fix any  $t \in I$ . Then, according to (7) and writing  $\lambda_j := (t + t_j)^{-1}$ , we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \lambda_1^{3/2} & \lambda_2^{3/2} & \dots & \lambda_k^{3/2} \\ \beta_1 \lambda_1^{5/2} & \beta_1 \lambda_2^{5/2} & \dots & \beta_1 \lambda_k^{5/2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1} \lambda_1^{k+1/2} & \beta_{k-1} \lambda_2^{k+1/2} & \dots & \beta_{k-1} \lambda_k^{k+1/2} \end{pmatrix} \\ &= \beta_1 \beta_2 \dots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \dots \lambda_k^{3/2} \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \end{pmatrix} \\ &= \beta_1 \beta_2 \dots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \dots \lambda_k^{3/2} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i), \end{aligned}$$

where the last equality follows of the well known Vandermonde determinant identity. Therefore  $W \neq 0$  if and only if  $\lambda_i \neq \lambda_j$ ,  $i \neq j$ , which in turn is equivalent to  $t_i \neq t_j$ ,  $i \neq j$ .  $\square$

*Proof of Theorem 3.2.* The condition  $f(t, L) \subset L$  for all  $t$  is equivalent to

$$\sum_{i=1}^n \frac{m_i r(t) Q(\theta(t)) q_i}{(r(t)^2 |q_i|^2 + z^2)^{3/2}} = 0, \quad (8)$$

for every  $t, z \in \mathbb{R}$ .

Let  $D = \{|q_i| : i = 1, \dots, n\}$ . Suppose that  $D = \{s_1, \dots, s_k\}$ , with  $s_i \neq s_j$  for  $i \neq j$ , and  $\{1, \dots, n\} = F_1 \cup \dots \cup F_k$ , where if  $i \in F_j$  then  $|q_i| = s_j$ . Then, multiplying equation (8) by  $r(t)^2 Q^{-1}(\theta(t))$  and writing  $\zeta = (z/r(t))^2$  we have that (8) is equivalent to

$$\sum_{j=1}^k \left\{ \frac{1}{(s_j^2 + \zeta)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0.$$

According to Lemma 3.3, the last equation is equivalent to (6).  $\square$

## 4 Admissible collisionless configurations for $n \leq 4$

In this section we find all admissible collisionless configuration with  $n \leq 4$ . We observe that the collisionless condition for a configuration  $q_1, \dots, q_n$  implies that  $q_i \neq 0$  for  $i = 1, \dots, n$ . From the fact that the center of masses is an excluded position we obtain

$$F_r \neq \emptyset \Rightarrow \#F_r \geq 2. \quad (9)$$

Trivially, a configuration of two point masses  $m_1$  and  $m_2$  is admissible if and only if  $m_1 = m_2$ .

If  $n = 3$ , from (9), the configuration consists of three equidistant bodies from the origin. Therefore, it must to be the Lagrangian equilateral triangle configuration. Now, by equation (6) and an elementary geometrical reasoning we have that  $m_1 = m_2 = m_3$ .

The case  $n = 4$  is more interesting. We will use the following result which can be seen in [18].

“Let  $q$  be a planar configuration. For each pair,  $i, j$ , the line containing  $q_i$  and  $q_j$  together with its perpendicular bisector form axes which divide the plane into four quadrants. The union of the first and third quadrants is an hourglass shaped region which will be called a ‘cone’; similarly, the second and fourth quadrants together form another cone. The phrase ‘open cone’ refers to a cone minus the axes.”

**Theorem 4.1** (Perpendicular Bisector Theorem). *Let  $q$  be a planar central configuration and let  $q_i$  and  $q_j$  be any two of its points. Then if one of the two open cones determined by the line through  $q_i$  and  $q_j$  and its perpendicular bisector contains points of the configuration, so does the other one.*

Next we characterize all the 4-body admissible collisionless configurations.

**Theorem 4.2.** *Let  $q$  be a 4-body central configuration. Then  $q$  is admissible and collisionless if and only if, for a suitable enumeration of bodies,  $q_1 = -q_3$ ,  $q_2 = -q_4$ ,  $m_1 = m_3$ ,  $m_2 = m_4$ , and  $q$  is of some of the following mutually exclusive types:*

**CCcl.** *collinear,*

**CCr.** *a rhombus with  $r_{13} < r_{24}$  and  $m_1 > m_2$ ,*

**CCs.** *a square with four equal masses.*

*Proof.* Since (9) we have two cases.

*Case 1.*  $m_1 \geq m_2$ ,  $|q_1| \neq |q_2|$ ,  $|q_1| = |q_3|$  and  $|q_2| = |q_4|$ . Now (6) implies that  $m_1 = m_3$ ,  $m_2 = m_4$ ,  $q_1 = -q_3$  and  $q_2 = -q_4$ . We divide the plane in two cones  $C_i$ ,  $i = 1, 2$ , by means of the line  $P$  joining  $q_1$  and  $q_3$  together with its perpendicular bisector  $M$ . From Theorem 4.1, if  $q_2$  is in  $C_1$ , then  $q_4$  is in  $C_2$ , and vice versa. This is a contradiction with the fact that  $q_2 = -q_4$ . Then  $q_2, q_4 \in P$

or  $q_2, q_4 \in M$ , i.e.  $q$  is collinear or is a rhombus with equal masses in opposite vertices. In the first case,  $q$  is of CCcl type. In the second case, if  $m_1 > m_2$ , was proved in [15, Eqs. (3.44) and (3.45)] that  $r_{13} < r_{24}$ , hence  $q$  is of CCr type. From [24, Corollary 2] if  $m_1 = m_2$  then the configuration is a square witch is a contradiction with the fact that  $|q_1| \neq |q_2|$ .

*Case 2.*  $|q_1| = |q_2| = |q_3| = |q_4|$ . In this situation, was proved in [9] that the configuration is the equal mass square.  $\square$

## 5 Massless particle motion

In this section we analyze all possible motions for the massless particle  $x_0$ . In particular we will see that all motion is periodic or is a scape trayectory. We will find that there exists  $T_0$ -periodic solutions for all  $T_0$  in an interval  $(\sigma(q), +\infty)$ . This fact implies that there exists an infinity quantity of periodic solutions for the entire  $n + 1$ -body system.

In this and next sections we suppose that the primaries bodies are in a  $T$ -periodic rigid motion asociate to an admisible collisionless  $q$ , i.e  $r(t) \equiv 1$  and from the remark follows (4),  $\theta(t) = \sqrt{\lambda}t$ . Without loss of generality, we have assumed that  $\theta(0) = 0$ . For the particle, we assume that it is moving on  $L$ . Therefore, we assume that  $x_0(t) = (0, 0, z(t))$ . According to Theorem 3.2,  $x_0$  is solution of (5), if and only if  $z(t)$  is solution of the autonomous equation

$$\ddot{z} = - \sum_{i=1}^n \frac{m_i z}{(s_i^2 + z^2)^{3/2}}, \quad (10)$$

where  $s_i = |q_i|$ .

The second order equation (10) is consevative, therefore solutions conserve the energy

$$E(z, v) := \frac{|v|^2}{2} - \sum_{i=1}^n \frac{m_i}{(s_i^2 + z^2)^{\frac{1}{2}}}, \quad (11)$$

i.e.  $E(z(t), \dot{z}(t))$  is constant.

Following [16] we introduce the next concepts.

**Definition 5.1.** A solution  $z(t)$  of (10) such that  $\lim_{t \rightarrow \infty} z(t) = \infty$  is called *hyperbolic* for  $t \rightarrow \infty$  when  $\lim_{t \rightarrow \infty} \dot{z}(t) = z_\infty \neq 0$  and is called *parabolic* if  $\lim_{t \rightarrow \infty} \dot{z}(t) = 0$ .

The following theorem characterize all the possible motions for the massless particle.

**Theorem 5.2.** We assume that  $q$  is a admisible collisionless configuration and the primaries are in a rigid motion. Every solution of (10) if of the some following types:

1. Hyperbolic, when  $E > 0$ ,
2. Parabolic, when  $E = 0$ ,

3. *Periodic, when  $E_{min} := -\sum_{i=1}^n \frac{m_i}{s_i} < E < 0$ .*

4. *Equilibrium solution when  $E = E_{min}$ .*

*Proof.* We follow a standard argument for hamiltonian systems (see [?]).

We plot the level sets of  $S(E) = \{(z, v) : E(z, v) = E\}$ , in the phase space  $(z, v)$ , for different values of  $E$ . An elementary analysis shows that

- If  $E \geq 0$  then  $S(E)$  is the union of two bounded graphs. They are symmetric with respect to  $z$ -axis, each of which is contained in some semiplane  $v > 0$  or  $v < 0$ . The  $v$ -positive branch is the graph of a function  $v(E, z)$ , which is decreasing with respect to  $|z|$ . Moreover,  $\lim_{|z| \rightarrow \infty} v(E, z) = \sqrt{2E}$ .
- For every  $E \geq E_{min}$ , the energy curve cut the  $v$ -axis at  $\pm(2E + 2\sum_{i=1}^n m_i s_i^{-1})^{\frac{1}{2}}$ .
- If  $E_{min} < E < 0$  then  $S(E)$  is a simple closed curve symmetric with respect to  $z$  and  $v$  axes.
- An energy curve cut the  $z$ -axis, only in the case that  $E < 0$ , at  $\pm z_E$ , where  $z_E$  is the only positive solution of  $-\sum_{i=1}^n m_i (s_i^2 + z_E^2)^{-\frac{1}{2}} = E$ .

In the figure 2 we show the phase portrait for two equal masses primaries.

The function  $\varphi(t) = (z(t), \dot{z}(t))$  solves the system  $\dot{\varphi}(t) = F(\varphi(t))$ , where  $F(z, v) = (v, -\sum_{i=1}^n m_i z (s_i^2 + z^2)^{-3/2})$ . The only stationary point of  $F$  is  $(z, v) = (0, 0)$ . Therefore, the level surfaces  $S(E)$ , with  $E \neq E_{min}$ , not contains stationary points. A well know argument implies that the trayectories  $t \mapsto (z(t), \dot{z}(t))$  fill completely the energy curves.

We note that if  $E \geq 0$  and  $v(E, 0) > 0$  ( $v(E, 0) < 0$ ) then  $z(t)$  is increasing (decreasing) with respect to  $t$ . For all the above if  $E \geq 0$  then  $|z(t)| \rightarrow \infty$  when  $t \rightarrow \infty$ . Moreover  $\lim_{t \rightarrow \infty} \dot{z}(t) = \pm\sqrt{2E}$ . From this we conclude that the trayectories are hyperbolic when  $E > 0$  and it is parabolic in the case  $E = 0$ .

In the case that  $E_{min} < E < 0$  we have that the trayectories are contained in a closed curve, therefore it is periodic orbit.

Finally if  $E = E_{min}$  clearly we have that  $z(t) \equiv 0$ .  $\square$

**Theorem 5.3.** *We denote by  $T_0(E)$  the minimal period for a solution of (10) with  $E_{min} < E < 0$ . Then*

1. *for  $z_E$  the only positive solution of  $-\sum_{i=1}^n m_i (s_i^2 + z_E^2)^{-\frac{1}{2}} = E$*

$$T_0(E) = 2^{3/2} \int_0^{z_E} \left( E + \sum_{i=1}^n m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz, \quad (12)$$

2.  *$T_0(E)$  is an increasing function.*

3.  *$T_0((E_{min}, 0)) = (T_{min}, +\infty)$ , where  $T_{min} = 2\pi \left( \sum_{i=1}^n \frac{m_i}{s_i^3} \right)^{-1/2}$ .*



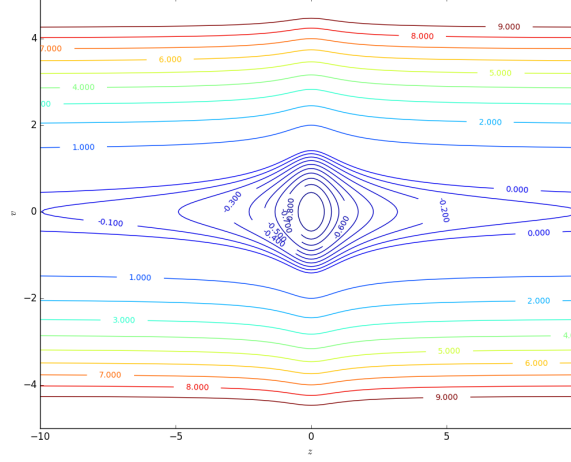


Figure 3: Energy level for two equal mass primaries

*Proof.* Let  $E_{min} < E < 0$  and let  $z(t)$  be a solution with  $z(0) = 0$  and  $\dot{z}(0) = \sqrt{2(E - E_{min})}$ . Then  $E(z(0), \dot{z}(0)) = E$ . Therefore  $z(t)$  is  $T_0(E)$ -periodic. As a consequence of the symmetries of the equation we have that  $z(T_0(E)/4) = z_E$ . Then, taking account (11) we have that

$$\begin{aligned} \frac{T_0}{4} &= \int_0^{T_0/4} \left( E + \sum_{i=1}^n m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} z'(t) dt \\ &= \frac{1}{\sqrt{2}} \int_0^{z_E} \left( E + \sum_{i=1}^n m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz, \end{aligned}$$

and we have proved item 1. In order to prove item 2 we note that

$$\begin{aligned} 2^{-3/2} T_0(E) &= \int_0^{z_E} \left( \sum_{i=1}^n m_i \left( (s_i^2 + z^2)^{-\frac{1}{2}} - (s_i^2 + z_E^2)^{-\frac{1}{2}} \right) \right)^{-\frac{1}{2}} dz \\ &= \int_0^{z_E} (z_E^2 - z^2)^{-\frac{1}{2}} f(z, z_E) dz \\ &= \int_0^1 (1 - u^2)^{-\frac{1}{2}} f(z_E u, z_E) du, \end{aligned}$$

where

$$f(z, z_E) = \left( \sum_{i=1}^n m_i \left\{ (s_i^2 + z^2)(s_i^2 + z_E^2) \right\}^{-\frac{1}{2}} \left\{ (s_i^2 + z^2)^{\frac{1}{2}} + (s_i^2 + z_E^2)^{\frac{1}{2}} \right\}^{-1} \right)^{-\frac{1}{2}}.$$

We note that  $f(z_E u, z_E)$  is an increasing function with respect to  $z_E$  for  $u \in [0, 1]$  fix. This implies item 2.

On the other hand

$$\lim_{z_E \rightarrow 0} f(z_E u, z_E) = \left( \sum_{i=1}^n \frac{m_i}{2s_i^3} \right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{z_E \rightarrow \infty} f(z_E u, z_E) = \infty.$$

Therefore, from the dominated convergence theorem and monotone convergence theorem we have that

$$\lim_{z_E \rightarrow 0} T_0 = 2\pi \left( \sum_{i=1}^n \frac{m_i}{s_i^3} \right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{z_E \rightarrow \infty} T_0 = \infty.$$

Finally, since  $T_0 = T_0(z_E)$  is continuous and increasing respect to  $z_E$  we conclude the affirmation in the item 3.  $\square$

*Remark 1.* It is possible to use the classical theory of hamiltonian systems (see [2]) to derive the formula (12) (see [1] for a related problem).

*Remark 2.* Let us to show a second proof of item 3 of Theorem 5.3.

The inequality  $T_0 > T_{min}$  is consequence of comparison Sturm's theorem applied to equations  $z'' + q_1(z)z = 0$ , where  $q_1(z) = \sum_{i=1}^n m_i (s_i^2 + z^2)^{-3/2}$ , and  $z'' + (\sum_{i=1}^n m_i s_i^{-3})z = 0$ . This prove that  $T_0((E_{min}, 0)) \subset (T_{min}, +\infty)$ .

For the reverse inclusion we follow arguments of [30] and [12], based on variational principles.

Let  $T_0 > T_{min}$ . We consider the action integral

$$\mathcal{I}(z) = \int_0^{T_0} \frac{1}{2} |z'|^2 + \sum_{i=1}^n \frac{m_i}{\sqrt{s_i^2 + z^2}} dt,$$

Then  $T_0$ -periodic solutions of (10) are critical points of  $\mathcal{I}$  in the space  $H^1(\mathbb{T}, \mathbb{R})$ , where  $\mathbb{T} = \mathbb{R}/T_0\mathbb{Z}$ , of the functions absolutely continuous,  $T_0$ -periodic with  $z' \in L^2(\mathbb{T}, \mathbb{R})$  (see [17, Cor. 1.1]). We prove the existence of critical points by means of the direct method of calculus of variations, i.e. we will prove that  $\mathcal{I}$  has minimum. The functional  $\mathcal{I}$  is not coercive in  $H^1(\mathbb{T}, \mathbb{R})$ , this deficiency is drawn with symmetry techniques (see [29]). The group  $\mathbb{Z}_2$  acts on  $H^1(\mathbb{T}, \mathbb{R})$  according to the following assignments  $(\bar{0} \cdot z)(t) = z(t)$  and  $(\bar{1} \cdot z)(t) = -z(t + \frac{T_0}{2})$ . The functional  $\mathcal{I}$  is  $\mathbb{Z}_2$ -invariant, i.e.  $\mathcal{I}(g \cdot z) = \mathcal{I}(z)$ . We define the space of all  $\mathbb{Z}_2$ -symmetric (this symmetry is called the italian symmetry) functions

$$\Lambda(\mathbb{T}, \mathbb{R}) := \{z \in H^1(\mathbb{T}, \mathbb{R}) | \forall g \in \mathbb{Z}_2 : z = g \cdot z\}.$$

The functional  $\mathcal{I}$  restricted to  $\Lambda$  is coercive. This fact follows from an obvious adaptation of proposition 4.1 of [29]. We note that  $F(z) := \sum_{i=1}^n m_i (s_i^2 + z^2)^{-\frac{1}{2}}$  satisfies the condition (A) in [17, p. 12], then  $\mathcal{I}$  is continuously differentiable and weakly lower semicontinuous on  $H^1(\mathbb{T}, \mathbb{R})$  (see [17, p. 13]). Therefore  $\mathcal{I}$  has a minimum  $z_0$  in  $\Lambda(\mathbb{T}, \mathbb{R})$ . Then by the Palais' principle symmetric criticality,  $z_0$  is a critical point of  $\mathcal{I}$  in  $H^1(\mathbb{T}, \mathbb{R})$  (see [29] and [22]).

We use the second variation  $\delta^2\mathcal{I}$  in order to show that  $z_0 \neq 0$ . It is well known (see [10, Th. 1.3.1]) that if  $z_0$  is a minimum of  $\mathcal{I}$  on  $H^1(\mathbb{T}, \mathbb{R})$  then  $\delta^2\mathcal{I}(z_0, \varphi) \geq 0$  for all  $\varphi \in H^1(\mathbb{T}, \mathbb{R})$ . In our case

$$\delta^2\mathcal{I}(0, \varphi) = \int_0^{T_0} |\varphi'|^2 - \sum_{i=1}^n \frac{m_i}{s_i^3} \varphi^2 dt,$$

(see [10, Eq. 1.3.6]). In particular for  $\varphi(t) = \sin(2\pi t/T_0)$  it follows from  $T_0 > T_{min}$  that

$$\delta^2\mathcal{I}(0, \varphi) = \left( \frac{4\pi^2}{T_0^2} - \sum_{i=1}^n \frac{m_i}{s_i^3} \right) \frac{T_0}{2} < 0. \quad (13)$$

It is sufficient to guarantee that  $z_0 \equiv 0$  is not a minimum.

This second proof, unlike the first one, does not guarantee that  $T_0$  is the minimum period for  $z_0$ . It could happen that  $z_0$  had period  $T_0/m$ , with natural  $m \in \mathbb{N}$ . Because of Italian symmetry this  $m$  should be odd.

**Corollary 5.4.** *The complete  $n+1$ -body has a infinity quantity of periodic solutions.*

*Proof.* Let  $l/m$  be a rational number with  $l/m > T_{min}$ . Then, there exists a solution of the entire system with period  $lT$ .  $\square$

## 6 Synchronous solutions and pyramidal CC

If the equation (10) has a  $T$ -periodic solution, we say that the solution is *synchronous*. In [12] was studied the existence of synchronous solutions for  $n$  equal mass primary bodies in a regular polygon configuration.

In this section we establish a relation between the existence of synchronous solutions and the concept of pyramidal central configuration (see [7, 8, 21]).

**Definition 6.1.** *A central configuration of  $n+1$  mass point  $q_1, \dots, q_{n+1}$  in  $\mathbb{R}^3$  is called a pyramidal central configuration (PCC) if and only if  $n$  points, we say  $q_1, \dots, q_n$ , are in some plane  $\Pi$  and  $q_{n+1}$  is not.*

The following lemma was proved in [21] (see also [8])

**Lemma 6.2** ([21], Lemma 2.1). *Let  $q_0, \dots, q_n$  be a PCC such that  $m_0$  is off the plane containing  $m_1, \dots, m_n$ . If  $m_0 > 0$  then  $m_0$  is equidistant from  $m_1, \dots, m_n$ .*

We remark that the condition  $m_0 > 0$  is important in the previous Lemma. We will show below examples of two PCC with  $m_0 = 0$  which do not satisfy the conclusion of Lemma 6.2.

**Proposition 6.3.** *We assume that  $q = q_1, \dots, q_n$  is an admissible collisionless configuration and that the primaries are in a rigid motion. Then, there is a synchronous solution if and only if there exists  $c \in \mathbb{R}$  such that the points  $q_0, \dots, q_n$  associated to the masses  $m_0, \dots, m_n$ , with  $q_0 = (0, 0, c)$  and  $m_0 = 0$ , form a PCC.*

*Proof.* We start assuming that there exist a synchronous solution. As a consequence of the Theorem 5.3(3) and the fact that  $T^2 = 4\pi^2/\lambda$  we have that

$$\lambda < \sum_{i=1}^n \frac{m_i}{s_i^3}. \quad (14)$$

Since  $\sum_{i=1}^n m_i (s_i^2 + c^2)^{-3/2} \rightarrow 0$ , when  $c \rightarrow +\infty$ , there exists  $c \in \mathbb{R}$  such that  $\sum_{i=1}^n m_i (s_i^2 + c^2)^{-3/2} = \lambda$ . Therefore

$$-\sum_{i=1}^n \frac{m_i c}{(s_i^2 + c^2)^{3/2}} = -\lambda c. \quad (15)$$

As  $q_1, \dots, q_n$  is an admissible configuration then

$$\sum_{i=1}^n \frac{m_i q_i}{(s_i^2 + c^2)^{3/2}} = (0, 0). \quad (16)$$

We define  $q_{n+1} := (0, 0, c)$  and  $m_{n+1} = 0$ . The equations (15), (16) and the facts that  $q_1, \dots, q_n$  is a CC, with constant  $\lambda$ , implies that  $q_1, \dots, q_{n+1}$  and  $m_1, \dots, m_{n+1}$  form a PCC.

The proof of the reciprocal statement follows in a direct way.  $\square$

**Corollary 6.4.** *We assume that  $q = q_1, \dots, q_n$  is an admissible collisionless configuration and that the primaries are in a rigid motion. Then, there is a synchronous solution if and only if*

$$\sum_{i < j} \frac{m_i m_j}{r_{ij}} < \left( \sum_{i=1}^n \frac{m_i}{s_i^3} \right) \left( \sum_{i=1}^n m_i s_i^2 \right). \quad (17)$$

*Proof.* The result is consequence of (14) and the fact that  $T^2 = 4\pi^2 \sum_{i=1}^n m_i |q_i|^2 / U$  (see [13, p. 109]).  $\square$

The sufficiency of the condition  $n \leq 472$  in the following corollary was proved in [12].

**Corollary 6.5.** *We suppose that  $q$  is the equal masses regular polygon configuration (this is an admissible CC). Then there exists a synchronous solution if and only if  $2 \leq n \leq 472$ .*

*Proof.* In this case  $s_1 = s_2 = \dots = s_n =: r$  and  $m_1 = m_2 = \dots = m_n =: m$ . Then, from the law of cosines we obtain

$$\sum_{i < j} \frac{m_i m_j}{r_{ij}} = \frac{nm^2}{4r} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)}.$$

Therefore the condition (17) is equivalent to

$$\frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)} < 4. \quad (18)$$

This inequality was also derived by Li, J. et al. in [12], where the authors proved (performing computer calculations) that inequality (18) holds true for  $2 \leq n \leq 472$ . Let us prove that any other  $n$  does not satisfies (18).

Using that  $1/\sin(x)$  is a convex function on  $[0, \pi]$  and the composite trapezoid rule (see [11]) we have that

$$\begin{aligned} \int_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} \frac{1}{\sin(x)} dx &\leq \frac{\pi}{2n} \left\{ \frac{1}{\sin(\frac{\pi}{n})} + \frac{1}{\sin(\frac{n-1}{n}\pi)} + 2 \sum_{j=2}^{n-2} \frac{1}{\sin(j\frac{\pi}{n})} \right\} \\ &= \frac{\pi}{n} \sum_{j=1}^{n-2} \frac{1}{\sin(j\frac{\pi}{n})}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)} &\geq \frac{1}{\pi} \int_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} \frac{1}{\sin(x)} dx + \frac{1}{n \sin\left(\frac{n-1}{n}\pi\right)} \\ &= \frac{1}{2\pi} \log\left(\frac{1 - \cos(x)}{1 + \cos(x)}\right) \Big|_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} + \frac{1}{n \sin\left(\frac{\pi}{n}\right)} \\ &= \frac{1}{\pi} \left\{ \log\left(\frac{1 + \cos(\frac{\pi}{n})}{1 - \cos(\frac{\pi}{n})}\right) + \frac{\pi/n}{\sin\left(\frac{\pi}{n}\right)} \right\} \\ &=: f\left(\frac{\pi}{n}\right). \end{aligned}$$

It is easy to see that  $f(x)$  is a decreasing function on  $(0, \pi/2)$ . Moreover  $f(\pi/842) \approx 4.0006 > 4$ . Therefore, if  $n \geq 842$  then  $n$  does not satisfies inequality (18). The validity of the inequality (18), for  $n \leq 841$ , is easily checked using computer. This gives the result that the inequality holds for all  $n \leq 472$ .  $\square$

Our next objective is to verify that condition (17) is satisfied for all admissible CC of 3-body or 4-body. Since we have already proved, in Corollary 6.5, that (17) holds for a equilateral triangle and square configurations of equal masses bodies, it only rest to prove, in virtue of Theorem 4.2, the following result.

**Theorem 6.6.** *The central configurations CCcl and CCr satisfy condition (17).*

*Proof.* Let's start by analyzing the central configuration CCr. We can suppose without loss of generality that  $q_1 = -q_3 = (0, y)$  for  $y > 0$ ,  $q_2 = -q_4 = (1, 0)$ . Recall that  $y < 1$ . The condition (17) becomes

$$\frac{m_1^2}{2y} + \frac{4m_1m_2}{\sqrt{1+y^2}} + \frac{m_2^2}{2} < \left(\frac{2m_1}{y^3} + 2m_2\right)(2m_1y^2 + 2m_2).$$

As  $m_1^2/(2y) < 4m_1^2/y$ ,  $m_2^2/2 < 4m_2^2$  and  $4m_1m_2/\sqrt{1+y^2} < 4m_1m_2/y^3$  (since  $y < 1$ ), we have that the inequality holds.

Now we consider the central configuration CCl. We can suppose without loss of generality that  $q_1 = -q_3 = 1$ ,  $q_2 = -q_4 = x$  with  $0 < x < 1$ , and  $m_1 = m_3 = \mu$ ,  $m_2 = m_4 = 1 - \mu$ , with  $0 < \mu < 1$ . Then the inequality (17) becomes

$$\frac{2\mu(1-\mu)}{1-x} + \frac{2\mu(1-\mu)}{1+x} + \frac{\mu^2}{2} + \frac{(1-\mu)^2}{2x} < 4\mu^2 + 4\mu(1-\mu)x^2 + \frac{4\mu(1-\mu)}{x^3} + \frac{4(1-\mu)^2}{x}.$$

As  $\mu^2/2 < 4\mu^2$  and  $(1-\mu)^2/(2x) < 4(1-\mu)^2/x$  (without taking into account the term  $4\mu(1-\mu)x^2$ ) we just have to show that

$$\frac{2\mu(1-\mu)}{1-x} + \frac{2\mu(1-\mu)}{1+x} < \frac{4\mu(1-\mu)}{x^3},$$

and this is equivalent to see that

$$\frac{x^3}{1-x^2} < 1. \quad (19)$$

The values of  $x$  involved in the above inequality are such that the configuration for the vector mass  $(\mu, 1-\mu, 1-\mu, \mu)$  is central, by Moulton [20], fixed a mass  $\mu$  there is only one value  $x$  satisfying this condition. So, we can define  $x(\mu)$  as such value of  $x$ . We note that  $x^3/(1-x^2)$  is an increasing function with respect to  $x \in (0, 1)$  and it is less than 1 for  $x \in (0, 3/4)$ . Hence, if we could prove that  $x(\mu)$  is a decreasing function and

$$\lim_{\mu \rightarrow 0} x(\mu) < 3/4 \quad (20)$$

we would have justified (19).

Let's first prove that  $x(\mu)$  is a decreasing function. Replacing  $q_j^0$ ,  $m_j$  and eliminating  $\lambda$  from the equations (2) we get

$$\frac{\mu}{4} - \frac{\mu}{x(x+1)^2} + \frac{\mu}{x(-x+1)^2} + \frac{-\mu+1}{(x+1)^2} + \frac{-\mu+1}{(-x+1)^2} - \frac{1}{x^3} \left( -\frac{\mu}{4} + \frac{1}{4} \right) = 0$$

which is equivalent to

$$\mu = -\frac{8x^5 - x^4 + 8x^3 + 2x^2 - 1}{(x-1)(x+1)(x^5 - 9x^3 + x^2 - 1)}.$$

Therefore

$$\frac{d\mu}{dx} = \frac{x^2(16x^9 - 3x^8 + 32x^7 + 12x^6 - 304x^5 - 2x^4 + 44x^2 - 51)}{(x-1)^2(x+1)^2(x^5 - 9x^3 + x^2 - 1)^2}.$$

Clearly  $d\mu/dx < 0$  for  $x \in (0, 1)$ . Which, in turn, implies that  $x$  is decreasing respect to  $\mu$ .

Let's see now that (20) holds. When  $\mu$  goes to 0,  $x(\mu)$  converges to the only solution,  $x(0)$ , in the interval  $(0, 1)$  of the equation  $8x(0)^5 - x(0)^4 + 8x(0)^3 + 2x(0)^2 - 1 = 0$ . Then  $8x(0)^3 - 1 < 0$  which implies that  $x(0) < 3/4$  as we wanted to prove.  $\square$

*Remark 3.* As consequence of previous results there exist five-body  $PCC's$  with  $m_1, \dots, m_4$  in a CCcl or CCr configuration and  $m_0 = 0$  is in the line perpendicular to plane containing  $m_1, \dots, m_4$  passing by the mass center. These are examples of  $PCC's$  which does not verify the conclusion of Lemma 6.2.

The configurations CCcl and CCr are bases of a PCC, with a vertex out of the plane, which do not verify the conclusion of Lemma 6.2.

**Corollary 6.7.** *For all admissible CC of 3-body or 4-body problem  $P$  has a synchronous solution.*

## 7 Non admissible central configurations

The following result shows a necessary condition for that a non-admissible CC allows a solution of the problem  $P$ .

**Theorem 7.1.** *We suppose that  $q$  is a non admissible CC and that the primaries are in a homographic motion, i.e. equation (1) is satisfied. Assume that the massless particle is moving on the  $z$ -axis with position vector  $x_0(t) = (0, 0, z(t))$ . Then, some of the following statement is satisfies:*

1. *The massless particle is in a stationary motion and*

$$\sum_{i=1}^n \frac{m_i q_i}{s_i^3} = 0, \quad (21)$$

*i.e. the positions  $0, q_1, \dots, q_n$  and the masses  $0, m_1, \dots, m_n$  are in a CC.*

2. *The  $n + 1$ -body system is in a homothetic motion. i.e.  $Q(\theta(t))$  in the equation (1) is the identity matrix and  $z(t) = cr(t)$ , for some constant  $c$ . Moreover, the configuration  $q_0, \dots, q_n$  is a PCC, where  $q_0 = (0, 0, c)$  and  $m_0 = 0$ .*

*Proof.* We recall the definition of the function  $f$  and line  $L$  from the Section 3.

The fact of the massless particle is moving on  $L$ , is equivalent to the condition  $f(t, x_0(t)) \in L$  for all  $t$ , which, instead, is equivalent to the equality

$$\sum_{i=1}^n \frac{m_i r(t) Q(\theta(t)) q_i}{(r(t)^2 |q_i|^2 + z(t)^2)^{3/2}} = 0, \quad (22)$$

for every  $t \in \mathbb{R}$ .

With the same notation and following similar reasoning that in the proof of Theorem 3.2, we prove that

$$\sum_{j=1}^k \left\{ \frac{1}{(s_j^2 + (z(t)/r(t))^2)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0. \quad (23)$$

If  $z(t)/r(t)$  would be a non constant function then previous equation and Lemma 3.3 would imply that  $q$  is admissible, which is a contradiction. Hence there exist  $c \in \mathbb{R}$  such that  $z(t) = cr(t)$ . Now, we have two cases.

*Case 1:*  $c = 0$ . Then  $z \equiv 0$  and (21) follows from (22).

*Case 2:*  $c \neq 0$ . From the motion equation for the massless particle and the Kepler equations (4), and the fact that  $z(t) = cr(t)$  we have that

$$-\frac{1}{r(t)^2} \sum_{i=1}^n \frac{m_i}{(s_i^2 + c^2)^{3/2}} = -\frac{\lambda}{r(t)^2} + r(t)\dot{\theta}(t)^2. \quad (24)$$

The second equality in (4) implies the Kepler's second law, i.e. there exists  $d \in \mathbb{R}$  such that  $r^2\dot{\theta} \equiv d$ . Replacing  $\dot{\theta}$  in equation (24) and multiplying by  $r(t)^3$  we obtain

$$-r(t) \left( \sum_{i=1}^n \frac{m_i}{(s_i^2 + c^2)^{3/2}} + \lambda \right) = d^2. \quad (25)$$

Therefore, if  $d \neq 0$  then  $\dot{r}(t) \equiv 0$ , and this implies  $\dot{z}(t) \equiv 0$ . As  $z(t)$  is a constant function and it solves equation (10), then  $z(t) \equiv 0$ . Hence we are in the case 1 again. Consequently we suppose  $d = 0$ . Therefore  $\theta(t) \equiv cte$  and the motion is homothetic. From (23) and (25) we deduce that in this new situation equation (15) and (16) hold. Therefore as in the proof of Proposition 6.3.  $\square$

*Example 1.* We present an example of a  $n+1$ -body system satisfying the situation described in the item 1 of the Theorem 7.1, i.e.  $q$  is a non admissible CC and  $z(t) \equiv 0$ .

Let's consider a particular Euler's linear central configuration formed by three collinear bodies of mass  $m_1 = 4 - \mu$ ,  $m_2 = 2 + \mu$ ,  $m_3 = 1$ , where  $0 < \mu < 1$ , and  $q_1 = 0$ ,  $q_2 = 1$  and  $q_3 = 1 + r$  their respective positions, with  $r$  such that

$$p(r, \mu) = 0,$$

where  $p(r, \mu) = 6r^5 + (-\mu + 16)r^4 + (-2\mu + 14)r^3 + (-\mu - 5)r^2 + (-2\mu - 7)r - \mu - 3$ . Note that as  $p(0, \mu) = -\mu - 3$  and  $P(1, \mu) = -7\mu + 21$  then, for all  $0 < \mu < 1$ ,  $r \in (0, 1)$ . In this case the center of mass  $C$  is equal to  $\frac{\mu}{7} + \frac{r}{7} + \frac{3}{7}$ , so  $C \in (0, 1)$ .

We denote with  $x$  the point between 0 and 1 where the sum of the forces that the primary bodies make on a massless particle located in that position is equal to zero. For this value of  $x$  we have to  $f(x) = 0$ , where  $f(x) = -\frac{4-\mu}{x^2} + \frac{\mu+2}{(-x+1)^2} + \frac{1}{(r-x+1)^2}$ . Note that the left side of the previous equation is an increasing function that tends to  $-\infty$  when  $x$  goes to 0, and tends to  $+\infty$  when  $x$  goes to 1, so there is a unique point  $x \in (0, 1)$ , such that the equality holds.

Then, if we want to have a trivial solution of the problem (5) then necessarily  $C$  has to be equal to  $x$ . Let's see that there exists  $\mu \in (0, 1)$  such that  $C = x$ , i.e.  $f(C) = 0$ . For this purpose, since  $C$  is a continuous function with respect to  $\mu$ , we need to see that there exists a value  $\mu_1 \in (0, 1)$  such that  $f(C) < 0$  and  $\mu_2 \in (0, 1)$  such that  $f(C) > 0$ . The function  $f(x)$  can be factorized as

$$f(x) = \frac{Nf(x)}{Df(x)},$$

where  $Nf(x) = 2\mu r^2 x^2 - 2\mu r^2 x + \mu r^2 - 4\mu r x^3 + 8\mu r x^2 - 6\mu r x + 2\mu r + 2\mu x^4 - 6\mu x^3 + 7\mu x^2 - 4\mu x + \mu - 2r^2 x^2 + 8r^2 x - 4r^2 + 4r x^3 - 20r x^2 + 24r x - 8r - x^4 + 10x^3 - 21x^2 + 16x - 4$



and  $Df(x) = x^2(x-1)^2(r-x+1)^2$ . Note that  $Df(x) > 0$  for all  $x \in (0, 1)$ . If we consider  $\mu = 0$  and compute  $Nf(C)$  we have that

$$Nf(C) = \frac{r^4}{2401} + \frac{1514r^3}{2401} + \frac{2245r^2}{2401} + \frac{1110r}{2401} + \frac{333}{2401} > 0,$$

on the other, hand if  $\mu = 1$  then

$$Nf(C) = -\frac{71r^4}{2401} + \frac{1486r^3}{2401} + \frac{401r^2}{2401} - \frac{1480r}{2401} - \frac{592}{2401} < 0,$$

because  $0 < r < 1$ .

*Remark 4.* The following question is posed. Is there some non admissible central configuration  $q$  such that the  $n + 1$ -body system perform the motion described in Theorem 7.1(2)?

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