

Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

Sonia Acinas *

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_1)$$

where $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d \geq 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u : [0, T] \rightarrow \mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t, x, y) := \frac{|y|^p}{p} + F(t, x), \quad (1)$$

*SECyT-UNRC and FCEyN-UNLPam

†SECyT-UNRC, FCEyN-UNLPam and CONICET

2010 AMS Subject Classification. Primary: . Secondary: .

Keywords and phrases. .

for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (P_1) , for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (1) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (P_1) is reduced to the p -laplacian system

$$\begin{cases} \frac{d}{dt}(u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P_2)$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t, x)$ is differentiable with respect to x for a.e. $t \in [0, T]$ and the following conditions are verified:

(C) F and its gradient ∇F , with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

(A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (2)$$

In this inequality we assume that the function $a : [0, +\infty) \rightarrow [0, +\infty)$ is continuous and non decreasing and $0 \leq b \in L^1([0, T], \mathbb{R})$.

In [Acinas et al., 2015] it was treated the case of a lagrangian \mathcal{L} which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi, F}(t, x, y) = \Phi(|y|) + F(t, x), \quad (3)$$

where Φ is an N -function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (P_1) .

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$, i.e. functions such that $\Phi(x)$ depends on the direction of x , unlike the radial case where $\Phi(x) = \Phi(|x|)$. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of p -laplacian equations as the following example shows.

Example 1. Let $1 < p_1, p_2 < \infty$. We define $\Phi_{p_1, p_2} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|}{p_1} + \frac{|y_2|}{p_2}.$$

Suppose the following Lagrange function

$$\mathcal{L}(t, x, y) = \Phi_{p_1, p_2}(y) + F(t, x).$$

Then the equations (P_1) becomes

$$\begin{cases} \frac{d}{dt}(|u'_1|^{p_1-2}u'_1) = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt}(|u'_2|^{p_2-2}u'_2) = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_3)$$

These equations are known in the literature as (p_1, p_2) -Laplacian system, see [Yang and Chen, 2013, Pasca and Wang, 2016, Yang and Chen, 2012, Pasca, 2010, Paşca and Tang, 2010, Pasca and Tang, 2011].

On the other hand, anisotropic Orlicz-Sobolev spaces allow to simplify the writing, and they provide the natural frame of statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question was what motivated us to use these spaces.

Hereafter we denote by \mathbb{R}^+ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \rightarrow +\infty$, when $|x| \rightarrow +\infty$. Additionally, we assume that the Young's functions which we deal with, satisfy that $\Phi(x) > 0$ when $x \neq 0$. Following [Schappacher, 2005] we say that Φ is *coercive* if

$$\lim_{|x| \rightarrow \infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function Φ , we define function $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$A_\Phi(s) = \min \{ \Phi(x) \mid \|x\| = s \}, \quad (4)$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.1. *The function A_Φ has the following properties:*

1. A_Φ is continuous,
2. $A_\Phi(s)/s$ is increasing,
3. A_Φ is the greatest radial minorant of $\Phi(x)$,
4. Φ is coercive if and only if A_Φ is.

Proof. It is well known that finite and convex functions defined in finite dimensional vectorial spaces are locally Lipschitz functions (see [Clarke, 2013]). This fact imply item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ ,

$$\frac{A_\Phi(r)}{r} \leq \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leq \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily. □

We also say that $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ satisfies the Δ_2^∞ -condition, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geq 0$ such that

$$\Phi(2x) \leq KH(x), \quad (5)$$

for every $|x| \geq M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \rightarrow \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \quad (6)$$

We denote by $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$, with $d \geq 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on \mathbb{R}^d and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions \mathcal{M} , as for other similar sets, we will omit the reference to codomain \mathbb{R}^d when $d = 1$.

Given an N -function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Here $|\cdot|$ is the euclidean norm of \mathbb{R}^d . Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (7)$$

The *Orlicz space* $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$ is the linear hull of C^Φ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (8)$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left(\frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in \mathbb{R}^d between u and v .

The subspace $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0, T], \mathbb{R}^d)$ of all \mathbb{R}^d -valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of E^Φ , which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0, T]$, $n = 1, 2, \dots$ then $\|\chi_{E_n} u\| \rightarrow 0$ when $|E_n| \rightarrow 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \quad (9)$$

Like in [Krasnosel'skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (10)$$

for any positive r (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ , E^Φ , $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

Definition 2.2. Let $u_n, u \in L^\Phi([0, T], \mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^\infty([0, T], \mathbb{R})$, $n = 1, 2, \dots$, such that $0 \leq \alpha_n(t) \leq \alpha_{n+1}(t)$, $\alpha_n(t) \rightarrow 1$ a.e., when $n \rightarrow \infty$ and $u_n(t) = \alpha_n(t)u(t)$.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leq C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow [L^\Phi]^*$, where a function $v \in L^{\Phi^*}$ is associated to $\xi_v \in [L^\Phi]^*$ being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \quad (11)$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $[L^\Phi]^*$ which can be identified with L^{Φ^*} .

Proposition 2.3. Let $F \in [L^\Phi([0, T], \mathbb{R}^d)]^*$. Then the following statements are equivalent

1. $\xi \in L^{\Phi^*}([0, T], \mathbb{R}^d)$
2. ξ satisfies the monotone convergence property, which is if u_n converges monotonically to u then $\langle \xi, u_n \rangle \rightarrow \langle \xi, u \rangle$.

If $\Phi \in \Delta_2^\infty$ and Φ is coercive then $L^{\Phi^*}([0, T], \mathbb{R}^d) = [L^\Phi([0, T], \mathbb{R}^d)]^*$ (see [Desch and Grimmer, 2001, Thm. 2.9, Thm. 2.10]).

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\}.$$

$W^1 L^\Phi([0, T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (12)$$

And, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \quad (13)$$

We will use repeatedly the decomposition $u = \bar{u} + \tilde{u}$ for a function $u \in L^1([0, T])$ where $\bar{u} = \frac{1}{T} \int_0^T u(t) \, dt$ and $\tilde{u} = u - \bar{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.4. *Let $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ be a Young's function and let $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$. Let $G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined by (4). Then*

1. *For every $s, t \in [0, T]$, $s \neq t$,*

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| G^{-1} \left(\frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq G^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. *We have $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$ and*

$$\|\tilde{u}\|_{L^\infty} \leq T G^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. *If Φ is coercive then the space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0, T], \mathbb{R}^d)$.*

Proof. By the absolutely continuity of u , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi \left(\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \right) &\leq \Phi \left(\frac{1}{|s - t|} \int_s^t \frac{|u'(r)|}{\|u'\|_{L^\Phi}} dr \right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi \left(\frac{|u'(r)|}{\|u'\|_{L^\Phi}} \right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By ?? and ?? we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq G^{-1} \left(\frac{1}{|s - t|} \right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha G^{-1}(1/\alpha)$ is an increasing function with respect to $\alpha \in (0, \infty)$ we have

$$|u(t) - \bar{u}| \leq \|u'\|_{L^\Phi} T G^{-1} \left(\frac{1}{T} \right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi \left(\frac{\bar{u}}{\|u\|_{L^\Phi}} \right) \leq \frac{1}{T} \int_0^T \Phi \left(\frac{u(s)}{\|u\|_{L^\Phi}} \right) ds \leq \frac{1}{T}$$

Then by ?? and ??

$$|\bar{u}| \leq G^{-1} \left(\frac{1}{T} \right) \|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \\ &\leq G^{-1}\left(\frac{1}{T}\right)\max\{1, T\}\|u\|_{W^1 L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1 L^\Phi([0, T], \mathbb{R}^d)$. From (Morrey's inequality) and ?? we infer that u_n are equicontinuous. Furthermore (Sobolev's inequality) implies that u_n is bounded in $C([0, T], \mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence u_{n_k} and $u \in C([0, T], \mathbb{R}^d)$ with $u_{n_k} \rightarrow u$ in $C([0, T], \mathbb{R}^d)$. □

3 Superposition operators in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateaux differentiability of action integrals associated to lagrangian functions defined in Sobolev-Orlicz spaces.

Henceforth we assume that f is a *Carathéodory function*,

- (C) f is measurable with respect to $t \in [0, T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

Definition 3.1. For $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ we denote by \mathbf{f} the Nemytskii (o superposition) operator defined for functions $u : [0, T] \rightarrow \mathbb{R}^d$ by

$$\mathbf{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined in anisotropic Orlicz spaces of vectorial functions. For the proofs of these results and additional discussions see [Płuciennik, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

Theorem 3.2. We assume that f satisfies condition ((C)). Then

1. Measurability. The operator \mathbf{f} maps measurable function into measurable functions
2. Extensibility.? If
3. Continuity.? If

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

References

- [Acinas et al., 2015] Acinas, S., Buri, L., Giubergia, G., Mazzone, F., and Schwindt, E. (2015). Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698.
- [Clarke, 2013] Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Graduate Texts in Mathematics.
- [Desch and Grimmer, 2001] Desch, W. and Grimmer, R. (2001). On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc.*, (353):5095–5120.
- [Krasnosel'skiĭ and Rutickiĭ, 1961] Krasnosel'skiĭ, M. A. and Rutickiĭ, J. B. (1961). *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen.
- [Mawhin and Willem, 1989] Mawhin, J. and Willem, M. (1989). *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York.
- [Pasca, 2010] Pasca, D. (2010). Periodic solutions of a class of nonautonomous second order differential systems with (q, p) -laplacian. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 17(5):841–851.
- [Paşca and Tang, 2010] Paşca, D. and Tang, C.-L. (2010). Some existence results on periodic solutions of nonautonomous second-order differential systems with (q, p) -laplacian. *Applied Mathematics Letters*, 23(3):246–251.
- [Pasca and Tang, 2011] Pasca, D. and Tang, C.-L. (2011). Some existence results on periodic solutions of ordinary (q, p) -laplacian systems. *Journal of applied mathematics & informatics*, 29(1_2):39–48.
- [Pasca and Wang, 2016] Pasca, D. and Wang, Z. (2016). On periodic solutions of nonautonomous second order hamiltonian systems with (q, p) -laplacian. *Electronic Journal of Qualitative Theory of Differential Equations*, 2016(106):1–9.
- [Płuciennik, 1985a] Płuciennik, R. (1985a). Boundedness of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Bull. Pol. Acad. Sci., Math.*, 33:531–540.
- [Płuciennik, 1985b] Płuciennik, R. (1985b). On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 25:321–337.
- [Płuciennik, 1987] Płuciennik, R. (1987). The superposition operator in Musielak-Orlicz spaces of vector-valued functions. Abstract analysis, Proc. 14th Winter Sch., Srní/Czech. 1986, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14, 411–417 (1987).
- [Schappacher, 2005] Schappacher, G. (2005). A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386.

References

- [Skaff, 1969] Skaff, M. S. (1969). Vector valued orlicz spaces. ii. *Pacific J. Math.*, 28(2):413–430.
- [Tang, 1995] Tang, C.-L. (1995). Periodic solutions of non-autonomous second-order systems with γ -quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675.
- [Tang, 1998] Tang, C.-L. (1998). Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270.
- [Tang and Wu, 2001] Tang, C. L. and Wu, X.-P. (2001). Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397.
- [Tang and Zhang, 2010] Tang, X. and Zhang, X. (2010). Periodic solutions for second-order Hamiltonian systems with a p -Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113.
- [Tian and Ge, 2007] Tian, Y. and Ge, W. (2007). Periodic solutions of non-autonomous second-order systems with a p -Laplacian. *Nonlinear Anal.*, 66(1):192–203.
- [Wu and Tang, 1999] Wu, X.-P. and Tang, C.-L. (1999). Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235.
- [Yang and Chen, 2012] Yang, X. and Chen, H. (2012). Periodic solutions for a non-linear (q, p) -laplacian dynamical system with impulsive effects. *Journal of Applied Mathematics and Computing*, 40(1-2):607–625.
- [Yang and Chen, 2013] Yang, X. and Chen, H. (2013). Existence of periodic solutions for sublinear second order dynamical system with (q, p) -laplacian. *Mathematica Slovaca*, 63(4):799–816.
- [Zhao and Wu, 2004] Zhao, F. and Wu, X. (2004). Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434.