

# Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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## Abstract

## 1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_1)$$

where  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 1$ , is called the *Lagrange function* or *lagrangian* and the unknown function  $u : [0, T] \rightarrow \mathbb{R}^d$  is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t, x, y) := \frac{|y|^p}{p} + F(t, x), \quad (1)$$

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for  $1 < p < \infty$ . For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem  $(P_1)$ , for the lagrangian  $\mathcal{L}_{2,F}$ , through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (1) for arbitrary  $1 < p < \infty$  were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case  $(P_1)$  is reduced to the  $p$ -laplacian system

$$\begin{cases} \frac{d}{dt}(u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P_2)$$

In this context, it is customary to call  $F$  a *potential function*, and it is assumed that  $F(t, x)$  is differentiable with respect to  $x$  for a.e.  $t \in [0, T]$  and the following conditions are verified:

(C)  $F$  and its gradient  $\nabla F$ , with respect to  $x \in \mathbb{R}^d$ , are Carathéodory functions, i.e. they are measurable functions with respect to  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ , and they are continuous functions with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

(A) For a.e.  $t \in [0, T]$ , it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (2)$$

In this inequality we assume that the function  $a : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and non decreasing and  $0 \leq b \in L^1([0, T], \mathbb{R})$ .

In [Acinas et al., 2015] it is treated the case of a lagrangian  $\mathcal{L}$  which is lower bounded by a Lagrange function like

$$\mathcal{L}_{\Phi, F}(t, x, y) = \Phi(|y|) + F(t, x), \quad (3)$$

where  $\Phi$  is an  $N$ -function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on  $F$  (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of  $(P_1)$ .

## 2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , i.e. functions such that  $\Phi(x)$  depends on the direction of  $x$ , unlike the radial case where  $\Phi(x) = \Phi(|x|)$ . References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of  $p$ -laplacian equations as the following example shows.

**Example 1.** Let  $1 < p_1, p_2 < \infty$ . We define  $\Phi_{p_1, p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|}{p_1} + \frac{|y_2|}{p_2}.$$

where  $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^n$ . And, we consider the following Lagrange function

$$\mathcal{L}(t, x, y) = \Phi_{p_1, p_2}(y) + F(t, x).$$

Then the equations  $(P_1)$  become

$$\begin{cases} \frac{d}{dt}(|u'_1|^{p_1-2}u'_1) = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt}(|u'_2|^{p_2-2}u'_2) = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_3)$$

In the literature, these equations are known as  $(p_1, p_2)$ -Laplacian system, see [Yang and Chen, 2013, Pasca and Wang, 2016, Yang and Chen, 2012, Pasca, 2010, Paşca and Tang, 2010, Pasca and Tang, 2011].

On the other hand, anisotropic Orlicz-Sobolev spaces allow us to simplify the writing, and they provide the natural frame for statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question motivated us to use these spaces.

Hereafter we denote by  $\mathbb{R}^+$  the set of all non negative real numbers. A function  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is called an *Young's function* if  $\Phi$  is convex,  $\Phi(0) = 0$ ,  $\Phi(-x) = \Phi(x)$  and  $\Phi(x) \rightarrow +\infty$ , when  $|x| \rightarrow +\infty$ . Additionally, we assume that Young's functions which we deal with, satisfy that  $\Phi(x) > 0$  when  $x \neq 0$ . Following [Schappacher, 2005] we say that  $\Phi$  is an  $N_\infty$ -function if

$$\lim_{|x| \rightarrow \infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function  $\Phi$ , we define function  $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by

$$A_\Phi(s) = \min \{ \Phi(x) \mid |x| = s \}, \quad (4)$$

Let us establish some elementary properties of  $A_\Phi$  that we will use in this article.

**Proposition 2.1.** *The function  $A_\Phi$  has the following properties:*

1.  $A_\Phi$  is continuous,
2.  $A_\Phi(s)/s$  is increasing,
3.  $A_\Phi(|x|)$  is the greatest radial minorant of  $\Phi(x)$ ,
4.  $\Phi$  is  $N_\infty$  if and only if  $A_\Phi$  is.

*Proof.* It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [Clarke, 2013]). This fact implies item 1 immediately.

In order to prove item 2, suppose  $0 < r < s$  and  $x \in \mathbb{R}^d$  with  $A_\Phi(s) = \Phi(x)$ . Then, from the definition of  $A_\Phi$  and the convexity of  $\Phi$ ,

$$\frac{A_\Phi(r)}{r} \leq \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leq \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily. □

**Example 2.** We compute  $A_\Phi$  for the function  $\Phi = \Phi_{p_1, p_2}$  given in Example (2). We apply the method of Lagrange multipliers to solve the problem

$$G(r) = \min \{ \Phi(x, y) : |(x, y)|_2^2 = r^2 \}$$

The first order equations are

$$\begin{cases} |x|^{p_1-2}x + \lambda x &= 0 \\ |y|^{p_2-2}y + \lambda y &= 0 \\ |x|^2 + |y|^2 &= r^2 \end{cases}$$

These equations are solved, among others, by the following sets (if  $n > 1$  infinite) of critical values: a)  $|x| = r$ ,  $y = 0$  and  $\lambda = -r^{p_1-2}$  and b)  $x = 0$ ,  $|y| = r$  and  $\lambda = -r^{p_2-2}$ . Associated with these critical points we have the following critical values: a)  $r^{p_1}/p_1$  and b)  $r^{p_2}/p_2$ .

Now, suppose that  $x \neq 0$  and  $y \neq 0$  then  $|x|^2 + |y|^2 = r^2$  and  $|y| = |x|^{\frac{p_1-2}{p_2-2}}$  and  $\lambda = -|x|^{p_1-2}$ .

We have to split the analysis in several cases.

Now, we consider  $p_1 \leq 2$  and  $p_2 \leq 2$  with of them different to 2.

There exists  $(z, w)$  such that  $zx^t + wy^t = 0$  ( $z=-y$ ,  $w=x$ ) where  $H = |\lambda||y|^2|x|^2[(p_1 - 2)|x|^{-2} + (p_2 - 2)|y|^{-2}] < 0$

**(aclarar algo de H, poner un nombre adecuado y cambiar el formato de letra)**

Then, by the second order criteria [?, Thm....], at  $(x, y)$  there cannot be a minimum. Therefore, the minima occur at  $x = 0$  or  $y = 0$ .

The remaining cases can be treated with similar techniques.

Finally, we conclude that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leq A_\Phi \leq K_2 \max\{r^{p_1}, r^{p_2}\}$$

with  $K_1, K_2 > 0$ .

We also say that  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$  satisfies the  $\Delta_2^\infty$ -condition, denoted by  $\Phi \in \Delta_2^\infty$ , if there exist constants  $K > 0$  and  $M \geq 0$  such that

$$\Phi(2x) \leq KH(x), \tag{5}$$

for every  $|x| \geq M$ .

If  $\Phi$  is a Young's function we define its *Fenchel conjugate*  $\Phi^* : \mathbb{R}^d \rightarrow \mathbb{R}^+$  by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \tag{6}$$

We denote by  $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$ , with  $d \geq 1$ , the set of all measurable functions (i.e. functions which are limits of simple functions) defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $u = (u_1, \dots, u_d)$  for  $u \in \mathcal{M}$ . For the set of functions  $\mathcal{M}$ , as for other similar sets, we will omit the reference to codomain  $\mathbb{R}^d$  when  $d = 1$ .

Given an  $N$ -function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Here  $|\cdot|$  is the euclidean norm of  $\mathbb{R}^d$ . Now, we introduce the *Orlicz class*  $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$  by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (7)$$

The *Orlicz space*  $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$  is the linear hull of  $C^\Phi$ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (8)$$

The Orlicz space  $L^\Phi$  equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left( \frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space. By  $u \cdot v$  we denote the usual dot product in  $\mathbb{R}^d$  between  $u$  and  $v$ .

The subspace  $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$  is defined as the closure in  $L^\Phi$  of the subspace  $L^\infty([0, T], \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1])  $u \in E^\Phi$  if and only if  $\rho_\Phi(\lambda u) < \infty$  for any  $\lambda > 0$ . The equality  $L^\Phi = E^\Phi$  is true if and only if  $\Phi \in \Delta_2^\infty$  (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of  $E^\Phi$ , which is particularly useful for us, is that  $u \in E^\Phi$  if and only if  $u$  has *absolutely continuous norm*, i.e. if  $E_n \subset [0, T]$ ,  $n = 1, 2, \dots$  then  $\|\chi_{E_n} u\| \rightarrow 0$  when  $|E_n| \rightarrow 0$ .

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if  $u \in L^\Phi$  and  $v \in L^\Psi$  then  $u \cdot v \in L^1$  and

$$\int_0^T v \cdot u \, dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \quad (9)$$

Like in [Krasnosel'skiĭ and Rutickiĭ, 1961] we will consider the subset  $\Pi(E^\Phi, r)$  of  $L^\Phi$  given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class  $C^\Phi$  by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset r C^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (10)$$

for any positive  $r$  (see [Schappacher, 2005, Thm. 5.6]). If  $\Phi \in \Delta_2^\infty$ , then the sets  $L^\Phi$ ,  $E^\Phi$ ,  $\Pi(E^\Phi, r)$  and  $C^\Phi$  are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

**Definition 2.2.** Let  $u_n, u \in L^\Phi([0, T], \mathbb{R}^d)$ . We say that  $u_n$  converges monotonically to  $u$  if there exists  $\alpha_n \in L^\infty([0, T], \mathbb{R})$ ,  $n = 1, 2, \dots$ , such that  $0 \leq \alpha_n(t) \leq \alpha_{n+1}(t)$ ,  $\alpha_n(t) \rightarrow 1$  a.e., when  $n \rightarrow \infty$  and  $u_n(t) = \alpha_n(t)u(t)$ .

As usual, if  $(X, \|\cdot\|_X)$  is a normed space and  $(Y, \|\cdot\|_Y)$  is a linear subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when there exists  $C > 0$  such that  $\|y\|_X \leq C\|y\|_Y$  for any  $y \in Y$ . With this notation, Hölder's inequality states that  $L^\Psi \hookrightarrow [L^\Phi]^*$ , where a function  $v \in L^\Psi$  is associated to  $\xi_v \in [L^\Phi]^*$  being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \quad (11)$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of  $[L^\Phi]^*$  which can be identified with  $L^\Psi$ .

**Proposition 2.3.** *Let  $F \in [L^\Phi([0, T], \mathbb{R}^d)]^*$ . Then the following statements are equivalent*

1.  $\xi \in L^\Psi([0, T], \mathbb{R}^d)$
2.  $\xi$  satisfies the monotone convergence property, which is if  $u_n$  converges monotonically to  $u$  then  $\langle \xi, u_n \rangle \rightarrow \langle \xi, u \rangle$ .

If  $\Phi \in \Delta_2^\infty$  and  $\Phi$  is  $N_\infty$  then  $L^\Psi([0, T], \mathbb{R}^d) = [L^\Phi([0, T], \mathbb{R}^d)]^*$  (see [Desch and Grimmer, 2001, Thm. 2.9, Thm. 2.10]).

We define the *Sobolev-Orlicz space*  $W^1 L^\Phi$  by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\}.$$

$W^1 L^\Phi([0, T], \mathbb{R}^d)$  is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (12)$$

And, we introduce the following subspaces of  $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \quad (13)$$

We will use repeatedly the decomposition  $u = \bar{u} + \tilde{u}$  for a function  $u \in L^1([0, T])$  where  $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$  and  $\tilde{u} = u - \bar{u}$ .

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

**Lemma 2.4.** *Let  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  be a Young's function and let  $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$ . Let  $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be the function defined by (4). Then*

1. For every  $s, t \in [0, T]$ ,  $s \neq t$ ,

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| A_\Phi^{-1} \left( \frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq A_\Phi^{-1} \left( \frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. We have  $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$  and

$$\|\tilde{u}\|_{L^\infty} \leq T A_\Phi^{-1} \left( \frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. If  $\Phi$  is  $N_\infty$  then the space  $W^1 L^\Phi([0, T], \mathbb{R}^d)$  is compactly embedded in the space of continuous functions  $C([0, T], \mathbb{R}^d)$ .

*Proof.* By the absolutely continuity of  $u$ , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|}\right) &\leq \Phi\left(\frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr\right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By Proposition 2.1(3) we have  $A_\Phi^{-1}\Phi(x) \geq |x|$ , therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq A_\Phi^{-1}\left(\frac{1}{|s - t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.1 (2) and we have

$$\begin{aligned} |u(t) - \bar{u}| &= \left| \frac{1}{T} \int_0^T u(t) - u(s) ds \right| \\ &\leq \frac{1}{T} \int_0^T |u(t) - u(s)| ds \\ &\leq \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right) \end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of  $\|u\|_{L^\Phi}$ , we obtain

$$\Phi\left(\frac{\bar{u}}{\|u\|_{L^\Phi}}\right) \leq \frac{1}{T} \int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leq \frac{1}{T}$$

Then by Proposition 2.1(3)

$$|\bar{u}| \leq A_\Phi^{-1}\left(\frac{1}{T}\right) \|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq A_\Phi^{-1}\left(\frac{1}{T}\right) \|u\|_{L^\Phi} + T A_\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \\ &\leq A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \end{aligned}$$

In order to prove item 3, we take a bounded sequence  $u_n$  in  $W^1 L^\Phi([0, T], \mathbb{R}^d)$ . Since  $\Phi$  is  $N_\infty$ , from Proposition 2.1(4) we obtain  $s A_\Phi^{-1}(1/s) \rightarrow 0$  when  $s \rightarrow 0$ . Therefore (Morrey's inequality) implies that  $u_n$  are equicontinuous. Furthermore (??) implies that  $u_n$  is bounded in  $C([0, T], \mathbb{R}^d)$ . Therefore by the Arzela-Ascoli Theorem we obtain a subsequence  $n_k$  and  $u \in C([0, T], \mathbb{R}^d)$  with  $u_{n_k} \rightarrow u$  in  $C([0, T], \mathbb{R}^d)$ .  $\square$

**Lemma 2.5.** *Let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence of functions in  $\Pi(E^\Phi, 1)$  converging to  $u \in \Pi(E^\Phi, 1)$  in the  $L^\Phi$ -norm. Then, there exist a subsequence  $u_{n_k}$  and a real valued function  $h \in L^1([0, T], \mathbb{R})$  such that  $u_{n_k} \rightarrow u$  a.e. and  $\Phi(u_{n_k}) \leq h$  a.e.*

*Proof.* Since  $d(u, E^\Phi) < 1$  and  $u_n$  converges to  $u$ , there exists  $u_0 \in E^\Phi$ , a subsequence of  $u_n$  (again denoted  $u_n$ ) and  $0 < r < 1$  such that  $d(u_n, u_0) < r$ . Let  $\lambda_0 \in (r, 1)$ . By extracting more subsequences, if necessary, we can assume that  $u_n \rightarrow u$  a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^\Phi} < \frac{1 - \lambda_0}{2^n}, \quad \text{for } n \geq 1.$$

We can assume  $\lambda_n > 0$  for every  $n = 0, \dots$

Let  $\lambda := 1 - \sum_{n=0}^{\infty} \lambda_n$  and define  $h : [0, T] \rightarrow \mathbb{R}$  by

$$h(x) = \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^{\infty} \lambda_n \Phi\left(\frac{u_{n+1} - u_n}{\lambda_n}\right). \quad (14)$$

Note that  $\sum_{n=0}^{\infty} \lambda_n + \lambda = 1$ , therefore for any  $n = 1, \dots$

$$\begin{aligned} \Phi(u_n) &= \Phi\left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j}\right) \\ &\leq \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1} \lambda_j \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) \leq h \end{aligned}$$

Since  $u_0 \in E^\Phi \subset C^\Phi$  and  $E^\Phi$  is a subspace we have that  $\Phi(u_0/\lambda) \in L^1([0, T], \mathbb{R})$ . On the other hand  $\|u_{n+1} - u_n\|_{L^\Phi} \leq \lambda_n$ , therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) dt \leq 1.$$

Then  $h \in L^1([0, T], \mathbb{R})$ . □

### 3 Differentiability Gateaux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateaux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a *Carathéodory function*, i.e.

- (C)  $f$  is measurable with respect to  $t \in [0, T]$  for every  $x \in \mathbb{R}^d$ , and  $f$  is a continuous function with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .



**Definition 3.1.** For  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  we denote by  $\mathbf{f}$  the Nemytskii (o superposition) operator defined for functions  $u : [0, T] \rightarrow \mathbb{R}^d$  by

$$\mathbf{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [Krasnosel'skii et al., 2011] for scalar functions and [Pluciennik, 1987, Pluciennik, 1985b, Pluciennik, 1985a] for the generalization to  $\mathbb{R}^d$ -valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

**Theorem 3.2.** We assume that  $f$  satisfies condition ((C)) and that  $\Phi_1, \Phi_2 : \mathbb{R}^d \rightarrow [0, +\infty)$  are anisotropic Young functions. Then

1. Measurability. The operator  $\mathbf{f}$  maps measurable function into measurable functions
2. Extensibility. If the operator  $\mathbf{f}$  acts from the ball  $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} \mid \|u\|_{L^{\Phi_1}} < r\}$  into the space  $L^{\Phi_2}$  or the space  $E^{\Phi_2}$  then  $\mathbf{f}$  can be extended from  $\Pi(E^{\Phi_1}, r)$  into space  $L^{\Phi_2}$  or  $E^{\Phi_2}$ , respectively.
3. Continuity. If the operator  $\mathbf{f}$  acts from  $\Pi(E^{\Phi_1}, r)$  into space  $E^{\Phi_2}$ , then  $\mathbf{f}$  is continuous.

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

We assume that the Lagrangian  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is Carathéodory and differentiable function satisfying

$$|\mathcal{L}(t, x, y)| + |D_x \mathcal{L}(t, x, y)| + \Psi(D_y \mathcal{L}(t, x, y)) \leq a(|x|) (b(t) + \Phi(y)), \quad (15)$$

where  $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ ,  $b \in L_1^1([0, T])$ ,  $\Phi$  and  $\Psi$  are  $N_\infty$ -functions (complementary???? o en el teorema o nunca?)

Next, we deal with the differentiability of the action integral

$$I(\mathbf{u}) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t)) dt. \quad (16)$$

**Theorem 3.3.** Let  $\mathcal{L}$  be a differentiable Carathéodory function satisfying (15). Then the following statements hold:

1. The action integral given by (16) is finitely defined on  $\mathcal{E}^\Phi(\lambda) := W^1 L^\Phi \cap \{u \mid \dot{u} \in \Pi(E^\Phi, 1)\}$ .
2. The function  $I$  is Gâteaux differentiable on  $\mathcal{E}^\Phi(\lambda)$  and its derivative  $I'$  is demi-continuous from  $\mathcal{E}^\Phi(\lambda)$  into  $[W^1 L^\Phi]^*$ . Moreover,  $I'$  is given by the following expression

$$\langle I'(u), v \rangle = \int_0^T \{D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v}\} dt. \quad (17)$$

3. If  $\Psi \in \Delta_2$  then  $I'$  is continuous from  $\mathcal{E}^\Phi(\lambda)$  into  $[W^1 L^\Phi]^*$  when both spaces are equipped with the strong topology.

*Proof.* Let  $u \in \mathcal{E}^\Phi(\lambda)$ . As

$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \quad (18)$$

and (10), then  $\Phi(\dot{u}(t)) \in L^1$ . Now,

$$|\mathcal{L}(\cdot, u, \dot{u})| \leq A(\|u\|_{W^1 L^\Phi})(b + \Phi(\dot{u})) \in L^1,$$

by Corollary 2.3 in [Acinas et al., 2015] and (15). Thus item (1) is proved.

Aquí conviene escribir lo de arriba o decir directamente que se prueba como en teorema del [Acinas et al., 2015]??? Y  $\mathcal{E}^\Phi(\lambda)$  contiene  $\lambda$  y ahora estamos trabajando con  $\lambda = 1$ , qué hacemos???

We split up the proof of item 2 into four steps.

*Step 1.* The non linear operator  $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$  is continuous from  $\mathcal{E}^\Phi(\lambda)$  into  $L^1([0, T])$  with the strong topology on both sets.

If  $u \in \mathcal{E}^\Phi(\lambda)$ , from (15) and (18), we obtain

$$|D_x \mathcal{L}(\cdot, u, \dot{u})| \leq A(\|u\|_{W^1 L^\Phi})(b + \Phi(\dot{u})) \in L^1. \quad (19)$$

Se podría poner número a la primera ecuación de la demo y decir que se razona igual???

Let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence of functions in  $\mathcal{E}^\Phi(\lambda)$  and let  $u \in \mathcal{E}^\Phi(\lambda)$  such that  $u_n \rightarrow u$  in  $W^1 L^\Phi$ . By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leq T A_\Phi^{-1} \left( \frac{1}{T} \right) \|u_n - u\|_{L^\Phi}$$

then  $u_n \rightarrow u$  uniformly. As  $\dot{u}_n \rightarrow \dot{u} \in \mathcal{E}^\Phi(\lambda)$ , by Lemma 2.5, there exist a subsequence of  $\dot{u}_{n_k}$  (again denoted  $\dot{u}_{n_k}$ ) and a function  $h \in \Pi(E^\Phi, 1)$  such that  $\dot{u}_{n_k} \rightarrow \dot{u}$  a.e. and  $\Phi(\dot{u}_{n_k}) \leq h$  a.e.

Since  $u_{n_k}$ ,  $k = 1, 2, \dots$ , is a strong convergent sequence in  $W^1 L^\Phi$ , it is a bounded sequence in  $\cdot$ . According to Lemma 2.4 and Corollary 2.3 in [Acinas et al., 2015], there exists  $M > 0$  such that  $\|a(u_{n_k})\|_{L^\infty} \leq M$ ,  $k = 1, 2, \dots$ . From the previous facts and (19), we get

$$|D_x \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leq a(|u_{n_k}|)(b(t) + \Phi(\dot{u}_{n_k})) \leq M(b(t) + h) \in L^1.$$

On the other hand, by the continuous differentiability of  $\mathcal{L}$ , we have

$$D_x \mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow D_x \mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

*Step 2.* The non linear operator  $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$  is continuous from  $\mathcal{E}^\Phi(\lambda)$  with the strong topology into  $[L^\Phi]^*$  with the weak\* topology.

Let  $u \in \mathcal{E}^\Phi(\lambda)$ . From (15) and Corollary 2.3 in [Acinas et al., 2015] it follows that

$$\Psi(D_y \mathcal{L}(t, u, \dot{u})) \leq a(|u|)(b(t) + \Phi(\dot{u})) \in L^1 \quad (20)$$

then

$$D_y \mathcal{L}(t, x, y) \in C^\Psi. \quad (21)$$

Note that (19), (21) and the imbeddings  $W^1 L^\Phi \hookrightarrow L^\infty$  and  $L^\Psi \hookrightarrow [L^\Phi]^*$  imply that the second member of (17) defines an element of  $[W^1 L^\Phi]^*$ .

Let  $u_n, u \in \mathcal{E}^\Phi(\lambda)$  such that  $u_n \rightarrow u$  in the norm of  $W^1 L^\Phi$ . We must prove that  $D_y \mathcal{L}(\cdot, u_n, \dot{u}_n) \xrightarrow{w^*} D_y \mathcal{L}(\cdot, u, \dot{u})$ . On the contrary, there exist  $v \in L^\Phi$ ,  $\epsilon > 0$  and a subsequence of  $\{u_n\}$  (denoted  $\{u_n\}$  for simplicity) such that

$$|\langle D_y \mathcal{L}(\cdot, u_n, \dot{u}_n), v \rangle - \langle D_y \mathcal{L}(\cdot, u, \dot{u}), v \rangle| \geq \epsilon. \quad (22)$$

We have  $u_n \rightarrow u$  in  $L^\Phi$  and  $\dot{u}_n \rightarrow \dot{u}$  in  $L^\Phi$ . By Lemma 2.5, there exist a subsequence  $u_{n_k}$  and a function  $h \in \Pi(E^\Phi, 1)$  such that  $u_{n_k} \rightarrow u$  a.e.,  $\dot{u}_{n_k} \rightarrow \dot{u}$  a.e. and  $\Phi(\dot{u}_{n_k}) \leq h$  a.e. As in the previous step, since  $u_n$  is a convergent sequence, Corollary 2.3 in [Acinas et al., 2015] implies that  $a(|u_n(t)|)$  is uniformly bounded by a certain constant  $M > 0$ . Therefore, with  $u_{n_k}$  instead of  $u$ , inequality (20) becomes

$$\Psi(D_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) \leq M(b(t) + h(t)) \in L_1. \quad (23)$$

Consequently, as  $v \in L^\Phi$  there exists  $\lambda > 0$  such that  $\Phi(\frac{v}{\lambda}) \in L^1$ , then arregar!!!! and employing Hölder's inequality, we obtain that

$$\sup_k |D_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot v| \in L_1^1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v \, dt \rightarrow \int_0^T D_y \mathcal{L}(t, u, \dot{u}) \cdot v \, dt \quad (24)$$

which contradicts the inequality (22). This completes the proof of step 2.  $\square$

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