Periodic solutions of Euler-Lagrange equations and "sublinear" pontentials in an Orlicz-Sobolev space setting

Sonia Acinas *
Instituto de Matemática Aplicada San Luis (IMASL)
Universidad Nacional de San Luis and CONICET
Ejército de los Andes 950, (D5700HDW) San Luis, Argentina
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper, we obtain existence results of periodic solutions of hamiltonian systems in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$. We employ the direct method of calculus of variations and we consider a potential function F satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t)$, with $b_1, b_2 \in L^1$ and certain N-functions Φ_0.

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1}$$

* SECyT-UNRC, UNSL and CONICET
† SECyT-UNRC and CONICET
2010 AMS Subject Classification. Primary: . Secondary: .
Keywords and phrases. .

where $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary equations*.

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \mathcal{L}(t, u(t), u'(t))\, dt. \tag{2}$$

This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) = \frac{|y|^p}{p} + F(t,x), \tag{3}$$

for $1 < p < \infty$. For example, the classic book [1] deals mainly with problem (1) for the lagrangian $\mathcal{L}_{2,F}$ and through various methods: direct, dual action, minimax, etc. The results in [1] were extended and improved in several articles, see [2, 3, 4, 5, 6] to cite some examples. Lagrange functions (3) for arbitrary $1 < p < \infty$ are considered in [7, 8] and in this case (1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions hold:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$ for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$,

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t).$$

In this inequality, it is assumed that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [9] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x),$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [9] it was also assumed a condition of *bounded oscillation* on F (see [9],[10]). In this paper we shall study the condition of *sublinearity* (see [3, 4, 6, 8, 11]) on ∇F for the lagrangian $\mathcal{L}_{\Phi,F}$, or more generally for lagrangians which are lower bounded by $\mathcal{L}_{\Phi,F}$.

The paper is organized as follows. In section 2, we give preliminaries facts on N-functions and Orlicz-Sobolev spaces of functions. Section 3 is devoted to the main result of this work and an auxiliary lemma. Section 4 contains the proofs and section 5 provides an application of our result to a concrete case.

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions. Classic references for these topics are [12, 13, 14].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is convex and it also satisfies that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0} \frac{\Phi(t)}{t} = 0.$$

In addition, in this paper for the sake of simplicity we assume that Φ is differentiable and we call φ the derivative of Φ. On these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism whose inverse will be denoted by ψ. We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then, Ψ is an N-function which is called the *complementary function* of Φ.

We recall that an N-function $\Phi(u)$ has *principal part* $f(u)$ if $\Phi(u) = f(u)$ for large values of the argument (see [13, p. 16] and [13, Sec. 7] for properties of principal part).

There exist several orders and equivalence relations between N-functions (see [14, Sec. 2.2]). Following [14, Def. 1, pp. 15-16] we say that the N-function Φ_2 is *stronger* than the N-function Φ_1, in symbols $\Phi_1 \prec \Phi_2$, if there exist $a > 0$ and $x_0 \geqslant 0$ such that

$$\Phi_1(x) \leqslant \Phi_2(ax), \quad x \geqslant x_0. \tag{4}$$

The N-functions Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) when $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$. We say that Φ_2 is *essentially stronger* than Φ_1 ($\Phi_1 \lll \Phi_2$) if and only if for every $a > 0$ there exists $x_0 = x_0(a) \geqslant 0$ such that (4) holds. Finally, we say that Φ_2 is *completely stronger* than Φ_1 ($\Phi_1 \lessdot \Phi_2$) if and only if for every $a > 0$ there exist $K = K(a) > 0$ and $x_0 = x_0(a) \geqslant 0$ such that

$$\Phi_1(x) \leqslant K\Phi_2(ax), \quad x \geqslant x_0.$$

We also say that a non decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the *Δ_2^∞-condition*, denoted by $\eta \in \Delta_2^\infty$, if there exist constants $K > 0$ and $x_0 \geqslant 0$ such that

$$\eta(2x) \leqslant K\eta(x), \tag{5}$$

for every $x \geqslant x_0$. We note that $\eta \in \Delta_2^\infty$ if and only if $\eta \lessdot \eta$. If $x_0 = 0$, the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the *Δ_2-condition* ($\eta \in \Delta_2$). If there exists $x_0 > 0$ such that inequality (5) holds for $x \leqslant x_0$, we will say that Φ satisfies the *Δ_2^0-condition* ($\Phi \in \Delta_2^0$).

We denote by α_η and β_η the so called *Matuszewska-Orlicz indices* of the function η, which are defined next. Given an increasing, unbounded, continuous function $\eta : [0, +\infty) \to [0, +\infty)$ such that $\eta(0) = 0$, we define

$$\alpha_\eta := \lim_{t\to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\eta(tu)}{\eta(u)}\right)}{\log(t)}, \quad \beta_\eta := \lim_{t\to+\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\eta(tu)}{\eta(u)}\right)}{\log(t)}.$$

It is known that the previous limits exist and $0 \leqslant \alpha_\eta \leqslant \beta_\eta \leqslant +\infty$ (see [15, p. 84]). The relation $\beta_\eta < +\infty$ holds true if and only if $\eta \in \Delta_2$ ([15, Thm. 11.7]). If (Φ, Ψ) is a complementary pair of N-functions then

$$\frac{1}{\alpha_\Phi} + \frac{1}{\beta_\Psi} = 1, \tag{6}$$

(see [15, Cor. 11.6]). Therefore $1 \leqslant \alpha_\Phi \leqslant \beta_\Phi \leqslant \infty$.

If η is an increasing function that satisfies the Δ_2-condition, then η is controlled by above and below by power functions ([16, Sec. 1], [17, Eq. (2.3)-(2.4)] and [15, Thm. 11.13]). More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\eta, \epsilon)$ such that, for every $t, u \geqslant 0$,

$$K^{-1} \min\left\{t^{\beta_\eta+\epsilon}, t^{\alpha_\eta-\epsilon}\right\}\eta(u) \leqslant \eta(tu) \leqslant K \max\left\{t^{\beta_\eta+\epsilon}, t^{\alpha_\eta-\epsilon}\right\}\eta(u). \tag{7}$$

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}.$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}.$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup\left\{\int_0^T u \cdot v\, dt \Big| \rho_\Psi(v) \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The following inequality holds for any $u \in L^\Phi$

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k}\left\{1 + \rho_\Phi(ku)\right\}, \quad \text{for every } k > 0. \tag{8}$$

In fact, $\|u\|_{L^\Phi}$ is the infimum for $k > 0$ of the right hand side in above expression (see [13, Thm. 10.5] and [18]).

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [13, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}.$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset \left[L^\Phi\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt,$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2^\infty$, the relation $L^\Psi = \left[L^\Phi\right]^*$ will not be satisfied. In general, it is true that $\left[E^\Phi\right]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* W^1L^Φ (see [12]) by

$$W^1L^\Phi := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi\}.$$

W^1L^Φ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{9}$$

And, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty \hookrightarrow L^\Phi \hookrightarrow L^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. It is said that $u : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t-s|). \tag{10}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions that satisfy (10) for some $C > 0$. This is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t-s|)}.$$

The following simple embedding lemma, whose proof can be found in [9], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ *and for every* $u \in W^1L^\Phi$
$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|) \qquad \text{(Morrey's inequality)},$$
$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}.$$

2. *For every* $u \in W^1L^\Phi$ *we have* $\widetilde{u} \in L^\infty_d$ *and*
$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}.$$

3 Main result

We begin with a lemma which establishes the coercivity of the modular function $\rho_\Phi(u)$ with respect to certain functions of the Orlicz norm $\Phi_0(\|u\|_{L^\Phi}|)$. This lemma generalizes [9, Lemma 5.2] in two directions. Namely, the power function is replaced by a more general one Φ_0 and the Δ_2-condition on Ψ is relaxed to Δ_2^∞. It is worth noting that the second improvement is more important that the first one. And, we present the result here since the lemma introduces a function Φ^* that will play a significant role in our main theorem.

Lemma 3.1. *Let Φ, Ψ be complementary N-functions with $\Psi \in \Delta_2^\infty$. Then, there exists an N-function Φ^* with $\Phi^* \prec \Phi$, such that for every N-function Φ_0 that satisfies $\Phi_0 \lll \Phi^*$ and for every $k > 0$, we have*

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} = \infty. \tag{11}$$

Reciprocally, if (11) *holds for some N-function Φ_0, then $\Psi \in \Delta_2^\infty$.*

We point out that this lemma can be applied to more cases than [9, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively, then (11) holds for Φ_0. On the other hand, $\Phi_0(|u|)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

As in [9] we will consider general Lagrange functions $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the structure conditions

$$|\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right), \tag{A_1}$$

$$|D_x\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right), \tag{A_2}$$

$$|D_y\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(c(t) + \varphi\left(\frac{|y|}{\lambda} + f(t)\right)\right), \tag{A_3}$$

where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function, φ is the right continuous derivative of Φ, $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$. We denote by $\mathfrak{A}(a,b,c,\lambda,f,\Phi)$ the set of all Lagrange functions satisfying (A_1), (A_2) and (A_3).

In [9] it was shown that if $\mathcal{L} \in \mathfrak{A}(a,b,c,\lambda,f,\Phi)$ then there exists the Gateâux derivative of the integral functional I, defined in (2), on the subspace $W^1E^\Phi([0,T],\mathbb{R}^d)$. We observe that the condition (A) on the lagrangian $\mathcal{L}_{\Phi,F}$ is equivalent to say that $\mathcal{L}_{\Phi,F} \in \mathfrak{A}(a,b,0,1,0,\Phi)$.

Unlike what is usual in the literature, we do not assume the lagrangian $\mathcal{L}$ split into two terms, one of them function of y and the other one function of (t,x). We only suppose that $\mathcal{L}$ is lower bounded by a function of this type. More precisely, we assume that for every $(t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{L} \geqslant \mathcal{L}_{\Phi,F}, \quad \text{with } F \text{ satisfying (A) and (C), and } \Phi \text{ being } N\text{-function.} \qquad (A_4)$$

Moreover, as usual we suppose that the time integral of F satisfies certain coercivity condition, see (A_6) below. However, all these hypotheses are not enough. It is also necessary to assume extra conditions on the potential F. Several hypotheses were tested in the past years. The so called *subconvexity* of F was tried in [4, 2, 6] for semilinear equations and in [19, 8] for p-laplacian systems. A boundedness on the oscillation of F was studied in [9]. In [3, 8] the authors dealt with the p-laplacian case where the potentials F satisfied

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^\alpha + b_2(t),$$

where $b_1, b_2 \in L^1([0,T])$ and $\alpha < p$. Such potentials F were called *sublinear nonlinearities*. In this paper, we are interested in studying this type of potentials, but with more general bounds on ∇F which include N-functions instead of power functions; namely, we will consider inequalities like

$$|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t), \qquad (A_5)$$

with Φ_0 a differentiable N-function and $b_1, b_2 \in L^1([0,T],\mathbb{R})$.

Theorem 3.2. *Let Φ be an N-function whose complementary function Ψ satisfies the Δ_2^∞-condition and suppose that Φ^* is given by Lemma 3.1. Let F be a potential that satisfies (C), (A) and the following conditions:*

1. *(A_5) for some N-function Φ_0 such that $\Phi_0 \lll \Phi^*$.*

2. $$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Psi_2(\Phi_0'(2|x|))} = +\infty, \qquad (A_6)$$

   for some N-function Ψ_2 with complementary function Φ_2 satisfying $\Phi_0 \lll \Phi_2 \lll \Phi^*$.

Now, if the lagrangian $\mathcal{L}(t,x,y)$ is strictly convex with respect to $y \in \mathbb{R}^d$, $\mathcal{L} \in \mathfrak{A}(a,b,c,\lambda,f,\Phi)$, $D_y\mathcal{L}(0,x,y) = D_y\mathcal{L}(T,x,y)$ and (A_4) holds, then the problem (1) *has at least a solution which minimizes the action integral I on $W^1E^\Phi_T$.*

4 Proofs

Lemma 4.1. *E^Φ is weak* closed in L^Φ.*

Proof. From [14, Thm. 7, p. 110] we have that $L^\Phi = \left[E^\Psi\right]^*$. Then, L^Φ is a dual and therefore we are allowed to speak about the weak* topology of L^Φ. Besides, E^Φ is separable (see [14, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [14, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because if $u_n \in S$ and $u_n \overset{*}{\rightharpoonup} u \in L^\Phi$, then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \overset{*}{\rightharpoonup} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ is separable and $L^\Phi = \left[E^\Psi\right]^*$, the closed ball of $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ of L^Φ is weak* metrizable (see [20, Thm. 5.1, p. 138]). Thus, S is closed with respect to the weak* topology. Now, by Krein-Smulian theorem, [20, Cor. 12.6, p. 165] implies that E^Φ is weak* closed. □

The following result is analogous to some lemmata in $W^{1,p}$, see [19].

Lemma 4.2. *If $\|u\|_{W^1L^\Phi} \to \infty$, then $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u} + \tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{12}$$

It is known that $L^\infty \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L^\infty$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1 \|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2 \|u'\|_{L^\Phi}. \tag{13}$$

Therefore, from (12), (13) and (9), we get

$$\|u\|_{W^1L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. □

Lemma 4.3. *Let Φ be a not necessarily differentiable N-function and let φ be the right continuous derivative of Φ. Then $\Phi \in \Delta_2^\infty$ ($\Phi \in \Delta_2$) iff $\varphi \in \Delta_2^\infty$ ($\varphi \in \Delta_2$).*

Proof. It is consequence of [15, Thm. 11.7] and [15, Rem. 5, p. 87]. □

The following lemma improves the result on the comment at the beginning of [13, p. 24].

Lemma 4.4. *Let Ψ be an N-function satisfying the Δ_2^∞-condition. Then there exists an N-function Ψ^* such that $\Psi^* \in \Delta_2$, $\Psi \leqslant \Psi^*$ and for every $a > 1$ there exists $x_0 = x_0(a) \geqslant 0$ such that $\Psi^*(x) \leqslant a\Psi(x)$, for every $x \geqslant x_0$. In other words, every Δ_2 near infinity N-function is equivalent to a Δ_2 N-function.*

Proof. We can assume that the Δ_2-condition on Ψ fails near to 0. Consequently, from Lemma 4.3, we have that the right continuous derivative ψ of Ψ does not satisfy the Δ_2^0-condition. Therefore, we obtain a sequence of positive numbers x_n, $n = 1, 2, \ldots$, such that $x_n \to 0$,

$$2x_{n+1} < x_n < 2x_n \quad \text{and} \quad \psi(2x_n) > 2\psi(x_n). \tag{14}$$

We define ψ^* inductively on n on the interval $[2x_n, +\infty)$ of the following way. We put $\psi^*(x) = \psi(x)$ when $x \in [2x_1, +\infty)$. Suppose ψ^* defined on $[2x_n, +\infty)$ and we set ψ^* on $[2x_{n+1}, 2x_n)$ by

$$\psi^*(x) = \begin{cases} \max\left\{\psi(x), \frac{\psi^*(2x_n)}{2x_n}(x - x_n) + \frac{\psi^*(2x_n)}{2}\right\}, & \text{if } x_n \leqslant x < 2x_n \\ \frac{\psi^*(2x_n)}{2} & \text{if } 2x_{n+1} \leqslant x < x_n \end{cases}$$

Moreover, we define $\psi^*(0) = 0$.

Next, we will use induction again to prove that

1. $\psi^*(x_n) = \frac{1}{2}\psi^*(2x_n)$,
2. ψ^* is non-decreasing $[2x_n, +\infty)$,
3. $\psi \leqslant \psi^*$ in $[2x_n, +\infty)$.

We suppose $n = 1$. Then items 2 and 3 are obvious. From (14) we have

$$\psi(x_1) < \frac{1}{2}\psi(2x_1) = \frac{1}{2}\psi^*(2x_1).$$

This inequality implies 1.

Clearly ψ^* is non decreasing on each interval $[2x_{n+1}, x_n)$ and $[x_n, 2x_n)$. And, since ψ is right continuous, then ψ^* is continuous at x_n. Therefore, ψ^* is non decreasing on $[2x_{n+1}, 2x_n)$. Suppose $x \in [2x_{n+1}, 2x_n)$ and $y \geqslant 2x_n$. From the definition of ψ^*, inductive hypothesis, item 3 and item 2, we obtain

$$\psi^*(x) \leqslant \max\{\psi(2x_n), \psi^*(2x_n)\} = \psi^*(2x_n) \leqslant \psi^*(y).$$

This proves item 2 on the interval $[2x_{n+1}, +\infty)$. Inequality in item 3 holds by inductive hypothesis on $[2x_n, +\infty)$ and it is obvious for $x \in [x_n, 2x_n)$. If $x \in [2x_{n+1}, x_n)$, then $\psi(x) \leqslant \psi(x_n) \leqslant \psi^*(x_n) = \psi^*(x)$. This proves 3 on the interval $[2x_{n+1}, +\infty)$.

Now, using (14) and the already proved item 3 for $n + 1$, we deduce

$$\psi(x_{n+1}) < \frac{1}{2}\psi(2x_{n+1}) \leqslant \frac{1}{2}\psi^*(2x_{n+1}).$$

Then

$$\psi^*(x_{n+1}) = \max\left\{\psi(x_{n+1}), \frac{1}{2}\psi^*(2x_{n+1})\right\} = \frac{1}{2}\psi^*(2x_{n+1}),$$

i.e. we have proved item 1.

We note that

$$\psi^*(x_{n+1}) = \frac{1}{2}\psi^*(2x_{n+1}) \leqslant \psi^*(x_n).$$

Consequently $\psi(x) \to 0$ when $x \to 0$. Therefore ψ^* is right continuous at 0 and indeed it is right continuous on $[0, +\infty)$. Moreover, since $\psi(x) = \psi^*(x)$ for $x \geqslant 2x_1$ being ψ the right continuous derivative of an N-function, $\psi^*(x) \to +\infty$ when $x \to +\infty$. In this way,

$$\Psi^*(x) := \int_0^x \psi^*(t)dt$$

defines an N-function.

Let's see that ψ^* satisfies the Δ_2-condition. It is sufficient to prove that ψ^* satisfies the Δ_2^0-condition. To this end, suppose that $x \leqslant x_1$ and take $n \in \mathbb{N}$ such that $x_{n+1} \leqslant x \leqslant x_n$. Then

$$\psi^*(2x) \leqslant \psi^*(2x_n) = 2\psi^*(2x_{n+1}) = 4\psi^*(x_{n+1}) \leqslant 4\psi^*(x).$$

Thus, $\Psi^* \in \Delta_2$ and $\Psi \leqslant \Psi^*$.

It remains to show the inequality $\Psi^*(x) \leqslant a\Psi(x)$, for every $a > 1$ and sufficiently large x. We take x_0 sufficiently large to have

$$\frac{1}{a-1}\int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x_0).$$

Therefore, if $x > \max\{x_0, 2x_1\}$ then

$$\Psi^*(x) = \Psi(x) + \int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x) + (a-1)\Psi(x) = a\Psi(x).$$

The last assessment of the lemma is consequence of $\Psi(ax) > a\Psi(x)$ when $a > 1$. □

The following lemma is essentially known, this is basically a consequence of the fact that $\Psi \in \Delta_2^\infty$ if and only if $\Psi \prec \Psi$, [14, Prop. 4, p. 20] and [14, Cor. 10, p. 30]. However, we prefer to include an alternative proof, because we do not see clearly that the results of [14] contemplates the case of the following lemma with N-functions satisfying the Δ_2-condition.

Lemma 4.5. *Let Φ, Ψ be complementary functions. The next statements are equivalent:*

1. $\Psi \in \Delta_2$ *(*$\Psi \in \Delta_2^\infty$*).*

2. *There exists an N-function Φ^* such that*

$$\Phi(rs) \geqslant \Phi^*(r)\Phi(s) \quad \textit{for every } r \geqslant 1, s \geqslant 0\, (r \geqslant 1, s \geqslant 1). \tag{15}$$

Proof. In virtue of the comment that precedes the lemma, we only consider the case $\Psi \in \Delta_2$.

1)⇒2). As a consequence of the Δ_2-condition on Ψ, (6) and (7), we get for every $1 < \nu < \alpha_\Phi$ a constant $K = K_\nu > 0$ such that

$$\Phi(rs) \geqslant K r^\nu \Phi(s),$$

for any $1 < \nu < \alpha_\Phi$, $s \geqslant 0$ and $r > 1$. This proves (15) with $\Phi^*(r) = kr^\nu$, which is an N-function.

2)⇒1) Next, we follow [14, p. 32, Prop. 13] and [14, p. 29, Prop. 9]. Assume that

$$\Phi^*(r)\Phi(s) \leqslant \Phi(rs) \;\; r > 1,\; s \geqslant 0.$$

Let $u = \Phi^*(r) \geqslant \Phi^*(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [14, p. 13, Prop. 1] and (15), for $u \geqslant \Phi^*(1)$ and $v > 0$ we have

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant \Phi^{*-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{\Psi^{*-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi^{*-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi^*(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi^*(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi^{*-1}(\Phi^*(1))\}$ we get that $\Psi \in \Delta_2$. □

Remark 1. Note that if Φ^* satisfies (15) then $\Phi^* \prec \Phi$.

Proof Lemma 3.1 At first, we assume that $\Psi \in \Delta_2$. Let Φ^* be an N-function satisfying (15). By the inequality (8), for $r > 1$ we have

$$\int_0^T \Phi(|u|)\,dt \geqslant \Phi^*(r)\int_0^T \Phi(r^{-1}|u|)\,dt \geqslant \Phi^*(r)\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$, as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$. From [14, Thm. 2 (b)(v), p. 16], there exists an N-function Φ_0 such that $\Phi_0 \lll \Phi^*$ and we get

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi^*\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(k\|u\|_{L^\Phi})} = \infty. \tag{16}$$

If $\Psi \in \Delta_2^\infty$ but $\Psi \notin \Delta_2$, we use Lemma 4.4. Then, there exists an N-function Ψ_1 such that $\Psi_1 \in \Delta_2$ and $\Psi_1 \sim \Psi \leqslant \Psi_1$. Let Φ_1 be the complementary function of Ψ_1. Then $\Phi \sim \Phi_1 \leqslant \Phi$ (see [13, Thm. 3.1]) and $\|\cdot\|_{L^\Phi}$ and $\|\cdot\|_{L^{\Phi_1}}$ are equivalent norms (see [13, Thm. 13.2 and Thm. 13.3]). By (16), there exists an N-function Φ_0 satisfying (11) with the N-function Φ_1 instead of Φ. Let $C > 0$ be a constant such that $\|\cdot\|_{L^\Phi} \leqslant C\|\cdot\|_{L^{\Phi_1}}$. Then

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi_1(|u|)dt}{\Phi_0(kC\|u\|_{L^{\Phi_1}})} = +\infty.$$

Finally, if Φ_0 is an N-function, then $\Phi_0(x) \geqslant \alpha|x|$ for α small enough and $|x| > 1$. Therefore (11) holds for $\Phi_0(x) = |x|$, then [9, Lemma 5.2] implies $\Psi \in \Delta_2^\infty$. □

Definition 4.6. *We define the functionals $J_{C,\varphi} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\varphi} : \mathbb{R}^n \to \mathbb{R}$, where $C > 0$ and $\varphi : [0, +\infty) \to [0, +\infty)$, by*

$$J_{C,\varphi}(u) := \rho_\Phi(u) - C\varphi(\|u\|_{L^\Phi}),$$

and

$$H_{C,\varphi}(x) := \int_0^T F(t,x)dt - C\varphi(2|x|),$$

respectively.

Next, we give the proof of Theorem 3.2. Here, we will amend a mistake that was made in the end of the proof of [9, Thm. 6.2], where it was wrongly assumed that the minimum of I was on the domain of differentiability of I. We avoid this problem minimizing I on $W^1E_T^\Phi$, which is a weak* closed subspace contained in that domain.
Proof. Theorem 3.2 . By the decomposition $u = \overline{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (A_5), we have

$$\begin{aligned} &\left|\int_0^T F(t,u) - F(t,\overline{u})\,dt\right| = \left|\int_0^T\int_0^1 \nabla F(t,\overline{u}+s\tilde{u}(t))\cdot\tilde{u}(t)\,ds\,dt\right| \\ &\leqslant \int_0^T\int_0^1 b_1(t)\Phi_0'(|\overline{u}+s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt + \int_0^T\int_0^1 b_2(t)|\tilde{u}(t)|\,ds\,dt \\ &=: I_1 + I_2. \end{aligned} \tag{17}$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty} \leqslant C_1\|u'\|_{L^\Phi}, \tag{18}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

Note that, since Φ_0' is an increasing function and $\Phi_0'(x) \geqslant 0$ for $x \geqslant 0$, $\Phi_0'(a+b) \leqslant \Phi_0'(2a) + \Phi_0'(2b)$ for every $a, b \geqslant 0$. In this way, we have

$$\Phi_0'(|\overline{u}+s\tilde{u}(t)|) \leqslant \Phi_0'(2|\overline{u}|) + \Phi_0'(2\|\tilde{u}\|_{L^\infty}), \tag{19}$$

for every $s \in [0,1]$. Now, inequality (19), Hölder's and Sobolev-Wirtinger's inequalities imply that

$$\begin{aligned} I_1 &\leqslant \Phi_0'(2|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} + \Phi_0'(2\|\tilde{u}\|_{L^\infty})\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} \\ &\leqslant C_2\left\{\Phi_0'(2|\overline{u}|)\|u'\|_{L^\Phi} + \Phi_0'(C_3\|u'\|_{L^\Phi})\|u'\|_{L^\Phi}\right\}, \end{aligned} \tag{20}$$

where $C_2 = C_2(T, \|b_1\|_{L^1})$ and $C_3 = C_3(T)$. Next, by Young's inequality with complementary functions Φ_1 and Ψ_1, we obtain

$$\Phi_0'(2|\overline{u}|)\|u'\|_{L^\Phi} \leqslant \Psi_1(\Phi_0'(2|\overline{u}|)) + \Phi_1(\|u'\|_{L^\Phi}). \tag{21}$$

We have that any N-function Φ_0 satisfies the inequality $x\Phi_0'(x) \leqslant \Phi_0(2x)$ (see [14, p. 17]). Moreover, since $\Phi_0 \lll \Phi_1$ there exists $x_0 = x_0(\Phi_0, \Phi_1, T) \geqslant 0$ such that $\Phi_0(2C_3 x) \leqslant \Phi_1(x)$, for every $x \geqslant x_0$. Therefore, $\Phi_0(2C_3 x) \leqslant \Phi_1(x) + C_4$, with $C_4 = \Phi_0(2x_0)$. The previous observations imply that

$$\Phi_0'(C_3\|u'\|_{L^\Phi})\|u'\|_{L^\Phi} \leqslant C_3^{-1}(\Phi_1(\|u'\|_{L^\Phi}) + C_4). \tag{22}$$

From (20), (21), (22) and (18), we have

$$I_1 + I_2 \leqslant C_5\left\{\Psi_1(\Phi_0'(2|\overline{u}|)) + \Phi_1(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi} + 1\right\}, \tag{23}$$

with C_5 depending on $\Phi_0, \Phi_1, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$.

Using (17) and (23), we get

$$\begin{aligned}
I(u) &\geqslant \rho_\Phi(u') + \int_0^T F(t,u)\,dt \\
&= \rho_\Phi(u') + \int_0^T [F(t,u) - F(t,\overline{u})]\,dt + \int_0^T F(t,\overline{u})\,dt \\
&\geqslant \rho_\Phi(u') - C_5\Phi_1(\|u'\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_5\Psi_1(\Phi_0'(2|\overline{u}|)) - C_5 \\
&\geqslant \rho_\Phi(u') - C_5\Phi_1(\|u'\|_{L^\Phi}) + H_{C_5,\Psi_1\circ\Phi_0}(\overline{u}) - C_5 \\
&= J_{C_5,\Phi_0}(u') + H_{C_5,\Psi_1\circ\Phi_0}(\overline{u}) - C_5.
\end{aligned}$$

Let u_n be a sequence in W^1L^Φ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n, $I(u_n)$ is upper bounded, i.e. there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 4.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence is necessary, still denoted u_n, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.1 implies that the functional $J_{C_5,\Phi_0}(u')$ is coercive; and, by (A_6), the functional $H_{C_5,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_5,\Phi_0}(u_n') \to \infty$ or $H_{C_5,\Phi_0}(\overline{u}_n) \to \infty$. From the condition (A) on F, we have that on a bounded set the functional $H_{C_5,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_5,\Phi_0}(u_n') \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u)|u \in W^1E_T^\Phi\}$. Since $I(u_n)$, $n = 1, 2, \ldots$, is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in W^1E^Φ. Hence, by virtue of [9, Cor. 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T-periodic continuous (therefore in E_T^Φ) function u. As $u_n' \in E^\Phi$ is a norm bounded sequence in L^Φ, there exists a subsequence (again denoted by u_n') such that u_n' converges to a function $v \in L^\Phi$ in the weak* topology of L^Φ. Since E^Φ is weak* closed, by Lemma 4.1,

$v \in E^\Phi$. From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \xi' \cdot u \, dt = \lim_{n\to\infty} \int_0^T \xi' \cdot u_n \, dt = -\lim_{n\to\infty} \int_0^T \xi \cdot u_n' \, dt = -\int_0^T \xi \cdot v \, dt,$$

for every T-periodic function $\xi \in C^\infty([0,T], \mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [1, p. 6]) and $u \in W^1 E_T^\Phi$.

Now, taking into account the relations $\left[L^1\right]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that u_n' converges to u' in the weak topology of L^1. Consequently, from the semicontinuity of I (see [9, Lemma 6.1]) we get

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{v \in W^1 E_T^\Phi} I(v).$$

Hence $u \in W^1 E_T^\Phi$ is a minimun and, since I is Gâteaux differentiable on $W^1 E^\Phi$ (see [9, Thm. 3.2]), therefore $I'(u) \in (W^1 E_T^\Phi)^\perp$. Thus,

$$\int_0^T D_y \mathcal{L}(t, u(t), u'(t)) \cdot v'(t) dt = -\int_0^T D_x \mathcal{L}(t, u(t), u'(t)) \cdot v(t) dt,$$

for every $v \in W^1 E_T^\Phi$. From [9, Eq. (26)] we have $D_y \mathcal{L}(t, u(t), u'(t)) \in L^\Psi([0,T], \mathbb{R}^n) \hookrightarrow L^1([0,T], \mathbb{R}^n)$; and, from [9, Eq. (24)], it follows that $D_x \mathcal{L}(t, u(t), u'(t)) \in L^1([0,T], \mathbb{R}^n)$. Consequently, from [1, p. 6] (note that $W^1 E_T^\Phi$ includes the periodic test functions) we obtain the absolutely continuity of $D_y \mathcal{L}(t, u(t), u'(t))$ and that the differential equations in (1) are satisfied. The strict convexity of $\mathcal{L}(t, x, y)$ with respect to y and the T-periodicity with respect to t imply the boundary conditions in (1) (see [9, Thm. 4.1]). □

5 An example

In this section we develop an application of our main result so that the reader can appreciate the innovations that brings.

One of the main novelties of our work is that we obtain existence of solutions for lagrangian functions $\mathcal{L}(t, x, y)$ that do not satisfy a power-like grow condition in y.

In fact, it is possible to apply Theorem 3.2 to lagrangians $\mathcal{L} = \mathcal{L}(t, x, y)$ with exponential grow on the variable y. For example, suppose that

$$\mathcal{L}(t, x, y) = f(y) + F(t, x),$$

with $f : \mathbb{R}^n \to \mathbb{R}$ differentiable, strictly convex and $f(y) \geqslant e^{|y|}$. We define for $n \geqslant 1$

$$\Phi(y) = e^y - \sum_{i=0}^{n-1} \frac{y^i}{i!}$$

It is easy to see that $\Phi : [0, +\infty) \to [0, +\infty)$ is an N-function. From [15, Ex. 3, p. 85] we know that $\alpha_\Phi = n$. As consequence of (6) we have $\beta_\Psi = \frac{n}{n-1} < \infty$ and consequently $\Psi \in \Delta_2$. From (7), for every $1 < p < n$ there exists $C_p > 0$ such that

$$\Phi(rs) \geqslant C_p r^p \Phi(s), \quad r > 1, s > 0.$$

Then, the complementary pair (Φ, Ψ) and the N-function $\Phi^*(r) := r^p$ satisfy Lemma 3.1 for every $1 < p < n$. Now, we fix arbitrary real numbers $1 < p_0 < p_1 < p < n$ and we consider $\Phi_i = r^{p_i}$, $i = 0, 1$. Therefore $\Phi_0 \lll \Phi_1 \lll \Phi^*$. The conditions (A_5) and (A_6) become

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^{p_0-1} + b_2(t), \quad b_1, b_2 \in L^1([0,T]), \tag{24}$$

and

$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{|x|^{(p_0-1)q_1}} = +\infty, \quad q_1 = p_1/(p_1-1), \tag{25}$$

respectively. Since n is an arbitrary positive integer, the pair p_0 and p_1 of real numbers with $1 < p_0 < p_1$ is also arbitrary. For clarity, assume that $F(t,x) = b(t)|x|^\sigma$, for some $1 < \sigma < \infty$ and $b \in L^1([0,T])$. We note that this F satisfies (A) and (C). Now, we choose any $1 < p_0$ with $p_0 - 1 < \sigma < p_0$ and we take p_1 with $p_1 > \sigma(\sigma - p_0 + 1)^{-1}$. Then, (24) and (25) hold. In conclusion, the system (1) has solution for the Lagrangian given by $\mathcal{L}(t,x,y) = f(y) + b(t)|x|^\sigma$ for any $1 < \sigma$.

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant PROICO 30412.

References

[1] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989.

[2] C.-L. Tang, Periodic solutions of non-autonomous second-order systems with γ-quasisubadditive potential, Journal of Mathematical Analysis and Applications 189 (3) (1995) 671–675.

[3] C.-L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, Proc. Amer. Math. Soc. 126 (11) (1998) 3263–3270.

[4] X.-P. Wu, C.-L. Tang, Periodic solutions of a class of non-autonomous second-order systems, J. Math. Anal. Appl. 236 (2) (1999) 227–235.

[5] C. L. Tang, X.-P. Wu, Periodic solutions for second order systems with not uniformly coercive potential, J. Math. Anal. Appl. 259 (2) (2001) 386–397.

[6] F. Zhao, X. Wu, Periodic solutions for a class of non-autonomous second order systems, J. Math. Anal. Appl. 296 (2) (2004) 422–434.

[7] Y. Tian, W. Ge, Periodic solutions of non-autonomous second-order systems with a p-Laplacian, Nonlinear Anal. 66 (1) (2007) 192–203.

[8] X. Tang, X. Zhang, Periodic solutions for second-order Hamiltonian systems with a p-Laplacian, Ann. Univ. Mariae Curie-Skłodowska Sect. A 64 (1) (2010) 93–113.

[9] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, E. Schwindt, Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, Nonlinear Analysis, TMA. 125 (2015) 681 – 698.

[10] K. Zhu, Analysis on Fock Spaces, Graduate Texts in Mathematics, Springer, 2012.

[11] F. Zhao, X. Wu, Existence and multiplicity of periodic solution for non-autonomous second-order systems with linear nonlinearity, Nonlinear Analysis: Theory, Methods & Applications 60 (2) (2005) 325–335.

[12] R. Adams, J. Fournier, Sobolev spaces, Elsevier/Academic Press, Amsterdam, 2003.

[13] M. A. Krasnosel′skiĭ, J. B. Rutickiĭ, Convex functions and Orlicz spaces, P. Noordhoff Ltd., Groningen, 1961.

[14] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Vol. 146, Marcel Dekker, Inc., New York, 1991.

[15] L. Maligranda, Orlicz spaces and interpolation, Vol. 5 of Seminários de Matemática [Seminars in Mathematics], Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[16] J. Gustavsson, J. Peetre, Interpolation of orlicz spaces, Studia Mathematica 60 (1) (1977) 33–59.
URL `http://eudml.org/doc/218150`

[17] A. Fiorenza, M. Krbec, Indices of Orlicz spaces and some applications, Commentationes Mathematicae Universitatis Carolinae 38 (3) (1997) 433–452.

[18] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general, Indag. Math. (N.S.) 11 (4) (2000) 573–585.

[19] B. Xu, C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems, J. Math. Anal. Appl. 333 (2) (2007) 1228–1236.

[20] J. B. Conway, A Course in Functional Analysis, Springer, USA, 1977.