Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$, of hamiltonian systems with a potential function F satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t)$, with $b_1(t), b_2(t) \in L^1$ and for certain N-functions Φ_0.

1 Introduction

The purpose of this paper is to study the existence of periodic solution for the following non-autonomous second-order systems:

$$\begin{cases} \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right) = \nabla F(t,u(t)) \quad \text{a.e. } t \in (0,T) \\ u(0)-u(T) = u'(0)-u'(T) = 0 \end{cases} \tag{1}$$

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where $T > 0$, $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous and Φ is an differentiable N-function (see preliminaries section for definitions). Furthermore, the *potential* $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfy the following conditions

(C) F and its gradient ∇F are Carathéodory functions, i.e. they are measurable functions with respecto to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$ we have that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t) \tag{2}$$

In these inequalities we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

We call the differential operator.

$$L_\Phi[u] = \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right)$$

the Φ-*laplacian operator*. If $\Phi(x) = |x|^p$, $1 < p < \infty$, L_Φ is the well known p-laplacian operator.

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associeted to the *action integral*

$$I(u) = \int_0^T \Phi(|u'(t)|) + F(t,u(t))\, dt. \tag{3}$$

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [1, 2, 3]. For Orlicz spaces of vector valued functions, see [4] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an N-*function* if Φ is convex and satisfies that

$$\lim_{t\to+\infty}\frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0}\frac{\Phi(t)}{t} = 0$$

In addition, in this paper, we asssume that Φ is differentiable, and we call φ to the derivative of Φ. With these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism, with inverse ψ. We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then Ψ is a N-function which is called the *complementary function* of Φ.

There exists several order relations between N-functions (see [3, Section 2.2]). Following [3, Def. 1, p.15] we said that the N-function Φ_2 is *essentially stronger* than the N-function Φ_1 ($\Phi_1 \lll \Phi_2$) if and only if there exists $x_0 \geqslant 0$ such that $\Phi_1(x) \leqslant \Phi_2(ax)$, for every $a > 0$ and $x \geqslant x_0$.

We say that a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Δ_2-*condition*, denoted by $\eta \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geqslant 0$ such that

$$\eta(2t) \leqslant K\eta(t) \tag{4}$$

for every $t \geqslant t_0$. If $t_0 = 0$, we say that a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Δ_2-*condition globally* ($\eta \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}_d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C_d^\Phi = C_d^\Phi([0,T], \mathbb{R}^d)$ is given by

$$C_d^\Phi := \{u \in \mathcal{M}_d | \rho_\Phi(u) < \infty\}. \tag{5}$$

The *Orlicz space* $L^\Phi = L_d^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M}_d | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{6}$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup \left\{ \int_0^T u \cdot v\, dt \middle| \rho_\Psi(v) \leqslant 1 \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [2, Thm. 10.5] and [5]). For every $u \in L^\Phi$,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(ku)\}. \tag{7}$$

In particular

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k} \{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \tag{8}$$

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L_d^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [2, Th. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L_1^1$ and

$$\int_0^T v \cdot u\, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \tag{9}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset \left[L^\Phi\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt \tag{10}$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi = \left[L^\Phi\right]^*$ will not hold. In general, it is true that $\left[E^\Phi\right]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* W^1L^Φ (see [1]) by

$$W^1L^\Phi := \{u | u \text{ is absolutely continuous in } [0,T] \text{ and } u' \in L^\Phi\}.$$

W^1L^Φ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{11}$$

We introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{12}$$

For a function $u \in L^1_d([0,T])$, we write $u = \overline{u} + \widetilde{u}$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty_d \hookrightarrow L^\Phi \hookrightarrow L^1_d$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $u : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t - s|). \tag{13}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (13) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [6, 7, 8, 9, 10]. The next simple lemma, whose proof can be found in [11], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ *and for every* $u \in W^1L^\Phi$

$$|u(t)-u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|), \qquad \text{(Morrey inequality)}. \tag{14}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev inequality)}. \tag{15}$$

2. *For every* $u \in W^1L^\Phi$ *we have* $\widetilde{u} \in L_d^\infty$ *and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|\dot{u}\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger inequality)}. \tag{16}$$

3 Lagrangians satisfying sublinear nonlinearity type conditions

Lemma 3.1. *E^Φ is weak* closed in L^Φ.*

Proof. From [3, Thm. 7, p. 110] we have that $L^\Phi = \left[E^\Psi\right]^*$. Then, L^Φ is a dual and therefore we are allowed to speak about the weak* topology of L^Φ. Besides, E^Φ is separable (see [3, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [3, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because is $u_n \in S$ and $u_n \xrightarrow{*} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \xrightarrow{*} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ is separable and $L^\Phi = \left[E^\Psi\right]^*$, the ball of L^Φ $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ is weak* metrizable (see [12, Thm. 5.1, p. 138]). Thus, S is closed respect to the weak* topology. Now, by the Krein-Smulian Theorem, [12, Cor. 12.6, p. 165] implies that E^Φ is weak* closed. □

The following result is analogous to some lemmata in $W^1L_d^p$, see [13].

Lemma 3.2. *If $\|u\|_{W^1L^\Phi} \to \infty$, then $(|\overline{u}| + \|\dot{u}\|_{L^\Phi}) \to \infty$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u}+\tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{17}$$

It is known that $L_d^\infty \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L_d^\infty$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1\|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2\|u'\|_{L^\Phi}. \tag{18}$$

Therefore, from (17), (18) and (11), we get

$$\|u\|_{W^1L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. □

Lemma 3.3. *Let Φ, Ψ complementary functions. The next statements are equivalent:*

1. *$\Psi \in \Delta_2$ globally.*
2. *There exists an N-function Φ_1 such that*

$$\Phi(rs) \geqslant \Phi_1(r)\Phi(s) \text{ for every } r \geqslant 1, \quad s \geqslant 0. \tag{19}$$

Proof. 1)⇒2) In virtue of the Δ_2-condition on Ψ, [14, Thm. 11.7] and [14, Cor. 11.6] (see also [15, Eq. (2.8)]), we get constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geqslant K r^\nu \Phi(s) \tag{20}$$

for any $1 < \nu < \alpha_\Phi$, $s \geqslant 0$ and $r > 1$. This proves (19) with $\Phi_1(r) = kr^\nu$, which is is an N-function.

2)⇒1) Next, we follow [3, p. 32, Prop. 13] and [3, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leqslant \Phi(rs) \;\; r > 1, \; s \geqslant 0.$$

Let $u = \Phi_1(r) \geqslant \Phi_1(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [3, p. 13, Prop. 1] and (19), we have for $u \geqslant \Phi_1(1)$ and $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant \Phi_1^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi_1(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi_1(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$ we get that $\Psi \in \Delta_2$ globally. □

The following lemma generalizes [11, Lemma 5.2].

Lemma 3.4. *Let Φ, Ψ be complementary N-functions with $\Psi \in \Delta_2$ globally. Let Φ_1 be any N-function satisfying* (19)*. Then*

$$\lim_{\|u\|_{L^\Phi} \to \infty} \frac{\int_0^T \Phi(|u|)\, dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \tag{21}$$

for every Φ_0 with $\Phi_0 \lll \Phi_1$).
If (21) *holds for some N-function Φ_0, then $\Psi \in \Delta_2$ (at ∞).*

Proof. By the assumptions on Φ and Φ_1 and the inequality (8), we have, for $r > 1$,

$$\frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \Phi_1(r)\frac{\int_0^T \Phi(r^{-1}|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})}\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$ and as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$. $\Phi_0 = o(\Phi_1)$ at ∞, and we get

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

The last equality follows of [3, Th. 2 (b),p. 16].

Finally, if Φ_0 is an N-function, then $\Phi_0(u) \geqslant k|u|$ for k small enough and $|u| > 1$. Therefore (21) holds for $\Phi_0(u) = |u|$, then [11, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞. □

Remark 1. We point out that this lemma can be applied to more cases than [11, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [2, p. 16] and [2, Section 7] for the definition and properties of principal part). Then (21) holds for Φ_0, however $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

Definition 3.5. *We define the functionals* $J_{C,\Phi_0} : L^\Phi \to (-\infty, +\infty]$ *and* $H_{C,\Phi_0} : \mathbb{R}^n \to \mathbb{R}$*, where* $C > 0$ *and* Φ_0 *is an* N*-function, by*

$$J_{C,\Phi_0}(u) := \rho_\Phi(u) - C\Phi_0(\|u\|_{L^\Phi}), \tag{22}$$

and

$$H_{C,\Phi_0}(x) := \int_0^T F(t,x)dt - C\Phi_0(|x|), \tag{23}$$

respectively.

In [16] and [17] was considered, for the p-laplacian case, potentials F satisfying the inequality

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^\alpha + b_2(t),$$

where $b_1, b_2 \in L_1^1$ and α is any power less than p. Thus, they said F is a sublinear nonlinearity. In this paper, we will consider bounds on ∇F of a more general type.

Definition 3.6. *Let* Φ_0 *a differentiable* N*-function. We said that* $G : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfies a* Φ_0-grow condition *if*

$$|G(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t), \tag{24}$$

with $b_1, b_2 \in L^1([0,T],\mathbb{R})$.

Theorem 3.7. *Let Φ be a N-function whose complementary function Ψ is Δ_2-globally. We suppose that the N-function Φ_1 satisfies* (19)*, F satisfies (C), (A) and that ∇F satisfies a Φ_0-grow condition, for some Δ_2-globally N-function Φ_0 such that $\Phi_0 \ll \Phi_1$. Furthermore, we suppose that*

$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Phi_0(|x|)} = +\infty. \tag{25}$$

Then the problem (1) *has at least a solution which minimizes the action integral I on $W^1E_T^{\Phi}$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (24), we have

$$\begin{aligned}
&\left|\int_0^T F(t,u) - F(t,\overline{u})\,dt\right| = \left|\int_0^T \int_0^1 \nabla F(t,\overline{u} + s\tilde{u}(t))\cdot\tilde{u}(t)\,ds\,dt\right| \\
&\leqslant \int_0^T \int_0^1 b_1(t)\Phi_0'(|\overline{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt + \int_0^T \int_0^1 b_2(t)|\tilde{u}(t)|\,ds\,dt \\
&=: I_1 + I_2.
\end{aligned} \tag{26}$$

On the one hand, by Hölder's and Sobolev-Wirtinger inequality, we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty} \leqslant C_1\|\dot{u}\|_{L^\Phi}, \tag{27}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

On the other hand, since $\Phi_0 \in \Delta_2$ globally, then $\Phi_0' \in \Delta_2$ globally and consequently Φ_0' is a quasi-subadditive function, i.e. there exists $C(\Phi_0) > 0$ such that $\Phi_0'(a+b) \leqslant C(\Phi_0')(\Phi_0'(a) + \Phi_0'(b))$ for every $a, b \geqslant 0$. In this way, we have

$$\Phi_0'(|\overline{u} + s\tilde{u}(t)|) \leqslant C(\Phi_0)[\Phi_0'(|\overline{u}|) + \Phi_0'(\|\tilde{u}\|_{L^\infty})], \tag{28}$$

for every $s \in [0,1]$.

Now, inequality (28), Hölder's and Sobolev-Wirtinger inequality, the monotonicity, the subadditivity and the Δ_2-condition on Φ_0', imply that

$$\begin{aligned}
I_1 &\leqslant C(\Phi_0')\left\{\Phi_0'(|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} + \|b_1\|_{L^1}\Phi_0'(\|\tilde{u}\|_{L^\infty})\|\tilde{u}\|_{L^\infty}\right\} \\
&\leqslant C_2\left\{\Phi_0'(|\overline{u}|)\|u'\|_{L^\Phi} + \Phi_0'(\|\dot{u}\|_{L^\Phi})\|\dot{u}\|_{L^\Phi}\right\},
\end{aligned} \tag{29}$$

where $C_2 = C_2(\Phi_0', T, \|b_1\|_{L^1})$.

Next, by Young's inequality with complementary functions Φ_0 and Ψ_0 and the fact that $\Phi_0 \in \Delta_2$ globally, Young's equality [2, Eq. 2.7-2.8] and [3, Th. 3-(ii), p. 23], we get

$$\begin{aligned}
\Phi_0'(|\overline{u}|)\|u'\|_{L^\Phi} &\leqslant \Psi_0(\Phi_0'(|\overline{u}|)) + \Phi_0(\|u'\|_{L^\Phi}) \\
&\leqslant |\overline{u}|\Phi_0'(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \\
&\leqslant C(\Phi_0)\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi})
\end{aligned} \tag{30}$$

and

$$\Phi_0'(\|\dot{u}\|_{L^\Phi})\|\dot{u}\|_{L^\Phi} \leqslant C(\Phi_0)\Phi_0(\|\dot{u}\|_{L^\Phi}), \tag{31}$$

with $C(\Phi_0)$ the constant that comes from the Δ_2-condition on Φ_0.

From (29), (30), (31) and (27), we have

$$\begin{aligned} I_1 + I_2 &\leqslant C_3\left\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi}\right\} \\ &\leqslant C_4\left\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + 1\right\}, \end{aligned} \tag{32}$$

with C_3 and C_4 depending on $\Phi_0, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$. The last inequality follows from the fact that Φ_0 is an N-function, then there exists $C > 0$ such that $\Phi_0(x) \geqslant Cx$ for every $x \geqslant 1$. Thus $x \leqslant C\Phi_0(x) + 1$ for every $x \geqslant 0$.

In the subsequent estimates, we use (26), (32), the fact that $\Phi_0 \in \Delta_2$ and we get

$$\begin{aligned} I(u) &\geqslant \alpha_0\rho_\Phi\left(\frac{u'}{\Lambda}\right) + \int_0^T F(t,u)\,dt \\ &= \alpha_0\rho_\Phi\left(\frac{u'}{\Lambda}\right) + \int_0^T \left[F(t,u) - F(t,\overline{u})\right]\,dt + \int_0^T F(t,\overline{u})\,dt \\ &\geqslant \alpha_0\rho_\Phi\left(\frac{u'}{\Lambda}\right) - C_4\Phi_0(\|\dot{u}\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_4\Phi_0(|\overline{u}|) - C_4 \\ &\geqslant \alpha_0\rho_\Phi\left(\frac{u'}{\Lambda}\right) - C_4\Phi_0(\|\dot{u}\|_{L^\Phi}) + H_{C_4,\Phi_0}(\overline{u}) - C_4 \\ &\geqslant \alpha_0\rho_\Phi\left(\frac{u'}{\Lambda}\right) - C_5\Phi_0\left(\frac{\|\dot{u}\|_{L^\Phi}}{\Lambda}\right) + H_{C_4,\Phi_0}(\overline{u}) - C_4 \\ &= \alpha_0 J_{C_6,\Phi_0}\left(\frac{\dot{u}}{\Lambda}\right) + H_{C_4,\Phi_0}(\overline{u}) - C_4, \end{aligned} \tag{33}$$

where $C_5 = C_5(\Phi_0, \Lambda, C_4)$ and $C_6 = \frac{C_5}{\alpha_0}$.

Let u_n be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n, $I(u_n)$ is upper bounded, i.e., there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 3.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence, still denoted u_n, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.4 implies that the functional $J_{C_6,\Phi_0}(\frac{\dot{u}}{\Lambda})$ is coercive; and, by (25), the functional $H_{C_4,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_6,\Phi_0}(\frac{\dot{u}_n}{\Lambda}) \to \infty$ or $H_{C_4,\Phi_0}(\overline{u}_n) \to \infty$. From the (A) condition for F, we have that on a bounded set the functional $H_{C_4,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_6,\Phi_0}(\frac{\dot{u}_n}{\Lambda}) \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u) | u \in W^1E_T^\Phi\}$. Since $I(u_n)$, $n = 1, 2, \dots$ is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in W^1E^Φ. Hence, in virtue of Corollary [11, Corollary 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T-periodic continuous (therefore in E_T^Φ) function u. As $u_n' \in E^\Phi$ is a norm bounded sequence in L^Φ, there exists a subsequence (again denoted by u_n') such that u_n' converges to a function $v \in L^\Phi$ in the weak* topology of L^Φ. Since E^Φ is weak* closed, by Lemma 3.1, $v \in E^\Phi$. From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \dot{\boldsymbol{\xi}} \cdot u \, dt = \lim_{n\to\infty} \int_0^T \dot{\boldsymbol{\xi}} \cdot u_n \, dt = -\lim_{n\to\infty} \int_0^T \boldsymbol{\xi} \cdot u_n' \, dt = -\int_0^T \boldsymbol{\xi} \cdot v \, dt$$

for every T-periodic function $\boldsymbol{\xi} \in C^\infty([0,T], \mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [18, p. 6]) and $u \in E_T^\Phi$.

Now, taking into account the relations $\left[L^1\right]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that u_n' converges to u' in the weak topology of L^1. Consequently, the semicontinuity lemma (see [11, Lemma 6.1]) implies that

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{u \in W^1 E_T^\Phi} I(u).$$

Hence $u \in W^1 E_T^\Phi$ is a minimun and since I is Gâteaux differentiable on $W^1 E^\Phi$ (see [11, Th. 3.2]) we have and therefore $I'(u) \in (W^1 E_T^\Phi)^\perp$. Therefore

$$\int_0^T \frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t) \cdot v'(t) dt = -\int_0^T \nabla F(t, u(t)) \cdot v(t) dt,$$

for every $v \in W^1 E_T^\Phi$. Since $\frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t) \in L^\Psi([0,T], \mathbb{R}^n) \hookrightarrow L^1([0,T], \mathbb{R}^n)$ and $\nabla F(t, u(t)) \in L^1([0,T], \mathbb{R}^n)$.

Then, from [11, Lemma 2.4] we obtain $u'(t)\Phi'(|u'(t)|)/|u'(t)| \in L^\Psi([0,T], \mathbb{R}^n) \hookrightarrow L^1([0,T], \mathbb{R}^n)$ and from condition (A) and $u \in L^\infty$ we have $\nabla F(t, u(t)) \in L^1([0,T], \mathbb{R}^n)$. Consequently, from [18, p. 6] we obtain that the differential equations in (1) is verified and $u'(0)\Phi'(|u'(0)|)/|u'(0)| = u'(T)\Phi'(|u'(T)|)/|u'(T)|$ holds. Thus $u'(0) = u'(T)$. □

4 Examples

The employment of N-functions instead of power functions in inequalities like (24) will allow us to extend some results of [16] and [17], no only to the Φ-laplacian operator, but even in the case of p-laplacian operator we get bounds that may be more sharp that those in [16, 17]. More precisely, in [17, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions for the p-laplacian operator under la assumption

$$|\nabla F| \leqslant b_1(t)|x|^\alpha + b_2(t).$$

for any $\alpha \in (0, p-1)$

Assuming $\|b_1\|_{L^1}$ small enough, in [19, 17] coercivity was obtained even for the limit value $\mu = p$ in inequality (24).

OJO que μ no aparece en (24)!!!!. Quizás debería decir $\Phi_0'(x) = x^p$. O, mecionarse la ecuación anterior donde aparece $\alpha < p$, no μ.

This result leans on the fact that

$$\|u\|_{L^\Phi}^{\alpha_\Phi} = O\left(\int_0^T \Phi(|u|)\,dt\right) \quad \text{for } \|u\|_{L^\Phi} \to \infty, \tag{34}$$

when $\Phi(u) = |u|^p$. Nevertheless, it is no longer the case for any N-function Φ as the following example shows.

In this section, from now on we will suppose that

$$\Phi(u) = \begin{cases} \frac{p-1}{p} u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function Φ.

Theorem 4.1. *If $p \geqslant \frac{1+\sqrt{2}}{2}$, then Φ is an N-function.*

Proof. We have

$$\varphi(u) = \Phi'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that Φ' is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that φ_1 is increasing on $[0, e]$ and φ_2 is increasing on $[e, \infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p > 1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an icreasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Besides $\varphi_2(u) \to \infty$ and $\varphi_1(u) \to 0$ as $u \to \infty$ and $u \to 0$ respectively, provided that $p > 1$. Hence, Φ is an N-function. $\square$

Theorem 4.2. *For every $\varepsilon > 0$, there exists a positive constant $C = C(p, \varepsilon)$ such that*

$$C^{-1} t^{p-\varepsilon} \Phi(u) \leqslant \Phi(tu) \leqslant C t^p \Phi(u) \quad t \geqslant 1, u > 0, \tag{35}$$

Proof. If $u \leqslant tu \leqslant e$, then $\Phi(tu) = t^p \Phi(u)$ and (35) holds with $C = 1$.

If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geqslant 1$, we have $\Phi(tu) \leqslant t^p u^p = \frac{p}{p-1} t^p \Phi(u)$. Thus, the second inequality of (35) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$, then $f((tu)^p) \geqslant f(e^p) = e^p/p$. Now,

$$\begin{aligned} \Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geqslant \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} \\ &\geqslant \frac{p-1}{p} \frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p. \end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t+1}$ on the interval $[1,+\infty)$ then

$$\Phi(tu) \geqslant \frac{p-1}{p}\varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,$$

which is the first inequality of (35) with $C = \frac{p}{p-1}\varepsilon^{-1}e^{-1+\varepsilon}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v}, \tag{36}$$

where $v := u^p$ and $v \geqslant e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on (α,∞) and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p,\infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every $v \geqslant e^p$. In this way, from (36), we have

$$\Phi(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and the second inequality of (35) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s+A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, the function f attains a minimum on $[1,\infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$, f is increasing on $[1,\infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geqslant \varepsilon$ in any case, therefore

$$\Phi(tu) \geqslant \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geqslant \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi(u).$$

Therefore, (35) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. $\square$

Remark 2. The inequality

$$\Phi(tu) \geqslant Ct^p\Phi(u)$$

is false for every C because for every $u \geqslant e$ we have

$$\lim_{t\to\infty}\frac{\Phi(tu)}{t^p\Phi(u)} = 0$$

Theorem 4.3. $\alpha_\Phi = \beta_\Phi = p$

Proof. From (**??**) and (35), we get

$$\beta_\Phi = \lim_{t\to\infty} \frac{\log\left[\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} \leqslant \lim_{t\to\infty} \frac{\log C + p\log t}{\log t} = p.$$

On the other hand, employing (**??**) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t\to 0^+} \frac{\log\left[\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} = \lim_{s\to\infty} \frac{\log\left[\sup\limits_{v>0} \frac{\Phi(v)}{\Phi(sv)}\right]^{-1}}{\log s} = \lim_{s\to\infty} \frac{\log\left[\inf\limits_{v>0} \frac{\Phi(sv)}{\Phi(v)}\right]}{\log s}$$

where $v := tu$ and $s := \frac{1}{t}$. Then, using (35), for every $\varepsilon > 0$ we have

$$\alpha_\Phi = \lim_{s\to\infty} \frac{\log\left[\inf\limits_{v>0} \frac{\Phi(sv)}{\Phi(v)}\right]}{\log s} \geqslant \lim_{s\to\infty} \frac{\log C + (p-\varepsilon)\log s}{\log s} \geqslant p - \varepsilon,$$

therefore $\alpha_\Phi \geqslant p$.

Finally, as $\alpha_\Phi \leqslant \beta_\Phi \leqslant p$, we get $\alpha_\Phi = \beta_\Phi = p$. □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\,dx \geqslant C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^{p}$$

is false.

In fact, if we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\,dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geqslant C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geqslant Ct^p$ would also be true; however, this last inequality is false.

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