Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt}\nabla_y \mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t\in(0,T), \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases} \tag{$\boldsymbol{P}$}$$

where the function $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$ (called the *Lagrange function* or *lagrangian*) satisfies that it is measurable in t for each $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and continuously differentiable in (x,y) for almost every $t\in[0,T]$. The unknown function $u:[0,T]\to\mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the article [18] for definitions and main results on anisotropic Orlicz spaces, see also [3]. These spaces allow us to unify and extend previous results on existence of solutions for systems like ($\boldsymbol{P}$).

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Through this article we say that a function $\Phi : \mathbb{R}^d \to [0, +\infty)$ belongs to N_∞ class if Φ is convex, $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$ and $\Phi(-y) = \Phi(y)$, and

$$\lim_{|y|\to\infty} \frac{\Phi(y)}{|y|} = +\infty. \tag{1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From [4, Cor. 2.35] an N_∞ function is continuous.

Associated to Φ we have the *complementary function* Ψ which is defined in $\xi \in \mathbb{R}^d$ as

$$\Psi(\xi) = \sup_{y\in\mathbb{R}^d} y \cdot \xi - \Phi(y), \tag{2}$$

then, from the continuity of Φ and (1), we have that $\Psi : \mathbb{R}^d \to [0, \infty)$. It is easy to see that Ψ is a convex function such that $\Psi(0) = 0$, $\Psi(-\xi) = \Psi(\xi)$ [9, Ch. 2]. Moreover, Ψ satisfies (1) (see [18, Thm. 2.2]). i.e. Ψ is an N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\Phi_p(y) := |y|^p/p$, for $1 < p < \infty$. In this case $\Psi(\xi) = |\xi|^q/q$, $q = p/(p-1)$.

Example 1.2. If $\Phi : \mathbb{R} \to [0, +\infty)$ is a N_∞ function on $\mathbb{R}$ then $\overline{\Phi}(y) = \Phi(|y|)$ is a N_∞ function on $\mathbb{R}^d$. In this example, as in the previous one, the function Φ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\Phi(y)$ depends on the direction of y. For example, if $1 < p_1, p_2 < \infty$, we define $\Phi_{p_1,p_2} : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ by

$$\Phi_{p_1,p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then Φ_{p_1,p_2} is an N_∞ function. In this case the complementary function is Φ_{q_1,q_2} with $q_i = p_i/(p_i - 1)$.

More generally, if $\Phi_k : \mathbb{R}^d \to [0, +\infty)$, $k = 1, \ldots, n$, are N_∞ functions, then $\Phi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to [0, +\infty)$ defined by $\Phi(y_1, \ldots, y_n) = \Phi_1(y_1) + \cdots + \Phi_n(y_n)$ is an N_∞ function. These functions are truly anisotropic, i.e. $|x| = |y|$ does not imply that $\Phi(x) = \Phi(y)$.

Example 1.4. If $\Phi : \mathbb{R} \to [0, +\infty)$ is an N_∞ function and $O \in GL(d, \mathbb{R})$, then $\Phi(y) = \Phi(Oy)$ is an N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by power functions if it does not satisfy the Δ_2 condition (see xxxxx). For example $\Phi : \mathbb{R}^d :\to [0, +\infty)$ defined by $\Phi(y) = \exp(|y|) - 1$ is an N_∞ function.

The appearance of Orlicz Spaces in this paper is due to the fact that we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi(\nabla_y \mathcal{L}) \leqslant a(x) \left\{ b(t) + \Phi\left(\frac{y}{\lambda}\right) \right\}, \tag{$\boldsymbol{S}$}$$

for a.e. $t \in [0, T]$, where $a \in C\left(\mathbb{R}^d, [0, +\infty)\right)$, $b \in L^1\left([0, T], [0, +\infty)\right)$.

Our condition ($\boldsymbol{S}$) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition ($\boldsymbol{S}$) is equivalent to the structure condition in [9, Th. 1.4]. If Φ is a radial N_∞ function such that Ψ satisfies that Δ_2 function then ($\boldsymbol{S}$) is essentially equivalent????? to conditions [1, Eq. (2)-(4)] (see xxxx mas abajo). If Φ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ then inequality ($\boldsymbol{S}$) is related to estructure conditions like [24, Lemma 3.1, Eq. (3.1)]. As can be seen, condition ($\boldsymbol{S}$) is a more compact expression than [24, Lemma 3.1, Eq. (3.1)] and moreover weaker, because ($\boldsymbol{S}$) does not imply a control of $|D_{y_1} L|$ independent of y_2. We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi,F}(t,x,y) \coloneqq \Phi(y) + F(t,x). \tag{3}$$

Here the function $F(t,x)$, which is often referred to a potential, be differentiable with respect to x for a.e. $t \in [0,T]$. Moreover F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla_x F(t,x)| \leqslant a(x)b(t). \tag{4}$$

where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$ and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi,F}$ satisfies condition ($\boldsymbol{S}$). In order to prove this, the only non trivial fact that we should establish is that $\Psi(\nabla_y \mathcal{L}) \leqslant a(x)\left\{b(t) + \Phi\left(y/\lambda\right)\right\}$. But, from inequality xxxx below, $\Psi(\nabla_y \mathcal{L}) = \Psi\left(\nabla\Phi(y)\right) \leqslant \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi,F}$ leads to the system

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{$\boldsymbol{P_\Phi}$}$$

Problem ($\boldsymbol{P_\Phi}$) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [9] deals mainly with problem ($\boldsymbol{P}$) for the lagrangian $\mathcal{L}_{\Phi,F}$ with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [9] were extended and improved in several articles, see [22, 20, 25, 21, 28] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [24, 23], among other papers. In this case, ($\boldsymbol{P_\Phi}$) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{$\boldsymbol{P_p}$}$$

If Φ is as in Example 1.3 and $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, then the equations ($\boldsymbol{P_\Phi}$) become

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) & \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} , \tag{$\boldsymbol{P_{p_1,p_2}}$}$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as (p_1, p_2)-Laplacian system, see [27, 13, 26, 10, 11, 12, 7].

In conclusion, the problem ($\boldsymbol{P}$) subject to conditions ($\boldsymbol{S}$) contains several problems that have been considered by many authors in the past.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^n \to [0, +\infty)$. References for these topics are [5, 18, 19].

We say that $\Phi : \mathbb{R}^d \to [0, +\infty)$ satisfies the *Δ_2^∞-condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant K\Phi(x), \tag{5}$$

for every $|x| \geqslant M$.

We denote by $\mathcal{M} := \mathcal{M}\left([0, T], \mathbb{R}^d\right)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi\left([0, T], \mathbb{R}^d\right)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{6}$$

The *Orlicz space* $L^\Phi = L^\Phi\left([0, T], \mathbb{R}^d\right)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{7}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi\left([0, T], \mathbb{R}^d\right)$ is defined as the closure in L^Φ of the subspace $L^\infty\left([0, T], \mathbb{R}^d\right)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [18, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [18, Thm. 5.2]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [18, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^\Psi}. \tag{8}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

We consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{9}$$

for any positive r. This relation is a trivial generalization of [18, Thm. 5.6]. If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow \left[L^\Psi\right]^*$, where a function $v \in L^\Phi$ is associated to $\xi_v \in \left[L^\Psi\right]^*$ being

$$\xi_v(u) = \langle \xi_v, u\rangle = \int_0^T v \cdot u \, dt, \tag{10}$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [18].

Proposition 2.1. *If Ψ satisfies the Δ_2^∞-condition then $L^\Phi\left([0,T], \mathbb{R}^d\right) = \left[L^\Psi\left([0,T], \mathbb{R}^d\right)\right]^*$.*

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi\left([0,T], \mathbb{R}^d\right) := \left\{u | u \in AC\left([0,T], \mathbb{R}^d\right) \text{ and } u' \in L^\Phi\left([0,T], \mathbb{R}^d\right)\right\},$$

where $AC\left([0,T], \mathbb{R}^d\right)$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $[0,T]$. The space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{11}$$

We introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E_T^\Phi &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{12}$$

In order to find a modulus of continuity for functions in W^1L^Φ, and from there, to obtain compact embedding of W^1L^Φ, we define the function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, |x| = s\right\}, \tag{13}$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.2. *The function A_Φ has the following properties:*

1. *A_Φ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*

3. $A_\Phi(|x|)$ *is the* greatest radial minorant *of* $\Phi(x)$,

4. Φ *is* N_∞ *if and only if* $\lim_{s\to+\infty} A_\Phi(s)/s = +\infty$.

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [4]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

Example 2.1. We compute A_Φ for the function $\Phi = \Phi_{p_1,p_2}$ given in Example (1.3). We apply the method of Lagrange multipliers (see [8, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1,p_2}(y_1,y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2}y_1 + \lambda y_1 &= 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 &= 0 \\ |y_1|^2 + |y_2|^2 &= r^2 \end{cases} \tag{14}$$

These equations are solved, among others, by the following two sets of critical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2.

We deal with $p_1 \leqslant 2$ and $p_2 \leqslant 2$ being one of them (suppose p_2) different from 2. The remaining cases can be treated with similar techniques.

If (y_1,y_2) solve (14) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1,y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi,\eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let L be the Lagrangian associated to the constrained problem: $L(y_1,y_2,\lambda) = \Phi(y_1,y_2) + \lambda H(y_1,y_2)$ being $H = 0$ the constraint. We must analyze the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2 H$ on the subspace M. By elementary computations we have for $(\xi,\eta) \in M$

$$(\xi,\eta)^t\mathcal{H}(\xi,\eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1-2) + (p_2-2)|y_2|^{-2}],$$

on the subspace M. We note that $(-y_2,y_1) \in M$ and $(-y_2,y_1)^t\mathcal{H}(-y_2,y_1) < 0$. Then, by second order necessary conditions [8, p.333], at (y_1,y_2) there cannot be a minimum. Therefore, the minima occur at $y_1 = 0$ or $y_2 = 0$, then

$$A_\Phi(x,y) = \min\{r^{p_1}/p_1, r^{p_2}/p_2\}.$$

More generally, it holds that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leqslant A_\Phi \leqslant K_2 \min\{r^{p_1}, r^{p_2}\}$$

with $K_1, K_2 > 0$, for every $1 < p_1, p_2 < \infty$.

As it is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (13)*. Then*

1. *For every $s, t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s-t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty\left([0,T], \mathbb{R}^d\right)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If Φ is N_∞ then the space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is compactly embedded in the space of continuous functions $C\left([0,T], \mathbb{R}^d\right)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned}\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}\,dr\right) \\ &\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leqslant \frac{1}{|s-t|}.\end{aligned}$$

By Proposition 2.2(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned}|u(t)-\overline{u}| &= \left|\frac{1}{T}\int_0^T u(t)-u(s)\,ds\right|\\ &\leqslant \frac{1}{T}\int_0^T |u(t)-u(s)|\,ds\\ &\leqslant \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right)\end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right)\leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right)ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.2(3)

$$|\overline{u}|\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$. Since Φ is N_∞, from Proposition 2.2(4) we obtain $sA_\Phi^{-1}(1/s)\to 0$ when $s\to 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (**??**) implies that u_n is bounded in $C\left([0,T],\mathbb{R}^d\right)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u\in C\left([0,T],\mathbb{R}^d\right)$ with $u_{n_k}\to u$ in $C\left([0,T],\mathbb{R}^d\right)$. □

QUIZAS LO QUE VIENE TENDRIA QUE IR CON LAS OTRAS DESIGUALDADES... O MODIFICAR EL LEMA ANTERIOR....

We get an anisotropic version of Sobolev-Wirtinger inequality, as follows.

Proposition 2.4. *If $u\in W^1L^\Phi$, then*

$$\Phi(\tilde{u}(t))\leqslant \int_0^T \Phi(\max\{1,T\}u'(r))\,dr. \tag{15}$$

Proof. Writing $\tilde{u}(t) = u(t) - \overline{u}$ and applying Jensen's inequality, we get

$$\begin{aligned}\Phi(\tilde{u}(t)) &= \Phi\left(\frac{1}{T}\int_0^T u(t) - u(s)\,ds\right)\\ &\leqslant \frac{1}{T}\int_0^T \Phi(u(t)-u(s))\,ds \leqslant \frac{1}{T}\int_0^T \Phi\left(\int_s^t |t-s|u'(r)\frac{dr}{|t-s|}\right)ds\\ &\leqslant \frac{1}{T}\int_0^T \frac{1}{|t-s|}\int_s^t \Phi\left(|t-s|u'(r)\right)dr\,ds\end{aligned}$$

As Φ is convex, we have

$$\frac{1}{|t-s|}\Phi(|t-s|u'(r)) \leqslant \begin{cases} \Phi(u'(r)) & if \quad |t-s| \leqslant 1\\ \Phi(Tu'(r)) & if \quad 1 < |t-s| < T\end{cases}$$

Then $\frac{1}{|t-s|}\Phi\left(|t-s|u'(r)\right) \leqslant \Phi(\max\{1,T\}u'(r))$ and (15) follows. □

Lemma 2.5. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi,1)$ converging to $u \in \Pi(E^\Phi,1)$ in the L^Φ-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0,T],\mathbb{R})$ such that $u_{n_k} \to u$ a.e. and $\Phi(u_{n_k}) \leqslant h$ a.e.*

Proof. Since $d(u,E^\Phi) < 1$ and u_n converges to u, there exists $u_0 \in E^\Phi$, a subsequence of u_n (again denoted u_n) and $0 < r < 1$ such that $d(u_n,u_0) < r$. Let $\lambda_0 \in (r,1)$. By extracting more subsequences, if necessary, we can assume that $u_n \to u$ a.e. and

$$\lambda_n := \|u_{n+1}-u_n\|_{L^\Phi} < \frac{1-\lambda_0}{2^n}, \quad \text{for } n \geqslant 1.$$

We can assume $\lambda_n > 0$ for every $n = 0,\ldots$.

Let $\lambda := 1 - \sum_{n=0}^\infty \lambda_n$ and define $h : [0,T] \to \mathbb{R}$ by

$$h(x) = \lambda\Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^\infty \lambda_n\Phi\left(\frac{u_{n+1}-u_n}{\lambda_n}\right). \tag{16}$$

Note that $\sum_{n=0}^\infty \lambda_n + \lambda = 1$, therefore for any $n = 1,\ldots$

$$\begin{aligned}\Phi(u_n) &= \Phi\left(\lambda\frac{u_0}{\lambda} + \sum_{j=0}^{n-1}\lambda_j\frac{u_{j+1}-u_j}{\lambda_j}\right)\\ &\leqslant \lambda\Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1}\lambda_j\Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right) \leqslant h\end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0,T],\mathbb{R})$. On the other hand $\|u_{n+1}-u_n\|_{L^\Phi} \leqslant \lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right)dt \leqslant 1.$$

Then $h \in L^1([0,T],\mathbb{R})$.

□

DEMOSTRACION ALTERNATIVA PARA EL LEMA y con λ

Lemma 2.6. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi,\lambda)$ converging to $u \in \Pi(E^\Phi,\lambda)$ in the L^Φ-norm. Then, there exist a subsequence $\Phi(\frac{u_{n_k}}{\lambda})$ and a function $h \in L^1([0,T],\mathbb{R})$ such that $\Phi(\frac{u_{n_k}}{\lambda}) \to \Phi(\frac{u}{\lambda})$ a.e. and $\Phi(\frac{u_{n_k}}{\lambda}) \leqslant h$ a.e.*

Proof. As $u \in \Pi(E^\Phi,\lambda)$, we consider $\Lambda \in (0,\lambda)$. In this way, $d(u,E^\Phi) < \Lambda < \lambda$ and, taking into account (9), $\Phi(\frac{u}{\lambda}) \in L^1([0,T],\mathbb{R})$.

Applying [3, Lemma 3.1] with $x+y = \frac{u_n}{\lambda}$, $x = \frac{u}{\lambda}$, $k = \frac{\lambda}{\Lambda}$, $0<\epsilon<\frac{\Lambda}{\lambda}$ and $C_\epsilon = \frac{\Lambda}{\epsilon(\lambda-\Lambda)}$, we have

$$\begin{aligned}&\int_0^T \left|\Phi\left(\frac{u_n}{\lambda}\right) - \Phi\left(\frac{u}{\lambda}\right)\right| dt\\ &\leqslant \epsilon \int_0^T \left|\Phi\left(\frac{u}{\lambda}\right) - \frac{\lambda}{\Lambda}\Phi\left(\frac{u}{\lambda}\right)\right| dt + 2\int_0^T \Phi\left(\frac{\Lambda}{\epsilon(\lambda-\Lambda)}\frac{u_n-u}{\lambda}\right) dt.\end{aligned} \tag{17}$$

Let $\eta>0$. Since $\Phi(\frac{u}{\lambda}) \in L^1([0,T,\mathbb{R}])$, we can choose ϵ such that

$$\epsilon \int_0^T \left|\Phi\left(\frac{u}{\lambda}\right) - \frac{\lambda}{\Lambda}\Phi\left(\frac{u}{\lambda}\right)\right| dt < \eta. \tag{18}$$

From the fact that $u_n \to u$ in the L^Φ-norm, there exists n_0 such that $\|u_n-u\|_{L^\Phi} < \epsilon^2(\lambda-\Lambda)$ for every $n \geqslant n_0$. Then, by the convexity of Φ and the definition of Orlicz norm, we get

$$\int_0^T \Phi\left(\frac{\Lambda}{\epsilon(\lambda-\Lambda)}\frac{u_n-u}{\lambda}\right) \leqslant \epsilon\frac{\Lambda}{\lambda}\int_0^T \Phi\left(\frac{u_n-u}{\epsilon^2(\lambda-\Lambda)}\right) < \epsilon \tag{19}$$

Thus, from (17), (18) and (19), we obtain that $\Phi(\frac{u_n}{\lambda})$ converges to $\Phi(\frac{u}{\lambda})$ in the L^1-norm. Now, [2, Thm. 4.9] implies that there exist a subsequence $\Phi(\frac{u_{n_k}}{\lambda})$ and a function $h \in L^1([0,T],\mathbb{R})$ such that $\Phi(\frac{u_{n_k}}{\lambda}) \to \Phi(\frac{u}{\lambda})$ a.e. and $\Phi(\frac{u_{n_k}}{\lambda}) \leqslant h$ a.e. □

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that $f:[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$is a *Carathéodory function*, i.e.

(C) f is measurable with respect to $t \in [0,T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

Definition 3.1. *For $f:[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u:[0,T]\to\mathbb{R}^d$ by*

$$\boldsymbol{f}u(t) = f(t,u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [6] for scalar functions and [16, 15, 14] for the generalization to $\mathbb{R}^d$-valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

Theorem 3.2. *We assume that f satisfies condition* ((C)) *and that $\Phi_1, \Phi_2 : \mathbb{R}^d \to [0, +\infty)$ are anisotropic Young functions. Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps measurable function into measurable functions*

2. Extensibility. *If the operator $\boldsymbol{f}$ acts from the ball $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} | \|u\|_{L^{\Phi_1}} < r\}$ into the space L^{Φ_2} or the space E^{Φ_2} then $\boldsymbol{f}$ can be extended from $\Pi(E^{\Phi_1}, r)$ into space L^{Φ_2} or E^{Φ_2}, respectively.*

3. Continuity. *If the operator $\boldsymbol{f}$ acts from $\Pi(E^{\Phi_1}, r)$ into space E^{Φ_2}, then $\boldsymbol{f}$ is continuous.*

Given a continuous function $a \in C(\mathbb{R}^n, \mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

Lemma 3.3. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

Proof. Let $A \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\| \leqslant s, x \in \mathbb{R}^d} |a(x)|$. If $u \in W^1L_d^\Phi$ then, by Sobolev's inequality, for a.e. $t \in [0,T]$

$$a(u(t)) \leqslant \alpha(\|u\|_{L^\infty}) \leqslant \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\right) =: A(\|u\|_{W^1L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t))\, dt. \tag{20}$$

Theorem 3.4. *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying ($\boldsymbol{S}$). Then the following statements hold:*

1. *The action integral given by* (20) *is finitely defined on $\mathcal{E}^\Phi := W^1L^\Phi \cap \{u | \dot{u} \in \Pi(E^\Phi, 1)\}$.*

2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$. Moreover, I' is given by the following expression*
$$\langle I'(u), v\rangle = \int_0^T \left\{D_x\mathcal{L}\big(t,u,\dot{u}\big)\cdot v + \nabla_y\mathcal{L}\big(t,u,\dot{u}\big)\cdot \dot{v}\right\}\, dt. \tag{21}$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^\Phi$. As
$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \tag{22}$$
and (9), then $\Phi(\dot{u}(t)) \in L^1$. Now,
$$|\mathcal{L}(\cdot,u,\dot{u})| + |\nabla_x\mathcal{L}(\cdot,u,\dot{u})| + \Psi(\nabla_y\mathcal{L}(\cdot,u,\dot{u})) \leqslant A(\|u\|_{W^1L^\Phi})(b+\Phi(\dot{u})) \in L^1, \tag{23}$$
by ($\boldsymbol{S}$) and Lemma 3.3. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto \nabla_x\mathcal{L}(t,u,\dot{u})$ is continuous from $\mathcal{E}^\Phi$ into $L^1([0,T])$ with the strong topology on both sets.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^\Phi$ and let $u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in W^1L^Φ. By (Sobolev's inequality), we have
$$|u_n(t) - u(t)| \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u_n - u\|_{L^\Phi}$$
then $u_n \to u$ uniformly. As $\dot{u}_n \to \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.5, there exist a subsequence of $\dot{u}_{n_k}$ (again denoted $\dot{u}_{n_k}$) and a function $h \in L^1([0,T],\mathbb{R})$ such that $\dot{u}_{n_k} \to \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leqslant h$ a.e.

Since u_{n_k}, $k = 1,2,\ldots$, is a strong convergent sequence in W^1L^Φ, it is a bounded sequence in W^1L^Φ. According to item (3) of Lemma 2.3, there exists $M > 0$ such that $\|\boldsymbol{a}(u_{n_k})\|_{L^\infty} \leqslant M$, $k = 1,2,\ldots$. From the previous facts and (23), we get
$$|\nabla_x\mathcal{L}(\cdot,u_{n_k},\dot{u}_{n_k})| \leqslant a(|u_{n_k}|)(b+\Phi(\dot{u}_{n_k})) \leqslant M(b+h) \in L^1.$$
On the other hand, by the continuous differentiability of $\mathcal{L}$, we have
$$\nabla_x\mathcal{L}(t,u_{n_k}(t),\dot{u}_{n_k}(t)) \to \nabla_x\mathcal{L}(t,u(t),\dot{u}(t)) \quad \text{for a.e. } t \in [0,T].$$
Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto \nabla_y\mathcal{L}(t,u,\dot{u})$ is continuous from $\mathcal{E}^\Phi$ with the strong topology into $\left[L^\Phi\right]^*$ with the weak topology.*

Let $u \in \mathcal{E}^\Phi$. From (23) it follows that
$$\nabla_y\mathcal{L}(\cdot,u,\dot{u}) \in C^\Psi. \tag{24}$$

Note that (23), (24) and the imbeddings $W^1L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$ imply that the second member of (21) defines an element of $\left[W^1L^\Phi\right]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in the norm of W^1L^Φ. We must prove that $\nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n) \overset{w^*}{\rightharpoonup} \nabla_y \mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle \nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n), v\rangle - \langle \nabla_y \mathcal{L}(\cdot, u, \dot{u}), v\rangle| \geqslant \epsilon. \tag{25}$$

We have $u_n \to u$ in L^Φ and $\dot{u}_n \to \dot{u}$ in L^Φ. By Lemma 2.5, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T], \mathbb{R})$ such that $u_n \to u$ uniformly, $\dot{u}_n \to \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leqslant h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.3 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (23) with u_n instead of u, we have

$$\Psi(\nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n)) \leqslant M(b + h) \in L^1. \tag{26}$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (26), we have

$$\begin{aligned}
&\lambda \nabla_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\
&\leqslant \lambda \left[\Psi(\nabla_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right)\right] \\
&\leqslant \lambda M(b + h) + \lambda \Phi\left(\frac{v}{\lambda}\right) \in L^1
\end{aligned} \tag{27}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T \nabla_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v\, dt \to \int_0^T \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot v\, dt \tag{28}$$

which contradicts the inequality (25). This completes the proof of step 2.

Step 3. We will prove (21). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1L^\Phi$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leqslant s_0 := \min\{\left(1 - d(\dot{u}, E^\Phi)\right)/\|v\|_{W^1L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d\left(\dot{u} + s\dot{v}, E^\Phi\right) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.4 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$.

We also have $\|u + sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 3.3, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leqslant M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leqslant s_0\lambda^{-1}$, from the convexity and the parity of Φ, we get

$$\begin{aligned}
&\Phi(\dot{u} + s\dot{v}) = \Phi\left((1 - s_0)\frac{\dot{u}}{1 - s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right) \\
&\leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1
\end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1 - s_0}, E^\Phi\right) = \frac{1}{1 - s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1-s_0} \in C^\Phi$.

Now, applying (23), (27), the fact that $v \in L^\infty$ and $\dot{v} \in L^\Phi$, we get

$$
\begin{aligned}
|D_s H(s,t)| &= \left| \nabla_x \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + \lambda \nabla_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \frac{\dot{v}}{\lambda} \right| \\
&\leqslant M\left[b(t) + \Phi(\dot{u} + s\dot{v})\right]|v| \\
&\quad + \lambda \left[\Psi(\nabla_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right)\right] \\
&\leqslant M\left\{\left[b(t) + \Phi(\dot{u} + s\dot{v})\right]|v|\right\} + \lambda M[b(t) + \Phi(\dot{u} + s\dot{v})] + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \\
&= M[b(t) + \Phi(\dot{u} + s\dot{v})](|v| + \lambda) + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1.
\end{aligned} \tag{29}
$$

Consequently, I has a directional derivative and

$$\langle I'(u), v\rangle = \frac{d}{ds} I(u + sv)\Big|_{s=0} = \int_0^T \left\{\nabla_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v}\right\} \, dt.$$

Moreover, from the previous formula, (23), (24), and Lemma 2.3, we obtain

$$|\langle I'(u), v\rangle| \leqslant \|\nabla_x \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|\nabla_y \mathcal{L}\|_{L^\Psi} \|\dot{v}\|_{L^\Phi} \leqslant C\|v\|_{W^1 L^\Phi}$$

with a appropriate constant C.

This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : \mathcal{E}^\Phi \to \left[W^1 L^\Phi_d\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \to u$ in the norm of $W^1 L^\Phi$ and $v \in W^1 L^\Phi$, then

$$
\begin{aligned}
\langle I'(u_n), v\rangle &= \int_0^T \left\{\nabla_x \mathcal{L}\left(t, u_n, \dot{u}_n\right) \cdot v + \nabla_y \mathcal{L}\left(t, u_n, \dot{u}_n\right) \cdot \dot{v}\right\} \, dt \\
&\to \int_0^T \left\{\nabla_x \mathcal{L}\left(t, u, \dot{u}\right) \cdot v + \nabla_y \mathcal{L}\left(t, u, \dot{u}\right) \cdot \dot{v}\right\} \, dt \\
&= \langle I'(u), v\rangle .
\end{aligned}
$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$ are norm continuous from $\mathcal{E}^\Phi$ into L^1 and L^Ψ, respectively.

The continuity of the first map has already been proved in step 1.

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1 L^\Phi} \to 0$.

Applying Lemma 2.5 to $\dot{u}_n$, there exists a subsequence (denoted $\dot{u}_n$ for simplicity) such that $\dot{u}_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot{u}_n) \leqslant h$ and $\dot{u}_n \to \dot{u}$ a.e.

Then, by (27) we have $\Psi(v_n) \leqslant m(t) \in L^1$ being $v_n := \nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n)$ and $m(t) := M(b + h)$. In addition, from the continuous differentiability of $\mathcal{L}$, we have that $v_n \to v$ a.e. where $\nabla_y \mathcal{L}(\cdot, u, \dot{u})$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(\lambda x) \leqslant c(|\lambda|)\Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})\Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \to 0$ a.e. as $n \to \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})K\Psi(v_n) + \Psi(v)) \leqslant c(|\lambda|^{-1})K[m(t) + \Psi(v)]) \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda})\, dt \to 0$ for every $\lambda > 0$. Thus, $v_n \to v$ in L^Ψ.

The continuity of I' follows from the continuity of $\nabla_x \mathcal{L}$ and $\nabla_y \mathcal{L}$ using the formula (21). $\square$

ALGO DE ESTO ES NECESARIO SI VAMOS A USAR LAS RELACIONES $\prec$ y $\lll$, QUIZAS EN LOS PRELIMINARES

There exist several orders and equivalence relations between N-functions (see [17, Sec. 2.2]). Following [17, Def. 1, pp. 15-16] we say that the N-function Φ_2 is *stronger* than the N-function Φ_1, in symbols $\Phi_1 \prec \Phi_2$, if there exist $a > 0$ and $x_0 \geqslant 0$ such that

$$\Phi_1(x) \leqslant \Phi_2(ax), \quad x \geqslant x_0. \tag{30}$$

The N-functions Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) when $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$. We say that Φ_2 is *essentially stronger* than Φ_1 ($\Phi_1 \lll \Phi_2$) if and only if for every $a > 0$ there exists $x_0 = x_0(a) \geqslant 0$ such that (30) holds. Finally, we say that Φ_2 is *completely stronger* than Φ_1 ($\Phi_1 \prec\!\!\!\prec \Phi_2$) if and only if for every $a > 0$ there exist $K = K(a) > 0$ and $x_0 = x_0(a) \geqslant 0$ such that

We assume that there exist an N_∞ function Φ_0 such that $\Phi_0 \lll \Phi$, a function $a \in L^1([0,T], \mathbb{R})$ such that $a \geqslant 1$ and a constant $M > 0$ such that

$$\Psi_0(a^{-1}(t)\nabla_x F) \leqslant \Phi_0(x),\ |x| \geqslant M \tag{Sub}$$

Proposition 3.5. *Let $1 < p < \infty$ and suppose that F satisfies (A). Then,*

$$|\nabla_x F| \leqslant a(t)|x|^{p_0'-1} + b(t),\ for\ p_0' < p, \tag{31}$$

if and only if F satisfies (Sub) *with $\Phi_0(x) = |x|^{p'}$ for all $p' \in (1, p)$.*

Proof. Suppose that (Sub) holds with $\Phi_0(x) = |x|^{p'}$ for $1 < p' < p - 1$. Then,

$$(a^{-1}(t)|\nabla_x F|)^{q_0'} \leqslant |x|^{p_0'},\ for\ |x| \geqslant M$$

As F sastisfies (A), we have $|\nabla_x F| \leqslant b(t) \in L^1$ for $|x| \leqslant M$.

PONEMOS SOLO b O Kb?? No confunde?

Then,

$$|\nabla_x F| \leqslant a|x|^{\frac{p_0'}{q_0'}} + b(t) \leqslant a|x|^{p_0'-1} + b(t)$$

Now, we assume that F satisfies (31). If $|x| \geqslant 1$, we have

$$|\nabla_x F| \leqslant a(t)|x|^{p_0'-1} + b(t) \leqslant (a(t) + b(t))|x|^{\frac{p_0'}{q_0'}}$$

which is condition (Sub) with $a(t) + b(t)$ instead of $a(t)$. $\square$

Theorem 3.6. *Let Φ be an N-function whose complementary function Ψ satisfies the Δ_2^∞-condition.*

Let F be a potential that satisfies (C), (A) and the following conditions:

1. (Sub) *for some* N_∞ *function* Φ_0 *such that* $\Phi_0 \ll \Phi$

2.
$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Psi_0(2|x|)} = +\infty, \tag{32}$$
for some N*-function* Ψ_0 *complementary function of* Φ_0.

EN LO QUE SIGUE, PONEMOS LA NUEVA CONDICION DE ESCTRUCTURA O DEFINIMOS UN CONJUNTO DONDE SE VERIFICAN ESAS CONDICIONES???

Now, if the lagrangian $\mathcal{L}(t,x,y)$ *is strictly convex with respect to* $y \in \mathbb{R}^d$, $\mathcal{L} \in \mathfrak{A}(a,b,c,\lambda,f,\Phi)$, $D_y\mathcal{L}(0,x,y) = D_y\mathcal{L}(T,x,y)$ *and* (**??**) *holds, then the problem* ($\boldsymbol{P}$) *has at least a solution which minimizes the action integral* I *on* $W^1E^\Phi_T$.

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Proof. By the decomposition $u = \overline{u} + \tilde{u}$, Young's inequality, (Sub), the convexity of Φ_0 and (15), we obtain

$$\begin{aligned}
&\left|\int_0^T F(t,u) - F(t,\overline{u})\,dt\right| \leqslant \int_0^T \int_0^1 |\nabla_x F(t,\overline{u}+s\tilde{u})\tilde{u}|\,ds\,dt\\
&= \int_0^T a(t)\int_0^1 a^{-1}(t)|\nabla_x F(t,\overline{u}+s\tilde{u})\tilde{u}|\,ds\,dt\\
&\leqslant \int_0^T a(t)\int_0^1 \Psi_0(a^{-1}(t)\nabla_x F(t,\overline{u}+s\tilde{u})) + \Phi_0(\tilde{u}(t))\,ds\,dt\\
&\leqslant \int_0^T a(t)\left[\int_0^1 \Phi_0(\overline{u}+s\tilde{u})\,ds + \Phi_0(\tilde{u}(t))\right]dt\\
&\leqslant \int_0^T a(t)(\Phi_0(2\overline{u}) + 2\Phi_0(2\tilde{u}))\,dt\\
&\leqslant C_1\Phi_0(2\overline{u}) + \int_0^T a(t)2\Phi_0(2\tilde{u})\,dt\\
&\leqslant C_1\Phi_0(2\overline{u}) + C_2\int_0^T \Phi_0(C_T u'(t))\,dt
\end{aligned} \tag{33}$$

Then, by (33) we get

$$\begin{aligned}
&\int_0^T \Phi(u') + F(t,u)\,dt = \int_0^T \{\Phi(u') + [F(t,u) - F(t,\overline{u})] + F(t,\overline{u})\}\,dt\\
&\geqslant \int_0^T [\Phi(u') - C_2\Phi_0(C_T u'(t))]\,dt - C_1\Phi_0(2\overline{u}) + \int_0^T F(t,\overline{u})\,dt
\end{aligned} \tag{34}$$

If $\|u\|_{W^1L^\Phi} \to \infty$, then $\|u'\|_{L^\Phi} \to \infty$ or $|\overline{u}| \to \infty$.
Now, if $|\overline{u}| \to \infty$, by hypothesis we have $\frac{1}{\Psi_0(2\overline{u})}\int_0^\infty F(t,\overline{u})\,dt \to \infty$.
If $\|u_n\|_{L^\Phi} \to \infty$ for a sequence u_n.
Let $M > 0$ such that $\frac{\Phi(x)}{C_2\Phi_0(C_T x)} \geqslant 2$ for $|x| \geqslant M$.
Set $A_n := \{t \in [0,T] : |u_n'(t)| \geqslant M\}$.
Now, if $\|u_n'\|_{L^\Phi} \to \infty$ then $\|u_n'\chi_{A_n}\|_{L^\Phi} \to \infty$ or $\|u_n'\chi_{A_n^C}\|_{L^\Phi} \to \infty$.

As Φ is continuous, then Φ is bounded on $\overline{B}_r(0) = \{x \in \mathbb{R}^n : |x| \leqslant r\}$

Let $M_r := \max\limits_{\overline{B}_r(0)} \Phi(x)$.

As $M_r \to 0$ when $r \to 0$, we can choose r such that $M_r T \leqslant 1$.

Then $\int_0^T \Phi(\frac{u}{\frac{1}{r}\|u\|_{L^\infty}})\,dt \leqslant M_r T \leqslant 1$ and consequently $\|u\|_{L^\Phi} \leqslant r^{-1}\|u\|_{L^\infty}$, i.e. $L^\infty \hookrightarrow L^\Phi$. Thus, $\|u_n'\chi_{A_n^C}\|_{L^\Phi}$ is bounded and $\|u_n'\chi_{A_n}\|_{L^\Phi} \to \infty$.

By Ameniya norm, we have $\|u_n'\|_{L^\Phi} \leqslant 1 + \int_{A_n} \Phi(u_n')\,dt$, then

$$\int_{A_n} \Phi(u_n')\,dt \to \infty. \tag{35}$$

Now, as $\frac{\Phi(x)}{2} \geqslant C_2\Phi_0$ for $|x| \geqslant M$, we have

$$\begin{aligned}
&\int_0^T [\Phi(u_n') - C_2\Phi_0(C_T u_n')]\,dt \\
&\geqslant \int_{A_n} [\Phi(u_n') - C_2\Phi_0(C_T u_n')]\,dt - C_2\int_{[0,T]/A_n} \Phi_0(C_T u_n')\,dt \\
&\geqslant \frac{1}{2}\int_{A_n} \Phi(u_n')\,dt - C_2 T \max\{\Phi_0(x) : \|x\| \leqslant C_T M\}.
\end{aligned} \tag{36}$$

Finally, by (35), we conclude that $\int_0^T [\Phi(u_n') - C_2\Phi_0(C_T u_n')]\,dt \to \infty$

□

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