Clarke dual method for Hamiltonian systems with non standard grow

(In alphabetical order)

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Abstract

In this paper we consider the problem of finding periodic solutions of certain Hamiltonian systems .....blablabla

1 Main problem

Let $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We are looking for periodic solutions of the Hamiltonian system

$$\begin{cases} \dot{q}(t) & = D_pH(t, q(t), p(t)) \\ \dot{p}(t) & = -D_qH(t, q(t), p(t)) \\ p(0) & = p(T) \ , \ q(0) = q(T) \end{cases} \tag{1}$$

for $t \in [0,T]$. I think that, like in [7], is better to present the Hamiltonian problem as the main problem

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An alternative writing of (1) using the combined variable $u = (q, p)$ and the canonical symplectic matrix

$$J = \begin{pmatrix} 0 & I_{d\times d} \\ -I_{d\times d} & 0 \end{pmatrix}$$

is the following

$$\dot{u} = J\nabla H(t, u(t)) \tag{2}$$

or equivalently

$$J\dot{u} = -\nabla H(t, u(t)) \tag{3}$$

where ∇H is the gradient of H with respect to the combined variable.

2 Preliminaries

We will use some basic concepts of convex analysis that we list below.

Let $\Gamma_0(\mathbb{R}^d) = \{F : \mathbb{R}^d \to (-\infty, +\infty]$
convex, lower semicontinous functions with non-empty effective domain.$\}$

The Fenchel conjugate of F is given by

$$F^\star(p) = \sup_{q\in\mathbb{R}^d} \langle p, q\rangle - F(q)$$

The Fenchel conjugate satisfies the following properties:

1. $F^\star \in \Gamma_0(\mathbb{R}^d)$
2. If $F \leqslant G$, then $G^\star \leqslant F^\star$.
3. If $G(q) = \alpha F(\beta q) + \sigma$ with $\alpha, \beta, \sigma > 0$ then $G^\star(p) = \alpha F^\star(\frac{p}{\beta\alpha}) - \sigma$

Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a differentiable, convex function such that $\Phi(0) = 0$, $\Phi(q) > 0$ if $q \neq 0$, $\Phi(-q) = \Phi(q)$, and

$$\lim_{|q|\to\infty} \frac{\Phi(q)}{|q|} = +\infty, \tag{4}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From now on, we say that Φ is an G-function if Φ satisfies the previous properties.

We write $\Phi^\star$ for the Fenchel conjugate of Φ.

We do not assume that Φ and Φ' satisfy the Δ_2-condition.

We denote by $\partial F(q)$ the subdifferential of F in the sense of convex analysis (see [2, 3])

The next result is a generalization of [6, Prop. 2.2, p.34]

Proposition 2.1. *Let $F \in \Gamma_0(\mathbb{R}^d)$. Suppose that there exist an anisotropic function Φ and non negative constants β, γ such that*

$$-\beta \leqslant F(q) \leqslant \Phi(q) + \gamma, \text{ for all } q \in \mathbb{R}^d. \tag{5}$$

Now, if $p \in \partial F(q)$ then

$$\Phi^\star(p) \leqslant \Phi(2q) + 2(\beta+\gamma). \tag{6}$$

Proof. If $p \in \partial F(q)$, from [6, Thm. 2.2, p.33],

$$F^\star(p) = \langle p, q\rangle - F(q) \tag{7}$$

Conjugating (5), we have

$$F^\star(p) \geqslant \Phi^\star(p) - \gamma. \tag{8}$$

From Young's inequality, we get

$$\langle p, q\rangle = \frac{1}{2}\langle p, 2q\rangle \leqslant \frac{1}{2}\Phi^\star(p) + \frac{1}{2}\Phi(2q) \tag{9}$$

By eqs. (5) and (7) to (9), we get

$$\Phi^\star(p) \leqslant \frac{1}{2}\Phi^\star(p) + \frac{1}{2}\Phi(2q) + \beta + \gamma$$

which implies (6) □

Remark 1. Inequality (6) is a few better than the corresponding in [6, Prop. 2.2] because the the case of power function we obtain $(\beta+\gamma)^{1/p}$, meanwhile in [6] appears $(\beta+\gamma)^{1/(p-1)}$.

3 Optimal bounds for a symplectic bilinear form

We consider the Euclidean space $\mathbb{R}^{2d}$ equipped with the standard symplectic structure given by bilinear canonical symplectic 2-form

$$\Omega(u, v) := \langle Ju, v\rangle.$$

As Jakub observed we can not consider any G-function on the symplectic manifold $\mathbb{R}^{2n}$. I thinkthat the following can be the appropriate form of the G-function defined on the symplectic manifold $\mathbb{R}^{2n}$

Definition 3.1. *Let Φ a G-function defined in the symplectic manifold $\mathbb{R}^{2n}$. We say that Φ is a* symplectic G-function *if*

$$\Phi(Ju) = \Phi^\star(u). \tag{10}$$

Example 3.1. Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a G-function. Then the G-function

$$\Phi(u) = \Phi(q, p) := \Phi(q) + \Phi^\star(p).$$

is a symplectic G-function.

PROBLEM 0: It is the previous the general form of any symplectic G-function? It is possible to find other example of these functions?

We note that if Φ is symplectic then

$$\nabla\Phi(Ju) = J\Phi^\star(u). \tag{11}$$

Here we are agreeing that $\nabla\Phi$ is a column vector.

As a consequence of (10), the matrix J induce a isometry between the spaces $L^\Phi([0,T],\mathbb{R}^{2d})$ and $L^{\Phi^\star}([0,T],\mathbb{R}^{2d})$. Therefore we candefine a bilinear form $\overline{\Omega}$ on $L^\Phi([0,T],\mathbb{R}^{2d})$ of the following way

$$\overline{\Omega}(u,v) := \int_0^T \Omega(u,v)dt, \quad u,v \in L^\Phi([0,T],\mathbb{R}^{2d})$$

We consider the following functional

$$\Theta(u) := \overline{\Omega}(u,\dot{u}).$$

We are interested in to find bounds of the quadratic functional Θ of the following type

$$\theta(u) \geqslant -C\int_0^T \Phi\left(\dot{u}\right)dt, \tag{12}$$

for $u \in W^1L^\Phi([0,T],\mathbb{R}^{2d})$. It is important to get the best constant C in previous inequality because this constant imposes restrictions to the Hamiltonian H.

If $\Phi(q) = |q|^2/2$ was proved in [6, Prop. 3.2] (12) holds width $C = T/\pi$. Below we prove that this is the optimal constant satisfying (12). Meanwhile in [9, Lem. 3.3] was proved that $C_\Phi = 2T$ satisfies (12) when $\Phi(q) = |q|^\alpha/\alpha$, $1 < \alpha < \infty$. Since this constant is not equal to T/π when $\alpha = 2$, it is not optimal.

Proposition 3.2. *Let Φ be any symplectic G-function. Then* (12) *holds for and $C = 2T^{-1}$ for every $u \in W^1L^\Phi([0,T],\mathbb{R}^{2d})$.*

Proof. Let $u \in W^1L^\Phi([0,T],\mathbb{R}^{2d})$. As is usual we write $u = \tilde{u} + \overline{u}$ where

$$\overline{u} = \frac{1}{T}\int_0^T u(t)dt.$$

From [8, Lem. 2.4] we have that

$$\int_0^T \Phi(\tilde{u})dt \leqslant \int_0^T \Phi(T\dot{u})dt.$$

Then by Young's inequality and using (10)

$$\begin{aligned}
\int_0^T \Omega\left(\dot{u},u\right)dt &= T\int_0^T \left\langle J\dot{u}, T^{-1}\tilde{u}\right\rangle dt \\
&\geqslant -T\left\{\int_0^T \Phi^\star(J\dot{u})dt + \int_0^T \Phi(T^{-1}\tilde{u})dt\right\} \\
&\geqslant -2T\left\{\int_0^T \Phi(\dot{u})dt\right\}
\end{aligned}$$

□

Clearly the cosntant $2/T$ is far to be optimal. A possible way of improve C is consider other average $\overline{u}$. The mean value that it was used is the standard condered in the literature. But this value is apropriate for el Hilbert setting $\Phi(q) = |q|^2/2$. In this case, the value of $\overline{u}$ is the nearest (in the L^2-norm) constant vector to u. For a arbitrary G function, it seem more reasonable consider the nearest constant vector to u respect to the Φ-integral, i.e.

$$\int_0^T \Phi(u - \overline{u})dt \leqslant \int_0^T \Phi(u - u_0)dt, \quad \text{for every } u_0 \in \mathbb{R}^{2n}$$

Equivalently $\overline{u}$ is characterizate by

$$\int_0^T \nabla\Phi(u - \overline{u})dt = 0.$$

There is not a explicit formula as in the Hilbert setting.
PROBLEM 1. We can get a better constant taking this $\overline{u}$???

We call to the best constant in (12) C_Φ, i.e.

$$C_\Phi = -\inf\left\{ \frac{\int_0^T \langle J\dot{u}, u\rangle dt}{\int_0^T \Phi(\dot{u})dt} \,\middle|\, u \in W^1L^\Phi\left([0,T], \mathbb{R}^{2d}\right)\right\} \tag{13}$$

Proposition 3.3. *The relation $C_\Phi = C_{\Phi^\star}$ holds for every symplectic Φ.*

Proof. Since Φ is symplectic if $u = Jv$

$$\frac{\int_0^T \langle J\dot{u}, u\rangle dt}{\int_0^T \Phi(\dot{u})dt} = \frac{\int_0^T \langle -\dot{v}, Jv\rangle dt}{\int_0^T \Phi(J\dot{v})dt} = \frac{\int_0^T \langle J\dot{v}, v\rangle dt}{\int_0^T \Phi^\star(\dot{v})dt}.$$

Using that $u \mapsto Ju$ is invertible from $W^1L^{\Phi^\star}([0,T], \mathbb{R}^{2d})$ into $W^1L^\Phi([0,T], \mathbb{R}^{2d})$ the statement follows taking infimum in previous equality.

□

For the following result we need the theory of indices of G-functions, see [4, 5] for a complete treatment in the case of N-functions defined on $\mathbb{R}$. The results are easily extended to the anisotropic setting. We denote by α_Φ and β_Φ the so called *Matuszewska-Orlicz indices* of the function Φ, which are defined next

$$\alpha_\Phi := \lim_{t\to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right)}{\log(t)}, \quad \beta_\Phi := \lim_{t\to +\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right)}{\log(t)}. \tag{14}$$

We have that $1 \leqslant \alpha_\Phi \leqslant \beta_\Phi \leqslant +\infty$. The relation $\beta_\Phi < \infty$ holds true if and only if Φ is a Δ_2-function. The indices satisfy the following relation

$$\frac{1}{\alpha_\Phi} + \frac{1}{\beta_{\Phi^\star}} = 1. \tag{15}$$

Therefore if $\Phi^\star$ is a Δ_2-function (I mean Δ_2 as globally Δ_2) then $\alpha_\Phi > 1$.

We observe that if Φ is symplectic then $\Phi \in \Delta_2$ implies $\Phi^\star \in \Delta_2$. It is well known that if Φ and $\Phi^\star$ are Δ_2-function, then Φ is controlled by above and below by power functions. More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\Phi, \epsilon)$ and p_0, p_1 with $1 < \alpha_\Phi - \epsilon < p_1 \leqslant p_2 < \beta_\Phi + \epsilon < \infty$ such that, for every $t, u \geqslant 0$,

$$K^{-1} \min\left\{t^{p_2}, t^{p_1}\right\}\Phi(u) \leqslant \Phi(tu) \leqslant K \max\left\{t^{p_2}, t^{p_1}\right\}\Phi(u). \tag{16}$$

We recall the following result of [1].

Lemma 3.4. *Let Φ be a G-functions. If $\Phi^\star \in \Delta_2$ globally, then for any $0 < \mu < \alpha_\Phi$,*

$$\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right) dt}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} = +\infty. \tag{17}$$

Theorem 3.5. *Suppose that $u \in W^1L_T^\Phi([0,T], \mathbb{R}^{2d})$ attains the minimum in* (13)*, then $\lambda = 2/C_\Phi$ is the first eigenvalue and u the corresponding eigenfunction of the following problem.*

$$\begin{cases} \frac{d}{dt}\nabla\Phi^\star(\dot{u}) + \lambda\nabla\Phi^\star(\lambda u) = 0 \\ u(0) = u(T),\ \int_0^T \nabla\Phi^\star(\lambda u) dt = 0 \end{cases} \tag{Eig}$$

Proof. □

4 Differentiability of Hamiltonian dual action

Theorem 4.1. *Suppose that $\Phi : \mathbb{R}^{2d} \to [0, +\infty)$ is a differentiable G-function, not necessarily symplectic. Additionally*

1. *$H : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}$ is measurable in t, continuously differentiable with respect to u.*
2. *there exist $\beta, \gamma \in L^1([0,T], \mathbb{R})$, $\Lambda > \lambda > 0$ such that*

$$\Phi^\star\left(\frac{u}{\Lambda}\right) - \beta(t) \leqslant H(t,u) \leqslant \Phi^\star\left(\frac{u}{\lambda}\right) + \gamma(t) \tag{18}$$

Then there exists Λ_0 such that the dual action

$$\chi(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + H^\star(t, \dot{v}) dt \tag{19}$$

is continuously differentiable in $W^1L_T^\Phi([0,T], \mathbb{R}^{2d}) \cap \{u | d(\dot{u}, L^\infty) < \Lambda_0\}$.

If v is a critical point of χ with $d(\dot{v}, L^\infty) < \Lambda_0$, the function defined by $u(t) = \nabla H^\star(t, \dot{v})$ solves

$$\begin{cases} \dot{u} &= J\nabla H(t,u) \\ u(t) &= u(T) \end{cases}$$

Proof. Conjugating 2 we obtain

$$\Phi(\lambda u) - \gamma(t) \leqslant H^\star(t,v) \leqslant \Phi(\Lambda v) + \beta(t) \tag{20}$$

Since $H^\star$ is smooth, we have $\partial_v H^\star(t,v) = \{\nabla_v H^\star(t,v)\}$. Applying Proposition 2.1 with $F = H^\star$, $\Phi(\Lambda v)$ instead of $\Phi(u)$ and $u = \nabla H^\star(t,v) \in \partial_v H(t,v)$, inequality (18) becomes

$$\Phi^\star\left(\frac{\nabla H^\star(t,v)}{\Lambda}\right) \leqslant \Phi(2\Lambda v) + 2(\beta+\gamma). \tag{21}$$

which will be the main inequality in the proof.

We are planning to obtain the structure condition (**??**) of [8] which guarantees differentiability.

We consider the Lagrangian

$$\mathcal{L}(t,v,\xi) = \frac{1}{2}\langle J\xi, v\rangle + H^\star(t,\xi) \tag{22}$$

and we have to prove that there exist $\Lambda_0 > \lambda_0 > 0$ such that

$$|\mathcal{L}| + |\nabla_v \mathcal{L}| + \Phi^\star\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) \leqslant a(v)\left(b(t) + \Phi\left(\frac{\xi}{\Lambda_0}\right)\right) \tag{23}$$

We start with $|\mathcal{L}|$. From (20),

$$|\mathcal{L}| \leqslant \frac{1}{2}|\langle J\xi, v\rangle| + H^\star(t,\xi) \leqslant \frac{1}{2}|\xi||v| + \Phi(\Lambda\xi) + \beta(t).$$

Since $\frac{\Phi(x)}{|x|} \to \infty$ as $|x| \to \infty$, there exists $C > 0$ such that $|x| \leqslant \Phi(x) + C$ for all $x \in \mathbb{R}^d$. Then,

$$|\mathcal{L}| \leqslant \frac{1}{2}\frac{|v|}{\Lambda}\left(\Phi(\Lambda\xi) + C\right) + \Phi(\Lambda\xi) + \beta(t) \leqslant \max\left\{\Lambda, \frac{|v|}{2\Lambda}\right\}\left[\Phi(\Lambda\xi) + C + \beta(t)\right]$$

which is an estimate like the second member of (23).

Now, we treat $|\nabla_v \mathcal{L}|$ and we get

$$|\nabla_v \mathcal{L}| = \frac{1}{2}|J\xi| \leqslant |\xi| \leqslant \frac{1}{2\Lambda}(\Phi(\Lambda\xi) + C). \tag{24}$$

which is also an estimate of the desired type.

Finally, we deal with $\Phi(\nabla_\xi \mathcal{L}\lambda_0)$. As $\Phi^\star$ is a convex, even function, we have

$$\Phi^\star\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) = \Phi^\star\left(\frac{-\frac{1}{2}Jv}{\lambda_0} + \frac{\nabla H^\star(t,\xi)}{\lambda_0}\right) \leqslant \frac{1}{2}\Phi^\star\left(\frac{Jv}{\lambda_0}\right) + \frac{1}{2}\Phi^\star\left(\frac{2\nabla H^\star(t,\xi)}{\lambda_0}\right).$$

We choose $\frac{2}{\lambda_0} = \frac{1}{\Lambda}$ with Λ as in (21) and we finally have

$$\begin{aligned}\Phi^\star\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) \leqslant \Phi^\star\left(\frac{Jv}{2\Lambda}\right) + \Phi(2\Lambda\xi) + 2(\beta+\gamma) = \\ \max\left\{\Phi^\star\left(\frac{Jv}{2\Lambda}\right), 1\right\}\left[\Phi(2\Lambda\xi) + 2(\beta+\gamma)\right]\end{aligned} \tag{25}$$

which is a bound like the second member of (23).

Therefore, from (23), (24), (25) and choosing the worst functions a and b, we obtain condition (**??**).

Next, [8, Thm. 4.5] implies differentiability of χ in a set like $W^1L^\Phi_T([0,T],\mathbb{R}^d)\cap\{u|d(\dot{u},L^\infty)<\lambda_0\}$.

If $v\in W^1L^\Phi_T([0,T],\mathbb{R}^d)$ is a critical point of χ with $d(\dot{v},L^\infty)<\lambda_0$ then, from equations (21) of [8] we obtain

$$0=\int_0^T \frac{1}{2}\langle J\dot{v},h\rangle-\frac{1}{2}\langle\dot{h},Jv\rangle+\langle\nabla H^\star(t,\dot{v}),\dot{h}\rangle.$$

The rest of the proof follows as in [6].

□

5 Existence periodic solutions Hamiltonian system

The following theorem extend to a quite general function Φ the result in [6, Th. 3.1] formulated for $\Phi_2(u)=|u|^2/2$. Even more, our result improves a little bit [6, Th. 3.1] in the sense that we obtain existence for Φ_2 when the functions, introduced below, l and γ are in L^2 and L^1 respectively instead that L^4 and L^2 which is assumed in [6, Th. 3.1]. This little improvement is due to the observation in Remark 1.

Theorem 5.1. *Suppose that Φ is a symplectic G-function and*

H1) Exists $\xi\in L^{\Phi^\star}([0,T],\mathbb{R}^{2d})$ such that

$$H(t,u)\geqslant\langle\xi(t),u\rangle.$$

H2) There exists $\alpha\in(0,C_\Phi)$ (C_Φ is defined in (13)) such that for every $(t,u)\in[0,T]\times\mathbb{R}^{2d}$

$$H(t,u)\leqslant\frac{1}{\alpha}\Phi(\alpha u)+\gamma(t).$$

H3)

$$\int_0^T H(t,u)dt\to+\infty,\quad \textit{when } |u|\to+\infty.$$

Then xxxxxxxxxxxxxxxxxx

Proof. Let δ be a positive number such that $\alpha+\delta<C_\Phi^{-1}$. Note that from (16) we have that

$$\frac{K}{\alpha+\delta}\Phi\left((\alpha+\delta)u\right)-\frac{1}{\alpha}\Phi\left(\alpha u\right)\geqslant\frac{K_1}{\alpha}\Phi\left(\alpha u\right),$$

where $K_1=[(\alpha+\delta)/\alpha]^{p_1-1}-1>0$ and K,p_2 are the constants in (16). The constat K_1 depends only on α,δ and Φ.

We define

$$H_\delta(t,u) = \frac{K_1}{\alpha}\Phi\left(\alpha u\right) + H(t,u)$$

Let $\lambda = \min\{1, K_1/2\}$. Then by H1), Young inequality and since $0 < \lambda \leqslant 1$ we have

$$\begin{aligned} H_\delta(t,u) &\geqslant \frac{K_1}{\alpha}\Phi\left(\alpha u\right) - \left|\frac{1}{\alpha}\left\langle \frac{\xi(t)}{\lambda}, \lambda\alpha u(t)\right\rangle\right| \\ &\geqslant \frac{K_1}{\alpha}\Phi\left(\alpha u\right) - \frac{1}{\alpha}\Phi^\star\left(\frac{\xi(t)}{\lambda}\right) - \frac{1}{\alpha}\Phi\left(\lambda\alpha u(t)\right) \qquad (26) \\ &\geqslant \frac{K_1}{2\alpha}\Phi\left(\alpha u\right) - \frac{1}{\alpha}\Phi^\star\left(\frac{\xi(t)}{\lambda}\right) \end{aligned}$$

On the other hand

$$H_\delta(t,u) \leqslant \frac{K}{\alpha+\delta}\Phi\left((\alpha+\delta)u\right) + \gamma(t) \qquad (27)$$

The perturbed Hamiltonian H_δ verifies the assumptions of Theorem 4.1. Moreover, since $\Phi \in \Delta_2 \cap \nabla_2$ we have that the dual action

$$\chi_\delta(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + H_\delta^\star(t,\dot{v})dt \qquad (28)$$

is continuously differentiable in $W^1L_T^{\Phi^\star}([0,T],\mathbb{R}^{2d})$.

On the other hand

$$\chi_\delta(v) \geqslant \left(\frac{1}{\alpha+\delta} - C_\Phi\right)\int_0^T \Phi(v)dt - \gamma_0$$

$\square$

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