

# Periodic solutions for a Sitnikov restricted $n + 1$ -body problem with primaries in rigid motion

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## Abstract

## 1 Introduction

In this paper we discuss existence of periodic solutions for the following restricted nonplanar Newtonian  $n + 1$ -body problem  $P$  (see figure 1):

- $P_1$  We have  $n$  primary bodies of masses  $m_1, \dots, m_n$  and an additional massless body.
- $P_2$  The primary bodies are in a central configuration rigid motion (see [11, Section 2.9]). This motion is carried out in a plane  $\Pi$ .
- $P_3$  The massless particle is moving on the perpendicular line to  $\Pi$  passing through the center of masses.

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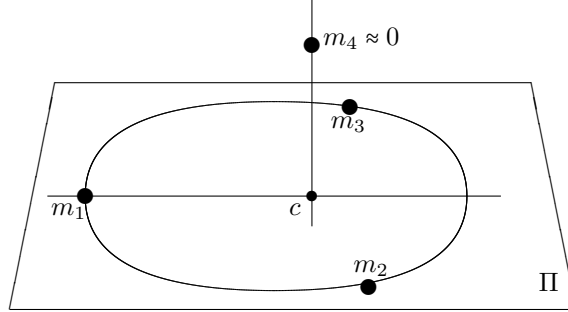


Figure 1: Four-body problem with three primaries

Problems like the one presented have been extensively discussed in the literature. In [16] K. Sitnikov considered the problem of two body in a Keplerian motion and a massless particle moving in the perpendicular line to the orbital plane passing through the center of masses. Sitnikov obtained deep results about existence of solutions, some of them periodic (see [13, III(5)]). Since then many other authors have studied Sitnikov problem, for instance Liu, Zhou, and Sun [9], Hagel and Trenkler [6], Dvorak [5], Dankowicz and Holmes [4], Llibre, Meyer and Soler [10], Chesley [3], Jiménez-Lara a and Escalona-Buendía [7], Llibre and Ortega [12], Pérez, Jiménez and Lacomba [15].

Problems like the Sitnikov problem for four bodies were addressed more recently. In [17] Soulis, Papadakis and Bountis studied existence, linear stability and bifurcations for a problem similar to  $P$ , where in place to have a Eulerian collinear configuration they had a Lagrangian equilateral triangle configuration for the primaries bodies, which are supposed to have the same mass  $m_1 = m_2 = m_3$ . Later, In [1] Baltagiannis nad Papadakis considered more general masses and in [14] Pandey and Ahmad extend the analysis started in [17] to the case when the primaries are oblate (not mass points). In [19], Zhao and Zhang proved existence of periodic solutions for a problem similar to the one dealt with in [17]. They used a variational approach. In the present paper we extend the analysis in [19] to the case of a collinear central configuration for the primaries. In [8] Li, Zhang and Zhao studied a special type of restricted circular  $N+1$ -body problem with equal masses for the primaries.

Given that in our problem the primaries are no longer equidistant and their relative position is determined by a polynomial equation of fifth degree, the calculations involved here are tedious to reproduce completely and difficult for that the reader to check them by hand. For this reason we have prepared a jupyter-notebook (see [2]) with some of these calculations. With a little knowledge of Python-Sympy (see [18]) the reader can check and reproduce them easily.

## 2 Preliminaries and Main Results

We start considering  $n$  bodies,  $n > 2$ , of masses  $m_1, \dots, m_n$  moving in a Euclidean 3-dimensional space according to Newton's laws of motion. We assume that  $q_1(t), \dots, q_n(t)$  are the coordinates (column vectors) of the bodies in some inertial Cartesian coordinate system. We denote by  $r_{ij} = |q_i - q_j|$  the Euclidean distance between  $q_i$  and  $q_j$ . We can suppose without any loss of generality, we can assume the center of mass  $c := \sum_j m_j q_j / M$  ( $M := \sum_j m_j$ ) is fixed at the origin ( $c = 0$ ).

We assume that these bodies are in a *rigid motion*. We recall that a *rigid motion*, is a solution of motions equations with  $r_{ij}$  constant. It is known (see -buscar referencias) that a rigid motion is performed in a plane  $\Pi$ . We assume that  $\Pi$  is the plane determined by the first two coordinates axes. Then a rigid motion has the form

$$q_j(t) = Q(\nu t) q_j^0,$$

where

$$Q(\nu t) = \begin{pmatrix} \cos(\nu t) & -\sin(\nu t) & 0 \\ \sin(\nu t) & \cos(\nu t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and  $q_j^0 \in \Pi$ ,  $j = 1, \dots, n$  are vectors in a planar *central configuration* (CC) in  $\mathbb{R}^3$ , i.e. there exists  $\lambda \in \mathbb{R}$  such that

$$\nabla_j U(q_1^0, \dots, q_n^0) + \lambda m_j q_j^0 = 0, \quad j = 1, \dots, n.$$

where the *potential function*  $U$  is defined by:

$$U(x) = \sum_{i < j} \frac{m_i m_j}{r_{ij}}, \quad (1)$$

and  $\nabla_j$  denotes the 3-dimensional partial gradient with respect to  $q_j$ . According to [11, Eq. (2.16)] we have  $\nu^2 = \lambda$ . Then the primaries bodies perform a periodic motion with period  $T := 2\pi/\nu$ .

We suppose that we have a massless particle with coordinates  $q(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ . This particle does not disturb the rigid motion of primaries. We want to find conditions under which this particle perform a  $T$ -periodic motion on the third axis of coordinates.

The particle  $q$  satisfies the Newtonian equations of motion

$$\ddot{q} = \sum_{i=1}^n \frac{m_i (q_i - q)}{|q_i - q|^3}, \quad (2)$$

**Theorem 2.1.** *There exists a non-stationary solution of (2) with  $x(t) = y(t) = 0$  if and only if  $q_1^0, \dots, q_n^0$  satisfy that for any  $r > 0$ , such that the set*

$$F_r := \{i : |q_i^0| = r\}$$

*is non empty, that*

$$\sum_{i \in F_r} m_i q_i^0 = 0. \quad (3)$$

i.e. every maximal equidistant from origin set of bodies has center of mass equal to 0.

We note that in order to get collisionless solutions of problem  $P$  we need that no primary body is located in the center of mass. We call this configurations *admissible*. In the following theorem we characterize all 3-4 bodies CC satisfying condition in Theorem 2.1 and without bodies in the center of mass.

**Theorem 2.2.** *The only 3-body admissible CC is the equilateral triangle with three equal masses. In the case of 4-body, an admissible CC has two pairs of equal masses ( $m_1 = m_2$  and  $m_3 = m_4$ ), and satisfies some of the following properties: it is collinear and symmetric around the center of mass or it is a rhombus with the equal masses in opposite vertices.*

### 3 Proofs

**Lemma 3.1.** *For  $c > 0$  we define the function  $y_c(t) := (c + t)^{-3/2}$ . If  $0 < t_1 < t_2 < \dots < t_k$  then the functions  $y_j(t) := y_{t_j}(t)$  are linearly independent on each open interval  $I \subset \mathbb{R}^+$ .*

*Proof.* It is sufficient to prove that Wronskian

$$W := W(y_1, \dots, y_k)(t) = \det \begin{pmatrix} y_1 & \dots & y_k \\ \frac{dy_1}{dt} & \dots & \frac{dy_k}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1}{dt^{k-1}} & \dots & \frac{d^{k-1}y_k}{dt^{k-1}} \end{pmatrix}$$

is not null on  $I$ .

Using induction is easy to show that

$$\frac{d^i y_c}{dt^i} = \beta_i y_c^{\frac{2i+3}{3}}, \quad \text{for some } \beta_i \neq 0, \text{ and for all } i = 1, \dots \quad (4)$$

Fix any  $t \in I$ . Then, according to (4) and writing  $\lambda_j := (t + t_j)^{-1}$ , we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \lambda_1^{3/2} & \lambda_2^{3/2} & \dots & \lambda_k^{3/2} \\ \beta_1 \lambda_1^{5/2} & \beta_1 \lambda_2^{5/2} & \dots & \beta_1 \lambda_k^{5/2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1} \lambda_1^{k+1/2} & \beta_{k-1} \lambda_2^{k+1/2} & \dots & \beta_{k-1} \lambda_k^{k+1/2} \end{pmatrix} \\ &= \beta_1 \beta_2 \dots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \dots \lambda_k^{3/2} \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \end{pmatrix} \\ &= \beta_1 \beta_2 \dots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \dots \lambda_k^{3/2} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i), \end{aligned}$$

where the last equality follows of the well known Vandermonde determinant identity. Therefore  $W \neq 0$  if and only if  $\lambda_i \neq \lambda_j$ ,  $i \neq j$ , which in turn is equivalent to  $t_i \neq t_j$ ,  $i \neq j$ .  $\square$

*Proof Theorem 2.1.* We use a rotating coordinate system where the primaries are fixed. Concretely we put

$$\xi = Q(-\nu t)q.$$

In this system the motion equations are

$$\ddot{\xi} + 2\nu B\dot{\xi} + \nu^2 C\xi = \sum_{i=1}^n \frac{m_i(q_i^0 - \xi)}{|q_i^0 - \xi|^3}, \quad (5)$$

where

$$B := \begin{pmatrix} J & 0_{2 \times 1} \\ 0_{1 \times 2} & 0 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -I_2 & 0_{2 \times 1} \\ 0_{1 \times 2} & 0 \end{pmatrix},$$

where  $0_{n \times m}$  and  $I_n$  denote the null  $n \times m$  matrix and the identity  $n \times n$  matrix respectively. Assuming that the massless particle is moving on the  $z$ -axis then  $\xi = q = (0, 0, z)$  and the Coriolis and centrifugal forces,  $2\nu B\dot{\xi}$  and  $\nu^2 C\xi$  respectively, are null. Therefore, taking account in the first two equation in (5) and identifying the vectors  $q_i^0$ ,  $i = 1, \dots, n$  with vectors in  $\mathbb{R}^2$ , we have

$$\sum_{i=1}^n \frac{m_i q_i^0}{|q_i^0 - \xi|^3} = 0.$$

Let  $D = \{|q_i^0| : i = 1, \dots, n\}$ . Suppose that  $D = \{r_1, \dots, r_k\}$ , with  $r_i \neq r_j$  for  $i \neq j$ , and  $\{1, \dots, n\} = F_1 \cup \dots \cup F_k$ , where if  $i \in F_j$  then  $|q_i^0| = r_j$ . Then

$$\sum_{j=1}^k \left\{ \frac{1}{(r_j^2 + z^2)^{3/2}} \sum_{i \in F_j} m_i q_i^0 \right\} = 0.$$

Since we are considering a non-stationary solution, we have that  $z(t)$  is not constant. Therefore there exists an interval  $I \subset \mathbb{R}^+$  where

$$\sum_{j=1}^k \left\{ \frac{1}{(r_j^2 + s)^{3/2}} \sum_{i \in F_j} m_i q_i^0 \right\} = 0, \quad s \in I.$$

Then, according to Lemma 3.1, we obtain (3).  $\square$

*Proof Theorem 2.2.* For the case of 3-bodies, we note that the Theorem 2.1 and the fact that the center of masses is an excluded position imply that if  $F_r$  is not empty then  $\#F_r \geq 2$ . Hence an admissible 3-body CC consists of three equidistant bodies. Therefore, it must to be the Lagrangian equilateral triangle configuration. Now, equation (3) implies that every bodies has the same mass.

In the caso of 4-bodies  $\square$

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## References

- [1] AN Baltagiannis and KE Papadakis. Families of periodic orbits in the restricted four-body problem. *Astrophysics and Space Science*, 336(2):357–367, 2011.
- [2] Gastón Beltritti, Fernando Mazzone, and Martina Oviedo. Auxiliary calculations. URL: <https://github.com/fdmazzone/ArchivosProyecto/blob/master/Mecanica%20Celeste/CalculosPaper.ipynb> [cited 29.12.2016].
- [3] Steven R Chesley. A global analysis of the generalized Sitnikov problem. *Celestial Mechanics and Dynamical Astronomy*, 73(1-4):291–302, 1999.
- [4] Harry Dankowicz and Philip Holmes. The existence of transverse homoclinic points in the sitnikov problem. *Journal of differential equations*, 116(2):468–483, 1995.
- [5] R Dvorak. Numerical results to the sitnikov-problem. In *Qualitative and Quantitative Behaviour of Planetary Systems*, pages 71–80. Springer, 1993.
- [6] Johannes Hagel and Thomas Trenkler. A computer aided analysis of the sitnikov problem. In *Qualitative and Quantitative Behaviour of Planetary Systems*, pages 81–98. Springer, 1993.
- [7] Lidia Jiménez-Lara and Adolfo Escalona-Buendía. Symmetries and bifurcations in the sitnikov problem. *Celestial Mechanics and Dynamical Astronomy*, 79(2):97–117, 2001.
- [8] Fengying Li, Shiqing Zhang, and Xiaoxiao Zhao. The characterization of the variational minimizers for spatial restricted  $N + 1$ -body problems. *Abstract and Applied Analysis*, 2013(Article ID 845795), 2013.
- [9] Jie Liu, Ji-lin Zhou, and Yi-sui Sun. Numerical research on the sitnikov problem. *Chinese astronomy and astrophysics*, 15(3):339–344, 1991.
- [10] Jaume Llibre, Kenneth R Meyer, and Jaume Soler. Bridges between the generalized sitnikov family and the lyapunov family of periodic orbits. *Journal of Differential Equations*, 154(1):140–156, 1999.
- [11] Jaume Llibre, Richard Moeckel, and Carles Simó. *Central Configurations, Periodic Orbits, and Hamiltonian Systems*. Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, 2015, nov 2015.
- [12] Jaume Llibre and Rafael Ortega. On the families of periodic orbits of the sitnikov problem. *SIAM Journal on Applied Dynamical Systems*, 7(2):561–576, 2008.

## References

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- [13] J. Moser. *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Celestial Mechanics*. Annals Mathematics Studies. Princeton University Press, 1973.
- [14] LP Pandey and I Ahmad. Periodic orbits and bifurcations in the sitnikov four-body problem when all primaries are oblate. *Astrophysics and Space Science*, 345(1):73–83, 2013.
- [15] Hugo Jiménez Pérez and Ernesto A Lacomba. On the periodic orbits of the circular double sitnikov problem. *Comptes Rendus Mathématique*, 347(5):333–336, 2009.
- [16] K Sitnikov. The existence of oscillatory motions in the three-body problem. In *Dokl. Akad. Nauk SSSR*, volume 133, pages 303–306, 1960.
- [17] PS Soulis, KE Papadakis, and T Bountis. Periodic orbits and bifurcations in the sitnikov four-body problem. *Celestial Mechanics and Dynamical Astronomy*, 100(4):251–266, 2008.
- [18] SymPy Development Team. *SymPy: Python library for symbolic mathematics*, 2016. URL: <http://www.sympy.org>.
- [19] Xiaoxiao Zhao and Shiqing Zhang. Nonplanar periodic solutions for spatial restricted 3-body and 4-body problems. *Boundary Value Problems*, 2015(1):1, 2015.