Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}, \tag{1}$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary equations*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y)=\frac{|y|^p}{p}+F(t,x), \tag{2}$$

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for $1 < p < \infty$. For example, the classic book [**?**] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [**?**] were extended and improved in several articles, see [**?**, **?**, **?**, **?**, **?**] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [**?**, **?**] and in this case (1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t, u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{4}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [**?**] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{5}$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [**?**] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([**?**, Ch. 3]) to obtain solutions of (1).

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces). References for these topics are [**?**, **?**, **?**].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$.

Following [**?**] we say that Φ is *coercive* if

$$\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = +\infty.$$

We define the function G by

$$G(s) = \min\{\Phi(x) : |x| \leqslant s\}, \tag{6}$$

where Φ is a Young's function.

We point out that the function $G(|x|)$ has the following properties, as reader can easily check,

(G1) G is monotonous increasing, continuous and $G(s) \to \infty$ as $s \to \infty$.

(G2) G is the *greatest radial minorant* of $\Phi(x)$, i.e. $G(|x|) \leqslant \Phi(x)$ and $G(|x|)$ is the biggest radial function with this property.

(G3) There exists G^{-1} and $G^{-1}(\Phi(x)) \geqslant |x|$.

(G4) As $\Phi(\alpha x)/\alpha$ is increasing with respect to α for evey $x > 0$, $G(\alpha s)/\alpha$ is also increasing with respect to α for every $s > 0$. Alternatively $\beta G^{-1}(t/\beta)$ is an increasing function with respect to β for every $t > 0$.

(G5) In the event that Φ is coervive, then G is also coercive. Alternatively $G^{-1}(s)/s \to 0$ when $s \to +\infty$.

We also say that $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ satisfies the Δ_2^∞-*condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant KH(x), \tag{7}$$

for every $|x| \geqslant M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \to \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x\in\mathbb{R}^d} x \cdot y - \Phi(x) \tag{8}$$

We denote by $\mathcal{M} := \mathcal{M}([0,T],\mathbb{R}^d)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1,\dots,u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T],\mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{9}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{10}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [**?**, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [**?**, Thm. 5.2]). Another alternative characterization of E^Φ, which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \ldots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [**?**, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \tag{11}$$

Like in [**?**] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{12}$$

for any positive r (see [**?**, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [**?**] we introduce the next definition.

Definition 2.1. *Let $u_n, u \in L^\Phi([0,T],\mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^\infty([0,T],\mathbb{R})$, $n = 1, 2, \ldots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t)u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow \left[L^\Phi\right]^*$, where a function $v \in L^{\Phi^*}$ is associated to $\xi_v \in \left[L^\Phi\right]^*$ being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \tag{13}$$

In [**?**, Thm 2.9] it was characterized a subspace of $\left[L^\Phi\right]^*$ which can be identified with L^{Φ^*}.

Proposition 2.2. *Let $F \in \left[L^\Phi([0,T],\mathbb{R}^d)\right]^*$. Then the following statements are equivalent*

1. $\xi \in L^{\Phi^*}([0,T],\mathbb{R}^d)$
2. *ξ satisfies the* monotone convergence property*, which is if u_n converges monotonically to u then $\langle \xi, u_n \rangle \to \langle \xi, u \rangle$.*

If $\Phi \in \Delta_2^\infty$ and Φ is coercive then $L^{\Phi^*}([0,T],\mathbb{R}^d) = \left[L^\Phi([0,T],\mathbb{R}^d)\right]^*$ (see [**?**, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi([0,T],\mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi([0,T],\mathbb{R}^d)\}.$$

$W^1L^\Phi([0,T],\mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{14}$$

And, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{15}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \to [0,+\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (6)*. Then*

1. *For every $s,t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi}|s-t|G^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant G^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If Φ is coercive then the space $W^1L^\Phi([0,T],\mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0,T],\mathbb{R}^d)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right) \\ &\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}. \end{aligned}$$

By (G1) and (G3) we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant G^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha G^{-1}(1/\alpha)$ is an increasing function with respect to $\alpha \in (0,\infty)$ we have

$$|u(t)-\overline{u}| \leqslant \|u'\|_{L^\Phi} T G^{-1}\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then by (G1) and (G3)

$$|\overline{u}| \leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}
\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty} \\
&\leqslant G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \\
&\leqslant G^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}
\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^\Phi([0,T],\mathbb{R}^d)$. From (Morrey's inequality) and (G5) we infer that u_n are equicontinuous. Furthermore (Sobolev's inequality) implies that u_n is bounded in $C([0,T],\mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0,T],\mathbb{R}^d)$ with $u_{n_k} \to u$ in $C([0,T],\mathbb{R}^d)$.

□

3 Superposition operators in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anistropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined in Sobolev-Orlicz spaces.

Henceforth we assume that f is a *Carathéodory function*,

(C) f is measurable with respect to $t \in [0,T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

Definition 3.1. *For $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0,T] \to \mathbb{R}^d$ by*

$$\boldsymbol{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators definied in anisotropic Orlicz spaces of vectorial functions. For the proofs of these results and additional discussions see [**?**, **?**, **?**].

Theorem 3.2. *We assume that f satisfies condition* ((C)). *Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps masurable function into measurable functions*
2. Extensibility.? *If*
3. Continuity.? *If*

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