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The Sitnikov problem for several primary bodies configurations

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Abstract In this paper we address an $n+1$-body gravitational problem governed by the Newton's laws, where n primary bodies orbit on a plane Π and an additional massless particle moves on the perpendicular line to Π passing through the center of mass of the primary bodies. We find a condition for the described configuration to be possible. The case that the primaries are in a rigid motion we classify all the motions of the massless particle. We study the situation when the massless particle has a periodic motion with the same minimal period than primary bodies. We show that this fact is related to the existence of a certain pyramidal central configuration.

1 Introduction

In this paper we study the following restricted Newtonian $n+1$-body problem P (see figure 1):

P_1 We have n primary bodies of masses $m_1, \dots, m_n$ and an additional massless particle.
P_2 The primary bodies are in a homographic motion (see [16, Section 2.9]). This motion is carried out in a plane Π.
P_3 The massless particle is moving on the perpendicular line to Π passing through the center of mass of the primary bodies.

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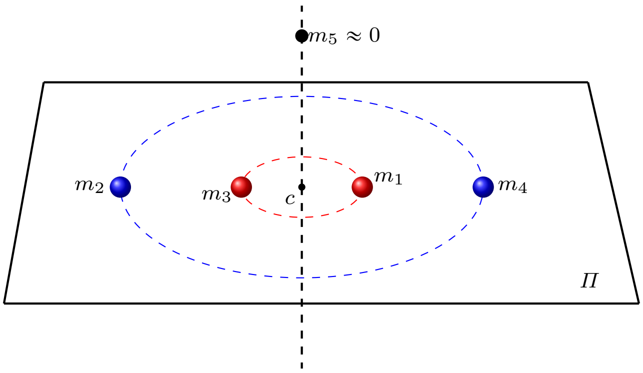

Fig. 1 Five-body problem with primaries in a collinear configuration

Problems like the one presented above have been extensively discussed in the literature. In [32] K. Sitnikov considered the problem of two bodies in a Keplerian elliptic motion and a massless particle moving on the perpendicular line to the orbital plane passing through the center of mass. Sitnikov obtained deep results about existence of solutions, for small $e > 0$, with a chaotic behavior (see [23, III(5)]). Periodic solutions for a Sitnikov configuration were considered in [8,9,17, 29].

Generalized circular Sitnikov problems, i.e. when there are $n \geqslant 3$ primaries in a relative equilibrium motion, were addressed more recently. In [33] Soulis, Papadakis and Bountis studied existence, linear stability and bifurcations for a problem similar to P. They considered a Lagrangian equilateral triangle configuration for the primary bodies, which were supposed to have the same mass $m_1 = m_2 = m_3$. In [6] Papadakis and Bountis extended the results of [33] to n primaries ($n \geqslant 3$) in a polygonal equal masses configuration. Later, in [27], Pandey and Ahmad generalized the analysis started in [33] to the case with oblate primaries. In [15] Li, Zhang and Zhao studied a special type of restricted circular $n+1$-body problem with equal masses for the primaries in a regular polygonal configuration. Periodic solutions for generalized Sitnikov problems with primaries performing no rigid motions were studied in [29,30]. We emphasize that in [6,15,27,29,30,33] it is assumed that the primary bodies are in the vertices of a regular polygon. As far as we know, the first non-polygonal configuration of primary bodies was considered in [19] where Marchesin and Vidal studied the problem P for a rigid motion of primaries in a rhomboidal configuration. In [4] Bakker and Simmons studied scape regions for the massless particle in a similar problem to P where the primaries performe certain type of periodic orbits including non homographic motions.

In the present paper, after introducing preliminary facts in Section 2, we obtain in Section 3 necessary and sufficient conditions on the configuration of primary bodies in order the z-axis to be invariant for the flow associated to the motion equations of the massless particle. For this type of configurations, that we call *admissible*, the Sitnikov problem has sense. The conclusions of Section 3 are obtained basically by elementary linear algebra arguments. We consider that the main contribution of Section 3 is to expand the variety of problems of Sitnikov type. In Section 4, we find all admissible configurations for $n \leqslant 4$ primaries. The

Perpendicular Bisector Theorem of Moeckel (see [21]) is an important help to solve this question. In Section 5 we describe all possible motions of the massless particle when the primaries are in a relative equilibrium (or rigid) motion. In this direction, we observe that only scape (both parabolic and hyperbolic) and periodic motions are possible. We also give in Theorem 5 a formula expressing the period of solutions by means of integrals. We prove in Corollary 1 that the complete $n+1$-body system has infinite quantity of periodic solutions. We solve some problems raised in Section 5 by two alternative techniques: i) elementary arguments, by using energy conservation ([2, Ch. 2]) and ii) variational techniques inspired in [15,34, 35]. In Section 6 we discuss the situation when the entire system has a solution with the same period as the motion of primaries. We call it *synchronous solution*. Surprisingly, the existence of synchronous solutions is related to the existence of certain pyramidal central configurations (for the definition of this concept see [10, 11,25]). Finally, in the last section, we study certain non admissible configurations which provide some particular solutions of problem P.

In this paper we generalize and extend some previously obtained results. For example, the results in Section 5, obtained for admissible configurations, generalize some results in [19] established for rhomboidal configurations. In Section 6 we prove that there exists synchronous solutions for primaries in a regular polygonal equal mass configuration if and only if $2 \leqslant n \leqslant 472$. The sufficiency of this fact was established in [15].

2 Preliminaries

We start considering n mass points, $n > 2$, of masses $m_1, \ldots, m_n$ moving in a Euclidean 3-dimensional space according to Newton's laws of motion. We assume that $x_1(t), \ldots, x_n(t)$ are the coordinates of the bodies in some inertial Cartesian coordinate system. We can suppose, without any loss of generality, that the center of mass $C := \sum_j m_j x_j / M$ ($M := \sum_j m_j$) is fixed at the origin ($C = 0$).

Initially we suppose that the bodies are in a *planar homographic motion* on the plane Π (see [16]), where Π is the plane determined by the first two coordinates axes. Concretely, we are assuming that

$$x_j(t) = r(t)Q(\theta(t))q_j, \tag{1}$$

where

$$Q(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $q_j \in \Pi$, $j = 1, \ldots, n$ are vectors in a planar *central configuration* (CC) in Π. We recall the following definition of this concept (see [16]).

Definition 1 Let $q = (q_1, \ldots, q_n)$ be an n-tuple of positions in $\mathbb{R}^3$ and let $m = (m_1, \ldots, m_n)$ be a vector of masses. We say that (q, m) is a central configuration if there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_j U(q_1, \ldots, q_n) + \lambda m_j q_j = 0, \quad j = 1, \ldots, n, \tag{2}$$

where

$$U(q_1,\ldots,q_n)=\sum_{i<j}\frac{m_im_j}{r_{ij}}, \tag{3}$$

$r_{ij}=|q_i-q_j|$ and ∇_j denotes the 3-dimensional partial gradient with respect to q_j.

From [16, Eq. (2.16)], the functions $r(t)$ and $\theta(t)$ solve the two-dimensional Kepler problem in polar coordinates, which is

$$\begin{aligned} \ddot r(t)-r(t)\dot\theta(t)^2 &= -\tfrac{\lambda}{r(t)^2} \\ \tfrac{d}{dt}\left[r(t)^2\dot\theta(t)\right] &= 0. \end{aligned} \tag{4}$$

It may be the case that the solutions of (4) are defined only on a proper subset of $\mathbb{R}$. We denote by $\mathcal{O}$ the domain of the solutions r and θ. In the particular case of *rigid motion*, we have $\mathcal{O}=\mathbb{R}$, $r(t)\equiv 1$ and $\theta(t)=\sqrt{\lambda}t+\theta(0)$. In this case the primary bodies perform a periodic motion with minimal period $T:=2\pi/\sqrt{\lambda}$.

Let $x_0(t)$ be the position of the massless particle. According to the Newtonian equations of motion, x_0 satisfies

$$\ddot x_0=\sum_{i=1}^n\frac{m_i(x_i-x_0)}{|x_i-x_0|^3}=:f(t,x_0). \tag{5}$$

In the previous equation, we assume that we know the positions of the primaries. Therefore, this equation plus initial conditions determine the position of the particle completely.

3 Admissible configurations

Henceforth, we denote by L the coordinate z axis. A necessary and sufficient condition for that L be invariant under the flow associated to the non autonomous system (5) is $f(t,L)\subset L$ for all $t\in\mathcal{O}$, i.e. L is f*-invariant* for every $t\in\mathcal{O}$. This fact follows by applying [7, Th. 1] to the first order autonomous system

$$\begin{cases} \frac{ds}{dt} &= 1 \\ \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= f(s,x) \end{cases}$$

which is equivalent to equation (5). In addition, the following observations must be taken into account: i) the autonomous vector field $F(s,x,v)=(1,v,f(s,x))$ satisfies $F(\mathcal{O}\times L\times L)\subset\mathcal{O}\times L\times L$ if and only if $f(t,L)\subset L$ for all $t\in\mathcal{O}$ and ii) if $A\subset\mathbb{R}^d$ is a subspace, $x\in A$ and $v\in\mathbb{R}^d$ then $d(x+hv,A)/h\to 0$, when $h\to 0$ if and only if $v\in A$. In the last assertion d denotes the distance function.

Definition 2 We say that a central configuration (q,m) is *admissible* if and only if

1. $q_i\neq 0$, for $i=1,\ldots,n$.

2. For any $r > 0$, if the set
$$F_r := \{i : |q_i| = r\}$$
is non empty, then
$$\sum_{i \in F_r} m_i q_i = 0, \tag{6}$$
i.e. every maximal set of bodies which are equidistant from origin has center of mass equal to 0.

Remark 1 In the previous definition we introduce the condition $q_i \neq 0$ in order to avoid some collisions between the primaries and the particle.

Theorem 1 *L is f-invariant for every $t \in \mathcal{O}$ if and only if (q, m) is admissible.*

For the proof of the previous theorem we need the following result.

Lemma 1 *For $c > 0$ we define the function $y_c(t) := (c + t)^{-3/2}$. If $0 < t_1 < t_2 < \ldots < t_k$ then the functions $y_j(t) := y_{t_j}(t)$ are linearly independent on each open interval $I \subset \mathbb{R}^+$.*

Proof It is sufficient to prove that the Wronskian

$$W := W(y_1, \ldots, y_k)(t) = \det \begin{pmatrix} y_1 & \cdots & y_k \\ \frac{dy_1}{dt} & \cdots & \frac{dy_k}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1}{dt^{k-1}} & \cdots & \frac{d^{k-1}y_k}{dt^{k-1}} \end{pmatrix}$$

is not null on I.

Using induction, it is easy to show that
$$\frac{d^i y_c}{dt^i} = \beta_i y_c^{\frac{2i+3}{3}}, \quad \text{for some } \beta_i \neq 0, \text{ and for all } i = 1, \ldots. \tag{7}$$
Fix any $t \in I$. Then, according to (7) and writing $\lambda_j := (t + t_j)^{-1}$, we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \lambda_1^{3/2} & \lambda_2^{3/2} & \cdots & \lambda_k^{3/2} \\ \beta_1 \lambda_1^{5/2} & \beta_1 \lambda_2^{5/2} & \cdots & \beta_1 \lambda_k^{5/2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1}\lambda_1^{k+1/2} & \beta_{k-1}\lambda_2^{k+1/2} & \cdots & \beta_{k-1}\lambda_k^{k+1/2} \end{pmatrix} \\ &= \beta_1\beta_2 \cdots \beta_{k-1} \lambda_1^{3/2}\lambda_2^{3/2} \cdots \lambda_k^{3/2} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \\ &= \beta_1\beta_2 \cdots \beta_{k-1} \lambda_1^{3/2}\lambda_2^{3/2} \cdots \lambda_k^{3/2} \prod_{1 \leqslant i < j \leqslant n} (\lambda_j - \lambda_i), \end{aligned}$$

where the last equality follows from the well known Vandermonde determinant identity. Therefore, $W \neq 0$ if and only if $\lambda_i \neq \lambda_j$, $i \neq j$, which in turn is equivalent to $t_i \neq t_j$, $i \neq j$. □

Proof (Proof of Theorem 1) The condition $f(t, L) \subset L$ for all $t \in \mathcal{O}$ is equivalent to

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{(r(t)^2 |q_i|^2 + z^2)^{3/2}} = 0 \in \mathbb{R}^2, \tag{8}$$

for every $t \in \mathcal{O}$ and $z \in \mathbb{R}$.

Let $D = \{|q_i| : i = 1, \dots, n\}$. Suppose that $D = \{s_1, \dots, s_k\}$, with $s_i \neq s_j$ for $i \neq j$. Therefore $\{1, \dots, n\} = F_{s_1} \cup \cdots \cup F_{s_k}$. Then, multiplying equation (8) by $r(t)^2 Q^{-1}(\theta(t))$ and writing $\zeta = (z/r(t))^2$ we have that (8) is equivalent to

$$\sum_{j=1}^{k} \left\{ \frac{1}{(s_j^2 + \zeta)^{3/2}} \sum_{i \in F_{s_j}} m_i q_i \right\} = 0.$$

According to Lemma 1, the last equation is equivalent to (6). □

4 Admissible configurations for $n \leqslant 4$

In this section, we find all admissible configurations with $n \leqslant 4$. Since the center of mass is an excluded position, an admissible configuration satisfies

$$\# F_r \neq 1. \tag{9}$$

It is a trivial fact that two point masses m_1 and m_2 configuration is admissible if and only if $m_1 = m_2$.

From (9), a 3-body admissible configuration consists of equidistant bodies from the origin. Therefore, it must be the Lagrangian equilateral triangle. Now, by equation (6) and an elementary geometrical reasoning, we have $m_1 = m_2 = m_3$.

The case $n = 4$ is more interesting. We include Definition 3 and Theorem 2, which were introduced for the first time in [21], for the reader's convenience.

Definition 3 Let q be a planar configuration. For each pair, i, j, the line containing q_i and q_j together with its perpendicular bisector form axes which divide the plane into four quadrants. The union of the first and third quadrants is an hourglass shaped region which will be called a 'cone'; similarly, the second and fourth quadrants together form another cone. The phrase 'open cone' refers to a cone minus the axes.

Theorem 2 (Perpendicular Bisector Theorem) *Let (q, m) be a planar central configuration and let q_i and q_j be any two of its points. Then if one of the two open cones determined by the line through q_i and q_j and its perpendicular bisector contains points of the configuration, so does the other one.*

Next, we characterize all the 4-body admissible configurations.

Theorem 3 *Let (q, m) be a 4-body central configuration. Then (q, m) is admissible if and only if, $q_i \neq 0$ and for a suitable enumeration of bodies, $q_1 = -q_3$, $q_2 = -q_4$, $m_1 = m_3$, $m_2 = m_4$, and (q, m) is of some of the following mutually exclusive types:*

CCcl. collinear,

CCr. *a rhombus with* $r_{13} < r_{24}$ *and* $m_1 > m_2$*,*
CCs. *a square with four equal masses.*

Remark 2 In [31] central configurations of type CCcl were studied; while, CCr configurations were treated in [18] and [28].

Proof From (9) we have to consider two cases.

Case 1. $m_1 \geqslant m_2$, $|q_1| \neq |q_2|$, $|q_1| = |q_3|$ and $|q_2| = |q_4|$. Now (6) implies that $m_1 = m_3$, $m_2 = m_4$, $q_1 = -q_3$ and $q_2 = -q_4$. We divide the plane into two open cones C_i, $i = 1, 2$, by means of the line P joining q_1 and q_3 together with its perpendicular bisector M. From Theorem 2, if q_2 is in C_1, then q_4 is in C_2, and vice versa. This is a contradiction with the fact that $q_2 = -q_4$. Then $q_2, q_4 \in P$ or $q_2, q_4 \in M$, i.e. q is collinear or a rhombus with equal masses in opposite vertices. In the first case, (q, m) is of CCcl type. In the second case, if $m_1 > m_2$, was proved in [18, Eqs. (3.44) and (3.45)] that $r_{13} < r_{24}$. Hence (q, m) is of CCr type. From [28, Corollary 2] if $m_1 = m_2$ then the configuration is a square which is a contradiction with the fact that $|q_1| \neq |q_2|$.

Case 2. $|q_1| = |q_2| = |q_3| = |q_4|$. In this situation, in [12] it was proved that the configuration is the equal mass square. □

5 Massless particle motion

In this section and in Section 6, we will suppose that the primary bodies are in a T-periodic rigid motion associated to an admissible CC (q, m), i.e $r(t) \equiv 1$ and according to remark that follows equation (4), $\theta(t) = \sqrt{\lambda}t$ (w.l.o.g we assume that $\theta(0) = 0$). To the particle, we suppose that it is moving on L, i.e. $x_0(t) = (0, 0, z(t))$. From Theorem 1, x_0 is solution of (5), if and only if $z(t)$ is solution of the autonomous equation

$$\ddot{z} = -\sum_{i=1}^{n} \frac{m_i z}{(s_i^2 + z^2)^{3/2}}, \tag{10}$$

where $s_i = |q_i|$.

We will analyze all possible motions for the massless particle x_0. In particular, we will see that every motion is either periodic or a scape trajectory. We will find that there exist T_0-periodic solutions for all T_0 on an interval $(\sigma(q, m), +\infty)$. This fact implies that there exists an infinity quantity of periodic solutions for the entire $n + 1$-body system.

The second order equation (10) is conservative, therefore solutions conserve the energy

$$E(z, v) := \frac{|v|^2}{2} - \sum_{i=1}^{n} \frac{m_i}{\left(s_i^2 + z^2\right)^{\frac{1}{2}}}, \tag{11}$$

i.e. $E(z(t), \dot{z}(t))$ is constant.

Following [3] (see also [19]) we introduce the next concepts.

Definition 4 (Chazy, 1922) Let $z(t)$ be a solution of (10) such that $\lim_{t\to\infty} z(t) = \infty$. Then $z(t)$ is called:

– hyperbolic when there exists $\lim_{t\to\infty} \dot{z}(t)$ and it is not null,

– parabolic if $\lim_{t\to\infty} \dot z(t) = 0$.

The following theorem characterizes all the possible motions for the massless particle.

Theorem 4 *We assume that* (q, m) *is an admissible configuration and the primaries are in a rigid motion. Every solution of* (10) *is of some of the following types:*

1. *Hyperbolic, when* $E > 0$,
2. *Parabolic, when* $E = 0$,
3. *Periodic, when* $E_{min} := -\sum_{i=1}^n \frac{m_i}{s_i} < E < 0$.
4. *Equilibrium solution when* $E = E_{min}$.

Proof We follow a standard argument for Hamiltonian systems (see [2]).

We consider the level sets $S(E) = \{(z, v) : E(z, v) = E\}$ on the phase space (z, v). An elementary analysis shows that

– If $E \geqslant 0$ then $S(E)$ is the union of two bounded graphs. They are symmetric with respect to z-axis, each of which is contained in some semiplane $v > 0$ or $v < 0$. The v-positive branch is the graph of a function $v(E, z)$, which is decreasing with respect to $|z|$. Moreover, $\lim_{|z|\to\infty} v(E, z) = \sqrt{2E}$.
– For every $E \geqslant E_{min}$, the energy curve $S(E)$ cuts the v-axis at the value $\pm(2E + 2\sum_{i=1}^n m_i s_i^{-1})^{\frac{1}{2}}$.
– If $E_{min} < E < 0$ then $S(E)$ is a simple closed curve symmetric with respect to z and v axes.
– An energy curve cuts the z-axis, only in the case that $E < 0$, at the point $\pm z_E$, where z_E is the only positive solution of $-\sum_{i=1}^n m_i(s_i^2 + z_E^2)^{-\frac{1}{2}} = E$.

In Figure 1, we show the phase portrait for a rhomboidal configuration with masses $m_1 = m_3 = 1$ and $m_2 = m_4 = 0.5$.

The function $\varphi(t) = (z(t), \dot z(t))$ solves the system $\dot\varphi(t) = F(\varphi(t))$, where $F(z, v) = (v, -\sum_{i=1}^n m_i z(s_i^2 + z^2)^{-3/2})$. It is easy to show that the vector field F has a bounded Jacobian DF. Therefore $F(z, v)$ is a global Lipschitz function on $\mathbb{R}^2$. This fact and [5, Th. B.1] imply that the trajectories $t \mapsto (z(t), \dot z(t))$ are defined for every time. On the other hand, since $\dot z = v$, the motion along trajectories is in clockwise direction. The only fixed point of F is $(z, v) = (0, 0)$. Therefore, the level surfaces $S(E)$, with $E \neq E_{min}$, do not contain stationary points. Then the $\lim_{t\to\infty} \varphi(t)$ does not exists. As consequence the map $t \mapsto \varphi(t)$ fills completely one connected component of its corresponding energy curve.

We observe that any solution z crosses the v-axis. On the other hand, if $E \geqslant 0$ and $v(E, 0) > 0$ ($v(E, 0) < 0$) then $z(t)$ is increasing (decreasing) with respect to t. If $z(t)$ remained bounded when $t \to +\infty$, then there would be the limit $\zeta_\infty := \lim_{t\to\infty} z(t)$. This would imply that $(\zeta_\infty, 0)$ would be a fixed point of F, which is a contradiction. As a consequence, if $E \geqslant 0$ then $|z(t)| \to \infty$ when $t \to +\infty$. Moreover $\lim_{t\to+\infty} \dot z(t) = \pm\sqrt{2E}$. From this fact, we conclude that the trajectory is hyperbolic when $E > 0$ and it is parabolic in the case that $E = 0$.

In the case that $E_{min} < E < 0$, we have that the trajectory is contained in a closed curve; therefore, it is a periodic orbit.

Finally, if $E = E_{min}$ we clearly have that $z(t) \equiv 0$. □

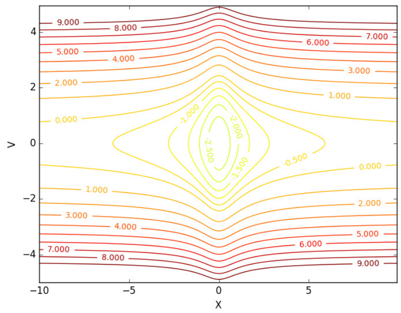

Fig. 2 Energy level for a rhomboidal configuration with masses $m_1 = m_3 = 1$ and $m_2 = m_4 = 0.5$.

Theorem 5 *We denote by $T_0(E)$ the minimal period for a solution of (10) with $E_{min} < E < 0$. Then*

1.

$$T_0(E) = 2^{3/2} \int_0^{z_E} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz, \tag{12}$$

where z_E is the only positive solution of $-\sum_{i=1}^n m_i(s_i^2 + z_E^2)^{-\frac{1}{2}} = E$,

2. $T_0(E)$ is an increasing function.

3. $T_0\left((E_{min}, 0)\right) = (T_{min}, +\infty)$, where $T_{min} = 2\pi \left(\sum_{i=1}^n \frac{m_i}{s_i^3} \right)^{-1/2}$.

Proof Let $E_{min} < E < 0$ and let $z(t)$ be the only solution with $z(0) = 0$, $\dot{z}(0) > 0$ and energy equals to E. Therefore $z(t)$ is $T_0(E)$-periodic. As a consequence of the symmetries of the equation, we have that $z(T_0(E)/4) = z_E$. Then, taking account of (11), we have

$$\begin{aligned} \frac{T_0}{4} &= \frac{1}{\sqrt{2}} \int_0^{T_0/4} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} \dot{z} dt \\ &= \frac{1}{\sqrt{2}} \int_0^{z_E} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz, \end{aligned}$$

and we have proved item *1*. In order to prove item *2*, we note that

$$\begin{aligned} 2^{-3/2} T_0(E) &= \int_0^{z_E} \left(\sum_{i=1}^{n} m_i \left((s_i^2 + z^2)^{-\frac{1}{2}} - (s_i^2 + z_E^2)^{-\frac{1}{2}} \right) \right)^{-\frac{1}{2}} dz \\ &= \int_0^{z_E} \left(z_E^2 - z^2 \right)^{-\frac{1}{2}} f(z, z_E) dz \\ &= \int_0^{1} \left(1 - u^2 \right)^{-\frac{1}{2}} f(z_E u, z_E) du, \end{aligned}$$

where

$$f(z,z_E)=\left(\sum_{i=1}^{n} m_i\left\{(s_i^2+z^2)(s_i^2+z_E^2)\right\}^{-\frac{1}{2}}\left\{(s_i^2+z^2)^{\frac{1}{2}}+(s_i^2+z_E^2)^{\frac{1}{2}}\right\}^{-1}\right)^{-\frac{1}{2}}.$$

We point out that $f(z_Eu,z_E)$ is an increasing function with respect to z_E for $u\in[0,1]$ fix. This assertion implies item 2.

On the other hand,

$$\lim_{z_E\to 0} f(z_Eu,z_E)=\left(\sum_{i=1}^{n}\frac{m_i}{2s_i^3}\right)^{-\frac{1}{2}} \quad\text{and}\quad \lim_{z_E\to+\infty} f(z_Eu,z_E)=+\infty.$$

Thus, from the Dominated Convergence Theorem and Monotone Convergence Theorem, we have

$$\lim_{E\to E_{min}} T_0=\lim_{z_E\to 0} T_0=2\pi\left(\sum_{i=1}^{n}\frac{m_i}{s_i^3}\right)^{-\frac{1}{2}} \quad\text{and}\quad \lim_{E\to 0} T_0=\lim_{z_E\to+\infty} T_0=+\infty.$$

Finally, since $T_0=T_0(z_E)$ is continuous and increasing with respect to z_E, we conclude the statement of item 3. ◻

Remark 3 It is possible to use the classical theory of Hamiltonian systems (see [2]) to derive the formula (12) (see [1] for this approach in a related problem).

Remark 4 Let us show a second proof of item 3 of Theorem 5.

The inequality $T_0>T_{min}$ is a consequence of comparison Sturm's theorem applied to equations $\ddot z+h(z)z=0$, where $h(z)=\sum_{i=1}^{n} m_i\left(s_i^2+z^2\right)^{-3/2}$, and $\ddot z+\left(\sum_{i=1}^{n} m_i s_i^{-3}\right)z=0$. This proves that $T_0\left((E_{min},0)\right)\subset(T_{min},+\infty)$.

For the reverse inclusion, we follow arguments of [35] and [15] based on variational principles.

Let $T_0>T_{min}$. We consider the action integral

$$\mathcal{I}(z)=\int_0^{T_0}\frac{1}{2}|\dot z|^2+\sum_{i=1}^{n}\frac{m_i}{\sqrt{s_i^2+z^2}}dt,$$

Then T_0-periodic solutions of (10) are critical points of $\mathcal{I}$ in the space $H^1(\mathbb{T},\mathbb{R})$ of the functions which are absolutely continuous, T_0-periodic with $\dot z\in L^2(\mathbb{T},\mathbb{R})$ and being $\mathbb{T}=\mathbb{R}/T_0\mathbb{Z}$ (see [20, Cor. 1.1]). We prove the existence of critical points by means of the direct method of calculus of variations, i.e. we will prove that $\mathcal{I}$ has a minimum. The functional $\mathcal{I}$ is not coercive in $H^1(\mathbb{T},\mathbb{R})$. This deficiency is overcome with symmetry techniques (see [34]). The group $\mathbb{Z}_2$ acts on $H^1(\mathbb{T},\mathbb{R})$ according to the following assignments $(\bar 0\cdot z)(t)=z(t)$ and $(\bar 1\cdot z)(t)=-z(t+\frac{T_0}{2})$. The symmetry involved in previous definition is called *Italian Symmetry*. The functional $\mathcal{I}$ is $\mathbb{Z}_2$-invariant, i.e. $\mathcal{I}(g\cdot z)=\mathcal{I}(z)$. We define the space of all $\mathbb{Z}_2$-symmetric functions

$$\Lambda(\mathbb{T},\mathbb{R}):=\left\{z\in H^1(\mathbb{T},\mathbb{R})|\forall g\in\mathbb{Z}_2: z=g\cdot z\right\}.$$

The functional $\mathcal{I}$ restricted to Λ is coercive. This fact follows from an obvious adaptation of Proposition 4.1 of [34]. We note that $F(z):=\sum_{i=1}^{n} m_i(s_i^2+z^2)^{-\frac{1}{2}}$ satisfies the condition (A) in [20, p. 12], then $\mathcal{I}$ is continuously differentiable and weakly lower semicontinuous on $H^1(\mathbb{T},\mathbb{R})$ (see [20, p. 13]). Therefore $\mathcal{I}$ has a minimum z_0 in $\Lambda(\mathbb{T},\mathbb{R})$. Then by the Palais' principle of symmetric criticality, z_0 is a critical point of $\mathcal{I}$ in $H^1(\mathbb{T},\mathbb{R})$ (see [34] and [26]).

We use the second variation $\delta^2\mathcal{I}$ in order to show that $z_0 \not\equiv 0$. It is well known (see [13, Th. 1.3.1]) that if z_0 is a minimum of $\mathcal{I}$ on $H^1(\mathbb{T},\mathbb{R})$ then $\delta^2\mathcal{I}(z_0,\varphi) \geqslant 0$ for all $\varphi \in H^1(\mathbb{T},\mathbb{R})$. In our case,

$$\delta^2\mathcal{I}(0,\varphi) = \int_0^{T_0} |\dot{\varphi}|^2 - \sum_{i=1}^n \frac{m_i}{s_i^3}\varphi^2 dt,$$

(see [13, Eq. 1.3.6]). In particular, if $\varphi(t) = \sin(2\pi t/T_0)$ it follows from $T_0 > T_{min}$ that

$$\delta^2\mathcal{I}(0,\varphi) = \left(\frac{4\pi^2}{T_0^2} - \sum_{i=1}^n \frac{m_i}{s_i^3}\right)\frac{T_0}{2} < 0. \tag{13}$$

It is sufficient to guarantee that $z_0 \equiv 0$ is not a minimum. □

This second proof, unlike the first one, does not prove that T_0 is the minimum period of z_0. It could happen that z_0 had period T_0/m, with natural $m \in \mathbb{N}$. Because of Italian symmetry this m should be odd.

Corollary 1 *The complete $n+1$-body system has an infinity quantity of periodic solutions.*

Proof We recall that T denotes minimal period of the primaries. Let l/m be a positive rational number with $Tl/m > T_{min}$. Then, there exists a solution of the entire system with period lT. □

6 Synchronous solutions and pyramidal CC

If the equation (10) has a T-periodic solution, we say that the solution is *synchronous*. In [15] was studied the problem of existence of synchronous solutions for n equal mass primary bodies in a regular polygon configuration.

In this section we establish a relation between the existence of synchronous solutions and the concept of pyramidal central configuration (see [10, 11, 25]).

Definition 5 A central configuration of $n+1$ mass point $q_0,\dots,q_n$ in $\mathbb{R}^3$ is called a pyramidal central configuration (PCC) if and only if n points, we say $q_1,\dots,q_n$, are in some plane Π and $q_0 \notin \Pi$.

The following lemma was proved in [25] (see also [11]).

Lemma 2 ([25], Lemma 2.1) *Let $q_0,\dots,q_n$ be a PCC such that m_0 is off the plane containing $m_1,\dots,m_n$. If $m_0 > 0$ then m_0 is equidistant from $m_1,\dots,m_n$.*

We remark that the condition $m_0 > 0$ is important in the previous lemma. In the examples below, we will show two PCC with $m_0 = 0$ which do not satisfy the conclusion of Lemma 2.

Proposition 1 *We assume that $q = q_1,\dots,q_n$ is an admissible configuration and that the primaries are in a rigid motion. Then, there is a synchronous solution if and only if there exists $c \in \mathbb{R}$ such that the points $(0,0,c), q_1,\dots,q_n$ associated to the masses $0, m_1,\dots,m_n$ form a PCC.*

Proof We start assuming that there exist a synchronous solution. As a consequence of the Theorem 5(3) and the fact that $T^2 = 4\pi^2/\lambda$, we get

$$\lambda < \sum_{i=1}^{n} \frac{m_i}{s_i^3}. \tag{14}$$

Since $\sum_{i=1}^{n} m_i \left(s_i^2 + c^2\right)^{-3/2} \to 0$, when $c \to +\infty$, there exists $c \in \mathbb{R}$ such that $\sum_{i=1}^{n} m_i \left(s_i^2 + c^2\right)^{-3/2} = \lambda$. Therefore

$$-\sum_{i=1}^{n} \frac{m_i c}{\left(s_i^2 + c^2\right)^{3/2}} = -\lambda c. \tag{15}$$

As $q_1, \ldots, q_n$ is an admissible configuration, then

$$\sum_{i=1}^{n} \frac{m_i q_i}{\left(s_i^2 + c^2\right)^{3/2}} = (0, 0). \tag{16}$$

The equations (15), (16) and the fact that $q_1, \ldots, q_n$ is a CC with constant λ, complete the proof. The proof of the reciprocal statement follows in a direct way. □

Corollary 2 *We assume that (q, m) is an admissible configuration and the primaries are in a rigid motion. Then, there is a synchronous solution if and only if*

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} < \left(\sum_{i=1}^{n} \frac{m_i}{s_i^3}\right)\left(\sum_{i=1}^{n} m_i s_i^2\right). \tag{17}$$

Proof The result is a consequence of (14) and the fact that $T^2 = 4\pi^2 \sum_{i=1}^{n} m_i s_i^2 / U$ (see [16, p. 109]). □

Remark 5 Let (q, m) be an admissible CC with constant $\lambda > 0$ satisfying (17) and let r, μ be positive numbers. Then $(rq, \mu m)$ is a CC with constant $\lambda \mu r^3$, and (17) remains unchanged. In virtue of previous observation, we can assume that any length and any mass take any desired value. The equation (10) has a synchronous solution if and only if the same equation with $(rq, \mu m)$ instead of (q, m) has a synchronous solution.

The sufficiency of the condition $n \leqslant 472$ in the following corollary was proved in [15].

Corollary 3 *We suppose that (q, m) is the equal masses regular polygon configuration (this is an admissible CC). Then, there exists a synchronous solution if and only if $2 \leqslant n \leqslant 472$.*

Proof In this case $s_1 = s_2 = \cdots = s_n =: r$ and $m_1 = m_2 = \cdots = m_n =: M$. Then, from the law of cosines, we obtain

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} = \frac{nM^2}{4r} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)}.$$

Therefore, the condition (17) is equivalent to

$$\frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{j\pi}{n}\right)} < 4. \tag{18}$$

This inequality was also derived by Li, J. et al. in [15], where the authors proved (performing computer calculations) that inequality (18) holds true for $2 \leqslant n \leqslant 472$. Let us prove that any other n does not satisfy (18).

Using that $1/\sin(x)$ is a convex function on $[0, \pi]$ and the composite trapezoid rule (see [14]), we have

$$\begin{aligned}\int_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi}\frac{1}{\sin(x)}dx &\leqslant \frac{\pi}{2n}\left\{\frac{1}{\sin(\frac{\pi}{n})}+\frac{1}{\sin(\frac{n-1}{n}\pi)}+2\sum_{j=2}^{n-2}\frac{1}{\sin(j\frac{\pi}{n})}\right\}\\ &= \frac{\pi}{n}\sum_{j=1}^{n-2}\frac{1}{\sin(j\frac{\pi}{n})}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{j\pi}{n}\right)} &\geqslant \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\frac{\pi(n-1)}{n}}\frac{1}{\sin(x)}dx+\frac{1}{n\sin\left(\frac{n-1}{n}\pi\right)}\\ &= \frac{1}{2\pi}\log\left(\frac{1-\cos(x)}{1+\cos(x)}\right)\Bigg|_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi}+\frac{1}{n\sin\left(\frac{\pi}{n}\right)}\\ &= \frac{1}{\pi}\left\{\log\left(\frac{1+\cos(\frac{\pi}{n})}{1-\cos(\frac{\pi}{n})}\right)+\frac{\pi/n}{\sin\left(\frac{\pi}{n}\right)}\right\}\\ &=: f\left(\frac{\pi}{n}\right).\end{aligned}$$

It is easy to see that $f(x)$ is a decreasing function on $(0, \pi/2)$. Moreover $f(\pi/842) \approx 4.0006 > 4$. Thus, if $n \geqslant 842$ then n does not satisfy inequality (18). The validity of the inequality (18), for $n \leqslant 841$ can be easily checked using computer. This gives the result that the inequality holds only for $n \leqslant 472$. □

Our next goal is to verify that condition (17) is satisfied for all admissible CC of 3-body or 4-body. Since that (17) holds for an equilateral triangle and square configurations of equal masses bodies, it only rests to prove, in virtue of Theorem 3, the following result.

Theorem 6 *The central configurations CCcl and CCr satisfy condition* (17).

Proof Let's start by analyzing the central configuration CCr. From Remark 5, we can suppose without loss of generality that $q_1 = -q_3 = (0, y)$ for $0 < y < 1$, $q_2 = -q_4 = (1, 0)$. The condition (17) becomes

$$\frac{m_1^2}{2y}+\frac{4m_1m_2}{\sqrt{1+y^2}}+\frac{m_2^2}{2} < \left(\frac{2m_1}{y^3}+2m_2\right)\left(2m_1y^2+2m_2\right).$$

As $m_1^2/(2y) < 4m_1^2/y$, $m_2^2/2 < 4m_2^2$ and $4m_1m_2/\sqrt{1+y^2} < 4m_1m_2/y^3$ (since $y < 1$), we have that the inequality holds.

Now we consider the central configuration CCl. From Remark 5 again, we can suppose that $q_1 = -q_3 = 1$, $q_2 = -q_4 = x$ with $0 < x < 1$, and $m_1 = m_3 = \mu$, $m_2 = m_4 = 1-\mu$, with $0 < \mu < 1$. Then, inequality (17) becomes

$$\frac{2\mu(1-\mu)}{1-x} + \frac{2\mu(1-\mu)}{1+x} + \frac{\mu^2}{2} + \frac{(1-\mu)^2}{2x} < 4\mu^2 + 4\mu(1-\mu)x^2 + \frac{4\mu(1-\mu)}{x^3} + \frac{4(1-\mu)^2}{x}.$$

As $\mu^2/2 < 4\mu^2$ and $(1-\mu)^2/(2x) < 4(1-\mu)^2/x$, it is sufficient to show that

$$\frac{2\mu(1-\mu)}{1-x} + \frac{2\mu(1-\mu)}{1+x} < \frac{4\mu(1-\mu)}{x^3},$$

and, this is equivalent to see that

$$\frac{x^3}{1-x^2} < 1. \tag{19}$$

The values of x involved in the inequality above are such that the configuration of positions $(-1,-x,x,1)$ and masses $(\mu, 1-\mu, 1-\mu, \mu)$ is central. It was shown in [24] that given a mass μ there is only one value of x satisfying this condition (see also [31]). Consequently, we can define $x(\mu)$ as such value of x. We note that $h(x) = x^3/(1-x^2)$ is an increasing function with respect to $x \in (0,1)$ and $h(x) < 1$ for $x \in (0, 3/4)$. Hence, if we could prove that $x(\mu)$ is a decreasing function and

$$\lim_{\mu\to 0} x(\mu) < 3/4, \tag{20}$$

we would have justified (19).

Let's first prove that $x(\mu)$ is a decreasing function. Eliminating λ from the equations (2) and replacing q_j and m_j by their expressions in x and μ, we get

$$\frac{\mu}{4} - \frac{\mu}{x\,(x+1)^2} + \frac{\mu}{x\,(-x+1)^2} + \frac{-\mu+1}{(x+1)^2} + \frac{-\mu+1}{(-x+1)^2} - \frac{1}{x^3}\left(-\frac{\mu}{4} + \frac{1}{4}\right) = 0,$$

which is equivalent to

$$\mu = -\frac{8x^5 - x^4 + 8x^3 + 2x^2 - 1}{(x-1)\,(x+1)\,(x^5 - 9x^3 + x^2 - 1)}.$$

Therefore

$$\frac{d\mu}{dx} = \frac{x^2\left(16x^9 - 3x^8 + 32x^7 + 12x^6 - 304x^5 - 2x^4 + 44x^2 - 51\right)}{(x-1)^2\,(x+1)^2\,(x^5 - 9x^3 + x^2 - 1)^2}.$$

Since $44x^2 < 51$ and $16x^9 + 32x^7 + 12x^6 < 304x^5$ for $x \in (0,1)$, then $d\mu/dx < 0$ on the interval $(0,1)$. Which, in turn, implies that x is decreasing with respect to μ.

Let's see now that (20) holds. When μ goes to 0, $x(\mu)$ converges to the only solution on the interval $(0,1)$ of equation $8x(0)^5 - x(0)^4 + 8x(0)^3 + 2x(0)^2 - 1 = 0$. Then, $8x(0)^3 - 1 < 0$ which implies that $x(0) < 3/4$ as we wanted to prove. □

Remark 6 As a consequence of previous results, there exist five-body $PCC's$ with $m_1, \dots, m_4$ in a CCcl or CCr configuration and the mass $m_0 = 0$ is in the perpendicular line to the plane containing $m_1, \dots, m_4$ and passing by the center of mass. These are examples of $PCC's$ which do not verify the conclusion of Lemma 2.

Corollary 4 *For all admissible CC of 3-body or 4-body, the problem P has a synchronous solution.*

7 Non admissible central configurations

The following result shows when a non-admissible CC has a solution of the problem P.

Theorem 7 *We suppose that (q, m) is a non admissible CC with $q_i \neq 0$ and that the primaries are in a homographic motion, i.e. equation (1) is satisfied. Assume that the massless particle is moving on the z-axis with position vector $x_0(t) = (0, 0, z(t))$. Then, one and only one of the following statements is satisfied:*

1. *The massless particle is in a stationary motion and*

$$\sum_{i=1}^{n} \frac{m_i q_i}{s_i^3} = 0, \tag{21}$$

i.e. the positions $0, q_1, \ldots, q_n$ and the masses $0, m_1, \ldots, m_n$ are in a CC.

2. *The $n+1$-body system is in a homothetic motion. i.e. $Q(\theta(t))$ in the equation (1) is the identity matrix and $z(t) = cr(t)$, for some constant c. Moreover, the configuration $q_0, \ldots, q_n$ is a PCC, where $q_0 = (0, 0, c)$ and $m_0 = 0$.*

Proof We recall the definition of the function f and line L from Section 3.

The fact that the massless particle is moving on L is equivalent to the condition $f(t, x_0(t)) \in L$ for all $t \in \mathcal{O}$, which is equivalent to the equality

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{(r(t)^2 |q_i|^2 + z(t)^2)^{3/2}} = 0, \tag{22}$$

for every $t \in \mathcal{O}$.

With the same notation and reasoning as in the proof of Theorem 1, we prove that

$$\sum_{j=1}^{k} \left\{ \frac{1}{(s_j^2 + (z(t)/r(t))^2)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0. \tag{23}$$

If $z(t)/r(t)$ would be a non constant function then previous equation and Lemma 1 would imply that q is admissible, which is a contradiction. Hence, there exists $c \in \mathbb{R}$ such that $z(t) = cr(t)$. Now, we have two cases.

Case 1: $c = 0$. Then $z \equiv 0$ and (21) follows from (22).

Case 2: $c \neq 0$. From equation (10), the Kepler equations (4) and the fact that $z(t) = cr(t)$, we have

$$-\frac{1}{r(t)^2} \sum_{i=1}^{n} \frac{m_i}{(s_i^2 + c^2)^{3/2}} = -\frac{\lambda}{r(t)^2} + r(t)\dot{\theta}(t)^2. \tag{24}$$

The second equality in (4) implies the Kepler's second law, i.e. there exists $d \in \mathbb{R}$ such that $r^2 \dot{\theta} \equiv d$. Replacing $\dot{\theta}$ in equation (24) and multiplying by $r(t)^3$, we obtain

$$-r(t) \left(\sum_{i=1}^{n} \frac{m_i}{(s_i^2 + c^2)^{3/2}} - \lambda \right) = d^2. \tag{25}$$

Therefore, if $d \neq 0$ then $\dot{r}(t) \equiv 0$, and this implies $\dot{z}(t) \equiv 0$. As $z(t)$ is a constant function and it solves equation (10), then $z(t) \equiv 0$. Hence we are in case 1 again.

Consequently we suppose $d = 0$. Therefore $\theta(t)$ is a constant function and the motion is homothetic. From (23) and (25), we deduce that in this new situation equations (15) and (16) hold. This fact, as in the proof of Proposition 1, implies the desired result.

□

Example 1 We present an example of a $3+1$-body system satisfying the situation described in item 1 of Theorem 7, i.e. (q, m) is a non admissible CC and $z(t) \equiv 0$. For this purpose, it is sufficient to find a 4-body CC with a zero mass body located in the center of mass.

We start with an Euler's collinear central configuration formed by three primary bodies of masses $m_1 = 4 - \mu$, $m_2 = 2 + \mu$ and $m_3 = 1$, where $0 < \mu < 1$, and positions, with respect to a convenient 1-dimensional coordinate system, given by $q_1 = 0$, $q_2 = 1$ and $q_3 = 1 + r$. It is known (see [22]) that r is the only positive solution of

$$p(r, \mu) := 6r^5 + (16 - \mu)\, r^4 + (14 - 2\mu)\, r^3 - (\mu + 5)\, r^2 - (2\mu + 7)\, r - \mu - 3 = 0.$$

Since $p(0, \mu) = -\mu - 3$ and $p(1, \mu) = -7\mu + 21$, then $r = r(\mu) \in (0, 1)$, for all $0 < \mu < 1$. Therefore, as the center of mass $C = C(\mu)$ is equal to $(\mu + r + 3)/7$, we obtain $C \in (0, 1)$.

We consider a massless particle with coordinate x. The acceleration resulting from the action of the gravitational field is equal to

$$f(x) = -\frac{4-\mu}{x^2} + \frac{\mu+2}{(-x+1)^2} + \frac{1}{(r-x+1)^2}.$$

Note that the right hand side of the previous equation is an increasing function that tends to $-\infty$ when x goes to 0, and tends to $+\infty$ when x goes to 1, so there is a unique point $\bar{x} = \bar{x}(\mu) \in (0, 1)$ such that the equality $f(\bar{x}) = 0$ holds. This point is an equilibrium for the gravitational field generated for the primaries.

Let's see that there exists $\mu \in (0, 1)$ such that $C(\mu) = \bar{x}$, i.e. $f(C) = 0$. For this purpose, since C is a continuous function with respect to μ, it is sufficient to show that f changes its sign on $(0, 1)$. The function $f(x)$ can be written as

$$f(x) = \frac{g(x)}{h(x)},$$

where $h(x) = x^2\,(x-1)^2\,(r-x+1)^2$. Note that $h(x) > 0$ for all $x \in (0, 1)$. If we consider $\mu = 0$ and compute $g(C)$, we have

$$g(C) = \frac{r^4}{2401} + \frac{1514r^3}{2401} + \frac{2245r^2}{2401} + \frac{1110r}{2401} + \frac{333}{2401} > 0.$$

On the other hand, if $\mu = 1$ then

$$g(C) = -\frac{71r^4}{2401} + \frac{1486r^3}{2401} + \frac{401r^2}{2401} - \frac{1480r}{2401} - \frac{592}{2401} < 0,$$

because $0 < r < 1$.

Remark 7 The following question is posed. Is there some non admissible central configuration (q, m) such that the $n+1$-body system perform the motion described in Theorem 7(2)?

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