Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

Sonia Acinas *
Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}, \qquad (P_1)$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) := \frac{|y|^p}{p} + F(t,x), \qquad (1)$$

*SECyT-UNRC and FCEyN-UNLPam
†SECyT-UNRC, FCEyN-UNLPam and CONICET
2010 AMS Subject Classification. Primary: . Secondary: .
Keywords and phrases. .

for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (P_1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (1) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (P_1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t, u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P_2}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{2}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [Acinas et al., 2015] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{3}$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (P_1).

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions $\Phi : \mathbb{R}^n \to \mathbb{R}_+$, i.e. functions such that $\Phi(x)$ depends on the direction of x, unlike the radial case where $\Phi(x) = \Phi(|x|)$. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of p-laplacian equations as the following example shows.

Example 1. Let $1 < p_1, p_2 < \infty$. We define $\Phi_{p_1,p_2} : \mathbb{R}^n \to \mathbb{R}_+$ by

$$\Phi_{p_1,p_2}(y_1,y_2) := \frac{|y_1|}{p_1} + \frac{|y_2|}{p_2}.$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. Suppose the following Lagrange function

$$\mathcal{L}(t,x,y) = \Phi_{p_1,p_2}(y) + F(t,x).$$

Then the equations (P_1) becomes

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) \quad \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) \quad \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \qquad (P_3)$$

EN CONSTRUCCION!!!!...FALTA ORDENAR, REVISAR, AGREGAR O QUITAR....

Las F de arriba....qué hacemos????

We apply the method of Lagrange multipliers to solve the problem

$$G(r) = \min\{\Phi(x,y) : \frac{1}{2}\|(x,y)\|_2^2 = \frac{r^2}{2}\}$$

then $F(x,y) = \Phi(x,y) - \lambda\frac{1}{2}\|(x,y)\|_2^2$.

Now, the first order equations are

$$\begin{cases} |x|^{p_1-2}x + \lambda x &= 0 \\ |y|^{p_2-2}y + \lambda y &= 0 \\ |x|^2 + |y|^2 &= r^2 \end{cases}$$

The critics points are $(x, 0, -|x|^{p_1-2})$ and $(0, y, -|y|^{p_2-2})$ with $|x| = r$ and $|y| = r$ respectively.

In case of $2 \leqslant p_1 \leqslant p_2$, the critic values are $\Phi(x,0) = \frac{r^{p_1}}{p_1}$ and $\Phi(0,y) = \frac{r^{p_2}}{p_2}$

Now, suppose that $x \neq 0$ and $y \neq 0$ then $|x|^2 + |y|^2 = r^2$ and $|y| = |x|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|x|^{p_1-2}$.

If r is bigger enough then $G(r) \sim r^{p_1}$. And, if r is small enough we have $G(r) \sim r^{p_2}$????

If $p_1 \leqslant 2$ and $p_2 \leqslant 2$ with one of them different to 2, the second order criterium [**?**, Thm. —] implies that the minimum appear in $x = 0$ or $y = 0$.

These equations are known in the literature as (p_1, p_2)-Laplacian system, see [Yang and Chen, 2013, Pasca and Wang, 2016, Yang and Chen, 2012, Pasca, 2010, Paşca and Tang, 2010, Pasca and Tang, 2011].

On the other hand, anisotropic Orlicz-Sobolev spaces allow to simplify the writing, and they provide the natural frame of statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question was what motivated us to use these spaces.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$. Additionally, we assume that the Young's functions which we deal with, satisfy that $\Phi(x) > 0$ when $x \neq 0$. Following [Schappacher, 2005] we say that Φ is an N_∞-function if

$$\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function Φ, we define function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, \|x\| = s\right\}, \tag{4}$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.1. *The function A_Φ has the following properties:*

1. *A_Φ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *$A_\Phi(|x|)$ is the* greatest radial minorant *of $\Phi(x)$,*
4. *Φ is N_∞ if and only if A_Φ is.*

Proof. It is well known that finite and convex functions defined in finite dimensional vectorial spaces are locally Lipschitz functions (see [Clarke, 2013]). This fact imply item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi\left(x\right)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

We also say that $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ satisfies the Δ_2^∞*-condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant KH(x), \tag{5}$$

for every $|x| \geqslant M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \to \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x\in\mathbb{R}^d} x \cdot y - \Phi(x) \tag{6}$$

We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\; dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{7}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{8}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of E^Φ, which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \ldots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \tag{9}$$

Like in [Krasnosel′skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{10}$$

for any positive r (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

Definition 2.2. *Let $u_n, u \in L^\Phi([0,T],\mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^\infty([0,T],\mathbb{R})$, $n = 1, 2, \ldots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t) u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow \left[L^\Phi\right]^*$, where a function $v \in L^{\Phi^*}$ is associated to $\xi_v \in \left[L^\Phi\right]^*$ being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \tag{11}$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $\left[L^\Phi\right]^*$ which can be identified with L^{Φ^*}.

Proposition 2.3. *Let $F \in \left[L^{\Phi}([0,T],\mathbb{R}^d)\right]^*$. Then the following statements are equivalent*

1. $\xi \in L^{\Phi^*}([0,T],\mathbb{R}^d)$
2. ξ *satisfies the* monotone convergence property*, which is if u_n converges monotonically to u then $\langle \xi, u_n\rangle \to \langle \xi, u\rangle$.*

If $\Phi \in \Delta_2^{\infty}$ and Φ is N_∞ then $L^{\Phi^*}([0,T],\mathbb{R}^d) = \left[L^{\Phi}([0,T],\mathbb{R}^d)\right]^*$ (see [Desch and Grimmer, 2001, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi([0,T],\mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi([0,T],\mathbb{R}^d)\}.$$

$W^1L^\Phi([0,T],\mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{12}$$

And, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{13}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.4. *Let $\Phi : \mathbb{R}^d \to [0,+\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (4)*. Then*

1. *For every $s,t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s-t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If Φ is N_∞ then the space $W^1L^\Phi([0,T],\mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0,T],\mathbb{R}^d)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned}\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right)\\ &\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}.\end{aligned}$$

By Proposition 2.1(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.1 (2) and we have

$$\begin{aligned}|u(t)-\overline{u}| &= \left|\frac{1}{T}\int_0^T u(t)-u(s)\,ds\right|\\ &\leqslant \frac{1}{T}\int_0^T |u(t)-u(s)|\,ds\\ &\leqslant \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right)\end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right)ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.1(3)

$$|\overline{u}| \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^\Phi([0,T],\mathbb{R}^d)$. Since Φ is N_∞, from Proposition 2.1(4) we obtain $sA_\Phi^{-1}(1/s) \to 0$ when $s \to 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (Sobolev's inequality) implies that u_n is bounded in $C([0,T],\mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0,T],\mathbb{R}^d)$ with $u_{n_k} \to u$ in $C([0,T],\mathbb{R}^d)$.

□

Lemma 2.5. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi,1)$ converging to $u\in\Pi(E^\Phi,1)$ in the L^Φ-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h\in L^1([0,T],\mathbb{R})$ such that $u_{n_k}\to u \quad$ a.e. and $\Phi(u_{n_k})\leqslant h \quad$ a.e.*

Proof. Since $d(u,E^\Phi)<1$ and u_n converges to u, there exists $u_0\in E^\Phi$, a subsequence of u_n (again denoted u_n) and $0<r<1$ such that $d(u_n,u_0)<r$. Let $\lambda_0\in(r,1)$. By extracting more subsequences, if necessary, we can assume that $u_n\to u$ a.e. and

$$\lambda_n := \|u_{n+1}-u_n\|_{L^\Phi} < \frac{1-\lambda_0}{2^n}, \quad \text{for } n\geqslant 1.$$

We can assume $\lambda_n>0$ for every $n=0,\ldots$.

Let $\lambda := 1-\sum_{n=0}^{\infty}\lambda_n$ and define $h:[0,T]\to\mathbb{R}$ by

$$h(x) = \lambda\Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^{\infty}\lambda_n\Phi\left(\frac{u_{n+1}-u_n}{\lambda_n}\right). \tag{14}$$

Note that $\sum_{n=0}^{\infty}\lambda_n+\lambda=1$, therefore for any $n=1,\ldots$

$$\begin{aligned}\Phi(u_n) &= \Phi\left(\lambda\frac{u_0}{\lambda} + \sum_{j=0}^{n-1}\lambda_j\frac{u_{j+1}-u_j}{\lambda_j}\right)\\ &\leqslant \lambda\Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1}\lambda_j\Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right)\leqslant h\end{aligned}$$

Since $v\in E^\Phi\subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda)\in L^1([0,T],\mathbb{R})$. On the other hand $\|u_{n+1}-u_n\|_{L^\Phi}\leqslant\lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right)dt\leqslant 1.$$

Then $h\in L^1([0,T],\mathbb{R})$.

□

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anistropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined in Sobolev-Orlicz spaces.

Henceforth we assume that $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$is a *Carathéodory function*, i.e.

(C) f is measurable with respect to $t\in[0,T]$ for every $x\in\mathbb{R}^d$, and f is a continuous function with respect to $x\in\mathbb{R}^d$ for a.e. $t\in[0,T]$.

Definition 3.1. *For $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0,T] \to \mathbb{R}^d$ by*

$$\boldsymbol{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators definied in anisotropic Orlicz spaces of vectorial functions. For the proofs see [Krasnosel'skii et al., 2011] for scalar functions and [Płuciennik, 1987, Płuciennik, 1985b, Płuciennik, 1985a] for the generalization to $\mathbb{R}^d$-valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

Theorem 3.2. *We assume that f satisfies condition* ((C)) *and that $\Phi_1, \Phi_2 : \mathbb{R}^d \to [0, +\infty)$ are anisotropic Young functions. Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps masurable function into measurable functions*
2. Extensibility. *If the operator $\boldsymbol{f}$ acts from the ball $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} | \|u\|_{L^{\Phi_1}} < r\}$ into the space L^{Φ_2} or the space E^{Φ_2} then $\boldsymbol{f}$ can be extended to $\Pi(E^{\Phi_1}, r)$ into space L^{Φ_2} or E^{Φ_2} respectively.*
3. Continuity. *If the operator $\boldsymbol{f}$ acts from $\Pi(E^{\Phi_1}, r)$ into space E^{Φ_2}, then $\boldsymbol{f}$ is continous.*

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

References

[Acinas et al., 2015] Acinas, S., Buri, L., Giubergia, G., Mazzone, F., and Schwindt, E. (2015). Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698.

[Clarke, 2013] Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Graduate Texts in Mathematics.

[Desch and Grimmer, 2001] Desch, W. and Grimmer, R. (2001). On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc*, (353):5095–5120.

[Krasnosel'skii et al., 2011] Krasnosel'skii, M., Zabreyko, P., Pustylnik, E., and Sobolevski, P. (2011). *Integral operators in spaces of summable functions*. Mechanics: Analysis. Springer Netherlands.

[Krasnosel′skiĭ and Rutickiĭ, 1961] Krasnosel′skiĭ, M. A. and Rutickiĭ, J. B. (1961). *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen.

[Mawhin and Willem, 1989] Mawhin, J. and Willem, M. (1989). *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York.

[Pasca, 2010] Pasca, D. (2010). Periodic solutions of a class of nonautonomous second order differential systems with (q, p)-laplacian. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 17(5):841–851.

[Paşca and Tang, 2010] Paşca, D. and Tang, C.-L. (2010). Some existence results on periodic solutions of nonautonomous second-order differential systems with (q, p)-laplacian. *Applied Mathematics Letters*, 23(3):246–251.

[Pasca and Tang, 2011] Pasca, D. and Tang, C.-L. (2011). Some existence results on periodic solutions of ordinary (q, p)-laplacian systems. *Journal of applied mathematics & informatics*, 29(1_2):39–48.

[Pasca and Wang, 2016] Pasca, D. and Wang, Z. (2016). On periodic solutions of nonautonomous second order hamiltonian systems with (q, p)-laplacian. *Electronic Journal of Qualitative Theory of Differential Equations*, 2016(106):1–9.

[Płuciennik, 1985a] Płuciennik, R. (1985a). Boundedness of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Bull. Pol. Acad. Sci., Math.*, 33:531â540.

[Płuciennik, 1985b] Płuciennik, R. (1985b). On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 25:321â337.

[Płuciennik, 1987] Płuciennik, R. (1987). The superposition operator in Musielak-Orlicz spaces of vector-valfued functions. Abstract analysis, Proc. 14th Winter Sch., Srní/Czech. 1986, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14, 411-417 (1987).

[Schappacher, 2005] Schappacher, G. (2005). A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386.

[Skaff, 1969] Skaff, M. S. (1969). Vector valued orlicz spaces. ii. *Pacific J. Math.*, 28(2):413–430.

[Tang, 1995] Tang, C.-L. (1995). Periodic solutions of non-autonomous second-order systems with γ-quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675.

[Tang, 1998] Tang, C.-L. (1998). Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270.

[Tang and Wu, 2001] Tang, C. L. and Wu, X.-P. (2001). Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397.

[Tang and Zhang, 2010] Tang, X. and Zhang, X. (2010). Periodic solutions for second-order Hamiltonian systems with a p-Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113.

[Tian and Ge, 2007] Tian, Y. and Ge, W. (2007). Periodic solutions of non-autonomous second-order systems with a p-Laplacian. *Nonlinear Anal.*, 66(1):192–203.

[Wu and Tang, 1999] Wu, X.-P. and Tang, C.-L. (1999). Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235.

[Yang and Chen, 2012] Yang, X. and Chen, H. (2012). Periodic solutions for a non-linear (q, p)-laplacian dynamical system with impulsive effects. *Journal of Applied Mathematics and Computing*, 40(1-2):607–625.

[Yang and Chen, 2013] Yang, X. and Chen, H. (2013). Existence of periodic solutions for sublinear second order dynamical system with (q, p)-laplacian. *Mathematica Slovaca*, 63(4):799–816.

[Zhao and Wu, 2004] Zhao, F. and Wu, X. (2004). Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434.