Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt}\nabla_y\mathcal{L}(t,u(t),u'(t)) = \nabla_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t\in(0,T),\\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases} \qquad (\boldsymbol{P})$$

where the function $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$ (called the *Lagrange function* or *lagrangian*) satisfying that it is measurable in t for each $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and continuously differentiable in (x,y) for almost every $t\in[0,T]$. The unknown function $u:[0,T]\to\mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the articles [18] for definitions and main results on anisotropic Orlicz spaces. These spaces allow us to unify and extend previous results on existences of solutions for systems like ($\boldsymbol{P}$).

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Through this article we say that a function $\Phi : \mathbb{R}^d \to [0, +\infty)$ is of N_∞ class if Φ is convex, $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$ and $\Phi(-y) = \Phi(y)$, and

$$\lim_{|y|\to\infty} \frac{\Phi(y)}{|y|} = +\infty. \tag{1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From [6, Cor. 2.35] a N_∞ function is continuous.

Associated to Φ we have the *complementary function* Ψ which is defined in $\xi \in \mathbb{R}^d$ as

$$\Psi(\xi) = \sup_{y\in\mathbb{R}^d} y \cdot \xi - \Phi(y) \tag{2}$$

then, from the continuity of Φ and (1), we have that $\Psi : \mathbb{R}^d \to [0, \infty)$. Moreover, it is easy to see that Ψ is a convex function such that $\Psi(0) = 0$, $\Psi(-\xi) = \Psi(\xi)$ [13, Chapter 2]. Moreover Ψ satisfies (1) (see [18, Th. 2.2]). i.e. Ψ is N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\Phi_p(y) := |y|^p/p$, for $1 < p < \infty$. In this case $\Psi(\xi) = |\xi|^q/q$, $q = p/(p-1)$.

Example 1.2. If $\Phi : \mathbb{R} \to [0, +\infty)$ is a N_∞ function on $\mathbb{R}$ then $\overline{\Phi}(y) = \Phi(|y|)$ is a N_∞ function on $\mathbb{R}^d$. In this example, as in the previous one, the function Φ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\Phi(y)$ depends on the direction of y. For example, if $1 < p_1, p_2 < \infty$, we define $\Phi_{p_1,p_2} : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ by

$$\Phi_{p_1,p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then Φ_{p_1,p_2} is a N_∞ function. In this case the complementary function is the function Φ_{q_1,q_2} with $q_i = p_i/(p_i - 1)$.

More generally, if $\Phi_k : \mathbb{R}^d \to [0, +\infty)$, $k = 1, \dots, n$, are N_∞ functions, then $\Phi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to [0, +\infty)$ defined by $\Phi(y_1, \dots, y_n) = \Phi_1(y_1) + \cdots + \Phi_n(y_n)$ is a N_∞ function. These functions are truly anisotropic, i.e. $|x| = |y|$ does not imply that $\Phi(x) = \Phi(y)$.

Example 1.4. If $\Phi : \mathbb{R} \to [0, +\infty)$ is a N_∞ function and $O \in GL(d, \mathbb{R})$, then $\Phi(y) = \Phi(Oy)$ is a N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by powers if it does not satisfy the Δ_2 condition (see xxxxx). For example $\Phi : \mathbb{R}^d :\to [0, +\infty)$ defined by $\Phi(y) = \exp(|y|) - 1$ is N_∞ function.

The occurrence of Orlicz Spaces in this paper obeys to we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi\left(\frac{\nabla_y \mathcal{L}}{\lambda}\right) \leqslant a(x)\left\{b(t) + \Phi\left(\frac{y}{\Lambda}\right)\right\}, \tag{$\boldsymbol{S}$}$$

for a.e. $t \in [0, T]$, where $a \in C\left(\mathbb{R}^d, [0, +\infty)\right)$, $b \in L^1([0, T], [0, +\infty))$ and $\Lambda, \lambda > 0$.

Our condition ($\boldsymbol{S}$) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition ($\boldsymbol{S}$) is equivalent to the structure condition in [13, Th. 1.4]. If Φ is a radial N_∞ function such that Ψ satisfies that Δ_2 function then ($\boldsymbol{S}$) is essentially equivalent????? to conditions [1, Eq. (2)-(4)] (see xxxx mas abajo). If Φ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ then inequality ($\boldsymbol{S}$) is related to estructure conditions like [24, Lemma 3.1, Eq. (3.1)]. As can be seen, condition ($\boldsymbol{S}$) is a more compact expression than [24, Lemma 3.1, Eq. (3.1)] and moreover weaker, because ($\boldsymbol{S}$) does not imply a control of $|D_{y_1}L|$ independent of y_2. We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi,F}(t,x,y) := \Phi(y) + F(t,x). \tag{3}$$

Here Φ is assumed differentiable and the function $F(t,x)$, which is often referred to potential, is differentiable with respect to x for a.e. $t \in [0,T]$. Moreover F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla_x F(t,x)| \leqslant a(x)b(t). \tag{4}$$

where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$ and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi,F}$ satisfies condition ($\boldsymbol{S}$). In order to prove this, the only non trivial fact that we should to establishis is that $\Psi(\nabla_y\mathcal{L}) \leqslant a(x)\left\{b(t) + \Phi\left(y/\Lambda\right)\right\}$. But, from inequality (7) below, $\Psi(\nabla_y\mathcal{L}) = \Psi\left(\nabla\Phi(y)\right) \leqslant \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi,F}$ leads to the problem

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} . \tag{P_Φ}$$

The differential equation in the previous problem is called the *anisotropic Φ-Laplacian equation*.

Problem (P_Φ) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [13] deals mainly with problem (P_Φ) with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [13] were extended and improved in several articles, see [22, 20, 26, 21, 29] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [24, 23], among other papers, and in this case (P_Φ) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P_p}$$

If Φ is as in Example 1.3 and $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, then the equations (P_Φ) become

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) \quad \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) \quad \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \qquad (P_{p_1,p_2})$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as (p_1, p_2)-Laplacian system, see [28, 17, 27, 14, 15, 16, 11].

In conclusion, the problem ($\boldsymbol{P}$) with conditions ($\boldsymbol{S}$) contains several problems that have been considered by many authors in the past. Moreover, our results still improve some results on (p_1, p_2)-lamplacian since our structure conditions are less restrictive even in that particular case.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^n \to [0, +\infty)$. References for these topics are [7, 18, 19, 3, 5, 2, 25, 9]. For the theory of convex functions to see [6]. Note that, unlike in [9], we do not require that N_∞ functions be superlinear near from 0, i.e. $\Phi(x)/|x| \to 0$ when $|x| \to 0$. However, most of the results proved in [9] do not depend on this property.

If Φ is a N_∞ function then from convexity and $\Phi(0) = 0$ we obtain that

$$\Phi(\lambda x) \leqslant \lambda \Phi(x), \quad \lambda \in [0,1], x \in \mathbb{R}^d. \tag{5}$$

One of the greatest difficulties when dealing with anisotropic Orlicz spaces is the lack of monotony with respect to the Euclidean norm, i.e. $|x| \leqslant |y|$ does not imply $\Phi(x) \leqslant \Phi(y)$. This problem is avoided if we consider functions whose values on a sphere are comparable (see[19]). However, from (5), we see that N_∞ functions have the following form of radial monotony:

$$0 < \lambda_1 \leqslant \lambda_2, x \in \mathbb{R}^d \Rightarrow \Phi(\lambda_1 x) \leqslant \Phi(\lambda_2 x). \tag{6}$$

The mutually complementary functions N_∞ functions Φ and Ψ satisfy the following importan relations (see [6]): for any $x, y \in \mathbb{R}^d$

$$x \cdot y \leqslant \Phi(x) + \Psi(y) \qquad \text{(Fenchel's Inequality)}$$
$$x \cdot \nabla\Phi(x) = \Phi(x) + \Psi(\nabla\Phi(x)) \qquad \text{(Fenchel's Identity)}$$

In (Fenchel's Identity) we assume Φ differentiable. More generality (Fenchel's Identity) holds when $\nabla\Phi(y)$ is replaced by elements in the subdifferential $\partial\Phi(y)$ of Φ (see [6, Ex. 4.27]).

The following inequality will be useful, it is consequence of that $(d/dt)\Phi(tx)$ is an non decreasing function of t.

$$\Psi(\nabla\Phi(x)) \leqslant x\cdot\nabla\Phi(x) \leqslant \int_1^2 \frac{d}{dt}\Phi(tx)dt \leqslant \int_0^2 \frac{d}{dt}\Phi(tx)dt = \Phi(2x). \tag{7}$$

We say that $\Phi:\mathbb{R}^d\to[0,+\infty)$ satisfies the Δ_2^∞-*condition*, denoted by $\Phi\in\Delta_2^\infty$, if there exists a constant $C>0$ such that

$$\Phi(2x)\leqslant C\Phi(x)+1, \tag{8}$$

for every x.

Throughout this article, we denote by $C=C(\lambda_1,\dots,\lambda_n)$ a positive constant that may depend on T and Φ, or other N_∞ functions, and the parameters $\lambda_1,\dots,\lambda_n$. We assume that the value that C represents may change in different occurrences in the same chain of inequalities.

If Φ is a Δ_2^∞ function then Φ satisfies the following properties

- *Quasi-subadditivity.* There exists $C>0$ such that for every $x,y\in\mathbb{R}^d$

$$\Phi(x+y)\leqslant C(\Phi(x)+\Phi(y))+1. \tag{9}$$

- For any $\lambda>0$ there exists $C:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\Phi(\lambda x)\leqslant C(\lambda)\Phi(x)+1. \tag{10}$$

- Φ is bounded by powers functions (see [7, Proof Lemma 2.4] and [4, Prop. 1]), i.e. there exists $1<p<\infty$, $C>0$ and $r\geqslant 0$ such that

$$\Phi(x)\leqslant C|x|^p,\quad |x|\geqslant r_0.$$

We consider that one of the most important aspects in considering N_∞ functions is that it accounts for the Lagrange functions that present faster growth than powers, for example an exponential growth. Hence we consider it important to avoid imposing hypothesis that Φ to be Δ_2. For some results we will need that Ψ to be Δ_2.

Let Φ_1 and Φ_2 be N_∞ functions. Following to [25] we write $\Phi_1 \dashv \Phi_2$ if there exists non negative numbers k and C such that

$$\Phi_1(x)\leqslant C+\Phi_2(kx),\quad x\in\mathbb{R}^d. \tag{11}$$

For example if Φ is Δ_2 then there exist $p\in(1,+\infty)$ such that $\Phi\dashv|x|^p$. If for every $k>0$ there exists $C=C(k)>0$ such that (11) holds we write $\Phi_1\lll\Phi_2$.

If $\Phi_1\dashv\Phi_2$ then $\Psi_2\dashv\Psi_1$ as the following simple computation proves

$$\begin{aligned}\Psi_1(k\xi) &\geqslant \sup\{k\xi\cdot x-\Phi_2(kx)-C\}\\ &=\sup\{\xi\cdot x-\Phi_2(x)\}-C\\ &=\Psi_2(\xi)-C.\end{aligned}$$

If Ψ is Δ_2^∞ then Φ satisfies that for every $0<r<1$ there exists $\lambda=\lambda(r)>0$ and $C=C(r)>0$ such that for every $x\in\mathbb{R}^d$

$$\Phi(x) \leqslant \frac{r}{\lambda}\Phi(\lambda x) + C. \tag{12}$$

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Note that if $\lambda > 0$ satisfies (12) and $\lambda' \geqslant \lambda$ then λ' also satisfies (12).

As a consequence of the previous result, we obtain that if a Lagrange function $\mathcal{L}$ satisfies structure condition ($\boldsymbol{S}$) and $\Phi \dashv \Phi_0$ then $\mathcal{L}$ satisfies ($\boldsymbol{S}$) with Φ_0 instead to Φ and possibly with other functions b, a and constants Λ and λ.

We denote by $\mathcal{M} := \mathcal{M}\left([0,T],\mathbb{R}^d\right)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1,\dots,u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi\left([0,T],\mathbb{R}^d\right)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{13}$$

The *Orlicz space* $L^\Phi = L^\Phi\left([0,T],\mathbb{R}^d\right)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{14}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi\left([0,T],\mathbb{R}^d\right)$ is defined as the closure in L^Φ of the subspace $L^\infty\left([0,T],\mathbb{R}^d\right)$ of all $\mathbb{R}^d$-valued essentially bounded functions. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [18, Cor. 5.1]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [18, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^\Psi}. \tag{15}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

We consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by the following inclusions

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{16}$$

for any positive r. This relation is a trivial generalization of [18, Thm. 5.6]. If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow \left[L^\Psi\right]^*$, where a function $v \in L^\Phi$ is associated to $\xi_v \in \left[L^\Psi\right]^*$ being

$$\langle \xi_v, u\rangle = \int_0^T v \cdot u \, dt, \tag{17}$$

We suppose $u \in L^\infty([0,T], \mathbb{R}^d)$. Since Φ is continuous, Φ is bounded on $\overline{B}_r(0) = \{x \in \mathbb{R}^n : |x| \leqslant r\}$. Let $M_r := \max_{\overline{B}_r(0)} \Phi(x)$. As $M_r \to 0$ when $r \to 0$, we can choose r such that $M_r T \leqslant 1$. Then

$$\int_0^T \Phi\left(\frac{ru}{\|u\|_{L^\infty}}\right) dt \leqslant M_r T \leqslant 1$$

and consequently $\|u\|_{L^\Phi} \leqslant r^{-1}\|u\|_{L^\infty}$, i.e. $L^\infty \hookrightarrow L^\Phi$.

We highlight the following result (see [9, Th. 3.3]).

Proposition 2.1. $L^\Phi\left([0,T], \mathbb{R}^d\right) = \left[E^\Psi\left([0,T], \mathbb{R}^d\right)\right]^*$.

Consequently $L^\Phi\left([0,T], \mathbb{R}^d\right)$ can be equipped with the weak* topology induced by $E^\Psi\left([0,T], \mathbb{R}^d\right)$.

We define the *Sobolev-Orlicz space* $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ by

$$W^1L^\Phi\left([0,T], \mathbb{R}^d\right) := \left\{u | u \in AC\left([0,T], \mathbb{R}^d\right) \text{ and } u' \in L^\Phi\left([0,T], \mathbb{R}^d\right)\right\},$$

where $AC\left([0,T], \mathbb{R}^d\right)$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $[0,T]$. The space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{18}$$

Anisotropic Sobolev-Orlicz spaces were treated in [3, 5, 2, 25]. Usually functions in Sobolev spaces are required to be weakly differentiable. In the particular and simplest case of functions of one variable, the weak differentiability implies absolute continuity. Hence we can assume $u \in AC\left([0,T], \mathbb{R}^d\right)$ for functions $u \in W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$.

As is well known, an active research topic in mathematical analysis are the Sobolev and Poincare inequalities. This topic have also been treated in the framework of anisotropic Orlicz-Sobolev mainly in [3, 5, 25] for several variables functions and in [2] for functions of one single variable, Φ and Ψ functions of Δ_2^∞ class. We do not know a reference for the embedding of Sobolev-Orlicz anisotropic spaces in the space of continuous functions when Φ or Ψ are not Δ_2^∞. Below we present the results that we will require in this article and we show in detail the case of the incrustation in the space of continuous functions in the simple case of function of one variable.

In order to find a modulus of continuity for functios in W^1L^Φ, and from there, to obtain compact embedding of W^1L^Φ, we define the function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, |x| = s\right\}, \tag{19}$$

Let us establish some elementary properties of A_Φ.

Proposition 2.2. *The function A_Φ has the following properties:*

1. *A_Φ is continuous and $A_\Phi(s) > 0$, when $s > 0$,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *$A_\Phi(|x|)$ is the* greatest radial minorant *of $\Phi(x)$,*
4. *Φ is N_∞ if and only if $\lim_{s\to+\infty} A_\Phi(s)/s = +\infty$.*

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [6]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

Example 2.1. Let $\Phi = \Phi_{p_1,p_2}$ be the function given in Example (1.3). We show that

$$K \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\} \leqslant A_\Phi(r) \leqslant \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\}$$

for some $K > 0$, for every $1 < p_1, p_2 < \infty$. The second inequality follows directly from definition of A_Φ. For the first inequality, we note that $|(y_1, y_2)| = r$ implies that $|y_1| \geqslant r/2$ or $|y_2| \geqslant r/2$. Then

$$\Phi_{p_1,p_2}(y_1, y_2) \geqslant \min\{2^{-p_1}, 2^{-p_2}\} \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\}. \tag{20}$$

Let us in a little digression to show that

$$A_\Phi(r) = \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\},$$

when $1 < p_1, p_2 \leqslant 2$. We apply the method of Lagrange multipliers (see [12, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1,p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2}y_1 + \lambda y_1 & = & 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 & = & 0 \\ |y_1|^2 + |y_2|^2 & = & r^2 \end{cases} \tag{21}$$

These equations are solved, among others, by the following two sets of citical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2.

If (y_1, y_2) solve (21) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1, y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let L be the Lagrangian associated to the constrained problem: $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) + \lambda H(y_1, y_2)$ being $H = 0$ the constraint. We must analize the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2 H$ on the subspace M. By elementary computations we have for $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace M. We can assume that $p_1 < 2$ or $p_2 < 2$, otherwise the statement we intend to prove would be trivial. Under this assumption, we note that $(-y_2, y_1) \in M$ and $(-y_2, y_1)^t \mathcal{H}(-y_2, y_1) < 0$. Then, by second order necessary conditions [12, p.333], there cannot be a minimum at (y_1, y_2). Therefore follows (20).

Note that A_Φ is not necessarily convex. In some applications we overcome this lack by taking the *greatest convex minorant* of A_Φ. This function is defined as follows

$$B_\Phi(s) = \sup\{\varphi(s) | \varphi \text{ es convex and } \varphi \leqslant A_\Phi\}. \tag{22}$$

The function B_Φ is the greatest convex radial minorant of Φ. Moreover B_Φ is N^∞ function. To justify this statement it would only be necessary to prove that $B_{\Phi(s)} > 0$, when $s > 0$ and $B_\Phi(s)/s \to \infty$. We write, for $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$. From Proposition 2.2(2) we obtain that $A_\Phi(s) \geqslant A_\Phi(s_0)(s - s_0)^+/s_0 > 0$, for $s > s_0$. From this it follows that $B_\Phi(s) \geqslant A_\Phi(s_0)(s - s_0)^+/s_0 > 0$, when $s > 0$ and that $\liminf_{s\to\infty} B_\Phi(s)/s \geqslant A_\Phi(s_0)/s_0$. Hence, from Proposition 2.2(2) we obtain that $B_\Phi(s)/s \to \infty$.

Recall that a function $w : [0, +\infty) \to [0, +\infty)$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example $w(s) = sA_\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $u : [0, T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t - s|). \tag{23}$$

We denote by $C^w([0, T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (23) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t\neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

As is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0, T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1L^\Phi\left([0, T], \mathbb{R}^d\right)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (19). *Then*

1. *For every* $s, t \in [0, T]$, $s \neq t$,

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s-t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have* $\widetilde{u} \in L^\infty\left([0,T], \mathbb{R}^d\right)$ *and*

$$\Phi\left(\tilde{u}(t)\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(T u'(r)\right) dr. \qquad \text{(Poincaré-Wirtinger's inequality)}$$

3. *If* Φ *is* N_∞ *then the space* $W^1 L^\Phi\left([0,T], \mathbb{R}^d\right)$ *is compactly embedded in the space of continuous functions* $C([0,T], \mathbb{R}^d)$.

Proof. By the absolutely continuity of u, Jensen's inequality, the definition of the Luxemburg norm and following a similar argument that in the deduction of [2, Th. 4.5], we have

$$\begin{aligned}
\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr\right) \\
&\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leqslant \frac{1}{|s-t|}.
\end{aligned}$$

By Proposition 2.2(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then item 1 holds.

Applying Jensen's inequality two times, we get

$$\begin{aligned}
\Phi(\tilde{u}(t)) &= \Phi\left(\frac{1}{T}\int_0^T \left(u(t)-u(s)\right) ds\right) \\
&\leqslant \frac{1}{T}\int_0^T \Phi(u(t)-u(s)) ds \\
&\leqslant \frac{1}{T}\int_0^T \Phi\left(\int_s^t |t-s| u'(r) \frac{dr}{|t-s|}\right) ds \\
&\leqslant \frac{1}{T}\int_0^T \frac{1}{|t-s|}\int_s^t \Phi\left(|t-s| u'(r)\right) dr ds
\end{aligned}$$

From (5) we have that $\Phi(rx)/r$ is increasing with respecto to $r > 0$ for $x \in \mathbb{R}^d$ fix. Therefore, previous inequality implies (Poincaré-Wirtinger's inequality). If we apply this inequality to the function $\left(T\|u'\|_{L^\Phi}\right)^{-1} u$ we obtain

$$\Phi\left(\frac{\tilde{u}(t)}{T\|u'\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leqslant \frac{1}{T}.$$

Using Proposition 2.2(3) we obtain $\tilde{u} \in L^\infty$ and

$$|\tilde{u}(t)| \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}. \tag{24}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.2(3)

$$|\overline{u}| \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (24) we get

$$\|u\|_{L^\infty} \leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1, T\}\|u\|_{W^1L^\Phi}$$

Morrey's inequality and Sobolev's inequality imply that there exist $C_T > 0$ with

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} \leqslant C_T\|u\|_{W^1L^\Phi},$$

i.e. $W^1L^\Phi\left([0,T],\mathbb{R}^d\right) \hookrightarrow C^w([0,T],\mathbb{R}^d)$. As a consequence of Arzela-Ascoli Theorem the embedding $C^w\left([0,T],\mathbb{R}^d\right) \hookrightarrow C([0,T],\mathbb{R}^d)$ is compact. This was proved in [8, Prop. 5.13] for the case $w(s) = |s|^\alpha$ with $0 < \alpha \leqslant 1$. For w arbitrary, the proof follows with some obvious modifications. Consequently the embedding $W^1L^\Phi\left([0,T],\mathbb{R}^d\right) \hookrightarrow C([0,T],\mathbb{R}^d)$ is compact. □

Remark 1. As consequence of Lemma 2.3 we obtain that

$$\|u\|'_{W^1L^\Phi} = |\overline{u}| + \|u'\|_{L^\Phi},$$

define a equivalent norm to $\|\cdots\|_{W^1L^\Phi}$ on $W^1L^\Phi([0,T],\mathbb{R}^d)$. This affirmation is proved as follows. On the one hand, by Hölder inequality (15)

$$|\overline{u}| \leqslant \frac{2}{T}\|1\|_{L^\Psi}\|u\|_{L^\Phi}.$$

On the other hand, from the embedding $L^\infty \hookrightarrow L^\Phi$ and (24)

$$\|u\|_{L^\Phi} \leqslant C|\overline{u}| + C\|\tilde{u}\|_{L^\infty} \leqslant C\|u\|'_{W^1L^\Phi}.$$

Corollary 2.4. *Every bounded sequence $\{\boldsymbol{u}_n\}$ in $W^1L^\Phi([0,T],\mathbb{R}^d)$ has an uniformly convergent subsequence.*

Given a function $a : \mathbb{R}^d \to \mathbb{R}$, we define the composition operator $\boldsymbol{a} : \mathcal{M} \to \mathcal{M}$ by $\boldsymbol{a}(u)(x) = a(u(x))$. We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

Lemma 2.5. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : [0,+\infty) \to [0,+\infty)$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

The following theorem will be used repeatedly. We adapted the proof of [1, Lemma 2.5] to the anisotropic case. For an alternative approach see [2].

Lemma 2.6. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions converging to $u \in \Pi(E^\Phi, \lambda)$ in the L^Φ-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0,T],\mathbb{R})$ such that $u_{n_k} \to u$ a.e. and $\Phi(u_{n_k}/\lambda) \leqslant h$ a.e.*

Proof. Since $d(u, E^\Phi) < \lambda$ and u_n converges to u, there exists a subsequence of u_n (again denoted u_n), $\overline{\lambda} \in (0,\lambda)$ and $u_0 \in E^\Phi$ such that $d(u_n, u_0) < \overline{\lambda}$, $n = 1, \ldots$. As a consequence of (1), we obtain that $L^\Phi\left([0,T],\mathbb{R}^d\right) \hookrightarrow L^1\left([0,T],\mathbb{R}^d\right)$. This fact impies that u_n converges in measure to u. Therefore, we can to extract a subsequence (denoted u_n) such that $u_n \to u$ a.e. and

$$\lambda_n := \|u_n - u_{n-1}\|_{L^\Phi} < \frac{\lambda - \overline{\lambda}}{2^{n-1}}, \quad \text{for } n \geqslant 2.$$

We can assume $\lambda_n > 0$ for every $n = 1, \ldots$. We write $\lambda_1 := \|u_1 - u_0\|_{L^\Phi}$ and $\lambda_0 := \lambda - \sum_{n=1}^\infty \lambda_n$ and define $h : [0,T] \to \mathbb{R}$ by

$$h(t) = \frac{\lambda_0}{\lambda}\Phi\left(\frac{u_0}{\lambda_0}\right) + \sum_{j=0}^{\infty} \frac{\lambda_{j+1}}{\lambda}\Phi\left(\frac{u_{j+1} - u_j}{\lambda_{j+1}}\right). \tag{25}$$

Since $\Phi(0) = 0$ and from the convexity of Φ we have for any $n = 1, \ldots$

$$\begin{aligned}\Phi\left(\frac{u_n}{\lambda}\right) &= \Phi\left(\frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \frac{u_{j+1} - u_j}{\lambda}\right) \\ &\leqslant \frac{\lambda_0}{\lambda}\Phi\left(\frac{u_0}{\lambda_0}\right) + \sum_{j=0}^{n-1} \frac{\lambda_{j+1}}{\lambda}\Phi\left(\frac{u_{j+1} - u_j}{\lambda_{j+1}}\right) \leqslant h\end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda_0) \in L^1([0,T],\mathbb{R})$. On the other hand $\|u_{j+1} - u_j\|_{L^\Phi} = \lambda_{j+1}$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_{j+1}}\right) dt \leqslant 1.$$

Then $h \in L^1([0,T],\mathbb{R})$.

□

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), u'(t))\, dt. \tag{26}$$

In this direction, the following theorem is our main result. Its proof follows the same lines as [1, Th. 3.2] but with some modifications by the lack of monotony of Φ with respect to the euclidean norm and the fact that we do not have the notion of absolutely continuous norm.

Theorem 3.1. *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying ($\boldsymbol{S}$). Then the following statements hold:*

1. *The action integral given by* (26) *is finitely defined on the set $\mathcal{E}^{\Phi}_{\Lambda} := W^1L^{\Phi} \cap \{u|u' \in \Pi(E^{\Phi}, \Lambda)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}^{\Phi}_{\Lambda}$ and its derivative I' is demicontinuous from $\mathcal{E}^{\Phi}_{\Lambda}$ into $\left[W^1L^{\Phi}\right]^*$, i.e. I' is continuous when $\mathcal{E}^{\Phi}_{\Lambda}$ is equipped with the strong topology and $\left[W^1L^{\Phi}\right]^*$ with the weak* topology. Moreover, I' is given by the following expression*

$$\langle I'(u), v\rangle = \int_0^T \left\{\nabla_x\mathcal{L}\big(t,u,u'\big)\cdot v + \nabla_y\mathcal{L}\big(t,u,u'\big)\cdot v'\right\}\,dt. \tag{27}$$

3. *If $\Psi \in \Delta_2^{\infty}$ then I' is continuous from $\mathcal{E}^{\Phi}_{\Lambda}$ into $\left[W^1L^{\Phi}\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^{\Phi}_{\Lambda}$. From (16) we obtain $\Phi(u'(t)/\Lambda) \in L^1$. Now, from ($\boldsymbol{S}$) and Lemma 2.5 we have

$$\begin{aligned} |\mathcal{L}(t,u(t),u'(t))| + |\nabla_x\mathcal{L}(t,u(t),u'(t))| + \Psi\left(\frac{\nabla_y\mathcal{L}(t,u,u')}{\lambda}\right) \\ \leqslant A(\|u\|_{W^1L^{\Phi}})\left\{b(t) + \Phi\left(\frac{u'(t)}{\Lambda}\right)\right\} \in L^1, \end{aligned} \tag{28}$$

for every $u \in \mathcal{E}^{\Phi}_{\Lambda}$. Thus item (1) is proved integrating this inequality.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto \nabla_x\mathcal{L}(\cdot,u,u')$ is continuous from $\mathcal{E}^{\Phi}_{\Lambda}$ into $L^1([0,T])$ with the strong topology on both sets.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^{\Phi}_{\Lambda}$ and let $u \in \mathcal{E}^{\Phi}_{\Lambda}$ such that $u_n \to u$ in W^1L^{Φ}. By (Sobolev's inequality), $u_n \to u$ uniformly. As $u'_n \to u' \in \mathcal{E}^{\Phi}_{\Lambda}$, by Lemma 2.6, there exist a subsequence of u'_n (again denoted u'_n) and a function $h \in L^1([0,T],\mathbb{R})$ such that $u'_n \to u'$ a.e. and $\Phi(u'_n/\Lambda) \leqslant h$ a.e.

Since u_n, $n = 1,2,\dots$, is a strong convergent sequence in W^1L^{Φ}, it is a bounded sequence in W^1L^{Φ}. According to Lemma (2.5), there exists $M > 0$ such that $\|\boldsymbol{a}(u_n)\|_{L^\infty} \leqslant M$, $n = 1,2,\dots$. From the previous facts and (28), we get

$$|\nabla_x\mathcal{L}(\cdot,u_n,u'_n)| \leqslant a(u_n)\left\{b + \Phi\left(\frac{u'_n}{\Lambda}\right)\right\} \leqslant M(b+h) \in L^1.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$\nabla_x\mathcal{L}(t,u_{n_k}(t),u'_{n_k}(t)) \to \nabla_x\mathcal{L}(t,u(t),u'(t)) \quad \text{for a.e. } t \in [0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.
Step 2. The non linear operator $u \mapsto \nabla_y \mathcal{L}(\cdot, u, u')$ *is continuous from* $\mathcal{E}^{\Phi}_{\Lambda}$ *with the strong topology into* $\left[L^{\Phi}\right]^*$ *with the weak** *topology.*

Let $u \in \mathcal{E}^{\Phi}_{\Lambda}$. From (28) it follows that

$$\nabla_y \mathcal{L}(\cdot, u, u') \in \lambda C^{\Psi}\left([0,T], \mathbb{R}^d\right) \subset L^{\Psi}\left([0,T], \mathbb{R}^d\right) \subset \left[L^{\Phi}\left([0,T], \mathbb{R}^d\right)\right]^*. \tag{29}$$

Let $u_n, u \in \mathcal{E}^{\Phi}_{\Lambda}$ such that $u_n \to u$ in the norm of W^1L^{Φ}. We must prove that $\nabla_y \mathcal{L}(\cdot, u_n, u_n') \xrightharpoonup{w^*} \nabla_y \mathcal{L}(\cdot, u, u')$. On the contrary, there exist $v \in L^{\Phi}$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle \nabla_y \mathcal{L}(\cdot, u_n, u_n'), v\rangle - \langle \nabla_y \mathcal{L}(\cdot, u, u'), v\rangle| \geqslant \epsilon. \tag{30}$$

We have $u_n \to u$ in L^{Φ} and $u_n' \to u'$ in L^{Φ}. By Lemma 2.6, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T], \mathbb{R})$ such that $u_n \to u$ uniformly, $u_n' \to u'$ a.e. and $\Phi(u_n'/\lambda) \leqslant h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 2.5 implies that $a(u_n(t))$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (28) with u_n instead of u, we have

$$\Psi\left(\frac{\nabla_y \mathcal{L}(\cdot, u_n, u_n')}{\lambda}\right) \leqslant M(b+h) =: h_1 \in L^1. \tag{31}$$

As $v \in L^{\Phi}$ there exists $\lambda_v > 0$ such that $\Phi(v/\lambda_v) \in L^1$. Now, by Young inequality and (31), we have

$$\begin{aligned}\nabla_y \mathcal{L}(\cdot, u_n, u_n') \cdot v(t) &\leqslant \lambda\lambda_v \left[\Psi\left(\frac{\nabla_y \mathcal{L}(\cdot, u_n, u_n')}{\lambda}\right) + \Phi\left(\frac{v}{\lambda_v}\right)\right] \\ &\leqslant \lambda\lambda_v M(b+h) + \lambda\lambda_v \Phi\left(\frac{v}{\lambda_v}\right) \in L^1\end{aligned} \tag{32}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T \nabla_y \mathcal{L}(t, u_n, u_n') \cdot v dt \to \int_0^T \nabla_y \mathcal{L}(t, u, u') \cdot v\, dt \tag{33}$$

which contradicts the inequality (30). This completes the proof of step 2.

Step 3. We will prove (27). Note that (28), (29) and the imbeddings $W^1L^{\Phi} \hookrightarrow L^{\infty}$ and $L^{\Psi} \hookrightarrow \left[L^{\Phi}\right]^*$ imply that the second member of (27) defines an element of $\left[W^1L^{\Phi}\right]^*$.

The proof follows similar lines as [13, Thm. 1.4]. For $u \in \mathcal{E}^{\Phi}_{\Lambda}$ and $0 \neq v \in W^1L^{\Phi}$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), u'(t) + sv'(t)).$$

For $|s| \leqslant s_0 := \left(\Lambda - d(u', E^{\Phi})\right)/\|v\|_{W^1L^{\Phi}}$, using triangle inequality we get $d\left(u' + sv', E^{\Phi}\right) < \Lambda$ and thus $u' + sv' \in \Pi(E^{\Phi}, \Lambda)$. These facts imply, in virtue of Theorem 3.1 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$.

We write $s_1 := \min\{s_0, 1 - d(u', E^\Phi)/\Lambda\}$. Let $\lambda_v > 0$ such that $\Phi(v'/\lambda_v) \in L^1$. As $u' \in \Pi(E^\Phi, \Lambda)$ then

$$d\left(\frac{u'}{(1-s_1)\Lambda}, E^\Phi\right) = \frac{1}{(1-s_1)\Lambda} d(u', E^\Phi) < 1$$

and therefore $(1-s_1)^{-1}\Lambda^{-1}u' \in C^\Phi$. Hence, if $v' \in L^\Phi$ and $|s| \leqslant s_1\Lambda\lambda_v^{-1}$, from the convexity and the parity of Φ, we get

$$\begin{aligned}\Phi\left(\frac{u'+sv'}{\Lambda}\right) &\leqslant (1-s_1)\Phi\left(\frac{u'}{(1-s_1)\Lambda}\right) + s_1\Phi\left(\frac{s}{s_1\Lambda}v'\right)\\ &\leqslant (1-s_1)\Phi\left(\frac{u'}{(1-s_1)\Lambda}\right) + s_1\Phi\left(\frac{v'}{\lambda_v}\right) \in L^1\end{aligned} \tag{34}$$

We also have $\|u+sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 2.5, there exists $M > 0$, independent of s, such that $\|a(u+sv)\|_{L^\infty} \leqslant M$. Now, applying Young's Inequality, (28), the fact that $v \in L^\infty$, (34) and $\Phi(v'/\lambda_v) \in L_1$, we get

$$\begin{aligned}|D_sH(s,t)| &= |\nabla_x\mathcal{L}(t, u+sv, u'+sv')\cdot v + \nabla_y\mathcal{L}(t, u+sv, u'+sv')\cdot v'|\\ &\leqslant M\left\{b(t) + \Phi\left(\frac{u'+sv'}{\Lambda}\right)\right\}|v|\\ &\quad + \lambda\lambda_v\left\{\Psi\left(\frac{\nabla_y\mathcal{L}(t, u+sv, u'+sv')}{\lambda}\right) + \Phi\left(\frac{v'}{\lambda_v}\right)\right\}\\ &\leqslant M\left\{b(t) + \Phi\left(\frac{u'+sv'}{\Lambda}\right)\right\}(|v| + \lambda\lambda_v) + \lambda\lambda_v\Phi\left(\frac{v'}{\lambda}\right) \in L^1.\end{aligned} \tag{35}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v\rangle = \frac{d}{ds}I(u+sv)\Big|_{s=0} = \int_0^T \{\nabla_x\mathcal{L}(t,u,u')\cdot v + \nabla_y\mathcal{L}(t,u,u')\cdot v'\}\ dt.$$

Moreover, from the previous formula, (28), (29), and Lemma 2.3, we obtain

$$|\langle I'(u), v\rangle| \leqslant \|\nabla_x\mathcal{L}\|_{L^1}\|v\|_{L^\infty} + \|\nabla_y\mathcal{L}\|_{L^\Psi}\|v'\|_{L^\Phi} \leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant C. This completes the proof of the Gâteaux differentiability of I. The previous steps imply the demicontinuity of the operator $I' : \mathcal{E}^\Phi_\Lambda \to \left[W^1L^\Phi_d\right]^*$.

In order to prove item 3, it is necessary to see that the maps $u \mapsto \nabla_x\mathcal{L}(t,u,u')$ and $u \mapsto \nabla_y\mathcal{L}(t,u,u')$ are norm continuous from $\mathcal{E}^\Phi_\Lambda$ into L^1 and L^Ψ, respectively. It remains to the continuity of the second map. To this purpose, we take $u_n, u \in \mathcal{E}^\Phi_\Lambda$, $n = 1, 2, \ldots$, with $\|u_n - u\|_{W^1L^\Phi} \to 0$. As before, we can deduce there exist a subsequence (denoted u_n' for simplicity) and $h_1 \in L^1$ such that (32) holds and $u_n \to u$ a.e. Since $\Psi \in \Delta_2$,

$$\Psi(\nabla_y\mathcal{L}(\cdot, u_n, u_n')) \leqslant c(\lambda)\Psi\left(\frac{\nabla_y\mathcal{L}(\cdot, u_n, u_n')}{\lambda}\right) + 1 \leqslant c(\lambda)h_1 + 1 =: h_2 \in L^1. \tag{36}$$

Then, from the quasi-subadditivity of Ψ we have

$$\Psi\left(\nabla_y\mathcal{L}(\cdot,u_n,u_n')-\nabla_y\mathcal{L}(\cdot,u,u')\right)\leqslant K(h_2+\Psi(\nabla_y\mathcal{L}(\cdot,u,u')))+1.$$

Now, by Dominated Convergence Theorem, we obtain that $\nabla_y\mathcal{L}(\cdot,u_n,u_n')$ is ρ_Ψ modular convergent to $\nabla_y\mathcal{L}\left(\cdot,u,u'\right))$. Since Ψ is Δ_2^∞, modular convergence implies norm convergence (see [19]). □

4 Existence of minimizers

For simplicity, from now on we will consider Lagrangian functions of the form (3), i.e. $\mathcal{L}=\mathcal{L}_{\Phi,F}$. However, the results of this section extend without difficulty to any lagrangian $\mathcal{L}$ with $\mathcal{L}\geqslant\mathcal{L}_{\Phi,F}$ (see [1]).

It is well known that an important ingredient in the direct method of calculus of variations is the coercivity of action integrals. To obtain coercivity for integral I defined in (26) with $\mathcal{L}=\mathcal{L}_{\Phi,F}$, it is necessary to impose more restrictions on the potential F.

There are several restrictions that were explored in the past. The one we will study in this article is based on what is known in the literature as sublinearity (see [20, 26, 29] for the laplacian, [23, 10] for the p-laplacian and [28, 11, 14, 15] for (p,q)-laplacian). In this article we will use another denomination for the sublinearity.

We say that F satisfies condition (B) if there exist an N_∞ function Φ_0, with $\Phi_0\lll\Phi$, a function $d\in L^1([0,T],\mathbb{R})$, witht $d\geqslant1$, such that

$$\Psi(d^{-1}(t)\nabla_xF)\leqslant\Phi_0(x)+1 \tag{B}$$

Let us take a moment to show that condition (B) encompasses the sublinearity condition, for example, as it is formulated in [11] for the (p_1,p_2)-Laplacian. We write $p'=p/(p-1)$ for the Lebesgue conjugate exponent of p. In [11, Th. 1.1.] Li, Ou and Tang considered a potential $F:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ satisfying (C) and (A). As we have said, with these hypotheses, the Lagragian $\mathcal{L}=\Phi_{p_1,p_2}+F$ satisfies the structure condition ($\boldsymbol{S}$) and the Euler-Lagrange equations for $\mathcal{L}$ are the (p_1,p_2)-Laplacian equations. Additionally in [11] was used the condition (H1) below.

(H1) There exists $f_i,g_i,h_i\in L^1([0,T],\mathbb{R}_+)$, $\alpha_1\in[0,p_1-1)$, $\alpha_2\in[0,p_2-1)$, $\beta_1\in[0,p_2/p_1')$ and $\beta_2\in[0,p_1/p_2')$ such that

$$|\nabla_{x_1}F(t,x_1,x_2)|\leqslant f_1(t)|x_1|^{\alpha_1}+f_2(t)|x_2|^{\beta_1}+h_1(t)$$
$$|\nabla_{x_2}F(t,x_1,x_2)|\leqslant f_2(t)|x_2|^{\alpha_2}+g_2(t)|x_1|^{\beta_2}+h_2(t)$$

It is easy to prove that ((H1))⇒(B) where $\Phi_0(x_1,x_2)=\Phi_{\overline{p}_1,\overline{p}_2}(x_2,x_2)=|x_1|^{\overline{p}_1}/\overline{p}_1+|x_2|^{\overline{p}_2}/\overline{p}_2$, where $\overline{p}_i$, $i=1,2$, are taken so that $\alpha_i+1\leqslant\overline{p}_i<p_i$, $\beta_1<\overline{p}_2/\overline{p}_1'$, $\beta_2<\overline{p}_1/\overline{p}_2'$ and $d=C(1+\sum_i\{f_i+g_i+h_i\})\in L^1$, for $C>0$ large enough.

The following is our main result.

Theorem 4.1. *Let Φ be an N_∞-function whose complementary function Ψ satisfies the Δ_2^∞-condition. Let F be a potential that satisfies (C), (A),(B) and the following condition*

$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\, dt}{\Phi_0(2x)} = +\infty. \tag{37}$$

Then the problem (P_Φ) has at least a solution which minimizes the action integral I on $W^1E_T^\Phi$.

Proof. Step 1. The action integral is coercive.

Let λ be any positive number with $\lambda > 2\max\{T,1\}$. Since $\Phi_0 \lll \Phi$ we obtain $C(\lambda) > 0$ such that

$$\Phi_0(x) \leqslant \Phi\left(\frac{x}{2\lambda}\right) + C(\lambda), \quad x \in \mathbb{R}^d. \tag{38}$$

By the decomposition $u = \overline{u}+\tilde{u}$, the absolutely continuity of $F(t,x+sy)$ with respect to $s \in \mathbb{R}$, Young's inequality, (B), the convexity of Φ_0, (6), (38), (Poincaré-Wirtinger's inequality) we obtain

$$\begin{aligned}
J &:= \left| \int_0^T F(t,u) - F(t,\overline{u})dt \right| \\
&\leqslant \int_0^T \int_0^1 |\nabla_x F(t,\overline{u}+s\tilde{u})\tilde{u}| ds dt \\
&\leqslant \lambda \int_0^T d(t) \int_0^1 \Psi\left(d^{-1}(t)\nabla_x F(t,\overline{u}+s\tilde{u})\right) + \Phi\left(\frac{\tilde{u}}{\lambda}\right) ds dt \\
&\leqslant \lambda \int_0^T d(t) \left[\int_0^1 \frac{1}{2}\Phi_0(2\overline{u}) + \frac{1}{2}\Phi_0(2\tilde{u})ds + \Phi\left(\frac{\tilde{u}}{\lambda}\right) + 1 ds\right] dt \\
&\leqslant \lambda \int_0^T d(t) \left[\int_0^1 \Phi_0(2\overline{u}) + 2\Phi\left(\frac{\tilde{u}}{\lambda}\right) + C(\lambda) ds\right] dt \\
&\leqslant C_1\Phi_0(2\overline{u}) + \lambda C_2 \int_0^T \Phi\left(\frac{Tu'(s)}{\lambda}\right) ds + C_1
\end{aligned} \tag{39}$$

where $C_2 = C_2(\|d\|_{L^1})$ and $C_1 = C_1(\|d\|_{L^1}, \lambda)$. From (12)

Then, by (39) we get

$$\begin{aligned}
&\int_0^T \Phi(u') + F(t,u)dt = \int_0^T \{\Phi(u') + [F(t,u) - F(t,\overline{u})] + F(t,\overline{u})\}dt \\
&\geqslant \int_0^T [\Phi(u') - C_2\Phi_0(C_T u')]dt - C_1\Phi_0(2\overline{u}) + \int_0^T F(t,\overline{u})dt - C_3
\end{aligned} \tag{40}$$

We suppose that $u_n \in W^1L^\Phi$ with $\|u_n W^1L^\Phi \to \infty$. From Remark 1, we have that $\|u_u'\|_{L^\Phi} \to \infty$ or $|\overline{u}_n| \to \infty$.

Now, if $|\overline{u}| \to \infty$, by hypothesis we have $\frac{1}{\Psi_0(2\overline{u})}\int_0^\infty F(t,\overline{u})dt \to \infty$.

If $\|u_n\|_{L^\Phi} \to \infty$ for a sequence u_n.

Let $M > 0$ such that $\frac{\Phi(x)}{C_2\Phi_0(C_T x)} \geqslant 2$ for $|x| \geqslant M$.

Set $A_n := \{t \in [0,T] : |u_n'(t)| \geqslant M\}$.

Now, if $\|u_n'\|_{L^\Phi} \to \infty$ then $\|u_n'\chi_{A_n}\|_{L^\Phi} \to \infty$ or $\|u_n'\chi_{A_n^C}\|_{L^\Phi} \to \infty$.

Thus, $\|u_n'\chi_{A_n^C}\|_{L^\Phi}$ is bounded and $\|u_n'\chi_{A_n}\|_{L^\Phi} \to \infty$.
By Ameniya norm, we have $\|u_n'\|_{L^\Phi} \leqslant 1 + \int_{A_n} \Phi(u_n')dt$, then

$$\int_{A_n} \Phi(u_n')dt \to \infty. \tag{41}$$

Now, as $\frac{\Phi(x)}{2} \geqslant C_2\Phi_0$ for $|x| \geqslant M$, we have

$$\begin{aligned}
&\int_0^T [\Phi(u_n') - C_2\Phi_0(C_T u_n')]dt \\
&\geqslant \int_{A_n} [\Phi(u_n') - C_2\Phi_0(C_T u_n')]dt - C_2 \int_{[0,T]/A_n} \Phi_0(C_T u_n')dt \\
&\geqslant \frac{1}{2}\int_{A_n} \Phi(u_n')dt - C_2 T \max\{\Phi_0(x) : \|x\| \leqslant C_T M\}.
\end{aligned} \tag{42}$$

Finally, by (41), we conclude that $\int_0^T [\Phi(u_n') - C_2\Phi_0(C_T u_n')]dt \to \infty$
Step 2. If $u_n \to u$ uniformly and $u_n' \rightharpoonup u'$ in $L^1([0,T],\mathbb{R}^d)$ then $I(u) \leqslant \liminf_{n\to\infty} I(u_n)$.

We assume that $u_n \rightharpoonup u$ in $W^1L^\Phi([0,T],\mathbb{R}^d)$. From the embedding $L^\Phi([0,T],\mathbb{R}^d) \hookrightarrow L^1([0,T],\mathbb{R}^d)$ we have the $u_n' \rightharpoonup u'$ in $L^1([0,T],\mathbb{R}^d)$ and from

$\square$

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