Periodic solutions of Euler-Lagrange equations and "sublinear" pontentials in an Orlicz-Sobolev space setting

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Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$, of hamiltonian systems with a potential function F satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t)$, with $b_1, b_2 \in L^1$ and for certain N-functions Φ_0.

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t\in(0,T) \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases}, \tag{1}$$

where $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous. In other words, we

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are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary equations*. This topic was deeply addressed???(studied, treated????) for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) = \frac{|y|^p}{p} + F(t,x), \tag{2}$$

for $1 < p < \infty$. For example, the classic book [1] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [1] were extended and improved in several articles, see [2, 3, 4, 5, 6] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [7, 8], in this case (1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions

- **(C)** F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

- **(A)** For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{4}$$

  In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [9] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{5}$$

with Φ an N-function (see section 2 for the definition of this concept). In the paper [9] was assumed on F a condition of *bounded oscilation* (see xxxxx below). In this paper we shall study the condition of *sublinearity* (see [3, 4, 6, 8, 10]) on ∇F for the lagrangian $\mathcal{L}_{\Phi,F}$, or more generally for lagrangians which are lower bounded by $\mathcal{L}_{\Phi,F}$.

The problem (1) comes from a variational one, that is, the equation in (**??**) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \mathcal{L}(t,u(t),u'(t))\,dt. \tag{6}$$

The paper is organized as follows. In section 2 we give preliminaries facts on N-functions, Orlicz-Sobolev spaces of functions and we establish our main results. Section 4 contains the proofs and section 5 examples of applications of our results to concrete cases. Finally, in section 6, we compare the condition of sublinearity considered in this paper with the condition of bounded oscillations studied in [9].

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions. Classic references for these topics are [11, 12, 13].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is convex and it also satisfies that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0} \frac{\Phi(t)}{t} = 0$$

In addition, in this paper for sake of simplicity we assume that Φ is differentiable and we call φ the derivative of Φ. On these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism whose inverse will be denoted by ψ. We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then, Ψ is an N-function which is called the *complementary function* of Φ.

We recall that an N-function $\Phi(u)$ has *principal part* $f(u)$ if $\Phi(u) = f(u)$ for large values of the argument (see [12, p. 16] and [12, Sec. 7] for properties of principal part)

There exist several orders and equivalence relations between N-functions (see [13, Sec. 2.2]). Following [13, Def. 1, pp. 15-16] we say that the N-function Φ_2 is *stronger* than the N-function Φ_1, in symbols $\Phi_1 \prec \Phi_2$, if there exist $a > 0$ and $x_0 \geqslant 0$ such that

$$\Phi_1(x) \leqslant \Phi_2(ax), \quad x \geqslant x_0. \tag{7}$$

The N-functions Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) when $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$. We say that Φ_2 is *essentially stronger* than Φ_1 ($\Phi_1 \lll \Phi_2$) if and only if for every $a > 0$ there exists $x_0 = x_0(a) \geqslant 0$ such that (7) holds. Finally we say that Φ_2 is *completely stronger* than Φ_1 ($\Phi_1 \curlyeqprec \Phi_2$) if and only if for every $a > 0$ there exist $K = K(a) > 0$ and $x_0 = x_0(a) \geqslant 0$ such that

$$\Phi_1(x) \leqslant K\Phi_2(ax), \quad x \geqslant x_0. \tag{8}$$

We also say that a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the *Δ_2^∞-condition*, denoted by $\eta \in \Delta_2^\infty$, if there exist constants $K > 0$ and $x_0 \geqslant 0$ such that

$$\eta(2x) \leqslant K\eta(x), \tag{9}$$

for every $x \geqslant x_0$. We note that η is Δ_2^∞ if and only if $\eta \curlyeqprec \eta$. If $x_0 = 0$, the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the Δ_2 ($\eta \in \Delta_2$). If there exists $x_0 > 0$ such that inequality (9) holds for $x \leqslant x_0$ we say that Φ satisfies the Δ_2^0-condition ($\Phi \in \Delta_2^0$).

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C^\Phi = C^\Phi([0,T],\mathbb{R}^d)$ is defined by

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{10}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{11}$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup\left\{ \int_0^T u \cdot v \, dt \Big| \rho_\Psi(v) \leqslant 1 \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The following inequality holds for any $u \in L^\Phi$

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k}\{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \tag{12}$$

In fact, $\|u\|_{L^\Phi}$ is the infimum for $k > 0$ of the r.h.s in above expression (see [12, Thm. 10.5] and [14]).

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ holds if and only if $\Phi \in \Delta_2^\infty$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [12, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \tag{13}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset \left[L^\Phi\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \tag{14}$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2^\infty$, the relation $L^\Psi = \left[L^\Phi\right]^*$ will not hold. In general, it is true that $\left[E^\Phi\right]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* W^1L^Φ (see [11]) by

$$W^1L^\Phi := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi\}.$$

W^1L^Φ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{15}$$

We introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{16}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty \hookrightarrow L^\Phi \hookrightarrow L^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $u : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t-s|). \tag{17}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (17) for some $C > 0$. This is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t-s|)}.$$

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The next simple embedding lemma, whose proof can be found in [9], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1L^\Phi \hookrightarrow C^w([0,T], \mathbb{R}^d)$ *and for every* $u \in W^1L^\Phi$

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|) \qquad \textit{(Morrey's inequality),} \tag{18}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality).} \tag{19}$$

2. *For every* $u \in W^1L^\Phi$ *we have* $\widetilde{u} \in L^\infty_d$ *and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \textit{(Sobolev-Wirtinger's inequality).} \tag{20}$$

3 Main results

We begin stating the following lemma which establishes the coercivity of the modular function $\rho_\Phi(u)$ with respect to certain functions of the Orlicz norm $\Phi_0(\|u\|_{L^\Phi}|)$. This lemma generalizes [9, Lemma 5.2] in two directions. The first, we consider function

Φ_0 more general than in [9, Lemma 5.2]. The second one and more important, the condition Δ_2-global on Ψ used in [9, Lemma 5.2] is relaxed to Δ_2^∞. We need to present this result here since the lemma introduces the function Φ^* that will play an important role in our main theorem.

Lemma 3.1. *Let Φ, Ψ be complementary N-functions with $\Psi \in \Delta_2^\infty$. Then there exists an N-function Φ^*, with $\Phi^* \prec \Phi$, such that for every $\Phi_0 \lll \Phi^*$ and $k > 0$*

$$\lim_{\|u\|_{L^\Phi} \to \infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} = \infty. \tag{21}$$

Reciprocally, if (21) *holds for some N-function Φ_0, then $\Psi \in \Delta_2^\infty$.*

We point out that this lemma can be applied to more cases than [9, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively, then (21) holds for Φ_0. On the other hand, $\Phi_0(|u|)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

As in [9] we will consider general lagrangian functions $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the structure conditions

$$|\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right) \tag{A_1}$$

$$|D_x\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right) \tag{A_2}$$

$$|D_y\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(c(t) + \varphi\left(\frac{|y|}{\lambda} + f(t)\right)\right), \tag{A_3}$$

where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function, φ is the right continuous derivative of Φ, $b \in L_1^1([0,T])$, $c \in L_1^\Psi([0,T])$ and $f \in E_1^\Phi([0,T])$. We denote by $\mathfrak{A}(a,b,c,\lambda,f,\Phi)$ the set of all Lagrange functions satisfying (A_1),(A_2) and (A_3).

It was shown in [9] that if $\mathcal{L} \in \mathfrak{A}(a,b,c,\lambda,f,\Phi)$ then there exists the Gateâux derivative $I'(u)$ of the integral functional I, defined in (6), on the subspace $W^1E^\Phi([0,T],\mathbb{R}^d)$. We note that for the lagrangian $\mathcal{L}_{\Phi,F}$ the condition (A) is equivalent to $\mathcal{L}_{\Phi,F} \in \mathfrak{A}(a,b,0,1,0,\Phi)$.

Unlike what is usual in the literature, we do not assume that the Lagrangian $\mathcal{L}$ split in two terms one of them function of y and the other one function of (t,x). We only suppose that $\mathcal{L}$ is lower bounded by a function of this type. More precisely, we assume that for every $(t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{L} \geqslant \mathcal{L}_{\Phi,F}, \quad F \text{ satisfying (A) and (C), } \Phi \text{ being } N\text{-function.} \tag{A_4}$$

Moreover, as usual, it is needed to suppose that the time integral of F satisfies certain coercivity condition, see (A_5) below. However, all these hypotheses are not enough. It is necessary to assume extra conditions on the potential F. Several hypotheses were tested in the past years. The so-called *subconvexity* of F was tried in [4, 2, 6] for semilinear equations and in [15, 8] for p-laplacian systems. A boundedness on the oscillation of F was studied in [9]. In [3, 8]) the authors considered, for the p-laplacian case, potentials F satisfying the inequality

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^\alpha + b_2(t), \tag{22}$$

where $b_1, b_2 \in L^1([0,T])$ and $\alpha < p$. The potentials F satisfying these inequalities were called by the authors *sublinear nonlinearities*. In this paper, we are interested in studying this type of potential, but with more general bounds on ∇F which include N-functions instead of power functions, namely, we will consider inequalities for F of the following type

$$|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t), \tag{A_6}$$

with Φ_0 a differentiable N-function and $b_1, b_2 \in L^1([0,T],\mathbb{R})$.

Theorem 3.2. *Let Φ be an N-function, whose complementary function Ψ satisfies $\Psi \in \Delta_2^\infty$ and suppose that Φ^* satisfies Lemma 3.1. Assume that the potential F satisfies (C), (A) and the conditions*

1. (A_6) *for some N-function Φ_0 such that $\Phi_0 \lll \Phi^*$.*

2. $$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Psi_2(\Phi_0'(2|x|))} = +\infty, \tag{A_5}$$

   for some N-function Ψ_2 whose complementary function Φ_2 satisfies $\Phi_0 \lll \Phi_2 \lll \Phi^*$.

Then, for every Lagrangian $\mathcal{L}(t,x,y)$ such that $D_y\mathcal{L}(0,x,y) = D_y\mathcal{L}(T,x,y)$, stritly convex with respecto to $y \in \mathbb{R}^d$, $\mathcal{L} \in \mathfrak{A}(a,b,c,\lambda,f,\Phi)$, and satisfying (A_4) the problem (1) *has at least a solution which minimizes the action integral I on $W^1E_T^\Phi$.*

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4 Proofs

Lemma 4.1. *E^Φ is weak* closed in L^Φ.*

Proof. From [13, Thm. 7, p. 110] we have that $L^\Phi = \left[E^\Psi\right]^*$. Then, L^Φ is a dual and therefore we are allowed to speak about the weak* topology of L^Φ. Besides, E^Φ is separable (see [13, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [13, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because if $u_n \in S$ and $u_n \overset{*}{\rightharpoonup} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \overset{*}{\rightharpoonup} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ is separable and $L^\Phi = \left[E^\Psi\right]^*$, the closed ball of $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ of L^Φ is weak* metrizable (see [16, Thm. 5.1, p. 138]). Thus, S is closed with respect to the weak* topology. Now, by Krein-Smulian theorem, [16, Cor. 12.6, p. 165] implies that E^Φ is weak* closed. □

The following result is analogous to some lemmata in $W^{1,p}$, see [15].

Lemma 4.2. *If $\|u\|_{W^1L^\Phi} \to \infty$, then $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u} + \tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{23}$$

It is known that $L^\infty \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L^\infty_d$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1\|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2\|u'\|_{L^\Phi}. \tag{24}$$

Therefore, from (23), (24) and (15), we get

$$\|u\|_{W^1L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. □

Lemma 4.3. *Let Φ be a (not necessary differentiable) N-function and φ the right continuous derivative of Φ. Then $\Phi \in \Delta_2^\infty$ ($\Phi \in \Delta_2$) iff $\varphi \in \Delta_2^\infty$ ($\varphi \in \Delta_2$).*

Proof. It is consequence of $[17, Th.11.7]$ and [17, Remark 5, p. 87].

□

The following lemma improves the result on the comment at the beginning of [12, p. 24].

Lemma 4.4. *Let Ψ be an N-function satisfying the Δ_2^∞ condition. Then there exists an N-function Ψ^* such that $\Psi^* \in \Delta_2$, $\Psi \leqslant \Psi^*$ and for every $a > 1$ there exists $x_0 = x_0(a) \geqslant 0$ such that $\Psi^*(x) \leqslant a\Psi(x)$, for every $x \geqslant x_0$. In other words, every Δ_2 near infinity N-function is equivalent to a Δ_2 globally N-function.*

Proof. We can assume that the Δ_2 condition for Ψ fails near to 0. Consequently, from Lemma 4.3, we have that the right continuous derivative ψ of Ψ is not Δ_2 near to 0. Therefore, we obtain a sequence x_n, $n = 1, 2, \dots$ of positive numbers with $x_n \to 0$, and

$$2x_{n+1} < x_n < 2x_n \quad \text{and} \quad \psi(2x_n) > 2\psi(x_n). \tag{25}$$

We define ψ^* inductively on n on the interval $[2x_n, +\infty)$ of the following way. We put $\psi^*(x) = \psi(x)$ when $x \in [2x_1, +\infty)$. Suppose ψ^* defined in $[2x_n, +\infty)$. We define ψ^* in $[2x_{n+1}, 2x_n)$ by

$$\psi^*(x) = \begin{cases} \max\left\{\psi(x), \frac{\psi^*(2x_n)}{2x_n}(x - x_n) + \frac{\psi^*(2x_n)}{2}\right\}, & \text{if } x_n \leqslant x < 2x_n \\ \frac{\psi^*(2x_n)}{2} & \text{if } 2x_{n+1} \leqslant x < x_n \end{cases}$$

Moreover we define $\psi^*(0) = 0$.

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Next, we will use induction again to prove that

1. $\psi^*(x_n) = \frac{1}{2}\psi^*(2x_n)$,
2. ψ^* is non-decreasing $[2x_n, +\infty)$,
3. $\psi \leqslant \psi^*$ in $[2x_n, +\infty)$

We suppose $n = 1$. Then items 2 and 3 are obvious. From (25) we have

$$\psi(x_1) < \frac{1}{2}\psi(2x_1) = \frac{1}{2}\psi^*(2x_1).$$

This inequality implies 1.

Clearly ψ^* is non-decreasing on each interval $[2x_{n+1}, x_n)$ and $[x_n, 2x_n)$. We note that since ψ is right continuous, ψ^* is continuous at x_n. Therefore ψ^* is non-decreasing on $[2x_{n+1}, 2x_n)$. Suppose $x \in [2x_{n+1}, 2x_n)$ and $y \geqslant 2x_n$. From the definition of ψ^*, inductive hypothesis item 3 and item 2 we obtain

$$\psi^*(x) \leqslant \max\{\psi(2x_n), \psi^*(2x_n)\} = \psi^*(2x_n) \leqslant \psi^*(y).$$

This proves item 2 in the interval $[2x_{n+1}, +\infty)$. Inequality in item 3 holds by inductive hypothesis in $[2x_n, +\infty)$ and is obvious for $x \in [x_n, 2x_n)$. If $x \in [2x_{n+1}, x_n)$, then $\psi(x) \leqslant \psi(x_n) \leqslant \psi^*(x_n) = \psi^*(x)$. This proves 3 in the interval $[2x_{n+1}, +\infty)$

Now, using (25) and the alreadyr proved items 3 for $n+1$ we deduce

$$\psi(x_{n+1}) < \frac{1}{2}\psi(2x_{n+1}) \leqslant \frac{1}{2}\psi^*(2x_{n+1})$$

Then

$$\psi^*(x_{n+1}) = \max\left\{\psi(x_{n+1}), \frac{1}{2}\psi^*(2x_{n+1})\right\} = \frac{1}{2}\psi^*(2x_{n+1}),$$

i.e. we have proved item 1.

We note that

$$\psi^*(x_{n+1}) = \frac{1}{2}\psi^*(2x_{n+1}) \leqslant \psi^*(x_n).$$

Consequently $\psi(x) \to 0$ when $x \to 0$. Therefore ψ^* is right continuous at 0 and, in fact, right continuous on $[0, +\infty)$. Moreover, since $\psi(x) = \psi^*(x)$ for $x \geqslant 2x_1$ being ψ the right continuous derivative of aN N-function, $\psi^*(x) \to +\infty$ when $x \to +\infty$. In this way

$$\Psi^*(x) := \int_0^x \psi^*(t)dt.$$

define a N function.

Let see that ψ^* is Δ_2 globally. For it is sufficient to prove that ψ^* satisfies the Δ_2 conditions near of 0. For this end, suppose $x \leqslant x_1$ and take $n \in \mathbb{N}$ such that $x_{n+1} \leqslant x \leqslant x_n$. Then

$$\psi^*(2x) \leqslant \psi^*(2x_n) = 2\psi^*(2x_{n+1}) = 4\psi^*(x_{n+1}) \leqslant 4\psi^*(x).$$

Consequently Ψ^* is Δ_2 globally and $\Psi \leqslant \Psi^*$.

It remains to show the inequality $\Psi^*(x) \leqslant a\Psi(x)$, for every $a > 1$ and sufficiently large x. We take x_0 sufficiently large for that

$$\frac{1}{a-1}\int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x_0).$$

Therefore, if $x > \max\{x_0, 2x_1\}$ then

$$\Psi^*(x) = \Psi(x) + \int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x) + (a-1)\Psi(x) = a\Psi(x).$$

The last assessment of the lemma is consequence of $\Psi(ax) > a\Psi(x)$ when $a > 1$.

□

The following lemma is essentially known, this is basically consequence of the fact that $\Psi \in \Delta_2^\infty$ if and only if $\Psi \prec \Psi$, of [13, Prop. 4, p. 20] and [13, Cor. 10, p. 30]. However, we prefer to includ an alternative proof, because we do not see clearly that the previous results of [13] contemplates the case in the following lemma with the N-functions satisfying the Δ_2 condition globally.

Lemma 4.5. *Let Φ, Ψ be complementary functions. The next statements are equivalent:*

1. *$\Psi \in \Delta_2$ ($\Psi \in \Delta_2^\infty$).*
2. *There exists an N-function Φ^* such that*

$$\Phi(rs) \geqslant \Phi^*(r)\Phi(s) \quad \textit{for every } r \geqslant 1, s \geqslant 0\,(r \geqslant 1, s \geqslant 1). \tag{26}$$

Proof. In virtue of the comment that precedes the statement of the lemma, we only condider the case $\Psi \in \Delta_2$.

1)⇒2). By virtue of the Δ_2-condition on Ψ, [17, Thm. 11.7] and [17, Cor. 11.6] (see also [18, Eq. (2.8)]), we get constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geqslant Kr^\nu \Phi(s), \tag{27}$$

for any $1 < \nu < \alpha_\Phi$, $s \geqslant 0$ and $r > 1$. This proves (26) with $\Phi^*(r) = kr^\nu$, which is an N-function.

2)⇒1) Next, we follow [13, p. 32, Prop. 13] and [13, p. 29, Prop. 9]. Assume that

$$\Phi^*(r)\Phi(s) \leqslant \Phi(rs) \;\; r > 1, \;\; s \geqslant 0.$$

Let $u = \Phi^*(r) \geqslant \Phi^*(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [13, p. 13, Prop. 1] and (26), we have for $u \geqslant \Phi^*(1)$ and $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant {\Phi^*}^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{{\Psi^*}^{-1}(u)\Psi^{-1}(v)},$$

then

$${\Psi^*}^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi^*(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi^*(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi^{*-1}(\Phi^*(1))\}$ we get that $\Psi \in \Delta_2$ globally. □

Remark 1. We note that if Φ^* satisfies (26) then $\Phi^* \prec \Phi$.

Proof Lemma 3.1 At first we assume that Φ is Δ_2-globally. Let Φ^* be an N function satisfying (26). By the inequality (12), for $r > 1$ we have

$$\int_0^T \Phi(|u|)\,dt \geqslant \Phi^*(r)\int_0^T \Phi(r^{-1}|u|)\,dt \geqslant \Phi^*(r)\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$, as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$. From [13, Thm. 2 (b)(v), p. 16] we get

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi^*\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(k\|u\|_{L^\Phi})} = \infty.$$

If $\Psi \in \Delta_2^\infty$, but $\Psi \notin \Delta_2$, we use Lemma 4.4. Then there exists a Δ_2 N-function Ψ_1, with $\Psi_1 \sim \Psi \leqslant \Psi_1$. Let Φ_1 the complementary function of Ψ_1. Then $\Phi \sim \Phi_1 \leqslant \Phi$ (see [12, Thm. 3.1]) and $\|\cdot\|_{L^\Phi}$ and $\|\cdot\|_{L^{\Phi_1}}$ are equivalents norms (see [12, Thm. 13.2 and Thm. 13.3]). By the previously proved WHAT????, there exists Φ_0 satisfying (21) with $\Phi = \Phi_1$. Let $C > 0$ be such that $\|\cdot\|_{L^\Phi} \leqslant C\|\cdot\|_{L^{\Phi_1}}$. Then

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(k\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi_1(|u|)dt}{\Phi_0(kC\|u\|_{L^{\Phi_1}})} = +\infty.$$

Finally, if Φ_0 is an N-function, then $\Phi_0(x) \geqslant \alpha|x|$ for α small enough and $|x| > 1$. Therefore (21) holds for $\Phi_0(x) = |x|$, then [9, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞. □

Definition 4.6. *We define the functionals $J_{C,\varphi} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\varphi} : \mathbb{R}^n \to \mathbb{R}$, where $C > 0$ and $\varphi : [0,+\infty) \to [0,+\infty)$ by*

$$J_{C,\varphi}(u) \coloneqq \rho_\Phi(u) - C\varphi(\|u\|_{L^\Phi}), \tag{28}$$

and

$$H_{C,\varphi}(x) \coloneqq \int_0^T F(t,x)dt - C\varphi(2|x|), \tag{29}$$

respectively.

Next we give the proof of the Theorem 3.2. Here we correct an error in the end of the proof of [9, Thm. 6.2] where was wrongly assumed that the minimum of I is in the domain of differentiability of I. We avoid this problem by minimizing I on $W^1E_T^\Phi$ which is a subspace weak* closed contained that domain.

Proof. Theorem 3.2 . By the decomposition $u = \overline{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (A_6), we have

$$\begin{aligned} &\left|\int_0^T F(t,u) - F(t,\overline{u})\, dt\right| = \left|\int_0^T \int_0^1 \nabla F(t,\overline{u} + s\tilde{u}(t)) \cdot \tilde{u}(t)\, ds\, dt\right| \\ &\leqslant \int_0^T \int_0^1 b_1(t)\Phi_0'(|\overline{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\, ds\, dt + \int_0^T \int_0^1 b_2(t)|\tilde{u}(t)|\, ds\, dt \\ &=: I_1 + I_2. \end{aligned} \tag{30}$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty} \leqslant C_1\|u'\|_{L^\Phi}, \tag{31}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

We note that, since Φ_0' is increasing function and $\Phi_0'(x) \geqslant 0$ for $x \geqslant 0$, then $\Phi_0'(a + b) \leqslant \Phi_0'(2a) + \Phi_0'(2b)$ for every $a, b \geqslant 0$. In this way, we have

$$\Phi_0'(|\overline{u} + s\tilde{u}(t)|) \leqslant \Phi_0'(2|\overline{u}|) + \Phi_0'(2\|\tilde{u}\|_{L^\infty}), \tag{32}$$

for every $s \in [0, 1]$. Now, inequality (32), Hölder's and Sobolev-Wirtinger's inequalities imply that

$$\begin{aligned} I_1 &\leqslant \Phi_0'(2|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} + \Phi_0'(2\|\tilde{u}\|_{L^\infty})\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} \\ &\leqslant C_2\left\{\Phi_0'(2|\overline{u}|)\|u'\|_{L^\Phi} + \Phi_0'(C_3\|u'\|_{L^\Phi})\|u'\|_{L^\Phi}\right\}, \end{aligned} \tag{33}$$

where $C_2 = C_2(T, \|b_1\|_{L^1})$ and $C_3 = C_3(T)$. Next, by Young's inequality with complementary functions Φ_2 and Ψ_2

$$\Phi_0'(2|\overline{u}|)\|u'\|_{L^\Phi} \leqslant \Psi_2(\Phi_0'(2|\overline{u}|)) + \Phi_2(\|u'\|_{L^\Phi}). \tag{34}$$

We have that any N-function Φ_0 satisfies the inequality $x\Phi_0'(x) \leqslant \Phi_0(2x)$ (see [13, p. 17]). Moreover, since $\Phi_0 \lll \Phi_2$ there exists $x_0 = x_0(\Phi_0, \Phi_2, T) \geqslant 0$ such that $\Phi_0(2C_3x) \leqslant \Phi_2(x)$, for every $x \geqslant x_0$. Therefore, $\Phi_0(2C_3x) \leqslant \Phi_2(x) + C_4$, with $C_4 = \Phi_0(2x_0)$. The previous observations imply

$$\Phi_0'(C_3\|u'\|_{L^\Phi})\|u'\|_{L^\Phi} \leqslant C_3^{-1}(\Phi_2(\|u'\|_{L^\Phi}) + C_4). \tag{35}$$

From (33), (34), (35) and (31), we have

$$I_1 + I_2 \leqslant C_5\left\{\Psi_2(\Phi_0'(2|\overline{u}|)) + \Phi_2(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi} + 1\right\} \tag{36}$$

with C_5 depending on $\Phi_0, \Phi_2, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$.

Using (30) and (36) we get

$$\begin{aligned} I(u) &\geqslant \rho_\Phi(u') + \int_0^T F(t,u)\,dt \\ &= \rho_\Phi(u') + \int_0^T \left[F(t,u) - F(t,\overline{u})\right]\,dt + \int_0^T F(t,\overline{u})\,dt \\ &\geqslant \rho_\Phi(u') - C_5\Phi_2(\|u'\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_5\Psi_2(\Phi_0'(2|\overline{u}|)) - C_5 \\ &\geqslant \rho_\Phi(u') - C_5\Phi_2(\|u'\|_{L^\Phi}) + H_{C_5,\Psi_2\circ\Phi_0}(\overline{u}) - C_5 \\ &= J_{C_5,\Phi_0}(u') + H_{C_5,\Psi_2\circ\Phi_0}(\overline{u}) - C_5. \end{aligned} \tag{37}$$

Let u_n be a sequence in W^1L^Φ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n, $I(u_n)$ is upper bounded, i.e. there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 4.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence is necessary, still denoted u_n, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.1 implies that the functional $J_{C_5,\Phi_0}(u')$ is coercive; and, by (A_5), the functional $H_{C_5,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_5,\Phi_0}(u_n') \to \infty$ or $H_{C_5,\Phi_0}(\overline{u}_n) \to \infty$. From the condition (A) on F, we have that on a bounded set the functional $H_{C_5,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_5,\Phi_0}(u_n') \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u)|u \in W^1E_T^\Phi\}$. Since $I(u_n)$, $n = 1,2,\ldots$, is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in W^1E^Φ. Hence, by virtue of [9, Cor. 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T-periodic continuous (therefore in E_T^Φ) function u. As $u_n' \in E^\Phi$ is a norm bounded sequence in L^Φ, there exists a subsequence (again denoted by u_n') such that u_n' converges to a function $v \in L^\Phi$ in the weak* topology of L^Φ. Since E^Φ is weak* closed, by Lemma 4.1, $v \in E^\Phi$. From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \xi' \cdot u\,dt = \lim_{n\to\infty}\int_0^T \xi'\cdot u_n\,dt = -\lim_{n\to\infty}\int_0^T \xi\cdot u_n'\,dt = -\int_0^T \xi\cdot v\,dt$$

for every T-periodic function $\xi \in C^\infty([0,T],\mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [1, p. 6]) and $u \in W^1E_T^\Phi$.

Now, taking into account the relations $\left[L^1\right]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that u_n' converges to u' in the weak topology of L^1. Consequently, from the semicontinuity of I (see [9, Lemma 6.1]) we get

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{v\in W^1E_T^\Phi} I(v).$$

Hence $u \in W^1E_T^\Phi$ is a minimun and, since I is Gâteaux differentiable on W^1E^Φ (see [9, Thm. 3.2]), therefore $I'(u) \in (W^1E_T^\Phi)^\perp$. Thus,

$$\int_0^T D_y\mathcal{L}(t,u(t),u'(t))\cdot v'(t)dt = -\int_0^T D_x\mathcal{L}(t,u(t),u'(t))\cdot v(t)dt,$$

for every $v \in W^1E_T^\Phi$.

From [9, Eq. (26)] we have $D_y\mathcal{L}(t,u(t),u'(t)) \in L^\Psi([0,T],\mathbb{R}^n) \hookrightarrow L^1([0,T],\mathbb{R}^n)$; and, from [9, Eq. (24)], it follows that $D_x\mathcal{L}(t,u(t),u'(t)) \in L^1([0,T],\mathbb{R}^n)$. Consequently, from [1, p. 6] we obtain that the differential equations in (1) are verified.

Now we consider a genral lagragian $\mathcal{L}$. In Theorem 3.2 we have just seen that the action integral $\int_0^T \Phi(|u'(t)|) + F(t,u(t))\,dt$ is coercive, then the functional $\mathcal{L}$ does so.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u)|u \in W^1E_T^\Phi\}$. Now, $\{u_n\}$ is bounded due to the coercivity of $\mathcal{L}$.

By [9, Thm. 3.2, Lemma 6.1] we have that I is differentiable on W^1E^Φ and lower semi-continuous, respectively.

Next, we find the minimum of I by means of a similar argument to the one developed on the last part of the proof of Theorem 3.2, changing $u'(t)\Phi'(|u'(t)|)/|u'(t)|$ by $D_y\mathcal{L}(t,u(t),u'(t))$ which also belongs to L^Ψ (see [9, Eq. (26)]).

Finally, the strict convexity of $\mathcal{L}(t,x,y)$ at y and the injectivity of the function $D_y\mathcal{L}(T,u(T),\cdot)$ imply that $u'(0) = u'(T)$. □

5 Examples

In this section we developed some applications of our main results so that the reader can appreciate the innovations that bring.

One of the main novelties of our work is that we obtain existence of solutions for lagrangian functions $\mathcal{L}(t,x,y)$ that not satisfy a power like grow condition in y. is not necessarily a power function.

Example 1.We can applied Theorem 3.2 to Lagrangians $\mathcal{L} = \mathcal{L}(t,x,y)$ with an exponential grow in y. For example, suppose

$$\mathcal{L}(t,x,y) = f(y) + F(t,x),$$

with $f(y) \geqslant e^{|y|} - |y| - 1$. The complementary function $\Psi(x)$ of the N-function $\Phi(x) = e^x - x - 1$ is Δ_2 globally (see [12, p. 28]). We suppose that Φ_1 satisfies (26). Taking $s \to 0$ in (26) we obtain that $\Phi_1(r) \leqslant r^2$. On the other hand, is not hard to prove that $\Phi_1(r) = r^2$ satisfies (26). Hence, if $1 < \alpha < \beta < 2$ and $\Phi_0(x) = |x|^\alpha$ and $\Phi_2(x) = |x|^\beta$ then $\Phi_0 \lll \Phi_2 \lll \Phi_1$, we obtain existence of periodic solutions when F satisfies

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^{\alpha-1} + b_2(t) \quad \text{and} \quad \lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{|x|^\gamma} = +\infty,$$

with $\alpha \in (1,2)$ and $\gamma \in (2\alpha - 2, \alpha)$.

Example 2. We want to emphasize that, even in the case of p-laplacian operator (3), our results extend previous ones (see [3, 8]), because we get bounds that may be sharper than those in [3, 8]. For example, in [8, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions of (3) under the assumption (22) for any $\alpha \in (0, p-1)$. Meanwhile, our Theorem 3.2 implies existence for the potential

$$F_0(t,x) = |x|^p / \ln(2 + |x|)^2.$$

We note that this F does not satisfy (22) for any $\alpha < p-1$. Next we will show an N-function Φ_0 satisfying the hypothesis of Theorem 3.2 for this potential F_0.

We define

$$\Phi_0(u) = \begin{cases} \frac{p-1}{p}u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function Φ_0.

Theorem 5.1. *If $p \geqslant \frac{1+\sqrt{2}}{2}$, then Φ_0 is a differentiable N-function. The N-function Φ_0 satisfies that for every $\varepsilon > 0$, there exists a positive constant $C = C(p,\varepsilon)$ such that*

$$C^{-1}t^{p-\varepsilon}\Phi_0(u) \leqslant \Phi_0(tu) \leqslant Ct^p\Phi_0(u) \quad t \geqslant 1, u > 0, \tag{38}$$

Proof. We have

$$\varphi(u) = \Phi_0'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that Φ_0' is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that φ_1 is increasing on $[0,e]$ and φ_2 is increasing on $[e,\infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p > 1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u) > 0$ on $[e,\infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an increasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Moreover $\varphi_2(u) \to \infty$ and $\varphi_1(u) \to 0$ as $u \to \infty$ and $u \to 0$ respectively, provided that $p > 1$. Hence, Φ_0 is an N-function.

Next we will prove (38). If $u \leqslant tu \leqslant e$, then $\Phi_0(tu) = t^p\Phi_0(u)$ and (38) holds with $C = 1$. If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geqslant 1$, we have $\Phi_0(tu) \leqslant t^p u^p = \frac{p}{p-1}t^p\Phi_0(u)$. Thus, the second inequality of (38) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e,\infty)$, then $f((tu)^p) \geqslant f(e^p) = e^p/p$. Now,

$$\begin{aligned} \Phi_0(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} \\ &\geqslant \frac{p-1}{p}\frac{t^\varepsilon}{\log t + 1}t^{p-\varepsilon}u^p. \end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t + 1}$ on the interval $[1,+\infty)$ then

$$\Phi_0(tu) \geqslant \frac{p-1}{p}\varepsilon e^{1-\varepsilon}t^{p-\varepsilon}u^p,$$

which is the first inequality of (38) with $C = \frac{p}{p-1}\varepsilon^{-1}e^{-1+\varepsilon}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi_0(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v}, \tag{39}$$

where $v := u^p$ and $v \geqslant e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on (α, ∞) and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p, \infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every $v \geqslant e^p$. In this way, from (39), we have

$$\Phi_0(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and the second inequality of (38) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi_0(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, the function f attains a minimum on $[1, \infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$, f is increasing on $[1, \infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geqslant \varepsilon$ in any case, therefore

$$\Phi_0(tu) \geqslant \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geqslant \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi_0(u).$$

Finally, (38) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. $\square$

Remark 2. The inequality

$$\Phi_0(tu) \geqslant Ct^p\Phi_0(u)$$

is false for every C because for every $u \geqslant e$ we have

$$\lim_{t\to\infty}\frac{\Phi_0(tu)}{t^p\Phi_0(u)} = 0$$

We note that Φ_0 and F_0 satisfy (A_5). For the p-laplacian operator we have that $\Phi(|u|) = |u|^p/p$. Then we can take $\Phi_1 = \Phi$ in (26). Clearly $\Phi_0 \lll \Phi_1$.

6 On the *BO* condition

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