

# Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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## Abstract

## 1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (1)$$

where  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 1$ , is called the *Lagrange function* or *lagrangian* and the unknown function  $u : [0, T] \rightarrow \mathbb{R}^d$  is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t, x, y) := \frac{|y|^p}{p} + F(t, x), \quad (2)$$

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for  $1 < p < \infty$ . For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (1), for the lagrangian  $\mathcal{L}_{2,F}$ , through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (2) for arbitrary  $1 < p < \infty$  were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (1) is reduced to the  $p$ -laplacian system

$$\begin{cases} \frac{d}{dt}(u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (3)$$

In this context, it is customary to call  $F$  a *potential function*, and it is assumed that  $F(t, x)$  is differentiable with respect to  $x$  for a.e.  $t \in [0, T]$  and the following conditions are verified:

(C)  $F$  and its gradient  $\nabla F$ , with respect to  $x \in \mathbb{R}^d$ , are Carathéodory functions, i.e. they are measurable functions with respect to  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ , and they are continuous functions with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

(A) For a.e.  $t \in [0, T]$ , it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (4)$$

In this inequality we assume that the function  $a : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and non decreasing and  $0 \leq b \in L^1([0, T], \mathbb{R})$ .

In [Acinas et al., 2015] it was treated the case of a lagrangian  $\mathcal{L}$  which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi, F}(t, x, y) = \Phi(|y|) + F(t, x), \quad (5)$$

where  $\Phi$  is an  $N$ -function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on  $F$  (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (1).

## 2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , i.e. functions such that  $\Phi(x)$  depends on the direction of  $x$ , unlike the radial case where  $\Phi(x) = \Phi(|x|)$ . References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of  $p$ -laplacian equations as the following example shows.

**Example 1.** Let  $1 < p_1, p_2 < \infty$ . We define  $\Phi_{p_1, p_2} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|}{p_1} + \frac{|y_2|}{p_2}.$$

Suppose the following Lagrange function

$$\mathcal{L}(t, x, y) = \Phi_{p_1, p_2}(y) + F(t, x).$$

Then the equations (1) becomes

$$\begin{cases} \frac{d}{dt}(|u_1'|^{p_1-2}u_1') = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt}(|u_2'|^{p_2-2}u_2') = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (6)$$

On the other hand, anisotropic Orlicz-Sobolev spaces allow to simplify the writing, and they provide the natural frame of statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question was what motivated us to use these spaces

Hereafter we denote by  $\mathbb{R}^+$  the set of all non negative real numbers. A function  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is called an *Young's function* if  $\Phi$  is convex,  $\Phi(0) = 0$ ,  $\Phi(-x) = \Phi(x)$  and  $\Phi(x) \rightarrow +\infty$ , when  $|x| \rightarrow +\infty$ . Additionally we assume that the Young's functions which we deal with, satisfy that  $\Phi(x) > 0$  when  $x \neq 0$ .

Following [Schappacher, 2005] we say that  $\Phi$  is *coercive* if

$$\lim_{|x| \rightarrow \infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function  $\Phi$ , we define the function  $\hat{\Phi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by

$$\hat{\Phi}(s) = \sup\{|x| : \Phi(x) \leq s\}, \quad (7)$$

We point out that the function  $G(|x|)$  has the following properties, as reader can easily check,

- (G1)  $G$  is monotonous increasing, continuous and  $G(s) \rightarrow \infty$  as  $s \rightarrow \infty$ .
- (G2)  $G$  is the *greatest radial minorant* of  $\Phi(x)$ , i.e.  $G(|x|) \leq \Phi(x)$  and  $G(|x|)$  is the biggest radial function with this property.
- (G3) There exists  $G^{-1}$  and  $G^{-1}(\Phi(x)) \geq |x|$ .
- (G4) As  $\Phi(\alpha x)/\alpha$  is increasing with respect to  $\alpha$  for every  $x > 0$ ,  $G(\alpha s)/\alpha$  is also increasing with respect to  $\alpha$  for every  $s > 0$ . Alternatively  $\beta G^{-1}(t/\beta)$  is an increasing function with respect to  $\beta$  for every  $t > 0$ .
- (G5) In the event that  $\Phi$  is coercive, then  $G$  is also coercive. Alternatively  $G^{-1}(s)/s \rightarrow 0$  when  $s \rightarrow +\infty$ .

We also say that  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$  satisfies the  $\Delta_2^\infty$ -condition, denoted by  $\Phi \in \Delta_2^\infty$ , if there exist constants  $K > 0$  and  $M \geq 0$  such that

$$\Phi(2x) \leq KH(x), \quad (8)$$

for every  $|x| \geq M$ .

If  $\Phi$  is a Young's function we define its *Fenchel conjugate*  $\Phi^* : \mathbb{R}^d \rightarrow \mathbb{R}^+$  by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \quad (9)$$

We denote by  $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$ , with  $d \geq 1$ , the set of all measurable functions (i.e. functions which are limits of simple functions) defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $u = (u_1, \dots, u_d)$  for  $u \in \mathcal{M}$ . For the set of functions  $\mathcal{M}$ , as for other similar sets, we will omit the reference to codomain  $\mathbb{R}^d$  when  $d = 1$ .

Given an  $N$ -function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Here  $|\cdot|$  is the euclidean norm of  $\mathbb{R}^d$ . Now, we introduce the *Orlicz class*  $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$  by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (10)$$

The *Orlicz space*  $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$  is the linear hull of  $C^\Phi$ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (11)$$

The Orlicz space  $L^\Phi$  equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left( \frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space. By  $u \cdot v$  we denote the usual dot product in  $\mathbb{R}^d$  between  $u$  and  $v$ .

The subspace  $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$  is defined as the closure in  $L^\Phi$  of the subspace  $L^\infty([0, T], \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1])  $u \in E^\Phi$  if and only if  $\rho_\Phi(\lambda u) < \infty$  for any  $\lambda > 0$ . The equality  $L^\Phi = E^\Phi$  is true if and only if  $\Phi \in \Delta_2^\infty$  (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of  $E^\Phi$ , which is particularly useful for us, is that  $u \in E^\Phi$  if and only if  $u$  has *absolutely continuous norm*, i.e. if  $E_n \subset [0, T]$ ,  $n = 1, 2, \dots$  then  $\|\chi_{E_n} u\| \rightarrow 0$  when  $|E_n| \rightarrow 0$ .

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if  $u \in L^\Phi$  and  $v \in L^{\Phi^*}$  then  $u \cdot v \in L^1$  and

$$\int_0^T v \cdot u dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \quad (12)$$

Like in [Krasnosel'skiĭ and Rutickiĭ, 1961] we will consider the subset  $\Pi(E^\Phi, r)$  of  $L^\Phi$  given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class  $C^\Phi$  by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset r C^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (13)$$

for any positive  $r$  (see [Schappacher, 2005, Thm. 5.6]). If  $\Phi \in \Delta_2^\infty$ , then the sets  $L^\Phi$ ,  $E^\Phi$ ,  $\Pi(E^\Phi, r)$  and  $C^\Phi$  are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

**Definition 2.1.** Let  $u_n, u \in L^\Phi([0, T], \mathbb{R}^d)$ . We say that  $u_n$  converges monotonically to  $u$  if there exists  $\alpha_n \in L^\infty([0, T], \mathbb{R})$ ,  $n = 1, 2, \dots$ , such that  $0 \leq \alpha_n(t) \leq \alpha_{n+1}(t)$ ,  $\alpha_n(t) \rightarrow 1$  a.e., when  $n \rightarrow \infty$  and  $u_n(t) = \alpha_n(t)u(t)$ .

As usual, if  $(X, \|\cdot\|_X)$  is a normed space and  $(Y, \|\cdot\|_Y)$  is a linear subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when there exists  $C > 0$  such that  $\|y\|_X \leq C\|y\|_Y$  for any  $y \in Y$ . With this notation, Hölder's inequality states that  $L^{\Phi^*} \hookrightarrow [L^\Phi]^*$ , where a function  $v \in L^{\Phi^*}$  is associated to  $\xi_v \in [L^\Phi]^*$  being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \quad (14)$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of  $[L^\Phi]^*$  which can be identified with  $L^{\Phi^*}$ .

**Proposition 2.2.** Let  $F \in [L^\Phi([0, T], \mathbb{R}^d)]^*$ . Then the following statements are equivalent

1.  $\xi \in L^{\Phi^*}([0, T], \mathbb{R}^d)$
2.  $\xi$  satisfies the monotone convergence property, which is if  $u_n$  converges monotonically to  $u$  then  $\langle \xi, u_n \rangle \rightarrow \langle \xi, u \rangle$ .

If  $\Phi \in \Delta_2^\infty$  and  $\Phi$  is coercive then  $L^{\Phi^*}([0, T], \mathbb{R}^d) = [L^\Phi([0, T], \mathbb{R}^d)]^*$  (see [Desch and Grimmer, 2001, Thm. 2.9, Thm. 2.10]).

We define the *Sobolev-Orlicz space*  $W^1 L^\Phi$  by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\}.$$

$W^1 L^\Phi([0, T], \mathbb{R}^d)$  is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (15)$$

And, we introduce the following subspaces of  $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \quad (16)$$

We will use repeatedly the decomposition  $u = \bar{u} + \tilde{u}$  for a function  $u \in L^1([0, T])$  where  $\bar{u} = \frac{1}{T} \int_0^T u(t) \, dt$  and  $\tilde{u} = u - \bar{u}$ .

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

**Lemma 2.3.** Let  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  be a Young's function and let  $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$ . Let  $G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be the function defined by (7). Then

1. For every  $s, t \in [0, T]$ ,  $s \neq t$ ,

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| G^{-1} \left( \frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq G^{-1} \left( \frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. We have  $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$  and

$$\|\tilde{u}\|_{L^\infty} \leq TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. If  $\Phi$  is coercive then the space  $W^1L^\Phi([0, T], \mathbb{R}^d)$  is compactly embedded in the space of continuous functions  $C([0, T], \mathbb{R}^d)$ .

*Proof.* By the absolutely continuity of  $u$ , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi}|s - t|}\right) &\leq \Phi\left(\frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr\right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By (G1) and (G3) we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi}|s - t|} \leq G^{-1}\left(\frac{1}{|s - t|}\right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that  $\alpha G^{-1}(1/\alpha)$  is an increasing function with respect to  $\alpha \in (0, \infty)$  we have

$$|u(t) - \bar{u}| \leq \|u'\|_{L^\Phi} TG^{-1}\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of  $\|u\|_{L^\Phi}$ , we obtain

$$\Phi\left(\frac{\bar{u}}{\|u\|_{L^\Phi}}\right) \leq \frac{1}{T} \int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leq \frac{1}{T}$$

Then by (G1) and (G3)

$$|\bar{u}| \leq G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq G^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TG^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \\ &\leq G^{-1}\left(\frac{1}{T}\right)\max\{1, T\}\|u\|_{W^1L^\Phi} \end{aligned}$$

In order to prove item 3, we take a bounded sequence  $u_n$  in  $W^1 L^\Phi([0, T], \mathbb{R}^d)$ . From (Morrey's inequality) and (G5) we infer that  $u_n$  are equicontinuous. Furthermore (Sobolev's inequality) implies that  $u_n$  is bounded in  $C([0, T], \mathbb{R}^d)$ . Therefore by the Arzela-Ascoli Theorem we obtain a subsequence  $n_k$  and  $u \in C([0, T], \mathbb{R}^d)$  with  $u_{n_k} \rightarrow u$  in  $C([0, T], \mathbb{R}^d)$ . □

### 3 Superposition operators in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateaux differentiability of action integrals associated to lagrangian functions defined in Sobolev-Orlicz spaces.

Henceforth we assume that  $f$  is a *Carathéodory function*,

- (C)  $f$  is measurable with respect to  $t \in [0, T]$  for every  $x \in \mathbb{R}^d$ , and  $f$  is a continuous function with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

**Definition 3.1.** For  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  we denote by  $\mathbf{f}$  the Nemytskii (o superposition) operator defined for functions  $u : [0, T] \rightarrow \mathbb{R}^d$  by

$$\mathbf{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined in anisotropic Orlicz spaces of vectorial functions. For the proofs of these results and additional discussions see [Płuciennik, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

**Theorem 3.2.** We assume that  $f$  satisfies condition ((C)). Then

1. Measurability. The operator  $\mathbf{f}$  maps measurable function into measurable functions
2. Extensibility.? If
3. Continuity.? If

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