Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

Sonia Acinas *
Instituto de Matemática Aplicada San Luis (CONICET-UNSL)
(5700) San Luis, Argentina
Universidad Nacional de La Pampa
(6300) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper we....

1 Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t\in(0,T) \\ \boldsymbol{u}(0)-\boldsymbol{u}(T)=\dot{\boldsymbol{u}}(0)-\dot{\boldsymbol{u}}(T)=0 \end{cases} \tag{1}$$

* SECyT-UNRC, UNSL and CONICET
† SECyT-UNRC and CONICET
2010 AMS Subject Classification. Primary: . Secondary: .
Keywords and phrases. .

where $T > 0$, $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{2}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{3}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(c(t) + \varphi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right). \tag{4}$$

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ. The non negative functions b, c and f satisfy that $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$, where the Banach spaces $L^1_1([0,T])$, $L^\Psi_1([0,T])$ and $E^\Phi_1([0,T])$ will be defined later.

It is well known that problem (**??**) comes from a variational one, that is, a solution of (**??**) is a critical point of the *action integral*

$$I(\boldsymbol{u}) = \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt. \tag{5}$$

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [**?**, **?**, **?**]. For Orlicz spaces of vector valued functions, see [**?**] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \text{for } t \geq 0,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the Δ_2*-condition*, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{6}$$

for every $t \geq t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2*-condition globally* ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0,T])$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $\boldsymbol{u} = (u_1, \dots, u_d)$ for $\boldsymbol{u} \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_0^T \Phi(|\boldsymbol{u}|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C_d^\Phi = C_d^\Phi([0,T])$ is given by

$$C_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \rho_\Phi(\boldsymbol{u}) < \infty\}\,. \tag{7}$$

The *Orlicz space* $L_d^\Phi = L_d^\Phi([0,T])$ is the linear hull of C_d^Φ; equivalently,

$$L_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \exists \lambda > 0 : \rho_\Phi(\lambda \boldsymbol{u}) < \infty\}\,. \tag{8}$$

The Orlicz space L_d^Φ equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup\left\{\int_0^T \boldsymbol{u}\cdot\boldsymbol{v}\, dt \big| \rho_\Psi(\boldsymbol{v}) \leqslant 1\right\},$$

is a Banach space. By $\boldsymbol{u}\cdot\boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^d$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [**?**, Thm. 10.5] and [**?**]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k}\left\{1 + \rho_\Phi(k\boldsymbol{u})\right\}. \tag{9}$$

The subspace $E_d^\Phi = E_d^\Phi([0,T])$ is defined as the closure in L_d^Φ of the subspace L_d^∞ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E_d^Φ is the only one maximal subspace contained in the Orlicz class C_d^Φ, i.e. $\boldsymbol{u} \in E_d^\Phi$ if and only if $\rho_\Phi(\lambda\boldsymbol{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [**?**, Th. 9.3]). Namely, if $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$ then $\boldsymbol{u}\cdot\boldsymbol{v} \in L_1^1$ and

$$\int_0^T \boldsymbol{v}\cdot\boldsymbol{u}\, dt \leqslant \|\boldsymbol{u}\|_{L^\Phi}\|\boldsymbol{v}\|_{L^\Psi}. \tag{10}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x\rangle = x^*(x)$. Hölder's inequality shows that $L_d^\Psi \subset \left[L_d^\Phi\right]^*$, where the pairing $\langle \boldsymbol{v}, \boldsymbol{u}\rangle$ is defined by

$$\langle \boldsymbol{v}, \boldsymbol{u}\rangle = \int_0^T \boldsymbol{v}\cdot\boldsymbol{u}\, dt \tag{11}$$

with $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L_d^\Psi = \left[L_d^\Phi\right]^*$ will not hold. In general, it is true that $\left[E_d^\Phi\right]^* = L_d^\Psi$.

Like in [**?**], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ given by

$$\Pi(E_d^\Phi, r) := \{\boldsymbol{u} \in L_d^\Phi | d(\boldsymbol{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \tag{12}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L_d^Φ, E_d^Φ, $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

We define the *Sobolev-Orlicz space* $W^1 L_d^\Phi$ (see [**?**]) by

$$W^1 L_d^\Phi := \{\boldsymbol{u} | \boldsymbol{u} \text{ is absolutely continuous and } \dot{\boldsymbol{u}} \in L_d^\Phi\}.$$

$W^1 L_d^\Phi$ is a Banach space when equipped with the norm

$$\|\boldsymbol{u}\|_{W^1 L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{13}$$

For a function $\boldsymbol{u} \in L_d^1([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t)\,dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L_d^\Psi \hookrightarrow \left[L_d^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L_d^\infty \hookrightarrow L_d^\Phi \hookrightarrow L_d^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant Cw(|t-s|). \tag{14}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (**??**) for some $C > 0$ and it is a Banach space with norm

$$\|\boldsymbol{u}\|_{C^w([0,T],\mathbb{R}^d)} := \|\boldsymbol{u}\|_{L^\infty} + \sup_{t \neq s} \frac{|\boldsymbol{u}(t) - \boldsymbol{u}(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [**?**, **?**, **?**, **?**, **?**]. The next simple lemma, whose proof can be found in [**?**], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1 L^\Phi \hookrightarrow C^w([0,T], \mathbb{R}^d)$ and for every $\boldsymbol{u} \in W^1 L^\Phi$*

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi} w(|t-s|), \tag{15}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|\boldsymbol{u}\|_{W^1 L^\Phi} \tag{16}$$

2. *For every $\boldsymbol{u} \in W^1 L^\Phi$ we have $\widetilde{\boldsymbol{u}} \in L_d^\infty$ and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Sobolev's inequality).} \tag{17}$$

The following result is analogous to some lemmata in $W^1L^p_d$, see [**?**].

Lemma 2.2. *If* $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$*, then* $(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \to \infty$.

Proof. By the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$ and some elementary operations, we get

$$\|\boldsymbol{u}\|_{L^\Phi} = \|\overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant \|\overline{\boldsymbol{u}}\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi} = |\overline{\boldsymbol{u}}|\|1\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi}. \tag{18}$$

It is known that $L^\infty_d \hookrightarrow L^\Phi_d$, i.e., there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{\boldsymbol{u}} \in L^\infty_d$ we have

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C_1\|\tilde{\boldsymbol{u}}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain the Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ suvh that

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C_2\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{19}$$

Therefore, from (**??**), (**??**) and (**??**), we get

$$\|\boldsymbol{u}\|_{W^1L^\Phi} \leqslant C_3(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \to \infty$. □

We present a definition that will be useful later.

Definition 2.3. *A function* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a* Carathéodory *function if for fixed* $(\boldsymbol{x},\boldsymbol{y})$ *the map* $t \mapsto \mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ *is measurable and for fixed* t *the map* $(\boldsymbol{x},\boldsymbol{y}) \mapsto \mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ *is continuous for almost everywhere* $t \in [0,T]$. *We say that* $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ *is* differentiable Carathéodory *if in addition* $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ *is continuously differentiable with respect to* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *for almost everywhere* $t \in [0,T]$.

In [**?**] we proved the next results.

Theorem 2.4. *Let* $\mathcal{L}$ *be a differentiable Carathéodory function satisfying* (**??**)*,* (**??**) *and* (**??**)*. Then the following statements hold:*

1. *The action integral given by* (**??**) *is finitely defined on* $\mathcal{E}^\Phi_d(\lambda) := W^1L^\Phi_d \cap \{\boldsymbol{u} | \dot{\boldsymbol{u}} \in \Pi(E^\Phi_d, \lambda)\}$.
2. *The function* I *is Gâteaux differentiable on* $\mathcal{E}^\Phi_d(\lambda)$ *and its derivative* I' *is demicontinuous from* $\mathcal{E}^\Phi_d(\lambda)$ *into* $\left[W^1L^\Phi_d\right]^*$*. Moreover,* I' *is given by the following expression*

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\big(t,\boldsymbol{u},\dot{\boldsymbol{u}}\big)\cdot\boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t,\boldsymbol{u},\dot{\boldsymbol{u}}\big)\cdot\dot{\boldsymbol{v}}\right\}\, dt. \tag{20}$$

3. *If* $\Psi \in \Delta_2$ *then* I' *is continuous from* $\mathcal{E}^\Phi_d(\lambda)$ *into* $\left[W^1L^\Phi_d\right]^*$ *when both spaces are equipped with the strong topology.*

In [?] we derive the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T-periodic functions. We denote by $W^1L^\Phi_T$ the subspace of $W^1L^\Phi_d$ containing all T-periodic functions. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f:\mathbb{R}^d\to\mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) < \frac{1}{2}\left(f\left(\boldsymbol{x}\right)+f\left(\boldsymbol{y}\right)\right)$ for $\boldsymbol{x}\neq\boldsymbol{y}$. It is well known that if f is a strictly convex and differentiable function, then $D_{\boldsymbol{x}}f:\mathbb{R}^d\to\mathbb{R}^d$ is a one-to-one map (see, e.g. [?, Thm. 12.17]).

Theorem 2.5. *Let $\boldsymbol{u}\in\mathcal{E}^\Phi_d(\lambda)$ be a T-periodic function. The following statements are equivalent:*

1. $I'(\boldsymbol{u})\in\left(W^1L^\Phi_T\right)^\perp$.
2. $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and* $\boldsymbol{u}$ *solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t\in(0,T)\\ \boldsymbol{u}(0)-\boldsymbol{u}(T) = D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T)) = 0. \end{cases} \tag{21}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t,x,y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T)) = 0$ is equivalent to $\dot{\boldsymbol{u}}(0)=\dot{\boldsymbol{u}}(T)$.

Habría que ver si el lugar de los índices es el adecuado. Copié lo que teníamos en el primer trabajo.

Next, we enumerate some definitions and results from the theory of convex functions. We suggest [?, ?, ?, ?, ?] for definitions, proofs and additional details.

We denote by α_φ and β_φ the so called *Matuszewska-Orlicz indices* of the function φ, which are defined next. Given an increasing, unbounded, continuous function $\varphi:[0,+\infty)\to[0,+\infty)$ such that $\varphi(0)=0$ we define

$$\alpha_\varphi := \lim_{t\to0^+}\frac{\log\left(\sup\limits_{u>0}\frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)},\quad \beta_\varphi := \lim_{t\to+\infty}\frac{\log\left(\sup\limits_{u>0}\frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)}. \tag{22}$$

We have that $0\leqslant\alpha_\varphi\leqslant\beta_\varphi\leqslant+\infty$. The relation $\beta_\varphi<\infty$ holds true if and only if φ is a Δ_2-function. If φ is a homeomorphism we have that

$$\alpha_{\varphi^{-1}} = \frac{1}{\beta_\varphi}. \tag{23}$$

Moreover $\varphi\in\mathcal{F}$ implies $\alpha_\varphi\geq1$. As a consequence, φ^{-1} is a Δ_2-function.

It is well known that if φ is an increasing Δ_2-function, φ is controlled by above and below by power functions. More concretely, for every $\epsilon>0$ there exists a constant $K=K(\varphi,\epsilon)$ such that, for every $t,u\geq0$,

$$K^{-1}\min\left\{t^{\beta_\varphi+\epsilon},t^{\alpha_\varphi-\epsilon}\right\}\varphi(u)\leqslant\varphi(tu)\leqslant K\max\left\{t^{\beta_\varphi+\epsilon},t^{\alpha_\varphi-\epsilon}\right\}\varphi(u). \tag{24}$$

We define the following functionals $J_{C,\mu} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\sigma} : \mathbb{R}^n \to \mathbb{R}$, with $C, \nu, \sigma > 0$, by

$$J_{C,\nu}(\boldsymbol{u}) := \rho_\Phi(\boldsymbol{u}) - C\|\boldsymbol{u}\|_{L^\Phi}^\nu, \tag{25}$$

and

$$H_{C,\sigma}(\boldsymbol{x}) = \int_0^T F(t,\boldsymbol{x})dt - C|\boldsymbol{x}|^\sigma, \tag{26}$$

respectively.

We will use the classic notations big O and little o [**?**]. The following lemma, was essentially proved in [**?**].

Lemma 2.6. *Let Φ and Ψ be complementary N-functions and let $\boldsymbol{u} \in L^\Phi$. Then:*

1. $\|\boldsymbol{u}\|_{L^\Phi} = O(\rho_\Phi(\boldsymbol{u}))$.
2. *If $\Psi \in \Delta_2$ globally, then there exists a constant $\alpha_\Phi > 1$ such that, for any $0 < \mu < \alpha_\Phi$,*
$$\|\boldsymbol{u}\|_{L^\Phi}^\mu = o(\rho_\Phi(\boldsymbol{u})) \tag{27}$$
as $\|\boldsymbol{u}\|_{L^\Phi} \to \infty$.
3. *If (**??**) holds for $\mu \geq 1$ then $\Psi \in \Delta_2$.*

Proof. **??**. In virtue of [**?**, Lemma 5.2(1)] we have that $\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to+\infty} J_{C,1}(\boldsymbol{u}) = +\infty$ for $C < 1$. Then for sufficiently large $\|\boldsymbol{u}\|_{L^\Phi}$ we have that $J_{C,1}(\boldsymbol{u}) > 0$. This implies $\rho_\Phi(\boldsymbol{u}) \leqslant C^{-1}\|\boldsymbol{u}\|_{L^\Phi}$.

??. Taking account of [**?**, Lemma 5.2(2)] we have that $\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to+\infty} J_{C,1}(\boldsymbol{u}) = +\infty$ for every $C > 0$. Therefore for any $n \in \mathbb{N}$ there exists $k_n > 0$ such that for $\|\boldsymbol{u}\|_{L^\Phi} > k_n$ we have $J_{n,\mu}(\boldsymbol{u}) > 1$. Then $\rho_\Phi(\boldsymbol{u})/\|\boldsymbol{u}\|_{L^\Phi}^\mu > n$.

??. The statement **??** represents a parcial reciprocal of **??**. The difference rests on the fact that the Δ_2 condition for Ψ in item **??** is global, while in item **??** is for large values. We can suppose $\mu = 1$. By (**??**) we obtain for any $C > 1$ that

$$\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to+\infty} J_{C,1}(\boldsymbol{u}) = \lim_{\|\boldsymbol{u}\|_{L^\Phi}\to+\infty} \|\boldsymbol{u}\|_{L^\Phi}\left(\frac{\rho_\Phi(\boldsymbol{u})}{\|\boldsymbol{u}\|_{L^\Phi}} - C\right) = +\infty.$$

Therefore, using [**?**, Lemma 5.2(3)], we conclude $\Psi \in \Delta_2$.

□

3 Lagrangians with sublinear "nonlinearity"????

Like in [**?**] we consider Lagrangians $\mathcal{L}$ which are lower bounded as follows

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geq \alpha_0\Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t,\boldsymbol{x}). \tag{28}$$

Based on [**?**] we say that F satisfies the condition (A) if $F(t,\boldsymbol{x})$ is a Carathéodory function and F is continuously differentiable with respect to $\boldsymbol{x}$. Moreover, the next inequality holds

$$|F(t,\boldsymbol{x})|+|D_{\boldsymbol{x}}F(t,\boldsymbol{x})|\leqslant a(|\boldsymbol{x}|)b_0(t),\quad \text{for a.e. } t\in[0,T],\forall \boldsymbol{x}\in\mathbb{R}^d. \tag{29}$$

Now, we have another result about coercivity of I assuming some conditions on the nonlinearity ∇F.

Theorem 3.1. *Let $\mathcal{L}$ be a lagrangian function satisfying* (**??**)*,* (**??**)*,* (**??**)*,* (**??**) *and F satisfies condition (A). We assume the following conditions:*

1. $\Psi\in\Delta_2$.
2. *There exist non negative functions $b_1,b_2\in L^1_1$ and a constant $1<\mu<\alpha_\Phi$ such that for any $\boldsymbol{x}\in\mathbb{R}^d$ and a.e. $t\in[0,T]$*
$$|\nabla F(t,\boldsymbol{x})|\leqslant b_1(t)|\boldsymbol{x}|^{\mu-1}+b_2(t). \tag{30}$$
3. *There exists a real positive number σ such that $\sigma>(\mu-1)\beta_\Psi$ and*
$$|\boldsymbol{x}|^\sigma=o\left(\int_0^T F(t,\boldsymbol{x})\,dt\right)\quad as\quad |\boldsymbol{x}|\to\infty. \tag{31}$$

Then the action integral I is coercive.

Proof. By the decomposition $u=\overline{u}+\tilde{\boldsymbol{u}}$, Mean Value Theorem, Cauchy-Schwarz's inequality and (**??**), we have

$$\begin{aligned}
&\left|\int_0^T F(t,\boldsymbol{u})-F(t,\overline{\boldsymbol{u}})\,dt\right|=\left|\int_0^T\int_0^1\nabla F(t,\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t))\cdot\tilde{\boldsymbol{u}}(t)\,ds\,dt\right|\\
&\leqslant\int_0^T\int_0^1 b_1(t)|\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t)|^{\mu-1}|\tilde{\boldsymbol{u}}(t)|\,ds\,dt+\int_0^T\int_0^1 b_2(t)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt\\
&=I_1+I_2.
\end{aligned}\tag{32}$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate I_2 as follows

$$I_2\leqslant\|b_2\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty}\leqslant C_1\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{33}$$

where $C_1=C_1(\|b_2\|_{L^1},T)$. On the other hand, as $s\in[0,1]$, we have

$$|\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t)|^{\mu-1}\leqslant C(\mu)(|\overline{\boldsymbol{u}}|^{\mu-1}+\|\tilde{\boldsymbol{u}}\|^{\mu-1}_{L^\infty}). \tag{34}$$

where $C(\mu)=2^{\mu-2}$, for $\mu\geq 2$ and $C(\mu)=1$, for $1<\mu<2$. Now, inequality (**??**), Hölder's inequality and Sobolev's inequality imply that

$$\begin{aligned}
I_1&\leqslant C(\mu)\left(|\overline{\boldsymbol{u}}|^{\mu-1}\int_0^T b_1(t)|\tilde{\boldsymbol{u}}(t)|\,dt+\|\tilde{\boldsymbol{u}}\|^{\mu-1}_{L^\infty}\int_0^T b_1(t)|\tilde{\boldsymbol{u}}(t)|\,dt\right)\\
&\leqslant C(\mu)\left\{|\overline{\boldsymbol{u}}|^{\mu-1}\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty}+\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|^{\mu}_{L^\infty}\right\}\\
&\leqslant C_2\left\{|\overline{\boldsymbol{u}}|^{\mu-1}\|\dot{\boldsymbol{u}}\|_{L^\Phi}+\|\dot{\boldsymbol{u}}\|^{\mu}_{L^\Phi}\right\},
\end{aligned}\tag{35}$$

where $C_2 = C_2(\mu, T, \|b_1\|_{L^1})$. Let μ' be a positive constant such that $1 < \mu \leqslant \mu' < \alpha_\Phi$. Next, using Young's inequality with conjugate exponents μ' and $\frac{\mu'}{\mu'-1}$ we get

$$|\overline{\boldsymbol{u}}|^{\mu-1}\|\dot{\boldsymbol{u}}\|_{L^\Phi} \leqslant \frac{(\mu'-1)}{\mu'}|\overline{\boldsymbol{u}}|^{\boldsymbol{\sigma}} + \frac{1}{\mu'}\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} \tag{36}$$

where $\sigma = \frac{(\mu-1)\mu'}{\mu'-1}$. We point out that σ is an arbitrary positive constant bigger than $(\mu-1)b_\Psi$.

From (**??**), (**??**), (**??**) and the inequality $x^{r_1} \leqslant x^{r_2} + 1$, for any $x \geq 0$ and $r_1 \leqslant r_2$, we have

$$\begin{aligned} I_1 + I_2 &\leqslant C_3\Big\{|\overline{\boldsymbol{u}}|^\sigma + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}\Big\} \\ &\leqslant C_3\Big\{|\overline{\boldsymbol{u}}|^\sigma + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + 1\Big\} \end{aligned} \tag{37}$$

with $C_3 = C_3(\mu, T, \|b_1\|_{L^1}, \mu')$. In the subsequent estimates, we use the decomposition $u = \overline{u} + \tilde{\boldsymbol{u}}$, (**??**), (**??**), (**??**) and we get

$$\begin{aligned} I(\boldsymbol{u}) &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t,\boldsymbol{u})\,dt \\ &= \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T \left[F(t,\boldsymbol{u}) - F(t,\overline{\boldsymbol{u}})\right]\,dt + \int_0^T F(t,\overline{\boldsymbol{u}})\,dt \\ &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C_3\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + \int_0^T F(t,\overline{\boldsymbol{u}})\,dt - C_3|\overline{\boldsymbol{u}}|^\sigma - C_3 \\ &= \alpha_0 J_{C_4,\mu'}\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + H_{C_3,\sigma}(\overline{\boldsymbol{u}}) - C_3, \end{aligned} \tag{38}$$

where $C_4 = \Lambda^{\mu'}C_3/\alpha_0$.

Let $\boldsymbol{u}_n$ be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(\boldsymbol{u}_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by $\boldsymbol{u}_n$, $I(\boldsymbol{u}_n)$ is upper bounded, i.e., there exists $M > 0$ such that $|I(\boldsymbol{u}_n)| \leqslant M$. As $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$, from Lemma **??**, we have $|\overline{\boldsymbol{u}}_n| + \|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Then, there exists a subsequence of $\{\boldsymbol{u}_n\}$, still denoted by $\boldsymbol{u}_n$, which is not bounded. Then, $|\overline{\boldsymbol{u}}_n| \to \infty$ or $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Now, Lemma **??** implies that the functional $J_{C_4,\mu'}(\frac{\dot{\boldsymbol{u}}}{\Lambda})$ is coercive, and, by (**??**), the functional $H(\overline{\boldsymbol{u}})$ is also coercive, then $J_{C_4,\mu'}(\frac{\dot{\boldsymbol{u}}_n}{\Lambda}) \to \infty$ or $H(\overline{\boldsymbol{u}}_n) \to \infty$. From (**??**), we have that on a bounded set the functional $H(\overline{\boldsymbol{u}}_n)$ is lower bounded; and, $J_{C_4,\mu'}(\frac{\dot{\boldsymbol{u}}_n}{\Lambda})$ is also lower bounded because the modular $\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)$ is always bigger than zero. Therefore, $I(\boldsymbol{u}_n) \to \infty$ as $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(\boldsymbol{u}_n)$. □

Leer y ver si es coherente lo anterior, si conviene trabajar siempre con u_n o habría que usar la notación de subsucesiones explícita!!!

Falta leer y corregir la sección que sigue del caso límite!!!

4 Limit case $\mu = \alpha_\Phi$

Coercivity was obtained even in the limit case $\mu = 1$ and $\mu = p$ (see [**?**, **?**]) assuming additional conditions on one of the L^1 functions that bound the nonlinearity. These results lean on the fact that in L^p spaces the norm and the modular coincides, that is, $\|\cdot\|_p^p = O(\int_0^T |\cdot|^p\,dt)$. In Orlicz spaces, $\|\cdot\|_{L^\Phi}^\mu$ can be upper controlled by a modular provided that $\mu < \alpha_\Phi$ for any N-function Φ. Nevertheless, the limit case does not hold for any Φ, i.e. in general $\|\cdot\|_{L^\Phi}^{\alpha_\Phi} = O(\int_0^T \Phi(|u|)\,dt)$ is false as can be seen as follows.

Let Φ be an N-function that satisfies the Δ_2-condition. We claim that the inequality $\Phi(tu) \geq t^{\alpha_\Phi}\Phi(u)$ for any $u > 0$ and for any $t \geq 1$ is false.

With the aim of proving the above assertion, we define the function

$$\Phi(u) = \begin{cases} \frac{p-1}{p}u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$.

Me parece que habría que decir algo que aclare los que se hace a continuación. Por ejemplo:

Next, we will study the behaviour of Φ/ establish some properties of Φ/...

Theorem 4.1. *If $p \geq \frac{1+\sqrt{2}}{2}$, then Φ is an N-function.*

Proof. We have

$$\Phi'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u < e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) & := & \varphi_2(u) & \text{if } u > e \end{cases}$$

and Φ is differentiable at e because $\varphi_1(e) = \varphi_2(e) = (p-1)e^{p-1}$.

We will see that Φ' is increasing when $p \geq \frac{1+\sqrt{2}}{2}$. Then, it is enough to see that φ_1 is increasing on $[0, e]$ and φ_2 is increasing on $[e, \infty)$ for $p \geq \frac{1+\sqrt{2}}{2}$.

Tendríamos que ver que φ_1, φ_2 son crecientes y que $\varphi_2 \to \infty$ cuando $u \to \infty$???? o basta con ver que φ_2 es creciente????

φ_1 is an increasing function for $p > 1$ and $\varphi_1(u) \to 0$ as $u \to 0$.

On the other hand, $\varphi_2(u) \to \infty$ as $u \to \infty$ provided that $p > 1$. And, $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if

$$\left(p^2 - p - \frac{2p}{\log u} + \frac{1}{\log u} + \frac{2}{\log^2 u}\right) > 0. \tag{39}$$

Now, as (**??**) holds if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$, then φ_2 is an icreasing function when $p \geq \frac{1+\sqrt{2}}{2}$. $\square$

Conviene agregar más cuentas en la prueba anterior, es decir, que la fórmula (??) es una cuadrática y bla....???

Theorem 4.2. *There exists a constant $C > 0$ such that*

$$\Phi(tu) \leqslant ct^p\Phi(u) \;\; t \geq 1, u > 0. \tag{40}$$

For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, p)$ such that

$$\Phi(tu) \geq Ct^{p-\varepsilon}\Phi(u) \;\; t \geq 1, u > 0. \tag{41}$$

Proof. In order to prove (**??**), we analyze three cases.

If $u \leqslant tu \leqslant e$, then $\Phi(tu) = t^p\Phi(u)$ and (**??**) holds with $C = 1$.

If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geq 1$, we have $\Phi(tu) \leqslant t^pu^p = \frac{p}{p-1}t^p\Phi(u)$. Thus, (**??**) holds with $C = \frac{p}{p-1}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) \leqslant \frac{t^pu^p}{\log(tu)} \leqslant \frac{t^pu^p}{\log(u)} = \frac{pt^pv}{\log v} \tag{42}$$

where $v := u^p$ and $v \geq e^p$.

If $\alpha > 0$, the function $f(x) = \frac{x}{x-\alpha}$ is decreasing on (α, ∞). And, the function $g(v) = \frac{pv}{\log v}$ is decreasing on $[e^p, \infty)$. Therefore, $f \circ g$ is decreasing on $[e^p, \infty)$ and we have

$$(f \circ g)(v) = \frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant e^p - \frac{e^p}{p} = \frac{p}{p-1}$$

for every $v \geq e^p$.

In this way, from (**??**), we have

$$\Phi(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and (**??**) holds with $C = \frac{p}{p-1}$.

Now, we will prove (**??**).

If $u \leqslant tu \leqslant e$, (**??**) is immediate because $t^p \geq t^{p-\varepsilon}$ for every $t \geq 1$, $p > 1$ and ε **sufficiently small????**

If $u \leqslant e \leqslant tu$, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$ then $f(t) \geq e$ for every $t \geq e$. **Habría que mirar en $[e^p, \infty)$ para que $f(t) \geq \frac{e^p}{p}$???? O, como f es creciente y $e^p \leqslant (tu)^p$ entonces $f((tu)^p) \geq f(e^p)$???**

Now,

$$\Phi(tu) = \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} = \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \geq$$
$$\frac{p-1}{p}\frac{(tu)^p}{\log(tu)} \geq \frac{p-1}{p}\frac{t^\varepsilon}{\log t + 1}t^{p-\varepsilon}u^p.$$

Since $f(t) = \frac{t^\varepsilon}{\log t+1}$ attains its minimum value $\varepsilon e^{1-\varepsilon}$ at $e^{\frac{1-\varepsilon}{\varepsilon}}$, then

$$\Phi(tu) \geq \frac{p-1}{p}\varepsilon e^{1-\varepsilon}t^{p-\varepsilon}u^p.$$

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) = \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} + \frac{1}{p}\frac{(tu)^p}{\log(tu)} - \frac{e^p}{p} \geq \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log(u)^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s+1}$ with $s \geq 1$ and $A \geq \varepsilon$. Then, the function f attains a minimum at $s = e^{1-A}$; but, s has to be bigger than 1, then it is necessary that $\varepsilon \leqslant A \leqslant 1$. And, the minimum value is $f(e^{1-A}) = Ae^{1-A} \geq \varepsilon$. If $A \geq 1$, f attains the minimum at $s = 1$??? and $f(1) = 1$. Then, $f \geq \varepsilon$ and therefore

$$\Phi(tu) \geq \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geq \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi(u).$$

□

La prueba anterior está rara.

Remark 4.3. *The inequality*

$$\Phi(tu) \geq Ct^p\Phi(u)$$

is false for every C because for every u $\geq e$ *we have*

$$\lim_{t\to\infty}\frac{\Phi(tu)}{t^p\Phi(u)} = \lim_{t\to\infty}\frac{\frac{u^p}{\log(tu)} - \frac{e^p}{pt^p}}{\frac{u^p}{\log u} - \frac{e^p}{p}} = 0$$

El límite intermedio de la fórmula anterior se podría quitar.

Theorem 4.4. $\alpha_\Phi = \beta_\Phi = p$

Proof. From (**??**) and (**??**), we get

$$\beta_\Phi = \lim_{t\to\infty}\frac{\log\left[\sup\limits_{u>0}\frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} \leqslant \lim_{t\to\infty}\frac{\log C + p\log t}{\log t} = p.$$

On the other hand, employing (**??**) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t\to 0^+}\frac{\log\left[\sup\limits_{u>0}\frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} = \lim_{s\to\infty}\frac{\log\left[\sup\limits_{v>0}\frac{\Phi(v)}{\Phi(sv)}\right]^{-1}}{\log s} = \lim_{s\to\infty}\frac{\log\left[\inf\limits_{v>0}\frac{\Phi(sv)}{\Phi(v)}\right]}{\log s}$$

where $v := tu$ and $s := \frac{1}{t}$. Then, using (**??**), for every $\varepsilon > 0$ we have

$$\alpha_\Phi = \lim_{s\to\infty}\frac{\log\left[\inf\limits_{v>0}\frac{\Phi(sv)}{\Phi(v)}\right]}{\log s} \geq \lim_{s\to\infty}\frac{\log C + (p-\varepsilon)\log s}{\log s} \geq p - \varepsilon,$$

therefore $\alpha_\Phi \geq p$.

Finally, as $\alpha_\Phi \leqslant \beta_\Phi \leqslant p$, we get $\alpha_\Phi = \beta_\Phi = p$. □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\,dx \geq C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^p$$

is false.

In fact, if we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\,dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geq C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geq Ct^p$ would also be true; however, this last inequality is false.

5 Bounding by power-behaviour functions

Theorem 5.1. *Let $\mathcal{L}$ be a lagrangian function satisfying* (**??**)*,* (**??**)*,* (**??**)*,* (**??**) *and F satisfies condition (A). We assume the following conditions:*

1. $\Psi \in \Delta_2$.
2. *There exist non negative functions $b_1, b_2 \in L_1^1$ and $f(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})|\boldsymbol{x}|^{\alpha_\Phi - 1}$ which is non-decreasing, sub additive and such that for any $\boldsymbol{x} \in \mathbb{R}^d$ and a.e. $t \in [0, T]$*
$$|\nabla F(t, \boldsymbol{x})| \leqslant b_1(t) f(|\boldsymbol{x}|) + b_2(t). \tag{43}$$
3. *There exists a real positive number σ such that $\sigma > (\alpha_\Phi - 1)\beta_\Psi$ and*
$$|\boldsymbol{x}|^\sigma = o\left(\int_0^T F(t, \boldsymbol{x})\,dt\right) \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{44}$$

En las cuentas, aparece la función ε!!! ¿sirve aún la fórmula anterior???

Then the action integral I is coercive.

Proof. By the decomposition $u = \overline{u} + \tilde{\boldsymbol{u}}$, Mean Value Theorem, Cauchy-Schwarz's inequality and (**??**), we have

$$\begin{aligned}
&\left|\int_0^T F(t, \boldsymbol{u}) - F(t, \overline{\boldsymbol{u}})\,dt\right| = \left|\int_0^T \int_0^1 \nabla F(t, \overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)) \cdot \tilde{\boldsymbol{u}}(t)\,ds\,dt\right| \\
&\leqslant \int_0^T \int_0^1 b_1(t) f(|\overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)|)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt + \int_0^T \int_0^1 b_2(t)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt \\
&= I_1 + I_2.
\end{aligned} \tag{45}$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} \leqslant C_1\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{46}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$. On the other hand, as $s \in [0, 1]$, we have

$$f(|\overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)|) \leqslant C(f)(f(|\overline{\boldsymbol{u}}|) + f(\|\tilde{\boldsymbol{u}}\|_{L^\infty})). \tag{47}$$

where $C(f)$ springs from the subadditivity of f. Now, inequality (**??**), Hölder's inequality, Sobolev's inequality and **the properties of** f**(no decrecimiento y subaditividad)** imply that

$$\begin{aligned} I_1 &\leqslant C(f)\left\{ f(|\overline{\boldsymbol{u}}|)\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} + \|b_1\|_{L^1} f(\|\tilde{\boldsymbol{u}}\|_{L^\infty})\|\tilde{\boldsymbol{u}}\|_{L^\infty} \right\} \\ &\leqslant C_2\left\{ f(|\overline{\boldsymbol{u}}|)\|\dot{\boldsymbol{u}}\|_{L^\Phi} + f(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi} \right\} \\ &= C_2\left\{ f(|\overline{\boldsymbol{u}}|)\|\dot{\boldsymbol{u}}\|_{L^\Phi} + \varepsilon(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\alpha_\Phi} \right\} \end{aligned} \tag{48}$$

where $C_2 = C_2(f, T, \|b_1\|_{L^1})$.

Let μ be a positive constant such that $1 < \mu < \alpha_\Phi$. Next, using Young's inequality with conjugate exponents μ and $\frac{\mu}{\mu-1}$ we get

$$\begin{aligned} f(|\overline{\boldsymbol{u}}|)\|\dot{\boldsymbol{u}}\|_{L^\Phi} &= [\varepsilon(|\overline{\boldsymbol{u}}|)|\overline{\boldsymbol{u}}|^{\alpha_\Phi - 1}]\|\dot{\boldsymbol{u}}\|_{L^\Phi} \\ &\leqslant \frac{(\mu-1)}{\mu}[\varepsilon(|\overline{\boldsymbol{u}}|)]^{\frac{\mu}{\mu-1}}|\overline{\boldsymbol{u}}|^{(\alpha_\Phi-1)\frac{\mu}{\mu-1}} + \frac{1}{\mu}\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} \\ &= \frac{1}{\gamma}[\varepsilon(|\overline{\boldsymbol{u}}|)]^{\gamma}|\overline{\boldsymbol{u}}|^{(\alpha_\Phi-1)\gamma} + \frac{1}{\mu}\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} \\ &= \frac{1}{\gamma}[\varepsilon(|\overline{\boldsymbol{u}}|)]^{\gamma}|\overline{\boldsymbol{u}}|^{\sigma} + \frac{1}{\mu}\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} \end{aligned} \tag{49}$$

where $\gamma = \frac{\mu}{\mu-1}$ and $\sigma = (\alpha_\Phi - 1)\gamma$. We point out that $\sigma = (\alpha_\Phi - 1)\gamma$ is an arbitrary positive constant bigger than $(\alpha_\Phi - 1)b_\Psi$.

($\mu < \alpha_\Phi$ then $\sigma = \frac{\mu}{\mu-1} > b_\Psi$, because $\frac{1}{\mu} + \frac{1}{b_\Psi} > \frac{1}{\alpha_\Phi} + \frac{1}{b_\Psi} = 1$, then $\frac{1}{b_\Psi} > \frac{\mu-1}{\mu} = \frac{1}{\gamma}$)

From (**??**), (**??**) and (**??**), we have

$$I_1 + I_2 \leqslant C_3\left\{ [\varepsilon(|\overline{\boldsymbol{u}}|)]^{\gamma}|\overline{\boldsymbol{u}}|^{\sigma} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} + \varepsilon(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\alpha_\Phi} \right\} \tag{50}$$

with $C_3 = C_3(\mu, T, \|b_1\|_{L^1})$, $\mu < \alpha_\Phi$ and $\sigma > b_\Psi(\alpha_\Phi - 1)$.

In the subsequent estimates, we use the decomposition $u = \overline{u} + \tilde{\boldsymbol{u}}$, (**??**), (**??**), (**??**) and we get

$$\begin{aligned} I(\boldsymbol{u}) &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t, \boldsymbol{u})\,dt \\ &= \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T [F(t, \boldsymbol{u}) - F(t, \overline{\boldsymbol{u}})]\,dt + \int_0^T F(t, \overline{\boldsymbol{u}})\,dt \\ &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C_3\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} + \int_0^T F(t, \overline{\boldsymbol{u}})\,dt - C_3[\varepsilon(|\overline{\boldsymbol{u}}|)]^{\gamma}|\overline{\boldsymbol{u}}|^{\sigma} - C_3\varepsilon(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\alpha_\Phi} \\ &= \alpha_0 J_{C_4,\mu}\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t, \overline{\boldsymbol{u}})\,dt - C_3[\varepsilon(|\overline{\boldsymbol{u}}|)]^{\gamma}|\overline{\boldsymbol{u}}|^{\sigma} - C_3\varepsilon(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\alpha_\Phi}, \end{aligned} \tag{51}$$

where $C_4 = \Lambda^{\mu'} C_3/\alpha_0$.

De acá en adelante, es copia del final de la otra demo. Debe adaptarse a la fórmula anterior, si es que ella logra sobrevivir....

Let $\boldsymbol{u}_n$ be a sequence in $\mathcal{E}_d^{\Phi}(\lambda)$ with $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(\boldsymbol{u}_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by $\boldsymbol{u}_n$, $I(\boldsymbol{u}_n)$ is upper bounded, i.e., there exists $M > 0$ such that $|I(\boldsymbol{u}_n)| \leqslant M$. As $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$, from Lemma **??**, we have $|\overline{\boldsymbol{u}}_n| + \|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Then, there exists a subsequence of $\{\boldsymbol{u}_n\}$, still denoted by $\boldsymbol{u}_n$, which is not bounded. Then, $|\overline{\boldsymbol{u}}_n| \to \infty$ or $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Now, Lemma **??** implies that the functional $J_{C_4,\mu'}(\frac{\dot{\boldsymbol{u}}}{\Lambda})$ is coercive, and, by (**??**), the functional $H(\overline{\boldsymbol{u}})$ is also coercive, then $J_{C_4,\mu'}(\frac{\dot{\boldsymbol{u}}_n}{\Lambda}) \to \infty$ or $H(\overline{\boldsymbol{u}}_n) \to \infty$. From (**??**), we have that on a bounded set the functional $H(\overline{\boldsymbol{u}}_n)$ is lower bounded; and, $J_{C_4,\mu'}(\frac{\dot{\boldsymbol{u}}_n}{\Lambda})$ is also lower bounded because the modular $\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)$ is always bigger than zero. Therefore, $I(\boldsymbol{u}_n) \to \infty$ as $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(\boldsymbol{u}_n)$. $\square$

probando

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.