Periodic solutions of Euler-Lagrange equations with "sublinear nonlinearity" in an Orlicz-Sobolev space setting

Sonia Acinas *
Instituto de Matemática Aplicada San Luis (IMASL)
Universidad Nacional de San Luis and CONICET
Ejército de los Andes 950, (D5700HDW) San Luis, Argentina
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$, of hamiltonian systems with a potential function F satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\varphi_0(|x|) + b_2(t)$, with $b_1(t), b_2(t) \in L^1$ and for certain functions φ_0.

1 Introduction

The purpose of this paper is to study the existence of periodic solution for the following non-autonomous second-order systems:

$$\begin{cases} \frac{d}{dt}\left(u'(t)\frac{\varphi(|u'|)}{|u'|}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0)-u(T) = u'(0)-u'(T) = 0 \end{cases} \tag{1}$$

*SECyT-UNRC, UNSL and CONICET
†SECyT-UNRC and CONICET

2010 AMS Subject Classification. Primary: . Secondary: .
Keywords and phrases. .

where $T > 0$, $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions

(C) $\mathcal{L}$ and its derivatives $D_x\mathcal{L}$ and $D_y\mathcal{L}$ are Carathéodory functions, i.e. they are measurable functions with respecto to $t \in [0,T]$, for every $x, y \in \mathbb{R}^d$, and continuous functions with respect to $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$ we have that

$$|\mathcal{L}(t,x,y)| + |D_x\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right)\right), \tag{A1}$$

$$|D_y\mathcal{L}(t,x,y)| \leqslant a(|x|)\left(c(t) + \varphi\left(\frac{|y|}{\lambda} + f(t)\right)\right). \tag{A2}$$

Where, in these inequalities we assume that $a : [0,+\infty) \to [0,+\infty)$ is nondecreasing, $\lambda > 0$, Φ is an N-function (see preliminaries section for definitions), φ is the right continuous derivative of Φ. The non negative functions b, c and f satisfy that $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$, where the Banach spaces $L^1_1([0,T])$, $L^\Psi_1([0,T])$ and $E^\Phi_1([0,T])$ will be defined in the preliminaries section.

(LB) We assume that there exist a Caratheodory function $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$

$$\mathcal{L}(t,x,y) \geqslant \alpha_0 \Phi\left(\frac{|y|}{\Lambda}\right) + F(t,x). \tag{2}$$

If $\mathcal{L}$ is given by the right hand side in (27) and $\Phi(u) = |u|^2$, then the ODE $\ddot{u} = \nabla F(t,u(t))$ in (1) is quasilinear, being $\nabla F(t,u(t))$ the nonlinearity. Following the literature, we refer to ∇F as the non linearity even when we assume in (27) just the inequality. In [1] and [2] the authors considered, for the p-laplacian case, non linearities satisfying the inequality

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(u) = \int_0^T \mathcal{L}(t,u(t),u'(t))\,dt. \tag{3}$$

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [3, 4, 5]. For Orlicz spaces of vector valued functions, see [6] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \quad \text{for } t \geqslant 0,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the Δ_2-*condition*, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geqslant 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{4}$$

for every $t \geqslant t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2-*condition globally* ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0, T])$ the set of all measurable functions defined on $[0, T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C^\Phi_d = C^\Phi_d([0, T])$ is given by

$$C^\Phi_d := \{u \in \mathcal{M}_d | \rho_\Phi(u) < \infty\}. \tag{5}$$

The *Orlicz space* $L^\Phi_d = L^\Phi_d([0, T])$ is the linear hull of C^Φ_d; equivalently,

$$L^\Phi_d := \{u \in \mathcal{M}_d | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{6}$$

The Orlicz space L^Φ_d equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup\left\{\int_0^T u \cdot v\, dt \big| \rho_\Psi(v) \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [4, Thm. 10.5] and [7]). For every $u \in L^\Phi$,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(ku)\}. \tag{7}$$

The subspace $E^\Phi_d = E^\Phi_d([0, T])$ is defined as the closure in L^Φ_d of the subspace L^∞_d of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ_d is the only one maximal subspace contained in the Orlicz class C^Φ_d, i.e. $u \in E^\Phi_d$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [4, Th. 9.3]). Namely, if $u \in L^\Phi_d$ and $v \in L^\Psi_d$ then $u \cdot v \in L^1_1$ and

$$\int_0^T v \cdot u\, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \tag{8}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi_d \subset \left[L^\Phi_d\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt \tag{9}$$

with $u \in L^\Phi_d$ and $v \in L^\Psi_d$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi_d = \left[L^\Phi_d\right]^*$ will not hold. In general, it is true that $\left[E^\Phi_d\right]^* = L^\Psi_d$.

Like in [4], we will consider the subset $\Pi(E^\Phi_d, r)$ of L^Φ_d given by

$$\Pi(E^\Phi_d, r) := \{u \in L^\Phi_d | d(u, E^\Phi_d) < r\}.$$

This set is related to the Orlicz class C^Φ_d by means of inclusions, namely,

$$\Pi(E^\Phi_d, r) \subset rC^\Phi_d \subset \overline{\Pi(E^\Phi_d, r)} \tag{10}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L^Φ_d, E^Φ_d, $\Pi(E^\Phi_d, r)$ and C^Φ_d are equal.

We define the *Sobolev-Orlicz space* $W^1L^\Phi_d$ (see [3]) by

$$W^1L^\Phi_d := \{u | u \text{ is absolutely continuous and } u' \in L^\Phi_d\}.$$

$W^1L^\Phi_d$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{11}$$

For a function $u \in L^1_d([0, T])$, we write $u = \overline{u} + \widetilde{u}$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi_d \hookrightarrow \left[L^\Phi_d\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty_d \hookrightarrow L^\Phi_d \hookrightarrow L^1_d$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $u : [0, T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t - s|). \tag{12}$$

We denote by $C^w([0, T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (12) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [8, 9, 10, 11, 12]. The next simple lemma, whose proof can be found in [13], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ *and for every* $u \in W^1L^\Phi$

$$|u(t)-u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|), \tag{13}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \tag{14}$$

2. *For every* $u \in W^1L^\Phi$ *we have* $\widetilde{u} \in L^\infty_d$ *and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|\dot{u}\|_{L^\Phi} \qquad \textit{(Sobolev's inequality).} \tag{15}$$

The following result is analogous to some lemmata in $W^1L^p_d$, see [14].

Lemma 2.2. *If* $\|u\|_{W^1L^\Phi} \to \infty$*, then* $(|\overline{u}| + \|\dot{u}\|_{L^\Phi}) \to \infty$.

Proof. By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u}+\tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{16}$$

It is known that $L^\infty_d \hookrightarrow L^\Phi_d$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L^\infty_d$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1\|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2\|u'\|_{L^\Phi}. \tag{17}$$

Therefore, from (16), (17) and (11), we get

$$\|u\|_{W^1L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. □

We present a definition that will be useful later.

Definition 2.3. *A function* $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ *is a* Carathéodory *function if for fixed* (x,y) *the map* $t \mapsto \mathcal{L}(t,x,y)$ *is measurable and for fixed* t *the map* $(x,y) \mapsto \mathcal{L}(t,x,y)$ *is continuous for almost everywhere* $t \in [0,T]$*. We say that* $\mathcal{L}(t,x,y)$ *is* differentiable Carathéodory *if in addition* $\mathcal{L}(t,x,y)$ *is continuously differentiable with respect to* x *and* y *for almost everywhere* $t \in [0,T]$.

In [13] we proved the next results.

Theorem 2.4. *Let* $\mathcal{L}$ *be a differentiable Carathéodory function satisfying* (A1)*,* (**??**) *and* (A2)*. Then the following statements hold:*

1. *The action integral given by* (3) *is finitely defined on* $\mathcal{E}^\Phi_d(\lambda) := W^1L^\Phi_d \cap \{u|u' \in \Pi(E^\Phi_d,\lambda)\}$.

2. *The function I is Gâteaux differentiable on $\mathcal{E}_d^\Phi(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^\Phi(\lambda)$ into $\left[W^1L_d^\Phi\right]^*$. Moreover, I' is given by the following expression*

$$\langle I'(u), v\rangle = \int_0^T \left\{D_x\mathcal{L}\big(t,u,u'\big)\cdot v + D_y\mathcal{L}\big(t,u,u'\big)\cdot v'\right\}\,dt. \tag{18}$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^\Phi(\lambda)$ into $\left[W^1L_d^\Phi\right]^*$ when both spaces are equipped with the strong topology.*

In [13] we derive the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T-periodic functions. We denote by $W^1L_T^\Phi$ the subspace of $W^1L_d^\Phi$ containing all T-periodic functions. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f:\mathbb{R}^d\to\mathbb{R}$ is called *strictly convex* if $f\left(\frac{x+y}{2}\right) < \frac{1}{2}\left(f(x)+f(y)\right)$ for $x\neq y$. It is well known that if f is a strictly convex and differentiable function, then $D_xf:\mathbb{R}^d\to\mathbb{R}^d$ is a one-to-one map (see, e.g. [15, Thm. 12.17]).

The following theorem is a slight modification of [13, Th. 4.1] and it is proved in a similar way.

Theorem 2.5. *Let $u\in W^1E_T^\Phi$. The following statements are equivalent:*

1. $I'(u)\in\left(W^1E_T^\Phi\right)^\perp$.
2. $D_y\mathcal{L}(t,u(t),u'(t))$ *is an absolutely continuous function and u solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t\in(0,T)\\ u(0)-u(T) = D_y\mathcal{L}(0,u(0),u'(0)) - D_y\mathcal{L}(T,u(T),u'(T)) = 0. \end{cases} \tag{19}$$

Moreover if $D_y\mathcal{L}(t,x,y)$ is T-periodic with respect to the variable t and strictly convex with respect to y, then $D_y\mathcal{L}(0,u(0),\boldsymbol{u'}(0)) - D_y\mathcal{L}(T,u(T),u'(T)) = 0$ is equivalent to $u'(0)=u'(T)$.

HABRÍA QUE ARREGLAR EL TEOREMA ANTERIOR CAMBIANDO $W^1L_T^\Phi$ por $W^1E_T^\Phi$??????

Habría que ver si el lugar de los índices es el adecuado. Copié lo que teníamos en el primer trabajo.

Next, we enumerate some definitions and results from the theory of convex functions. We suggest [16, 17, 4, 18, 5] for definitions, proofs and additional details.

We denote by α_φ and β_φ the so-called *Matuszewska-Orlicz indices* of the function φ, which are defined next. Given an increasing, unbounded, continuous function $\varphi:[0,+\infty)\to[0,+\infty)$ such that $\varphi(0)=0$ we define

$$\alpha_\varphi := \lim_{t\to0^+}\frac{\log\left(\sup\limits_{u>0}\frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)},\quad \beta_\varphi := \lim_{t\to+\infty}\frac{\log\left(\sup\limits_{u>0}\frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)}. \tag{20}$$

We have that $0 \leqslant \alpha_\varphi \leqslant \beta_\varphi \leqslant +\infty$. The relation $\beta_\varphi < \infty$ holds true if and only if φ satisfies the Δ_2-condition. If φ is a homeomorphism we have that

$$\alpha_{\varphi^{-1}} = \frac{1}{\beta_\varphi}. \tag{21}$$

Moreover $\varphi \in \mathcal{F}$ implies $\alpha_\varphi \geqslant 1$. As a consequence, φ^{-1} satisfies the Δ_2-function.

It is well known that if φ is an increasing function that satisfies the Δ_2-condition, φ is controlled by above and below by power functions. More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\varphi, \epsilon)$ such that, for every $t, u \geqslant 0$,

$$K^{-1} \min\left\{t^{\beta_\varphi+\epsilon}, t^{\alpha_\varphi-\epsilon}\right\}\varphi(u) \leqslant \varphi(tu) \leqslant K \max\left\{t^{\beta_\varphi+\epsilon}, t^{\alpha_\varphi-\epsilon}\right\}\varphi(u). \tag{22}$$

3 Lagrangians satisfying sublinear nonlinearity type conditions

Lemma 3.1. *Let Φ, Ψ complementary functions. The next statements are equivalent:*

1. $\Psi \in \Delta_2$ *globally.*
2. *There exists an N-function $\Phi_1 \in \Delta_2$ such that*

$$\Phi(rs) \geqslant \Phi_1(r)\Phi(s) \ \text{ for every } \ r \geqslant 1, \quad s \geqslant 0. \tag{23}$$

Proof. 1)⇒2) As $\Psi \in \Delta_2$ globally, there exist $k > 0$ and $\nu > 1$ such that

$$\Phi(rs) \geqslant kr^\nu \Phi(s) \ \ r \geqslant 1, \ s > 0,$$

which is (23) with $\Phi_1(r) = kr^\nu$ that is an N-function satisfying the Δ_2-condition.

2)⇒1) Next, we follow [5, p. 32, Prop. 13] and [5, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leqslant \Phi(rs) \ \ r > 1, \ s \geqslant 0.$$

Let $u = \Phi_1(r) \geqslant \Phi_1(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [5, p. 13, Prop. 1] and (23), we have for $u \geqslant \Phi_1(1)$ and $v \geqslant 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant \Phi_1^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi_1(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi_1(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$ we get that $\Psi \in \Delta_2$ globally. □

The following lemma generalizes [13, Lemma 5.2].

Lemma 3.2. *Let Φ, Ψ be N-functions and suppose that $\Psi \in \Delta_2$ globally. Then*

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \tag{24}$$

for every Φ_0 with $\Phi_0 = o(\Phi_1)$ at ∞ where Φ_1 is any N-function satisfying (23). *Reciprocally if* (24) *holds for some N-function Φ_0, then $\Psi \in \Delta_2$ (at ∞).*

Proof. By the assumptions on Φ and Φ_1 and the identity (7), we have

$$\frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \Phi_1(r)\frac{\int_0^T \Phi(r^{-1}|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})}\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$ and as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$. Next, we use the fact that $\Phi_1 \in \Delta_2$ and $\Phi_0 = o(\Phi_1)$ at ∞, and we get

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} \geqslant C \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1(\|u\|_{L^\Phi})}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

The last assertion of the lemma follows from the fact that if Φ_0 is an N-function, then $\Phi_0(u) \geqslant ku$ for k small enough. Therefore (24) holds for $\Phi_0(u) = |u|$, then [13, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞. □

Remark 1. We point out that this lemma can be applied to more cases than [13, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [4, p. 16] and [4, Section 7] for the definition and properties of principal part). Then (24) holds for Φ_0, however $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

We define the following functionals $J_{C,\Phi_0} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\Phi_0} : \mathbb{R}^n \to \mathbb{R}$, where $C > 0$ and Φ_0 is an N-function, by

$$J_{C,\Phi_0}(u) := \rho_\Phi(u) - C\Phi_0\left(\|u\|_{L^\Phi}\right), \tag{25}$$

and

$$H_{C,\Phi_0}(x) := \int_0^T F(t,x)dt - C\Phi_0(|x|), \tag{26}$$

respectively.

Like in [13] we consider Lagrangians $\mathcal{L}$ which are lower bounded as follows

$$\mathcal{L}(t,x,y) \geqslant \alpha_0\Phi\left(\frac{|y|}{\Lambda}\right) + F(t,x). \tag{27}$$

If $\mathcal{L}$ is given by the right hand side in (27) and $\Phi(u) = |u|^2$, then the ODE $\ddot{u} = \nabla F(t,u(t))$ in (1) is quasilinear, being $\nabla F(t,u(t))$ the nonlinearity. Following the literature, we refer to ∇F as the non linearity even when we assume in (27) just the inequality. In [1] and [2] the authors considered, for the p-laplacian case, non linearities satisfying the inequality

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^{\alpha} + b_2(t),$$

where $b_1, b_2 \in L_1^1$ and α is any power less than p. Thus, they said F is a sublinear nonlinearity. In this paper, we consider the following type of bounds for the nonlinearity

$$|\nabla F(t,x)| \leqslant b_1(t)\varphi_0(|x|) + b_2(t), \tag{28}$$

where $\varphi_0 = \Phi_0'$ with Φ_0 an N-function. The employment of N-functions instead of power functions in inequalities like (28) will allow us to extend some results of [1] and [2] even in the p-laplacian case.

Based on [19] we say that F satisfies the condition (A) if $F(t,x)$ is a Carathéodory function and F is continuously differentiable with respect to x. Moreover, the next inequality holds

$$|F(t,x)| + |D_xF(t,x)| \leqslant a(|x|)b_0(t), \quad \text{for a.e. } t \in [0,T], \forall x \in \mathbb{R}^d. \tag{29}$$

The following theorem establishes coercivity of I assuming sublinear conditions on the nonlinearity ∇F.

Theorem 3.3. *Let $\mathcal{L}$ be a lagrangian function satisfying* (A1)*,* (**??**)*,* (A2)*,* (27) *and suppose that F satisfies condition (A). We assume the following conditions:*

1. $\Psi \in \Delta_2$.
2. *Inequality* (28) *with $b_1, b_2 \in L_1^1$, $\varphi_0 = \Phi_0'$ where Φ_0 is a differentiable N-function that satisfies the Δ_2-condition globally such that $\Phi_0 = o(\Phi_1)$ at ∞ and Φ_1 verifies* (23)*.*
3. $$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Phi_0(|x|)} = +\infty. \tag{30}$$

Then the action integral I is coercive.

Proof. By the decomposition $u = \overline{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (28), we have

$$\begin{aligned}
&\left|\int_0^T F(t,u) - F(t,\overline{u})\,dt\right| = \left|\int_0^T \int_0^1 \nabla F(t,\overline{u} + s\tilde{u}(t)) \cdot \tilde{u}(t)\,ds\,dt\right| \\
&\leqslant \int_0^T \int_0^1 b_1(t)\varphi_0(|\overline{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt + \int_0^T \int_0^1 b_2(t)|\tilde{u}(t)|\,ds\,dt \\
&= I_1 + I_2.
\end{aligned} \tag{31}$$

On the one hand, by Hölder's and Sobolev's inequalities, we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty} \leqslant C_1\|\dot{u}\|_{L^\Phi}, \tag{32}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

On the other hand, since $\Phi_0 \in \Delta_2$ globally, then $\varphi_0 \in \Delta_2$ globally and consequently φ_0 is a quasi-subadditive function, i.e. there exists $C(\varphi_0) > 0$ such that $\varphi_0(a+b) \leqslant C(\varphi_0)(\varphi_0(a) + \varphi_0(b))$ for every $a, b \geqslant 0$. In this way, we have

$$\varphi_0(|\overline{u} + s\tilde{u}(t)|) \leqslant C(\varphi_0)[\varphi_0(|\overline{u}|) + \varphi_0(\|\tilde{u}\|_{L^\infty})], \tag{33}$$

for every $s \in [0,1]$.

Now, inequality (33), Hölder's and Sobolev's inequalities, the monotonicity, the subadditivity and the Δ_2-condition on φ_0, imply that

$$\begin{aligned} I_1 &\leqslant C(\varphi_0)\left\{\varphi_0(|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} + \|b_1\|_{L^1}\varphi_0(\|\tilde{u}\|_{L^\infty})\|\tilde{u}\|_{L^\infty}\right\} \\ &\leqslant C_2\left\{\varphi_0(|\overline{u}|)\|u'\|_{L^\Phi} + \varphi_0(\|\dot{u}\|_{L^\Phi})\|\dot{u}\|_{L^\Phi}\right\}, \end{aligned} \tag{34}$$

where $C_2 = C_2(\varphi_0, T, \|b_1\|_{L^1})$.

Next, by Young's inequality with complementary functions Φ_0 and Ψ_0 and the fact that $\Phi_0 \in \Delta_2$ globally, Young's equality [4, Eq. 2.7-2.8] and [5, Th. 3-(ii), p. 23], we get

$$\begin{aligned} \varphi_0(|\overline{u}|)\|u'\|_{L^\Phi} &\leqslant \Psi_0(\varphi_0(|\overline{u}|)) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leqslant |\overline{u}|\varphi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leqslant C(\Phi_0)\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \end{aligned} \tag{35}$$

and

$$\varphi_0(\|\dot{u}\|_{L^\Phi})\|\dot{u}\|_{L^\Phi} \leqslant C(\Phi_0)\Phi_0(\|\dot{u}\|_{L^\Phi}), \tag{36}$$

with $C(\Phi_0)$ the constant that comes from the Δ_2-condition on Φ_0.

From (34), (35), (36) and (32), we have

$$\begin{aligned} I_1 + I_2 &\leqslant C_3\left\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi}\right\} \\ &\leqslant C_4\left\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + 1\right\}, \end{aligned} \tag{37}$$

with C_3 and C_4 depending on $\Phi_0, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$. The last inequality follows from the fact that Φ_0 is an N-function, then there exists $C > 0$ such that $\Phi_0(x) \geqslant Cx$ for every $x \geqslant 1$. Thus $x \leqslant C\Phi_0(x) + 1$ for every $x \geqslant 0$.

In the subsequent estimates, we use (27), (31), (37), the fact that $\Phi_0 \in \Delta_2$ and we get

$$\begin{aligned}
I(u) &\geqslant \alpha_0 \rho_\Phi \left(\frac{u'}{\Lambda}\right) + \int_0^T F(t,u)\,dt \\
&= \alpha_0 \rho_\Phi \left(\frac{u'}{\Lambda}\right) + \int_0^T \left[F(t,u) - F(t,\overline{u})\right]\,dt + \int_0^T F(t,\overline{u})\,dt \\
&\geqslant \alpha_0 \rho_\Phi \left(\frac{u'}{\Lambda}\right) - C_4 \Phi_0(\|\dot{u}\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_4 \Phi_0(|\overline{u}|) - C_4 \\
&\geqslant \alpha_0 \rho_\Phi \left(\frac{u'}{\Lambda}\right) - C_4 \Phi_0(\|\dot{u}\|_{L^\Phi}) + H_{C_4,\Phi_0}(\overline{u}) - C_4 \\
&\geqslant \alpha_0 \rho_\Phi \left(\frac{u'}{\Lambda}\right) - C_5 \Phi_0\left(\frac{\|\dot{u}\|_{L^\Phi}}{\Lambda}\right) + H_{C_4,\Phi_0}(\overline{u}) - C_4 \\
&= \alpha_0 J_{C_6,\Phi_0}\left(\frac{\dot{u}}{\Lambda}\right) + H_{C_4,\Phi_0}(\overline{u}) - C_4,
\end{aligned} \tag{38}$$

where $C_5 = C_5(\Phi_0, \Lambda, C_4)$ and $C_6 = \frac{C_5}{\alpha_0}$.

Let u_n be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n, $I(u_n)$ is upper bounded, i.e., there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 2.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence, still denoted u_n, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.2 implies that the functional $J_{C_6,\Phi_0}(\frac{\dot{u}}{\Lambda})$ is coercive; and, by (30), the functional $H_{C_4,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_6,\Phi_0}(\frac{\dot{u}_n}{\Lambda}) \to \infty$ or $H_{C_4,\Phi_0}(\overline{u}_n) \to \infty$. From (29), we have that on a bounded set the functional $H_{C_4,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_6,\Phi_0}(\frac{\dot{u}_n}{\Lambda}) \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$. □

4 Main result

In order to find conditions for the lower semicontinuity of I, we perform a little adaptation of a result of [20].

Lemma 4.1. *Let $\mathcal{L}(t,x,y)$ be a differentiable Carathéodory function. Suppose that F satisfies the condition (A) and the inequality*

$$\mathcal{L}(t,x,y) \geqslant \Phi\left(|y|\right) + F(t,x), \tag{39}$$

where Φ is an N-function. In addition, suppose that $\mathcal{L}(t,x,\cdot)$ is convex in $\mathbb{R}^d$ for each $(t,x) \in [0,T] \times \mathbb{R}^d$. Let $\{u_n\} \subset W^1L^\Phi$ be a sequence such that u_n converges uniformly to a function $u \in W^1L^\Phi$ and u_n' converges in the weak topology of L_d^1 to u'. Then

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n). \tag{40}$$

Proof. First, we point out that (39) and (29) imply that I is defined on W^1L^Φ taking values on the interval $(-\infty, +\infty]$. Let $\{u_n\}$ be a sequence satisfying the assumptions of the theorem. We define the differentiable Carathéodory function $\hat{\mathcal{L}} = \mathcal{L} - F$ and we denote by $\hat{I}$ its associated action integral. Using [20, Thm. 2.1, p. 243], we get

$$\int_0^T \hat{\mathcal{L}}(t, u, u')\, dt \leqslant \liminf_{n\to\infty} \int_0^T \hat{\mathcal{L}}(t, u_n, u_n')\, dt. \tag{41}$$

Taking account of the uniform convergence of u_n and the fact that F is a Carathéodory function, we obtain that $F(t, u_n(t)) \to F(t, u(t))$ a.e. $t \in [0, T]$. Since the sequence u_n is uniformly bounded, from (29) follows that there exists $g \in L^1_1([0, T])$ such that $|F(t, u_n(t))| \leqslant g(t)$. Now, by the Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty} \int_0^T F(t, u_n(t))\, dt = \int_0^T F(t, u(t))\, dt. \tag{42}$$

Finally, as a consequence of (41) and (42), we obtain (40). □

Lemma 4.2. *E^Φ_d is weak* closed in L^Φ_d.*

Proof. From [5, Thm. 7, p. 110] we have that $L^\Phi_d = \left[E^\Psi_d\right]^*$. Then, L^Φ_d is a dual and therefore we are allowed to speak about the weak* topology of L^Φ_d. Besides, E^Φ_d is separable (see [5, Thm. 1, p. 87]). Let $S = E^\Phi_d \cap \{u \in L^\Phi_d | \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [5, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because is $u_n \in S$ and $u_n \overset{*}{\rightharpoonup} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \overset{*}{\rightharpoonup} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ_d is separable and $L^\Phi_d = \left[E^\Psi_d\right]^*$, the ball of L^Φ $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ is weak* metrizable (see [21, Thm. 5.1, p. 138]). Thus, S is closed respect to the weak* topology. Now, by the Krein-Smulian Theorem, [21, Cor. 12.6, p. 165] implies that E^Φ_d is weak* closed. □

Gathering our previous results we obtain existence of solutions.
Let $W^1E^\Phi_T = W^1L^\Phi_T \cap W^1E^\Phi_d$.

Theorem 4.3. *Let Φ and Ψ be complementary N-functions. Suppose that the differentiable Carathéodory function $\mathcal{L}(t, x, y)$ is strictly convex at y, $D_y\mathcal{L}$ is T-periodic with respect to t. In addition, assume the same hypothesis than Theorem 3.3. Then, problem* (1) *has a solution.*

Proof. Let $\{u_n\} \subset W^1E^\Phi_T$ be a minimizing sequence for the problem $\inf\{I(u) | u \in W^1E^\Phi_T\}$. Since $I(u_n)$, $n = 1, 2, \ldots$ is upper bounded, Theorem 3.3 implies that $\{u_n\}$ is norm bounded in $W^1E^\Phi_d$. Hence, in virtue of Corollary [13, Corollary 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T-periodic continuous function u. Then, u is bounded and $u \in E^\Phi_d$.

As $u_n' \in E^\Phi_d \subset L^\Phi_d$, there exists a subsequence (again denoted by u_n') such that u_n' converges to a function $v \in L^\Phi_d$ in the weak* topology of L^Φ_d. Since E^Φ_d is weak* closed, by Lemma 4.2, $v \in E^\Phi_d$.

From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \dot{\boldsymbol{\xi}} \cdot u\, dt = \lim_{n\to\infty} \int_0^T \dot{\boldsymbol{\xi}} \cdot u_n\, dt = -\lim_{n\to\infty} \int_0^T \boldsymbol{\xi} \cdot u_n'\, dt = -\int_0^T \boldsymbol{\xi} \cdot v\, dt$$

for every T-periodic function $\boldsymbol{\xi} \in C^\infty([0,T],\mathbb{R}^d) \subset E_d^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [19, p. 6]) and $u \in E_T^\Phi$.

Now, taking into account the relations $\left[L_d^1\right]^* = L_d^\infty \subset E_d^\Psi$ and $L_d^\Phi \subset L_d^1$, we have that u_n' converges to u' in the weak topology of L_d^1. Consequently, Lemma 4.1 applied to the N-function $\alpha_0\Phi\left(|\cdot|/\Lambda\right)$ implies that

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{u\in W^1E_T^\Phi} I(u).$$

As $u \in W^1E_T^\Phi \subset \mathcal{E}_d^\Phi(\lambda)$ then $I(u) > -\infty$, hence, u is a minimun and therefore $I'(u) \in (W^1E_T^\Phi)^\perp$. Finally, invoking Theorem 2.5, the proof concludes. □

5 Limit case $\mu = \alpha_\Phi$

Assuming $\|b_1\|_{L^1}$ small enough, in [22, 2] coercivity was obtained even for the limit value $\mu = p$ in inequality (28).

OJO que μ no aparece en (28)!!!!. Quizás debería decir $\varphi_0(x) = x^p$. O, mecionarse la ecuación anterior donde aparece $\alpha < p$, no μ.

This result leans on the fact that

$$\|u\|_{L^\Phi}^{\alpha_\Phi} = O\left(\int_0^T \Phi(|u|)\, dt\right) \quad \text{for } \|u\|_{L^\Phi} \to \infty, \tag{43}$$

when $\Phi(u) = |u|^p$. Nevertheless, it is no longer the case for any N-function Φ as the following example shows.

In this section, from now on we will suppose that

$$\Phi(u) = \begin{cases} \frac{p-1}{p} u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function Φ.

Theorem 5.1. *If $p \geqslant \frac{1+\sqrt{2}}{2}$, then Φ is an N-function.*

Proof. We have

$$\varphi(u) = \Phi'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}\left(p - \frac{1}{\log u}\right) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that Φ' is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that φ_1 is increasing on $[0,e]$ and φ_2 is increasing on $[e,\infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p>1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u)>0$ on $[e,\infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an icreasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Besides $\varphi_2(u)\to\infty$ and $\varphi_1(u)\to 0$ as $u\to\infty$ and $u\to 0$ respectively, provided that $p>1$. Hence, Φ is an N-function. □

Theorem 5.2. *For every $\varepsilon>0$, there exists a positive constant $C=C(p,\varepsilon)$ such that*

$$C^{-1}t^{p-\varepsilon}\Phi(u) \leqslant \Phi(tu) \leqslant Ct^p\Phi(u) \quad t\geqslant 1, u>0, \tag{44}$$

Proof. If $u\leqslant tu\leqslant e$, then $\Phi(tu)=t^p\Phi(u)$ and (44) holds with $C=1$.

If $u\leqslant e\leqslant tu$, as $\frac{e^p}{p}>0$ and $\log(tu)\geqslant 1$, we have $\Phi(tu)\leqslant t^pu^p=\frac{p}{p-1}t^p\Phi(u)$. Thus, the second inequality of (44) holds with $C=\frac{p}{p-1}$. On the other hand, as $f(t)=\frac{t}{\log t}$ is increasing on $[e,\infty)$, then $f((tu)^p)\geqslant f(e^p)=e^p/p$. Now,

$$\begin{aligned}\Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p}-\frac{e^p}{p}\\ &= \frac{(p-1)(tu)^p}{\log(tu)^p}+\frac{(tu)^p}{\log(tu)^p}-\frac{e^p}{p}\\ &\geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)}\\ &\geqslant \frac{p-1}{p}\frac{t^\varepsilon}{\log t+1}t^{p-\varepsilon}u^p.\end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t\mapsto\frac{t^\varepsilon}{\log t+1}$ on the interval $[1,+\infty)$ then

$$\Phi(tu)\geqslant\frac{p-1}{p}\varepsilon e^{1-\varepsilon}t^{p-\varepsilon}u^p,$$

which is the first inequality of (44) with $C=\frac{p}{p-1}\varepsilon^{-1}e^{-1+\varepsilon}$.

If $e\leqslant u\leqslant tu$, then

$$\Phi(tu)\leqslant\frac{t^pu^p}{\log(tu)}\leqslant\frac{t^pu^p}{\log(u)}=\frac{pt^pv}{\log v}, \tag{45}$$

where $v:=u^p$ and $v\geqslant e^p$. If $\alpha>0$, the function $x\mapsto\frac{x}{x-\alpha}$ is decreasing on (α,∞) and the function $v\mapsto\frac{pv}{\log v}$ is increasing on $[e^p,\infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v}-\frac{e^p}{p}}\leqslant\frac{e^p}{e^p-\frac{e^p}{p}}=\frac{p}{p-1}$$

for every $v\geqslant e^p$. In this way, from (45), we have

$$\Phi(tu)\leqslant\frac{pt^p}{p-1}\left(\frac{pv}{\log v}-\frac{e^p}{p}\right)=\frac{pt^p}{p-1}\left(\frac{u^p}{\log u}-\frac{e^p}{p}\right)$$

and the second inequality of (44) holds with $C=\frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, the function f attains a minimum on $[1,\infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$, f is increasing on $[1,\infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geqslant \varepsilon$ in any case, therefore

$$\Phi(tu) \geqslant \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geqslant \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi(u).$$

Therefore, (44) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. □

Remark 2. The inequality

$$\Phi(tu) \geqslant Ct^p\Phi(u)$$

is false for every C because for every $u \geqslant e$ we have

$$\lim_{t\to\infty}\frac{\Phi(tu)}{t^p\Phi(u)} = 0$$

Theorem 5.3. $\alpha_\Phi = \beta_\Phi = p$

Proof. From (20) and (44), we get

$$\beta_\Phi = \lim_{t\to\infty}\frac{\log\left[\sup\limits_{u>0}\frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} \leqslant \lim_{t\to\infty}\frac{\log C + p\log t}{\log t} = p.$$

On the other hand, employing (20) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t\to 0^+}\frac{\log\left[\sup\limits_{u>0}\frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} = \lim_{s\to\infty}\frac{\log\left[\sup\limits_{v>0}\frac{\Phi(v)}{\Phi(sv)}\right]^{-1}}{\log s} = \lim_{s\to\infty}\frac{\log\left[\inf\limits_{v>0}\frac{\Phi(sv)}{\Phi(v)}\right]}{\log s}$$

where $v := tu$ and $s := \frac{1}{t}$. Then, using (44), for every $\varepsilon > 0$ we have

$$\alpha_\Phi = \lim_{s\to\infty}\frac{\log\left[\inf\limits_{v>0}\frac{\Phi(sv)}{\Phi(v)}\right]}{\log s} \geqslant \lim_{s\to\infty}\frac{\log C + (p-\varepsilon)\log s}{\log s} \geqslant p - \varepsilon,$$

therefore $\alpha_\Phi \geqslant p$.

Finally, as $\alpha_\Phi \leqslant \beta_\Phi \leqslant p$, we get $\alpha_\Phi = \beta_\Phi = p$. □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\,dx \geqslant C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^p$$

is false.

In fact, if we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\,dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geqslant C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geqslant Ct^p$ would also be true; however, this last inequality is false.

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

References

[1] C.-L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, Proc. Amer. Math. Soc. 126 (11) (1998) 3263–3270.

[2] X. Tang, X. Zhang, Periodic solutions for second-order Hamiltonian systems with a p-Laplacian, Ann. Univ. Mariae Curie-Skłodowska Sect. A 64 (1) (2010) 93–113.

[3] R. Adams, J. Fournier, Sobolev spaces, Elsevier/Academic Press, Amsterdam, 2003.

[4] M. A. Krasnosel′skiĭ, J. B. Rutickiĭ, Convex functions and Orlicz spaces, P. Noordhoff Ltd., Groningen, 1961.

[5] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Vol. 146, Marcel Dekker, Inc., New York, 1991.

[6] G. Schappacher, A notion of Orlicz spaces for vector valued functions, Appl. Math. 50 (4) (2005) 355–386.

[7] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general, Indag. Math. (N.S.) 11 (4) (2000) 573–585.

[8] A. Cianchi, A fully anisotropic Sobolev inequality, Pacific J. Math. 196 (2) (2000) 283–295.

[9] A. Cianchi, Some results in the theory of Orlicz spaces and applications to variational problems, in: Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), Acad. Sci. Czech Repub., Prague, 1999, pp. 50–92.

[10] N. Clavero, Optimal Sobolev embeddings and Function Spaces, `http://www.maia.ub.edu/~soria/sobolev1.pdf`, last accessed: 2014-12-22. (2011).

[11] D. Edmunds, R. Kerman, L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, J. Funct. Anal. 170 (2) (2000) 307–355.

[12] R. Kerman, L. Pick, Optimal Sobolev imbeddings, Forum Math. 18 (4) (2006) 535–570.

[13] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, E. Schwindt, Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, Nonlinear Analysis, TMA. 125 (2015) 681 – 698.

[14] B. Xu, C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems, J. Math. Anal. Appl. 333 (2) (2007) 1228–1236.

[15] R. T. Rockafellar, R. Wets, Variational analysis, Springer-Verlag, Berlin, 1998.

[16] A. Fiorenza, M. Krbec, Indices of Orlicz spaces and some applications, Comment. Math. Univ. Carolin. 38 (3) (1997) 433–451.

[17] J. Gustavsson, J. Peetre, Interpolation of Orlicz spaces, Studia Mathematica 60 (1) (1977) 33–59.
URL `http://eudml.org/doc/218150`

[18] L. Maligranda, Orlicz spaces and interpolation, Vol. 5 of Seminários de Matemática [Seminars in Mathematics], Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[19] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989.

[20] I. Ekeland, R. Témam, Convex analysis and variational problems, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[21] J. B. Conway, A Course in Functional Analysis, Springer, USA, 1977.

[22] F. Zhao, X. Wu, Periodic solutions for a class of non-autonomous second order systems, J. Math. Anal. Appl. 296 (2) (2004) 422–434.