Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}, \tag{1}$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary equations*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) = \frac{|y|^p}{p} + F(t,x), \tag{2}$$

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for $1 < p < \infty$. For example, the classic book [**?**] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [**?**] were extended and improved in several articles, see [**?**, **?**, **?**, **?**, **?**] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [**?**, **?**] and in this case (1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t, x)$ is differentiable with respect to x for a.e. $t \in [0, T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

(A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leqslant a(|x|)b(t). \tag{4}$$

In this inequality we assume that the function $a : [0, +\infty) \to [0, +\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0, T], \mathbb{R})$.

In [**?**] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t, x, y) = \Phi(|y|) + F(t, x), \tag{5}$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [**?**] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([**?**, Ch. 3]) to obtain solutions of (1).

2 Preliminaries

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces) and other brief introduction to superposition operators between these spaces. References for these topics are [**?**, **?**, **?**] and [**?**, **?**, **?**, **?**].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$.

Following [**?**] we say that Φ is *coercive* if

$$\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = +\infty.$$

We define the function the F by

$$F(s) = \sup\{|x| : \Phi(x) \leqslant s\}, \tag{6}$$

where Φ is a Young's function.

As $\alpha\Phi(\frac{x}{\alpha})$ is decreasing with respect to α, we get that the function $\alpha F(\frac{x}{\alpha})$ is increasing with respect to α. That is, if $0 < \alpha \leqslant \beta$, we have

$$\alpha F\left(\frac{s}{\alpha}\right) = \alpha \sup\left\{|x| : \Phi(x) \leqslant \frac{s}{\alpha}\right\} = \sup\left\{\alpha|x| : \alpha\Phi(x) \leqslant s\right\} =$$
$$\sup\left\{|y| : \alpha\Phi\left(\frac{y}{\alpha}\right) \leqslant s\right\} \leqslant \sup\left\{|y| : \beta\Phi\left(\frac{y}{\beta}\right) \leqslant s\right\} =$$
$$\sup\left\{\beta|x| : \beta\Phi(x) \leqslant s\right\} = \sup \beta\left\{|x| : \Phi(x) \leqslant \frac{x}{\beta}\right\} =$$
$$\beta F\left(\frac{x}{\beta}\right).$$

We note that for every $K > 0$, if $|x| > F(K)$ then $\Phi(x) > K$ and therefore we see that $|x| \leqslant F(\Phi(x))$. If $d = 1$ then $F = \Phi^{-1}$.

We also say that a non decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Δ_2^∞-*condition*, denoted by $\eta \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\eta(2x) \leqslant K\eta(x), \tag{7}$$

for every $|x| \geqslant M$.

If Φ is a Young function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \to \mathbb{R}_+$by:

$$\Phi^*(y) = \sup_{x\in\mathbb{R}^d} x \cdot y - \Phi(x) \tag{8}$$

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T],\mathbb{R}^d)$ the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1,\ldots,u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T],\mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{9}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T],\mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{10}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The subspace $E^\Phi = E^\Phi([0,T],\mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T],\mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [**?**, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [**?**, Thm. 5.2]). Another alternative characterization of E^Φ, which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1,2,\ldots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [**?**, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \tag{11}$$

Like in [**?**] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{12}$$

for any positive r (see [**?**, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [**?**] we introduce the next definition.

Definition 2.1. *Let* $u_n, uL^\Phi([0,T],\mathbb{R}^d)$. *We say that* u_n *converges monotonically to* u *if there exists* $\alpha_n \in \|_{L^\infty}([0,T],\mathbb{R}^d)$, $n = 1,2,\ldots$, *such that* $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ *a.e., when* $n \to \infty$ *and* $u_n(t) = \alpha_n(t)u(t)$.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow \left[L^\Phi\right]^*$, where a function $v \in L^{\Phi^*}$ is associated to $F_v \in \left[L^\Phi\right]^*$ where

$$F_v(u) := \langle v, u\rangle = \int_0^T v \cdot u \, dt, \tag{13}$$

In [**?**, Thm 2.9] it was characterized a subspace of $\left[L^\Phi\right]^*$ which is identified with L^{Φ^*}. Namely $L^{\Phi^*} = P^{\Phi^*}([0,T],\mathbb{R}^d)$ where $F \in P^{\Phi^*}([0,T],\mathbb{R}^d)$ if and only if $F \in \left[L^\Phi\right]^*$ and satisfying the *monotone convergence property*, which is if u_n converges monotonically to u then $F(u_n) \to F(u)$.

If $\Phi \in \Delta_2^\infty$ and Φ is coercive then $L^{\Phi^*}([0,T],\mathbb{R}^d) = \left[L^\Phi([0,T],\mathbb{R}^d)\right]^*$ is satisfied (see [**?**, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi([0,T],\mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi([0,T],\mathbb{R}^d)\}.$$

$W^1L^\Phi([0,T],\mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{14}$$

And, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E^\Phi_T &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{15}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.2. *Let $\Phi : \mathbb{R}^d \to [0,+\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (6)*. Then*

1. *For every $s,t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi}|s-t|F\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant F\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant TF\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *The space $W^1L^\Phi([0,T],\mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0,T],\mathbb{R}^d)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right) \\ &\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}. \end{aligned}$$

By (6) we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant F\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha F(1/\alpha)$ is an increasing function with respect to $\alpha \in [0,\infty)$ we have

$$|u(t) - \overline{u}| \leqslant \|u'\|_{L^\Phi}TF\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then

$$|\overline{u}| \leqslant F\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}\\ &\leqslant F\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TF\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}\\ &\leqslant F\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

We take a bounded sequence u_n in $W^1L^\Phi([0,T],\mathbb{R}^d)$ and suppose that u_n has not convergent subsequence.

□

3 Superposition operators in anisotropic Orlicz spaces

Vamos escribiendo lo que queremos...(de acuerdo a mis apuntes y sin ver las hojitas de la semana pasada)

For $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ we denote by $\mathfrak{f}$ the Nemytskii (o superposition) operator defined for functions $u:[0,T]\to\mathbb{R}^d$ by

$$\mathfrak{f}u(t) = f(t,u(t))$$

Referencias y alguna propiedad interesante medibles en medibles? [**?**, **?**]

Theorem 3.1. *Let $\Phi_1,\Phi_2,\dots,\Phi_n$ be N-functions. Assume that M is another N-functions that satisfy the Δ_2-condition. We write $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ with $x_i\in\mathbb{R}^d$, $y_i\in\mathbb{R}^d$. Let $f(t,x_1,\dots,x_n,y_1,\dots,y_n)$ be a function Charathéodory? with $f:[0,T]\times(\mathbb{R}^d)^n\times(\mathbb{R}^d)^n\to\mathbb{R}^{d'}$.*

Suppose that $a:(\mathbb{R}^d)^n\to[0,+\infty)$ is a bounded function on bounded sets and $b\in L^M([0,T])$, for a.e. $t\in[0,T]$ such that

$$|f| \leqslant a(x)[b(t) + \sum_{i=1}^n M^{-1}(\Phi_i(|y_i|))], \tag{16}$$

then

$$\mathfrak{f}:\left(\prod_{i=1}^n L^\infty([0,T],\mathbb{R}^d)\right)\times\left(\prod_{i=1}^n \Pi(E^{\Phi_i}([0,T],\mathbb{R}^d),\lambda=1)\right)\to L^M.$$

Proof. If $(u,v) \in \left(\prod_{i=1}^{n} L^\infty([0,T],\mathbb{R}^d)\right) \times \left(\prod_{i=1}^{n} \Pi(E_d^{\Phi_i}, \lambda = 1)\right)$. By [**?**, Thm. 17.6] (y otras cosas), we get

$$|\mathfrak{f}u(t)| = |f(t,u(t),v(t))| \leqslant M_a[b_j(t) + \sum_{i=1}^{n} M_j^{-1}(\Phi_i(|v_i(t)|))] \in L_1^{M_j}.$$

□

We define the space X by $X = \{v = (v_1,v_2) : v_1 \in W^1L_T^{\Phi_1}, v_2 \in W^1L_T^{\Phi_2}\}$ and $X^* = \{v = (v_1,v_2) : v_1 \in (W^1L_T^{\Phi_1})^*, v_2 \in (W^1L_T^{\Phi_2})^*\}$ where $(W^1L_T^{\Phi_i})^*$ stands for the conjugate space of $W^1L_T^{\Phi_i}$ for $i = 1,2$.

Corollary 3.2. *We will consider the Lagrange function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(t,x_1,x_2,y_1,y_2) \to \mathcal{L}(t,x_1,x_2,y_1,y_2)$ which is measurable in t for each $(x_1,x_2,y_1,y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x_1,x_2,y_1,y_2) for almost every $t \in [0,T]$.*

Let $x = (x_1,x_2)$, $y = (y_1,y_2)$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^d$ and let

$$I(x) = \int_0^T \mathcal{L}(t,x,y)\, dt \tag{17}$$

If there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $i = 1,2$, $b \in L_1^1([0,T])$, $j = 1,\dots,d'$ for a.e. $t \in [0,T]$ and every $(x_1,x_2,y_1,y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying the structure conditions

The nonlinear operator $(x_1,x_2) \mapsto D_x\mathcal{L}(t,x_1,y_1,y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \cdots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into $L^1([0,T])$ with the strong topology on both sets.

The nonlinear operator $(x_1,x_2) \mapsto D_y\mathcal{L}(t,x_1,y_1,y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \cdots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into X with the weak topology.*

The function I is Gâteaux differentiable on $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^*. Moreover, I' is given by the following expression

$$\begin{aligned}\langle I'(x), w\rangle = \int_0^T [&(D_{x_1}\mathcal{L}(t,x_1(t),x_2(t),y_1(t),y_2(t)),w_1(t))+\\ &(D_{x_2}\mathcal{L}(t,x_1(t),x_2(t),y_1(t),y_2(t)),w_2(t))+\\ &(D_{y_1}\mathcal{L}(t,x_1(t),x_2(t),y_1(t),y_2(t)),w_1'(t))+\\ &(D_{y_2}\mathcal{L}(t,x_1(t),x_2(t),y_1(t),y_2(t)),w_2'(t))]\, dt\end{aligned} \tag{18}$$

If $\Phi^* \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^* when both spaces are equipped with the strong topology.

We denote by $\mathfrak{A}(a,b,c,\lambda,f,\Phi)$ the set of all Lagrange functions satisfying (**??**), (**??**) and (**??**).

Proof. **OJO!!!! Es algo que teníamos del trabajo anterior!!! con algunas adaptaciones a 2 variables sin controlar y a lo bruto!!!!!**

Let $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$.
Step 1. The non linear operator $(x_1, x_2) \mapsto (D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2), D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2))$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ *into* $L_d^1([0,T])\times L_d^1([0,T])$ *with the strong topology on both sets.*

If $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$, from (**??**) and (**??**), we obtain Let $\{x_n = (x_{1n}, x_{2n})\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ and let $x = (x_1, x_2) \in \mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ such that $x_n \to x$ in X. From $x_{in} \to x_i$ in L^{Φ_i}, there exists a subsequence x_{in_k} such that $x_{in_k} \to x_i$ a.e.; and, as $x_{in} \to x_i \in \mathcal{E}_d^{\Phi}(\lambda)$, by Lemma **??**, there exist a subsequence of x_{in_k} (again denoted x_{in_k}) and a function $h_i \in \Pi(E_1^{\Phi}, \lambda))$ such that $x_{in_k} \to u_i$ a.e. and $|x_{in_k}| \leqslant h_i$ a.e. Since x_{in_k}, $k = 1, 2, \dots$, is a strong convergent sequence in $W^1L_d^{\Phi_i}$, it is a bounded sequence in $W^1L_d^{\Phi_i}$. According to Lemma 2.2 and Corollary **??**, there exist $M_i > 0$ such that $\|\boldsymbol{a}(x_{in_k})\|_{L^\infty} \leqslant M_i$, $k = 1, 2, \dots$. From the previous facts and (**??**), we get

$$|D_{x_i}\mathcal{L}(\cdot, x_{1n_k}, x_{2n_k}, y_{1n_k}, y_{2n_k})| \leqslant M_i(b + \Phi_i(|h_i|)) \in L_1^1 \;\; i = 1, 2.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$D_{x_i}\mathcal{L}(t, x_{in_k}(t), y_{in_k}(t)) \to D_{x_i}\mathcal{L}(t, x_i(t), y_i(t)) \quad \text{for a.e. } t \in [0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.
Step 2. The non linear operator $(x_1, x_2) \mapsto (D_{y_1}\mathcal{L}(t, x_1, y_1, D_{y_2}\mathcal{L}(t, x_2, y_2)$ *is continuous from* $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ *with the strong topology into* X *with the weak* topology.*

Note that (**??**), (**??**) and the imbeddings $W^1L_d^{\Phi} \hookrightarrow L_d^{\infty}$ and $L_d^{\Phi^*} \hookrightarrow [L^{\Phi}]^*$ imply that the second member of (18) defines an element in $[W^1L_d^{\Phi}]^*$.

Let $(x_{1n}, x_{2n}) \in \mathcal{E}_d^{\Phi}(\lambda)$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$ in the norm of X. We must prove that $D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}) \overset{w^*}{\rightharpoonup} D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2)$ para $i = 1, 2$. On the contrary, there exist $v = (v_1, v_2) \in L^{\Phi_1} \times L^{\Phi_2}$, $\epsilon > 0$ and a subsequence of $\{x_n\}$ (denoted $\{x_n\}$ for simplicity) such that

$$|\langle D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}, y_{1n}, y_{2n}),\ v\rangle - \langle D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2, v\rangle| \geqslant \epsilon. \tag{19}$$

We have $x_n \to x$ in X and $y_n \to y$ in X. By Lemma **??**, there exist a subsequence x_{n_k} and a function $h \in \Pi(E_1^{\Phi_1}, \lambda) \times \Pi(E_1^{\Phi_2}, \lambda)$ such that $x_{n_k} \to x$ a.e., $y_{n_k} \to y$ a.e. and $|y_{n_k}| \leqslant h$ a.e. As in the previous step, since x_n is a convergent sequence, the Corollary **??** implies that $a(|y_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, with x_{n_k} instead of x, inequality (**??**) becomes Consequently, as $v \in L^{\Phi}$ and employing Hölder's inequality, we obtain that

$$\sup_k |D_{\boldsymbol{y}}\mathcal{L}(\cdot, u_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot v| \in L_1^1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, u_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot \boldsymbol{v}\, dt \to \int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, u, \dot{\boldsymbol{u}}) \cdot \boldsymbol{v}\, dt \tag{20}$$

which contradicts the inequality (19). This completes the proof of step 2.

Step 3. We will prove (18). The proof follows similar lines as [**?**, Thm. 1.4]. For $u \in \mathcal{E}^{\Phi}_d(\lambda)$ and $\mathbf{0} \neq v \in W^1L^{\Phi}_d$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), \dot{\boldsymbol{u}}(t) + s\dot{\boldsymbol{v}}(t)).$$

From [**?**, Lemma 10.1] (or [**?**, Thm. 5.5]) we obtain that if $|u| \leqslant |v|$ then $d(u, E^{\Phi}) \leqslant d(v, E^{\Phi})$. Therefore, for $|s| \leqslant s_0 := \left(\lambda - d(\dot{\boldsymbol{u}}, E^{\Phi})\right)/\|v\|_{W^1L^{\Phi}}$ we have

$$d\left(\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}, E^{\Phi}\right) \leqslant d\left(|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}|, E^{\Phi}_1\right) \leqslant d\left(|\dot{\boldsymbol{u}}|, E^{\Phi}_1\right) + s\|\dot{\boldsymbol{v}}\|_{L^{\Phi}} < \lambda.$$

Thus $\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}} \in \Pi(E^{\Phi}, \lambda)$ and $|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}| \in \Pi(E^{\Phi}_1, \lambda)$. These facts imply, in virtue of Theorem **??** item **??**, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$. And, using Corollary **??**, we also see that

$$\|a(|u + sv|)\|_{L^{\infty}} \leqslant A(\|u + sv\|_{W^1L^{\Phi}}) \leqslant A(\|u\|_{W^1L^{\Phi}} + s_0\|v\|_{W^1L^{\Phi}}) =: M$$

Now, applying Chain Rule, (**??**), (**??**) the monotonicity of φ and Φ, the fact that $v \in L^{\infty}_d$ and $\dot{\boldsymbol{v}} \in L^{\Phi}$ and Hölder's inequality, we get

$$\begin{aligned} |D_s H(s,t)| &= |D_{\boldsymbol{x}}\mathcal{L}(t, u + sv, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot v + D_{\boldsymbol{y}}\mathcal{L}(t, u + sv, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \dot{\boldsymbol{v}}| \\ &\leqslant M\left[\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|v| \right. \\ &\qquad \left. + \left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|\dot{\boldsymbol{v}}|\right] \in L^1_1. \end{aligned} \tag{21}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v\rangle = \frac{d}{ds}I(u + sv)\Big|_{s=0} = \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}(t, u, \dot{\boldsymbol{u}}) \cdot v + D_{\boldsymbol{y}}\mathcal{L}(t, u, \dot{\boldsymbol{u}}) \cdot \dot{\boldsymbol{v}}\}\ dt.$$

Moreover, from (**??**), (**??**), Lemma 2.2 and the previous formula, we obtain

$$|\langle I'(u), v\rangle| \leqslant \|D_{\boldsymbol{x}}\mathcal{L}\|_{L^1}\|v\|_{L^{\infty}} + \|D_{\boldsymbol{y}}\mathcal{L}\|_{L^{\Phi^*}}\|\dot{\boldsymbol{v}}\|_{L^{\Phi}} \leqslant C\|v\|_{W^1L^{\Phi}}$$

with a appropriate constant C. This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : \mathcal{E}^{\Phi}_d(\lambda) \to \left[W^1L^{\Phi}_d\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, u, \dot{\boldsymbol{u}})$ and $u \mapsto D_{\boldsymbol{y}}\mathcal{L}(t, u, \dot{\boldsymbol{u}})$. Indeed, if $u_n, u \in \mathcal{E}^{\Phi}_d(\lambda)$ with $u_n \to u$ in the norm of $W^1L^{\Phi}_d$ and $v \in W^1L^{\Phi}_d$, then

$$\begin{aligned} \langle I'(u_n), v\rangle &= \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t, u_n, \dot{\boldsymbol{u}}_n\right) \cdot v + D_{\boldsymbol{y}}\mathcal{L}\left(t, u_n, \dot{\boldsymbol{u}}_n\right) \cdot \dot{\boldsymbol{v}}\}\ dt \\ &\to \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t, u, \dot{\boldsymbol{u}}\right) \cdot v + D_{\boldsymbol{y}}\mathcal{L}\left(t, u, \dot{\boldsymbol{u}}\right) \cdot \dot{\boldsymbol{v}}\}\ dt \\ &= \langle I'(u), v\rangle. \end{aligned}$$

In order to prove item **??**, it is necessary to see that the maps $u \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, u, \dot{\boldsymbol{u}})$ and $u \mapsto D_{\boldsymbol{y}}\mathcal{L}(t, u, \dot{\boldsymbol{u}})$ are norm continuous from $\mathcal{E}^{\Phi}_d(\lambda)$ into L^1_d and $L^{\Phi^*}_d$ respectively.

The continuity of the first map has already been proved in step 1. Let $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$. Therefore, there exist a subsequence $u_{n_k} \in \mathcal{E}_d^\Phi(\lambda)$ and a function $h \in \Pi(E_1^\Phi, \lambda)$ such that (**??**) holds true. And, as $\Phi^* \in \Delta_2$ then the right hand side of (**??**) belongs to $E_1^{\Phi^*}$. Now, invoking Lemma **??**, we prove that from any sequence u_n which converges to u in $W^1L_d^\Phi$ we can extract a subsequence such that $D_{\boldsymbol{y}}\mathcal{L}(t, u_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t, u, \dot{\boldsymbol{u}})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of I' follows from the continuity of $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ using the formula (18). □

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