# Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

Sonia Acinas *
Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

**Abstract**

## 1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) \quad \text{a.e. } t \in (0,T), \\ u(0)-u(T) = u'(0)-u'(T) = 0, \end{cases} \tag{$\boldsymbol{P}$}$$

where the function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$ (called the *Lagrange function* or *lagrangian*) satisfying that it is measurable in $t$ for each $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in $(x,y)$ for almost every $t \in [0,T]$. The unknown function $u : [0,T] \to \mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the articles [15] for definitions and main results on anisotropic Orlicz spaces, see also [2]. These spaces allow us to unify and extend previous results on existences of solutions for systems like ($\boldsymbol{P}$).

---

* SECyT-UNRC and FCEyN-UNLPam
† SECyT-UNRC, FCEyN-UNLPam and CONICET
**2010 AMS Subject Classification.** Primary: . Secondary: .
**Keywords and phrases.** .

Through this article we say that a function $\Phi : \mathbb{R}^d \to [0, +\infty)$ is of $N_\infty$ class if $\Phi$ is convex, $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$ and $\Phi(-y) = \Phi(y)$, and

$$\lim_{|y|\to\infty} \frac{\Phi(y)}{|y|} = +\infty. \tag{1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From [6, Cor. 2.35] a $N_\infty$ function is continuous.

Associated to $\Phi$ we have the *complementary function* $\Psi$ which is defined in $\xi \in \mathbb{R}^d$ as

$$\Psi(\xi) = \sup_{y\in\mathbb{R}^d} y \cdot \xi - \Phi(y) \tag{2}$$

then, from the continuity of $\Phi$ and (1), we have that $\Psi : \mathbb{R}^d \to [0, \infty)$. Moreover, it is easy to see that $\Psi$ is a convex function such that $\Psi(0) = 0$, $\Psi(-\xi) = \Psi(\xi)$ [10, Chapter 2]. Moreover $\Psi$ satisfies (1) (see [15, Th. 2.2]). i.e. $\Psi$ is $N_\infty$ function.

Some examples of $N_\infty$ functions are the following.

*Example 1.1.* $\Phi_p(y) := |y|^p/p$, for $1 < p < \infty$. In this case $\Psi(\xi) = |\xi|^q/q$, $q = p/(p-1)$.

*Example 1.2.* If $\Phi : \mathbb{R} \to [0, +\infty)$ is a $N_\infty$ function on $\mathbb{R}$ then $\overline{\Phi}(y) = \Phi(|y|)$ is a $N_\infty$ function on $\mathbb{R}^d$. In this example, as in the previous one, the function $\Phi$ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of $y$ and not on its direction. These cases are not authentically anisotropic.

*Example 1.3.* An anisotropic function $\Phi(y)$ depends on the direction of $y$. For example, if $1 < p_1, p_2 < \infty$, we define $\Phi_{p_1,p_2} : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ by

$$\Phi_{p_1,p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then $\Phi_{p_1,p_2}$ is a $N_\infty$ function. In this case the complementary function is the function $\Phi_{q_1,q_2}$ with $q_i = p_i/(p_i - 1)$.

More generally, if $\Phi_k : \mathbb{R}^d \to [0, +\infty)$, $k = 1, \ldots, n$, are $N_\infty$ functions, then $\Phi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to [0, +\infty)$ defined by $\Phi(y_1, \ldots, y_n) = \Phi_1(y_1) + \cdots + \Phi_n(y_n)$ is a $N_\infty$ function. These functions are truly anisotropic, i.e. $|x| = |y|$ does not imply that $\Phi(x) = \Phi(y)$.

*Example 1.4.* If $\Phi : \mathbb{R} \to [0, +\infty)$ is a $N_\infty$ function and $O \in GL(d, \mathbb{R})$, then $\Phi(y) = \Phi(Oy)$ is a $N_\infty$ function.

*Example 1.5.* An anisotropic $N_\infty$ function is not necessarily controlled by powers if it does not satisfy the $\Delta_2$ condition (see xxxxx). For example $\Phi : \mathbb{R}^d :\to [0, +\infty)$ defined by $\Phi(y) = \exp(|y|) - 1$ is $N_\infty$ function.

The occurrence of Orlicz Spaces in this paper obeys to we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi\left(\frac{\nabla_y \mathcal{L}}{\lambda}\right) \leqslant a(x) \left\{ b(t) + \Phi\left(\frac{y}{\Lambda}\right) \right\}, \tag{$\boldsymbol{S}$}$$

for a.e. $t \in [0, T]$, where $a \in C\left(\mathbb{R}^d, [0, +\infty)\right)$, $b \in L^1\left([0, T], [0, +\infty)\right)$ and $\Lambda, \lambda > 0$.

Our condition ($\boldsymbol{S}$) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition ($\boldsymbol{S}$) is equivalent to the structure condition in [10, Th. 1.4]. If $\Phi$ is a radial $N_\infty$ function such that $\Psi$ satisfies that $\Delta_2$ function then ($\boldsymbol{S}$) is essentially equivalent????? to conditions [1, Eq. (2)-(4)] (see xxxx mas abajo). If $\Phi$ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ then inequality ($\boldsymbol{S}$) is related to estructure conditions like [21, Lemma 3.1, Eq. (3.1)]. As can be seen, condition ($\boldsymbol{S}$) is a more compact expression than [21, Lemma 3.1, Eq. (3.1)] and moreover weaker, because ($\boldsymbol{S}$) does not imply a control of $|D_{y_1}L|$ independent of $y_2$. We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi,F}(t,x,y) := \Phi(y) + F(t,x). \tag{3}$$

Here the function $F(t,x)$, which is often referred to potential, be differentiable with respect to $x$ for a.e. $t \in [0,T]$. Moreover $F$ satisfies the following conditions:

**(C)** $F$ and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

**(A)** For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla_x F(t,x)| \leqslant a(x)b(t). \tag{4}$$

where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$ and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi,F}$ satisfies condition ($\boldsymbol{S}$). In order to prove this, the only non trivial fact that we should to establishis is that $\Psi(\nabla_y\mathcal{L}) \leqslant a(x)\left\{b(t) + \Phi\left(y/\lambda\right)\right\}$. But, from inequality xxxx below, $\Psi(\nabla_y\mathcal{L}) = \Psi\left(\nabla\Phi(y)\right) \leqslant \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi,F}$ leads to the system

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{$\boldsymbol{P_\Phi}$}$$

Problem ($\boldsymbol{P_\Phi}$) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [10] deals mainly with problem ($\boldsymbol{P}$), for the lagrangian $\mathcal{L}_{\Phi,F}$, with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [10] were extended and improved in several articles, see [19, 17, 23, 18, 26] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [21, 20], among other papers, and in this case ($\boldsymbol{P_\Phi}$) is reduced to the $p$-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{$\boldsymbol{P_p}$}$$

If $\Phi$ is as in Example 1.3 and $F : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, then the equations ($\boldsymbol{P_\Phi}$) become

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) & \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}, \tag{$\boldsymbol{P_{p_1,p_2}}$}$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as $(p_1, p_2)$-Laplacian system, see [25, 14, 24, 11, 12, 13, 8].

In conclusion, the problem ($\boldsymbol{P}$) with conditions ($\boldsymbol{S}$) contains several problems that have been considered by many authors in the past. Moreover, our results still improve some results on $(p_1, p_2)$-lamplacian since our structure conditions are less restrictive even in that particular case.

# 2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic $N_\infty$ functions $\Phi : \mathbb{R}^n \to [0, +\infty)$. References for these topics are [7, 15, 16, 3, 5, 2, 22].

If $\Phi$ is a $N_\infty$ function then from convexity and $\Phi(0) = 0$ we obtain that

$$\Phi(\lambda x) \leqslant \lambda \Phi(x), \quad \lambda \in [0, 1], x \in \mathbb{R}^d. \tag{5}$$

One of the greatest difficulties when dealing with anisotropic Orlicz spaces is the lack of monotony with respect to the Euclidean norm, i.e. $|x| \leqslant |y|$ does not imply $\Phi(x) \leqslant \Phi(y)$. This problem is avoided if we consider functions whose values on a sphere are comparable (see[16]). However, from (5), we see that $N_\infty$ functions have the following form of radial monotony: $|x| \leqslant |y|$ and $y = \lambda x$ imply $\Phi(x) \leqslant \Phi(y)$.

We say that $\Phi : \mathbb{R}^d \to [0, +\infty)$ satisfies the $\Delta_2^\infty$-*condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant K\Phi(x), \tag{6}$$

for every $|x| \geqslant M$. If $\Phi$ es a $\Delta_2$ function then $\Phi$ is bounded by powers functions (see [7, Proof Lemma 2.4] and [4, Prop. 1]), i.e. there exists $1 < p < \infty$, $C > 0$ and $r \geqslant 0$ such that

$$\Phi(x) \leqslant C|x|^p, \quad |x| \geqslant r_0.$$

We consider that one of the most important aspects in considering $N_\infty$ functions is that it accounts for the Lagrange functions that present faster growth than powers, for example an exponential growth. Hence we consider it important to avoid imposing hypothesis that $\Phi$ to be $\Delta_2$. For some results we will need that $\Psi$ to be $\Delta_2$.

Let $\Phi_1$ and $\Phi_2$ be $N_\infty$ functions. Following to [22] we write $\Phi_1 \dashv \Phi_2$ if there exists non negative numbers $k$ and $C$ such that

$$\Phi_1(x) \leqslant C + \Phi_2(kx), \quad x \in \mathbb{R}^d. \tag{7}$$

For example if $\Phi$ is $\Delta_2$ then there exist $p \in (1, +\infty)$ such that $\Phi \dashv |x|^p$. If for every $k > 0$ there exists $C = C(k) > 0$ such that (7) holds we write $\Phi_1 \lll \Phi_2$.

If $\Phi_1 \dashv \Phi_2$ then $\Psi_2 \dashv \Psi_1$ as the following simple computation proves

$$\begin{aligned}
\Psi_1(k\xi) &\geqslant \sup \{k\xi \cdot x - \Phi_2(kx) - C\} \\
&= \sup \{\xi \cdot x - \Phi_2(x)\} - C \\
&= \Psi_2(\xi) - C.
\end{aligned}$$

As a consequence of the previous result, we obtain that if a Lagrange function $\mathcal{L}$ satisfies structure condition ($\boldsymbol{S}$) and $\Phi \dashv \Phi_0$ then $\mathcal{L}$ satisfies ($\boldsymbol{S}$) with $\Phi_0$ instead to $\Phi$ with other functions $b$, $a$ and constant $\Lambda$ and $\lambda$.

We denote by $\mathcal{M} := \mathcal{M}\left([0,T],\mathbb{R}^d\right)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1,\ldots,u_d)$ for $u \in \mathcal{M}$.

Given an $N_\infty$ function $\Phi$ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi\left([0,T],\mathbb{R}^d\right)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{8}$$

The *Orlicz space* $L^\Phi = L^\Phi\left([0,T],\mathbb{R}^d\right)$ is the linear hull of $C^\Phi$; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists\lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{9}$$

The Orlicz space $L^\Phi$ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi\left([0,T],\mathbb{R}^d\right)$ is defined as the closure in $L^\Phi$ of the subspace $L^\infty\left([0,T],\mathbb{R}^d\right)$ of all $\mathbb{R}^d$-valued essentially bounded functions. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [15, Cor. 5.1]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [15, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^\Psi}. \tag{10}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between $u$ and $v$.

We consider the subset $\Pi(E^\Phi, r)$ of $L^\Phi$ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class $C^\Phi$ by the following inclusions

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{11}$$

for any positive $r$. This relation is a trivial generalization of [15, Thm. 5.6]. If $\Phi \in \Delta_2^\infty$, then the sets $L^\Phi$, $E^\Phi$, $\Pi(E^\Phi, r)$ and $C^\Phi$ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of $X$, we write $Y \hookrightarrow X$ and we say that $Y$ is *embedded* in $X$ when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow \left[L^\Psi\right]^*$, where a function $v \in L^\Phi$ is associated to $\xi_v \in \left[L^\Psi\right]^*$ being

$$\langle \xi_v, u\rangle = \int_0^T v \cdot u \, dt, \tag{12}$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [15].

**Proposition 2.1.** *If $\Psi$ satisfies the $\Delta_2^\infty$-condition then $L^\Phi\left([0,T],\mathbb{R}^d\right) = \left[L^\Psi\left([0,T],\mathbb{R}^d\right)\right]^*$.*

Consequently if $\Psi$ satisfies the $\Delta_2^\infty$-condition then $L^\Phi\left([0,T],\mathbb{R}^d\right)$ can be equipped with the weak* topology.

We define the *Sobolev-Orlicz space* $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$ by

$$W^1L^\Phi\left([0,T],\mathbb{R}^d\right) := \left\{u | u \in AC\left([0,T],\mathbb{R}^d\right) \text{ and } u' \in L^\Phi\left([0,T],\mathbb{R}^d\right)\right\},$$

where $AC\left([0,T],\mathbb{R}^d\right)$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $[0,T]$. The space $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{13}$$

Anisotropic Sobolev-Orlicz spaces were treated in [3, 5, 2, 22]. Usually functions in Sobolev spaces are required to be weakly differentiable. In the particular and simplest case of functions of a variable, the weak differentiability implies absolute continuity. Hence we can assume $u \in AC\left([0,T],\mathbb{R}^d\right)$ for functions $u \in W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$.

As is well known, an active research topic in mathematical analysis are the Sobolev and Poincare inequalities. This topic have also been treated in the framework of Anisotropic Orlicz-Sobolev mainly in [3, 5, 22] for several variables functions and in [2] for functions of one single variable, $\Phi$ and $\Psi$ functions of $\Delta_2^\infty$ class. We do not know a reference for the embedding of Sobolev-Orlicz anisotropic spaces in the space of continuous functions when $\Phi$ or $\Psi$ are not $\Delta_2^\infty$. Below we present the results that we will require in this article and we show in detail the case of the incrustation in the space of continuous functions in the simple case of function of one variable.

In order to find a modulus of continuity for functios in $W^1L^\Phi$, and from there, to obtain compact embedding of $W^1L^\Phi$, we define the function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, |x| = s\right\}, \tag{14}$$

Let us establish some elementary properties of $A_\Phi$.

**Proposition 2.2.** *The function $A_\Phi$ has the following properties:*

1. *$A_\Phi$ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *$A_\Phi(|x|)$ is the* greatest radial minorant *of $\Phi(x)$,*
4. *$\Phi$ is $N_\infty$ if and only if $\lim_{s\to+\infty} A_\Phi(s)/s = +\infty$.*

*Proof.* It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [6]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of $A_\Phi$ and the convexity of $\Phi$,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi\left(x\right)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

*Example 2.1.* Let $\Phi = \Phi_{p_1,p_2}$ be the function given in Example (1.3). We show that

$$K \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\} \leqslant A_\Phi(r) \leqslant \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\}$$

for some $K > 0$, for every $1 < p_1, p_2 < \infty$. The second inequality follows directly from definition of $A_\Phi$. For the first inequality, we note that $|(y_1, y_2)| = r$ implies that $|y_1| \geqslant r/2$ or $|y_2| \geqslant r/2$. Then

$$\Phi_{p_1,p_2}(y_1, y_2) \geqslant \min\{2^{-p_1}, 2^{-p_2}\} \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\}. \tag{15}$$

Let us in a little digression to show that

$$A_\Phi(r) = \min\left\{\frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2}\right\},$$

when $1 < p_1, p_2 \leqslant 2$. We apply the method of Lagrange multipliers (see [9, Ch. 11]) to solve the next minimization problem subject to constraints

$$\left\{\begin{array}{l} \text{minimize } \Phi_{p_1,p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{array}\right. .$$

The first order conditions are

$$\left\{\begin{array}{rcl} |y_1|^{p_1-2}y_1 + \lambda y_1 & = & 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 & = & 0 \\ |y_1|^2 + |y_2|^2 & = & r^2 \end{array}\right. \tag{16}$$

These equations are solved, among others, by the following two sets of citical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) $r^{p_1}/p_1$ and b) $r^{p_2}/p_2$.

If $(y_1, y_2)$ solve (16) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1, y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let $L$ be the Lagrangian associated to the constrained problem: $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) +$

$\lambda H(y_1, y_2)$ being $H = 0$ the constraint. We must analize the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2 H$ on the subspace $M$. By elementary computations we have for $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace $M$. We can assume that $p_1 < 2$ or $p_2 < 2$, otherwise the statement we intend to prove would be trivial. Under this assumption, we note that $(-y_2, y_1) \in M$ and $(-y_2, y_1)^t \mathcal{H}(-y_2, y_1) < 0$. Then, by second order necessary conditions [9, p.333], there cannot be a minimum at $(y_1, y_2)$. Therefore follows (15).

As is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

**Lemma 2.3.** *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1 L^\Phi\left([0,T], \mathbb{R}^d\right)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (14). *Then*

1. *For every $s, t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s - t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty\left([0,T], \mathbb{R}^d\right)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If $\Phi$ is $N_\infty$ then the space $W^1 L^\Phi\left([0,T], \mathbb{R}^d\right)$ is compactly embedded in the space of continuous functions $C\left([0,T], \mathbb{R}^d\right)$.*

*Proof.* By the absolutely continuity of $u$, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s-t|}\right) \leqslant \Phi\left(\frac{1}{|s-t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr\right) \leqslant \frac{1}{|s-t|} \int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leqslant \frac{1}{|s-t|}.$$

By Proposition 2.2(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned}|u(t) - \overline{u}| &= \left|\frac{1}{T}\int_0^T u(t) - u(s)\,ds\right|\\ &\leqslant \frac{1}{T}\int_0^T |u(t) - u(s)|\,ds\\ &\leqslant \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right)\end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.2(3)

$$|\overline{u}| \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1, T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence $u_n$ in $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$. Since $\Phi$ is $N_\infty$, from Proposition 2.2(4) we obtain $sA_\Phi^{-1}(1/s) \to 0$ when $s \to 0$. Therefore (Morrey's inequality) implies that $u_n$ are equicontinuous. Furthermore (**??**) implies that $u_n$ is bounded in $C\left([0,T],\mathbb{R}^d\right)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence $n_k$ and $u \in C\left([0,T],\mathbb{R}^d\right)$ with $u_{n_k} \to u$ in $C\left([0,T],\mathbb{R}^d\right)$. □

**Lemma 2.4.** *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi, 1)$ converging to $u \in \Pi(E^\Phi, 1)$ in the $L^\Phi$-norm. Then, there exist a subsequence $u_{n_k}$ and a real valued function $h \in L^1([0,T],\mathbb{R})$ such that $u_{n_k} \to u$ a.e. and $\Phi(u_{n_k}) \leqslant h$ a.e.*

*Proof.* Since $d(u, E^\Phi) < 1$ and $u_n$ converges to $u$, there exists $u_0 \in E^\Phi$, a subsequence of $u_n$ (again denoted $u_n$) and $0 < r < 1$ such that $d(u_n, u_0) < r$. Let $\lambda_0 \in (r, 1)$. By extracting more subsequences, if necessary, we can assume that $u_n \to u$ a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^\Phi} < \frac{1-\lambda_0}{2^n}, \quad \text{for } n \geqslant 1.$$

We can assume $\lambda_n > 0$ for every $n = 0, \ldots$.

Let $\lambda := 1 - \sum_{n=0}^{\infty} \lambda_n$ and define $h : [0,T] \to \mathbb{R}$ by

$$h(x) = \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^{\infty} \lambda_n \Phi\left(\frac{u_{n+1} - u_n}{\lambda_n}\right). \tag{17}$$

Note that $\sum_{n=0}^{\infty} \lambda_n + \lambda = 1$, therefore for any $n = 1, \dots$

$$\begin{aligned} \Phi(u_n) &= \Phi\left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j}\right) \\ &\leqslant \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1} \lambda_j \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) \leqslant h \end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and $E^\Phi$ is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0,T],\mathbb{R})$. On the other hand $\|u_{n+1} - u_n\|_{L^\Phi} \leqslant \lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) dt \leqslant 1.$$

Then $h \in L^1([0,T],\mathbb{R})$.

□

# 3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

Given a continuous function $a \in C(\mathbb{R}^n,\mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

**Lemma 3.1.** *If $a \in C(\mathbb{R}^d,\mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

*Proof.* Let $A \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\|\leqslant s, x\in\mathbb{R}^d} |a(x)|$. If $u \in W^1L_d^\Phi$ then, by Sobolev's inequality, for a.e. $t \in [0,T]$

$$a(u(t)) \leqslant \alpha(\|u\|_{L^\infty}) \leqslant \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\right) =: A(\|u\|_{W^1L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t))\, dt. \tag{18}$$

**Theorem 3.2.** *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying ($\boldsymbol{S}$). Then the following statements hold:*

1. *The action integral given by* (18) *is finitely defined on $\mathcal{E}^{\Phi} := W^1L^{\Phi} \cap \{u|\dot{u} \in \Pi(E^{\Phi}, 1)\}$.*
2. *The function $I$ is Gâteaux differentiable on $\mathcal{E}^{\Phi}$ and its derivative $I'$ is demicontinuous from $\mathcal{E}^{\Phi}$ into $\left[W^1L^{\Phi}\right]^*$. Moreover, $I'$ is given by the following expression*
$$\langle I'(u), v\rangle = \int_0^T \left\{D_x\mathcal{L}\big(t, u, \dot{u}\big)\cdot v + D_y\mathcal{L}\big(t, u, \dot{u}\big)\cdot \dot{v}\right\}\, dt. \tag{19}$$
3. *If $\Psi \in \Delta_2$ then $I'$ is continuous from $\mathcal{E}^{\Phi}$ into $\left[W^1L^{\Phi}\right]^*$ when both spaces are equipped with the strong topology.*

*Proof.* Let $u \in \mathcal{E}^{\Phi}$. As
$$\dot{u} \in \Pi(E^{\Phi}, 1) \subset C_1^{\Phi} \tag{20}$$
and (11), then $\Phi(\dot{u}(t)) \in L^1$. Now,
$$|\mathcal{L}(\cdot, u, \dot{u})| + |D_x\mathcal{L}(\cdot, u, \dot{u})| + \Psi(D_y\mathcal{L}(\cdot, u, \dot{u})) \leqslant A(\|u\|_{W^1L^{\Phi}})(b + \Phi(\dot{u})) \in L^1, \tag{21}$$
by ($\boldsymbol{S}$) and Lemma 3.1. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

*Step 1. The non linear operator $u \mapsto D_x\mathcal{L}(t, u, \dot{u})$ is continuous from $\mathcal{E}^{\Phi}$ into $L^1([0, T])$ with the strong topology on both sets.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^{\Phi}$ and let $u \in \mathcal{E}^{\Phi}$ such that $u_n \to u$ in $W^1L^{\Phi}$. By (Sobolev's inequality), we have
$$|u_n(t) - u(t)| \leqslant TA_{\Phi}^{-1}\left(\frac{1}{T}\right)\|u_n - u\|_{L^{\Phi}}$$
then $u_n \to u$ uniformly. As $\dot{u}_n \to \dot{u} \in \mathcal{E}^{\Phi}$, by Lemma 2.4, there exist a subsequence of $\dot{u}_{n_k}$ (again denoted $\dot{u}_{n_k}$) and a function $h \in L^1([0, T], \mathbb{R})$ such that $\dot{u}_{n_k} \to \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leqslant h$ a.e.

Since $u_{n_k}$, $k = 1, 2, \ldots$, is a strong convergent sequence in $W^1L^{\Phi}$, it is a bounded sequence in $W^1L^{\Phi}$. According to item (3) of Lemma 2.3, there exists $M > 0$ such that $\|\boldsymbol{a}(u_{n_k})\|_{L^\infty} \leqslant M$, $k = 1, 2, \ldots$. From the previous facts and (21), we get
$$|D_x\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leqslant a(|u_{n_k}|)(b + \Phi(\dot{u}_{n_k})) \leqslant M(b + h) \in L^1.$$
On the other hand, by the continuous differentiability of $\mathcal{L}$, we have
$$D_x\mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \to D_x\mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0, T].$$
Applying the Dominated Convergence Theorem we conclude the proof of step 1.

*Step 2. The non linear operator $u \mapsto D_y\mathcal{L}(t, u, \dot{u})$ is continuous from $\mathcal{E}^{\Phi}$ with the strong topology into $\left[L^{\Phi}\right]^*$ with the weak* topology.*

Let $u \in \mathcal{E}^\Phi$. From (21) it follows that

$$D_y\mathcal{L}(\cdot, u, \dot{u}) \in C^\Psi. \tag{22}$$

Así? o conviene poner la cota de $\Psi(D_y)$ explícitamente???

Note that (21), (22) and the imbeddings $W^1L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$ imply that the second member of (19) defines an element of $\left[W^1L^\Phi\right]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in the norm of $W^1L^\Phi$. We must prove that $D_y\mathcal{L}(\cdot, u_n, \dot{u}_n) \overset{w^*}{\rightharpoonup} D_y\mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle D_y\mathcal{L}(\cdot, u_n, \dot{u}_n), v\rangle - \langle D_y\mathcal{L}(\cdot, u, \dot{u}), v\rangle| \geqslant \epsilon. \tag{23}$$

We have $u_n \to u$ in $L^\Phi$ and $\dot{u}_n \to \dot{u}$ in $L^\Phi$. By Lemma 2.4, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T], \mathbb{R})$ such that $u_n \to u$ uniformly, $\dot{u}_n \to \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leqslant h$ a.e. As in the previous step, since $u_n$ is a convergent sequence, Lemma 3.1 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (21) with $u_n$ instead of $u$, we have

$$\Psi(D_y\mathcal{L}(\cdot, u_n, \dot{u}_n)) \leqslant M(b + h) \in L^1. \tag{24}$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (24), we have

$$\begin{aligned} &\lambda D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ &\leqslant \lambda\left[\Psi(D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right)\right] \\ &\leqslant \lambda M(b + h) + \lambda\Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \tag{25}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y\mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v\, dt \to \int_0^T D_y\mathcal{L}(t, u, \dot{u}) \cdot v\, dt \tag{26}$$

which contradicts the inequality (23). This completes the proof of step 2.

*Step 3.* We will prove (19). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1L^\Phi$, we define the function

$$H(s,t) \coloneqq \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leqslant s_0 \coloneqq \min\{\left(1 - d(\dot{u}, E^\Phi)\right)/\|v\|_{W^1L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d\left(\dot{u} + s\dot{v}, E^\Phi\right) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.2 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$.

We also have $\|u + sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 3.1, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leqslant M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leqslant s_0\lambda^{-1}$, from the convexity and the parity of $\Phi$, we get

$$\Phi(\dot{u} + s\dot{v}) = \Phi\left((1-s_0)\frac{\dot{u}}{1-s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leqslant (1-s_0)\Phi\left(\frac{\dot{u}}{1-s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right)$$
$$\leqslant (1-s_0)\Phi\left(\frac{\dot{u}}{1-s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1-s_0}, E^\Phi\right) = \frac{1}{1-s_0}d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1-s_0} \in C^\Phi$.

Now, applying (21), (25), the fact that $v \in L^\infty$ and $\dot{v} \in L^\Phi$, we get

$$\begin{aligned}
|D_sH(s,t)| &= \left|D_x\mathcal{L}(t, u+sv, \dot{u}+s\dot{v})\cdot v + \lambda D_y\mathcal{L}(t, u+sv, \dot{u}+s\dot{v})\cdot\frac{\dot{v}}{\lambda}\right| \\
&\leqslant M\left[b(t) + \Phi(\dot{u}+s\dot{v})\right]|v| \\
&\quad + \lambda\left[\Psi(D_y\mathcal{L}(t, u+sv, \dot{u}+s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right)\right] \\
&\leqslant M\left\{\left[b(t) + \Phi(\dot{u}+s\dot{v})\right]|v|\right\} + \lambda M[b(t) + \Phi(\dot{u}+s\dot{v})] + \lambda\Phi\left(\frac{\dot{v}}{\lambda}\right) \\
&= M[b(t) + \Phi(\dot{u}+s\dot{v})](|v|+\lambda) + \lambda\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1.
\end{aligned} \tag{27}$$

Consequently, $I$ has a directional derivative and

$$\langle I'(u), v\rangle = \frac{d}{ds}I(u+sv)\Big|_{s=0} = \int_0^T \left\{D_x\mathcal{L}(t,u,\dot{u})\cdot v + D_y\mathcal{L}(t,u,\dot{u})\cdot\dot{v}\right\}\,dt.$$

Moreover, from the previous formula, (21), (22), and Lemma 2.3, we obtain

$$|\langle I'(u), v\rangle| \leqslant \|D_x\mathcal{L}\|_{L^1}\|v\|_{L^\infty} + \|D_y\mathcal{L}\|_{L^\Psi}\|\dot{v}\|_{L^\Phi} \leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant $C$.

This completes the proof of the Gâteaux differentiability of $I$.

*Step 4. The operator $I' : \mathcal{E}^\Phi \to \left[W^1L^\Phi_d\right]^*$ is demicontinuous.* This is a consequence of the continuity of the mappings $u \mapsto D_x\mathcal{L}(t,u,\dot{u})$ and $u \mapsto D_y\mathcal{L}(t,u,\dot{u})$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \to u$ in the norm of $W^1L^\Phi$ and $v \in W^1L^\Phi$, then

$$\begin{aligned}
\langle I'(u_n), v\rangle &= \int_0^T \left\{D_x\mathcal{L}\left(t, u_n, \dot{u}_n\right)\cdot v + D_y\mathcal{L}\left(t, u_n, \dot{u}_n\right)\cdot\dot{v}\right\}\,dt \\
&\to \int_0^T \left\{D_x\mathcal{L}\left(t, u, \dot{u}\right)\cdot v + D_y\mathcal{L}\left(t, u, \dot{u}\right)\cdot\dot{v}\right\}\,dt \\
&= \langle I'(u), v\rangle\,.
\end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto D_x\mathcal{L}(t,u,\dot{u})$ and $u \mapsto D_y\mathcal{L}(t,u,\dot{u})$ are norm continuous from $\mathcal{E}^\Phi$ into $L^1$ and $L^\Psi$, respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de $D_x$ aquí!!!

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$.

Applying Lemma 2.4 to $\dot{u}_n$, there exists a subsequence (denoted $\dot{u}_n$ for simplicity) such that $\dot{u}_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot{u}_n) \leqslant h$ and $\dot{u}_n \to \dot{u}$ a.e.

Then, by (25) we have $\Psi(v_n) \leqslant m(t) \in L^1$ being $v_n := D_y\mathcal{L}(\cdot, u_n, \dot{u}_n)$ and $m(t) := M(b+h)$. In addition, from the continuous differentiability of $\mathcal{L}$, we have that $v_n \to v$ a.e. where $D_y\mathcal{L}(\cdot, u, \dot{u})$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \to ???$ such that $\Psi(\lambda x) \leqslant c(|\lambda|)\Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})\Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \to 0$ a.e. as $n \to \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})K\Psi(v_n) + \Psi(v)) \leqslant c(|\lambda|^{-1})K[m(t) + \Psi(v)]) \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda})\,dt \to 0$ for every $\lambda > 0$. Thus, $v_n \to v$ in $L^\Psi$.

The continuity of $I'$ follows from the continuity of $D_x\mathcal{L}$ and $D_y\mathcal{L}$ using the formula (19). $\square$

# Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

# References


[1] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, and E. Schwindt. Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698, 2015.

[2] M Chamra and J Maksymiuk. Anisotropic orlicz-sobolev spaces of vector valued functions and lagrange equations. *arXiv preprint arXiv:1702.08683*, 2017.

[3] A. Cianchi. A fully anisotropic Sobolev inequality. *Pacific J. Math.*, 196(2):283–295, 2000.

[4] A. Cianchi. Local boundedness of minimizers of anisotropic functionals. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(2):147–168, 2000.

[5] Andrea Cianchi et al. Optimal orlicz-sobolev embeddings. *Revista Matemática Iberoamericana*, 20(2):427–474, 2004.

[6] F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Graduate Texts in Mathematics. 2013.

[7] W. Desch and R. Grimmer. On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc*, (353):5095–5120, 2001.

[8] Chun Li, Zeng-Qi Ou, and Chun-Lei Tang. Periodic solutions for non-autonomous second-order differential systems with (q, p)-laplacian. *Electronic Journal of Differential Equations*, 2014(64):1–13, 2014.

[9] David G Luenberger and Yinyu Ye. *Linear and nonlinear programming*, volume 228. Springer, 2015.

[10] J. Mawhin and M. Willem. *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York, 1989.

[11] Daniel Pasca. Periodic solutions of a class of nonautonomous second order differential systems with (q, p)-laplacian. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 17(5):841–851, 2010.

[12] Daniel Paşca and Chun-Lei Tang. Some existence results on periodic solutions of nonautonomous second-order differential systems with $(q, p)$-laplacian. *Applied Mathematics Letters*, 23(3):246–251, 2010.

[13] Daniel Pasca and Chun-Lei Tang. Some existence results on periodic solutions of ordinary (q, p)-laplacian systems. *Journal of applied mathematics & informatics*, 29(1_2):39–48, 2011.

[14] Daniel Pasca and Zhiyong Wang. On periodic solutions of nonautonomous second order hamiltonian systems with $(q, p)$-laplacian. *Electronic Journal of Qualitative Theory of Differential Equations*, 2016(106):1–9, 2016.

[15] G. Schappacher. A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386, 2005.

[16] M. S. Skaff. Vector valued Orlicz spaces. II. *Pacific J. Math.*, 28(2):413–430, 1969.

[17] C.-L. Tang. Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270, 1998.

[18] C. L. Tang and X.-P. Wu. Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397, 2001.

[19] Chun-Lei Tang. Periodic solutions of non-autonomous second-order systems with $\gamma$-quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675, 1995.

[20] X. Tang and X. Zhang. Periodic solutions for second-order Hamiltonian systems with a $p$-Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113, 2010.

[21] Y. Tian and W. Ge. Periodic solutions of non-autonomous second-order systems with a $p$-Laplacian. *Nonlinear Anal.*, 66(1):192–203, 2007.

[22] Neil Trudinger. An imbedding theorem for $H^0(G, \Omega)$-spaces. *Studia Mathematica*, 50(1):17–30, 1974.

[23] X.-P. Wu and C.-L. Tang. Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235, 1999.

[24] Xiaoxia Yang and Haibo Chen. Periodic solutions for a nonlinear (q, p)-laplacian dynamical system with impulsive effects. *Journal of Applied Mathematics and Computing*, 40(1-2):607–625, 2012.

[25] Xiaoxia Yang and Haibo Chen. Existence of periodic solutions for sublinear second order dynamical system with (q, p)-laplacian. *Mathematica Slovaca*, 63(4):799–816, 2013.

[26] F. Zhao and X. Wu. Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434, 2004.