

Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

Sonia Acinas *

Instituto de Matemática Aplicada San Luis (CONICET-UNSL)
(5700) San Luis, Argentina
Universidad Nacional de La Pampa
(6300) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper we....

1 Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) = D_x \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) & \text{a.e. } t \in (0, T) \\ \mathbf{u}(0) - \mathbf{u}(T) = \dot{\mathbf{u}}(0) - \dot{\mathbf{u}}(T) = 0 \end{cases} \quad (1)$$

*SECyT-UNRC, UNSL and CONICET

†SECyT-UNRC and CONICET

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where $T > 0$, $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left(b(t) + \Phi \left(\frac{|\mathbf{y}|}{\lambda} + f(t) \right) \right), \quad (2)$$

$$|D_{\mathbf{x}} \mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left(b(t) + \Phi \left(\frac{|\mathbf{y}|}{\lambda} + f(t) \right) \right), \quad (3)$$

$$|D_{\mathbf{y}} \mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left(c(t) + \varphi \left(\frac{|\mathbf{y}|}{\lambda} + f(t) \right) \right). \quad (4)$$

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N -function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ . The non negative functions b, c and f satisfy that $b \in L_1^1([0, T])$, $c \in L_1^\Psi([0, T])$ and $f \in E_1^\Phi([0, T])$, where the Banach spaces $L_1^1([0, T])$, $L_1^\Psi([0, T])$ and $E_1^\Phi([0, T])$ will be defined later.

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(\mathbf{u}) = \int_0^T \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) dt. \quad (5)$$

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [1, 2, 3]. For Orlicz spaces of vector valued functions, see [4] and the references therein.

Hereafter we denote by \mathbb{R}^+ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called an N -function if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \text{for } t \geq 0,$$

where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t) \leq s} t$. The function ψ satisfies the same properties as the function φ , therefore we have an N -function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ .

We say that Φ satisfies the Δ_2 -condition, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leq K\Phi(t) \quad (6)$$

for every $t \geq t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2 -condition globally ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0, T])$ the set of all measurable functions defined on $[0, T]$ with values on \mathbb{R}^d and we write $\mathbf{u} = (u_1, \dots, u_d)$ for $\mathbf{u} \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in \mathbb{R}^d .

Given an N -function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\mathbf{u}) := \int_0^T \Phi(|\mathbf{u}|) dt.$$

Here $|\cdot|$ is the euclidean norm of \mathbb{R}^d . The *Orlicz class* $C_d^\Phi = C_d^\Phi([0, T])$ is given by

$$C_d^\Phi := \{\mathbf{u} \in \mathcal{M}_d \mid \rho_\Phi(\mathbf{u}) < \infty\}. \quad (7)$$

The *Orlicz space* $L_d^\Phi = L_d^\Phi([0, T])$ is the linear hull of C_d^Φ ; equivalently,

$$L_d^\Phi := \{\mathbf{u} \in \mathcal{M}_d \mid \exists \lambda > 0 : \rho_\Phi(\lambda \mathbf{u}) < \infty\}. \quad (8)$$

The Orlicz space L_d^Φ equipped with the *Orlicz norm*

$$\|\mathbf{u}\|_{L^\Phi} := \sup \left\{ \int_0^T \mathbf{u} \cdot \mathbf{v} dt \mid \rho_\Psi(\mathbf{v}) \leq 1 \right\},$$

is a Banach space. By $\mathbf{u} \cdot \mathbf{v}$ we denote the usual dot product in \mathbb{R}^d between \mathbf{u} and \mathbf{v} . The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [2, Thm. 10.5] and [5]). For every $\mathbf{u} \in L^\Phi$,

$$\|\mathbf{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(k\mathbf{u})\}. \quad (9)$$

The subspace $E_d^\Phi = E_d^\Phi([0, T])$ is defined as the closure in L_d^Φ of the subspace L_d^∞ of all \mathbb{R}^d -valued essentially bounded functions. It is shown that E_d^Φ is the only one maximal subspace contained in the Orlicz class C_d^Φ , i.e. $\mathbf{u} \in E_d^\Phi$ if and only if $\rho_\Phi(\lambda \mathbf{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [2, Th. 9.3]). Namely, if $\mathbf{u} \in L_d^\Phi$ and $\mathbf{v} \in L_d^\Psi$ then $\mathbf{u} \cdot \mathbf{v} \in L_1^1$ and

$$\int_0^T \mathbf{v} \cdot \mathbf{u} dt \leq \|\mathbf{u}\|_{L^\Phi} \|\mathbf{v}\|_{L^\Psi}. \quad (10)$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L_d^\Psi \subset [L_d^\Phi]^*$, where the pairing $\langle \mathbf{v}, \mathbf{u} \rangle$ is defined by

$$\langle \mathbf{v}, \mathbf{u} \rangle = \int_0^T \mathbf{v} \cdot \mathbf{u} dt \quad (11)$$

with $\mathbf{u} \in L_d^\Phi$ and $\mathbf{v} \in L_d^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L_d^\Psi = [L_d^\Phi]^*$ will not hold. In general, it is true that $[E_d^\Phi]^* = L_d^\Psi$.

Like in [2], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ given by

$$\Pi(E_d^\Phi, r) := \{\mathbf{u} \in L_d^\Phi \mid d(\mathbf{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \quad (12)$$

for any positive r . If $\Phi \in \Delta_2$, then the sets L_d^Φ , E_d^Φ , $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

We define the *Sobolev-Orlicz space* $W^1 L_d^\Phi$ (see [1]) by

$$W^1 L_d^\Phi := \{\mathbf{u} | \mathbf{u} \text{ is absolutely continuous and } \dot{\mathbf{u}} \in L_d^\Phi\}.$$

$W^1 L_d^\Phi$ is a Banach space when equipped with the norm

$$\|\mathbf{u}\|_{W^1 L^\Phi} = \|\mathbf{u}\|_{L^\Phi} + \|\dot{\mathbf{u}}\|_{L^\Phi}.$$

For a function $\mathbf{u} \in L_d^1([0, T])$, we write $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$ where $\bar{\mathbf{u}} = \frac{1}{T} \int_0^T \mathbf{u}(t) dt$ and $\tilde{\mathbf{u}} = \mathbf{u} - \bar{\mathbf{u}}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \rightarrow X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leq C\|y\|_Y$. With this notation, Hölder's inequality states that $L_d^\Psi \hookrightarrow [L_d^\Phi]^*$; and, it is easy to see that for every N -function Φ we have that $L_d^\infty \hookrightarrow L_d^\Phi \hookrightarrow L_d^1$.

Recall that a function $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N -function Φ . We say that $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|\mathbf{u}(t) - \mathbf{u}(s)| \leq Cw(|t - s|). \quad (13)$$

We denote by $C^w([0, T], \mathbb{R}^d)$ the space of w -Hölder continuous functions. This is the space of all functions satisfying (13) for some $C > 0$ and it is a Banach space with norm

$$\|\mathbf{u}\|_{C^w([0, T], \mathbb{R}^d)} := \|\mathbf{u}\|_{L^\infty} + \sup_{t \neq s} \frac{|\mathbf{u}(t) - \mathbf{u}(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [6, 7, 8, 9, 10]. The next simple lemma is essentially known and we will use it systematically. For the sake of completeness, we include a brief proof of it.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1 L^\Phi \hookrightarrow C^w([0, T], \mathbb{R}^d)$ and for every $\mathbf{u} \in W^1 L^\Phi$

$$|\mathbf{u}(t) - \mathbf{u}(s)| \leq \|\dot{\mathbf{u}}\|_{L^\Phi} w(|t - s|), \quad (14)$$

$$\|\mathbf{u}\|_{L^\infty} \leq \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|\mathbf{u}\|_{W^1 L^\Phi} \quad (15)$$

2. For every $\mathbf{u} \in W^1 L^\Phi$ we have $\tilde{\mathbf{u}} \in L_d^\infty$ and

$$\|\tilde{\mathbf{u}}\|_{L^\infty} \leq T\Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\mathbf{u}}\|_{L^\Phi} \quad (\text{Sobolev's inequality}). \quad (16)$$

The next result is analogous to some lemmata in $W^1 L_d^p$, see [11].

Lemma 2.2. *If $\|\mathbf{u}\|_{W^1 L^\Phi} \rightarrow \infty$, then $(|\overline{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi}) \rightarrow \infty$.*

Proof. We have

$$\|\mathbf{u}\|_{L^\Phi} = \|\overline{\mathbf{u}} + \tilde{\mathbf{u}}\|_{L^\Phi} \leq \|\overline{\mathbf{u}}\|_{L^\Phi} + \|\tilde{\mathbf{u}}\|_{L^\Phi} = \|\overline{\mathbf{u}}\|_{L^\Phi} + \|\tilde{\mathbf{u}}\|_{L^\Phi}$$

We know that Holder's inequality implies that $L_d^\infty \hookrightarrow L_d^\Phi$, that is, there exists $C > 0$ such that for any $\tilde{\mathbf{u}} \in L_d^\infty$ we have

$$\|\tilde{\mathbf{u}}\|_{L^\Phi} \leq C \|\tilde{\mathbf{u}}\|_{L^\infty}$$

and, applying Sobolev's inequality to the previous formula, we get

$$\|\tilde{\mathbf{u}}\|_{L^\Phi} \leq C \|\dot{\mathbf{u}}\|_{L^\Phi}$$

La desigualdad anterior sería del tipo Wirtinger's que no tenemos enunciada en ningún lado.

Therefore,

$$\|\mathbf{u}\|_{L^\Phi} \leq C(|\overline{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi}) \quad (17)$$

As $\|\mathbf{u}\|_{W^1 L^\Phi} = \|\mathbf{u}\|_{L^\Phi} + \|\dot{\mathbf{u}}\|_{L^\Phi}$, then

$$\|\mathbf{u}\|_{W^1 L^\Phi} \leq C(|\overline{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi})$$

and by hypothesis $\|\mathbf{u}\|_{W^1 L^\Phi} \rightarrow \infty$, then $|\overline{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi} \rightarrow \infty$. \square

Esta definición va así o requiere modificaciones/adaptaciones???

Definition 2.3. *We say that a function $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Carathéodory function if for fixed (\mathbf{x}, \mathbf{y}) the map $t \mapsto \mathcal{L}(t, \mathbf{x}, \mathbf{y})$ is measurable and for fixed t the map $(\mathbf{x}, \mathbf{y}) \mapsto \mathcal{L}(t, \mathbf{x}, \mathbf{y})$ is continuously differentiable for almost everywhere $t \in [0, T]$.*

In [12] we proved the next results.

Theorem 2.4. *Let \mathcal{L} be a Carathéodory function satisfying (2), (3) and (4). Then the following statements hold:*

1. *The action integral given by (5) is finitely defined on $\mathcal{E}_d^\Phi(\lambda) := W^1 L_d^\Phi \cap \{\mathbf{u} | \dot{\mathbf{u}} \in \Pi(E_d^\Phi, \lambda)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}_d^\Phi(\lambda)$ and its derivative I' is demi-continuous from $\mathcal{E}_d^\Phi(\lambda)$ into $[W^1 L_d^\Phi]^*$. Moreover, I' is given by the following expression*

$$\langle I'(\mathbf{u}), \mathbf{v} \rangle = \int_0^T \{D_{\mathbf{x}} \mathcal{L}(t, \mathbf{u}, \dot{\mathbf{u}}) \cdot \mathbf{v} + D_{\mathbf{y}} \mathcal{L}(t, \mathbf{u}, \dot{\mathbf{u}}) \cdot \dot{\mathbf{v}}\} dt. \quad (18)$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^\Phi(\lambda)$ into $[W^1 L_d^\Phi]^*$ when both spaces are equipped with the strong topology.*

In [12] we derived the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T -periodic functions. We denote by $W^1 L_T^\Phi$ the subspace of $W^1 L_d^\Phi$ containing all T -periodic functions. As usual, when Y is a subspace of the Banach space X , we denote by Y^\perp the *annihilator subspace* of X^* , i.e. the subspace that consists of all bounded linear functions which are identically zero on Y .

We recall that a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is called *strictly convex* if $f\left(\frac{x+y}{2}\right) < \frac{1}{2}(f(x) + f(y))$ for $x \neq y$. It is well known that if f is a strictly convex and differentiable function, then $D_x f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a one-to-one map (see, e.g. [13, Thm. 12.17]).

Theorem 2.5. *Let $\mathbf{u} \in \mathcal{E}_d^\Phi(\lambda)$ be a T -periodic function. The following statements are equivalent:*

1. $I'(\mathbf{u}) \in (W^1 L_T^\Phi)^\perp$.
2. $D_y \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t))$ is an absolutely continuous function and \mathbf{u} solves the following boundary value problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) = D_x \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) & \text{a.e. } t \in (0, T) \\ \mathbf{u}(0) - \mathbf{u}(T) = D_y \mathcal{L}(0, \mathbf{u}(0), \dot{\mathbf{u}}(0)) - D_y \mathcal{L}(T, \mathbf{u}(T), \dot{\mathbf{u}}(T)) = 0. \end{cases} \quad (19)$$

Moreover if $D_y \mathcal{L}(t, x, y)$ is T -periodic with respect to the variable t and strictly convex with respect to y , then $D_y \mathcal{L}(0, \mathbf{u}(0), \dot{\mathbf{u}}(0)) - D_y \mathcal{L}(T, \mathbf{u}(T), \dot{\mathbf{u}}(T)) = 0$ is equivalent to $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}(T)$.

DECIR ALGO DE LOS ÍNDICES ACÁ O EN LA INTRO...???

Por acá aparecía la NOTACIÓN DE $J_{C,\nu}$, que usamos en alguna demostración más adelante. Me parece que no es necesaria, sólo hay que corregir la prueba donde se la usaba!!!!

Lemma 2.6. *Let Φ and Ψ be complementary N -functions. Then:*

1. $\|\mathbf{u}\|_{L^\Phi} = O(\rho_\Phi(\mathbf{u}))$.
2. If $\Psi \in \Delta_2$ globally, then there exists a constant $\alpha_\Phi > 1$ such that, for any $0 < \mu < \alpha_\Phi$,

$$\|\mathbf{u}\|_{L^\Phi}^\mu = o(\rho_\Phi(\mathbf{u})). \quad (20)$$

Reciprocally, if (20) holds for $\mu \geq 1$ then $\Psi \in \Delta_2$.

Based on [14] we say that F satisfies the condition (A) if $F(t, \mathbf{x})$ is a Carathéodory function and F is continuously differentiable with respect to \mathbf{x} . Moreover, the next inequality holds

$$|F(t, \mathbf{x})| + |D_x F(t, \mathbf{x})| \leq a(|\mathbf{x}|)b_0(t), \quad \text{for a.e. } t \in [0, T], \forall \mathbf{x} \in \mathbb{R}^d. \quad (21)$$

3 Lagrangians with sublinear nonlinearity

Like in [12] we consider Lagrangians \mathcal{L} which are lower bounded as follows

$$\mathcal{L}(t, \mathbf{x}, \mathbf{y}) \geq \alpha_0 \Phi \left(\frac{|\mathbf{y}|}{\Lambda} \right) + F(t, \mathbf{x}). \quad (22)$$

Now, we have another result about coercivity of I assuming some conditions on the ∇F .

Theorem 3.1. *Let \mathcal{L} be a lagrangian function satisfying (2), (3), (4), (22) and F satisfies condition (A). We assume the following conditions:*

1. $\Psi \in \Delta_2$.
2. *There exist non negative functions $b_1, b_2 \in L^1_1$ and a constant $1 < \mu < \alpha_\Phi$ such that for any $\mathbf{x} \in \mathbb{R}^d$ and a.e. $t \in [0, T]$*

$$|\nabla F(t, \mathbf{x})| \leq b_1(t)|\mathbf{x}|^{\mu-1} + b_2(t). \quad (23)$$

3. *There exists a real positive number σ such that $\sigma > (\mu - 1)\beta_\Psi$ and*

$$|\mathbf{x}|^\sigma = o \left(\int_0^T F(t, \mathbf{x}) dt \right) \text{ as } |\mathbf{x}| \rightarrow \infty. \quad (24)$$

Then the action integral I is coercive.

Proof. By the decomposition $u = \bar{u} + \tilde{u}$, Mean Value Theorem, Cauchy-Schwarz inequality and (23), we have

$$\begin{aligned} \left| \int_0^T F(t, \mathbf{u}) - F(t, \bar{\mathbf{u}}) dt \right| &= \left| \int_0^T \int_0^1 \nabla F(t, \bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)) \cdot \tilde{\mathbf{u}}(t) ds dt \right| \\ &\leq \int_0^T \int_0^1 b_1(t) |\bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)|^{\mu-1} |\tilde{\mathbf{u}}(t)| ds dt + \int_0^T \int_0^1 b_2(t) |\tilde{\mathbf{u}}(t)| ds dt \\ &= I_1 + I_2. \end{aligned} \quad (25)$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate I_2 as follows

$$I_2 \leq \|b_2\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty} \leq C_1 \|\dot{\tilde{\mathbf{u}}}\|_{L^\Phi}. \quad (26)$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$. On the other hand, as $s \in [0, 1]$, we have

$$|\bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)|^{\mu-1} \leq C(\mu)(|\bar{\mathbf{u}}|^{\mu-1} + \|\tilde{\mathbf{u}}\|_{L^\infty}^{\mu-1}). \quad (27)$$

where $C(\mu) = 2^{\mu-2}$, for $\mu \geq 2$ and $C(\mu) = 1$, for $1 < \mu < 2$. Now, inequality (27), Hölder's inequality and Sobolev's inequality imply that

$$\begin{aligned} I_1 &\leq C(\mu) \left(|\bar{\mathbf{u}}|^{\mu-1} \int_0^T b_1(t) |\tilde{\mathbf{u}}(t)| dt + \|\tilde{\mathbf{u}}\|_{L^\infty}^{\mu-1} \int_0^T b_1(t) |\tilde{\mathbf{u}}(t)| dt \right) \\ &\leq C(\mu) \left\{ |\bar{\mathbf{u}}|^{\mu-1} \|b_1\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty} + \|b_1\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty}^\mu \right\} \\ &\leq C_2 \left\{ |\bar{\mathbf{u}}|^{\mu-1} \|\dot{\mathbf{u}}\|_{L^\Phi} + \|\dot{\mathbf{u}}\|_{L^\Phi}^\mu \right\}, \end{aligned} \quad (28)$$

where $C_2 = C_2(\mu, T, \|b_1\|_{L^1})$. Let μ' be a positive constant such that $1 < \mu \leq \mu' < \alpha_\Phi$. Next, using Young's inequality with conjugate exponents μ' and $\frac{\mu'}{\mu'-1}$ we get

$$|\bar{\mathbf{u}}|^{\mu-1} \|\dot{\mathbf{u}}\|_{L^\Phi} \leq \frac{(\mu'-1)}{\mu'} |\bar{\mathbf{u}}|^\sigma + \frac{1}{\mu'} \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} \quad (29)$$

where $\sigma = \frac{(\mu-1)\mu'}{\mu'-1}$. We note that σ is an arbitrary positive constant bigger than $(\mu-1)b_\Psi$.

From (28), (29), (26) and the inequality $x^{r_1} \leq x^{r_2} + 1$, for any $x \geq 0$ and $r_1 \leq r_2$ we have

$$\begin{aligned} I_1 + I_2 &\leq C_3 \left\{ |\bar{\mathbf{u}}|^\sigma + \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + \|\dot{\mathbf{u}}\|_{L^\Phi}^\mu + \|\dot{\mathbf{u}}\|_{L^\Phi} \right\} \\ &\leq C_3 \left\{ |\bar{\mathbf{u}}|^\sigma + \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + 1 \right\} \end{aligned} \quad (30)$$

with $C_3 = C_3(\mu, T, \|b_1\|_{L^1}, \mu')$. In the subsequent estimates, we use the decomposition $u = \bar{u} + \tilde{u}$, (22), (25), (30) and we get

$$\begin{aligned} I(\mathbf{u}) &\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\mathbf{u}}}{\Lambda} \right) + \int_0^T F(t, \mathbf{u}) dt \\ &= \alpha_0 \rho_\Phi \left(\frac{\dot{\mathbf{u}}}{\Lambda} \right) + \int_0^T [F(t, \mathbf{u}) - F(t, \bar{\mathbf{u}})] dt + \int_0^T F(t, \bar{\mathbf{u}}) dt \\ &\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\mathbf{u}}}{\Lambda} \right) - C_3 \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + \int_0^T F(t, \bar{\mathbf{u}}) dt - C_3 |\bar{\mathbf{u}}|^\sigma - C_3. \end{aligned} \quad (31)$$

In this way, from (31) and (32)

$$\begin{aligned} I(\mathbf{u}) &\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\mathbf{u}}}{\Lambda} \right) - C \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + \int_0^T F(t, \bar{\mathbf{u}}) dt - K |\bar{\mathbf{u}}|^\sigma - C \\ &= \alpha_0 J_{C, \mu'}(\dot{\mathbf{u}}) + \gamma(\bar{\mathbf{u}}) - C. \end{aligned}$$

Let \mathbf{u}_n be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$ and we have to prove that $I(\mathbf{u}_n) \rightarrow \infty$.

On the contrary, suppose that for a subsequence, still denoted by \mathbf{u}_n , $I(\mathbf{u}_n)$ is upper

bounded, that is, there exists $M > 0$ such that $|I(\mathbf{u}_n)| \leq M$. As $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$, from Lemma 2.2, we have $|\overline{\mathbf{u}}_n| + \|\dot{\mathbf{u}}_n\|_{L^\Phi} \rightarrow \infty$. Then, there exists subsequence of the subsequence $\{\mathbf{u}_n\}$, still denoted by \mathbf{u}_n , which is not bounded. Then, $\overline{\mathbf{u}}_n \rightarrow \infty$ or $\|\dot{\mathbf{u}}_n\|_{L^\Phi} \rightarrow \infty$. Now, as the functionals $J_{C,\mu'}(\dot{\mathbf{u}})$ and $\gamma(\overline{\mathbf{u}})$ are coercive, then $J_{C,\mu'}(\dot{\mathbf{u}}_n) \rightarrow \infty$ or $\gamma(\overline{\mathbf{u}}_n) \rightarrow \infty$. By (21), the functional $\gamma(\overline{\mathbf{u}}_n)$ is lower bounded and $J_{C,\mu'}(\dot{\mathbf{u}}_n)$ is also lower bounded on a bounded set because the modular $\rho_\Phi\left(\frac{u}{\Lambda}\right)$ is always bigger than zero. Therefore, $I(\mathbf{u}_n) \rightarrow \infty$ as $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$ which contradicts the initial assumption on the behavior of $I(\mathbf{u}_n)$. \square

REVISAR LA PRUEBA ANTERIOR Y MEJORAR LA ESCRITURA, y adaptar quitando J si fuera el caso!!!!

4 Limit case $\mu = \alpha_\Phi$

In [] coercivity was obtained even in the limit case $\mu = 1$ and $\mu = p$ assuming additional conditions on ... This was possible because in L^p spaces, the norm and the modular coincides, that is, $\|\cdot\|_p^p = O(\int_0^T |\cdot|^p dt)$. In Orlicz spaces, $\|\cdot\|_{L^\Phi}^\mu$ can be upper controlled by a modular provided that $\mu < \alpha_\Phi$ for any N -function Φ . But, the limit case does not hold for any Φ , i.e. in general $\|\cdot\|_{L^\Phi}^{\alpha_\Phi} = O(\int_0^T \Phi(|u|) dt)$ is false as can be seen as follows.

Let $\Phi, \Psi \in \Delta_2$, then the next inequality $\Phi(tu) \geq t^{\alpha_\Phi} \Phi(u)$ for any $u > 0$ and for any $t \geq 1$ is false.

$$\text{In fact, let } \Phi(u) = \begin{cases} \frac{p-1}{p} u^p & u \leq e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

Theorem 4.1. *If $p \geq \frac{1+\sqrt{2}}{2}$, then Φ is an N -function.*

Proof. We have

$$\Phi'(u) = \begin{cases} (p-1)u^{p-1} =: \varphi_1(u) & u < e \\ \frac{u^{p-1}}{\log u} (p - \frac{1}{\log u}) := \varphi_2(u) & u > e \end{cases}$$

and Φ is differentiable at e because $\varphi_1(e) = \varphi_2(e) = (p-1)e^{p-1}$.

Tendríamos que ver que φ_1, φ_2 son crecientes y que $\varphi_2 \rightarrow \infty$ cuando $u \rightarrow \infty$ o basta con ver que φ_2 es creciente????

φ_1 is an increasing function provided that $p > 1$ and $\varphi_1(u) \rightarrow 0$ as $u \rightarrow 0$.

In addition, $\varphi_2(u) \rightarrow \infty$ as $u \rightarrow \infty$ provided that $p > 1$. And

$$0 < \varphi_2'(u) = \frac{u^{p-2}}{\log u} \left(p^2 - p - \frac{2p}{\log u} + \frac{1}{\log u} + \frac{2}{\log^2 u} \right)$$

on $[e, \infty)$ if and only if

$$\left(p^2 - p - \frac{2p}{\log u} + \frac{1}{\log u} + \frac{2}{\log^2 u} \right) > 0.$$

If we take $\alpha := \frac{1}{\log u}$, then we need

$$2\alpha^2 + (1 - 2p)\alpha + (p^2 - p) \geq 0$$

which is true if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore, φ_2 is an increasing function when $p \geq \frac{1+\sqrt{2}}{2}$. \square

Theorem 4.2. *There exists a constant $C > 0$ such that*

$$\Phi(tu) \leq ct^p \Phi(u) \quad t \geq 1, u > 0. \quad (32)$$

For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, p)$ such that

$$\Phi(tu) \geq Ct^{p-\varepsilon} \Phi(u) \quad t \geq 1, u > 0. \quad (33)$$

Proof. Resumir la prueba \square

Remark 4.3. *The inequality*

$$\Phi(tu) \geq Ct^p \Phi(u)$$

is false for every C because for every $u \geq e$ we have

$$\lim_{t \rightarrow \infty} \frac{\Phi(tu)}{t^p \Phi(u)} = 0$$

Theorem 4.4. $\alpha_\Phi = \beta_\Phi = p$

Proof. Resumir la prueba. \square

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|) dx \geq C \|u\|_{L^\Phi}^{\alpha_\Phi} = C \|u\|_{L^\Phi}^p$$

is false.

If we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|) dx = C_2 \Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geq C \|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geq C t^p$ were also true but this last inequality is false.

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No sé por qué pero parece que funciona..., en realidad quité el .bib... Y recién ahora me doy cuenta que las referencias se ordenan de acuerdo al orden de mención/aparición....

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