Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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Abstract

In this paper we....

1 Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) & \text{a.e. } t \in (0, T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(T) = 0 \end{cases} \tag{1}$$

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where $T > 0$, $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\frac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{2}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\frac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{3}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(c(t) + \varphi\left(\frac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right). \tag{4}$$

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ and the non negative functions b, c and f belong to certain Banach spaces that will be introduced later.

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(\boldsymbol{u}) = \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt. \tag{5}$$

Variational problems and hamiltonian systems have been studied extensively. Classic references of these subjects are [21, 26, 13]. Problems like (1) have maintained the interest of researchers as the recent literature on the topic testifies. For lagrangian functions of the type $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{|\boldsymbol{y}|^2}{2} + F(t,\boldsymbol{x})$ many solvability conditions have been given expanding the results of [21]. In [28] the function F was split up into two potentials, one of them with a property of subadditivity and the other with a bounded gradient. In [27] it was required a certain sublinearity condition on the gradient of the potential F; and, in [32] it was considered a potential F given by a sum of a subconvex function and a subquadratic one. In [29] the uniform coercivity of $\int_0^T F(t,\boldsymbol{x})\, dt$ was replaced by local coercivity of F in some positive measure subset of $[0,T]$. In [35], the authors took a similar potential to that in [32] getting new solvability conditions and they also studied the case in which the two potentials do not have any convexity.

The Lagrangian $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{|\boldsymbol{y}|^p}{p} + F(t,\boldsymbol{x})$ for $p > 1$ was treated in more recent papers. By using the dual least action principle, in [31] it was performed the extension of some results given in [21]; and, in [30] the authors improved the work done in [32]. On the other hand, by the minimax methods in critical point theory some existence theorems were obtained. In [33] it was employed a subquadratic potential F in Rabinowitz's sense and in [34] F was taken as in [27].

Another source of problems, close to our proposal, is the one in which a *p-laplacian-like* operator is involved. Assuming that the function φ is a homeomorphism from $\mathbb{R}^d$ into itself, it is considered the differential operator $\boldsymbol{u} \mapsto (\varphi(\boldsymbol{u}'))'$. In [4, 5, 6, 19, 20], using the Leray-Schauder degree theory, some existence results of solutions of equations like $(\varphi(\boldsymbol{u}'))' = \boldsymbol{f}(t,\boldsymbol{u}(t),\boldsymbol{u}'(t))$ were obtained under different boundary conditions (periodic, Dirichlet, von Neumann) and where $\boldsymbol{f}$ is not necessarily a gradient. We point out that our approach differs from that of previous articles because we tackle the direct method of the calculus of variations.

In the Orlicz-Sobolev space setting, in [14] a constrained minimization problem associated to the existence of eigenvalues for certain differential operators involving

N-functions was studied. Slightly away from the problems to be treated in this paper, we can mention [8, 10] where A. Cianchi dealt with the regularity of minimizers of action integrals defined on several variable functions.

In this article we consider lagrangian functions defined on Orlicz-Sobolev spaces W^1L^Φ (see [3, 17, 22, 23]) and we use the direct method of calculus of variations. The exposition is organized as follows. In Section 2 we enumerate results related to Orlicz spaces, Orlicz-Sobolev spaces and composition operators. Almost all results in this section are essentially known. Conditions (2), (3) and (4) are the means to ensure that I is finitely defined on a non trivial subset of $W^1L^\Phi_d$ and I is Gâteaux differentiable in this subset. We develop these issues in Theorem 2.5 of Section **??**. In Section **??** we prove that critical points of (5) are solutions of (1). Conditions to guarantee the coercitivity of action integrals are discussed in Section 3. Finally, our main theorem about existence of solutions of (1) is introduced and proved in Section **??**.

We lay emphasis on that we use Δ_2-condition only when necessary in a certain sense (see, for example, Lemma 3.2).

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [3, 17, 22]. For Orlicz spaces of vector valued functions, see [25] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \text{for } t \geq 0,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the *Δ_2-condition*, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{6}$$

for every $t \geq t_0$. If $t_0 = 0$, we say that Φ satisfies the *Δ_2-condition globally* ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0, T])$ the set of all measurable functions defined on $[0, T]$ with values on $\mathbb{R}^d$ and we write $\boldsymbol{u} = (u_1, \ldots, u_d)$ for $\boldsymbol{u} \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_0^T \Phi(|\boldsymbol{u}|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C_d^\Phi = C_d^\Phi([0,T])$ is given by

$$C_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \rho_\Phi(\boldsymbol{u}) < \infty\}. \tag{7}$$

The *Orlicz space* $L_d^\Phi = L_d^\Phi([0,T])$ is the linear hull of C_d^Φ; equivalently,

$$L_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \exists \lambda > 0 : \rho_\Phi(\lambda \boldsymbol{u}) < \infty\}. \tag{8}$$

The Orlicz space L_d^Φ equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup \left\{ \int_0^T \boldsymbol{u} \cdot \boldsymbol{v}\, dt \Big| \rho_\Psi(\boldsymbol{v}) \leqslant 1 \right\},$$

is a Banach space. By $\boldsymbol{u} \cdot \boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^d$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [17, Thm. 10.5] and [15]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(k\boldsymbol{u})\}. \tag{9}$$

The subspace $E_d^\Phi = E_d^\Phi([0,T])$ is defined as the closure in L_d^Φ of the subspace L_d^∞ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E_d^Φ is the only one maximal subspace contained in the Orlicz class C_d^Φ, i.e. $\boldsymbol{u} \in E_d^\Phi$ if and only if $\rho_\Phi(\lambda \boldsymbol{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces. Namely, if $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$ then $\boldsymbol{u} \cdot \boldsymbol{v} \in L_1^1$ and

$$\int_0^T \boldsymbol{v} \cdot \boldsymbol{u}\, dt \leqslant \|\boldsymbol{u}\|_{L^\Phi} \|\boldsymbol{v}\|_{L^\Psi}. \tag{10}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L_d^\Psi \subset \left[L_d^\Phi\right]^*$, where the pairing $\langle \boldsymbol{v}, \boldsymbol{u} \rangle$ is defined by

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = \int_0^T \boldsymbol{v} \cdot \boldsymbol{u}\, dt \tag{11}$$

with $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L_d^\Psi = \left[L_d^\Phi\right]^*$ will not hold. In general, it is true that $\left[E_d^\Phi\right]^* = L_d^\Psi$.

Like in [17], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ given by

$$\Pi(E_d^\Phi, r) := \{\boldsymbol{u} \in L_d^\Phi | d(\boldsymbol{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \tag{12}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L_d^Φ, E_d^Φ, $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

We define the *Sobolev-Orlicz space* $W^1L_d^\Phi$ (see [3]) by

$$W^1L_d^\Phi := \{\boldsymbol{u} | \boldsymbol{u} \text{ is absolutely continuous and } \boldsymbol{u}, \dot{\boldsymbol{u}} \in L_d^\Phi\}.$$

$W^1L_d^\Phi$ is a Banach space when equipped with the norm

$$\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}.$$

For a function $\boldsymbol{u} \in L_d^1([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t)\, dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L_d^\Psi \hookrightarrow \left[L_d^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L_d^\infty \hookrightarrow L_d^\Phi \hookrightarrow L_d^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant Cw(|t-s|). \tag{13}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (13) for some $C > 0$ and it is a Banach space with norm

$$\|\boldsymbol{u}\|_{C^w([0,T],\mathbb{R}^d)} := \|\boldsymbol{u}\|_{L^\infty} + \sup_{t \neq s} \frac{|\boldsymbol{u}(t) - \boldsymbol{u}(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [7, 9, 11, 12, 16]. The next simple lemma is essentially known and we will use it systematically. For the sake of completeness, we include a brief proof of it.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T], \mathbb{R}^d)$ and for every $\boldsymbol{u} \in W^1L^\Phi$*

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi} w(|t-s|), \tag{14}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|\boldsymbol{u}\|_{W^1L^\Phi} \tag{15}$$

2. *For every $\boldsymbol{u} \in W^1L^\Phi$ we have $\widetilde{\boldsymbol{u}} \in L_d^\infty$ and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Sobolev's inequality).} \tag{16}$$

Proof. For $0 \leqslant s \leqslant t \leqslant T$, we get

$$\begin{aligned} |\boldsymbol{u}(t) - \boldsymbol{u}(s)| &\leqslant \int_s^t |\dot{\boldsymbol{u}}(\tau)| \quad d\tau \\ &\leqslant \|\chi_{[s,t]}\|_{L^\Psi} \|\dot{\boldsymbol{u}}\|_{L^\Phi} \\ &= \|\dot{\boldsymbol{u}}\|_{L^\Phi} (t-s)\Phi^{-1}\left(\frac{1}{t-s}\right), \end{aligned} \tag{17}$$

using Hölder's inequality and [17, Eq. (9.11)]. This proves the inequality (14).

Since u_i is continuous, from Mean Value Theorem for integrals, there exists $s_i \in [0, T]$ such that $u_i(s_i) = \overline{u}_i$. Using this s_i value in (14) with u_i instead of $\boldsymbol{u}$ and taking into account that $s\Phi^{-1}(1/s)$ is increasing, we obtain Sobolev's inequality for each u_i. The inequality (16) follows easily from the corresponding result for each component of $\boldsymbol{u}$.

On the other hand, again by Hölder's inequality and [17, Eq. (9.11)], we have

$$|\overline{\boldsymbol{u}}| = \frac{1}{T} \int\limits_0^T |\boldsymbol{u}(s)| ds \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \|\boldsymbol{u}\|_{L^\Phi}. \tag{18}$$

From (16), (18) and the fact that $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$, we obtain (15). This completes the proof of item 1. □

The next result is analogous to some lemmata in $W^1L^p_d$, see Xu-Tang-2007 y otros...

Lemma 2.2. *If* $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$*, then* $(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \to \infty$.

Proof. We have

$$\|\boldsymbol{u}\|_{L^\Phi} = \|\overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant \|\overline{\boldsymbol{u}}\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi} = |\overline{\boldsymbol{u}}|\|1\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi}$$

We know that Holder's inequality implies that $L^\infty_d \hookrightarrow L^\Phi_d$, that is, there exists $C > 0$ such that for any $\tilde{\boldsymbol{u}} \in L^\infty_d$ we have

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C\|\tilde{\boldsymbol{u}}\|_{L^\infty}$$

and, applying Sobolev's inequality to the previous formula, we get

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C\|\dot{\boldsymbol{u}}\|_{L^\Phi}$$

La desigualdad anterior sería del tipo Wirtinger's que no tenemos enunciada en ningún lado.

Therefore,

$$\|\boldsymbol{u}\|_{L^\Phi} \leqslant C(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \tag{19}$$

As $\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}$, then

$$\|\boldsymbol{u}\|_{W^1L^\Phi} \leqslant C(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi})$$

and by hypothesis $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$, then $|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi} \to \infty$. □

Given a continuous function $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, we define the composition operator $\boldsymbol{a} : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(\boldsymbol{u})(t) = a(|\boldsymbol{u}(t)|)$. We will often use the following elementary consequence of Lemma 2.1.

Corollary 2.3. *If $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi_d \to L^\infty_1([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(\boldsymbol{u})\|_{L^\infty([0,T])} \leqslant A(\|\boldsymbol{u}\|_{W^1L^\Phi})$.*

Proof. Let $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing majorant of a, for example $\alpha(s) := \sup_{0\leqslant t\leqslant s} a(t)$. If $\boldsymbol{u} \in W^1L^\Phi_d$ then, by Lemma 2.1,

$$a(|\boldsymbol{u}(t)|) \leqslant \alpha(\|\boldsymbol{u}\|_{L^\infty}) \leqslant \alpha\left(\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi}\right) =: A(\|\boldsymbol{u}\|_{W^1L^\Phi}).$$

□

Definition 2.4. *We say that a function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function if for fixed $(\boldsymbol{x}, \boldsymbol{y})$ the map $t \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is measurable and for fixed t the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is continuously differentiable for almost everywhere $t \in [0,T]$.*

In [1] we proved the next result.

Theorem 2.5. *Let $\mathcal{L}$ be a Carathéodory function satisfying* (2), (3) *and* (4). *Then the following statements hold:*

1. *The action integral given by* (5) *is finitely defined on $\mathcal{E}^\Phi_d(\lambda) := W^1L^\Phi_d \cap \{\boldsymbol{u} | \dot{\boldsymbol{u}} \in \Pi(E^\Phi_d, \lambda)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi_d(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi_d(\lambda)$ into $\left[W^1L^\Phi_d\right]^*$. Moreover, I' is given by the following expression*
$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \dot{\boldsymbol{v}}\right\}\, dt. \tag{20}$$
3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}^\Phi_d(\lambda)$ into $\left[W^1L^\Phi_d\right]^*$ when both spaces are equipped with the strong topology.*

In [1] we derived the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T-periodic functions. We denote by $W^1L^\Phi_T$ the subspace of $W^1L^\Phi_d$ containing all T-periodic functions. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) < \frac{1}{2}\left(f(\boldsymbol{x}) + f(\boldsymbol{y})\right)$ for $\boldsymbol{x} \neq \boldsymbol{y}$. It is well known that if f is a strictly convex and differentiable function, then $D_{\boldsymbol{x}}f : \mathbb{R}^d \to \mathbb{R}^d$ is a one-to-one map (see, e.g. [24, Thm. 12.17]).

Theorem 2.6. *Let $\boldsymbol{u} \in \mathcal{E}^\Phi_d(\lambda)$ be a T-periodic function. The following statements are equivalent:*

1. $I'(\boldsymbol{u}) \in \left(W^1 L_T^{\Phi}\right)^{\perp}$.

2. $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and* $\boldsymbol{u}$ *solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t \in (0,T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0. \end{cases} \tag{21}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t, x, y)$ *is* T*-periodic with respect to the variable* t *and strictly convex with respect to* $\boldsymbol{y}$*, then* $D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0$ *is equivalent to* $\dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}(T)$.

3 Coercivity discussion

DECIR ALGO DE LOS ÍNDICES ACÁ O EN LA INTRO...???? We recall a usual definition in the context of calculus of variations.

Definition 3.1. *Let* X *be a Banach space and let* D *be an unbounded subset of* X*. Suppose that* $J : D \subset X \to \mathbb{R}$*. We say that* J *is* coercive *if* $J(u) \to +\infty$ *when* $\|\boldsymbol{u}\|_X \to +\infty$.

It is well known that coercivity is a useful ingredient in the process of establishing existence of minima. Therefore, we are interested in finding conditions which ensure the coercivity of the action integral I acting on $\mathcal{E}_d^{\Phi}(\lambda)$. For this purpose, we need to introduce the following extra condition on the lagrangian function $\mathcal{L}$

$$\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y}) \geq \alpha_0 \Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t, \boldsymbol{x}), \tag{22}$$

where $\alpha_0, \Lambda > 0$ and $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, i.e. $F(t, \boldsymbol{x})$ is measurable with respect to t for every fixed $\boldsymbol{x} \in \mathbb{R}^d$ and it is continuous at $\boldsymbol{x}$ for a.e. $t \in [0,T]$. We observe that, from (22) and (2), we have $F(t, \boldsymbol{x}) \leqslant a(|\boldsymbol{x}|)b_0(t)$ with $b_0(t) := b(t) + \Phi(f(t)) \in L_1^1([0,T])$. In order to guarantee that integral $\int_0^T F(t, \boldsymbol{u})\, dt$ is finite for $\boldsymbol{u} \in W^1L^{\Phi}$, we need to assume

$$|F(t, \boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t \in [0,T] \quad \text{and for every } \boldsymbol{x} \in \mathbb{R}^d. \tag{23}$$

As we shall see in Theorem 3.3, when $\mathcal{L}$ satisfies (2), (3), (4), (22) and (23), the coercivity of the action integral I is related to the coercivity of the functional

$$J_{C,\nu}(\boldsymbol{u}) := \rho_{\Phi}\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^{\Phi}}^{\nu}, \tag{24}$$

for $C, \nu > 0$. If $\Phi(x) = |x|^p/p$ then $J_{C,\nu}$ is clearly coercive for $\nu < p$. For more general Φ the situation is more interesting as it will be shown in the following lemma.

Lemma 3.2. *Let* Φ *and* Ψ *be complementary* N*-functions. Then:*

1. *If* $C\Lambda < 1$*, then* $J_{C,1}$ *is coercive.*

2. *If $\Psi \in \Delta_2$ globally, then there exists a constant $\alpha_\Phi > 1$ such that, for any $0 < \mu < \alpha_\Phi$,*
$$\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to\infty} \frac{\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)}{\|\boldsymbol{u}\|_{L^\Phi}^{\mu}} = +\infty. \tag{25}$$
In particular, the functional $J_{C,\mu}$ is coercive for every $C > 0$ and $0 < \mu < a_\Phi$. The constant α_Φ is one of the so-called Matuszewska-Orlicz indices *(see [18, Ch. 11]).*

3. *If $J_{C,1}$ is coercive with $C\Lambda > 1$, then $\Psi \in \Delta_2$.*

Proof. By (9) we have
$$(1 - C\Lambda)\|\boldsymbol{u}\|_{L^\Phi} + C\Lambda\|\boldsymbol{u}\|_{L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} \leqslant \Lambda + \Lambda\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right),$$
then
$$\frac{(1 - C\Lambda)}{\Lambda}\|\boldsymbol{u}\|_{L^\Phi} - 1 \leqslant \rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^\Phi} = J_{C,1}(\boldsymbol{u}).$$
This inequality shows that $J_{C,1}$ is coercive and therefore item 1 is proved.

In virtue of [2, Eq. (2.8)], the Δ_2-condition on Ψ, [18, Thm. 11.7] and [18, Cor. 11.6], we obtain constants $K > 0$ and $\alpha_\Phi > 1$ such that
$$\Phi(rs) \geq Kr^\nu\Phi(s) \tag{26}$$
for any $0 < \nu < \alpha_\Phi$, $s \geq 0$ and $r > 1$.

Let $1 < \mu < \nu < \alpha_\Phi$ and let $r > \Lambda$ be a constant that will be specified later. Then, from (26) and (9), we get
$$\begin{aligned}\frac{\int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right)\, dt}{\|\boldsymbol{u}\|_{L^\Phi}^{\mu}} &\geq K\left(\frac{r}{\Lambda}\right)^\nu \frac{\int_0^T \Phi(r^{-1}|\boldsymbol{u}|)\, dt}{\|\boldsymbol{u}\|_{L^\Phi}^{\mu}} \\ &\geq K\left(\frac{r}{\Lambda}\right)^\nu \frac{r^{-1}\|\boldsymbol{u}\|_{L^\Phi} - 1}{\|\boldsymbol{u}\|_{L^\Phi}^{\mu}}.\end{aligned}$$

We choose $r = \|\boldsymbol{u}\|_{L^\Phi}/2$. Since $\|\boldsymbol{u}\|_{L^\Phi} \to +\infty$ we can assume $\|\boldsymbol{u}\|_{L^\Phi} > 2\Lambda$. Thus $r > \Lambda$ and
$$\frac{\int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right) dt}{\|\boldsymbol{u}\|_{L^\Phi}^{\mu}} \geq \frac{K}{2^\nu\Lambda^\nu}\|\boldsymbol{u}\|_{L^\Phi}^{\nu-\mu} \to +\infty \quad \text{as } \|\boldsymbol{u}\|_{L^\Phi} \to +\infty,$$
because $\nu > \mu$.

With the aim of proving item 3, we suppose that $\Psi \notin \Delta_2$. By [17, Thm. 4.1], there exists a sequence of real numbers r_n such that $r_n \to \infty$ and
$$\lim_{n\to\infty} \frac{r_n\psi(r_n)}{\Psi(r_n)} = +\infty. \tag{27}$$

Now, we choose $\boldsymbol{u}_n$ such that $|\boldsymbol{u}_n| = \Lambda\psi(r_n)\chi_{[0,\frac{1}{\Psi(r_n)}]}$. Then, by [17, Eq. (9.11)], we get

$$\|\boldsymbol{u}_n\|_{L^\Phi} = \Lambda\frac{\psi(r_n)}{\Psi(r_n)}\Psi^{-1}(\Psi(r_n)) = \Lambda\frac{r_n\psi(r_n)}{\Psi(r_n)} \to \infty, \quad \text{as} \quad n\to\infty.$$

And, using Young's equality (see [17, Eq. (2.7)]), we have

$$\begin{aligned} J_{C,1}(\boldsymbol{u}_n) &= \int_0^T \Phi\left(\frac{|\boldsymbol{u}_n|}{\Lambda}\right) dt - C\|\boldsymbol{u}_n\|_{L^\Phi} \\ &= \frac{1}{\Psi(r_n)}\left[\Phi(\psi(r_n)) - C\Lambda r_n\psi(r_n)\right] \\ &= \frac{1}{\Psi(r_n)}\left[r_n\psi(r_n) - \Psi(r_n) - C\Lambda r_n\psi(r_n)\right] \\ &= \frac{(1-C\Lambda)r_n\psi(r_n)}{\Psi(r_n)} - 1. \end{aligned}$$

From (27) and the condition $C\Lambda > 1$, we obtain $J_{C,1}(\boldsymbol{u}_n) \to -\infty$, which contradicts the coercivity of $J_{C,1}$. □

In [1] we established coercivity of action integrals under different assumptions as follows.

Theorem 3.3. *Let $\mathcal{L}$ be a lagrangian function satisfying* (2), (3), (4), (22) *and* (23)*. We assume the following conditions:*

1. *There exist a non negative function $b_1 \in L^1_1$ and a constant $\mu > 0$ such that for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^d$ and a.e. $t \in [0,T]$*

$$|F(t,\boldsymbol{x_2}) - F(t,\boldsymbol{x_1})| \leqslant b_1(t)(1 + |\boldsymbol{x_2} - \boldsymbol{x_1}|^\mu). \tag{28}$$

   We suppose that $\mu < \alpha_\Phi$, with α_Φ as in Lemma 3.2, in the case that $\Psi \in \Delta_2$; and, we suppose $\mu = 1$ if Ψ is an arbitrary N-function.

2. $$\int_0^T F(t,\boldsymbol{x})\,dt \to \infty \quad \textit{as} \quad |\boldsymbol{x}| \to \infty. \tag{29}$$

3. $\Psi \in \Delta_2$ *or, alternatively,* $\alpha_0^{-1}T\Phi^{-1}(1/T)\|b_1\|_{L^1}\Lambda < 1$.

Then the action integral I is coercive.

Based on [21] we say that F satisfies the condition (A) if $F(t,\boldsymbol{x})$ is a Carathéodory function, F verifies (23) and F is continuously differentiable with respect to $\boldsymbol{x}$. Moreover, the next inequality holds

$$|D_{\boldsymbol{x}}F(t,\boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t\in[0,T] \text{ and for every } \boldsymbol{x}\in\mathbb{R}^d. \tag{30}$$

The following result was proved in [21, p. 18].

Lemma 3.4. *Suppose that F satisfies condition (A) and* (29)*, $F(t,\cdot)$ is differentiable and convex a.e. $t\in[0,T]$. Then, there exists $\boldsymbol{x}_0\in\mathbb{R}^d$ such that*

$$\int_0^T D_{\boldsymbol{x}}F(t,\boldsymbol{x}_0)\,dt=0. \tag{31}$$

Theorem 3.5. *Let $\mathcal{L}$ be as in Theorem 3.3 and let F be as in Lemma 3.4. Moreover, assume that $\Psi\in\Delta_2$ or, alternatively $\alpha_0^{-1}T\Phi^{-1}\left(1/T\right)a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}\Lambda<1$, with a and b_0 as in* (23) *and $\boldsymbol{x}_0\in\mathbb{R}^d$ any point satisfying* (31)*. Then I is coercive.*

4 New result

The symbol C with various subscripts will stand for a constant, not necessarily the same at each occurrence, which can depend (unless otherwise stated) on the constants previously displayed such as $\|b_1\|_{L^1},\|b_2\|_{L^1},T,\mu,\mu'$.

Now, we have another result about coercivity of I assuming some conditions on the nonlinearity ∇F.

Theorem 4.1. *Let $\mathcal{L}$ be a lagrangian function satisfying* (2), (3), (4), (22) *and* (23)*. We assume the following conditions:*

1. $\Psi\in\Delta_2$.
2. *There exist non negative functions $b_1,b_2\in L^1_1$ and a constant $1<\mu<\alpha_\Phi$ such that for any $\boldsymbol{x}\in\mathbb{R}^d$ and a.e. $t\in[0,T]$*

$$|\nabla F(t,\boldsymbol{x})|\leqslant b_1(t)|\boldsymbol{x}|^{\mu-1}+b_2(t). \tag{32}$$

3. *There exists a real positive number σ such that $\sigma>(\mu-1)\beta_\Psi$ and*

$$|\boldsymbol{x}|^\sigma=o\left(\int_0^T F(t,\boldsymbol{x})\,dt\right)\quad \text{as}\quad |\boldsymbol{x}|\to\infty. \tag{33}$$

Then the action integral I is coercive.

Proof. By the decomposition $u=\overline{u}+\tilde{\boldsymbol{u}}$, Mean Value Theorem, Cauchy-Schwarz inequality and (32), we have

$$\begin{aligned}&\left|\int_0^T F(t,\boldsymbol{u})-F(t,\overline{\boldsymbol{u}})\,dt\right|=\left|\int_0^T\int_0^1\nabla F(t,\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t))\cdot\tilde{\boldsymbol{u}}(t)\,ds\,dt\right|\\&\leqslant\int_0^T\int_0^1 b_1(t)|\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t)|^{\mu-1}|\tilde{\boldsymbol{u}}(t)|\,ds\,dt+\int_0^T\int_0^1 b_2(t)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt\\&=I_1+I_2.\end{aligned} \tag{34}$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} \leqslant C\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{35}$$

On the other hand, as $\overline{\boldsymbol{u}} \in \mathbb{R}$ and $s \in [0,1]$, we have

$$|\overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)|^{\mu-1} \leqslant C(|\overline{\boldsymbol{u}}|^{\mu-1} + \|\tilde{\boldsymbol{u}}\|_{L^\infty}^{\mu-1}). \tag{36}$$

Now, inequality (36), Hölder's inequality and Sobolev's inequality imply that

$$\begin{aligned} I_1 &\leqslant C\left(|\overline{\boldsymbol{u}}|^{\mu-1}\int_0^T b_1(t)|\tilde{\boldsymbol{u}}(t)|\,dt + \|\tilde{\boldsymbol{u}}\|_{L^\infty}^{\mu-1}\int_0^T b_1(t)|\tilde{\boldsymbol{u}}(t)|\,dt\right) \\ &\leqslant C\left(|\overline{\boldsymbol{u}}|^{\mu-1}\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} + \|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty}^{\mu}\right) \\ &\leqslant C\left(|\overline{\boldsymbol{u}}|^{\mu-1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu}\right). \end{aligned} \tag{37}$$

Let μ' be a positive constant such that $1 < \mu \leqslant \mu' < \alpha_\Phi$. Next, using Young's inequality with conjugate exponents μ' and $\frac{\mu'}{\mu'-1}$ and Sobolev's inequality, we get

$$|\overline{\boldsymbol{u}}|^{\mu-1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} \leqslant \frac{(\mu'-1)}{\mu'}|\overline{\boldsymbol{u}}|^{\boldsymbol{\sigma}} + \frac{1}{\mu'}\|\tilde{\boldsymbol{u}}\|_{L^\infty}^{\mu'} \tag{38}$$

where $\sigma = \frac{(\mu-1)\mu'}{\mu'-1}$ is a positive constant such that $\sigma > (\mu-1)b_\Psi$.

From (37),(38) and (35), we have

$$I_1 + I_2 \leqslant C(|\overline{\boldsymbol{u}}|^{\sigma} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}). \tag{39}$$

In the subsequent estimates, we use the decomposition $u = \overline{u} + \tilde{\boldsymbol{u}}$, (22), (34), (39) and we get

$$\begin{aligned} I(\boldsymbol{u}) &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t,\boldsymbol{u})\,dt \\ &= \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T \left[F(t,\boldsymbol{u}) - F(t,\overline{\boldsymbol{u}})\right]\,dt + \int_0^T F(t,\overline{\boldsymbol{u}})\,dt \\ &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C(\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \\ &+ \int_0^T F(t,\overline{\boldsymbol{u}})\,dt - C|\overline{\boldsymbol{u}}|^{\sigma}. \end{aligned} \tag{40}$$

As $1 < \mu \leqslant \mu'$, we have $\|\dot{\boldsymbol{u}}\|_{L^\Phi} \leq \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + 1$ and $\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} \leq \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + 1$, then

$$-C(\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \geq -C(3\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + 2). \tag{41}$$

In this way, from (40) and (41)

$$\begin{aligned} I(\boldsymbol{u}) &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} + \int_0^T F(t,\overline{\boldsymbol{u}})\,dt - K|\overline{\boldsymbol{u}}|^{\sigma} - C \\ &= \alpha_0 J_{C,\mu'}(\dot{\boldsymbol{u}}) + \gamma(\overline{\boldsymbol{u}}) - C. \end{aligned}$$

Let $\boldsymbol{u}_n$ be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(\boldsymbol{u}_n) \to \infty$.
On the contrary, suppose that for a subsequence, still denoted by $\boldsymbol{u}_n$, $I(\boldsymbol{u}_n)$ is upper bounded, that is, there exists $M > 0$ such that $|I(\boldsymbol{u}_n)| \leqslant M$. As $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$, from Lemma 2.2, we have $|\overline{\boldsymbol{u}}_n| + \|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Then, there exists subsequence of the subsequence $\{\boldsymbol{u}_n\}$, still denoted by $\boldsymbol{u}_n$, which is not bounded. Then, $\overline{\boldsymbol{u}}_n \to \infty$ or $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Now, as the functionals $J_{C,\mu'(\dot{\boldsymbol{u}})}$ and $\gamma(\overline{\boldsymbol{u}})$ are coercive, then $J_{C,\mu'(\dot{\boldsymbol{u}}_n)} \to \infty$ or $\gamma(\overline{\boldsymbol{u}}_n) \to \infty$. By (23), the functional $\gamma(\overline{\boldsymbol{u}}_n)$ is lower bounded and $J_{C,\mu'(\dot{\boldsymbol{u}}_n)}$ is also lower bounded on a bounded set because the modular $\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)$ is always bigger than zero. Therefore, $I(\boldsymbol{u}_n) \to \infty$ as $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ which contradits the initial assumption on the behavior of $I(\boldsymbol{u}_n)$. □

REVISAR LA PRUEBA ANTERIOR Y MEJORAR LA ESCRITURA!!!!

5 Limit case $\mu = \alpha_\Phi$

In [] coercivity was obtained even in the limit case $\mu = 1$ and $\mu = p$ assuming additional conditions on ... This was possible because in L^p spaces, the norm and the modular coincides, that is, $\|\cdot\|_p^p = O(\int_0^T |\cdot|^p\, dt)$. In Orlicz spaces, $\|\cdot\|_{L^\Phi}^\mu$ can be upper controled by a modular provided that $\mu < \alpha_\Phi$ for any N-function Φ. But, the limit case does not hold for any Φ, i.e. in general $\|\cdot\|_{L^\Phi}^{\alpha_\Phi} = O(\int_0^T \Phi(|u|)\, dt)$ is false as can be seen as follows.

Let $\Phi, \Psi \in \Delta_2$, then the next inequality $\Phi(tu) \geq t^{\alpha_\Phi}\Phi(u)$ for any $u > 0$ and for any $t \geq 1$ is false.

In fact, let $\Phi(u) = \begin{cases} \frac{p-1}{p}u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$

Theorem 5.1. *If* $p \geq \frac{1+\sqrt{2}}{2}$*, then* Φ *is an* N*-function.*

Proof. Resumir la prueba.... □

Theorem 5.2. *There exists a constant* $C > 0$ *such that*

$$\Phi(tu) \leqslant ct^p\Phi(u) \;\; t \geq 1, u > 0. \tag{42}$$

For every $\varepsilon > 0$ *there exists a constant* $C = C(\varepsilon, p)$ *such that*

$$\Phi(tu) \geq Ct^{p-\varepsilon}\Phi(u) \;\; t \geq 1, u > 0. \tag{43}$$

Proof. Resumir la prueba □

Remark 5.3. *The inequality*

$$\Phi(tu) \geq Ct^p\Phi(u)$$

is false for every C *because for every* $u \geq e$ *we have*

$$\lim_{t\to\infty} \frac{\Phi(tu)}{t^p\Phi(u)} = 0$$

Theorem 5.4. $\alpha_\Phi = \beta_\Phi = p$

Proof. Resumir la prueba. □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\,dx \geq C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^p$$

is false.

If we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\,dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geq C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geq Ct^p$ were also true but this last inequality is false.

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