Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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Abstract

In this paper we....

ALGO ASÍ HABÍA QUE ESCRIBIR ACÁ O EN LA INTRO....

The results of this paper improve on the classic ones obtained in [1] and [2] with $|\nabla F(t,\boldsymbol{x})| \leqslant f(t)\boldsymbol{x}^{\alpha} + g(t)$ for $\alpha < 1$ and $\alpha < p$ respectively. Here, the bounds of the sublinearity are bigger than those considered in the aforementioned works.

1 Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}} \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) & \text{a.e. } t \in (0, T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(T) = 0 \end{cases} \tag{1}$$

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where $T > 0$, $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L}$: $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$
\begin{aligned}
|\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| &\leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), && (2)\\
|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| &\leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), && (3)\\
|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| &\leqslant a(|\boldsymbol{x}|)\left(c(t) + \varphi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right). && (4)
\end{aligned}
$$

In these inequalities we assume that $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ. The non negative functions b, c and f satisfy that $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$, where the Banach spaces $L^1_1([0,T])$, $L^\Psi_1([0,T])$ and $E^\Phi_1([0,T])$ will be defined later.

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(\boldsymbol{u}) = \int_0^T \mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\,dt. \tag{5}$$

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [3, 4, 5]. For Orlicz spaces of vector valued functions, see [6] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \quad \text{for } t \geqslant 0,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty}\varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the Δ_2-*condition*, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geqslant 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{6}$$

for every $t \geqslant t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2-*condition globally* ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0,T])$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $\boldsymbol{u} = (u_1,\ldots,u_d)$ for $\boldsymbol{u} \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_0^T \Phi(|\boldsymbol{u}|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C_d^\Phi = C_d^\Phi([0,T])$ is given by

$$C_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \rho_\Phi(\boldsymbol{u}) < \infty\}\,. \tag{7}$$

The *Orlicz space* $L_d^\Phi = L_d^\Phi([0,T])$ is the linear hull of C_d^Φ; equivalently,

$$L_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \exists \lambda > 0 : \rho_\Phi(\lambda \boldsymbol{u}) < \infty\}\,. \tag{8}$$

The Orlicz space L_d^Φ equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup\left\{\int_0^T \boldsymbol{u} \cdot \boldsymbol{v}\, dt \big| \rho_\Psi(\boldsymbol{v}) \leqslant 1\right\},$$

is a Banach space. By $\boldsymbol{u} \cdot \boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^d$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [4, Thm. 10.5] and [7]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k}\{1 + \rho_\Phi(k\boldsymbol{u})\}\,. \tag{9}$$

The subspace $E_d^\Phi = E_d^\Phi([0,T])$ is defined as the closure in L_d^Φ of the subspace L_d^∞ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E_d^Φ is the only one maximal subspace contained in the Orlicz class C_d^Φ, i.e. $\boldsymbol{u} \in E_d^\Phi$ if and only if $\rho_\Phi(\lambda\boldsymbol{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [4, Th. 9.3]). Namely, if $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$ then $\boldsymbol{u} \cdot \boldsymbol{v} \in L_1^1$ and

$$\int_0^T \boldsymbol{v} \cdot \boldsymbol{u}\, dt \leqslant \|\boldsymbol{u}\|_{L^\Phi} \|\boldsymbol{v}\|_{L^\Psi}. \tag{10}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x\rangle = x^*(x)$. Hölder's inequality shows that $L_d^\Psi \subset \left[L_d^\Phi\right]^*$, where the pairing $\langle \boldsymbol{v}, \boldsymbol{u}\rangle$ is defined by

$$\langle \boldsymbol{v}, \boldsymbol{u}\rangle = \int_0^T \boldsymbol{v} \cdot \boldsymbol{u}\, dt \tag{11}$$

with $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L_d^\Psi = \left[L_d^\Phi\right]^*$ will not hold. In general, it is true that $\left[E_d^\Phi\right]^* = L_d^\Psi$.

Like in [4], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ given by

$$\Pi(E_d^\Phi, r) := \{\boldsymbol{u} \in L_d^\Phi | d(\boldsymbol{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \tag{12}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L_d^Φ, E_d^Φ, $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

We define the *Sobolev-Orlicz space* $W^1L_d^\Phi$ (see [3]) by

$$W^1L_d^\Phi := \{\boldsymbol{u}|\boldsymbol{u} \text{ is absolutely continuous and } \dot{\boldsymbol{u}} \in L_d^\Phi\}.$$

$W^1L_d^\Phi$ is a Banach space when equipped with the norm

$$\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{13}$$

For a function $\boldsymbol{u} \in L_d^1([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t)\,dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L_d^\Psi \hookrightarrow \left[L_d^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L_d^\infty \hookrightarrow L_d^\Phi \hookrightarrow L_d^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant Cw(|t-s|). \tag{14}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (14) for some $C > 0$ and it is a Banach space with norm

$$\|\boldsymbol{u}\|_{C^w([0,T],\mathbb{R}^d)} := \|\boldsymbol{u}\|_{L^\infty} + \sup_{t\neq s} \frac{|\boldsymbol{u}(t) - \boldsymbol{u}(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [8, 9, 10, 11, 12]. The next simple lemma, whose proof can be found in [13], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T], \mathbb{R}^d)$ and for every $\boldsymbol{u} \in W^1L^\Phi$*

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi} w(|t-s|), \tag{15}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|\boldsymbol{u}\|_{W^1L^\Phi} \tag{16}$$

2. *For every $\boldsymbol{u} \in W^1L^\Phi$ we have $\widetilde{\boldsymbol{u}} \in L_d^\infty$ and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Sobolev's inequality).} \tag{17}$$

The following result is analogous to some lemmata in $W^1L^p_d$, see [14].

Lemma 2.2. *If* $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$, *then* $(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \to \infty$.

Proof. By the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$ and some elementary operations, we get

$$\|\boldsymbol{u}\|_{L^\Phi} = \|\overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant \|\overline{\boldsymbol{u}}\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi} = |\overline{\boldsymbol{u}}|\|1\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi}. \tag{18}$$

It is known that $L^\infty_d \hookrightarrow L^\Phi_d$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{\boldsymbol{u}} \in L^\infty_d$ we have

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C_1 \|\tilde{\boldsymbol{u}}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain the Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C_2 \|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{19}$$

Therefore, from (18), (19) and (13), we get

$$\|\boldsymbol{u}\|_{W^1L^\Phi} \leqslant C_3(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \to \infty$. □

We present a definition that will be useful later.

Definition 2.3. *A function* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a* Carathéodory *function if for fixed* $(\boldsymbol{x}, \boldsymbol{y})$ *the map* $t \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ *is measurable and for fixed* t *the map* $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ *is continuous for almost everywhere* $t \in [0,T]$. *We say that* $\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ *is* differentiable Carathéodory *if in addition* $\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ *is continuously differentiable with respect to* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *for almost everywhere* $t \in [0,T]$.

In [13] we proved the next results.

Theorem 2.4. *Let* $\mathcal{L}$ *be a differentiable Carathéodory function satisfying* (2), (3) *and* (4). *Then the following statements hold:*

1. *The action integral given by* (5) *is finitely defined on* $\mathcal{E}^\Phi_d(\lambda) := W^1L^\Phi_d \cap \{\boldsymbol{u} | \dot{\boldsymbol{u}} \in \Pi(E^\Phi_d, \lambda)\}$.
2. *The function* I *is Gâteaux differentiable on* $\mathcal{E}^\Phi_d(\lambda)$ *and its derivative* I' *is demicontinuous from* $\mathcal{E}^\Phi_d(\lambda)$ *into* $\left[W^1L^\Phi_d\right]^*$. *Moreover,* I' *is given by the following expression*

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{ D_{\boldsymbol{x}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \dot{\boldsymbol{v}} \right\} \, dt. \tag{20}$$

3. *If* $\Psi \in \Delta_2$ *then* I' *is continuous from* $\mathcal{E}^\Phi_d(\lambda)$ *into* $\left[W^1L^\Phi_d\right]^*$ *when both spaces are equipped with the strong topology.*

In [13] we derive the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T-periodic functions. We denote by $W^1L^\Phi_T$ the subspace of $W^1L^\Phi_d$ containing all T-periodic functions. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) < \frac{1}{2}\left(f(\boldsymbol{x}) + f(\boldsymbol{y})\right)$ for $\boldsymbol{x} \neq \boldsymbol{y}$. It is well known that if f is a strictly convex and differentiable function, then $D_{\boldsymbol{x}} f : \mathbb{R}^d \to \mathbb{R}^d$ is a one-to-one map (see, e.g. [15, Thm. 12.17]).

Theorem 2.5. *Let $\boldsymbol{u} \in \mathcal{E}^\Phi_d(\lambda)$ be a T-periodic function. The following statements are equivalent:*

1. $I'(\boldsymbol{u}) \in \left(W^1L^\Phi_T\right)^\perp$.
2. $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and* $\boldsymbol{u}$ *solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad \textit{a.e. } t \in (0, T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0. \end{cases} \tag{21}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t, x, y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0$ is equivalent to $\dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}(T)$.

HABRÍA QUE ARREGLAR EL TEOREMA ANTERIOR CAMBIANDO $W^1L^\Phi_T$ por $W^1E^\Phi_T$??????

Habría que ver si el lugar de los índices es el adecuado. Copié lo que teníamos en el primer trabajo.

Next, we enumerate some definitions and results from the theory of convex functions. We suggest [16, 17, 4, 18, 5] for definitions, proofs and additional details.

We denote by α_φ and β_φ the so-called *Matuszewska-Orlicz indices* of the function φ, which are defined next. Given an increasing, unbounded, continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(0) = 0$ we define

$$\alpha_\varphi := \lim_{t\to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)}, \quad \beta_\varphi := \lim_{t\to +\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)}. \tag{22}$$

We have that $0 \leqslant \alpha_\varphi \leqslant \beta_\varphi \leqslant +\infty$. The relation $\beta_\varphi < \infty$ holds true if and only if φ satisfies the Δ_2-condition. If φ is a homeomorphism we have that

$$\alpha_{\varphi^{-1}} = \frac{1}{\beta_\varphi}. \tag{23}$$

Moreover $\varphi \in \mathcal{F}$ implies $\alpha_\varphi \geqslant 1$. As a consequence, φ^{-1} is a Δ_2-function.

It is well known that if φ is an increasing function that satisfies the Δ_2-condition, φ is controlled by above and below by power functions. More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\varphi, \epsilon)$ such that, for every $t, u \geqslant 0$,

$$K^{-1} \min\left\{t^{\beta_\varphi+\epsilon}, t^{\alpha_\varphi-\epsilon}\right\}\varphi(u) \leqslant \varphi(tu) \leqslant K \max\left\{t^{\beta_\varphi+\epsilon}, t^{\alpha_\varphi-\epsilon}\right\}\varphi(u). \tag{24}$$

3 Lagrangians satisfying sublinear nonlinearity type conditions

Lemma 3.1. *Let Φ, Ψ complementary functions. The next statements are equivalent:*

1. *$\Psi \in \Delta_2$ globally.*
2. *There exists an N-function $\Phi_1 \in \Delta_2$ such that*

$$\Phi(rs) \geqslant \Phi_1(r)\Phi(s) \ \text{ for every } \ r \geqslant 1, \quad s \geqslant 0. \tag{25}$$

Proof. 1)⇒2) As $\Psi \in \Delta_2$ globally, there exist $k > 0$ and $\nu > 1$ such that

$$\Phi(rs) \geqslant kr^{\nu}\Phi(s) \ \ r \geqslant 1, \ s > 0,$$

which is (25) with $\Phi_1(r) = kr^{\nu}$ that is an N-function satisfying the Δ_2-condition.

2)⇒1) Next, we follow [5, p. 32, Prop. 13] and [5, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leqslant \Phi(rs) \ \ r > 1, \ s \geqslant 0.$$

Let $u = \Phi_1(r) \geqslant \Phi_1(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [5, p. 13, Prop. 1] and (25), we have for $u \geqslant \Phi_1(1)$ and $v \geqslant 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant \Phi_1^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi_1(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi_1(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$ we get that $\Psi \in \Delta_2$ globally. $\square$

The following lemma generalizes [13, Lemma 5.2].

Lemma 3.2. *Let Φ, Ψ be N-functions and suppose that $\Psi \in \Delta_2$ globally. Then*

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \tag{26}$$

for every Φ_0 with $\Phi_0 = o(\Phi_1)$ at ∞ where Φ_1 is any N-function satisfying (25)*. Reciprocally if* (26) *holds for some N-function Φ_0, then $\Psi \in \Delta_2$ (at ∞).*

Proof. By the assumptions on Φ and Φ_1 and the identity (9), we have

$$\frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \Phi_1(r)\frac{\int_0^T \Phi(r^{-1}|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})}\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$, as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$, we use the fact that $\Phi_1 \in \Delta_2$ and $\Phi_0 = o(\Phi_1)$ at ∞, and therefore we get

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} \geqslant C \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1(\|u\|_{L^\Phi})}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

The last assertion of the lemma follows from the fact that if Φ_0 is an N-function, then $\Phi_0(u) \geqslant ku$ for k small enough. Therefore (26) holds for $\Phi_0(u) = |u|$, then [13, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞. $\square$

Remark 1. We point out that this lemma can be applied to more cases than [13, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [4, p. 16] and [4, Section 7] for the definition and properties of principal part). Then (26) holds for Φ_0, however $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

We define the following functionals $J_{C,\Phi_0} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\Phi_0} : \mathbb{R}^n \to \mathbb{R}$ where $C > 0$ and Φ_0 an N-function, by

$$J_{C,\Phi_0}(\boldsymbol{u}) := \rho_\Phi(\boldsymbol{u}) - C\Phi_0\left(\|\boldsymbol{u}\|_{L^\Phi}\right), \tag{27}$$

and

$$H_{C,\Phi_0}(\boldsymbol{x}) := \int_0^T F(t,\boldsymbol{x})dt - C\Phi_0(|\boldsymbol{x}|), \tag{28}$$

respectively.

Like in [13] we consider Lagrangians $\mathcal{L}$ which are lower bounded as follows

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geqslant \alpha_0 \Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t,\boldsymbol{x}). \tag{29}$$

If $\mathcal{L}$ is given by the right hand side in (29) and $\Phi(u) = |u|^2$, then the ODE $\ddot{\boldsymbol{u}} = \nabla F(t,\boldsymbol{u}(t))$ in (1) is quasilinear, being $\nabla F(t,\boldsymbol{u}(t))$ the nonlinearity. Following the literature, we refer to ∇F as the non linearity even when we assume in (29) just the inequality. In [1] and [2] the authors considered for p-laplacian non linearities satisfying the inequality

$$|\nabla F(t,\boldsymbol{x})| \leqslant b_1(t)|\boldsymbol{x}|^\alpha + b_2(t),$$

where $b_1, b_2 \in L^1_1$ and α is any power less than p. Thus they said F is a sublinear nonlinearity. In this paper, we consider the following type of bounds for the nonlinearity

$$|\nabla F(t,\boldsymbol{x})| \leqslant b_1(t)\varphi_0(|\boldsymbol{x}|) + b_2(t), \tag{30}$$

where $\varphi_0 = \Phi_0'$ with Φ_0 an N-function. The employment of N-functions instead of power functions in inequalities likes (30) will allow us to extend some results of [1] and [2] even in the p-Laplacian case.

Based on [19] we say that F satisfies the condition (A) if $F(t,\boldsymbol{x})$ is a Carathéodory function and F is continuously differentiable with respect to $\boldsymbol{x}$. Moreover, the next inequality holds

$$|F(t,\boldsymbol{x})| + |D_{\boldsymbol{x}}F(t,\boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t\in[0,T], \forall \boldsymbol{x}\in\mathbb{R}^d. \tag{31}$$

The following theorem establishes coercivity of I assuming sublinear conditions on the nonlinearity ∇F.

Theorem 3.3. *Let $\mathcal{L}$ be a lagrangian function satisfying* (2), (3), (4), (29) *and F satisfies condition (A). We assume the following conditions:*

1. $\Psi\in\Delta_2$.
2. *Inequality* (30) *with $b_1, b_2 \in L^1_1$, $\varphi_0 = \Phi_0'$ where Φ_0 is a differentiable N-function that satisfies the Δ_2-condition globally such that $\Phi_0 = o(\Phi_1)$ at ∞ and Φ_1 verifies* (25).
3. $$\lim_{|x|\to\infty} \frac{\int_0^T F(t,\boldsymbol{x})\,dt}{\Phi_0(|\boldsymbol{x}|)} = +\infty. \tag{32}$$

Then the action integral I is coercive.

Proof. By the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, Cauchy-Schwarz's inequality and (30), we have

$$\begin{aligned}
&\left|\int_0^T F(t,\boldsymbol{u}) - F(t,\overline{\boldsymbol{u}})\,dt\right| = \left|\int_0^T\int_0^1 \nabla F(t,\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t))\cdot\tilde{\boldsymbol{u}}(t)\,ds\,dt\right|\\
&\leqslant \int_0^T\int_0^1 b_1(t)\varphi_0(|\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t)|)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt + \int_0^T\int_0^1 b_2(t)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt\\
&= I_1 + I_2.
\end{aligned} \tag{33}$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} \leqslant C_1\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{34}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

On the other hand, since $\Phi_0 \in \Delta_2$ globally, then $\varphi_0\in\Delta_2$ globally and consequently φ_0 is a quasi-subadditive function, i.e. there exists $C(\varphi_0) > 0$ such that $\varphi_0(a+b) \leqslant C(\varphi_0)(\varphi_0(a)+\varphi_0(b))$ for every $a,b\geqslant 0$. In this way, we have

$$\varphi_0(|\overline{\boldsymbol{u}}+s\tilde{\boldsymbol{u}}(t)|) \leqslant C(\varphi_0)[\varphi_0(|\overline{\boldsymbol{u}}|) + \varphi_0(\|\tilde{\boldsymbol{u}}\|_{L^\infty})]. \tag{35}$$

for every $s\in[0,1]$.

Now, inequality (35), Hölder's inequality, Sobolev's inequality, the monotonicity, the subadditivity and the Δ_2-condition on φ_0 imply that

$$\begin{aligned}
I_1 &\leqslant C(\varphi_0)\left\{\varphi_0(|\overline{\boldsymbol{u}}|)\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} + \|b_1\|_{L^1}\varphi_0(\|\tilde{\boldsymbol{u}}\|_{L^\infty})\|\tilde{\boldsymbol{u}}\|_{L^\infty}\right\}\\
&\leqslant C_2\left\{\varphi_0(|\overline{\boldsymbol{u}}|)\|\dot{\boldsymbol{u}}\|_{L^\Phi} + \varphi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi}\right\}
\end{aligned} \tag{36}$$

where $C_2 = C_2(\varphi_0, T, \|b_1\|_{L^1})$.

Next, by Young's inequality with complementary functions Φ_0 and Ψ_0 and the fact that $\Phi_0 \in \Delta_2$ globally, Young's equality [4, Eq. 2.7-2.8] and [5, Th. 3-(ii), p. 23], we get

$$\begin{aligned}\varphi_0(|\overline{\boldsymbol{u}}|)\|\dot{\boldsymbol{u}}\|_{L^\Phi} &\leqslant \Psi_0(\varphi_0(|\overline{\boldsymbol{u}}|)) + \Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}) \\ &\leqslant |\overline{\boldsymbol{u}}|\varphi_0(|\overline{\boldsymbol{u}}|) + \Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}) \\ &\leqslant C(\Phi_0)\Phi_0(|\overline{\boldsymbol{u}}|) + \Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\end{aligned} \tag{37}$$

and

$$\varphi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi})\|\dot{\boldsymbol{u}}\|_{L^\Phi} \leqslant C(\Phi_0)\Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}), \tag{38}$$

with $C(\Phi_0)$ the constant that comes from the Δ_2-condition on Φ_0.

From (36), (37), (38) and (34), we have

$$\begin{aligned}I_1 + I_2 &\leqslant C_3\Big\{\Phi_0(|\overline{\boldsymbol{u}}|) + \Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}) + \|\dot{\boldsymbol{u}}\|_{L^\Phi}\Big\} \\ &\leqslant C_4\Big\{\Phi_0(|\overline{\boldsymbol{u}}|) + \Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}) + 1\Big\}\end{aligned} \tag{39}$$

with C_3 and C_4 depending on $\Phi_0, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$. The last inequality follows from the fact that Φ_0 is an N-function, then there exists $C > 0$ such that $\Phi_0(x) \geqslant Cx$ for every $x \geqslant 1$. Thus $x \leqslant C\Phi_0(x) + 1$ for every $x \geqslant 0$.

In the subsequent estimates, we use (29), (33), (39), the fact that $\Phi_0 \in \Delta_2$ and we get

$$\begin{aligned}I(\boldsymbol{u}) &\geqslant \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t,\boldsymbol{u})\,dt \\ &= \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T \left[F(t,\boldsymbol{u}) - F(t,\overline{\boldsymbol{u}})\right]\,dt + \int_0^T F(t,\overline{\boldsymbol{u}})\,dt \\ &\geqslant \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C_4\Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}) + \int_0^T F(t,\overline{\boldsymbol{u}})\,dt - C_4\Phi_0(|\overline{\boldsymbol{u}}|) - C_4 \\ &\geqslant \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C_4\Phi_0(\|\dot{\boldsymbol{u}}\|_{L^\Phi}) + H_{C_4,\Phi_0}(\overline{\boldsymbol{u}}) - C_4 \\ &\geqslant \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - C_5\Phi_0\left(\frac{\|\dot{\boldsymbol{u}}\|_{L^\Phi}}{\Lambda}\right) + H_{C_4,\Phi_0}(\overline{\boldsymbol{u}}) - C_4 \\ &= \alpha_0 J_{C_6,\Phi_0}\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + H_{C_4,\Phi_0}(\overline{\boldsymbol{u}}) - C_4\end{aligned} \tag{40}$$

where $C_5 = C_5(\Phi_0, \Lambda, C_4)$ and $C_6 = \frac{C_5}{\alpha_0}$.

Let $\boldsymbol{u}_n$ be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(\boldsymbol{u}_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by $\boldsymbol{u}_n$, $I(\boldsymbol{u}_n)$ is upper bounded, i.e., there exists $M > 0$ such that $|I(\boldsymbol{u}_n)| \leqslant M$. As $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$, from Lemma 2.2, we have $|\overline{\boldsymbol{u}}_n| + \|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Passing to a subsequence, still denoted $\boldsymbol{u}_n$, we can assume that $|\overline{\boldsymbol{u}}_n| \to \infty$ or $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$.

Now, Lemma 3.2 implies that the functional $J_{C_6,\Phi_0}(\frac{\dot{\boldsymbol{u}}}{\Lambda})$ is coercive, and, by (**??**), the functional $H_{C_4,\Phi_0}(\overline{\boldsymbol{u}})$ is also coercive, then $J_{C_6,\Phi_0}(\frac{\dot{\boldsymbol{u}}_n}{\Lambda}) \to \infty$ or $H_{C_4,\Phi_0}(\overline{\boldsymbol{u}}_n) \to \infty$. From (31), we have that on a bounded set the functional $H_{C_4,\Phi_0}(\overline{\boldsymbol{u}}_n)$ is lower bounded and also $J_{C_6,\Phi_0}(\frac{\dot{\boldsymbol{u}}_n}{\Lambda}) \geqslant 0$. Therefore, $I(\boldsymbol{u}_n) \to \infty$ as $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(\boldsymbol{u}_n)$. □

4 Limit case $\mu = \alpha_\Phi$

Assuming $\|b_1\|_{L^1}$ small enough, in [20, 2] even was obtained coercitivity for the limit value $\mu = p$ in inequality (30). This result lean on the fact that when $\Phi(u) = |u|^p$,

$$\|u\|_{L^\Phi}^{\alpha_\Phi} = O\left(\int_0^T \Phi(|u|)\,dt\right), \quad \text{for } \|u\|_{L^\Phi} \to \infty. \tag{41}$$

However, it is no longer the case for any N-function Φ as the example below shows

From now on in this section, we will suppose that

$$\Phi(u) = \begin{cases} \frac{p-1}{p}u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this Φ.

Theorem 4.1. *If $p \geqslant \frac{1+\sqrt{2}}{2}$, then Φ is an N-function.*

Proof. We have

$$\varphi(u) = \Phi'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that Φ' is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough see that φ_1 is increasing on $[0, e]$ and φ_2 is increasing on $[e, \infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p > 1$. On the other hand, an elementary function analysis shows that $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an icreasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Besides $\varphi_2(u) \to \infty$ and $\varphi_1(u) \to 0$ as $u \to \infty$ and $u \to 0$ respectively, provided that $p > 1$. Hence Φ is N-function.

□

Theorem 4.2. *For every $\varepsilon > 0$, there exists a positive constant $C = C(p, \varepsilon)$ such that*

$$C^{-1}t^{p-\varepsilon}\Phi(u) \leqslant \Phi(tu) \leqslant Ct^p\Phi(u) \quad t \geqslant 1, u > 0. \tag{42}$$

Proof. If $u \leqslant tu \leqslant e$, then $\Phi(tu) = t^p\Phi(u)$ and (42) holds with $C = 1$.

If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geqslant 1$, we have $\Phi(tu) \leqslant t^p u^p = \frac{p}{p-1} t^p \Phi(u)$. Thus, the last inequality in (42) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$, then $f((tu)^p) \geqslant f(e^p) = e^p/p$. Now,

$$\begin{aligned}\Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} \\ &\geqslant \frac{p-1}{p}\frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p.\end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t+1}$ on the interval $[1, +\infty)$ then

$$\Phi(tu) \geqslant \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,$$

which is the first inequality of (42) with $C = \frac{p}{p-1}\varepsilon^{-1} e^{-1+\varepsilon}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v} \tag{43}$$

where $v := u^p$ and $v \geqslant e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on (α, ∞) and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p, \infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every $v \geqslant e^p$. In this way, from (43), we have

$$\Phi(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and the second inequality of (42) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, then, the function f attains a minimum on $[1, \infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$, f is increasing on $[1, \infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geqslant \varepsilon$ in any case, therefore

$$\Phi(tu) \geqslant \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geqslant \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi(u).$$

Therefore (42) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. $\square$

Remark 2. The inequality

$$\Phi(tu) \geqslant Ct^p\Phi(u)$$

is false for every C because for every $u \geqslant e$ we have

$$\lim_{t\to\infty} \frac{\Phi(tu)}{t^p\Phi(u)} = 0$$

Theorem 4.3. $\alpha_\Phi = \beta_\Phi = p$

Proof. From (22) and (42), we get

$$\beta_\Phi = \lim_{t\to\infty} \frac{\log\left[\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} \leqslant \lim_{t\to\infty} \frac{\log C + p\log t}{\log t} = p.$$

On the other hand, employing (22) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t\to 0^+} \frac{\log\left[\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} = \lim_{s\to\infty} \frac{\log\left[\sup\limits_{v>0} \frac{\Phi(v)}{\Phi(sv)}\right]^{-1}}{\log s} = \lim_{s\to\infty} \frac{\log\left[\inf\limits_{v>0} \frac{\Phi(sv)}{\Phi(v)}\right]}{\log s}$$

where $v := tu$ and $s := \frac{1}{t}$. Then, using (**??**), for every $\varepsilon > 0$ we have

$$\alpha_\Phi = \lim_{s\to\infty} \frac{\log\left[\inf\limits_{v>0} \frac{\Phi(sv)}{\Phi(v)}\right]}{\log s} \geqslant \lim_{s\to\infty} \frac{\log C + (p-\varepsilon)\log s}{\log s} \geqslant p - \varepsilon,$$

therefore $\alpha_\Phi \geqslant p$.

Finally, as $\alpha_\Phi \leqslant \beta_\Phi \leqslant p$, we get $\alpha_\Phi = \beta_\Phi = p$. □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\,dx \geqslant C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^p$$

is false.

In fact, if we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\,dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geqslant C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geqslant Ct^p$ would also be true; however, this last inequality is false.

5 Main result

Lo que sigue estaba en el trabajo anterior, pero debería adaptarse con E_T^Φ?????

In order to find conditions for the lower semicontinuity of I, we perform a little adaptation of a result of [21].

Lemma 5.1. *Let $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ be a differentiable Carathéodory function. Suppose that F satisfies the condition (A) and the inequality*

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geqslant \Phi(|\boldsymbol{y}|) + F(t,\boldsymbol{x}), \tag{44}$$

where Φ is an N-function. In addtion, suppose that $\mathcal{L}(t,\boldsymbol{x},\cdot)$ is convex in $\mathbb{R}^d$ for each $(t,\boldsymbol{x}) \in [0,T]\times\mathbb{R}^d$. Let $\{\boldsymbol{u}_n\} \subset W^1L^\Phi$ be a sequence such that $\boldsymbol{u}_n$ converges uniformly to a function $\boldsymbol{u} \in W^1L^\Phi$ and $\dot{\boldsymbol{u}}_n$ converges in the weak topology of L^1_d to $\dot{\boldsymbol{u}}$. Then

$$I(\boldsymbol{u}) \leqslant \liminf_{n\to\infty} I(\boldsymbol{u}_n). \tag{45}$$

Proof. First, we point out that (44) and (31) imply that I is defined on W^1L^Φ taking values on the interval $(-\infty,+\infty]$. Let $\{\boldsymbol{u}_n\}$ be a sequence satisfying the assumptions of the theorem. We define the differentiable Carathéodory function $\hat{\mathcal{L}} = \mathcal{L} - F$ and we denote by $\hat{I}$ its associated action integral. Using [21, Thm. 2.1, p. 243], we get

$$\int_0^T \hat{\mathcal{L}}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\,dt \leqslant \liminf_{n\to\infty}\int_0^T \hat{\mathcal{L}}(t,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n)\,dt. \tag{46}$$

Taking account of the uniform convergence of $\boldsymbol{u}_n$ and the fact that F is a Carathéodory function, we obtain that $F(t,\boldsymbol{u}_n(t)) \to F(t,\boldsymbol{u}(t))$ a.e. $t\in[0,T]$. Since the sequence $\boldsymbol{u}_n$ is uniformly bounded, from (31) follows that there exists $g \in L^1_1([0,T])$ such that $|F(t,\boldsymbol{u}_n(t))| \leqslant g(t)$. Now, by the Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty}\int_0^T F(t,\boldsymbol{u}_n(t))\,dt = \int_0^T F(t,\boldsymbol{u}(t))\,dt. \tag{47}$$

Finally, as a consequence of (46) and (47), we obtain (45). □

El lema que sigue es nuevo!!!

Lemma 5.2. *E^Φ_d is weak* closed in L^Φ_d.*

Proof. From [5, Thm. 7, p. 110] we have that $L^\Phi_d = \left[E^\Psi_d\right]^*$. Then, L^Φ_d is a dual and therefore we are allowed to speak about the weak* topology of L^Φ_d. Besides, E^Φ_D is separable (see [5, Thm. 1, p. 87]). Let $S = E^\Phi_d \cap \{u \in L^\Phi_d | \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [5, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because is $u_n \in S$ and $u_n \xrightarrow{*} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \xrightarrow{*} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ_d is separable and $L^\Phi_d = \left[E^\Psi_d\right]^*$, the ball of L^Φ $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ is weak* metrizable (see [22, Thm. 5.1, p. 138]). Thus, S with the weak * topology. Now, by the Krein-Smulian Theorem [22, Cor. 12.6, p. 165] implies that E^Φ_d is weak* closed. □

Gathering our previous results we obtain existence of solutions under four different sets of assumptions. We enunciate all these alternatives in the following theorem.

Let $W^1E^\Phi_T = W^1L^\Phi_T \cap W^1E^\Phi_d$.

Theorem 5.3. *Let Φ and Ψ be complementary N-functions. Suppose that the differentiable Carathéodory function $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ is strictly convex at $\boldsymbol{y}$, $D_{\boldsymbol{y}}\mathcal{L}$ is T-periodic with respect to T and* (2), (3), (4), (29), (31) *and* (32) *are satisfied. In addition, assume the same hypothesis than Theorem 3.3. Then, problem* (1) *has a solution.*

Proof. First of all, we prove that I is, lower bounded on $W^1L^\Phi_T$. The condition (32) imply that there exists $M>0$

$$\int_0^T F(t,|x|)dt \geqslant 0, \quad \text{when } |x|>M.$$

Then from (29) and (31)

$$\begin{aligned} I(\boldsymbol{u}) &\geqslant \int_0^T F(t,\boldsymbol{u}(t))dt \\ &= \int_{|\boldsymbol{u}|\leqslant M} F(t,\boldsymbol{u}(t))dt + \int_{|\boldsymbol{u}|>M} F(t,\boldsymbol{u}(t))dt \\ &\geqslant -\|a(\boldsymbol{u})\|_{L^\infty([0,T]\cap\{|\boldsymbol{u}|\leqslant M\})}\|b_0\|_{L^1} = -C_M\|b_0\|_{L^1}, \end{aligned}$$

where $C_M := \sup_{s\in[0,M]}|a(s)|$.

Let $\{\boldsymbol{u}_n\}\subset W^1E^\Phi_T$ (**acordarse de definirlo**) be a minimizing sequence for the problem $\min\{I(\boldsymbol{u})|\boldsymbol{u}\in W^1E^\Phi_T\}$. Since $I(\boldsymbol{u}_n)$, $n=1,2,\dots$ is bounded, Theorem 3.3 implies that $\{\boldsymbol{u}_n\}$ is norm bounded in $W^1E^\Phi_d$. Hence, in virtue of Corollary [13, Corollary 2.2], we can assume that $\boldsymbol{u}_n$ converges uniformly to a T-periodic continuous function $\boldsymbol{u}$. Then, $\boldsymbol{u}$ is bounded and $\boldsymbol{u}\in E^\Phi_d$.

As $\dot{\boldsymbol{u}}_n\in E^\Phi_d\subset L^\Phi_d$, there exists a subsequence (again denoted by $\dot{\boldsymbol{u}}_n$) such that $\dot{\boldsymbol{u}}_n$ converges to a function $\boldsymbol{v}\in L^\Phi_d$ in the weak* topology of L^Φ_d. Since E^Φ_d is weak* closed, by Lemma 5.2, $v\in E^\Phi_d$.

From this fact and the uniform convergence of $\boldsymbol{u}_n$ to $\boldsymbol{u}$, we obtain that

$$\int_0^T \dot{\boldsymbol{\xi}}\cdot\boldsymbol{u}\,dt = \lim_{n\to\infty}\int_0^T \dot{\boldsymbol{\xi}}\cdot\boldsymbol{u}_n\,dt = -\lim_{n\to\infty}\int_0^T \boldsymbol{\xi}\cdot\dot{\boldsymbol{u}}_n\,dt = -\int_0^T \boldsymbol{\xi}\cdot\boldsymbol{v}\,dt$$

for every T-periodic function $\boldsymbol{\xi}\in C^\infty([0,T],\mathbb{R}^d)\subset E^\Psi_d$. Thus $\boldsymbol{v}=\dot{\boldsymbol{u}}$ a.e. $t\in[0,T]$ (see [19, p. 6]) and $\boldsymbol{u}\in E^\Phi_T$.

Now, taking into account the relations $\left[L^1_d\right]^* = L^\infty_d\subset E^\Psi_d$ and $L^\Phi_d\subset L^1_d$, we have that $\dot{\boldsymbol{u}}_n$ converges to $\dot{\boldsymbol{u}}$ in the weak*??? topology of L^1_d. Consequently, Lemma 5.1 applied to the N-function $\alpha_0\Phi\left(|\cdot|/\Lambda\right)$ implies that

$$I(\boldsymbol{u})\leqslant\liminf_{n\to\infty} I(\boldsymbol{u}_n) = \min_{\boldsymbol{u}\in W^1E^\Phi_T} I(\boldsymbol{u}).$$

Hence, $\boldsymbol{u}$ is a minimun and therefore a critical point of I. Finally, invoking Theorem 2.5, the proof concludes. □

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References

[1] C.-L. Tang, Periodic solutions for nonautonomous second order systems with sublinear nonlinearity, Proc. Amer. Math. Soc. 126 (11) (1998) 3263–3270.

[2] X. Tang, X. Zhang, Periodic solutions for second-order Hamiltonian systems with a p-Laplacian, Ann. Univ. Mariae Curie-Skłodowska Sect. A 64 (1) (2010) 93–113.

[3] R. Adams, J. Fournier, Sobolev spaces, Elsevier/Academic Press, Amsterdam, 2003.

[4] M. A. Krasnosel′skiĭ, J. B. Rutickiĭ, Convex functions and Orlicz spaces, P. Noordhoff Ltd., Groningen, 1961.

[5] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Vol. 146, Marcel Dekker, Inc., New York, 1991.

[6] G. Schappacher, A notion of Orlicz spaces for vector valued functions, Appl. Math. 50 (4) (2005) 355–386.

[7] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general, Indag. Math. (N.S.) 11 (4) (2000) 573–585.

[8] A. Cianchi, A fully anisotropic Sobolev inequality, Pacific J. Math. 196 (2) (2000) 283–295.

[9] A. Cianchi, Some results in the theory of Orlicz spaces and applications to variational problems, in: Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), Acad. Sci. Czech Repub., Prague, 1999, pp. 50–92.

[10] N. Clavero, Optimal Sobolev embeddings and Function Spaces, `http://www.maia.ub.edu/~soria/sobolev1.pdf`, last accessed: 2014-12-22. (2011).

[11] D. Edmunds, R. Kerman, L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, J. Funct. Anal. 170 (2) (2000) 307–355.

[12] R. Kerman, L. Pick, Optimal Sobolev imbeddings, Forum Math. 18 (4) (2006) 535–570.

[13] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, E. Schwindt, Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, Nonlinear Analysis, TMA.(to appear).

[14] B. Xu, C.-L. Tang, Some existence results on periodic solutions of ordinary p-Laplacian systems, J. Math. Anal. Appl. 333 (2) (2007) 1228–1236.

[15] R. T. Rockafellar, R. Wets, Variational analysis, Springer-Verlag, Berlin, 1998.

[16] A. Fiorenza, M. Krbec, Indices of Orlicz spaces and some applications, Comment. Math. Univ. Carolin. 38 (3) (1997) 433–451.

[17] J. Gustavsson, J. Peetre, Interpolation of Orlicz spaces, Studia Mathematica 60 (1) (1977) 33–59.
URL http://eudml.org/doc/218150

[18] L. Maligranda, Orlicz spaces and interpolation, Vol. 5 of Seminários de Matemática [Seminars in Mathematics], Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[19] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989.

[20] F. Zhao, X. Wu, Periodic solutions for a class of non-autonomous second order systems, J. Math. Anal. Appl. 296 (2) (2004) 422–434.

[21] I. Ekeland, R. Témam, Convex analysis and variational problems, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[22] J. B. Conway, A Course in Functional Analysis, Springer, USA, 1977.