Periodic solutions of Euler-Lagrange equations with "sublinear nonlinearity" in an Orlicz-Sobolev space setting

Sonia Acinas *
Instituto de Matemática Aplicada San Luis (IMASL)
Universidad Nacional de San Luis and CONICET
Ejército de los Andes 950, (D5700HDW) San Luis, Argentina
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$, of hamiltonian systems with a potential function F satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t)$, with $b_1, b_2 \in L^1$ and for certain N-functions Φ_0.

1 Introduction

The purpose of this paper is to study the existence of periodic solution for the following non-autonomous second-order systems:

$$\begin{cases} \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right) = \nabla F(t,u(t)) \quad \text{a.e. } t \in (0,T) \\ u(0)-u(T) = u'(0)-u'(T) = 0 \end{cases} \tag{1}$$

* SECyT-UNRC, UNSL and CONICET
† SECyT-UNRC and CONICET
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where $T > 0$, $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous and Φ is a differentiable N-function (see section Preliminaries for definitions). Furthermore, the *potential* $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

(C) F and its gradient ∇F are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{2}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

We will call the differential operator

$$L_\Phi[u] = \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right)$$

the Φ*-laplacian operator*. If $\Phi(x) = |x|^p/p$, $1 < p < \infty$, L_Φ is the well known p-laplacian operator. In this case, we have the *Dirichlet problem* for the p-laplacian

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{3}$$

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \Phi(|u'(t)|) + F(t,u(t))\, dt. \tag{4}$$

ESCRIBIR ALGO SOBRE EL RESULTADO PRINCIPAL!!!!

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [1, 2, 3]. For Orlicz spaces of vector valued functions, see [4] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is convex and satisfies that

$$\lim_{t\to+\infty}\frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0}\frac{\Phi(t)}{t} = 0$$

In addition, in this paper we assume that Φ is differentiable and we call φ the derivative of Φ. On these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism whose inverse is ψ.

We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then, Ψ is an N-function which is called the *complementary function* of Φ.

There exist several order relations between N-functions (see [3, Sec. 2.2]). Following [3, Def. 1, p. 15] we say that the N-function Φ_2 is *essentially stronger* than the N-function Φ_1 ($\Phi_1 \ll \Phi_2$) if and only if there exists $x_0 \geqslant 0$ such that $\Phi_1(x) \leqslant \Phi_2(ax)$, for every $a > 0$ and $x \geqslant x_0$.

We also say that a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Δ_2-*condition*, denoted by $\eta \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geqslant 0$ such that

$$\eta(2t) \leqslant K\eta(t), \tag{5}$$

for every $t \geqslant t_0$. If $t_0 = 0$, a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the Δ_2-*condition globally* ($\eta \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ is defined by

$$C_d^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{6}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{7}$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup \left\{ \int_0^T u \cdot v \, dt \middle| \rho_\Psi(v) \leqslant 1 \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [2, Thm. 10.5] and [5]). For every $u \in L^\Phi$,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(ku)\}. \tag{8}$$

In particular

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k} \{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \tag{9}$$

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L_d^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [2, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \tag{10}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset \left[L^\Phi\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \tag{11}$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi = \left[L^\Phi\right]^*$ will not hold. In general, it is true that $\left[E^\Phi\right]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ (see [1]) by

$$W^1 L^\Phi := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi\}.$$

$W^1 L^\Phi$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{12}$$

Moreover, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E^\Phi_T &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{13}$$

For a function $u \in L^1_d([0,T])$, we write $u = \overline{u} + \widetilde{u}$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty_d \hookrightarrow L^\Phi \hookrightarrow L^1_d$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $u : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant C w(|t-s|). \tag{14}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (14) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [6, 7, 8, 9, 10]. The next simple lemma, whose proof can be found in [11], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ and for every $u \in W^1L^\Phi$*

$$|u(t)-u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|) \qquad \textit{(Morrey's inequality)}, \tag{15}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality)}. \tag{16}$$

2. *For every $u \in W^1L^\Phi$ we have $\widetilde{u} \in L^\infty_d$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \textit{(Sobolev-Wirtinger's inequality)}. \tag{17}$$

3 Lagrangians satisfying sublinear nonlinearity type conditions

Lemma 3.1. *E^Φ is weak* closed in L^Φ.*

Proof. From [3, Thm. 7, p. 110] we have that $L^\Phi = \left[E^\Psi\right]^*$. Then, L^Φ is a dual and therefore we are allowed to speak about the weak* topology of L^Φ. Besides, E^Φ is separable (see [3, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [3, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because if $u_n \in S$ and $u_n \xrightarrow{*} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \xrightarrow{*} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ is separable and $L^\Phi = \left[E^\Psi\right]^*$, the ball of L^Φ $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ is weak* metrizable (see [12, Thm. 5.1, p. 138]). Thus, S is closed with respect to the weak* topology. Now, by Krein-Smulian theorem, [12, Cor. 12.6, p. 165] implies that E^Φ is weak* closed. □

The following result is analogous to some lemmata in $W^{1,p}$, see [13].

Lemma 3.2. *If $\|u\|_{W^1L^\Phi} \to \infty$, then $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u} + \tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{18}$$

It is known that $L^\infty_d \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L^\infty_d$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1\|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2 \|u'\|_{L^\Phi}. \tag{19}$$

Therefore, from (18), (19) and (12), we get

$$\|u\|_{W^1 L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1 L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. ☐

Lemma 3.3. *Let Φ, Ψ be complementary functions. The next statements are equivalent:*

1. *$\Psi \in \Delta_2$ globally.*
2. *There exists an N-function Φ_1 such that*

$$\Phi(rs) \geqslant \Phi_1(r)\Phi(s) \text{ for every } r \geqslant 1, \quad s \geqslant 0. \tag{20}$$

Proof. 1)⇒2) By virtue of the Δ_2-condition on Ψ, [14, Thm. 11.7] and [14, Cor. 11.6] (see also [15, Eq. (2.8)]), we get constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geqslant K r^\nu \Phi(s), \tag{21}$$

for any $1 < \nu < \alpha_\Phi$, $s \geqslant 0$ and $r > 1$. This proves (20) with $\Phi_1(r) = kr^\nu$, which is an N-function.

2)⇒1) Next, we follow [3, p. 32, Prop. 13] and [3, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leqslant \Phi(rs) \quad r > 1, \ s \geqslant 0.$$

Let $u = \Phi_1(r) \geqslant \Phi_1(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [3, p. 13, Prop. 1] and (20), we have for $u \geqslant \Phi_1(1)$ and $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant \Phi_1^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi_1(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi_1(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$ we get that $\Psi \in \Delta_2$ globally. ☐

The following lemma generalizes [11, Lemma 5.2].

Lemma 3.4. *Let Φ, Ψ be complementary N-functions with $\Psi \in \Delta_2$ globally. Let Φ_1 be any N-function satisfying* (20). *Then*

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \tag{22}$$

for every Φ_0 with $\Phi_0 \lll \Phi_1$).

If (22) *holds for some N-function Φ_0, then $\Psi \in \Delta_2$ (at ∞).*

Proof. By the assumptions on Φ and Φ_1 and inequality (9), for $r > 1$ we have

$$\frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \Phi_1(r)\frac{\int_0^T \Phi(r^{-1}|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})}\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$ and as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$ and by [3, Thm. 2 (b), p. 16].

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

Finally, if Φ_0 is an N-function, then $\Phi_0(u) \geqslant k|u|$ for k small enough and $|u| > 1$. Therefore (22) holds for $\Phi_0(u) = |u|$, then [11, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞. $\square$

Remark 1. We point out that this lemma can be applied to more cases than [11, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [2, p. 16] and [2, Sec. 7] for the definition and properties of principal part), then (22) holds for Φ_0. However, $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

Definition 3.5. *We define the functionals $J_{C,\Phi_0} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\Phi_0} : \mathbb{R}^n \to \mathbb{R}$, where $C > 0$ and Φ_0 is an N-function, by*

$$J_{C,\Phi_0}(u) := \rho_\Phi(u) - C\Phi_0(\|u\|_{L^\Phi}), \tag{23}$$

and

$$H_{C,\Phi_0}(x) := \int_0^T F(t,x)dt - C\Phi_0(|x|), \tag{24}$$

respectively.

In [16] and [17] the authors considered, for the p-laplacian case, potentials F satisfying the inequality

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^\alpha + b_2(t), \tag{25}$$

where $b_1, b_2 \in L^1_1$ and $\alpha < p$. Thus, they called F a sublinear nonlinearity. In this paper, we will consider bounds on ∇F of a more general type.

Definition 3.6. *Let Φ_0 be a differentiable N-function. We say that $G : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies a* Φ_0-grow condition *if*

$$|G(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t), \tag{26}$$

with $b_1, b_2 \in L^1([0,T],\mathbb{R})$.

Theorem 3.7. *Let Φ be an N-function whose complementary function Ψ satisfies the Δ_2 condition globally. Assume that the N-function Φ_1 satisfies* (20)*, F satisfies (C) and (A), and ∇F satisfies a Φ_0-grow condition for some Δ_2-globally N-function Φ_0 such that $\Phi_0 \ll \Phi_1$. Furthermore, we suppose that*

$$\lim_{|x|\to\infty} \frac{\int_0^T F(t,x)\,dt}{\Phi_0(|x|)} = +\infty. \tag{27}$$

Then, the problem (1) *has at least a solution which minimizes the action integral I on $W^1E_T^\Phi$.*

Proof. By the decomposition $u = \overline{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (26), we have

$$\begin{aligned} &\left|\int_0^T F(t,u) - F(t,\overline{u})\,dt\right| = \left|\int_0^T \int_0^1 \nabla F(t, \overline{u} + s\tilde{u}(t))\cdot \tilde{u}(t)\,ds\,dt\right| \\ &\leqslant \int_0^T\int_0^1 b_1(t)\Phi_0'(|\overline{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt + \int_0^T\int_0^1 b_2(t)|\tilde{u}(t)|\,ds\,dt \\ &=: I_1 + I_2. \end{aligned} \tag{28}$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty} \leqslant C_1\|u'\|_{L^\Phi}, \tag{29}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

On the other hand, since $\Phi_0 \in \Delta_2$ globally, then $\Phi_0' \in \Delta_2$ globally and consequently Φ_0' is a quasi-subadditive function, i.e. there exists $C(\Phi_0) > 0$ such that $\Phi_0'(a+b) \leqslant C(\Phi_0')(\Phi_0'(a) + \Phi_0'(b))$ for every $a, b \geqslant 0$. In this way, we have

$$\Phi_0'(|\overline{u} + s\tilde{u}(t)|) \leqslant C(\Phi_0)[\Phi_0'(|\overline{u}|) + \Phi_0'(\|\tilde{u}\|_{L^\infty})], \tag{30}$$

for every $s \in [0,1]$.

Now, inequality (30), Hölder's and Sobolev-Wirtinger's inequalities, the monotonicity, the subadditivity and the Δ_2-condition on Φ_0', imply that

$$\begin{aligned} I_1 &\leqslant C(\Phi_0')\left\{\Phi_0'(|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty} + \|b_1\|_{L^1}\Phi_0'(\|\tilde{u}\|_{L^\infty})\|\tilde{u}\|_{L^\infty}\right\} \\ &\leqslant C_2\left\{\Phi_0'(|\overline{u}|)\|u'\|_{L^\Phi} + \Phi_0'(\|u'\|_{L^\Phi})\|u'\|_{L^\Phi}\right\}, \end{aligned} \tag{31}$$

where $C_2 = C_2(\Phi_0', T, \|b_1\|_{L^1})$.

Next, by Young's inequality with complementary functions Φ_0 and Ψ_0 and the fact that $\Phi_0 \in \Delta_2$ globally, Young's equality [2, Eq. 2.7-2.8] and [3, Thm. 3-(ii), p. 23], we get

$$\begin{aligned} \Phi_0'(|\overline{u}|)\|u'\|_{L^\Phi} &\leqslant \Psi_0(\Phi_0'(|\overline{u}|)) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leqslant |\overline{u}|\Phi_0'(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leqslant C(\Phi_0)\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \end{aligned} \tag{32}$$

and

$$\Phi_0'(\|u'\|_{L^\Phi})\|u'\|_{L^\Phi} \leqslant C(\Phi_0)\Phi_0(\|u'\|_{L^\Phi}), \tag{33}$$

with $C(\Phi_0)$ the constant that comes from the Δ_2-condition on Φ_0.

From (31), (32), (33) and (29), we have

$$\begin{aligned} I_1 + I_2 &\leqslant C_3\Big\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi}\Big\} \\ &\leqslant C_4\Big\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + 1\Big\}, \end{aligned} \tag{34}$$

with C_3 and C_4 depending on $\Phi_0, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$. The last inequality follows from the fact that Φ_0 is an N-function, then there exists $C > 0$ such that $\Phi_0(x) \geqslant Cx$ for every $x \geqslant 1$. Thus $x \leqslant C\Phi_0(x) + 1$ for every $x \geqslant 0$.

In the subsequent estimates, we use (28), (34), the fact that $\Phi_0 \in \Delta_2$ and we get

$$\begin{aligned} I(u) &= \rho_\Phi(u') + \int_0^T F(t,u)\,dt \\ &= \rho_\Phi(u') + \int_0^T [F(t,u) - F(t,\overline{u})]\,dt + \int_0^T F(t,\overline{u})\,dt \\ &\geqslant \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_4\Phi_0(|\overline{u}|) - C_4 \\ &\geqslant \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + H_{C_4,\Phi_0}(\overline{u}) - C_4 \\ &= J_{C_4,\Phi_0}(u') + H_{C_4,\Phi_0}(\overline{u}) - C_4. \end{aligned} \tag{35}$$

Let u_n be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n, $I(u_n)$ is upper bounded, i.e. there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 3.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence is necessary, still denoted u_n, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.4 implies that the functional $J_{C_4,\Phi_0}(u')$ is coercive; and, by (27), the functional $H_{C_4,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_4,\Phi_0}(u_n') \to \infty$ or $H_{C_4,\Phi_0}(\overline{u}_n) \to \infty$. From the condition (A) on F, we have that on a bounded set the functional $H_{C_4,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_4,\Phi_0}(u_n') \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u)|u \in W^1E_T^\Phi\}$. Since $I(u_n)$, $n = 1, 2, \ldots$, is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in W^1E^Φ. Hence, by virtue of [11, Cor. 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T-periodic continuous (therefore in E^Φ_T) function u. As $u'_n \in E^\Phi$ is a norm bounded sequence in L^Φ, there exists a subsequence (again denoted by u'_n) such that u'_n converges to a function $v \in L^\Phi$ in the weak* topology of L^Φ. Since E^Φ is weak* closed, by Lemma 3.1, $v \in E^\Phi$. From this fact and the uniform convergence of u_n to u, we obtain that

$$\int_0^T \xi' \cdot u\, dt = \lim_{n\to\infty} \int_0^T \xi' \cdot u_n\, dt = -\lim_{n\to\infty} \int_0^T \xi \cdot u'_n\, dt = -\int_0^T \xi \cdot v\, dt$$

for every T-periodic function $\xi \in C^\infty([0,T],\mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [18, p. 6]) and $u \in W^1E^\Phi_T$.

Now, taking into account the relations $\left[L^1\right]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that u'_n converges to u' in the weak topology of L^1. Consequently, from the semicontinuity of I (see [11, Lemma 6.1]) we get

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{v\in W^1E^\Phi_T} I(v).$$

Hence $u \in W^1E^\Phi_T$ is a minimun and, since I is Gâteaux differentiable on W^1E^Φ (see [11, Thm. 3.2]), therefore $I'(u) \in (W^1E^\Phi_T)^\perp$. Thus,

$$\int_0^T \frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t)\cdot v'(t)dt = -\int_0^T \nabla F(t,u(t))\cdot v(t)dt,$$

for every $v \in W^1E^\Phi_T$.

From [11, Lemma 2.4] we have $u'(t)\Phi'(|u'(t)|)/|u'(t)| \in L^\Psi([0,T],\mathbb{R}^n) \hookrightarrow L^1([0,T],\mathbb{R}^n)$; and, from condition (A) and the fact that $u \in L^\infty$, it follows that $\nabla F(t,u(t)) \in L^1([0,T],\mathbb{R}^n)$. Consequently, from [18, p. 6] we obtain that the differential equations in (1) are verified and $u'(0)\Phi'(|u'(0)|)/|u'(0)| = u'(T)\Phi'(|u'(T)|)/|u'(T)|$ holds. Thus $u'(0) = u'(T)$. □

3.1 More general action integrals

Muuuuuy en construcción.....Fernando: no sé si era la idea que tenías en mente.....

Let us consider the following problem

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) & \text{a.e. } t \in (0,T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(T) = 0 \end{cases} \tag{36}$$

where $T > 0$, $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ is a Carathéodory and differentiable function satisfying the conditions

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geqslant \Phi(|\boldsymbol{y}|) + F(t,\boldsymbol{x}), \tag{37}$$

$$|\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{38}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{39}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(c(t) + \varphi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right). \tag{40}$$

and where the potential F satisfies conditions (A) and (C).

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ. The non negative functions b, c and f satisfy that $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$, where the Banach spaces $L^1_1([0,T])$, $L^\Psi_1([0,T])$ and $E^\Phi_1([0,T])$.

Now, performing some slight modifications in the proof of Theorem 3.7 (in first line of (35), $=$ has to be changed by $\geqslant$ and Thm. 4.1 of [11] can be applied to prove $u'(0) = u'(T)$ exchanging W^1L^Φ by $W^1E^\Phi_T$), we also get that problem (36) has at least one solution which minimizes the action integral

$$I(\boldsymbol{u}) = \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt. \tag{41}$$

4 Examples growing conditions with power functions

The employment of N-functions instead of power functions in inequalities like (26) will allow us to extend some results of [16] and [17] a Φ-laplacian operators with N-functions Φ which grow faster that power functions, for example with a exponential grow. Furthermore, we extend some results of [16] and [17] even in the case of p-laplacian operator (3), because we get bounds that may be more sharp that those in [16, 17]. More precisely, in [17, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions of (3) under la assumption (25) for any $\alpha \in (0, p-1)$. Meanwhile, our Theorem 3.7 implies existence for the potential

$$F(t,x) = |x|^p / \sqrt{\ln(2 + |x|)}.$$

We note that this F does not satisfies (25) for any $\alpha < p - 1$. Next we will show a N-function Φ satisfying the hypothesis of Theorem 3.7 for the potential F.

The function Φ

As it has been seen in the proof of Lemma 3.3 and in [11, Lemma 5.2], the function Φ_0 in Lemma 3.3 may be assumed a power function $\Phi_0(x) = |x|^\mu$ with $0 < \mu < \alpha_\Phi$ and α_Φ is a Matuszewska-Orlicz indices (see [14, Ch 11]). These indices are defined by

$$\alpha_\Phi := \lim_{t \to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right)}{\log(t)}, \quad \beta_\Phi := \lim_{t \to +\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right)}{\log(t)}. \tag{42}$$

Hence, following the same lines that in the proof of Theorem 3.7, using [11, Lemma 5.2] instead of our Lemma 3.3, we can assume that $\Phi_0(x) = |x|^\mu$ with $0 < \mu < \alpha_\Phi$ in Theorem 3.7.

Assuming $\|b_1\|_{L^1}$ small enough, in [19, 17] coercivity was obtained even for the limit value $\alpha = p - 1$ in inequality (26).

This result leans on the fact that

$$\|u\|^{\alpha_\Phi}_{L^\Phi} = O\left(\int_0^T \Phi(|u|)\, dt\right) \quad \text{for } \|u\|_{L^\Phi} \to \infty, \tag{43}$$

when $\Phi(u) = |u|^p$. Nevertheless, it is no longer the case for any N-function Φ as the following example shows.

In this section, from now on we will suppose that

$$\Phi(u) = \begin{cases} \frac{p-1}{p}u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function Φ.

Theorem 4.1. *If $p \geqslant \frac{1+\sqrt{2}}{2}$, then Φ is an N-function.*

Proof. We have

$$\varphi(u) = \Phi'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that Φ' is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that φ_1 is increasing on $[0, e]$ and φ_2 is increasing on $[e, \infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p > 1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an icreasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Besides $\varphi_2(u) \to \infty$ and $\varphi_1(u) \to 0$ as $u \to \infty$ and $u \to 0$ respectively, provided that $p > 1$. Hence, Φ is an N-function. $\square$

Theorem 4.2. *For every $\varepsilon > 0$, there exists a positive constant $C = C(p, \varepsilon)$ such that*

$$C^{-1}t^{p-\varepsilon}\Phi(u) \leqslant \Phi(tu) \leqslant Ct^p\Phi(u) \quad t \geqslant 1, u > 0, \tag{44}$$

Proof. If $u \leqslant tu \leqslant e$, then $\Phi(tu) = t^p\Phi(u)$ and (44) holds with $C = 1$.

If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geqslant 1$, we have $\Phi(tu) \leqslant t^p u^p = \frac{p}{p-1}t^p\Phi(u)$. Thus, the second inequality of (44) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$, then $f((tu)^p) \geqslant f(e^p) = e^p/p$. Now,

$$\begin{aligned} \Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} \\ &\geqslant \frac{p-1}{p}\frac{t^\varepsilon}{\log t + 1}t^{p-\varepsilon}u^p. \end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t+1}$ on the interval $[1, +\infty)$ then

$$\Phi(tu) \geqslant \frac{p-1}{p}\varepsilon e^{1-\varepsilon}t^{p-\varepsilon}u^p,$$

which is the first inequality of (44) with $C = \frac{p}{p-1}\varepsilon^{-1}e^{-1+\varepsilon}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v}, \tag{45}$$

where $v := u^p$ and $v \geqslant e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on (α, ∞) and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p, \infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every $v \geqslant e^p$. In this way, from (45), we have

$$\Phi(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and the second inequality of (44) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, the function f attains a minimum on $[1, \infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$, f is increasing on $[1, \infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geqslant \varepsilon$ in any case, therefore

$$\Phi(tu) \geqslant \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geqslant \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi(u).$$

Therefore, (44) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. □

Remark 2. The inequality

$$\Phi(tu) \geqslant Ct^p\Phi(u)$$

is false for every C because for every $u \geqslant e$ we have

$$\lim_{t\to\infty}\frac{\Phi(tu)}{t^p\Phi(u)} = 0$$

Theorem 4.3. $\alpha_\Phi = \beta_\Phi = p$

Proof. From (42) and (44), we get

$$\beta_\Phi = \lim_{t\to\infty}\frac{\log\left[\sup_{u>0}\frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} \leqslant \lim_{t\to\infty}\frac{\log C + p\log t}{\log t} = p.$$

On the other hand, employing (42) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t\to 0^+} \frac{\log\left[\sup\limits_{u>0}\frac{\Phi(tu)}{\Phi(u)}\right]}{\log t} = \lim_{s\to\infty} \frac{\log\left[\sup\limits_{v>0}\frac{\Phi(v)}{\Phi(sv)}\right]^{-1}}{\log s} = \lim_{s\to\infty} \frac{\log\left[\inf\limits_{v>0}\frac{\Phi(sv)}{\Phi(v)}\right]}{\log s}$$

where $v := tu$ and $s := \frac{1}{t}$. Then, using (44), for every $\varepsilon > 0$ we have

$$\alpha_\Phi = \lim_{s\to\infty} \frac{\log\left[\inf\limits_{v>0}\frac{\Phi(sv)}{\Phi(v)}\right]}{\log s} \geqslant \lim_{s\to\infty} \frac{\log C + (p-\varepsilon)\log s}{\log s} \geqslant p - \varepsilon,$$

therefore $\alpha_\Phi \geqslant p$.

Finally, as $\alpha_\Phi \leqslant \beta_\Phi \leqslant p$, we get $\alpha_\Phi = \beta_\Phi = p$. $\square$

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\, dx \geqslant C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^p$$

is false.

In fact, if we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\, dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geqslant C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geqslant Ct^p$ would also be true; however, this last inequality is false.

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