

Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1 L^\Phi([0, T])$, of hamiltonian systems with a potential function F satisfying the inequality $|\nabla F(t, x)| \leq b_1(t)\Phi'_0(|x|) + b_2(t)$, with $b_1, b_2 \in L^1$ and for certain N -functions Φ_0 .

1 Introduction

The purpose of this paper is to study the existence of periodic solution for the following non-autonomous second-order systems:

$$\begin{cases} \frac{d}{dt} \left(u'(t) \frac{\Phi'(|u'|)}{|u'|} \right) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \quad (1)$$

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where $T > 0$, $u : [0, T] \rightarrow \mathbb{R}^d$ is absolutely continuous and Φ is a differentiable N -function (see section Preliminaries for definitions). Furthermore, the *potential* $F : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the following conditions:

(C) F and its gradient ∇F are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

(A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (2)$$

In this inequality we assume that the function $a : [0, +\infty) \rightarrow [0, +\infty)$ is continuous and nondecreasing and $0 \leq b \in L^1([0, T], \mathbb{R})$.

We will call the differential operator

$$L_\Phi[u] = \frac{d}{dt} \left(u'(t) \frac{\Phi'(|u'|)}{|u'|} \right)$$

the Φ -laplacian operator. If $\Phi(x) = |x|^p$, $1 < p < \infty$, L_Φ is the well known p -laplacian operator.

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \Phi(|u'(t)|) + F(t, u(t)) dt. \quad (3)$$

ESCRIBIR ALGO SOBRE EL RESULTADO PRINCIPAL!!!!

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [1, 2, 3]. For Orlicz spaces of vector valued functions, see [4] and the references therein.

Hereafter we denote by \mathbb{R}^+ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called an N -function if Φ is convex and satisfies that

$$\lim_{t \rightarrow +\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$$

In addition, in this paper we assume that Φ is differentiable and we call φ the derivative of Φ . On these assumptions, $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a homeomorphism whose inverse is ψ . We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then, Ψ is an N -function which is called the *complementary function* of Φ .

There exist several order relations between N -functions (see [3, Sec. 2.2]). Following [3, Def. 1, p. 15] we say that the N -function Φ_2 is *essentially stronger* than the N -function Φ_1 ($\Phi_1 \ll \Phi_2$) if and only if there exists $x_0 \geq 0$ such that $\Phi_1(x) \leq \Phi_2(ax)$, for every $a > 0$ and $x \geq x_0$.

We also say that a function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the Δ_2 -condition, denoted by $\eta \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geq 0$ such that

$$\eta(2t) \leq K\eta(t), \quad (4)$$

for every $t \geq t_0$. If $t_0 = 0$, a function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to satisfy the Δ_2 -condition globally ($\eta \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0, T]$ with values on \mathbb{R}^d and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$.

Given an N -function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|) dt.$$

Here $|\cdot|$ is the euclidean norm of \mathbb{R}^d . The *Orlicz class* $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$ is defined by

$$C_d^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (5)$$

The *Orlicz space* $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$ is the linear hull of C^Φ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (6)$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup \left\{ \int_0^T u \cdot v dt \mid \rho_\Psi(v) \leq 1 \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in \mathbb{R}^d between u and v . The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [2, Thm. 10.5] and [5]). For every $u \in L^\Phi$,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(ku)\}. \quad (7)$$

In particular

$$\|u\|_{L^\Phi} \leq \frac{1}{k} \{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \quad (8)$$

The subspace $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L_d^\infty([0, T], \mathbb{R}^d)$ of all \mathbb{R}^d -valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ , i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [2, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u dt \leq \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \quad (9)$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset [L^\Phi]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \quad (10)$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi = [L^\Phi]^*$ will not hold. In general, it is true that $[E^\Phi]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ (see [1]) by

$$W^1 L^\Phi := \{u \mid u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi\}.$$

$W^1 L^\Phi$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (11)$$

Moreover, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi \mid u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi \mid u(0) = u(T)\}. \end{aligned} \quad (12)$$

For a function $u \in L_d^1([0, T])$, we write $u = \bar{u} + \tilde{u}$ where $\bar{u} = \frac{1}{T} \int_0^T u(t) \, dt$ and $\tilde{u} = u - \bar{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \rightarrow X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leq C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow [L^\Phi]^*$; and, it is easy to see that for every N -function Φ we have that $L_d^\infty \hookrightarrow L^\Phi \hookrightarrow L_d^1$.

Recall that a function $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N -function Φ . We say that $u : [0, T] \rightarrow \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leq Cw(|t - s|). \quad (13)$$

We denote by $C^w([0, T], \mathbb{R}^d)$ the space of w -Hölder continuous functions. This is the space of all functions satisfying (13) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0, T], \mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [6, 7, 8, 9, 10]. The next simple lemma, whose proof can be found in [11], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1 L^\Phi \hookrightarrow C^w([0, T], \mathbb{R}^d)$ and for every $u \in W^1 L^\Phi$

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} w(|t - s|) \quad (\text{Morrey's inequality}), \quad (14)$$

$$\|u\|_{L^\infty} \leq \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality}). \quad (15)$$

2. For every $u \in W^1 L^\Phi$ we have $\tilde{u} \in L_d^\infty$ and

$$\|\tilde{u}\|_{L^\infty} \leq T \Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality}). \quad (16)$$

3 Lagrangians satisfying sublinear nonlinearity type conditions

Lemma 3.1. E^Φ is weak* closed in L^Φ .

Proof. From [3, Thm. 7, p. 110] we have that $L^\Phi = [E^\Psi]^*$. Then, L^Φ is a dual and therefore we are allowed to speak about the weak* topology of L^Φ . Besides, E^Φ is separable (see [3, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi \mid \|u\|_{L^\Phi} \leq 1\}$, then S is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [3, Cor. 5, p. 148] S is weak* sequentially compact. Thus, S is weak* sequentially closed because if $u_n \in S$ and $u_n \xrightarrow{*} u \in L^\Phi$ then the weak* sequential compactness implies the existence of $v \in S$ and a subsequence u_{n_k} such that $u_{n_k} \xrightarrow{*} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As E^Ψ is separable and $L^\Phi = [E^\Psi]^*$, the ball of L^Φ $\{u \in L^\Phi \mid \|u\|_{L^\Phi} \leq 1\}$ is weak* metrizable (see [12, Thm. 5.1, p. 138]). Thus, S is closed with respect to the weak* topology. Now, by Krein-Smulian theorem, [12, Cor. 12.6, p. 165] implies that E^Φ is weak* closed. \square

The following result is analogous to some lemmata in $W^{1,p}$, see [13].

Lemma 3.2. If $\|u\|_{W^1 L^\Phi} \rightarrow \infty$, then $(|\bar{u}| + \|u'\|_{L^\Phi}) \rightarrow \infty$.

Proof. By the decomposition $u = \bar{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\bar{u} + \tilde{u}\|_{L^\Phi} \leq \|\bar{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\bar{u}| \|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \quad (17)$$

It is known that $L_d^\infty \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L_d^\infty$ we have

$$\|\tilde{u}\|_{L^\Phi} \leq C_1 \|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leq C_2 \|u'\|_{L^\Phi}. \quad (18)$$

Therefore, from (17), (18) and (11), we get

$$\|u\|_{W^1 L^\Phi} \leq C_3 (|\bar{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1 L^\Phi} \rightarrow \infty$ we conclude that $(|\bar{u}| + \|u'\|_{L^\Phi}) \rightarrow \infty$. \square

Lemma 3.3. *Let Φ, Ψ be complementary functions. The next statements are equivalent:*

1. $\Psi \in \Delta_2$ globally.
2. There exists an N -function Φ_1 such that

$$\Phi(rs) \geq \Phi_1(r)\Phi(s) \text{ for every } r \geq 1, s \geq 0. \quad (19)$$

Proof. 1) \Rightarrow 2) By virtue of the Δ_2 -condition on Ψ , [14, Thm. 11.7] and [14, Cor. 11.6] (see also [15, Eq. (2.8)]), we get constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geq Kr^\nu \Phi(s), \quad (20)$$

for any $1 < \nu < \alpha_\Phi$, $s \geq 0$ and $r > 1$. This proves (19) with $\Phi_1(r) = kr^\nu$, which is an N -function.

2) \Rightarrow 1) Next, we follow [3, p. 32, Prop. 13] and [3, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leq \Phi(rs) \text{ } r > 1, s \geq 0.$$

Let $u = \Phi_1(r) \geq \Phi_1(1)$ and $v = \Phi(s) \geq 0$. By a well known inequality [3, p. 13, Prop. 1] and (19), we have for $u \geq \Phi_1(1)$ and $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leq \Phi^{-1}(uv) \leq \Phi_1^{-1}(u)\Phi^{-1}(v) \leq \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leq 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geq \Psi_1^{-1}(\Phi_1(1))$ and $y = \Psi^{-1}(v) \geq 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leq \Psi_1(x)\Psi(y).$$

Now, taking $x \geq \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$ we get that $\Psi \in \Delta_2$ globally. \square

The following lemma generalizes [11, Lemma 5.2].

Lemma 3.4. *Let Φ, Ψ be complementary N -functions with $\Psi \in \Delta_2$ globally. Let Φ_1 be any N -function satisfying (19). Then*

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \quad (21)$$

for every Φ_0 with $\Phi_0 \ll \Phi_1$.

If (21) holds for some N -function Φ_0 , then $\Psi \in \Delta_2$ (at ∞).

Proof. By the assumptions on Φ and Φ_1 and inequality (8), for $r > 1$ we have

$$\frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \Phi_1(r) \frac{\int_0^T \Phi(r^{-1}|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})} \{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$ and as $\|u\|_{L^\Phi} \rightarrow \infty$ we can assume $r > 1$ and by [3, Thm. 2 (b), p. 16].

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

Finally, if Φ_0 is an N -function, then $\Phi_0(u) \geq k|u|$ for k small enough and $|u| > 1$. Therefore (21) holds for $\Phi_0(u) = |u|$, then [11, Lemma 5.2] implies $\Psi \in \Delta_2$ at ∞ . \square

Remark 1. We point out that this lemma can be applied to more cases than [11, Lemma 5.2]. For example, if $\Phi(u) = u^2$, Φ_1 and Φ_0 are N -functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [2, p. 16] and [2, Sec. 7] for the definition and properties of principal part), then (21) holds for Φ_0 . However, $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

Definition 3.5. We define the functionals $J_{C,\Phi_0} : L^\Phi \rightarrow (-\infty, +\infty]$ and $H_{C,\Phi_0} : \mathbb{R}^n \rightarrow \mathbb{R}$, where $C > 0$ and Φ_0 is an N -function, by

$$J_{C,\Phi_0}(u) := \rho_\Phi(u) - C\Phi_0(\|u\|_{L^\Phi}), \quad (22)$$

and

$$H_{C,\Phi_0}(x) := \int_0^T F(t, x) dt - C\Phi_0(|x|), \quad (23)$$

respectively.

In [16] and [17] the authors considered, for the p -laplacian case, potentials F satisfying the inequality

$$|\nabla F(t, x)| \leq b_1(t)|x|^\alpha + b_2(t),$$

where $b_1, b_2 \in L_1^1$ and $\alpha < p$. Thus, they called F a sublinear nonlinearity. In this paper, we will consider bounds on ∇F of a more general type.

Definition 3.6. Let Φ_0 be a differentiable N -function. We say that $G : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies a Φ_0 -grow condition if

$$|G(t, x)| \leq b_1(t)\Phi_0'(|x|) + b_2(t), \quad (24)$$

with $b_1, b_2 \in L^1([0, T], \mathbb{R})$.

Theorem 3.7. Let Φ be an N -function whose complementary function Ψ satisfies the Δ_2 condition globally. Assume that the N -function Φ_1 satisfies (19), F satisfies (C)

and (A), and ∇F satisfies a Φ_0 -grow condition for some Δ_2 -globally N -function Φ_0 such that $\Phi_0 \ll \Phi_1$. Furthermore, we suppose that

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\Phi_0(|x|)} = +\infty. \quad (25)$$

Then, the problem (1) has at least a solution which minimizes the action integral I on $W^1 E_T^\Phi$.

Proof. By the decomposition $u = \bar{u} + \tilde{u}$, Cauchy-Schwarz's inequality and (24), we have

$$\begin{aligned} \left| \int_0^T F(t, u) - F(t, \bar{u}) dt \right| &= \left| \int_0^T \int_0^1 \nabla F(t, \bar{u} + s\tilde{u}(t)) \cdot \tilde{u}(t) ds dt \right| \\ &\leq \int_0^T \int_0^1 b_1(t) \Phi'_0(|\bar{u} + s\tilde{u}(t)|) |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 b_2(t) |\tilde{u}(t)| ds dt \\ &=: I_1 + I_2. \end{aligned} \quad (26)$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate I_2 as follows

$$I_2 \leq \|b_2\|_{L^1} \|\tilde{u}\|_{L^\infty} \leq C_1 \|u'\|_{L^\Phi}, \quad (27)$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$.

On the other hand, since $\Phi_0 \in \Delta_2$ globally, then $\Phi'_0 \in \Delta_2$ globally and consequently Φ'_0 is a quasi-subadditive function, i.e. there exists $C(\Phi_0) > 0$ such that $\Phi'_0(a + b) \leq C(\Phi'_0)(\Phi'_0(a) + \Phi'_0(b))$ for every $a, b \geq 0$. In this way, we have

$$\Phi'_0(|\bar{u} + s\tilde{u}(t)|) \leq C(\Phi_0)[\Phi'_0(|\bar{u}|) + \Phi'_0(\|\tilde{u}\|_{L^\infty})], \quad (28)$$

for every $s \in [0, 1]$.

Now, inequality (28), Hölder's and Sobolev-Wirtinger's inequalities, the monotonicity, the subadditivity and the Δ_2 -condition on Φ'_0 , imply that

$$\begin{aligned} I_1 &\leq C(\Phi'_0) \left\{ \Phi'_0(|\bar{u}|) \|b_1\|_{L^1} \|\tilde{u}\|_{L^\infty} + \|b_1\|_{L^1} \Phi'_0(\|\tilde{u}\|_{L^\infty}) \|\tilde{u}\|_{L^\infty} \right\} \\ &\leq C_2 \left\{ \Phi'_0(|\bar{u}|) \|u'\|_{L^\Phi} + \Phi'_0(\|u'\|_{L^\Phi}) \|u'\|_{L^\Phi} \right\}, \end{aligned} \quad (29)$$

where $C_2 = C_2(\Phi'_0, T, \|b_1\|_{L^1})$.

Next, by Young's inequality with complementary functions Φ_0 and Ψ_0 and the fact that $\Phi_0 \in \Delta_2$ globally, Young's equality [2, Eq. 2.7-2.8] and [3, Thm. 3-(ii), p. 23], we get

$$\begin{aligned} \Phi'_0(|\bar{u}|) \|u'\|_{L^\Phi} &\leq \Psi_0(\Phi'_0(|\bar{u}|)) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leq |\bar{u}| \Phi'_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leq C(\Phi_0) \Phi_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \end{aligned} \quad (30)$$

and

$$\Phi'_0(\|u'\|_{L^\Phi})\|u'\|_{L^\Phi} \leq C(\Phi_0)\Phi_0(\|u'\|_{L^\Phi}), \quad (31)$$

with $C(\Phi_0)$ the constant that comes from the Δ_2 -condition on Φ_0 .

From (29), (30), (31) and (27), we have

$$\begin{aligned} I_1 + I_2 &\leq C_3 \left\{ \Phi_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi} \right\} \\ &\leq C_4 \left\{ \Phi_0(|\bar{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + 1 \right\}, \end{aligned} \quad (32)$$

with C_3 and C_4 depending on $\Phi_0, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$. The last inequality follows from the fact that Φ_0 is an N -function, then there exists $C > 0$ such that $\Phi_0(x) \geq Cx$ for every $x \geq 1$. Thus $x \leq C\Phi_0(x) + 1$ for every $x \geq 0$.

In the subsequent estimates, we use (26), (32), the fact that $\Phi_0 \in \Delta_2$ and we get

$$\begin{aligned} I(u) &= \rho_\Phi(u') + \int_0^T F(t, u) dt \\ &= \rho_\Phi(u') + \int_0^T [F(t, u) - F(t, \bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\ &\geq \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + \int_0^T F(t, \bar{u}) dt - C_4\Phi_0(|\bar{u}|) - C_4 \\ &\geq \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + H_{C_4, \Phi_0}(\bar{u}) - C_4 \\ &= J_{C_4, \Phi_0}(u') + H_{C_4, \Phi_0}(\bar{u}) - C_4. \end{aligned} \quad (33)$$

Let u_n be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$ and we have to prove that $I(u_n) \rightarrow \infty$. On the contrary, suppose that for a subsequence, still denoted by u_n , $I(u_n)$ is upper bounded, i.e. there exists $M > 0$ such that $|I(u_n)| \leq M$. As $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$, from Lemma 3.2, we have $|\bar{u}_n| + \|u'_n\|_{L^\Phi} \rightarrow \infty$. Passing to a subsequence is necessary, still denoted u_n , we can assume that $|\bar{u}_n| \rightarrow \infty$ or $\|u'_n\|_{L^\Phi} \rightarrow \infty$. Now, Lemma 3.4 implies that the functional $J_{C_4, \Phi_0}(u')$ is coercive; and, by (25), the functional $H_{C_4, \Phi_0}(\bar{u})$ is also coercive, then $J_{C_4, \Phi_0}(u'_n) \rightarrow \infty$ or $H_{C_4, \Phi_0}(\bar{u}_n) \rightarrow \infty$. From the condition (A) on F , we have that on a bounded set the functional $H_{C_4, \Phi_0}(\bar{u}_n)$ is lower bounded and also $J_{C_4, \Phi_0}(u'_n) \geq 0$. Therefore, $I(u_n) \rightarrow \infty$ as $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1 E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u) | u \in W^1 E_T^\Phi\}$. Since $I(u_n)$, $n = 1, 2, \dots$, is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in $W^1 E^\Phi$. Hence, by virtue of [11, Cor. 2.2], we can assume, taking a subsequence if necessary, that u_n converges uniformly to a T -periodic continuous (therefore in E_T^Φ) function u . As $u'_n \in E^\Phi$ is a norm bounded sequence in L^Φ , there exists a subsequence (again denoted by u'_n) such that u'_n converges to a function $v \in L^\Phi$ in the weak* topology of L^Φ . Since E^Φ is weak* closed, by Lemma 3.1, $v \in E^\Phi$. From this fact and the uniform convergence of u_n to u , we obtain that

$$\int_0^T \xi' \cdot u dt = \lim_{n \rightarrow \infty} \int_0^T \xi' \cdot u_n dt = - \lim_{n \rightarrow \infty} \int_0^T \xi \cdot u'_n dt = - \int_0^T \xi \cdot v dt$$

for every T -periodic function $\xi \in C^\infty([0, T], \mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0, T]$ (see [18, p. 6]) and $u \in W^1 E_T^\Phi$.

Now, taking into account the relations $[L^1]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that u'_n converges to u' in the weak topology of L^1 . Consequently, from the semicontinuity of I (see [11, Lemma 6.1]) we get

$$I(u) \leq \liminf_{n \rightarrow \infty} I(u_n) = \inf_{v \in W^1 E_T^\Phi} I(v).$$

Hence $u \in W^1 E_T^\Phi$ is a minimum and, since I is Gâteaux differentiable on $W^1 E^\Phi$ (see [11, Thm. 3.2]), therefore $I'(u) \in (W^1 E_T^\Phi)^\perp$. Thus,

$$\int_0^T \frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t) \cdot v'(t) dt = - \int_0^T \nabla F(t, u(t)) \cdot v(t) dt,$$

for every $v \in W^1 E_T^\Phi$.

From [11, Lemma 2.4] we have $u'(t)\Phi'(|u'(t)|)/|u'(t)| \in L^\Psi([0, T], \mathbb{R}^n) \hookrightarrow L^1([0, T], \mathbb{R}^n)$; and, from condition (A) and the fact that $u \in L^\infty$, it follows that $\nabla F(t, u(t)) \in L^1([0, T], \mathbb{R}^n)$. Consequently, from [18, p. 6] we obtain that the differential equations in (1) are verified and $u'(0)\Phi'(|u'(0)|)/|u'(0)| = u'(T)\Phi'(|u'(T)|)/|u'(T)|$ holds. Thus $u'(0) = u'(T)$. \square

3.1 More general action integrals

Muuuuuy en construcción.....Fernando: no sé si era la idea que tenías en mente.....

Let us consider the following problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) = D_x \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) & \text{a.e. } t \in (0, T) \\ \mathbf{u}(0) - \mathbf{u}(T) = \dot{\mathbf{u}}(0) - \dot{\mathbf{u}}(T) = 0 \end{cases} \quad (34)$$

where $T > 0$, $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Carathéodory and differentiable function satisfying the conditions

$$\mathcal{L}(t, \mathbf{x}, \mathbf{y}) \geq \Phi(|\mathbf{y}|) + F(t, \mathbf{x}), \quad (35)$$

$$|\mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left(b(t) + \Phi\left(\frac{|\mathbf{y}|}{\lambda} + f(t)\right) \right), \quad (36)$$

$$|D_x \mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left(b(t) + \Phi\left(\frac{|\mathbf{y}|}{\lambda} + f(t)\right) \right), \quad (37)$$

$$|D_y \mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left(c(t) + \varphi\left(\frac{|\mathbf{y}|}{\lambda} + f(t)\right) \right). \quad (38)$$

and where the potential F satisfies conditions (A) and (C).

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N -function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ . The non negative functions b, c and f satisfy that $b \in L_1^1([0, T])$, $c \in L_1^\Psi([0, T])$ and $f \in E_1^\Phi([0, T])$, where the Banach spaces $L_1^1([0, T])$, $L_1^\Psi([0, T])$ and $E_1^\Phi([0, T])$.

Now, performing some slight modifications in the proof of Theorem 3.7 (in first line of (33), $=$ has to be changed by \geq and Thm. 4.1 of [11] can be applied to prove

$u'(0) = u'(T)$ exchanging $W^1 L^\Phi$ by $W^1 E_T^\Phi$, we also get that problem (34) has at least one solution which minimizes the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t)) dt. \quad (39)$$

4 Growing conditions with power functions

As it has been seen in the proof of Lemma 3.3 and in [11, Lemma 5.2], the function Φ_0 in Lemma 3.3 may be assumed a power function $\Phi_0(x) = |x|^\mu$ with $0 < \mu < \alpha_\Phi$ and α_Φ is a Matuszewska-Orlicz indices (see [14, Ch 11]). These indices are defined by

$$\alpha_\Phi := \lim_{t \rightarrow 0^+} \frac{\log \left(\sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right)}{\log(t)}, \quad \beta_\Phi := \lim_{t \rightarrow +\infty} \frac{\log \left(\sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right)}{\log(t)}. \quad (40)$$

Hence, following the same lines that in the proof of Theorem 3.7, using [11, Lemma 5.2] instead of our Lemma 3.3, we can assume that $\Phi_0(x) = |x|^\mu$ with $0 < \mu < \alpha_\Phi$ in Theorem 3.7.

The employment of N -functions instead of power functions in inequalities like (24) will allow us to extend some results of [16] and [17], not only to the Φ -laplacian operator, but even in the case of p -laplacian operator we get bounds that may be more sharp than those in [16, 17]. More precisely, in [17, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions for the p -laplacian operator under the assumption

$$|\nabla F| \leq b_1(t)|x|^\alpha + b_2(t).$$

for any $\alpha \in (0, p-1)$.

In this section, from now on we will suppose that

$$\Phi(u) = \begin{cases} \frac{p-1}{p} u^p & u \leq e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function Φ .

Theorem 4.1. *If $p \geq \frac{1+\sqrt{2}}{2}$, then Φ is an N -function.*

Proof. We have

$$\varphi(u) = \Phi'(u) = \begin{cases} (p-1)u^{p-1} & := \varphi_1(u) \quad \text{if } u \leq e \\ \frac{u^{p-1}}{\log u} \left(p - \frac{1}{\log u} \right) & := \varphi_2(u) \quad \text{if } u \geq e \end{cases}$$

First let us see that Φ' is increasing when $p \geq \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that φ_1 is increasing on $[0, e]$ and φ_2 is increasing on $[e, \infty)$ for every $p \geq \frac{1+\sqrt{2}}{2}$. Clearly φ_1 is an increasing function for $p > 1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore φ_2 is an increasing function when $p \geq \frac{1+\sqrt{2}}{2}$.

Besides $\varphi_2(u) \rightarrow \infty$ and $\varphi_1(u) \rightarrow 0$ as $u \rightarrow \infty$ and $u \rightarrow 0$ respectively, provided that $p > 1$. Hence, Φ is an N -function. \square

Theorem 4.2. *For every $\varepsilon > 0$, there exists a positive constant $C = C(p, \varepsilon)$ such that*

$$C^{-1}t^{p-\varepsilon}\Phi(u) \leq \Phi(tu) \leq Ct^p\Phi(u) \quad t \geq 1, u > 0, \quad (41)$$

Proof. If $u \leq tu \leq e$, then $\Phi(tu) = t^p\Phi(u)$ and (41) holds with $C = 1$.

If $u \leq e \leq tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geq 1$, we have $\Phi(tu) \leq t^p u^p = \frac{p}{p-1} t^p \Phi(u)$. Thus, the second inequality of (41) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$, then $f((tu)^p) \geq f(e^p) = e^p/p$. Now,

$$\begin{aligned} \Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} \\ &\geq \frac{p-1}{p} \frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p. \end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t + 1}$ on the interval $[1, +\infty)$ then

$$\Phi(tu) \geq \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,$$

which is the first inequality of (41) with $C = \frac{p}{p-1} \varepsilon^{-1} e^{-1+\varepsilon}$.

If $e \leq u \leq tu$, then

$$\Phi(tu) \leq \frac{t^p u^p}{\log(tu)} \leq \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v}, \quad (42)$$

where $v := u^p$ and $v \geq e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on (α, ∞) and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p, \infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leq \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every $v \geq e^p$. In this way, from (42), we have

$$\Phi(tu) \leq \frac{pt^p}{p-1} \left(\frac{pv}{\log v} - \frac{e^p}{p} \right) = \frac{pt^p}{p-1} \left(\frac{u^p}{\log u} - \frac{e^p}{p} \right)$$

and the second inequality of (41) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi(tu) \geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} = \frac{p-1}{p} \frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)} \frac{t^{p-\varepsilon} u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geq 1$ and $A \geq \varepsilon$. If $A \leq 1$, the function f attains a minimum on $[1, \infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geq \varepsilon$. If $A > 1$,

f is increasing on $[1, \infty)$ and its minimum value is $f(1) = 1$. Then, $f(s) \geq \varepsilon$ in any case, therefore

$$\Phi(tu) \geq \frac{p-1}{p} \varepsilon \frac{t^{p-\varepsilon} u^p}{\log u} \geq \frac{p-1}{p} \varepsilon t^{p-\varepsilon} \Phi(u).$$

Therefore, (41) holds with $C = \frac{p}{\varepsilon(p-1)}$, because this C is the biggest constant that we have obtained in each case under consideration. \square

Remark 2. The inequality

$$\Phi(tu) \geq C t^p \Phi(u)$$

is false for every C because for every $u \geq e$ we have

$$\lim_{t \rightarrow \infty} \frac{\Phi(tu)}{t^p \Phi(u)} = 0$$

Theorem 4.3. $\alpha_\Phi = \beta_\Phi = p$

Proof. From (40) and (41), we get

$$\beta_\Phi = \lim_{t \rightarrow \infty} \frac{\log \left[\sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} \leq \lim_{t \rightarrow \infty} \frac{\log C + p \log t}{\log t} = p.$$

On the other hand, employing (40) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t \rightarrow 0^+} \frac{\log \left[\sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} = \lim_{s \rightarrow \infty} \frac{\log \left[\sup_{v>0} \frac{\Phi(v)}{\Phi(sv)} \right]^{-1}}{\log s} = \lim_{s \rightarrow \infty} \frac{\log \left[\inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s}$$

where $v := tu$ and $s := \frac{1}{t}$. Then, using (41), for every $\varepsilon > 0$ we have

$$\alpha_\Phi = \lim_{s \rightarrow \infty} \frac{\log \left[\inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s} \geq \lim_{s \rightarrow \infty} \frac{\log C + (p - \varepsilon) \log s}{\log s} \geq p - \varepsilon,$$

therefore $\alpha_\Phi \geq p$.

Finally, as $\alpha_\Phi \leq \beta_\Phi \leq p$, we get $\alpha_\Phi = \beta_\Phi = p$. \square

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|) dx \geq C \|u\|_{L^\Phi}^{\alpha_\Phi} = C \|u\|_{L^\Phi}^p$$

is false.

In fact, if we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1 t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|) dx = C_2 \Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geq C \|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geq C t^p$ would also be true; however, this last inequality is false.

Assuming $\|b_1\|_{L^1}$ small enough, in [19, 17] coercivity was obtained even for the limit value $\alpha = p - 1$ in inequality (24).

This result leans on the fact that

$$\|u\|_{L^\Phi}^{\alpha_\Phi} = O\left(\int_0^T \Phi(|u|) dt\right) \quad \text{for } \|u\|_{L^\Phi} \rightarrow \infty, \quad (43)$$

when $\Phi(u) = |u|^p$. Nevertheless, it is no longer the case for any N -function Φ as the following example shows.

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References

- [1] R. Adams, J. Fournier, Sobolev spaces, Elsevier/Academic Press, Amsterdam, 2003.
- [2] M. A. Krasnosel'skiĭ, J. B. Rutickiĭ, Convex functions and Orlicz spaces, P. Noordhoff Ltd., Groningen, 1961.
- [3] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Vol. 146, Marcel Dekker, Inc., New York, 1991.
- [4] G. Schappacher, A notion of Orlicz spaces for vector valued functions, Appl. Math. 50 (4) (2005) 355–386.
- [5] H. Hudzik, L. Maligranda, Amemiya norm equals Orlicz norm in general, Indag. Math. (N.S.) 11 (4) (2000) 573–585.
- [6] A. Cianchi, A fully anisotropic Sobolev inequality, Pacific J. Math. 196 (2) (2000) 283–295.
- [7] A. Cianchi, Some results in the theory of Orlicz spaces and applications to variational problems, in: Nonlinear analysis, function spaces and applications, Vol. 6 (Prague, 1998), Acad. Sci. Czech Repub., Prague, 1999, pp. 50–92.
- [8] N. Clavero, Optimal Sobolev embeddings and Function Spaces, <http://www.maia.ub.edu/~soria/sobolev1.pdf>, last accessed: 2014-12-22. (2011).
- [9] D. Edmunds, R. Kerman, L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, J. Funct. Anal. 170 (2) (2000) 307–355.
- [10] R. Kerman, L. Pick, Optimal Sobolev imbeddings, Forum Math. 18 (4) (2006) 535–570.

References

- [11] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, E. Schwindt, Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting, *Nonlinear Analysis, TMA.* 125 (2015) 681 – 698.
- [12] J. B. Conway, *A Course in Functional Analysis*, Springer, USA, 1977.
- [13] B. Xu, C.-L. Tang, Some existence results on periodic solutions of ordinary p -Laplacian systems, *J. Math. Anal. Appl.* 333 (2) (2007) 1228–1236.
- [14] L. Maligranda, Orlicz spaces and interpolation, Vol. 5 of *Seminários de Matemática [Seminars in Mathematics]*, Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.
- [15] S. Acinas, G. Giubergia, F. Mazzone, E. Schwindt, On estimates for the period of solutions of equations involving the ϕ -Laplace operator, *J. Abstr. Differ. Equ. Appl.* 5 (1) (2014) 21–34.
- [16] C.-L. Tang, Periodic solutions for nonautonomous second order systems with sub-linear nonlinearity, *Proc. Amer. Math. Soc.* 126 (11) (1998) 3263–3270.
- [17] X. Tang, X. Zhang, Periodic solutions for second-order Hamiltonian systems with a p -Laplacian, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 64 (1) (2010) 93–113.
- [18] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, New York, 1989.
- [19] F. Zhao, X. Wu, Periodic solutions for a class of non-autonomous second order systems, *J. Math. Anal. Appl.* 296 (2) (2004) 422–434.