

Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_1)$$

where $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d \geq 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u : [0, T] \rightarrow \mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange systems of ordinary equations*.

This topic was deeply addressed for the several types of *Lagrange functions*. For example,

$$\mathcal{L}_{p,F}(t, x, y) := \frac{|y|^p}{p} + F(t, x), \quad (1)$$

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for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (P_1) , for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (1) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (P_1) is reduced to the p -laplacian system

$$\begin{cases} \frac{d}{dt} (u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P_2)$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t, x)$ is differentiable with respect to x for a.e. $t \in [0, T]$ and the following conditions are verified:

(C) F and its gradient ∇F , with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

(A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (2)$$

In this inequality we assume that the function $a : [0, +\infty) \rightarrow [0, +\infty)$ is continuous and non decreasing and $0 \leq b \in L^1([0, T], \mathbb{R})$.

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of p -laplacian equations as the following example shows.

Example 1. Let $1 < p_1, p_2 < \infty$. We define $\Phi_{p_1, p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

where $|\cdot|$ is the Euclidean norm on \mathbb{R}^n . And, we consider the following Lagrange function

$$\mathcal{L}(t, x, y) = \Phi_{p_1, p_2}(y) + F(t, x).$$

where $y = (y_1, y_2), x = (x_1, x_2) \in \mathbb{R}^{2n}$...o algo asf????

Then the equations (P_1) become

$$\begin{cases} \frac{d}{dt} (|u'_1|^{p_1-2} u'_1) = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt} (|u'_2|^{p_2-2} u'_2) = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_3)$$

In the literature, these equations are known as (p_1, p_2) -Laplacian system, see [Yang and Chen, 2013, Pasca and Wang, 2016, Yang and Chen, 2012, Pasca, 2010, Paşca and Tang, 2010, Pasca and Tang, 2011].

In [Acinas et al., 2015] it is treated the case of a lagrangian \mathcal{L} which is lower bounded by a Lagrange function like

$$\mathcal{L}_{\Phi, F}(t, x, y) := \Phi(|y|) + F(t, x), \quad (3)$$

where Φ is an N -function (see section 2 for the definition of this concept).

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$, i.e. functions such that $\Phi(x)$ depends on the direction of x , unlike the radial case where $\Phi(x) = \Phi(|x|)$. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

On the other hand, anisotropic Orlicz-Sobolev spaces allow us to simplify the writing, and they provide the natural frame for statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question motivated us to use these spaces.

Hereafter we denote by \mathbb{R}^+ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \rightarrow +\infty$, when $|x| \rightarrow +\infty$. Additionally, we assume that Young's functions which we deal with, satisfy that $\Phi(x) > 0$ when $x \neq 0$. Following [Schappacher, 2005] we say that Φ is an N_∞ -function if

$$\lim_{|x| \rightarrow \infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function Φ , we define function $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$A_\Phi(s) = \min \{ \Phi(x) \mid |x| = s \}, \quad (4)$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.1. *The function A_Φ has the following properties:*

1. A_Φ is continuous,
2. $A_\Phi(s)/s$ is increasing,
3. $A_\Phi(|x|)$ is the greatest radial minorant of $\Phi(x)$,
4. Φ is N_∞ if and only if A_Φ is.

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [Clarke, 2013]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ ,

$$\frac{A_\Phi(r)}{r} \leq \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leq \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily. □

Example 2. We compute A_Φ for the function $\Phi = \Phi_{p_1, p_2}$ given in Example (1).

We apply the method of Lagrange multipliers to solve the next minimization problem subject to a constraint

$$G(r) = \min \{ \Phi(x, y) : |(x, y)|_2^2 = r^2 \}$$

The first order conditions are

$$\begin{cases} |x|^{p_1-2}x + \lambda x &= 0 \\ |y|^{p_2-2}y + \lambda y &= 0 \\ |x|^2 + |y|^2 &= r^2 \end{cases}$$

These equations can be solved, among others, by the following two sets of critical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2 .

We deal with $p_1 \leq 2$ and $p_2 \leq 2$ being one of them (suppose p_2) different from 2. The remaining cases can be treated with similar techniques.

If (x, y) is a critical point with $x \neq 0$ and $y \neq 0$ then $|y| = |x|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|x|^{p_1-2}$. In order to employ second order conditions for constrained problems, we have to consider the tangent plane at the point $(x, y) \in \mathbb{R}^{2n}$, i.e. $M = \{(z, w) \in \mathbb{R}^{2n} : zx^T + wy^T = 0\}$. We also consider the Lagrangian associated to the constrained problem: $L(x, y, \lambda) = \Phi(x, y) + \lambda H(x, y)$ being $H = 0$ the constraint. And, we are interested in knowing how to behave the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2H$ with respect to the vector $(x, y) \in \mathbb{R}^{2n}$ on the subspace M .

We compute

$$(z, w)\mathcal{H}(z, w)^T = |\lambda|(zx^T)^2[|x|^{-2}(p_1-2) + (p_2-2)|y|^{-2}],$$

on the subspace M . We note that $(-y, x) \in M$ and $(-y, x)\mathcal{H}(-y, x)^T < 0$.

Then, by second order necessary conditions [Luenberger and Ye, 2015, p.333], at (x, y) there cannot be a minimum. Therefore, the minima occur at $x = 0$ or $y = 0$, then $A_\Phi(x, y) = \min\{r^{p_1}/p_1, r^{p_2}/p_2\}$.

More generally, it holds that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leq A_\Phi \leq K_2 \max\{r^{p_1}, r^{p_2}\}$$

with $K_1, K_2 > 0$, for every $1 < p_1, p_2 < \infty$.

We also say that $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ satisfies the Δ_2^∞ -condition, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geq 0$ such that

$$\Phi(2x) \leq K\Phi(x), \tag{5}$$

for every $|x| \geq M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \rightarrow \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \tag{6}$$

We denote by $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$, with $d \geq 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on \mathbb{R}^d and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions \mathcal{M} , as for other similar sets, we will omit the reference to codomain \mathbb{R}^d when $d = 1$.

Given an N -function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Here $|\cdot|$ is the euclidean norm of \mathbb{R}^d . Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (7)$$

The *Orlicz space* $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$ is the linear hull of C^Φ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (8)$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left(\frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in \mathbb{R}^d between u and v .

The subspace $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0, T], \mathbb{R}^d)$ of all \mathbb{R}^d -valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of E^Φ , which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0, T]$, $n = 1, 2, \dots$ then $\|\chi_{E_n} u\| \rightarrow 0$ when $|E_n| \rightarrow 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \quad (9)$$

Like in [Krasnosel'skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset r C^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (10)$$

for any positive r (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ , E^Φ , $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

Definition 2.2. Let $u_n, u \in L^\Phi([0, T], \mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^\infty([0, T], \mathbb{R})$, $n = 1, 2, \dots$, such that $0 \leq \alpha_n(t) \leq \alpha_{n+1}(t)$, $\alpha_n(t) \rightarrow 1$ a.e., when $n \rightarrow \infty$ and $u_n(t) = \alpha_n(t)u(t)$.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leq C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow [L^\Phi]^*$, where a function $v \in L^\Psi$ is associated to $\xi_v \in [L^\Phi]^*$ being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \quad (11)$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $[L^\Phi]^*$ which can be identified with L^Ψ .

Proposition 2.3. Let $F \in [L^\Phi([0, T], \mathbb{R}^d)]^*$. Then the following statements are equivalent

1. $\xi \in L^\Psi([0, T], \mathbb{R}^d)$
2. ξ satisfies the monotone convergence property, which is if u_n converges monotonically to u then $\langle \xi, u_n \rangle \rightarrow \langle \xi, u \rangle$.

If $\Phi \in \Delta_2^\infty$ and Φ is N_∞ then $L^\Psi([0, T], \mathbb{R}^d) = [L^\Phi([0, T], \mathbb{R}^d)]^*$ (see [Desch and Grimmer, 2001, Thm. 2.9, Thm. 2.10]).

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\}.$$

$W^1 L^\Phi([0, T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (12)$$

And, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \quad (13)$$

We will use repeatedly the decomposition $u = \bar{u} + \tilde{u}$ for a function $u \in L^1([0, T])$ where $\bar{u} = \frac{1}{T} \int_0^T u(t) \, dt$ and $\tilde{u} = u - \bar{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.4. Let $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ be a Young's function and let $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$. Let $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined by (4). Then

1. For every $s, t \in [0, T]$, $s \neq t$,

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| A_\Phi^{-1} \left(\frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq A_\Phi^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. We have $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$ and

$$\|\tilde{u}\|_{L^\infty} \leq TA_\Phi^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. If Φ is N_∞ then the space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0, T], \mathbb{R}^d)$.

Proof. By the absolutely continuity of u , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi \left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|} \right) &\leq \Phi \left(\frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr \right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi \left(\frac{u'(r)}{\|u'\|_{L^\Phi}} \right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By Proposition 2.1(3) we have $A_\Phi^{-1} \Phi(x) \geq |x|$, therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq A_\Phi^{-1} \left(\frac{1}{|s - t|} \right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.1 (2) and we have

$$\begin{aligned} |u(t) - \bar{u}| &= \left| \frac{1}{T} \int_0^T u(t) - u(s) ds \right| \\ &\leq \frac{1}{T} \int_0^T |u(t) - u(s)| ds \\ &\leq \|u'\|_{L^\Phi} TA_\Phi^{-1} \left(\frac{1}{T} \right) \end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi \left(\frac{\bar{u}}{\|u\|_{L^\Phi}} \right) \leq \frac{1}{T} \int_0^T \Phi \left(\frac{u(s)}{\|u\|_{L^\Phi}} \right) ds \leq \frac{1}{T}$$

Then by Proposition 2.1(3)

$$|\bar{u}| \leq A_\Phi^{-1} \left(\frac{1}{T} \right) \|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq A_\Phi^{-1} \left(\frac{1}{T} \right) \|u\|_{L^\Phi} + TA_\Phi^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \\ &\leq A_\Phi^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1 L^\Phi([0, T], \mathbb{R}^d)$. Since Φ is N_∞ , from Proposition 2.1(4) we obtain $sA_\Phi^{-1}(1/s) \rightarrow 0$ when $s \rightarrow 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (??) implies that u_n is bounded in $C([0, T], \mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0, T], \mathbb{R}^d)$ with $u_{n_k} \rightarrow u$ in $C([0, T], \mathbb{R}^d)$. \square

Lemma 2.5. *Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi, 1)$ converging to $u \in \Pi(E^\Phi, 1)$ in the L^Φ -norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0, T], \mathbb{R})$ such that $u_{n_k} \rightarrow u$ a.e. and $\Phi(u_{n_k}) \leq h$ a.e.*

Proof. Since $d(u, E^\Phi) < 1$ and u_n converges to u , there exists $u_0 \in E^\Phi$, a subsequence of u_n (again denoted u_n) and $0 < r < 1$ such that $d(u_n, u_0) < r$. Let $\lambda_0 \in (r, 1)$. By extracting more subsequences, if necessary, we can assume that $u_n \rightarrow u$ a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^\Phi} < \frac{1 - \lambda_0}{2^n}, \quad \text{for } n \geq 1.$$

We can assume $\lambda_n > 0$ for every $n = 0, \dots$

Let $\lambda := 1 - \sum_{n=0}^\infty \lambda_n$ and define $h : [0, T] \rightarrow \mathbb{R}$ by

$$h(x) = \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^\infty \lambda_n \Phi\left(\frac{u_{n+1} - u_n}{\lambda_n}\right). \quad (14)$$

Note that $\sum_{n=0}^\infty \lambda_n + \lambda = 1$, therefore for any $n = 1, \dots$

$$\begin{aligned} \Phi(u_n) &= \Phi\left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j}\right) \\ &\leq \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1} \lambda_j \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) \leq h \end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0, T], \mathbb{R})$. On the other hand $\|u_{n+1} - u_n\|_{L^\Phi} \leq \lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) dt \leq 1.$$

Then $h \in L^1([0, T], \mathbb{R})$. \square

3 Differentiability Gateaux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateaux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a *Carathéodory function*, i.e.

- (C) f is measurable with respect to $t \in [0, T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

Definition 3.1. For $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ we denote by \mathbf{f} the Nemytskii (o superposition) operator defined for functions $u : [0, T] \rightarrow \mathbb{R}^d$ by

$$\mathbf{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [Krasnosel'skii et al., 2011] for scalar functions and [Pluciennik, 1987, Pluciennik, 1985b, Pluciennik, 1985a] for the generalization to \mathbb{R}^d -valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

Theorem 3.2. We assume that f satisfies condition ((C)) and that $\Phi_1, \Phi_2 : \mathbb{R}^d \rightarrow [0, +\infty)$ are anisotropic Young functions. Then

1. Measurability. The operator \mathbf{f} maps measurable function into measurable functions
2. Extensibility. If the operator \mathbf{f} acts from the ball $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} \mid \|u\|_{L^{\Phi_1}} < r\}$ into the space L^{Φ_2} or the space E^{Φ_2} then \mathbf{f} can be extended from $\Pi(E^{\Phi_1}, r)$ into space L^{Φ_2} or E^{Φ_2} , respectively.
3. Continuity. If the operator \mathbf{f} acts from $\Pi(E^{\Phi_1}, r)$ into space E^{Φ_2} , then \mathbf{f} is continuous.

Given a continuous function $a \in C(\mathbb{R}^n, \mathbb{R}^+)$, we define the composition operator $\alpha : \mathcal{M}_d \rightarrow \mathcal{M}_d$ by $\alpha(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [Acinas et al., 2015].

Lemma 3.3. If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\alpha : W^1 L^\Phi \rightarrow L^\infty([0, T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|\alpha(u)\|_{L^\infty([0, T])} \leq A(\|u\|_{W^1 L^\Phi})$.

Proof. Let $A \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\| \leq s, x \in \mathbb{R}^d} |a(x)|$. If $u \in W^1 L_d^\Phi$ then, by Sobolev's inequality, for a.e. $t \in [0, T]$

$$a(u(t)) \leq \alpha(\|u\|_{L^\infty}) \leq \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi}\right) =: A(\|u\|_{W^1 L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

We assume that the Lagrangian $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is Carathéodory and differentiable function satisfying

$$|\mathcal{L}(t, x, y)| + |D_x \mathcal{L}(t, x, y)| + \Psi(D_y \mathcal{L}(t, x, y)) \leq a(|x|)(b(t) + \Phi(y)), \quad (15)$$

where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1_1([0, T])$, Φ and Ψ are N_∞ -functions (complementary???? o en el teorema o nunca?)

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t)) dt. \quad (16)$$

Theorem 3.4. *Let \mathcal{L} be a differentiable Carathéodory function satisfying (15). Then the following statements hold:*

1. *The action integral given by (16) is finitely defined on $\mathcal{E}^\Phi := W^1 L^\Phi \cap \{u | \dot{u} \in \Pi(E^\Phi, 1)\}$.*
2. *The function I is Gâteaux differentiable on \mathcal{E}^Φ and its derivative I' is demicontinuous from \mathcal{E}^Φ into $[W^1 L^\Phi]^*$. Moreover, I' is given by the following expression*

$$\langle I'(u), v \rangle = \int_0^T \{D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v}\} dt. \quad (17)$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from \mathcal{E}^Φ into $[W^1 L^\Phi]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^\Phi$. As

$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \quad (18)$$

and (10), then $\Phi(\dot{u}(t)) \in L^1$. Now,

$$|\mathcal{L}(\cdot, u, \dot{u})| + |D_x \mathcal{L}(\cdot, u, \dot{u})| + \Psi(D_y \mathcal{L}(\cdot, u, \dot{u})) \leq A(\|u\|_{W^1 L^\Phi})(b + \Phi(\dot{u})) \in L^1, \quad (19)$$

by (15) and Lemma 3.3. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ is continuous from \mathcal{E}^Φ into $L^1([0, T])$ with the strong topology on both sets.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in \mathcal{E}^Φ and let $u \in \mathcal{E}^\Phi$ such that $u_n \rightarrow u$ in $W^1 L^\Phi$. By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leq T A_\Phi^{-1} \left(\frac{1}{T} \right) \|u_n - u\|_{L^\Phi}$$

then $u_n \rightarrow u$ uniformly. As $\dot{u}_n \rightarrow \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.5, there exist a subsequence of \dot{u}_{n_k} (again denoted \dot{u}_{n_k}) and a function $h \in L^1([0, T], \mathbb{R})$ such that $\dot{u}_{n_k} \rightarrow \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leq h$ a.e.

Since u_{n_k} , $k = 1, 2, \dots$, is a strong convergent sequence in $W^1 L^\Phi$, it is a bounded sequence in $W^1 L^\Phi$. According to item (3) of Lemma 2.4, there exists $M > 0$ such that $\|a(u_{n_k})\|_{L^\infty} \leq M$, $k = 1, 2, \dots$. From the previous facts and (19), we get

$$|D_x \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leq a(|u_{n_k}|)(b + \Phi(\dot{u}_{n_k})) \leq M(b + h) \in L^1.$$

On the other hand, by the continuous differentiability of \mathcal{L} , we have

$$D_x \mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow D_x \mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$ is continuous from \mathcal{E}^Φ with the strong topology into $[L^\Phi]^*$ with the weak* topology.

Let $u \in \mathcal{E}^\Phi$. From (19) it follows that

$$D_y \mathcal{L}(\cdot, u, \dot{u}) \in C^\Psi. \quad (20)$$

Así? o conviene poner la cota de $\Psi(D_y)$ explícitamente???

Note that (19), (20) and the imbeddings $W^1 L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow [L^\Phi]^*$ imply that the second member of (17) defines an element of $[W^1 L^\Phi]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \rightarrow u$ in the norm of $W^1 L^\Phi$. We must prove that $D_y \mathcal{L}(\cdot, u_n, \dot{u}_n) \xrightarrow{w^*} D_y \mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle D_y \mathcal{L}(\cdot, u_n, \dot{u}_n), v \rangle - \langle D_y \mathcal{L}(\cdot, u, \dot{u}), v \rangle| \geq \epsilon. \quad (21)$$

We have $u_n \rightarrow u$ in L^Φ and $\dot{u}_n \rightarrow \dot{u}$ in L^Φ . By Lemma 2.5, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0, T], \mathbb{R})$ such that $u_n \rightarrow u$ uniformly, $\dot{u}_n \rightarrow \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leq h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.3 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (19) with u_n instead of u , we have

$$\Psi(D_y \mathcal{L}(\cdot, u_n, \dot{u}_n)) \leq M(b + h) \in L^1. \quad (22)$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (22), we have

$$\begin{aligned} & \lambda D_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ & \leq \lambda \left[\Psi(D_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right) \right] \\ & \leq \lambda M(b + h) + \lambda \Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \quad (23)$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v \, dt \rightarrow \int_0^T D_y \mathcal{L}(t, u, \dot{u}) \cdot v \, dt \quad (24)$$

which contradicts the inequality (21). This completes the proof of step 2.

Step 3. We will prove (17). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1 L^\Phi$, we define the function

$$H(s, t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leq s_0 := \min\{(1 - d(\dot{u}, E^\Phi)) / \|v\|_{W^1 L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d(\dot{u} + s\dot{v}, E^\Phi) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.4 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leq s_0$.

We also have $\|u + sv\|_{W^1 L^\Phi} \leq \|u\|_{W^1 L^\Phi} + s_0 \|v\|_{W^1 L^\Phi}$; then, by Lemma 3.3, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leq M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leq s_0 \lambda^{-1}$, from the convexity and the parity of Φ , we get

$$\begin{aligned} \Phi(\dot{u} + s\dot{v}) &= \Phi\left((1-s_0)\frac{\dot{u}}{1-s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leq (1-s_0)\Phi\left(\frac{\dot{u}}{1-s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right) \\ &\leq (1-s_0)\Phi\left(\frac{\dot{u}}{1-s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1 \end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1-s_0}, E^\Phi\right) = \frac{1}{1-s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1-s_0} \in C^\Phi$.

Now, applying (19), (23), the fact that $v \in L^\infty$ and $\dot{v} \in L^\Phi$, we get

$$\begin{aligned} |D_s H(s, t)| &= \left| D_x \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + \lambda D_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \frac{\dot{v}}{\lambda} \right| \\ &\leq M [b(t) + \Phi(\dot{u} + s\dot{v})] |v| \\ &\quad + \lambda \left[\Psi(D_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right) \right] \\ &\leq M \{ [b(t) + \Phi(\dot{u} + s\dot{v})] |v| \} + \lambda M [b(t) + \Phi(\dot{u} + s\dot{v})] + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \\ &= M [b(t) + \Phi(\dot{u} + s\dot{v})] (|v| + \lambda) + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1. \end{aligned} \tag{25}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v \rangle = \frac{d}{ds} I(u + sv) \Big|_{s=0} = \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt.$$

Moreover, from the previous formula, (19), (20), and Lemma 2.4, we obtain

$$|\langle I'(u), v \rangle| \leq \|D_x \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|D_y \mathcal{L}\|_{L^\Psi} \|\dot{v}\|_{L^\Phi} \leq C \|v\|_{W^1 L^\Phi}$$

with a appropriate constant C .

This completes the proof of the Gâteaux differentiability of I .

Step 4. The operator $I' : \mathcal{E}^\Phi \rightarrow [W^1 L_d^\Phi]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \rightarrow u$ in the norm of $W^1 L^\Phi$ and $v \in W^1 L^\Phi$, then

$$\begin{aligned} \langle I'(u_n), v \rangle &= \int_0^T \{ D_x \mathcal{L}(t, u_n, \dot{u}_n) \cdot v + D_y \mathcal{L}(t, u_n, \dot{u}_n) \cdot \dot{v} \} dt \\ &\rightarrow \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt \\ &= \langle I'(u), v \rangle. \end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$ are norm continuous from \mathcal{E}^Φ into L^1 and L^Ψ , respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de D_x aquí!!!

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1 L^\Phi} \rightarrow 0$.

Applying Lemma 2.5 to \dot{u}_n , there exists a subsequence (denoted \dot{u}_n for simplicity) such that $\dot{u}_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot{u}_n) \leq h$ and $\dot{u}_n \rightarrow \dot{u}$ a.e.

Then, by (23) we have $\Psi(v_n) \leq m(t) \in L^1$ being $v_n := D_y \mathcal{L}(\cdot, u_n, \dot{u}_n)$ and $m(t) := M(b + h)$. In addition, from the continuous differentiability of \mathcal{L} , we have that $v_n \rightarrow v$ a.e. where $D_y \mathcal{L}(\cdot, u, \dot{u})$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\Psi(\lambda x) \leq c(|\lambda|)\Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1})\Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \rightarrow 0$ a.e. as $n \rightarrow \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1})K\Psi(v_n) + \Psi(v) \leq c(|\lambda|^{-1})K[m(t) + \Psi(v)] \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda}) dt \rightarrow 0$ for every $\lambda > 0$. Thus, $v_n \rightarrow v$ in L^Ψ .

The continuity of I' follows from the continuity of $D_x \mathcal{L}$ and $D_y \mathcal{L}$ using the formula (17). \square

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