Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

Sonia Acinas *
Instituto de Matemática Aplicada San Luis (CONICET-UNSL)
(5700) San Luis, Argentina
Universidad Nacional de La Pampa
(6300) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper we....

1 Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) & \text{a.e. } t\in(0,T) \\ \boldsymbol{u}(0)-\boldsymbol{u}(T)=\dot{\boldsymbol{u}}(0)-\dot{\boldsymbol{u}}(T)=0 \end{cases} \tag{1}$$

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† SECyT-UNRC and CONICET
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where $T > 0$, $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{2}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right), \tag{3}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(c(t) + \varphi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + f(t)\right)\right). \tag{4}$$

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ. The non negative functions b, c and f satisfy that $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$, where the Banach spaces $L^1_1([0,T])$, $L^\Psi_1([0,T])$ and $E^\Phi_1([0,T])$ will be defined later.

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(\boldsymbol{u}) = \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt. \tag{5}$$

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [1, 2, 3]. For Orlicz spaces of vector valued functions, see [4] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \text{for } t \geq 0,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the Δ_2*-condition*, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{6}$$

for every $t \geq t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2*-condition globally* ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0,T])$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $\boldsymbol{u} = (u_1, \ldots, u_d)$ for $\boldsymbol{u} \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_0^T \Phi(|\boldsymbol{u}|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C_d^\Phi = C_d^\Phi([0,T])$ is given by

$$C_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \rho_\Phi(\boldsymbol{u}) < \infty\}\,. \tag{7}$$

The *Orlicz space* $L_d^\Phi = L_d^\Phi([0,T])$ is the linear hull of C_d^Φ; equivalently,

$$L_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \exists \lambda > 0 : \rho_\Phi(\lambda \boldsymbol{u}) < \infty\}\,. \tag{8}$$

The Orlicz space L_d^Φ equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup\left\{ \int_0^T \boldsymbol{u} \cdot \boldsymbol{v}\, dt \big| \rho_\Psi(\boldsymbol{v}) \leqslant 1 \right\},$$

is a Banach space. By $\boldsymbol{u} \cdot \boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^d$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [2, Thm. 10.5] and [5]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(k\boldsymbol{u})\}\,. \tag{9}$$

The subspace $E_d^\Phi = E_d^\Phi([0,T])$ is defined as the closure in L_d^Φ of the subspace L_d^∞ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E_d^Φ is the only one maximal subspace contained in the Orlicz class C_d^Φ, i.e. $\boldsymbol{u} \in E_d^\Phi$ if and only if $\rho_\Phi(\lambda \boldsymbol{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [2, Th. 9.3]). Namely, if $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$ then $\boldsymbol{u} \cdot \boldsymbol{v} \in L_1^1$ and

$$\int_0^T \boldsymbol{v} \cdot \boldsymbol{u}\, dt \leqslant \|\boldsymbol{u}\|_{L^\Phi} \|\boldsymbol{v}\|_{L^\Psi}. \tag{10}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x\rangle = x^*(x)$. Hölder's inequality shows that $L_d^\Psi \subset \left[L_d^\Phi\right]^*$, where the pairing $\langle \boldsymbol{v}, \boldsymbol{u}\rangle$ is defined by

$$\langle \boldsymbol{v}, \boldsymbol{u}\rangle = \int_0^T \boldsymbol{v} \cdot \boldsymbol{u}\, dt \tag{11}$$

with $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L_d^\Psi = \left[L_d^\Phi\right]^*$ will not hold. In general, it is true that $\left[E_d^\Phi\right]^* = L_d^\Psi$.

Like in [2], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ given by

$$\Pi(E_d^\Phi, r) := \{\boldsymbol{u} \in L_d^\Phi | d(\boldsymbol{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \tag{12}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L_d^Φ, E_d^Φ, $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

We define the *Sobolev-Orlicz space* $W^1L_d^\Phi$ (see [1]) by

$$W^1L_d^\Phi := \{\boldsymbol{u} | \boldsymbol{u} \text{ is absolutely continuous and } \dot{\boldsymbol{u}} \in L_d^\Phi\}.$$

$W^1L_d^\Phi$ is a Banach space when equipped with the norm

$$\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}.$$

For a function $\boldsymbol{u} \in L_d^1([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t)\, dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L_d^\Psi \hookrightarrow \left[L_d^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L_d^\infty \hookrightarrow L_d^\Phi \hookrightarrow L_d^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant Cw(|t-s|). \tag{13}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (13) for some $C > 0$ and it is a Banach space with norm

$$\|\boldsymbol{u}\|_{C^w([0,T],\mathbb{R}^d)} := \|\boldsymbol{u}\|_{L^\infty} + \sup_{t \neq s} \frac{|\boldsymbol{u}(t) - \boldsymbol{u}(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [6, 7, 8, 9, 10]. The next simple lemma is essentially known and we will use it systematically. For the sake of completeness, we include a brief proof of it.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T], \mathbb{R}^d)$ and for every $\boldsymbol{u} \in W^1L^\Phi$*

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi} w(|t-s|), \tag{14}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|\boldsymbol{u}\|_{W^1L^\Phi} \tag{15}$$

2. *For every $\boldsymbol{u} \in W^1L^\Phi$ we have $\widetilde{\boldsymbol{u}} \in L_d^\infty$ and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Sobolev's inequality)}. \tag{16}$$

The next result is analogous to some lemmata in $W^1L^p_d$, see [11].

Lemma 2.2. *If* $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$*, then* $(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \to \infty$.

Proof. We have

$$\|\boldsymbol{u}\|_{L^\Phi} = \|\overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant \|\overline{\boldsymbol{u}}\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi} = |\overline{\boldsymbol{u}}|\|1\|_{L^\Phi} + \|\tilde{\boldsymbol{u}}\|_{L^\Phi}$$

We know that Holder's inequality implies that $L^\infty_d \hookrightarrow L^\Phi_d$, that is, there exists $C > 0$ such that for any $\tilde{\boldsymbol{u}} \in L^\infty_d$ we have

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C\|\tilde{\boldsymbol{u}}\|_{L^\infty}$$

and, applying Sobolev's inequality to the previous formula, we get

$$\|\tilde{\boldsymbol{u}}\|_{L^\Phi} \leqslant C\|\dot{\boldsymbol{u}}\|_{L^\Phi}$$

La desigualdad anterior sería del tipo Wirtinger's que no tenemos enunciada en ningún lado.
Therefore,

$$\|\boldsymbol{u}\|_{L^\Phi} \leqslant C(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}) \tag{17}$$

As $\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}$, then

$$\|\boldsymbol{u}\|_{W^1L^\Phi} \leqslant C(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi})$$

and by hypothesis $\|\boldsymbol{u}\|_{W^1L^\Phi} \to \infty$, then $|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi} \to \infty$. □

Esta definición va así o requiere modificaciones/adaptaciones???

Definition 2.3. *We say that a function* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a Carathéodory function if for fixed* $(\boldsymbol{x}, \boldsymbol{y})$ *the map* $t \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ *is measurable and for fixed* t *the map* $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ *is continuously differentiable for almost everywhere* $t \in [0,T]$.

In [12] we proved the next results.

Theorem 2.4. *Let* $\mathcal{L}$ *be a Carathéodory function satisfying* (2), (3) *and* (4). *Then the following statements hold:*

1. *The action integral given by* (5) *is finitely defined on* $\mathcal{E}^\Phi_d(\lambda) := W^1L^\Phi_d \cap \{\boldsymbol{u}|\dot{\boldsymbol{u}} \in \Pi(E^\Phi_d, \lambda)\}$.
2. *The function* I *is Gâteaux differentiable on* $\mathcal{E}^\Phi_d(\lambda)$ *and its derivative* I' *is demicontinuous from* $\mathcal{E}^\Phi_d(\lambda)$ *into* $\left[W^1L^\Phi_d\right]^*$. *Moreover,* I' *is given by the following expression*

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \dot{\boldsymbol{v}}\right\}\, dt. \tag{18}$$

3. *If* $\Psi \in \Delta_2$ *then* I' *is continuous from* $\mathcal{E}^\Phi_d(\lambda)$ *into* $\left[W^1L^\Phi_d\right]^*$ *when both spaces are equipped with the strong topology.*

In [12] we derived the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T-periodic functions. We denote by $W^1L^\Phi_T$ the subspace of $W^1L^\Phi_d$ containing all T-periodic functions. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f:\mathbb{R}^d\to\mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) < \frac{1}{2}\left(f\left(\boldsymbol{x}\right)+f\left(\boldsymbol{y}\right)\right)$ for $\boldsymbol{x}\neq\boldsymbol{y}$. It is well known that if f is a strictly convex and differentiable function, then $D_{\boldsymbol{x}}f:\mathbb{R}^d\to\mathbb{R}^d$ is a one-to-one map (see, e.g. [13, Thm. 12.17]).

Theorem 2.5. *Let $\boldsymbol{u}\in\mathcal{E}^\Phi_d(\lambda)$ be a T-periodic function. The following statements are equivalent:*

1. $I'(\boldsymbol{u})\in\left(W^1L^\Phi_T\right)^\perp$.
2. $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and $\boldsymbol{u}$ solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \quad a.e.\ t\in(0,T) \\ \boldsymbol{u}(0)-\boldsymbol{u}(T) = D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T)) = 0. \end{cases} \tag{19}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t,x,y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T)) = 0$ is equivalent to $\dot{\boldsymbol{u}}(0)=\dot{\boldsymbol{u}}(T)$.

DECIR ALGO DE LOS ÍNDICES ACÁ O EN LA INTRO...????

Lemma 2.6. *Let Φ and Ψ be complementary N-functions. Then:*

1. $\|\boldsymbol{u}\|_{L^\Phi} = O\left(\rho_\Phi\left(\boldsymbol{u}\right)\right)$.
2. *If $\Psi\in\Delta_2$ globally, then there exists a constant $\alpha_\Phi>1$ such that, for any $0<\mu<\alpha_\Phi$,*

$$\|\boldsymbol{u}\|^\mu_{L^\Phi} = o\left(\rho_\Phi\left(\boldsymbol{u}\right)\right). \tag{20}$$

   Reciprocally, if (20) *holds for $\mu\geq 1$ then $\Psi\in\Delta_2$.*

Based on [14] we say that F satisfies the condition (A) if $F(t,\boldsymbol{x})$ is a Carathéodory function and F is continuously differentiable with respect to $\boldsymbol{x}$. Moreover, the next inequality holds

$$|F(t,\boldsymbol{x})|+|D_{\boldsymbol{x}}F(t,\boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t\in[0,T], \forall\boldsymbol{x}\in\mathbb{R}^d. \tag{21}$$

3 Lagrangians with sublinear nonlinearity

We define the following functionals $J_{C,\mu}:L^\Phi\to(-\infty,+\infty]$ and $H_{C,\sigma}:\mathbb{R}^n\to\mathbb{R}$, with $C,\nu,\sigma>0$, by

$$J_{C,\nu}(\boldsymbol{u}) := \rho_\Phi\left(\boldsymbol{u}\right) - C\|\boldsymbol{u}\|^\nu_{L^\Phi}, \tag{22}$$

and

$$H_{C,\sigma}(\boldsymbol{x}) = \int_0^T F(t,\boldsymbol{x})dt - C|\boldsymbol{x}|^\sigma \tag{23}$$

respectively.

Like in [12] we consider Lagrangians $\mathcal{L}$ which are lower bounded as follows

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geq \alpha_0 \Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t,\boldsymbol{x}). \tag{24}$$

Now, we have another result about coercivity of I assuming some conditions on the ∇F.

Theorem 3.1. *Let $\mathcal{L}$ be a lagrangian function satisfying* (2), (3), (4), (24) *and F satisfies condition (A). We assume the following conditions:*

1. $\Psi \in \Delta_2$.
2. *There exist non negative functions $b_1, b_2 \in L^1_1$ and a constant $1 < \mu < \alpha_\Phi$ such that for any $\boldsymbol{x} \in \mathbb{R}^d$ and a.e. $t \in [0,T]$*
$$|\nabla F(t,\boldsymbol{x})| \leqslant b_1(t)|\boldsymbol{x}|^{\mu-1} + b_2(t). \tag{25}$$
3. *There exists a real positive number σ such that $\sigma > (\mu-1)\beta_\Psi$ and*
$$|\boldsymbol{x}|^\sigma = o\left(\int_0^T F(t,\boldsymbol{x})\,dt\right) \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{26}$$

Then the action integral I is coercive.

Proof. By the decomposition $u = \overline{u} + \tilde{\boldsymbol{u}}$, Mean Value Theorem, Cauchy-Schwarz inequality and (25), we have

$$\begin{aligned}
&\left|\int_0^T F(t,\boldsymbol{u}) - F(t,\overline{\boldsymbol{u}})\,dt\right| = \left|\int_0^T \int_0^1 \nabla F(t,\overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)) \cdot \tilde{\boldsymbol{u}}(t)\,ds\,dt\right| \\
&\leqslant \int_0^T \int_0^1 b_1(t)|\overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)|^{\mu-1}|\tilde{\boldsymbol{u}}(t)|\,ds\,dt + \int_0^T \int_0^1 b_2(t)|\tilde{\boldsymbol{u}}(t)|\,ds\,dt \\
&= I_1 + I_2.
\end{aligned} \tag{27}$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate I_2 as follows

$$I_2 \leqslant \|b_2\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty} \leqslant C_1\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{28}$$

where $C_1 = C_1(\|b_2\|_{L^1}, T)$. On the other hand, as $s \in [0,1]$, we have

$$|\overline{\boldsymbol{u}} + s\tilde{\boldsymbol{u}}(t)|^{\mu-1} \leqslant C(\mu)(|\overline{\boldsymbol{u}}|^{\mu-1} + \|\tilde{\boldsymbol{u}}\|^{\mu-1}_{L^\infty}). \tag{29}$$

where $C(\mu)=2^{\mu-2}$, for $\mu\geq 2$ and $C(\mu)=1$, for $1<\mu<2$. Now, inequality (29), Hölder's inequality and Sobolev's inequality imply that

$$\begin{aligned}
I_1 &\leqslant C(\mu)\left(|\overline{\boldsymbol{u}}|^{\mu-1}\int_0^T b_1(t)|\tilde{\boldsymbol{u}}(t)|\,dt+\|\tilde{\boldsymbol{u}}\|_{L^\infty}^{\mu-1}\int_0^T b_1(t)|\tilde{\boldsymbol{u}}(t)|\,dt\right)\\
&\leqslant C(\mu)\left\{|\overline{\boldsymbol{u}}|^{\mu-1}\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty}+\|b_1\|_{L^1}\|\tilde{\boldsymbol{u}}\|_{L^\infty}^{\mu}\right\}\\
&\leqslant C_2\left\{|\overline{\boldsymbol{u}}|^{\mu-1}\|\dot{\boldsymbol{u}}\|_{L^\Phi}+\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu}\right\},
\end{aligned} \tag{30}$$

where $C_2=C_2(\mu,T,\|b_1\|_{L^1})$. Let μ' be a positive constant such that $1<\mu\leqslant\mu'<\alpha_\Phi$. Next, using Young's inequality with conjugate exponents μ' and $\frac{\mu'}{\mu'-1}$ we get

$$|\overline{\boldsymbol{u}}|^{\mu-1}\|\dot{\boldsymbol{u}}\|_{L^\Phi}\leqslant\frac{(\mu'-1)}{\mu'}|\overline{\boldsymbol{u}}|^{\boldsymbol{\sigma}}+\frac{1}{\mu'}\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'} \tag{31}$$

where $\sigma=\frac{(\mu-1)\mu'}{\mu'-1}$. We note that σ is an arbitrary positive constant bigger than $(\mu-1)b_\Psi$.

From (30),(31), (28) and the inequality $x^{r_1}\leqslant x^{r_2}+1$, for any $x\geq 0$ and $r_1\leqslant r_2$ we have

$$\begin{aligned}
I_1+I_2 &\leqslant C_3\left\{|\overline{\boldsymbol{u}}|^{\sigma}+\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'}+\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu}+\|\dot{\boldsymbol{u}}\|_{L^\Phi}\right\}\\
&\leqslant C_3\left\{|\overline{\boldsymbol{u}}|^{\sigma}+\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'}+1\right\}
\end{aligned} \tag{32}$$

with $C_3=C_3(\mu,T,\|b_1\|_{L^1},\mu')$. In the subsequent estimates, we use the decomposition $u=\overline{u}+\tilde{\boldsymbol{u}}$, (24), (27), (32) and we get

$$\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)+\int_0^T F(t,\boldsymbol{u})\,dt\\
&=\alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)+\int_0^T\left[F(t,\boldsymbol{u})-F(t,\overline{\boldsymbol{u}})\right]\,dt+\int_0^T F(t,\overline{\boldsymbol{u}})\,dt\\
&\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)-C_3\|\dot{\boldsymbol{u}}\|_{L^\Phi}^{\mu'}+\int_0^T F(t,\overline{\boldsymbol{u}})\,dt-C_3|\overline{\boldsymbol{u}}|^{\sigma}-C_3\\
&=\alpha_0 J_{C_4,\mu'}\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)+H_{C_3,\sigma}(\overline{\boldsymbol{u}})-C_3,
\end{aligned} \tag{33}$$

where $C_4=\Lambda^{\mu'}C_3/\alpha_0$.

Let $\boldsymbol{u}_n$ be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n\|_{W^1L^\Phi}\to\infty$ and we have to prove that $I(\boldsymbol{u}_n)\to\infty$.

On the contrary, suppose that for a subsequence, still denoted by $\boldsymbol{u}_n$, $I(\boldsymbol{u}_n)$ is upper bounded, that is, there exists $M>0$ such that $|I(\boldsymbol{u}_n)|\leqslant M$. As $\|\boldsymbol{u}_n\|_{W^1L^\Phi}\to\infty$, from Lemma 2.2, we have $|\overline{\boldsymbol{u}}_n|+\|\dot{\boldsymbol{u}}_n\|_{L^\Phi}\to\infty$. Then, there exists subsequence of the subsequence $\{\boldsymbol{u}_n\}$, still denoted by $\boldsymbol{u}_n$, which is not bounded. Then, $\overline{\boldsymbol{u}}_n \to \infty$ or $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Now, as the functionals $J_{C,\mu'(\dot{\boldsymbol{u}})}$ and $\gamma(\overline{\boldsymbol{u}})$ are coercive, then $J_{C,\mu'(\dot{\boldsymbol{u}}_n)} \to \infty$ or $\gamma(\overline{\boldsymbol{u}}_n) \to \infty$. By (21), the functional $\gamma(\overline{\boldsymbol{u}}_n)$ is lower bounded and $J_{C,\mu'(\dot{\boldsymbol{u}}_n)}$ is also lower bounded on a bounded set because the modular $\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)$ is always bigger than zero. Therefore, $I(\boldsymbol{u}_n) \to \infty$ as $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$ which contradits the initial assumption on the behavior of $I(\boldsymbol{u}_n)$. $\square$

REVISAR LA PRUEBA ANTERIOR Y MEJORAR LA ESCRITURA, y adaptar quitando J si fuera el caso!!!!

4 Limit case $\mu = \alpha_\Phi$

In [] coercivity was obtained even in the limit case $\mu = 1$ and $\mu = p$ assuming additional conditions on ... This was possible because in L^p spaces, the norm and the modular coincides, that is, $\|\cdot\|_p^p = O(\int_0^T |\cdot|^p\, dt)$. In Orlicz spaces, $\|\cdot\|_{L^\Phi}^\mu$ can be upper controled by a modular provided that $\mu < \alpha_\Phi$ for any N-function Φ. But, the limit case does not hold for any Φ, i.e. in general $\|\cdot\|_{L^\Phi}^{\alpha_\Phi} = O(\int_0^T \Phi(|u|)\, dt)$ is false as can be seen as follows.

Let $\Phi, \Psi \in \Delta_2$, then the next inequality $\Phi(tu) \geq t^{\alpha_\Phi}\Phi(u)$ for any $u > 0$ and for any $t \geq 1$ is false.

In fact, let $\Phi(u) = \begin{cases} \frac{p-1}{p}u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$ with $p > 1$.

Theorem 4.1. *If $p \geq \frac{1+\sqrt{2}}{2}$, then Φ is an N-function.*

Proof. We have

$$\Phi'(u) = \begin{cases} (p-1)u^{p-1} =: \varphi_1(u) & u < e \\ \frac{u^{p-1}}{\log u}(p - \frac{1}{\log u}) := \varphi_2(u) & u > e \end{cases}$$

and Φ is differentiable at e because $\varphi_1(e) = \varphi_2(e) = (p-1)e^{p-1}$.

Tendríamos que ver que φ_1, φ_2 son crecientes y que $\varphi_2 \to \infty$ cuando $u \to \infty$???? o basta con ver que φ_2 es creciente????

φ_1 is an increasing function provided that $p > 1$ and $\varphi_1(u) \to 0$ as $u \to 0$.

In addition, $\varphi_2(u) \to \infty$ as $u \to \infty$ provided that $p > 1$. And

$$0 < \varphi_2'(u) = \frac{u^{p-2}}{\log u}\left(p^2 - p - \frac{2p}{\log u} + \frac{1}{\log u} + \frac{2}{\log^2 u}\right)$$

on $[e, \infty)$ if and only if

$$\left(p^2 - p - \frac{2p}{\log u} + \frac{1}{\log u} + \frac{2}{\log^2 u}\right) > 0.$$

If we take $\alpha := \frac{1}{\log u}$, then we need

$$2\alpha^2 + (1-2p)\alpha + (p^2 - p) \geq 0$$

which is true if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore, φ_2 is an icreasing function when $p \geq \frac{1+\sqrt{2}}{2}$. □

Theorem 4.2. *There exists a constant $C > 0$ such that*

$$\Phi(tu) \leqslant ct^p\Phi(u) \;\; t \geq 1, u > 0. \tag{34}$$

For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, p)$ such that

$$\Phi(tu) \geq Ct^{p-\varepsilon}\Phi(u) \;\; t \geq 1, u > 0. \tag{35}$$

Proof. In order to prove (**??**), we will analyze three cases.

If $u \leqslant tu \leqslant e$, then $\Phi(tu) = t^p\Phi(u)$ and (**??**) holds with $C = 1$.

If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geq 1$, we have $\Phi(tu) \leqslant t^p u^p = \frac{p}{p-1}t^p\Phi(u)$. Thus, (**??**) holds with $C = \frac{p}{p-1}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v} \tag{36}$$

where $v := u^p$ and $v \geq e^p$.

If $\alpha > 0$, the function $f(x) = \frac{x}{x-\alpha}$ is decreasing on (α, ∞). And, the function $g(v) = \frac{pv}{\log v}$ is decreasing on $[e^p, \infty)$. Therefore, $f \circ g$ is decreasing on $[e^p, \infty)$ and we have

$$(f \circ g)(v) = \frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant e^p - \frac{e^p}{p} = \frac{p}{p-1}$$

for every $v \geq e^p$.

In this way, from (**??**), we have

$$\Phi(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and (**??**) holds with $C = \frac{p}{p-1}$.

Now, we will prove (**??**).

If $u \leqslant tu \leqslant e$, (**??**) is immediate because $t^p \geq t^{p-\varepsilon}$ for every $t \geq 1$, $p > 1$ and ε **sufficiently small????**

If $u \leqslant e \leqslant tu$, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$ then $f(t) \geq e$ for every $t \geq e$. Habría que mirar en $[e^p, \infty)$ para que $f(t) \geq \frac{e^p}{p}$???? O, como f es creciente y $e^p \leqslant (tu)^p$ entonces $f((tu)^p) \geq f(e^p)$??? Now,

$$\Phi(tu) = \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} = \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \geq \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} \geq \frac{p-1}{p}\frac{t^\varepsilon}{\log t + 1}t^{p-\varepsilon}u^p$$

Since the function $f(t) = \frac{t^\varepsilon}{\log t + 1}$ attains a minimum at $e^{\frac{1-\varepsilon}{\varepsilon}}$ and its minimum value is $\varepsilon e^{1-\varepsilon}$, then

$$\Phi(tu) \geq \frac{p-1}{p}\varepsilon e^{1-\varepsilon}t^{p-\varepsilon}u^p.$$

If $e \leqslant u \leqslant tu$, then

$$\Phi(tu) = \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} + \frac{1}{p}\frac{(tu)^p}{\log(tu)} - \frac{e^p}{p} \geq \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log(u)^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s+1}$ with $s \geq 1$ and $A \geq \varepsilon$. Then, the function f attains a minimum at $s = e^{1-A}$; but, s has to be bigger than 1, then it is necessary that $\varepsilon \leqslant A \leqslant 1$. And, the minimum value is $f(e^{1-A}) = Ae^{1-A} \geq \varepsilon$. If $A \geq 1$, f attains the minimum at $s =???$ and $f(1) = 1$. Then, $f \geq \varepsilon$ and therefore

$$\Phi(tu) \geq \frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u} \geq \frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi(u)$$

□

Remark 4.3. *The inequality*

$$\Phi(tu) \geq Ct^p\Phi(u)$$

is false for every C because for every $u \geq e$ we have

$$\lim_{t\to\infty}\frac{\Phi(tu)}{t^p\Phi(u)} = \lim_{t\to\infty}\frac{\frac{u^p}{\log(tu)} - \frac{e^p}{pt^p}}{\frac{u^p}{\log u} - \frac{e^p}{p}} = 0$$

Theorem 4.4. $\alpha_\Phi = \beta_\Phi = p$

Proof. Resumir la prueba. □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|)\,dx \geq C\|u\|_{L^\Phi}^{\alpha_\Phi} = C\|u\|_{L^\Phi}^p$$

is false.

If we take $u \equiv t > 0$, then $\|u\|_{L^\Phi}^p = C_1t^p$ where $C_1 = \|1\|_{L^\Phi}$ and $\int_0^T \Phi(|u|)\,dx = C_2\Phi(t)$ with $C_2 = T$. Then, if $\rho_\Phi(u) \geq C\|u\|_{L^\Phi}^p$ were true, then $\Phi(t) \geq Ct^p$ were also true but this last inequality is false.

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