

# Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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## Abstract

In this paper we....

ALGO ASÍ HABÍA QUE ESCRIBIR ACÁ O EN LA INTRO....

The results of this paper improve on the classic ones obtained in [1] and [2] with  $|\nabla F(t, \mathbf{x})| \leq f(t)\mathbf{x}^\alpha + g(t)$  for  $\alpha < 1$  and  $\alpha < p$  respectively. Here, the bounds of the sublinearity are bigger than those considered in the aforementioned works.

## 1 Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) = D_x \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) & \text{a.e. } t \in (0, T) \\ \mathbf{u}(0) - \mathbf{u}(T) = \dot{\mathbf{u}}(0) - \dot{\mathbf{u}}(T) = 0 \end{cases} \quad (1)$$

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where  $T > 0$ ,  $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^d$  is absolutely continuous and the *Lagrangian*  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left( b(t) + \Phi \left( \frac{|\mathbf{y}|}{\lambda} + f(t) \right) \right), \quad (2)$$

$$|D_{\mathbf{x}} \mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left( b(t) + \Phi \left( \frac{|\mathbf{y}|}{\lambda} + f(t) \right) \right), \quad (3)$$

$$|D_{\mathbf{y}} \mathcal{L}(t, \mathbf{x}, \mathbf{y})| \leq a(|\mathbf{x}|) \left( c(t) + \varphi \left( \frac{|\mathbf{y}|}{\lambda} + f(t) \right) \right). \quad (4)$$

In these inequalities we assume that  $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ ,  $\lambda > 0$ ,  $\Phi$  is an  $N$ -function (see section Preliminaries for definitions),  $\varphi$  is the right continuous derivative of  $\Phi$ . The non negative functions  $b, c$  and  $f$  satisfy that  $b \in L_1^1([0, T])$ ,  $c \in L_1^\Psi([0, T])$  and  $f \in E_1^\Phi([0, T])$ , where the Banach spaces  $L_1^1([0, T])$ ,  $L_1^\Psi([0, T])$  and  $E_1^\Phi([0, T])$  will be defined later.

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(\mathbf{u}) = \int_0^T \mathcal{L}(t, \mathbf{u}(t), \dot{\mathbf{u}}(t)) dt. \quad (5)$$

## 2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [3, 4, 5]. For Orlicz spaces of vector valued functions, see [6] and the references therein.

Hereafter we denote by  $\mathbb{R}^+$  the set of all non negative real numbers. A function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called an  $N$ -function if  $\Phi$  is given by

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \text{for } t \geq 0,$$

where  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a right continuous non decreasing function satisfying  $\varphi(0) = 0$ ,  $\varphi(t) > 0$  for  $t > 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = +\infty$ .

Given a function  $\varphi$  as above, we consider the so-called right inverse function  $\psi$  of  $\varphi$  which is defined by  $\psi(s) = \sup_{\varphi(t) \leq s} t$ . The function  $\psi$  satisfies the same properties as the function  $\varphi$ , therefore we have an  $N$ -function  $\Psi$  such that  $\Psi' = \psi$ . The function  $\Psi$  is called the *complementary function* of  $\Phi$ .

We say that  $\Phi$  satisfies the  $\Delta_2$ -condition, denoted by  $\Phi \in \Delta_2$ , if there exist constants  $K > 0$  and  $t_0 \geq 0$  such that

$$\Phi(2t) \leq K\Phi(t) \quad (6)$$

for every  $t \geq t_0$ . If  $t_0 = 0$ , we say that  $\Phi$  satisfies the  $\Delta_2$ -condition globally ( $\Phi \in \Delta_2$  globally).

Let  $d$  be a positive integer. We denote by  $\mathcal{M}_d := \mathcal{M}_d([0, T])$  the set of all measurable functions defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $\mathbf{u} = (u_1, \dots, u_d)$  for  $\mathbf{u} \in \mathcal{M}_d$ . In this paper we adopt the convention that bold symbols denote points in  $\mathbb{R}^d$ .

Given an  $N$ -function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M}_d \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(\mathbf{u}) := \int_0^T \Phi(|\mathbf{u}|) dt.$$

Here  $|\cdot|$  is the euclidean norm of  $\mathbb{R}^d$ . The *Orlicz class*  $C_d^\Phi = C_d^\Phi([0, T])$  is given by

$$C_d^\Phi := \{\mathbf{u} \in \mathcal{M}_d \mid \rho_\Phi(\mathbf{u}) < \infty\}. \quad (7)$$

The *Orlicz space*  $L_d^\Phi = L_d^\Phi([0, T])$  is the linear hull of  $C_d^\Phi$ ; equivalently,

$$L_d^\Phi := \{\mathbf{u} \in \mathcal{M}_d \mid \exists \lambda > 0 : \rho_\Phi(\lambda \mathbf{u}) < \infty\}. \quad (8)$$

The Orlicz space  $L_d^\Phi$  equipped with the *Orlicz norm*

$$\|\mathbf{u}\|_{L^\Phi} := \sup \left\{ \int_0^T \mathbf{u} \cdot \mathbf{v} dt \mid \rho_\Psi(\mathbf{v}) \leq 1 \right\},$$

is a Banach space. By  $\mathbf{u} \cdot \mathbf{v}$  we denote the usual dot product in  $\mathbb{R}^d$  between  $\mathbf{u}$  and  $\mathbf{v}$ . The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [4, Thm. 10.5] and [7]). For every  $\mathbf{u} \in L^\Phi$ ,

$$\|\mathbf{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(k\mathbf{u})\}. \quad (9)$$

The subspace  $E_d^\Phi = E_d^\Phi([0, T])$  is defined as the closure in  $L_d^\Phi$  of the subspace  $L_d^\infty$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that  $E_d^\Phi$  is the only one maximal subspace contained in the Orlicz class  $C_d^\Phi$ , i.e.  $\mathbf{u} \in E_d^\Phi$  if and only if  $\rho_\Phi(\lambda \mathbf{u}) < \infty$  for any  $\lambda > 0$ .

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [4, Th. 9.3]). Namely, if  $\mathbf{u} \in L_d^\Phi$  and  $\mathbf{v} \in L_d^\Psi$  then  $\mathbf{u} \cdot \mathbf{v} \in L_1^1$  and

$$\int_0^T \mathbf{v} \cdot \mathbf{u} dt \leq \|\mathbf{u}\|_{L^\Phi} \|\mathbf{v}\|_{L^\Psi}. \quad (10)$$

If  $X$  and  $Y$  are Banach spaces such that  $Y \subset X^*$ , we denote by  $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{R}$  the bilinear pairing map given by  $\langle x^*, x \rangle = x^*(x)$ . Hölder's inequality shows that  $L_d^\Psi \subset [L_d^\Phi]^*$ , where the pairing  $\langle \mathbf{v}, \mathbf{u} \rangle$  is defined by

$$\langle \mathbf{v}, \mathbf{u} \rangle = \int_0^T \mathbf{v} \cdot \mathbf{u} dt \quad (11)$$

with  $\mathbf{u} \in L_d^\Phi$  and  $\mathbf{v} \in L_d^\Psi$ . Unless  $\Phi \in \Delta_2$ , the relation  $L_d^\Psi = [L_d^\Phi]^*$  will not hold. In general, it is true that  $[E_d^\Phi]^* = L_d^\Psi$ .

Like in [4], we will consider the subset  $\Pi(E_d^\Phi, r)$  of  $L_d^\Phi$  given by

$$\Pi(E_d^\Phi, r) := \{\mathbf{u} \in L_d^\Phi \mid d(\mathbf{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class  $C_d^\Phi$  by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \quad (12)$$

for any positive  $r$ . If  $\Phi \in \Delta_2$ , then the sets  $L_d^\Phi$ ,  $E_d^\Phi$ ,  $\Pi(E_d^\Phi, r)$  and  $C_d^\Phi$  are equal.

We define the *Sobolev-Orlicz space*  $W^1 L_d^\Phi$  (see [3]) by

$$W^1 L_d^\Phi := \{\mathbf{u} | \mathbf{u} \text{ is absolutely continuous and } \dot{\mathbf{u}} \in L_d^\Phi\}.$$

$W^1 L_d^\Phi$  is a Banach space when equipped with the norm

$$\|\mathbf{u}\|_{W^1 L^\Phi} = \|\mathbf{u}\|_{L^\Phi} + \|\dot{\mathbf{u}}\|_{L^\Phi}. \quad (13)$$

For a function  $\mathbf{u} \in L_d^1([0, T])$ , we write  $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$  where  $\bar{\mathbf{u}} = \frac{1}{T} \int_0^T \mathbf{u}(t) dt$  and  $\tilde{\mathbf{u}} = \mathbf{u} - \bar{\mathbf{u}}$ .

As usual, if  $(X, \|\cdot\|_X)$  is a Banach space and  $(Y, \|\cdot\|_Y)$  is a subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when the restricted identity map  $i_Y : Y \rightarrow X$  is bounded. That is, there exists  $C > 0$  such that for any  $y \in Y$  we have  $\|y\|_X \leq C\|y\|_Y$ . With this notation, Hölder's inequality states that  $L_d^\Psi \hookrightarrow [L_d^\Phi]^*$ ; and, it is easy to see that for every  $N$ -function  $\Phi$  we have that  $L_d^\infty \hookrightarrow L_d^\Phi \hookrightarrow L_d^1$ .

Recall that a function  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called a *modulus of continuity* if  $w$  is a continuous increasing function which satisfies  $w(0) = 0$ . For example, it can be easily shown that  $w(s) = s\Phi^{-1}(1/s)$  is a modulus of continuity for every  $N$ -function  $\Phi$ . We say that  $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^d$  has modulus of continuity  $w$  when there exists a constant  $C > 0$  such that

$$|\mathbf{u}(t) - \mathbf{u}(s)| \leq Cw(|t - s|). \quad (14)$$

We denote by  $C^w([0, T], \mathbb{R}^d)$  the space of  $w$ -Hölder continuous functions. This is the space of all functions satisfying (14) for some  $C > 0$  and it is a Banach space with norm

$$\|\mathbf{u}\|_{C^w([0, T], \mathbb{R}^d)} := \|\mathbf{u}\|_{L^\infty} + \sup_{t \neq s} \frac{|\mathbf{u}(t) - \mathbf{u}(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [8, 9, 10, 11, 12]. The next simple lemma, whose proof can be found in [13], will be used systematically.

**Lemma 2.1.** *Let  $w(s) := s\Phi^{-1}(1/s)$ . Then, the following statements hold:*

1.  $W^1 L^\Phi \hookrightarrow C^w([0, T], \mathbb{R}^d)$  and for every  $\mathbf{u} \in W^1 L^\Phi$

$$|\mathbf{u}(t) - \mathbf{u}(s)| \leq \|\dot{\mathbf{u}}\|_{L^\Phi} w(|t - s|), \quad (15)$$

$$\|\mathbf{u}\|_{L^\infty} \leq \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|\mathbf{u}\|_{W^1 L^\Phi} \quad (16)$$

2. For every  $\mathbf{u} \in W^1 L^\Phi$  we have  $\tilde{\mathbf{u}} \in L_d^\infty$  and

$$\|\tilde{\mathbf{u}}\|_{L^\infty} \leq T\Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\mathbf{u}}\|_{L^\Phi} \quad (\text{Sobolev's inequality}). \quad (17)$$

The following result is analogous to some lemmata in  $W^1 L_d^p$ , see [14].

**Lemma 2.2.** *If  $\|\mathbf{u}\|_{W^1 L^\Phi} \rightarrow \infty$ , then  $(|\bar{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi}) \rightarrow \infty$ .*

*Proof.* By the decomposition  $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$  and some elementary operations, we get

$$\|\mathbf{u}\|_{L^\Phi} = \|\bar{\mathbf{u}} + \tilde{\mathbf{u}}\|_{L^\Phi} \leq \|\bar{\mathbf{u}}\|_{L^\Phi} + \|\tilde{\mathbf{u}}\|_{L^\Phi} = |\bar{\mathbf{u}}| \|1\|_{L^\Phi} + \|\tilde{\mathbf{u}}\|_{L^\Phi}. \quad (18)$$

It is known that  $L_d^\infty \hookrightarrow L_d^\Phi$ , i.e., there exists  $C_1 = C_1(T) > 0$  such that for any  $\tilde{\mathbf{u}} \in L_d^\infty$  we have

$$\|\tilde{\mathbf{u}}\|_{L^\Phi} \leq C_1 \|\tilde{\mathbf{u}}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain the Wirtinger's inequality, that is there exists  $C_2 = C_2(T) > 0$  such that

$$\|\tilde{\mathbf{u}}\|_{L^\Phi} \leq C_2 \|\dot{\mathbf{u}}\|_{L^\Phi}. \quad (19)$$

Therefore, from (18), (19) and (13), we get

$$\|\mathbf{u}\|_{W^1 L^\Phi} \leq C_3 (|\bar{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi})$$

where  $C_3 = C_3(T)$ . Finally, as  $\|\mathbf{u}\|_{W^1 L^\Phi} \rightarrow \infty$  we conclude that  $(|\bar{\mathbf{u}}| + \|\dot{\mathbf{u}}\|_{L^\Phi}) \rightarrow \infty$ .  $\square$

We present a definition that will be useful later.

**Definition 2.3.** *A function  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a Carathéodory function if for fixed  $(\mathbf{x}, \mathbf{y})$  the map  $t \mapsto \mathcal{L}(t, \mathbf{x}, \mathbf{y})$  is measurable and for fixed  $t$  the map  $(\mathbf{x}, \mathbf{y}) \mapsto \mathcal{L}(t, \mathbf{x}, \mathbf{y})$  is continuous for almost everywhere  $t \in [0, T]$ . We say that  $\mathcal{L}(t, \mathbf{x}, \mathbf{y})$  is differentiable Carathéodory if in addition  $\mathcal{L}(t, \mathbf{x}, \mathbf{y})$  is continuously differentiable with respect to  $\mathbf{x}$  and  $\mathbf{y}$  for almost everywhere  $t \in [0, T]$ .*

In [13] we proved the next results.

**Theorem 2.4.** *Let  $\mathcal{L}$  be a differentiable Carathéodory function satisfying (2), (3) and (4). Then the following statements hold:*

1. *The action integral given by (5) is finitely defined on  $\mathcal{E}_d^\Phi(\lambda) := W^1 L_d^\Phi \cap \{\mathbf{u} | \dot{\mathbf{u}} \in \Pi(E_d^\Phi, \lambda)\}$ .*
2. *The function  $I$  is Gâteaux differentiable on  $\mathcal{E}_d^\Phi(\lambda)$  and its derivative  $I'$  is demi-continuous from  $\mathcal{E}_d^\Phi(\lambda)$  into  $[W^1 L_d^\Phi]^*$ . Moreover,  $I'$  is given by the following expression*

$$\langle I'(\mathbf{u}), \mathbf{v} \rangle = \int_0^T \{D_{\mathbf{x}} \mathcal{L}(t, \mathbf{u}, \dot{\mathbf{u}}) \cdot \mathbf{v} + D_{\mathbf{y}} \mathcal{L}(t, \mathbf{u}, \dot{\mathbf{u}}) \cdot \dot{\mathbf{v}}\} dt. \quad (20)$$

3. *If  $\Psi \in \Delta_2$  then  $I'$  is continuous from  $\mathcal{E}_d^\Phi(\lambda)$  into  $[W^1 L_d^\Phi]^*$  when both spaces are equipped with the strong topology.*

In [13] we derive the Euler-Lagrange equations associated to critical points of action integrals on the subspace of  $T$ -periodic functions. We denote by  $W^1 L_T^\Phi$  the subspace of  $W^1 L_d^\Phi$  containing all  $T$ -periodic functions. As usual, when  $Y$  is a subspace of the Banach space  $X$ , we denote by  $Y^\perp$  the *annihilator subspace* of  $X^*$ , i.e. the subspace that consists of all bounded linear functions which are identically zero on  $Y$ .

We recall that a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is called *strictly convex* if  $f\left(\frac{x+y}{2}\right) < \frac{1}{2}(f(x) + f(y))$  for  $x \neq y$ . It is well known that if  $f$  is a strictly convex and differentiable function, then  $D_x f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a one-to-one map (see, e.g. [15, Thm. 12.17]).

**Theorem 2.5.** *Let  $u \in \mathcal{E}_d^\Phi(\lambda)$  be a  $T$ -periodic function. The following statements are equivalent:*

1.  $I'(u) \in (W^1 L_T^\Phi)^\perp$ .
2.  $D_y \mathcal{L}(t, u(t), \dot{u}(t))$  is an absolutely continuous function and  $u$  solves the following boundary value problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), \dot{u}(t)) = D_x \mathcal{L}(t, u(t), \dot{u}(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = D_y \mathcal{L}(0, u(0), \dot{u}(0)) - D_y \mathcal{L}(T, u(T), \dot{u}(T)) = 0. \end{cases} \quad (21)$$

Moreover if  $D_y \mathcal{L}(t, x, y)$  is  $T$ -periodic with respect to the variable  $t$  and strictly convex with respect to  $y$ , then  $D_y \mathcal{L}(0, u(0), \dot{u}(0)) - D_y \mathcal{L}(T, u(T), \dot{u}(T)) = 0$  is equivalent to  $\dot{u}(0) = \dot{u}(T)$ .

**Habría que ver si el lugar de los índices es el adecuado. Copié lo que teníamos en el primer trabajo.**

Next, we enumerate some definitions and results from the theory of convex functions. We suggest [16, 17, 4, 18, 5] for definitions, proofs and additional details.

We denote by  $\alpha_\varphi$  and  $\beta_\varphi$  the so called *Matuszewska-Orlicz indices* of the function  $\varphi$ , which are defined next. Given an increasing, unbounded, continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\varphi(0) = 0$  we define

$$\alpha_\varphi := \lim_{t \rightarrow 0^+} \frac{\log \left( \sup_{u>0} \frac{\varphi(tu)}{\varphi(u)} \right)}{\log(t)}, \quad \beta_\varphi := \lim_{t \rightarrow +\infty} \frac{\log \left( \sup_{u>0} \frac{\varphi(tu)}{\varphi(u)} \right)}{\log(t)}. \quad (22)$$

We have that  $0 \leq \alpha_\varphi \leq \beta_\varphi \leq +\infty$ . The relation  $\beta_\varphi < \infty$  holds true if and only if  $\varphi$  is a  $\Delta_2$ -function. If  $\varphi$  is a homeomorphism we have that

$$\alpha_{\varphi^{-1}} = \frac{1}{\beta_\varphi}. \quad (23)$$

Moreover  $\varphi \in \mathcal{F}$  implies  $\alpha_\varphi \geq 1$ . As a consequence,  $\varphi^{-1}$  is a  $\Delta_2$ -function.

It is well known that if  $\varphi$  is an increasing  $\Delta_2$ -function,  $\varphi$  is controlled by above and below by power functions. More concretely, for every  $\epsilon > 0$  there exists a constant  $K = K(\varphi, \epsilon)$  such that, for every  $t, u \geq 0$ ,

$$K^{-1} \min \{t^{\beta_\varphi + \epsilon}, t^{\alpha_\varphi - \epsilon}\} \varphi(u) \leq \varphi(tu) \leq K \max \{t^{\beta_\varphi + \epsilon}, t^{\alpha_\varphi - \epsilon}\} \varphi(u). \quad (24)$$

**Lemma 2.6.** *Let  $\Phi, \Psi$  complementary functions. The next statements are equivalent:*

1.  $\Psi \in \Delta_2$  globally.
2. There exists an  $N$ -function  $\Phi_1 \in \Delta_2$  such that

$$\Phi(rs) \geq \Phi_1(r)\Phi(s) \text{ for every } r \geq 1, s \geq 0. \quad (25)$$

*Proof.* 1) $\Rightarrow$ 2) As  $\Psi \in \Delta_2$  globally, there exist  $k > 0$  and  $\nu > 1$  such that

$$\Phi(rs) \geq kr^\nu \Phi(s) \text{ } r \geq 1, s > 0,$$

which is (25) with  $\Phi_1(r) = kr^\nu$  that is an  $N$ -function that satisfies the  $\Delta_2$ -condition.

2) $\Rightarrow$ 1) Next, we follow [5, p. 32, Prop. 13] and [5, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leq \Phi(rs) \text{ } r > 1, s \geq 0.$$

Let  $u = \Phi_1(r) \geq \Phi_1(1)$  and  $v = \Phi(s) \geq 0$ . By the well known inequality [5, p. 13, Prop. 1] and (25) we have for  $u \geq \Phi_1(1)$  and  $v \geq 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leq \Phi^{-1}(uv) \leq \Phi_1^{-1}(u)\Phi^{-1}(v) \leq \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leq 4\Psi^{-1}(uv).$$

If we take  $x = \Psi_1^{-1}(u) \geq \Psi_1^{-1}(\Phi_1(1))$  and  $y = \Psi^{-1}(v) \geq 0$ , then

$$\Psi\left(\frac{xy}{4}\right) \leq \Psi_1(x)\Psi(y).$$

Now, taking  $x \geq \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$  we get that  $\Psi \in \Delta_2$  globally.  $\square$

The following Lemma generalizes [13, Lemma 5.2].

**Lemma 2.7.** *Let  $\Phi$  be  $N$ -functions and suppose that  $\Psi \in \Delta_2$  globally. Then*

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \quad (26)$$

*for every  $\Phi_0$  with  $\Phi_0 = o(\Phi_1)$  at  $\infty$  where  $\Phi_1$  is any  $N$ -function satisfying (25).*

*Reciprocally if (26) holds for some  $N$ -function  $\Phi_0$  then  $\Psi \in \Delta_2$  (at  $\infty$ ).*

*Proof.* By the assumptions on  $\Phi$  and  $\Phi_1$  and identity (9), we have

$$\frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \Phi_1(r) \frac{\int_0^T \Phi(r^{-1}|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})} \{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose  $r = \frac{\|u\|_{L^\Phi}}{2}$ , as  $\|u\|_{L^\Phi} \rightarrow \infty$  we can assume  $r > 1$ , we use the fact that  $\Phi_1 \in \Delta_2$  and  $\Phi_0 = o(\Phi_1)$  at  $\infty$ , and therefore we get

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(\|u\|_{L^\Phi})} \geq \lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} \geq C \lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\Phi_1(\|u\|_{L^\Phi})}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

The last affirmation of the Lemma follows from the fact that if  $\Phi_0$  is  $N$ -function then  $\Phi_0(u) \geq ku$  for  $k$  small enough. Therefore (26) holds for  $\Phi_0(u) = |u|$ , then [13, Lemma 5.2] implies  $\Psi \in \Delta_2$  at  $\infty$ .  $\square$

*Remark 1.* We point out that this Lemma can be applied to more cases than [13, Lemma 5.2]. For example, if  $\Phi(u) = u^2$ ,  $\Phi_1$  and  $\Phi_0$  are  $N$ -functions with principal parts equal to  $u^2/\log u$  and  $u^2/(\log u)^2$  respectively (see [4, p. 16] and [4, Section 7] for the definition and properties of principal part). Then (26) holds for  $\Phi_0$ , however  $\Phi_0(u)$  is not dominated for any power function  $|u|^\alpha$  for every  $\alpha < 2$ .

We define the following functionals  $J_{C,\nu} : L^\Phi \rightarrow (-\infty, +\infty]$  and  $H_{C,\sigma} : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $C, \nu, \sigma > 0$ , by

$$J_{C,\nu}(\mathbf{u}) := \rho_\Phi(\mathbf{u}) - C\|\mathbf{u}\|_{L^\Phi}^\nu, \quad (27)$$

and

$$H_{C,\sigma}(\mathbf{x}) = \int_0^T F(t, \mathbf{x}) dt - C|\mathbf{x}|^\sigma, \quad (28)$$

**Debería ir algo así, en lugar de la anterior!!!**

$$H_{C,\Phi_0}(\mathbf{x}) = \int_0^T F(t, \mathbf{x}) dt - C\Phi_0(|\mathbf{x}|) \quad (29)$$

respectively.

### 3 Lagrangians satisfying sublinear nonlinearity type conditions

Like in [13] we consider Lagrangians  $\mathcal{L}$  which are lower bounded as follows

$$\mathcal{L}(t, \mathbf{x}, \mathbf{y}) \geq \alpha_0 \Phi\left(\frac{|\mathbf{y}|}{\Lambda}\right) + F(t, \mathbf{x}). \quad (30)$$

If  $\mathcal{L}$  is given by right hand side in (3) and  $\Phi(u) = |u|^2$  then the ode in (1) is quasi-linear, which is linear in  $\dot{\mathbf{u}}$  and non linear in  $\mathbf{u}$ , being  $F(t, \mathbf{u}(t))$  the nonlinearity. Following the literature, we refer to  $F$  as the non linearity yet we assume in just the inequality. In [1] and [2] the authors considered for  $p$ -laplacian non linearities satisfying the inequality

$$|\nabla F(t, \mathbf{x})| \leq b_1(t)|\mathbf{x}|^\alpha + b_2(t),$$

where  $b_1, b_2 \in L_1^1$  and  $\alpha$  is any power less than  $p$ .

Based on [19] we say that  $F$  satisfies the condition (A) if  $F(t, \mathbf{x})$  is a Carathéodory function and  $F$  is continuously differentiable with respect to  $\mathbf{x}$ . Moreover, the next inequality holds

$$|F(t, \mathbf{x})| + |D_{\mathbf{x}}F(t, \mathbf{x})| \leq a(|\mathbf{x}|)b_0(t), \quad \text{for a.e. } t \in [0, T], \forall \mathbf{x} \in \mathbb{R}^d. \quad (31)$$



Now, we have another result about coercivity of  $I$  assuming some conditions on the nonlinearity  $\nabla F$ .

**Theorem 3.1.** *Let  $\mathcal{L}$  be a lagrangian function satisfying (2), (3), (4), (3) and  $F$  satisfies condition (A). We assume the following conditions:*

1.  $\Psi \in \Delta_2$ .

2. *There exist non negative functions  $b_1, b_2 \in L^1_1$  such that for any  $\mathbf{x} \in \mathbb{R}^d$  and a.e.  $t \in [0, T]$*

$$|\nabla F(t, \mathbf{x})| \leq b_1(t)\varphi_0(|\mathbf{x}|) + b_2(t) \quad (32)$$

*being  $\varphi_0 = \Phi'_0$  and  $\Phi_0 \in \Delta_2$  globally that satisfies  $\Phi_0 = o(\Phi_1)$  at  $\infty$  and where  $\Phi(rs) \geq \Phi_1(r)\Phi(s)$ .*

3.

$$\Phi_0(|\mathbf{x}|) = o\left(\int_0^T F(t, \mathbf{x}) dt\right) \text{ as } |\mathbf{x}| \rightarrow \infty. \quad (33)$$

Then the action integral  $I$  is coercive.

*Proof.* By the decomposition  $u = \bar{u} + \tilde{u}$ , Mean Value Theorem, Cauchy-Schwarz's inequality and (43), we have

$$\begin{aligned} \left| \int_0^T F(t, \mathbf{u}) - F(t, \bar{\mathbf{u}}) dt \right| &= \left| \int_0^T \int_0^1 \nabla F(t, \bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)) \cdot \tilde{\mathbf{u}}(t) ds dt \right| \\ &\leq \int_0^T \int_0^1 b_1(t) f(|\bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)|) |\tilde{\mathbf{u}}(t)| ds dt + \int_0^T \int_0^1 b_2(t) |\tilde{\mathbf{u}}(t)| ds dt \\ &= I_1 + I_2. \end{aligned} \quad (34)$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate  $I_2$  as follows

$$I_2 \leq \|b_2\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty} \leq C_1 \|\dot{\mathbf{u}}\|_{L^\Phi}. \quad (35)$$

where  $C_1 = C_1(\|b_2\|_{L^1}, T)$ .

On the other hand, since  $\Phi_0 \in \Delta_2$  globally, then  $\varphi \in \Delta_2$  globally and consequently  $\varphi_0$  is a subadditive function, i.e. there exists  $k > 0$  such that  $\varphi_0(a+b) \leq k(\varphi_0(a) + \varphi_0(b))$  for every  $a, b \geq 0$ . In this way, we have

$$\varphi_0(|\bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)|) \leq C(\varphi_0)[\varphi_0(|\bar{\mathbf{u}}|) + \varphi_0(\|\tilde{\mathbf{u}}\|_{L^\infty})]. \quad (36)$$

where  $s \in [0, 1]$  and  $C(\varphi_0)$  springs from the subadditivity of  $\varphi_0$ .

Now, inequality (47), Hölder's inequality, Sobolev's inequality, and the monotonicity and the subadditivity of  $\varphi_0$  imply that

$$\begin{aligned} I_1 &\leq C(\varphi_0) \left\{ \varphi_0(|\bar{\mathbf{u}}|) \|b_1\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty} + \|b_1\|_{L^1} \varphi_0(\|\tilde{\mathbf{u}}\|_{L^\infty}) \|\tilde{\mathbf{u}}\|_{L^\infty} \right\} \\ &\leq C_2 \left\{ \varphi_0(|\bar{\mathbf{u}}|) \|\dot{\mathbf{u}}\|_{L^\Phi} + \varphi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) \|\dot{\mathbf{u}}\|_{L^\Phi} \right\} \end{aligned} \quad (37)$$

where  $C_2 = C_2(\varphi_0, T, \|b_1\|_{L^1})$ .

Next, by Young's inequality with complementary functions  $\Phi_0$  and  $\Psi_0$  and the fact that  $\Phi_0 \in \Delta_2$  globally, we get

$$\begin{aligned} \varphi_0(|\bar{u}|) \|\dot{\mathbf{u}}\|_{L^\Phi} &\leq \Psi_0(\varphi_0(|\bar{u}|) + \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi})) \\ &= |\bar{u}| \varphi_0(|\bar{u}|) + \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) \\ &\leq C(\Phi_0) \Phi_0(|\bar{u}|) + \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) \end{aligned} \quad (38)$$

and

$$\varphi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) \|\dot{\mathbf{u}}\|_{L^\Phi} \leq C(\Phi_0) \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}), \quad (39)$$

with  $C(\Phi_0)$  the constant that comes from the  $\Delta_2$  condition on  $\Phi_0$ .

From (48), (49), (39) and (46), we have

$$I_1 + I_2 \leq C_3 \left\{ \Phi_0(|\bar{u}|) + \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) + \|\dot{\mathbf{u}}\|_{L^\Phi} \right\} \quad (40)$$

with  $C_3 = C_3(\Phi_0, T, \|b_1\|_{L^1}, \|b_2\|_{L^1})$ .

We recall that as  $\Phi_0$  is an  $N$ -function, there exists  $C > 0$  such that  $\Phi_0(x) \geq Cx$  for every  $x \geq 1$ ; and for  $0 \leq x \leq 1$ , we also have  $x \leq 1 + C\Phi_0(x)$ . Thus,

$$x \leq C\Phi_0(x) + 1 \text{ for every } x \geq 0. \quad (41)$$

$C = \Phi_0(1)$ ????, en las notas pero no lo veo claro

In the subsequent estimates, we use the decomposition  $u = \bar{u} + \tilde{u}$ , (3), (45), (50), (41), the fact that  $\Phi_0 \in \Delta_2$  and we get

$$\begin{aligned} I(\mathbf{u}) &\geq \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) + \int_0^T F(t, \mathbf{u}) dt \\ &= \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) + \int_0^T [F(t, \mathbf{u}) - F(t, \bar{\mathbf{u}})] dt + \int_0^T F(t, \bar{\mathbf{u}}) dt \\ &\geq \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) - C_3 \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) + \int_0^T F(t, \bar{\mathbf{u}}) dt - C_3 \Phi_0(|\bar{u}|) - C_3 \|\dot{\mathbf{u}}\|_{L^\Phi} \\ &\geq \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) - C_4 \Phi_0(\|\dot{\mathbf{u}}\|_{L^\Phi}) + H_{C_3, \Phi_0}(\bar{\mathbf{u}}) - C_3 \\ &\geq \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) - C_5 \Phi_0 \left( \frac{\|\dot{\mathbf{u}}\|_{L^\Phi}}{\Lambda} \right) + H_{C_3, \Phi_0}(\bar{\mathbf{u}}) - C_3 \\ &= \alpha_0 J_{C_6, \Phi_0} \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) + H_{C_3, \Phi_0}(\bar{\mathbf{u}}) - C_3 \end{aligned} \quad (42)$$

where  $C_4 = (C + 1)C_3$ ,  $C_5 = C_5(\Phi_0, \Lambda, C_4)$  and  $C_6 = \frac{C_5}{\alpha_0}$ .

**De acá en adelante, es copia del final de la otra demo. Debe adaptarse a la fórmula anterior, si es que ella logra sobrevivir....**

Let  $\mathbf{u}_n$  be a sequence in  $\mathcal{E}_d^\Phi(\lambda)$  with  $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$  and we have to prove that  $I(\mathbf{u}_n) \rightarrow \infty$ . On the contrary, suppose that for a subsequence, still denoted by

$\mathbf{u}_n$ ,  $I(\mathbf{u}_n)$  is upper bounded, i.e., there exists  $M > 0$  such that  $|I(\mathbf{u}_n)| \leq M$ . As  $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$ , from Lemma 2.2, we have  $|\bar{\mathbf{u}}_n| + \|\dot{\mathbf{u}}_n\|_{L^\Phi} \rightarrow \infty$ . Then, there exists a subsequence of  $\{\mathbf{u}_n\}$ , still denoted by  $\mathbf{u}_n$ , which is not bounded. Then,  $|\bar{\mathbf{u}}_n| \rightarrow \infty$  or  $\|\dot{\mathbf{u}}_n\|_{L^\Phi} \rightarrow \infty$ . Now, Lemma ?? implies that the functional  $J_{C_4, \mu'}(\frac{\dot{\mathbf{u}}}{\Lambda})$  is coercive, and, by (44), the functional  $H(\bar{\mathbf{u}})$  is also coercive, then  $J_{C_4, \mu'}(\frac{\dot{\mathbf{u}}_n}{\Lambda}) \rightarrow \infty$  or  $H(\bar{\mathbf{u}}_n) \rightarrow \infty$ . From (31), we have that on a bounded set the functional  $H(\bar{\mathbf{u}}_n)$  is lower bounded; and,  $J_{C_4, \mu'}(\frac{\dot{\mathbf{u}}_n}{\Lambda})$  is also lower bounded because the modular  $\rho_\Phi(\frac{\dot{\mathbf{u}}}{\Lambda})$  is always bigger than zero. Therefore,  $I(\mathbf{u}_n) \rightarrow \infty$  as  $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$  which contradicts the initial assumption on the behavior of  $I(\mathbf{u}_n)$ .  $\square$

### 3.1 versión potencias por si acaso...

**Theorem 3.2.** *Let  $\mathcal{L}$  be a lagrangian function satisfying (2), (3), (4), (3) and  $F$  satisfies condition (A). We assume the following conditions:*

1.  $\Psi \in \Delta_2$ .
2. *There exist non negative functions  $b_1, b_2 \in L^1_1$  and a constant  $1 < \mu < \alpha_\Phi$  such that for any  $\mathbf{x} \in \mathbb{R}^d$  and a.e.  $t \in [0, T]$*

$$|\nabla F(t, \mathbf{x})| \leq b_1(t)|\mathbf{x}|^{\mu-1} + b_2(t). \quad (43)$$

3. *There exists a real positive number  $\sigma$  such that  $\sigma > (\mu - 1)\beta_\Psi$  and*

$$|\mathbf{x}|^\sigma = o\left(\int_0^T F(t, \mathbf{x}) dt\right) \text{ as } |\mathbf{x}| \rightarrow \infty. \quad (44)$$

*Then the action integral  $I$  is coercive.*

*Proof.* By the decomposition  $u = \bar{u} + \tilde{u}$ , Mean Value Theorem, Cauchy-Schwarz's inequality and (43), we have

$$\begin{aligned} \left| \int_0^T F(t, \mathbf{u}) - F(t, \bar{\mathbf{u}}) dt \right| &= \left| \int_0^T \int_0^1 \nabla F(t, \bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)) \cdot \tilde{\mathbf{u}}(t) ds dt \right| \\ &\leq \int_0^T \int_0^1 b_1(t)|\bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)|^{\mu-1} |\tilde{\mathbf{u}}(t)| ds dt + \int_0^T \int_0^1 b_2(t)|\tilde{\mathbf{u}}(t)| ds dt \\ &= I_1 + I_2. \end{aligned} \quad (45)$$

On the one hand, by Hölder's inequality and Sobolev's inequality, we estimate  $I_2$  as follows

$$I_2 \leq \|b_2\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty} \leq C_1 \|\dot{\mathbf{u}}\|_{L^\Phi}. \quad (46)$$

where  $C_1 = C_1(\|b_2\|_{L^1}, T)$ . On the other hand, as  $s \in [0, 1]$ , we have

$$|\bar{\mathbf{u}} + s\tilde{\mathbf{u}}(t)|^{\mu-1} \leq C(\mu)(|\bar{\mathbf{u}}|^{\mu-1} + \|\tilde{\mathbf{u}}\|_{L^\infty}^{\mu-1}). \quad (47)$$

where  $C(\mu) = 2^{\mu-2}$ , for  $\mu \geq 2$  and  $C(\mu) = 1$ , for  $1 < \mu < 2$ . Now, inequality (47), Hölder's inequality and Sobolev's inequality imply that

$$\begin{aligned} I_1 &\leq C(\mu) \left( |\bar{\mathbf{u}}|^{\mu-1} \int_0^T b_1(t) |\tilde{\mathbf{u}}(t)| dt + \|\tilde{\mathbf{u}}\|_{L^\infty}^{\mu-1} \int_0^T b_1(t) |\tilde{\mathbf{u}}(t)| dt \right) \\ &\leq C(\mu) \left\{ |\bar{\mathbf{u}}|^{\mu-1} \|b_1\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty} + \|b_1\|_{L^1} \|\tilde{\mathbf{u}}\|_{L^\infty}^\mu \right\} \\ &\leq C_2 \left\{ |\bar{\mathbf{u}}|^{\mu-1} \|\dot{\mathbf{u}}\|_{L^\Phi} + \|\dot{\mathbf{u}}\|_{L^\Phi}^\mu \right\}, \end{aligned} \quad (48)$$

where  $C_2 = C_2(\mu, T, \|b_1\|_{L^1})$ . Let  $\mu'$  be a positive constant such that  $1 < \mu \leq \mu' < \alpha_\Phi$ . Next, using Young's inequality with conjugate exponents  $\mu'$  and  $\frac{\mu'}{\mu'-1}$  we get

$$|\bar{\mathbf{u}}|^{\mu-1} \|\dot{\mathbf{u}}\|_{L^\Phi} \leq \frac{(\mu'-1)}{\mu'} |\bar{\mathbf{u}}|^\sigma + \frac{1}{\mu'} \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} \quad (49)$$

where  $\sigma = \frac{(\mu-1)\mu'}{\mu'-1}$ . We point out that  $\sigma$  is an arbitrary positive constant bigger than  $(\mu-1)b_\Psi$ .

From (48), (49), (46) and the inequality  $x^{r_1} \leq x^{r_2} + 1$ , for any  $x \geq 0$  and  $r_1 \leq r_2$ , we have

$$\begin{aligned} I_1 + I_2 &\leq C_3 \left\{ |\bar{\mathbf{u}}|^\sigma + \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + \|\dot{\mathbf{u}}\|_{L^\Phi}^\mu + \|\dot{\mathbf{u}}\|_{L^\Phi} \right\} \\ &\leq C_3 \left\{ |\bar{\mathbf{u}}|^\sigma + \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + 1 \right\} \end{aligned} \quad (50)$$

with  $C_3 = C_3(\mu, T, \|b_1\|_{L^1}, \mu')$ . In the subsequent estimates, we use the decomposition  $u = \bar{u} + \tilde{u}$ , (3), (45), (50) and we get

$$\begin{aligned} I(\mathbf{u}) &\geq \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) + \int_0^T F(t, \mathbf{u}) dt \\ &= \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) + \int_0^T [F(t, \mathbf{u}) - F(t, \bar{\mathbf{u}})] dt + \int_0^T F(t, \bar{\mathbf{u}}) dt \\ &\geq \alpha_0 \rho_\Phi \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) - C_3 \|\dot{\mathbf{u}}\|_{L^\Phi}^{\mu'} + \int_0^T F(t, \bar{\mathbf{u}}) dt - C_3 |\bar{\mathbf{u}}|^\sigma - C_3 \\ &= \alpha_0 J_{C_4, \mu'} \left( \frac{\dot{\mathbf{u}}}{\Lambda} \right) + H_{C_3, \sigma}(\bar{\mathbf{u}}) - C_3, \end{aligned} \quad (51)$$

where  $C_4 = \Lambda^{\mu'} C_3 / \alpha_0$ .

Let  $\mathbf{u}_n$  be a sequence in  $\mathcal{E}_d^\Phi(\lambda)$  with  $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$  and we have to prove that  $I(\mathbf{u}_n) \rightarrow \infty$ . On the contrary, suppose that for a subsequence, still denoted by  $\mathbf{u}_n$ ,  $I(\mathbf{u}_n)$  is upper bounded, i.e., there exists  $M > 0$  such that  $|I(\mathbf{u}_n)| \leq M$ . As  $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$ , from Lemma 2.2, we have  $|\bar{\mathbf{u}}_n| + \|\dot{\mathbf{u}}_n\|_{L^\Phi} \rightarrow \infty$ . Then, there exists a subsequence of  $\{\mathbf{u}_n\}$ , still denoted by  $\mathbf{u}_n$ , which is not bounded. Then,  $|\bar{\mathbf{u}}_n| \rightarrow \infty$  or

$\|\dot{\mathbf{u}}_n\|_{L^\Phi} \rightarrow \infty$ . Now, Lemma ?? implies that the functional  $J_{C_4, \mu'}(\frac{\dot{\mathbf{u}}}{\Lambda})$  is coercive, and, by (44), the functional  $H(\bar{\mathbf{u}})$  is also coercive, then  $J_{C_4, \mu'}(\frac{\dot{\mathbf{u}}_n}{\Lambda}) \rightarrow \infty$  or  $H(\bar{\mathbf{u}}_n) \rightarrow \infty$ . From (31), we have that on a bounded set the functional  $H(\bar{\mathbf{u}}_n)$  is lower bounded; and,  $J_{C_4, \mu'}(\frac{\dot{\mathbf{u}}_n}{\Lambda})$  is also lower bounded because the modular  $\rho_\Phi(\frac{\dot{\mathbf{u}}}{\Lambda})$  is always bigger than zero. Therefore,  $I(\mathbf{u}_n) \rightarrow \infty$  as  $\|\mathbf{u}_n\|_{W^1 L^\Phi} \rightarrow \infty$  which contradicts the initial assumption on the behavior of  $I(\mathbf{u}_n)$ .  $\square$

**Leer y ver si es coherente lo anterior, si conviene trabajar siempre con  $u_n$  o habría que usar la notación de subsucesiones explícita!!!**

**Falta leer y corregir la sección que sigue del caso límite!!!**

## 4 Limit case $\mu = \alpha_\Phi$

Coercivity was obtained even in the limit case  $\mu = 1$  and  $\mu = p$  (see [20, 2]) assuming additional conditions on one of the  $L^1$  functions that bound the nonlinearity. These results lean on the fact that in  $L^p$  spaces the norm and the modular coincides, that is,  $\|\cdot\|_p^p = O(\int_0^T |\cdot|^p dt)$ . In Orlicz spaces,  $\|\cdot\|_{L^\Phi}^\mu$  can be upper controlled by a modular provided that  $\mu < \alpha_\Phi$  for any  $N$ -function  $\Phi$ . Nevertheless, the limit case does not hold for any  $\Phi$ , i.e. in general  $\|\cdot\|_{L^\Phi}^{\alpha_\Phi} = O(\int_0^T \Phi(|u|) dt)$  is false as can be seen as follows.

Let  $\Phi$  be an  $N$ -function that satisfies the  $\Delta_2$ -condition. We claim that the inequality  $\Phi(tu) \geq t^{\alpha_\Phi} \Phi(u)$  for any  $u > 0$  and for any  $t \geq 1$  is false.

With the aim of proving the above assertion, we define the function

$$\Phi(u) = \begin{cases} \frac{p-1}{p} u^p & u \leq e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with  $p > 1$ .

**Me parece que habría que decir algo que aclare los que se hace a continuación.**

**Por ejemplo:**

**Next, we will study the behaviour of  $\Phi$ / establish some properties of  $\Phi$ /...**

**Theorem 4.1.** *If  $p \geq \frac{1+\sqrt{2}}{2}$ , then  $\Phi$  is an  $N$ -function.*

*Proof.* We have

$$\Phi'(u) = \begin{cases} (p-1)u^{p-1} & := \varphi_1(u) \quad \text{if } u < e \\ \frac{u^{p-1}}{\log u} (p - \frac{1}{\log u}) & := \varphi_2(u) \quad \text{if } u > e \end{cases}$$

and  $\Phi$  is differentiable at  $e$  because  $\varphi_1(e) = \varphi_2(e) = (p-1)e^{p-1}$ .

We will see that  $\Phi'$  is increasing when  $p \geq \frac{1+\sqrt{2}}{2}$ . Then, it is enough to see that  $\varphi_1$  is increasing on  $[0, e]$  and  $\varphi_2$  is increasing on  $[e, \infty)$  for  $p \geq \frac{1+\sqrt{2}}{2}$ .

**Tendríamos que ver que  $\varphi_1, \varphi_2$  son crecientes y que  $\varphi_2 \rightarrow \infty$  cuando  $u \rightarrow \infty$ ??? o basta con ver que  $\varphi_2$  es creciente???**

$\varphi_1$  is an increasing function for  $p > 1$  and  $\varphi_1(u) \rightarrow 0$  as  $u \rightarrow 0$ .

On the other hand,  $\varphi_2(u) \rightarrow \infty$  as  $u \rightarrow \infty$  provided that  $p > 1$ . And,  $\varphi'_2(u) > 0$  on  $[e, \infty)$  if and only if

$$\left(p^2 - p - \frac{2p}{\log u} + \frac{1}{\log u} + \frac{2}{\log^2 u}\right) > 0. \quad (52)$$

Now, as (52) holds if and only if  $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$ , then  $\varphi_2$  is an increasing function when  $p \geq \frac{1+\sqrt{2}}{2}$ .  $\square$

**Conviene agregar más cuentas en la prueba anterior, es decir, que la fórmula (52) es una cuadrática y bla....???**

**Theorem 4.2.** *There exists a constant  $C > 0$  such that*

$$\Phi(tu) \leq ct^p \Phi(u) \quad t \geq 1, u > 0. \quad (53)$$

*For every  $\varepsilon > 0$  there exists a constant  $C = C(\varepsilon, p)$  such that*

$$\Phi(tu) \geq Ct^{p-\varepsilon} \Phi(u) \quad t \geq 1, u > 0. \quad (54)$$

*Proof.* In order to prove (53), we analyze three cases.

If  $u \leq tu \leq e$ , then  $\Phi(tu) = t^p \Phi(u)$  and (53) holds with  $C = 1$ .

If  $u \leq e \leq tu$ , as  $\frac{e^p}{p} > 0$  and  $\log(tu) \geq 1$ , we have  $\Phi(tu) \leq t^p u^p = \frac{p}{p-1} t^p \Phi(u)$ . Thus, (53) holds with  $C = \frac{p}{p-1}$ .

If  $e \leq u \leq tu$ , then

$$\Phi(tu) \leq \frac{t^p u^p}{\log(tu)} \leq \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v} \quad (55)$$

where  $v := u^p$  and  $v \geq e^p$ .

If  $\alpha > 0$ , the function  $f(x) = \frac{x}{x-\alpha}$  is decreasing on  $(\alpha, \infty)$ . And, the function  $g(v) = \frac{pv}{\log v}$  is decreasing on  $[e^p, \infty)$ . Therefore,  $f \circ g$  is decreasing on  $[e^p, \infty)$  and we have

$$(f \circ g)(v) = \frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leq e^p - \frac{e^p}{p} = \frac{p}{p-1}$$

for every  $v \geq e^p$ .

In this way, from (55), we have

$$\Phi(tu) \leq \frac{pt^p}{p-1} \left( \frac{pv}{\log v} - \frac{e^p}{p} \right) = \frac{pt^p}{p-1} \left( \frac{u^p}{\log u} - \frac{e^p}{p} \right)$$

and (53) holds with  $C = \frac{p}{p-1}$ .

Now, we will prove (54).

If  $u \leq tu \leq e$ , (54) is immediate because  $t^p \geq t^{p-\varepsilon}$  for every  $t \geq 1$ ,  $p > 1$  and  $\varepsilon$  sufficiently small????

If  $u \leq e \leq tu$ , as  $f(t) = \frac{t}{\log t}$  is increasing on  $[e, \infty)$  then  $f(t) \geq e$  for every  $t \geq e$ .

**Habría que mirar en  $[e^p, \infty)$  para que  $f(t) \geq \frac{e^p}{p}$ ???? O, como  $f$  es creciente y**

$e^p \leq (tu)^p$  entonces  $f((tu)^p) \geq f(e^p)$ ???

Now,

$$\begin{aligned}\Phi(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} = \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \geq \\ &\quad \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} \geq \frac{p-1}{p} \frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p.\end{aligned}$$

Since  $f(t) = \frac{t^\varepsilon}{\log t + 1}$  attains its minimum value  $\varepsilon e^{1-\varepsilon}$  at  $e^{\frac{1-\varepsilon}{\varepsilon}}$ , then

$$\Phi(tu) \geq \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p.$$

If  $e \leq u \leq tu$ , then

$$\Phi(tu) = \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} + \frac{1}{p} \frac{(tu)^p}{\log(tu)} - \frac{e^p}{p} \geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} = \frac{p-1}{p} \frac{t^\varepsilon \log(u)^\varepsilon}{\log(t^\varepsilon u^\varepsilon)} \frac{t^{p-\varepsilon} u^p}{\log u}$$

Let  $f(s) = \frac{sA}{\log s + 1}$  with  $s \geq 1$  and  $A \geq \varepsilon$ . Then, the function  $f$  attains a minimum at  $s = e^{1-A}$ ; but,  $s$  has to be bigger than 1, then it is necessary that  $\varepsilon \leq A \leq 1$ . And, the minimum value is  $f(e^{1-A}) = Ae^{1-A} \geq \varepsilon$ . If  $A \geq 1$ ,  $f$  attains the minimum at  $s = 1$ ?? and  $f(1) = 1$ . Then,  $f \geq \varepsilon$  and therefore

$$\Phi(tu) \geq \frac{p-1}{p} \varepsilon \frac{t^{p-\varepsilon} u^p}{\log u} \geq \frac{p-1}{p} \varepsilon t^{p-\varepsilon} \Phi(u).$$

□

**La prueba anterior está rara.**

*Remark 2.* The inequality

$$\Phi(tu) \geq Ct^p \Phi(u)$$

is false for every  $C$  because for every  $u \geq e$  we have

$$\lim_{t \rightarrow \infty} \frac{\Phi(tu)}{t^p \Phi(u)} = \lim_{t \rightarrow \infty} \frac{\frac{u^p}{\log(tu)} - \frac{e^p}{pt^p}}{\frac{u^p}{\log u} - \frac{e^p}{p}} = 0$$

**El límite intermedio de la fórmula anterior se podría quitar.**

**Theorem 4.3.**  $\alpha_\Phi = \beta_\Phi = p$

*Proof.* From (22) and (53), we get

$$\beta_\Phi = \lim_{t \rightarrow \infty} \frac{\log \left[ \sup_{u > 0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} \leq \lim_{t \rightarrow \infty} \frac{\log C + p \log t}{\log t} = p.$$

On the other hand, employing (22) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t \rightarrow 0^+} \frac{\log \left[ \sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} = \lim_{s \rightarrow \infty} \frac{\log \left[ \sup_{v>0} \frac{\Phi(v)}{\Phi(sv)} \right]^{-1}}{\log s} = \lim_{s \rightarrow \infty} \frac{\log \left[ \inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s}$$

where  $v := tu$  and  $s := \frac{1}{t}$ . Then, using (54), for every  $\varepsilon > 0$  we have

$$\alpha_\Phi = \lim_{s \rightarrow \infty} \frac{\log \left[ \inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s} \geq \lim_{s \rightarrow \infty} \frac{\log C + (p - \varepsilon) \log s}{\log s} \geq p - \varepsilon,$$

therefore  $\alpha_\Phi \geq p$ .

Finally, as  $\alpha_\Phi \leq \beta_\Phi \leq p$ , we get  $\alpha_\Phi = \beta_\Phi = p$ . □

Now, we are able to see that

$$\rho_\Phi(u) = \int_0^T \Phi(|u|) dx \geq C \|u\|_{L^\Phi}^{\alpha_\Phi} = C \|u\|_{L^\Phi}^p$$

is false.

In fact, if we take  $u \equiv t > 0$ , then  $\|u\|_{L^\Phi}^p = C_1 t^p$  where  $C_1 = \|1\|_{L^\Phi}$  and  $\int_0^T \Phi(|u|) dx = C_2 \Phi(t)$  with  $C_2 = T$ . Then, if  $\rho_\Phi(u) \geq C \|u\|_{L^\Phi}^p$  were true, then  $\Phi(t) \geq C t^p$  would also be true; however, this last inequality is false.

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