Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting

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Introduction

This work is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,\mathbf{u}(t),\dot{\mathbf{u}}(t)) = D_\mathbf{x}\mathcal{L}(t,\mathbf{u}(t),\dot{\mathbf{u}}(t)) & \text{a.e. } t\in(0,T) \\ \mathbf{u}(0)-\mathbf{u}(T)=\dot{\mathbf{u}}(0)-\dot{\mathbf{u}}(T)=0 \end{cases} \tag{1}$$

where $T>0$, $\mathbf{u}:[0,T]\to\mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t,\mathbf{x},\mathbf{y})| \leqslant a(|\mathbf{x}|)\left(b(t)+\Phi\left(\frac{|\mathbf{y}|}{\lambda}+f(t)\right)\right) \tag{2}$$

$$|D_\mathbf{x}\mathcal{L}(t,\mathbf{x},\mathbf{y})| \leqslant a(|\mathbf{x}|)\left(b(t)+\Phi\left(\frac{|\mathbf{y}|}{\lambda}+f(t)\right)\right), \tag{3}$$

$$|D_\mathbf{y}\mathcal{L}(t,\mathbf{x},\mathbf{y})| \leqslant a(|\mathbf{x}|)\left(c(t)+\varphi\left(\frac{|\mathbf{y}|}{\lambda}+f(t)\right)\right). \tag{4}$$

In these inequalities we assume that $\lambda>0$ and (see below for definitions)

- $a\in C(\mathbb{R}^+,\mathbb{R}^+)$.
- Φ is an N-function.
- $b\in L^1([0,T])$ and $c\in L^\Psi_1$, where Ψ is the complementary N-function of Φ.
- $f\in E^\Phi_1$.

Definitions

- Φ is an N-function if
$$\Phi(t)=\int_0^t\varphi(\tau)\,d\tau,\quad \text{for } t\geq 0,$$
where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a right continuous, non decreasing function satisfying $\varphi(0)=0$, $\varphi(t)>0$ for $t>0$ and $\lim_{t\to\infty}\varphi(t)=+\infty$. We denote by L^Φ_d the Orlicz space associated to the N-function Φ of functions defined on $[0,T]$ taking values in $\mathbb{R}^d$.
- Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s)=\sup_{\varphi(t)\leqslant s}t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi'=\psi$. The function Ψ is called the *complementary function* of Φ.
- For $\mathbf{u}:[0,T]\to\mathbb{R}^d$ measurable we define the *modular function*
$$\rho_\Phi(\mathbf{u}):=\int_0^T\Phi(|\mathbf{u}|)\,dt.$$
- The *Orlicz class* $C^\Phi_d=C^\Phi_d([0,T])$ is given by
$$C^\Phi_d:=\{\mathbf{u}|\rho_\Phi(\mathbf{u})<\infty\}. \tag{5}$$
- The *Orlicz space* $L^\Phi_d=L^\Phi_d([0,T])$ is the linear hull of C^Φ_d; equivalently,
$$L^\Phi_d:=\{\mathbf{u}|\exists\lambda>0:\rho_\Phi(\lambda\mathbf{u})<\infty\}. \tag{6}$$
- The Orlicz space L^Φ_d equipped with the *Orlicz norm*
$$\|\mathbf{u}\|_{L^\Phi}:=\sup\left\{\int_0^T\mathbf{u}\cdot\mathbf{v}\,dt\Big|\rho_\Psi(\mathbf{v})\leqslant 1\right\},$$
is a Banach space.
- The subspace $E^\Phi_d=E^\Phi_d([0,T])$ is defined as the closure in L^Φ_d of the subspace L^∞_d of all $\mathbb{R}^d$-valued essentially bounded functions.
- We define the *Sobolev-Orlicz space* $W^1L^\Phi_d$ by
$$W^1L^\Phi_d:=\{\mathbf{u}|\mathbf{u}\text{ is absolutely continuous and }\mathbf{u},\dot{\mathbf{u}}\in L^\Phi_d\}.$$
$W^1L^\Phi_d$ is a Banach space when it is equipped with the norm
$$\|\mathbf{u}\|_{W^1L^\Phi}=\|\mathbf{u}\|_{L^\Phi}+\|\dot{\mathbf{u}}\|_{L^\Phi}.$$

Methodology

Our approach will be through the direct method of the calculus of variations, i.e. we will find solutions of (1) exhibiting extreme points of the *action integral*

$$I(\mathbf{u})=\int_0^T\mathcal{L}(t,\mathbf{u}(t),\dot{\mathbf{u}}(t))\,dt. \tag{7}$$

Differentiability of action integrals in Orlicz spaces

As a first step we need to show that the action integral is well defined and differentiable on a subset of a Banach space. For this purpose we introduce the sets

$$\Pi(E^\Phi_d,r):=\{\mathbf{u}\in L^\Phi_d|d(\mathbf{u},E^\Phi_d)<r\}$$

and

$$\mathcal{E}^\Phi_d(\lambda):=W^1L^\Phi_d\cap\{\mathbf{u}|\dot{\mathbf{u}}\in\Pi(E^\Phi_d,\lambda)\}.$$

Theorem

Let $\mathcal{L}$ be a Carathéodory function satisfying (2), (3) *and* (4). *Then the following statements hold:*

- *The action integral given by* (7) *is finitely defined on $\mathcal{E}^\Phi_d(\lambda)$.*
- *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi_d(\lambda)$ and its derivative I' is continuous from $\mathcal{E}^\Phi_d(\lambda)$ into $\left[W^1L^\Phi_d\right]^*$. Here we consider $\mathcal{E}^\Phi_d(\lambda)$ equipped with the strong topology and $\left[W^1L^\Phi_d\right]^*$ with the weak* topology. Moreover, I' is given by the following expression*
$$\langle I'(\mathbf{u}),\mathbf{v}\rangle=\int_0^T\{D_\mathbf{x}\mathcal{L}(t,\mathbf{u},\dot{\mathbf{u}})\cdot\mathbf{v}+D_\mathbf{y}\mathcal{L}(t,\mathbf{u},\dot{\mathbf{u}})\cdot\dot{\mathbf{v}}\}\,dt. \tag{8}$$
- *If $\Psi\in\Delta_2$ then I' is continuous from $\mathcal{E}^\Phi_d(\lambda)$ into $\left[W^1L^\Phi_d\right]^*$ when both spaces are equipped with the strong topology.*

Critical points

Now we are in condition to show that solutions of (1) are critical points of I on $W^1L^\Phi_T$, where $W^1L^\Phi_T$ denotes the subspace of W^1L^Φ consisting of T-periodic functions.

Theorem

Let $\mathbf{u}\in\mathcal{E}^\Phi_d(\lambda)$ be a T-periodic function. The following statements are equivalent:

- $I'(\mathbf{u})\in\left(W^1L^\Phi_T\right)^\perp$.
- *$D_\mathbf{y}\mathcal{L}(t,\mathbf{u}(t),\dot{\mathbf{u}}(t))$ is an absolutely continuous function and $\mathbf{u}$ solves the following boundary value problem*
$$\begin{cases}\frac{d}{dt}D_y\mathcal{L}(t,\mathbf{u}(t),\dot{\mathbf{u}}(t))=D_\mathbf{x}\mathcal{L}(t,\mathbf{u}(t),\dot{\mathbf{u}}(t)) & \text{a.e. } t\in(0,T)\\ \mathbf{u}(0)-\mathbf{u}(T)=D_y\mathcal{L}(0,\mathbf{u}(0),\dot{\mathbf{u}}(0))-D_y\mathcal{L}(T,\mathbf{u}(T),\dot{\mathbf{u}}(T))=0.\end{cases} \tag{9}$$

Moreover if $D_\mathbf{y}\mathcal{L}(t,x,y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\mathbf{y}$, then $D_y\mathcal{L}(0,\mathbf{u}(0),\dot{\mathbf{u}}(0))-D_y\mathcal{L}(T,\mathbf{u}(T),\dot{\mathbf{u}}(T))=0$ is equivalent to $\dot{\mathbf{u}}(0)=\dot{\mathbf{u}}(T)$.

Coercivity

As usual, in order to establish coercivity of our action integral we need to suppose an adequate bound from below of the lagrangian function. In this respect we assume that

$$\mathcal{L}(t,\mathbf{x},\mathbf{y})\geq\alpha_0\Phi\left(\frac{|\mathbf{y}|}{\Lambda}\right)+F(t,\mathbf{x}), \tag{10}$$

where $\alpha_0,\Lambda>0$ and $F:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is a Carathéodory function, i.e. $F(t,\mathbf{x})$ is measurable with respect to t for every fixed $\mathbf{x}\in\mathbb{R}^d$ and F is continuous at $\mathbf{x}$ for a.e. $t\in[0,T]$. We need to assume

$$|F(t,\mathbf{x})|\leqslant a(|\mathbf{x}|)b_0(t),\quad\text{for a.e. } t\in[0,T]\quad\text{and for every }\mathbf{x}\in\mathbb{R}^d, \tag{11}$$

where $b_0\in L^1_1$.
We say that F satisfies the condition (A) if $F(t,\mathbf{x})$ is a Carathéodory function, F verifies (11) and F is continuously differentiable with respect to $\mathbf{x}$. Moreover, the next inequality holds

$$|D_\mathbf{x}F(t,\mathbf{x})|\leqslant a(|\mathbf{x}|)b_0(t),\quad\text{for a.e. } t\in[0,T]\text{ and for every }\mathbf{x}\in\mathbb{R}^d. \tag{12}$$

We can show that the coercivity of the action integral I is obtained if the functional

$$J_{C,\nu}(\mathbf{u}):=\rho_\Phi\left(\frac{\mathbf{u}}{\Lambda}\right)-C\|\mathbf{u}\|^\nu_{L^\Phi}, \tag{13}$$

is coercive for $C,\nu>0$.
If $\Phi(x)=|x|^p/p$ then $J_{C,\nu}$ is clearly coercive for $\nu<p$. For more general Φ the situation is more interesting.

Lemma

Let Φ and Ψ be complementary N-functions. Then:

- *If $C\Lambda<1$, then $J_{C,1}$ is coercive.*
- *If $\Psi\in\Delta_2$ globally, then there exists a constant $\alpha_\Phi>1$ such that, for any $0<\mu<\alpha_\Phi$,*
$$\lim_{\|\mathbf{u}\|_{L^\Phi}\to\infty}\frac{\rho_\Phi\left(\frac{\mathbf{u}}{\Lambda}\right)}{\|\mathbf{u}\|^\mu_{L^\Phi}}=+\infty. \tag{14}$$
In particular, the functional $J_{C,\mu}$ is coercive for every $C>0$ and $0<\mu<a_\Phi$. The constant α_Φ is one of the so-called Matuszewska-Orlicz indices.
- *If $J_{C,1}$ is coercive with $C\Lambda>1$, then $\Psi\in\Delta_2$.*

Theorem (Coercivity I)

Let $\mathcal{L}$ be a lagrangian function satisfying (2), (3), (4), (10) *and* (11). *We assume the following conditions:*

- *There exist a non negative function $b_1\in L^1_1$ and a constant $\mu>0$ such that for any $\mathbf{x_1},\mathbf{x_2}\in\mathbb{R}^d$ and a.e. $t\in[0,T]$*
$$|F(t,\mathbf{x_2})-F(t,\mathbf{x_1})|\leqslant b_1(t)(1+|\mathbf{x_2}-\mathbf{x_1}|^\mu). \tag{15}$$
We suppose that $\mu<\alpha_\Phi$, with α_Φ as in previous lemma, in the case that $\Psi\in\Delta_2$; and, we suppose $\mu=1$ if Ψ is an arbitrary N-function.
- $$\int_0^T F(t,\mathbf{x})\,dt\to\infty\quad\text{as}\quad|\mathbf{x}|\to\infty. \tag{16}$$
- *$\Psi\in\Delta_2$ or, alternatively, $\alpha_0^{-1}T\Phi^{-1}(1/T)\|b_1\|_{L^1}\Lambda<1$.*

Then the action integral I is coercive.

We can formulate an alternative coercivity result. We need to mention the following technical fact.

Lemma (Mawhin-Willem)

Suppose that F satisfies condition (A) and (16), *$F(t,\cdot)$ is differentiable and convex a.e. $t\in[0,T]$. Then, there exists $\mathbf{x}_0\in\mathbb{R}^d$ such that*

$$\int_0^T D_\mathbf{x}F(t,\mathbf{x}_0)\,dt=0. \tag{17}$$

Theorem (Coercivity II)

Let $\mathcal{L}$ be as in Theorem (Coercivity I) and let F be as in previous lemma. Moreover, assume that $\Psi\in\Delta_2$ or, alternatively $\alpha_0^{-1}T\Phi^{-1}(1/T)\,a(|\mathbf{x}_0|)\|b_0\|_{L^1}\Lambda<1$, with a and b_0 as in (11) *and $\mathbf{x}_0\in\mathbb{R}^d$ any point satisfying* (17). *Then I is coercive.*

Main results

Let Φ and Ψ be complementary N-functions. Suppose that the Carathéodory function $\mathcal{L}(t,\mathbf{x},\mathbf{y})$ is strictly convex at $\mathbf{y}$, $D_y\mathcal{L}$ is T-periodic with respect to T and (2), (3), (4), (10), (11) and (16) are satisfied. In addition, assume that some of the following statements hold (we recall the definitions and properties of α_0, b_1, $\mathbf{x}_0$ and b_0 from (10), (15), (17) and (12) respectively):

- $\Psi\in\Delta_2$ and (15).
- (15) and $\alpha_0^{-1}T\Phi^{-1}(1/T)\|b_1\|_{L^1}\Lambda<1$.
- $\Psi\in\Delta_2$, F satisfies condition (A) and $F(t,\cdot)$ is convex a.e. $t\in[0,T]$.
- As previous item but with $\alpha_0^{-1}T\Phi^{-1}(1/T)\,a(|\mathbf{x}_0|)\|b_0\|_{L^1}\Lambda<1$ instead of $\Psi\in\Delta_2$.

Then, problem (1) has a solution.