# Periodic solutions of Euler-Lagrange equations with “sublinear nonlinearity” in an Orlicz-Sobolev space setting

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**Abstract**

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space $W^1L^\Phi([0,T])$, of hamiltonian systems with a potential function $F$ satisfying the inequality $|\nabla F(t,x)| \leqslant b_1(t)\Phi_0'(|x|) + b_2(t)$, with $b_1, b_2 \in L^1$ and for certain $N$-functions $\Phi_0$.

## 1 Introduction

The purpose of this paper is to study the existence of periodic solution for the following non-autonomous second-order systems:

$$\begin{cases} \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right) = \nabla F(t,u(t)) \quad \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{1}$$

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where $T > 0$, $u : [0,T] \to \mathbb{R}^d$ is absolutely continuous and $\Phi$ is a differentiable $N$-function (see section Preliminaries for definitions). Furthermore, the *potential* $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

**(C)** $F$ and its gradient $\nabla F$ are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

**(A)** For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{2}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

We will call the differential operator

$$L_\Phi[u] = \frac{d}{dt}\left(u'(t)\frac{\Phi'(|u'|)}{|u'|}\right)$$

the $\Phi$-*laplacian operator*. If $\Phi(x) = |x|^p/p$, $1 < p < \infty$, $L_\Phi$ is the well known $p$-laplacian operator. In this case, we have the *Dirichlet problem* for the $p$-laplacian

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{3}$$

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \Phi(|u'(t)|) + F(t,u(t))\, dt. \tag{4}$$

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The main result of this article is Theorem 3.7 which establishes conditions to guarantee existence of solutions of the problem (1) by minimization of functional (41). We point out that the hypothesis of Theorem 3.7 are generalizations of those given in [1, 2, 3, 4] about the sublinearity.

# 2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [5, 6, 7]. For Orlicz spaces of vector valued functions, see [8] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an $N$-*function* if $\Phi$ is convex and satisfies that

$$\lim_{t\to+\infty}\frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0}\frac{\Phi(t)}{t} = 0$$

In addition, in this paper we assume that $\Phi$ is differentiable and we call $\varphi$ the derivative of $\Phi$. On these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism whose inverse is $\psi$. We denote by $\Psi$ the primitive of $\psi$ that satisfies $\Psi(0) = 0$. Then, $\Psi$ is an $N$-function which is called the *complementary function* of $\Phi$.

There exist several order relations between $N$-functions (see [7, Sec. 2.2]). Following [7, Def. 1, p. 15] we say that the $N$-function $\Phi_2$ is *essentially stronger* than the $N$-function $\Phi_1$ ($\Phi_1 \ll \Phi_2$) if and only if there exists $x_0 \geqslant 0$ such that $\Phi_1(x) \leqslant \Phi_2(ax)$, for every $a > 0$ and $x \geqslant x_0$.

We also say that a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the $\Delta_2$-*condition*, denoted by $\eta \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geqslant 0$ such that

$$\eta(2t) \leqslant K\eta(t), \tag{5}$$

for every $t \geqslant t_0$. If $t_0 = 0$, a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the $\Delta_2$-*condition globally* ($\eta \in \Delta_2$ globally).

Let $d$ be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$.

Given an $N$-function $\Phi$ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ is defined by

$$C_d^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{6}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of $C^\Phi$; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{7}$$

The Orlicz space $L^\Phi$ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup \left\{ \int_0^T u \cdot v\, dt \middle| \rho_\Psi(v) \leqslant 1 \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between $u$ and $v$. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [6, Thm. 10.5] and [9]). For every $u \in L^\Phi$,

$$\|u\|_{L^\Phi} = \inf_{k>0} \frac{1}{k} \{1 + \rho_\Phi(ku)\}. \tag{8}$$

In particular

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k} \{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \tag{9}$$

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in $L^\Phi$ of the subspace $L_d^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that $E^\Phi$ is the only one maximal subspace contained in the Orlicz class $C^\Phi$, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [6, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \tag{10}$$

If $X$ and $Y$ are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset \left[L^\Phi\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \tag{11}$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi = \left[L^\Phi\right]^*$ will not hold. In general, it is true that $\left[E^\Phi\right]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ (see [5]) by

$$W^1 L^\Phi := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi\}.$$

$W^1 L^\Phi$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{12}$$

Moreover, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E^\Phi_T &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{13}$$

For a function $u \in L^1_d([0, T])$, we write $u = \overline{u} + \widetilde{u}$ where $\overline{u} = \frac{1}{T} \int_0^T u(t) \, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \| \cdot \|_X)$ is a Banach space and $(Y, \| \cdot \|_Y)$ is a subspace of $X$, we write $Y \hookrightarrow X$ and we say that $Y$ is *embedded* in $X$ when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C \|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every $N$-function $\Phi$ we have that $L^\infty_d \hookrightarrow L^\Phi \hookrightarrow L^1_d$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if $w$ is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every $N$-function $\Phi$. We say that $u : [0, T] \to \mathbb{R}^d$ has modulus of continuity $w$ when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant C w(|t - s|). \tag{14}$$

We denote by $C^w([0, T], \mathbb{R}^d)$ the space of $w$-Hölder continuous functions. This is the space of all functions satisfying (14) for some $C > 0$ and it is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [10, 11, 12, 13, 14]. The next simple lemma, whose proof can be found in [15], will be used systematically.

**Lemma 2.1.** *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ and for every $u \in W^1L^\Phi$*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|) \qquad \textit{(Morrey's inequality),} \tag{15}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1,T\} \|u\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality).} \tag{16}$$

2. *For every $u \in W^1L^\Phi$ we have $\widetilde{u} \in L^\infty_d$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \qquad \textit{(Sobolev-Wirtinger's inequality).} \tag{17}$$

# 3 Lagrangians satisfying sublinear nonlinearity type conditions

**Lemma 3.1.** *$E^\Phi$ is weak* closed in $L^\Phi$.*

*Proof.* From [7, Thm. 7, p. 110] we have that $L^\Phi = \left[E^\Psi\right]^*$. Then, $L^\Phi$ is a dual and therefore we are allowed to speak about the weak* topology of $L^\Phi$. Besides, $E^\Phi$ is separable (see [7, Thm. 1, p. 87]). Let $S = E^\Phi \cap \{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$, then $S$ is closed in the norm $\|\cdot\|_{L^\Phi}$. Now, according to [7, Cor. 5, p. 148] $S$ is weak* sequentially compact. Thus, $S$ is weak* sequentially closed because if $u_n \in S$ and $u_n \xrightarrow{*} u \in L^\Phi$ then the weak* sequentially compactness implies the existence of $v \in S$ and a subsequence $u_{n_k}$ such that $u_{n_k} \xrightarrow{*} v$. Finally, by the uniqueness of the limit, we get $u = v \in S$. As $E^\Psi$ is separable and $L^\Phi = \left[E^\Psi\right]^*$, the ball of $L^\Phi$ $\{u \in L^\Phi | \|u\|_{L^\Phi} \leqslant 1\}$ is weak* metrizable (see [16, Thm. 5.1, p. 138]). Thus, $S$ is closed with respect to the weak* topology. Now, by Krein-Smulian theorem, [16, Cor. 12.6, p. 165] implies that $E^\Phi$ is weak* closed. □

The following result is analogous to some lemmata in $W^{1,p}$, see [17].

**Lemma 3.2.** *If $\|u\|_{W^1L^\Phi} \to \infty$, then $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$.*

*Proof.* By the decomposition $u = \overline{u} + \tilde{u}$ and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\overline{u} + \tilde{u}\|_{L^\Phi} \leqslant \|\overline{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\overline{u}|\|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \tag{18}$$

It is known that $L^\infty_d \hookrightarrow L^\Phi$, i.e. there exists $C_1 = C_1(T) > 0$ such that for any $\tilde{u} \in L^\infty_d$ we have

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_1 \|\tilde{u}\|_{L^\infty};$$

and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists $C_2 = C_2(T) > 0$ such that

$$\|\tilde{u}\|_{L^\Phi} \leqslant C_2 \|u'\|_{L^\Phi}. \tag{19}$$

Therefore, from (18), (19) and (12), we get

$$\|u\|_{W^1L^\Phi} \leqslant C_3(|\overline{u}| + \|u'\|_{L^\Phi})$$

where $C_3 = C_3(T)$. Finally, as $\|u\|_{W^1L^\Phi} \to \infty$ we conclude that $(|\overline{u}| + \|u'\|_{L^\Phi}) \to \infty$. □

**Lemma 3.3.** *Let $\Phi, \Psi$ be complementary functions. The next statements are equivalent:*

1. *$\Psi \in \Delta_2$ globally.*
2. *There exists an $N$-function $\Phi_1$ such that*

$$\Phi(rs) \geqslant \Phi_1(r)\Phi(s) \text{ for every } r \geqslant 1, \quad s \geqslant 0. \tag{20}$$

*Proof.* 1)⇒2) By virtue of the $\Delta_2$-condition on $\Psi$, [18, Thm. 11.7] and [18, Cor. 11.6] (see also [19, Eq. (2.8)]), we get constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geqslant K r^\nu \Phi(s), \tag{21}$$

for any $1 < \nu < \alpha_\Phi$, $s \geqslant 0$ and $r > 1$. This proves (20) with $\Phi_1(r) = kr^\nu$, which is an $N$-function.

2)⇒1) Next, we follow [7, p. 32, Prop. 13] and [7, p. 29, Prop. 9]. Assume that

$$\Phi_1(r)\Phi(s) \leqslant \Phi(rs) \;\; r > 1, \; s \geqslant 0.$$

Let $u = \Phi_1(r) \geqslant \Phi_1(1)$ and $v = \Phi(s) \geqslant 0$. By a well known inequality [7, p. 13, Prop. 1] and (20), we have for $u \geqslant \Phi_1(1)$ and $v > 0$

$$\frac{uv}{\Psi^{-1}(uv)} \leqslant \Phi^{-1}(uv) \leqslant \Phi_1^{-1}(u)\Phi^{-1}(v) \leqslant \frac{4uv}{\Psi_1^{-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi_1^{-1}(u)\Psi^{-1}(v) \leqslant 4\Psi^{-1}(uv).$$

If we take $x = \Psi_1^{-1}(u) \geqslant \Psi_1^{-1}(\Phi_1(1))$ and $y = \Psi^{-1}(v) \geqslant 0$, then

$$\Psi\left(\frac{xy}{4}\right) \leqslant \Psi_1(x)\Psi(y).$$

Now, taking $x \geqslant \max\{8, \Psi_1^{-1}(\Phi_1(1))\}$ we get that $\Psi \in \Delta_2$ globally. □

The following lemma generalizes [15, Lemma 5.2].

**Lemma 3.4.** *Let $\Phi, \Psi$ be complementary $N$-functions with $\Psi \in \Delta_2$ globally. Let $\Phi_1$ be any $N$-function satisfying* (20)*. Then*

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} = \infty, \tag{22}$$

*for every $\Phi_0$ with $\Phi_0 \lll \Phi_1$).*
*If* (22) *holds for some $N$-function $\Phi_0$, then $\Psi \in \Delta_2$ (at $\infty$).*

*Proof.* By the assumptions on $\Phi$ and $\Phi_1$ and inequality (9), for $r > 1$ we have

$$\frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \Phi_1(r)\frac{\int_0^T \Phi(r^{-1}|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \frac{\Phi_1(r)}{\Phi_0(\|u\|_{L^\Phi})}\{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose $r = \frac{\|u\|_{L^\Phi}}{2}$ and as $\|u\|_{L^\Phi} \to \infty$ we can assume $r > 1$ and by [7, Thm. 2 (b), p. 16].

$$\lim_{\|u\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi(|u|)\,dt}{\Phi_0(\|u\|_{L^\Phi})} \geqslant \lim_{\|u\|_{L^\Phi}\to\infty} \frac{\Phi_1\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(\|u\|_{L^\Phi})} = \infty.$$

Finally, if $\Phi_0$ is an $N$-function, then $\Phi_0(u) \geqslant k|u|$ for $k$ small enough and $|u| > 1$. Therefore (22) holds for $\Phi_0(u) = |u|$, then [15, Lemma 5.2] implies $\Psi \in \Delta_2$ at $\infty$. $\square$

*Remark* 1. We point out that this lemma can be applied to more cases than [15, Lemma 5.2]. For example, if $\Phi(u) = u^2$, $\Phi_1$ and $\Phi_0$ are $N$-functions with principal parts equal to $u^2/\log u$ and $u^2/(\log u)^2$ respectively (see [6, p. 16] and [6, Sec. 7] for the definition and properties of principal part), then (22) holds for $\Phi_0$. However, $\Phi_0(u)$ is not dominated for any power function $|u|^\alpha$ for every $\alpha < 2$.

**Definition 3.5.** *We define the functionals $J_{C,\Phi_0} : L^\Phi \to (-\infty, +\infty]$ and $H_{C,\Phi_0} : \mathbb{R}^n \to \mathbb{R}$, where $C > 0$ and $\Phi_0$ is an $N$-function, by*

$$J_{C,\Phi_0}(u) := \rho_\Phi(u) - C\Phi_0(\|u\|_{L^\Phi}), \tag{23}$$

*and*

$$H_{C,\Phi_0}(x) := \int_0^T F(t,x)dt - C\Phi_0(|x|), \tag{24}$$

*respectively.*

In [20] and [4] the authors considered, for the $p$-laplacian case, potentials $F$ satisfying the inequality

$$|\nabla F(t,x)| \leqslant b_1(t)|x|^\alpha + b_2(t), \tag{25}$$

where $b_1, b_2 \in L^1_1$ and $\alpha < p$. Thus, they called $F$ a sublinear nonlinearity. In this paper, we will consider bounds on $\nabla F$ of a more general type.

**Definition 3.6.** *Let $\Phi_0$ be a differentiable $N$-function. We say that $G:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ satisfies a $\Phi_0$-*grow condition *if*

$$|G(t,x)|\leqslant b_1(t)\Phi_0'(|x|)+b_2(t), \tag{26}$$

*with $b_1,b_2\in L^1([0,T],\mathbb{R})$.*

**Theorem 3.7.** *Let $\Phi$ be an $N$-function whose complementary function $\Psi$ satisfies the $\Delta_2$ condition globally. Assume that the $N$-function $\Phi_1$ satisfies* (20)*, $F$ satisfies (C) and (A), and $\nabla F$ satisfies a $\Phi_0$-grow condition for some $\Delta_2$-globally $N$-function $\Phi_0$ such that $\Phi_0\lll\Phi_1$. Furthermore, we suppose that*

$$\lim_{|x|\to\infty}\frac{\int_0^T F(t,x)\,dt}{\Phi_0(|x|)}=+\infty. \tag{27}$$

*Then, the problem* (1) *has at least a solution which minimizes the action integral $I$ on $W^1E^\Phi_T$.*

*Proof.* By the decomposition $u=\overline{u}+\tilde{u}$, Cauchy-Schwarz's inequality and (26), we have

$$\begin{aligned}
\left|\int_0^T F(t,u)-F(t,\overline{u})\,dt\right| &= \left|\int_0^T\int_0^1\nabla F(t,\overline{u}+s\tilde{u}(t))\cdot\tilde{u}(t)\,ds\,dt\right|\\
&\leqslant \int_0^T\int_0^1 b_1(t)\Phi_0'(|\overline{u}+s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt+\int_0^T\int_0^1 b_2(t)|\tilde{u}(t)|\,ds\,dt\\
&=: I_1+I_2.
\end{aligned}\tag{28}$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate $I_2$ as follows

$$I_2\leqslant\|b_2\|_{L^1}\|\tilde{u}\|_{L^\infty}\leqslant C_1\|u'\|_{L^\Phi}, \tag{29}$$

where $C_1=C_1(\|b_2\|_{L^1},T)$.

On the other hand, since $\Phi_0\in\Delta_2$ globally, then $\Phi_0'\in\Delta_2$ globally and consequently $\Phi_0'$ is a quasi-subadditive function, i.e. there exists $C(\Phi_0)>0$ such that $\Phi_0'(a+b)\leqslant C(\Phi_0')(\Phi_0'(a)+\Phi_0'(b))$ for every $a,b\geqslant 0$. In this way, we have

$$\Phi_0'(|\overline{u}+s\tilde{u}(t)|)\leqslant C(\Phi_0)[\Phi_0'(|\overline{u}|)+\Phi_0'(\|\tilde{u}\|_{L^\infty})], \tag{30}$$

for every $s\in[0,1]$.

Now, inequality (30), Hölder's and Sobolev-Wirtinger's inequalities, the monotonicity, the subadditivity and the $\Delta_2$-condition on $\Phi_0'$, imply that

$$\begin{aligned}
I_1 &\leqslant C(\Phi_0')\left\{\Phi_0'(|\overline{u}|)\|b_1\|_{L^1}\|\tilde{u}\|_{L^\infty}+\|b_1\|_{L^1}\Phi_0'(\|\tilde{u}\|_{L^\infty})\|\tilde{u}\|_{L^\infty}\right\}\\
&\leqslant C_2\left\{\Phi_0'(|\overline{u}|)\|u'\|_{L^\Phi}+\Phi_0'(\|u'\|_{L^\Phi})\|u'\|_{L^\Phi}\right\},
\end{aligned}\tag{31}$$

where $C_2=C_2(\Phi_0',T,\|b_1\|_{L^1})$.

Next, by Young's inequality with complementary functions $\Phi_0$ and $\Psi_0$ and the fact that $\Phi_0 \in \Delta_2$ globally, Young's equality [6, Eq. 2.7-2.8] and [7, Thm. 3-(ii), p. 23], we get

$$\begin{aligned} \Phi_0'(|\overline{u}|)\|u'\|_{L^\Phi} &\leqslant \Psi_0(\Phi_0'(|\overline{u}|)) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leqslant |\overline{u}|\Phi_0'(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \\ &\leqslant C(\Phi_0)\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) \end{aligned} \tag{32}$$

and

$$\Phi_0'(\|u'\|_{L^\Phi})\|u'\|_{L^\Phi} \leqslant C(\Phi_0)\Phi_0(\|u'\|_{L^\Phi}), \tag{33}$$

with $C(\Phi_0)$ the constant that comes from the $\Delta_2$-condition on $\Phi_0$.

From (31), (32), (33) and (29), we have

$$\begin{aligned} I_1 + I_2 &\leqslant C_3\left\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi}\right\} \\ &\leqslant C_4\left\{\Phi_0(|\overline{u}|) + \Phi_0(\|u'\|_{L^\Phi}) + 1\right\}, \end{aligned} \tag{34}$$

with $C_3$ and $C_4$ depending on $\Phi_0, T, \|b_1\|_{L^1}$ and $\|b_2\|_{L^1}$. The last inequality follows from the fact that $\Phi_0$ is an $N$-function, then there exists $C > 0$ such that $\Phi_0(x) \geqslant Cx$ for every $x \geqslant 1$. Thus $x \leqslant C\Phi_0(x) + 1$ for every $x \geqslant 0$.

In the subsequent estimates, we use (28), (34), the fact that $\Phi_0 \in \Delta_2$ and we get

$$\begin{aligned} I(u) &= \rho_\Phi(u') + \int_0^T F(t,u)\,dt \\ &= \rho_\Phi(u') + \int_0^T \left[F(t,u) - F(t,\overline{u})\right]\,dt + \int_0^T F(t,\overline{u})\,dt \\ &\geqslant \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + \int_0^T F(t,\overline{u})\,dt - C_4\Phi_0(|\overline{u}|) - C_4 \\ &\geqslant \rho_\Phi(u') - C_4\Phi_0(\|u'\|_{L^\Phi}) + H_{C_4,\Phi_0}(\overline{u}) - C_4 \\ &= J_{C_4,\Phi_0}(u') + H_{C_4,\Phi_0}(\overline{u}) - C_4. \end{aligned} \tag{35}$$

Let $u_n$ be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|u_n\|_{W^1L^\Phi} \to \infty$ and we have to prove that $I(u_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by $u_n$, $I(u_n)$ is upper bounded, i.e. there exists $M > 0$ such that $|I(u_n)| \leqslant M$. As $\|u_n\|_{W^1L^\Phi} \to \infty$, from Lemma 3.2, we have $|\overline{u}_n| + \|u_n'\|_{L^\Phi} \to \infty$. Passing to a subsequence is necessary, still denoted $u_n$, we can assume that $|\overline{u}_n| \to \infty$ or $\|u_n'\|_{L^\Phi} \to \infty$. Now, Lemma 3.4 implies that the functional $J_{C_4,\Phi_0}(u')$ is coercive; and, by (27), the functional $H_{C_4,\Phi_0}(\overline{u})$ is also coercive, then $J_{C_4,\Phi_0}(u_n') \to \infty$ or $H_{C_4,\Phi_0}(\overline{u}_n) \to \infty$. From the condition (A) on $F$, we have that on a bounded set the functional $H_{C_4,\Phi_0}(\overline{u}_n)$ is lower bounded and also $J_{C_4,\Phi_0}(u_n') \geqslant 0$. Therefore, $I(u_n) \to \infty$ as $\|u_n\|_{W^1L^\Phi} \to \infty$ which contradicts the initial assumption on the behavior of $I(u_n)$.

Let $\{u_n\} \subset W^1E_T^\Phi$ be a minimizing sequence for the problem $\inf\{I(u) | u \in W^1E_T^\Phi\}$. Since $I(u_n)$, $n = 1, 2, \ldots$, is upper bounded, the previous part of the proof shows that $\{u_n\}$ is norm bounded in $W^1E^\Phi$. Hence, by virtue of [15, Cor. 2.2], we can assume, taking a subsequence if necessary, that $u_n$ converges uniformly to a $T$-periodic continuous (therefore in $E^\Phi_T$) function $u$. As $u_n' \in E^\Phi$ is a norm bounded sequence in $L^\Phi$, there exists a subsequence (again denoted by $u_n'$) such that $u_n'$ converges to a function $v \in L^\Phi$ in the weak* topology of $L^\Phi$. Since $E^\Phi$ is weak* closed, by Lemma 3.1, $v \in E^\Phi$. From this fact and the uniform convergence of $u_n$ to $u$, we obtain that

$$\int_0^T \xi' \cdot u \, dt = \lim_{n\to\infty} \int_0^T \xi' \cdot u_n \, dt = -\lim_{n\to\infty} \int_0^T \xi \cdot u_n' \, dt = -\int_0^T \xi \cdot v \, dt$$

for every $T$-periodic function $\xi \in C^\infty([0,T], \mathbb{R}^d) \subset E^\Psi$. Thus $v = u'$ a.e. $t \in [0,T]$ (see [21, p. 6]) and $u \in W^1E^\Phi_T$.

Now, taking into account the relations $\left[L^1\right]^* = L^\infty \subset E^\Psi$ and $L^\Phi \subset L^1$, we have that $u_n'$ converges to $u'$ in the weak topology of $L^1$. Consequently, from the semicontinuity of $I$ (see [15, Lemma 6.1]) we get

$$I(u) \leqslant \liminf_{n\to\infty} I(u_n) = \inf_{v \in W^1E^\Phi_T} I(v).$$

Hence $u \in W^1E^\Phi_T$ is a minimun and, since $I$ is Gâteaux differentiable on $W^1E^\Phi$ (see [15, Thm. 3.2]), therefore $I'(u) \in (W^1E^\Phi_T)^\perp$. Thus,

$$\int_0^T \frac{\Phi'(|u'(t)|)}{|u'(t)|} u'(t) \cdot v'(t) dt = -\int_0^T \nabla F(t, u(t)) \cdot v(t) dt,$$

for every $v \in W^1E^\Phi_T$.

From [15, Lemma 2.4] we have $u'(t)\Phi'(|u'(t)|)/|u'(t)| \in L^\Psi([0,T], \mathbb{R}^n) \hookrightarrow L^1([0,T], \mathbb{R}^n)$; and, from condition (A) and the fact that $u \in L^\infty$, it follows that $\nabla F(t, u(t)) \in L^1([0,T], \mathbb{R}^n)$. Consequently, from [21, p. 6] we obtain that the differential equations in (1) are verified and $u'(0)\Phi'(|u'(0)|)/|u'(0)| = u'(T)\Phi'(|u'(T)|)/|u'(T)|$ holds. Thus $u'(0) = u'(T)$. $\square$

**Corollary 3.8.** *Let $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory, differentiable,* **strictly convex???** *function such that*

$$\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y}) \geqslant \Phi(|\boldsymbol{y}|) + F(t, \boldsymbol{x}), \tag{36}$$

$$|\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left( b(t) + \Phi\left( \tfrac{|\boldsymbol{y}|}{\lambda} + f(t) \right) \right), \tag{37}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left( b(t) + \Phi\left( \tfrac{|\boldsymbol{y}|}{\lambda} + f(t) \right) \right), \tag{38}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left( c(t) + \varphi\left( \tfrac{|\boldsymbol{y}|}{\lambda} + f(t) \right) \right). \tag{39}$$

*where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, $\Phi$ is an $N$-function, $\varphi$ is the right continuous derivative of $\Phi$, $b \in L^1_1([0,T])$, $c \in L^\Psi_1([0,T])$ and $f \in E^\Phi_1([0,T])$. In addition, the potential $F$ satisfies conditions (A) and (C)* **and convex???** *and $\Psi \in \Delta_2$ ????.*

*Then, the problem*

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad a.e.\ t \in (0,T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(T) = 0 \end{cases} \tag{40}$$

*has at least one solution* $\boldsymbol{u} : [0, T] \to \mathbb{R}^d$ *absolutely continuous, which minimizes the action integral*

$$I(u) = \int_0^T \mathcal{L}(t, u(t), u'(t))\, dt. \tag{41}$$

**Quizás hay demasiada información en el enunciado y se puede quitar algo....**

*Proof.* In Theorem 3.7 we have seen that $\Phi(|y|) + F(t, x)$ is coercive, then the function $\mathcal{L}$ do so. The differentiability and lower semicontinuity of the $I$ follow from Thm. 3.2 and Lemma 6.1 of [15], respectively. Last, applying Thm. 4.1 and Thm. 6.2???? of [15], we get the existence of $u$ and the initial conditions on $u$. □

# 4 Examples

The employment of $N$-functions instead of power functions in inequalities like (26) will allow us to extend some results of [20] and [4] a $\Phi$-laplacian operators with $N$-functions $\Phi$ which grow faster that power functions, for example with a exponential grow.

Furthermore, we want to emphasize that, even in the case of $p$-laplacian operator (3), our results extend previous one (see [20, 4]), because we get bounds that may be more sharp that those in [20, 4]. More precisely, in [4, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions of (3) under la assumption (25) for any $\alpha \in (0, p-1)$. Meanwhile, our Theorem 3.7 implies existence for the potential

$$F_0(t, x) = |x|^p / \ln(2 + |x|)^2.$$

We note that this $F$ does not satisfies (25) for any $\alpha < p - 1$. Next we will show a $N$-function $\Phi_0$ satisfying the hypothesis of Theorem 3.7 for this potential $F_0$.

We define

$$\Phi_0(u) = \begin{cases} \frac{p-1}{p} u^p & u \leqslant e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with $p > 1$. Next, we will establish some properties of this function $\Phi_0$.

**Theorem 4.1.** *If* $p \geqslant \frac{1+\sqrt{2}}{2}$*, then* $\Phi_0$ *is an differentiable* $N$*-function. The* $N$*-function* $\Phi_0$ *satisfies that for every* $\varepsilon > 0$*, there exists a positive constant* $C = C(p, \varepsilon)$ *such that*

$$C^{-1} t^{p-\varepsilon} \Phi_0(u) \leqslant \Phi_0(tu) \leqslant C t^p \Phi_0(u) \quad t \geqslant 1, u > 0, \tag{42}$$

*Proof.* We have

$$\varphi(u) = \Phi_0'(u) = \begin{cases} (p-1)u^{p-1} & := & \varphi_1(u) & \text{if } u \leqslant e \\ \frac{u^{p-1}}{\log u}\left(p - \frac{1}{\log u}\right) & := & \varphi_2(u) & \text{if } u \geqslant e \end{cases}$$

First let us see that $\Phi_0'$ is increasing when $p \geqslant \frac{1+\sqrt{2}}{2}$. For this purpose, since $\varphi_1(e) = \varphi_2(e)$, it is enough to see that $\varphi_1$ is increasing on $[0, e]$ and $\varphi_2$ is increasing on $[e, \infty)$ for every $p \geqslant \frac{1+\sqrt{2}}{2}$. Clearly $\varphi_1$ is an increasing function for $p > 1$. On the other hand, an elementary analysis of the function shows that $\varphi_2'(u) > 0$ on $[e, \infty)$ if and only if $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$. Therefore $\varphi_2$ is an icreasing function when $p \geqslant \frac{1+\sqrt{2}}{2}$.

Besides $\varphi_2(u) \to \infty$ and $\varphi_1(u) \to 0$ as $u \to \infty$ and $u \to 0$ respectively, provided that $p > 1$. Hence, $\Phi_0$ is an $N$-function.

Next we will prove (42). If $u \leqslant tu \leqslant e$, then $\Phi_0(tu) = t^p\Phi_0(u)$ and (42) holds with $C = 1$. If $u \leqslant e \leqslant tu$, as $\frac{e^p}{p} > 0$ and $\log(tu) \geqslant 1$, we have $\Phi_0(tu) \leqslant t^p u^p = \frac{p}{p-1} t^p \Phi_0(u)$. Thus, the second inequality of (42) holds with $C = \frac{p}{p-1}$. On the other hand, as $f(t) = \frac{t}{\log t}$ is increasing on $[e, \infty)$, then $f((tu)^p) \geqslant f(e^p) = e^p/p$. Now,

$$\begin{aligned}
\Phi_0(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\
&= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\
&\geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} \\
&\geqslant \frac{p-1}{p}\frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p.
\end{aligned}$$

Since $\varepsilon e^{1-\varepsilon}$ is the minimum value of $t \mapsto \frac{t^\varepsilon}{\log t + 1}$ on the interval $[1, +\infty)$ then

$$\Phi_0(tu) \geqslant \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,$$

which is the first inequality of (42) with $C = \frac{p}{p-1}\varepsilon^{-1}e^{-1+\varepsilon}$.

If $e \leqslant u \leqslant tu$, then

$$\Phi_0(tu) \leqslant \frac{t^p u^p}{\log(tu)} \leqslant \frac{t^p u^p}{\log(u)} = \frac{p t^p v}{\log v}, \tag{43}$$

where $v := u^p$ and $v \geqslant e^p$. If $\alpha > 0$, the function $x \mapsto \frac{x}{x-\alpha}$ is decreasing on $(\alpha, \infty)$ and the function $v \mapsto \frac{pv}{\log v}$ is increasing on $[e^p, \infty)$. Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leqslant \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$

for every $v \geqslant e^p$. In this way, from (43), we have

$$\Phi_0(tu) \leqslant \frac{pt^p}{p-1}\left(\frac{pv}{\log v} - \frac{e^p}{p}\right) = \frac{pt^p}{p-1}\left(\frac{u^p}{\log u} - \frac{e^p}{p}\right)$$

and the second inequality of (42) holds with $C = \frac{p}{p-1}$. For the first inequality we have, as it was proved previously,

$$\Phi_0(tu) \geqslant \frac{p-1}{p}\frac{(tu)^p}{\log(tu)} = \frac{p-1}{p}\frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)}\frac{t^{p-\varepsilon}u^p}{\log u}$$

Let $f(s) = \frac{sA}{\log s + A}$ with $s \geqslant 1$ and $A \geqslant \varepsilon$. If $A \leqslant 1$, the function $f$ attains a minimum on $[1, \infty)$ at $s = e^{1-A}$ and the minimum value is $f(e^{1-A}) = Ae^{1-A} \geqslant \varepsilon$. If $A > 1$,

$f$ is increasing on $[1,\infty)$ and its minimum value is $f(1)=1$. Then, $f(s)\geqslant\varepsilon$ in any case, therefore

$$\Phi_0(tu)\geqslant\frac{p-1}{p}\varepsilon\frac{t^{p-\varepsilon}u^p}{\log u}\geqslant\frac{p-1}{p}\varepsilon t^{p-\varepsilon}\Phi_0(u).$$

Therefore, (42) holds with $C=\frac{p}{\varepsilon(p-1)}$, because this $C$ is the biggest constant that we have obtained in each case under consideration. □

*Remark* 2. The inequality

$$\Phi_0(tu)\geqslant Ct^p\Phi_0(u)$$

is false for every $C$ because for every $u\geqslant e$ we have

$$\lim_{t\to\infty}\frac{\Phi_0(tu)}{t^p\Phi_0(u)}=0$$

We note that $\Phi_0$ and $F_0$ satisfy (27). For the $p$-laplacian operator we have that $\Phi(|u|)=|u|^p/p$. Then we can take $Phi_1=\Phi$ in (20). Clearly $\Phi_0\lll\Phi_1$.

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