Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t \in (0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}, \tag{1}$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary*

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2010 AMS Subject Classification. Primary: . Secondary: .
Keywords and phrases. .

equations. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t,x,y) = \frac{|y|^p}{p} + F(t,x), \tag{2}$$

for $1 < p < \infty$. For example, the classic book [9] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [9] were extended and improved in several articles, see [19, 17, 22, 18, 23] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [21, 20] and in this case (1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{4}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In [1] it was treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t,x,y) = \Phi(|y|) + F(t,x), \tag{5}$$

where Φ is an N-function (see section 2 for the definition of this concept). In the paper [1] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([9, Ch. 3]) to obtain solutions of (1).

2 Preliminaries

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and other brief introduction to superposition operators between these spaces. References for these topics are [15, 16, 3] and [13, 10, 12, 11].

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is convex and it also satisfies that

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to 0} \frac{\Phi(t)}{t} = 0.$$

In addition, in this paper for the sake of simplicity we assume that Φ is differentiable and we call φ the derivative of Φ. On these assumptions, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism whose inverse will be denoted by ψ. We denote by Ψ the primitive of ψ that satisfies $\Psi(0) = 0$. Then, Ψ is an N-function which is called the *complementary function* of Φ.

We recall that an N-function $\Phi(u)$ has *principal part* $f(u)$ if $\Phi(u) = f(u)$ for large values of the argument (see [6, p. 16] and [6, Sec. 7] for properties of principal part).

There exist several orders and equivalence relations between N-functions (see [14, Sec. 2.2]). Following [14, Def. 1, pp. 15-16] we say that the N-function Φ_2 is *stronger* than the N-function Φ_1, in symbols $\Phi_1 \prec \Phi_2$, if there exist $a > 0$ and $x_0 \geqslant 0$ such that

$$\Phi_1(x) \leqslant \Phi_2(ax), \quad x \geqslant x_0. \tag{6}$$

The N-functions Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) when $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$. We say that Φ_2 is *essentially stronger* than Φ_1 ($\Phi_1 \lll \Phi_2$) if and only if for every $a > 0$ there exists $x_0 = x_0(a) \geqslant 0$ such that (6) holds. Finally, we say that Φ_2 is *completely stronger* than Φ_1 ($\Phi_1 \prec\!\!\!\prec \Phi_2$) if and only if for every $a > 0$ there exist $K = K(a) > 0$ and $x_0 = x_0(a) \geqslant 0$ such that

$$\Phi_1(x) \leqslant K\Phi_2(ax), \quad x \geqslant x_0. \tag{7}$$

We also say that a non decreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the Δ_2^∞*-condition*, denoted by $\eta \in \Delta_2^\infty$, if there exist constants $K > 0$ and $x_0 \geqslant 0$ such that

$$\eta(2x) \leqslant K\eta(x), \tag{8}$$

for every $x \geqslant x_0$. We note that $\eta \in \Delta_2^\infty$ if and only if $\eta \prec\!\!\!\prec \eta$. If $x_0 = 0$, the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the Δ_2*-condition* ($\eta \in \Delta_2$). If there exists $x_0 > 0$ such that inequality (8) holds for $x \leqslant x_0$, we will say that Φ satisfies the Δ_2^0*-condition* ($\Phi \in \Delta_2^0$).

We denote by α_η and β_η the so called *Matuszewska-Orlicz indices* of the function η, which are defined next. Given an increasing, unbounded, continuous function $\eta : [0, +\infty) \to [0, +\infty)$ such that $\eta(0) = 0$, we define

$$\alpha_\eta := \lim_{t \to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\eta(tu)}{\eta(u)}\right)}{\log(t)}, \quad \beta_\eta := \lim_{t \to +\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\eta(tu)}{\eta(u)}\right)}{\log(t)}. \tag{9}$$

It is known that the previous limits exist and $0 \leqslant \alpha_\eta \leqslant \beta_\eta \leqslant +\infty$ (see [8, p. 84]). The relation $\beta_\eta < +\infty$ holds true if and only if $\eta \in \Delta_2$ ([8, Thm. 11.7]). If (Φ, Ψ) is a complementary pair of N-functions then

$$\frac{1}{\alpha_\Phi} + \frac{1}{\beta_\Psi} = 1, \tag{10}$$

(see [8, Cor. 11.6]). Therefore $1 \leqslant \alpha_\Phi \leqslant \beta_\Phi \leqslant \infty$.

If η is an increasing function that satisfies the Δ_2-condition, then η is controlled by above and below by power functions ([4, Sec. 1], [**?**, Eq. (2.3)-(2.4)] and [8, Thm.

11.13]). More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\eta, \epsilon)$ such that, for every $t, u \geqslant 0$,

$$K^{-1} \min\left\{t^{\beta_\eta+\epsilon}, t^{\alpha_\eta-\epsilon}\right\}\eta(u) \leqslant \eta(tu) \leqslant K \max\left\{t^{\beta_\eta+\epsilon}, t^{\alpha_\eta-\epsilon}\right\}\eta(u). \tag{11}$$

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$ the set of all measurable functions defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{12}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{13}$$

The Orlicz space L^Φ equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup\left\{\int_0^T u \cdot v\, dt \middle| \rho_\Psi(v) \leqslant 1\right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The following inequality holds for any $u \in L^\Phi$

$$\|u\|_{L^\Phi} \leqslant \frac{1}{k}\{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \tag{14}$$

In fact, $\|u\|_{L^\Phi}$ is the infimum for $k > 0$ of the right hand side in above expression (see [6, Thm. 10.5] and [5]).

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ is the only one maximal subspace contained in the Orlicz class C^Φ, i.e. $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [6, Thm. 9.3]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \tag{15}$$

Like in [6], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ given by

$$\Pi(E_d^\Phi, r) := \{\boldsymbol{u} \in L_d^\Phi | d(\boldsymbol{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, namely,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \tag{16}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L_d^Φ, E_d^Φ, $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

Let $\mathcal{E}_d^{\Phi_i}(\lambda) := W^1 L_d^{\Phi_i} \cap \{u | \dot{u} \in \Pi(E_d^{\Phi_i}, \lambda)\}$.

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi \subset \left[L^\Phi\right]^*$, where the pairing $\langle v, u \rangle$ is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \tag{17}$$

with $u \in L^\Phi$ and $v \in L^\Psi$. Unless $\Phi \in \Delta_2^\infty$, the relation $L^\Psi = \left[L^\Phi\right]^*$ will not be satisfied. In general, it is true that $\left[E^\Phi\right]^* = L^\Psi$.

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ (see [2]) by

$$W^1 L^\Phi := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi\}.$$

$W^1 L^\Phi$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{18}$$

And, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{19}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\, dt$ and $\widetilde{u} = u - \overline{u}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty \hookrightarrow L^\Phi \hookrightarrow L^1$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. It is said that $u : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|u(t) - u(s)| \leqslant Cw(|t-s|). \tag{20}$$

We denote by $C^w([0,T], \mathbb{R}^d)$ the space of w-Hölder continuous functions that satisfy (20) for some $C > 0$. This is a Banach space with norm

$$\|u\|_{C^w([0,T],\mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t-s|)}.$$

The following simple embedding lemma, whose proof can be found in [1], will be used systematically.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. *$W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ and for every $u \in W^1L^\Phi$*

$$|u(t)-u(s)| \leqslant \|u'\|_{L^\Phi} w(|t-s|) \qquad \textit{(Morrey's inequality)}, \tag{21}$$

$$\|u\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right) \max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality)}. \tag{22}$$

2. *For every $u \in W^1L^\Phi$ we have $\widetilde{u} \in L^\infty_d$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \textit{(Sobolev-Wirtinger's inequality)}. \tag{23}$$

3 Once upon a time...

Vamos escribiendo lo que queremos...(de acuerdo a mis apuntes y sin ver las hojitas de la semana pasada)

For $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ we denote by $\mathfrak{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0,T] \to \mathbb{R}^d$ by

$$\mathfrak{f}u(t) = f(t,u(t))$$

Referencias y alguna propiedad interesante medibles en medibles? [7, 6]

Theorem 3.1. *Let $\Phi_1, \Phi_2, \dots, \Phi_n$ be N-functions. Assume that M is another N-functions that satisfy the Δ_2-condition. We write $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ with $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}^d$. Let $f(t,x_1,\dots,x_n,y_1,\dots,y_n)$ be a function Chatratheodory? with $f : [0,T] \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathbb{R}^{d'}$.*

Suppose that $a : (\mathbb{R}^d)^n \to [0,+\infty)$ is a bounded function on bounded sets and $b \in L^M([0,T])$, for a.e. $t \in [0,T]$ such that

$$|f| \leqslant a(x)[b(t) + \sum_{i=1}^n M^{-1}(\Phi_i(|y_i|))], \tag{24}$$

then

$$\mathfrak{f} : \left(\prod_{i=1}^n L^\infty([0,T],\mathbb{R}^d)\right) \times \left(\prod_{i=1}^n \Pi(E^{\Phi_i}([0,T],\mathbb{R}^d), \lambda = 1)\right) \to L^M.$$

Proof. If $(u,v) \in \left(\prod_{i=1}^n L^\infty([0,T],\mathbb{R}^d)\right) \times \left(\prod_{i=1}^n \Pi(E_d^{\Phi_i}, \lambda = 1)\right)$. By [6, Thm. 17.6] (y otras cosas), we get

$$|\mathfrak{f}u(t)| = |f(t,u(t),v(t))| \leqslant M_a[b_j(t) + \sum_{i=1}^n M_j^{-1}(\Phi_i(|v_i(t)|))] \in L_1^{M_j}.$$

□

We define the space X by $X = \{v = (v_1, v_2) : v_1 \in W^1 L_T^{\Phi_1}, v_2 \in W^1 L_T^{\Phi_2}\}$ and $X^* = \{v = (v_1, v_2) : v_1 \in (W^1 L_T^{\Phi_1})^*, v_2 \in (W^1 L_T^{\Phi_2})^*\}$ where $(W^1 L_T^{\Phi_i})^*$ stands for the conjugate space of $W^1 L_T^{\Phi_i}$ for $i = 1, 2$.

Corollary 3.2. *We will consider the Lagrange function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \to \mathcal{L}(t, x_1, x_2, y_1, y_2)$ which is measurable in t for each $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x_1, x_2, y_1, y_2) for almost every $t \in [0,T]$.*

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^d$ and let

$$I(x) = \int_0^T \mathcal{L}(t, x, y)\, dt \tag{25}$$

If there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2$, $b \in L_1^1([0,T])$, $j = 1, \ldots, d'$ for a.e. $t \in [0,T]$ and every $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying the structure conditions

$$|\mathcal{L}(t,x,y)| + \sum_{i=1}^{2} |D_{x_i}\mathcal{L}(t,x,y)| \leqslant a(|x|)(b(t) + \Phi_1(|y_1|) + \Phi_2(|y_2|)), \tag{26}$$

$$|D_{y_i}\mathcal{L}(t,x,y)| \leqslant a(|x|)(c_i(t) + \textstyle\sum_{j=1}^{n} \Psi_i^{-1}(\Phi_j(|y_j|))\ i = 1, 2. \tag{27}$$

The nonlinear operator $(x_1, x_2) \mapsto D_x\mathcal{L}(t, x_1, y_1, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \cdots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into $L^1([0,T])$ with the strong topology on both sets.

The nonlinear operator $(x_1, x_2) \mapsto D_y\mathcal{L}(t, x_1, y_1, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \cdots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into X with the weak* topology.

The function I is Gâteaux differentiable on $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^*. Moreover, I' is given by the following expression

$$\begin{aligned} \langle I'(x), w\rangle = \int_0^T [&(D_{x_1}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1(t)) + \\ &(D_{x_2}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2(t)) + \\ &(D_{y_1}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1'(t)) + \\ &(D_{y_2}\mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2'(t))]\, dt \end{aligned} \tag{28}$$

If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^* when both spaces are equipped with the strong topology.

We denote by $\mathfrak{A}(a, b, c, \lambda, f, \Phi)$ the set of all Lagrange functions satisfying (**??**), (**??**) and (**??**).

Proof. **OJO!!!! Es algo que teníamos del trabajo anterior!!! con algunas adaptaciones a 2 variables sin controlar y a lo bruto!!!!!**

Let $\boldsymbol{u} \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$.

Step 1. The non linear operator $(x_1, x_2) \mapsto (D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2), D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2))$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into $L_d^1([0,T]) \times L_d^1([0,T])$ with the strong topology on both sets.

If $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$, from (**??**) and (**??**), we obtain Let $\{x_n = (x_{1n}, x_{2n})\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ and let $x = (x_1, x_2) \in \mathcal{E}_d^{\Phi_1}(\lambda)\times\mathcal{E}_d^{\Phi_2}(\lambda)$ such that $x_n \to x$ in X. From $x_{in} \to x_i$ in L^{Φ_i}, there exists a subsequence x_{in_k} such that $x_{in_k} \to x_i$ a.e.; and, as $x_{in} \to x_i \in \mathcal{E}_d^{\Phi}(\lambda)$, by Lemma **??**, there exist a subsequence of x_{in_k} (again denoted x_{in_k}) and a function $h_i \in \Pi(E_1^{\Phi}, \lambda))$ such that $x_{in_k} \to u_i$ a.e. and $|x_{in_k}| \leqslant h_i$ a.e. Since x_{in_k}, $k = 1, 2, \dots$, is a strong convergent sequence in $W^1L_d^{\Phi_i}$, it is a bounded sequence in $W^1L_d^{\Phi_i}$. According to Lemma 2.1 and Corollary **??**, there exist $M_i > 0$ such that $\|\boldsymbol{a}(x_{in_k})\|_{L^\infty} \leqslant M_i$, $k = 1, 2, \dots$. From the previous facts and (**??**), we get

$$|D_{x_i}\mathcal{L}(\cdot, x_{1n_k}, x_{2n_k}, y_{1n_k}, y_{2n_k})| \leqslant M_i(b + \Phi_i(|h_i|)) \in L_1^1 \;\; i = 1, 2.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$D_{x_i}\mathcal{L}(t, x_{in_k}(t), y_{in_k}(t)) \to D_{x_i}\mathcal{L}(t, x_i(t), y_i(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.
Step 2. The non linear operator $(x_1, x_2) \mapsto (D_{y_1}\mathcal{L}(t, x_1, y_1, D_{y_2}\mathcal{L}(t, x_2, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ with the strong topology into X with the weak* topology.

Note that (**??**), (**??**) and the imbeddings $W^1L_d^{\Phi} \hookrightarrow L_d^\infty$ and $L_d^\Psi \hookrightarrow \left[L_d^\Phi\right]^*$ imply that the second member of (28) defines an element in $\left[W^1L_d^\Phi\right]^*$.

Let $(x_{1n}, x_{2n}) \in \mathcal{E}_d^{\Phi}(\lambda)$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$ in the norm of X. We must prove that $D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}) \overset{w^*}{\rightharpoonup} D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2)$ para $i = 1, 2$. On the contrary, there exist $v = (v_1, v_2) \in L^{\Phi_1} \times L^{\Phi_2}$, $\epsilon > 0$ and a subsequence of $\{x_n\}$ (denoted $\{x_n\}$ for simplicity) such that

$$|\langle D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}, y_{1n}, y_{2n}),\ v\rangle - \langle D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2, v\rangle| \geqslant \epsilon. \tag{29}$$

We have $x_n \to x$ in X and $y_n \to y$ in X. By Lemma **??**, there exist a subsequence x_{n_k} and a function $h \in \Pi(E_1^{\Phi_1}, \lambda) \times \Pi(E_1^{\Phi_2}, \lambda)$ such that $x_{n_k} \to x$ a.e., $y_{n_k} \to y$ a.e. and $|y_{n_k}| \leqslant h$ a.e. As in the previous step, since x_n is a convergent sequence, the Corollary **??** implies that $a(|y_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, with x_{n_k} instead of x, inequality (**??**) becomes

$$|D_{y_i}\mathcal{L}(\cdot, x_{n_k}, y_{n_k})| \leqslant M_i(c_i + \varphi_i(h_i) + \Psi_i^{-1}(\Phi_j(|y_j|))) \in L_1^{\Psi_i}. \tag{30}$$

Consequently, as $v \in L_d^\Phi$ and employing Hölder's inequality, we obtain that

$$\sup_k |D_{\boldsymbol{y}}\mathcal{L}(\cdot, \boldsymbol{u}_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot v| \in L_1^1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot \boldsymbol{v}\, dt \to \int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}}) \cdot \boldsymbol{v}\, dt \tag{31}$$

which contradicts the inequality (29). This completes the proof of step 2.

Step 3. We will prove (28). The proof follows similar lines as [9, Thm. 1.4]. For $\boldsymbol{u} \in \mathcal{E}_d^{\Phi}(\lambda)$ and $\mathbf{0} \neq \boldsymbol{v} \in W^1 L_d^{\Phi}$, we define the function

$$H(s,t) := \mathcal{L}(t, \boldsymbol{u}(t) + s\boldsymbol{v}(t), \dot{\boldsymbol{u}}(t) + s\dot{\boldsymbol{v}}(t)).$$

From [6, Lemma 10.1] (or [15, Thm. 5.5]) we obtain that if $|\boldsymbol{u}| \leqslant |\boldsymbol{v}|$ then $d(\boldsymbol{u}, E_d^{\Phi}) \leqslant d(\boldsymbol{v}, E_d^{\Phi})$. Therefore, for $|s| \leqslant s_0 := \left(\lambda - d(\dot{\boldsymbol{u}}, E_d^{\Phi})\right)/\|\boldsymbol{v}\|_{W^1L^{\Phi}}$ we have

$$d\left(\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}, E_d^{\Phi}\right) \leqslant d\left(|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}|, E_1^{\Phi}\right) \leqslant d\left(|\dot{\boldsymbol{u}}|, E_1^{\Phi}\right) + s\|\dot{\boldsymbol{v}}\|_{L^{\Phi}} < \lambda.$$

Thus $\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}} \in \Pi(E_d^{\Phi}, \lambda)$ and $|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}| \in \Pi(E_1^{\Phi}, \lambda)$. These facts imply, in virtue of Theorem **??** item **??**, that $I(\boldsymbol{u} + s\boldsymbol{v})$ is well defined and finite for $|s| \leqslant s_0$. And, using Corollary **??**, we also see that

$$\|a(|\boldsymbol{u} + s\boldsymbol{v}|)\|_{L^\infty} \leqslant A(\|\boldsymbol{u} + s\boldsymbol{v}\|_{W^1L^{\Phi}}) \leqslant A(\|\boldsymbol{u}\|_{W^1L^{\Phi}} + s_0\|\boldsymbol{v}\|_{W^1L^{\Phi}}) =: M$$

Now, applying Chain Rule, (**??**), (**??**) the monotonicity of φ and Φ, the fact that $\boldsymbol{v} \in L_d^\infty$ and $\dot{\boldsymbol{v}} \in L_d^{\Phi}$ and Hölder's inequality, we get

$$\begin{aligned} |D_s H(s,t)| &= |D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u} + s\boldsymbol{v}, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u} + s\boldsymbol{v}, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \dot{\boldsymbol{v}}| \\ &\leqslant M\left[\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|\boldsymbol{v}|\right. \\ &\quad \left. + \left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|\dot{\boldsymbol{v}}|\right] \in L_1^1. \end{aligned} \tag{32}$$

Consequently, I has a directional derivative and

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \frac{d}{ds} I(\boldsymbol{u} + s\boldsymbol{v})\Big|_{s=0} = \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}}) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}}) \cdot \dot{\boldsymbol{v}}\right\}\, dt.$$

Moreover, from (**??**), (**??**), Lemma 2.1 and the previous formula, we obtain

$$|\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle| \leqslant \|D_{\boldsymbol{x}}\mathcal{L}\|_{L^1}\|\boldsymbol{v}\|_{L^\infty} + \|D_{\boldsymbol{y}}\mathcal{L}\|_{L^{\Psi}}\|\dot{\boldsymbol{v}}\|_{L^{\Phi}} \leqslant C\|\boldsymbol{v}\|_{W^1L^{\Phi}}$$

with a appropriate constant C. This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : \mathcal{E}_d^{\Phi}(\lambda) \to \left[W^1 L_d^{\Phi}\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$. Indeed, if $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}_d^{\Phi}(\lambda)$ with $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of $W^1L_d^{\Phi}$ and $\boldsymbol{v} \in W^1 L_d^{\Phi}$, then

$$\begin{aligned} \langle I'(\boldsymbol{u}_n), \boldsymbol{v}\rangle &= \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\left(t, \boldsymbol{u}_n, \dot{\boldsymbol{u}}_n\right) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t, \boldsymbol{u}_n, \dot{\boldsymbol{u}}_n\right) \cdot \dot{\boldsymbol{v}}\right\}\, dt \\ &\to \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\left(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\right) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\right) \cdot \dot{\boldsymbol{v}}\right\}\, dt \\ &= \langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle. \end{aligned}$$

In order to prove item **??**, it is necessary to see that the maps $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ are norm continuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into L_d^1 and L_d^{Ψ} respectively. The continuity of the first map has already been proved in step 1. Let $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{W^1L^\Phi} \to 0$. Therefore, there exist a subsequence $\boldsymbol{u}_{n_k} \in \mathcal{E}_d^\Phi(\lambda)$ and a function $h \in \Pi(E_1^\Phi, \lambda)$ such that (30) holds true. And, as $\Psi \in \Delta_2$ then the right hand side of (30) belongs to E_1^Ψ. Now, invoking Lemma **??**, we prove that from any sequence $\boldsymbol{u}_n$ which converges to $\boldsymbol{u}$ in $W^1L_d^\Phi$ we can extract a subsequence such that $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of I' follows from the continuity of $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ using the formula (28). □

Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

References

[1] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, and E. Schwindt. Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698, 2015.

[2] R. Adams and J. Fournier. *Sobolev spaces*. Elsevier/Academic Press, Amsterdam, 2003.

[3] W. Desch and R. Grimmer. On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc*, (353):5095–5120, 2001.

[4] Jan Gustavsson and Jaak Peetre. Interpolation of orlicz spaces. *Studia Mathematica*, 60(1):33–59, 1977.

[5] H. Hudzik and L. Maligranda. Amemiya norm equals Orlicz norm in general. *Indag. Math. (N.S.)*, 11(4):573–585, 2000.

[6] M. A. Krasnosel′skiĭ and Ja. B. Rutickiĭ. *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen, 1961.

[7] M.A. Krasnosel'skii, P.P. Zabreyko, E.I. Pustylnik, and P.E. Sobolevski. *Integral operators in spaces of summable functions*. Mechanics: Analysis. Springer Netherlands, 2011.

[8] L. Maligranda. *Orlicz spaces and interpolation*, volume 5 of *Seminários de Matemática [Seminars in Mathematics]*. Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[9] J. Mawhin and M. Willem. *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York, 1989.

[10] Nguen Hong Thai. The superposition operator in the Orlicz spaces of vector functions. *Dokl. Akad. Nauk BSSR*, 31:197â200, 1987.

[11] Ryszard Płuciennik. Boundedness of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Bull. Pol. Acad. Sci., Math.*, 33:531â540, 1985.

[12] Ryszard Płuciennik. On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 25:321â337, 1985.

[13] Ryszard Płuciennik. The superposition operator in Musielak-Orlicz spaces of vector-valfued functions. Abstract analysis, Proc. 14th Winter Sch., Srní/Czech. 1986, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14, 411-417 (1987)., 1987.

[14] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146. Marcel Dekker, Inc., New York, 1991.

[15] G. Schappacher. A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386, 2005.

[16] M. S. Skaff. Vector valued orlicz spaces. ii. *Pacific J. Math.*, 28(2):413–430, 1969.

[17] C.-L. Tang. Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270, 1998.

[18] C. L. Tang and X.-P. Wu. Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397, 2001.

[19] Chun-Lei Tang. Periodic solutions of non-autonomous second-order systems with γ-quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675, 1995.

[20] X. Tang and X. Zhang. Periodic solutions for second-order Hamiltonian systems with a p-Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113, 2010.

[21] Y. Tian and W. Ge. Periodic solutions of non-autonomous second-order systems with a p-Laplacian. *Nonlinear Anal.*, 66(1):192–203, 2007.

[22] X.-P. Wu and C.-L. Tang. Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235, 1999.

[23] F. Zhao and X. Wu. Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434, 2004.