Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

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Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,u(t),u'(t)) = D_x\mathcal{L}(t,u(t),u'(t)) & \text{a.e. } t\in(0,T) \\ u(0)-u(T)=u'(0)-u'(T)=0, \end{cases}, \qquad (P_1)$$

where $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $d\geqslant 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u:[0,T]\to\mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions* of *Euler-Lagrange systems of ordinary equations*.

This topic was deeply addressed for the several types of *Lagrange functions*.

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For example,

$$\mathcal{L}_{p,F}(t,x,y) := \frac{|y|^p}{p} + F(t,x), \tag{1}$$

for $1 < p < \infty$. For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (P_1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (1) for arbitrary $1 < p < \infty$ were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (P_1) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{P_2}$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t,x)$ is differentiable with respect to x for a.e. $t \in [0,T]$ and the following conditions are verified:

(C) F and its gradient ∇F, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t). \tag{2}$$

In this inequality we assume that the function $a : [0,+\infty) \to [0,+\infty)$ is continuous and non decreasing and $0 \leqslant b \in L^1([0,T],\mathbb{R})$.

In the framework of anisotropic Sobolev-Orlicz spaces, we can study system of p-laplacian equations as the following example shows.

Example 1. Let $1 < p_1, p_2 < \infty$. We define $\Phi_{p_1,p_2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$\Phi_{p_1,p_2}(y_1,y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. And, we consider the following Lagrange function

$$\mathcal{L}(t,x,y) = \Phi_{p_1,p_2}(y) + F(t,x).$$

where $y = (y_1,y_2), x = (x_1,x_2) \in \mathbb{R}^{2n}$...o algo así?????

Then the equations (P_1) become

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) & \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}, \tag{P_3}$$

In the literature, these equations are known as (p_1,p_2)-Laplacian system, see [Yang and Chen, 2013, Pasca and Wang, 2016, Yang and Chen, 2012, Pasca, 2010, Paşca and Tang, 2010, Pasca and Tang, 2011].

In [Acinas et al., 2015] it is treated the case of a lagrangian $\mathcal{L}$ which is lower bounded by a Lagrange function like

$$\mathcal{L}_{\Phi,F}(t,x,y) := \Phi(|y|) + F(t,x), \tag{3}$$

where Φ is an N-function (see section 2 for the definition of this concept).

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic Young functions $\Phi : \mathbb{R}^n \to \mathbb{R}_+$, i.e. functions such that $\Phi(x)$ depends on the direction of x, unlike the radial case where $\Phi(x) = \Phi(|x|)$. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001].

On the other hand, anisotropic Orlicz-Sobolev spaces allow us to simplify the writing, and they provide the natural frame for statements of the type [Tian and Ge, 2007, Lemma 3.1]. This type of question motivated us to use these spaces.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \to +\infty$, when $|x| \to +\infty$. Additionally, we assume that Young's functions which we deal with, satisfy that $\Phi(x) > 0$ when $x \neq 0$. Following [Schappacher, 2005] we say that Φ is an N_∞-function if

$$\lim_{|x|\to\infty} \frac{\Phi(x)}{|x|} = +\infty.$$

Given a Young's function Φ, we define function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, |x| = s\right\}, \tag{4}$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.1. *The function A_Φ has the following properties:*

1. *A_Φ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *$A_\Phi(|x|)$ is the* greatest radial minorant *of $\Phi(x)$,*
4. *Φ is N_∞ if and only if A_Φ is.*

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [Clarke, 2013]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

Example 2. We compute A_Φ for the function $\Phi = \Phi_{p_1,p_2}$ given in Example (1).

We apply the method of Lagrange multipliers to solve the next minimization problem subject to a constraint

$$G(r) = \min\left\{\Phi(x,y) : |(x,y)|_2^2 = r^2\right\}$$

.

The first order conditions are

$$\begin{cases} |x|^{p_1-2}x + \lambda x &= 0 \\ |y|^{p_2-2}y + \lambda y &= 0 \\ |x|^2 + |y|^2 &= r^2 \end{cases}$$

These equations can be solved, among others, by the following sets (if $n > 1$ infinite) of citical values: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2.

Now, suppose that $x \neq 0$ and $y \neq 0$ then $|x|^2 + |y|^2 = r^2$ and $|y| = |x|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|x|^{p_1-2}$.

In order to employ second order conditions for constrained problems, we define the tangent plane at the point $(x,y) \in \mathbb{R}^{2n}$ as $M = \{(z,w) \in \mathbb{R}^{2n} : zx^T + wy^T = 0\}$. We also consider the Lagrangian $\mathcal{L}(x,y,\lambda) = \Phi(x,y) + \lambda H(x,y)$ being $H = 0$ the constraint. And, we are interested in knowing how to behave the matrix of second partial derivatives of $\mathcal{L}$ with respect to the vector $(x,y) \in \mathbb{R}^{2n}$, given by $\mathcal{H} = D^2\Phi + \lambda D^2 H$, on the subspace M.

We have to split the analysis in several cases according to the values of p_1 and p_2.

First, we deal with $p_1 \leqslant 2$ and $p_2 \leqslant 2$ being one of them different from 2.

We compute

$$(z,w)\mathcal{H}(z,w)^T = |\lambda|(zx^T)^2[|x|^{-2})(p_1-2) + (p_2-2)|y|^{-2}],$$

on the subspace M.

As there exists $(z,w) = (-y,x)$ such that $zx^t + wy^t = 0$, we obtain that
$(-y,x)\mathcal{H}(-y,x)^T = |\lambda||y|^2|x|^2[(p_1-2)|x|^{-2} + (p_2-2)|y|^{-2}] < 0.$

Then, by second order necessary conditions [Luenberger and Ye, 2015, p.333], at (x,y) there cannot be a minimum. Therefore, the minima occur at $x = 0$ or $y = 0$.

The remaining cases can be treated with similar techniques.

Finally, we conclude that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leqslant A_\Phi \leqslant K_2 \max\{r^{p_1}, r^{p_2}\}$$

with $K_1, K_2 > 0$.

We also say that $\Phi : \mathbb{R}^d \to \mathbb{R}^+$ satisfies the Δ_2^∞*-condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant K\Phi(x), \tag{5}$$

for every $|x| \geqslant M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \to \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \tag{6}$$

We denote by $\mathcal{M} := \mathcal{M}([0,T], \mathbb{R}^d)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0,T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions $\mathcal{M}$, as for other similar sets, we will omit the reference to codomain $\mathbb{R}^d$ when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0,T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{7}$$

The *Orlicz space* $L^\Phi = L^\Phi([0,T], \mathbb{R}^d)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{8}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \middle| \rho_\Phi \left(\frac{v}{\lambda} \right) dt \leqslant 1 \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

The subspace $E^\Phi = E^\Phi([0,T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of E^Φ, which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0,T]$, $n = 1, 2, \dots$ then $\|\chi_{E_n} u\| \to 0$ when $|E_n| \to 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u\, dt \leqslant 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \tag{9}$$

Like in [Krasnosel′skiĭ and Rutickiĭ, 1961] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{10}$$

for any positive r (see [Schappacher, 2005, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

Definition 2.2. *Let $u_n, u \in L^\Phi([0,T],\mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^\infty([0,T],\mathbb{R})$, $n = 1,2,\dots$, such that $0 \leqslant \alpha_n(t) \leqslant \alpha_{n+1}(t)$, $\alpha_n(t) \to 1$ a.e., when $n \to \infty$ and $u_n(t) = \alpha_n(t)u(t)$.*

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$, where a function $v \in L^\Psi$ is associated to $\xi_v \in \left[L^\Phi\right]^*$ being

$$\xi_v(u) = \langle \xi_v, u\rangle = \int_0^T v \cdot u \, dt, \tag{11}$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of $\left[L^\Phi\right]^*$ which can be identified with L^Ψ.

Proposition 2.3. *Let $F \in \left[L^\Phi([0,T],\mathbb{R}^d)\right]^*$. Then the following statements are equivalent*

1. *$\xi \in L^\Psi([0,T],\mathbb{R}^d)$*
2. *ξ satisfies the* monotone convergence property*, which is if u_n converges monotonically to u then $\langle \xi, u_n\rangle \to \langle \xi, u\rangle$.*

If $\Phi \in \Delta_2^\infty$ and Φ is N_∞ then $L^\Psi([0,T],\mathbb{R}^d) = \left[L^\Phi([0,T],\mathbb{R}^d)\right]^*$ (see [Desch and Grimmer, 2001, Thm. 2.9 , Thm. 2.10]).

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi([0,T],\mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0,T] \text{ and } u' \in L^\Phi([0,T],\mathbb{R}^d)\}.$$

$W^1L^\Phi([0,T],\mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{12}$$

And, we introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\},\\ W^1E_T^\Phi &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{13}$$

We will use repeatedly the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.4. *Let $\Phi : \mathbb{R}^d \to [0,+\infty)$ be a Young's function and let $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (4)*. Then*

1. *For every $s,t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi} |s-t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1,T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have* $\widetilde{u} \in L^\infty([0,T],\mathbb{R}^d)$ *and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If* Φ *is* N_∞ *then the space* $W^1L^\Phi([0,T],\mathbb{R}^d)$ *is compactly embedded in the space of continuous functions* $C([0,T],\mathbb{R}^d)$.

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned}\Phi\left(\frac{u(t)-u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) &\leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right)\\ &\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}.\end{aligned}$$

By Proposition 2.1(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.1 (2) and we have

$$\begin{aligned}|u(t)-\overline{u}| &= \left|\frac{1}{T}\int_0^T u(t)-u(s)\,ds\right|\\ &\leqslant \frac{1}{T}\int_0^T |u(t)-u(s)|\,ds\\ &\leqslant \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right)\end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right)ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.1(3)

$$|\overline{u}| \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi}\\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^\Phi([0,T],\mathbb{R}^d)$. Since Φ is N_∞, from Proposition 2.1(4) we obtain $sA_\Phi^{-1}(1/s)\to 0$ when $s\to 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (**??**) implies that u_n is bounded in $C([0,T],\mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u\in C([0,T],\mathbb{R}^d)$ with $u_{n_k}\to u$ in $C([0,T],\mathbb{R}^d)$.

□

Lemma 2.5. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi,1)$ converging to $u\in\Pi(E^\Phi,1)$ in the L^Φ-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h\in L^1([0,T],\mathbb{R})$ such that $u_{n_k}\to u$ a.e. and $\Phi(u_{n_k})\leqslant h$ a.e.*

Proof. Since $d(u,E^\Phi)<1$ and u_n converges to u, there exists $u_0\in E^\Phi$, a subsequence of u_n (again denoted u_n) and $0<r<1$ such that $d(u_n,u_0)<r$. Let $\lambda_0\in(r,1)$. By extracting more subsequences, if necessary, we can assume that $u_n\to u$ a.e. and

$$\lambda_n := \|u_{n+1}-u_n\|_{L^\Phi} < \frac{1-\lambda_0}{2^n}, \quad \text{for } n\geqslant 1.$$

We can assume $\lambda_n>0$ for every $n=0,\ldots$.

Let $\lambda := 1-\sum_{n=0}^\infty \lambda_n$ and define $h:[0,T]\to\mathbb{R}$ by

$$h(x)=\lambda\Phi\left(\frac{u_0}{\lambda}\right)+\sum_{n=0}^{\infty}\lambda_n\Phi\left(\frac{u_{n+1}-u_n}{\lambda_n}\right). \tag{14}$$

Note that $\sum_{n=0}^\infty \lambda_n+\lambda=1$, therefore for any $n=1,\ldots$

$$\begin{aligned}\Phi(u_n) &= \Phi\left(\lambda\frac{u_0}{\lambda}+\sum_{j=0}^{n-1}\lambda_j\frac{u_{j+1}-u_j}{\lambda_j}\right)\\ &\leqslant \lambda\Phi\left(\frac{u_0}{\lambda}\right)+\sum_{j=0}^{n-1}\lambda_j\Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right)\leqslant h\end{aligned}$$

Since $u_0\in E^\Phi\subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda)\in L^1([0,T],\mathbb{R})$. On the other hand $\|u_{n+1}-u_n\|_{L^\Phi}\leqslant\lambda_n$, therefore

$$\int_0^T\Phi\left(\frac{u_{j+1}-u_j}{\lambda_j}\right)dt\leqslant 1.$$

Then $h\in L^1([0,T],\mathbb{R})$.

□

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anistropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that $f:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$is a *Carathéodory function*, i.e.

(C) f is measurable with respect to $t \in [0,T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

Definition 3.1. *For $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0,T] \to \mathbb{R}^d$ by*

$$\boldsymbol{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [Krasnosel'skii et al., 2011] for scalar functions and [Płuciennik, 1987, Płuciennik, 1985b, Płuciennik, 1985a] for the generalization to $\mathbb{R}^d$-valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

Theorem 3.2. *We assume that f satisfies condition ((C)) and that $\Phi_1, \Phi_2 : \mathbb{R}^d \to [0,+\infty)$ are anisotropic Young functions. Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps measurable function into measurable functions*
2. Extensibility. *If the operator $\boldsymbol{f}$ acts from the ball $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} | \|u\|_{L^{\Phi_1}} < r\}$ into the space L^{Φ_2} or the space E^{Φ_2} then $\boldsymbol{f}$ can be extended from $\Pi(E^{\Phi_1}, r)$ into space L^{Φ_2} or E^{Φ_2}, respectively.*
3. Continuity. *If the operator $\boldsymbol{f}$ acts from $\Pi(E^{\Phi_1}, r)$ into space E^{Φ_2}, then $\boldsymbol{f}$ is continuous.*

Given a continuous function $a \in C(\mathbb{R}^n, \mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [Acinas et al., 2015].

Lemma 3.3. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

Quizás no sea necesaria la prueba, si dejamos el comentario de arriba???

Proof. Let $A \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\| \leqslant s, x \in \mathbb{R}^d} |a(x)|$. If $u \in W^1L^\Phi_d$ then, by Sobolev's inequality,

$$a(u(x)) \leqslant \alpha(\|u\|_{L^\infty}) \leqslant \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\right) =: A(\|u\|_{W^1L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

We assume that the *Lagrangian* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is Carathéodory and differentiable function satisfying

$$|\mathcal{L}(t,x,y)| + |D_x\mathcal{L}(t,x,y)| + \Psi(D_y\mathcal{L}(t,x,y)) \leqslant a(|x|)\,(b(t) + \Phi(y)), \tag{15}$$

where $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1_1([0,T])$, Φ and Ψ are N_∞-functions (complementary???? o en el teorema o nunca?)

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t))\, dt. \tag{16}$$

PEDIMOS $\Psi \in \Delta_2$ de entrada y eliminamos la parte de la demicontinuidad????

Theorem 3.4. *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying* (15)*. Then the following statements hold:*

1. *The action integral given by* (16) *is finitely defined on $\mathcal{E}^\Phi := W^1L^\Phi \cap \{u | \dot{u} \in \Pi(E^\Phi, 1)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$. Moreover, I' is given by the following expression*
$$\langle I'(u), v\rangle = \int_0^T \left\{D_x\mathcal{L}\big(t,u,\dot{u}\big)\cdot v + D_y\mathcal{L}\big(t,u,\dot{u}\big)\cdot \dot{v}\right\}\, dt. \tag{17}$$
3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^\Phi$. As

$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \tag{18}$$

and (10), then $\Phi(\dot{u}(t)) \in L^1$. Now,

$$|\mathcal{L}(\cdot, u, \dot{u})| + |D_x\mathcal{L}(\cdot, u, \dot{u})| + \Psi(D_y\mathcal{L}(\cdot, u, \dot{u})) \leqslant A(\|u\|_{W^1L^\Phi})(b + \Phi(\dot{u})) \in L^1, \tag{19}$$

by (15) and Lemma 3.3. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto D_x\mathcal{L}(t, u, \dot{u})$ is continuous from $\mathcal{E}^\Phi$ into $L^1([0,T])$ with the strong topology on both sets.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^\Phi$ and let $u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in W^1L^Φ. By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u_n - u\|_{L^\Phi}$$

then $u_n \to u$ uniformly. As $\dot{u}_n \to \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.5, there exist a subsequence of $\dot{u}_{n_k}$ (again denoted $\dot{u}_{n_k}$) and a function $h \in L^1([0,T], \mathbb{R})$ such that $\dot{u}_{n_k} \to \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leqslant h$ a.e.

Since u_{n_k}, $k = 1, 2, \ldots$, is a strong convergent sequence in W^1L^Φ, it is a bounded sequence in W^1L^Φ. According to item (3) of Lemma 2.4, there exists $M > 0$ such that $\|\boldsymbol{a}(u_{n_k})\|_{L^\infty} \leqslant M$, $k = 1, 2, \ldots$. From the previous facts and (19), we get

$$|D_x\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leqslant a(|u_{n_k}|)(b + \Phi(\dot{u}_{n_k})) \leqslant M(b + h) \in L^1.$$

On the other hand, by the continuous differentiability of $\mathcal{L}$, we have

$$D_x\mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \to D_x\mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.
Step 2. The non linear operator $u \mapsto D_y\mathcal{L}(t,u,\dot{u})$ is continuous from $\mathcal{E}^\Phi$ with the strong topology into $\left[L^\Phi\right]^*$ with the weak topology.*

Let $u \in \mathcal{E}^\Phi$. From (19) it follows that

$$D_y\mathcal{L}(\cdot, u, \dot{u}) \in C^\Psi. \tag{20}$$

Así? o conviene poner la cota de $\Psi(D_y)$ explícitamente???

Note that (19), (20) and the imbeddings $W^1L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$ imply that the second member of (17) defines an element of $\left[W^1L^\Phi\right]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in the norm of W^1L^Φ. We must prove that $D_y\mathcal{L}(\cdot, u_n, \dot{u}_n) \overset{w^*}{\rightharpoonup} D_y\mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle D_y\mathcal{L}(\cdot, u_n, \dot{u}_n), v\rangle - \langle D_y\mathcal{L}(\cdot, u, \dot{u}), v\rangle| \geqslant \epsilon. \tag{21}$$

We have $u_n \to u$ in L^Φ and $\dot{u}_n \to \dot{u}$ in L^Φ. By Lemma 2.5, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T], \mathbb{R})$ such that $u_n \to u$ uniformly, $\dot{u}_n \to \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leqslant h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.3 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (19) with u_n instead of u, we have

$$\Psi(D_y\mathcal{L}(\cdot, u_n, \dot{u}_n)) \leqslant M(b + h) \in L^1. \tag{22}$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (22), we have

$$\begin{aligned} &\lambda D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ &\leqslant \lambda\left[\Psi(D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right)\right] \\ &\leqslant \lambda M(b+h) + \lambda\Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \tag{23}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y\mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v\, dt \to \int_0^T D_y\mathcal{L}(t, u, \dot{u}) \cdot v\, dt \tag{24}$$

which contradicts the inequality (21). This completes the proof of step 2.

Step 3. We will prove (17). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1L^\Phi$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leqslant s_0 := \min\{(1 - d(\dot{u}, E^\Phi))/\|v\|_{W^1L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d(\dot{u} + s\dot{v}, E^\Phi) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.4 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$.

We also have $\|u + sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 3.3, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leqslant M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leqslant s_0\lambda^{-1}$, from the convexity and the parity of Φ, we get

$$\begin{aligned}
&\Phi(\dot{u} + s\dot{v}) = \Phi\left((1 - s_0)\frac{\dot{u}}{1 - s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right)\\
&\leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1
\end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1 - s_0}, E^\Phi\right) = \frac{1}{1 - s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1-s_0} \in C^\Phi$.

Now, applying (19), (23), the fact that $v \in L^\infty$ and $\dot{v} \in L^\Phi$, we get

$$\begin{aligned}
|D_sH(s,t)| &= \left|D_x\mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + \lambda D_y\mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \frac{\dot{v}}{\lambda}\right|\\
&\leqslant M\left\{[b(t) + \Phi(\dot{u} + s\dot{v})]\,|v|\right\}\\
&+ \lambda\left[\Psi(D_y\mathcal{L}(t, u + sv, \dot{u} + s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right)\right]\\
&\leqslant M\left\{[b(t) + \Phi(\dot{u} + s\dot{v})]\,|v|\right\} + \lambda M[b(t) + \Phi(\dot{u} + s\dot{v})] + \lambda\Phi\left(\frac{\dot{v}}{\lambda}\right)\\
&= M[b(t) + \Phi(\dot{u} + s\dot{v})](|v| + \lambda) + \lambda\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1.
\end{aligned} \tag{25}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v\rangle = \frac{d}{ds}I(u + sv)\Big|_{s=0} = \int_0^T \{D_x\mathcal{L}(t, u, \dot{u}) \cdot v + D_y\mathcal{L}(t, u, \dot{u}) \cdot \dot{v}\}\ dt.$$

Moreover, from the previous formula, (19), (20), and Lemma 2.4, we obtain

$$|\langle I'(u), v\rangle| \leqslant \|D_x\mathcal{L}\|_{L^1}\|v\|_{L^\infty} + \|D_y\mathcal{L}\|_{L^\Psi}\|\dot{v}\|_{L^\Phi} \leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant C.

This completes the proof of the Gâteaux differentiability of I.

LO QUE SIGUE NO IRÍA ???? PORQUE TOMARÍAMOS $\Psi \in \Delta_2$

Step 4. The operator $I' : \mathcal{E}^\Phi \to \left[W^1L^\Phi_d\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto D_x\mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y\mathcal{L}(t, u, \dot{u})$.

Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \to u$ in the norm of W^1L^Φ and $v \in W^1L^\Phi$, then

$$\begin{aligned}\langle I'(u_n), v\rangle &= \int_0^T \{D_x\mathcal{L}(t, u_n, \dot{u}_n)\cdot v + D_y\mathcal{L}(t, u_n, \dot{u}_n)\cdot \dot{v}\}\ dt \\ &\to \int_0^T \{D_x\mathcal{L}(t, u, \dot{u})\cdot v + D_y\mathcal{L}(t, u, \dot{u})\cdot \dot{v}\}\ dt \\ &= \langle I'(u), v\rangle .\end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto D_x\mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y\mathcal{L}(t, u, \dot{u})$ are norm continuous from $\mathcal{E}^\Phi$ into L^1 and L^Ψ, respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de D_x aquí!!!

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$.

Applying Lemma 2.5 to $\dot{u}_n$, there exists a subsequence (denoted $\dot{u}_n$ for simplicity) such that $\dot{u}_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot{u}_n) \leqslant h$ and $\dot{u}_n \to \dot{u}$ a.e.

Then, by (23) we have $\Psi(v_n) \leqslant m(t) \in L^1$ being $v_n := D_y\mathcal{L}(\cdot, u_n, \dot{u}_n)$ and $m(t) := M(b+h)$. In addition, from the continuous differentiability of $\mathcal{L}$, we have that $v_n \to v$ a.e. where $D_y\mathcal{L}(\cdot, u, \dot{u})$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \to ???$ such that $\Psi(\lambda x) \leqslant c(|\lambda|)\Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|)\Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \to 0$ a.e. as $n \to \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda| K\Psi(v_n) + \Psi(v)) \leqslant c(|\lambda| K[m(t) + \Psi(v)]) \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda})\, dt \to 0$ for every $\lambda > 0$. Thus, $v_n \to v$ in L^Ψ.

The continuity of I' follows from the continuity of $D_x\mathcal{L}$ and $D_y\mathcal{L}$ using the formula (17). □

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