A generalized Sitnikov problem with primary bodies in both homographic and rigid motions

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Abstract

1 Introduction

In this paper we study the following restricted Newtonian $n+1$-body problem P (see figure 1):

- P_1 We have n primary bodies of masses $m_1, \dots, m_n$ and an additional massless particle.
- P_2 The primary bodies are in a homographic motion (see [14, Section 2.9]). This motion is carried out in a plane Π.
- P_3 The massless particle is moving on the perpendicular line to Π passing through the center of mass of the primary bodies.

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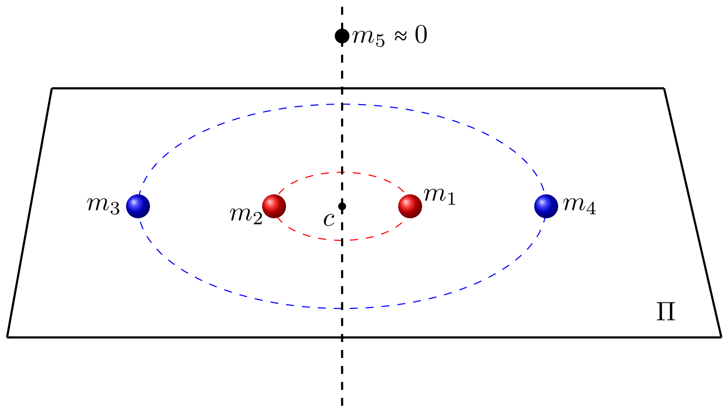

Figure 1: Six-body problem

Problems like the one presented above have been extensively discussed in the literature. In [29] K. Sitnikov considered the problem of two body in a Keplerian elliptic motion and a massless particle moving in the perpendicular line to the orbital plane passing through the center of mass. Sitnikov obtained deep results about existence of solutions, for small $e > 0$, with a chaotic behavior (see [21, III(5)]). Periodic solutions for a Sitnikov configuration were considered in [6, 7, 15, 27].

Generalized circular Sitnikov problems, i.e. we have $n \geq 3$ primaries in a relative equilibrium motion, were addressed more recently. In [30] Soulis, Papadakis and Bountis studied existence, linear stability and bifurcations for a problem similar to P. They considered a Lagrangian equilateral triangle configuration for the primary bodies, which were supposed to have the same mass $m_1 = m_2 = m_3$. In [5] Papadakis and Bountis extended the results of [30] to n primaries ($n \geq 3$) in a poligonal equal masses configuration. Later in [25], Pandey and Ahmad generalized analysis started in [30] to the case with oblate primaries, i.e with space extended bodies. In [13] Li, Zhang and Zhao studied a special type of restricted circular $n+1$-body problem with equal masses for the primaries in a regular polygon configuration. Periodic solutions for generalized Sitnikov problems with primaries performing no rigid motions were studied in [27, 28]. In the previous papers the primary bodies are in the vertices of a regular polygon. In [17] Marchesin and Vidal studied the problem P for a rigid motion of primaries in a rhomboidal configuration. In [4] Bakker and Simmons studied scape regions for the massless particle in a problem similar to P for a certain type of periodic orbits for primaries including non homographic motions.

In the present paper, after introducing preliminaries facts in Section 2, we obtain in Section 3 necessary and sufficient conditions on the configuration of primary bodies in order that the z-axis to be invariant for the flow associated to the motion equations of the massless particle. For this type of configurations, that we call *balanced*, the Sitnikov problem has sense. In Section 4 we will find all balanced configurations for $n \leq 4$ primaries. The Section 5 is devoted to describe all possible motion of the massless particle when the primaries are in a relative equilibrium (or rigid) motion. In this direction we observe that only are possible scape (both parabolic and hiperbolic) and periodic motions. Moreover, we will give a formula expressing the period of solutions by means of integrals. We prove in Corollary 5.4 that the complete $n+1$-body system has infinite quantity of periodic solutions. In Section 6 we discuss the situation when the entire system has a solution with the same period that the rigid motion of primaries. We call it synchronous solution. Surprisingly this fact is related to the existence of certain pyramidal central configurations (for the definition of this concept see [8, 9, 23]). Finally, in the last section, we study certain non balanced configurations which allows some particular solutions of problem P.

In this paper we generalize and extend some results previously obtained. For example, our results in Section 5 concerning to balanced configurations generalize the results in [17] established for rhomboidal configurations. In Section 6 we prove that there exists synchronous solutions for primaries in a regular poligonal equal mass configuration if and only if $2 \leq n \leq 472$. The sufficient of this fact was established in [13].

2 Preliminaries

We start considering n mass points, $n > 2$, of masses $m_1, \dots, m_n$ moving in a Euclidean 3-dimensional space according to Newton's laws of motion. We assume that $x_1(t), \dots, x_n(t)$ are the coordinates of the bodies in some inertial Cartesian coordinate system. We can suppose, without any loss generality, that the center of mass $c := \sum_j m_j x_j / M$ $(M := \sum_j m_j)$ is fixed at the origin $(c = 0)$.

Initially we assume that the bodies are in a *planar homographic motion* on the plane Π (see [14]). That means, assuming that Π is the plane determined by the first two coordinates axes, that

$$x_j(t) = r(t)Q(\theta(t))q_j, \tag{1}$$

where

$$Q(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $q_j \in \Pi$, $j = 1, \dots, n$ are vectors in a planar *central configuration* (CC) in Π. We recall the following definition of this concept (see [14]).

Definition 2.1. *Let $q = (q_1, \dots, q_n)$ be a n-tuple of positions in $\mathbb{R}^3$ and let $m = (m_1, \dots, m_n)$ be a vector of masses. We say that (q, m) is a central configuration if there exists $\lambda \in \mathbb{R}$ such that*

$$\nabla_j U(q_1, \dots, q_n) + \lambda m_j q_j = 0, \quad j = 1, \dots, n. \tag{2}$$

where the potential function U *is defined by:*

$$U(q_1,\dots,q_n)=\sum_{i<j}\frac{m_im_j}{r_{ij}}, \tag{3}$$

where $r_{ij}=|q_i-q_j|$ *and* ∇_j *denotes the* 3*-dimensional partial gradient with respect to* q_j*.*

According to [14, Eq. (2.16)] the functions $r(t)$ and $\theta(t)$ solves the two-dimensional Kepler problem in polar coordinates, i.e.

$$\begin{aligned} \ddot r(t)-r(t)\dot\theta(t)^2 &= -\tfrac{\lambda}{r(t)^2}\\ \tfrac{d}{dt}\left[r(t)^2\dot\theta(t)\right] &= 0. \end{aligned} \tag{4}$$

In the particular case of *rigid motion*, we have $r(t)\equiv 1$ and $\theta(t)=\sqrt{\lambda}t+\theta(0)$. Therefore the primary bodies perform a periodic motion with period $T:=2\pi/\sqrt{\lambda}$.

Let $x_0(t)$ be the position of the massless particle. According to the Newtonian equations of motion x_0 satisfies

$$\ddot x_0=\sum_{i=1}^n\frac{m_i(x_i-x_0)}{|x_i-x_0|^3}=:f(t,x_0). \tag{5}$$

In the previous equation, we assume know the positions of the primaries. Therefore, this equation plus convenient initial conditions completely detemines the position of the particle.

3 Balanced configurations

Henceforth we denote by L the coordinate z axis. It is well know that a necessary and sufficient condition for that L be invariant under the flow associated to the non autonomous system (5), is that $f(t,L)\subset L$ for all t, i.e. L is f*-invariant* for every t.

Definition 3.1. *We say that a central configuration* (q,m) *is* balanced *if and only if* (q,m) *satisfies that, for any* $r>0$*, such that the set*

$$F_r:=\{i:|q_i|=r\}$$

is non empty, then

$$\sum_{i\in F_r}m_iq_i=0. \tag{6}$$

i.e. every maximal set of bodies which are equidistant from origin has center of mass equal to 0*.*

Theorem 3.2. *L is f-invariant for every t if and only if (q,m) is balanced.*

For the proof of the previous theorem we need the following result.

Lemma 3.3. *For $c > 0$ we define the function $y_c(t) := (c+t)^{-3/2}$. If $0 < t_1 < t_2 < \ldots < t_k$ then the functions $y_j(t) := y_{t_j}(t)$ are linearly independent on each open interval $I \subset \mathbb{R}^+$.*

Proof. It is sufficient to prove that Wronskian

$$W := W(y_1, \ldots, y_k)(t) = \det \begin{pmatrix} y_1 & \cdots & y_k \\ \frac{dy_1}{dt} & \cdots & \frac{dy_k}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1}{dt^{k-1}} & \cdots & \frac{d^{k-1}y_k}{dt^{k-1}} \end{pmatrix}$$

is not null on I.

Using induction is easy to show that

$$\frac{d^i y_c}{dt^i} = \beta_i y_c^{\frac{2i+3}{3}}, \quad \text{for some } \beta_i \neq 0, \text{ and for all } i = 1, \ldots. \tag{7}$$

Fix any $t \in I$. Then, according to (7) and writing $\lambda_j := (t+t_j)^{-1}$, we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \lambda_1^{3/2} & \lambda_2^{3/2} & \cdots & \lambda_k^{3/2} \\ \beta_1 \lambda_1^{5/2} & \beta_1 \lambda_2^{5/2} & \cdots & \beta_1 \lambda_k^{5/2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1} \lambda_1^{k+1/2} & \beta_{k-1} \lambda_2^{k+1/2} & \cdots & \beta_{k-1} \lambda_k^{k+1/2} \end{pmatrix} \\ &= \beta_1 \beta_2 \cdots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \cdots \lambda_k^{3/2} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \\ &= \beta_1 \beta_2 \cdots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \cdots \lambda_k^{3/2} \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i), \end{aligned}$$

where the last equality follows of the well known Vandermonde determinant identity. Therefore $W \neq 0$ if and only if $\lambda_i \neq \lambda_j$, $i \neq j$, which in turn is equivalent to $t_i \neq t_j$, $i \neq j$. □

Proof of Theorem 3.2. The condition $f(t, L) \subset L$ for all t is equivalent to

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{(r(t)^2 |q_i|^2 + z^2)^{3/2}} = 0, \tag{8}$$

for every $t, z \in \mathbb{R}$.

Let $D = \{|q_i| : i = 1, \ldots, n\}$. Suppose that $D = \{s_1, \ldots, s_k\}$, with $s_i \neq s_j$ for $i \neq j$. Therefore $\{1, \ldots, n\} = F_{s_1} \cup \cdots \cup F_{s_k}$. Then, multiplying equation (8) by $r(t)^2 Q^{-1}(\theta(t))$ and writing $\zeta = (z/r(t))^2$ we have that (8) is equivalent to

$$\sum_{j=1}^{k} \left\{ \frac{1}{(s_j^2 + \zeta)^{3/2}} \sum_{i \in F_{s_j}} m_i q_i \right\} = 0.$$

According to Lemma 3.3, the last equation is equivalent to (6). □

4 Balanced collisionless configurations for $n \leq 4$

In this section we find all balanced collisionless configurations with $n \leq 4$. In this context we understand that $q_1 \ldots, q_n$ is a collisionless configuration when $q_i \neq 0$ for $i = 1, \ldots, n$. We note that despite this fact the system can have collisions in the case that the primaries have a homothetic collapsing motion. From the fact that the center of mass is an excluded position we obtain

$$F_r \neq \varnothing \Rightarrow \# F_r \geq 2. \tag{9}$$

Trivially, a configuration of two point masses m_1 and m_2 is balanced if and only if $m_1 = m_2$.

From (9), a 3-body configuration consists of equidistant bodies from the origin. Therefore, it must to be the Lagrangian equilateral triangle configuration. Now, by equation (6) and an elementary geometrical reazoning we have that $m_1 = m_2 = m_3$.

The case $n = 4$ is more interesting. We include Definition 4.1 and Theorem 4.2, which are presented for the first time in [20], for the reader's convenience.

Definition 4.1. *Let q be a planar configuration. For each pair, i, j, the line containing q_i and q_j together with its perpendicular bisector form axes which divide the plane into four quadrants. The union of the first and third quadrants is an hourglass shaped region which will be called a 'cone'; similarly, the second and fourth quadrants together form another cone. The phrase 'open cone' refers to a cone minus the axes.*

Theorem 4.2 (Perpendicular Bisector Theorem)**.** *Let (q, m) be a planar central configuration and let q_i and q_j be any two of its points. Then if one of the two open cones determined by the line through q_i and q_j and its perpendicular bisector contains points of the configuration, so does the other one.*

Next we characterize all the 4-body balanced collisionless configurations.

Theorem 4.3. *Let (q, m) be a 4-body central configuration. Then (q, m) is balanced and collisionless if and only if, for a suitable enumeration of bodies, $q_1 = -q_3$, $q_2 = -q_4$, $m_1 = m_3$, $m_2 = m_4$, and q is of some of the following mutually exclusive types:*

CCcl. *collinear,*

CCr. *a rhombus with $r_{13} < r_{24}$ and $m_1 > m_2$,*

CCs. *a square with four equal masses.*

Proof. From (9) we have to consider two cases.

Case 1. $m_1 \geq m_2$, $|q_1| \neq |q_2|$, $|q_1| = |q_3|$ and $|q_2| = |q_4|$. Now (6) implies that $m_1 = m_3$, $m_2 = m_4$, $q_1 = -q_3$ and $q_2 = -q_4$. We divide the plane in two cones C_i, $i = 1, 2$, by means of the line P joining q_1 and q_3 together with its perpendicular bisector M. From Theorem 4.2, if q_2 is in C_1, then q_4 is in C_2, and vice versa.

This is a contradiction with the fact that $q_2 = -q_4$. Then $q_2, q_4 \in P$ or $q_2, q_4 \in M$, i.e. q is collinear or is a rhombus with equal masses in opposite vertices. In the first case, (q, m) is of CCcl type. In the second case, if $m_1 > m_2$, was proved in [16, Eqs. (3.44) and (3.45)] that $r_{13} < r_{24}$. Hence (q, m) is of CCr type. From [26, Corollary 2] if $m_1 = m_2$ then the configuration is a square witch is a contradiction with the fact that $|q_1| \neq |q_2|$.

Case 2. $|q_1| = |q_2| = |q_3| = |q_4|$. In this situation, was proved in [10] that the configuration is the equal mass square. □

5 Massless particle motion

In this and next sections we suppose that the primary bodies are in a T-periodic rigid motion associated to a balanced collisionless q, i.e $r(t) \equiv 1$ and from the remark follows (4), $\theta(t) = \sqrt{\lambda}t$. Without loss of generality, we have assumed here that $\theta(0) = 0$. For the particle, we suppose that it is moving on L, i.e. $x_0(t) = (0, 0, z(t))$. According to Theorem 3.2, x_0 is solution of (5), if and only if $z(t)$ is solution of the autonomous equation

$$\ddot{z} = -\sum_{i=1}^{n} \frac{m_i z}{(s_i^2 + z^2)^{3/2}}, \tag{10}$$

where $s_i = |q_i|$.

We will analyze all possible motions for the massless particle x_0. In particular we will see that all motion is periodic or is a scape trayectory. We will find that there exists T_0-periodic solutions for all T_0 in an interval $(\sigma(q), +\infty)$. This fact implies that there exists an infinity quantity of periodic solutions for the entire $n+1$-body system.

The second order equation (10) is consevative, therefore solutions conserve the energy

$$E(z, v) := \frac{|v|^2}{2} - \sum_{i=1}^{n} \frac{m_i}{\left(s_i^2 + z^2\right)^{\frac{1}{2}}}, \tag{11}$$

i.e. $E(z(t), \dot{z}(t))$ is constant.

Following [3] (see also [17]) we introduce the next concepts.

Definition 5.1 (Chazy, 1922)**.** *A solution $z(t)$ of* (10) *such that $\lim_{t\to\infty} z(t) = \infty$ is called hyperbolic for $t \to \infty$ when $\lim_{t\to\infty} \dot{z}(t) = z_\infty \neq 0$ and is called parabolic if $\lim_{t\to\infty} \dot{z}(t) = 0$.*

The following theorem characterize all the possible motions for the massless particle.

Theorem 5.2. *We assume that q is a balanced collisionless configuration and the primaries are in a rigid motion. Every solution of* (10) *is of some of the following types:*

1. *Hyperbolic, when $E > 0$,*

2. *Parabolic, when* $E = 0$,

3. *Periodic, when* $E_{min} := -\sum_{i=1}^{n} \frac{m_i}{s_i} < E < 0$.

4. *Equilibrium solution when* $E = E_{min}$.

Proof. We follow a standard argument for hamiltonian systems (see [2]).

We consider the level sets of $S(E) = \{(z, v) : E(z, v) = E\}$, in the phase space (z, v), for different values of E. An elementary analysis shows that

- If $E \geq 0$ then $S(E)$ is the union of two bounded graphs. They are symmetric with respect to z-axis, each of which is contained in some semiplane $v > 0$ or $v < 0$. The v-positive branch is the graph of a function $v(E, z)$, which is decreasing with respect to $|z|$. Moreover, $\lim_{|z|\to\infty} v(E, z) = \sqrt{2E}$.

- For every $E \geq E_{min}$, the energy curve cut the v-axis at $\pm(2E+2\sum_{i=1}^{n} m_i s_i^{-1})^{\frac{1}{2}}$.

- If $E_{min} < E < 0$ then $S(E)$ is a simple closed curve symmetric with respecto to z and v axes.

- An energy curve cut the z-axis, only in the case that $E < 0$, at $\pm z_E$, where z_E is the only positive solution of $-\sum_{i=1}^{n} m_i (s_i^2 + z_E^2)^{-\frac{1}{2}} = E$.

In the figure 1 we show the phase portrait for a rhomboidal configuration with masses $m_1 = m_3 = 1$ and $m_2 = m_4 = 0.5$.

The function $\varphi(t) = (z(t), \dot{z}(t))$ solves the system $\dot{\varphi}(t) = F(\varphi(t))$, where $F(z, v) = (v, -\sum_{i=1}^{n} m_i z(s_i^2 + z^2)^{-3/2})$. The only fixed point of F is $(z, v) = (0, 0)$. Therefore, the level surfaces $S(E)$, with $E \neq E_{min}$, do not contain stationary points. A well know argument implies that the trayectories $t \mapsto (z(t), \dot{z}(t))$ fill completely the energy curves.

We observe that any solution z crosses the v-axis. On the other hand if $E \geq 0$ and $v(E, 0) > 0$ ($v(E, 0) < 0$) then $z(t)$ is increasing (decreasing) with respect to t. If $z(t)$ remained bounded when $t \to +\infty$, then there would be the limit $\zeta_\infty := \lim_{t\to\infty} z(t)$. This would imply that $(\zeta_\infty, 0)$ is a fixed point of F, which is a contradiction. As a consequence, if $E \geq 0$ then $|z(t)| \to \infty$ when $t \to +\infty$. Moreover $\lim_{t\to+\infty} \dot{z}(t) = \pm\sqrt{2E}$. From this we conclude that the trayectory is hyperbolic when $E > 0$ and it is parabolic in the case $E = 0$.

In the case that $E_{min} < E < 0$ we have that the trayectory is contained in a closed curve, therefore it is a periodic orbit.

Finally if $E = E_{min}$ clearly we have that $z(t) \equiv 0$. □

Theorem 5.3. *We denote by* $T_0(E)$ *the minimal period for a solution of* (10) *with* $E_{min} < E < 0$. *Then*

1. *for* z_E *the only positive solution of* $-\sum_{i=1}^{n} m_i (s_i^2 + z_E^2)^{-\frac{1}{2}} = E$

$$T_0(E) = 2^{3/2} \int_0^{z_E} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz, \tag{12}$$

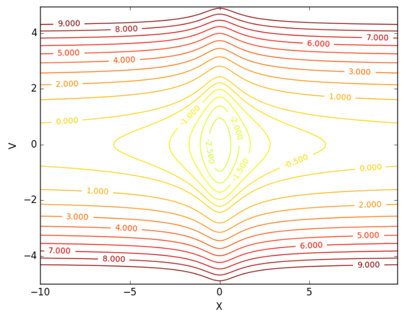

Figure 2: Energy level for a rhomboidal configuration with masses $m_1 = m_3 = 1$ and $m_2 = m_4 = 0.5$

2. *$T_0(E)$ is an increasing function.*

3. *$T_0((E_{min}, 0)) = (T_{min}, +\infty)$, where $T_{min} = 2\pi \left(\sum_{i=1}^{n} \frac{m_i}{s_i^3}\right)^{-1/2}$.*

Proof. Let $E_{min} < E < 0$ and let $z(t)$ be the only increasing solution with $z(0) = 0$, $\dot z(0) > 0$ and energy equal to E. Therefore $z(t)$ is $T_0(E)$-periodic. As a consequence of the symmetries of the equation we have that $z(T_0(E)/4) = z_E$. Then, taking account of (11) we have that

$$\begin{aligned} \frac{T_0}{4} &= \int_0^{T_0/4} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \dot z(t) dt \\ &= \frac{1}{\sqrt{2}} \int_0^{z_E} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}}\right)^{-\frac{1}{2}} dz, \end{aligned}$$

and we have proved item *1*. In order to prove item *2* we note that

$$\begin{aligned} 2^{-3/2} T_0(E) &= \int_0^{z_E} \left(\sum_{i=1}^{n} m_i \left((s_i^2 + z^2)^{-\frac{1}{2}} - (s_i^2 + z_E^2)^{-\frac{1}{2}}\right)\right)^{-\frac{1}{2}} dz \\ &= \int_0^{z_E} \left(z_E^2 - z^2\right)^{-\frac{1}{2}} f(z, z_E) dz \\ &= \int_0^{1} \left(1 - u^2\right)^{-\frac{1}{2}} f(z_E u, z_E) du, \end{aligned}$$

where

$$f(z, z_E) = \left(\sum_{i=1}^{n} m_i \left\{(s_i^2 + z^2)(s_i^2 + z_E^2)\right\}^{-\frac{1}{2}} \left\{(s_i^2 + z^2)^{\frac{1}{2}} + (s_i^2 + z_E^2)^{\frac{1}{2}}\right\}^{-1}\right)^{-\frac{1}{2}}.$$

We note that $f(z_E u, z_E)$ is a increasing function with respect to z_E for $u \in [0, 1]$ fix. This implies item 2.

On the other hand

$$\lim_{z_E\to 0} f(z_E u, z_E) = \left(\sum_{i=1}^{n}\frac{m_i}{2s_i^3}\right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{z_E\to+\infty} f(z_E u, z_E) = +\infty.$$

Therefore, from the dominated convergence theorem and monotone convergence theorem we have that

$$\lim_{E\to E_{min}} T_0 = \lim_{z_E\to 0} T_0 = 2\pi\left(\sum_{i=1}^{n}\frac{m_i}{s_i^3}\right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{E\to 0} T_0 = \lim_{z_E\to+\infty} T_0 = +\infty.$$

Finally, since $T_0 = T_0(z_E)$ is continuous and increasing respect to z_E, we conclude the afirmation in the item 3. □

Remark 1. It si posible to use the classical theory of hamiltonian systems (see [2]) to derive the formula (12) (see [1] for this approach in a related problem).

Remark 2. Let us to show a second proof of item 3 of Theorem 5.3.

The inequality $T_0 > T_{min}$ is consequence of comparison Sturm's theorem applied to equations $\ddot{z} + h(z)z = 0$, where $h(z) = \sum_{i=1}^{n} m_i\left(s_i^2+z^2\right)^{-3/2}$, and $\ddot{z} + \left(\sum_{i=1}^{n} m_i s_i^{-3}\right)z = 0$. This prove that $T_0\left((E_{min},0)\right) \subset (T_{min},+\infty)$.

For the reverse inclusion we follow arguments of [32] and [13], based on variational principles.

Let $T_0 > T_{min}$. We consider the action integral

$$\mathcal{I}(z) = \int_0^{T_0} \frac{1}{2}|\dot{z}|^2 + \sum_{i=1}^{n}\frac{m_i}{\sqrt{s_i^2+z^2}}dt,$$

Then T_0-periodic solutions of (10) are critical points of $\mathcal{I}$ in the space $H^1(\mathbb{T},\mathbb{R})$, where $\mathbb{T} = \mathbb{R}/T_0\mathbb{Z}$, of the functions absolutely continuous, T_0-periodic with $\dot{z} \in L^2(\mathbb{T},\mathbb{R})$ (see [18, Cor. 1.1]). We prove the existence of critical points by means of the direct method of calculus of variations, i.e. we will prove that $\mathcal{I}$ has minimum. The functional $\mathcal{I}$ is not coercive in $H^1(\mathbb{T},\mathbb{R})$. This deficiency is drawn with symmetry techniques (see [31]). The group $\mathbb{Z}_2$ acts on $H^1(\mathbb{T},\mathbb{R})$ according to the following assignments $(\bar{0}\cdot z)(t) = z(t)$ and $(\bar{1}\cdot z)(t) = -z(t+\frac{T_0}{2})$. The functional $\mathcal{I}$ is $\mathbb{Z}_2$-invariant, i.e. $\mathcal{I}(g\cdot z) = \mathcal{I}(z)$. We define the space of all $\mathbb{Z}_2$-symmetric (this simmetry is called the italian simmetry) funcions

$$\Lambda(\mathbb{T},\ mathbbR) := \left\{z \in H^1(\mathbb{T},\mathbb{R}) | \forall g \in \mathbb{Z}_2 : z = g\cdot z\right\}.$$

The funciontal $\mathcal{I}$ restricted to Λ is coercive. This fact follows from an obvious adaptation of Proposition 4.1 of [31]. We note that $F(z) := \sum_{i=1}^{n} m_i(s_i^2+z^2)^{-\frac{1}{2}}$ satisfies the condition (A) in [18, p. 12], then $\mathcal{I}$ is continuously differentiable and weakly lower semicontinuous on $H^1(\mathbb{T},\mathbb{R})$ (see [18, p. 13]). Therefore $\mathcal{I}$ has a minimum z_0 in $\Lambda(\mathbb{T},\mathbb{R})$. Then by the Palais' principle symmetric criticality, z_0 is a critical point of $\mathcal{I}$ in $H^1(\mathbb{T},\mathbb{R})$ (see [31] and [24]).

We use the second variation $\delta^2\mathcal{I}$ in order to show that $z_0 \not\equiv 0$. It is well known (see [11, Th. 1.3.1]) that if z_0 is a minimum of $\mathcal{I}$ on $H^1(\mathbb{T},\mathbb{R})$ then $\delta^2\mathcal{I}(z_0,\varphi) \geq 0$ for all $\varphi \in H^1(\mathbb{T},\mathbb{R})$. In our case

$$\delta^2\mathcal{I}(0,\varphi) = \int_0^{T_0} |\dot{\varphi}|^2 - \sum_{i=1}^{n} \frac{m_i}{s_i^3}\varphi^2 dt,$$

(see [11, Eq. 1.3.6]). In particular for $\varphi(t) = \sin(2\pi t/T_0)$ it follows from $T_0 > T_{min}$ that

$$\delta^2\mathcal{I}(0,\varphi) = \left(\frac{4\pi^2}{T_0^2} - \sum_{i=1}^{n}\frac{m_i}{s_i^3}\right)\frac{T_0}{2} < 0. \tag{13}$$

It is sufficient to guarantee that $z_0 \equiv 0$ is not a minimum.

This second proof, unlike the first one, does not prove that T_0 is the minimum period for z_0. It could happen that z_0 had period T_0/m, with natural $m \in \mathbb{N}$. Because of Italian symmetry this m should be odd.

Corollary 5.4. *The complete $n+1$-body system has a infinity quantity of periodic solutions.*

Proof. Let l/m be a positive rational number with $Tl/m > T_{min}$. Then, there exists a solution of the entire system with period lT. □

6 Synchronous solutions and pyramidal CC

If the equation (10) has a T-periodic solution, we say that the solution is *synchronous*. In [13] was studied the existence of synchronous solutions for n equal mass primary bodies in a regular polygon configuration.

In this section we establish a relation between the existence of synchronous solutions and the concept of pyramidal central configuration (see [8, 9, 23]).

Definition 6.1. *A central configuration of $n+1$ mass point $q_0,\dots,q_n$ in $\mathbb{R}^3$ is called a pyramidal central configuration (PCC) if and only if n points, we say $q_1,\dots,q_n$, are in some plane Π and $q_0 \notin \Pi$.*

The following lemma was proved in [23] (see also [9]).

Lemma 6.2 ([23], Lemma 2.1)**.** *Let $q_0,\dots,q_n$ be a PCC such that m_0 is off the plane containing $m_1,\dots,m_n$. If $m_0 > 0$ then m_0 is equidistant from $m_1,\dots,m_n$.*

We remark that the condition $m_0 > 0$ is important in the previous Lemma. We will show below examples of two PCC with $m_0 = 0$ which do not satisfy the conclusion of Lemma 6.2.

Proposition 6.3. *We assume that $q = q_1,\dots,q_n$ is a balanced collisionless configuration and that the primaries are in a rigid motion. Then, there is a synchronous solution if and only if there exists $c \in \mathbb{R}$ such that the points $q_0,\dots,q_n$ associated to the masses $m_0,\dots,m_n$, with $q_0 = (0,0,c)$ and $m_0 = 0$, form a PCC.*

Proof. We start assuming that there exist a synchronous solution. As a consequence of the Theorem 5.3(3) and the fact that $T^2 = 4\pi^2/\lambda$ we have that

$$\lambda < \sum_{i=1}^{n} \frac{m_i}{s_i^3}. \tag{14}$$

Since $\sum_{i=1}^{n} m_i \left(s_i^2 + c^2\right)^{-3/2} \to 0$, when $c \to +\infty$, there exists $c \in \mathbb{R}$ such that $\sum_{i=1}^{n} m_i \left(s_i^2 + c^2\right)^{-3/2} = \lambda$. Therefore

$$-\sum_{i=1}^{n} \frac{m_i c}{\left(s_i^2 + c^2\right)^{3/2}} = -\lambda c. \tag{15}$$

As $q_1, \dots, q_n$ is a balanced configuration then

$$\sum_{i=1}^{n} \frac{m_i q_i}{\left(s_i^2 + c^2\right)^{3/2}} = (0, 0). \tag{16}$$

We define $q_0 := (0, 0, c)$ and $m_0 = 0$. The equations (15), (16) and the facts that $q_1, \dots, q_n$ is a CC, with constant λ, implies that $q_0, \dots, q_n$ and $m_0, \dots, m_n$ form a PCC.

The proof of the reciprocal statement follows in a direct way. $\square$

Corollary 6.4. *We assume that (q, m) is a balanced collisionless configuration and that the primaries are in a rigid motion. Then, there is a synchronous solution if and only if*

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} < \left(\sum_{i=1}^{n} \frac{m_i}{s_i^3}\right)\left(\sum_{i=1}^{n} m_i s_i^2\right). \tag{17}$$

Proof. The result is consequence of (14) and the fact that $T^2 = 4\pi^2 \sum_{i=1}^{n} m_i |q_i|^2 / U$ (see [14, p. 109]). $\square$

Remark 3. We observe that if (q, m) is a balanced CC with constant $\lambda > 0$, which satisfies (17) and if $r, \mu > 0$ then $(rq, \mu m)$ is a CC with constant $\lambda \mu r^3$, and (17) remains unchanged for the new positions rq and the new masses μm. Consequently we can assume that a given length has any desired value. Then the equation (10) has a synchronous solutions if and only if the same equation, with $(rq, \mu m)$ instead (q, m), has a synchronous solution.

The sufficiency of the condition $n \le 472$ in the following corollary was proved in [13].

Corollary 6.5. *We suppose that q is the equal masses regular polygon configuration (this is an balanced CC). Then there exists a synchronous solution if and only if* $2 \le n \le 472$.

Proof. In this case $s_1 = s_2 = \cdots = s_n =: r$ and $m_1 = m_2 = \cdots = m_n =: m$. Then, from the law of cosines we obtain

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} = \frac{nm^2}{4r} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)}.$$

Therefore the condition (17) is equivalent to

$$\frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)} < 4. \tag{18}$$

This inequality was also derived by Li, J. et al. in [13], where the authors proved (performing computer calculations) that inequality (18) holds true for $2 \le n \le 472$. Let us prove that any other n does not satisfies (18).

Using that $1/\sin(x)$ is a convex function on $[0,\pi]$ and the composite trapezoid rule (see [12]) we have that

$$\begin{aligned} \int_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} \frac{1}{\sin(x)} dx &\le \frac{\pi}{2n} \left\{ \frac{1}{\sin(\frac{\pi}{n})} + \frac{1}{\sin(\frac{n-1}{n}\pi)} + 2 \sum_{j=2}^{n-2} \frac{1}{\sin(j\frac{\pi}{n})} \right\} \\ &= \frac{\pi}{n} \sum_{j=1}^{n-2} \frac{1}{\sin(j\frac{\pi}{n})}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)} &\ge \frac{1}{\pi} \int_{\frac{\pi}{n}}^{\frac{\pi(n-1)}{n}} \frac{1}{\sin(x)} dx + \frac{1}{n \sin\left(\frac{n-1}{n}\pi\right)} \\ &= \frac{1}{2\pi} \log\left(\frac{1-\cos(x)}{1+\cos(x)} \right)\Bigg|_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} + \frac{1}{n \sin\left(\frac{\pi}{n}\right)} \\ &= \frac{1}{\pi} \left\{ \log\left(\frac{1+\cos(\frac{\pi}{n})}{1-\cos(\frac{\pi}{n})} \right) + \frac{\pi/n}{\sin\left(\frac{\pi}{n}\right)} \right\} \\ &=: f\left(\frac{\pi}{n}\right). \end{aligned}$$

It is easy to see that $f(x)$ is a decreasing function on $(0,\pi/2)$. Moreover $f(\pi/842) \approx 4.0006 > 4$. Therefore, if $n \ge 842$ then n does not satisfy inequality (18). The validity of the inequality (18), for $n \le 841$, is easily checked using computer. This gives the result that the inequality holds only for $n \le 472$. □

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Our next objective is to verify that condition (17) is satisfied for all balanced CC of 3-body or 4-body. Since we have already proved, in Corollary 6.5, that (17) holds for a equilateral triangle and square configurations of equal masses bodies, it only rest to prove, in virtue of Theorem 4.3, the following result.

Theorem 6.6. *The central configurations CCcl and CCr satisfy condition* (17).

Proof. Let's start by analyzing the central configuration CCr. From the Remark 3, we can suppose without loss of generality that $q_1 = -q_3 = (0, y)$ for $0 < y < 1$, $q_2 = -q_4 = (1, 0)$. The condition (17) becomes

$$\frac{m_1^2}{2y} + \frac{4m_1m_2}{\sqrt{1+y^2}} + \frac{m_2^2}{2} < \left(\frac{2m_1}{y^3} + 2m_2\right)\left(2m_1y^2 + 2m_2\right).$$

As $m_1^2/(2y) < 4m_1^2/y$, $m_2^2/2 < 4m_2^2$ and $4m_1m_2/\sqrt{1+y^2} < 4m_1m_2/y^3$ (since $y < 1$), we have that the inequality holds.

Now we consider the central configuration CCl. From Remark 3 again, we can suppose that $q_1 = -q_3 = 1$, $q_2 = -q_4 = x$ with $0 < x < 1$, and $m_1 = m_3 = \mu$, $m_2 = m_4 = 1 - \mu$, with $0 < \mu < 1$. Then the inequality (17) becomes

$$\frac{2\mu(1-\mu)}{1-x} + \frac{2\mu(1-\mu)}{1+x} + \frac{\mu^2}{2} + \frac{(1-\mu)^2}{2x} < 4\mu^2 + 4\mu(1-\mu)x^2 + \frac{4\mu(1-\mu)}{x^3} + \frac{4(1-\mu)^2}{x}.$$

As $\mu^2/2 < 4\mu^2$ and $(1-\mu)^2/(2x) < 4(1-\mu)^2/x$ it is sufficient to show that

$$\frac{2\mu(1-\mu)}{1-x} + \frac{2\mu(1-\mu)}{1+x} < \frac{4\mu(1-\mu)}{x^3},$$

and this is equivalent to see that

$$\frac{x^3}{1-x^2} < 1. \tag{19}$$

The values of x involved in the above inequality are such that the configuration of positions $(-1, -x, x, 1)$ and masses $(\mu, 1-\mu, 1-\mu, \mu)$ is central. By Moulton [22], fixed a mass μ there is only one value of x satisfiying this condition. So, we can define $x(\mu)$ as such value of x. We note that $h(x) = x^3/(1-x^2)$ is an increasing function with respect to $x \in (0,1)$ and $h(x) \le 1$ for $x \in (0, 3/4)$. Hence, if we could prove that $x(\mu)$ is a decreasing function and

$$\lim_{\mu \to 0} x(\mu) < 3/4 \tag{20}$$

we would have justified (19).

Let's first prove that $x(\mu)$ is a decreasing function. Eliminating λ from the equations (2) and replacing q_j and m_j by their expressions in x and μ we get

$$\frac{\mu}{4} - \frac{\mu}{x\left(x+1\right)^2} + \frac{\mu}{x\left(-x+1\right)^2} + \frac{-\mu+1}{\left(x+1\right)^2} + \frac{-\mu+1}{\left(-x+1\right)^2} - \frac{1}{x^3}\left(-\frac{\mu}{4} + \frac{1}{4}\right) = 0$$

which is equivalent to

$$\mu = -\frac{8x^5 - x^4 + 8x^3 + 2x^2 - 1}{(x-1)(x+1)(x^5 - 9x^3 + x^2 - 1)}.$$

Therefore

$$\frac{d\mu}{dx} = \frac{x^2\left(16x^9 - 3x^8 + 32x^7 + 12x^6 - 304x^5 - 2x^4 + 44x^2 - 51\right)}{\left(x-1\right)^2\left(x+1\right)^2\left(x^5 - 9x^3 + x^2 - 1\right)^2}.$$

Since $44x^2 < 51$ and $16x^9 + 32x^7 + 12x^6 < 304x^5$ for $x \in (0,1)$ then $d\mu/dx < 0$ in the interval $(0,1)$. Which, in turn, implies that x is decreasing respect to μ.

Let's see now that (20) holds. When μ goes to 0, $x(\mu)$ converges to the only solution, $x(0)$, in the interval $(0,1)$, of the equation $8x(0)^5 - x(0)^4 + 8x(0)^3 + 2x(0)^2 - 1 = 0$. Then $8x(0)^3 - 1 < 0$ which implies that $x(0) < 3/4$ as we wanted to prove. □

Remark 4. As consequence of previous results there exist five-body $PCC's$ with $m_1, \ldots, m_4$ in a CCcl or CCr configuration and the mass $m_0 = 0$ is in the line perpendicular to the plane containing $m_1, \ldots, m_4$ and passing by the center of mass. These are examples of $PCC's$ wich does not verify the conclusion of Lemma 6.2.

Corollary 6.7. *For all balanced CC of 3-body or 4-body, the problem P has a synchronous solution.*

7 Non balanced central configurations

The following result shows a necessary condition for that a non-balanced CC allows a solution of the problem P.

Theorem 7.1. *We suppose that (q,m) is a non balanced CC and that the primaries are in a homographic motion, i.e. equation (1) is satisfied. Assume that the massless particle is moving on the z-axis with position vector $x_0(t) = (0,0,z(t))$. Then, some of the following statements are satisfied:*

1. *The massless particle is in a stationary motion and*

$$\sum_{i=1}^{n} \frac{m_i q_i}{s_i^3} = 0, \tag{21}$$

i.e. the positions $0, q_1, \ldots, q_n$ and the masses $0, m_1, \ldots, m_n$ are in a CC.

2. *The $n+1$-body system is in a homothetic motion. i.e. $Q(\theta(t))$ in the equation (1) is the identity matrix and $z(t) = cr(t)$, for some constant c. Moreover, the configuration $q_0, \ldots, q_n$ is a PCC, where $q_0 = (0,0,c)$ and $m_0 = 0$.*

Proof. We recall the definition of the function f and line L from the Section 3.

The fact that the massless particle is moving on L, is equivalent to the condition $f(t, x_0(t)) \in L$ for all t, which, instead, is equivalent to the equality

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{\left(r(t)^2 |q_i|^2 + z(t)^2\right)^{3/2}} = 0, \tag{22}$$

for every $t \in \mathbb{R}$.

With the same notation and following similar reasoning that in the proof of Theorem 3.2, we prove that

$$\sum_{j=1}^{k}\left\{\frac{1}{(s_j^2+(z(t)/r(t))^2)^{3/2}}\sum_{i\in F_j} m_i q_i\right\}=0. \tag{23}$$

If $z(t)/r(t)$ would be a non constant function then previous equation and Lemma 3.3 would imply that q is balanced, which is a contradiction. Hence there exist $c\in\mathbb{R}$ such that $z(t)=cr(t)$. Now, we have two cases.

Case 1: $c=0$. Then $z\equiv 0$ and (21) follows from (22).

Case 2: $c\neq 0$. From equation (10), the Kepler equations (4), and the fact that $z(t)=cr(t)$ we have that

$$-\frac{1}{r(t)^2}\sum_{i=1}^{n}\frac{m_i}{(s_i^2+c^2)^{3/2}}=-\frac{\lambda}{r(t)^2}+r(t)\dot{\theta}(t)^2. \tag{24}$$

The second equality in (4) implies the Kepler's second law, i.e. there exists $d\in\mathbb{R}$ such that $r^2\dot{\theta}\equiv d$. Replacing $\dot{\theta}$ in equation (24) and multiplying by $r(t)^3$ we obtain

$$-r(t)\left(\sum_{i=1}^{n}\frac{m_i}{(s_i^2+c^2)^{3/2}}+\lambda\right)=d^2. \tag{25}$$

Therefore, if $d\neq 0$ then $\dot{r}(t)\equiv 0$, and this implies $\dot{z}(t)\equiv 0$. As $z(t)$ is a constant function and it solves equation (10), then $z(t)\equiv 0$. Hence we are in the case 1 again. Consequently we suppose $d=0$. Therefore $\theta(t)\equiv cte$ and the motion is homothetic. From (23) and (25) we deduce that in this new situation equation (15) and (16) hold. This, as in the proof of Proposition 6.3, implies the desired result. □

Example 1. We present an example of a $3+1$-body system satisfiying the situation described in the item 1 of the Theorem 7.1, i.e. q is a non balanced CC and $z(t)\equiv 0$. For this, it is sufficient to find a 4-body CC with a zero mass body located in the center of mass.

We start with a Euler's collinear central configuration formed by three primary bodies of masses $m_1=4-\mu$, $m_2=2+\mu$ and $m_3=1$, where $0<\mu<1$, and positions, respect to a convenient 1-dimensional coordinate system, given by $q_1=0$, $q_2=1$ and $q_3=1+r$. It is know (see [19]) that r is the only positive solution of

$$p(r,\mu):=6r^5+(16-\mu)\,r^4+(14-2\mu)\,r^3-(\mu+5)\,r^2-(2\mu+7)\,r-\mu-3=0.$$

Since $p(0,\mu)=-\mu-3$ and $p(1,\mu)=-7\mu+21$ then $r\in(0,1)$, for all $0<\mu<1$. In this case the center of mass C is equal to $(\mu+r+3)/7$, so $C\in(0,1)$.

We consider a massless particle with coordinate x. The acceleration resulting from the action of the gravitational field is equal to

$$f(x)=-\frac{4-\mu}{x^2}+\frac{\mu+2}{(-x+1)^2}+\frac{1}{(r-x+1)^2}.$$

Note that the left hand side of the previous equation is an increasing function that tends to $-\infty$ when x goes to 0, and tends to $+\infty$ when x goes to 1, so there is a unique point $\bar{x} \in (0,1)$ such that the equality $f(\bar{x}) = 0$. This point is an equilibrium for the gravitational field generated for the primaries.

Now if we want to have a trivial solution of the problem (5) then necessarily C has to be equal to x. Let's see that there exists $\mu \in (0,1)$ such that $C = x$, i.e. $f(C) = 0$. For this purpose, since C is a continuous function with respect to μ, we need to see that there exists a value $\mu_1 \in (0,1)$ such that $f(C) < 0$ and $\mu_2 \in (0,1)$ such that $f(C) > 0$. The function $f(x)$ can be factorized as

$$f(x) = \frac{Nf(x)}{Df(x)},$$

where $Nf(x) = 2\mu r^2x^2 - 2\mu r^2 x + \mu r^2 - 4\mu r x^3 + 8\mu r x^2 - 6\mu r x + 2\mu r + 2\mu x^4 - 6\mu x^3 + 7\mu x^2 - 4\mu x + \mu - 2r^2x^2 + 8r^2x - 4r^2 + 4rx^3 - 20rx^2 + 24rx - 8r - x^4 + 10x^3 - 21x^2 + 16x - 4$ and $Df(x) = x^2(x-1)^2(r-x+1)^2$. Note that $Df(x) > 0$ for all $x \in (0,1)$. If we consider $\mu = 0$ and compute $Nf(C)$ we have that

$$Nf(C) = \frac{r^4}{2401} + \frac{1514r^3}{2401} + \frac{2245r^2}{2401} + \frac{1110r}{2401} + \frac{333}{2401} > 0,$$

on the other, hand if $\mu = 1$ then

$$Nf(C) = -\frac{71r^4}{2401} + \frac{1486r^3}{2401} + \frac{401r^2}{2401} - \frac{1480r}{2401} - \frac{592}{2401} < 0,$$

because $0 < r < 1$.

Remark 5. The following question is posed. Is there some non balanced central configuration q such that the $n+1$-body system perform the motion described in Theorem 7.1(2)?

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