

# Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

Sonia Acinas \*

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales  
Universidad Nacional de La Pampa  
(L6300CLB) Santa Rosa, La Pampa, Argentina  
sonia.acinas@gmail.com

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales  
Universidad Nacional de Río Cuarto  
(5800) Río Cuarto, Córdoba, Argentina,  
fmazzone@exa.unrc.edu.ar

## Abstract

## 1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt} \nabla_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P)$$

where the function  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 1$  (called the *Lagrange function* or *lagrangian*) satisfies that it is measurable in  $t$  for each  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$  and continuously differentiable in  $(x, y)$  for almost every  $t \in [0, T]$ . The unknown function  $u : [0, T] \rightarrow \mathbb{R}^d$  is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the article [17] for definitions and main results on anisotropic Orlicz spaces, see also [2]. These spaces allow us to unify and extend previous results on existence of solutions for systems like  $(P)$ .

---

\*SECyT-UNRC and FCEyN-UNLPam

†SECyT-UNRC, FCEyN-UNLPam and CONICET

**2010 AMS Subject Classification.** Primary: . Secondary: .

**Keywords and phrases.** .

Through this article we say that a function  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  belongs to  $N_\infty$  class if  $\Phi$  is convex,  $\Phi(0) = 0$ ,  $\Phi(y) > 0$  if  $y \neq 0$  and  $\Phi(-y) = \Phi(y)$ , and

$$\lim_{|y| \rightarrow \infty} \frac{\Phi(y)}{|y|} = +\infty. \quad (1)$$

where  $|\cdot|$  denotes the euclidean norm on  $\mathbb{R}^d$ . From [3, Cor. 2.35] an  $N_\infty$  function is continuous.

Associated to  $\Phi$  we have the *complementary function*  $\Psi$  which is defined in  $\xi \in \mathbb{R}^d$  as

$$\Psi(\xi) = \sup_{y \in \mathbb{R}^d} y \cdot \xi - \Phi(y), \quad (2)$$

then, from the continuity of  $\Phi$  and (1), we have that  $\Psi : \mathbb{R}^d \rightarrow [0, \infty)$ . It is easy to see that  $\Psi$  is a convex function such that  $\Psi(0) = 0$ ,  $\Psi(-\xi) = \Psi(\xi)$  [8, Ch. 2]. Moreover,  $\Psi$  satisfies (1) (see [17, Thm. 2.2]), i.e.  $\Psi$  is an  $N_\infty$  function.

Some examples of  $N_\infty$  functions are the following.

*Example 1.1.*  $\Phi_p(y) := |y|^p/p$ , for  $1 < p < \infty$ . In this case  $\Psi(\xi) = |\xi|^q/q$ ,  $q = p/(p-1)$ .

*Example 1.2.* If  $\Phi : \mathbb{R} \rightarrow [0, +\infty)$  is a  $N_\infty$  function on  $\mathbb{R}$  then  $\Phi(y) = \Phi(|y|)$  is a  $N_\infty$  function on  $\mathbb{R}^d$ . In this example, as in the previous one, the function  $\Phi$  is *radial*, i.e. the value of  $\Phi(y)$  depends on the norm of  $y$  and not on its direction. These cases are not authentically anisotropic.

*Example 1.3.* An anisotropic function  $\Phi(y)$  depends on the direction of  $y$ . For example, if  $1 < p_1, p_2 < \infty$ , we define  $\Phi_{p_1, p_2} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty)$  by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then  $\Phi_{p_1, p_2}$  is an  $N_\infty$  function. In this case the complementary function is  $\Phi_{q_1, q_2}$  with  $q_i = p_i/(p_i - 1)$ .

More generally, if  $\Phi_k : \mathbb{R}^d \rightarrow [0, +\infty)$ ,  $k = 1, \dots, n$ , are  $N_\infty$  functions, then  $\Phi : \mathbb{R}^d \times \dots \times \mathbb{R}^d \rightarrow [0, +\infty)$  defined by  $\Phi(y_1, \dots, y_n) = \Phi_1(y_1) + \dots + \Phi_n(y_n)$  is an  $N_\infty$  function. These functions are truly anisotropic, i.e.  $|x| = |y|$  does not imply that  $\Phi(x) = \Phi(y)$ .

*Example 1.4.* If  $\Phi : \mathbb{R} \rightarrow [0, +\infty)$  is an  $N_\infty$  function and  $O \in GL(d, \mathbb{R})$ , then  $\Phi(y) = \Phi(Oy)$  is an  $N_\infty$  function.

*Example 1.5.* An anisotropic  $N_\infty$  function is not necessarily controlled by power functions if it does not satisfy the  $\Delta_2$  condition (see xxxxx). For example  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  defined by  $\Phi(y) = \exp(|y|) - 1$  is an  $N_\infty$  function.

The appearance of Orlicz Spaces in this paper is due to the fact that we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi(\nabla_y \mathcal{L}) \leq a(x) \left\{ b(t) + \Phi\left(\frac{y}{\lambda}\right) \right\}, \quad (S)$$

for a.e.  $t \in [0, T]$ , where  $a \in C(\mathbb{R}^d, [0, +\infty))$ ,  $b \in L^1([0, T], [0, +\infty))$ .

Our condition (S) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when  $\Phi(x)$  is as in Example

1.1, then the condition  $(S)$  is equivalent to the structure condition in [8, Th. 1.4]. If  $\Phi$  is a radial  $N_\infty$  function such that  $\Psi$  satisfies that  $\Delta_2$  function then  $(S)$  is essentially equivalent to conditions [1, Eq. (2)-(4)] (see xxxx mas abajo). If  $\Phi$  is as in Example 1.3 and  $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$  is a lagrangian with  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  then inequality  $(S)$  is related to estructure conditions like [23, Lemma 3.1, Eq. (3.1)]. As can be seen, condition  $(S)$  is a more compact expression than [23, Lemma 3.1, Eq. (3.1)] and moreover weaker, because  $(S)$  does not imply a control of  $|D_{y_1} L|$  independent of  $y_2$ . We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi, F}(t, x, y) := \Phi(y) + F(t, x). \quad (3)$$

Here the function  $F(t, x)$ , which is often referred to a potential, be differentiable with respect to  $x$  for a.e.  $t \in [0, T]$ . Moreover  $F$  satisfies the following conditions:

(C)  $F$  and its gradient  $\nabla_x F$ , with respect to  $x \in \mathbb{R}^d$ , are Carathéodory functions, i.e. they are measurable functions with respect to  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ , and they are continuous functions with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

(A) For a.e.  $t \in [0, T]$ , it holds that

$$|F(t, x)| + |\nabla_x F(t, x)| \leq a(x)b(t). \quad (4)$$

where  $a \in C(\mathbb{R}^d, [0, +\infty))$  and  $0 \leq b \in L^1([0, T], \mathbb{R})$ .

The lagrangian  $\mathcal{L}_{\Phi, F}$  satisfies condition  $(S)$ . In order to prove this, the only non trivial fact that we should establish is that  $\Psi(\nabla_y \mathcal{L}) \leq a(x) \{b(t) + \Phi(y/\lambda)\}$ . But, from inequality xxxx below,  $\Psi(\nabla_y \mathcal{L}) = \Psi(\nabla \Phi(y)) \leq \Phi(2y)$ .

The laplacian  $\mathcal{L}_{\Phi, F}$  leads to the system

$$\begin{cases} \frac{d}{dt} \nabla \Phi(u'(t)) = \nabla_x F(t, u(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_\Phi)$$

Problem  $(P_\Phi)$  contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [8] deals mainly with problem  $(P)$  for the lagrangian  $\mathcal{L}_{\Phi, F}$  with  $\Phi(x) = |x|^2/2$ , through various methods: direct, dual action, minimax, etc. The results in [8] were extended and improved in several articles, see [21, 19, 24, 20, 27] to cite some examples. The case  $\Phi(y) = |y|^p/p$ , for arbitrary  $1 < p < \infty$  were considered in [23, 22], among other papers. In this case,  $(P_\Phi)$  is reduced to the  $p$ -laplacian system

$$\begin{cases} \frac{d}{dt} (u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P_p)$$

If  $\Phi$  is as in Example 1.3 and  $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a Carathéodory function, then the equations  $(P_\Phi)$  become

$$\begin{cases} \frac{d}{dt} (|u'_1|^{p_1-2} u'_1) = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt} (|u'_2|^{p_2-2} u'_2) = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_{p_1, p_2})$$

where  $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$  and  $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$ . In the literature, these equations are known as  $(p_1, p_2)$ -Laplacian system, see [26, 12, 25, 9, 10, 11, 6].

In conclusion, the problem  $(P)$  subject to conditions  $(S)$  contains several problems that have been considered by many authors in the past.

## 2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic  $N_\infty$  functions  $\Phi : \mathbb{R}^n \rightarrow [0, +\infty)$ . References for these topics are [4, 17, 18].

We say that  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  satisfies the  $\Delta_2^\infty$ -condition, denoted by  $\Phi \in \Delta_2^\infty$ , if there exist constants  $K > 0$  and  $M \geq 0$  such that

$$\Phi(2x) \leq K\Phi(x), \quad (5)$$

for every  $|x| \geq M$ .

We denote by  $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$ , with  $d \geq 1$ , the set of all measurable functions (i.e. functions which are limits of simple functions) defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $u = (u_1, \dots, u_d)$  for  $u \in \mathcal{M}$ .

Given an  $N_\infty$  function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Now, we introduce the *Orlicz class*  $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$  by setting

$$C^\Phi := \{u \in \mathcal{M} \mid \rho_\Phi(u) < \infty\}. \quad (6)$$

The *Orlicz space*  $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$  is the linear hull of  $C^\Phi$ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} \mid \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (7)$$

The Orlicz space  $L^\Phi$  equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left( \frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space.

The subspace  $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$  is defined as the closure in  $L^\Phi$  of the subspace  $L^\infty([0, T], \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that (see [17, Thm. 5.1])  $u \in E^\Phi$  if and only if  $\rho_\Phi(\lambda u) < \infty$  for any  $\lambda > 0$ . The equality  $L^\Phi = E^\Phi$  is true if and only if  $\Phi \in \Delta_2^\infty$  (see [17, Thm. 5.2]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [17, Thm. 7.2]). Namely, if  $u \in L^\Phi$  and  $v \in L^\Psi$  then  $u \cdot v \in L^1$  and

$$\int_0^T v \cdot u dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \quad (8)$$

By  $u \cdot v$  we denote the usual dot product in  $\mathbb{R}^d$  between  $u$  and  $v$ .

We consider the subset  $\Pi(E^\Phi, r)$  of  $L^\Phi$  given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi \mid d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class  $C^\Phi$  by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (9)$$

for any positive  $r$ . This relation is a trivial generalization of [17, Thm. 5.6]. If  $\Phi \in \Delta_2^\infty$ , then the sets  $L^\Phi$ ,  $E^\Phi$ ,  $\Pi(E^\Phi, r)$  and  $C^\Phi$  are equal.

As usual, if  $(X, \|\cdot\|_X)$  is a normed space and  $(Y, \|\cdot\|_Y)$  is a linear subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when there exists  $C > 0$  such that  $\|y\|_X \leq C\|y\|_Y$  for any  $y \in Y$ . With this notation, Hölder's inequality states that  $L^\Phi \hookrightarrow [L^\Psi]^*$ , where a function  $v \in L^\Phi$  is associated to  $\xi_v \in [L^\Psi]^*$  being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \quad (10)$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [17].

**Proposition 2.1.** *If  $\Psi$  satisfies the  $\Delta_2^\infty$ -condition then  $L^\Phi([0, T], \mathbb{R}^d) = [L^\Psi([0, T], \mathbb{R}^d)]^*$ .*

We define the *Sobolev-Orlicz space*  $W^1 L^\Phi$  by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u \mid u \in AC([0, T], \mathbb{R}^d) \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\},$$

where  $AC([0, T], \mathbb{R}^d)$  denotes the space of all  $\mathbb{R}^d$  valued absolutely continuous functions defined on  $[0, T]$ . The space  $W^1 L^\Phi([0, T], \mathbb{R}^d)$  is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (11)$$

We introduce the following subspaces of  $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi \mid u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi \mid u(0) = u(T)\}. \end{aligned} \quad (12)$$

In order to find a modulus of continuity for functions in  $W^1 L^\Phi$ , and from there, to obtain compact embedding of  $W^1 L^\Phi$ , we define the function  $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by

$$A_\Phi(s) = \min \{\Phi(x) \mid |x| = s\}, \quad (13)$$

Let us establish some elementary properties of  $A_\Phi$  that we will use in this article.

**Proposition 2.2.** *The function  $A_\Phi$  has the following properties:*

1.  $A_\Phi$  is continuous,
2.  $A_\Phi(s)/s$  is increasing.

3.  $A_\Phi(|x|)$  is the greatest radial minorant of  $\Phi(x)$ ,
4.  $\Phi$  is  $N_\infty$  if and only if  $\lim_{s \rightarrow +\infty} A_\Phi(s)/s = +\infty$ .

*Proof.* It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [3]). This fact implies item 1 immediately.

In order to prove item 2, suppose  $0 < r < s$  and  $x \in \mathbb{R}^d$  with  $A_\Phi(s) = \Phi(x)$ . Then, from the definition of  $A_\Phi$  and the convexity of  $\Phi$ ,

$$\frac{A_\Phi(r)}{r} \leq \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leq \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily. □

*Example 2.1.* We compute  $A_\Phi$  for the function  $\Phi = \Phi_{p_1, p_2}$  given in Example (1.3). We apply the method of Lagrange multipliers (see [7, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1, p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2}y_1 + \lambda y_1 &= 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 &= 0 \\ |y_1|^2 + |y_2|^2 &= r^2 \end{cases} \quad (14)$$

These equations are solved, among others, by the following two sets of critical points: a)  $|x| = r$ ,  $y = 0$  and  $\lambda = -r^{p_1-2}$  and b)  $x = 0$ ,  $|y| = r$  and  $\lambda = -r^{p_2-2}$ . These sets are infinite when  $d > 1$ . Associated with these critical points we have the following critical values: a)  $r^{p_1}/p_1$  and b)  $r^{p_2}/p_2$ .

We deal with  $p_1 \leq 2$  and  $p_2 \leq 2$  being one of them (suppose  $p_2$ ) different from 2. The remaining cases can be treated with similar techniques.

If  $(y_1, y_2)$  solve (14) with  $y_1 \neq 0$  and  $y_2 \neq 0$  then  $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$  and  $\lambda = -|y_1|^{p_1-2}$ . We use second order conditions for constrained problems. We have to consider the tangent plane at the point  $(y_1, y_2) \in \mathbb{R}^{2n}$ , i.e.  $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$ . Let  $L$  be the Lagrangian associated to the constrained problem:  $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) + \lambda H(y_1, y_2)$  being  $H = 0$  the constraint. We must analyze the positivity of the quadratic form associated to the matrix of second partial derivatives  $\mathcal{H} = D^2\Phi + \lambda D^2H$  on the subspace  $M$ . By elementary computations we have for  $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace  $M$ . We note that  $(-y_2, y_1) \in M$  and  $(-y_2, y_1)^t \mathcal{H}(-y_2, y_1) < 0$ . Then, by second order necessary conditions [7, p.333], at  $(y_1, y_2)$  there cannot be a minimum. Therefore, the minima occur at  $y_1 = 0$  or  $y_2 = 0$ , then

$$A_\Phi(x, y) = \min\{r^{p_1}/p_1, r^{p_2}/p_2\}.$$

More generally, it holds that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leq A_\Phi \leq K_2 \min\{r^{p_1}, r^{p_2}\}$$

with  $K_1, K_2 > 0$ , for every  $1 < p_1, p_2 < \infty$ .

As is customary, we will use the decomposition  $u = \bar{u} + \tilde{u}$  for a function  $u \in L^1([0, T])$  where  $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$  and  $\tilde{u} = u - \bar{u}$ .

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

**Lemma 2.3.** *Let  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  be a Young's function and let  $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$ . Let  $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be the function defined by (13). Then*

1. *For every  $s, t \in [0, T]$ ,  $s \neq t$ ,*

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| A_\Phi^{-1} \left( \frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq A_\Phi^{-1} \left( \frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. *We have  $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$  and*

$$\|\tilde{u}\|_{L^\infty} \leq T A_\Phi^{-1} \left( \frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. *If  $\Phi$  is  $N_\infty$  then the space  $W^1 L^\Phi([0, T], \mathbb{R}^d)$  is compactly embedded in the space of continuous functions  $C([0, T], \mathbb{R}^d)$ .*

*Proof.* By the absolute continuity of  $u$ , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi \left( \frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|} \right) &\leq \Phi \left( \frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr \right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi \left( \frac{u'(r)}{\|u'\|_{L^\Phi}} \right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By Proposition 2.2(3) we have  $A_\Phi^{-1} \Phi(x) \geq |x|$ , therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq A_\Phi^{-1} \left( \frac{1}{|s - t|} \right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned} |u(t) - \bar{u}| &= \left| \frac{1}{T} \int_0^T u(t) - u(s) ds \right| \\ &\leq \frac{1}{T} \int_0^T |u(t) - u(s)| ds \\ &\leq \|u'\|_{L^\Phi} T A_\Phi^{-1} \left( \frac{1}{T} \right) \end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of  $\|u\|_{L^\Phi}$ , we obtain

$$\Phi \left( \frac{\bar{u}}{\|u\|_{L^\Phi}} \right) \leq \frac{1}{T} \int_0^T \Phi \left( \frac{u(s)}{\|u\|_{L^\Phi}} \right) ds \leq \frac{1}{T}$$

Then by Proposition 2.2(3)

$$|\bar{u}| \leq A_\Phi^{-1} \left( \frac{1}{T} \right) \|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq A_\Phi^{-1} \left( \frac{1}{T} \right) \|u\|_{L^\Phi} + T A_\Phi^{-1} \left( \frac{1}{T} \right) \|u'\|_{L^\Phi} \\ &\leq A_\Phi^{-1} \left( \frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \end{aligned}$$

In order to prove item 3, we take a bounded sequence  $u_n$  in  $W^1 L^\Phi([0, T], \mathbb{R}^d)$ . Since  $\Phi$  is  $N_\infty$ , from Proposition 2.2(4) we obtain  $s A_\Phi^{-1}(1/s) \rightarrow 0$  when  $s \rightarrow 0$ . Therefore (Morrey's inequality) implies that  $u_n$  are equicontinuous. Furthermore (??) implies that  $u_n$  is bounded in  $C([0, T], \mathbb{R}^d)$ . Therefore by the Arzela-Ascoli Theorem we obtain a subsequence  $n_k$  and  $u \in C([0, T], \mathbb{R}^d)$  with  $u_{n_k} \rightarrow u$  in  $C([0, T], \mathbb{R}^d)$ .  $\square$

QUIZAS LO QUE VIENE TENDRIA QUE IR CON LAS OTRAS DESIGUALDADES... O MODIFICAR EL LEMA ANTERIOR....

We get an anisotropic version of Sobolev-Wirtinger inequality, as follows.

**Proposition 2.4.** *If  $u \in W^1 L^\Phi$ , then*

$$\Phi(\tilde{u}(t)) \leq \int_0^T \Phi(\max\{1, T\} u'(r)) dr. \quad (15)$$



*Proof.* Writing  $\tilde{u}(t) = u(t) - \bar{u}$  and applying Jensen's inequality, we get

$$\begin{aligned}\Phi(\tilde{u}(t)) &= \Phi\left(\frac{1}{T} \int_0^T u(t) - u(s) ds\right) \\ &\leq \frac{1}{T} \int_0^T \Phi(u(t) - u(s)) ds \leq \frac{1}{T} \int_0^T \Phi\left(\int_s^t |t-s| u'(r) \frac{dr}{|t-s|}\right) ds \\ &\leq \frac{1}{T} \int_0^T \frac{1}{|t-s|} \int_s^t \Phi(|t-s| u'(r)) dr ds\end{aligned}$$

As  $\Phi$  is convex, we have

$$\frac{1}{|t-s|} \Phi(|t-s| u'(r)) \leq \begin{cases} \Phi(u'(r)) & \text{if } |t-s| \leq 1 \\ \Phi(T u'(r)) & \text{if } 1 < |t-s| < T \end{cases}$$

Then  $\frac{1}{|t-s|} \Phi(|t-s| u'(r)) \leq \Phi(\max\{1, T\} u'(r))$  and (15) follows.  $\square$

**Lemma 2.5.** *Let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence of functions in  $\Pi(E^\Phi, 1)$  converging to  $u \in \Pi(E^\Phi, 1)$  in the  $L^\Phi$ -norm. Then, there exist a subsequence  $u_{n_k}$  and a real valued function  $h \in L^1([0, T], \mathbb{R})$  such that  $u_{n_k} \rightarrow u$  a.e. and  $\Phi(u_{n_k}) \leq h$  a.e.*

*Proof.* Since  $d(u, E^\Phi) < 1$  and  $u_n$  converges to  $u$ , there exists  $u_0 \in E^\Phi$ , a subsequence of  $u_n$  (again denoted  $u_n$ ) and  $0 < r < 1$  such that  $d(u_n, u_0) < r$ . Let  $\lambda_0 \in (r, 1)$ . By extracting more subsequences, if necessary, we can assume that  $u_n \rightarrow u$  a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^\Phi} < \frac{1 - \lambda_0}{2^n}, \quad \text{for } n \geq 1.$$

We can assume  $\lambda_n > 0$  for every  $n = 0, \dots$

Let  $\lambda := 1 - \sum_{n=0}^\infty \lambda_n$  and define  $h : [0, T] \rightarrow \mathbb{R}$  by

$$h(x) = \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^\infty \lambda_n \Phi\left(\frac{u_{n+1} - u_n}{\lambda_n}\right). \quad (16)$$

Note that  $\sum_{n=0}^\infty \lambda_n + \lambda = 1$ , therefore for any  $n = 1, \dots$

$$\begin{aligned}\Phi(u_n) &= \Phi\left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j}\right) \\ &\leq \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1} \lambda_j \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) \leq h\end{aligned}$$

Since  $u_0 \in E^\Phi \subset C^\Phi$  and  $E^\Phi$  is a subspace we have that  $\Phi(u_0/\lambda) \in L^1([0, T], \mathbb{R})$ . On the other hand  $\|u_{n+1} - u_n\|_{L^\Phi} \leq \lambda_n$ , therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) dt \leq 1.$$

Then  $h \in L^1([0, T], \mathbb{R})$ .  $\square$

### 3 Differentiability Gateaux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateaux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a *Carathéodory function*, i.e.

- (C)  $f$  is measurable with respect to  $t \in [0, T]$  for every  $x \in \mathbb{R}^d$ , and  $f$  is a continuous function with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

**Definition 3.1.** For  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  we denote by  $\mathbf{f}$  the Nemytskii (o superposition) operator defined for functions  $u : [0, T] \rightarrow \mathbb{R}^d$  by

$$\mathbf{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [5] for scalar functions and [15, 14, 13] for the generalization to  $\mathbb{R}^d$ -valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

**Theorem 3.2.** We assume that  $f$  satisfies condition ((C)) and that  $\Phi_1, \Phi_2 : \mathbb{R}^d \rightarrow [0, +\infty)$  are anisotropic Young functions. Then

1. Measurability. The operator  $\mathbf{f}$  maps measurable function into measurable functions
2. Extensibility. If the operator  $\mathbf{f}$  acts from the ball  $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} \mid \|u\|_{L^{\Phi_1}} < r\}$  into the space  $L^{\Phi_2}$  or the space  $E^{\Phi_2}$  then  $\mathbf{f}$  can be extended from  $\Pi(E^{\Phi_1}, r)$  into space  $L^{\Phi_2}$  or  $E^{\Phi_2}$ , respectively.
3. Continuity. If the operator  $\mathbf{f}$  acts from  $\Pi(E^{\Phi_1}, r)$  into space  $E^{\Phi_2}$ , then  $\mathbf{f}$  is continuous.

Given a continuous function  $a \in C(\mathbb{R}^n, \mathbb{R}^+)$ , we define the composition operator  $a : \mathcal{M}_d \rightarrow \mathcal{M}_d$  by  $\mathbf{a}(u)(x) = a(u(x))$ .

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

**Lemma 3.3.** If  $a \in C(\mathbb{R}^d, \mathbb{R}^+)$  then  $\mathbf{a} : W^1 L^\Phi \rightarrow L^\infty([0, T])$  is bounded. More concretely, there exists a non decreasing function  $A : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\|\mathbf{a}(u)\|_{L^\infty([0, T])} \leq A(\|u\|_{W^1 L^\Phi})$ .

*Proof.* Let  $A \in C(\mathbb{R}^+, \mathbb{R}^+)$  be a non decreasing, continuous function defined by  $\alpha(s) := \sup_{\|x\| \leq s, x \in \mathbb{R}^d} |a(x)|$ . If  $u \in W^1 L^\Phi$  then, by Sobolev's inequality, for a.e.  $t \in [0, T]$

$$a(u(t)) \leq \alpha(\|u\|_{L^\infty}) \leq \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi}\right) =: A(\|u\|_{W^1 L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t)) dt. \quad (17)$$

**Theorem 3.4.** *Let  $\mathcal{L}$  be a differentiable Carathéodory function satisfying (S). Then the following statements hold:*

1. *The action integral given by (17) is finitely defined on  $\mathcal{E}^\Phi := W^1 L^\Phi \cap \{u | \dot{u} \in \Pi(E^\Phi, 1)\}$ .*
2. *The function  $I$  is Gâteaux differentiable on  $\mathcal{E}^\Phi$  and its derivative  $I'$  is demicontinuous from  $\mathcal{E}^\Phi$  into  $[W^1 L^\Phi]^*$ . Moreover,  $I'$  is given by the following expression*

$$\langle I'(u), v \rangle = \int_0^T \{D_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v}\} dt. \quad (18)$$

3. *If  $\Psi \in \Delta_2$  then  $I'$  is continuous from  $\mathcal{E}^\Phi$  into  $[W^1 L^\Phi]^*$  when both spaces are equipped with the strong topology.*

*Proof.* Let  $u \in \mathcal{E}^\Phi$ . As

$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \quad (19)$$

and (9), then  $\Phi(\dot{u}(t)) \in L^1$ . Now,

$$|\mathcal{L}(\cdot, u, \dot{u})| + |\nabla_x \mathcal{L}(\cdot, u, \dot{u})| + \Psi(\nabla_y \mathcal{L}(\cdot, u, \dot{u})) \leq A(\|u\|_{W^1 L^\Phi})(b + \Phi(\dot{u})) \in L^1, \quad (20)$$

by (S) and Lemma 3.3. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

*Step 1.* The non linear operator  $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$  is continuous from  $\mathcal{E}^\Phi$  into  $L^1([0, T])$  with the strong topology on both sets.

Let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence of functions in  $\mathcal{E}^\Phi$  and let  $u \in \mathcal{E}^\Phi$  such that  $u_n \rightarrow u$  in  $W^1 L^\Phi$ . By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leq T A_\Phi^{-1} \left( \frac{1}{T} \right) \|u_n - u\|_{L^\Phi}$$

then  $u_n \rightarrow u$  uniformly. As  $\dot{u}_n \rightarrow \dot{u} \in \mathcal{E}^\Phi$ , by Lemma 2.5, there exist a subsequence of  $\dot{u}_{n_k}$  (again denoted  $\dot{u}_{n_k}$ ) and a function  $h \in L^1([0, T], \mathbb{R})$  such that  $\dot{u}_{n_k} \rightarrow \dot{u}$  a.e. and  $\Phi(\dot{u}_{n_k}) \leq h$  a.e.

Since  $u_{n_k}, k = 1, 2, \dots$ , is a strong convergent sequence in  $W^1 L^\Phi$ , it is a bounded sequence in  $W^1 L^\Phi$ . According to item (3) of Lemma 2.3, there exists  $M > 0$  such that  $\|a(u_{n_k})\|_{L^\infty} \leq M, k = 1, 2, \dots$ . From the previous facts and (20), we get

$$|\nabla_x \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leq a(|u_{n_k}|)(b + \Phi(\dot{u}_{n_k})) \leq M(b + h) \in L^1.$$

On the other hand, by the continuous differentiability of  $\mathcal{L}$ , we have

$$\nabla_x \mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow \nabla_x \mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

*Step 2.* The non linear operator  $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$  is continuous from  $\mathcal{E}^\Phi$  with the strong topology into  $[L^\Phi]^*$  with the weak\* topology.

Let  $u \in \mathcal{E}^\Phi$ . From (20) it follows that

$$\nabla_y \mathcal{L}(\cdot, u, \dot{u}) \in C^\Psi. \quad (21)$$

Note that (20), (21) and the imbeddings  $W^1 L^\Phi \hookrightarrow L^\infty$  and  $L^\Psi \hookrightarrow [L^\Phi]^*$  imply that the second member of (18) defines an element of  $[W^1 L^\Phi]^*$ .

Let  $u_n, u \in \mathcal{E}^\Phi$  such that  $u_n \rightarrow u$  in the norm of  $W^1 L^\Phi$ . We must prove that  $\nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n) \xrightarrow{w^*} \nabla_y \mathcal{L}(\cdot, u, \dot{u})$ . On the contrary, there exist  $v \in L^\Phi$ ,  $\epsilon > 0$  and a subsequence of  $\{u_n\}$  (denoted  $\{u_n\}$  for simplicity) such that

$$|\langle \nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n), v \rangle - \langle \nabla_y \mathcal{L}(\cdot, u, \dot{u}), v \rangle| \geq \epsilon. \quad (22)$$

We have  $u_n \rightarrow u$  in  $L^\Phi$  and  $\dot{u}_n \rightarrow \dot{u}$  in  $L^\Phi$ . By Lemma 2.5, there exist a subsequence of  $\{u_n\}$  (again denoted  $\{u_n\}$  for simplicity) and a function  $h \in L^1([0, T], \mathbb{R})$  such that  $u_n \rightarrow u$  uniformly,  $\dot{u}_n \rightarrow \dot{u}$  a.e. and  $\Phi(\dot{u}_n) \leq h$  a.e. As in the previous step, since  $u_n$  is a convergent sequence, Lemma 3.3 implies that  $a(|u_n(t)|)$  is uniformly bounded by a certain constant  $M > 0$ . Therefore, from inequality (20) with  $u_n$  instead of  $u$ , we have

$$\Psi(\nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n)) \leq M(b + h) \in L^1. \quad (23)$$

As  $v \in L^\Phi$  there exists  $\lambda > 0$  such that  $\Phi(\frac{v}{\lambda}) \in L^1$ . Now, by Young inequality and (23), we have

$$\begin{aligned} & \lambda \nabla_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ & \leq \lambda \left[ \Psi(\nabla_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right) \right] \\ & \leq \lambda M(b + h) + \lambda \Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \quad (24)$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T \nabla_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v \, dt \rightarrow \int_0^T \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot v \, dt \quad (25)$$

which contradicts the inequality (22). This completes the proof of step 2.

*Step 3.* We will prove (18). For  $u \in \mathcal{E}^\Phi$  and  $0 \neq v \in W^1 L^\Phi$ , we define the function

$$H(s, t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For  $|s| \leq s_0 := \min\{(1 - d(\dot{u}, E^\Phi)) / \|v\|_{W^1 L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$ , using triangle inequality we get  $d(\dot{u} + s\dot{v}, E^\Phi) < 1$  and thus  $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$ . These facts imply, in virtue of Theorem 3.4 item 1, that  $I(u + sv)$  is well defined and finite for  $|s| \leq s_0$ .

We also have  $\|u + sv\|_{W^1 L^\Phi} \leq \|u\|_{W^1 L^\Phi} + s_0 \|v\|_{W^1 L^\Phi}$ ; then, by Lemma 3.3, there exists  $M > 0$  such that  $\|a(u + sv)\|_{L^\infty} \leq M$ .

Let  $\lambda > 0$  such that  $\Phi(\frac{\dot{u}}{\lambda}) \in L^1$ . On the other hand, if  $\dot{v} \in L^\Phi$  and  $|s| \leq s_0 \lambda^{-1}$ , from the convexity and the parity of  $\Phi$ , we get

$$\begin{aligned} \Phi(\dot{u} + s\dot{v}) &= \Phi\left((1-s_0)\frac{\dot{u}}{1-s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leq (1-s_0)\Phi\left(\frac{\dot{u}}{1-s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right) \\ &\leq (1-s_0)\Phi\left(\frac{\dot{u}}{1-s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1 \end{aligned}$$

As  $\dot{u} \in \Pi(E^\Phi, 1)$  then

$$d\left(\frac{\dot{u}}{1-s_0}, E^\Phi\right) = \frac{1}{1-s_0}d(\dot{u}, E^\Phi) < 1$$

and therefore  $\frac{\dot{u}}{1-s_0} \in C^\Phi$ .

Now, applying (20), (24), the fact that  $v \in L^\infty$  and  $\dot{v} \in L^\Phi$ , we get

$$\begin{aligned} |D_s H(s, t)| &= \left| \nabla_x \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + \lambda \nabla_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \frac{\dot{v}}{\lambda} \right| \\ &\leq M[b(t) + \Phi(\dot{u} + s\dot{v})]|v| \\ &\quad + \lambda \left[ \Psi(\nabla_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right) \right] \\ &\leq M\{[b(t) + \Phi(\dot{u} + s\dot{v})]|v|\} + \lambda M[b(t) + \Phi(\dot{u} + s\dot{v})] + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \\ &= M[b(t) + \Phi(\dot{u} + s\dot{v})](|v| + \lambda) + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1. \end{aligned} \tag{26}$$

Consequently,  $I$  has a directional derivative and

$$\langle I'(u), v \rangle = \frac{d}{ds} I(u + sv) \Big|_{s=0} = \int_0^T \{ \nabla_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt.$$

Moreover, from the previous formula, (20), (21), and Lemma 2.3, we obtain

$$|\langle I'(u), v \rangle| \leq \|\nabla_x \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|\nabla_y \mathcal{L}\|_{L^\Psi} \|\dot{v}\|_{L^\Phi} \leq C \|v\|_{W^1 L^\Phi}$$

with a appropriate constant  $C$ .

This completes the proof of the Gateaux differentiability of  $I$ .

*Step 4.* The operator  $I' : \mathcal{E}^\Phi \rightarrow [W^1 L_d^\Phi]^*$  is demicontinuous. This is a consequence of the continuity of the mappings  $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$  and  $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$ . Indeed, if  $u_n, u \in \mathcal{E}^\Phi$  with  $u_n \rightarrow u$  in the norm of  $W^1 L^\Phi$  and  $v \in W^1 L^\Phi$ , then

$$\begin{aligned} \langle I'(u_n), v \rangle &= \int_0^T \{ \nabla_x \mathcal{L}(t, u_n, \dot{u}_n) \cdot v + \nabla_y \mathcal{L}(t, u_n, \dot{u}_n) \cdot \dot{v} \} dt \\ &\rightarrow \int_0^T \{ \nabla_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt \\ &= \langle I'(u), v \rangle. \end{aligned}$$

In order to prove item 3, it is necessary to see that the maps  $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$  and  $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$  are norm continuous from  $\mathcal{E}^\Phi$  into  $L^1$  and  $L^\Psi$ , respectively.

The continuity of the first map has already been proved in step 1.

Let  $u_n, u \in \mathcal{E}^\Phi$  with  $\|u_n - u\|_{W^1 L^\Phi} \rightarrow 0$ .

Applying Lemma 2.5 to  $\dot{u}_n$ , there exists a subsequence (denoted  $\dot{u}_n$  for simplicity) such that  $\dot{u}_n \in L^\Phi$  and a function  $h \in L^1$  such that  $\Psi(\dot{u}_n) \leq h$  and  $\dot{u}_n \rightarrow \dot{u}$  a.e.

Then, by (24) we have  $\Psi(v_n) \leq m(t) \in L^1$  being  $v_n := \nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n)$  and  $m(t) := M(b + h)$ . In addition, from the continuous differentiability of  $\mathcal{L}$ , we have that  $v_n \rightarrow v$  a.e. where  $\nabla_y \mathcal{L}(\cdot, u, \dot{u})$ .

As  $\Psi \in \Delta_2$ , there exists  $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\Psi(\lambda x) \leq c(|\lambda|)\Psi(x)$ . Then,  $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1})\Psi(v_n - v)$  for every  $\lambda \in \mathbb{R}$ .

Therefore,  $\Psi(\frac{v_n - v}{\lambda}) \rightarrow 0$  a.e. as  $n \rightarrow \infty$  and  $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1})K\Psi(v_n) + \Psi(v) \leq c(|\lambda|^{-1})K[m(t) + \Psi(v)] \in L^1$ .

Now, by Dominated Convergence Theorem, we get  $\int \Psi(\frac{v_n - v}{\lambda}) dt \rightarrow 0$  for every  $\lambda > 0$ . Thus,  $v_n \rightarrow v$  in  $L^\Psi$ .

The continuity of  $I'$  follows from the continuity of  $\nabla_x \mathcal{L}$  and  $\nabla_y \mathcal{L}$  using the formula (18).  $\square$

ALGO DE ESTO ES NECESARIO SI VAMOS A USAR LAS RELACIONES  $<$  y  $\ll$ , QUIZAS EN LOS PRELIMINARES

There exist several orders and equivalence relations between  $N$ -functions (see [16, Sec. 2.2]). Following [16, Def. 1, pp. 15-16] we say that the  $N$ -function  $\Phi_2$  is *stronger* than the  $N$ -function  $\Phi_1$ , in symbols  $\Phi_1 < \Phi_2$ , if there exist  $a > 0$  and  $x_0 \geq 0$  such that

$$\Phi_1(x) \leq \Phi_2(ax), \quad x \geq x_0. \quad (27)$$

The  $N$ -functions  $\Phi_1$  and  $\Phi_2$  are *equivalent* ( $\Phi_1 \sim \Phi_2$ ) when  $\Phi_1 < \Phi_2$  and  $\Phi_2 < \Phi_1$ . We say that  $\Phi_2$  is *essentially stronger* than  $\Phi_1$  ( $\Phi_1 \ll \Phi_2$ ) if and only if for every  $a > 0$  there exists  $x_0 = x_0(a) \geq 0$  such that (27) holds. Finally, we say that  $\Phi_2$  is *completely stronger* than  $\Phi_1$  ( $\Phi_1 \prec \Phi_2$ ) if and only if for every  $a > 0$  there exist  $K = K(a) > 0$  and  $x_0 = x_0(a) \geq 0$  such that

We assume that there exist an  $N_\infty$  function  $\Phi_0$  such that  $\Phi_0 \ll \Phi$ , a function  $a \in L^1([0, T], \mathbb{R})$  such that  $a \geq 1$  and a constant  $M > 0$  such that

$$\Psi_0(a^{-1}(t)\nabla_x F) \leq \Phi_0(x), \quad |x| \geq M \quad (\text{Sub})$$

**Proposition 3.5.** *Let  $1 < p < \infty$  and suppose that  $F$  satisfies (A). Then,*

$$|\nabla_x F| \leq a(t)|x|^{p'_0-1} + b(t), \quad \text{for } p'_0 < p, \quad (28)$$

*if and only if  $F$  satisfies (Sub) with  $\Phi_0(x) = |x|^{p'}$  for all  $p' \in (1, p)$ .*

*Proof.* Suppose that (Sub) holds with  $\Phi_0(x) = |x|^{p'}$  for  $1 < p' < p - 1$ . Then,

$$(a^{-1}(t)|\nabla_x F|)^{q'_0} \leq |x|^{p'_0}, \quad \text{for } |x| \geq M$$

As  $F$  satisfies (A), we have  $|\nabla_x F| \leq b(t) \in L^1$  for  $|x| \leq M$ .

PONEMOS SOLO b O Kb?? No confunde?

Then,

$$|\nabla_x F| \leq a|x|^{\frac{p'_0}{q'_0}} + b(t) \leq a|x|^{p'_0-1} + b(t)$$

Now, we assume that  $F$  satisfies (28). If  $|x| \geq 1$ , we have

$$|\nabla_x F| \leq a(t)|x|^{p'_0-1} + b(t) \leq (a(t) + b(t))|x|^{\frac{p'_0}{q'_0}}$$

which is condition (Sub) with  $a(t) + b(t)$  instead of  $a(t)$ .  $\square$

**Theorem 3.6.** *Let  $\Phi$  be an  $N$ -function whose complementary function  $\Psi$  satisfies the  $\Delta_2^\infty$ -condition.*

*Let  $F$  be a potential that satisfies (C), (A) and the following conditions:*

1. (Sub) for some  $N_\infty$  function  $\Phi_0$  such that  $\Phi_0 \ll \Phi$
- 2.

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\Psi_0(2|x|)} = +\infty, \quad (29)$$

for some  $N$ -function  $\Psi_0$  complementary function of  $\Phi_0$ .

EN LO QUE SIGUE, PONEMOS LA NUEVA CONDICION DE ESCTRUCTURA O DEFINIMOS UN CONJUNTO DONDE SE VERIFICAN ESAS CONDICIONES???

Now, if the lagrangian  $\mathcal{L}(t, x, y)$  is strictly convex with respect to  $y \in \mathbb{R}^d$ ,  $\mathcal{L} \in \mathfrak{A}(a, b, c, \lambda, f, \Phi)$ ,  $D_y \mathcal{L}(0, x, y) = D_y \mathcal{L}(T, x, y)$  and (??) holds, then the problem (P) has at least a solution which minimizes the action integral  $I$  on  $W^1 E_T^\Phi$ .

FALTA UNA BUENA REVISION DE LO ESCRITO EN LA PRUEBA

*Proof.* By the decomposition  $u = \bar{u} + \tilde{u}$ , Young's inequality, (Sub), the convexity of  $\Phi_0$  and (15), we obtain

$$\begin{aligned} & \left| \int_0^T F(t, u) - F(t, \bar{u}) dt \right| \leq \int_0^T \int_0^1 |\nabla_x F(t, \bar{u} + s\tilde{u})\tilde{u}| ds dt \\ &= \int_0^T a(t) \int_0^1 a^{-1}(t) |\nabla_x F(t, \bar{u} + s\tilde{u})\tilde{u}| ds dt \\ &\leq \int_0^T a(t) \int_0^1 \Psi_0(a^{-1}(t) \nabla_x F(t, \bar{u} + s\tilde{u})) + \Phi_0(\tilde{u}(t)) ds dt \\ &\leq \int_0^T a(t) \left[ \int_0^1 \Phi_0(\bar{u} + s\tilde{u}) ds + \Phi_0(\tilde{u}(t)) \right] dt \\ &\leq \int_0^T a(t) (\Phi_0(2\bar{u}) + 2\Phi_0(2\tilde{u})) dt \\ &\leq C_1 \Phi_0(2\bar{u}) + \int_0^T a(t) 2\Phi_0(2\tilde{u}) dt \\ &\leq C_1 \Phi_0(2\bar{u}) + C_2 \int_0^T \Phi_0(C_T u'(t)) dt \end{aligned} \quad (30)$$

Then, by (30) we get

$$\begin{aligned} & \int_0^T \Phi(u') + F(t, u) dt = \int_0^T \{ \Phi(u') + [F(t, u) - F(t, \bar{u})] + F(t, \bar{u}) \} dt \\ &\geq \int_0^T [\Phi(u') - C_2 \Phi_0(C_T u'(t))] dt - C_1 \Phi_0(2\bar{u}) + \int_0^T F(t, \bar{u}) dt \end{aligned} \quad (31)$$

If  $\|u\|_{W^1 L^\Phi} \rightarrow \infty$ , then  $\|u'\|_{L^\Phi} \rightarrow \infty$  or  $|\bar{u}| \rightarrow \infty$ .  
 Now, if  $|\bar{u}| \rightarrow \infty$ , by hypothesis we have  $\frac{1}{\Psi_0(2\bar{u})} \int_0^\infty F(t, \bar{u}) dt \rightarrow \infty$ .  
 If  $\|u_n\|_{L^\Phi} \rightarrow \infty$  for a sequence  $u_n$ .  
 Let  $M > 0$  such that  $\frac{\Phi(x)}{C_2 \Phi_0(C_T x)} \geq 2$  for  $|x| \geq M$ .  
 Set  $A_n := \{t \in [0, T] : |u'_n(t)| \geq M\}$ .  
 Now, if  $\|u'_n\|_{L^\Phi} \rightarrow \infty$  then  $\|u'_n \chi_{A_n}\|_{L^\Phi} \rightarrow \infty$  or  $\|u'_n \chi_{A_n^c}\|_{L^\Phi} \rightarrow \infty$ .  
 As  $\Phi$  is continuous, then  $\Phi$  is bounded on  $\overline{B_r}(0) = \{x \in \mathbb{R}^n : |x| \leq r\}$ .  
 Let  $M_r := \max_{\overline{B_r}(0)} \Phi(x)$ .  
 As  $M_r \rightarrow 0$  when  $r \rightarrow 0$ , we can choose  $r$  such that  $M_r T \leq 1$ .  
 Then  $\int_0^T \Phi(\frac{1}{r} u) dt \leq M_r T \leq 1$  and consequently  $\|u\|_{L^\Phi} \leq r^{-1} \|u\|_{L^\infty}$ , i.e.  $L^\infty \hookrightarrow L^\Phi$ . Thus,  $\|u'_n \chi_{A_n^c}\|_{L^\Phi}$  is bounded and  $\|u'_n \chi_{A_n}\|_{L^\Phi} \rightarrow \infty$ .  
 By Ameniya norm, we have  $\|u'_n\|_{L^\Phi} \leq 1 + \int_{A_n} \Phi(u'_n) dt$ , then

$$\int_{A_n} \Phi(u'_n) dt \rightarrow \infty. \quad (32)$$

Now, as  $\frac{\Phi(x)}{2} \geq C_2 \Phi_0$  for  $|x| \geq M$ , we have

$$\begin{aligned} & \int_0^T [\Phi(u'_n) - C_2 \Phi_0(C_T u'_n)] dt \\ & \geq \int_{A_n} [\Phi(u'_n) - C_2 \Phi_0(C_T u'_n)] dt - C_2 \int_{[0, T] \setminus A_n} \Phi_0(C_T u'_n) dt \\ & \geq \frac{1}{2} \int_{A_n} \Phi(u'_n) dt - C_2 T \max\{\Phi_0(x) : \|x\| \leq C_T M\}. \end{aligned} \quad (33)$$

Finally, by (32), we conclude that  $\int_0^T [\Phi(u'_n) - C_2 \Phi_0(C_T u'_n)] dt \rightarrow \infty$

□

## Acknowledgments

The authors are partially supported by a UNRC grant number 18/C417. The first author is partially supported by a UNSL grant number 22/F223.

## References

- [1] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, and E. Schwindt. Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698, 2015.
- [2] M Chamra and J Maksymiuk. Anisotropic orlicz-sobolev spaces of vector valued functions and lagrange equations. *arXiv preprint arXiv:1702.08683*, 2017.
- [3] F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Graduate Texts in Mathematics. 2013.



## References

---

- [4] W. Desch and R. Grimmer. On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc.*, (353):5095–5120, 2001.
- [5] M.A. Krasnosel'skii, P.P. Zabreyko, E.I. Pustyl'nik, and P.E. Sobolevski. *Integral operators in spaces of summable functions*. Mechanics: Analysis. Springer Netherlands, 2011.
- [6] Chun Li, Zeng-Qi Ou, and Chun-Lei Tang. Periodic solutions for non-autonomous second-order differential systems with  $(q, p)$ -laplacian. *Electronic Journal of Differential Equations*, 2014(64):1–13, 2014.
- [7] David G Luenberger and Yinyu Ye. *Linear and nonlinear programming*, volume 228. Springer, 2015.
- [8] J. Mawhin and M. Willem. *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York, 1989.
- [9] Daniel Pasca. Periodic solutions of a class of nonautonomous second order differential systems with  $(q, p)$ -laplacian. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 17(5):841–851, 2010.
- [10] Daniel Paşca and Chun-Lei Tang. Some existence results on periodic solutions of nonautonomous second-order differential systems with  $(q, p)$ -laplacian. *Applied Mathematics Letters*, 23(3):246–251, 2010.
- [11] Daniel Pasca and Chun-Lei Tang. Some existence results on periodic solutions of ordinary  $(q, p)$ -laplacian systems. *Journal of applied mathematics & informatics*, 29(1\_2):39–48, 2011.
- [12] Daniel Pasca and Zhiyong Wang. On periodic solutions of nonautonomous second order hamiltonian systems with  $(q, p)$ -laplacian. *Electronic Journal of Qualitative Theory of Differential Equations*, 2016(106):1–9, 2016.
- [13] Ryszard Pluciennik. Boundedness of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Bull. Pol. Acad. Sci., Math.*, 33:531–540, 1985.
- [14] Ryszard Pluciennik. On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 25:321–337, 1985.
- [15] Ryszard Pluciennik. The superposition operator in Musielak-Orlicz spaces of vector-valued functions. Abstract analysis, Proc. 14th Winter Sch., Srní/Czech. 1986, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14, 411–417 (1987)., 1987.
- [16] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146. Marcel Dekker, Inc., New York, 1991.
- [17] G. Schappacher. A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386, 2005.

- [18] M. S. Skaff. Vector valued Orlicz spaces. II. *Pacific J. Math.*, 28(2):413–430, 1969.
- [19] C.-L. Tang. Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270, 1998.
- [20] C. L. Tang and X.-P. Wu. Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397, 2001.
- [21] Chun-Lei Tang. Periodic solutions of non-autonomous second-order systems with  $\gamma$ -quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675, 1995.
- [22] X. Tang and X. Zhang. Periodic solutions for second-order Hamiltonian systems with a  $p$ -Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113, 2010.
- [23] Y. Tian and W. Ge. Periodic solutions of non-autonomous second-order systems with a  $p$ -Laplacian. *Nonlinear Anal.*, 66(1):192–203, 2007.
- [24] X.-P. Wu and C.-L. Tang. Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235, 1999.
- [25] Xiaoxia Yang and Haibo Chen. Periodic solutions for a nonlinear  $(q, p)$ -laplacian dynamical system with impulsive effects. *Journal of Applied Mathematics and Computing*, 40(1-2):607–625, 2012.
- [26] Xiaoxia Yang and Haibo Chen. Existence of periodic solutions for sublinear second order dynamical system with  $(q, p)$ -laplacian. *Mathematica Slovaca*, 63(4):799–816, 2013.
- [27] F. Zhao and X. Wu. Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434, 2004.