

Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

Sonia Acinas *

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (1)$$

where $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d \geq 1$, is called the *Lagrange function* or *lagrangian* and the unknown function $u : [0, T] \rightarrow \mathbb{R}^d$ is absolutely continuous. In other words, we are interested in finding *periodic weak solutions of Euler-Lagrange system of ordinary equations*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t, x, y) = \frac{|y|^p}{p} + F(t, x), \quad (2)$$

*SECyT-UNRC and FCEyN-UNLPam

†SECyT-UNRC, FCEyN-UNLPam and CONICET

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for $1 < p < \infty$. For example, the classic book [?] deals mainly with problem (1), for the lagrangian $\mathcal{L}_{2,F}$, through various methods: direct, dual action, minimax, etc. The results in [?] were extended and improved in several articles, see [?, ?, ?, ?, ?] to cite some examples. Lagrange functions (2) for arbitrary $1 < p < \infty$ were considered in [?, ?] and in this case (1) is reduced to the p -laplacian system

$$\begin{cases} \frac{d}{dt}(u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (3)$$

In this context, it is customary to call F a *potential function*, and it is assumed that $F(t, x)$ is differentiable with respect to x for a.e. $t \in [0, T]$ and the following conditions are verified:

(C) F and its gradient ∇F , with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

(A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (4)$$

In this inequality we assume that the function $a : [0, +\infty) \rightarrow [0, +\infty)$ is continuous and non decreasing and $0 \leq b \in L^1([0, T], \mathbb{R})$.

In [?] it was treated the case of a lagrangian \mathcal{L} which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi, F}(t, x, y) = \Phi(|y|) + F(t, x), \quad (5)$$

where Φ is an N -function (see section 2 for the definition of this concept). In the paper [?] it was assumed a condition of *bounded oscillation* on F (see xxxxx below). In this paper we apply the dual method ([?, Ch. 3]) to obtain solutions of (1).

2 Preliminaries

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces) and other brief introduction to superposition operators between these spaces. References for these topics are [?, ?, ?] and [?, ?, ?, ?].

Hereafter we denote by \mathbb{R}^+ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is called an *Young's function* if Φ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\Phi(x) \rightarrow +\infty$, when $|x| \rightarrow +\infty$.

Following [?] we say that Φ is *coercive* if

$$\lim_{|x| \rightarrow \infty} \frac{\Phi(x)}{|x|} = +\infty.$$

We define the function G by

$$G(s) = \min\{\Phi(x) : |x| = s\}, \quad (6)$$

where Φ is a Young's function.

We point out that the function G is the greatest radial minorant of Φ , then $G(|x|) \leq \Phi(x)$.

We can see that G is monotonous, continuous and $G(s) \rightarrow \infty$ as $s \rightarrow \infty$. **Agregar algunas pruebas de lo anterior, cómo sale la monotonía ????**

Therefore, there exists its inverse function G^{-1} and we get that $G^{-1}(\Phi(x)) \geq |x|$.

As $\frac{\Phi(\alpha x)}{\alpha}$ is increasing with respect to α , $\frac{G(\alpha s)}{\alpha}$ is also increasing with respect to α . In fact, if $0 < \alpha \leq \beta$, we have

$$\begin{aligned} \frac{G(\alpha s)}{\alpha} &= \frac{\min\{\Phi(x) : |x| = \alpha s\}}{\alpha} = \min\left\{\frac{\Phi(x)}{\alpha} : \frac{|x|}{\alpha} = s\right\} = \\ &= \min\left\{\frac{\Phi(\alpha y)}{\alpha} : |y| = s\right\} \leq \min\left\{\frac{\Phi(\beta y)}{\beta} : |y| = s\right\} = \\ &= \min\left\{\frac{\Phi(x)}{\beta} : \frac{|x|}{\beta} = s\right\} = \frac{\min\{\Phi(x) : |x| = \beta s\}}{\beta} = \frac{G(\beta s)}{\beta}. \end{aligned}$$

Performing the change of variables $\alpha s = G^{-1}(\frac{t}{\beta})$ on $\frac{G(\alpha s)}{\alpha}$, we can see that $\beta G^{-1}(\frac{t}{\beta})$ is an increasing function with respect to β . **(en las cuentas, usamos que G es creciente?????)**

And, in the event that Φ is an N-function, we obtain that $\frac{G(\alpha s)}{\alpha} \rightarrow \infty$ as $\alpha \rightarrow \infty$.

We also say that a non decreasing function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the Δ_2^∞ -condition, denoted by $\eta \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geq 0$ such that

$$\eta(2x) \leq K\eta(x), \quad (7)$$

for every $|x| \geq M$.

If Φ is a Young's function we define its *Fenchel conjugate* $\Phi^* : \mathbb{R}^d \rightarrow \mathbb{R}^+$ by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \quad (8)$$

Let d be a positive integer. We denote by $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$ the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on \mathbb{R}^d and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$. For the set of functions \mathcal{M} , as for other similar sets, we will omit the reference to codomain \mathbb{R}^d when $d = 1$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Here $|\cdot|$ is the euclidean norm of \mathbb{R}^d . Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (9)$$

The *Orlicz space* $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$ is the linear hull of C^Φ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (10)$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left(\frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space. By $u \cdot v$ we denote the usual dot product in \mathbb{R}^d between u and v .

The subspace $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0, T], \mathbb{R}^d)$ of all \mathbb{R}^d -valued essentially bounded functions. It is shown that (see [?, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [?, Thm. 5.2]). Another alternative characterization of E^Φ , which is particularly useful for us, is that $u \in E^\Phi$ if and only if u has *absolutely continuous norm*, i.e. if $E_n \subset [0, T]$, $n = 1, 2, \dots$ then $\|\chi_{E_n} u\| \rightarrow 0$ when $|E_n| \rightarrow 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [?, Thm. 4.1]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^*}$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \quad (11)$$

Like in [?] we will consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset r C^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (12)$$

for any positive r (see [?, Thm. 5.6]). If $\Phi \in \Delta_2^\infty$, then the sets L^Φ , E^Φ , $\Pi(E^\Phi, r)$ and C^Φ are equal.

Following to [?] we introduce the next definition.

Definition 2.1. Let $u_n, u \in L^\Phi([0, T], \mathbb{R}^d)$. We say that u_n converges monotonically to u if there exists $\alpha_n \in L^\infty([0, T], \mathbb{R}^d)$, $n = 1, 2, \dots$, such that $0 \leq \alpha_n(t) \leq \alpha_{n+1}(t)$, $\alpha_n(t) \rightarrow 1$ a.e., when $n \rightarrow \infty$ and $u_n(t) = \alpha_n(t)u(t)$.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leq C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\Phi^*} \hookrightarrow [L^\Phi]^*$, where a function $v \in L^{\Phi^*}$ is associated to $F_v \in [L^\Phi]^*$ where

$$F_v(u) := \langle v, u \rangle = \int_0^T v \cdot u \, dt, \quad (13)$$

In [?, Thm 2.9] it was characterized a subspace of $[L^\Phi]^*$ which is identified with L^{Φ^*} . Namely $L^{\Phi^*} = P^{\Phi^*}([0, T], \mathbb{R}^d)$ where $F \in P^{\Phi^*}([0, T], \mathbb{R}^d)$ if and only if $F \in [L^\Phi]^*$ and satisfying the *monotone convergence property*, which is if u_n converges monotonically to u then $F(u_n) \rightarrow F(u)$.

If $\Phi \in \Delta_2^\infty$ and Φ is coercive then $L^{\Phi^*}([0, T], \mathbb{R}^d) = [L^\Phi([0, T], \mathbb{R}^d)]^*$ is satisfied (see [?, Thm. 2.9, Thm. 2.10]).

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\}.$$

$W^1 L^\Phi([0, T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (14)$$

And, we introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \quad (15)$$

We will use repeatedly the decomposition $u = \bar{u} + \tilde{u}$ for a function $u \in L^1([0, T])$ where $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u} = u - \bar{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.2. *Let $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ be a Young's function and let $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined by (6). Then*

1. *For every $s, t \in [0, T]$, $s \neq t$,*

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| G^{-1} \left(\frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq G^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. *We have $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$ and*

$$\|\tilde{u}\|_{L^\infty} \leq T G^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. *If Φ is coercive then the space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0, T], \mathbb{R}^d)$.*

Proof. By the absolute continuity of u , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi \left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|} \right) &\leq \Phi \left(\frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr \right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi \left(\frac{u'(r)}{\|u'\|_{L^\Phi}} \right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By (6)???? we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq G^{-1} \left(\frac{1}{|s - t|} \right),$$

then 1 holds.

Morrey's inequality implies Sobolev-Wirtinger's inequality according to the following argument. Taking into account that $\alpha G^{-1}(1/\alpha)$ is an increasing function with respect to $\alpha \in [0, \infty)$ we have

$$|u(t) - \bar{u}| \leq \|u'\|_{L^\Phi} T G^{-1}\left(\frac{1}{T}\right),$$

and Sobolev-Wirtinger's inequality follows easily.

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\bar{u}}{\|u\|_{L^\Phi}}\right) \leq \frac{1}{T} \int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leq \frac{1}{T}$$

Then

$$|\bar{u}| \leq G^{-1}\left(\frac{1}{T}\right) \|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq G^{-1}\left(\frac{1}{T}\right) \|u\|_{L^\Phi} + T G^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \\ &\leq G^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \end{aligned}$$

We take a bounded sequence u_n in $W^1 L^\Phi([0, T], \mathbb{R}^d)$ and suppose that u_n has not convergent subsequence. From (Morrey's inequality) and ?????? we infer that u_n are equicontinuous. Furthermore (Sobolev's inequality) implies that u_n is bounded in $C([0, T], \mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0, T], \mathbb{R}^d)$ with $u_{n_k} \rightarrow u$ in $C([0, T], \mathbb{R}^d)$. □

3 Superposition operators in anisotropic Orlicz spaces

For $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ we denote by \mathfrak{f} the Nemytskii (o superposition) operator defined for functions $u : [0, T] \rightarrow \mathbb{R}^d$ by

$$\mathfrak{f}u(t) = f(t, u(t))$$

Referencias y alguna propiedad interesante medibles en medibles? [?, ?]

Theorem 3.1. *Let $\Phi_1, \Phi_2, \dots, \Phi_n$ be N -functions. Assume that M is another N -functions that satisfy the Δ_2 -condition. We write $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ with $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}^d$. Let $f(t, x_1, \dots, x_n, y_1, \dots, y_n)$ be a function Charathéodory? with $f : [0, T] \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \rightarrow \mathbb{R}^{d'}$.*

Suppose that $a : (\mathbb{R}^d)^n \rightarrow [0, +\infty)$ is a bounded function on bounded sets and $b \in L^M([0, T])$, for a.e. $t \in [0, T]$ such that

$$|f| \leq a(x)[b(t) + \sum_{i=1}^n M^{-1}(\Phi_i(|y_i|))], \quad (16)$$

then

$$f : \left(\prod_{i=1}^n L^\infty([0, T], \mathbb{R}^d) \right) \times \left(\prod_{i=1}^n \Pi(E^{\Phi_i}([0, T], \mathbb{R}^d), \lambda = 1) \right) \rightarrow L^M.$$

Proof. If $(u, v) \in \left(\prod_{i=1}^n L^\infty([0, T], \mathbb{R}^d) \right) \times \left(\prod_{i=1}^n \Pi(E^{\Phi_i}_d, \lambda = 1) \right)$. By [?, Thm. 17.6] (y otras cosas), we get

$$|fu(t)| = |f(t, u(t), v(t))| \leq M_a[b_j(t) + \sum_{i=1}^n M_j^{-1}(\Phi_i(|v_i(t)|))] \in L_1^{M_j}.$$

□

We define the space X by $X = \{v = (v_1, v_2) : v_1 \in W^1 L_T^{\Phi_1}, v_2 \in W^1 L_T^{\Phi_2}\}$ and $X^* = \{v = (v_1, v_2) : v_1 \in (W^1 L_T^{\Phi_1})^*, v_2 \in (W^1 L_T^{\Phi_2})^*\}$ where $(W^1 L_T^{\Phi_i})^*$ stands for the conjugate space of $W^1 L_T^{\Phi_i}$ for $i = 1, 2$.

Corollary 3.2. We will consider the Lagrange function $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \rightarrow \mathcal{L}(t, x_1, x_2, y_1, y_2)$ which is measurable in t for each $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x_1, x_2, y_1, y_2) for almost every $t \in [0, T]$.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}^d$ and let

$$I(x) = \int_0^T \mathcal{L}(t, x, y) dt \quad (17)$$

If there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $i = 1, 2$, $b \in L_1^1([0, T])$, $j = 1, \dots, d'$ for a.e. $t \in [0, T]$ and every $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying the structure conditions

The nonlinear operator $(x_1, x_2) \mapsto D_x \mathcal{L}(t, x_1, y_1, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \dots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into $L^1([0, T])$ with the strong topology on both sets.

The nonlinear operator $(x_1, x_2) \mapsto D_y \mathcal{L}(t, x_1, y_1, y_2)$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \dots \times \mathcal{E}_d^{\Phi_n}(\lambda)$ with the strong topology into X with the weak* topology.

The function I is Gâteaux differentiable on $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^* . Moreover, I' is given by the following expression

$$\begin{aligned} \langle I'(x), w \rangle = \int_0^T & [(D_{x_1} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1(t)) + \\ & (D_{x_2} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2(t)) + \\ & (D_{y_1} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w'_1(t)) + \\ & (D_{y_2} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w'_2(t))] dt \end{aligned} \quad (18)$$

If $\Phi^* \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into X^* when both spaces are equipped with the strong topology.

We denote by $\mathfrak{A}(a, b, c, \lambda, f, \Phi)$ the set of all Lagrange functions satisfying (??), (??) and (??).

Proof. **OJO!!!! Es algo que teníamos del trabajo anterior!!! con algunas adaptaciones a 2 variables sin controlar y a lo bruto!!!!**

Let $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$.

Step 1. The non linear operator $(x_1, x_2) \mapsto (D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2), D_{x_1}\mathcal{L}(t, x_1, x_2, y_1, y_2))$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ into $L_d^1([0, T]) \times L_d^1([0, T])$ with the strong topology on both sets.

If $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$, from (??) and (??), we obtain Let $\{x_n = (x_{1n}, x_{2n})\}_{n \in \mathbb{N}}$ be a sequence of functions in $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ and let $x = (x_1, x_2) \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ such that $x_n \rightarrow x$ in X . From $x_{in} \rightarrow x_i$ in L^{Φ_i} , there exists a subsequence x_{in_k} such that $x_{in_k} \rightarrow x_i$ a.e.; and, as $x_{in} \rightarrow x_i \in \mathcal{E}_d^{\Phi}(\lambda)$, by Lemma ??, there exist a subsequence of x_{in_k} (again denoted x_{in_k}) and a function $h_i \in \Pi(E_1^{\Phi}, \lambda)$ such that $x_{in_k} \rightarrow u_i$ a.e. and $|x_{in_k}| \leq h_i$ a.e. Since $x_{in_k}, k = 1, 2, \dots$, is a strong convergent sequence in $W^1 L_d^{\Phi_i}$, it is a bounded sequence in $W^1 L_d^{\Phi_i}$. According to Lemma 2.2 and Corollary ??, there exist $M_i > 0$ such that $\|a(x_{in_k})\|_{L^\infty} \leq M_i, k = 1, 2, \dots$. From the previous facts and (??), we get

$$|D_{x_i}\mathcal{L}(\cdot, x_{1n_k}, x_{2n_k}, y_{1n_k}, y_{2n_k})| \leq M_i(b + \Phi_i(|h_i|)) \in L_1^1 \quad i = 1, 2.$$

On the other hand, by the continuous differentiability of \mathcal{L} , we have

$$D_{x_i}\mathcal{L}(t, x_{in_k}(t), y_{in_k}(t)) \rightarrow D_{x_i}\mathcal{L}(t, x_i(t), y_i(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $(x_1, x_2) \mapsto (D_{y_1}\mathcal{L}(t, x_1, y_1, D_{y_2}\mathcal{L}(t, x_2, y_2))$ is continuous from $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ with the strong topology into X with the weak topology.*

Note that (??), (??) and the imbeddings $W^1 L_d^{\Phi} \hookrightarrow L_d^\infty$ and $L_d^{\Phi^*} \hookrightarrow [L^{\Phi}]^*$ imply that the second member of (18) defines an element in $[W^1 L_d^{\Phi}]^*$.

Let $(x_{1n}, x_{2n}) \in \mathcal{E}_d^{\Phi}(\lambda)$ such that $(x_{1n}, x_{2n}) \rightarrow (x_1, x_2)$ in the norm of X . We must prove that $D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}) \xrightarrow{w^*} D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2)$ para $i = 1, 2$. On the contrary, there exist $v = (v_1, v_2) \in L^{\Phi_1} \times L^{\Phi_2}$, $\epsilon > 0$ and a subsequence of $\{x_n\}$ (denoted $\{x_n\}$ for simplicity) such that

$$|\langle D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}, y_{1n}, y_{2n}), v \rangle - \langle D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2, v) \rangle| \geq \epsilon. \quad (19)$$

We have $x_n \rightarrow x$ in X and $y_n \rightarrow y$ in X . By Lemma ??, there exist a subsequence x_{n_k} and a function $h \in \Pi(E_1^{\Phi_1}, \lambda) \times \Pi(E_1^{\Phi_2}, \lambda)$ such that $x_{n_k} \rightarrow x$ a.e., $y_{n_k} \rightarrow y$ a.e. and $|y_{n_k}| \leq h$ a.e. As in the previous step, since x_n is a convergent sequence, the Corollary ?? implies that $a(|y_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, with x_{n_k} instead of x , inequality (??) becomes Consequently, as $v \in L^{\Phi}$ and employing Hölder's inequality, we obtain that

$$\sup_k |D_{y_i}\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot v| \in L_1^1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\mathbf{y}} \mathcal{L}(t, u_{n_k}, \dot{\mathbf{u}}_{n_k}) \cdot \mathbf{v} \, dt \rightarrow \int_0^T D_{\mathbf{y}} \mathcal{L}(t, u, \dot{\mathbf{u}}) \cdot \mathbf{v} \, dt \quad (20)$$

which contradicts the inequality (19). This completes the proof of step 2.

Step 3. We will prove (18). The proof follows similar lines as [?, Thm. 1.4]. For $u \in \mathcal{E}_d^\Phi(\lambda)$ and $\mathbf{0} \neq v \in W^1 L_d^\Phi$, we define the function

$$H(s, t) := \mathcal{L}(t, u(t) + sv(t), \dot{\mathbf{u}}(t) + s\dot{\mathbf{v}}(t)).$$

From [?, Lemma 10.1] (or [?, Thm. 5.5]) we obtain that if $|u| \leq |v|$ then $d(u, E^\Phi) \leq d(v, E^\Phi)$. Therefore, for $|s| \leq s_0 := (\lambda - d(\dot{\mathbf{u}}, E^\Phi)) / \|v\|_{W^1 L^\Phi}$ we have

$$d(\dot{\mathbf{u}} + s\dot{\mathbf{v}}, E^\Phi) \leq d(|\dot{\mathbf{u}}| + s|\dot{\mathbf{v}}|, E_1^\Phi) \leq d(|\dot{\mathbf{u}}|, E_1^\Phi) + s\|\dot{\mathbf{v}}\|_{L^\Phi} < \lambda.$$

Thus $\dot{\mathbf{u}} + s\dot{\mathbf{v}} \in \Pi(E^\Phi, \lambda)$ and $|\dot{\mathbf{u}}| + s|\dot{\mathbf{v}}| \in \Pi(E_1^\Phi, \lambda)$. These facts imply, in virtue of Theorem ?? item ??, that $I(u + sv)$ is well defined and finite for $|s| \leq s_0$. And, using Corollary ??, we also see that

$$\|a(|u + sv|)\|_{L^\infty} \leq A(\|u + sv\|_{W^1 L^\Phi}) \leq A(\|u\|_{W^1 L^\Phi} + s_0\|v\|_{W^1 L^\Phi}) =: M$$

Now, applying Chain Rule, (??), (??) the monotonicity of φ and Φ , the fact that $v \in L_d^\infty$ and $\dot{\mathbf{v}} \in L^\Phi$ and Hölder's inequality, we get

$$\begin{aligned} |D_s H(s, t)| &= |D_{\mathbf{x}} \mathcal{L}(t, u + sv, \dot{\mathbf{u}} + s\dot{\mathbf{v}}) \cdot v + D_{\mathbf{y}} \mathcal{L}(t, u + sv, \dot{\mathbf{u}} + s\dot{\mathbf{v}}) \cdot \dot{\mathbf{v}}| \\ &\leq M \left[\left(b(t) + \Phi \left(\frac{|\dot{\mathbf{u}}| + s_0|\dot{\mathbf{v}}|}{\lambda} + f(t) \right) \right) |v| \right. \\ &\quad \left. + \left(c(t) + \varphi \left(\frac{|\dot{\mathbf{u}}| + s_0|\dot{\mathbf{v}}|}{\lambda} + f(t) \right) \right) |\dot{\mathbf{v}}| \right] \in L_1^1. \end{aligned} \quad (21)$$

Consequently, I has a directional derivative and

$$\langle I'(u), v \rangle = \frac{d}{ds} I(u + sv) \Big|_{s=0} = \int_0^T \{ D_{\mathbf{x}} \mathcal{L}(t, u, \dot{\mathbf{u}}) \cdot v + D_{\mathbf{y}} \mathcal{L}(t, u, \dot{\mathbf{u}}) \cdot \dot{\mathbf{v}} \} \, dt.$$

Moreover, from (??), (??), Lemma 2.2 and the previous formula, we obtain

$$|\langle I'(u), v \rangle| \leq \|D_{\mathbf{x}} \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|D_{\mathbf{y}} \mathcal{L}\|_{L^{\Phi^*}} \|\dot{\mathbf{v}}\|_{L^\Phi} \leq C \|v\|_{W^1 L^\Phi}$$

with a appropriate constant C . This completes the proof of the Gâteaux differentiability of I .

Step 4. The operator $I' : \mathcal{E}_d^\Phi(\lambda) \rightarrow [W^1 L_d^\Phi]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto D_{\mathbf{x}} \mathcal{L}(t, u, \dot{\mathbf{u}})$ and $u \mapsto D_{\mathbf{y}} \mathcal{L}(t, u, \dot{\mathbf{u}})$. Indeed, if $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$ with $u_n \rightarrow u$ in the norm of $W^1 L_d^\Phi$ and $v \in W^1 L_d^\Phi$, then

$$\begin{aligned} \langle I'(u_n), v \rangle &= \int_0^T \{ D_{\mathbf{x}} \mathcal{L}(t, u_n, \dot{\mathbf{u}}_n) \cdot v + D_{\mathbf{y}} \mathcal{L}(t, u_n, \dot{\mathbf{u}}_n) \cdot \dot{\mathbf{v}} \} \, dt \\ &\rightarrow \int_0^T \{ D_{\mathbf{x}} \mathcal{L}(t, u, \dot{\mathbf{u}}) \cdot v + D_{\mathbf{y}} \mathcal{L}(t, u, \dot{\mathbf{u}}) \cdot \dot{\mathbf{v}} \} \, dt \\ &= \langle I'(u), v \rangle. \end{aligned}$$

In order to prove item ??, it is necessary to see that the maps $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$ are norm continuous from $\mathcal{E}_d^\Phi(\lambda)$ into L_d^1 and $L_d^{\Phi^*}$ respectively. The continuity of the first map has already been proved in step 1. Let $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$ with $\|u_n - u\|_{W^1 L^\Phi} \rightarrow 0$. Therefore, there exist a subsequence $u_{n_k} \in \mathcal{E}_d^\Phi(\lambda)$ and a function $h \in \Pi(E_1^\Phi, \lambda)$ such that (??) holds true. And, as $\Phi^* \in \Delta_2$ then the right hand side of (??) belongs to $E_1^{\Phi^*}$. Now, invoking Lemma ??, we prove that from any sequence u_n which converges to u in $W^1 L_d^\Phi$ we can extract a subsequence such that $D_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \rightarrow D_y \mathcal{L}(t, u, \dot{u})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of I' follows from the continuity of $D_x \mathcal{L}$ and $D_y \mathcal{L}$ using the formula (18). \square

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