Euler-Lagrange equations in an Orlicz-Sobolev space setting

Sonia Acinas [*]
Instituto de Matemática Aplicada San Luis (CONICET-UNSL)
(5700) San Luis, Argentina
Universidad Nacional de La Pampa
(6300) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Leopoldo Buri [†]
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
lburi@exa.unrc.edu.ar

Graciela Giubergia [‡]
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
ggiubergia@exa.unrc.edu.ar

Fernando D. Mazzone [§]
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Erica L. Schwindt[¶]
Université d'Orléans, Laboratoire MAPMO, CNRS, UMR 7349,
Fédération Denis Poisson, FR 2964,
Bâtiment de Mathématiques, BP 6759, 45067 Orléans Cedex 2, France,
leris98@gmail.com

S. Acinas, L. Buri, G. Giubergia, F. Mazzone and E. Schwindt

Abstract

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1 Introduction

2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. We refer to [2, 12, 19] for additional details and proofs. In the first two references scalar valued functions are considered, however the generalization of the results to vector valued functions, which is enumerated below, is direct. Last one reference considers vector valued functions.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an N-*function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \text{for } t \geq 0,$$

where $\varphi : \mathbb{R}^+ \to: \mathbb{R}^+$ is a right continuous nondecreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$. Throughout this paper, we assume that φ is a continuous function.

Given a function φ as above, we also consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties that function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the Δ_2-*condition*, denoted by $\Phi \in \Delta_2$, if there exists a constant $K > 0$ and a $t_0 \geq 0$ such that $\Phi(2t) \leqslant K\Phi(t)$ for every $t \geq t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2-*condition globally* ($\Phi \in \Delta_2$ globally).

In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$ and plain symbols indicate scalars.

For d positive integer we denote by $\mathcal{M}_d := \mathcal{M}_d([0,T])$ the set of all measurable functions defined in $[0,T]$ with values in $\mathbb{R}^d$. Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_0^T \Phi(|\boldsymbol{u}|)\, dt.$$

*SECyT-UNRC, UNSL and CONICET
†SECyT-UNRC
‡SECyT-UNRC and CONICET
§SECyT-UNRC and CONICET
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Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C_d^\Phi = C_d^\Phi([0,T])$ is defined by

$$C_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \rho_\Phi(\boldsymbol{u}) < \infty\}. \tag{1}$$

The *Orlicz space* $L_d^\Phi = L_d^\Phi([0,T])$ is the linear hull of C_d^Φ. Equivalently

$$L_d^\Phi := \{\boldsymbol{u} \in \mathcal{M}_d | \exists\lambda > 0 : \rho_\Phi(\lambda\boldsymbol{u}) < \infty\}. \tag{2}$$

The Orlicz space L_d^Φ equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup\left\{\int_0^T \boldsymbol{u}\cdot\boldsymbol{v}\,dt \big| \rho_\Psi(\boldsymbol{v}) \leqslant 1\right\},$$

is a Banach space. By $\boldsymbol{u}\cdot\boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^d$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. Sometimes the following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [12, Thm. 10.5] and [9]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k}\{1 + \rho_\Phi(k\boldsymbol{u})\}. \tag{3}$$

The subspace $E_d^\Phi = E_d^\Phi([0,T])$ is defined as the closure in L_d^Φ of the subspace L_d^∞ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E_d^Φ is the only one maximal subspace contained in the Orlicz class C_d^Φ, i.e. $\boldsymbol{u} \in E_d^\Phi$ if and only if $\rho_\Phi(\lambda\boldsymbol{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in the setting of Orlicz spaces (see [12, Thm. 9.3]). Namely, if $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$ then $\boldsymbol{u}\cdot\boldsymbol{v} \in L_1^1$ and

$$\int_0^T \boldsymbol{v}\cdot\boldsymbol{u}\,dt \leqslant \|\boldsymbol{u}\|_{L^\Phi}\|\boldsymbol{v}\|_{L^\Psi}. \tag{4}$$

If X and Y are Banach spaces, with $Y \subset X^*$ we denote by $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$ to the bilinear pairing map given by $\langle x^*, x\rangle = x^*(x)$. Hölder's inequality shows that $L_d^\Psi \subset \left[L_d^\Phi\right]^*$, where the pairing $\langle\boldsymbol{u},\boldsymbol{v}\rangle$, $\boldsymbol{u} \in L_d^\Phi$ and $\boldsymbol{v} \in L_d^\Psi$, is defined by

$$\langle\boldsymbol{v},\boldsymbol{u}\rangle = \int_0^T \boldsymbol{u}\cdot\boldsymbol{v}\,dt. \tag{5}$$

Unless $\Phi \in \Delta_2$, the relation $L_d^\Psi = \left[L_d^\Phi\right]^*$ will not hold. In general, it is true that $\left[E_d^\Phi\right]^* = L_d^\Psi$.

Like in [12], we will consider the subset $\Pi(E_d^\Phi, r)$ of L_d^Φ defined by

$$\Pi(E_d^\Phi, r) := \{\boldsymbol{u} \in L_d^\Phi | d(\boldsymbol{u}, E_d^\Phi) < r\}.$$

This set is related to the Orlicz class C_d^Φ by means of inclusions, that is,

$$\Pi(E_d^\Phi, r) \subset rC_d^\Phi \subset \overline{\Pi(E_d^\Phi, r)} \tag{6}$$

for any positive r. The proof of this fact, and similar ones, is given by real valued function in [12], the extension to $\mathbb{R}^d$-valued functions does not involve any difficulty. If $\Phi \in \Delta_2$, then the sets L_d^Φ, E_d^Φ, $\Pi(E_d^\Phi, r)$ and C_d^Φ are equal.

We define the *Sobolev-Orlicz space* $W^1L^\Phi_d$ (see [2]) by

$$W^1L^\Phi_d := \{\boldsymbol{u}|\boldsymbol{u} \text{ is absolutely continuous and } \boldsymbol{u}, \dot{\boldsymbol{u}} \in L^\Phi_d\}.$$

$W^1L^\Phi_d$ is a Banach space when it is equipped with the norm

$$\|\boldsymbol{u}\|_{W^1L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}.$$

For a function $\boldsymbol{u} \in L^1_d([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$, where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t)\,dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the setting of Orlicz-Sobolev spaces, see for example [4, 5, 6, 7, 11]. The following simple lemma is well known and we will use it systematically. For the sake of completeness, we include a brief proof of it.

Lemma 2.1. *Let $\boldsymbol{u} \in W^1L^\Phi_d$. Then $\boldsymbol{u} \in L^\infty_d([0,T])$ and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Wirtinger's inequality)} \tag{7}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi} \qquad \textit{(Sobolev's inequality)} \tag{8}$$

Proof. Since $\boldsymbol{u}$ is continuous, from the Mean Value Theorem there exists τ such that $\boldsymbol{u}(\tau) = \overline{\boldsymbol{u}}$. From Hölder's inequality and the norm of a characteristic function (see [12, Eq. (9.11)]), we have

$$|\boldsymbol{u}(t) - \overline{\boldsymbol{u}}| \leqslant \int_\tau^t |\dot{\boldsymbol{u}}(s)|ds \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi}\|1\|_{L^\Psi} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|\dot{\boldsymbol{u}}\|_{L^\Phi}. \tag{9}$$

Thus, the last inequality proves Wirtinger's inequality.

On the other hand, again by Hölder's inequality and [12, Eq. (9.11)], we obtain

$$|\overline{\boldsymbol{u}}| \leqslant \frac{1}{T}\int_0^T |\boldsymbol{u}(s)|ds \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\|\boldsymbol{u}\|_{L^\Phi}. \tag{10}$$

From (9), (10) and the fact that $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$, we obtain (8). □

If $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subespace of X, as usual we write $Y \hookrightarrow X$ and we say that Y is *embeeded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Lemma 2.1 states $W^1L^\Phi_d \hookrightarrow L^\infty_d$ and Hölder's inequality states that $L^\Psi_d \hookrightarrow \left[L^\Phi_d\right]^*$.

Given a continuous function $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, we define the composition operator $\boldsymbol{a} : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(\boldsymbol{u})(t) = a(|\boldsymbol{u}(t)|)$. We will often use the following elementary consequence of the previous lemma.

Corollary 2.2. *If $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi_d \to L^\infty_1([0,T])$ is bounded. More concretely, there exists a non decreasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(\boldsymbol{u})\|_{L^\infty([0,T])} \leqslant c(\|\boldsymbol{u}\|_{W^1L^\Phi})$. La letra c es confusa!!! Cambiar por A*

Proof. Let $\alpha \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non-decreasing majorant of a, for example $\alpha(s) := \sup_{0\leqslant t\leqslant s} a(t)$. If $\boldsymbol{u} \in W^1L^\Phi_d$ then, by Lemma 2.1,

$$a(|\boldsymbol{u}(t)|) \leqslant \alpha(\|\boldsymbol{u}\|_{L^\infty}) \leqslant a\left(\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi}\right) =: c(\|\boldsymbol{u}\|_{W^1L^\Phi}).$$

□

The following lemma is an inmediate consequence of principles related to operators of Nemitskii type, see [12, §17].

Lemma 2.3. *The composition operator $\boldsymbol{\varphi}$ acts from $\Pi(E^\Phi_d, 1)$ into C^Ψ_1.*

Proof. As a consequence of [12, Lemma 9.1] we have that $\boldsymbol{\varphi}\left(B_{L^\Phi}(0,1)\right) \subset C^\Psi_1$, where $B_X(\boldsymbol{u}_0, r)$ is the open ball with center $\boldsymbol{u}_0$ and radius $r > 0$ in the space X. Therefore, applying [12, Lemma 17.1] we deduce that $\boldsymbol{\varphi}$ acts from $\Pi(E^\Phi_d, 1)$ into C^Ψ_1. □

We also need the following technical lemma.

Lemma 2.4. *Let $\lambda > 0$ and let $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi_d, \lambda)$ converging to $\boldsymbol{u} \in \Pi(E^\Phi_d, \lambda)$ in the L^Φ-norm. Then, there exist a subsequence $\boldsymbol{u}_{n_k}$ and a real valued function $h \in \Pi\left(E^\Phi_1([0,T]), \lambda\right)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e. and $|\boldsymbol{u}_{n_k}| \leqslant h$ a.e.*

Proof. Let $r := d(\boldsymbol{u}, E^\Phi_d)$, $r < \lambda$. As $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$, there exists a subsequence (n_k) such that

$$\|\boldsymbol{u}_{n_k} - \boldsymbol{u}\|_{L^\Phi} < \frac{\lambda - r}{2} \quad \text{and} \quad \|\boldsymbol{u}_{n_k} - \boldsymbol{u}_{n_{k+1}}\|_{L^\Phi} < 2^{-(k+1)}(\lambda - r).$$

Let $h : [0,T] \to \mathbb{R}$ defined by

$$h(x) = |\boldsymbol{u}_{n_1}(x)| + \sum_{k=2}^{\infty} |\boldsymbol{u}_{n_k}(x) - \boldsymbol{u}_{n_{k-1}}(x)|. \tag{11}$$

As a consequence of [12, Lemma 10.1] we have that $d(\boldsymbol{v}, E^\Phi_d) = d(|\boldsymbol{v}|, E^\Phi_1)$ for any $\boldsymbol{v} \in L^\Phi_d$. Now

$$d(|\boldsymbol{u}_{n_1}|, E^\Phi_1) = d(\boldsymbol{u}_{n_1}, E^\Phi_d) \leqslant d(\boldsymbol{u}_{n_1}, \boldsymbol{u}) + d(\boldsymbol{u}, E^\Phi_d) < \frac{\lambda + r}{2}.$$

Then

$$d(h, E^\Phi_1) \leqslant d(h, |\boldsymbol{u}_{n_1}|) + d(|\boldsymbol{u}_{n_1}|, E^\Phi_1) < \lambda.$$

Therefore, $h \in \Pi(E^\Phi_1, \lambda)$ and $|h| < \infty$ a.e. We conclude that the series $\boldsymbol{u}_{n_1}(x) + \sum_{k=2}^{\infty}(\boldsymbol{u}_{n_k}(x) - \boldsymbol{u}_{n_{k-1}}(x))$ is absolutely convergent a.e. and this fact implies that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e. The inequality $|\boldsymbol{u}_{n_k}| \leqslant h$ follows straightforwardly from the definition of h. □

A common obstacle with Orlicz spaces, that distinguishes it from L^p spaces, is that a sequence $\boldsymbol{u}_n \in L_d^\Phi$ which is uniformly bounded by $h \in L_1^\Phi$ and a.e. convergent to $\boldsymbol{u}$ is not necessarily norm convergent. Fortunately, the subspace E_d^Φ has that property.

Lemma 2.5. *Suppose that $\boldsymbol{u}_n \in L_d^\Phi$ is a sequence such that $\boldsymbol{u}_n \to \boldsymbol{u}$ a.e. and suppose that there exist $h \in E_1^\Phi$ with $|\boldsymbol{u}_n| \leqslant h$ a.e. then $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^\Phi} \to 0$.*

Proof. [15, p. 84] and [12, Thm. 10.3]. □

We recall some useful concepts.

Definition 2.6. *Given a function $I : U \to \mathbb{R}$ where U is an open set of a Banach space X, we say that I has a Gâteaux derivative at $\boldsymbol{u} \in U$ if there exists $\boldsymbol{u}^* \in X^*$ such that for every $\boldsymbol{v} \in X$*

$$\lim_{s\to 0} \frac{I(\boldsymbol{u} + s\boldsymbol{v}) - I(\boldsymbol{u})}{s} = \langle \boldsymbol{u}^*, \boldsymbol{v}\rangle.$$

See [3] for details.

Definition 2.7. *Let X be a Banach space and let $D \subset X$. A non linear operator $T : D \to X^*$ is called* demicontinuous *if it is continuous when X is equipped with the strong topology and X^* with the weak* topology (see [10]).*

3 Differentiability of action integrals in Orlicz spaces

Definition 3.1. *We say that a function $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function if for fixed $(\boldsymbol{x}, \boldsymbol{y})$ the map $t \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is measurable and for fixed t the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is continuously differentiable for almost everywhere $t \in [0, T]$.*

In this paper, we will consider lagrangian functions satisfying the following structure conditions. We assume that there exists $\lambda > 0$ and non negative functions $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L_1^1([0, T])$, $c \in L_1^\Psi([0, T])$ and $d \in E_1^\Phi$ such that

$$|\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + d(t)\right)\right), \tag{12}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left(b(t) + \Phi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + d(t)\right)\right), \tag{13}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left(c(t) + \varphi\left(\tfrac{|\boldsymbol{y}|}{\lambda} + d(t)\right)\right). \tag{14}$$

Cambiar $d(t)$ por $f(t)$.

Remark 1. These conditions are a direct generalization of the conditions [14, Eq (a), p. 10].

In the following comments we discuss the relevance of the function d in the inequalities (12), (13) and (14). In particular, we are interested in to see when it is possible to find, for every $d \in E_1^\Phi$, a function $b \in L_1^1$and $C > 0$ such that for every $s > 0$

$$\Phi(s + d(t)) \leqslant C\Phi(s) + b(t). \tag{15}$$

In that case we can suppose $d = 0$ in (12) and (13). The same considerations should be done with $\varphi(s + d(t))$.

Remark 2. As a direct consequence of convexity, we can bound the term $\Phi(s + d(t))$, $d \in E_1^\Phi$, by the expression $\frac{1}{2}\Phi(2s) + b(t)$ where $b(t) := \frac{1}{2}\Phi(2d(t)) \in L_1^1$. Therefore, we can always assume $d = 0$ in (12) and (13) at the price of making smaller the value of λ.

Remark 3. If $\Phi \in \Delta_2$ then we can asssume $d = 0$ keeping the same value of λ. This is consequence of that a non decreasing Δ_2 function $G : \mathbb{R}^+ \to \mathbb{R}^+$ is quasi-subadditive. In fact, we suppose $s_1 \leqslant s_2$, then

$$G(s_1 + s_2) \leqslant G(2s_2) \leqslant KG(s_2) \leqslant K\left(G(s_1) + G(s_2)\right).$$

Moreover, if Φ is Δ_2 then φ is also Δ_2, as the following simple argument shows

$$2s\varphi(2s) \leqslant \alpha\Phi(2s) \leqslant K\Phi(s) \leqslant Ks\varphi(s)$$

Here we have used [12, Th. 4.1], the Δ_2 condition for Φ and the inequality $\Phi(s) \leqslant s\varphi(s)$ valid for any N-function. Therefore if Φ is Δ_2 we have that

$$\Phi\left(s + d(t)\right) \leqslant K\Phi(s) + K\Phi\left(d(t)\right) = K\Phi(s) + b_1(t),$$

where $b_1(t) = b(t) + K\Phi\left(d(t)\right) \in L_1^1([0, T])$. A similar fact holds with φ instead Φ namely

$$\varphi\left(s + d(t)\right) \leqslant c_1(t) + \varphi\left(s\right),$$

where, as consequence of Lemma 2.3 and the Δ_2 condition for Φ, we have $c_1(t) := K\varphi\left(d(t)\right) \in L_1^\Psi$.

Remark 4. If $\Phi \notin \Delta_2$ then (15) may be true or not. For example, we consider the N-function $\Phi(s) = e^s - s - 1$ which is not a Δ_2 function. We have that $E_1^\Phi = L_1^\infty$. In fact, if $d \in E_1^\Phi$ then from the inequality $1/2e^s \leqslant \Phi(s) + 1$ and since $pd \in C_1^\Phi$, for every $p > 0$, we have that $\int_0^T e^{pd(t)}dt < \infty$, for every $p > 0$. This imply $d \in L_1^\infty$. Therefore,

$$\Phi(s + d(t)) \leqslant e^{s+d(t)} \leqslant 2\|e^d\|_{L^\infty}\Phi(s) + 2\|e^d\|_{L^\infty}.$$

On the other hand, we consider the N-function $\Phi(s) = e^{s^2} - 1$. Suppose that $0 \leqslant d \in E_1^\Phi$ and $b \in L_1^1$ satisfy (15). Then

$$e^{s^2}e^{2sd(t)} \leqslant \Phi(s + d(t)) + 1 \leqslant Ce^{s^2} + b(t).$$

Dividing by e^{s^2} and taking the limit $s \to \infty$ we obtain that $d = 0$ a.e.. In other words, if $d \neq 0$ on a set with positive measure, the bounds in (12) and (13) are essentially bigger than a bound of the type $a(|\boldsymbol{x}|)\left(b(t) + \Phi\left(|\boldsymbol{y}|/\lambda\right)\right)$.

Theorem 3.2. *Let $\mathcal{L}$ be a Carathéodory function satisfying* (12)*,* (13) *and* (14)*. Then the following statements hold*

1. *The* action integral
$$I(\boldsymbol{u}) := \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\,dt \tag{16}$$
is finitely defined in $\mathcal{E}_d^\Phi(\lambda) := W^1L_d^\Phi \cap \{\boldsymbol{u} | \dot{\boldsymbol{u}} \in \Pi(E_d^\Phi, \lambda)\}$.

2. *The function I is Gâteaux differentiable on $\mathcal{E}_d^{\Phi}(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into $\left[W^1L_d^{\Phi}\right]^*$. Moreover I' is given by the following expression*

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{ D_{\boldsymbol{x}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big)\cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big)\cdot \dot{\boldsymbol{v}} \right\}\, dt. \tag{17}$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into $\left[W^1L_d^{\Phi}\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Since $\lambda\Pi(E_d^{\Phi}, r) = \Pi(E_d^{\Phi}, \lambda r)$, we have $\dot{\boldsymbol{u}}/\lambda \in \Pi(E_d^{\Phi}, 1)$. Thus, as $d \in E_1^{\Phi}$ and attending to (6), we get

$$|\dot{\boldsymbol{u}}|/\lambda + d \in \Pi(E_1^{\Phi}, 1) \subset C_1^{\Phi}. \tag{18}$$

By Corollary 2.2 we get a constant $c = c(\|\boldsymbol{u}\|_{W^1L^{\Phi}})$ such that $a(|\boldsymbol{u}(t)|) \leqslant c$, $t \in [0, T]$. Thus,

$$|\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})| \leqslant c\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}(t)|}{\lambda} + d(t)\right)\right) \in L_1^1.$$

This fact proves item 1.

We split the proof of 2 in three steps.

Step 1. The non linear operator $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ is continuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into $L_d^1([0, T])$ with the strong topology on both sets.

Let $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}_d^{\Phi}(\lambda)$, and $\boldsymbol{u} \in \mathcal{E}_d^{\Phi}(\lambda)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $W^1L_d^{\Phi}$. Then $\boldsymbol{u}_n \to \boldsymbol{u}$ in L_d^{Φ} and $\dot{\boldsymbol{u}}_n \to \dot{\boldsymbol{u}}$ in L_d^{Φ}. By Lemma 2.4 there exist a subsequence $\boldsymbol{u}_{n_k}$ and a function $h \in \Pi(E_1^{\Phi}, \lambda))$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k} \to \dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}| \leqslant h$ a.e.. Since $\boldsymbol{u}_{n_k}$, $k = 1, 2, \ldots$ is a strong convergent sequence in $W^1L_n^{\Phi}$, it is a bounded sequence in $W^1L_d^{\Phi}$. According to Lemma 2.1 and Corollary 2.2, there exists $M > 0$ such that $\|\boldsymbol{a}(\boldsymbol{u}_{n_k})\|_{L^\infty} \leqslant M$, $k = 1, 2, \ldots$. From the previous facts, (13) and (18) we get

$$|D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}_{n_k}(t), \dot{\boldsymbol{u}}_{n_k}(t))| \leqslant M\left(b(t) + \Phi\left(\frac{|h|}{\lambda} + d(t)\right)\right) \in L_1^1. \tag{19}$$

On the other hand, by the Carathéodory condition, we have

$$D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u}_{n_k}(t), \dot{\boldsymbol{u}}_{n_k}(t)) \to D_x\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad \text{for a.e } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $\boldsymbol{u} \mapsto D_y\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}})$ is continuous from $\mathcal{E}_d^{\Phi}(\lambda)$ with the strong topology into $\left[L_d^{\Phi}\right]^*$ with the weak* topology.

Let $\boldsymbol{u} \in \mathcal{E}_d^{\Phi}(\lambda)$. From (18), Lemma 2.3 and Corollary 2.2, it follows that

$$\varphi\left(\frac{|\dot{\boldsymbol{u}}|}{\lambda} + d\right) \in C_1^{\Psi} \tag{20}$$

and $\boldsymbol{a}(|\boldsymbol{u}|) \in L^{\infty}_1$. Therefore, in virtue of (14) we get

$$|D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))| \leqslant c(\|\boldsymbol{u}\|_{W^1L^\Phi}) \left(c(t) + \varphi \left(\frac{|\dot{\boldsymbol{u}}(t)|}{\lambda} + d(t) \right) \right) \in L^{\Psi}_1. \tag{21}$$

We note that (19), (21), the imbeddings $W^1L^{\Phi}_d \hookrightarrow L^{\infty}_d$ and $L^{\Psi}_d \hookrightarrow \left[L^{\Phi}_d\right]^*$ imply that the second member of (17) defines an element in $\left[W^1L^{\Phi}_d\right]^*$.

Let $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}^{\Phi}_d(\lambda)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of $W^1L^{\Phi}_d$. We must prove that $D_y\mathcal{L}(\cdot, \boldsymbol{u}_n, \dot{\boldsymbol{u}_n}) \overset{w^*}{\rightharpoonup} D_y\mathcal{L}(\cdot, \boldsymbol{u}, \dot{\boldsymbol{u}})$. On the contrary, if there exist $\boldsymbol{v} \in L^{\Phi}_d$, $\epsilon > 0$ and a subsequence of $\{\boldsymbol{u}_n\}$ (denoted $\{\boldsymbol{u}_n\}$ for simplicity) such that

$$|\langle D_{\boldsymbol{y}}\mathcal{L}(\cdot, \boldsymbol{u}_n, \dot{\boldsymbol{u}}_{\boldsymbol{n}}), \boldsymbol{v}\rangle - \langle D_{\boldsymbol{y}}\mathcal{L}(\cdot, \boldsymbol{u}, \dot{\boldsymbol{u}}), \boldsymbol{v}\rangle| \geq \epsilon. \tag{22}$$

We have $\boldsymbol{u}_n \to \boldsymbol{u}$ in L^{Φ}_d and $\dot{\boldsymbol{u}}_n \to \dot{\boldsymbol{u}}$ in L^{Φ}_d. By Lemma 2.4, there exist a subsequence $\boldsymbol{u}_{n_k}$ and a function $h \in \Pi(E^{\Phi}_1, \lambda)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k} \to \dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}| \leqslant h$ a.e. As in the previous step, since $\boldsymbol{u}_n$ is a convergent sequence, the Corollary 2.2 implies that $a(|\boldsymbol{u}_n(t)|)$ is uniformly bounded by a certain constant C. Therefore, from (14), (20), the fact that $c \in L^{\Psi}_1$ and Hölder's inequality, we obtain

$$|D_y\mathcal{L}(\cdot, \boldsymbol{u}_n, \dot{\boldsymbol{u}}_n) \cdot \boldsymbol{v}| \leqslant C \left(c + \varphi \left(\frac{h}{\lambda} + d \right) \right) |\boldsymbol{v}| \in L^1_1.$$

From the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}_{n_k}, \dot{\boldsymbol{u}}_{n_k}) \cdot \boldsymbol{v}\, dt \to \int_0^T D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}, \dot{\boldsymbol{u}}) \cdot \boldsymbol{v}\, dt \tag{23}$$

which contradicts the inequality (22). This completes the proof of step 2.

Step 3. Finally we prove 2. The proof follows similar lines that [14, Thm. 1.4]. For $\boldsymbol{u} \in \mathcal{E}^{\Phi}_d(\lambda)$ and $0 \neq \boldsymbol{v} \in W^1L^{\Phi}_d$ we define the function

$$f(s,t) := \mathcal{L}(t, \boldsymbol{u}(t) + s\boldsymbol{v}(t), \dot{\boldsymbol{u}}(t) + s\dot{\boldsymbol{v}}(t)).$$

From [12, Thm. 10.1] we obtain that if $|\boldsymbol{u}| \leqslant |\boldsymbol{v}|$ then $d(\boldsymbol{u}, E^{\Phi}_d) \leqslant d(\boldsymbol{v}, E^{\Phi}_d)$. Therefore, for $|s| \leqslant s_0 := \left(\lambda - d(\dot{\boldsymbol{u}}, E^{\Phi}_d)\right) / \|\boldsymbol{v}\|_{W^1L^\Phi}$ we have

$$d\left(\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}, E^{\Phi}_d\right) \leqslant d\left(|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}|, E^{\Phi}_1\right) \leqslant d\left(|\dot{\boldsymbol{u}}|, E^{\Phi}_1\right) + s\|\dot{\boldsymbol{v}}\|_{L^\Phi} < \lambda.$$

As a consequence $\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}} \in \Pi(E^{\Phi}_d, \lambda)$ and $|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}| \in \Pi(E^{\Phi}_1, \lambda)$. These facts imply, in virtue of Theorem 3.2 item 1, that $I(\boldsymbol{u} + s\boldsymbol{v})$ is well defined and finite for $|s| \leqslant s_0$. Using Corollary 2.2 we see that

$$\|a(|\boldsymbol{u} + s\boldsymbol{v}|)\|_{L^\infty} \leqslant c(\|\boldsymbol{u} + s\boldsymbol{v}\|_{W^1L^\Phi}) \leqslant c(\|\boldsymbol{u}\|_{W^1L^\Phi} + s_0\|\boldsymbol{v}\|_{W^1L^\Phi}).$$

Consequently, applying Chain Rule, (13), (14), the previous inequality and the monotonicity of φ and Φ, we obtain

$$\begin{aligned} |D_s f(s,t)| &= |D_{\boldsymbol{x}}\mathcal{L}(t, \boldsymbol{u} + s\boldsymbol{v}, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u} + s\boldsymbol{v}, \dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \dot{\boldsymbol{v}}| \\ &\leqslant c \left[\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + d \right) \right) |\boldsymbol{v}| \right. \\ &\quad \left. + \left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + d \right) \right) |\dot{\boldsymbol{v}}|. \right] \end{aligned} \tag{24}$$

Invoking (19), (21) with $|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|$ instead $\dot{\boldsymbol{u}}$ and taking account of $\boldsymbol{v} \in L^\infty_d$ and $\dot{\boldsymbol{v}} \in L^\Phi_d$, we show that there exists a function $g \in L^1_1([0,T],\mathbb{R}^+)$ such that $|D_s f(s,t)| \leqslant g(t)$. Consequently, I has a directional derivative and

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \frac{d}{ds} I(\boldsymbol{u} + s\boldsymbol{v})\big|_{s=0} = \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}}) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}}) \cdot \dot{\boldsymbol{v}}\}\ dt.$$

Moreover, from (19), (21), Lemma 2.1 and previous formula

$$|\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle| \leqslant C\|\boldsymbol{v}\|_{L^\infty} + C\|\dot{\boldsymbol{v}}\|_{L^\Phi} \leqslant C\|\boldsymbol{v}\|_{W^1L^\Phi}.$$

decir en inglés que c es cualquier constante Si!!!.

This completes the proof of the Gâteaux differentiability of I. Finally, the demicontinuity of $I' : \mathcal{E}^\Phi_d(\lambda) \to \left[W^1L^\Phi_d\right]^*$ is a consequence of the continuity of the mappings $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$. Indeed, if $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}^\Phi_d(\lambda)$ with $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of $W^1L^\Phi_d$ and $\boldsymbol{v} \in W^1L^\Phi_d$, then

$$\begin{aligned}\langle I'(\boldsymbol{u}_n), \boldsymbol{v}\rangle &= \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n\right) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n\right) \cdot \dot{\boldsymbol{v}}\}\ dt \\ &\to \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t,\boldsymbol{u},\dot{\boldsymbol{u}}\right) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t,\boldsymbol{u},\dot{\boldsymbol{u}}\right) \cdot \dot{\boldsymbol{v}}\}\ dt \\ &= \langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle .\end{aligned}$$

In order to prove 3, it is necessary to see that the maps $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(\cdot,\boldsymbol{u}(\cdot),\dot{\boldsymbol{u}}(\cdot))$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u}(\cdot),\dot{\boldsymbol{u}}(\cdot))$ are norm continuous from $\mathcal{E}^\Phi_d(\lambda)$ into L^1_d and L^Ψ_d respectively. The continuity of the first map has already been proved in step 1. We will prove the continuity of the second map developing a similar argument to the one given in step 2 of item 2. We consider $\boldsymbol{u}_n$ and $\boldsymbol{u}$ in $\mathcal{E}^\Phi_d(\lambda)$ with $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{W^1L^\Phi} \to 0$. By Lemma 2.4, there exist a subsequence $\boldsymbol{u}_{n_k} \in \mathcal{E}^\Phi_d(\lambda)$ and a function $h \in \Pi(E^\Phi_1, \lambda)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k} \to \dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}| \leqslant h$ a.e. Then, since $\mathcal{L}$ is a Carathéodory function we have $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}_{n_k}(t),\dot{\boldsymbol{u}}_{n_k}(t)) \to D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))$ a.e. $t \in [0,T]$. By (14) and the fact that $\Psi \in \Delta_2$, we obtain

$$\begin{aligned}|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}_{n_k}(t),\dot{\boldsymbol{u}}_{n_k}(t))| &\leqslant a(|\boldsymbol{u}_{n_k}(t)|)\left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}_{n_k}(t)|}{\lambda} + d(t)\right)\right) \\ &\leqslant C\left(c(t) + \varphi\left(\frac{|h(t)|}{\lambda} + d(t)\right)\right) \in L^\Psi_1 = E^\Psi_1.\end{aligned}$$

Therefore, invoking Lemma 2.5, we have proved that from any sequence $\boldsymbol{u}_n$ which converges to $\boldsymbol{u}$ in $W^1L^\Phi_d$ we can extract a subsequence such that $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of I' follows from the continuity of $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ by using the formula (17). $\square$

4 Critical points and Euler-Lagrange equations

In this section we derive the Euler-Lagrange equations associated to critical points of action integrals. We denote by $W^1L^\Phi_T$ the subspace of $W^1L^\Phi_d$ containing all T-periodic functions. Similarly we consider the subspaces E^Φ_T, L^Φ_T. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f:\mathbb{R}^d\to\mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right)<\frac{1}{2}\left(f\left(\boldsymbol{x}\right)+f\left(\boldsymbol{y}\right)\right)$ for $\boldsymbol{x}\neq\boldsymbol{y}$. It is well known that if f is a strictly convex and differentiable function, then $D_{\boldsymbol{x}}f:\mathbb{R}^d\to\mathbb{R}^d$ is a one-to-one map (see, for instance [16, Thm. 12.17]).

Theorem 4.1. *Let $\boldsymbol{u}\in\mathcal{E}^\Phi_d(\lambda)$. The following statements are equivalent*

1. $I'(\boldsymbol{u})\in\left(W^1L^\Phi_T\right)^\perp$.
2. $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and* $\boldsymbol{u}$ *solve the following boundary value problem*

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))=D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \quad \text{a.e. } t\in(0,T)\\ \boldsymbol{u}(0)-\boldsymbol{u}(T)=D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0))-D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T))=0. \end{cases} \tag{25}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t,x,y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0))-D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T))=0$ is equivalent to $\dot{\boldsymbol{u}}(0)=\dot{\boldsymbol{u}}(T)$.

Proof. The condition $I'(\boldsymbol{u})\in\left(W^1L^\Phi_T\right)^\perp$ and (17) imply

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\cdot\dot{\boldsymbol{v}}(t)\,dt=-\int_0^T D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\cdot\boldsymbol{v}(t)\,dt.$$

Using [14, pp. 6–7] we obtain that $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))$ is absolutely continuous and T-periodic, therefore it is differentiable a.e. on $[0,T]$ and the first equality of (25) holds true. This complete the proof of *1.* implies *2.* The proof of *2.* implies *1.* follows easily from (17) and (25).

The last part of the theorem is a consequence of that $D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T))=D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0))=D_{\boldsymbol{y}}\mathcal{L}(T,u(T),\dot{\boldsymbol{u}}(0))$ and the injectivity of $D_{\boldsymbol{y}}\mathcal{L}(T,u(T),\cdot)$. □

5 Coercivity discussion

We recall the following usual definition in the context of calculus of variations.

Definition 5.1. *Let X be a Banach space and let D be an unbounded subset of X. Suppose $J:D\subset X\to\mathbb{R}$. We say that J is* coercive *if $J(u)\to+\infty$ when $\|\boldsymbol{u}\|_X\to+\infty$.*

It is well known that coercivity is a useful ingredient in order to establish existence of minima. Therefore, we are interested in finding conditions which ensure the coercivity of the action integral I acting on $\mathcal{E}_d^{\Phi}(\lambda)$. For this purpose, we need to introduce the following extra condition on lagrangian function $\mathcal{L}$

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geq \alpha_0 \Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t,\boldsymbol{x}), \tag{26}$$

where $\alpha_0, \Lambda > 0$ and $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a Caratheodory function, i.e. $F(t,\boldsymbol{x})$ is measurable respect to t for every fixed $\boldsymbol{x} \in \mathbb{R}^n$ and it is continuous in $\boldsymbol{x}$ for a.e. $t \in [0,T]$. We note that in virtue of (26) and (12) we have that $F(t,\boldsymbol{x}) \leqslant a(|\boldsymbol{x}|)b_0(t)$, with $b_0(t) := b(t)+\Phi(d(t)) \in L_1^1([0,T])$. In order to ensure that integral $\int_0^T F(t,\boldsymbol{u})dt$ be finite for $\boldsymbol{u} \in W^1L^{\Phi}$, we need to assume

$$|F(t,\boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t \in [0,T] \quad \text{and for every } \boldsymbol{x} \in \mathbb{R}^d. \tag{27}$$

As we shall see in Theorem 5.3, when $\mathcal{L}$ satisfies (12), (13), (14), (26) and (27), the coercivity of the action integral I is related to the coercivity of the functional

$$J_{C,\nu}(\boldsymbol{u}) := \rho_{\Phi}\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^{\Phi}}^{\nu}, \tag{28}$$

for $C, \nu > 0$. If $\Phi(x) = |x|^p/p$ then $J_{C,\nu}$ is clearly coercive for $\nu < p$. For more general Φ the situation is more interesting as it will be shown in the following lemma.

Lemma 5.2. *Let Φ and Ψ be complementary N-functions. Then:*

1. *If $C\Lambda < 1$ then $J_{C,1}$ is coercive.*

2. *If $\Psi \in \Delta_2$ globally, then there exists a constant $\alpha_{\Phi} > 1$ such that, for any $0 < \mu < \alpha_{\Phi}$,*
$$\lim_{\|\boldsymbol{u}\|_{L^{\Phi}} \to \infty} \frac{\rho_{\Phi}\left(\frac{\boldsymbol{u}}{\Lambda}\right)}{\|\boldsymbol{u}\|_{L^{\Phi}}^{\mu}} = +\infty. \tag{29}$$
In particular, the functional $J_{C,\mu}$ is coercive for every $C > 0$ and $0 < \mu < a_{\Phi}$. The constant α_{Φ} is one of the so called Matuszewska-Orlicz indices *(see [13, Ch. 11]).*

3. *If $J_{C,1}$ is coercive with $C\Lambda > 1$, then $\Psi \in \Delta_2$.*

Proof. By (3) we have

$$(1 - C\Lambda)\|\boldsymbol{u}\|_{L^{\Phi}} + C\Lambda\|\boldsymbol{u}\|_{L^{\Phi}} = \|\boldsymbol{u}\|_{L^{\Phi}} \leqslant \Lambda + \Lambda\rho_{\Phi}\left(\frac{\boldsymbol{u}}{\Lambda}\right),$$

then

$$\frac{(1 - C\Lambda)}{\Lambda}\|\boldsymbol{u}\|_{L^{\Phi}} - 1 \leqslant \rho_{\Phi}\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^{\Phi}} = J_{C,1}(\boldsymbol{u}).$$

This shows that $J_{C,1}$ is coercive and therefore item 1. is proved.

In virtue of [1, Eq. (2.8)], the Δ_2-condition for Ψ, [13, Thm. 11.7] and [13, Cor. 11.6] we obtain a constant $K>0$ and $\alpha_\Phi>1$ such that for any $0<\nu<\alpha_\Phi$, $s\geq 0$ and $r>1$

$$\Phi(rs)\geq Kr^\nu\Phi(s). \tag{30}$$

Let $1<\mu<\alpha_\Phi$ and we consider a constant $r>\Lambda$ that will be specified later. Then, from (30) and (3), we get

$$\begin{aligned}\frac{\int_0^T\Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right)\,dt}{\|\boldsymbol{u}\|^\mu_{L^\Phi}}&\geq K\left(\frac{r}{\Lambda}\right)^\nu\frac{\int_0^T\Phi(r^{-1}|\boldsymbol{u}|)\,dt}{\|\boldsymbol{u}\|^\mu_{L^\Phi}}\\&\geq K\left(\frac{r}{\Lambda}\right)^\nu\frac{r^{-1}\|\boldsymbol{u}\|_{L^\Phi}-1}{\|\boldsymbol{u}\|^\mu_{L^\Phi}}.\end{aligned}$$

We choose $r=\|\boldsymbol{u}\|_{L^\Phi}/2$. Since $\|\boldsymbol{u}\|_{L^\Phi}\to+\infty$ we can assume $\|\boldsymbol{u}\|_{L^\Phi}>2\Lambda$. Thus $r>\Lambda$ and

$$\frac{\int_0^T\Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right)dt}{\|\boldsymbol{u}\|^\mu_{L^\Phi}}\geq\frac{K}{2^\nu\Lambda^\nu}\|\boldsymbol{u}\|^{\nu-\mu}_{L^\Phi}\to+\infty\quad\text{for }\|\boldsymbol{u}\|_{L^\Phi}\to+\infty,$$

because $\nu>\mu$.

In order to prove the last item, we assume that $\Psi\notin\Delta_2$. By [12, Thm. 4.1], there exists a sequence of real numbers r_n such that $r_n\to\infty$ and

$$\lim_{n\to\infty}\frac{r_n\psi(r_n)}{\Psi(r_n)}=+\infty. \tag{31}$$

Now, we choose $\boldsymbol{u}_n$ such that $|\boldsymbol{u}_n|=\Lambda\psi(r_n)\chi_{[0,\frac{1}{\Psi(r_n)}]}$. Then, by [12, Eq. (9.11)], we get

$$\|\boldsymbol{u}_n\|_{L^\Phi}=\Lambda\frac{\psi(r_n)}{\Psi(r_n)}\Psi^{-1}(\Psi(r_n))=\Lambda\frac{r_n\psi(r_n)}{\Psi(r_n)}\to\infty,\quad\text{as}\quad n\to\infty.$$

Next, using Young's equality (see [12, Eq. (2.7)]), we have

$$\begin{aligned}J_{C,1}(\boldsymbol{u}_n)&=\int_0^T\Phi\left(\frac{|\boldsymbol{u}_n|}{\Lambda}\right)dt-C\|\boldsymbol{u}_n\|_{L^\Phi}\\&=\frac{1}{\Psi(r_n)}\left[\Phi(\psi(r_n))-C\Lambda r_n\psi(r_n)\right]\\&=\frac{1}{\Psi(r_n)}\left[r_n\psi(r_n)-\Psi(r_n)-C\Lambda r_n\psi(r_n)\right]\\&=\frac{(1-C\Lambda)r_n\psi(r_n)}{\Psi(r_n)}-1.\end{aligned}$$

From (31) and the condition $C\Lambda>1$, we obtain $J_{C,1}(\boldsymbol{u}_n)\to-\infty$, which is a contradiction. $\square$

Next, we present two theorems that establish coercivity of action integrals.

Theorem 5.3. *Let $\mathcal{L}$ be a Lagrangian function satisfying* (12)*,* (13)*,* (14)*,* (26) *and* (27)*. We assume the following conditions:*

1. *There exist a non negative function $b_1 \in L^1_1$ and a constant $\mu > 0$ such that for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^d$ and a.e. $t \in [0, T]$*

$$|F(t, \boldsymbol{x_2}) - F(t, \boldsymbol{x_1})| \leqslant b_1(t)(1 + |\boldsymbol{x_2} - \boldsymbol{x_1}|^{\mu}). \tag{32}$$

We suppose that $\mu < \alpha_\Phi$, with α_Φ as in Lemma 5.2, in the case that $\Psi \in \Delta_2$; and $\mu = 1$ if Ψ is an arbitrary N-function.

2. $$\int_0^T F(t, \boldsymbol{x})\, dt \to \infty \quad as \quad |\boldsymbol{x}| \to \infty. \tag{33}$$

3. $\Psi \in \Delta_2$ *or, alternatively,* $\alpha_0^{-1} T \Phi^{-1}(1/T) \|b_1\|_{L^1} \Lambda < 1$.

Then the action integral I is coercive.

Remark 5. (32) es más débil que las hipótesis anteriores...

Proof. In the following estimates, we will use (26), the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, Hölder's inequality and Wirtinger's inequality (7). Namely,

$$\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t, \boldsymbol{u})\, dt \\
&= \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T [F(t, \boldsymbol{u}) - F(t, \overline{\boldsymbol{u}})]\, dt + \int_0^T F(t, \overline{\boldsymbol{u}})\, dt \\
&\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - \int_0^T b_1(t)(1 + |\tilde{\boldsymbol{u}}(t)|^{\mu})\, dt + \int_0^T F(t, \overline{\boldsymbol{u}})\, dt \\
&\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - \|b_1\|_{L^1}(1 + \|\tilde{\boldsymbol{u}}\|^{\mu}_{L^\infty}) + \int_0^T F(t, \overline{\boldsymbol{u}})\, dt \\
&\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - \|b_1\|_{L^1}\left(1 + \left[T\Phi^{-1}\left(\frac{1}{T}\right)\right]^{\mu} \|\dot{\boldsymbol{u}}\|^{\mu}_{L^\Phi}\right) \\
&\quad + \int_0^T F(t, \overline{\boldsymbol{u}})\, dt \\
&= \alpha_0 J_{C,\mu}(\dot{\boldsymbol{u}}) - \|b_1\|_{L^1} + \int_0^T F(t, \overline{\boldsymbol{u}})\, dt,
\end{aligned} \tag{34}$$

where $C = \alpha_0^{-1}\left[T\Phi^{-1}(1/T)\right]^{\mu}\|b_1\|_{L^1}$. Suppose that $\boldsymbol{u}_n$ is a sequence in $\mathcal{E}^\Phi_d(\lambda)$ such that the sequence $\overline{\boldsymbol{u}}_n$ is bounded in $\mathbb{R}^d$ and $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$. Then the Wirtinger's inequality implies that $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$. Therefore, one of the following statements holds true $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$ or $|\overline{\boldsymbol{u}}_n| \to \infty$. On the other hand, (27) and (33) imply that the integral $\int_0^T F(t, \overline{\boldsymbol{u}}_n)\, dt$ is lower bounded. These observations, the lower bound of I (34), assumption 3 in Theorem 5.3 and Lemma 5.2 imply the desired result. □

Following [14] we say that F satisfies the condition (A) if it verifies (27) and

Following [14] we said that F satisfies the condition (A) if $F(t,\boldsymbol{x})$ is a Caratheodory function, F satisfies (27) and F is continuously differentiable with respect to $\boldsymbol{x}$. Moreover, the following inequality holds

$$|D_{\boldsymbol{x}}F(t,\boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t\in[0,T] \text{ and for every } \boldsymbol{x}\in\mathbb{R}^d, \tag{35}$$

The following result was proved in [14, p. 18].

Lemma 5.4. *Suppose that F satisfies condition (A) and* (33)*, $F(t,\cdot)$ is differentiable and convex a.e. $t\in[0,T]$. Then there exists $\boldsymbol{x}_0\in\mathbb{R}^d$ such that*

$$\int_0^T D_{\boldsymbol{x}}F(t,\boldsymbol{x}_0)\,dt = 0. \tag{36}$$

Theorem 5.5. *Let $\mathcal{L}$ be as in Theorem 5.3 and let F be as in Lemma 5.4. In addition, assume that $\Psi\in\Delta_2$ or, alternatively $\alpha_0^{-1}T\Phi^{-1}\left(1/T\right)a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}\Lambda<1$, with a and b_0 as in* (27) *and $\boldsymbol{x}_0\in\mathbb{R}^d$ any point satisfying* (36)*. Then I is coercive.*

Proof. Using (26), [14, Eq. (18), p.17], the decomposition $\boldsymbol{u}=\overline{\boldsymbol{u}}+\tilde{\boldsymbol{u}}$, (36), (4) and Wirtinger's inequality (7), we get

$$\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)+\int_0^T F(t,\boldsymbol{x}_0)\,dt+\int_0^T D_{\boldsymbol{x}}F(t,\boldsymbol{x}_0)\cdot(\boldsymbol{u}-\boldsymbol{x}_0)\,dt\\
&=\alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)+\int_0^T F(t,\boldsymbol{x}_0)\,dt+\int_0^T D_{\boldsymbol{x}}F(t,\boldsymbol{x}_0)\cdot\widetilde{\boldsymbol{u}}\,dt\\
&\quad+\int_0^T D_{\boldsymbol{x}}F(t,x_0)\cdot(\overline{\boldsymbol{u}}-\boldsymbol{x}_0)\,dt\\
&=\alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)+\int_0^T F(t,\boldsymbol{x}_0)\,dt+\int_0^T D_{\boldsymbol{x}}F(t,\boldsymbol{x}_0)\cdot\widetilde{\boldsymbol{u}}\,dt\\
&\geq \alpha_0\rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right)-a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}-a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}T\Phi^{-1}\left(\frac{1}{T}\right)\|\dot{\boldsymbol{u}}\|_{L^\Phi}\\
&=\alpha_0 J_{C,1}(\dot{\boldsymbol{u}})-a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}
\end{aligned}\tag{37}$$

with $C:=\alpha_0^{-1}a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}T\Phi^{-1}(1/T)$.

Let α be as in Corollary 2.2, then it is a non decreasing majorant of a. Using that $F(t,\overline{\boldsymbol{u}}/2)\leqslant(1/2)F(t,\boldsymbol{u})+(1/2)F(t,-\widetilde{\boldsymbol{u}})$ and taking into account that Φ is a non negative function, inequality (27), Hölder's inequality, Corollary 2.2 and Wirtinger's inequality, we obtain

$$
\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - \int_0^T F(t, -\widetilde{\boldsymbol{u}})\, dt \\
&\geq 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - \|b_0\|_{L^1} \|\boldsymbol{a}(\tilde{\boldsymbol{u}})\|_{L^\infty} \\
&\geq 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - \|b_0\|_{L^1} \alpha(\|\tilde{\boldsymbol{u}}\|_{L^\infty}) \\
&\geq 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - C\alpha(C\|\dot{\boldsymbol{u}}\|_{L^\Phi}).
\end{aligned}
\tag{38}
$$

Let $\boldsymbol{u}_n$ be a sequence in $W^1L^\Phi_d$ such that $\|\boldsymbol{u}_n\|_{W^1L^\Phi} \to \infty$. We need to consider two situations:
i) If $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi} \to \infty$ then, from (37) and Lemma 5.2, we have $I(\boldsymbol{u}_n) \to \infty$.
ii) If $\|\dot{\boldsymbol{u}}_n\|_{L^\Phi}$ is bounded and $\|\boldsymbol{u}_n\|_{L^\Phi} \to \infty$, then we obtain $\overline{\boldsymbol{u}}_n \to \infty$, reasoning in a similar way to that developed in the proof of Theorem 5.3. This fact together with (38) finishes the proof.

□

6 Weak lower semicontinuity of actions integrals

Lemma 6.1. *If the sequence $\{\boldsymbol{u}_k\}_{k\geq 1}$ converges weakly to $\boldsymbol{u}$ in W^1L^Φ, then $\{\boldsymbol{u}_k\}_{k\geq 1}$ converges uniformly to $\boldsymbol{u}$ on $[0, T]$.*

Proof. By Lemma 2.1, the injection of W^1L^Φ in L^∞ is continuous. Since $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in W^1L^Φ it follows that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $C(0, T; \mathbb{R}^n)$. Since $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in W^1L^Φ, we know that $\{\boldsymbol{u}_k\}_{k\geq 1}$ is bounded in W^1L^Φ and, hence by (**??**) in $C(0, T; \mathbb{R}^n)$. Moreover, the sequence $\{\boldsymbol{u}_k\}_{k\geq 1}$ is equi-uniformly continuous since, for $0 \leqslant s \leqslant t \leqslant T$, we have

$$
\begin{aligned}
|\boldsymbol{u}_k(t) - \boldsymbol{u}_k(s)| \leqslant \int_s^t |\dot{\boldsymbol{u}}_k(\tau)| \;\; d\tau &\leqslant \|t - s\|_{L^\Psi} \|\dot{\boldsymbol{u}}_k\|_{L^\Phi} \\
&\leqslant \|t - s\|_{L^\Psi} \|\boldsymbol{u}_k\|_{W^1L^\Phi} \leqslant C\|t - s\|_{L^\Psi}.
\end{aligned}
$$

By Arzela-Ascoli theorem, $\{\boldsymbol{u}_k\}_{k\geq 1}$ is relatively compact in $C(0, T; \mathbb{R}^n)$. By the uniqueness of the weak limit in $C(0, T; \mathbb{R}^n)$, every uniformly convergent subsequence of $\{\boldsymbol{u}_k\}_{k\geq 1}$ converges to $\boldsymbol{u}$. Thus, $\{\boldsymbol{u}_k\}_{k\geq 1}$ converges uniformly on $[0, T]$.

□

Theorem 6.2. *We suppose that $\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is a Charateodory functions satisfying* (12)-(14). *Moreover we assume $\mathcal{L}(t, \boldsymbol{x}, \cdot)$ is convex for each $t, \boldsymbol{x}$. We suppose that Φ, Ψ are Δ_2 functions. Then the functional* (16) *is weakly lower semicontinuous (w.l.s.c.).*

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