

# Periodic solutions of Euler-Lagrange equations and “sublinear” potentials in an Orlicz-Sobolev space setting

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## Abstract

In this paper we obtain existence of periodic solutions, in the Orlicz-Sobolev space  $W^1 L^\Phi([0, T])$ , of hamiltonian systems with a potential function  $F$  satisfying the inequality  $|\nabla F(t, x)| \leq b_1(t)\Phi'_0(|x|) + b_2(t)$ , with  $b_1, b_2 \in L^1$  and for certain  $N$ -functions  $\Phi_0$ .

## 1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (1)$$

where  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 1$ , is called the *Lagrange function* or *lagrangian* and the unknown function  $u : [0, T] \rightarrow \mathbb{R}^d$  is absolutely continuous. In other words, we

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are interested in finding *periodic weak solutions* of *Euler-Lagrange system of ordinary equations*. This topic was deeply addressed (studied, treated) for the *Lagrange function*

$$\mathcal{L}_{p,F}(t, x, y) = \frac{|y|^p}{p} + F(t, x), \quad (2)$$

for  $1 < p < \infty$ . For example, the classic book [1] deals mainly with problem (1), for the lagrangian  $\mathcal{L}_{2,F}$ , through various methods: direct, dual action, minimax, etc. The results in [1] were extended and improved in several articles, see [2, 3, 4, 5, 6] to cite some examples. Lagrange functions (2) for arbitrary  $1 < p < \infty$  were considered in [7, 8] and in this case (1) is reduced to the  $p$ -laplacian system

$$\begin{cases} \frac{d}{dt} (u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (3)$$

In this context, it is customary to call  $F$  a *potential function*, and it is assumed that  $F(t, x)$  is differentiable with respect to  $x$  for a.e.  $t \in [0, T]$  and the following conditions are verified:

(C)  $F$  and its gradient  $\nabla F$ , with respect to  $x \in \mathbb{R}^d$ , are Carathéodory functions, i.e. they are measurable functions with respect to  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ , and they are continuous functions with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

(A) For a.e.  $t \in [0, T]$ , it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (4)$$

In this inequality we assume that the function  $a : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and nondecreasing and  $0 \leq b \in L^1([0, T], \mathbb{R})$ .

In [9] it was treated the case of a lagrangian  $\mathcal{L}$  which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi,F}(t, x, y) = \Phi(|y|) + F(t, x), \quad (5)$$

where  $\Phi$  is an  $N$ -function (see section 2 for the definition of this concept). In the paper [9] it was assumed a condition of *bounded oscillation* on  $F$  (see xxxxx below). In this paper we shall study the condition of *sublinearity* (see [3, 4, 6, 8, 10]) on  $\nabla F$  for the lagrangian  $\mathcal{L}_{\Phi,F}$ , or more generally for lagrangians which are lower bounded by  $\mathcal{L}_{\Phi,F}$ .

The problem (1) comes from a variational one, that is, the equation in (1) is the Euler-Lagrange equation associated to the *action integral*

$$I(u) = \int_0^T \mathcal{L}(t, u(t), u'(t)) dt. \quad (6)$$

The paper is organized as follows. In section 2 we give preliminaries facts on  $N$ -functions, Orlicz-Sobolev spaces of functions and we establish our main results. Section 4 contains the proofs and section 5 provides examples of applications of our results to concrete cases. Finally, in section 6, we compare the condition of sublinearity considered in this paper with the condition of bounded oscillations studied in [9].

## 2 Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions. Classic references for these topics are [11, 12, 13].

Hereafter we denote by  $\mathbb{R}^+$  the set of all non negative real numbers. A function  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called an *N-function* if  $\Phi$  is convex and it also satisfies that

$$\lim_{t \rightarrow +\infty} \frac{\Phi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0.$$

In addition, in this paper for the sake of simplicity we assume that  $\Phi$  is differentiable and we call  $\varphi$  the derivative of  $\Phi$ . On these assumptions,  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a homeomorphism whose inverse will be denoted by  $\psi$ . We denote by  $\Psi$  the primitive of  $\psi$  that satisfies  $\Psi(0) = 0$ . Then,  $\Psi$  is an *N-function* which is called the *complementary function* of  $\Phi$ .

We recall that an *N-function*  $\Phi(u)$  has *principal part*  $f(u)$  if  $\Phi(u) = f(u)$  for large values of the argument (see [12, p. 16] and [12, Sec. 7] for properties of principal part).

There exist several orders and equivalence relations between *N-functions* (see [13, Sec. 2.2]). Following [13, Def. 1, pp. 15-16] we say that the *N-function*  $\Phi_2$  is *stronger* than the *N-function*  $\Phi_1$ , in symbols  $\Phi_1 < \Phi_2$ , if there exist  $a > 0$  and  $x_0 \geq 0$  such that

$$\Phi_1(x) \leq \Phi_2(ax), \quad x \geq x_0. \quad (7)$$

The *N-functions*  $\Phi_1$  and  $\Phi_2$  are *equivalent* ( $\Phi_1 \sim \Phi_2$ ) when  $\Phi_1 < \Phi_2$  and  $\Phi_2 < \Phi_1$ . We say that  $\Phi_2$  is *essentially stronger* than  $\Phi_1$  ( $\Phi_1 \ll \Phi_2$ ) if and only if for every  $a > 0$  there exists  $x_0 = x_0(a) \geq 0$  such that (7) holds. Finally, we say that  $\Phi_2$  is *completely stronger* than  $\Phi_1$  ( $\Phi_1 \prec \Phi_2$ ) if and only if for every  $a > 0$  there exist  $K = K(a) > 0$  and  $x_0 = x_0(a) \geq 0$  such that

$$\Phi_1(x) \leq K \Phi_2(ax), \quad x \geq x_0. \quad (8)$$

We also say that a non decreasing function  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies the  $\Delta_2^\infty$ -condition, denoted by  $\eta \in \Delta_2^\infty$ , if there exist constants  $K > 0$  and  $x_0 \geq 0$  such that

$$\eta(2x) \leq K \eta(x), \quad (9)$$

for every  $x \geq x_0$ . We note that  $\eta \in \Delta_2^\infty$  if and only if  $\eta \prec \eta$ . If  $x_0 = 0$ , the function  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is said to satisfy the  $\Delta_2$ -condition ( $\eta \in \Delta_2$ ). If there exists  $x_0 > 0$  such that inequality (9) holds for  $x \leq x_0$  we say that  $\Phi$  satisfies the  $\Delta_2^0$ -condition ( $\Phi \in \Delta_2^0$ ).

We denote by  $\alpha_\eta$  and  $\beta_\eta$  the so called *Matuszewska-Orlicz indices* of the function  $\eta$ , which are defined next. Given an increasing, unbounded, continuous function  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\eta(0) = 0$ , we define

$$\alpha_\eta := \lim_{t \rightarrow 0^+} \frac{\log \left( \sup_{u > 0} \frac{\eta(tu)}{\eta(u)} \right)}{\log(t)}, \quad \beta_\eta := \lim_{t \rightarrow +\infty} \frac{\log \left( \sup_{u > 0} \frac{\eta(tu)}{\eta(u)} \right)}{\log(t)}. \quad (10)$$

It is known that the previous limits exist and  $0 \leq \alpha_\eta \leq \beta_\eta \leq +\infty$  (see [14, p. 84]). The relation  $\beta_\eta < +\infty$  holds true if and only if  $\eta \in \Delta_2$  ([14, Thm. 11.7]). If  $(\Phi, \Psi)$  is a complementary pair of  $N$ -functions then

$$\frac{1}{\alpha_\Phi} + \frac{1}{\beta_\Psi} = 1, \quad (11)$$

(see [14, Cor. 11.6]). Therefore  $1 \leq \alpha_\Phi \leq \beta_\Phi \leq \infty$ .

If  $\eta$  is an increasing that satisfies the  $\Delta_2$ -condition, then  $\eta$  is controlled by above and below by power functions ([15, Sec. 1], [16, Eq. (2.3)-(2.4)] and [14, Thm. 11.13]). More concretely, for every  $\epsilon > 0$  there exists a constant  $K = K(\eta, \epsilon)$  such that, for every  $t, u \geq 0$ ,

$$K^{-1} \min \{t^{\beta_\eta + \epsilon}, t^{\alpha_\eta - \epsilon}\} \eta(u) \leq \eta(tu) \leq K \max \{t^{\beta_\eta + \epsilon}, t^{\alpha_\eta - \epsilon}\} \eta(u). \quad (12)$$

Let  $d$  be a positive integer. We denote by  $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$  the set of all measurable functions defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $u = (u_1, \dots, u_d)$  for  $u \in \mathcal{M}$ . For the set of functions  $\mathcal{M}$ , as for other similar sets, we will omit the reference to codomain  $\mathbb{R}^d$  when  $d = 1$ .

Given an  $N$ -function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(u) := \int_0^T \Phi(|u|) dt.$$

Here  $|\cdot|$  is the euclidean norm of  $\mathbb{R}^d$ . The *Orlicz class*  $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$  is defined by

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (13)$$

The *Orlicz space*  $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$  is the linear hull of  $C^\Phi$ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (14)$$

The Orlicz space  $L^\Phi$  equipped with the *Orlicz norm*

$$\|u\|_{L^\Phi} := \sup \left\{ \int_0^T u \cdot v dt \mid \rho_\Psi(v) \leq 1 \right\},$$

is a Banach space. By  $u \cdot v$  we denote the usual dot product in  $\mathbb{R}^d$  between  $u$  and  $v$ .

The following inequality holds for any  $u \in L^\Phi$

$$\|u\|_{L^\Phi} \leq \frac{1}{k} \{1 + \rho_\Phi(ku)\}, \quad \text{for every } k > 0. \quad (15)$$

In fact,  $\|u\|_{L^\Phi}$  is the infimum for  $k > 0$  of the r.h.s in above expression (see [12, Thm. 10.5] and [17]).

The subspace  $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$  is defined as the closure in  $L^\Phi$  of the subspace  $L^\infty([0, T], \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that  $E^\Phi$  is the only one maximal subspace contained in the Orlicz class  $C^\Phi$ , i.e.  $u \in E^\Phi$  if and only if  $\rho_\Phi(\lambda u) < \infty$  for any  $\lambda > 0$ . The equality  $L^\Phi = E^\Phi$  holds if and only if  $\Phi \in \Delta_2^\infty$ .

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [12, Thm. 9.3]). Namely, if  $u \in L^\Phi$  and  $v \in L^\Psi$  then  $u \cdot v \in L^1$  and

$$\int_0^T v \cdot u \, dt \leq \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \quad (16)$$

If  $X$  and  $Y$  are Banach spaces such that  $Y \subset X^*$ , we denote by  $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{R}$  the bilinear pairing map given by  $\langle x^*, x \rangle = x^*(x)$ . Hölder's inequality shows that  $L^\Psi \subset [L^\Phi]^*$ , where the pairing  $\langle v, u \rangle$  is defined by

$$\langle v, u \rangle = \int_0^T v \cdot u \, dt, \quad (17)$$

with  $u \in L^\Phi$  and  $v \in L^\Psi$ . Unless  $\Phi \in \Delta_2^\infty$ , the relation  $L^\Psi = [L^\Phi]^*$  will not hold. In general, it is true that  $[L^\Phi]^* = L^\Psi$ .

We define the *Sobolev-Orlicz space*  $W^1 L^\Phi$  (see [11]) by

$$W^1 L^\Phi := \{u \mid u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi\}.$$

$W^1 L^\Phi$  is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (18)$$

We introduce the following subspaces of  $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi \mid u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi \mid u(0) = u(T)\}. \end{aligned} \quad (19)$$

We will use repeatedly the decomposition  $u = \bar{u} + \tilde{u}$  for a function  $u \in L^1([0, T])$  where  $\bar{u} = \frac{1}{T} \int_0^T u(t) \, dt$  and  $\tilde{u} = u - \bar{u}$ .

As usual, if  $(X, \|\cdot\|_X)$  is a Banach space and  $(Y, \|\cdot\|_Y)$  is a subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when the restricted identity map  $i_Y : Y \rightarrow X$  is bounded. That is, there exists  $C > 0$  such that for any  $y \in Y$  we have  $\|y\|_X \leq C\|y\|_Y$ . With this notation, Hölder's inequality states that  $L^\Psi \hookrightarrow [L^\Phi]^*$ ; and, it is easy to see that for every  $N$ -function  $\Phi$  we have that  $L^\infty \hookrightarrow L^\Phi \hookrightarrow L^1$ .

Recall that a function  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is called a *modulus of continuity* if  $w$  is a continuous increasing function which satisfies  $w(0) = 0$ . For example, it can be easily shown that  $w(s) = s\Phi^{-1}(1/s)$  is a modulus of continuity for every  $N$ -function  $\Phi$ . We say that  $u : [0, T] \rightarrow \mathbb{R}^d$  has modulus of continuity  $w$  when there exists a constant  $C > 0$  such that

$$|u(t) - u(s)| \leq Cw(|t - s|). \quad (20)$$

We denote by  $C^w([0, T], \mathbb{R}^d)$  the space of  $w$ -Hölder continuous functions. This is the space of all functions satisfying (20) for some  $C > 0$ . This is a Banach space with norm

$$\|u\|_{C^w([0, T], \mathbb{R}^d)} := \|u\|_{L^\infty} + \sup_{t \neq s} \frac{|u(t) - u(s)|}{w(|t - s|)}.$$

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The next simple embedding lemma, whose proof can be found in [9], will be used systematically.

**Lemma 2.1.** *Let  $w(s) := s\Phi^{-1}(1/s)$ . Then, the following statements hold:*

1.  $W^1 L^\Phi \hookrightarrow C^w([0, T], \mathbb{R}^d)$  and for every  $u \in W^1 L^\Phi$

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} w(|t - s|) \quad (\text{Morrey's inequality}), \quad (21)$$

$$\|u\|_{L^\infty} \leq \Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality}). \quad (22)$$

2. For every  $u \in W^1 L^\Phi$  we have  $\tilde{u} \in L_d^\infty$  and

$$\|\tilde{u}\|_{L^\infty} \leq T\Phi^{-1}\left(\frac{1}{T}\right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality}). \quad (23)$$

### 3 Main results

We begin stating a lemma which establishes the coercivity of the modular function  $\rho_\Phi(u)$  with respect to certain functions of the Orlicz norm  $\Phi_0(\|u\|_{L^\Phi})$ . This lemma generalizes [9, Lemma 5.2] in two directions. The first, we consider function  $\Phi_0$  more general than in [9, Lemma 5.2]. The second one and more important, the condition  $\Delta_2$ -global on  $\Psi$  used in [9, Lemma 5.2] is relaxed to  $\Delta_2^\infty$ . We need to present this result here since the lemma introduces the function  $\Phi^*$  that will play an important role in our main theorem.

**Lemma 3.1.** *Let  $\Phi, \Psi$  be complementary  $N$ -functions with  $\Psi \in \Delta_2^\infty$ . Then there exists an  $N$ -function  $\Phi^*$ , with  $\Phi^* < \Phi$ , such that for every  $\Phi_0 \ll \Phi^*$  and  $k > 0$*

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(k\|u\|_{L^\Phi})} = \infty. \quad (24)$$

Reciprocally, if (24) holds for some  $N$ -function  $\Phi_0$ , then  $\Psi \in \Delta_2^\infty$ .

We point out that this lemma can be applied to more cases than [9, Lemma 5.2]. For example, if  $\Phi(u) = u^2$ ,  $\Phi_1$  and  $\Phi_0$  are  $N$ -functions with principal parts equal to  $u^2/\log u$  and  $u^2/(\log u)^2$  respectively, then (24) holds for  $\Phi_0$ . On the other hand,  $\Phi_0(|u|)$  is not dominated for any power function  $|u|^\alpha$  for every  $\alpha < 2$ .

As in [9] we will consider general Lagrange functions  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the structure conditions

$$|\mathcal{L}(t, x, y)| \leq a(|x|) \left( b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right) \right), \quad (A_1)$$

$$|D_x \mathcal{L}(t, x, y)| \leq a(|x|) \left( b(t) + \Phi\left(\frac{|y|}{\lambda} + f(t)\right) \right), \quad (A_2)$$

$$|D_y \mathcal{L}(t, x, y)| \leq a(|x|) \left( c(t) + \varphi\left(\frac{|y|}{\lambda} + f(t)\right) \right), \quad (A_3)$$

where  $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ ,  $\lambda > 0$ ,  $\Phi$  is an  $N$ -function,  $\varphi$  is the right continuous derivative of  $\Phi$ ,  $b \in L_1^1([0, T])$ ,  $c \in L_1^\Psi([0, T])$  and  $f \in E_1^\Phi([0, T])$ . We denote by  $\mathfrak{A}(a, b, c, \lambda, f, \Phi)$  the set of all Lagrange functions satisfying  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ .

It was shown in [9] that if  $\mathcal{L} \in \mathfrak{A}(a, b, c, \lambda, f, \Phi)$  then there exists the Gateaux derivative  $I'(u)$  of the integral functional  $I$ , defined in (6), on the subspace  $W^1 E^\Phi([0, T], \mathbb{R}^d)$ . We note that for the lagrangian  $\mathcal{L}_{\Phi, F}$  the condition (A) is equivalent to  $\mathcal{L}_{\Phi, F} \in \mathfrak{A}(a, b, 0, 1, 0, \Phi)$ .

Unlike what is usual in the literature, we do not assume the lagrangian  $\mathcal{L}$  split into two terms, one of them function of  $y$  and the other one function of  $(t, x)$ . We only suppose that  $\mathcal{L}$  is lower bounded by a function of this type. More precisely, we assume that for every  $(t, x, y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{L} \geq \mathcal{L}_{\Phi, F}, \quad \text{with } F \text{ satisfying (A) and (C), and } \Phi \text{ being } N\text{-function.} \quad (A_4)$$

Moreover, as usual it is needed to suppose that the time integral of  $F$  satisfies certain coercivity condition, see  $(A_5)$  below. However, all these hypotheses are not enough. It is necessary to assume extra conditions on the potential  $F$ . Several hypotheses were tested in the past years. The so called *subconvexity* of  $F$  was tried in [4, 2, 6] for semilinear equations and in [18, 8] for  $p$ -laplacian systems. A boundedness on the oscillation of  $F$  was studied in [9]. In [3, 8] the authors considered, for the  $p$ -laplacian case, potentials  $F$  satisfying the inequality

$$|\nabla F(t, x)| \leq b_1(t)|x|^\alpha + b_2(t), \quad (25)$$

where  $b_1, b_2 \in L^1([0, T])$  and  $\alpha < p$ . The potentials  $F$  satisfying (25) were called *sublinear nonlinearities*. In this paper, we are interested in studying this type of potentials, but with more general bounds on  $\nabla F$  which include  $N$ -functions instead of power functions, namely, we will consider inequalities like

$$|\nabla F(t, x)| \leq b_1(t)\Phi'_0(|x|) + b_2(t), \quad (A_6)$$

with  $\Phi_0$  a differentiable  $N$ -function and  $b_1, b_2 \in L^1([0, T], \mathbb{R})$ .

El siguiente enunciado tiene muchos satisfies y me parece que el resultado está enredado!!! Abajo va una versión intentando mejorar la redacción, por favor mirá si respecto el sentido del original. Quizás sean necesarios otros refinamientos!!!!!!!

**Theorem 3.2.** *Let  $\Phi$  be an  $N$ -function, whose complementary function  $\Psi$  satisfies  $\Psi \in \Delta_2^\infty$  and suppose that  $\Phi^*$  satisfies Lemma 3.1. Assume that the potential  $F$  satisfies (C), (A) and the conditions*

1.  $(A_6)$  for some  $N$ -function  $\Phi_0$  such that  $\Phi_0 \ll \Phi^*$ .

2.

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\Psi_1(\Phi'_0(2|x|))} = +\infty, \quad (A_5)$$

for some  $N$ -function  $\Psi_1$  whose complementary function  $\Phi_1$  satisfies  $\Phi_0 \ll \Phi_1 \ll \Phi^*$ .

Then, for every Lagrangian  $\mathcal{L}(t, x, y)$  such that  $D_y \mathcal{L}(0, x, y) = D_y \mathcal{L}(T, x, y)$ , strictly convex with respect to  $y \in \mathbb{R}^d$ ,  $\mathcal{L} \in \mathfrak{A}(a, b, c, \lambda, f, \Phi)$ , and satisfying  $(A_4)$  the problem (1) has at least a solution which minimizes the action integral  $I$  on  $W^1 E_T^\Phi$ .

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**Theorem 3.3.** *Let  $\Phi$  be an  $N$ -function, whose complementary function  $\Psi$  satisfies  $\Psi \in \Delta_2^\infty$ -condition and suppose that  $\Phi^*$  is given by Lemma 3.1. Moreover, assume that the potential  $F$  verifies (C), (A) and the following conditions:*

1.  $(A_6)$  for some  $N$ -function  $\Phi_0$  such that  $\Phi_0 \ll \Phi^*$ .

2.

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\Psi_2(\Phi_0'(2|x|))} = +\infty, \quad (A_5)$$

for some  $N$ -function  $\Psi_2$  whose complementary function  $\Phi_2$  satisfies  $\Phi_0 \ll \Phi_2 \ll \Phi^*$ .

Now, if the lagrangian  $\mathcal{L}(t, x, y)$  is strictly convex with respect to  $y \in \mathbb{R}^d$ ,  $\mathcal{L} \in \mathfrak{A}(a, b, c, \lambda, f, \Phi)$ ,  $D_y \mathcal{L}(0, x, y) = D_y \mathcal{L}(T, x, y)$  and  $(A_4)$  holds, then the problem (1) has at least a solution which minimizes the action integral  $I$  on  $W^1 E_T^\Phi$ .

## 4 Proofs

**Lemma 4.1.**  $E^\Phi$  is weak\* closed in  $L^\Phi$ .

*Proof.* From [13, Thm. 7, p. 110] we have that  $L^\Phi = [E^\Psi]^*$ . Then,  $L^\Phi$  is a dual and therefore we are allowed to speak about the weak\* topology of  $L^\Phi$ . Besides,  $E^\Phi$  is separable (see [13, Thm. 1, p. 87]). Let  $S = E^\Phi \cap \{u \in L^\Phi \mid \|u\|_{L^\Phi} \leq 1\}$ , then  $S$  is closed in the norm  $\|\cdot\|_{L^\Phi}$ . Now, according to [13, Cor. 5, p. 148]  $S$  is weak\* sequentially compact. Thus,  $S$  is weak\* sequentially closed because if  $u_n \in S$  and  $u_n \xrightarrow{*} u \in L^\Phi$ , then the weak\* sequential compactness implies the existence of  $v \in S$  and a subsequence  $u_{n_k}$  such that  $u_{n_k} \xrightarrow{*} v$ . Finally, by the uniqueness of the limit, we get  $u = v \in S$ . As  $E^\Psi$  is separable and  $L^\Phi = [E^\Psi]^*$ , the closed ball of  $\{u \in L^\Phi \mid \|u\|_{L^\Phi} \leq 1\}$  of  $L^\Phi$  is weak\* metrizable (see [19, Thm. 5.1, p. 138]). Thus,  $S$  is closed with respect to the weak\* topology. Now, by Krein-Smulian theorem, [19, Cor. 12.6, p. 165] implies that  $E^\Phi$  is weak\* closed.  $\square$

The following result is analogous to some lemmata in  $W^{1,p}$ , see [18].

**Lemma 4.2.** *If  $\|u\|_{W^1 L^\Phi} \rightarrow \infty$ , then  $(|\bar{u}| + \|u'\|_{L^\Phi}) \rightarrow \infty$ .*

*Proof.* By the decomposition  $u = \bar{u} + \tilde{u}$  and some elementary operations, we get

$$\|u\|_{L^\Phi} = \|\bar{u} + \tilde{u}\|_{L^\Phi} \leq \|\bar{u}\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi} = |\bar{u}| \|1\|_{L^\Phi} + \|\tilde{u}\|_{L^\Phi}. \quad (26)$$

It is known that  $L^\infty \hookrightarrow L^\Phi$ , i.e. there exists  $C_1 = C_1(T) > 0$  such that for any  $\tilde{u} \in L^\infty$  we have

$$\|\tilde{u}\|_{L^\Phi} \leq C_1 \|\tilde{u}\|_{L^\infty};$$



and, applying Sobolev's inequality, we obtain Wirtinger's inequality, that is there exists  $C_2 = C_2(T) > 0$  such that

$$\|\tilde{u}\|_{L^\Phi} \leq C_2 \|u'\|_{L^\Phi}. \quad (27)$$

Therefore, from (26), (27) and (18), we get

$$\|u\|_{W^1 L^\Phi} \leq C_3(|\bar{u}| + \|u'\|_{L^\Phi})$$

where  $C_3 = C_3(T)$ . Finally, as  $\|u\|_{W^1 L^\Phi} \rightarrow \infty$  we conclude that  $(|\bar{u}| + \|u'\|_{L^\Phi}) \rightarrow \infty$ .  $\square$

**Lemma 4.3.** *Let  $\Phi$  be a not necessarily differentiable  $N$ -function and let  $\varphi$  be the right continuous derivative of  $\Phi$ . Then  $\Phi \in \Delta_2^\infty$  ( $\Phi \in \Delta_2$ ) iff  $\varphi \in \Delta_2^\infty$  ( $\varphi \in \Delta_2$ ).*

*Proof.* It is consequence of [14, Thm. 11.7] and [14, Rem. 5, p. 87].  $\square$

The following lemma improves the result on the comment at the beginning of [12, p. 24].

**Lemma 4.4.** *Let  $\Psi$  be an  $N$ -function satisfying the  $\Delta_2^\infty$ -condition. Then there exists an  $N$ -function  $\Psi^*$  such that  $\Psi^* \in \Delta_2$ ,  $\Psi \leq \Psi^*$  and for every  $a > 1$  there exists  $x_0 = x_0(a) \geq 0$  such that  $\Psi^*(x) \leq a\Psi(x)$ , for every  $x \geq x_0$ . In other words, every  $\Delta_2$  near infinity  $N$ -function is equivalent to a  $\Delta_2$  globally  $N$ -function.*

*Proof.* We can assume that the  $\Delta_2$  condition on  $\Psi$  fails near to 0. Consequently, from Lemma 4.3, we have that the right continuous derivative  $\psi$  of  $\Psi$  does not satisfy the  $\Delta_2$ -condition near to 0. Therefore, we obtain a sequence of positive numbers  $x_n$ ,  $n = 1, 2, \dots$ , such that  $x_n \rightarrow 0$ ,

$$2x_{n+1} < x_n < 2x_n \quad \text{and} \quad \psi(2x_n) > 2\psi(x_n). \quad (28)$$

We define  $\psi^*$  inductively on  $n$  on the interval  $[2x_n, +\infty)$  of the following way. We put  $\psi^*(x) = \psi(x)$  when  $x \in [2x_1, +\infty)$ . Suppose  $\psi^*$  defined on  $[2x_n, +\infty)$ . We define  $\psi^*$  on  $[2x_{n+1}, 2x_n)$  by

$$\psi^*(x) = \begin{cases} \max \left\{ \psi(x), \frac{\psi^*(2x_n)}{2x_n}(x - x_n) + \frac{\psi^*(2x_n)}{2} \right\}, & \text{if } x_n \leq x < 2x_n \\ \frac{\psi^*(2x_n)}{2} & \text{if } 2x_{n+1} \leq x < x_n \end{cases}$$

Moreover we define  $\psi^*(0) = 0$ .

Aparece DEFINE MUUUUUUCHAS veces!!!! y la definici3n de la funci3n un poco enredada...

Next, we will use induction again to prove that

1.  $\psi^*(x_n) = \frac{1}{2}\psi^*(2x_n)$ ,
2.  $\psi^*$  is non-decreasing  $[2x_n, +\infty)$ ,
3.  $\psi \leq \psi^*$  in  $[2x_n, +\infty)$ .

We suppose  $n = 1$ . Then items 2 and 3 are obvious. From (28) we have

$$\psi(x_1) < \frac{1}{2}\psi(2x_1) = \frac{1}{2}\psi^*(2x_1).$$

This inequality implies 1.

Clearly  $\psi^*$  is non decreasing on each interval  $[2x_{n+1}, x_n]$  and  $[x_n, 2x_n]$ . We note that since  $\psi$  is right continuous, then  $\psi^*$  is continuous at  $x_n$ . Therefore,  $\psi^*$  is non decreasing on  $[2x_{n+1}, 2x_n]$ . Suppose  $x \in [2x_{n+1}, 2x_n]$  and  $y \geq 2x_n$ . From the definition of  $\psi^*$ , inductive hypothesis, item 3 and item 2, we obtain

$$\psi^*(x) \leq \max\{\psi(2x_n), \psi^*(2x_n)\} = \psi^*(2x_n) \leq \psi^*(y).$$

This proves item 2 on the interval  $[2x_{n+1}, +\infty)$ . Inequality in item 3 holds by inductive hypothesis on  $[2x_n, +\infty)$  and it is obvious for  $x \in [x_n, 2x_n]$ . If  $x \in [2x_{n+1}, x_n]$ , then  $\psi(x) \leq \psi(x_n) \leq \psi^*(x_n) = \psi^*(x)$ . This proves 3 on the interval  $[2x_{n+1}, +\infty)$ .

Now, using (28) and the already proved item 3 for  $n + 1$ , we deduce

$$\psi(x_{n+1}) < \frac{1}{2}\psi(2x_{n+1}) \leq \frac{1}{2}\psi^*(2x_{n+1}).$$

Then

$$\psi^*(x_{n+1}) = \max\left\{\psi(x_{n+1}), \frac{1}{2}\psi^*(2x_{n+1})\right\} = \frac{1}{2}\psi^*(2x_{n+1}),$$

i.e. we have proved item 1.

We note that

$$\psi^*(x_{n+1}) = \frac{1}{2}\psi^*(2x_{n+1}) \leq \psi^*(x_n).$$

Consequently  $\psi(x) \rightarrow 0$  when  $x \rightarrow 0$ . Therefore  $\psi^*$  is right continuous at 0 and indeed it is right continuous on  $[0, +\infty)$ . Moreover, since  $\psi(x) = \psi^*(x)$  for  $x \geq 2x_1$  being  $\psi$  the right continuous derivative of an  $N$ -function,  $\psi^*(x) \rightarrow +\infty$  when  $x \rightarrow +\infty$ . In this way

$$\Psi^*(x) := \int_0^x \psi^*(t)dt.$$

define an  $N$ -function.

Let's see that  $\psi^*$  satisfies the  $\Delta_2$ -condition. It is sufficient to prove that  $\psi^*$  satisfies the  $\Delta_2^0$  condition. To this end, suppose that  $x \leq x_1$  and take  $n \in \mathbb{N}$  such that  $x_{n+1} \leq x \leq x_n$ . Then

$$\psi^*(2x) \leq \psi^*(2x_n) = 2\psi^*(2x_{n+1}) = 4\psi^*(x_{n+1}) \leq 4\psi^*(x).$$

Thus,  $\Psi^* \in \Delta_2$  and  $\Psi \leq \Psi^*$ .

It remains to show the inequality  $\Psi^*(x) \leq a\Psi(x)$ , for every  $a > 1$  and sufficiently large  $x$ . We take  $x_0$  sufficiently to have

$$\frac{1}{a-1} \int_0^{2x_1} \psi^*(t) - \psi(t)dt < \Psi(x_0).$$

Therefore, if  $x > \max\{x_0, 2x_1\}$  then

$$\Psi^*(x) = \Psi(x) + \int_0^{2x_1} \psi^*(t) - \psi(t) dt < \Psi(x) + (a-1)\Psi(x) = a\Psi(x).$$

The last assessment of the lemma is consequence of  $\Psi(ax) > a\Psi(x)$  when  $a > 1$ .  $\square$

The following lemma is essentially known, this is basically consequence of the fact that  $\Psi \in \Delta_2^\infty$  if and only if  $\Psi \prec \Psi$ , of [13, Prop. 4, p. 20] and [13, Cor. 10, p. 30]. However, we prefer to include an alternative proof, because we do not see clearly that the results of [13] contemplates the case of the following lemma with  $N$ -functions satisfying the  $\Delta_2$ -condition.

**Lemma 4.5.** *Let  $\Phi, \Psi$  be complementary functions. The next statements are equivalent:*

1.  $\Psi \in \Delta_2$  ( $\Psi \in \Delta_2^\infty$ ).
2. *There exists an  $N$ -function  $\Phi^*$  such that*

$$\Phi(rs) \geq \Phi^*(r)\Phi(s) \quad \text{for every } r \geq 1, s \geq 0 \ (r \geq 1, s \geq 1). \quad (29)$$

*Proof.* In virtue of the comment that precedes the lemma, we only consider the case  $\Psi \in \Delta_2$ .

1) $\Rightarrow$ 2).

As a consequences of the  $\Delta_2$ -condition on  $\Psi$ , (11) and (12) we get for every  $1 < \nu < \alpha_\Phi$  a constant  $K = K_\nu > 0$  such that

$$\Phi(rs) \geq Kr^\nu \Phi(s), \quad (30)$$

for any  $1 < \nu < \alpha_\Phi$ ,  $s \geq 0$  and  $r > 1$ . This proves (29) with  $\Phi^*(r) = kr^\nu$ , which is an  $N$ -function.

2) $\Rightarrow$ 1) Next, we follow [13, p. 32, Prop. 13] and [13, p. 29, Prop. 9]. Assume that

$$\Phi^*(r)\Phi(s) \leq \Phi(rs) \quad r > 1, s \geq 0.$$

Let  $u = \Phi^*(r) \geq \Phi^*(1)$  and  $v = \Phi(s) \geq 0$ . By a well known inequality [13, p. 13, Prop. 1] and (29), for  $u \geq \Phi^*(1)$  and  $v > 0$  we have

$$\frac{uv}{\Psi^{-1}(uv)} \leq \Phi^{-1}(uv) \leq \Phi^{*-1}(u)\Phi^{-1}(v) \leq \frac{4uv}{\Psi^{*-1}(u)\Psi^{-1}(v)},$$

then

$$\Psi^{*-1}(u)\Psi^{-1}(v) \leq 4\Psi^{-1}(uv).$$

If we take  $x = \Psi_1^{-1}(u) \geq \Psi_1^{-1}(\Phi^*(1))$  and  $y = \Psi^{-1}(v) \geq 0$ , then

$$\Psi\left(\frac{xy}{4}\right) \leq \Psi^*(x)\Psi(y).$$

Now, taking  $x \geq \max\{8, \Psi^{*-1}(\Phi^*(1))\}$  we get that  $\Psi \in \Delta_2$ .  $\square$

*Remark 1.* We note that if  $\Phi^*$  satisfies (29) then  $\Phi^* < \Phi$ .

**Proof Lemma 3.1** At first, we assume that  $\Phi \in \Delta_2$ . Let  $\Phi^*$  be an  $N$  function satisfying (29). By the inequality (15), for  $r > 1$  we have

$$\int_0^T \Phi(|u|) dt \geq \Phi^*(r) \int_0^T \Phi(r^{-1}|u|) dt \geq \Phi^*(r) \{r^{-1}\|u\|_{L^\Phi} - 1\}.$$

Now, we choose  $r = \frac{\|u\|_{L^\Phi}}{2}$ , as  $\|u\|_{L^\Phi} \rightarrow \infty$  we can assume  $r > 1$ . From [13, Thm. 2 (b)(v), p. 16] we get

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(k\|u\|_{L^\Phi})} \geq \lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\Phi^*\left(\frac{\|u\|_{L^\Phi}}{2}\right)}{\Phi_0(k\|u\|_{L^\Phi})} = \infty.$$

If  $\Psi \in \Delta_2^\infty$  but  $\Psi \notin \Delta_2$ , we use Lemma 4.4. Then, there exists an  $N$ -function  $\Psi_1$  such that  $\Psi_1 \in \Delta_2$  and  $\Psi_1 \sim \Psi \leq \Psi_1$ . Let  $\Phi_1$  be the complementary function of  $\Psi_1$ . Then  $\Phi \sim \Phi_1 \leq \Phi$  (see [12, Thm. 3.1]) and  $\|\cdot\|_{L^\Phi}$  and  $\|\cdot\|_{L^{\Phi_1}}$  are equivalent norms (see [12, Thm. 13.2 and Thm. 13.3]). By the previously proved WHAT???, there exists  $\Phi_0$  satisfying (24) with  $\Phi = \Phi_1$ . Let  $C > 0$  be a constant such that  $\|\cdot\|_{L^\Phi} \leq C\|\cdot\|_{L^{\Phi_1}}$ . Then

$$\lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi(|u|) dt}{\Phi_0(k\|u\|_{L^\Phi})} \geq \lim_{\|u\|_{L^\Phi} \rightarrow \infty} \frac{\int_0^T \Phi_1(|u|) dt}{\Phi_0(kC\|u\|_{L^{\Phi_1}})} = +\infty.$$

Finally, if  $\Phi_0$  is an  $N$ -function, then  $\Phi_0(x) \geq \alpha|x|$  for  $\alpha$  small enough and  $|x| > 1$ . Therefore (24) holds for  $\Phi_0(x) = |x|$ , then [9, Lemma 5.2] implies  $\Psi \in \Delta_2^\infty$ .  $\square$

**Definition 4.6.** We define the functionals  $J_{C,\varphi} : L^\Phi \rightarrow (-\infty, +\infty]$  and  $H_{C,\varphi} : \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $C > 0$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  by

$$J_{C,\varphi}(u) := \rho_\Phi(u) - C\varphi(\|u\|_{L^\Phi}), \quad (31)$$

and

$$H_{C,\varphi}(x) := \int_0^T F(t, x) dt - C\varphi(2|x|), \quad (32)$$

respectively.

Next, we give the proof of the Theorem 3.3. Here, we will amend a mistake that was made in the end of the proof of [9, Thm. 6.2], where it was wrongly assumed that the minimum of  $I$  is on the domain of differentiability of  $I$ . We avoid this problem minimizing  $I$  on  $W^1 E_T^\Phi$ , which is a subspace weak\* closed contained in that domain.

**Proof. Theorem 3.3 .** By the decomposition  $u = \bar{u} + \tilde{u}$ , Cauchy-Schwarz's inequality and  $(A_6)$ , we have

$$\begin{aligned} & \left| \int_0^T F(t, u) - F(t, \bar{u}) dt \right| = \left| \int_0^T \int_0^1 \nabla F(t, \bar{u} + s\tilde{u}(t)) \cdot \tilde{u}(t) ds dt \right| \\ & \leq \int_0^T \int_0^1 b_1(t) \Phi'_0(|\bar{u} + s\tilde{u}(t)|) |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 b_2(t) |\tilde{u}(t)| ds dt \\ & =: I_1 + I_2. \end{aligned} \quad (33)$$

On the one hand, by Hölder's and Sobolev-Wirtinger's inequalities we estimate  $I_2$  as follows

$$I_2 \leq \|b_2\|_{L^1} \|\tilde{u}\|_{L^\infty} \leq C_1 \|u'\|_{L^\Phi}, \quad (34)$$

where  $C_1 = C_1(\|b_2\|_{L^1}, T)$ .

We note that, since  $\Phi'_0$  is an increasing function and  $\Phi'_0(x) \geq 0$  for  $x \geq 0$ , then  $\Phi'_0(a+b) \leq \Phi'_0(2a) + \Phi'_0(2b)$  for every  $a, b \geq 0$ . In this way, we have

$$\Phi'_0(|\bar{u} + s\tilde{u}(t)|) \leq \Phi'_0(2|\bar{u}|) + \Phi'_0(2\|\tilde{u}\|_{L^\infty}), \quad (35)$$

for every  $s \in [0, 1]$ . Now, inequality (35), Hölder's and Sobolev-Wirtinger's inequalities imply that

$$\begin{aligned} I_1 &\leq \Phi'_0(2|\bar{u}|) \|b_1\|_{L^1} \|\tilde{u}\|_{L^\infty} + \Phi'_0(2\|\tilde{u}\|_{L^\infty}) \|b_1\|_{L^1} \|\tilde{u}\|_{L^\infty} \\ &\leq C_2 \left\{ \Phi'_0(2|\bar{u}|) \|u'\|_{L^\Phi} + \Phi'_0(C_3 \|u'\|_{L^\Phi}) \|u'\|_{L^\Phi} \right\}, \end{aligned} \quad (36)$$

where  $C_2 = C_2(T, \|b_1\|_{L^1})$  and  $C_3 = C_3(T)$ . Next, by Young's inequality with complementary functions  $\Phi_1$  and  $\Psi_1$ , we obtain

$$\Phi'_0(2|\bar{u}|) \|u'\|_{L^\Phi} \leq \Psi_1(\Phi'_0(2|\bar{u}|)) + \Phi_1(\|u'\|_{L^\Phi}). \quad (37)$$

We have that any  $N$ -function  $\Phi_0$  satisfies the inequality  $x\Phi'_0(x) \leq \Phi_0(2x)$  (see [13, p. 17]). Moreover, since  $\Phi_0 \ll \Phi_1$  there exists  $x_0 = x_0(\Phi_0, \Phi_1, T) \geq 0$  such that  $\Phi_0(2C_3x) \leq \Phi_1(x)$ , for every  $x \geq x_0$ . Therefore,  $\Phi_0(2C_3x) \leq \Phi_1(x) + C_4$ , with  $C_4 = \Phi_0(2x_0)$ . The previous observations imply that

We have that any  $N$ -function  $\Phi_0$  satisfies the inequality  $x\Phi'_0(x) \leq \Phi_0(2x)$  (see [13, p. 17]). Moreover, since  $\Phi_0 \ll \Phi_2$  there exists  $x_0 = x_0(\Phi_0, \Phi_2, T) \geq 0$  such that  $\Phi_0(2C_3x) \leq \Phi_2(x)$ , for every  $x \geq x_0$ . Therefore,  $\Phi_0(2C_3x) \leq \Phi_2(x) + C_4$ , with  $C_4 = \Phi_0(2x_0)$ . The previous observations imply that

$$\Phi'_0(C_3 \|u'\|_{L^\Phi}) \|u'\|_{L^\Phi} \leq C_3^{-1} (\Phi_1(\|u'\|_{L^\Phi}) + C_4). \quad (38)$$

From (36), (37), (38) and (34), we have

$$I_1 + I_2 \leq C_5 \left\{ \Psi_1(\Phi'_0(2|\bar{u}|)) + \Phi_1(\|u'\|_{L^\Phi}) + \|u'\|_{L^\Phi} + 1 \right\}, \quad (39)$$

with  $C_5$  depending on  $\Phi_0, \Phi_1, T, \|b_1\|_{L^1}$  and  $\|b_2\|_{L^1}$ .

Using (33) and (39), we get

$$\begin{aligned} I(u) &\geq \rho_\Phi(u') + \int_0^T F(t, u) dt \\ &= \rho_\Phi(u') + \int_0^T [F(t, u) - F(t, \bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\ &\geq \rho_\Phi(u') - C_5 \Phi_1(\|u'\|_{L^\Phi}) + \int_0^T F(t, \bar{u}) dt - C_5 \Psi_1(\Phi'_0(2|\bar{u}|)) - C_5 \\ &\geq \rho_\Phi(u') - C_5 \Phi_1(\|u'\|_{L^\Phi}) + H_{C_5, \Psi_1 \circ \Phi_0}(\bar{u}) - C_5 \\ &= J_{C_5, \Phi_0}(u') + H_{C_5, \Psi_1 \circ \Phi_0}(\bar{u}) - C_5. \end{aligned} \quad (40)$$

Let  $u_n$  be a sequence in  $W^1 L^\Phi$  with  $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$  and we have to prove that  $I(u_n) \rightarrow \infty$ . On the contrary, suppose that for a subsequence, still denoted by  $u_n$ ,  $I(u_n)$  is upper bounded, i.e. there exists  $M > 0$  such that  $|I(u_n)| \leq M$ . As  $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$ , from Lemma 4.2, we have  $|\bar{u}_n| + \|u'_n\|_{L^\Phi} \rightarrow \infty$ . Passing to a subsequence is necessary, still denoted  $u_n$ , we can assume that  $|\bar{u}_n| \rightarrow \infty$  or  $\|u'_n\|_{L^\Phi} \rightarrow \infty$ . Now, Lemma 3.1 implies that the functional  $J_{C_5, \Phi_0}(u')$  is coercive; and, by  $(A_5)$ , the functional  $H_{C_5, \Phi_0}(\bar{u})$  is also coercive, then  $J_{C_5, \Phi_0}(u'_n) \rightarrow \infty$  or  $H_{C_5, \Phi_0}(\bar{u}_n) \rightarrow \infty$ . From the condition (A) on  $F$ , we have that on a bounded set the functional  $H_{C_5, \Phi_0}(\bar{u}_n)$  is lower bounded and also  $J_{C_5, \Phi_0}(u'_n) \geq 0$ . Therefore,  $I(u_n) \rightarrow \infty$  as  $\|u_n\|_{W^1 L^\Phi} \rightarrow \infty$  which contradicts the initial assumption on the behavior of  $I(u_n)$ .

Let  $\{u_n\} \subset W^1 E_T^\Phi$  be a minimizing sequence for the problem  $\inf \{I(u) | u \in W^1 E_T^\Phi\}$ . Since  $I(u_n)$ ,  $n = 1, 2, \dots$ , is upper bounded, the previous part of the proof shows that  $\{u_n\}$  is norm bounded in  $W^1 E^\Phi$ . Hence, by virtue of [9, Cor. 2.2], we can assume, taking a subsequence if necessary, that  $u_n$  converges uniformly to a  $T$ -periodic continuous (therefore in  $E_T^\Phi$ ) function  $u$ . As  $u'_n \in E^\Phi$  is a norm bounded sequence in  $L^\Phi$ , there exists a subsequence (again denoted by  $u'_n$ ) such that  $u'_n$  converges to a function  $v \in L^\Phi$  in the weak\* topology of  $L^\Phi$ . Since  $E^\Phi$  is weak\* closed, by Lemma 4.1,  $v \in E^\Phi$ . From this fact and the uniform convergence of  $u_n$  to  $u$ , we obtain that

$$\int_0^T \xi' \cdot u \, dt = \lim_{n \rightarrow \infty} \int_0^T \xi' \cdot u_n \, dt = - \lim_{n \rightarrow \infty} \int_0^T \xi \cdot u'_n \, dt = - \int_0^T \xi \cdot v \, dt,$$

for every  $T$ -periodic function  $\xi \in C^\infty([0, T], \mathbb{R}^d) \subset E^\Psi$ . Thus  $v = u'$  a.e.  $t \in [0, T]$  (see [1, p. 6]) and  $u \in W^1 E_T^\Phi$ .

Now, taking into account the relations  $[L^1]^* = L^\infty \subset E^\Psi$  and  $L^\Phi \subset L^1$ , we have that  $u'_n$  converges to  $u'$  in the weak topology of  $L^1$ . Consequently, from the semicontinuity of  $I$  (see [9, Lemma 6.1]) we get

$$I(u) \leq \liminf_{n \rightarrow \infty} I(u_n) = \inf_{v \in W^1 E_T^\Phi} I(v).$$

Hence  $u \in W^1 E_T^\Phi$  is a minimum and, since  $I$  is Gâteaux differentiable on  $W^1 E^\Phi$  (see [9, Thm. 3.2]), therefore  $I'(u) \in (W^1 E_T^\Phi)^\perp$ . Thus,

$$\int_0^T D_y \mathcal{L}(t, u(t), u'(t)) \cdot v'(t) dt = - \int_0^T D_x \mathcal{L}(t, u(t), u'(t)) \cdot v(t) dt,$$

for every  $v \in W^1 E_T^\Phi$ . From [9, Eq. (26)] we have  $D_y \mathcal{L}(t, u(t), u'(t)) \in L^\Psi([0, T], \mathbb{R}^n) \hookrightarrow L^1([0, T], \mathbb{R}^n)$ ; and, from [9, Eq. (24)], it follows that  $D_x \mathcal{L}(t, u(t), u'(t)) \in L^1([0, T], \mathbb{R}^n)$ . Consequently, from [1, p. 6] (note that  $W^1 E_T^\Phi$  includes the periodic test functions) we obtain the absolute continuity of  $D_y \mathcal{L}(t, u(t), u'(t))$  and that the differential equations in (1) are verified. The strict convexity of  $\mathcal{L}(t, x, y)$  with respect to  $y$  and the  $T$ -periodicity with respect to  $t$  imply the boundary conditions in (1) (see [9, Thm. 4.1]).  $\square$

## 5 Examples

In this section we developed some applications of our main results so that the reader can appreciate the innovations that bring.

One of the main novelties of our work is that we obtain existence of solutions for lagrangian functions  $\mathcal{L}(t, x, y)$  that not satisfy a power like grow condition in  $y$ . is not necessarily a power function.

*Example 1.* We can applied Theorem 3.3 to Lagrangians  $\mathcal{L} = \mathcal{L}(t, x, y)$  with an exponential grow in  $y$ . For example, suppose

$$\mathcal{L}(t, x, y) = f(y) + F(t, x),$$

with  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable, stritly convex and  $f(y) \geq e^{|y|}$ . We define for  $n \geq 1$

$$\Phi(y) = e^y - \sum_{i=0}^{n-1} \frac{y^i}{i!}$$

It is easy to see that  $\Phi : [0, +\infty) \rightarrow [0, +\infty)$  is  $N$ -function. From [14, Ex. 3, p. 85] we known that  $\alpha_\Phi = n$ . Therefore, as consequence of (11) we have  $\beta_\Psi = \frac{n}{n-1} < \infty$ . Consequently  $\Psi \in \Delta_2$ . From (12) we have that for every  $1 < p < n$  there exists  $C_p > 0$  such that

$$\Phi(rs) \geq C_p r^p \Phi(s), \quad r > 1, s > 0.$$

Then the complementary pair  $(\Phi, \Psi)$  and the  $N$ -function  $\Phi^*(r) := r^p$  satisfy Lemma 3.1 for every  $1 < p < n$ . Fix arbitrary real numbers  $1 < p_0 < p_1 < p < n$  and we consider  $\Phi_i = r^{p_i}$ ,  $i = 0, 1$ . We have that  $\Phi_0 \ll \Phi_1 \ll \Phi^*$ . The conditions  $(A_6)$  and  $(A_5)$  becomes

$$|\nabla F(t, x)| \leq b_1(t)|x|^{p_0-1} + b_2(t), \quad b_1, b_2 \in L^1([0, T]) \quad (41)$$

and

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{|x|^{(p_0-1)q_1}} = +\infty, \quad q_1 = p_1/(p_1 - 1) \quad (42)$$

respectively. Since  $n$  is an arbitrary positive integer, the pair  $p_0$  and  $p_1$  of numbers with  $1 < p_0 < p_1$  is also arbitrary. For clarity we suppose  $F(t, x) = b(t)|x|^\sigma$ , for some  $1 < \sigma < \infty$  and  $b \in L^1([0, T])$ . We note that this  $F$  satisfies ((A)) and ((C)). Now we can fix any  $1 < p_0$  with  $p_0 - 1 < \sigma < p_0$  and we took  $p_1$  with  $p_1 > \sigma(\sigma - p_0 + 1)^{-1}$ . Then (41) and (42) holds. In conclusion the system (1) has solution for the Lagrangian given by  $\mathcal{L}(t, x, y) = f(y) + b(t)|x|^\sigma$  for any  $1 < \sigma$ .

*Example 2.* We want to emphasize that, even in the case of  $p$ -laplacian operator (3), our results extend previous ones (see [3, 8]), because we get bounds that may be sharper than those in [3, 8]. For example, in [8, Th. 2.1] X. Tang and X. Zhang obtained existences of solutions of (3) under the assumption (25) for any  $\alpha \in (0, p - 1)$ . Meanwhile, our Theorem 3.3 implies existence for the potential

$$F_0(t, x) = |x|^p / \ln(2 + |x|)^2.$$

We note that this  $F$  does not satisfy (25) for any  $\alpha < p - 1$ . Next we will show an  $N$ -function  $\Phi_0$  satisfying the hypothesis of Theorem 3.3 for this potential  $F_0$ .

We define

$$\Phi_0(u) = \begin{cases} \frac{p-1}{p} u^p & u \leq e \\ \frac{u^p}{\log u} - \frac{e^p}{p} & u > e \end{cases}$$

with  $p > 1$ . Next, we will establish some properties of this function  $\Phi_0$ .

**Theorem 5.1.** *If  $p \geq \frac{1+\sqrt{2}}{2}$ , then  $\Phi_0$  is a differentiable  $N$ -function. The  $N$ -function  $\Phi_0$  satisfies that for every  $\varepsilon > 0$ , there exists a positive constant  $C = C(p, \varepsilon)$  such that*

$$C^{-1}t^{p-\varepsilon}\Phi_0(u) \leq \Phi_0(tu) \leq Ct^p\Phi_0(u) \quad t \geq 1, u > 0, \quad (43)$$

*Proof.* We have

$$\varphi(u) = \Phi'_0(u) = \begin{cases} (p-1)u^{p-1} & := \varphi_1(u) \quad \text{if } u \leq e \\ \frac{u^{p-1}}{\log u} \left(p - \frac{1}{\log u}\right) & := \varphi_2(u) \quad \text{if } u \geq e \end{cases}$$

First let us see that  $\Phi'_0$  is increasing when  $p \geq \frac{1+\sqrt{2}}{2}$ . For this purpose, since  $\varphi_1(e) = \varphi_2(e)$ , it is enough to see that  $\varphi_1$  is increasing on  $[0, e]$  and  $\varphi_2$  is increasing on  $[e, \infty)$  for every  $p \geq \frac{1+\sqrt{2}}{2}$ . Clearly  $\varphi_1$  is an increasing function for  $p > 1$ . On the other hand, an elementary analysis of the function shows that  $\varphi'_2(u) > 0$  on  $[e, \infty)$  if and only if  $p \notin (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$ . Therefore  $\varphi_2$  is an increasing function when  $p \geq \frac{1+\sqrt{2}}{2}$ .

Moreover  $\varphi_2(u) \rightarrow \infty$  and  $\varphi_1(u) \rightarrow 0$  as  $u \rightarrow \infty$  and  $u \rightarrow 0$  respectively, provided that  $p > 1$ . Hence,  $\Phi_0$  is an  $N$ -function.

Next we will prove (43). If  $u \leq tu \leq e$ , then  $\Phi_0(tu) = t^p\Phi_0(u)$  and (43) holds with  $C = 1$ . If  $u \leq e \leq tu$ , as  $\frac{e^p}{p} > 0$  and  $\log(tu) \geq 1$ , we have  $\Phi_0(tu) \leq t^p u^p = \frac{p}{p-1} t^p \Phi_0(u)$ . Thus, the second inequality of (43) holds with  $C = \frac{p}{p-1}$ . On the other hand, as  $f(t) = \frac{t}{\log t}$  is increasing on  $[e, \infty)$ , then  $f((tu)^p) \geq f(e^p) = e^p/p$ . Now,

$$\begin{aligned} \Phi_0(tu) &= \frac{p(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &= \frac{(p-1)(tu)^p}{\log(tu)^p} + \frac{(tu)^p}{\log(tu)^p} - \frac{e^p}{p} \\ &\geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} \\ &\geq \frac{p-1}{p} \frac{t^\varepsilon}{\log t + 1} t^{p-\varepsilon} u^p. \end{aligned}$$

Since  $\varepsilon e^{1-\varepsilon}$  is the minimum value of  $t \mapsto \frac{t^\varepsilon}{\log t + 1}$  on the interval  $[1, +\infty)$  then

$$\Phi_0(tu) \geq \frac{p-1}{p} \varepsilon e^{1-\varepsilon} t^{p-\varepsilon} u^p,$$

which is the first inequality of (43) with  $C = \frac{p}{p-1} \varepsilon^{-1} e^{-1+\varepsilon}$ .

If  $e \leq u \leq tu$ , then

$$\Phi_0(tu) \leq \frac{t^p u^p}{\log(tu)} \leq \frac{t^p u^p}{\log(u)} = \frac{pt^p v}{\log v}, \quad (44)$$

where  $v := u^p$  and  $v \geq e^p$ . If  $\alpha > 0$ , the function  $x \mapsto \frac{x}{x-\alpha}$  is decreasing on  $(\alpha, \infty)$  and the function  $v \mapsto \frac{pv}{\log v}$  is increasing on  $[e^p, \infty)$ . Therefore, we have

$$\frac{\frac{pv}{\log v}}{\frac{pv}{\log v} - \frac{e^p}{p}} \leq \frac{e^p}{e^p - \frac{e^p}{p}} = \frac{p}{p-1}$$



for every  $v \geq e^p$ . In this way, from (44), we have

$$\Phi_0(tu) \leq \frac{pt^p}{p-1} \left( \frac{pv}{\log v} - \frac{e^p}{p} \right) = \frac{pt^p}{p-1} \left( \frac{u^p}{\log u} - \frac{e^p}{p} \right)$$

and the second inequality of (43) holds with  $C = \frac{p}{p-1}$ . For the first inequality we have, as it was proved previously,

$$\Phi_0(tu) \geq \frac{p-1}{p} \frac{(tu)^p}{\log(tu)} = \frac{p-1}{p} \frac{t^\varepsilon \log u^\varepsilon}{\log(t^\varepsilon u^\varepsilon)} \frac{t^{p-\varepsilon} u^p}{\log u}$$

Let  $f(s) = \frac{sA}{\log s + A}$  with  $s \geq 1$  and  $A \geq \varepsilon$ . If  $A \leq 1$ , the function  $f$  attains a minimum on  $[1, \infty)$  at  $s = e^{1-A}$  and the minimum value is  $f(e^{1-A}) = Ae^{1-A} \geq \varepsilon$ . If  $A > 1$ ,  $f$  is increasing on  $[1, \infty)$  and its minimum value is  $f(1) = 1$ . Then,  $f(s) \geq \varepsilon$  in any case, therefore

$$\Phi_0(tu) \geq \frac{p-1}{p} \varepsilon \frac{t^{p-\varepsilon} u^p}{\log u} \geq \frac{p-1}{p} \varepsilon t^{p-\varepsilon} \Phi_0(u).$$

Finally, (43) holds with  $C = \frac{p}{\varepsilon(p-1)}$ , because this  $C$  is the biggest constant that we have obtained in each case under consideration.  $\square$

*Remark 2.* The inequality

$$\Phi_0(tu) \geq Ct^p \Phi_0(u)$$

is false for every  $C$  because for every  $u \geq e$  we have

$$\lim_{t \rightarrow \infty} \frac{\Phi_0(tu)}{t^p \Phi_0(u)} = 0$$

We note that  $\Phi_0$  and  $F_0$  satisfy  $(A_5)$ . For the  $p$ -laplacian operator we have that  $\Phi(|u|) = |u|^p/p$ . Then we can take  $\Phi_1 = \Phi$  in (29). Clearly  $\Phi_0 \ll \Phi_1$ .

**Theorem 5.2.**  $\alpha_\Phi = \beta_\Phi = p$

*Proof.* From (10) and (43), we get

$$\beta_\Phi = \lim_{t \rightarrow \infty} \frac{\log \left[ \sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} \leq \lim_{t \rightarrow \infty} \frac{\log C + p \log t}{\log t} = p.$$

On the other hand, employing (10) and performing some elementary calculations, we obtain

$$\alpha_\Phi = \lim_{t \rightarrow 0^+} \frac{\log \left[ \sup_{u>0} \frac{\Phi(tu)}{\Phi(u)} \right]}{\log t} = \lim_{s \rightarrow \infty} \frac{\log \left[ \sup_{v>0} \frac{\Phi(v)}{\Phi(sv)} \right]^{-1}}{\log s} = \lim_{s \rightarrow \infty} \frac{\log \left[ \inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s}$$

where  $v := tu$  and  $s := \frac{1}{t}$ . Then, using (??), for every  $\varepsilon > 0$  we have

$$\alpha_\Phi = \lim_{s \rightarrow \infty} \frac{\log \left[ \inf_{v>0} \frac{\Phi(sv)}{\Phi(v)} \right]}{\log s} \geq \lim_{s \rightarrow \infty} \frac{\log C + (p - \varepsilon) \log s}{\log s} \geq p - \varepsilon,$$

therefore  $\alpha_\Phi \geq p$ .

Finally, as  $\alpha_\Phi \leq \beta_\Phi \leq p$ , we get  $\alpha_\Phi = \beta_\Phi = p$ .  $\square$

## 6 On the *BO* condition

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