

Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P)$$

where the function $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d \geq 1$ (called the *Lagrange function* or *lagrangian*) satisfying that it is measurable in t for each $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x, y) for almost every $t \in [0, T]$. The unknown function $u : [0, T] \rightarrow \mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the articles [?] for definitions and main results on anisotropic Orlicz spaces, see also [?]. These spaces allow us to unify and extend previous results on existences of solutions for systems like (P).

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Through this article we say that a function $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ is of N_∞ class if Φ is convex, $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$ and $\Phi(-y) = \Phi(y)$, and

$$\lim_{|y| \rightarrow \infty} \frac{\Phi(y)}{|y|} = +\infty. \quad (1)$$

where $|\cdot|$ denotes the euclidean norm on \mathbb{R}^d . From [?, Cor. 2.35] a N_∞ function is continuous.

Associated to Φ we have the *complementary function* Ψ which is defined in $\xi \in \mathbb{R}^d$ as

$$\Psi(\xi) = \sup_{y \in \mathbb{R}^d} y \cdot \xi - \Phi(y) \quad (2)$$

then, from the continuity of Φ and (1), we have that $\Psi : \mathbb{R}^d \rightarrow [0, \infty)$. Moreover, it is easy to see that Ψ is a convex function such that $\Psi(0) = 0$, $\Psi(-\xi) = \Psi(\xi)$ [?, Chapter 2]. Moreover Ψ satisfies (1) (see [?, Th. 2.2]). i.e. Ψ is N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\Phi_p(y) := |y|^p/p$, for $1 < p < \infty$. In this case $\Psi(\xi) = |\xi|^q/q$, $q = p/(p-1)$.

Example 1.2. If $\Phi : \mathbb{R} \rightarrow [0, +\infty)$ is a N_∞ function on \mathbb{R} then $\Phi(y) = \Phi(|y|)$ is a N_∞ function on \mathbb{R}^d . In this example, as in the previous one, the function Φ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\Phi(y)$ depends on the direction of y . For example, if $1 < p_1, p_2 < \infty$, we define $\Phi_{p_1, p_2} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty)$ by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then Φ_{p_1, p_2} is a N_∞ function. In this case the complementary function is the function Φ_{q_1, q_2} with $q_i = p_i/(p_i - 1)$.

More generally, if $\Phi_k : \mathbb{R}^d \rightarrow [0, +\infty)$, $k = 1, \dots, n$, are N_∞ functions, then $\Phi : \mathbb{R}^d \times \dots \times \mathbb{R}^d \rightarrow [0, +\infty)$ defined by $\Phi(y_1, \dots, y_n) = \Phi_1(y_1) + \dots + \Phi_n(y_n)$ is a N_∞ function. These functions are truly anisotropic, i.e. $|x| = |y|$ does not imply that $\Phi(x) = \Phi(y)$.

Example 1.4. If $\Phi : \mathbb{R} \rightarrow [0, +\infty)$ is a N_∞ function and $O \in GL(d, \mathbb{R})$, then $\Phi(y) = \Phi(Oy)$ is a N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by powers if it does not satisfy the Δ_2 condition (see xxxxx). For example $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ defined by $\Phi(y) = \exp(|y|) - 1$ is N_∞ function.

The occurrence of Orlicz Spaces in this paper obeys to we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi(\nabla_y \mathcal{L}) \leq a(x) \left\{ b(t) + \Phi\left(\frac{y}{\lambda}\right) \right\}, \quad (S)$$

for a.e. $t \in [0, T]$, where $a \in C(\mathbb{R}^d, [0, +\infty))$, $b \in L^1([0, T], [0, +\infty))$.

Our condition (S) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition (S) is equivalent to the structure condition in [?, Th. 1.4]. If Φ is a radial N_∞ function such that Ψ satisfies that Δ_2 function then (S) is essentially equivalent to conditions [?, Eq. (2)-(4)] (see xxxx mas abajo). If Φ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ then inequality (S) is related to estructure conditions like [?, Lemma 3.1, Eq. (3.1)]. As can be seen, condition (S) is a more compact expression than [?, Lemma 3.1, Eq. (3.1)] and moreover weaker, because (S) does not imply a control of $|D_{y_1} \mathcal{L}|$ independent of y_2 . We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi, F}(t, x, y) := \Phi(y) + F(t, x). \quad (3)$$

Here the function $F(t, x)$, which is often referred to potential, be differentiable with respect to x for a.e. $t \in [0, T]$. Moreover F satisfies the following conditions:

- (C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

- (A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla_x F(t, x)| \leq a(x)b(t). \quad (4)$$

where $a \in C(\mathbb{R}^d, [0, +\infty))$ and $0 \leq b \in L^1([0, T], \mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi, F}$ satisfies condition (S). In order to prove this, the only non trivial fact that we should to establish is that $\Psi(\nabla_y \mathcal{L}) \leq a(x) \{b(t) + \Phi(y/\lambda)\}$. But, from inequality xxxx below, $\Psi(\nabla_y \mathcal{L}) = \Psi(\nabla \Phi(y)) \leq \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi, F}$ leads to the system

$$\begin{cases} \frac{d}{dt} \nabla \Phi(u'(t)) = \nabla_x F(t, u(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_\Phi)$$

Problem (P_Φ) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [?] deals mainly with problem (P) , for the lagrangian $\mathcal{L}_{\Phi, F}$, with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [?] were extended and improved in several articles, see [?, ?, ?, ?] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [?, ?], among other papers, and in this case (P_Φ) is reduced to the p -laplacian system

$$\begin{cases} \frac{d}{dt} (u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P_p)$$

If Φ is as in Example 1.3 and $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Carathéodory function, then the equations (P_Φ) become

$$\begin{cases} \frac{d}{dt} (|u'_1|^{p_1-2} u'_1) = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt} (|u'_2|^{p_2-2} u'_2) = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_{p_1, p_2})$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as (p_1, p_2) -Laplacian system, see [?, ?, ?, ?, ?, ?, ?].

In conclusion, the problem (P) with conditions (S) contains several problems that have been considered by many authors in the past. Moreover, our results still improve some results on (p_1, p_2) -laplacian since our structure conditions are less restrictive even in that particular case.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^n \rightarrow [0, +\infty)$. References for these topics are [?, ?, ?, ?, ?, ?, ?].

If Φ is a N_∞ function then from convexity and $\Phi(0) = 0$ we obtain that

$$\Phi(\lambda x) \leq \lambda \Phi(x), \quad \lambda \in [0, 1], x \in \mathbb{R}^d. \quad (5)$$

One of the greatest difficulties when dealing with anisotropic Orlicz spaces is the lack of monotony with respect to the Euclidean norm, i.e. $|x| \leq |y|$ does not imply $\Phi(x) \leq \Phi(y)$. This problem is avoided if we consider functions whose values on a sphere are comparable (see[?]). However, from (5), we see that N_∞ functions have the following form of radial monotony: $|x| \leq |y|$ and $y = \lambda x$ imply $\Phi(x) \leq \Phi(y)$.

We say that $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ satisfies the Δ_2^∞ -condition, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geq 0$ such that

$$\Phi(2x) \leq K \Phi(x), \quad (6)$$

for every $|x| \geq M$. If Φ is a Δ_2 function then Φ is bounded by powers functions (see [?, Proof Lemma 2.4] and [?, Prop. 1]), i.e. there exists $1 < p < \infty$, $C > 0$ and $r \geq 0$ such that

$$\Phi(x) \leq C|x|^p, \quad |x| \geq r_0.$$

We consider that one of the most important aspects in considering N_∞ functions is that it accounts for the Lagrange functions that present faster growth than powers, for example an exponential growth. Hence we consider it important to avoid imposing hypothesis that Φ to be Δ_2 . For some results we will need that Ψ to be Δ_2 .

Let Φ_1 and Φ_2 be N_∞ functions. Following to [?] we write $\Phi_1 \rightarrow \Phi_2$ if there exists non negative numbers k and r_0 such that

$$\Phi_1(x) \leq \Phi_2(kx), \quad |x| \geq r_0. \quad (7)$$

For example if Φ is Δ_2 then there exist $p \in (1, +\infty)$ such that $\Phi \rightarrow |x|^p$. If for every $k > 0$ there exists $r_0 = r_0(k) > 0$ such that (7) holds we write $\Phi_1 \ll \Phi_2$.

If a Lagrange function

We denote by $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$, with $d \geq 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on \mathbb{R}^d and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (8)$$

The *Orlicz space* $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$ is the linear hull of C^Φ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (9)$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left(\frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0, T], \mathbb{R}^d)$ of all \mathbb{R}^d -valued essentially bounded functions. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [?, Cor. 5.1]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [?, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \quad (10)$$

By $u \cdot v$ we denote the usual dot product in \mathbb{R}^d between u and v .

We consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by the following inclusions

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (11)$$

for any positive r . This relation is a trivial generalization of [?, Thm. 5.6]. If $\Phi \in \Delta_2^\infty$, then the sets L^Φ , E^Φ , $\Pi(E^\Phi, r)$ and C^Φ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leq C \|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow [L^\Psi]^*$, where a function $v \in L^\Psi$ is associated to $\xi_v \in [L^\Psi]^*$ being

$$\langle \xi_v, u \rangle = \int_0^T v \cdot u dt, \quad (12)$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [?].

Proposition 2.1. *If Ψ satisfies the Δ_2^∞ -condition then $L^\Phi([0, T], \mathbb{R}^d) = [L^\Psi([0, T], \mathbb{R}^d)]^*$.*

Consequently if Ψ satisfies the Δ_2^∞ -condition then $L^\Phi([0, T], \mathbb{R}^d)$ can be equipped with the weak* topology.

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \in AC([0, T], \mathbb{R}^d) \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\},$$

where $AC([0, T], \mathbb{R}^d)$ denotes the space of all \mathbb{R}^d valued absolutely continuous functions defined on $[0, T]$. The space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (13)$$

We introduce the following subspaces of $W^1 L^\Phi$

$$W^1 E^\Phi = \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \quad W^1 E_T^\Phi = \{u \in W^1 E^\Phi | u(0) = u(T)\}. \quad (14)$$

In order to find a modulus of continuity for functions in $W^1 L^\Phi$, and from there, to obtain compact embedding of $W^1 L^\Phi$, we define the function $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$A_\Phi(s) = \min \{\Phi(x) | |x| = s\}, \quad (15)$$

Let us establish some elementary properties of A_Φ .

Proposition 2.2. *The function A_Φ has the following properties:*

1. A_Φ is continuous,
2. $A_\Phi(s)/s$ is increasing,
3. $A_\Phi(|x|)$ is the greatest radial minorant of $\Phi(x)$,
4. Φ is N_∞ if and only if $\lim_{s \rightarrow +\infty} A_\Phi(s)/s = +\infty$.

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [?]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ ,

$$\frac{A_\Phi(r)}{r} \leq \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leq \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily. □

Example 2.1. Let $\Phi = \Phi_{p_1, p_2}$ be the function given in Example (1.3). We show that

$$K_1 \min \left\{ \frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2} \right\} \leq A_\Phi(r) \leq \min \left\{ \frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2} \right\}$$

for some $K > 0$, for every $1 < p_1, p_2 < \infty$. The second inequality follows directly from definition of A_Φ . For the first inequality, we note that $|(y_1, y_2)| = r$ implies that $|y_1| \geq r/2$ or $|y_2| \geq r/2$. Then

$$\Phi_{p_1, p_2}(y_1, y_2) \geq K \min \left\{ \frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2} \right\}, \quad (16)$$

with $K = \min\{2^{-p_1}, 2^{-p_2}\}$.

Let us in a little digression to show that

$$A_\Phi(r) = \min \left\{ \frac{r^{p_1}}{p_1}, \frac{r^{p_2}}{p_2} \right\},$$

when $1 < p_1, p_2 \leq 2$. We apply the method of Lagrange multipliers (see [?, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1, p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2} y_1 + \lambda y_1 &= 0 \\ |y_2|^{p_2-2} y_2 + \lambda y_2 &= 0 \\ |y_1|^2 + |y_2|^2 &= r^2 \end{cases} \quad (17)$$

These equations are solved, among others, by the following two sets of critical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2 .

If (y_1, y_2) solve (17) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1, y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let L be the Lagrangian associated to the constrained problem: $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) + \lambda H(y_1, y_2)$ being $H = 0$ the constraint. We must analyze the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2H$ on the subspace M . By elementary computations we have for $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace M . We can assume that $p_1 < 2$ or $p_2 < 2$, otherwise the statement we intend to prove would be trivial. Under this assumption, we note that $(-y_2, y_1) \in M$ and $(-y_2, y_1)^t \mathcal{H}(-y_2, y_1) < 0$. Then, by second order necessary conditions [?, p.333], there cannot be a minimum at (y_1, y_2) . Therefore follows (16).

As is customary, we will use the decomposition $u = \bar{u} + \tilde{u}$ for a function $u \in L^1([0, T])$ where $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u} = u - \bar{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ be a Young's function and let $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$. Let $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined by (15). Then*

1. *For every $s, t \in [0, T]$, $s \neq t$,*

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| A_\Phi^{-1} \left(\frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq A_\Phi^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. *We have $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$ and*

$$\|\tilde{u}\|_{L^\infty} \leq T A_\Phi^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. *If Φ is N_∞ then the space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0, T], \mathbb{R}^d)$.*

Proof. By the absolutely continuity of u , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi \left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|} \right) &\leq \Phi \left(\frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr \right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi \left(\frac{u'(r)}{\|u'\|_{L^\Phi}} \right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By Proposition 2.2(3) we have $A_\Phi^{-1} \Phi(x) \geq |x|$, therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq A_\Phi^{-1} \left(\frac{1}{|s - t|} \right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned} |u(t) - \bar{u}| &= \left| \frac{1}{T} \int_0^T u(t) - u(s) ds \right| \\ &\leq \frac{1}{T} \int_0^T |u(t) - u(s)| ds \\ &\leq \|u'\|_{L^\Phi} T A_\Phi^{-1} \left(\frac{1}{T} \right) \end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi \left(\frac{\bar{u}}{\|u\|_{L^\Phi}} \right) \leq \frac{1}{T} \int_0^T \Phi \left(\frac{u(s)}{\|u\|_{L^\Phi}} \right) ds \leq \frac{1}{T}$$

Then by Proposition 2.2(3)

$$|\bar{u}| \leq A_{\Phi}^{-1} \left(\frac{1}{T} \right) \|u\|_{L^{\Phi}}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^{\infty}} &\leq |\bar{u}| + \|\tilde{u}\|_{L^{\infty}} \\ &\leq A_{\Phi}^{-1} \left(\frac{1}{T} \right) \|u\|_{L^{\Phi}} + T A_{\Phi}^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^{\Phi}} \\ &\leq A_{\Phi}^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^{\Phi}} \end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1 L^{\Phi}([0, T], \mathbb{R}^d)$. Since Φ is N_{∞} , from Proposition 2.2(4) we obtain $s A_{\Phi}^{-1}(1/s) \rightarrow 0$ when $s \rightarrow 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (??) implies that u_n is bounded in $C([0, T], \mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0, T], \mathbb{R}^d)$ with $u_{n_k} \rightarrow u$ in $C([0, T], \mathbb{R}^d)$. \square

Lemma 2.4. *Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $\Pi(E^{\Phi}, 1)$ converging to $u \in \Pi(E^{\Phi}, 1)$ in the L^{Φ} -norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0, T], \mathbb{R})$ such that $u_{n_k} \rightarrow u$ a.e. and $\Phi(u_{n_k}) \leq h$ a.e.*

Proof. Since $d(u, E^{\Phi}) < 1$ and u_n converges to u , there exists $u_0 \in E^{\Phi}$, a subsequence of u_n (again denoted u_n) and $0 < r < 1$ such that $d(u_n, u_0) < r$. Let $\lambda_0 \in (r, 1)$. By extracting more subsequences, if necessary, we can assume that $u_n \rightarrow u$ a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^{\Phi}} < \frac{1 - \lambda_0}{2^n}, \quad \text{for } n \geq 1.$$

We can assume $\lambda_n > 0$ for every $n = 0, \dots$

Let $\lambda := 1 - \sum_{n=0}^{\infty} \lambda_n$ and define $h : [0, T] \rightarrow \mathbb{R}$ by

$$h(x) = \lambda \Phi \left(\frac{u_0}{\lambda} \right) + \sum_{n=0}^{\infty} \lambda_n \Phi \left(\frac{u_{n+1} - u_n}{\lambda_n} \right). \quad (18)$$

Note that $\sum_{n=0}^{\infty} \lambda_n + \lambda = 1$, therefore for any $n = 1, \dots$

$$\begin{aligned} \Phi(u_n) &= \Phi \left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j} \right) \\ &\leq \lambda \Phi \left(\frac{u_0}{\lambda} \right) + \sum_{j=0}^{n-1} \lambda_j \Phi \left(\frac{u_{j+1} - u_j}{\lambda_j} \right) \leq h \end{aligned}$$

Since $u_0 \in E^{\Phi} \subset C^{\Phi}$ and E^{Φ} is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0, T], \mathbb{R})$. On the other hand $\|u_{n+1} - u_n\|_{L^{\Phi}} \leq \lambda_n$, therefore

$$\int_0^T \Phi \left(\frac{u_{j+1} - u_j}{\lambda_j} \right) dt \leq 1.$$

Then $h \in L^1([0, T], \mathbb{R})$.

□

3 Differentiability Gateaux of action integrals in anisotropic Orlicz spaces

Given a continuous function $a \in C(\mathbb{R}^n, \mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \rightarrow \mathcal{M}_d$ by $\mathbf{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [?].

Lemma 3.1. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\mathbf{a} : W^1 L^\Phi \rightarrow L^\infty([0, T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|\mathbf{a}(u)\|_{L^\infty([0, T])} \leq A(\|u\|_{W^1 L^\Phi})$.*

Proof. Let $A \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\| \leq s, x \in \mathbb{R}^d} |a(x)|$. If $u \in W^1 L^\Phi_d$ then, by Sobolev's inequality, for a.e. $t \in [0, T]$

$$a(u(t)) \leq \alpha(\|u\|_{L^\infty}) \leq \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi}\right) =: A(\|u\|_{W^1 L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t)) dt. \quad (19)$$

Theorem 3.2. *Let \mathcal{L} be a differentiable Carathéodory function satisfying (S). Then the following statements hold:*

1. *The action integral given by (19) is finitely defined on $\mathcal{E}^\Phi := W^1 L^\Phi \cap \{u | \dot{u} \in \Pi(E^\Phi, 1)\}$.*
2. *The function I is Gateaux differentiable on \mathcal{E}^Φ and its derivative I' is demicontinuous from \mathcal{E}^Φ into $[W^1 L^\Phi]^*$. Moreover, I' is given by the following expression*

$$\langle I'(u), v \rangle = \int_0^T \{D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v}\} dt. \quad (20)$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from \mathcal{E}^Φ into $[W^1 L^\Phi]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^\Phi$. As

$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \quad (21)$$

and (11), then $\Phi(\dot{u}(t)) \in L^1$. Now,

$$|\mathcal{L}(\cdot, u, \dot{u})| + |D_x \mathcal{L}(\cdot, u, \dot{u})| + \Psi(D_y \mathcal{L}(\cdot, u, \dot{u})) \leq A(\|u\|_{W^1 L^\Phi})(b + \Phi(\dot{u})) \in L^1, \quad (22)$$

by (S) and Lemma 3.1. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ is continuous from \mathcal{E}^Φ into $L^1([0, T])$ with the strong topology on both sets.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in \mathcal{E}^Φ and let $u \in \mathcal{E}^\Phi$ such that $u_n \rightarrow u$ in $W^1 L^\Phi$. By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leq T A_\Phi^{-1} \left(\frac{1}{T} \right) \|u_n - u\|_{L^\Phi}$$

then $u_n \rightarrow u$ uniformly. As $\dot{u}_n \rightarrow \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.4, there exist a subsequence of \dot{u}_{n_k} (again denoted \dot{u}_{n_k}) and a function $h \in L^1([0, T], \mathbb{R})$ such that $\dot{u}_{n_k} \rightarrow \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leq h$ a.e.

Since $u_{n_k}, k = 1, 2, \dots$, is a strong convergent sequence in $W^1 L^\Phi$, it is a bounded sequence in $W^1 L^\Phi$. According to item (3) of Lemma 2.3, there exists $M > 0$ such that $\|a(u_{n_k})\|_{L^\infty} \leq M, k = 1, 2, \dots$. From the previous facts and (22), we get

$$|D_x \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leq a(|u_{n_k}|)(b + \Phi(\dot{u}_{n_k})) \leq M(b + h) \in L^1.$$

On the other hand, by the continuous differentiability of \mathcal{L} , we have

$$D_x \mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow D_x \mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$ is continuous from \mathcal{E}^Φ with the strong topology into $[L^\Phi]^$ with the weak* topology.*

Let $u \in \mathcal{E}^\Phi$. From (22) it follows that

$$D_y \mathcal{L}(\cdot, u, \dot{u}) \in C^\Psi. \quad (23)$$

Así? o conviene poner la cota de $\Psi(D_y)$ explícitamente???

Note that (22), (23) and the imbeddings $W^1 L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow [L^\Phi]^*$ imply that the second member of (20) defines an element of $[W^1 L^\Phi]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \rightarrow u$ in the norm of $W^1 L^\Phi$. We must prove that $D_y \mathcal{L}(\cdot, u_n, \dot{u}_n) \xrightarrow{w^*} D_y \mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi, \epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle D_y \mathcal{L}(\cdot, u_n, \dot{u}_n), v \rangle - \langle D_y \mathcal{L}(\cdot, u, \dot{u}), v \rangle| \geq \epsilon. \quad (24)$$

We have $u_n \rightarrow u$ in L^Φ and $\dot{u}_n \rightarrow \dot{u}$ in L^Φ . By Lemma 2.4, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0, T], \mathbb{R})$ such that $u_n \rightarrow u$ uniformly, $\dot{u}_n \rightarrow \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leq h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.1 implies that $a(|u_n(t)|)$ is uniformly bounded

by a certain constant $M > 0$. Therefore, from inequality (22) with u_n instead of u , we have

$$\Psi(D_y \mathcal{L}(\cdot, u_n, \dot{u}_n)) \leq M(b + h) \in L^1. \quad (25)$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (25), we have

$$\begin{aligned} & \lambda D_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ & \leq \lambda \left[\Psi(D_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right) \right] \\ & \leq \lambda M(b + h) + \lambda \Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \quad (26)$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v \, dt \rightarrow \int_0^T D_y \mathcal{L}(t, u, \dot{u}) \cdot v \, dt \quad (27)$$

which contradicts the inequality (24). This completes the proof of step 2.

Step 3. We will prove (20). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1 L^\Phi$, we define the function

$$H(s, t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leq s_0 := \min\{(1 - d(\dot{u}, E^\Phi)) / \|v\|_{W^1 L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d(\dot{u} + s\dot{v}, E^\Phi) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.2 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leq s_0$.

We also have $\|u + sv\|_{W^1 L^\Phi} \leq \|u\|_{W^1 L^\Phi} + s_0 \|v\|_{W^1 L^\Phi}$; then, by Lemma 3.1, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leq M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leq s_0 \lambda^{-1}$, from the convexity and the parity of Φ , we get

$$\begin{aligned} \Phi(\dot{u} + s\dot{v}) &= \Phi\left((1 - s_0) \frac{\dot{u}}{1 - s_0} + s_0 \frac{s}{s_0} \dot{v}\right) \leq (1 - s_0) \Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0 \Phi\left(\frac{s}{s_0} \dot{v}\right) \\ &\leq (1 - s_0) \Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0 \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1 \end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1 - s_0}, E^\Phi\right) = \frac{1}{1 - s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1 - s_0} \in C^\Phi$.

Now, applying (22), (26), the fact that $v \in L^\infty$ and $\dot{v} \in L^\Phi$, we get

$$\begin{aligned}
 |D_s H(s, t)| &= \left| D_x \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + \lambda D_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \frac{\dot{v}}{\lambda} \right| \\
 &\leq M [b(t) + \Phi(\dot{u} + s\dot{v})] |v| \\
 &\quad + \lambda \left[\Psi(D_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right) \right] \\
 &\leq M \{ [b(t) + \Phi(\dot{u} + s\dot{v})] |v| \} + \lambda M [b(t) + \Phi(\dot{u} + s\dot{v})] + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \\
 &= M [b(t) + \Phi(\dot{u} + s\dot{v})] (|v| + \lambda) + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1.
 \end{aligned} \tag{28}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v \rangle = \frac{d}{ds} I(u + sv) \Big|_{s=0} = \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt.$$

Moreover, from the previous formula, (22), (23), and Lemma 2.3, we obtain

$$|\langle I'(u), v \rangle| \leq \|D_x \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|D_y \mathcal{L}\|_{L^\Psi} \|\dot{v}\|_{L^\Phi} \leq C \|v\|_{W^1 L^\Phi}$$

with a appropriate constant C .

This completes the proof of the Gateaux differentiability of I .

Step 4. The operator $I' : \mathcal{E}^\Phi \rightarrow [W^1 L_d^\Phi]^$ is demicontinuous.* This is a consequence of the continuity of the mappings $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \rightarrow u$ in the norm of $W^1 L^\Phi$ and $v \in W^1 L^\Phi$, then

$$\begin{aligned}
 \langle I'(u_n), v \rangle &= \int_0^T \{ D_x \mathcal{L}(t, u_n, \dot{u}_n) \cdot v + D_y \mathcal{L}(t, u_n, \dot{u}_n) \cdot \dot{v} \} dt \\
 &\rightarrow \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt \\
 &= \langle I'(u), v \rangle.
 \end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$ are norm continuous from \mathcal{E}^Φ into L^1 and L^Ψ , respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de D_x aquí!!!

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1 L^\Phi} \rightarrow 0$.

Applying Lemma 2.4 to \dot{u}_n , there exists a subsequence (denoted \dot{u}_n for simplicity) such that $\dot{u}_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot{u}_n) \leq h$ and $\dot{u}_n \rightarrow \dot{u}$ a.e.

Then, by (26) we have $\Psi(v_n) \leq m(t) \in L^1$ being $v_n := D_y \mathcal{L}(\cdot, u_n, \dot{u}_n)$ and $m(t) := M(b + h)$. In addition, from the continuous differentiability of \mathcal{L} , we have that $v_n \rightarrow v$ a.e. where $D_y \mathcal{L}(\cdot, u, \dot{u})$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\Psi(\lambda x) \leq c(|\lambda|) \Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1}) \Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \rightarrow 0$ a.e. as $n \rightarrow \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1}) K \Psi(v_n) + \Psi(v) \leq c(|\lambda|^{-1}) K [m(t) + \Psi(v)] \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n-v}{\lambda}) dt \rightarrow 0$ for every $\lambda > 0$. Thus, $v_n \rightarrow v$ in L^Ψ .

The continuity of I' follows from the continuity of $D_x \mathcal{L}$ and $D_y \mathcal{L}$ using the formula (20). \square

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