

Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt} \nabla_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P)$$

where the function $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, $d \geq 1$ (called the *Lagrange function* or *lagrangian*) satisfies that it is measurable in t for each $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x, y) for almost every $t \in [0, T]$. The unknown function $u : [0, T] \rightarrow \mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the article [18] for definitions and main results on anisotropic Orlicz spaces, see also [2]. These spaces allow us to unify and extend previous results on existence of solutions for systems like (P) .

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Through this article we say that a function $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ belongs to N_∞ class if Φ is convex, $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$ and $\Phi(-y) = \Phi(y)$, and

$$\lim_{|y| \rightarrow \infty} \frac{\Phi(y)}{|y|} = +\infty. \quad (1)$$

where $|\cdot|$ denotes the euclidean norm on \mathbb{R}^d . From [6, Cor. 2.35] an N_∞ function is continuous.

Associated to Φ we have the *complementary function* Ψ which is defined in $\xi \in \mathbb{R}^d$ as

$$\Psi(\xi) = \sup_{y \in \mathbb{R}^d} y \cdot \xi - \Phi(y), \quad (2)$$

then, from the continuity of Φ and (1), we have that $\Psi : \mathbb{R}^d \rightarrow [0, \infty)$. It is easy to see that Ψ is a convex function such that $\Psi(0) = 0$, $\Psi(-\xi) = \Psi(\xi)$ [13, Ch. 2]. Moreover, Ψ satisfies (1) (see [18, Thm. 2.2]), i.e. Ψ is an N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\Phi_p(y) := |y|^p/p$, for $1 < p < \infty$. In this case $\Psi(\xi) = |\xi|^q/q$, $q = p/(p-1)$.

Example 1.2. If $\Phi : \mathbb{R} \rightarrow [0, +\infty)$ is a N_∞ function on \mathbb{R} then $\Phi(y) = \Phi(|y|)$ is a N_∞ function on \mathbb{R}^d . In this example, as in the previous one, the function Φ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\Phi(y)$ depends on the direction of y . For example, if $1 < p_1, p_2 < \infty$, we define $\Phi_{p_1, p_2} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty)$ by

$$\Phi_{p_1, p_2}(y_1, y_2) := \frac{|y_1|^{p_1}}{p_1} + \frac{|y_2|^{p_2}}{p_2}.$$

Then Φ_{p_1, p_2} is an N_∞ function. In this case the complementary function is Φ_{q_1, q_2} with $q_i = p_i/(p_i - 1)$.

More generally, if $\Phi_k : \mathbb{R}^d \rightarrow [0, +\infty)$, $k = 1, \dots, n$, are N_∞ functions, then $\Phi : \mathbb{R}^d \times \dots \times \mathbb{R}^d \rightarrow [0, +\infty)$ defined by $\Phi(y_1, \dots, y_n) = \Phi_1(y_1) + \dots + \Phi_n(y_n)$ is an N_∞ function. These functions are truly anisotropic, i.e. $|x| = |y|$ does not imply that $\Phi(x) = \Phi(y)$.

Example 1.4. If $\Phi : \mathbb{R} \rightarrow [0, +\infty)$ is an N_∞ function and $O \in GL(d, \mathbb{R})$, then $\Phi(y) = \Phi(Oy)$ is an N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by power functions if it does not satisfy the Δ_2 condition (see xxxxx). For example $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ defined by $\Phi(y) = \exp(|y|) - 1$ is an N_∞ function.

The appearance of Orlicz Spaces in this paper is due to the fact that we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}| + |\nabla_x \mathcal{L}| + \Psi(\nabla_y \mathcal{L}) \leq a(x) \left\{ b(t) + \Phi\left(\frac{y}{\lambda}\right) \right\}, \quad (S)$$

for a.e. $t \in [0, T]$, where $a \in C(\mathbb{R}^d, [0, +\infty))$, $b \in L^1([0, T], [0, +\infty))$.

Our condition (S) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition (S) is equivalent to the structure condition in [13, Th. 1.4]. If Φ is a radial N_∞ function such that Ψ satisfies that Δ_2 function then (S) is essentially equivalent???? to conditions [1, Eq. (2)-(4)] (see xxxx mas abajo). If Φ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ then inequality (S) is related to estructure conditions like [24, Lemma 3.1, Eq. (3.1)]. As can be seen, condition (S) is a more compact expression than [24, Lemma 3.1, Eq. (3.1)] and moreover weaker, because (S) does not imply a control of $|D_{y_1} L|$ independent of y_2 . We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi, F}(t, x, y) := \Phi(y) + F(t, x). \quad (3)$$

Here the function $F(t, x)$, which is often referred to a potential, be differentiable with respect to x for a.e. $t \in [0, T]$. Moreover F satisfies the following conditions:

- (C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0, T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

- (A) For a.e. $t \in [0, T]$, it holds that

$$|F(t, x)| + |\nabla_x F(t, x)| \leq a(x)b(t). \quad (4)$$

where $a \in C(\mathbb{R}^d, [0, +\infty))$ and $0 \leq b \in L^1([0, T], \mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi, F}$ satisfies condition (S) . In order to prove this, the only non trivial fact that we should establish is that $\Psi(\nabla_y \mathcal{L}) \leq a(x) \{b(t) + \Phi(y/\lambda)\}$. But, from inequality xxxx below, $\Psi(\nabla_y \mathcal{L}) = \Psi(\nabla \Phi(y)) \leq \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi, F}$ leads to the system

$$\begin{cases} \frac{d}{dt} \nabla \Phi(u'(t)) = \nabla_x F(t, u(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_\Phi)$$

Problem (P_Φ) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [13] deals mainly with problem (P) for the lagrangian $\mathcal{L}_{\Phi, F}$ with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [13] were extended and improved in several articles, see [22, 20, 26, 21, 29] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [24, 23], among other papers. In this case, (P_Φ) is reduced to the p -laplacian system

$$\begin{cases} \frac{d}{dt} (u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P_p)$$

If Φ is as in Example 1.3 and $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Carathéodory function, then the equations (P_Φ) become

$$\begin{cases} \frac{d}{dt} (|u'_1|^{p_1-2} u'_1) = F_{x_1}(t, u) & \text{a.e. } t \in (0, T) \\ \frac{d}{dt} (|u'_2|^{p_2-2} u'_2) = F_{x_2}(t, u) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (P_{p_1, p_2})$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as (p_1, p_2) -Laplacian system, see [28, 17, 27, 14, 15, 16, 11].

In conclusion, the problem (P) subject to conditions (S) contains several problems that have been considered by many authors in the past.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^n \rightarrow [0, +\infty)$. References for these topics are [7, 18, 19].

We say that $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ satisfies the Δ_2^∞ -condition, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geq 0$ such that

$$\Phi(2x) \leq K\Phi(x), \quad (5)$$

for every $|x| \geq M$.

We denote by $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$, with $d \geq 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on \mathbb{R}^d and we write $u = (u_1, \dots, u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$ by setting

$$C^\Phi := \{u \in \mathcal{M} \mid \rho_\Phi(u) < \infty\}. \quad (6)$$

The *Orlicz space* $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$ is the linear hull of C^Φ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} \mid \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (7)$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left(\frac{u}{\lambda} \right) \leq 1 \right. \right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$ is defined as the closure in L^Φ of the subspace $L^\infty([0, T], \mathbb{R}^d)$ of all \mathbb{R}^d -valued essentially bounded functions. It is shown that (see [18, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [18, Thm. 5.2]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [18, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^\Psi}. \quad (8)$$

By $u \cdot v$ we denote the usual dot product in \mathbb{R}^d between u and v .

We consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi \mid d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (9)$$

for any positive r . This relation is a trivial generalization of [18, Thm. 5.6]. If $\Phi \in \Delta_2^\infty$, then the sets L^Φ , E^Φ , $\Pi(E^\Phi, r)$ and C^Φ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X , we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leq C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow [L^\Psi]^*$, where a function $v \in L^\Phi$ is associated to $\xi_v \in [L^\Psi]^*$ being

$$\xi_v(u) = \langle \xi_v, u \rangle = \int_0^T v \cdot u \, dt, \quad (10)$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [18].

Proposition 2.1. *If Ψ satisfies the Δ_2^∞ -condition then $L^\Phi([0, T], \mathbb{R}^d) = [L^\Psi([0, T], \mathbb{R}^d)]^*$.*

We define the *Sobolev-Orlicz space* $W^1 L^\Phi$ by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u \mid u \in AC([0, T], \mathbb{R}^d) \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\},$$

where $AC([0, T], \mathbb{R}^d)$ denotes the space of all \mathbb{R}^d valued absolutely continuous functions defined on $[0, T]$. The space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (11)$$

We introduce the following subspaces of $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi \mid u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi \mid u(0) = u(T)\}. \end{aligned} \quad (12)$$

In order to find a modulus of continuity for functions in $W^1 L^\Phi$, and from there, to obtain compact embedding of $W^1 L^\Phi$, we define the function $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$A_\Phi(s) = \min \{\Phi(x) \mid |x| = s\}, \quad (13)$$

Let us establish some elementary properties of A_Φ that we will use in this article.

Proposition 2.2. *The function A_Φ has the following properties:*

1. A_Φ is continuous,
2. $A_\Phi(s)/s$ is increasing.

3. $A_\Phi(|x|)$ is the greatest radial minorant of $\Phi(x)$,
4. Φ is N_∞ if and only if $\lim_{s \rightarrow +\infty} A_\Phi(s)/s = +\infty$.

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [6]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ ,

$$\frac{A_\Phi(r)}{r} \leq \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leq \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily. □

Example 2.1. We compute A_Φ for the function $\Phi = \Phi_{p_1, p_2}$ given in Example (1.3). We apply the method of Lagrange multipliers (see [12, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1, p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2}y_1 + \lambda y_1 & = & 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 & = & 0 \\ |y_1|^2 + |y_2|^2 & = & r^2 \end{cases} \quad (14)$$

These equations are solved, among others, by the following two sets of critical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2 .

We deal with $p_1 \leq 2$ and $p_2 \leq 2$ being one of them (suppose p_2) different from 2. The remaining cases can be treated with similar techniques.

If (y_1, y_2) solve (14) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1, y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let L be the Lagrangian associated to the constrained problem: $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) + \lambda H(y_1, y_2)$ being $H = 0$ the constraint. We must analyze the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2H$ on the subspace M . By elementary computations we have for $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace M . We note that $(-y_2, y_1) \in M$ and $(-y_2, y_1)^t \mathcal{H}(-y_2, y_1) < 0$. Then, by second order necessary conditions [12, p.333], at (y_1, y_2) there cannot be a minimum. Therefore, the minima occur at $y_1 = 0$ or $y_2 = 0$, then

$$A_\Phi(x, y) = \min\{r^{p_1}/p_1, r^{p_2}/p_2\}.$$

More generally, it holds that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leq A_\Phi \leq K_2 \min\{r^{p_1}, r^{p_2}\}$$

with $K_1, K_2 > 0$, for every $1 < p_1, p_2 < \infty$.

As it is customary, we will use the decomposition $u = \bar{u} + \tilde{u}$ for a function $u \in L^1([0, T])$ where $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u} = u - \bar{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$ be a Young's function and let $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$. Let $A_\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined by (13). Then*

1. *For every $s, t \in [0, T]$, $s \neq t$,*

$$|u(t) - u(s)| \leq \|u'\|_{L^\Phi} |s - t| A_\Phi^{-1} \left(\frac{1}{|s - t|} \right) \quad (\text{Morrey's inequality})$$

$$\|u\|_{L^\infty} \leq A_\Phi^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \quad (\text{Sobolev's inequality})$$

2. *We have $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$ and*

$$\|\tilde{u}\|_{L^\infty} \leq T A_\Phi^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. *If Φ is N_∞ then the space $W^1 L^\Phi([0, T], \mathbb{R}^d)$ is compactly embedded in the space of continuous functions $C([0, T], \mathbb{R}^d)$.*

Proof. By the absolute continuity of u , Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} \Phi \left(\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \right) &\leq \Phi \left(\frac{1}{|s - t|} \int_s^t \frac{|u'(r)|}{\|u'\|_{L^\Phi}} dr \right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi \left(\frac{|u'(r)|}{\|u'\|_{L^\Phi}} \right) dr \leq \frac{1}{|s - t|}. \end{aligned}$$

By Proposition 2.2(3) we have $A_\Phi^{-1} \Phi(x) \geq |x|$, therefore we get

$$\frac{|u(t) - u(s)|}{\|u'\|_{L^\Phi} |s - t|} \leq A_\Phi^{-1} \left(\frac{1}{|s - t|} \right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned} |u(t) - \bar{u}| &= \left| \frac{1}{T} \int_0^T u(t) - u(s) ds \right| \\ &\leq \frac{1}{T} \int_0^T |u(t) - u(s)| ds \\ &\leq \|u'\|_{L^\Phi} T A_\Phi^{-1} \left(\frac{1}{T} \right) \end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi \left(\frac{\bar{u}}{\|u\|_{L^\Phi}} \right) \leq \frac{1}{T} \int_0^T \Phi \left(\frac{u(s)}{\|u\|_{L^\Phi}} \right) ds \leq \frac{1}{T}$$

Then by Proposition 2.2(3)

$$|\bar{u}| \leq A_\Phi^{-1} \left(\frac{1}{T} \right) \|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq |\bar{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leq A_\Phi^{-1} \left(\frac{1}{T} \right) \|u\|_{L^\Phi} + T A_\Phi^{-1} \left(\frac{1}{T} \right) \|u'\|_{L^\Phi} \\ &\leq A_\Phi^{-1} \left(\frac{1}{T} \right) \max\{1, T\} \|u\|_{W^1 L^\Phi} \end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1 L^\Phi([0, T], \mathbb{R}^d)$. Since Φ is N_∞ , from Proposition 2.2(4) we obtain $s A_\Phi^{-1}(1/s) \rightarrow 0$ when $s \rightarrow 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (??) implies that u_n is bounded in $C([0, T], \mathbb{R}^d)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C([0, T], \mathbb{R}^d)$ with $u_{n_k} \rightarrow u$ in $C([0, T], \mathbb{R}^d)$. \square

QUIZAS LO QUE VIENE TENDRIA QUE IR CON LAS OTRAS DESIGUALDADES... O MODIFICAR EL LEMA ANTERIOR....

We get an anisotropic version of Sobolev-Wirtinger inequality, as follows.

Proposition 2.4. *If $u \in W^1 L^\Phi$, then*

$$\Phi(\tilde{u}(t)) \leq \int_0^T \Phi(\max\{1, T\} u'(r)) dr. \quad (15)$$

Proof. Writing $\tilde{u}(t) = u(t) - \bar{u}$ and applying Jensen's inequality, we get

$$\begin{aligned}\Phi(\tilde{u}(t)) &= \Phi\left(\frac{1}{T} \int_0^T u(t) - u(s) ds\right) \\ &\leq \frac{1}{T} \int_0^T \Phi(u(t) - u(s)) ds \leq \frac{1}{T} \int_0^T \Phi\left(\int_s^t |t-s| u'(r) \frac{dr}{|t-s|}\right) ds \\ &\leq \frac{1}{T} \int_0^T \frac{1}{|t-s|} \int_s^t \Phi(|t-s| u'(r)) dr ds\end{aligned}$$

As Φ is convex, we have

$$\frac{1}{|t-s|} \Phi(|t-s| u'(r)) \leq \begin{cases} \Phi(u'(r)) & \text{if } |t-s| \leq 1 \\ \Phi(T u'(r)) & \text{if } 1 < |t-s| < T \end{cases}$$

Then $\frac{1}{|t-s|} \Phi(|t-s| u'(r)) \leq \Phi(\max\{1, T\} u'(r))$ and (15) follows. \square

Lemma 2.5. *Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi, 1)$ converging to $u \in \Pi(E^\Phi, 1)$ in the L^Φ -norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0, T], \mathbb{R})$ such that $u_{n_k} \rightarrow u$ a.e. and $\Phi(u_{n_k}) \leq h$ a.e.*

Proof. Since $d(u, E^\Phi) < 1$ and u_n converges to u , there exists $u_0 \in E^\Phi$, a subsequence of u_n (again denoted u_n) and $0 < r < 1$ such that $d(u_n, u_0) < r$. Let $\lambda_0 \in (r, 1)$. By extracting more subsequences, if necessary, we can assume that $u_n \rightarrow u$ a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^\Phi} < \frac{1 - \lambda_0}{2^n}, \quad \text{for } n \geq 1.$$

We can assume $\lambda_n > 0$ for every $n = 0, \dots$

Let $\lambda := 1 - \sum_{n=0}^\infty \lambda_n$ and define $h : [0, T] \rightarrow \mathbb{R}$ by

$$h(x) = \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^\infty \lambda_n \Phi\left(\frac{u_{n+1} - u_n}{\lambda_n}\right). \quad (16)$$

Note that $\sum_{n=0}^\infty \lambda_n + \lambda = 1$, therefore for any $n = 1, \dots$

$$\begin{aligned}\Phi(u_n) &= \Phi\left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j}\right) \\ &\leq \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1} \lambda_j \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) \leq h\end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0, T], \mathbb{R})$. On the other hand $\|u_{n+1} - u_n\|_{L^\Phi} \leq \lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) dt \leq 1.$$

Then $h \in L^1([0, T], \mathbb{R})$. \square

DEMOSTRACION ALTERNATIVA PARA EL LEMA y con λ

Lemma 2.6. *Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi, \lambda)$ converging to $u \in \Pi(E^\Phi, \lambda)$ in the L^Φ -norm. Then, there exist a subsequence $\Phi(\frac{u_{n_k}}{\lambda})$ and a function $h \in L^1([0, T], \mathbb{R})$ such that $\Phi(\frac{u_{n_k}}{\lambda}) \rightarrow \Phi(\frac{u}{\lambda})$ a.e. and $\Phi(\frac{u_{n_k}}{\lambda}) \leq h$ a.e.*

Proof. As $u \in \Pi(E^\Phi, \lambda)$, we consider $\Lambda \in (0, \lambda)$. In this way, $d(u, E^\Phi) < \Lambda < \lambda$ and, taking into account (9), $\Phi(\frac{u}{\lambda}) \in L^1([0, T], \mathbb{R})$.

Applying [2, Lemma 3.1] with $x+y = \frac{u_n}{\lambda}$, $x = \frac{u}{\lambda}$, $k = \frac{\lambda}{\Lambda}$, $0 < \epsilon < \frac{\Lambda}{\lambda}$ and $C_\epsilon = \frac{\Lambda}{\epsilon(\lambda-\Lambda)}$, we have

$$\begin{aligned} & \int_0^T \left| \Phi\left(\frac{u_n}{\lambda}\right) - \Phi\left(\frac{u}{\lambda}\right) \right| dt \\ & \leq \epsilon \int_0^T \left| \Phi\left(\frac{u}{\lambda}\right) - \frac{\lambda}{\Lambda} \Phi\left(\frac{u}{\lambda}\right) \right| dt + 2 \int_0^T \Phi\left(\frac{\Lambda}{\epsilon(\lambda-\Lambda)} \frac{u_n - u}{\lambda}\right) dt. \end{aligned} \quad (17)$$

Let $\eta > 0$. Since $\Phi(\frac{u}{\lambda}) \in L^1([0, T], \mathbb{R})$, we can choose ϵ such that

$$\epsilon \int_0^T \left| \Phi\left(\frac{u}{\lambda}\right) - \frac{\lambda}{\Lambda} \Phi\left(\frac{u}{\lambda}\right) \right| dt < \eta. \quad (18)$$

From the fact that $u_n \rightarrow u$ in the L^Φ -norm, there exists n_0 such that $\|u_n - u\|_{L^\Phi} < \epsilon^2(\lambda - \Lambda)$ for every $n \geq n_0$. Then, by the convexity of Φ and the definition of Orlicz norm, we get

$$\int_0^T \Phi\left(\frac{\Lambda}{\epsilon(\lambda-\Lambda)} \frac{u_n - u}{\lambda}\right) \leq \epsilon \frac{\Lambda}{\lambda} \int_0^T \Phi\left(\frac{u_n - u}{\epsilon^2(\lambda-\Lambda)}\right) < \epsilon \quad (19)$$

Thus, from (17), (18) and (19), we obtain that $\Phi(\frac{u_n}{\lambda})$ converges to $\Phi(\frac{u}{\lambda})$ in the L^1 -norm. Now, [?, Thm. 4.9] implies that there exist a subsequence $\Phi(\frac{u_{n_k}}{\lambda})$ and a function $h \in L^1([0, T], \mathbb{R})$ such that $\Phi(\frac{u_{n_k}}{\lambda}) \rightarrow \Phi(\frac{u}{\lambda})$ a.e. and $\Phi(\frac{u_{n_k}}{\lambda}) \leq h$ a.e. \square

3 Differentiability Gateaux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anisotropic Orlicz Spaces. We apply these results to obtain Gateaux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a *Carathéodory function*, i.e.

- (C) f is measurable with respect to $t \in [0, T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

Definition 3.1. For $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ we denote by \mathbf{f} the Nemytskii (o superposition) operator defined for functions $u : [0, T] \rightarrow \mathbb{R}^d$ by

$$\mathbf{f}u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [?] for scalar functions and [?, ?, ?] for the generalization to \mathbb{R}^d -valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

Theorem 3.2. *We assume that f satisfies condition ((C)) and that $\Phi_1, \Phi_2 : \mathbb{R}^d \rightarrow [0, +\infty)$ are anisotropic Young functions. Then*

1. *Measurability. The operator f maps measurable function into measurable functions*
2. *Extensibility. If the operator f acts from the ball $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} \mid \|u\|_{L^{\Phi_1}} < r\}$ into the space L^{Φ_2} or the space E^{Φ_2} then f can be extended from $\Pi(E^{\Phi_1}, r)$ into space L^{Φ_2} or E^{Φ_2} , respectively.*
3. *Continuity. If the operator f acts from $\Pi(E^{\Phi_1}, r)$ into space E^{Φ_2} , then f is continuous.*

Given a continuous function $a \in C(\mathbb{R}^n, \mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \rightarrow \mathcal{M}_d$ by $\mathbf{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [1].

Lemma 3.3. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\mathbf{a} : W^1 L^\Phi \rightarrow L^\infty([0, T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|\mathbf{a}(u)\|_{L^\infty([0, T])} \leq A(\|u\|_{W^1 L^\Phi})$.*

Proof. Let $A \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\| \leq s, x \in \mathbb{R}^d} |a(x)|$. If $u \in W^1 L_d^\Phi$ then, by Sobolev's inequality, for a.e. $t \in [0, T]$

$$a(u(t)) \leq \alpha(\|u\|_{L^\infty}) \leq \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right) \max\{1, T\} \|u\|_{W^1 L^\Phi}\right) =: A(\|u\|_{W^1 L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t)) dt. \quad (20)$$

Theorem 3.4. *Let \mathcal{L} be a differentiable Carathéodory function satisfying (S). Then the following statements hold:*

1. *The action integral given by (20) is finitely defined on $\mathcal{E}^\Phi := W^1 L^\Phi \cap \{u \mid \dot{u} \in \Pi(E^\Phi, 1)\}$.*

2. The function I is Gâteaux differentiable on \mathcal{E}^Φ and its derivative I' is demicontinuous from \mathcal{E}^Φ into $[W^1 L^\Phi]^*$. Moreover, I' is given by the following expression

$$\langle I'(u), v \rangle = \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt. \quad (21)$$

3. If $\Psi \in \Delta_2$ then I' is continuous from \mathcal{E}^Φ into $[W^1 L^\Phi]^*$ when both spaces are equipped with the strong topology.

Proof. Let $u \in \mathcal{E}^\Phi$. As

$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \quad (22)$$

and (9), then $\Phi(\dot{u}(t)) \in L^1$. Now,

$$|\mathcal{L}(\cdot, u, \dot{u})| + |\nabla_x \mathcal{L}(\cdot, u, \dot{u})| + \Psi(\nabla_y \mathcal{L}(\cdot, u, \dot{u})) \leq A(\|u\|_{W^1 L^\Phi})(b + \Phi(\dot{u})) \in L^1, \quad (23)$$

by (S) and Lemma 3.3. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$ is continuous from \mathcal{E}^Φ into $L^1([0, T])$ with the strong topology on both sets.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions in \mathcal{E}^Φ and let $u \in \mathcal{E}^\Phi$ such that $u_n \rightarrow u$ in $W^1 L^\Phi$. By (Sobolev's inequality), we have

$$|u_n(t) - u(t)| \leq T A_\Phi^{-1} \left(\frac{1}{T} \right) \|u_n - u\|_{L^\Phi}$$

then $u_n \rightarrow u$ uniformly. As $\dot{u}_n \rightarrow \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.5, there exist a subsequence of \dot{u}_{n_k} (again denoted \dot{u}_{n_k}) and a function $h \in L^1([0, T], \mathbb{R})$ such that $\dot{u}_{n_k} \rightarrow \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leq h$ a.e.

Since $u_{n_k}, k = 1, 2, \dots$, is a strong convergent sequence in $W^1 L^\Phi$, it is a bounded sequence in $W^1 L^\Phi$. According to item (3) of Lemma 2.3, there exists $M > 0$ such that $\|a(u_{n_k})\|_{L^\infty} \leq M, k = 1, 2, \dots$. From the previous facts and (23), we get

$$|\nabla_x \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})| \leq a(|u_{n_k}|)(b + \Phi(\dot{u}_{n_k})) \leq M(b + h) \in L^1.$$

On the other hand, by the continuous differentiability of \mathcal{L} , we have

$$\nabla_x \mathcal{L}(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow \nabla_x \mathcal{L}(t, u(t), \dot{u}(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$ is continuous from \mathcal{E}^Φ with the strong topology into $[L^\Phi]^*$ with the weak* topology.

Let $u \in \mathcal{E}^\Phi$. From (23) it follows that

$$\nabla_y \mathcal{L}(\cdot, u, \dot{u}) \in C^\Psi. \quad (24)$$

Note that (23), (24) and the imbeddings $W^1 L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow [L^\Phi]^*$ imply that the second member of (21) defines an element of $[W^1 L^\Phi]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \rightarrow u$ in the norm of $W^1 L^\Phi$. We must prove that $\nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n) \xrightarrow{w^*} \nabla_y \mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle \nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n), v \rangle - \langle \nabla_y \mathcal{L}(\cdot, u, \dot{u}), v \rangle| \geq \epsilon. \quad (25)$$

We have $u_n \rightarrow u$ in L^Φ and $\dot{u}_n \rightarrow \dot{u}$ in L^Φ . By Lemma 2.5, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0, T], \mathbb{R})$ such that $u_n \rightarrow u$ uniformly, $\dot{u}_n \rightarrow \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leq h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.3 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (23) with u_n instead of u , we have

$$\Psi(\nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n)) \leq M(b + h) \in L^1. \quad (26)$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (26), we have

$$\begin{aligned} & \lambda \nabla_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ & \leq \lambda \left[\Psi(\nabla_y \mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right) \right] \\ & \leq \lambda M(b + h) + \lambda \Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \quad (27)$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T \nabla_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v \, dt \rightarrow \int_0^T \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot v \, dt \quad (28)$$

which contradicts the inequality (25). This completes the proof of step 2.

Step 3. We will prove (21). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1 L^\Phi$, we define the function

$$H(s, t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leq s_0 := \min\{(1 - d(\dot{u}, E^\Phi)) / \|v\|_{W^1 L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d(\dot{u} + s\dot{v}, E^\Phi) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.4 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leq s_0$.

We also have $\|u + sv\|_{W^1 L^\Phi} \leq \|u\|_{W^1 L^\Phi} + s_0 \|v\|_{W^1 L^\Phi}$; then, by Lemma 3.3, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leq M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leq s_0 \lambda^{-1}$, from the convexity and the parity of Φ , we get

$$\begin{aligned} \Phi(\dot{u} + s\dot{v}) &= \Phi\left((1 - s_0) \frac{\dot{u}}{1 - s_0} + s_0 \frac{s}{s_0} \dot{v}\right) \leq (1 - s_0) \Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0 \Phi\left(\frac{s}{s_0} \dot{v}\right) \\ &\leq (1 - s_0) \Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0 \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1 \end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1 - s_0}, E^\Phi\right) = \frac{1}{1 - s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot{u}}{1-s_0} \in C^\Phi$.

Now, applying (23), (27), the fact that $v \in L^\infty$ and $\dot{v} \in L^\Phi$, we get

$$\begin{aligned}
 |D_s H(s, t)| &= \left| \nabla_x \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + \lambda \nabla_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \frac{\dot{v}}{\lambda} \right| \\
 &\leq M [b(t) + \Phi(\dot{u} + s\dot{v})] |v| \\
 &\quad + \lambda \left[\Psi(\nabla_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v})) + \Phi\left(\frac{\dot{v}}{\lambda}\right) \right] \\
 &\leq M \{ [b(t) + \Phi(\dot{u} + s\dot{v})] |v| \} + \lambda M [b(t) + \Phi(\dot{u} + s\dot{v})] + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \\
 &= M [b(t) + \Phi(\dot{u} + s\dot{v})] (|v| + \lambda) + \lambda \Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1.
 \end{aligned} \tag{29}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v \rangle = \frac{d}{ds} I(u + sv) \Big|_{s=0} = \int_0^T \{ \nabla_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt.$$

Moreover, from the previous formula, (23), (24), and Lemma 2.3, we obtain

$$|\langle I'(u), v \rangle| \leq \|\nabla_x \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|\nabla_y \mathcal{L}\|_{L^\Psi} \|\dot{v}\|_{L^\Phi} \leq C \|v\|_{W^1 L^\Phi}$$

with a appropriate constant C .

This completes the proof of the Gâteaux differentiability of I .

Step 4. The operator $I' : \mathcal{E}^\Phi \rightarrow [W^1 L_d^\Phi]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \rightarrow u$ in the norm of $W^1 L^\Phi$ and $v \in W^1 L^\Phi$, then

$$\begin{aligned}
 \langle I'(u_n), v \rangle &= \int_0^T \{ \nabla_x \mathcal{L}(t, u_n, \dot{u}_n) \cdot v + \nabla_y \mathcal{L}(t, u_n, \dot{u}_n) \cdot \dot{v} \} dt \\
 &\rightarrow \int_0^T \{ \nabla_x \mathcal{L}(t, u, \dot{u}) \cdot v + \nabla_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt \\
 &= \langle I'(u), v \rangle.
 \end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto \nabla_x \mathcal{L}(t, u, \dot{u})$ and $u \mapsto \nabla_y \mathcal{L}(t, u, \dot{u})$ are norm continuous from \mathcal{E}^Φ into L^1 and L^Ψ , respectively.

The continuity of the first map has already been proved in step 1.

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1 L^\Phi} \rightarrow 0$.

Applying Lemma 2.5 to \dot{u}_n , there exists a subsequence (denoted \dot{u}_n for simplicity) such that $\dot{u}_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot{u}_n) \leq h$ and $\dot{u}_n \rightarrow \dot{u}$ a.e.

Then, by (27) we have $\Psi(v_n) \leq m(t) \in L^1$ being $v_n := \nabla_y \mathcal{L}(\cdot, u_n, \dot{u}_n)$ and $m(t) := M(b + h)$. In addition, from the continuous differentiability of \mathcal{L} , we have that $v_n \rightarrow v$ a.e. where $\nabla_y \mathcal{L}(\cdot, u, \dot{u})$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\Psi(\lambda x) \leq c(|\lambda|) \Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1}) \Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \rightarrow 0$ a.e. as $n \rightarrow \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leq c(|\lambda|^{-1}) K \Psi(v_n) + \Psi(v) \leq c(|\lambda|^{-1}) K [m(t) + \Psi(v)] \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda}) dt \rightarrow 0$ for every $\lambda > 0$. Thus, $v_n \rightarrow v$ in L^Ψ .

The continuity of I' follows from the continuity of $\nabla_x \mathcal{L}$ and $\nabla_y \mathcal{L}$ using the formula (21). \square

ALGO DE ESTO ES NECESARIO SI VAMOS A USAR LAS RELACIONES \prec y \ll , QUIZAS EN LOS PRELIMINARES

There exist several orders and equivalence relations between N -functions (see [?, Sec. 2.2]). Following [?, Def. 1, pp. 15-16] we say that the N -function Φ_2 is *stronger* than the N -function Φ_1 , in symbols $\Phi_1 \prec \Phi_2$, if there exist $a > 0$ and $x_0 \geq 0$ such that

$$\Phi_1(x) \leq \Phi_2(ax), \quad x \geq x_0. \quad (30)$$

The N -functions Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) when $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$. We say that Φ_2 is *essentially stronger* than Φ_1 ($\Phi_1 \ll \Phi_2$) if and only if for every $a > 0$ there exists $x_0 = x_0(a) \geq 0$ such that (30) holds. Finally, we say that Φ_2 is *completely stronger* than Φ_1 ($\Phi_1 \prec \Phi_2$) if and only if for every $a > 0$ there exist $K = K(a) > 0$ and $x_0 = x_0(a) \geq 0$ such that

We assume that there exist an N_∞ function Φ_0 such that $\Phi_0 \ll \Phi$, a function $a \in L^1([0, T], \mathbb{R})$ such that $a \geq 1$ and a constant $M > 0$ such that

$$\Psi_0(a^{-1}(t)|\nabla_x F|) \leq \Phi_0(x), \quad |x| \geq M \quad (\text{Sub})$$

Proposition 3.5. *Let $1 < p < \infty$ and suppose that F satisfies (A). Then,*

$$|\nabla_x F| \leq a(t)|x|^{p'_0-1} + b(t), \quad \text{for } p'_0 < p, \quad (31)$$

if and only if F satisfies (Sub) with $\Phi_0(x) = |x|^{p'}$ for all $p' \in (1, p)$.

Proof. Suppose that (Sub) holds with $\Phi_0(x) = |x|^{p'}$ for $1 < p' < p - 1$. Then,

$$(a^{-1}(t)|\nabla_x F|)^{q'_0} \leq |x|^{p'_0}, \quad \text{for } |x| \geq M$$

As F satisfies (A), we have $|\nabla_x F| \leq b(t) \in L^1$ for $|x| \leq M$.

PONEMOS SOLO b O Kb?? No confunde?

Then,

$$|\nabla_x F| \leq a|x|^{\frac{p'_0}{q'_0}} + b(t) \leq a|x|^{p'_0-1} + b(t)$$

Now, we assume that F satisfies (31). If $|x| \geq 1$, we have

$$|\nabla_x F| \leq a(t)|x|^{p'_0-1} + b(t) \leq (a(t) + b(t))|x|^{\frac{p'_0}{q'_0}}$$

which is condition (Sub) with $a(t) + b(t)$ instead of $a(t)$. \square

Theorem 3.6. *Let Φ be an N -function whose complementary function Ψ satisfies the Δ_2^∞ -condition.*

Let F be a potential that satisfies (C), (A) and the following conditions:

1. (Sub) for some N_∞ function Φ_0 such that $\Phi_0 \ll \Phi$

2.

$$\lim_{|x| \rightarrow \infty} \frac{\int_0^T F(t, x) dt}{\Psi_0(2|x|)} = +\infty, \quad (32)$$

for some N -function Ψ_0 complementary function of Φ_0 .

EN LO QUE SIGUE, PONEMOS LA NUEVA CONDICION DE ESTRUCTURA
O DEFINIMOS UN CONJUNTO DONDE SE VERIFICAN ESAS CONDICIONES???

Now, if the lagrangian $\mathcal{L}(t, x, y)$ is strictly convex with respect to $y \in \mathbb{R}^d$, $\mathcal{L} \in \mathfrak{A}(a, b, c, \lambda, f, \Phi)$, $D_y \mathcal{L}(0, x, y) = D_y \mathcal{L}(T, x, y)$ and (??) holds, then the problem (P) has at least a solution which minimizes the action integral I on $W^1 E_T^\Phi$.

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Proof. By the decomposition $u = \bar{u} + \tilde{u}$, Young's inequality, (Sub), the convexity of Φ_0 and (15), we obtain

$$\begin{aligned} & \left| \int_0^T F(t, u) - F(t, \bar{u}) dt \right| \leq \int_0^T \int_0^1 |\nabla_x F(t, \bar{u} + s\tilde{u}) \tilde{u}| ds dt \\ &= \int_0^T a(t) \int_0^1 a^{-1}(t) |\nabla_x F(t, \bar{u} + s\tilde{u}) \tilde{u}| ds dt \\ &\leq \int_0^T a(t) \int_0^1 \Psi_0(a^{-1}(t) \nabla_x F(t, \bar{u} + s\tilde{u})) + \Phi_0(\tilde{u}(t)) ds dt \\ &\leq \int_0^T a(t) \left[\int_0^1 \Phi_0(\bar{u} + s\tilde{u}) ds + \Phi_0(\tilde{u}(t)) \right] dt \\ &\leq \int_0^T a(t) (\Phi_0(2\bar{u}) + 2\Phi_0(2\tilde{u})) dt \\ &\leq C_1 \Phi_0(2\bar{u}) + \int_0^T a(t) 2\Phi_0(2\tilde{u}) dt \\ &\leq C_1 \Phi_0(2\bar{u}) + C_2 \int_0^T \Phi_0(C_T u'(t)) dt \end{aligned} \quad (33)$$

Then, by (33) we get

$$\begin{aligned} & \int_0^T \Phi(u') + F(t, u) dt = \int_0^T \{ \Phi(u') + [F(t, u) - F(t, \bar{u})] + F(t, \bar{u}) \} dt \\ &\geq \int_0^T [\Phi(u') - C_2 \Phi_0(C_T u'(t))] dt - C_1 \Phi_0(2\bar{u}) + \int_0^T F(t, \bar{u}) dt \end{aligned} \quad (34)$$

If $\|u\|_{W^1 L^\Phi} \rightarrow \infty$, then $\|u'\|_{L^\Phi} \rightarrow \infty$ or $|\bar{u}| \rightarrow \infty$.

Now, if $|\bar{u}| \rightarrow \infty$, by hypothesis we have $\frac{1}{\Psi_0(2\bar{u})} \int_0^\infty F(t, \bar{u}) dt \rightarrow \infty$.

If $\|u_n\|_{L^\Phi} \rightarrow \infty$ for a sequence u_n .

Let $M > 0$ such that $\frac{\Phi(x)}{C_2 \Phi_0(C_T x)} \geq 2$ for $|x| \geq M$.

Set $A_n := \{t \in [0, T] : |u'_n(t)| \geq M\}$.

Now, if $\|u'_n\|_{L^\Phi} \rightarrow \infty$ then $\|u'_n \chi_{A_n}\|_{L^\Phi} \rightarrow \infty$ or $\|u'_n \chi_{A_n^c}\|_{L^\Phi} \rightarrow \infty$.

As Φ is continuous, then Φ is bounded on $\overline{B_r}(0) = \{x \in \mathbb{R}^n : |x| \leq r\}$

Let $M_r := \max_{\overline{B_r}(0)} \Phi(x)$.

As $M_r \rightarrow 0$ when $r \rightarrow 0$, we can choose r such that $M_r T \leq 1$.

Then $\int_0^T \Phi(\frac{u}{r\|u\|_{L^\infty}}) dt \leq M_r T \leq 1$ and consequently $\|u\|_{L^\Phi} \leq r^{-1}\|u\|_{L^\infty}$, i.e.

$L^\infty \hookrightarrow L^\Phi$. Thus, $\|u'_n \chi_{A_n^c}\|_{L^\Phi}$ is bounded and $\|u'_n \chi_{A_n}\|_{L^\Phi} \rightarrow \infty$.

By Amemiya norm, we have $\|u'_n\|_{L^\Phi} \leq 1 + \int_{A_n} \Phi(u'_n) dt$, then

$$\int_{A_n} \Phi(u'_n) dt \rightarrow \infty. \quad (35)$$

Now, as $\frac{\Phi(x)}{2} \geq C_2 \Phi_0$ for $|x| \geq M$, we have

$$\begin{aligned} & \int_0^T [\Phi(u'_n) - C_2 \Phi_0(C_T u'_n)] dt \\ & \geq \int_{A_n} [\Phi(u'_n) - C_2 \Phi_0(C_T u'_n)] dt - C_2 \int_{[0,T]/A_n} \Phi_0(C_T u'_n) dt \\ & \geq \frac{1}{2} \int_{A_n} \Phi(u'_n) dt - C_2 T \max\{\Phi_0(x) : \|x\| \leq C_T M\}. \end{aligned} \quad (36)$$

Finally, by (35), we conclude that $\int_0^T [\Phi(u'_n) - C_2 \Phi_0(C_T u'_n)] dt \rightarrow \infty$

□

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