Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting

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1 Introduction

This work is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) & \text{a.e. } t\in(0,T) \\ \boldsymbol{u}(0)-\boldsymbol{u}(T)=\dot{\boldsymbol{u}}(0)-\dot{\boldsymbol{u}}(T)=0 \end{cases} \tag{1}$$

where $T>0$, $\boldsymbol{u}:[0,T]\to\mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t)+\Phi\left(\frac{|\boldsymbol{y}|}{\lambda}+f(t)\right)\right), \tag{2}$$

$$|D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(b(t)+\Phi\left(\frac{|\boldsymbol{y}|}{\lambda}+f(t)\right)\right), \tag{3}$$

$$|D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})| \leqslant a(|\boldsymbol{x}|)\left(c(t)+\varphi\left(\frac{|\boldsymbol{y}|}{\lambda}+f(t)\right)\right). \tag{4}$$

In these inequalities we assume that $\lambda>0$ and

1. $a\in C(\mathbb{R}^+,\mathbb{R}^+)$,
2. Φ is a N-function, i.e. Φ is given by

$$\Phi(t)=\int_0^t\varphi(\tau)\,d\tau,\quad \text{for } t\geq 0,$$

   where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0)=0$, $\varphi(t)>0$ for $t>0$ and $\lim_{t\to\infty}\varphi(t)=+\infty$. We denote by L^Φ_d the Orlicz space associated to the N-function Φ of functions defined on $[0,T]$ taking values in $\mathbb{R}^d$.
3. $b\in L^1([0,T])$ and $c\in L^\Psi_1$, where Ψ is la complemnetary N-function of Φ.
4. $f\in E^\Phi_1$, where the subspace $E^\Phi_d=E^\Phi_d([0,T])$ is defined as the closure in L^Φ_d of the subspace L^∞_d of all $\mathbb{R}^d$-valued essentially bounded functions.

We introduce the *action integral*

$$I(\boldsymbol{u})=\int_0^T\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\,dt. \tag{5}$$

2 Differentiability of action integrals in Orlicz spaces

We define

$$\Pi(E^\Phi_d,r):=\{\boldsymbol{u}\in L^\Phi_d | d(\boldsymbol{u},E^\Phi_d)<r\}.$$

Theorem 2.1. *Let $\mathcal{L}$ be a Carathéodory function satisfying* (2), (3) *and* (4). *Then the following statements hold:*

1. *The action integral given by* (5) *is finitely defined on $\mathcal{E}^\Phi_d(\lambda):=W^1L^\Phi_d\cap\{\boldsymbol{u}|\dot{\boldsymbol{u}}\in\Pi(E^\Phi_d,\lambda)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi_d(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi_d(\lambda)$ into $\left[W^1L^\Phi_d\right]^*$. Moreover, I' is given by the following expression*

$$\langle I'(\boldsymbol{u}),\boldsymbol{v}\rangle=\int_0^T\left\{D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\boldsymbol{v}+D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\dot{\boldsymbol{v}}\right\}\;dt. \tag{6}$$

3. *If $\Psi\in\Delta_2$ then I' is continuous from $\mathcal{E}^\Phi_d(\lambda)$ into $\left[W^1L^\Phi_d\right]^*$ when both spaces are equipped with the strong topology.*

Theorem 2.2. *Let $\boldsymbol{u}\in\mathcal{E}^\Phi_d(\lambda)$ be a T-periodic function. The following statements are equivalent:*

(a) $I'(\boldsymbol{u})\in\left(W^1L^\Phi_T\right)^\perp$.

(b) *$D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))$ is an absolutely continuous function and $\boldsymbol{u}$ solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt}D_y\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))=D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) & \text{a.e. } t\in(0,T) \\ \boldsymbol{u}(0)-\boldsymbol{u}(T)=D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0))-D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T))=0. \end{cases} \tag{7}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t,x,y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0,\boldsymbol{u}(0),\dot{\boldsymbol{u}}(0))-D_{\boldsymbol{y}}\mathcal{L}(T,\boldsymbol{u}(T),\dot{\boldsymbol{u}}(T))=0$ is equivalent to $\dot{\boldsymbol{u}}(0)=\dot{\boldsymbol{u}}(T)$.

3 Coercivity discussion

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})\geq\alpha_0\Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right)+F(t,\boldsymbol{x}), \tag{8}$$

where $\alpha_0,\Lambda>0$ and $F:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is a Carathéodory function, i.e. $F(t,\boldsymbol{x})$ is measurable with respect to t for every fixed $\boldsymbol{x}\in\mathbb{R}^d$ and it is continuous at $\boldsymbol{x}$ for a.e. $t\in[0,T]$. We need to assume

$$|F(t,\boldsymbol{x})|\leqslant a(|\boldsymbol{x}|)b_0(t),\quad \text{for a.e. } t\in[0,T]\quad\text{and for every } \boldsymbol{x}\in\mathbb{R}^d. \tag{9}$$

The coercivity of the action integral I is related to the coercivity of the functional

$$J_{C,\nu}(\boldsymbol{u}):=\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)-C\|\boldsymbol{u}\|^\nu_{L^\Phi}, \tag{10}$$

for $C,\nu>0$. If $\Phi(x)=|x|^p/p$ then $J_{C,\nu}$ is clearly coercive for $\nu<p$. For more general Φ the situation is more interesting.

Lemma 3.1. *Let Φ and Ψ be complementary N-functions. Then:*

1. *If $C\Lambda<1$, then $J_{C,1}$ is coercive.*
2. *If $\Psi\in\Delta_2$ globally, then there exists a constant $\alpha_\Phi>1$ such that, for any $0<\mu<\alpha_\Phi$,*

$$\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to\infty}\frac{\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)}{\|\boldsymbol{u}\|^\mu_{L^\Phi}}=+\infty. \tag{11}$$

   In particular, the functional $J_{C,\mu}$ is coercive for every $C>0$ and $0<\mu<a_\Phi$. The constant α_Φ is one of the so-called Matuszewska-Orlicz indices *(see [?, Ch. 11]).*
3. *If $J_{C,1}$ is coercive with $C\Lambda>1$, then $\Psi\in\Delta_2$.*

Theorem 3.2. *Let $\mathcal{L}$ be a lagrangian function satisfying* (2), (3), (4), (8) *and* (9). *We assume the following conditions:*

1. *There exist a non negative function $b_1\in L^1_1$ and a constant $\mu>0$ such that for any $\boldsymbol{x_1},\boldsymbol{x_2}\in\mathbb{R}^d$ and a.e. $t\in[0,T]$*

$$|F(t,\boldsymbol{x_2})-F(t,\boldsymbol{x_1})|\leqslant b_1(t)(1+|\boldsymbol{x_2}-\boldsymbol{x_1}|^\mu). \tag{12}$$

   We suppose that $\mu<\alpha_\Phi$, with α_Φ as in Lemma 3.1, in the case that $\Psi\in\Delta_2$; and, we suppose $\mu=1$ if Ψ is an arbitrary N-function.
2.

$$\int_0^T F(t,\boldsymbol{x})\;dt\to\infty\quad as\quad |\boldsymbol{x}|\to\infty. \tag{13}$$

3. *$\Psi\in\Delta_2$ or, alternatively, $\alpha_0^{-1}T\Phi^{-1}(1/T)\|b_1\|_{L^1}\Lambda<1$.*

Then the action integral I is coercive.

Lemma 3.3. *Suppose that F satisfies condition (A) and* (13), *$F(t,\cdot)$ is differentiable and convex a.e. $t\in[0,T]$. Then, there exists $\boldsymbol{x}_0\in\mathbb{R}^d$ such that*

$$\int_0^T D_{\boldsymbol{x}}F(t,\boldsymbol{x}_0)\;dt=0. \tag{14}$$

Theorem 3.4. *Let $\mathcal{L}$ be as in Theorem 3.2 and let F be as in Lemma 3.3. Moreover, assume that $\Psi\in\Delta_2$ or, alternatively $\alpha_0^{-1}T\Phi^{-1}(1/T)\,a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}\Lambda<1$, with a and b_0 as in* (9) *and $\boldsymbol{x}_0\in\mathbb{R}^d$ any point satisfying* (14). *Then I is coercive.*

4 The main result

Theorem 4.1. *Let Φ and Ψ be complementary N-functions. Suppose that the Carathéodory function $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ is strictly convex at $\boldsymbol{y}$, $D_{\boldsymbol{y}}\mathcal{L}$ is T-periodic with respect to T and* (2), (3), (4), (8), (9) *and* (13) *are satisfied. In addition, assume that some of the following statements hold (we recall the definitions and properties of α_0, b_1, $\boldsymbol{x}_0$ and b_0 from* (8), (12), (14) *and* (**??**) *respectively):*

1. *$\Psi\in\Delta_2$ and* (12).
2. (12) *and $\alpha_0^{-1}T\Phi^{-1}(1/T)\|b_1\|_{L^1}\Lambda<1$.*
3. *$\Psi\in\Delta_2$, F satisfies condition (A) and $F(t,\cdot)$ is convex a.e. $t\in[0,T]$.*
4. *As item 3 but with $\alpha_0^{-1}T\Phi^{-1}(1/T)\,a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}\Lambda<1$ instead of $\Psi\in\Delta_2$.*

Then, problem (1) *has a solution.*