Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting

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1. Introduction

This paper is concerned with the existence of periodic solutions of the problem

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}} \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) & \text{a.e. } t \in (0,T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(T) = 0 \end{cases} \tag{1}$$

where $T > 0$, $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ is absolutely continuous and the *Lagrangian* $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$|\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left(b(t) + \Phi\left(\frac{|\boldsymbol{y}|}{\lambda} + f(t) \right) \right), \tag{2}$$

$$|D_{\boldsymbol{x}} \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left(b(t) + \Phi\left(\frac{|\boldsymbol{y}|}{\lambda} + f(t) \right) \right), \tag{3}$$

$$|D_{\boldsymbol{y}} \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})| \leqslant a(|\boldsymbol{x}|) \left(c(t) + \varphi\left(\frac{|\boldsymbol{y}|}{\lambda} + f(t) \right) \right). \tag{4}$$

In these inequalities we assume that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lambda > 0$, Φ is an N-function (see section Preliminaries for definitions), φ is the right continuous derivative of Φ and the non negative functions b, c and f belong to certain Banach spaces that will be introduced later.

It is well known that problem (1) comes from a variational one, that is, a solution of (1) is a critical point of the *action integral*

$$I(\boldsymbol{u}) = \int_0^T \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))\, dt. \tag{5}$$

Variational problems and hamiltonian systems have been studied extensively. Classic references of these subjects are [23, 28, 14]. Problems like (1) have maintained the interest of researchers as the recent literature on the topic testifies. For lagrangian functions of the type $\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y}) = \frac{|\boldsymbol{y}|^2}{2} + F(t, \boldsymbol{x})$ many solvability conditions have been given expanding the results of [23]. In [29] the function F was split up into two potentials, one of them with a property of subadditivity and the other with a bounded gradient. In [30] it was required a certain sublinearity condition on the gradient of the potential F; and, in [34] it was considered a potential F given by a sum of a subconvex function and a subquadratic one. In [31] the uniform coercivity of $\int_0^T F(t, \boldsymbol{x})\, dt$ was replaced by local coercivity of F in some positive measure subset of $[0,T]$. In [37], the authors took a similar potential to that in [34] getting new solvability conditions and they also studied the case in which the two potentials do not have any convexity.

Keywords and phrases: Periodic Solution, Orlicz-Sobolev Spaces, Euler-Lagrange, N-function, Critical Points.

The Lagrangian $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{|\boldsymbol{y}|^p}{p} + F(t,\boldsymbol{x})$ for $p > 1$ was treated in more recent papers. By using the dual least action principle, in [33] it was performed the extension of some results given in [23]; and, in [32] the authors improved the work done in [34]. On the other hand, by the minimax methods in critical point theory some existence theorems were obtained. In [35] it was employed a subquadratic potential F in Rabinowitz's sense and in [36] F was taken as in [30].

Another source of problems, close to our proposal, is the one in which a *p-laplacian-like* operator is involved. Assuming that the function φ is a homeomorphism from $\mathbb{R}^d$ into itself, it is considered the differential operator $\boldsymbol{u} \mapsto (\varphi(\boldsymbol{u}'))'$. In [4, 6, 5, 22, 21], using the Leray-Schauder degree theory, some existence results of solutions of equations like $(\varphi(\boldsymbol{u}'))' = \boldsymbol{f}(t, \boldsymbol{u}(t), \boldsymbol{u}'(t))$ were obtained under different boundary conditions (periodic, Dirichlet, von Neumann) and where $\boldsymbol{f}$ is not necessarily a gradient. We point out that our approach differs from that of previous articles because we tackle the direct method of the calculus of variations.

In the Orlicz-Sobolev space setting, in [15] a constrained minimization problem associated to the existence of eigenvalues for certain differential operators involving N-functions was studied. Slightly away from the problems to be treated in this paper, we can mention [9, 10] where A. Cianchi dealt with the regularity of minimizers of action integrals defined on several variable functions.

In this article we consider lagrangian functions defined on Orlicz-Sobolev spaces W^1L^Φ (see [2, 19, 24, 25]) and we use the direct method of calculus of variations. The exposition is organized as follows. In Section 2 we enumerate results related to Orlicz spaces, Orlicz-Sobolev spaces and composition operators. Almost all results in this section are essentially known. Conditions (2), (3) and (4) are the means to ensure that I is finitely defined on a non trivial subset of $W^1L^\Phi_d$ and I is Gâteaux differentiable in this subspace. We develop these issues in Theorem 3.2 of Section 3. In Section 4 we prove that critical points of (5) are solutions of (1). Conditions to guarantee the coercitivity of action integrals are discussed in Section 5. Finally, our main theorem about existence of solutions of (1) is introduced and proved in Section 6.

We lay emphasis on that we use Δ_2-condition only when necessary in a certain sense (see, for example, Lemma 5.2).

2. Preliminaries

For reader convenience, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions and a list of results that we will use throughout the article. Classic references for Orlicz spaces of real valued functions are [2, 19, 24]. For Orlicz spaces of vector valued functions, see [27] and the references therein.

Hereafter we denote by $\mathbb{R}^+$ the set of all non negative real numbers. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *N-function* if Φ is given by

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \text{for } t \geq 0,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right continuous non decreasing function satisfying $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$.

Given a function φ as above, we consider the so-called right inverse function ψ of φ which is defined by $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. The function ψ satisfies the same properties as the function φ, therefore we have an N-function Ψ such that $\Psi' = \psi$. The function Ψ is called the *complementary function* of Φ.

We say that Φ satisfies the Δ_2-*condition*, denoted by $\Phi \in \Delta_2$, if there exist constants $K > 0$ and $t_0 \geq 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \tag{6}$$

for every $t \geq t_0$. If $t_0 = 0$, we say that Φ satisfies the Δ_2-*condition globally* ($\Phi \in \Delta_2$ globally).

Let d be a positive integer. We denote by $\mathcal{M}_d := \mathcal{M}_d([0, T])$ the set of all measurable functions defined on $[0, T]$ with values on $\mathbb{R}^d$ and we write $\boldsymbol{u} = (u_1, \ldots, u_d)$ for $\boldsymbol{u} \in \mathcal{M}_d$. In this paper we adopt the convention that bold symbols denote points in $\mathbb{R}^d$.

Given an N-function Φ we define the *modular function* $\rho_\Phi : \mathcal{M}_d \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(\boldsymbol{u}) := \int_0^T \Phi(|\boldsymbol{u}|)\, dt.$$

Here $|\cdot|$ is the euclidean norm of $\mathbb{R}^d$. The *Orlicz class* $C^\Phi_d = C^\Phi_d([0, T])$ is given by

$$C^\Phi_d := \{\boldsymbol{u} \in \mathcal{M}_d | \rho_\Phi(\boldsymbol{u}) < \infty\}. \tag{7}$$

The *Orlicz space* $L^\Phi_d = L^\Phi_d([0, T])$ is the linear hull of C^Φ_d; equivalently,

$$L^\Phi_d := \{\boldsymbol{u} \in \mathcal{M}_d | \exists\lambda > 0 : \rho_\Phi(\lambda\boldsymbol{u}) < \infty\}. \tag{8}$$

The Orlicz space L^Φ_d equipped with the *Orlicz norm*

$$\|\boldsymbol{u}\|_{L^\Phi} := \sup\left\{\int_0^T \boldsymbol{u}\cdot\boldsymbol{v}\, dt \big| \rho_\Psi(\boldsymbol{v}) \leqslant 1\right\},$$

is a Banach space. By $\boldsymbol{u}\cdot\boldsymbol{v}$ we denote the usual dot product in $\mathbb{R}^d$ between $\boldsymbol{u}$ and $\boldsymbol{v}$. The following alternative expression for the norm, known as *Amemiya norm*, will be useful (see [19, Thm. 10.5] and [16]). For every $\boldsymbol{u} \in L^\Phi$,

$$\|\boldsymbol{u}\|_{L^\Phi} = \inf_{k>0} \frac{1}{k}\{1 + \rho_\Phi(k\boldsymbol{u})\}. \tag{9}$$

The subspace $E^\Phi_d = E^\Phi_d([0, T])$ is defined as the closure in L^Φ_d of the subspace L^∞_d of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that E^Φ_d is the only one maximal subspace contained in the Orlicz class C^Φ_d, i.e. $\boldsymbol{u} \in E^\Phi_d$ if and only if $\rho_\Phi(\lambda \boldsymbol{u}) < \infty$ for any $\lambda > 0$.

A generalized version of *Hölder's inequality* holds in Orlicz spaces. Namely, if $\boldsymbol{u} \in L^\Phi_d$ and $\boldsymbol{v} \in L^\Psi_d$ then $\boldsymbol{u} \cdot \boldsymbol{v} \in L^1_1$ and

$$\int_0^T \boldsymbol{v} \cdot \boldsymbol{u} \, dt \leqslant \|\boldsymbol{u}\|_{L^\Phi} \|\boldsymbol{v}\|_{L^\Psi}. \tag{10}$$

If X and Y are Banach spaces such that $Y \subset X^*$, we denote by $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ the bilinear pairing map given by $\langle x^*, x \rangle = x^*(x)$. Hölder's inequality shows that $L^\Psi_d \subset \left[L^\Phi_d\right]^*$, where the pairing $\langle \boldsymbol{v}, \boldsymbol{u} \rangle$ is defined by

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = \int_0^T \boldsymbol{v} \cdot \boldsymbol{u} \, dt \tag{11}$$

with $\boldsymbol{u} \in L^\Phi_d$ and $\boldsymbol{v} \in L^\Psi_d$. Unless $\Phi \in \Delta_2$, the relation $L^\Psi_d = \left[L^\Phi_d\right]^*$ will not hold. In general, it is true that $\left[E^\Phi_d\right]^* = L^\Psi_d$.

Like in [19], we will consider the subset $\Pi(E^\Phi_d, r)$ of L^Φ_d given by

$$\Pi(E^\Phi_d, r) := \{\boldsymbol{u} \in L^\Phi_d | d(\boldsymbol{u}, E^\Phi_d) < r\}.$$

This set is related to the Orlicz class C^Φ_d by means of inclusions, namely,

$$\Pi(E^\Phi_d, r) \subset r C^\Phi_d \subset \overline{\Pi(E^\Phi_d, r)} \tag{12}$$

for any positive r. If $\Phi \in \Delta_2$, then the sets L^Φ_d, E^Φ_d, $\Pi(E^\Phi_d, r)$ and C^Φ_d are equal.

We define the *Sobolev-Orlicz space* $W^1 L^\Phi_d$ (see [2]) by

$$W^1 L^\Phi_d := \{\boldsymbol{u} | \boldsymbol{u} \text{ is absolutely continuous and } \boldsymbol{u}, \dot{\boldsymbol{u}} \in L^\Phi_d\}.$$

$W^1 L^\Phi_d$ is a Banach space when equipped with the norm

$$\|\boldsymbol{u}\|_{W^1 L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} + \|\dot{\boldsymbol{u}}\|_{L^\Phi}.$$

For a function $\boldsymbol{u} \in L^1_d([0,T])$, we write $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}$ where $\overline{\boldsymbol{u}} = \frac{1}{T}\int_0^T \boldsymbol{u}(t) \, dt$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \overline{\boldsymbol{u}}$.

As usual, if $(X, \|\cdot\|_X)$ is a Banach space and $(Y, \|\cdot\|_Y)$ is a subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when the restricted identity map $i_Y : Y \to X$ is bounded. That is, there exists $C > 0$ such that for any $y \in Y$ we have $\|y\|_X \leqslant C\|y\|_Y$. With this notation, Hölder's inequality states that $L^\Psi_d \hookrightarrow \left[L^\Phi_d\right]^*$; and, it is easy to see that for every N-function Φ we have that $L^\infty_d \hookrightarrow L^\Phi_d \hookrightarrow L^1_d$.

Recall that a function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *modulus of continuity* if w is a continuous increasing function which satisfies $w(0) = 0$. For example, it can be easily shown that $w(s) = s\Phi^{-1}(1/s)$ is a modulus of continuity for every N-function Φ. We say that $\boldsymbol{u} : [0,T] \to \mathbb{R}^d$ has modulus of continuity w when there exists a constant $C > 0$ such that

$$|\boldsymbol{u}(t) - \boldsymbol{u}(s)| \leqslant C w(|t - s|). \tag{13}$$

We denote by $C^w([0,T],\mathbb{R}^d)$ the space of w-Hölder continuous functions. This is the space of all functions satisfying (13) for some $C>0$ and it is a Banach space with norm

$$\|\boldsymbol{u}\|_{C^w([0,T],\mathbb{R}^d)} := \|\boldsymbol{u}\|_{L^\infty} + \sup_{t\neq s}\frac{|\boldsymbol{u}(t)-\boldsymbol{u}(s)|}{w(|t-s|)}.$$

An important aspect of the theory of Sobolev spaces is related to embedding theorems. There is an extensive literature on this question in the Orlicz-Sobolev space setting, see for example [8, 7, 11, 13, 18]. The next simple lemma is essentially known and we will use it systematically. For the sake of completeness, we include a brief proof of it.

Lemma 2.1. *Let $w(s) := s\Phi^{-1}(1/s)$. Then, the following statements hold:*

1. $W^1L^\Phi \hookrightarrow C^w([0,T],\mathbb{R}^d)$ *and for every* $\boldsymbol{u}\in W^1L^\Phi$

$$|\boldsymbol{u}(t)-\boldsymbol{u}(s)| \leqslant \|\dot{\boldsymbol{u}}\|_{L^\Phi} w(|t-s|), \tag{14}$$

$$\|\boldsymbol{u}\|_{L^\infty} \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi} \quad \textit{(Sobolev's inequality)}. \tag{15}$$

2. *For every* $\boldsymbol{u}\in W^1L^\Phi$ *we have* $\widetilde{\boldsymbol{u}}\in L^\infty_d$ *and*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \leqslant T\Phi^{-1}\left(\frac{1}{T}\right)\|\dot{\boldsymbol{u}}\|_{L^\Phi} \qquad \textit{(Wirtinger's inequality)}. \tag{16}$$

Proof. For $0\leqslant s\leqslant t\leqslant T$, we get

$$\begin{aligned} |\boldsymbol{u}(t)-\boldsymbol{u}(s)| &\leqslant \int_s^t |\dot{\boldsymbol{u}}(\tau)| \quad d\tau \\ &\leqslant \|\chi_{[s,t]}\|_{L^\Psi}\|\dot{\boldsymbol{u}}\|_{L^\Phi} \\ &= \|\dot{\boldsymbol{u}}\|_{L^\Phi}(t-s)\Phi^{-1}\left(\frac{1}{t-s}\right), \end{aligned} \tag{17}$$

using Hölder's inequality and [19, Eq. (9.11)]. This proves the inequality (14).

Since u_i is continuous, from Mean Value Theorem for integrals, there exists $s_i\in[0,T]$ such that $u_i(s_i)=\overline{u}_i$. Using this s_i value in (14) with u_i instead of $\boldsymbol{u}$ and taking into account that $s\Phi^{-1}(1/s)$ is increasing, we obtain Wirtinger's inequality for each u_i. The inequality (16) follows easily from the corresponding result for each component of $\boldsymbol{u}$.

On the other hand, again by Hölder's inequality and [19, Eq. (9.11)], we have

$$|\overline{\boldsymbol{u}}| = \frac{1}{T}\int_0^T |\boldsymbol{u}(s)|ds \leqslant \Phi^{-1}\left(\frac{1}{T}\right)\|\boldsymbol{u}\|_{L^\Phi}. \tag{18}$$

From (16), (18) and the fact that $\boldsymbol{u}=\overline{\boldsymbol{u}}+\widetilde{\boldsymbol{u}}$, we obtain (15). This completes the proof of item 1. $\square$

Remark 1. As a consequence of the previous lemma, there exists a constant C, only dependent on T, such that

$$\|\boldsymbol{u}\|_{W^1L^\Phi} \leqslant C\left(|\overline{\boldsymbol{u}}| + \|\dot{\boldsymbol{u}}\|_{L^\Phi}\right) \tag{19}$$

for every $\boldsymbol{u} \in W^1L^\Phi_d$.

The Arzelà-Ascoli Theorem implies that $C^w([0,T],\mathbb{R}^d) \hookrightarrow C([0,T],\mathbb{R}^d)$ is a compact embedding (see [12, Ch. 5] for the case $w(s) = |s|^\alpha$ with $0 < \alpha \leqslant 1$; and, if w is arbitrary, the proof follows with some obvious modifications). Therefore we have the subsequent result.

Corollary 2.2. *Every bounded sequence $\{\boldsymbol{u}_n\}$ in $W^1L^\Phi_d$ has an uniformly convergent subsequence.*

Given a continuous function $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, we define the composition operator $\boldsymbol{a} : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(\boldsymbol{u})(t) = a(|\boldsymbol{u}(t)|)$. We will often use the following elementary consequence of Lemma 2.1.

Corollary 2.3. *If $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi_d \to L^\infty_1([0,T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(\boldsymbol{u})\|_{L^\infty([0,T])} \leqslant A(\|\boldsymbol{u}\|_{W^1L^\Phi})$.*

Proof. Let $\alpha \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a non decreasing majorant of a, for example $\alpha(s) := \sup_{0\leqslant t\leqslant s} a(t)$. If $\boldsymbol{u} \in W^1L^\Phi_d$ then, by Lemma 2.1,

$$a(|\boldsymbol{u}(t)|) \leqslant \alpha(\|\boldsymbol{u}\|_{L^\infty}) \leqslant \alpha\left(\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|\boldsymbol{u}\|_{W^1L^\Phi}\right) =: A(\|\boldsymbol{u}\|_{W^1L^\Phi}).$$

□

The next lemma is an immediate consequence of principles related to operators of Nemitskii type, see [19, §17].

Lemma 2.4. *The composition operator $\boldsymbol{\varphi}$ acts from $\Pi(E^\Phi_d,1)$ into C^Ψ_1.*

Proof. From [19, Lemma 9.1] we have that $\boldsymbol{\varphi}\left(B_{L^\Phi}(0,1)\right) \subset C^\Psi_1$, where $B_X(\boldsymbol{u}_0,r)$ is the open ball with center $\boldsymbol{u}_0$ and radius $r > 0$ in the space X. Now, applying [19, Lemma 17.1], we deduce that $\boldsymbol{\varphi}$ acts from $\Pi(E^\Phi_d,1)$ into C^Ψ_1. □

We also need the following technical lemma.

Lemma 2.5. *Let $\lambda > 0$ and let $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi_d,\lambda)$ converging to $\boldsymbol{u} \in \Pi(E^\Phi_d,\lambda)$ in the L^Φ-norm. Then, there exist a subsequence $\boldsymbol{u}_{n_k}$ and a real valued function $h \in \Pi\left(E^\Phi_1([0,T]),\lambda\right)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e. and $|\boldsymbol{u}_{n_k}| \leqslant h$ a.e.*

Proof. Let $r := d(\boldsymbol{u},E^\Phi_d)$, $r < \lambda$. As $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$, there exists a subsequence (n_k) such that

$$\|\boldsymbol{u}_{n_k} - \boldsymbol{u}\|_{L^\Phi} < \frac{\lambda - r}{2} \quad \text{and} \quad \|\boldsymbol{u}_{n_k} - \boldsymbol{u}_{n_{k+1}}\|_{L^\Phi} < 2^{-(k+1)}(\lambda - r).$$

Let $h : [0,T] \to \mathbb{R}$ be defined by

$$h(x) = |\boldsymbol{u}_{n_1}(x)| + \sum_{k=2}^{\infty} |\boldsymbol{u}_{n_k}(x) - \boldsymbol{u}_{n_{k-1}}(x)|. \tag{20}$$

As a consequence of [19, Lemma 10.1] (see [27, Thm. 5.5] for vector valued functions), we have that $d(\boldsymbol{v}, E_d^{\Phi}) = d(|\boldsymbol{v}|, E_1^{\Phi})$ for any $\boldsymbol{v} \in L_d^{\Phi}$. Now

$$d(|\boldsymbol{u}_{n_1}|, E_1^{\Phi}) = d(\boldsymbol{u}_{n_1}, E_d^{\Phi}) \leqslant d(\boldsymbol{u}_{n_1}, \boldsymbol{u}) + d(\boldsymbol{u}, E_d^{\Phi}) < \frac{\lambda + r}{2}.$$

Then

$$d(h, E_1^{\Phi}) \leqslant d(h, |\boldsymbol{u}_{n_1}|) + d(|\boldsymbol{u}_{n_1}|, E_1^{\Phi}) < \lambda.$$

Therefore, $h \in \Pi(E_1^{\Phi}, \lambda)$ and $|h| < \infty$ a.e. We conclude that the series $\boldsymbol{u}_{n_1}(x) + \sum_{k=2}^{\infty}(\boldsymbol{u}_{n_k}(x) - \boldsymbol{u}_{n_{k-1}}(x))$ is absolutely convergent a.e. and this fact implies that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e. The inequality $|\boldsymbol{u}_{n_k}| \leqslant h$ follows straightforwardly from the definition of h. $\square$

A common obstacle in Orlicz spaces, that distinguishes them from L^p spaces, is that a sequence $\boldsymbol{u}_n \in L_d^{\Phi}$ which is uniformly bounded by $h \in L_1^{\Phi}$ and a.e. convergent to $\boldsymbol{u}$ is not necessarily norm convergent. Fortunately, the subspace E_d^{Φ} has this property.

Lemma 2.6. *Suppose that $\boldsymbol{u}_n \in L_d^{\Phi}$ is a sequence such that $\boldsymbol{u}_n \to \boldsymbol{u}$ a.e. and assume that there exists $h \in E_1^{\Phi}$ with $|\boldsymbol{u}_n| \leqslant h$ a.e., then $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^{\Phi}} \to 0$.*

Proof. [24, p. 84] and [19, Thm. 10.3]. $\square$

We recall some useful concepts.

Definition 2.7. *Given a function $I : U \to \mathbb{R}$ where U is an open set of a Banach space X, we say that I has a Gâteaux derivative at $\boldsymbol{u} \in U$ if there exists $\boldsymbol{u}^* \in X^*$ such that for every $\boldsymbol{v} \in X$*

$$\lim_{s \to 0} \frac{I(\boldsymbol{u} + s\boldsymbol{v}) - I(\boldsymbol{u})}{s} = \langle \boldsymbol{u}^*, \boldsymbol{v} \rangle.$$

See [3] for details.

Definition 2.8. *Let X be a Banach space and let $D \subset X$. A non linear operator $T : D \to X^*$ is called* demicontinuous *if it is continuous when X is equipped with the strong topology and X^* with the weak* topology (see [17]).*

3. Differentiability of action integrals in Orlicz spaces

We take a moment for discussing the relevance of the function f in the inequalities (2), (3) and (4), which are a direct generalization of the conditions

[23, Eq (a), p. 10]. In particular, we are interested in seeing when for every $f \in E_1^{\Phi}$ there exist $b \in L_1^1$ and a constant $C > 0$ such that

$$\Phi(s + f(t)) \leqslant C\Phi(s) + b(t) \text{ for every } s > 0. \tag{21}$$

If (21) is true, then we can suppose $f = 0$ in the equations (2) and (3) and the same considerations should be done with $\varphi(s + f(t))$.

The convexity of Φ allows us to bound $\Phi(s+f(t))$ by the expression $\frac{1}{2}\Phi(2s) + b(t)$ where $b(t) := \frac{1}{2}\Phi(2f(t)) \in L_1^1$ and $f \in E_1^{\Phi}$. Therefore, we can always assume $f = 0$ in (2) and (3) at the price of making the value of λ smaller. In the special case that $\Phi \in \Delta_2$, the inequality (6) implies (21).

If $\Phi \notin \Delta_2$, then (21) may not be true. In fact, if we consider the N-function $\Phi(s) = e^s - s - 1$ which does not satisfy the Δ_2-condition and $f(t) = \ln|\ln(t)|$ for $t \in [0, e^{-1}]$, then (21) does not hold.

First, note that $f(t) \geq 0$ on $[0, e^{-1}]$ and

$$\int_0^{e^{-1}} \Phi(\lambda f(t))dt \leqslant \int_0^{e^{-1}} e^{\lambda f(t)}dt \leqslant \int_0^{e^{-1}} |\ln(t)|^{\lambda}dt < \infty$$

for $\lambda > 1$, hence $f \in E_1^{\Phi}$. Now, suppose that there exist $b \in L_1^1$ and $C > 0$ satisfying (21). From the inequality $1/2e^s \leqslant \Phi(s) + 1$, we obtain

$$\frac{1}{2}e^s e^{f(t)} \leqslant \Phi(s + f(t)) + 1 \leqslant C\Phi(s) + b(t) + 1;$$

next, dividing by e^s and taking $s \to \infty$, we get $\frac{1}{2}|\ln(t)| \leqslant \frac{1}{2}e^f(t) \leqslant C$ which is a contradiction.

Before addressing the main results of this section, we recall a definition.

Definition 3.1. *We say that a function $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function if for fixed $(\boldsymbol{x}, \boldsymbol{y})$ the map $t \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is measurable and for fixed t the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y})$ is continuously differentiable for almost everywhere $t \in [0, T]$.*

Theorem 3.2. *Let $\mathcal{L}$ be a Carathéodory function satisfying* (2)*,* (3) *and* (4)*. Then the following statements hold:*

1. *The action integral given by* (5) *is finitely defined on $\mathcal{E}_d^{\Phi}(\lambda) := W^1L_d^{\Phi} \cap \{\boldsymbol{u} | \dot{\boldsymbol{u}} \in \Pi(E_d^{\Phi}, \lambda)\}$.*
2. *The function I is Gâteaux differentiable on $\mathcal{E}_d^{\Phi}(\lambda)$ and its derivative I' is demicontinuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into $\left[W^1L_d^{\Phi}\right]^*$. Moreover, I' is given by the following expression*

$$\langle I'(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \left\{D_{\boldsymbol{x}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\big(t, \boldsymbol{u}, \dot{\boldsymbol{u}}\big) \cdot \dot{\boldsymbol{v}}\right\}\, dt. \tag{22}$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}_d^{\Phi}(\lambda)$ into $\left[W^1L_d^{\Phi}\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $\boldsymbol{u}\in\mathcal{E}^{\Phi}_{d}(\lambda)$. Since $\lambda\Pi(E^{\Phi}_{d},r)=\Pi(E^{\Phi}_{d},\lambda r)$, we have $\dot{\boldsymbol{u}}/\lambda\in\Pi(E^{\Phi}_{d},1)$. Thus, as $f\in E^{\Phi}_{1}$ and attending to (12), we get

$$|\dot{\boldsymbol{u}}|/\lambda+f\in\Pi(E^{\Phi}_{1},1)\subset C^{\Phi}_{1}. \tag{23}$$

By Corollary 2.3 and (2), we get

$$|\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}})|\leqslant A(\|\boldsymbol{u}\|_{W^{1}L^{\Phi}})\left(b+\Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\lambda}+f\right)\right)\in L^{1}_{1}.$$

This fact proves item 1.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $\boldsymbol{u}\mapsto D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ is continuous from $\mathcal{E}^{\Phi}_{d}(\lambda)$ into $L^{1}_{d}([0,T])$ with the strong topology on both sets.

If $\boldsymbol{u}\in\mathcal{E}^{\Phi}_{d}(\lambda)$, from (3) and (23), we obtain

$$|D_{\boldsymbol{x}}\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}})|\leqslant A(\|u\|_{W^{1}L^{\Phi}})\left(b+\Phi\left(\frac{|\dot{\boldsymbol{u}}|}{\lambda}+f\right)\right)\in L^{1}_{1}. \tag{24}$$

Let $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^{\Phi}_{d}(\lambda)$ and let $\boldsymbol{u}\in\mathcal{E}^{\Phi}_{d}(\lambda)$ such that $\boldsymbol{u}_n\to\boldsymbol{u}$ in $W^{1}L^{\Phi}_{d}$. From $\boldsymbol{u}_n\to\boldsymbol{u}$ in L^{Φ}_{d}, there exists a subsequence $\boldsymbol{u}_{n_k}$ such that $\boldsymbol{u}_{n_k}\to\boldsymbol{u}$ a.e.; and, as $\dot{\boldsymbol{u}}_n\to\dot{\boldsymbol{u}}\in\mathcal{E}^{\Phi}_{d}(\lambda)$, by Lemma 2.5, there exist a subsequence of $\boldsymbol{u}_{n_k}$ (again denoted $\boldsymbol{u}_{n_k}$) and a function $h\in\Pi(E^{\Phi}_{1},\lambda))$ such that $\dot{\boldsymbol{u}}_{n_k}\to\dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}|\leqslant h$ a.e. Since $\boldsymbol{u}_{n_k}$, $k=1,2,\ldots$, is a strong convergent sequence in $W^{1}L^{\Phi}_{d}$, it is a bounded sequence in $W^{1}L^{\Phi}_{d}$. According to Lemma 2.1 and Corollary 2.3, there exists $M>0$ such that $\|\boldsymbol{a}(\boldsymbol{u}_{n_k})\|_{L^\infty}\leqslant M$, $k=1,2,\ldots$. From the previous facts and (24), we get

$$|D_{\boldsymbol{x}}\mathcal{L}(\cdot,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k})|\leqslant M\left(b+\Phi\left(\frac{|h|}{\lambda}+f\right)\right)\in L^{1}_{1}.$$

On the other hand, by the Carathéodory condition, we have

$$D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}_{n_k}(t),\dot{\boldsymbol{u}}_{n_k}(t))\to D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t))\quad\text{for a.e. } t\in[0,T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $\boldsymbol{u}\mapsto D_{y}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ is continuous from $\mathcal{E}^{\Phi}_{d}(\lambda)$ with the strong topology into $\left[L^{\Phi}_{d}\right]^{*}$ with the weak* topology.

Let $\boldsymbol{u}\in\mathcal{E}^{\Phi}_{d}(\lambda)$. From (23) and Lemma 2.4, it follows that

$$\varphi\left(\frac{|\dot{\boldsymbol{u}}|}{\lambda}+f\right)\in C^{\Psi}_{1}; \tag{25}$$

and, Corollary 2.3 implies $\boldsymbol{a}(\boldsymbol{u})\in L^{\infty}_{1}$. Therefore, in virtue of (4) we get

$$|D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}})|\leqslant A(\|\boldsymbol{u}\|_{W^{1}L^{\Phi}})\left(c+\varphi\left(\frac{|\dot{\boldsymbol{u}}|}{\lambda}+f\right)\right)\in L^{\Psi}_{1}. \tag{26}$$

Note that (24), (26) and the imbeddings $W^1L^\Phi_d \hookrightarrow L^\infty_d$ and $L^\Psi_d \hookrightarrow \left[L^\Phi_d\right]^*$ imply that the second member of (22) defines an element in $\left[W^1L^\Phi_d\right]^*$.

Let $\boldsymbol{u}_n, \boldsymbol{u} \in \mathcal{E}^\Phi_d(\lambda)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of $W^1L^\Phi_d$. We must prove that $D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n) \stackrel{w^*}{\rightharpoonup} D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}})$. On the contrary, there exist $\boldsymbol{v} \in L^\Phi_d$, $\epsilon > 0$ and a subsequence of $\{\boldsymbol{u}_n\}$ (denoted $\{\boldsymbol{u}_n\}$ for simplicity) such that

$$|\langle D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n),\boldsymbol{v}\rangle - \langle D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u},\dot{\boldsymbol{u}}),\boldsymbol{v}\rangle| \geq \epsilon. \tag{27}$$

We have $\boldsymbol{u}_n \to \boldsymbol{u}$ in L^Φ_d and $\dot{\boldsymbol{u}}_n \to \dot{\boldsymbol{u}}$ in L^Φ_d. By Lemma 2.5, there exist a subsequence $\boldsymbol{u}_{n_k}$ and a function $h \in \Pi(E^\Phi_1,\lambda)$ such that $\boldsymbol{u}_{n_k} \to \boldsymbol{u}$ a.e., $\dot{\boldsymbol{u}}_{n_k} \to \dot{\boldsymbol{u}}$ a.e. and $|\dot{\boldsymbol{u}}_{n_k}| \leqslant h$ a.e. As in the previous step, since $\boldsymbol{u}_n$ is a convergent sequence, the Corollary 2.3 implies that $a(|\boldsymbol{u}_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, with $\boldsymbol{u}_{n_k}$ instead of $\boldsymbol{u}$, inequality (26) becomes

$$|D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k})| \leqslant M\left(c + \varphi\left(\frac{h}{\lambda} + f\right)\right) \in L^\Psi_1. \tag{28}$$

Consequently, as $v \in L^\Phi_d$ and employing Hölder's inequality, we obtain that

$$\sup_k |D_{\boldsymbol{y}}\mathcal{L}(\cdot,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k}) \cdot v| \in L^1_1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k}) \cdot \boldsymbol{v}\; dt \to \int_0^T D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}}) \cdot \boldsymbol{v}\; dt \tag{29}$$

which contradicts the inequality (27). This completes the proof of step 2.

Step 3. We will prove (22). The proof follows similar lines as [23, Thm. 1.4]. For $\boldsymbol{u} \in \mathcal{E}^\Phi_d(\lambda)$ and $\boldsymbol{0} \neq \boldsymbol{v} \in W^1L^\Phi_d$, we define the function

$$H(s,t) := \mathcal{L}(t,\boldsymbol{u}(t) + s\boldsymbol{v}(t),\dot{\boldsymbol{u}}(t) + s\dot{\boldsymbol{v}}(t)).$$

From [19, Lemma 10.1] (or [27, Thm. 5.5]) we obtain that if $|\boldsymbol{u}| \leqslant |\boldsymbol{v}|$ then $d(\boldsymbol{u},E^\Phi_d) \leqslant d(\boldsymbol{v},E^\Phi_d)$. Therefore, for $|s| \leqslant s_0 := \left(\lambda - d(\dot{\boldsymbol{u}},E^\Phi_d)\right)/\|\boldsymbol{v}\|_{W^1L^\Phi}$ we have

$$d\left(\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}},E^\Phi_d\right) \leqslant d\left(|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}|,E^\Phi_1\right) \leqslant d\left(|\dot{\boldsymbol{u}}|,E^\Phi_1\right) + s\|\dot{\boldsymbol{v}}\|_{L^\Phi} < \lambda.$$

Thus $\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}} \in \Pi(E^\Phi_d,\lambda)$ and $|\dot{\boldsymbol{u}}| + s|\dot{\boldsymbol{v}}| \in \Pi(E^\Phi_1,\lambda)$. These facts imply, in virtue of Theorem 3.2 item 1, that $I(\boldsymbol{u} + s\boldsymbol{v})$ is well defined and finite for $|s| \leqslant s_0$. And, using Corollary 2.3, we also see that

$$\|a(|\boldsymbol{u} + s\boldsymbol{v}|)\|_{L^\infty} \leqslant A(\|\boldsymbol{u} + s\boldsymbol{v}\|_{W^1L^\Phi}) \leqslant A(\|\boldsymbol{u}\|_{W^1L^\Phi} + s_0\|\boldsymbol{v}\|_{W^1L^\Phi}) =: M$$

Now, applying Chain Rule, (24), (26) the monotonicity of φ and Φ, the fact that $\boldsymbol{v} \in L^\infty_d$ and $\dot{\boldsymbol{v}} \in L^\Phi_d$ and Hölder's inequality, we get

$$\begin{aligned}
|D_sH(s,t)| &= |D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u} + s\boldsymbol{v},\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u} + s\boldsymbol{v},\dot{\boldsymbol{u}} + s\dot{\boldsymbol{v}}) \cdot \dot{\boldsymbol{v}}| \\
&\leqslant M\left[\left(b(t) + \Phi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|\boldsymbol{v}|\right. \\
&\quad \left. + \left(c(t) + \varphi\left(\frac{|\dot{\boldsymbol{u}}| + s_0|\dot{\boldsymbol{v}}|}{\lambda} + f(t)\right)\right)|\dot{\boldsymbol{v}}|\right] \in L^1_1.
\end{aligned} \tag{30}$$

Consequently, I has a directional derivative and

$$\langle I'(\boldsymbol{u}),\boldsymbol{v}\rangle = \frac{d}{ds}I(\boldsymbol{u}+s\boldsymbol{v})\big|_{s=0} = \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\cdot\dot{\boldsymbol{v}}\}\ dt.$$

Moreover, from (24), (26), Lemma 2.1 and the previous formula, we obtain

$$|\langle I'(\boldsymbol{u}),\boldsymbol{v}\rangle| \leqslant \|D_{\boldsymbol{x}}\mathcal{L}\|_{L^1}\|\boldsymbol{v}\|_{L^\infty} + \|D_{\boldsymbol{y}}\mathcal{L}\|_{L^\Psi}\|\dot{\boldsymbol{v}}\|_{L^\Phi} \leqslant C\|\boldsymbol{v}\|_{W^1L^\Phi}$$

with a appropriate constant C. This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : \mathcal{E}_d^\Phi(\lambda) \to \left[W^1L_d^\Phi\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$. Indeed, if $\boldsymbol{u}_n,\boldsymbol{u} \in \mathcal{E}_d^\Phi(\lambda)$ with $\boldsymbol{u}_n \to \boldsymbol{u}$ in the norm of $W^1L_d^\Phi$ and $\boldsymbol{v} \in W^1L_d^\Phi$, then

$$\begin{aligned}\langle I'(\boldsymbol{u}_n),\boldsymbol{v}\rangle &= \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n\right)\cdot\boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n\right)\cdot\dot{\boldsymbol{v}}\}\ dt \\ &\to \int_0^T \{D_{\boldsymbol{x}}\mathcal{L}\left(t,\boldsymbol{u},\dot{\boldsymbol{u}}\right)\cdot\boldsymbol{v} + D_{\boldsymbol{y}}\mathcal{L}\left(t,\boldsymbol{u},\dot{\boldsymbol{u}}\right)\cdot\dot{\boldsymbol{v}}\}\ dt \\ &= \langle I'(\boldsymbol{u}),\boldsymbol{v}\rangle\,.\end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $\boldsymbol{u} \mapsto D_{\boldsymbol{x}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ and $\boldsymbol{u} \mapsto D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ are norm continuous from $\mathcal{E}_d^\Phi(\lambda)$ into L_d^1 and L_d^Ψ respectively. The continuity of the first map has already been proved in step 1. Let $\boldsymbol{u}_n,\boldsymbol{u} \in \mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n-\boldsymbol{u}\|_{W^1L^\Phi} \to 0$. Therefore, there exist a subsequence $\boldsymbol{u}_{n_k} \in \mathcal{E}_d^\Phi(\lambda)$ and a function $h \in \Pi(E_1^\Phi,\lambda)$ such that (28) holds true. And, as $\Psi \in \Delta_2$ then the right hand side of (28) belongs to E_1^Ψ. Now, invoking Lemma 2.6, we prove that from any sequence $\boldsymbol{u}_n$ which converges to $\boldsymbol{u}$ in $W^1L_d^\Phi$ we can extract a subsequence such that $D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u}_{n_k},\dot{\boldsymbol{u}}_{n_k}) \to D_{\boldsymbol{y}}\mathcal{L}(t,\boldsymbol{u},\dot{\boldsymbol{u}})$ in the strong topology. The desired result is obtained by a standard argument.

The continuity of I' follows from the continuity of $D_{\boldsymbol{x}}\mathcal{L}$ and $D_{\boldsymbol{y}}\mathcal{L}$ using the formula (22). □

4. Critical points and Euler-Lagrange equations

In this section we derive the Euler-Lagrange equations associated to critical points of action integrals on the subspace of T-periodic functions. We denote by $W^1L_T^\Phi$ the subspace of $W^1L_d^\Phi$ containing all T-periodic functions. As usual, when Y is a subspace of the Banach space X, we denote by $Y^\perp$ the *annihilator subspace* of X^*, i.e. the subspace that consists of all bounded linear functions which are identically zero on Y.

We recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is called *strictly convex* if $f\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right) < \frac{1}{2}\left(f\left(\boldsymbol{x}\right)+f\left(\boldsymbol{y}\right)\right)$ for $\boldsymbol{x} \neq \boldsymbol{y}$. It is well known that if f is a strictly convex and differentiable function, then $D_{\boldsymbol{x}}f : \mathbb{R}^d \to \mathbb{R}^d$ is a one-to-one map (see, e.g. [26, Thm. 12.17]).

Theorem 4.1. *Let $\boldsymbol{u} \in \mathcal{E}_d^{\Phi}(\lambda)$ be a T-periodic function. The following statements are equivalent:*

1. $I'(\boldsymbol{u}) \in \left(W^1 L_T^{\Phi}\right)^{\perp}$.
2. $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ *is an absolutely continuous function and* $\boldsymbol{u}$ *solves the following boundary value problem*

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) = D_{\boldsymbol{x}} \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad a.e.\ t \in (0, T) \\ \boldsymbol{u}(0) - \boldsymbol{u}(T) = D_{\boldsymbol{y}} \mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}} \mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0. \end{cases} \tag{31}$$

Moreover if $D_{\boldsymbol{y}}\mathcal{L}(t, x, y)$ is T-periodic with respect to the variable t and strictly convex with respect to $\boldsymbol{y}$, then $D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) - D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = 0$ is equivalent to $\dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}(T)$.

Proof. The condition $I'(\boldsymbol{u}) \in \left(W^1 L_T^{\Phi}\right)^{\perp}$ and (22) imply

$$\int_0^T D_{\boldsymbol{y}} \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \cdot \dot{\boldsymbol{v}}(t)\ dt = -\int_0^T D_{\boldsymbol{x}} \mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \cdot \boldsymbol{v}(t)\ dt,$$

for every $\boldsymbol{v} \in W^1 L_T^{\Phi}$. By [23, pp. 6-7] we obtain that $D_{\boldsymbol{y}}\mathcal{L}(t, \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$ is absolutely continuous and T-periodic, therefore it is differentiable a.e. on $[0, T]$ and the first equality of (31) holds true. This completes the proof of 1 implies 2. The proof of 2 implies 1 follows easily from (22) and (31).

The last part of the theorem is a consequence of $D_{\boldsymbol{y}}\mathcal{L}(T, \boldsymbol{u}(T), \dot{\boldsymbol{u}}(T)) = D_{\boldsymbol{y}}\mathcal{L}(0, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) = D_{\boldsymbol{y}}\mathcal{L}(T, u(T), \dot{\boldsymbol{u}}(0))$ and the injectivity of $D_{\boldsymbol{y}}\mathcal{L}(T, u(T), \cdot)$. □

5. Coercivity discussion

We recall a usual definition in the context of calculus of variations.

Definition 5.1. *Let X be a Banach space and let D be an unbounded subset of X. Suppose that $J : D \subset X \to \mathbb{R}$. We say that J is* coercive *if $J(u) \to +\infty$ when $\|\boldsymbol{u}\|_X \to +\infty$.*

It is well known that coercivity is a useful ingredient in the process of establishing existence of minima. Therefore, we are interested in finding conditions which ensure the coercivity of the action integral I acting on $\mathcal{E}_d^{\Phi}(\lambda)$. For this purpose, we need to introduce the following extra condition on the lagrangian function $\mathcal{L}$

$$\mathcal{L}(t, \boldsymbol{x}, \boldsymbol{y}) \geq \alpha_0 \Phi\left(\frac{|\boldsymbol{y}|}{\Lambda}\right) + F(t, \boldsymbol{x}), \tag{32}$$

where $\alpha_0, \Lambda > 0$ and $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, i.e. $F(t, \boldsymbol{x})$ is measurable with respect to t for every fixed $\boldsymbol{x} \in \mathbb{R}^d$ and it is continuous at $\boldsymbol{x}$ for a.e. $t \in [0, T]$. We observe that, from (32) and (2), we have $F(t, \boldsymbol{x}) \leqslant a(|\boldsymbol{x}|) b_0(t)$

with $b_0(t) := b(t) + \Phi(f(t)) \in L_1^1([0,T])$. In order to guarantee that integral $\int_0^T F(t,\boldsymbol{u})\,dt$ is finite for $\boldsymbol{u} \in W^1L^\Phi$, we need to assume

$$|F(t,\boldsymbol{x})| \leqslant a(|\boldsymbol{x}|)b_0(t), \quad \text{for a.e. } t \in [0,T] \quad \text{and for every } \boldsymbol{x} \in \mathbb{R}^d. \tag{33}$$

As we shall see in Theorem 5.3, when $\mathcal{L}$ satisfies (2), (3), (4), (32) and (33), the coercivity of the action integral I is related to the coercivity of the functional

$$J_{C,\nu}(\boldsymbol{u}) := \rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^\Phi}^\nu, \tag{34}$$

for $C, \nu > 0$. If $\Phi(x) = |x|^p/p$ then $J_{C,\nu}$ is clearly coercive for $\nu < p$. For more general Φ the situation is more interesting as it will be shown in the following lemma.

Lemma 5.2. *Let Φ and Ψ be complementary N-functions. Then:*

1. *If $C\Lambda < 1$, then $J_{C,1}$ is coercive.*
2. *If $\Psi \in \Delta_2$ globally, then there exists a constant $\alpha_\Phi > 1$ such that, for any $0 < \mu < \alpha_\Phi$,*
$$\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to\infty} \frac{\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right)}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} = +\infty. \tag{35}$$
In particular, the functional $J_{C,\mu}$ is coercive for every $C > 0$ and $0 < \mu < a_\Phi$. The constant α_Φ is one of the so-called Matuszewska-Orlicz indices *(see [20, Ch. 11]).*
3. *If $J_{C,1}$ is coercive with $C\Lambda > 1$, then $\Psi \in \Delta_2$.*

Proof. By (9) we have

$$(1 - C\Lambda)\|\boldsymbol{u}\|_{L^\Phi} + C\Lambda\|\boldsymbol{u}\|_{L^\Phi} = \|\boldsymbol{u}\|_{L^\Phi} \leqslant \Lambda + \Lambda\rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right),$$

then

$$\frac{(1 - C\Lambda)}{\Lambda}\|\boldsymbol{u}\|_{L^\Phi} - 1 \leqslant \rho_\Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right) - C\|\boldsymbol{u}\|_{L^\Phi} = J_{C,1}(\boldsymbol{u}).$$

This inequality shows that $J_{C,1}$ is coercive and therefore item 1 is proved.

In virtue of [1, Eq. (2.8)], the Δ_2-condition on Ψ, [20, Thm. 11.7] and [20, Cor. 11.6], we obtain constants $K > 0$ and $\alpha_\Phi > 1$ such that

$$\Phi(rs) \geq Kr^\nu\Phi(s) \tag{36}$$

for any $0 < \nu < \alpha_\Phi$, $s \geq 0$ and $r > 1$.

Let $1 < \mu < \nu < \alpha_\Phi$ and let $r > \Lambda$ be a constant that will be specified later. Then, from (36) and (9), we get

$$\begin{aligned}
\frac{\int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right)\,dt}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} &\geq K\left(\frac{r}{\Lambda}\right)^\nu \frac{\int_0^T \Phi(r^{-1}|\boldsymbol{u}|)\,dt}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} \\
&\geq K\left(\frac{r}{\Lambda}\right)^\nu \frac{r^{-1}\|\boldsymbol{u}\|_{L^\Phi} - 1}{\|\boldsymbol{u}\|_{L^\Phi}^\mu}.
\end{aligned}$$

We choose $r = \|\boldsymbol{u}\|_{L^\Phi}/2$. Since $\|\boldsymbol{u}\|_{L^\Phi} \to +\infty$ we can assume $\|\boldsymbol{u}\|_{L^\Phi} > 2\Lambda$. Thus $r > \Lambda$ and

$$\frac{\int_0^T \Phi\left(\frac{|\boldsymbol{u}|}{\Lambda}\right) dt}{\|\boldsymbol{u}\|_{L^\Phi}^\mu} \geq \frac{K}{2^\nu \Lambda^\nu} \|\boldsymbol{u}\|_{L^\Phi}^{\nu-\mu} \to +\infty \quad \text{as } \|\boldsymbol{u}\|_{L^\Phi} \to +\infty,$$

because $\nu > \mu$.

With the aim of proving item 3, we suppose that $\Psi \notin \Delta_2$. By [19, Thm. 4.1], there exists a sequence of real numbers r_n such that $r_n \to \infty$ and

$$\lim_{n\to\infty} \frac{r_n \psi(r_n)}{\Psi(r_n)} = +\infty. \tag{37}$$

Now, we choose $\boldsymbol{u}_n$ such that $|\boldsymbol{u}_n| = \Lambda \psi(r_n) \chi_{[0, \frac{1}{\Psi(r_n)}]}$. Then, by [19, Eq. (9.11)], we get

$$\|\boldsymbol{u}_n\|_{L^\Phi} = \Lambda \frac{\psi(r_n)}{\Psi(r_n)} \Psi^{-1}(\Psi(r_n)) = \Lambda \frac{r_n \psi(r_n)}{\Psi(r_n)} \to \infty, \quad \text{as} \quad n \to \infty.$$

And, using Young's equality (see [19, Eq. (2.7)]), we have

$$\begin{aligned} J_{C,1}(\boldsymbol{u}_n) &= \int_0^T \Phi\left(\frac{|\boldsymbol{u}_n|}{\Lambda}\right) dt - C\|\boldsymbol{u}_n\|_{L^\Phi} \\ &= \frac{1}{\Psi(r_n)} \left[\Phi(\psi(r_n)) - C\Lambda r_n \psi(r_n)\right] \\ &= \frac{1}{\Psi(r_n)} \left[r_n \psi(r_n) - \Psi(r_n) - C\Lambda r_n \psi(r_n)\right] \\ &= \frac{(1 - C\Lambda) r_n \psi(r_n)}{\Psi(r_n)} - 1. \end{aligned}$$

From (37) and the condition $C\Lambda > 1$, we obtain $J_{C,1}(\boldsymbol{u}_n) \to -\infty$, which contradicts the coercivity of $J_{C,1}$. □

Next, we present two results that establish coercivity of action integrals under different assumptions.

Theorem 5.3. *Let $\mathcal{L}$ be a lagrangian function satisfying* (2), (3), (4), (32) *and* (33). *We assume the following conditions:*

1. *There exist a non negative function $b_1 \in L^1_1$ and a constant $\mu > 0$ such that for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^d$ and a.e. $t \in [0, T]$*

$$|F(t, \boldsymbol{x_2}) - F(t, \boldsymbol{x_1})| \leqslant b_1(t)(1 + |\boldsymbol{x_2} - \boldsymbol{x_1}|^\mu). \tag{38}$$

We suppose that $\mu < \alpha_\Phi$, with α_Φ as in Lemma 5.2, in the case that $\Psi \in \Delta_2$; and, we suppose $\mu = 1$ if Ψ is an arbitrary N-function.

2.

$$\int_0^T F(t, \boldsymbol{x})\, dt \to \infty \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{39}$$

3. $\Psi \in \Delta_2$ *or, alternatively,* $\alpha_0^{-1} T \Phi^{-1}(1/T) \|b_1\|_{L^1} \Lambda < 1$.

Then the action integral I is coercive.

Proof. In the subsequent estimates, we use (32), the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, Hölder's inequality and Wirtinger's inequality.

$$\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\boldsymbol{u}}}{\Lambda} \right) + \int_0^T F(t, \boldsymbol{u}) \, dt \\
&= \alpha_0 \rho_\Phi \left(\frac{\dot{\boldsymbol{u}}}{\Lambda} \right) + \int_0^T \left[F(t, \boldsymbol{u}) - F(t, \overline{\boldsymbol{u}}) \right] \, dt + \int_0^T F(t, \overline{\boldsymbol{u}}) \, dt \\
&\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\boldsymbol{u}}}{\Lambda} \right) - \int_0^T b_1(t)(1 + |\tilde{\boldsymbol{u}}(t)|^\mu) \, dt + \int_0^T F(t, \overline{\boldsymbol{u}}) \, dt \\
&\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\boldsymbol{u}}}{\Lambda} \right) - \|b_1\|_{L^1} (1 + \|\tilde{\boldsymbol{u}}\|_{L^\infty}^\mu) + \int_0^T F(t, \overline{\boldsymbol{u}}) \, dt \qquad (40) \\
&\geq \alpha_0 \rho_\Phi \left(\frac{\dot{\boldsymbol{u}}}{\Lambda} \right) - \|b_1\|_{L^1} \left(1 + \left[T \Phi^{-1} \left(\frac{1}{T} \right) \right]^\mu \|\dot{\boldsymbol{u}}\|_{L^\Phi}^\mu \right) \\
&\quad + \int_0^T F(t, \overline{\boldsymbol{u}}) \, dt \\
&= \alpha_0 J_{C,\mu}(\dot{\boldsymbol{u}}) - \|b_1\|_{L^1} + \int_0^T F(t, \overline{\boldsymbol{u}}) \, dt,
\end{aligned}$$

where $C = \alpha_0^{-1} \left[T \Phi^{-1}(1/T) \right]^\mu \|b_1\|_{L^1}$. Let $\boldsymbol{u}_n$ be a sequence in $\mathcal{E}_d^\Phi(\lambda)$ with $\|\boldsymbol{u}_n\|_{W^1 L^\Phi} \to \infty$ and we have to prove that $I(\boldsymbol{u}_n) \to \infty$. On the contrary, suppose that for a subsequence, still denoted by $\boldsymbol{u}_n$, $I(\boldsymbol{u}_n)$ is upper bounded. Then, from (19) and passing to a subsequence, we can assume that $\dot{\boldsymbol{u}}_n$ is unbounded in L_d^Φ or $\overline{\boldsymbol{u}}_n$ is unbounded in $\mathbb{R}^d$. On the other hand, (33) and (39) imply that the integral $\int_0^T F(t, \overline{\boldsymbol{u}}_n) \, dt$ is lower bounded. These observations, the lower bound of I given in (40), assumption 3 in Theorem 5.3 and Lemma 5.2 imply that $I(\boldsymbol{u}_n)$ is not upper bounded. This contradiction leads us to the desired result. □

Based on [23] we say that F satisfies the condition (A) if $F(t, \boldsymbol{x})$ is a Carathéodory function, F verifies (33) and F is continuously differentiable with respect to $\boldsymbol{x}$. Moreover, the next inequality holds

$$|D_{\boldsymbol{x}} F(t, \boldsymbol{x})| \leqslant a(|\boldsymbol{x}|) b_0(t), \quad \text{for a.e. } t \in [0, T] \text{ and for every } \boldsymbol{x} \in \mathbb{R}^d. \qquad (41)$$

The following result was proved in [23, p. 18].

Lemma 5.4. *Suppose that F satisfies condition (A) and (39), $F(t, \cdot)$ is differentiable and convex a.e. $t \in [0, T]$. Then, there exists $\boldsymbol{x}_0 \in \mathbb{R}^d$ such that*

$$\int_0^T D_{\boldsymbol{x}} F(t, \boldsymbol{x}_0) \, dt = 0. \qquad (42)$$

Theorem 5.5. *Let $\mathcal{L}$ be as in Theorem 5.3 and let F be as in Lemma 5.4. Moreover, assume that $\Psi \in \Delta_2$ or, alternatively $\alpha_0^{-1} T \Phi^{-1}(1/T)\, a(|\boldsymbol{x}_0|) \|b_0\|_{L^1} \Lambda < 1$, with a and b_0 as in (33) and $\boldsymbol{x}_0 \in \mathbb{R}^d$ any point satisfying (42). Then I is coercive.*

Proof. Using (32), [23, Eq. (18), p.17], the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, (42), (10) and Wirtinger's inequality, we get

$$
\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t, \boldsymbol{x}_0)\, dt + \int_0^T D_{\boldsymbol{x}} F(t, \boldsymbol{x}_0) \cdot (\boldsymbol{u} - \boldsymbol{x}_0)\, dt \\
&= \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t, \boldsymbol{x}_0)\, dt + \int_0^T D_{\boldsymbol{x}} F(t, \boldsymbol{x}_0) \cdot \widetilde{\boldsymbol{u}}\, dt \\
&\quad + \int_0^T D_{\boldsymbol{x}} F(t, x_0) \cdot (\overline{\boldsymbol{u}} - \boldsymbol{x}_0)\, dt \\
&= \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + \int_0^T F(t, \boldsymbol{x}_0)\, dt + \int_0^T D_{\boldsymbol{x}} F(t, \boldsymbol{x}_0) \cdot \widetilde{\boldsymbol{u}}\, dt \\
&\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) - a(|\boldsymbol{x}_0|)\|b_0\|_{L^1} - a(|\boldsymbol{x}_0|)\|b_0\|_{L^1} T \Phi^{-1}\left(\frac{1}{T}\right) \|\dot{\boldsymbol{u}}\|_{L^\Phi} \\
&= \alpha_0 J_{C,1}(\dot{\boldsymbol{u}}) - a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}
\end{aligned}
\tag{43}
$$

with $C := \alpha_0^{-1} a(|\boldsymbol{x}_0|)\|b_0\|_{L^1} T \Phi^{-1}(1/T)$.

Let α be as in Corollary 2.3, i.e. α is a non decreasing majorant of a. Using that $F(t, \overline{\boldsymbol{u}}/2) \leqslant (1/2) F(t, \boldsymbol{u}) + (1/2) F(t, -\widetilde{\boldsymbol{u}})$ and taking into account that Φ is a non negative function, inequality (33), Hölder's inequality, Corollary 2.3 and Wirtinger's inequality, we obtain

$$
\begin{aligned}
I(\boldsymbol{u}) &\geq \alpha_0 \rho_\Phi\left(\frac{\dot{\boldsymbol{u}}}{\Lambda}\right) + 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - \int_0^T F(t, -\widetilde{\boldsymbol{u}})\, dt \\
&\geq 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - \|b_0\|_{L^1} \|\boldsymbol{a}(\tilde{\boldsymbol{u}})\|_{L^\infty} \\
&\geq 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - \|b_0\|_{L^1} \alpha(\|\tilde{\boldsymbol{u}}\|_{L^\infty}) \\
&\geq 2\int_0^T F(t, \overline{\boldsymbol{u}}/2)\, dt - C_1 \alpha(C_2 \|\dot{\boldsymbol{u}}\|_{L^\Phi})
\end{aligned}
\tag{44}
$$

for certain constants $C_1, C_2 > 0$.

Finally, reasoning in a similar way to that developed in the end of the proof of Theorem 5.3 we have that $I(\boldsymbol{u}_n) \to \infty$. $\square$

6. Main result

In order to find conditions for the lower semicontinuity of I, we perform a little adaptation of a result of [14].

Lemma 6.1. *Let $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ and $F(t,\boldsymbol{x})$ be Carathéodory functions satisfying*

$$\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y}) \geq \Phi(|\boldsymbol{y}|) + F(t,\boldsymbol{x}), \tag{45}$$

where Φ is an N-function. In addtion, suppose that F satisfies inequality (33) *and $\mathcal{L}(t,\boldsymbol{x},\cdot)$ is convex in $\mathbb{R}^d$ for each $(t,\boldsymbol{x}) \in [0,T] \times \mathbb{R}^d$. Let $\{\boldsymbol{u}_n\} \subset W^1L^\Phi$ be a sequence such that $\boldsymbol{u}_n$ converges uniformly to a function $\boldsymbol{u} \in W^1L^\Phi$ and $\dot{\boldsymbol{u}}_n$ converges in the weak topology of L^1_d to $\dot{\boldsymbol{u}}$. Then*

$$I(\boldsymbol{u}) \leqslant \liminf_{n\to\infty} I(\boldsymbol{u}_n). \tag{46}$$

Proof. First, we point out that (45) and (33) imply that I is defined on W^1L^Φ taking values on the interval $(-\infty,+\infty]$. Let $\{\boldsymbol{u}_n\}$ be a sequence satisfying the assumptions of the theorem. We define the Carathéodory function $\hat{\mathcal{L}} = \mathcal{L} - F$ and we denote by $\hat{I}$ its associated action integral. Using [14, Thm. 2.1, p. 243], we get

$$\int_0^T \hat{\mathcal{L}}(t,\boldsymbol{u},\dot{\boldsymbol{u}})\ dt \leqslant \liminf_{n\to\infty} \int_0^T \hat{\mathcal{L}}(t,\boldsymbol{u}_n,\dot{\boldsymbol{u}}_n)\ dt. \tag{47}$$

Taking account of the uniform convergence of $\boldsymbol{u}_n$ and the fact that F is a Carathéodory function, we obtain that $F(t,\boldsymbol{u}_n(t)) \to F(t,\boldsymbol{u}(t))$ a.e. $t \in [0,T]$. Since the sequence $\boldsymbol{u}_n$ is uniformly bounded, from (33) follows that there exists $g \in L^1_1([0,T])$ such that $|F(t,\boldsymbol{u}_n(t))| \leqslant g(t)$. Now, by the Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty} \int_0^T F(t,\boldsymbol{u}_n(t))\ dt = \int_0^T F(t,\boldsymbol{u}(t))\ dt. \tag{48}$$

Finally, as a consequence of (47) and (48), we obtain (46).

□

Theorem 6.2. *Let Φ and Ψ be complementary N-functions. Suppose that the Carathéodory function $\mathcal{L}(t,\boldsymbol{x},\boldsymbol{y})$ is strictly convex at $\boldsymbol{y}$, $D_{\boldsymbol{y}}\mathcal{L}$ is T-periodic with respect to T and* (2), (3), (4), (32), (33) *and* (39) *are satisfied. In addition, assume that some of the following statements hold (we recall the definitions and properties of α_0, b_1, $\boldsymbol{x}_0$ and b_0 from* (32), (38), (42) *and* (41) *respectively):*

1. $\Psi \in \Delta_2$ *and* (38).
2. (38) *and* $\alpha_0^{-1} T \Phi^{-1}(1/T) \|b_1\|_{L^1} \Lambda < 1$.
3. $\Psi \in \Delta_2$, *F satisfies condition (A) and $F(t,\cdot)$ is convex a.e. $t \in [0,T]$.*
4. *As item 3 but with $\alpha_0^{-1} T \Phi^{-1}(1/T)\, a(|\boldsymbol{x}_0|)\|b_0\|_{L^1}\Lambda < 1$ instead of $\Psi \in \Delta_2$.*

Then, problem (1) *has a solution.*

Proof. First of all, note that (32), (33) and (39) imply that I is ,lower bounded on $W^1L^\Phi_T$. Let $\{\boldsymbol{u}_n\} \subset W^1L^\Phi_T$ be a minimizing sequence for the problem $\min\{I(\boldsymbol{u})|\boldsymbol{u} \in W^1L^\Phi_T\}$. Since $I(\boldsymbol{u}_n)$, $n = 1,2,\ldots$ is bounded, Theorem 5.3 (or Theorem 5.5 according to which of the items 1-4 hold true) implies that

$\{\boldsymbol{u}_n\}$ is norm bounded in $W^1L^\Phi_d$. Hence, in virtue of Corollary 2.2, we can assume that $\boldsymbol{u}_n$ converges uniformly to a T-periodic continuous function $\boldsymbol{u}$. The space L^Φ_d is a predual space, concretely $L^\Phi_d = \left[E^\Psi_d\right]^*$. Thus, by [24, Cor. 5, p. 148] and since $\dot{\boldsymbol{u}}_n$ is bounded in L^Φ_d, there exists a subsequence (again denoted by $\dot{\boldsymbol{u}}_n$) such that $\dot{\boldsymbol{u}}_n$ converges to a function $\boldsymbol{v} \in L^\Phi_d$ in the weak* topology of L^Φ_d. From this fact and the uniform convergence of $\boldsymbol{u}_n$ to $\boldsymbol{u}$, we obtain that

$$\int_0^T \dot{\boldsymbol{\xi}} \cdot \boldsymbol{u} \; dt = \lim_{n\to\infty} \int_0^T \dot{\boldsymbol{\xi}} \cdot \boldsymbol{u}_n \; dt = -\lim_{n\to\infty} \int_0^T \boldsymbol{\xi} \cdot \dot{\boldsymbol{u}}_n \; dt = -\int_0^T \boldsymbol{\xi} \cdot \boldsymbol{v} \; dt$$

for every T-periodic function $\boldsymbol{\xi} \in C^\infty([0,T], \mathbb{R}^d) \subset E^\Psi_d$. Thus $\boldsymbol{v} = \dot{\boldsymbol{u}}$ a.e. $t \in [0,T]$ (see [23, p. 6]) and $\boldsymbol{u} \in W^1L^\Phi_T$.

Now, taking into account the relations $\left[L^1_d\right]^* = L^\infty_d \subset E^\Psi_d$ and $L^\Phi_d \subset L^1_d$, we have that $\dot{\boldsymbol{u}}_n$ converges to $\dot{\boldsymbol{u}}$ in the weak topology of L^1_d. Consequently, Theorem 6.1 applied to the N-function $\alpha_0\Phi\left(|\cdot|/\Lambda\right)$ implies that

$$I(\boldsymbol{u}) \leqslant \liminf_{n\to\infty} I(\boldsymbol{u}_n) = \min_{\boldsymbol{u}\in W^1L^\Phi_T} I(\boldsymbol{u}).$$

Hence, $\boldsymbol{u}$ is a minimun and therefore a critical point of I. Finally, invoking Theorem 4.1, the proof concludes. □

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