Clarke dual method for Hamiltonian systems with non standard grow

(In alphabetical order)

Sonia Acinas *
Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de La Pampa
(L6300CLB) Santa Rosa, La Pampa, Argentina
sonia.acinas@gmail.com

Jakub Maksymiuk

Fernando D. Mazzone †
Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales
Universidad Nacional de Río Cuarto
(5800) Río Cuarto, Córdoba, Argentina,
fmazzone@exa.unrc.edu.ar

Abstract

In this paper we consider the problem of finding periodic solutions of certain Hamiltonian systems .....blablabla

1 Main problem

Let $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We are looking for periodic solutions of the Hamiltonian system

$$\begin{cases} \dot{q}(t) & = D_pH(t,q(t),p(t)) \\ \dot{p}(t) & = -D_qH(t,q(t),p(t)) \\ p(0) & = p(T)\ ,\ q(0) = q(T) \end{cases} \tag{1}$$

for $t \in [0,T]$. I think that, like in [7], is better to present the Hamiltonian problem as the main problem

*SECyT-UNRC, FCEyN-UNLPam and UNSL
†SECyT-UNRC, FCEyN-UNLPam and CONICET

2010 AMS Subject Classification. Primary: 34C25. Secondary: 34B15.
Keywords and phrases. Periodic Solutions, Orlicz Spaces, Euler-Lagrange, Critical Points.

An alternative writing of (1) using the combined variable $u = (q, p)$ and the canonical symplectic matrix

$$J = \begin{pmatrix} 0 & I_{d\times d} \\ -I_{d\times d} & 0 \end{pmatrix}$$

is the following

$$\dot{u} = J\nabla H(t, u(t)) \tag{2}$$

or equivalently

$$J\dot{u} = -\nabla H(t, u(t)) \tag{3}$$

where ∇H is the gradient of H with respect to the combined variable.

2 Preliminaries

We will use some basic concepts of convex analysis that we list below.

Let $\Gamma_0(\mathbb{R}^d) = \{F : \mathbb{R}^d \to (-\infty, +\infty]$
convex, lower semicontinous functions with non-empty effective domain.$\}$

The Fenchel conjugate of F is given by

$$F^\star(p) = \sup_{q\in\mathbb{R}^d} \langle p, q\rangle - F(q)$$

The Fenchel conjugate satisfies the following properties:

1. $F^\star \in \Gamma_0(\mathbb{R}^d)$
2. If $F \leqslant G$, then $G^\star \leqslant F^\star$.
3. If $G(q) = \alpha F(\beta q) + \sigma$ with $\alpha, \beta, \sigma > 0$ then $G^\star(p) = \alpha F^\star(\frac{p}{\beta\alpha}) - \sigma$

Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a differentiable, convex function such that $\Phi(0) = 0$, $\Phi(q) > 0$ if $q \neq 0$, $\Phi(-q) = \Phi(q)$, and

$$\lim_{|q|\to\infty} \frac{\Phi(q)}{|q|} = +\infty, \tag{4}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From now on, we say that Φ is an G-function if Φ satisfies the previous properties.

We write $\Phi^\star$ for the Fenchel conjugate of Φ.

We do not assume that Φ and Φ' satisfy the Δ_2-condition.

We denote by $\partial F(q)$ the subdifferential of F in the sense of convex analysis (see [2, 3])

The next result is a generalization of [6, Prop. 2.2, p.34]

Proposition 2.1. *Let $F \in \Gamma_0(\mathbb{R}^d)$. Suppose that there exist an anisotropic function Φ and non negative constants β, γ such that*

$$-\beta \leqslant F(q) \leqslant \Phi(q) + \gamma, \text{ for all } q \in \mathbb{R}^d. \tag{5}$$

Now, if $p \in \partial F(q)$ then

$$\Phi^\star(p) \leqslant \Phi(2q) + 2(\beta + \gamma). \tag{6}$$

Proof. If $p \in \partial F(q)$, from [6, Thm. 2.2, p.33],

$$F^\star(p) = \langle p, q\rangle - F(q) \tag{7}$$

Conjugating (5), we have

$$F^\star(p) \geqslant \Phi^\star(p) - \gamma. \tag{8}$$

From Young's inequality, we get

$$\langle p, q\rangle = \frac{1}{2}\langle p, 2q\rangle \leqslant \frac{1}{2}\Phi^\star(p) + \frac{1}{2}\Phi(2q) \tag{9}$$

By eqs. (5) and (7) to (9), we get

$$\Phi^\star(p) \leqslant \frac{1}{2}\Phi^\star(p) + \frac{1}{2}\Phi(2q) + \beta + \gamma$$

which implies (6) $\square$

Remark 1. Inequality (6) is a few better than the corresponding in [6, Prop. 2.2] because the the case of power function we obtain $(\beta+\gamma)^{1/p}$, meanwhile in [6] appears $(\beta+\gamma)^{1/(p-1)}$.

3 Optimal bounds for a symplectic bilinear form

We consider the Euclidean space $\mathbb{R}^{2n}$ equipped with the standard symplectic structure given by bilinear canonical symplectic 2-form

$$\Omega(u, v) := \langle Ju, v\rangle.$$

As Jakub observed we can not consider any G-function on the symplectic manifold $\mathbb{R}^{2n}$. I think that the following can be the appropriate form of the G-function defined on the symplectic manifold $\mathbb{R}^{2n}$

Definition 3.1. *Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}$ be an anisotropic function G-function. We say that Ψ is a* symplectic G-function *if $\Psi^*(Ju) \prec \Psi(u)$, i.e. there exists $C, k > 0$ such that*

$$\Psi^*(Ju) \leqslant \Psi(ku) + C$$

Given $u = (q, p) \in \mathbb{R}^{2n}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$, Jakub define $\hat{\Phi} : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\hat{\Phi}(q, p) = \Phi(q) + \Phi^*(p)$$

with $\Phi^* \prec \Phi$.

Proposition 3.2. *$\hat{\Phi}$ is symplectic.*

Proof. We have $\hat{\Phi}^*(q, p) = \Phi^*(q) + \Phi^*(p)$, then $\hat{\Phi}^*(Ju) = \Phi^*(p) + \Phi^*(-q) \prec \Phi(q) + \Phi(p) = \hat{\Phi}(u)$. $\square$

Fernando suggests $\overline{\Phi}(q, p) = \Phi(q) + \Phi^*(p)$.

Proposition 3.3. *$\overline{\Phi}$ is symplectic.*

Proof. We have $\Phi^*(q,p) = \Phi(p) + \Phi^*(q)$, then $\overline{\Phi}^*(Ju) = \Phi(-q) + \Phi^*(p) = \overline{\Phi}(u)$. □

Theorem 3.4. *J induces an embedding of $L^{\Psi^*}([0,T],\mathbb{R}^{2n})$ into $L^{\Psi}([0,T],\mathbb{R}^{2n})$ when Ψ is symplectic.*

Proof. As Ψ is symplectic, there exist k, c such that $\Psi^*(Ju) < \Psi(ku) + c$ then

$$\int \Psi^*\left(\frac{Ju}{k\lambda}\right) \leqslant cT + \int \Psi\left(\frac{u}{\lambda}\right) < \infty$$

If $\|u\|_{L^\Psi} = 1$, then $\int \Psi(u) \leqslant 1$ and

$$\int \Psi^*\left(\frac{Ju}{k}\right) \leqslant cT + 1$$

As Ψ^* is convex, we have

$$\int \Psi^*\left(\frac{Ju}{(cT+1)\lambda}\right) \leqslant \frac{1}{cT+1}\int \Psi^*\left(\frac{Ju}{k}\right) \leqslant 1$$

then

$$\|Ju\|_{L^{\Psi^*}} \leqslant (cT+1)k := c_0$$

Finally, for any u,

$$\|Ju\|_{L^{\Psi^*}} \leqslant c_0 \|u\|_{L^\Psi}.$$

□

Corollary 3.5. *If $\Omega(u,v) = \int Jv \cdot u$ and Ψ is symplectic, then Ω is well defined in $L^\psi \times L^\Psi$.*

Theorem 3.6. *Let Φ be a symplectic G-function. There exist C_Φ, C and $\Lambda > 0$ such that for every $u \in W^1_T L^\Phi([0,T],\mathbb{R}^{2n})$ we have*

$$\Omega(\dot{u}, u) := \int_0^T J\dot{u}\cdot u\, dt \geqslant -C_\Phi \int_0^T \Phi\left(\frac{\dot{u}}{\Lambda}\right) dt - C \tag{10}$$

Proof. Let $u \in W^1 L^\Phi([0,T],\mathbb{R}^{2n})$. As usual we write $u = \tilde{u} + \overline{u}$ where

$$\overline{u} = \frac{1}{T}\int_0^T u(t)dt.$$

From [8, Lem. 2.4] we have that

$$\int_0^T \Phi(\tilde{u})dt \leqslant \int_0^T \Phi(T\dot{u})dt. \tag{11}$$

By Young's inequality, the fact that a G-function is even and (11), we obtain

$$\int_0^T J\dot{u}\cdot u\,dt = \frac{k}{T}\int_0^T J\frac{T\dot{u}}{k}\cdot\tilde{u}\,dt \geqslant$$
$$-\frac{k}{T}\left\{\int_0^T \Phi^*\left(J\frac{T\dot{u}}{k}\right)dt + \int_0^T \Phi(\tilde{u})\,dt\right\} \geqslant$$
$$-\frac{k}{T}\left\{2\int_0^T \Phi(T\dot{u})\,dt + C\right\}$$

□

4 Differentiability of Hamiltonian dual action

Theorem 4.1. *Suppose that $\Phi : \mathbb{R}^{2n} \to [0, +\infty)$ is a differentiable G-function, not necessarily symplectic. Additionally*

1. *$H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is measurable in t, continuously differentiable with respect to u.*
2. *there exist $\beta, \gamma \in L^1([0,T],\mathbb{R})$, $\Lambda > \lambda > 0$ such that*

$$\Phi^\star\left(\frac{u}{\Lambda}\right) - \beta(t) \leqslant H(t,u) \leqslant \Phi^\star\left(\frac{u}{\lambda}\right) + \gamma(t) \tag{12}$$

Then there exists Λ_0 such that the dual action

$$\chi(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + H^\star(t,\dot{v})dt \tag{13}$$

is continuously differentiable in $W^1L_T^\Phi([0,T],\mathbb{R}^{2n}) \cap \{u | d(\dot{u}, L^\infty) < \Lambda_0\}$.

If v is a critical point of χ with $d(\dot{v}, L^\infty) < \Lambda_0$, the function defined by $u(t) = \nabla H^\star(t,\dot{v})$ solves

$$\begin{cases} \dot{u} &= J\nabla H(t,u) \\ u(t) &= u(T) \end{cases}$$

Proof. Conjugating 2 we obtain

$$\Phi(\lambda u) - \gamma(t) \leqslant H^\star(t,v) \leqslant \Phi(\Lambda v) + \beta(t) \tag{14}$$

Since $H^\star$ is smooth, we have $\partial_v H^\star(t,v) = \{\nabla_v H^\star(t,v)\}$. Applying Proposition 2.1 with $F = H^\star$, $\Phi(\Lambda v)$ instead of $\Phi(u)$ and $u = \nabla H^\star(t,v) \in \partial_v H(t,v)$, inequality (12) becomes

$$\Phi^\star\left(\frac{\nabla H^\star(t,v)}{\Lambda}\right) \leqslant \Phi(2\Lambda v) + 2(\beta+\gamma). \tag{15}$$

which will be the main inequality in the proof.

We are planning to obtain the structure condition (**??**) of [8] which guarantees differentiability.

We consider the Lagrangian

$$\mathcal{L}(t, v, \xi) = \frac{1}{2}\langle J\xi, v\rangle + H^\star(t, \xi) \tag{16}$$

and we have to prove that there exist $\Lambda_0 > \lambda_0 > 0$ such that

$$|\mathcal{L}| + |\nabla_v \mathcal{L}| + \Phi^\star\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) \leqslant a(v)\left(b(t) + \Phi\left(\frac{\xi}{\Lambda_0}\right)\right) \tag{17}$$

We start with $|\mathcal{L}|$. From (14),

$$|\mathcal{L}| \leqslant \frac{1}{2}|\langle J\xi, v\rangle| + H^\star(t, \xi) \leqslant \frac{1}{2}|\xi||v| + \Phi(\Lambda\xi) + \beta(t).$$

Since $\frac{\Phi(x)}{|x|} \to \infty$ as $|x| \to \infty$, there exists $C > 0$ such that $|x| \leqslant \Phi(x) + C$ for all $x \in \mathbb{R}^d$. Then,

$$|\mathcal{L}| \leqslant \frac{1}{2}\frac{|v|}{\Lambda}\left(\Phi(\Lambda\xi) + C\right) + \Phi(\Lambda\xi) + \beta(t) \leqslant \max\left\{\Lambda, \frac{|v|}{2\Lambda}\right\}\left[\Phi(\Lambda\xi) + C + \beta(t)\right]$$

which is an estimate like the second member of (17).

Now, we treat $|\nabla_v \mathcal{L}|$ and we get

$$|\nabla_v \mathcal{L}| = \frac{1}{2}|J\xi| \leqslant |\xi| \leqslant \frac{1}{2\Lambda}(\Phi(\Lambda\xi) + C). \tag{18}$$

which is also an estimate of the desired type.

Finally, we deal with $\Phi(\nabla_\xi \mathcal{L}\lambda_0)$. As $\Phi^\star$ is a convex, even function, we have

$$\Phi^\star\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) = \Phi^\star\left(\frac{-\frac{1}{2}Jv}{\lambda_0} + \frac{\nabla H^\star(t, \xi)}{\lambda_0}\right) \leqslant \frac{1}{2}\Phi^\star\left(\frac{Jv}{\lambda_0}\right) + \frac{1}{2}\Phi^\star\left(\frac{2\nabla H^\star(t, \xi)}{\lambda_0}\right).$$

We choose $\frac{2}{\lambda_0} = \frac{1}{\Lambda}$ with Λ as in (15) and we finally have

$$\begin{aligned}\Phi^\star\left(\frac{\nabla_\xi \mathcal{L}}{\lambda_0}\right) &\leqslant \Phi^\star\left(\frac{Jv}{2\Lambda}\right) + \Phi(2\Lambda\xi) + 2(\beta + \gamma) = \\ &\max\left\{\Phi^\star\left(\frac{Jv}{2\Lambda}\right), 1\right\}\left[\Phi(2\Lambda\xi) + 2(\beta + \gamma)\right]\end{aligned} \tag{19}$$

which is a bound like the second member of (17).

Therefore, from (17), (18), (19) and choosing the worst functions a and b, we obtain condition (**??**).

Next, [8, Thm. 4.5] implies differentiability of χ in a set like $W^1L^\Phi_T([0, T], \mathbb{R}^d) \cap \{u | d(\dot{u}, L^\infty) < \lambda_0\}$.

If $v \in W^1L^\Phi_T([0,T],\mathbb{R}^d)$ is a critical point of χ with $d(\dot v, L^\infty) < \lambda_0$ then, from equations (21) of [8] we obtain

$$0 = \int_0^T \frac{1}{2}\langle J\dot v, h\rangle - \frac{1}{2}\langle \dot h, Jv\rangle + \langle \nabla H^\star(t,\dot v), \dot h\rangle.$$

The rest of the proof follows as in [6].

□

5 Existence periodic solutions Hamiltonian system

The following theorem extend to a quite general function Φ the result in [6, Thm. 3.1] formulated for $\Phi_2(u) = |u|^2/2$. Even more, our result improves a little bit [6, Thm. 3.1] in the sense that we obtain existence for Φ_2 when the functions l and γ, introduced below, belong to L^2 and L^1 respectively instead of L^4 and L^2 as it is assumed in [6, Thm. 3.1]. This little improvement is due to the observation in Remark 1.

Theorem 5.1. *Suppose that Φ is a symplectic G-function and*

H1) Exists $\xi \in L^{\Phi^\star}([0,T],\mathbb{R}^{2n})$ such that for every $u \in \mathbb{R}^{2n}$ and a.e. $t \in [0,T]$

$$H(t,u) \geqslant \langle \xi(t), u\rangle.$$

H2) There exists Λ_0 (indicar dónde vive en función de $C_{\Phi^\star}$???) and $\alpha \in L^1$ such that, for every $(t,u) \in [0,T]\times\mathbb{R}^{2n}$ and a.e. $t \in [0,T]$, we have

$$H(t,u) \leqslant \Phi\left(\frac{u}{\Lambda_0}\right) + \alpha(t).$$

H3)

$$\int_0^T H(t,u)dt \to +\infty, \quad \text{when } |u| \to +\infty.$$

Then the problem xxxxx has at least one solution u such that

$$v(t) = -J\left[u(t) - \frac{1}{T}\int_0^T u(s)\,ds\right]$$

minimizes the dual action

$$\chi(v) = \int_0^T \frac{1}{2}\langle J\dot v, v\rangle + H^\star(t,\dot v)dt$$

Proof. Suppose that $0 < r < 1$ and $\epsilon < \frac{r}{\Lambda_0}$

We define

$$H_\epsilon(t,u) = H(t,u) + \Phi(\epsilon u)$$

By H1), Young's inequality and the convexity of Φ, we have

$$\begin{aligned} H_\epsilon(t,u) &\geqslant \langle \xi(t), u\rangle + \Phi(\epsilon u) \\ &\geqslant -\Phi^\star\left(\frac{\Lambda_0}{r\epsilon}\xi(t)\right) - \Phi(r\epsilon u) + \Phi(\epsilon u) \\ &\geqslant \Phi((1-r)\epsilon u) + \beta(t) \end{aligned} \tag{20}$$

where $\beta(t) := \frac{\Lambda_0}{r\epsilon}\xi(t) \in L^1$.

On the other hand

$$H_\epsilon(t,u) \leqslant \Phi\left(\frac{u}{\Lambda_0}\right) + \alpha(t) + \Phi(\epsilon u) \leqslant (1+r)\Phi\left(\frac{u}{\Lambda_0}\right) + \alpha(t) \tag{21}$$

From (20), (21) and properties of Fenchel conjugate, we get

$$(1+r)\Phi^\star\left(\frac{\Lambda_0 u}{1+r}\right) - \alpha(t) \leqslant H^\star_\epsilon(t,u) \leqslant \Phi^\star\left(\frac{u}{(1-r)\epsilon}\right) + \beta(t). \tag{22}$$

The perturbed Hamiltonian H_ϵ verifies the assumptions of Theorem 4.1, then the dual action χ_ϵ is continuously differentiable in $W^1L^\Phi_T([0,T],\mathbb{R}^{2n}) \cap \{u | d(\dot{u}, L^\infty) < \lambda_0\}$.

VER QUIEN EN λ_0???

Now, we deal with the coercivity of χ_ϵ given by

$$\chi_\epsilon(v) = \int_0^T \frac{1}{2}\langle J\dot{v}, v\rangle + H^\star_\epsilon(t,\dot{v})dt \tag{23}$$

From (22) and (10), we have

$$\chi_\epsilon(v) \geqslant -\frac{C_{\Phi^\star}}{2}\int_0^T \Phi^\star(T\dot{v})\,dt + (1+r)\int_0^T \Phi^\star\left(\frac{\Lambda_0\dot{v}}{1+r}\right)dt - \int_0^T \alpha(t)\,dt - \frac{C}{2}T$$

En el papel, no aparece el 2 dividiendo. No sería necesario, porque acotaríamos por -1, pero no recuerdo si esa era la idea o fue un olvido. En la cuenta final, no molesta genera problemas dejar el 2.

Let $\Lambda_0 \geqslant \max\{(1+r)T, C_{\Phi^\star}T\}$, then $\Lambda_0 > \max\{T, C_{\Phi^\star}T\}$, $\frac{\Lambda_0}{T} > 1$ and there exists $r > 0$ such that $\frac{\Lambda_0}{T} = (1+r)$.

Thus,

$$\begin{aligned} \chi_\epsilon(v) &\geqslant -\frac{C_{\Phi^\star}}{2}\int_0^T \Phi^\star(T\dot{v})\,dt + \frac{\Lambda_0}{T}\int_0^T \Phi^\star(T\dot{v})\,dt - \int_0^T \alpha(t)\,dt - \frac{C}{2}T \\ &= \left(-\frac{C_{\Phi^\star}}{2} + \frac{\Lambda_0}{T}\right)\int_0^T \Phi^\star(T\dot{v})\,dt - \int_0^T \alpha(t)\,dt - \frac{C}{2}T \\ &> \left(-\frac{C_{\Phi^\star}}{2} + C_{\Phi^\star}\right)\int_0^T \Phi^\star(T\dot{v})\,dt - \int_0^T \alpha(t)\,dt - \frac{C}{2}T \\ &= \frac{C_{\Phi^\star}}{2}\int_0^T \Phi^\star(T\dot{v})\,dt - \int_0^T \alpha(t)\,dt - \frac{C}{2}T \end{aligned}$$

No creo que sea necesario tanto detalle en la cuenta anterior, pero como no la habíamos escrito en el papel, la hice para ver cómo salía. □

Observaciones de último momento!

- Habría que definir si usamos n o d porque un poco de $\mathbb{R}^d$ y otro poco de $\mathbb{R}^n$.
- Un problema similar tenemos con Φ y Ψ.
- Ahora recuerdo que dijiste algo sobre NO escribir UN resultado como el Teorema 3.1, sino varios resultados individuales (difrenciabilidad, coercividad, minimizacin, etc)

6 Cota óptima y muchas otras cosas...

Quizás aquí haya cosas que deben colocarse antes, pero me generaban confusión en el medio y por eso las acumulé acá.

6.1 Caso L^p para cota óptima

Let $u \in H_T^{1,p}([0,T],\mathbb{R}^d)$. Then

$$\int_0^T |u-\overline{u}|^{p'}\,dt \leqslant C_p T^p \int_0^T |u'|^p\,dt \tag{24}$$

where the optimal constant satisfies

$$\begin{aligned} C_p^{-1} &:= \inf\left\{T^p \frac{\int_0^T |u'|^p\,dt}{\int_0^T |u-\overline{u}|^p\,dt} \middle| u \in H_T^{1,p}\right\} = \\ &\inf\left\{T^p \frac{\int_0^T |u'|^p\,dt}{\int_0^T |u|^p\,dt} \middle| u \in H_T^{1,p},\ \int_0^T u\,dt = 0\right\} \end{aligned} \tag{25}$$

Lemma 6.1. *C_p given by* (25) *is independent of T.*

Proof. Let $T \neq T'$. If u is a function such that

$$C_p^{-1}(T) + \epsilon > T^p \frac{\int_0^T |u'|^p\,dt}{\int_0^T |u|^p\,dt} \tag{26}$$

Performing the change of variable $s = \frac{T'}{T}t$ and calling $r = \frac{T}{T}$, we have

$$T^p \frac{\int_0^T |u'(t)|^p\,dt}{\int_0^T |u(t)|^p\,dt} = T^p \frac{\int_0^T |u'(rs)|^p\,ds}{\int_0^T |u(rs)|^p\,dt} = (T')^p \frac{\int_0^T |v'(s)|^p\,ds}{\int_0^T |v(s)|^p\,dt} \geqslant C_p^{-1}(T') \tag{27}$$

where $v(s) = u(rs)$. Therefore, $C_p^{-1}(T') \leqslant C_p^{-1}(T)$ and consequently $C_p(T') = C_p(T)$. □

Lemma 6.2.

$$C_p^{-1} = \inf\left\{T^p \int_0^T |u'|^p\,dt | u \in H_T^{1,p},\ \int_0^T u\,dt = 0, \int_0^T |u|^p\,dt = 1\right\} \tag{28}$$

Proof. The existence of a minimum follows as usual by means of a minimizing sequence.

More details....?

We employ the method of Lagrange multipliers to solve an optimization problem with constraints. We will look for critical points of

$$I = \int_0^T |u'|^p\,dt - \lambda \int_0^T |u|^p\,dt + \mu \cdot \int_0^T u\,dt, \quad u \in H_T^{1,p} \tag{29}$$

The Gâteaux derivative of the functional is given by

$$\begin{aligned} \left\langle I'(u), v\right\rangle = \int_0^T p|u'|^{p-2}u' \cdot v'\,dt - p\lambda \int_0^T |u|^{p-2}u \cdot v\,dt + \mu \int_0^T v\,dt = \\ \int_0^T \left\{\frac{d}{dt}(p|u'|^{p-2}u') - p\lambda|u|^{p-2}u + \mu\right\} \cdot v\,dt + p|u'|^{p-2}u' \cdot v|_0^T = 0 \end{aligned} \tag{30}$$

Since v is an arbitrary function, we choose v such that $v(0) = v(T) = 0$ and we obtain

$$\frac{d}{dt}(p|u'|^{p-2}u') - p\lambda|u|^{p-2}u + \mu = 0 \ \ a.e. \tag{31}$$

This fact implies that $p|u'|^{p-2}u' \cdot v|_0^T = 0\ \forall v \in H_T^{1,p}$, that is

$$[p|u'(T)|^{p-2}u'(T) - p|u'(0)|^{p-2}u'(0)] \cdot v(0) = 0. \tag{32}$$

Then $u'(T) = u'(0)$. Now, integrating (31), we get

$$p\lambda \int_0^T |u|^{p-2}u\,dt + \mu T = 0.$$

If $p = 2$ then $\mu = 0$ and

$$\begin{cases} u'' + \lambda u &= 0 \\ u(0) &= u(T) \\ \int_0^T u\,dt &= 0 \end{cases}$$

The normalized solution is $u(t) = \cos(\sqrt{\lambda}t)u_0 + \sin(\sqrt{\lambda}t)u_1$ with $u_0, u_1 \in \mathbb{R}^d$.

As $u(0) = u(T)$ and $u'(0) = u'(T)$, the function $u(t)$ has minimal period $\frac{2\pi}{\sqrt{\lambda}}$ and it solves the second order ODE $u'' + \lambda u = 0$

Then $u(0) = u(T)$, $u'(0) = u'(T)$ imply that the function u has period T.

Ésto ....

As $u \neq 0$, we have $k\frac{2\pi}{\sqrt{\lambda}} = T$ with $k = 1, 3, \ldots$. Then $\lambda = k^2\frac{4\pi^2}{T^2}$.

Now, if $u_k(t) = \cos(\frac{2k\pi}{T}t)u_0 + \sin(\frac{2k\pi}{T}t)u_1$, then

$$\begin{aligned}
1 &= \int_0^T |u_k|^2\,dt \\
&= \int_0^T \left[\cos\left(\frac{2k\pi}{T}t\right)\right]^2 dt|u_0|^2 + \int_0^T \left[\sin\left(\frac{2k\pi}{T}t\right)\right]^2 dt|u_1|^2 \\
&+ \int_0^T \cos\left(\frac{2k\pi}{T}t\right)\sin\left(\frac{2k\pi}{T}t\right) dt\ u_0\cdot u_1 \\
&= \frac{T}{2}(|u_0|^2+|u_1|^2)
\end{aligned}$$

and

$$\begin{aligned}
&T^2\int_0^T |u_k'|^2\,dt \\
&= T^2\left(\frac{2k\pi}{t}\right)^2\left\{\int_0^T\left[\sin\left(\frac{2k\pi t}{T}\right)\right]^2|u_0|^2 + \int_0^T\left[\cos\left(\frac{2k\pi t}{T}\right)\right]^2|u_1|^2+0\right\} \\
&= \left(\frac{2k\pi}{t}\right)^2\frac{T}{2}(|u_0|^2+|u_1|^2) \\
&= 4k^2\pi^2
\end{aligned}$$

The minimum occurs when $k = 1$ and we get $C_2^{-1} = 4\pi^2$. Then, $\int_0^T |u|^2\,dt \leqslant \frac{T^2}{4\pi^2}\int_0^T |u'|^2\,dt$

Or...

From $u'' + \lambda u = 0$, we have $u''u + \lambda u^2 = 0$ and integrating over $[0,T]$ we obtain $0 = \int_0^T u''u + \lambda\int_0^t u^2 = -\int_0^T (u')^2 + \lambda\int_0^T u^2 + u'u|_0^T = -\int_0^T (u')^2 + \lambda\int_0^T u^2 + u'(T)u(T) - u'(0)u(0) = -\int_0^T (u')^2 + \lambda\int_0^T u^2$, then $\frac{4\pi^2 k}{T^2} = \lambda = \frac{\int_0^T (u')^2}{\int_0^T u^2} = \frac{1}{C_2}$ The minimum value is attained at $k = 1$ and therefore $C_2 = \frac{T^2}{4\pi^2}$. □

6.2 L^Φ case where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is an anisotropic function

Now, we are looking for the optimal constant $C(\Phi, T)$ such that

$$\int_0^T \Phi(u - \overline{u_\Phi})\,dt \leqslant C(\Phi,T)\int_0^T \Phi(u')\,dt \quad u \in W^1L_T^\Phi \tag{33}$$

Then,

$$\int_0^T \Phi(u - \overline{u}_\Phi)\,dt \leqslant C(\Phi,T)\int_0^T \Phi(u-a)\,dt \quad \forall a \in \mathbb{R}^d \tag{34}$$

where $a = \overline{u}_\Phi$ is the unique vector of $\mathbb{R}^d$ such that

$$\int_0^T \nabla\Phi(u-a)\,dt = 0. \tag{35}$$

Thus, $C^{-1} = \inf\left\{\frac{\int_0^T \Phi(u')\,dt}{\int_0^T \Phi(u-\overline{u}_\Phi)}|u \in W^1L^\Phi_T\right\}$ Let $v = u - \overline{u}_\Phi$, $v' = u'$ and $\overline{v}_\Phi = 0$ then

$$\lambda := C^{-1} = \inf\left\{\frac{\int_0^T \Phi(u')\,dt}{\int_0^T \Phi(u)\,dt}|u \in W^1L^\Phi_T, \int_0^T \nabla\Phi(u)\,dt = 0\right\} \tag{36}$$

Let

$$L(u,u') = \frac{\int_0^T \Phi(u')\,dt}{\int_0^T \Phi(u)\,dt} - \mu\cdot\int_0^T \nabla\Phi(u)\,dt \tag{37}$$

with $\mu \in \mathbb{R}^d$. By Gâteaux derivative we have

$$0 = \frac{\int_0^T \Phi(u)\,dt \int_0^T \nabla\Phi(u')\cdot v'\,dt - \int_0^T \Phi(u')\,dt \int_0^T \nabla\Phi(u)\cdot v\,dt}{(\int_0^T \Phi(u)\,dt)^2} - \mu\int_0^T D^2\Phi(u)\cdot v\,dt \tag{38}$$

then

$$\begin{gathered} 0 = \int_0^T \nabla\Phi(u')\cdot v'\,dt - \lambda\int_0^T \nabla\Phi(u)\cdot v\,dt - \int_0^T \Phi(u)\,dt\mu\cdot\int_0^T D^2\Phi(u)\cdot v\,dt = \\ \int_0^T \left\{-\frac{d}{dt}\nabla\Phi(u') - \lambda\nabla\Phi(u)\right\}\cdot v\,dt + \nabla\Phi(u')\cdot v|_0^T - \int_0^T \Phi(u)\,dt\,\mu\cdot\int_0^T D^2\Phi(u)\cdot v\,dt = \\ \int_0^T \left\{-\frac{d}{dt}\nabla\Phi(u') - \lambda\nabla\Phi(u) - \int_0^T \Phi(u)\,dt\,\mu\cdot D^2\Phi(u)\right\}\cdot v\,dt + \nabla\Phi(u')\cdot v|_0^T \end{gathered} \tag{39}$$

$\forall v \in W^1L^\Phi_T$.

Now, we consider any $v \in W^1_0L^\Phi$ and we get

$$-\frac{d}{dt}\nabla\Phi(u') - \lambda\nabla\Phi(u) - \int_0^T \Phi(u)\,dt\,\mu\cdot D^2\Phi(u) = 0 \tag{40}$$

Then,

$$\nabla\Phi(u')v|_0^T = 0 \ \ \forall v \in W^1L^\Phi_T,$$

that is

$$\{\nabla\Phi(u'(T)) - \nabla\Phi(u'(0))\}\cdot v(0) = 0$$

for any $v \in W^1L^\Phi_T$. Thus, $\nabla\Phi(u'(T)) = \nabla\Phi(u'(0))$

As Φ is strictly convex, then $\nabla\Phi$ is injective and consequently $u'(T) = u'(0)$.

Post-multiplying (40) by μ^t and integrating over $[0,T]$, we get

$$0 = \int_0^T -\frac{d}{dt}\nabla\Phi(u')\,dt\cdot\mu^t - \lambda\int_0^T \nabla\Phi(u)\,dt\cdot\mu^t - \int_0^T \Phi(u)\,dt\int_0^T \mu\cdot D^2\Phi(u)\,dt\cdot\mu^t \tag{41}$$

with $\nabla\Phi(u'(T)) = \nabla\Phi(u'(0))$.

We know that $\mu\cdot D^2\Phi(u)\cdot\mu^t = 0$ iff $\mu = 0$. And, as $\int_0^T \Phi(u)\,dt \neq 0$, then $\int_0^T \Phi(u)\,dt\int_0^T \mu\cdot D^2\Phi(u)\cdot\mu^t\,dt\cdot = 0$ implies that $\mu = 0$.

Therefore,

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u') + \lambda\nabla\Phi(u) = 0 \\ u(0) = u(T),\ u'(0) = u'(T),\ \int_0^T \nabla\Phi(u)\,dt = 0 \end{cases} \tag{42}$$

If $\Phi(u) = \frac{|u|^p}{p}$ and $u \in \mathbb{R}$. We know that $T = \frac{4\pi(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})\lambda^{\frac{1}{p}}}k$ for $k = +1, +2, \ldots$.

Cuestiones a resolver:

1. Qué dejar? Caso p o caso Φ?
2. En el caso Φ? Analizamos la existencia en detalle?
3. Consideramos el caso $\Phi(x_1, \ldots, x_d) = \Phi(x_1) + \cdots + \Phi(x_d)$?

Example 6.1. Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a G-function. Then the G-function

$$\Phi(u) = \Phi(q,p) := \Phi(q) + \Phi^\star(p).$$

is a symplectic G-function.

PROBLEM 0: It is the previous the general form of any symplectic G-function? It is possible to find other example of these functions?

We note that if Φ is symplectic then

$$\nabla\Phi(Ju) = J\Phi^\star(u). \tag{43}$$

Here we are agreeing that $\nabla\Phi$ is a column vector.

As a consequence of (**??**), the matrix J induce a isometry between the spaces $L^\Phi([0,T],\mathbb{R}^{2d})$ and $L^{\Phi^\star}([0,T],\mathbb{R}^{2d})$. Therefore we candefine a bilinear form $\overline{\Omega}$ on $L^\Phi([0,T],\mathbb{R}^{2d})$ of the following way

$$\overline{\Omega}(u,v) := \int_0^T \Omega(u,v)dt, \quad u,v \in L^\Phi([0,T],\mathbb{R}^{2d})$$

We consider the following functional

$$\Theta(u) := \overline{\Omega}(u,\dot{u}).$$

We are interested in to find bounds of the quadratic functional Θ of the following type

$$\theta(u) \geqslant -C\int_0^T \Phi(\dot{u})\,dt, \tag{44}$$

for $u \in W^1L^\Phi([0,T],\mathbb{R}^{2d})$. It is important to get the best constant C in previous inequality because this constant imposes restrictions to the Hamiltonian H.

If $\Phi(q) = |q|^2/2$ was proved in [6, Prop. 3.2] (44) holds width $C = T/\pi$. Below we prove that this is the optimal constant satisfying (44). Meanwhile in [9, Lem. 3.3] was proved that $C_\Phi = 2T$ satisfies (44) when $\Phi(q) = |q|^\alpha/\alpha$, $1 < \alpha < \infty$. Since this constant is not equal to T/π when $\alpha = 2$, it is not optimal.

Proposition 6.3. *Let Φ be any symplectic G-function. Then* (44) *holds for and $C = 2T^{-1}$ for every $u \in W^1L^\Phi([0,T], \mathbb{R}^{2d})$.*

Proof. Let $u \in W^1L^\Phi([0,T], \mathbb{R}^{2d})$. As is usual we write $u = \tilde{u} + \overline{u}$ where

$$\overline{u} = \frac{1}{T}\int_0^T u(t)dt.$$

From [8, Lem. 2.4] we have that

$$\int_0^T \Phi(\tilde{u})dt \leqslant \int_0^T \Phi(T\dot{u})dt.$$

Then by Young's inequality and using (**??**)

$$\begin{aligned}\int_0^T \Omega\left(\dot{u}, u\right) dt &= T\int_0^T \left\langle J\dot{u}, T^{-1}\tilde{u}\right\rangle dt \\ &\geqslant -T\left\{\int_0^T \Phi^\star(J\dot{u})dt + \int_0^T \Phi(T^{-1}\tilde{u})dt\right\} \\ &\geqslant -2T\left\{\int_0^T \Phi(\dot{u})dt\right\}\end{aligned}$$

□

Clearly the cosntant $2/T$ is far to be optimal. A possible way of improve C is consider other average $\overline{u}$. The mean value that it was used is the standard condered in the literature. But this value is apropriate for el Hilbert setting $\Phi(q) = |q|^2/2$. In this case, the value of $\overline{u}$ is the nearest (in the L^2-norm) constant vector to u. For a arbitrary G function, it seem more reasonable consider the nearest constant vector to u respect to the Φ-integral, i.e.

$$\int_0^T \Phi(u - \overline{u})dt \leqslant \int_0^T \Phi(u - u_0)dt, \quad \text{for every } u_0 \in \mathbb{R}^{2n}$$

Equivalently $\overline{u}$ is characterizate by

$$\int_0^T \nabla\Phi(u - \overline{u})dt = 0.$$

There is not a explicit formula as in the Hilbert setting.
PROBLEM 1. We can get a better constant taking this $\overline{u}$???

We call to the best constant in (44) C_Φ, i.e.

$$C_\Phi = -\inf\left\{\frac{\int_0^T \langle J\dot{u}, u\rangle dt}{\int_0^T \Phi(\dot{u})dt} \,\middle|\, u \in W^1L^\Phi\left([0,T], \mathbb{R}^{2d}\right)\right\} \tag{45}$$

Proposition 6.4. *The relation $C_\Phi = C_{\Phi^\star}$ holds for every symplectic Φ.*

Proof. Since Φ is symplectic if $u = Jv$

$$\frac{\int_0^T \langle J\dot{u}, u\rangle dt}{\int_0^T \Phi(\dot{u})dt} = \frac{\int_0^T \langle -\dot{v}, Jv\rangle dt}{\int_0^T \Phi(J\dot{v})dt} = \frac{\int_0^T \langle J\dot{v}, v\rangle dt}{\int_0^T \Phi^\star(\dot{v})dt}.$$

Using that $u \mapsto Ju$ is invertible from $W^1L^{\Phi^\star}([0,T],\mathbb{R}^{2d})$ into $W^1L^{\Phi}([0,T],\mathbb{R}^{2d})$ the statement follows taking infimum in previous equality.

□

For the following result we need the theory of indices of G-functions, see [4, 5] for a complete treatment in the case of N-functions defined on $\mathbb{R}$. The results are easily extended to the anisotropic setting. We denote by α_Φ and β_Φ the so called *Matuszewska-Orlicz indices* of the function Φ, which are defined next

$$\alpha_\Phi := \lim_{t\to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right)}{\log(t)}, \quad \beta_\Phi := \lim_{t\to +\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\Phi(tu)}{\Phi(u)}\right)}{\log(t)}. \tag{46}$$

We have that $1 \leqslant \alpha_\Phi \leqslant \beta_\Phi \leqslant +\infty$. The relation $\beta_\Phi < \infty$ holds true if and only if Φ is a Δ_2-function. The indices satisfy the following relation

$$\frac{1}{\alpha_\Phi} + \frac{1}{\beta_{\Phi^\star}} = 1. \tag{47}$$

Therefore if $\Phi^\star$ is a Δ_2-function (I mean Δ_2 as globally Δ_2) then $\alpha_\Phi > 1$.

We observe that if Φ is symplectic then $\Phi \in \Delta_2$ implies $\Phi^\star \in \Delta_2$. It is well known that if Φ and $\Phi^\star$ are Δ_2-function, then Φ is controlled by above and below by power functions. More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\Phi, \epsilon)$ and p_0, p_1 with $1 < \alpha_\Phi - \epsilon < p_1 \leqslant p_2 < \beta_\Phi + \epsilon < \infty$ such that, for every $t, u \geqslant 0$,

$$K^{-1} \min\left\{t^{p_2}, t^{p_1}\right\}\Phi(u) \leqslant \Phi(tu) \leqslant K \max\left\{t^{p_2}, t^{p_1}\right\}\Phi(u). \tag{48}$$

We recall the following result of [1].

Lemma 6.5. *Let Φ be a G-function. If $\Phi^\star \in \Delta_2$ globally, then for any $0 < \mu < \alpha_\Phi$,*

$$\lim_{\|\boldsymbol{u}\|_{L^\Phi}\to\infty} \frac{\int_0^T \Phi\left(\frac{\boldsymbol{u}}{\Lambda}\right) dt}{\|\boldsymbol{u}\|_{L^\Phi}^{\mu}} = +\infty. \tag{49}$$

Theorem 6.6. *Suppose that $u \in W^1L_T^\Phi([0,T],\mathbb{R}^{2d})$ attains the minimum in (45), then $\lambda = 2/C_\Phi$ is the first eigenvalue and u the corresponding eigenfunction of the following problem.*

$$\begin{cases} \frac{d}{dt}\nabla\Phi^\star(\dot{u}) + \lambda\nabla\Phi^\star(\lambda u) = 0 \\ u(0) = u(T),\ \int_0^T \nabla\Phi^\star(\lambda u)dt = 0 \end{cases} \tag{Eig}$$

Proof. □

Acknowledgments

The authors are partially supported by UNRC and UNLPam grants.

References

[1] S. Acinas, L. Buri, G. Giubergia, F. Mazzone, and E. Schwindt. Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698, 2015.

[2] F. Clarke. *Optimization and Nonsmooth Analysis.* Classics in Applied Mathematics. SIAM, Philadelphia, 1990.

[3] F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control.* Graduate Texts in Mathematics. 2013.

[4] Alberto Fiorenza and Miroslav Krbec. Indices of Orlicz spaces and some applications. *Commentationes Mathematicae Universitatis Carolinae*, 38(3):433–452, 1997.

[5] L. Maligranda. *Orlicz spaces and interpolation*, volume 5 of *Seminários de Matemática [Seminars in Mathematics].* Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[6] J. Mawhin and M. Willem. *Critical point theory and Hamiltonian systems.* Springer-Verlag, New York, 1989.

[7] J. Mawhin and M. Willem. *Critical Point Theory and Hamiltonian Systems.* Applied Mathematical Sciences. Springer, 2010.

[8] Fernando D Mazzone and Sonia Acinas. Periodic solutions of euler-lagrange equations in an anisotropic orlicz-sobolev space setting. *arXiv preprint arXiv:1708.06657*, 2017.

[9] Y. Tian and W. Ge. Periodic solutions of non-autonomous second-order systems with a p-Laplacian. *Nonlinear Anal.*, 66(1):192–203, 2007.