Periodic solutions for a generalized Sitnikov problem with primaries in rigid motion

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Abstract

1 Introduction

In this paper we study the following restricted Newtonian $n+1$-body problem P (see figure 1):

P_1 We have n primary bodies of masses $m_1, \dots, m_n$ and an additional massless particle.

P_2 The primary bodies are in a central configuration rigid motion (see [10, Section 2.9]). This motion is periodic and it is carried out in a plane Π.

P_3 The massless particle is moving on the perpendicular line to Π passing through the center of masses.

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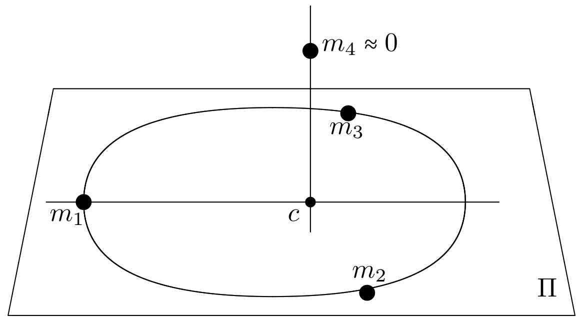

Figure 1: Four-body problem with three primaries

Problems like the one presented have been extensively discussed in the literature. In [24] K. Sitnikov considered the problem of two body in a Keplerian elliptic motion, for small enough eccentricity $e > 0$, and a massless particle moving in the perpendicular line to the orbital plane passing through the center of masses. Sitnikov obtained deep results about existence of solutions with a chaotic behavior (see [16, III(5)]). Periodic solutions for a Sitnikov configuration were considered in [21, 4, 5, 11].

Generalized circular Sitnikov problems ($n \geq 3$) were addressed more recently. In [25] Soulis, Papadakis and Bountis studied existence, linear stability and bifurcations for a problem similar to P. They considered a Lagrangian equilateral triangle configuration for the primaries bodies, which were supposed to have the same mass $m_1 = m_2 = m_3$. In [3] Papadakis and Bountis extend of results of [25] to n primaries ($n \geq 3$) in a poligonal equal mass configuration. Later in [19], Pandey and Ahmad extend the analysis started in [25] to the case when the primaries are oblate (not mass points). In [9] Li, Zhang and Zhao studied a special type of restricted circular $n+1$-body problem with equal masses for the primaries in a regular polygon configuration. In the previous papers the primary bodies are in the vertices of a regular polygon. In [13] Marchesin and Vidal studied the problem P for a rigid motion of primaries in a rhomboidal configuration. Periodic solutions for generalized Sitnikov problems with primaries performing no rigid motions were studied in [22, 21]. In [1] Bakker and Simmons studied scape regions for the massless particle in a problem similar to P for a certain type of periodic orbits for primaries.

In the present paper we extend the analysis in....FALTA

2 Preliminaries

We start considering n mass points, $n > 2$, of masses $m_1, \ldots, m_n$ moving in a Euclidean 3-dimensional space according to Newton's laws of motion. We assume that $x_1(t), \ldots, x_n(t)$ are the coordinates (column vectors) of the bodies in some inertial Cartesian coordinate system. We denote by $r_{ij} = |x_i - x_j|$

the Euclidean distance between x_i and x_j. We can suppose, without any loos generality, that the center of mass $c := \sum_j m_j x_j / M$ ($M := \sum_j m_j$) is fixed at the origin ($c = 0$).

Initially we assume that the bodies are in a *planar homographic motion* on the plane Π (see [10]). That means, assuming that Π is the plane determined by the first two coordinates axes, that

$$x_j(t) = r(t) Q(\theta(t)) q_j,$$

where

$$Q(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $q_j \in \Pi$, $j = 1, \ldots, n$ are vectors in a planar *central configuration* (CC) in $\mathbb{R}^3$, i.e. there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_j U(q_1, \ldots, q_n) + \lambda m_j q_j = 0, \quad j = 1, \ldots, n. \tag{1}$$

where the *potential function* U is defined by:

$$U(x) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \tag{2}$$

and ∇_j denotes the 3-dimensional partial gradient with respect to x_j. For simplicity we denote $q := (q_1, \ldots, q_n)$.

According to [10, Eq. (2.16)] the functions $r(t)$ and $\theta(t)$ solves the two-dimensional Kepler problems in polar coordinates, i.e.

$$\begin{aligned} \ddot{r}(t) - r(t)\dot{\theta}(t)^2 &= -\tfrac{\lambda}{r(t)^2} \\ r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) &= 0. \end{aligned} \tag{3}$$

In the particular case of *rigid motion*, we have $r(t) \equiv 1$ and $\theta(t) = \sqrt{\lambda} t + \theta(0)$. Therefore the primaries bodies perform a periodic motion with period $T := 2\pi/\nu$, where $\nu^2 = \lambda$.

Let $x_0(t)$ denotes the position of the massless particle. According to the Newtonian equations of motion q satisfies

$$\ddot{x}_0 = \sum_{i=1}^{n} \frac{m_i (x_i - x_0)}{|x_i - x_0|^3} =: f(t, x_0). \tag{4}$$

3 Admissible configurations

Henceforth we denote by L the coordinate z axis. It is well know that a necessary and sufficient condition for that L be invariant under the flow associated to the non autonomous system (4), is that $f(t, L) \subset L$ for all t, i.e. L is *f-invariant* for every t.

Definition 3.1. *We say that a central configuration q is* admissible *if and only if satisfies that for any $r > 0$, such that the set*

$$F_r := \{i : |q_i| = r\}$$

is non empty, then

$$\sum_{i \in F_r} m_i q_i = 0. \tag{5}$$

i.e. every maximal set of bodies which are equidistant from origin has center of mass equal to 0.

Theorem 3.2. *L is f-invariant for every t if and only q is admissible.*

For the proof of the previous theorem we need the following result

Lemma 3.3. *For $c > 0$ we define the function $y_c(t) := (c+t)^{-3/2}$. If $0 < t_1 < t_2 < \ldots < t_k$ then the functions $y_j(t) := y_{t_j}(t)$ are linearly independent on each open interval $\mathcal{I} \subset \mathbb{R}^+$.*

Proof. It is sufficient to prove that Wronskian

$$W := W(y_1, \ldots, y_k)(t) = \det \begin{pmatrix} y_1 & \cdots & y_k \\ \frac{dy_1}{dt} & \cdots & \frac{dy_k}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1}{dt^{k-1}} & \cdots & \frac{d^{k-1}y_k}{dt^{k-1}} \end{pmatrix}$$

is not null on $\mathcal{I}$.

Using induction is easy to show that

$$\frac{d^i y_c}{dt^i} = \beta_i y_c^{\frac{2i+3}{3}}, \quad \text{for some } \beta_i \neq 0, \text{ and for all } i = 1, \ldots. \tag{6}$$

Fix any $t \in I$. Then, according to (6) and writing $\lambda_j := (t + t_j)^{-1}$, we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \lambda_1^{3/2} & \lambda_2^{3/2} & \cdots & \lambda_k^{3/2} \\ \beta_1 \lambda_1^{5/2} & \beta_1 \lambda_2^{5/2} & \cdots & \beta_1 \lambda_k^{5/2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1} \lambda_1^{k+1/2} & \beta_{k-1} \lambda_2^{k+1/2} & \cdots & \beta_{k-1} \lambda_k^{k+1/2} \end{pmatrix} \\ &= \beta_1 \beta_2 \cdots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \cdots \lambda_k^{3/2} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \\ &= \beta_1 \beta_2 \cdots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \cdots \lambda_k^{3/2} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i), \end{aligned}$$

where the last equality follows of the well known Vandermonde determinant identity. Therefore $W \neq 0$ if and only if $\lambda_i \neq \lambda_j$, $i \neq j$, which in turn is equivalent to $t_i \neq t_j$, $i \neq j$. $\square$

Proof of Theorem 3.2. The condition $f(t, L) \subset L$ for all t is equivalent to

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{\left(r(t)^2 |q_i|^2 + z^2\right)^{3/2}} = 0, \tag{7}$$

for every $t, z \in \mathbb{R}$.

Let $D = \{|q_i| : i = 1, \ldots, n\}$. Suppose that $D = \{s_1, \ldots, s_k\}$, with $s_i \neq s_j$ for $i \neq j$, and $\{1, \ldots, n\} = F_1 \cup \cdots \cup F_k$, where if $i \in F_j$ then $|q_i| = s_i$. Then, multiplying equation (7) by $r(t)^2 Q^{-1}(\theta(t))$ and writing $\zeta = (z/r(t))^2$ we have that (7) is equivalent to

$$\sum_{j=1}^{k} \left\{ \frac{1}{(s_j^2 + \zeta)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0.$$

According to Lemma 3.3, the last equation is equivalent to (5). □

4 Admissible collisionless configurations for $n \leq 4$

In this section we find all admissible collisionless configuration with $n \leq 4$. We observe that the collisionless condition for a configuration $q_1 \ldots, q_n$ implies that $q_i \neq 0$ for $i = 1, \ldots, n$. From the fact that the center of masses is an excluded position we obtain

$$F_r \neq \varnothing \Rightarrow \# F_r \geq 2. \tag{8}$$

Trivially, a configuration of two point masses m_1 and m_2 is admissible if and only if $m_1 = m_2$.

If $n = 3$, from (8), the configuration consists of three equidistant bodies from the origin. Therefore, it must to be the Lagrangian equilateral triangle configuration. Now, by equation (5) and an elementary geometrical reazoning we have that $m_1 = m_2 = m_3$.

The case $n = 4$ is more interesting. We will use the following result which can be seen in [15].

"Let q be a planar configuration. For each pair, i, j, the line containing q_i and q_j together with its perpendicular bisector form axes which divide the plane into four quadrants. The union of the first and third quadrants is an hourglass shaped region which will be called a 'cone'; similarly, the second and fourth quadrants together form another cone. The phrase 'open cone' refers to a cone minus the axes."

Theorem 4.1 (Perpendicular Bisector Theorem). *Let q be a planar central configuration and let q_i and q_j be any two of its points. Then if one of the two open cones determined by the line through q_i and q_j and its perpendicular bisector contains points of the configuration, so does the other one.*

Next we characterize all the 4-body admisible collisionless configurations.

Theorem 4.2. *Let q be a 4-body central configuration. Then q is admisible and collisionless if and only if, for a suitable enumeration of bodies, $q_1 = -q_3$, $q_2 = -q_4$, $m_1 = m_3$, $m_2 = m_4$, and q is of some of the following mutually exclusive types:*

CCcl. *collinear,*

CCr. *a rhombus with $r_{13} < r_{24}$ and $m_1 > m_2$,*

CCs. *a square with four equal masses.*

Proof. Since (8) we have two cases.

Case 1. $m_1 \geq m_2$, $|q_1| \neq |q_2|$, $|q_1| = |q_3|$ and $|q_2| = |q_4|$. Now (5) implies that $m_1 = m_3$, $m_2 = m_4$, $q_1 = -q_3$ and $q_2 = -q_4$. We divide the plane in two cones C_i, $i = 1, 2$, by means of the line P joining q_1 and q_3 together with its perpendicular bisector M. From Theorem 4.1, if q_2 is in C_1, then q_4 is in C_2, and vice versa. This is a contradiction with the fact that $q_2 = -q_4$. Then $q_2, q_4 \in P$ or $q_2, q_4 \in M$, i.e. q is collinear or is a rhombus with equal masses in opposite vertices. In the first case, q is of CCcl type. In the second case, if $m_1 > m_2$, was proved in [12, Eqs. (3.44) and (3.45)] that $r_{13} < r_{24}$, hence q is of CCr type. From [20, Corollary 2] if $m_1 = m_2$ then the configuration is a square witch is a contradiction with the fact that $|q_1| \neq |q_2|$.

Case 2. $|q_1| = |q_2| = |q_3| = |q_4|$. In this situation, was proved in [6] that the configuration is the equal mass square. □

5 Massless particle motion

In this section we analyze all possible motions for the massless particle x_0. In particular we will see that all motion is periodic or is a scape trayectory. We will find that there exists T_0-periodic solutions for all T_0 in an interval $(\sigma(q), +\infty)$. This fact implies that there exists an infinity quantity of periodic solutions for the entire $n+1$-body system.

From now on we suppose that the primaries bodies are in a T-periodic rigid motion asociate to an admisible collisionless q, i.e $r(t) \equiv 1$ and from the remark follows (3), $\theta(t) = \sqrt{\lambda}t$. Without loss of generality, we have assumed that $\theta(0) = 0$. For the particle, we assume that it is moving on L. Therefore, we assume that $x_0(t) = (0, 0, z(t))$. According to Theorem 3.2, x_0 is solution of (4), if and only if $z(t)$ is solution of the autonomous equation

$$\ddot{z} = -\sum_{i=1}^{n} \frac{m_i z}{(s_i^2 + z^2)^{3/2}}, \tag{9}$$

where $s_i = |q_i|$.

The second order equation (9) is consevative, therefore solutions conserve the energy

$$E(z, v) := \frac{|v|^2}{2} - \sum_{i=1}^{n} \frac{m_i}{\left(s_i^2 + z^2\right)^{\frac{1}{2}}}, \tag{10}$$

i.e. $E(z(t), \dot{z}(t))$ is constant.

If the equation (9) has a T-periodic solution, we say that the solution is synchronous. Like as [9, 27], we are interesting in to find synchronous solutions.

Following [13] we introduce the next concepts.

Definition 5.1. *A solution $z(t)$ of* (9) *such that $\lim_{t\to\infty} z(t) = \infty$ is called hyperbolic for $t \to \infty$ when $\lim_{t\to\infty} \dot{z}(t) = z_\infty \neq 0$ and is called parabolic if $\lim_{t\to\infty} \dot{z}(t) = 0$.*

The following theorem characterize all the possible motions for the massless particle.

Theorem 5.2. *We assume that q is a admisible collisionless configuration. Every solution of* (9) *if of the some following types:*

1. *Hyperbolic, when $E > 0$,*
2. *Parabolic, when $E = 0$,*
3. *Periodic, when $E_{min} := -\sum_{i=1}^n \frac{m_i}{s_i} < E < 0$.*
4. *Equilibrium solution when $E = E_{min}$.*

Proof. We follow a standard argument for hamiltonian systems (see [23]).

We plot the level sets of $S(E) = \{(z,v) : E(z,v) = E\}$, in the phase space (z,v), for different values of E. An elementary analysis shows that

- If $E \geq 0$ then $S(E)$ is the union of two bounded graphs. They are symmetric with respect to z-axis, each of which is contained in some semiplane $v > 0$ or $v < 0$. The v-positive branch is the graph of a function $v(E,z)$, which is decreasing with respect to $|z|$. Moreover, $\lim_{|z|\to\infty} v(E,z) = \sqrt{2E}$.
- For every $E \geq E_{min}$, the energy curve cut the v-axis at $\pm(2E + 2\sum_{i=1}^n m_i s_i^{-1})^{\frac{1}{2}}$.
- If $E_{min} < E < 0$ then $S(E)$ is a simple closed curve symmetric with respecto to z and v axes.
- An energy curve cut the z-axis, only in the case that $E < 0$, at $\pm z_E$, where z_E is the only positive solution of $-\sum_{i=1}^n m_i(s_i^2 + z_E^2)^{-\frac{1}{2}} = E$.

In the figure 1 we show the phase portrait for two equal masses primaries.

The function $\varphi(t) = (z(t), \dot{z}(t))$ solves the system $\dot{\varphi}(t) = F(\varphi(t))$, where $F(z,v) = (v, -\sum_{i=1}^n m_i z(s_i^2 + z^2)^{-3/2})$. The only stationary point of F is $(z,v) = (0,0)$. Therefore, the level surfaces $S(E)$, with $E \neq E_{min}$, not contains stationary points. A well know argument implies that the trayectories $t \mapsto (z(t), \dot{z}(t))$ fill completely the energy curves.

We note that if $E \geq 0$ and $v(E,0) > 0$ ($v(E,0) < 0$) then $z(t)$ is increasing (decreasing) with respect to t. For all the above if $E \geq 0$ then $|z(t)| \to \infty$ when $t \to \infty$. Moreover $\lim_{t\to\infty} \dot{z}(t) = \pm\sqrt{2E}$. From this we conclude that the trayectory is hyperbolic when $E > 0$ and it is parabolic in the case $E = 0$.

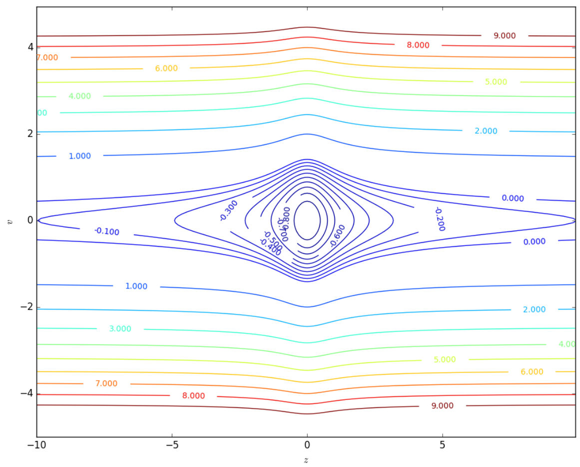

Figure 2: Energy level for two equal mass primaries

In the case that $E_{min} < E < 0$ we have that the trayectory is contained in a closed curve, therefore it is periodic orbit.

Finally if $E = E_{min}$ clearly we have that $z(t) \equiv 0$. □

Theorem 5.3. *We denote by $T_0(E)$ the minimal period for a solution of* (9) *with $E_{min} < E < 0$. Then*

1. *for z_E the only positive solution of $-\sum_{i=1}^n m_i(s_i^2 + z_E^2)^{-\frac{1}{2}} = E$*

$$T_0(E) = 2^{3/2} \int_0^{z_E} \left(E + \sum_{i=1}^n m_i(s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz,$$

2. *$T_0(E)$ is an increasing function.*

3. *$T_0\left((E_{min}, 0)\right) = (T_{min}, +\infty)$, where $T_{min} = 2\pi \left(\sum_{i=1}^n \frac{m_i}{s_i^3} \right)^{-1/2}$.*

Proof. Let $E_{min} < E < 0$ and let $z(t)$ be a solution with $z(0) = 0$ and $\dot{z}(0) = \sqrt{2(E - E_{min})}$. Then $E(z(0), \dot{z}(0)) = E$. Therefore $z(t)$ is $T_0(E)$-periodic. As a consequence of the symmetries of the equation we have that $z(T_0(E)/4) = z_E$. Then, taking account (10) we have that

$$\begin{aligned} \frac{T_0}{4} &= \int_0^{T_0/4} \left(E + \sum_{i=1}^n m_i(s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} z'(t)dt \\ &= \frac{1}{\sqrt{2}} \int_0^{z_E} \left(E + \sum_{i=1}^n m_i(s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} dz, \end{aligned}$$

and we have proved item *1*. In order to prove item *2* we note that

$$\begin{aligned}
2^{-3/2}T_0(E) &= \int_0^{z_E} \left(\sum_{i=1}^n m_i \left((s_i^2 + z^2)^{-\frac{1}{2}} - (s_i^2 + z_E^2)^{-\frac{1}{2}} \right) \right)^{-\frac{1}{2}} dz \\
&= \int_0^{z_E} \left(z_E^2 - z^2 \right)^{-\frac{1}{2}} f(z, z_E) dz \\
&= \int_0^1 \left(1 - u^2 \right)^{-\frac{1}{2}} f(z_E u, z_E) du,
\end{aligned}$$

where

$$f(z, z_E) = \left(\sum_{i=1}^n m_i \left\{ (s_i^2 + z^2)(s_i^2 + z_E^2) \right\}^{-\frac{1}{2}} \left\{ (s_i^2 + z^2)^{\frac{1}{2}} + (s_i^2 + z_E^2)^{\frac{1}{2}} \right\}^{-1} \right)^{-\frac{1}{2}}.$$

We note that $f(z_E u, z_E)$ is a increasing function with respect to z_E for $u \in [0, 1]$ fix. This implies item *2*.

On the other hand

$$\lim_{z_E \to 0} f(z_E u, z_E) = \left(\sum_{i=1}^n \frac{m_i}{2 s_i^3} \right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{z_E \to \infty} f(z_E u, z_E) = \infty.$$

Therefore, from the dominated convergence theorem and monotone convergence theorem we have that

$$\lim_{z_E \to 0} T_0 = 2\pi \left(\sum_{i=1}^n \frac{m_i}{s_i^3} \right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{z_E \to \infty} T_0 = \infty.$$

Finally, since $T_0 = T_0(z_E)$ is continuous and increasing respect to z_E we conclude the afirmation in the item *3*.

PONER ALTERNATIVAS!!!! □

Corollary 5.4. *There is a T-synchronous solution if and only if*

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} < \left(\sum_{i=1}^n \frac{m_i}{s_i^3} \right) \left(\sum_{i=1}^n m_i s_i^2 \right). \tag{11}$$

Proof. The result is consequence of Theorem 5.2 that $T^2 = 4\pi^2 \sum_{i=1}^n m_i |q_i|^2 / U$ (see [10, p. 109]). □

6 Main results

We observe that for all admissible CC there exists kT synchronous solution (and therefore a periodic solution of the complete $n+1$-problem) when $k \in \mathbb{N}$ is large enough.

The sufficiency of the condition $n \leq 472$ in the following corollary was proved in [9].

Corollary 6.1. *We suppose that* $q_1, \ldots, q_n$ *is the equal masses regular polygon configuration (this is an admisible CC). Then there exists a synchronous solution if and only if* $2 \le n \le 472$.

Our next objective is to verify that condition (11) is satisfied for all admissible CC of 3-body or 4-body. In virtue of Corollary 6.1 and Theorem **??** rest to prove that condition (11) holds true for the symmetric collinear 4-body CC, and for a non square CC forming a rhombus with equal masses in opposite vertices. Let's call these central configurations CCcl and CCr respectively.

Theorem 6.2. *The central configurations CCcl and CCr satisfy condition* (11).

Corollary 6.3. *For all admissible CC of 3-body or 4-body problem* P *has a* T*-synchronous solution.*

7 Proofs

Let us to show a second proof of item 2 of Theorem 5.2. For the sufficiency we follow arguments of [27] and [9], based on variational principles. For the necessity of condition (**??**) we use Sturm's comparison theorem.

Alternitave Proof Theorem 5.2 (2). First we prove that (**??**) is a necessary condition for the existence of a T_0-periodic solution. We assume that z is a T_0-periodic solution of (9). Using Sturm's Comparison Theorem (see [2]) with equations $z''+q_1(z)z = 0$, where $q_1(z) = \sum_{i=1}^n m_i \left(s_i^2 + z^2\right)^{-3/2}$, and $z''+\left(\sum_{i=1}^n m_i s_i^{-3}\right) z = 0$ we deduce (**??**).

Let $T_0 > 0$ satisfiying (**??**). We consider the action integral

$$\mathcal{I}(z) = \int_0^{T_0} \frac{1}{2}|z'|^2 + \sum_{i=1}^n \frac{m_i}{\sqrt{s_i^2 + z^2}} dt,$$

Then T_0-periodic solutions of (9) are critical points of $\mathcal{I}$ in the space $H^1(\mathbb{T}, \mathbb{R})$, where $\mathbb{T} = \mathbb{R}/T_0\mathbb{Z}$, of the functions absolutely continuous, T_0-periodic with $z' \in L^2(\mathbb{T}, \mathbb{R})$ (see [14, Cor. 1.1]). We prove the existence of critical points by means of the direct method of calculus of variations, i.e. we will prove that $\mathcal{I}$ has minimum. The functional $\mathcal{I}$ is not coercive in $H^1(\mathbb{T}, \mathbb{R})$, this deficiency is drawn with symmetry techniques (see [26]). The group $\mathbb{Z}_2$ acts on $H^1(\mathbb{T}, \mathbb{R})$ according to the following assignments $(\bar{0} \cdot z)(t) = z(t)$ and $(\bar{1} \cdot z)(t) = -z(t + \frac{T_0}{2})$. The functional $\mathcal{I}$ is $\mathbb{Z}_2$-invariant, i.e. $\mathcal{I}(g \cdot z) = \mathcal{I}(z)$. We define the space of all $\mathbb{Z}_2$-symmetric (this simmetry is called the italian simmetry) funcions

$$\Lambda(\mathbb{T}, \mathbb{R}) := \left\{ z \in H^1(\mathbb{T}, \mathbb{R}) | \forall g \in \mathbb{Z}_2 : z = g \cdot z \right\}.$$

The funciontal $\mathcal{I}$ restricted to Λ is coercive. This fact follows from an obvious adaptation of proposition 4.1 of [26]. We note that $F(z) := \sum_{i=1}^n m_i (s_i^2 + z^2)^{-\frac{1}{2}}$ satisfies the condition (A) in [14, p. 12], then $\mathcal{I}$ is continuously differentiable and weakly lower semicontinuous on $H^1(\mathbb{T}, \mathbb{R})$ (see [14, p. 13]). Therefore $\mathcal{I}$ has a minimum z_0 in $\Lambda(\mathbb{T},\mathbb{R})$. Then by the Palais' principle symmetric criticality, z_0 is a critical point of $\mathcal{I}$ in $H^1(\mathbb{T},\mathbb{R})$ (see [26] and [18]).

We use the second variation $\delta^2\mathcal{I}$ in order to show that $z_0 \not\equiv 0$. It is well known (see [7, Th. 1.3.1]) that if z_0 is a minimum of $\mathcal{I}$ on $H^1(\mathbb{T},\mathbb{R})$ then $\delta^2\mathcal{I}(z_0,\varphi) \geq 0$ for all $\varphi \in H^1(\mathbb{T},\mathbb{R})$. In our case

$$\delta^2\mathcal{I}(0,\varphi) = \int_0^{T_0} |\varphi'|^2 - \sum_{i=1}^{n} \frac{m_i}{r_i^3}\varphi^2 dt,$$

(see [7, Eq. 1.3.6]). In particular for $\varphi(t) = \sin(2\pi t/T_0)$ it follows from (**??**) that

$$\delta^2\mathcal{I}(0,\varphi) = \left(\frac{4\pi^2}{T_0^2} - \sum_{i=1}^{n} \frac{m_i}{r_i^3}\right)\frac{T_0}{2} < 0. \tag{12}$$

It is sufficient to guarantee that $z_0 \equiv 0$ is not a minimum. □

Remark 1. We note that this second proof of Theorem 5.2 (2), unlike the first one, does not guarantee that T_0 is the minimum period for z_0. It could happen that z_0 had period T_0/m, with natural $m \in \mathbb{N}$. Because of Italian symmetry this m should be odd.

Proof Corollary 6.1. In this case $s_1 = s_2 = \cdots = s_n =: r$ and $m_1 = m_2 = \cdots = m_n =: m$. Then, from the law of cosines we obtain

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} = \frac{nm^2}{4r} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)}.$$

Therefore the condition (11) is equivalent to

$$\frac{1}{n}\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)} < 4. \tag{13}$$

This inequality was also derived by Li, J. et al. in [9]. In [9] the authors noted (performing calculations with computer) that inequality (13) holds true for $2 \leq n \leq 472$. Let us prove that any other n does not satisfies (13).

Using that $1/\sin(x)$ is a convex function on $[0,\pi]$ and composite trapezoid rule (see [8]) we have that

$$\begin{aligned}\int_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} \frac{1}{\sin(x)}dx &\leq \frac{\pi}{2n}\left\{\frac{1}{\sin(\frac{\pi}{n})} + \frac{1}{\sin(\frac{n-1}{n}\pi)} + 2\sum_{j=2}^{n-2}\frac{1}{\sin(j\frac{\pi}{n})}\right\}\\ &= \frac{\pi}{n}\sum_{j=1}^{n-2}\frac{1}{\sin(j\frac{\pi}{n})}.\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{j\pi}{n}\right)} &\geq \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\frac{\pi(n-1)}{n}}\frac{1}{\sin(x)}dx+\frac{1}{n\sin\left(\frac{n-1}{n}\pi\right)}\\
&= \frac{1}{2\pi}\log\left(\frac{1-\cos(x)}{1+\cos(x)}\right)\Bigg|_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi}+\frac{1}{n\sin\left(\frac{\pi}{n}\right)}\\
&= \frac{1}{\pi}\left\{\log\left(\frac{1+\cos(\frac{\pi}{n})}{1-\cos(\frac{\pi}{n})}\right)+\frac{\pi/n}{\sin\left(\frac{\pi}{n}\right)}\right\}\\
&=: f\left(\frac{\pi}{n}\right).
\end{aligned}$$

It is easy to see that $f(x)$ is a decreasing function on $(0,\pi/2)$. Moreover $f(\pi/842)\approx 4.0006>4$. Therefore, if $n\geq 842$ then n does not satisfies inequality (13). The validity of the inequality (13), for $n\leq 841$, is easily checked using computer. This gives the result that the inequality holds for all $n\leq 472$. □

Proof Theorem 6.2. Let's start by analyzing the central configuration CCr. We can suppose without loss of generality that $q_1=-q_3=(0,y)$ for $y>0$, $q_2=-q_4=(1,0)$. Recall that $y<1$. The condition (11) becomes

$$\frac{m_1^2}{2y}+\frac{4m_1m_2}{\sqrt{1+y^2}}+\frac{m_2^2}{2}<\left(\frac{2m_1}{y^3}+2m_2\right)\left(2m_1y^2+2m_2\right).$$

As $m_1^2/(2y)<4m_1^2/y$, $m_2^2/2<4m_2^2$ and $4m_1m_2/\sqrt{1+y^2}<4m_1m_2/y^3$ (since $y<1$), we have that the inequality holds.

Now we consider the central configuration CCl. We can suppose without loss of generality that $q_1=-q_3=1$, $q_2=-q_4=x$ with $0<x<1$, and $m_1=m_3=\mu$, $m_2=m_4=1-\mu$, with $0<\mu<1$. Then the inequality (11) becomes

$$\frac{2\mu(1-\mu)}{1-x}+\frac{2\mu(1-\mu)}{1+x}+\frac{\mu^2}{2}+\frac{(1-\mu)^2}{2x}<4\mu^2+4\mu(1-\mu)x^2+\frac{4\mu(1-\mu)}{x^3}+\frac{4(1-\mu)^2}{x}.$$

As $\mu^2/2<4\mu^2$ and $(1-\mu)^2/(2x)<4(1-\mu)^2/x$ (without taking into account the term $4\mu(1-\mu)x^2$) we just have to show that

$$\frac{2\mu(1-\mu)}{1-x}+\frac{2\mu(1-\mu)}{1+x}<\frac{4\mu(1-\mu)}{x^3},$$

and this is equivalent to see that

$$\frac{x^3}{1-x^2}<1. \tag{14}$$

The values of x involved in the above inequality are such that the configuration for the vector mass $(\mu,1-\mu,1-\mu,\mu)$ is central, by Moulton [17], fixed a mass μ there is only one value x satisfiying this condition. So, we can define $x(\mu)$ as such value of x. We note that $x^3/(1-x^2)$ is an increasing function with respect to $x \in (0,1)$ and it is less than 1 for $x \in (0,3/4)$. Hence, if we could prove that $x(\mu)$ is a decreasing function and

$$\lim_{\mu\to 0} x(\mu) < 3/4 \tag{15}$$

we would have justified (14).

Let's first prove that $x(\mu)$ is a decreasing function. Replacing q_j^0, m_j and eliminating λ from the equations (1) we get

$$\frac{\mu}{4} - \frac{\mu}{x\,(x+1)^2} + \frac{\mu}{x\,(-x+1)^2} + \frac{-\mu+1}{(x+1)^2} + \frac{-\mu+1}{(-x+1)^2} - \frac{1}{x^3}\left(-\frac{\mu}{4}+\frac{1}{4}\right) = 0$$

which is equivalent to

$$\mu = -\frac{8x^5 - x^4 + 8x^3 + 2x^2 - 1}{(x-1)\,(x+1)\,(x^5 - 9x^3 + x^2 - 1)}.$$

Therefore

$$\frac{d\mu}{dx} = \frac{x^2\left(16x^9 - 3x^8 + 32x^7 + 12x^6 - 304x^5 - 2x^4 + 44x^2 - 51\right)}{(x-1)^2\,(x+1)^2\,(x^5 - 9x^3 + x^2 - 1)^2}.$$

Clearly $d\mu/dx < 0$ for $x \in (0,1)$. Which, in turn, implies that x is decreasing respect to μ.

Let's see now that (15) holds. When μ goes to 0, $x(\mu)$ converges to the only solution, $x(0)$, in the interval $(0,1)$ of the equation $8x(0)^5 - x(0)^4 + 8x(0)^3 + 2x(0)^2 - 1 = 0$. Then $8x(0)^3 - 1 < 0$ which implies that $x(0) < 3/4$ as we wanted to prove. □

8 An example

In this section we present an example that shows the condition of non-stacionarity in Theorem (3.2) is needed.

Proof Proposition **??**. Some of the following calculations were made with symbolic math software. Let's consider a particular Euler's linear central configuration formed by three collinear bodies of mass $m_1 = 4-\mu$, $m_2 = 2+\mu$, $m_3 = 1$, where $0<\mu<1$, and $q_1 = 0$, $q_2 = 1$ and $q_3 = 1+r$ their respective positions, with r such that

$$p(r,\mu) = 0,$$

where $p(r,\mu) = 6r^5 + (-\mu+16)\,r^4 + (-2\mu+14)\,r^3 + (-\mu-5)\,r^2 + (-2\mu-7)\,r - \mu - 3$. Note that as $p(0,\mu) = -\mu-3$ and $P(1,\mu) = -7\mu+21$ then, for all $0<\mu<1$, $r \in (0,1)$. In this case the center of mass C is equal to $\frac{\mu}{7} + \frac{r}{7} + \frac{3}{7}$, so $C \in (0,1)$.

We denote with x the point between 0 and 1 where the sum of the forces that the primary bodies make on a massless particle located in that position is equal to zero. For this value of x we have to $f(x) = 0$, where $f(x) = -\frac{4-\mu}{x^2} + \frac{\mu+2}{(-x+1)^2} + \frac{1}{(r-x+1)^2}$. Note that the left side of the previous equation is an increasing function that tends to $-\infty$ when x goes to 0, and tends to $+\infty$ when x goes to 1, so there is a unique point $x \in (0,1)$, such that the equality holds.

Then, if we want to have a trivial solution of the problem (4) then necessarily C has to be equal to x. Let's see that there exists $\mu \in (0,1)$ such that $C = x$, i.e. $f(C) = 0$. For this purpose, since C is a continuous function with respect to μ, we need to see that there exists a value $\mu_1 \in (0,1)$ such that $f(C) < 0$ and $\mu_2 \in (0,1)$ such that $f(C) > 0$. The function $f(x)$ can be factorized as

$$f(x) = \frac{Nf(x)}{Df(x)},$$

where $Nf(x) = 2\mu r^2x^2 - 2\mu r^2x + \mu r^2 - 4\mu r x^3 + 8\mu r x^2 - 6\mu r x + 2\mu r + 2\mu x^4 - 6\mu x^3 + 7\mu x^2 - 4\mu x + \mu - 2r^2x^2 + 8r^2x - 4r^2 + 4rx^3 - 20rx^2 + 24rx - 8r - x^4 + 10x^3 - 21x^2 + 16x - 4$ and $Df(x) = x^2(x-1)^2(r-x+1)^2$. Note that $Df(x) > 0$ for all $x \in (0,1)$. If we consider $\mu = 0$ and compute $Nf(C)$ we have that

$$Nf(C) = \frac{r^4}{2401} + \frac{1514r^3}{2401} + \frac{2245r^2}{2401} + \frac{1110r}{2401} + \frac{333}{2401} > 0,$$

on the other, hand if $\mu = 1$ then

$$Nf(C) = -\frac{71r^4}{2401} + \frac{1486r^3}{2401} + \frac{401r^2}{2401} - \frac{1480r}{2401} - \frac{592}{2401} < 0,$$

because $0 < r < 1$. □

9 Recortes

Proof Theorem 3.2. We use a rotating coordinate system where the primaries are fixed. Concretely we put

$$\xi = Q(-\nu t)q.$$

In this system the motion equations are

$$\ddot{\xi} + 2\nu B\dot{\xi} + \nu^2 C\xi = \sum_{i=1}^{n} \frac{m_i(q_i - \xi)}{|q_i - \xi|^3}, \tag{16}$$

where

$$B := \begin{pmatrix} J & 0_{2\times1} \\ 0_{1\times2} & 0 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -I_2 & 0_{2\times1} \\ 0_{1\times2} & 0 \end{pmatrix},$$

where $0_{n\times m}$ and $\mathcal{I}_n$ denote the null $n \times m$ matrix and the identity $n \times n$ matrix respectively. Assuming that the massless particle is moving on the z-axis then

$\xi = q = (0, 0, z)$ and the Coriolis and centrifugal forces, $2\nu B\dot{\xi}$ and $\nu^2 C\xi$ respectively, are null. Therefore, taking account in the first two equation in (16) and identifying the vectors q_i, $i = 1, \dots, n$ with vectors in $\mathbb{R}^2$, we have

$$\sum_{i=1}^{n} \frac{m_i q_i}{|q_i - \xi|^3} = 0.$$

Let $D = \{|q_i| : i = 1, \dots, n\}$. Suppose that $D = \{r_1, \dots, r_k\}$, with $r_i \neq r_j$ for $i \neq j$, and $\{1, \dots, n\} = F_1 \cup \dots \cup F_k$, where if $i \in F_j$ then $|q_i| = r_j$. Then

$$\sum_{j=1}^{k} \left\{ \frac{1}{(r_j^2 + z^2)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0.$$

Since we are considering a non-stationary solution, we have that $z(t)$ is not constant. Therefore there exists an interval $\mathcal{I} \subset \mathbb{R}^+$ where

$$\sum_{j=1}^{k} \left\{ \frac{1}{(r_j^2 + s)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0, \quad s \in I.$$

Then, according to Lemma 3.3, we obtain (5).

If (5) is satisfied then the force field F acting on the massless particle carries the z axis in itself. Therefore, from the existence and uniqueness theorem and other elementary properties of system of ODEs we obtain a solution of (4) with $x(t) = y(t) = 0$. $\square$

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