Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{$\boldsymbol{P}$}$$

where the function $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $d \geqslant 1$ (called the *Lagrange function* or *lagrangian*) satisfying that it is measurable in t for each $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in (x, y) for almost every $t \in [0, T]$. The unknown function $u : [0, T] \to \mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the articles [**?**] for definitions and main results on anisotropic Orlicz spaces, see also [**?**]. These spaces allow us to unify and extend previous results on existences of solutions for systems like ($\boldsymbol{P}$).

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Through this article we say that a function $\Phi:\mathbb{R}^d\to[0,+\infty)$ is of N_∞ class if Φ is convex, $\Phi(0)=0$, $\Phi(y)>0$ if $y\neq 0$ and $\Phi(-y)=\Phi(y)$, and

$$\lim_{|y|\to\infty}\frac{\Phi(y)}{|y|}=+\infty. \tag{1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From [?, Cor. 2.35] a N_∞ function is continuous.

Associated to Φ we have the *complementary function* Ψ which is defined in $\xi\in\mathbb{R}^d$ as

$$\Psi(\xi)=\sup_{y\in\mathbb{R}^d} y\cdot\xi-\Phi(y) \tag{2}$$

then, from the continuity of Φ and (1), we have that $\Psi:\mathbb{R}^d\to[0,\infty)$. Moreover, it is easy to see that Ψ is a convex function such that $\Psi(0)=0$, $\Psi(-\xi)=\Psi(\xi)$ [?, Chapter 2]. Moreover Ψ satisfies (1) (see [?, Th. 2.2]). i.e. Ψ is N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\Phi_p(y):=|y|^p/p$, for $1<p<\infty$. In this case $\Psi(\xi)=|\xi|^q/q$, $q=p/(p-1)$.

Example 1.2. If $\Phi:\mathbb{R}\to[0,+\infty)$ is a N_∞ function on $\mathbb{R}$ then $\overline{\Phi}(y)=\Phi(|y|)$ is a N_∞ function on $\mathbb{R}^d$. In this example, as in the previous one, the function Φ is *radial*, i.e. the value of $\Phi(y)$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\Phi(y)$ depends on the direction of y. For example, if $1<p_1,p_2<\infty$, we define $\Phi_{p_1,p_2}:\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty)$ by

$$\Phi_{p_1,p_2}(y_1,y_2):=\frac{|y_1|^{p_1}}{p_1}+\frac{|y_2|^{p_2}}{p_2}.$$

Then Φ_{p_1,p_2} is a N_∞ function. In this case the complementary function is the function Φ_{q_1,q_2} with $q_i=p_i/(p_i-1)$.

More generally, if $\Phi_k:\mathbb{R}^d\to[0,+\infty)$, $k=1,\dots,n$, are N_∞ functions, then $\Phi:\mathbb{R}^d\times\cdots\times\mathbb{R}^d\to[0,+\infty)$ defined by $\Phi(y_1,\dots,y_n)=\Phi_1(y_1)+\cdots+\Phi_n(y_n)$ is a N_∞ function. These functions are truly anisotropic, i.e. $|x|=|y|$ does not imply that $\Phi(x)=\Phi(y)$.

Example 1.4. If $\Phi:\mathbb{R}\to[0,+\infty)$ is a N_∞ function and $O\in GL(d,\mathbb{R})$, then $\Phi(y)=\Phi(Oy)$ is a N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by powers if it does not satisfy the Δ_2 condition (see xxxxx). For example $\Phi:\mathbb{R}^d:\to[0,+\infty)$ defined by $\Phi(y)=\exp(|y|)-1$ is N_∞ function.

The occurrence of Orlicz Spaces in this paper obeys to we will consider the following structure condition on the lagrangian:

$$|\mathcal{L}|+|\nabla_x\mathcal{L}|+\Psi(\nabla_y\mathcal{L})\leqslant a(x)\left\{b(t)+\Phi\left(\frac{y}{\lambda}\right)\right\}, \tag{$\boldsymbol{S}$}$$

for a.e. $t\in[0,T]$, where $a\in C\left(\mathbb{R}^d,[0,+\infty)\right)$, $b\in L^1\left([0,T],[0,+\infty)\right)$.

Our condition ($\boldsymbol{S}$) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\Phi(x)$ is as in Example

1.1, then the condition ($\boldsymbol{S}$) is equivalent to the structure condition in [**?**, Th. 1.4]. If Φ is a radial N_∞ function such that Ψ satisfies that Δ_2 function then ($\boldsymbol{S}$) is essentially equivalent????? to conditions [**?**, Eq. (2)-(4)] (see xxxx mas abajo). If Φ is as in Example 1.3 and $\mathcal{L} = \mathcal{L}(t, x_1, x_2, y_1, y_2)$ is a lagrangian with $\mathcal{L} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ then inequality ($\boldsymbol{S}$) is related to estructure conditions like [**?**, Lemma 3.1, Eq. (3.1)]. As can be seen, condition ($\boldsymbol{S}$) is a more compact expression than [**?**, Lemma 3.1, Eq. (3.1)] and moreover weaker, because ($\boldsymbol{S}$) does not imply a control of $|D_{y_1} L|$ independent of y_2. We will return to this point later.

An important example of lagrangian is giving by:

$$\mathcal{L}_{\Phi,F}(t,x,y) := \Phi(y) + F(t,x). \tag{3}$$

Here the function $F(t,x)$, which is often referred to potential, be differentiable with respect to x for a.e. $t \in [0,T]$. Moreover F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in [0,T]$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0,T]$.

(A) For a.e. $t \in [0,T]$, it holds that

$$|F(t,x)| + |\nabla_x F(t,x)| \leqslant a(x) b(t). \tag{4}$$

where $a \in C\left(\mathbb{R}^d, [0,+\infty)\right)$ and $0 \leqslant b \in L^1([0,T], \mathbb{R})$.

The lagrangian $\mathcal{L}_{\Phi,F}$ satisfies condition ($\boldsymbol{S}$). In order to prove this, the only non trivial fact that we should to establishis is that $\Psi(\nabla_y \mathcal{L}) \leqslant a(x)\left\{b(t) + \Phi\left(y/\lambda\right)\right\}$. But, from inequality xxxx below, $\Psi(\nabla_y \mathcal{L}) = \Psi\left(\nabla\Phi(y)\right) \leqslant \Phi(2y)$.

The laplacian $\mathcal{L}_{\Phi,F}$ leads to the system

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)) & \text{a.e. } t \in (0,T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{$\boldsymbol{P_\Phi}$}$$

Problem ($\boldsymbol{P_\Phi}$) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [**?**] deals mainly with problem ($\boldsymbol{P}$), for the lagrangian $\mathcal{L}_{\Phi,F}$, with $\Phi(x) = |x|^2/2$, through various methods: direct, dual action, minimax, etc. The results in [**?**] were extended and improved in several articles, see [**?**, **?**, **?**, **?**, **?**] to cite some examples. The case $\Phi(y) = |y|^p/p$, for arbitrary $1 < p < \infty$ were considered in [**?**, **?**], among other papers, and in this case ($\boldsymbol{P_\Phi}$) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\left(u'(t)|u'|^{p-2}\right) = \nabla F(t,u(t)) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{$\boldsymbol{P_p}$}$$

If Φ is as in Example 1.3 and $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, then the equations ($\boldsymbol{P_\Phi}$) become

$$\begin{cases} \frac{d}{dt}\left(|u_1'|^{p_1-2}u_1'\right) = F_{x_1}(t,u) & \text{a.e. } t \in (0,T) \\ \frac{d}{dt}\left(|u_2'|^{p_2-2}u_2'\right) = F_{x_2}(t,u) & \text{a.e. } t \in (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} , \tag{$\boldsymbol{P_{p_1,p_2}}$}$$

where $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as (p_1, p_2)-Laplacian system, see [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

In conclusion, the problem $(\boldsymbol{P})$ with conditions $(\boldsymbol{S})$ contains several problems that have been considered by many authors in the past.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\Phi : \mathbb{R}^n \to [0, +\infty)$. References for these topics are [**?**, **?**, **?**].

If Φ is a N_∞ function then from convexity and $\Phi(0) = 0$ we obtain that

$$\Phi(\lambda x) \leqslant \lambda \Phi(x), \quad \lambda \in [0, 1], x \in \mathbb{R}^d. \tag{5}$$

One of the greatest difficulties when dealing with anisotropic Orlicz spaces is the lack of monotony with respect to the Euclidean norm, i.e. $|x| \leqslant |y|$ does not imply $\Phi(x) \leqslant \Phi(y)$. This problem is avoided if we consider functions whose values on a sphere are comparable (see[**?**]). However, from (5), we see that N_∞ functions have the following form of radial monotony: $|x| \leqslant |y|$ and $y = \lambda x$ imply $\Phi(x) \leqslant \Phi(y)$.

We say that $\Phi : \mathbb{R}^d \to [0, +\infty)$ satisfies the Δ_2^∞*-condition*, denoted by $\Phi \in \Delta_2^\infty$, if there exist constants $K > 0$ and $M \geqslant 0$ such that

$$\Phi(2x) \leqslant K\Phi(x), \tag{6}$$

for every $|x| \geqslant M$. If Φ es a Δ_2 function then Φ is bounded by powers functions (see [**?**, Proof Lemma 2.4]), i.e. there exists $1 < p < \infty$, $C > 0$ and $M \geqslant 0$ such that

$$\Phi(x) \leqslant C|x|^p, \quad |x| \geqslant$$

We denote by $\mathcal{M} := \mathcal{M}\left([0, T], \mathbb{R}^d\right)$, with $d \geqslant 1$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $[0, T]$ with values on $\mathbb{R}^d$ and we write $u = (u_1, \ldots, u_d)$ for $u \in \mathcal{M}$.

Given an N_∞ function Φ we define the *modular function* $\rho_\Phi : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

Now, we introduce the *Orlicz class* $C^\Phi = C^\Phi\left([0, T], \mathbb{R}^d\right)$ by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \tag{7}$$

The *Orlicz space* $L^\Phi = L^\Phi\left([0, T], \mathbb{R}^d\right)$ is the linear hull of C^Φ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} | \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \tag{8}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \middle| \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1 \right\},$$

is a Banach space.

The subspace $E^\Phi = E^\Phi\left([0,T],\mathbb{R}^d\right)$ is defined as the closure in L^Φ of the subspace $L^\infty\left([0,T],\mathbb{R}^d\right)$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [**?**, Thm. 5.1]) $u \in E^\Phi$ if and only if $\rho_\Phi(\lambda u) < \infty$ for any $\lambda > 0$. The equality $L^\Phi = E^\Phi$ is true if and only if $\Phi \in \Delta_2^\infty$ (see [**?**, Thm. 5.2]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [**?**, Thm. 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^\Psi$ then $u \cdot v \in L^1$ and

$$\int_0^T v \cdot u \, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^\Psi}. \tag{9}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

We consider the subset $\Pi(E^\Phi, r)$ of L^Φ given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi | d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class C^Φ by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset rC^\Phi \subset \overline{\Pi(E^\Phi, r)} \tag{10}$$

for any positive r. This relation is a trivial generalization of [**?**, Thm. 5.6]. If $\Phi \in \Delta_2^\infty$, then the sets L^Φ, E^Φ, $\Pi(E^\Phi, r)$ and C^Φ are equal.

As usual, if $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > 0$ such that $\|y\|_X \leqslant C\|y\|_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^\Phi \hookrightarrow \left[L^\Psi\right]^*$, where a function $v \in L^\Phi$ is associated to $\xi_v \in \left[L^\Psi\right]^*$ being

$$\xi_v(u) = \langle \xi_v, u\rangle = \int_0^T v \cdot u \, dt, \tag{11}$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [**?**].

Proposition 2.1. *If Ψ satisfies the Δ_2^∞-condition then $L^\Phi\left([0,T],\mathbb{R}^d\right) = \left[L^\Psi\left([0,T],\mathbb{R}^d\right)\right]^*$.*

We define the *Sobolev-Orlicz space* W^1L^Φ by

$$W^1L^\Phi\left([0,T],\mathbb{R}^d\right) := \left\{u | u \in AC\left([0,T],\mathbb{R}^d\right) \text{ and } u' \in L^\Phi\left([0,T],\mathbb{R}^d\right)\right\},$$

where $AC\left([0,T],\mathbb{R}^d\right)$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $[0,T]$. The space $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{12}$$

We introduce the following subspaces of W^1L^Φ

$$\begin{aligned} W^1E^\Phi &= \{u \in W^1L^\Phi | u' \in E^\Phi\}, \\ W^1E_T^\Phi &= \{u \in W^1E^\Phi | u(0) = u(T)\}. \end{aligned} \tag{13}$$

In order to find a modulus of continuity for functios in W^1L^Φ, and from there, to obtain compact embedding of W^1L^Φ, we define the function $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_\Phi(s) = \min\left\{\Phi(x) \,\middle|\, |x| = s\right\}, \tag{14}$$

Let us establish some elementary properties of A_Φ.

Proposition 2.2. *The function A_Φ has the following properties:*

1. *A_Φ is continuous,*
2. *$A_\Phi(s)/s$ is increasing,*
3. *$A_\Phi(|x|)$ is the* greatest radial minorant *of $\Phi(x)$,*
4. *Φ is N_∞ if and only if $\lim_{s\to+\infty} A_\Phi(s)/s = +\infty$.*

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [**?**]). This fact implies item 1 immediately.

In order to prove item 2, suppose $0 < r < s$ and $x \in \mathbb{R}^d$ with $A_\Phi(s) = \Phi(x)$. Then, from the definition of A_Φ and the convexity of Φ,

$$\frac{A_\Phi(r)}{r} \leqslant \frac{\Phi\left(\frac{r}{s}x\right)}{r} \leqslant \frac{\Phi(x)}{s} = \frac{A_\Phi(s)}{s}.$$

Property in items 3 and 4 are obtained easily.

□

Example 2.1. We compute A_Φ for the function $\Phi = \Phi_{p_1,p_2}$ given in Example (1.3). We apply the method of Lagrange multipliers (see [**?**, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{cases} \text{minimize } \Phi_{p_1,p_2}(y_1, y_2) \\ \text{subject to } |y_1|^2 + |y_2|^2 = r^2 \end{cases}.$$

The first order conditions are

$$\begin{cases} |y_1|^{p_1-2}y_1 + \lambda y_1 &= 0 \\ |y_2|^{p_2-2}y_2 + \lambda y_2 &= 0 \\ |y_1|^2 + |y_2|^2 &= r^2 \end{cases} \tag{15}$$

These equations are solved, among others, by the following two sets of citical points: a) $|x| = r$, $y = 0$ and $\lambda = -r^{p_1-2}$ and b) $x = 0$, $|y| = r$ and $\lambda = -r^{p_2-2}$. These sets are infinite when $d > 1$. Associated with these critical points we have the following critical values: a) r^{p_1}/p_1 and b) r^{p_2}/p_2.

We deal with $p_1 \leqslant 2$ and $p_2 \leqslant 2$ being one of them (suppose p_2) different from 2. The remaining cases can be treated with similar techniques.

If (y_1, y_2) solve (15) with $y_1 \neq 0$ and $y_2 \neq 0$ then $|y_2| = |y_1|^{\frac{p_1-2}{p_2-2}}$ and $\lambda = -|y_1|^{p_1-2}$. We use second order conditions for constrained problems. We have to consider the tangent plane at the point $(y_1, y_2) \in \mathbb{R}^{2n}$, i.e. $M = \{(\xi, \eta) \in \mathbb{R}^{2n} : \xi y_1^t + \eta y_2^T = 0\}$. Let L be the Lagrangian associated to the constrained problem: $L(y_1, y_2, \lambda) = \Phi(y_1, y_2) + \lambda H(y_1, y_2)$ being $H = 0$ the constraint. We must analize the positivity of the quadratic form associated to the matrix of second partial derivatives $\mathcal{H} = D^2\Phi + \lambda D^2 H$ on the subspace M. By elementary computations we have for $(\xi, \eta) \in M$

$$(\xi, \eta)^t \mathcal{H}(\xi, \eta) = |\lambda|(\xi^t x)^2[|y_1|^{-2}(p_1 - 2) + (p_2 - 2)|y_2|^{-2}],$$

on the subspace M. We note that $(-y_2, y_1) \in M$ and $(-y_2, y_1)^t\mathcal{H}(-y_2, y_1) < 0$. Then, by second order necessary conditions [**?**, p.333], at (y_1, y_2) there cannot be a minimum. Therefore, the only minima occur at $y_1 = 0$ or $y_2 = 0$, then

$$A_\Phi(x, y) = \min\{r^{p_1}/p_1, r^{p_2}/p_2\}.$$

More generally, it holds that

$$K_1 \min\{r^{p_1}, r^{p_2}\} \leqslant A_\Phi \leqslant K_2 \min\{r^{p_1}, r^{p_2}\}$$

with $K_1, K_2 > 0$, for every $1 < p_1, p_2 < \infty$.

As is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$ where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a Young's function and let $u \in W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$. Let $A_\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by* (14). *Then*

1. *For every $s, t \in [0,T]$, $s \neq t$,*

$$|u(t) - u(s)| \leqslant \|u'\|_{L^\Phi}|s - t| A_\Phi^{-1}\left(\frac{1}{|s-t|}\right) \qquad \text{(Morrey's inequality)}$$

$$\|u\|_{L^\infty} \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1, T\}\|u\|_{W^1L^\Phi} \qquad \text{(Sobolev's inequality)}$$

2. *We have $\widetilde{u} \in L^\infty\left([0,T], \mathbb{R}^d\right)$ and*

$$\|\widetilde{u}\|_{L^\infty} \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If Φ is N_∞ then the space $W^1L^\Phi\left([0,T], \mathbb{R}^d\right)$ is compactly embedded in the space of continuous functions $C\left([0,T], \mathbb{R}^d\right)$.*

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi}|s-t|}\right) \leqslant \Phi\left(\frac{1}{|s-t|}\int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}}dr\right)$$
$$\leqslant \frac{1}{|s-t|}\int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right)dr \leqslant \frac{1}{|s-t|}.$$

By Proposition 2.2(3) we have $A_\Phi^{-1}\Phi(x) \geqslant |x|$, therefore we get

$$\frac{|u(t)-u(s)|}{\|u'\|_{L^\Phi}|s-t|} \leqslant A_\Phi^{-1}\left(\frac{1}{|s-t|}\right),$$

then 1 holds.

Now, we use Morrey's inequality and Proposition 2.2 (2) and we have

$$\begin{aligned}|u(t)-\overline{u}| &= \left|\frac{1}{T}\int_0^T u(t)-u(s)\,ds\right| \\ &\leqslant \frac{1}{T}\int_0^T |u(t)-u(s)|\,ds \\ &\leqslant \|u'\|_{L^\Phi} T A_\Phi^{-1}\left(\frac{1}{T}\right)\end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\|u\|_{L^\Phi}$, we obtain

$$\Phi\left(\frac{\overline{u}}{\|u\|_{L^\Phi}}\right) \leqslant \frac{1}{T}\int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leqslant \frac{1}{T}$$

Then by By Proposition 2.2(3)

$$|\overline{u}| \leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi}.$$

Therefore, from this and (Sobolev-Wirtinger's inequality) we get

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant |\overline{u}| + \|\tilde{u}\|_{L^\infty} \\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\|u\|_{L^\Phi} + TA_\Phi^{-1}\left(\frac{1}{T}\right)\|u'\|_{L^\Phi} \\ &\leqslant A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1,T\}\|u\|_{W^1L^\Phi}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$. Since Φ is N_∞, from Proposition 2.2(4) we obtain $sA_\Phi^{-1}(1/s) \to 0$ when $s \to 0$. Therefore (Morrey's inequality) implies that u_n are equicontinuous. Furthermore (**??**) implies that u_n is bounded in $C\left([0,T],\mathbb{R}^d\right)$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C\left([0,T],\mathbb{R}^d\right)$ with $u_{n_k} \to u$ in $C\left([0,T],\mathbb{R}^d\right)$. □

Lemma 2.4. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\Pi(E^\Phi,1)$ converging to $u \in \Pi(E^\Phi,1)$ in the L^Φ-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^1([0,T],\mathbb{R})$ such that $u_{n_k} \to u$ a.e. and $\Phi(u_{n_k}) \leqslant h$ a.e.*

Proof. Since $d(u, E^\Phi) < 1$ and u_n converges to u, there exists $u_0 \in E^\Phi$, a subsequence of u_n (again denoted u_n) and $0 < r < 1$ such that $d(u_n, u_0) < r$. Let $\lambda_0 \in (r, 1)$. By extracting more subsequences, if necessary, we can assume that $u_n \to u$ a.e. and

$$\lambda_n := \|u_{n+1} - u_n\|_{L^\Phi} < \frac{1 - \lambda_0}{2^n}, \quad \text{for } n \geqslant 1.$$

We can assume $\lambda_n > 0$ for every $n = 0, \ldots.$

Let $\lambda := 1 - \sum_{n=0}^\infty \lambda_n$ and define $h : [0, T] \to \mathbb{R}$ by

$$h(x) = \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{n=0}^\infty \lambda_n \Phi\left(\frac{u_{n+1} - u_n}{\lambda_n}\right). \tag{16}$$

Note that $\sum_{n=0}^\infty \lambda_n + \lambda = 1$, therefore for any $n = 1, \ldots$

$$\begin{aligned}
\Phi(u_n) &= \Phi\left(\lambda \frac{u_0}{\lambda} + \sum_{j=0}^{n-1} \lambda_j \frac{u_{j+1} - u_j}{\lambda_j}\right) \\
&\leqslant \lambda \Phi\left(\frac{u_0}{\lambda}\right) + \sum_{j=0}^{n-1} \lambda_j \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) \leqslant h
\end{aligned}$$

Since $u_0 \in E^\Phi \subset C^\Phi$ and E^Φ is a subspace we have that $\Phi(u_0/\lambda) \in L^1([0, T], \mathbb{R})$. On the other hand $\|u_{n+1} - u_n\|_{L^\Phi} \leqslant \lambda_n$, therefore

$$\int_0^T \Phi\left(\frac{u_{j+1} - u_j}{\lambda_j}\right) dt \leqslant 1.$$

Then $h \in L^1([0, T], \mathbb{R})$.

□

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

In this section we give a brief introduction to superposition operators between anistropic Orlicz Spaces. We apply these results to obtain Gateâux differentiability of action integrals associated to lagrangian functions defined on Sobolev-Orlicz spaces.

Henceforth we assume that $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$is a *Carathéodory function*, i.e.

(C) f is measurable with respect to $t \in [0, T]$ for every $x \in \mathbb{R}^d$, and f is a continuous function with respect to $x \in \mathbb{R}^d$ for a.e. $t \in [0, T]$.

Definition 3.1. *For $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ we denote by $\boldsymbol{f}$ the Nemytskii (o superposition) operator defined for functions $u : [0, T] \to \mathbb{R}^d$ by*

$$\boldsymbol{f} u(t) = f(t, u(t))$$

In the following Theorem we enumerate some known properties for superposition operators defined on anisotropic Orlicz spaces of vector functions. For the proofs see [**?**] for scalar functions and [**?**, **?**, **?**] for the generalization to $\mathbb{R}^d$-valued (moreover Banach spaces valued) functions in a anisotropic Orlicz Spaces (moreover modular anisotropic spaces).

Theorem 3.2. *We assume that f satisfies condition* ((C)) *and that $\Phi_1, \Phi_2 : \mathbb{R}^d \to [0, +\infty)$ are anisotropic Young functions. Then*

1. Measurability. *The operator $\boldsymbol{f}$ maps measurable function into measurable functions*
2. Extensibility. *If the operator $\boldsymbol{f}$ acts from the ball $B_{L^{\Phi_1}}(r) := \{u \in L^{\Phi_1} | \|u\|_{L^{\Phi_1}} < r\}$ into the space L^{Φ_2} or the space E^{Φ_2} then $\boldsymbol{f}$ can be extended from $\Pi(E^{\Phi_1}, r)$ into space L^{Φ_2} or E^{Φ_2}, respectively.*
3. Continuity. *If the operator $\boldsymbol{f}$ acts from $\Pi(E^{\Phi_1}, r)$ into space E^{Φ_2}, then $\boldsymbol{f}$ is continuous.*

Given a continuous function $a \in C(\mathbb{R}^n, \mathbb{R}^+)$, we define the composition operator $a : \mathcal{M}_d \to \mathcal{M}_d$ by $\boldsymbol{a}(u)(x) = a(u(x))$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [**?**].

Lemma 3.3. *If $a \in C(\mathbb{R}^d, \mathbb{R}^+)$ then $\boldsymbol{a} : W^1L^\Phi \to L^\infty([0, T])$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\boldsymbol{a}(u)\|_{L^\infty([0,T])} \leqslant A(\|u\|_{W^1L^\Phi})$.*

Proof. Let $A \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a non decreasing, continuous function defined by $\alpha(s) := \sup_{\|x\| \leqslant s, x \in \mathbb{R}^d} |a(x)|$. If $u \in W^1L^\Phi_d$ then, by Sobolev's inequality, for a.e. $t \in [0, T]$

$$a(u(t)) \leqslant \alpha(\|u\|_{L^\infty}) \leqslant \alpha\left(A_\Phi^{-1}\left(\frac{1}{T}\right)\max\{1, T\}\|u\|_{W^1L^\Phi}\right) =: A(\|u\|_{W^1L^\Phi}).$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$I(u) = \int_0^T \mathcal{L}(t, u(t), \dot{u}(t))\, dt. \tag{17}$$

Theorem 3.4. *Let $\mathcal{L}$ be a differentiable Carathéodory function satisfying* ($\boldsymbol{S}$)*. Then the following statements hold:*

1. *The action integral given by* (17) *is finitely defined on $\mathcal{E}^\Phi := W^1L^\Phi \cap \{u | \dot{u} \in \Pi(E^\Phi, 1)\}$.*

2. *The function I is Gâteaux differentiable on $\mathcal{E}^\Phi$ and its derivative I' is demicontinuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$. Moreover, I' is given by the following expression*
$$\langle I'(u), v\rangle = \int_0^T \left\{D_x\mathcal{L}\big(t,u,\dot{u}\big)\cdot v + D_y\mathcal{L}\big(t,u,\dot{u}\big)\cdot \dot{v}\right\}\, dt. \tag{18}$$

3. *If $\Psi \in \Delta_2$ then I' is continuous from $\mathcal{E}^\Phi$ into $\left[W^1L^\Phi\right]^*$ when both spaces are equipped with the strong topology.*

Proof. Let $u \in \mathcal{E}^\Phi$. As
$$\dot{u} \in \Pi(E^\Phi, 1) \subset C_1^\Phi \tag{19}$$
and (10), then $\Phi(\dot{u}(t)) \in L^1$. Now,
$$|\mathcal{L}(\cdot,u,\dot{u})| + |D_x\mathcal{L}(\cdot,u,\dot{u})| + \Psi(D_y\mathcal{L}(\cdot,u,\dot{u})) \leqslant A(\|u\|_{W^1L^\Phi})(b+\Phi(\dot{u})) \in L^1, \tag{20}$$
by ($\boldsymbol{S}$) and Lemma 3.3. Thus item (1) is proved.

We split up the proof of item 2 into four steps.

Step 1. The non linear operator $u \mapsto D_x\mathcal{L}(t,u,\dot{u})$ is continuous from $\mathcal{E}^\Phi$ into $L^1([0,T])$ with the strong topology on both sets.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}^\Phi$ and let $u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in W^1L^Φ. By (Sobolev's inequality), we have
$$|u_n(t) - u(t)| \leqslant T A_\Phi^{-1}\left(\frac{1}{T}\right)\|u_n - u\|_{L^\Phi}$$
then $u_n \to u$ uniformly. As $\dot{u}_n \to \dot{u} \in \mathcal{E}^\Phi$, by Lemma 2.4, there exist a subsequence of $\dot{u}_{n_k}$ (again denoted $\dot{u}_{n_k}$) and a function $h \in L^1([0,T],\mathbb{R})$ such that $\dot{u}_{n_k} \to \dot{u}$ a.e. and $\Phi(\dot{u}_{n_k}) \leqslant h$ a.e.

Since u_{n_k}, $k = 1,2,\dots$, is a strong convergent sequence in W^1L^Φ, it is a bounded sequence in W^1L^Φ. According to item (3) of Lemma 2.3, there exists $M > 0$ such that $\|\boldsymbol{a}(u_{n_k})\|_{L^\infty} \leqslant M$, $k = 1,2,\dots$. From the previous facts and (20), we get
$$|D_x\mathcal{L}(\cdot,u_{n_k},\dot{u}_{n_k})| \leqslant a(|u_{n_k}|)(b+\Phi(\dot{u}_{n_k})) \leqslant M(b+h) \in L^1.$$
On the other hand, by the continuous differentiability of $\mathcal{L}$, we have
$$D_x\mathcal{L}(t,u_{n_k}(t),\dot{u}_{n_k}(t)) \to D_x\mathcal{L}(t,u(t),\dot{u}(t)) \quad \text{for a.e. } t \in [0,T].$$
Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto D_y\mathcal{L}(t,u,\dot{u})$ is continuous from $\mathcal{E}^\Phi$ with the strong topology into $\left[L^\Phi\right]^*$ with the weak topology.*

Let $u \in \mathcal{E}^\Phi$. From (20) it follows that
$$D_y\mathcal{L}(\cdot,u,\dot{u}) \in C^\Psi. \tag{21}$$

Así? o conviene poner la cota de $\Psi(D_y)$ explícitamente???

Note that (20), (21) and the imbeddings $W^1L^\Phi \hookrightarrow L^\infty$ and $L^\Psi \hookrightarrow \left[L^\Phi\right]^*$ imply that the second member of (18) defines an element of $\left[W^1L^\Phi\right]^*$.

Let $u_n, u \in \mathcal{E}^\Phi$ such that $u_n \to u$ in the norm of W^1L^Φ. We must prove that $D_y\mathcal{L}(\cdot, u_n, \dot{u}_n) \overset{w^*}{\rightharpoonup} D_y\mathcal{L}(\cdot, u, \dot{u})$. On the contrary, there exist $v \in L^\Phi$, $\epsilon > 0$ and a subsequence of $\{u_n\}$ (denoted $\{u_n\}$ for simplicity) such that

$$|\langle D_y\mathcal{L}(\cdot, u_n, \dot{u}_n), v\rangle - \langle D_y\mathcal{L}(\cdot, u, \dot{u}), v\rangle| \geqslant \epsilon. \tag{22}$$

We have $u_n \to u$ in L^Φ and $\dot{u}_n \to \dot{u}$ in L^Φ. By Lemma 2.4, there exist a subsequence of $\{u_n\}$ (again denoted $\{u_n\}$ for simplicity) and a function $h \in L^1([0,T], \mathbb{R})$ such that $u_n \to u$ uniformly, $\dot{u}_n \to \dot{u}$ a.e. and $\Phi(\dot{u}_n) \leqslant h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma 3.3 implies that $a(|u_n(t)|)$ is uniformly bounded by a certain constant $M > 0$. Therefore, from inequality (20) with u_n instead of u, we have

$$\Psi(D_y\mathcal{L}(\cdot, u_n, \dot{u}_n)) \leqslant M(b + h) \in L^1. \tag{23}$$

As $v \in L^\Phi$ there exists $\lambda > 0$ such that $\Phi(\frac{v}{\lambda}) \in L^1$. Now, by Young inequality and (23), we have

$$\begin{aligned} &\lambda D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot \frac{v(t)}{\lambda} \\ &\leqslant \lambda \left[\Psi(D_y\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k})) + \Phi\left(\frac{v}{\lambda}\right)\right] \\ &\leqslant \lambda M(b + h) + \lambda\Phi\left(\frac{v}{\lambda}\right) \in L^1 \end{aligned} \tag{24}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_y\mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v\,dt \to \int_0^T D_y\mathcal{L}(t, u, \dot{u}) \cdot v\,dt \tag{25}$$

which contradicts the inequality (22). This completes the proof of step 2.

Step 3. We will prove (18). For $u \in \mathcal{E}^\Phi$ and $0 \neq v \in W^1L^\Phi$, we define the function

$$H(s,t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

For $|s| \leqslant s_0 := \min\{(1 - d(\dot{u}, E^\Phi))/\|v\|_{W^1L^\Phi}, 1 - d(\dot{u}, E^\Phi)\}$, using triangle inequality we get $d(\dot{u} + s\dot{v}, E^\Phi) < 1$ and thus $\dot{u} + s\dot{v} \in \Pi(E^\Phi, 1)$. These facts imply, in virtue of Theorem 3.4 item 1, that $I(u + sv)$ is well defined and finite for $|s| \leqslant s_0$.

We also have $\|u + sv\|_{W^1L^\Phi} \leqslant \|u\|_{W^1L^\Phi} + s_0\|v\|_{W^1L^\Phi}$; then, by Lemma 3.3, there exists $M > 0$ such that $\|a(u + sv)\|_{L^\infty} \leqslant M$.

Let $\lambda > 0$ such that $\Phi(\frac{\dot{v}}{\lambda}) \in L^1$. On the other hand, if $\dot{v} \in L^\Phi$ and $|s| \leqslant s_0\lambda^{-1}$, from the convexity and the parity of Φ, we get

$$\begin{aligned} \Phi(\dot{u} + s\dot{v}) &= \Phi\left((1 - s_0)\frac{\dot{u}}{1 - s_0} + s_0\frac{s}{s_0}\dot{v}\right) \leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{s}{s_0}\dot{v}\right) \\ &\leqslant (1 - s_0)\Phi\left(\frac{\dot{u}}{1 - s_0}\right) + s_0\Phi\left(\frac{\dot{v}}{\lambda}\right) \in L^1 \end{aligned}$$

As $\dot{u} \in \Pi(E^\Phi, 1)$ then

$$d\left(\frac{\dot{u}}{1 - s_0}, E^\Phi\right) = \frac{1}{1 - s_0} d(\dot{u}, E^\Phi) < 1$$

and therefore $\frac{\dot u}{1-s_0} \in C^\Phi$.

Now, applying (20), (24), the fact that $v \in L^\infty$ and $\dot v \in L^\Phi$, we get

$$\begin{aligned}
|D_s H(s,t)| &= \left| D_x\mathcal{L}(t, u+sv, \dot u + s\dot v)\cdot v + \lambda D_y\mathcal{L}(t,u+sv,\dot u + s\dot v)\cdot \frac{\dot v}{\lambda}\right| \\
&\leqslant M\left[b(t) + \Phi(\dot u + s\dot v)\right]|v| \\
&\quad + \lambda\left[\Psi(D_y\mathcal{L}(t,u+sv,\dot u+s\dot v)) + \Phi\left(\frac{\dot v}{\lambda}\right)\right] \\
&\leqslant M\left\{\left[b(t) + \Phi(\dot u + s\dot v)\right]|v|\right\} + \lambda M[b(t) + \Phi(\dot u + s\dot v)] + \lambda\Phi\left(\frac{\dot v}{\lambda}\right) \\
&= M[b(t) + \Phi(\dot u + s\dot v)](|v| + \lambda) + \lambda\Phi\left(\frac{\dot v}{\lambda}\right) \in L^1.
\end{aligned} \tag{26}$$

Consequently, I has a directional derivative and

$$\langle I'(u), v\rangle = \frac{d}{ds} I(u+sv)\Big|_{s=0} = \int_0^T \left\{D_x\mathcal{L}(t,u,\dot u)\cdot v + D_y\mathcal{L}(t,u,\dot u)\cdot \dot v\right\}\,dt.$$

Moreover, from the previous formula, (20), (21), and Lemma 2.3, we obtain

$$|\langle I'(u), v\rangle| \leqslant \|D_x\mathcal{L}\|_{L^1}\|v\|_{L^\infty} + \|D_y\mathcal{L}\|_{L^\Psi}\|\dot v\|_{L^\Phi} \leqslant C\|v\|_{W^1L^\Phi}$$

with a appropriate constant C.

This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : \mathcal{E}^\Phi \to \left[W^1L^\Phi_d\right]^*$ is demicontinuous. This is a consequence of the continuity of the mappings $u \mapsto D_x\mathcal{L}(t,u,\dot u)$ and $u \mapsto D_y\mathcal{L}(t,u,\dot u)$. Indeed, if $u_n, u \in \mathcal{E}^\Phi$ with $u_n \to u$ in the norm of W^1L^Φ and $v \in W^1L^\Phi$, then

$$\begin{aligned}
\langle I'(u_n), v\rangle &= \int_0^T \left\{D_x\mathcal{L}\left(t,u_n,\dot u_n\right)\cdot v + D_y\mathcal{L}\left(t,u_n,\dot u_n\right)\cdot \dot v\right\}\,dt \\
&\to \int_0^T \left\{D_x\mathcal{L}\left(t,u,\dot u\right)\cdot v + D_y\mathcal{L}\left(t,u,\dot u\right)\cdot \dot v\right\}\,dt \\
&= \langle I'(u), v\rangle.
\end{aligned}$$

In order to prove item 3, it is necessary to see that the maps $u \mapsto D_x\mathcal{L}(t,u,\dot u)$ and $u \mapsto D_y\mathcal{L}(t,u,\dot u)$ are norm continuous from $\mathcal{E}^\Phi$ into L^1 and L^Ψ, respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de D_x aquí!!!

Let $u_n, u \in \mathcal{E}^\Phi$ with $\|u_n - u\|_{W^1L^\Phi} \to 0$.

Applying Lemma 2.4 to $\dot u_n$, there exists a subsequence (denoted $\dot u_n$ for simplicity) such that $\dot u_n \in L^\Phi$ and a function $h \in L^1$ such that $\Psi(\dot u_n) \leqslant h$ and $\dot u_n \to \dot u$ a.e.

Then, by (24) we have $\Psi(v_n) \leqslant m(t) \in L^1$ being $v_n := D_y\mathcal{L}(\cdot, u_n, \dot u_n)$ and $m(t) := M(b+h)$. In addition, from the continuous differentiability of $\mathcal{L}$, we have that $v_n \to v$ a.e. where $D_y\mathcal{L}(\cdot, u, \dot u)$.

As $\Psi \in \Delta_2$, there exists $c : \mathbb{R}^+ \to ???$ such that $\Psi(\lambda x) \leqslant c(|\lambda|)\Psi(x)$. Then, $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})\Psi(v_n - v)$ for every $\lambda \in \mathbb{R}$.

Therefore, $\Psi(\frac{v_n - v}{\lambda}) \to 0$ a.e. as $n \to \infty$ and $\Psi(\frac{v_n - v}{\lambda}) \leqslant c(|\lambda|^{-1})K\Psi(v_n) + \Psi(v)) \leqslant c(|\lambda|^{-1})K[m(t) + \Psi(v)]) \in L^1$.

Now, by Dominated Convergence Theorem, we get $\int \Psi(\frac{v_n - v}{\lambda})\, dt \to 0$ for every $\lambda > 0$. Thus, $v_n \to v$ in L^Ψ.

The continuity of I' follows from the continuity of $D_x\mathcal{L}$ and $D_y\mathcal{L}$ using the formula (18). $\square$

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