

# Periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting by the dual least action principle

Sonia Acinas \*

Dpto. de Matemática, Facultad de Ciencias Exactas y Naturales  
Universidad Nacional de La Pampa  
(L6300CLB) Santa Rosa, La Pampa, Argentina  
sonia.acinas@gmail.com

Fernando D. Mazzone †

Dpto. de Matemática, Facultad de Ciencias Exactas, Físico-Químicas y Naturales  
Universidad Nacional de Río Cuarto  
(5800) Río Cuarto, Córdoba, Argentina,  
fmazzone@exa.unrc.edu.ar

## Abstract

## 1 Introduction

This paper deals with system of equations of the type:

$$\begin{cases} \frac{d}{dt} D_y \mathcal{L}(t, u(t), u'(t)) = D_x \mathcal{L}(t, u(t), u'(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (1)$$

where  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 1$ , is called the *Lagrange function* or *lagrangian* and the unknown function  $u : [0, T] \rightarrow \mathbb{R}^d$  is absolutely continuous. In other words, we are interested in finding *periodic weak solutions of Euler-Lagrange system of ordinary equations*. This topic was deeply addressed for the *Lagrange function*

$$\mathcal{L}_{p,F}(t, x, y) = \frac{|y|^p}{p} + F(t, x), \quad (2)$$

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\*SECyT-UNRC and FCEyN-UNLPam

†SECyT-UNRC, FCEyN-UNLPam and CONICET

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for  $1 < p < \infty$ . For example, the classic book [Mawhin and Willem, 1989] deals mainly with problem (1), for the lagrangian  $\mathcal{L}_{2,F}$ , through various methods: direct, dual action, minimax, etc. The results in [Mawhin and Willem, 1989] were extended and improved in several articles, see [Tang, 1995, Tang, 1998, Wu and Tang, 1999, Tang and Wu, 2001, Zhao and Wu, 2004] to cite some examples. Lagrange functions (2) for arbitrary  $1 < p < \infty$  were considered in [Tian and Ge, 2007, Tang and Zhang, 2010] and in this case (1) is reduced to the  $p$ -laplacian system

$$\begin{cases} \frac{d}{dt} (u'(t)|u'|^{p-2}) = \nabla F(t, u(t)) & \text{a.e. } t \in (0, T) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (3)$$

In this context, it is customary to call  $F$  a *potential function*, and it is assumed that  $F(t, x)$  is differentiable with respect to  $x$  for a.e.  $t \in [0, T]$  and the following conditions are verified:

(C)  $F$  and its gradient  $\nabla F$ , with respect to  $x \in \mathbb{R}^d$ , are Carathéodory functions, i.e. they are measurable functions with respect to  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ , and they are continuous functions with respect to  $x \in \mathbb{R}^d$  for a.e.  $t \in [0, T]$ .

(A) For a.e.  $t \in [0, T]$ , it holds that

$$|F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t). \quad (4)$$

In this inequality we assume that the function  $a : [0, +\infty) \rightarrow [0, +\infty)$  is continuous and non decreasing and  $0 \leq b \in L^1([0, T], \mathbb{R})$ .

In [Acinas et al., 2015] it was treated the case of a lagrangian  $\mathcal{L}$  which is lower bounded by a Lagrange function

$$\mathcal{L}_{\Phi, F}(t, x, y) = \Phi(|y|) + F(t, x), \quad (5)$$

where  $\Phi$  is an  $N$ -function (see section 2 for the definition of this concept). In the paper [Acinas et al., 2015] it was assumed a condition of *bounded oscillation* on  $F$  (see xxxxx below). In this paper we apply the dual method ([Mawhin and Willem, 1989, Ch. 3]) to obtain solutions of (1).

## 2 Preliminaries

In this section, we give a short introduction to known results on Orlicz and Orlicz-Sobolev spaces of vector valued functions (anisotropic Orlicz Spaces) and other brief introduction to superposition operators between these spaces. References for these topics are [Schappacher, 2005, Skaff, 1969, Desch and Grimmer, 2001] and [Płuciennik, 1987, Nguen Hong Thai, 1987, Płuciennik, 1985b, Płuciennik, 1985a].

Hereafter we denote by  $\mathbb{R}^+$  the set of all non negative real numbers. A function  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is called an *Young's function* if  $\Phi$  is convex,  $\Phi(0) = 0$ ,  $\Phi(-x) = \Phi(x)$  and  $\Phi(x) \rightarrow +\infty$ , when  $|x| \rightarrow +\infty$ .

Following [Schappacher, 2005] we say that  $\Phi$  is *coercive* if

$$\lim_{|x| \rightarrow \infty} \frac{\Phi(x)}{|x|} = +\infty.$$

The coercivity of a  $\Phi$  implies that for every  $K > 0$  there exists a function  $F := F(K)$  such that if  $|x| > F(K)$  then  $\Phi(x) \geq K$ .

It is possible to define a function  $F$  that satisfies the property above as follows:

$$F(s) = \sup\{|x| : \Phi(x) \leq s\} \quad (6)$$

We note that the function  $\alpha F\left(\frac{x}{\alpha}\right)$  is increasing with respect to  $\alpha$ . That is, if  $\alpha \leq \beta$ , we have

$$\begin{aligned} \alpha F\left(\frac{s}{\alpha}\right) &= \sup\left\{|x| : \Phi(x) \leq \frac{s}{\alpha}\right\} = \sup\{\alpha|x| : \alpha\Phi(x) \leq s\} = \\ &= \sup\left\{|y| : \alpha\Phi\left(\frac{y}{\alpha}\right) \leq s\right\} \leq \sup\left\{|y| : \beta\Phi\left(\frac{y}{\beta}\right) \leq s\right\} = \\ &= \sup\left\{\beta|x| : \beta\Phi(x) \leq s\right\} = \sup\left\{\beta|x| : \Phi(x) \leq \frac{s}{\beta}\right\} = \\ &= \beta F\left(\frac{s}{\beta}\right) \end{aligned}$$

It is easy to see that  $|x| \leq F(\Phi(x))$  and if  $\Phi$  is a scalar function, then  $F = \Phi^{-1}$ .

We also say that a non decreasing function  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfies the  $\Delta_2^\infty$ -condition, denoted by  $\eta \in \Delta_2^\infty$ , if there exist constants  $K > 0$  and  $M \geq 0$  such that

$$\eta(2x) \leq K\eta(x), \quad (7)$$

for every  $|x| \geq M$ .

If  $\Phi$  is a Young function we define its *Fenchel conjugate*  $\Phi^* : \mathbb{R}^d \rightarrow \mathbb{R}_+$  by:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^d} x \cdot y - \Phi(x) \quad (8)$$

Let  $d$  be a positive integer. We denote by  $\mathcal{M} := \mathcal{M}([0, T], \mathbb{R}^d)$  the set of all measurable functions (i.e. functions which are limits of simple functions) defined on  $[0, T]$  with values on  $\mathbb{R}^d$  and we write  $u = (u_1, \dots, u_d)$  for  $u \in \mathcal{M}$ . For the set of functions  $\mathcal{M}$ , as for other similar sets, we will omit the reference to codomain  $\mathbb{R}^d$  when  $d = 1$ .

Given an  $N$ -function  $\Phi$  we define the *modular function*  $\rho_\Phi : \mathcal{M} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

$$\rho_\Phi(u) := \int_0^T \Phi(u) dt.$$

Here  $|\cdot|$  is the euclidean norm of  $\mathbb{R}^d$ . Now, we introduce the *Orlicz class*  $C^\Phi = C^\Phi([0, T], \mathbb{R}^d)$  by setting

$$C^\Phi := \{u \in \mathcal{M} | \rho_\Phi(u) < \infty\}. \quad (9)$$

The Orlicz space  $L^\Phi = L^\Phi([0, T], \mathbb{R}^d)$  is the linear hull of  $C^\Phi$ ; equivalently,

$$L^\Phi := \{u \in \mathcal{M} \mid \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\}. \quad (10)$$

The Orlicz space  $L^\Phi$  equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf \left\{ \lambda \left| \rho_\Phi \left( \frac{v}{\lambda} \right) dt \leq 1 \right. \right\},$$

is a Banach space. By  $u \cdot v$  we denote the usual dot product in  $\mathbb{R}^d$  between  $u$  and  $v$ .

The subspace  $E^\Phi = E^\Phi([0, T], \mathbb{R}^d)$  is defined as the closure in  $L^\Phi$  of the subspace  $L^\infty([0, T], \mathbb{R}^d)$  of all  $\mathbb{R}^d$ -valued essentially bounded functions. It is shown that (see [Schappacher, 2005, Thm. 5.1])  $u \in E^\Phi$  if and only if  $\rho_\Phi(\lambda u) < \infty$  for any  $\lambda > 0$ . The equality  $L^\Phi = E^\Phi$  is true if and only if  $\Phi \in \Delta_2^\infty$  (see [Schappacher, 2005, Thm. 5.2]). Another alternative characterization of  $E^\Phi$ , which is particularly useful for us, is that  $u \in E^\Phi$  if and only if  $u$  has *absolutely continuous norm*, i.e. if  $E_n \subset [0, T]$ ,  $n = 1, 2, \dots$  then  $\|\chi_{E_n} u\| \rightarrow 0$  when  $|E_n| \rightarrow 0$ .

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [Skaff, 1969, Thm. 4.1]). Namely, if  $u \in L^\Phi$  and  $v \in L^{\Phi^*}$  then  $u \cdot v \in L^1$  and

$$\int_0^T v \cdot u \, dt \leq 2 \|u\|_{L^\Phi} \|v\|_{L^{\Phi^*}}. \quad (11)$$

Like in [Krasnosel'skiĭ and Rutickiĭ, 1961] we will consider the subset  $\Pi(E^\Phi, r)$  of  $L^\Phi$  given by

$$\Pi(E^\Phi, r) := \{u \in L^\Phi \mid d(u, E^\Phi) < r\}.$$

This set is related to the Orlicz class  $C^\Phi$  by means of inclusions, namely,

$$\Pi(E^\Phi, r) \subset r C^\Phi \subset \overline{\Pi(E^\Phi, r)} \quad (12)$$

for any positive  $r$  (see [Schappacher, 2005, Thm. 5.6]). If  $\Phi \in \Delta_2^\infty$ , then the sets  $L^\Phi$ ,  $E^\Phi$ ,  $\Pi(E^\Phi, r)$  and  $C^\Phi$  are equal.

Following to [Desch and Grimmer, 2001] we introduce the next definition.

**Definition 2.1.** Let  $u_n, u \in L^\Phi([0, T], \mathbb{R}^d)$ . We say that  $u_n$  converges monotonically to  $u$  if there exists  $\alpha_n \in L^\infty([0, T], \mathbb{R}^d)$ ,  $n = 1, 2, \dots$ , such that  $0 \leq \alpha_n(t) \leq \alpha_{n+1}(t)$ ,  $\alpha_n(t) \rightarrow 1$  a.e., when  $n \rightarrow \infty$  and  $u_n(t) = \alpha_n(t)u(t)$ .

As usual, if  $(X, \|\cdot\|_X)$  is a normed space and  $(Y, \|\cdot\|_Y)$  is a linear subspace of  $X$ , we write  $Y \hookrightarrow X$  and we say that  $Y$  is *embedded* in  $X$  when there exists  $C > 0$  such that  $\|y\|_X \leq C \|y\|_Y$  for any  $y \in Y$ . With this notation, Hölder's inequality states that  $L^{\Phi^*} \hookrightarrow [L^\Phi]^*$ , where a function  $v \in L^{\Phi^*}$  is associated to  $F_v \in [L^\Phi]^*$  where

$$F_v(u) := \langle v, u \rangle = \int_0^T v \cdot u \, dt, \quad (13)$$

In [Desch and Grimmer, 2001, Thm 2.9] it was characterized a subspace of  $[L^\Phi]^*$  which is identified with  $L^{\Phi^*}$ . Namely  $L^{\Phi^*} = P^{\Phi^*}([0, T], \mathbb{R}^d)$  where  $F \in P^{\Phi^*}([0, T], \mathbb{R}^d)$

if and only if  $F \in [L^\Phi]^*$  and satisfying the *monotone convergence property*, which is if  $u_n$  converges monotonically to  $u$  then  $F(u_n) \rightarrow F(u)$ .

If  $\Phi \in \Delta_2^\infty$  and  $\Phi$  is coercive then  $L^{\Phi^*}([0, T], \mathbb{R}^d) = [L^\Phi([0, T], \mathbb{R}^d)]^*$  is satisfied (see [Desch and Grimmer, 2001, Thm. 2.9, Thm. 2.10]).

We define the *Sobolev-Orlicz space*  $W^1 L^\Phi$  by

$$W^1 L^\Phi([0, T], \mathbb{R}^d) := \{u | u \text{ is absolutely continuous on } [0, T] \text{ and } u' \in L^\Phi([0, T], \mathbb{R}^d)\}.$$

$W^1 L^\Phi([0, T], \mathbb{R}^d)$  is a Banach space when equipped with the norm

$$\|u\|_{W^1 L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \quad (14)$$

And, we introduce the following subspaces of  $W^1 L^\Phi$

$$\begin{aligned} W^1 E^\Phi &= \{u \in W^1 L^\Phi | u' \in E^\Phi\}, \\ W^1 E_T^\Phi &= \{u \in W^1 E^\Phi | u(0) = u(T)\}. \end{aligned} \quad (15)$$

We will use repeatedly the decomposition  $u = \bar{u} + \tilde{u}$  for a function  $u \in L^1([0, T])$  where  $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$  and  $\tilde{u} = u - \bar{u}$ .

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

**Lemma 2.2.** *Let  $\Phi : \mathbb{R}^d \rightarrow [0, +\infty)$  be a Young's function and let  $u \in W^1 L^\Phi([0, T], \mathbb{R}^d)$ . Then*

1. *For every  $s, t \in [0, T]$ ,  $s \neq t$ ,*

$$\Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|}\right) \leq \frac{1}{|s - t|} \quad (\text{Morrey's inequality})$$

$$\left\| \Phi\left(\frac{u}{2 \max\{1, T\} \|u\|_{W^1 L^\Phi}}\right) \right\|_{L^\infty} \leq \frac{1}{T} \quad (\text{Sobolev's inequality})$$

2. *We have  $\tilde{u} \in L^\infty([0, T], \mathbb{R}^d)$  and*

$$\left\| \Phi\left(\frac{\tilde{u}}{\|u\|_{L^\Phi} T}\right) \right\|_{L^\infty} \leq \frac{1}{T} \quad (\text{Sobolev-Wirtinger's inequality})$$

3. *The space  $W^1 L^\Phi([0, T], \mathbb{R}^d)$  is compactly embedded in the space of continuous functions  $C([0, T], \mathbb{R}^d)$ .*

*Proof.* By the absolute continuity of  $u$  and Jensen's inequality we have

$$\begin{aligned} \Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} |s - t|}\right) &\leq \Phi\left(\frac{1}{|s - t|} \int_s^t \frac{u'(r)}{\|u'\|_{L^\Phi}} dr\right) \\ &\leq \frac{1}{|s - t|} \int_s^t \Phi\left(\frac{u'(r)}{\|u'\|_{L^\Phi}}\right) dr \leq \frac{1}{|s - t|}, \end{aligned}$$

which proves 1.

Morrey's inequality implies Sobolev-Wirtinger's inequality by the following argument. Taking into account that  $\alpha\Phi(x/\alpha)$  is a non increasing function with respect to  $\alpha \in [0, \infty)$  for every  $x \in \mathbb{R}^d$  we have

$$\Phi\left(\frac{u(t) - u(s)}{\|u'\|_{L^\Phi} T}\right) \leq \frac{1}{T}.$$

Dividing by  $T$  this inequality, integrating respect to  $s$  and using Jensen's inequality again

$$\Phi\left(\int_0^T \frac{u(t) - u(s)}{\|u'\|_{L^\Phi} T^2} ds\right) \leq \frac{1}{T}.$$

which implies

$$\Phi\left(\frac{u(t) - \bar{u}}{\|u'\|_{L^\Phi} T}\right) \leq \frac{1}{T}.$$

i.e. Sobolev-Wirtinger's inequality.

For the Sobolev's inequality we note that (we write  $T^* = \max\{1, T\}$ )

$$\begin{aligned} \Phi\left(\frac{u}{2T^*\|u\|_{W^1 L^\Phi}}\right) &\leq \frac{1}{2}\Phi\left(\frac{\tilde{u}}{T^*\|u\|_{W^1 L^\Phi}}\right) + \frac{1}{2}\Phi\left(\frac{\bar{u}}{T^*\|u\|_{W^1 L^\Phi}}\right) \\ &\leq \frac{1}{2}\Phi\left(\frac{\tilde{u}}{T^*\|u'\|_{L^\Phi}}\right) + \frac{1}{2}\Phi\left(\frac{\bar{u}}{T^*\|u\|_{L^\Phi}}\right) \\ &=: I_1 + I_2 \end{aligned}$$

Using Sobolev-Wirtinger's inequality, the inequality  $T^* \geq T$  and that  $\Phi$  is increasing function we get

$$2I_1 \leq \Phi\left(\frac{\tilde{u}}{T\|u'\|_{L^\Phi}}\right) \leq \frac{1}{T}$$

Using Jensen's inequality and that  $T^* \geq 1$  we have

$$2I_2 = \Phi\left(\frac{1}{T} \int_0^T \frac{u(s)}{T^*\|u\|_{L^\Phi}} ds\right) \leq \frac{1}{T} \int_0^T \Phi\left(\frac{u(s)}{\|u\|_{L^\Phi}}\right) ds \leq \frac{1}{T}$$

Then  $I_1 + I_2 \leq 1/T$ , which completes the proof of Sobolev's inequality.

Next we prove part 3 of the lemma. First we prove that there exists a non decreasing function  $F : (0, +\infty) \rightarrow (0, +\infty)$  such that  $\|u\|_{L^\infty} \leq F(\|\Phi(u)\|_{L^\infty})$ . In fact, since  $\Phi(x) \rightarrow +\infty$  when  $\|x\| \rightarrow +\infty$ , for every  $K > 0$  there exist  $G(K) > 0$  such that  $|x| \geq G(K)$  then  $\Phi(x) \geq K$ . Suppose that, for certain  $u$ ,  $\|u\|_{L^\infty} > G(\|\Phi(u)\|_{L^\infty})$ . Then there exists a set  $A \subset [0, T]$  with positive measure such  $|u(t)| > G(\|\Phi(u)\|_{L^\infty})$ , when  $t \in A$ . Then  $\Phi(u(t)) > \|\Phi(u)\|_{L^\infty}$ , for  $t \in A$ , which is a contradiction. Now we take  $F(K) := \sup\{G(s) | 0 < s \leq k\}$ .

We take a bounded sequence  $u_n$  in  $W^1 L^\Phi([0, T], \mathbb{R}^d)$  and suppose that  $u_n$  has not convergent subsequence.

□

### 3 Superposition operators in anisotropic Orlicz spaces

Vamos escribiendo lo que queremos...(de acuerdo a mis apuntes y sin ver las hojitas de la semana pasada)

For  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  we denote by  $\mathfrak{f}$  the Nemytskii (o superposition) operator defined for functions  $u : [0, T] \rightarrow \mathbb{R}^d$  by

$$\mathfrak{f}u(t) = f(t, u(t))$$

Referencias y alguna propiedad interesante medibles en medibles? [Krasnosel'skii et al., 2011, Krasnosel'skii and Rutickii, 1961]

**Theorem 3.1.** *Let  $\Phi_1, \Phi_2, \dots, \Phi_n$  be  $N$ -functions. Assume that  $M$  is another  $N$ -functions that satisfy the  $\Delta_2$ -condition. We write  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  with  $x_i \in \mathbb{R}^d$ ,  $y_i \in \mathbb{R}^d$ . Let  $f(t, x_1, \dots, x_n, y_1, \dots, y_n)$  be a function Charathéodory? with  $f : [0, T] \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \rightarrow \mathbb{R}^{d'}$ .*

*Suppose that  $a : (\mathbb{R}^d)^n \rightarrow [0, +\infty)$  is a bounded function on bounded sets and  $b \in L^M([0, T])$ , for a.e.  $t \in [0, T]$  such that*

$$|f| \leq a(x) \left[ b(t) + \sum_{i=1}^n M^{-1}(\Phi_i(|y_i|)) \right], \quad (16)$$

then

$$\mathfrak{f} : \left( \prod_{i=1}^n L^\infty([0, T], \mathbb{R}^d) \right) \times \left( \prod_{i=1}^n \Pi(E^{\Phi_i}([0, T], \mathbb{R}^d), \lambda = 1) \right) \rightarrow L^M.$$

*Proof.* If  $(u, v) \in \left( \prod_{i=1}^n L^\infty([0, T], \mathbb{R}^d) \right) \times \left( \prod_{i=1}^n \Pi(E^{\Phi_i}([0, T], \mathbb{R}^d), \lambda = 1) \right)$ . By [Krasnosel'skii and Rutickii, 1961, Thm. 17.6] (y otras cosas), we get

$$|\mathfrak{f}u(t)| = |f(t, u(t), v(t))| \leq M_a[b_j(t) + \sum_{i=1}^n M_j^{-1}(\Phi_i(|v_i(t)|))] \in L_1^{M_j}.$$

□

We define the space  $X$  by  $X = \{v = (v_1, v_2) : v_1 \in W^1 L_T^{\Phi_1}, v_2 \in W^1 L_T^{\Phi_2}\}$  and  $X^* = \{v = (v_1, v_2) : v_1 \in (W^1 L_T^{\Phi_1})^*, v_2 \in (W^1 L_T^{\Phi_2})^*\}$  where  $(W^1 L_T^{\Phi_i})^*$  stands for the conjugate space of  $W^1 L_T^{\Phi_i}$  for  $i = 1, 2$ .

**Corollary 3.2.** *We will consider the Lagrange function  $\mathcal{L} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $(t, x_1, x_2, y_1, y_2) \rightarrow \mathcal{L}(t, x_1, x_2, y_1, y_2)$  which is measurable in  $t$  for each  $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  and continuously differentiable in  $(x_1, x_2, y_1, y_2)$  for almost every  $t \in [0, T]$ .*

*Let  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$  with  $x_i \in \mathbb{R}^d$  and  $y_i \in \mathbb{R}^d$  and let*

$$I(x) = \int_0^T \mathcal{L}(t, x, y) dt \quad (17)$$

If there exist  $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ ,  $i = 1, 2$ ,  $b \in L^1([0, T])$ ,  $j = 1, \dots, d'$  for a.e.  $t \in [0, T]$  and every  $(x_1, x_2, y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  satisfying the structure conditions

The nonlinear operator  $(x_1, x_2) \mapsto D_x \mathcal{L}(t, x_1, y_1, y_2)$  is continuous from  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \dots \times \mathcal{E}_d^{\Phi_n}(\lambda)$  with the strong topology into  $L^1([0, T])$  with the strong topology on both sets.

The nonlinear operator  $(x_1, x_2) \mapsto D_y \mathcal{L}(t, x_1, y_1, y_2)$  is continuous from  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda) \times \dots \times \mathcal{E}_d^{\Phi_n}(\lambda)$  with the strong topology into  $X$  with the weak\* topology.

The function  $I$  is Gâteaux differentiable on  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  and its derivative  $I'$  is demicontinuous from  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  into  $X^*$ . Moreover,  $I'$  is given by the following expression

$$\begin{aligned} \langle I'(x), w \rangle = \int_0^T & [(D_{x_1} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_1(t)) + \\ & (D_{x_2} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w_2(t)) + \\ & (D_{y_1} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w'_1(t)) + \\ & (D_{y_2} \mathcal{L}(t, x_1(t), x_2(t), y_1(t), y_2(t)), w'_2(t))] dt \end{aligned} \quad (18)$$

If  $\Phi^* \in \Delta_2$  then  $I'$  is continuous from  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  into  $X^*$  when both spaces are equipped with the strong topology.

We denote by  $\mathfrak{A}(a, b, c, \lambda, f, \Phi)$  the set of all Lagrange functions satisfying (??), (??) and (??).

**Proof. OJO!!!! Es algo que teníamos del trabajo anterior!!! con algunas adaptaciones a 2 variables sin controlar y a lo bruto!!!!**

Let  $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ .

*Step 1.* The non linear operator  $(x_1, x_2) \mapsto (D_{x_1} \mathcal{L}(t, x_1, x_2, y_1, y_2), D_{x_1} \mathcal{L}(t, x_1, x_2, y_1, y_2))$  is continuous from  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  into  $L^1_d([0, T]) \times L^1_d([0, T])$  with the strong topology on both sets.

If  $u \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$ , from (??) and (??), we obtain Let  $\{x_n = (x_{1n}, x_{2n})\}_{n \in \mathbb{N}}$  be a sequence of functions in  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  and let  $x = (x_1, x_2) \in \mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  such that  $x_n \rightarrow x$  in  $X$ . From  $x_{in} \rightarrow x_i$  in  $L^{\Phi_i}$ , there exists a subsequence  $x_{in_k}$  such that  $x_{in_k} \rightarrow x_i$  a.e.; and, as  $x_{in} \rightarrow x_i \in \mathcal{E}_d^{\Phi}(\lambda)$ , by Lemma ??, there exist a subsequence of  $x_{in_k}$  (again denoted  $x_{in_k}$ ) and a function  $h_i \in \Pi(E_1^{\Phi}, \lambda)$  such that  $x_{in_k} \rightarrow u_i$  a.e. and  $|x_{in_k}| \leq h_i$  a.e. Since  $x_{in_k}$ ,  $k = 1, 2, \dots$ , is a strong convergent sequence in  $W^1 L_d^{\Phi_i}$ , it is a bounded sequence in  $W^1 L_d^{\Phi_i}$ . According to Lemma 2.2 and Corollary ??, there exist  $M_i > 0$  such that  $\|a(x_{in_k})\|_{L^\infty} \leq M_i$ ,  $k = 1, 2, \dots$ . From the previous facts and (??), we get

$$|D_{x_i} \mathcal{L}(\cdot, x_{1n_k}, x_{2n_k}, y_{1n_k}, y_{2n_k})| \leq M_i(b + \Phi_i(|h_i|)) \in L^1_1 \quad i = 1, 2.$$

On the other hand, by the continuous differentiability of  $\mathcal{L}$ , we have

$$D_{x_i} \mathcal{L}(t, x_{in_k}(t), y_{in_k}(t)) \rightarrow D_{x_i} \mathcal{L}(t, x_i(t), y_i(t)) \quad \text{for a.e. } t \in [0, T].$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.



*Step 2. The non linear operator  $(x_1, x_2) \mapsto (D_{y_1}\mathcal{L}(t, x_1, y_1), D_{y_2}\mathcal{L}(t, x_2, y_2))$  is continuous from  $\mathcal{E}_d^{\Phi_1}(\lambda) \times \mathcal{E}_d^{\Phi_2}(\lambda)$  with the strong topology into  $X$  with the weak\* topology.*

Note that (??), (??) and the imbeddings  $W^1 L_d^\Phi \hookrightarrow L_d^\infty$  and  $L_d^{\Phi^*} \hookrightarrow [L^\Phi]^*$  imply that the second member of (18) defines an element in  $[W^1 L_d^\Phi]^*$ .

Let  $(x_{1n}, x_{2n}) \in \mathcal{E}_d^\Phi(\lambda)$  such that  $(x_{1n}, x_{2n}) \rightarrow (x_1, x_2)$  in the norm of  $X$ . We must prove that  $D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}) \xrightarrow{w^*} D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2)$  para  $i = 1, 2$ . On the contrary, there exist  $v = (v_1, v_2) \in L^{\Phi_1} \times L^{\Phi_2}$ ,  $\epsilon > 0$  and a subsequence of  $\{x_n\}$  (denoted  $\{x_n\}$  for simplicity) such that

$$|\langle D_{y_i}\mathcal{L}(\cdot, x_{1n}, x_{2n}, y_1, y_2), v \rangle - \langle D_{y_i}\mathcal{L}(\cdot, x_1, x_2, y_1, y_2), v \rangle| \geq \epsilon. \quad (19)$$

We have  $x_n \rightarrow x$  in  $X$  and  $y_n \rightarrow y$  in  $X$ . By Lemma ??, there exist a subsequence  $x_{n_k}$  and a function  $h \in \Pi(E_1^{\Phi_1}, \lambda) \times \Pi(E_1^{\Phi_2}, \lambda)$  such that  $x_{n_k} \rightarrow x$  a.e.,  $y_{n_k} \rightarrow y$  a.e. and  $|y_{n_k}| \leq h$  a.e. As in the previous step, since  $x_n$  is a convergent sequence, the Corollary ?? implies that  $a(|y_n(t)|)$  is uniformly bounded by a certain constant  $M > 0$ . Therefore, with  $x_{n_k}$  instead of  $x$ , inequality (??) becomes Consequently, as  $v \in L^\Phi$  and employing Hölder's inequality, we obtain that

$$\sup_k |D_{\mathbf{y}}\mathcal{L}(\cdot, u_{n_k}, \dot{u}_{n_k}) \cdot v| \in L_1^1.$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\int_0^T D_{\mathbf{y}}\mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \cdot v \, dt \rightarrow \int_0^T D_{\mathbf{y}}\mathcal{L}(t, u, \dot{u}) \cdot v \, dt \quad (20)$$

which contradicts the inequality (19). This completes the proof of step 2.

*Step 3.* We will prove (18). The proof follows similar lines as [Mawhin and Willem, 1989, Thm. 1.4]. For  $u \in \mathcal{E}_d^\Phi(\lambda)$  and  $0 \neq v \in W^1 L_d^\Phi$ , we define the function

$$H(s, t) := \mathcal{L}(t, u(t) + sv(t), \dot{u}(t) + s\dot{v}(t)).$$

From [Krasnosel'skiĭ and Rutickiĭ, 1961, Lemma 10.1] (or [Schappacher, 2005, Thm. 5.5]) we obtain that if  $|u| \leq |v|$  then  $d(u, E^\Phi) \leq d(v, E^\Phi)$ . Therefore, for  $|s| \leq s_0 := (\lambda - d(\dot{u}, E^\Phi)) / \|v\|_{W^1 L^\Phi}$  we have

$$d(\dot{u} + s\dot{v}, E^\Phi) \leq d(|\dot{u}| + s|\dot{v}|, E_1^\Phi) \leq d(|\dot{u}|, E_1^\Phi) + s\|\dot{v}\|_{L^\Phi} < \lambda.$$

Thus  $\dot{u} + s\dot{v} \in \Pi(E^\Phi, \lambda)$  and  $|\dot{u}| + s|\dot{v}| \in \Pi(E_1^\Phi, \lambda)$ . These facts imply, in virtue of Theorem ?? item ??, that  $I(u + sv)$  is well defined and finite for  $|s| \leq s_0$ . And, using Corollary ??, we also see that

$$\|a(|u + sv|)\|_{L^\infty} \leq A(\|u + sv\|_{W^1 L^\Phi}) \leq A(\|u\|_{W^1 L^\Phi} + s_0\|v\|_{W^1 L^\Phi}) =: M$$

Now, applying Chain Rule, (??), (??) the monotonicity of  $\varphi$  and  $\Phi$ , the fact that  $v \in L_d^\infty$  and  $\dot{v} \in L^\Phi$  and Hölder's inequality, we get

$$\begin{aligned} |D_s H(s, t)| &= |D_x \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot v + D_y \mathcal{L}(t, u + sv, \dot{u} + s\dot{v}) \cdot \dot{v}| \\ &\leq M \left[ \left( b(t) + \Phi \left( \frac{|\dot{u}| + s_0|\dot{v}|}{\lambda} + f(t) \right) \right) |v| \right. \\ &\quad \left. + \left( c(t) + \varphi \left( \frac{|\dot{u}| + s_0|\dot{v}|}{\lambda} + f(t) \right) \right) |\dot{v}| \right] \in L_1^1. \end{aligned} \quad (21)$$

Consequently,  $I$  has a directional derivative and

$$\langle I'(u), v \rangle = \frac{d}{ds} I(u + sv) \Big|_{s=0} = \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt.$$

Moreover, from (??), (??), Lemma 2.2 and the previous formula, we obtain

$$|\langle I'(u), v \rangle| \leq \|D_x \mathcal{L}\|_{L^1} \|v\|_{L^\infty} + \|D_y \mathcal{L}\|_{L^{\Phi^*}} \|\dot{v}\|_{L^\Phi} \leq C \|v\|_{W^1 L^\Phi}$$

with a appropriate constant  $C$ . This completes the proof of the Gâteaux differentiability of  $I$ .

*Step 4.* The operator  $I' : \mathcal{E}_d^\Phi(\lambda) \rightarrow [W^1 L_d^\Phi]^*$  is demicontinuous. This is a consequence of the continuity of the mappings  $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$  and  $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$ . Indeed, if  $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$  with  $u_n \rightarrow u$  in the norm of  $W^1 L_d^\Phi$  and  $v \in W^1 L_d^\Phi$ , then

$$\begin{aligned} \langle I'(u_n), v \rangle &= \int_0^T \{ D_x \mathcal{L}(t, u_n, \dot{u}_n) \cdot v + D_y \mathcal{L}(t, u_n, \dot{u}_n) \cdot \dot{v} \} dt \\ &\rightarrow \int_0^T \{ D_x \mathcal{L}(t, u, \dot{u}) \cdot v + D_y \mathcal{L}(t, u, \dot{u}) \cdot \dot{v} \} dt \\ &= \langle I'(u), v \rangle. \end{aligned}$$

In order to prove item ??, it is necessary to see that the maps  $u \mapsto D_x \mathcal{L}(t, u, \dot{u})$  and  $u \mapsto D_y \mathcal{L}(t, u, \dot{u})$  are norm continuous from  $\mathcal{E}_d^\Phi(\lambda)$  into  $L_d^1$  and  $L_d^{\Phi^*}$  respectively. The continuity of the first map has already been proved in step 1. Let  $u_n, u \in \mathcal{E}_d^\Phi(\lambda)$  with  $\|u_n - u\|_{W^1 L^\Phi} \rightarrow 0$ . Therefore, there exist a subsequence  $u_{n_k} \in \mathcal{E}_d^\Phi(\lambda)$  and a function  $h \in \Pi(E_1^\Phi, \lambda)$  such that (??) holds true. And, as  $\Phi^* \in \Delta_2$  then the right hand side of (??) belongs to  $E_1^{\Phi^*}$ . Now, invoking Lemma ??, we prove that from any sequence  $u_n$  which converges to  $u$  in  $W^1 L_d^\Phi$  we can extract a subsequence such that  $D_y \mathcal{L}(t, u_{n_k}, \dot{u}_{n_k}) \rightarrow D_y \mathcal{L}(t, u, \dot{u})$  in the strong topology. The desired result is obtained by a standard argument.

The continuity of  $I'$  follows from the continuity of  $D_x \mathcal{L}$  and  $D_y \mathcal{L}$  using the formula (18).  $\square$

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## References

- [Acinas et al., 2015] Acinas, S., Buri, L., Giubergia, G., Mazzone, F., and Schwindt, E. (2015). Some existence results on periodic solutions of Euler-Lagrange equations in an Orlicz-Sobolev space setting. *Nonlinear Analysis, TMA.*, 125:681 – 698.
- [Desch and Grimmer, 2001] Desch, W. and Grimmer, R. (2001). On the well-posedness of constitutive laws involving dissipation potentials. *Trans. Amer. Math. Soc.*, (353):5095–5120.

## References

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- [Krasnosel'skii et al., 2011] Krasnosel'skii, M., Zabreyko, P., Pustynnik, E., and Sobolevski, P. (2011). *Integral operators in spaces of summable functions*. Mechanics: Analysis. Springer Netherlands.
- [Krasnosel'skiĭ and Rutickiĭ, 1961] Krasnosel'skiĭ, M. A. and Rutickiĭ, J. B. (1961). *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen.
- [Mawhin and Willem, 1989] Mawhin, J. and Willem, M. (1989). *Critical point theory and Hamiltonian systems*. Springer-Verlag, New York.
- [Nguen Hong Thai, 1987] Nguen Hong Thai (1987). The superposition operator in the Orlicz spaces of vector functions. *Dokl. Akad. Nauk BSSR*, 31:197â200.
- [Pluciennik, 1985a] Pluciennik, R. (1985a). Boundedness of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Bull. Pol. Acad. Sci., Math.*, 33:531â540.
- [Pluciennik, 1985b] Pluciennik, R. (1985b). On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 25:321â337.
- [Pluciennik, 1987] Pluciennik, R. (1987). The superposition operator in Musielak-Orlicz spaces of vector-valued functions. Abstract analysis, Proc. 14th Winter Sch., Srní/Czech. 1986, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14, 411-417 (1987).
- [Schappacher, 2005] Schappacher, G. (2005). A notion of Orlicz spaces for vector valued functions. *Appl. Math.*, 50(4):355–386.
- [Skaff, 1969] Skaff, M. S. (1969). Vector valued orlicz spaces. ii. *Pacific J. Math.*, 28(2):413–430.
- [Tang, 1995] Tang, C.-L. (1995). Periodic solutions of non-autonomous second-order systems with  $\gamma$ -quasisubadditive potential. *Journal of Mathematical Analysis and Applications*, 189(3):671–675.
- [Tang, 1998] Tang, C.-L. (1998). Periodic solutions for nonautonomous second order systems with sublinear nonlinearity. *Proc. Amer. Math. Soc.*, 126(11):3263–3270.
- [Tang and Wu, 2001] Tang, C. L. and Wu, X.-P. (2001). Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.*, 259(2):386–397.
- [Tang and Zhang, 2010] Tang, X. and Zhang, X. (2010). Periodic solutions for second-order Hamiltonian systems with a  $p$ -Laplacian. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, 64(1):93–113.
- [Tian and Ge, 2007] Tian, Y. and Ge, W. (2007). Periodic solutions of non-autonomous second-order systems with a  $p$ -Laplacian. *Nonlinear Anal.*, 66(1):192–203.

- [Wu and Tang, 1999] Wu, X.-P. and Tang, C.-L. (1999). Periodic solutions of a class of non-autonomous second-order systems. *J. Math. Anal. Appl.*, 236(2):227–235.
- [Zhao and Wu, 2004] Zhao, F. and Wu, X. (2004). Periodic solutions for a class of non-autonomous second order systems. *J. Math. Anal. Appl.*, 296(2):422–434.