Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Abstract

1 Introduction

In this paper we obtain existence of solutions for systems of equations of the type:

$$\begin{aligned} &\tfrac{d}{dt}D_yLt, ut, u't = D_xLt, ut, u't \quad \text{a.e. } t \in \not\approx, T,\\ &u\not\approx - uT = u'\not\approx - u'T = \not\approx, \end{aligned} \tag{$\boldsymbol{P}$}$$

where the function $L : (\not\approx, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $d \geqslant \not\subset$ (called the *Lagrange function* or *lagrangian*) satisfying that it is measurable in t for each $x, y \in \mathbb{R}^d \times \mathbb{R}^d$ and continuously differentiable in x, y for almost every $t \in (\not\approx, T]$. The unknown function $u : (\not\approx, T] \to \mathbb{R}^d$ is assumed absolutely continuous.

Our approach involves the direct method of the calculus of variations in the framework of *anisotropic Orlicz-Sobolev spaces*. We suggest the articles [**?**] for definitions and main results on anisotropic Orlicz spaces, see also [**?**]. These spaces allow us to unify and extend previous results on existences of solutions for systems like (**??**).

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Through this article we say that a function $\bullet : \mathbb{R}^d \to (\not\approx, +\infty$ is of N_∞ class if $\bullet$ is convex, $\bullet \not\approx = \not\approx$, $\bullet y > \not\approx$ if $y \not\approx$ and $\bullet -y = \bullet y$, and

$$\underset{\cup y \cup \to \infty}{\boxslash\boxtimes\wr} \frac{\bullet y}{\bigcup y \bigcup} = +\infty. \tag{1}$$

where $\bigcup \cdot \bigcup$ denotes the euclidean norm on $\mathbb{R}^d$. From [?, Cor. 2.35] a N_∞ function is continuous.

Associated to $\bullet$ we have the *complementary function* $/\!\!/$ which is defined in $\xi \in \mathbb{R}^d$ as

$$/\!\!/\xi = \underset{y \in \mathbb{R}^d}{\spadesuit\searrow\clubsuit}\, y \cdot \xi - \bullet y \tag{2}$$

then, from the continuity of $\bullet$ and (??), we have that $/\!\!/ : \mathbb{R}^d \to (\not\approx, \infty$. Moreover, it is easy to see that $/\!\!/$ is a convex function such that $/\!\!/ \not\approx = \not\approx$, $/\!\!/ -\xi = /\!\!/ \xi$ [?, Chapter 2]. Moreover $/\!\!/$ satisfies (??) (see [?, Th. 2.2]). i.e. $/\!\!/$ is N_∞ function.

Some examples of N_∞ functions are the following.

Example 1.1. $\bullet_p y := \bigcup y \bigcup^p \Uparrow p$, for $\not\subset < p < \infty$. In this case $/\!\!/\xi = \bigcup \xi \bigcup^q \Uparrow q$, $q = p \Uparrow p - \not\subset$.

Example 1.2. If $\bullet : \mathbb{R} \to (\not\approx, +\infty$ is a N_∞ function on $\mathbb{R}$ then $\overline{\bullet} y = \bullet \bigcup y \bigcup$ is a N_∞ function on $\mathbb{R}^d$. In this example, as in the previous one, the function $\bullet$ is *radial*, i.e. the value of $\bullet y$ depends on the norm of y and not on its direction. These cases are not authentically anisotropic.

Example 1.3. An anisotropic function $\bullet y$ depends on the direction of y. For example, if $\not\subset < p_{\not\subset}, p_{\not\supset} < \infty$, we define $\bullet_{p_{\not\subset}, p_{\not\supset}} : \mathbb{R}^d \times \mathbb{R}^d \to (\not\approx, +\infty$ by

$$\bullet_{p_{\not\subset}, p_{\not\supset}} y_{\not\subset}, y_{\not\supset} := \frac{\bigcup y_{\not\subset} \bigcup^{p_{\not\subset}}}{p_{\not\subset}} + \frac{\bigcup y_{\not\supset} \bigcup^{p_{\not\supset}}}{p_{\not\supset}}.$$

Then $\bullet_{p_{\not\subset}, p_{\not\supset}}$ is a N_∞ function. In this case the complementary function is the function $\bullet_{q_{\not\subset}, q_{\not\supset}}$ with $q_i = p_i \Uparrow p_i - \not\subset$.

More generally, if $\bullet_k : \mathbb{R}^d \to (\not\approx, +\infty$, $k = \not\subset, \dots, n$, are N_∞ functions, then $\bullet : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to (\not\approx, +\infty$ defined by $\bullet y_{\not\subset}, \dots, y_n = \bullet_{\not\subset} y_{\not\subset} + \cdots + \bullet_n y_n$ is a N_∞ function. These functions are truly anisotropic, i.e. $\bigcup x \bigcup = \bigcup y \bigcup$ does not imply that $\bullet x = \bullet y$.

Example 1.4. If $\bullet : \mathbb{R} \to (\not\approx, +\infty$ is a N_∞ function and $O \in GLd, \mathbb{R}$, then $\bullet y = \bullet O y$ is a N_∞ function.

Example 1.5. An anisotropic N_∞ function is not necessarily controlled by powers if it does not satisfy the $\natural$ condition (see xxxxx). For example $\bullet : \mathbb{R}^d :\to (\not\approx, +\infty$ defined by $\bullet y = \leftrightarrow\!\sqcap\!\clubsuit \bigcup y \bigcup - \not\subset$ is N_∞ function.

The occurrence of Orlicz Spaces in this paper obeys to we will consider the following structure condition on the lagrangian:

$$\bigcup L \bigcup + \bigcup \nabla_x L \bigcup + /\!/ \nabla_y L \leqslant ax\, bt + \bullet\, \frac{y}{\lambda} \ , \tag{$\boldsymbol{S}$}$$

for a.e. $t \in \big(\not\approx, T \big\rfloor$, where $a \in C\, \mathbb{R}^d, \big(\not\approx, +\infty$, $b \in L^{\not\subset}\ \big(\not\approx, T \big\rfloor, \big(\not\approx, +\infty$.

Our condition (**??**) includes structure conditions that have previously been considered in the literature. For example, it is easy to see that, when $\bullet x$ is as in Example **??**, then the condition (**??**) is equivalent to the structure condition in [**?**, Th. 1.4]. If $\bullet$ is a radial N_∞ function such that $/\!/$ satisfies that $\flat$ function then (**??**) is essentially equivalent????? to conditions [**?**, Eq. (2)-(4)] (see xxxx mas abajo). If $\bullet$ is as in Example **??** and $L = Lt, x_{\not\subset}, x_{\not\supset}, y_{\not\subset}, y_{\not\supset}$ is a lagrangian with $L : \big(\not\approx, T \big\rfloor \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ then inequality (**??**) is related to estructure conditions like [**?**, Lemma 3.1, Eq. (3.1)]. As can be seen, condition (**??**) is a more compact expression than [**?**, Lemma 3.1, Eq. (3.1)] and moreover weaker, because (**??**) does not imply a control of $\bigcup D_{y_{\not\subset}} L \bigcup$ independent of $y_{\not\supset}$. We will return to this point later.

An important example of lagrangian is giving by:

$$L_{\bullet,F} t, x, y := \bullet y + Ft, x. \tag{3}$$

Here the function Ft, x, which is often referred to potential, be differentiable with respect to x for a.e. $t \in \big(\not\approx, T \big\rfloor$. Moreover F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x \in \mathbb{R}^d$, are Carathéodory functions, i.e. they are measurable functions with respect to $t \in \big(\not\approx, T \big\rfloor$, for every $x \in \mathbb{R}^d$, and they are continuous functions with respect to $x \in \mathbb{R}^d$ for a.e. $t \in \big(\not\approx, T \big\rfloor$.

(A) For a.e. $t \in \big(\not\approx, T \big\rfloor$, it holds that

$$\bigcup Ft, x \bigcup + \bigcup \nabla_x Ft, x \bigcup \leqslant axbt. \tag{4}$$

where $a \in C\, \mathbb{R}^d, \big(\not\approx, +\infty$ and $\not\approx \leqslant b \in L^{\not\subset} \big(\not\approx, T \big\rfloor, \mathbb{R}$.

The lagrangian $L_{\bullet,F}$ satisfies condition (**??**). In order to prove this, the only non trivial fact that we should to establishis is that $/\!/ \nabla_y L \leqslant ax\, bt + \bullet\, y_{\Uparrow} \lambda$. But, from inequality xxxx below, $/\!/ \nabla_y L = /\!/\ \nabla \bullet y \leqslant \bullet \not\supset y$.

The laplacian $L_{\bullet,F}$ leads to the system

$$\begin{array}{l} \frac{d}{dt} \nabla \bullet u' t = \nabla_x Ft, ut \quad \text{a.e. } t \in \not\approx, T, \\ u \not\approx - uT = u' \not\approx - u' T = \not\approx, \end{array} \tag{$\boldsymbol{P_\bullet}$}$$

Problem (**??**) contains, as a particular case, many problems that are usually considered in the literature. For example, the classic book [**?**] deals mainly with problem (**??**), for the lagrangian $L_{\bullet,F}$, with $\bullet x = \bigcup x \bigcup^{\not\supset}_{\Uparrow} \not\supset$, through various methods: direct, dual action, minimax, etc. The results in [**?**] were extended and improved in several articles, see [**?**, **?**, **?**, **?**, **?**] to cite some examples. The case $\bullet y = \bigcup y \bigcup^{p} \Uparrow p$, for arbitrary $\not\subset < p < \infty$ were considered in [**?**, **?**], among other papers, and in this case (**??**) is reduced to the p-laplacian system

$$\begin{cases} \frac{d}{dt}\, u' t \bigcup u' \bigcup^{p-\not\supset} = \nabla F t, u t & \text{a.e. } t \in \not\approx, T \\ u\not\approx - uT = u'\not\approx - u'T = \not\approx. \end{cases} \qquad (\boldsymbol{P_p})$$

If $\bullet$ is as in Example **??** and $F : (\not\approx, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, then the equations (**??**) become

$$\begin{cases} \frac{d}{dt} \bigcup u'_{\not\subset} \bigcup^{p_{\not\subset}-\not\supset} u'_{\not\subset} = F_{x_{\not\subset}} t, u & \text{a.e. } t \in \not\approx, T \\ \frac{d}{dt} \bigcup u'_{\not\supset} \bigcup^{p_{\not\supset}-\not\supset} u'_{\not\supset} = F_{x_{\not\supset}} t, u & \text{a.e. } t \in \not\approx, T \\ u\not\approx - uT = u'\not\approx - u'T = \not\approx, \end{cases} \qquad (\boldsymbol{P_{p_{\not\subset},p_{\not\supset}}})$$

where $x = x_{\not\subset}, x_{\not\supset} \in \mathbb{R}^d \times \mathbb{R}^d$ and $ut = u_{\not\subset} t, u_{\not\supset} t \in \mathbb{R}^d \times \mathbb{R}^d$. In the literature, these equations are known as $p_{\not\subset}, p_{\not\supset}$-Laplacian system, see [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

In conclusion, the problem (**??**) with conditions (**??**) contains several problems that have been considered by many authors in the past. Moreover, our results still improve some results on $p_{\not\subset}, p_{\not\supset}$-lamplacian since our structure conditions are less restrictive even in that particular case.

2 Anisotropic Orlicz and Orlicz-Sobolev spaces

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N_∞ functions $\bullet : \mathbb{R}^n \to (\not\approx, +\infty$. References for these topics are [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

If $\bullet$ is a N_∞ function then from convexity and $\bullet\not\approx = \not\approx$ we obtain that

$$\bullet \lambda x \leqslant \lambda \bullet x, \quad \lambda \in (\not\approx, \not\subset], x \in \mathbb{R}^d. \tag{5}$$

One of the greatest difficulties when dealing with anisotropic Orlicz spaces is the lack of monotony with respect to the Euclidean norm, i.e. $\bigcup x \bigcup \leqslant \bigcup y \bigcup$ does not imply $\bullet x \leqslant \bullet y$. This problem is avoided if we consider functions whose values on a sphere are comparable (see[**?**]). However, from (**??**), we see that N_∞ functions have the following form of radial monotony: $\bigcup x \bigcup \leqslant \bigcup y \bigcup$ and $y = \lambda x$ imply $\bullet x \leqslant \bullet y$.

We say that $\bullet : \mathbb{R}^d \to (\not\approx, +\infty$ satisfies the $\natural^{\infty}_{\not\supset}$*-condition*, denoted by $\bullet \in \natural^{\infty}_{\not\supset}$, if there exist constants $K > \not\approx$ and $M \geqslant \not\approx$ such that

$$\bullet \not\supset x \leqslant K \bullet x, \tag{6}$$

for every $\bigcup x \bigcup \geqslant M$. If $\bullet$ es a $\natural_{\not\supset}$ function then $\bullet$ is bounded by powers functions (see [**?**, Proof Lemma 2.4] and [**?**, Prop. 1]), i.e. there exists $\not\subset < p < \infty$, $C > \not\approx$ and $r \geqslant \not\approx$ such that

$$\bullet x \leqslant C \bigcup x \bigcup^{p}, \quad \bigcup x \bigcup \geqslant r_{\not\approx}.$$

We consider that one of the most important aspects in considering N_∞ functions is that it accounts for the Lagrange functions that present faster growth than powers, for example an exponential growth. Hence we consider it important to avoid imposing hypothesis that $\bullet$ to be $\natural_{\not\supset}$. For some results we will need that // to be $\natural_{\not\supset}$.

Let $\bullet_{\not\subset}$ and $\bullet_{\not\supset}$ be N_∞ functions. Following to [**?**] we wrtite $\bullet \underset{\odot}{} \bullet_{\not\supset}$,

We denote by $M := M \left(\not\approx, T\right], \mathbb{R}^d$, with $d \geqslant \not\subset$, the set of all measurable functions (i.e. functions which are limits of simple functions) defined on $\left(\not\approx, T\right]$ with values on $\mathbb{R}^d$ and we write $u = u_{\not\subset}, \ldots, u_d$ for $u \in M$.

Given an N_∞ function $\bullet$ we define the *modular function* $\rho_\bullet : M \to \mathbb{R}^+ \cup +\infty$ by

$$\rho_\bullet u := \mathcal{R}_{\not\approx}^{T} \bullet u \; dt.$$

Now, we introduce the *Orlicz class* $C^\bullet = C^\bullet \left(\not\approx, T\right], \mathbb{R}^d$ by setting

$$C^\bullet := u \in M \bigcup \rho_\bullet u < \infty \; . \tag{7}$$

The *Orlicz space* $L^\bullet = L^\bullet \left(\not\approx, T\right], \mathbb{R}^d$ is the linear hull of $C^\bullet$; equivalently,

$$L^\bullet := u \in M \bigcup \exists \lambda > \not\approx : \rho_\bullet \lambda u < \infty \; . \tag{8}$$

The Orlicz space $L^\bullet$ equipped with the *Luxemburg norm*

$$\prod u \prod^{L^\bullet} := ⊞\lambda\rightsquigarrow \lambda \bigcup \rho_\bullet \frac{v}{\lambda} \; dt \leqslant \not\subset \; ,$$

is a Banach space.

The subspace $E^\bullet = E^\bullet \left(\not\approx, T\right], \mathbb{R}^d$ is defined as the closure in $L^\bullet$ of the subspace $L^\infty \left(\not\approx, T\right], \mathbb{R}^d$ of all $\mathbb{R}^d$-valued essentially bounded functions. It is shown that (see [**?**, Thm. 5.1]) $u \in E^\bullet$ if and only if $\rho_\bullet \lambda u < \infty$ for any $\lambda > \not\approx$. The equality $L^\bullet = E^\bullet$ is true if and only if $\bullet \in \natural_{\not\supset}^{\infty}$ (see [**?**, Thm. 5.2]).

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [**?**, Thm. 7.2]). Namely, if $u \in L^\bullet$ and $v \in L^{//}$ then $u \cdot v \in L^{\not\subset}$ and

$$\mathcal{R}_{\not\approx}^{T} v \cdot u \; dt \leqslant \not\supset \prod u \prod^{L^\bullet} \prod v \prod^{L^{//}}. \tag{9}$$

By $u \cdot v$ we denote the usual dot product in $\mathbb{R}^d$ between u and v.

We consider the subset $\Vdash E^\bullet, r$ of $L^\bullet$ given by

$$\Vdash E^\bullet, r := u \in L^\bullet \bigcup du, E^\bullet < r.$$

This set is related to the Orlicz class $C^{\bullet}$ by means of inclusions, namely,

$$\Vdash E^{\bullet}, r \subset rC^{\bullet} \subset \overline{\Vdash E^{\bullet}, r} \tag{10}$$

for any positive r. This relation is a trivial generalization of [**?**, Thm. 5.6]. If $\bullet \in \wp$, then the sets $L^{\bullet}$, $E^{\bullet}$, $\Vdash E^{\bullet}, r$ and $C^{\bullet}$ are equal.

As usual, if $X, \prod \cdot \prod_X$ is a normed space and $Y, \prod \cdot \prod_Y$ is a linear subspace of X, we write $Y \hookrightarrow X$ and we say that Y is *embedded* in X when there exists $C > \not\approx$ such that $\prod y \prod_X \leqslant C \prod y \prod_Y$ for any $y \in Y$. With this notation, Hölder's inequality states that $L^{\bullet} \hookrightarrow \left(L^{/\!/}\right\rfloor^*$, where a function $v \in L^{\bullet}$ is associated to $\xi_v \in \left(L^{/\!/}\right\rfloor^*$ being

$$\xi_v u = \coprod \xi_v, u \tilde{} = \mathcal{R}_{\not\approx}^T v \cdot u \, dt, \tag{11}$$

We highlight the following result that is a consequence of Theorems 7.1 and 7.3 in [**?**].

Proposition 2.1. *If $/\!/$ satisfies the $\wp$-condition then* $L^{\bullet}\ (\not\approx, T\rfloor, \mathbb{R}^d = \left(L^{/\!/}\ (\not\approx, T\rfloor, \mathbb{R}^d\right\rfloor^*$.

We define the *Sobolev-Orlicz space* $W^{\not\subset}L^{\bullet}$ by

$$W^{\not\subset}L^{\bullet}\ (\not\approx, T\rfloor, \mathbb{R}^d := u \bigcup u \in AC\ (\not\approx, T\rfloor, \mathbb{R}^d \text{ and } u' \in L^{\bullet}\ (\not\approx, T\rfloor, \mathbb{R}^d\ ,$$

where $AC\ (\not\approx, T\rfloor, \mathbb{R}^d$ denotes the space of all $\mathbb{R}^d$ valued absolutely continuous functions defined on $(\not\approx, T\rfloor$. The space $W^{\not\subset}L^{\bullet}\ (\not\approx, T\rfloor, \mathbb{R}^d$ is a Banach space when equipped with the norm

$$\prod u \prod_{W^{\not\subset}L^{\bullet}} = \prod u \prod_{L^{\bullet}} + \prod u' \prod_{L^{\bullet}}. \tag{12}$$

We introduce the following subspaces of $W^{\not\subset}L^{\bullet}$

$$\begin{aligned} W^{\not\subset}E^{\bullet} &= u \in W^{\not\subset}L^{\bullet} \bigcup u' \in E^{\bullet}, \\ W^{\not\subset}E^{\bullet}_T &= u \in W^{\not\subset}E^{\bullet} \bigcup u\not\approx = uT. \end{aligned} \tag{13}$$

In order to find a modulus of continuity for functios in $W^{\not\subset}L^{\bullet}$, and from there, to obtain compact embedding of $W^{\not\subset}L^{\bullet}$, we define the function $A_{\bullet} : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$A_{\bullet} s = \wr\boxplus\lambda \bullet x \bigcup \bigcup x \bigcup = s\ , \tag{14}$$

Let us establish some elementary properties of $A_{\bullet}$.

Proposition 2.2. *The function $A_{\bullet}$ has the following properties:*

1. *$A_{\bullet}$ is continuous,*

2. *$A_\bullet s_{\Uparrow} s$ is increasing,*
3. *$A_\bullet \bigcup x \bigcup$ is the* greatest radial minorant *of* $\bullet x$*,*
4. *$\bullet$ is N_∞ if and only if* $⊠⊞≀_{s\to+\infty} A_\bullet s_{\Uparrow} s = +\infty$.

Proof. It is well known that finite and convex functions defined on finite dimensional vector spaces are locally Lipschitz functions (see [**?**]). This fact implies item **??** immediately.

In order to prove item **??**, suppose $\not\approx < r < s$ and $x \in \mathbb{R}^d$ with $A_\bullet s = \bullet x$. Then, from the definition of $A_\bullet$ and the convexity of $\bullet$,

$$\frac{A_\bullet r}{r} \leqslant \frac{\bullet\ \frac{r}{s}x}{r} \leqslant \frac{\bullet\ x}{s} = \frac{A_\bullet s}{s}.$$

Property in items **??** and **??** are obtained easily. □

Example 2.1. We compute $A_\bullet$ for the function $\bullet = \bullet_{p_{\not\subset},p_{\not\supset}}$ given in Example (**??**). We apply the method of Lagrange multipliers (see [**?**, Ch. 11]) to solve the next minimization problem subject to constraints

$$\begin{array}{l} \text{minimize } \bullet_{p_{\not\subset},p_{\not\supset}} y_{\not\subset}, y_{\not\supset} \\ \text{subject to } \bigcup y_{\not\subset} \bigcup^{\not\supset} + \bigcup y_{\not\supset} \bigcup^{\not\supset} = r^{\not\supset} \end{array} \quad .$$

The first order conditions are

$$\begin{array}{rcl} \bigcup y_{\not\subset} \bigcup^{p_{\not\subset}-\not\supset} y_{\not\subset} + \lambda y_{\not\subset} & = & \not\approx \\ \bigcup y_{\not\supset} \bigcup^{p_{\not\supset}-\not\supset} y_{\not\supset} + \lambda y_{\not\supset} & = & \not\approx \\ \bigcup y_{\not\subset} \bigcup^{\not\supset} + \bigcup y_{\not\supset} \bigcup^{\not\supset} & = & r^{\not\supset} \end{array} \tag{15}$$

These equations are solved, among others, by the following two sets of citical points: a) $\bigcup x \bigcup = r$, $y = \not\approx$ and $\lambda = -r^{p_{\not\subset}-\not\supset}$ and b) $x = \not\approx$, $\bigcup y \bigcup = r$ and $\lambda = -r^{p_{\not\supset}-\not\supset}$. These sets are infinite when $d > \not\subset$. Associated with these critical points we have the following critical values: a) $r^{p_{\not\subset}}{}_{\Uparrow} p_{\not\subset}$ and b) $r^{p_{\not\supset}}{}_{\Uparrow} p_{\not\supset}$.

We deal with $p_{\not\subset} \leqslant \not\supset$ and $p_{\not\supset} \leqslant \not\supset$ being one of them (suppose $p_{\not\supset}$) different from 2. The remaining cases can be treated with similar techniques.

If $y_{\not\subset}, y_{\not\supset}$ solve (**??**) with $y_{\not\subset} \,/\!\!\!\not\approx$ and $y_{\not\supset} \,/\!\!\!\not\approx$ then $\bigcup y_{\not\supset} \bigcup = \bigcup y_{\not\subset} \bigcup^{\frac{p_{\not\subset}-\not\supset}{p_{\not\supset}-\not\supset}}$ and $\lambda = -\bigcup y_{\not\subset} \bigcup^{p_{\not\subset}-\not\supset}$.

We use second order conditions for constrained problems. We have to consider the tangent plane at the point $y_{\not\subset}, y_{\not\supset} \in \mathbb{R}^{\not\supset n}$, i.e. $M = \xi, \eta \in \mathbb{R}^{\not\supset n} : \xi y_{\not\subset}^t + \eta y_{\not\supset}^T = \not\approx$. Let L be the Lagrangian associated to the constrained problem: $L y_{\not\subset}, y_{\not\supset}, \lambda = \bullet y_{\not\subset}, y_{\not\supset} + \lambda H y_{\not\subset}, y_{\not\supset}$ being $H = \not\approx$ the constraint. We must analize the positivity of the quadratic form associated to the matrix of second partial derivatives $H = D^{\not\supset}\bullet + \lambda D^{\not\supset} H$ on the subspace M. By elementary computations we have for $\xi, \eta \in M$

$$\xi, \eta^t H \xi, \eta = \bigcup \lambda \bigcup \xi^t x^{\not\supset} \left(\bigcup y_{\not\subset} \bigcup^{-\not\supset} p_{\not\subset} - \not\supset + p_{\not\supset} - \not\supset \bigcup y_{\not\supset} \bigcup^{-\not\supset}\right],$$

on the subspace M. We note that $-y_{⊅}, y_{⊄} \in M$ and $-y_{⊅}, y_{⊄}{}^{t} H - y_{⊅}, y_{⊄} < ≉$. Then, by second order necessary conditions [**?**, p.333], at $y_{⊄}, y_{⊅}$ there cannot be a minimum. Therefore, the only minima occur at $y_{⊄} = ≉$ or $y_{⊅} = ≉$, then

$$A_{●} x, y = ≀⊞\lambda r^{p_{⊄}} {}_{⇑} p_{⊄}, r^{p_{⊅}} {}_{⇑} p_{⊅}.$$

More generally, it holds that

$$K_{⊄} ≀⊞\lambda r^{p_{⊄}}, r^{p_{⊅}} \leqslant A_{●} \leqslant K_{⊅} ≀⊞\lambda r^{p_{⊄}}, r^{p_{⊅}}$$

with $K_{⊄}, K_{⊅} > ≉$, for every $⊄ < p_{⊄}, p_{⊅} < \infty$.

As is customary, we will use the decomposition $u = \overline{u} + u$ for a function $u \in L^{⊄} \big(≉, T \big]$ where $\overline{u} = \frac{⊄}{T} \mathcal{R}_{≉}^{T} ut\, dt$ and $u = u - \overline{u}$.

The following lemma is an elementary generalization to anisotropic Sobolev-Orlicz spaces of known results of Sobolev spaces.

Lemma 2.3. *Let* $● : \mathbb{R}^d \to \big(≉, +\infty$ *be a Young's function and let* $u \in W^{⊄} L^{●} \big(≉, T \big], \mathbb{R}^d$. *Let* $A_{●} : \mathbb{R}^+ \to \mathbb{R}^+$ *be the function defined by* (**??**). *Then*

1. *For every* $s, t \in \big(≉, T \big], s \,\big/\, t$,

$$\bigcup^{ut - us}\bigcup \leqslant \prod^{u'}\prod^{L^{●}} \bigcup^{s-t}\bigcup A_{●}^{-⊄} \frac{⊄}{\bigcup^{s-t}\bigcup} \qquad \text{(Morrey's inequality)}$$

$$\prod^{u}\prod^{L^{\infty}} \leqslant A_{●}^{-⊄} \frac{⊄}{T} ≀⊄\Pi⊄, T \prod^{u}\prod^{W^{⊄}L^{●}} \qquad \text{(Sobolev's inequality)}$$

2. *We have* $u \in L^{\infty} \big(≉, T \big], \mathbb{R}^d$ *and*

$$\prod^{u}\prod^{L^{\infty}} \leqslant T A_{●}^{-⊄} \frac{⊄}{T} \prod^{u'}\prod^{L^{●}} \qquad \text{(Sobolev-Wirtinger's inequality)}$$

3. *If* $●$ *is* N_{∞} *then the space* $W^{⊄} L^{●} \big(≉, T \big], \mathbb{R}^d$ *is compactly embedded in the space of continuous functions* $C \big(≉, T \big], \mathbb{R}^d$.

Proof. By the absolutely continuity of u, Jensen's inequality and the definition of the Luxemburg norm, we have

$$\begin{aligned} ● \frac{ut - us}{\prod^{u'}\prod^{L^{●}} \bigcup^{s-t}\bigcup} &\leqslant ● \frac{⊄}{\bigcup^{s-t}\bigcup} \mathcal{R}_{s}^{t} \frac{u'r}{\prod^{u'}\prod^{L^{●}}} dr \\ &\leqslant \frac{⊄}{\bigcup^{s-t}\bigcup} \mathcal{R}_{s}^{t} ● \frac{u'r}{\prod^{u'}\prod^{L^{●}}}\, dr \leqslant \frac{⊄}{\bigcup^{s-t}\bigcup}. \end{aligned}$$

By Proposition **??**(**??**) we have $A_{\bullet}^{-\not\subset}\bullet x \geqslant \bigcup^{x}\bigcup$, therefore we get

$$\frac{\bigcup^{ut-us}\bigcup}{\prod^{u'}\prod^{L^{\bullet}}\bigcup^{s-t}\bigcup} \leqslant A_{\bullet}^{-\not\subset}\ \frac{\not\subset}{\bigcup^{s-t}\bigcup},$$

then **??** holds.

Now, we use **??** and Proposition **??** (**??**) and we have

$$\begin{aligned}\bigcup ut-\overline{u}\bigcup &= \bigcup \frac{\not\subset}{T}\mathcal{R}_{\not\approx}^{T}\, ut-us\, ds\bigcup \\ &\leqslant \frac{\not\subset}{T}\mathcal{R}_{\not\approx}^{T}\bigcup^{ut-us}\bigcup ds \\ &\leqslant \prod^{u'}\prod^{L^{\bullet}} TA_{\bullet}^{-\not\subset}\ \frac{\not\subset}{T}\end{aligned}$$

In order to prove the Sobolev's inequality, we note that, using Jensen's inequality and the definition of $\prod^{u}\prod^{L^{\bullet}}$, we obtain

$$\bullet\ \frac{\overline{u}}{\prod^{u}\prod^{L^{\bullet}}} \leqslant \frac{\not\subset}{T}\mathcal{R}_{\not\approx}^{T}\bullet\ \frac{us}{\prod^{u}\prod^{L^{\bullet}}}\ ds \leqslant \frac{\not\subset}{T}$$

Then by By Proposition **??**(**??**)

$$\bigcup^{\overline{u}}\bigcup \leqslant A_{\bullet}^{-\not\subset}\ \frac{\not\subset}{T}\prod^{u}\prod^{L^{\bullet}}.$$

Therefore, from this and (**??**) we get

$$\begin{aligned}\prod^{u}\prod^{L^{\infty}} &\leqslant \bigcup^{\overline{u}}\bigcup + \prod^{\mathbb{I}}\prod^{L^{\infty}} \\ &\leqslant A_{\bullet}^{-\not\subset}\ \frac{\not\subset}{T}\prod^{u}\prod^{L^{\bullet}} + TA_{\bullet}^{-\not\subset}\ \frac{\not\subset}{T}\prod^{u'}\prod^{L^{\bullet}} \\ &\leqslant A_{\bullet}^{-\not\subset}\ \frac{\not\subset}{T}\ \wr\not\subset\pi\not\subset, T\prod^{u}\prod^{W^{\not\subset}L^{\bullet}}\end{aligned}$$

In order to prove item 3, we take a bounded sequence u_n in $W^{\not\subset}L^{\bullet}\left(\not\approx, T\right]$, $\mathbb{R}^d$. Since $\bullet$ is N_∞, from Proposition **??**(**??**) we obtain $sA_{\bullet}^{-\not\subset}\not\subset_{\Uparrow} s \to \not\approx$ when $s \to \not\approx$. Therefore (**??**) implies that u_n are equicontinuous. Furthermore (**??**) implies that u_n is bounded in $C\left(\not\approx, T\right]$, $\mathbb{R}^d$. Therefore by the Arzela-Ascoli Theorem we obtain a subsequence n_k and $u \in C\left(\not\approx, T\right]$, $\mathbb{R}^d$ with $u_{n_k} \to u$ in $C\left(\not\approx, T\right]$, $\mathbb{R}^d$.

□

Lemma 2.4. *Let $u_{nn\in\mathbb{N}}$ be a sequence of functions in $\mathbb{E}^{\bullet}, \not\subset$ converging to $u \in \mathbb{E}^{\bullet}, \not\subset$ in the $L^{\bullet}$-norm. Then, there exist a subsequence u_{n_k} and a real valued function $h \in L^{\not\subset}\big(\not\approx, T\big], \mathbb{R}$ such that $u_{n_k} \to u$ a.e. and $\bullet u_{n_k} \leqslant h$ a.e.*

Proof. Since $du, E^{\bullet} < \not\subset$ and u_n converges to u, there exists $u_{\not\approx} \in E^{\bullet}$, a subsequence of u_n (again denoted u_n) and $\not\approx < r < \not\subset$ such that $du_n, u_{\not\approx} < r$. Let $\lambda_{\not\approx} \in r, \not\subset$. By extracting more subsequences, if necessary, we can assume that $u_n \to u$ a.e. and

$$\lambda_n := \prod u_{n+\not\subset} - u_n \prod_{L^{\bullet}} < \frac{\not\subset - \lambda_{\not\approx}}{\not\supset^n}, \quad \text{for } n \geqslant \not\subset.$$

We can assume $\lambda_n > \not\approx$ for every $n = \not\approx, \ldots$.

Let $\lambda := \not\subset - \mathcal{P}_{n=\not\approx}^{\infty} \lambda_n$ and define $h : \big(\not\approx, T\big] \to \mathbb{R}$ by

$$hx = \lambda \bullet \frac{u_{\not\approx}}{\lambda} + \mathcal{P}_{n=\not\approx}^{\infty} \lambda_n \bullet \frac{u_{n+\not\subset} - u_n}{\lambda_n} . \tag{16}$$

Note that $\mathcal{P}_{n=\not\approx}^{\infty} \lambda_n + \lambda = \not\subset$, therefore for any $n = \not\subset, \ldots$

$$\begin{aligned} \bullet u_n &= \bullet\ \lambda \frac{u_{\not\approx}}{\lambda} + \mathcal{P}_{j=\not\approx}^{n-\not\subset} \lambda_j \frac{u_{j+\not\subset} - u_j}{\lambda_j} \\ &\leqslant \lambda \bullet \frac{u_{\not\approx}}{\lambda} + \mathcal{P}_{j=\not\approx}^{n-\not\subset} \lambda_j \bullet \frac{u_{j+\not\subset} - u_j}{\lambda_j} \leqslant h \end{aligned}$$

Since $u_{\not\approx} \in E^{\bullet} \subset C^{\bullet}$ and $E^{\bullet}$ is a subspace we have that $\bullet u_{\not\approx} \Uparrow \lambda \in L^{\not\subset}\big(\not\approx, T\big], \mathbb{R}$. On the other hand $\prod u_{n+\not\subset} - u_n \prod_{L^{\bullet}} \leqslant \lambda_n$, therefore

$$\mathcal{R}_{\not\approx}^{T} \bullet \frac{u_{j+\not\subset} - u_j}{\lambda_j}\, dt \leqslant \not\subset.$$

Then $h \in L^{\not\subset}\big(\not\approx, T\big], \mathbb{R}$.

□

3 Differentiability Gateâux of action integrals in anisotropic Orlicz spaces

Given a continuous function $a \in C\mathbb{R}^n, \mathbb{R}^+$, we define the composition operator $a : M_d \to M_d$ by $\boldsymbol{a}ux = aux$.

We will often use the following result whose proof can be performed as that of Corollary 2.3 in [**?**].

Lemma 3.1. *If $a \in C\mathbb{R}^d, \mathbb{R}^+$ then $\boldsymbol{a} : W^{\not\subset}L^{\bullet} \to L^{\infty}\big(\not\approx, T\big]$ is bounded. More concretely, there exists a non decreasing function $A : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\prod \boldsymbol{a}u \prod_{L^{\infty}(\not\approx, T]} \leqslant A \prod u \prod_{W^{\not\subset}L^{\bullet}}$.*

Proof. Let $A \in C\mathbb{R}^+, \mathbb{R}^+$ be a non decreasing, continuous function defined by $\alpha s :=$ ♤⇘♧$\prod\prod^{x}{}^{\leqslant s, x\in\mathbb{R}^d} \bigcup^{ax}\bigcup$. If $u \in W^{\not\subset}L^{\bullet}_d$ then, by **??**, for a.e. $t \in (\not\approx, T\rfloor$

$$aut \leqslant \alpha \prod^{u}\prod^{L^\infty} \leqslant \alpha\ A^{-\not\subset}_{\bullet} \frac{\not\subset}{T} \wr\not\subset\Pi\not\subset, T \prod^{u}\prod^{W^{\not\subset}L^{\bullet}} =: A \prod^{u}\prod^{W^{\not\subset}L^{\bullet}}.$$

□

HABRÍA QUE VER DÓNDE SE UBICA LA CONDICIÓN DE ESTRUCTURA...QUIZÁS EN LA INTRODUCCIÓN?....

Next, we deal with the differentiability of the action integral

$$Iu = \mathcal{R}^T_{\not\approx} Lt, ut, \dot{u}t\ dt. \tag{17}$$

Theorem 3.2. *Let L be a differentiable Carathéodory function satisfying* (**??**). *Then the following statements hold:*

1. *The action integral given by* (**??**) *is finitely defined on* $E^{\bullet} := W^{\not\subset}L^{\bullet} \cap u \bigcup \dot{u} \in \Vdash E^{\bullet}, \not\subset$.
2. *The function I is Gâteaux differentiable on* $E^{\bullet}$ *and its derivative* I' *is demicontinuous from* $E^{\bullet}$ *into* $(W^{\not\subset}L^{\bullet}\rfloor^*$. *Moreover,* I' *is given by the following expression*
$$\coprod^{I'u, v^{\frown}} = \mathcal{R}^T_{\not\approx}\ D_xLt, u, \dot{u}\cdot v + D_yLt, u, \dot{u}\cdot\dot{v}\ \ dt. \tag{18}$$
3. *If* $/\!/ \in \natural$ *then* I' *is continuous from* $E^{\bullet}$ *into* $(W^{\not\subset}L^{\bullet}\rfloor^*$ *when both spaces are equipped with the strong topology.*

Proof. Let $u \in E^{\bullet}$. As

$$\dot{u} \in \Vdash E^{\bullet}, \not\subset \subset C^{\bullet}_{\not\subset} \tag{19}$$

and (**??**), then $\bullet\dot{u}t \in L^{\not\subset}$. Now,

$$\bigcup^{L\cdot, u, \dot{u}}\bigcup + \bigcup^{D_xL\cdot, u, \dot{u}}\bigcup + /\!/D_yL\cdot, u, \dot{u} \leqslant A \prod^{u}\prod^{W^{\not\subset}L^{\bullet}} b + \bullet\dot{u} \in L^{\not\subset}, \tag{20}$$

by (**??**) and Lemma **??**. Thus item (**??**) is proved.

We split up the proof of item **??** into four steps.

Step 1. The non linear operator $u \mapsto D_xLt, u, \dot{u}$ *is continuous from* $E^{\bullet}$ *into* $L^{\not\subset}(\not\approx, T\rfloor$ *with the strong topology on both sets.*

Let $u_{nn\in\mathbb{N}}$ be a sequence of functions in $E^{\bullet}$ and let $u \in E^{\bullet}$ such that $u_n \to u$ in $W^{\not\subset}L^{\bullet}$. By (**??**), we have

$$\bigcup^{u_nt - ut}\bigcup \leqslant TA^{-\not\subset}_{\bullet} \frac{\not\subset}{T} \prod^{u_n - u}\prod^{L^{\bullet}}$$

then $u_n \to u$ uniformly. As $\dot u_n \to \dot u \in E^\bullet$, by Lemma **??**, there exist a subsequence of $\dot u_{n_k}$ (again denoted $\dot u_{n_k}$) and a function $h \in L^{\not\subset}(\not\approx, T\rfloor, \mathbb{R}$ such that $\dot u_{n_k} \to \dot u$ a.e. and $\bullet \dot u_{n_k} \leqslant h$ a.e.

Since u_{n_k}, $k = \not\subset, \not\supset, \ldots$, is a strong convergent sequence in $W^{\not\subset}L^\bullet$, it is a bounded sequence in $W^{\not\subset}L^\bullet$. According to item (**??**) of Lemma **??**, there exists $M > \not\approx$ such that $\prod \boldsymbol{a} u_{n_k} \prod_{L^\infty} \leqslant M$, $k = \not\subset, \not\supset, \ldots$. From the previous facts and (**??**), we get

$$\bigcup D_x L\cdot, u_{n_k}, \dot u_{n_k} \bigcup \leqslant a \bigcup u_{n_k} \bigcup b + \bullet \dot u_{n_k} \leqslant Mb + h \in L^{\not\subset}.$$

On the other hand, by the continuous differentiability of L, we have

$$D_x Lt, u_{n_k} t, \dot u_{n_k} t \to D_x Lt, ut, \dot u t \quad \text{for a.e. } t \in (\not\approx, T\rfloor.$$

Applying the Dominated Convergence Theorem we conclude the proof of step 1.

Step 2. The non linear operator $u \mapsto D_y Lt, u, \dot u$ is continuous from $E^\bullet$ with the strong topology into $(L^\bullet\rfloor^*$ with the weak* topology.

Let $u \in E^\bullet$. From (**??**) it follows that

$$D_y L\cdot, u, \dot u \in C^{/\!/}. \tag{21}$$

Así? o conviene poner la cota de $/\!/ D_y$ explícitamente???

Note that (**??**), (**??**) and the imbeddings $W^{\not\subset}L^\bullet \hookrightarrow L^\infty$ and $L^{/\!/} \hookrightarrow (L^\bullet\rfloor^*$ imply that the second member of (**??**) defines an element of $(W^{\not\subset}L^\bullet\rfloor^*$.

Let $u_n, u \in E^\bullet$ such that $u_n \to u$ in the norm of $W^{\not\subset}L^\bullet$. We must prove that $D_y L\cdot, u_n, \dot u_n \xrightharpoonup{w^*} D_y L\cdot, u, \dot u$. On the contrary, there exist $v \in L^\bullet$, $\epsilon > \not\approx$ and a subsequence of u_n (denoted u_n for simplicity) such that

$$\bigcup \coprod D_y L\cdot, u_n, \dot u_n, v^{\sim} - \coprod D_y L\cdot, u, \dot u, v^{\sim} \bigcup \geqslant \epsilon. \tag{22}$$

We have $u_n \to u$ in $L^\bullet$ and $\dot u_n \to \dot u$ in $L^\bullet$. By Lemma **??**, there exist a subsequence of u_n (again denoted u_n for simplicity) and a function $h \in L^{\not\subset}(\not\approx, T\rfloor, \mathbb{R}$ such that $u_n \to u$ uniformly, $\dot u_n \to \dot u$ a.e. and $\bullet \dot u_n \leqslant h$ a.e. As in the previous step, since u_n is a convergent sequence, Lemma **??** implies that $a \bigcup u_n t \bigcup$ is uniformly bounded by a certain constant $M > \not\approx$. Therefore, from inequality (**??**) with u_n instead of u, we have

$$/\!/ D_y L\cdot, u_n, \dot u_n \leqslant Mb + h \in L^{\not\subset}. \tag{23}$$

As $v \in L^\bullet$ there exists $\lambda > \not\approx$ such that $\bullet \frac{v}{\lambda} \in L^{\not\subset}$. Now, by Young inequality and (**??**), we have

$$\begin{aligned} &\lambda D_y L\cdot, u_{n_k}, \dot u_{n_k} \cdot \frac{vt}{\lambda} \\ &\leqslant \lambda \left(/\!/ D_y L\cdot, u_{n_k}, \dot u_{n_k} + \bullet \frac{v}{\lambda} \right\rfloor \\ &\leqslant \lambda M b + h + \lambda \bullet \frac{v}{\lambda} \in L^{\not\subset} \end{aligned} \tag{24}$$

Finally, from the Lebesgue Dominated Convergence Theorem, we deduce

$$\mathcal{R}_{\not\approx}^{T} D_y Lt, u_{n_k}, \dot{u}_{n_k} \cdot v\, dt \to \mathcal{R}_{\not\approx}^{T} D_y Lt, u, \dot{u} \cdot v\, dt \tag{25}$$

which contradicts the inequality (**??**). This completes the proof of step 2.

Step 3. We will prove (**??**). For $u \in E^{\bullet}$ and $\not\approx v \in W^{\not\subset}L^{\bullet}$, we define the function

$$Hs, t := Lt, ut + svt, \dot{u}t + s\dot{v}t.$$

For $\bigcup s \bigcup \leqslant s_{\not\approx} := \wr\boxtimes\lambda\not\subset - d\dot{u}, E^{\bullet} \Uparrow \prod\prod v \prod\prod_{W^{\not\subset}L^{\bullet}}, \not\subset - d\dot{u}, E^{\bullet}$, using triangle inequality we get $d\ \dot{u} + s\dot{v}, E^{\bullet} < \not\subset$ and thus $\dot{u} + s\dot{v} \in \Vdash E^{\bullet}, \not\subset$. These facts imply, in virtue of Theorem **??** item **??**, that $Iu + sv$ is well defined and finite for $\bigcup s \bigcup \leqslant s_{\not\approx}$.

We also have $\prod u + sv \prod_{W^{\not\subset}L^{\bullet}} \leqslant \prod\prod u \prod\prod_{W^{\not\subset}L^{\bullet}} + s_{\not\approx} \prod\prod v \prod\prod_{W^{\not\subset}L^{\bullet}} \not\approx$ then, by Lemma **??**, there exists $M > \not\approx$ such that $\prod au + sv \prod_{L^{\infty}} \leqslant M$.

Let $\lambda > \not\approx$ such that $\bullet \frac{\dot{v}}{\lambda} \in L^{\not\subset}$. On the other hand, if $\dot{v} \in L^{\bullet}$ and $\bigcup s \bigcup \leqslant s_{\not\approx}\lambda^{-\not\subset}$, from the convexity and the parity of $\bullet$, we get

$$\begin{aligned}\bullet\dot{u} + s\dot{v} &= \bullet\ \not\subset - s_{\not\approx}\frac{\dot{u}}{\not\subset - s_{\not\approx}} + s_{\not\approx}\frac{s}{s_{\not\approx}}\dot{v} \leqslant \not\subset - s_{\not\approx}\bullet\ \frac{\dot{u}}{\not\subset - s_{\not\approx}} + s_{\not\approx}\bullet\ \frac{s}{s_{\not\approx}}\dot{v} \\ &\leqslant \not\subset - s_{\not\approx}\bullet\ \frac{\dot{u}}{\not\subset - s_{\not\approx}} + s_{\not\approx}\bullet\ \frac{\dot{v}}{\lambda} \in L^{\not\subset}\end{aligned}$$

As $\dot{u} \in \Vdash E^{\bullet}, \not\subset$ then

$$d\ \frac{\dot{u}}{\not\subset - s_{\not\approx}}, E^{\bullet} = \frac{\not\subset}{\not\subset - s_{\not\approx}} d\dot{u}, E^{\bullet} < \not\subset$$

and therefore $\frac{\dot{u}}{\not\subset - s_{\not\approx}} \in C^{\bullet}$.

Now, applying (**??**), (**??**), the fact that $v \in L^{\infty}$ and $\dot{v} \in L^{\bullet}$, we get

$$\begin{aligned}\bigcup D_s Hs, t \bigcup &= \bigcup D_x Lt, u + sv, \dot{u} + s\dot{v} \cdot v + \lambda D_y Lt, u + sv, \dot{u} + s\dot{v} \cdot \frac{\dot{v}}{\lambda} \bigcup \\ &\leqslant M\left(bt + \bullet\dot{u} + s\dot{v}\right\rfloor \bigcup v \bigcup \\ &\quad + \lambda\left(/\!/ D_y Lt, u + sv, \dot{u} + s\dot{v} + \bullet\ \frac{\dot{v}}{\lambda} \right\rfloor \\ &\leqslant M\left(bt + \bullet\dot{u} + s\dot{v}\right\rfloor \bigcup v \bigcup + \lambda M \left(bt + \bullet\dot{u} + s\dot{v}\right\rfloor + \lambda\bullet\ \frac{\dot{v}}{\lambda} \\ &= M\left(bt + \bullet\dot{u} + s\dot{v}\right\rfloor \bigcup v \bigcup + \lambda + \lambda\bullet\ \frac{\dot{v}}{\lambda} \in L^{\not\subset}.\end{aligned} \tag{26}$$

Consequently, I has a directional derivative and

$$\coprod I'u, v^{\frown} = \frac{d}{ds} Iu + sv \bigcup_{s=\not\approx} = \mathcal{R}_{\not\approx}^{T}\ D_x Lt, u, \dot{u} \cdot v + D_y Lt, u, \dot{u} \cdot \dot{v}\ \ dt.$$

Moreover, from the previous formula, (**??**), (**??**), and Lemma **??**, we obtain

$$\cup\amalg^{I'u,v\frown}\cup \leqslant \prod^{D_xL}\prod^{L^{\not\subset}}\prod^{v}\prod^{L^\infty} + \prod^{D_yL}\prod^{L^{/\!/}}\prod^{\dot v}\prod^{L^\bullet} \leqslant C\prod^{v}\prod^{W^{\not\subset}L^\bullet}$$

with a appropriate constant C.

This completes the proof of the Gâteaux differentiability of I.

Step 4. The operator $I' : E^\bullet \to \left(W^{\not\subset}L^\bullet_d\right]^*$ *is demicontinuous.* This is a consequence of the continuity of the mappings $u \mapsto D_xLt, u, \dot u$ and $u \mapsto D_yLt, u, \dot u$. Indeed, if $u_n, u \in E^\bullet$ with $u_n \to u$ in the norm of $W^{\not\subset}L^\bullet$ and $v \in W^{\not\subset}L^\bullet$, then

$$\begin{aligned} \coprod I'u_n, v\frown &= \mathcal{R}^T_{\not\approx}\ D_xL\ t, u_n, \dot u_n \cdot v + D_yL\ t, u_n, \dot u_n \cdot \dot v\ \ dt \\ &\to \mathcal{R}^T_{\not\approx}\ D_xL\ t, u, \dot u \cdot v + D_yL\ t, u, \dot u \cdot \dot v\ \ dt \\ &= \coprod I'u, v\frown . \end{aligned}$$

In order to prove item **??**, it is necessary to see that the maps $u \mapsto D_xLt, u, \dot u$ and $u \mapsto D_yLt, u, \dot u$ are norm continuous from $E^\bullet$ into $L^{\not\subset}$ and $L^{/\!/}$, respectively.

The continuity of the first map has already been proved in step 1.

Si eliminamos la demicontinuidad del segundo item, hay que copiar la continuidad de D_x aquí!!!

Let $u_n, u \in E^\bullet$ with $\prod^{u_n - u}\prod^{W^{\not\subset}L^\bullet} \to \not\approx$.

Applying Lemma **??** to $\dot u_n$, there exists a subsequence (denoted $\dot u_n$ for simplicity) such that $\dot u_n \in L^\bullet$ and a function $h \in L^{\not\subset}$ such that $/\!/\dot u_n \leqslant h$ and $\dot u_n \to \dot u$ a.e.

Then, by (**??**) we have $/\!/v_n \leqslant mt \in L^{\not\subset}$ being $v_n := D_yL\cdot, u_n, \dot u_n$ and $mt := Mb + h$. In addition, from the continuous differentiability of L, we have that $v_n \to v$ a.e. where $D_yL\cdot, u, \dot u$.

As $/\!/ \in \natural_{\not\supset}$, there exists $c : \mathbb{R}^+ \to \not\ni\not\ni\not\ni$ such that $/\!/\lambda x \leqslant c_{\bigcup}\lambda_{\bigcup}/\!/x$. Then, $/\!/\frac{v_n - v}{\lambda} \leqslant c_{\bigcup}\lambda_{\bigcup}^{-\not\subset}/\!/v_n - v$ for every $\lambda \in \mathbb{R}$.

Therefore, $/\!/\frac{v_n - v}{\lambda} \to \not\approx$ a.e. as $n \to \infty$ and $/\!/\frac{v_n - v}{\lambda} \leqslant c_{\bigcup}\lambda_{\bigcup}^{-\not\subset}K/\!/v_n + /\!/v \leqslant c_{\bigcup}\lambda_{\bigcup}^{-\not\subset}K_{(}mt + /\!/v_{\rfloor} \in L^{\not\subset}$.

Now, by Dominated Convergence Theorem, we get $\mathcal{R}\ /\!/\frac{v_n - v}{\lambda}\ dt \to \not\approx$ for every $\lambda > \not\approx$. Thus, $v_n \to v$ in $L^{/\!/}$.

The continuity of I' follows from the continuity of D_xL and D_yL using the formula (**??**). □

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