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# Lebesgue Measure and Integration

## *An Introduction*

*Frank Burk*

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**Frank Burk**

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*For Janet*

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# *Preface*

This book is intended for individuals seeking an understanding of Lebesgue measure and integration. As a consequence, it is not an encyclopedic reference, or a compendium, of the latest developments in this area of mathematics. Only the most fundamental concepts are presented: Lebesgue measure for $R$, Lebesgue integration for extended real-valued functions on $R$. No apologies are made for this approach, after all, it is the proper foundation for any general treatment of measure and integration. In fact, no claim to originality is made for any of the mathematics in this book, but we do accept full responsibility for any mistakes or blunders in its presentation. It is old mathematics after all (standard graduate fare for the last forty or fifty years), but particularly beautiful. It deserves a wider audience. Lebesgue measure and integration, presented properly, reveals mathematical creation in its highest form. Motivation has been the dominant concern, and understanding will be the final measure.

Where to begin? As a concession to understanding the subtleties of measure, and the effort required for such, I have taken the least upper bound axiom as a starting point. (Besides, it would be difficult, if not impossible, to improve on Landau's (1960) book, *Foundations of Analysis.*) The formal prerequisites are a basic calculus course and a course emphasizing what constitutes a proof, standard methods of proof, and the like. In reality, a curiosity for things mathematical and the "need to understand such," is both necessary and sufficient.

The arrangement of topics is standard. The historical struggle to give a rigorous definition of "area" and "area under a curve," resulting in Lebesgue measure and integration, is the subject of Chapter 1. (Tribute is paid to our mathematical ancestors by understanding and studying their results.) Mastery of this material is not necessary for subsequent chapters. After all, it is an "overview," written with the benefit of hindsight. The reader may return from time to time as she understands "measurable", "Borel", "Lebesgue Dominated Convergence," and so on. Mathematical concepts (undergraduate analysis) that are useful for the understanding of measure, measurable functions, and integration, are developed in Chapter 2. Chapter 3, measure theory, is the essence of this book. Here an elementary, but rigorous, treatment of Lebesgue measure, as a natural extension of "length of an interval" and as a subject of interest in and of itself, is presented. Set measurability is via Carathéodory's Condition. Measurable functions, motivated by the necessity of "measuring" inverse images of intervals as discussed by Lebesgue [Ma], are defined and developed in Chapter 4. The last chapter, Chapter 5, begins with the Riemann integral, developed from step functions. Replacing "step" with "simple" results in the Lebesgue integral for bounded functions on sets of finite measure. Some incisive observations and we have the celebrated convergence theorems that permit the interchange of "limit" and "integral", and justifies "Lebesgue" for those with such a need. (By the way, if at any time you are confused or lack a sense of direction, I apologize; for a solution, reread the master [Ma].) Finally, appendices A-E present other topics of beauty and inspiration to mathematicians, testament to the wonderful creativity of the human mind.

This book may be used in many ways: especially as a text for an undergraduate analysis course, first-year graduate students in statistics or probability, and other applied areas; a self-study guide to elementary analysis or as a refresher for comprehensive examinations; a supplement to the traditional real analysis course taken by beginning graduate students in mathematics.

I want to thank my good friend and colleague, Gene Meyer, for his countless hours of discussions and suggestions as to topics, and what would or would not be appropriate for a book of this nature. Accolades to Debora Naber. She had the arduous task of translating my handwriting into the final manuscript. I thank my parents, Glen and Helen Burk, whose constant encouragement has been a source of strength throughout my life. Finally, I thank my wonderful wife Janet, who somehow finds the time to encourage my dreams while rearing our five beautiful children—(Eric, Angela, Michael, Brandon, and Bryan.).

Even now there is a very wavering grasp of the true position of mathematics as an element in the history of thought. I will not go so far as to say that to construct a history of thought without profound study of the mathematical ideas of successive epochs is like omitting Hamlet from the play which is named after him. That would be claiming too much. But it is certainly analogous to cutting out the part of Ophelia. This simile is singularly exact. For Ophelia is quite essential to the play, she is very charming—and a little mad. Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

—Alfred North Whitehead

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learned as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the real and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world.

—Bertrand Russell

*Lebesgue Measure*
*and Integration*

# 1

# Historical Highlights

Some of the major discoveries in quadratures that culminated with the Lebesgue-Young integral are presented in this chapter. Our purpose is twofold:

1. We want to acknowledge our appreciation and gratitude to the thinkers of the past. It is hoped that the student will be motivated to continue these threads that distinguish civilization from barbarism.

> Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.
>
> —Roger Bacon

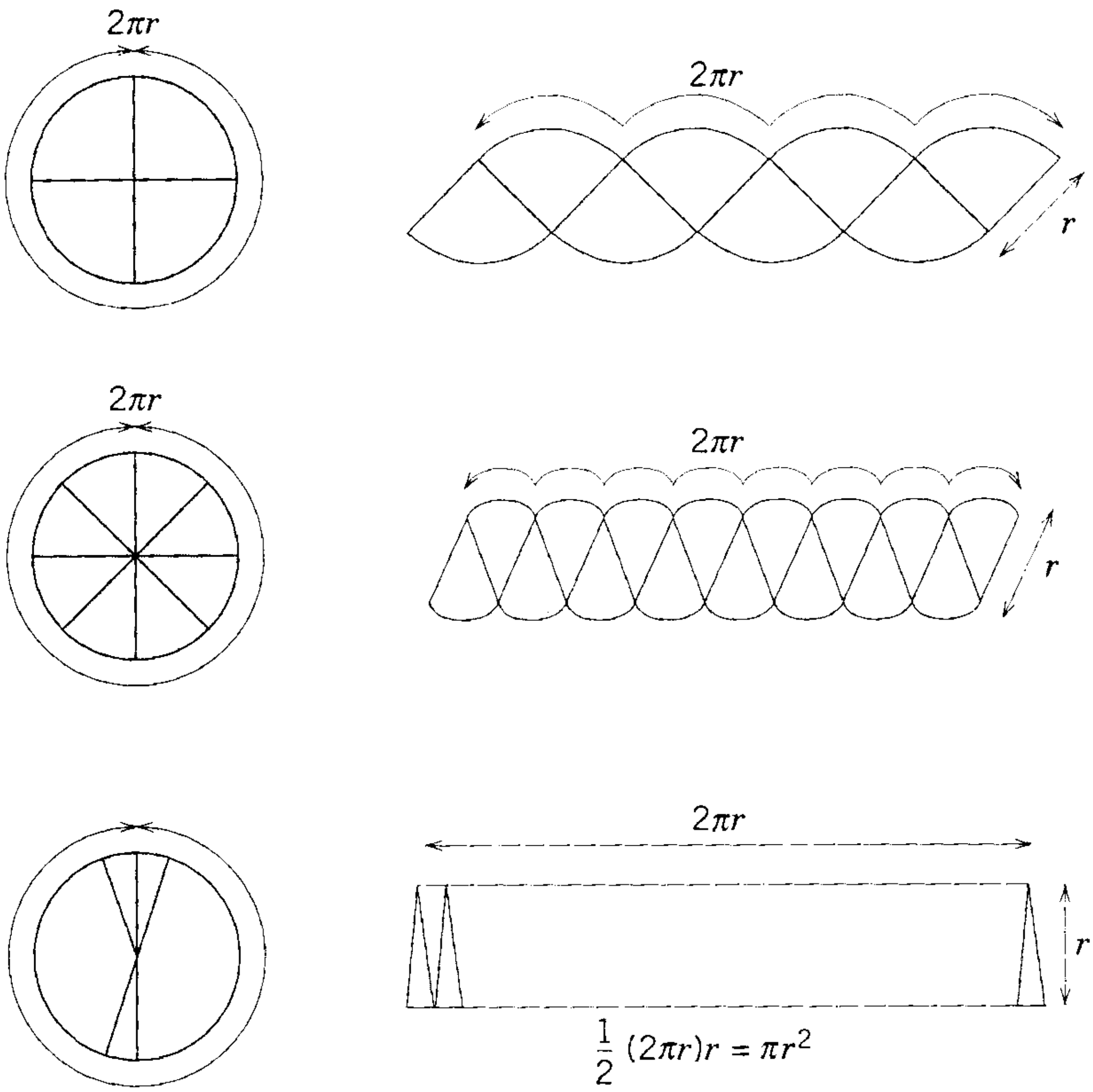
2. The student will see the process of mathematical creation and generalization as it applies to the development of the Lebesgue integral.

> Reason with a capital R = Sweet Reason, the newest and rarest thing in human life, the most delicate child of human history.
>
> —Edward Abbey

If this material is too difficult on the first reading, relax. It will make sense after Chapter 5.

## 1.1 REARRANGEMENTS

The figures below demonstrate the general idea of "rearranging"; in the first example, a circle rearranged into a parallelogram. This method has been known for hundreds of years.

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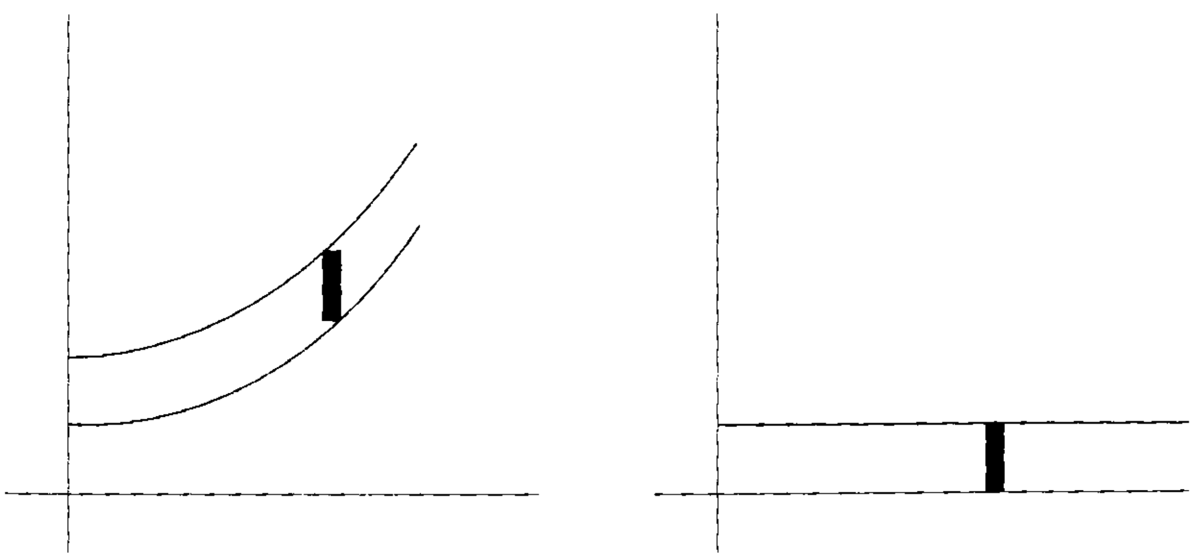

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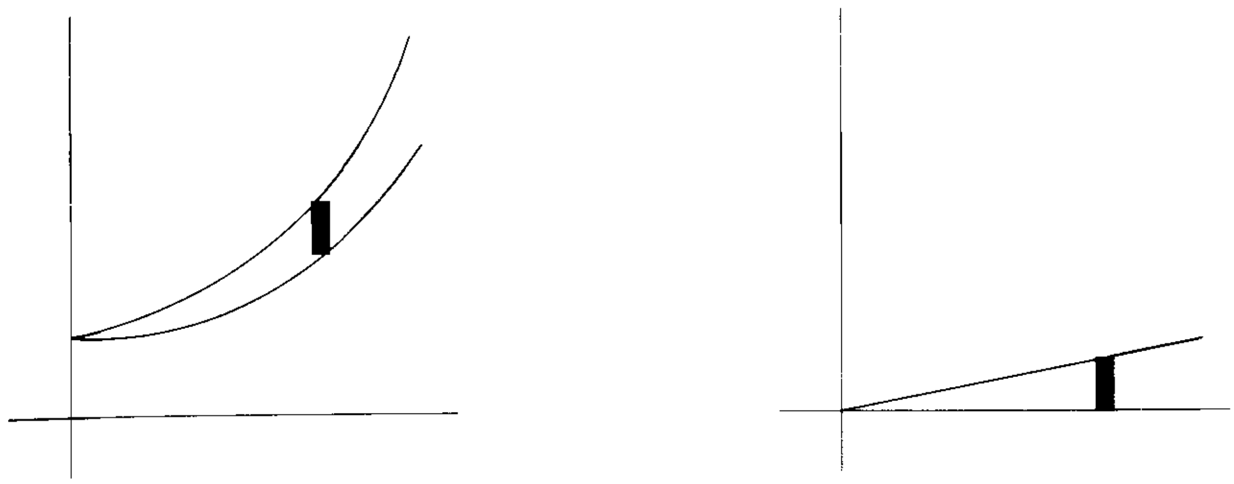

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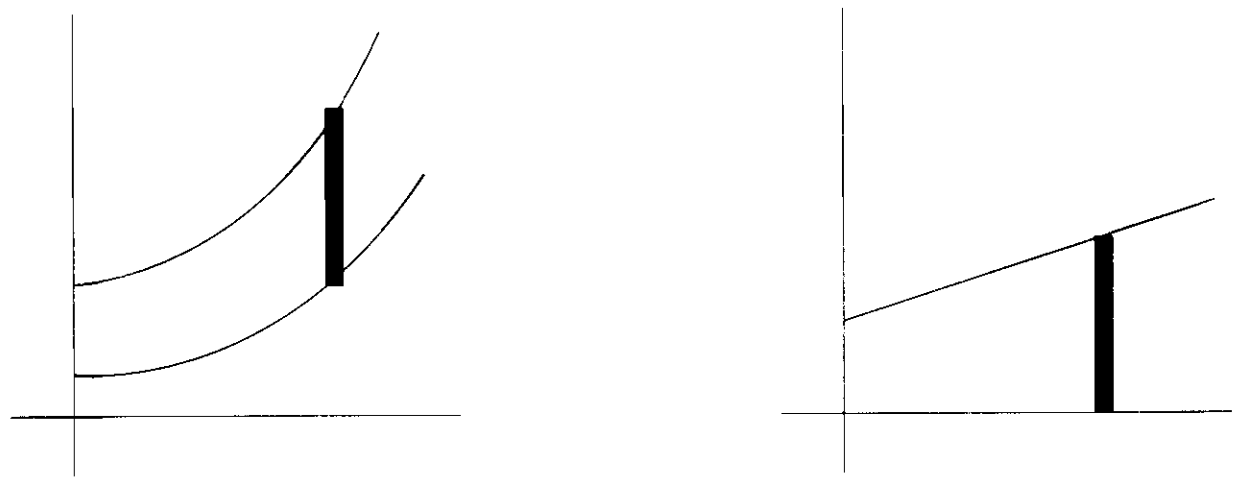

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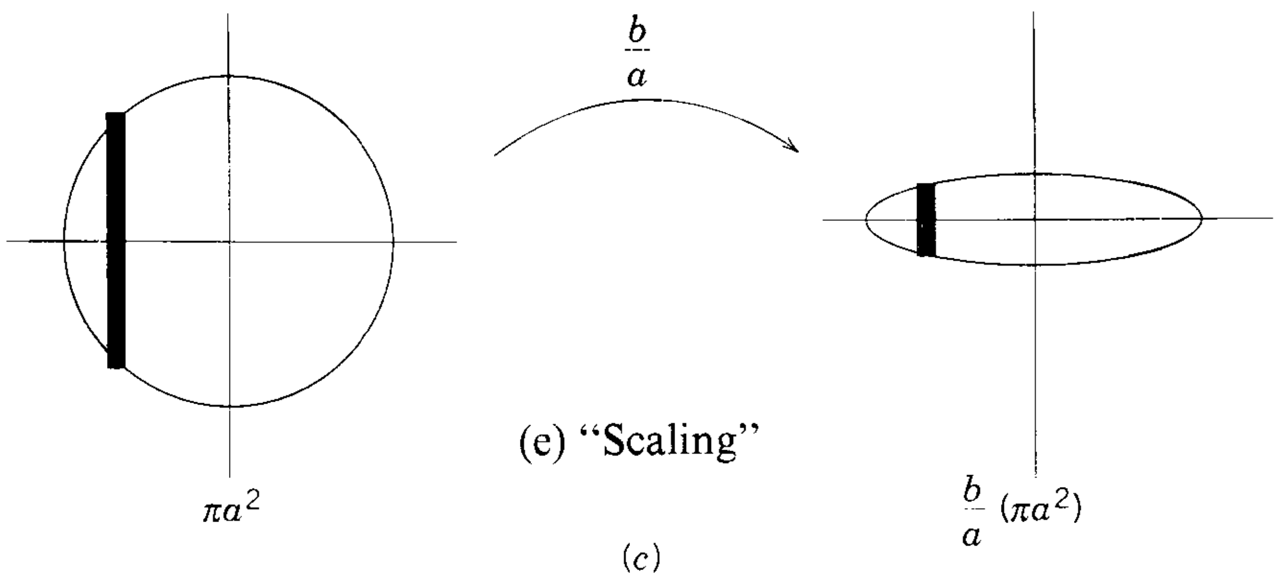



(e) "Scaling"

(c)

## 1.2 EUDOXUS (408–355 B.C.E.) AND THE METHOD OF EXHAUSTION

"Willingly would I burn to death like Phaeton, were this the price for reaching the sun and learning its shape, its size, and its substance."

—Eudoxus

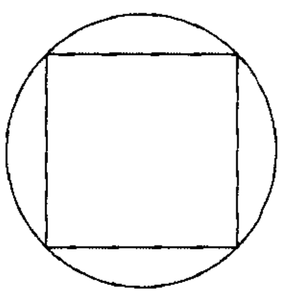
Eudoxus was responsible for the notion of approximating curved regions with polygonal regions: "truth" for polygonal regions implies "truth" for curved regions. This notion will be used to show that the areas of circles are to each other as the squares of their diameters, an obvious result for regular polygons. "Truth" was to be based on Eudoxus' Axiom:

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continuously, there will be left some magnitude which will be less than the lesser magnitude set out.

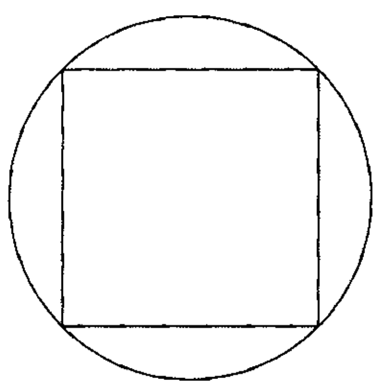
In modern terminology, let $M$ and $\epsilon > 0$ be given with $0 < \epsilon < M$. Then form: $M, M - rM = (1-r)M, (1-r)M - r(1-r)M = (1-r)^2 M, \ldots,$ where $1/2 < r \leq 1$. The axiom tells us that for $n$ sufficiently large, say $N$, $(1-r)^N M < \epsilon$, a consequence of the set of natural numbers not being bounded above.

Back to what we are trying to show: Let $c, C$ be circles with areas $a, A$ and diameters $d, D$, respectively. We want to show $a/A = d^2/D^2$, given that the result is true for polygons and given the Axiom of Eudoxus.

Assume $a/A > d^2/D^2$. Then we have $a^* < a$ so that $0 < a - a^*$ and $a^*/A = d^2/D^2$. Let $\epsilon < a - a^*$. Inscribe regular polygons of areas $p_n, P_n$ in circles $c, C$ and consider the areas $a - p_n$, $A - P_n$:



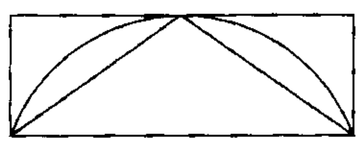
$a - p_n$



$A - P_n$

Now, double the number of sides. What is the relationship between $a - p_n$ and $a - p_{2n}$?

$a - p_n$



$a - p_{2n}$

Certainly $a - p_{2n} < 1/2(a - p_n)$. We are subtracting more than half at each stage of doubling the number of sides. From the Axiom of Eudoxus, we may determine $N$ so that

$$0 < a - p_N < \epsilon < a - a^*, \qquad \text{that is,}$$

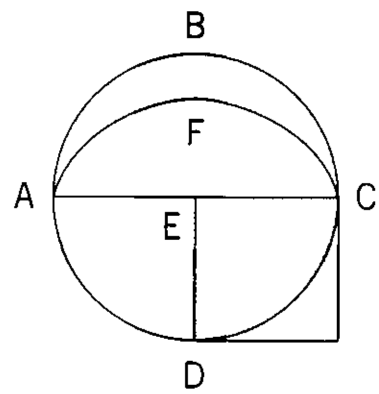
we have a regular inscribed polygon of $N$ sides, where area $p_N > a^*$. But, $p_N/P_N = d^2/D^2$ and since $a^*/A = d^2/D^2$, we have $p_N/P_N = a^*/A$, that is, $P_N > A$. This cannot be: $P_N$ is the area of an inscribed polygon to the circle $C$ of area $A$.

A similar argument shows that $a/A$ cannot be less than $d^2/D^2$:

*double reductio ad absurdum.*

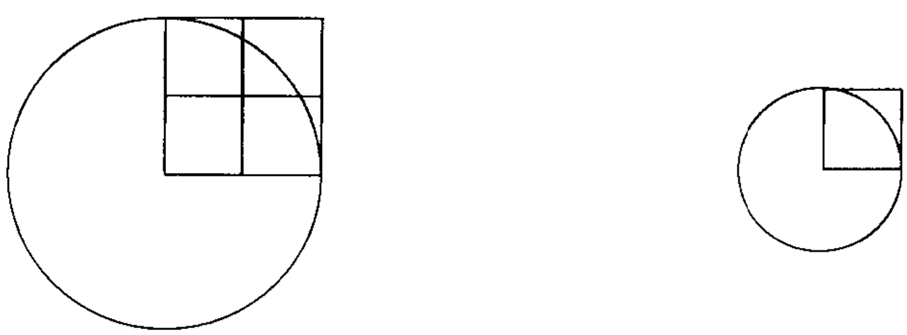
## 1.3 THE LUNE OF HIPPOCRATES (430 B.C.E.)

Hippocrates, a merchant of Athens, was one of the earliest individuals to find the area of a plane figure (lune) bounded by curves (circular arcs). The crescent-shaped region whose area is to be determined is shown below.



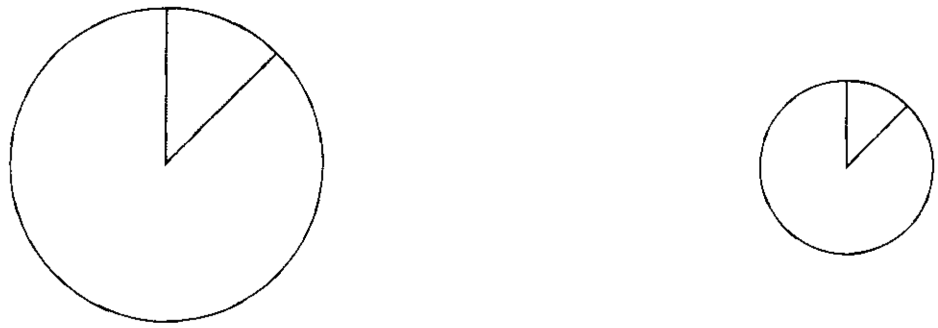

$ABC$ and $AFC$ are circular arcs with centers $E$ and $D$, respectively. Hippocrates showed that the area of the shaded region bounded by the circular arcs $ABC$ and $AFC$ is exactly the area of the shaded square whose side is the radius of the circle. The argument depends on the following assumption:

(a) The areas of two circles are to each other as the squares of the radii:



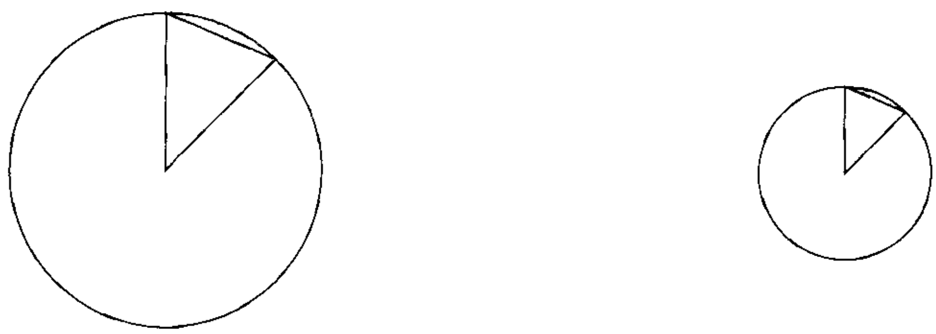
(a)

From this assumption we conclude that (b) the sectors of two circles with equal central angles are to each other as the squares of the radii:



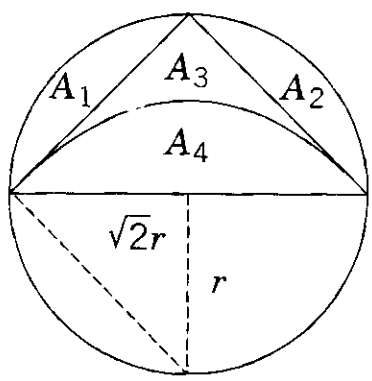
(b)

(c) The segments of two circles with equal central angles are to each other as the squares of the radii:



(c)

Hippocrates' argument proceeds as follows:



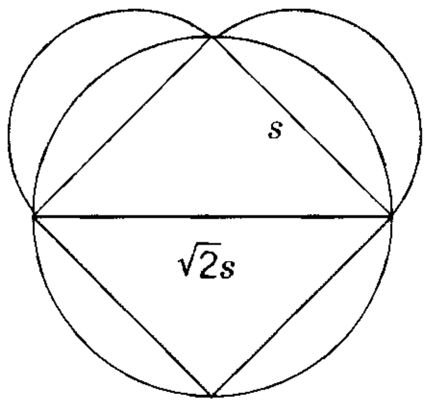

From (c), $A_1/A_4 = r^2/(\sqrt{2}r)^2 = 1/2$. Hence $A_1 = 1/2\,A_4$ and $A_2 = 1/2\,A_4$ and thus $A_1 + A_2 = A_4$.

$$\begin{aligned} \text{The area of the lune} &= A_1 + A_2 + A_3 \\ &= A_4 + A_3 \\ &= \text{area of triangle} \\ &= \tfrac{1}{2}(\sqrt{2}r)(\sqrt{2}r) \\ &= r^2 \\ &= \text{area of the square.} \end{aligned}$$

The reader may use similar reasoning on these "lunes":

1.

2.




He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.

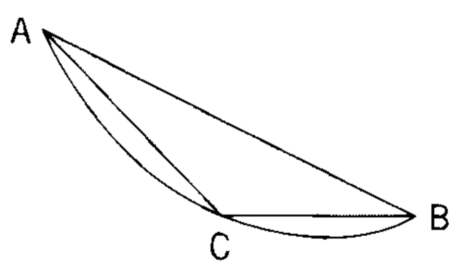
—Plato

## 1.4 ARCHIMEDES (287–212 B.C.E.)

It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius, ...

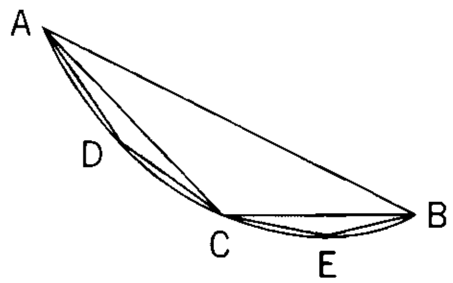
—Plutarch

This masterpiece of mathematical reasoning is due to one of the greatest intellects of all time, Archimedes of Syracuse. He shows that the area of the parabolic segment is 4/3 that of the inscribed triangle $ACB$. (The symbol $\triangle$ will denote "area of".)

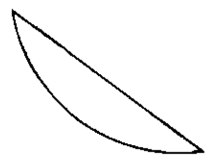


The argument proceeds as follows: the combined area of triangle $ADC$ and $BEC$ is one-fourth the area of triangle $ACB$, that is,

$$\triangle ADC + \triangle BEC = \frac{1}{4}\triangle ACB.$$




Repeating the process, trying to "exhaust" the area between the parabolic curve and the inscribed triangles, we have:

The area of the parabolic segment $=$ 

$$= \triangle ACB + \frac{1}{4}(\triangle ACB) + \frac{1}{4}\left(\frac{1}{4}(\triangle ACB)\right) + \cdots$$

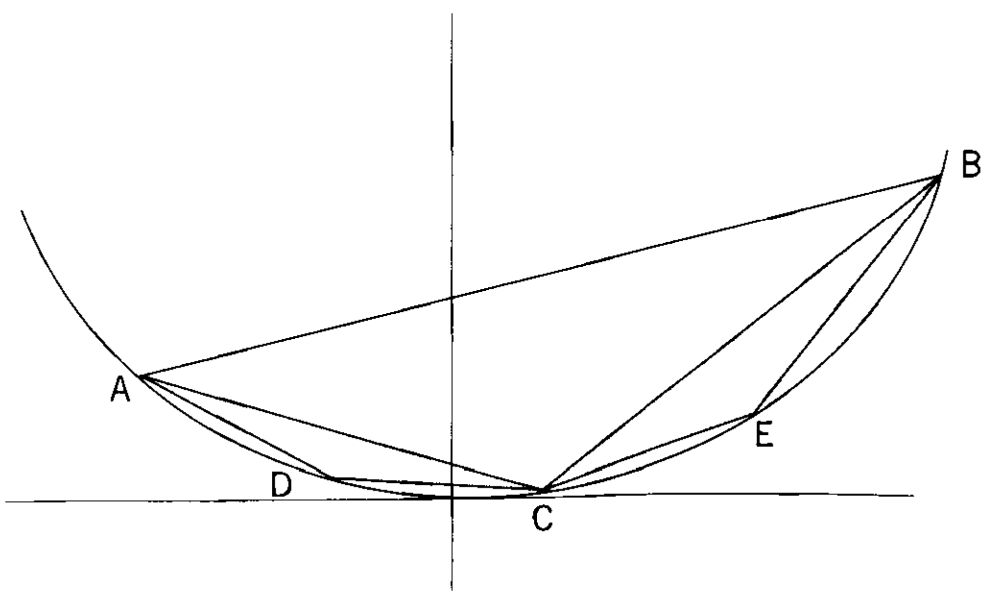
$$= \triangle ACB\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right)$$

$$= \frac{4}{3}\triangle ACB.$$

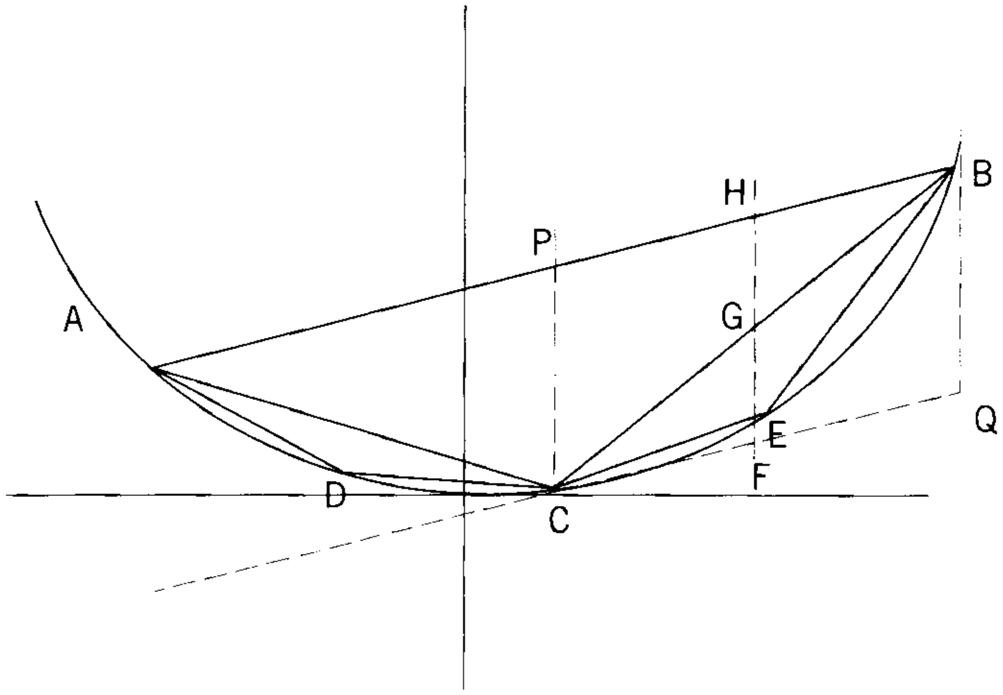
We argue that

$$\triangle ADC + \triangle BEC = \frac{1}{4}\triangle ACB$$

for the parabola $y = ax^2, \qquad a > 0.$



The reader should show the tangent line at $C$ is parallel to $AB$ and that the vertical line through $C$ bisects $AB$ at $P$. We need to show $\triangle BEC = 1/4 \triangle BCP$. Complete the parallelogram:




We note:

1. $\triangle CEG = \triangle BEG$ (equal height and base)
2. $\triangle HGB = \frac{1}{4}\triangle BCP$.

Thus, we must show

$$\triangle CEG + \triangle BEG = \triangle HGB,$$

or that

$$\triangle BEG = \frac{1}{2}\triangle HGB.$$

This will be accomplished by showing $FE = 1/4\, FH = 1/4\, QB$. Since

$$\begin{aligned} FE &= a((X_C + X_B)/2)^2 - \left[aX_C^2 + 2aX_C \times \frac{1}{2}(X_B - X_C)\right] \\ &= \frac{1}{4}a(X_B - X_C)^2, \end{aligned}$$

and

$$\begin{aligned} QB &= aX_B^2 - \left[aX_C^2 + 2aX_C(X_B - X_C)\right] \\ &= a(X_B - X_C)^2, \end{aligned}$$

we are done.

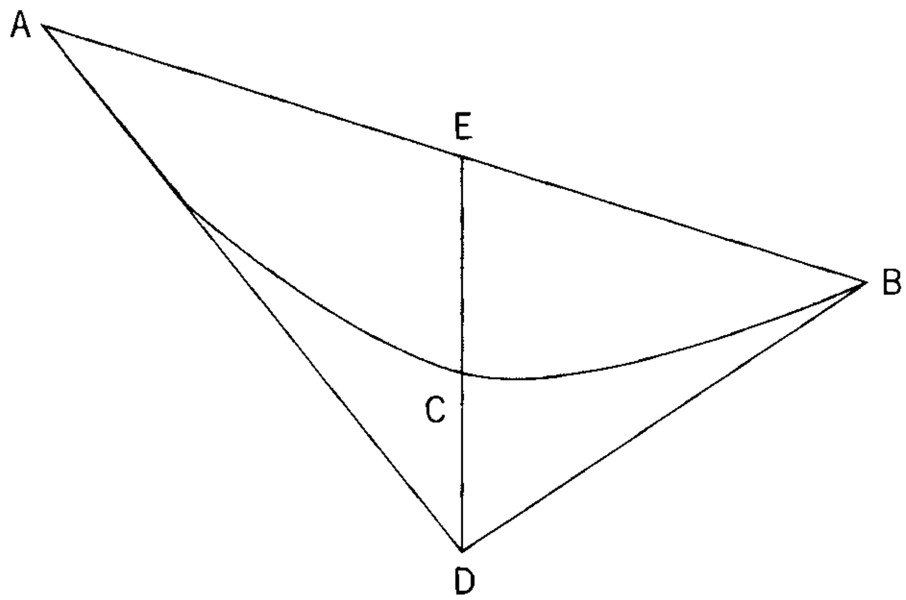
... there was far more imagination in the head of Archimedes than in that of Homer.

—Voltaire

Archimedes will be remembered when Aeschylus is forgotten because languages die and mathematical ideas do not.

—G.H. Hardy

The reader may show that the area of the parabolic segment is $2/3$ the area of the circumscribed triangle $ADB$ formed by the tangent lines to the parabola at $A$ and $B$ with base $AB(EC = CD)$.




## 1.5 PIERRE FERMAT (1601–1665): $\int_0^b x^{p/q}dx = b^{p/q+1}/(p/q+1)$

It appears that Fermat, the true inventor of the differential calculus, ...

—Laplace

The Italian mathematician Cavalieri demonstrated (1630's) that

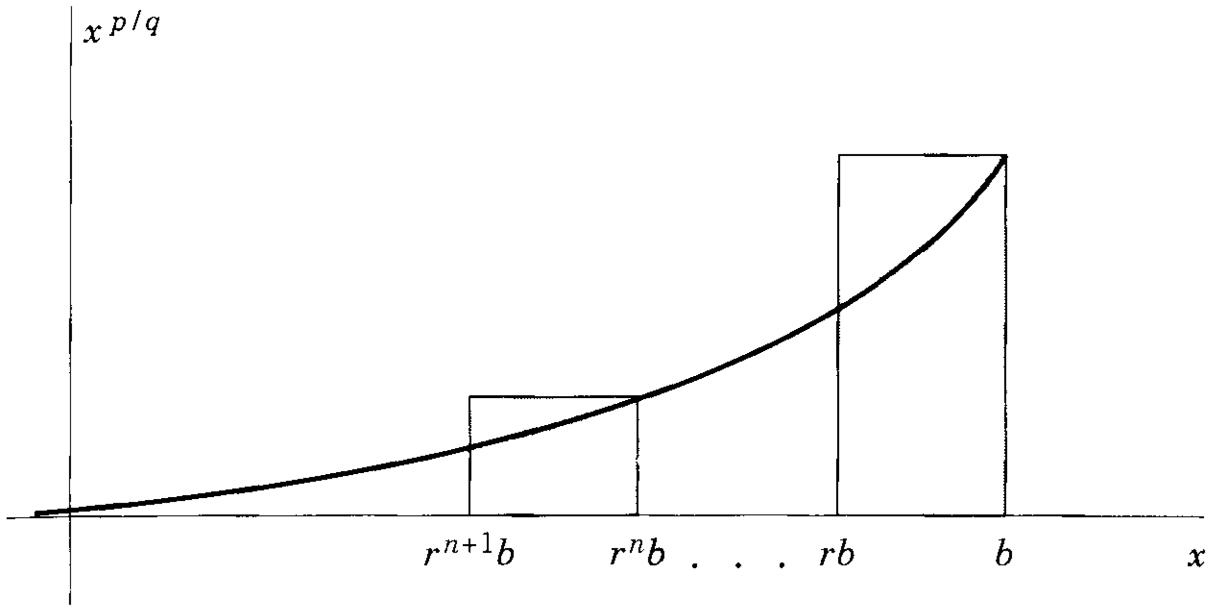
$$\int_0^b x^n\,dx = \frac{b^{n+1}}{n+1}$$

for $n = 1, 2, \ldots, 9$. But it was Fermat who was able to show

$$\int_0^b x^{\frac{p}{q}}dx = \frac{b^{p/q+1}}{\frac{p}{q}+1},$$

where $p/q$ is a positive rational number.

Fermat divided the interval $[0, b]$ into an **infinite** sequence of subintervals with endpoints (heretofore a finite number of subintervals of equal width) $br^n$, $0 < r < 1$, and erected a rectangle of height $(br^n)^{p/q}$ over the subinterval $[br^{n+1}, br^n]$ (see below).




Let $S_r$ denote the sum of the areas of the exterior rectangles. We have

$$\begin{aligned}
S_r &= (b - br)b^{\frac{p}{q}} + (br - br^2)(br)^{\frac{p}{q}} + \cdots + (br^n - br^{n+1})(br^n)^{\frac{p}{q}} + \cdots \\
&= b^{\frac{p}{q}+1}(1-r)\left[1 + r^{\frac{p}{q}+1} + r^{(\frac{p}{q}+1)2} + \cdots + r^{(\frac{p}{q}+1)n} + \cdots\right] \\
&= \frac{b^{\frac{p}{q}+1}(1-r)}{(1 - r^{\frac{p}{q}+1})} \\
&= b^{\frac{p}{q}+1}\frac{\left[1 - (r^{\frac{1}{q}})^q\right]}{(1 - r^{\frac{1}{q}})}\,\frac{(1 - r^{\frac{1}{q}})}{\left[1 - (r^{\frac{1}{q}})^{p+q}\right]} \\
&= b^{\frac{p}{q}+1}\frac{\left(1 + r^{\frac{1}{q}} + \cdots + r^{\frac{q-1}{q}}\right)}{\left(1 + r^{\frac{1}{q}} + \cdots + r^{\frac{p+q-1}{q}}\right)} \\
&\longrightarrow b^{\frac{p}{q}+1}\frac{q}{p+q} \qquad \text{as } r \to 1 \\
&= \frac{b^{\frac{p}{q}+1}}{\frac{p}{q}+1}.
\end{aligned}$$

. . . a master of masters.

—E.T. Bell

## 1.6 GOTTFRIED LEIBNITZ (1646–1716), ISSAC NEWTON (1642–1723)

> Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.
>
> —Gottfried Leibnitz

> Nature and Nature's laws lav hid in night; God said, "Let Newton be!" and all was light.
>
> —Alexander Pope

> The capital discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz.
>
> —Sophus Lie

During the seventeenth and eighteenth centuries the integral was thought of in a descriptive sense, as an antiderivative, due to the beautiful Fundamental Theorem of Calculus (FTC), as developed by Leibnitz and Newton. The ease of this method for specific functions probably induced a sense of euphoria, as generations of calculus students can attest to after struggling through Riemann sums. A particular function $f$ on $[a, b]$ was integrated by finding an antiderivative $F$ so that $F' = f$ or by finding a power series expansion and using the FTC to integrate termwise. The Leibnitz-Newton integral of $f$ was $F(b) - F(a)$, that is,

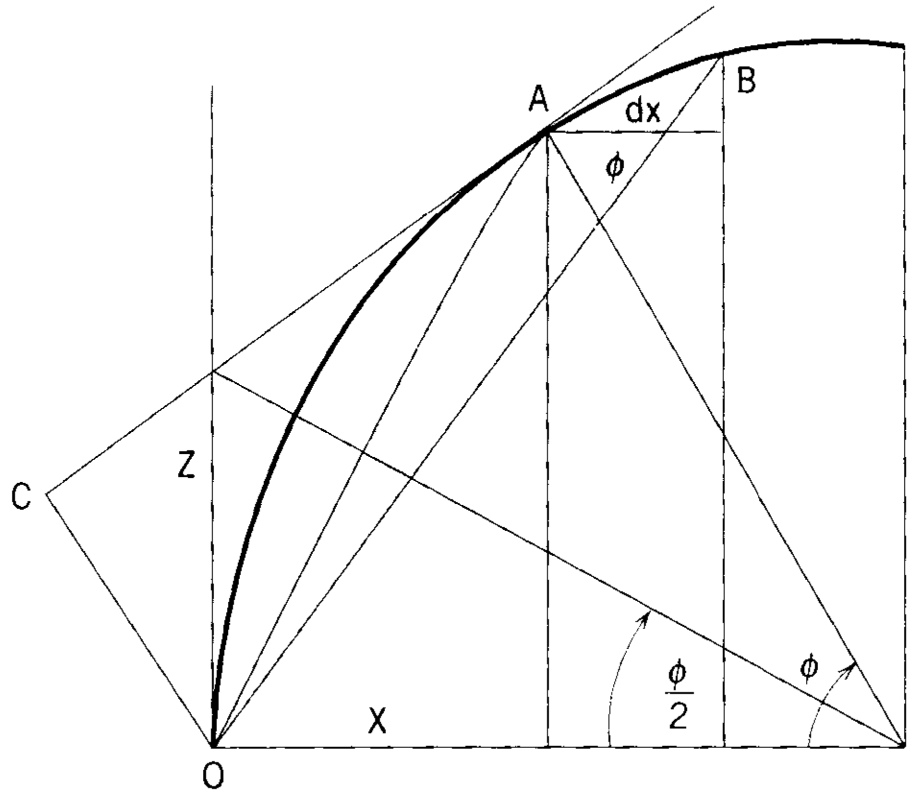
$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F' = f$.

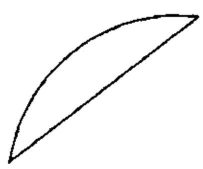
We give an argument of Leibnitz and a result of Newton to illustrate the power of these geniuses.

$$\text{Leibnitz}: \ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Take the quarter circle $(x-1)^2+y^2=1,\ 0\leq x\leq 1$, whose area is $\pi/4$:



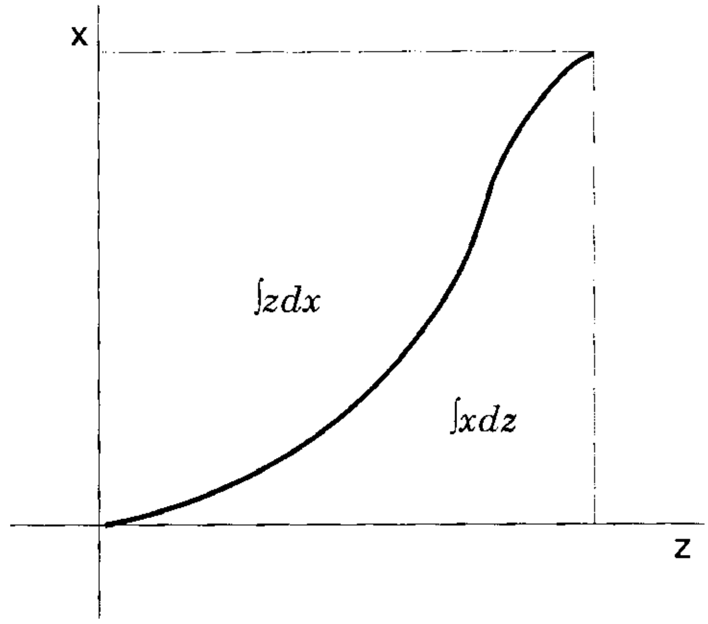

Leibnitz determines the area of the circular sector



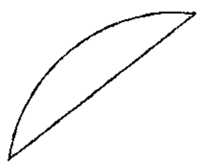
by dividing it into infinitesimal triangles $OAB$, $A$ and $B$ two close points on the circle, and summing. So, how to estimate the area of $OAB$, henceforth $\Delta OAB$. Construct the tangent to the circle at $A$, with a perpendicular at $C$ passing through the origin. Then $\Delta OAB \approx 1/2\, AB \times OC$. By similar triangles, $AB/dx = z/OC$, so $\Delta OAB = 1/2\, z\, dx$. Observe

$$x = 1-\cos\phi = 2\sin^2\frac{\phi}{2} \qquad \text{and} \qquad z=\tan\frac{\phi}{2}, \qquad \text{that is,}$$

$x = 2z^2/(1 + z^2)$. Leibnitz knew $xz = \int z\,dx + \int x\,dz$:




Hence



$$= \int_0^1 \frac{1}{2} z\,dx$$

$$= \frac{1}{2}\left[xz\ \Big|_0^1 - \int_0^1 x\,dz\right]$$

$$= \frac{1}{2}\left[1 - \int_0^1 \frac{2z^2}{1+z^2}\,dz\right]$$

$$= \frac{1}{2} - \int_0^1 z^2(1 - z^2 + z^4 - \cdots)dz \qquad \text{(long division)}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \qquad \text{(integrating termwise)},$$

and by adding $1/2$ to both sides,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \qquad !!$$

WOW!!

It would be difficult to name a man more remarkable for the greatness and universality of his intellectual powers than Leibnitz.

—John Stuart Mill

Note:

$$\frac{1}{8}\pi = \frac{1}{1\times 3} + \frac{1}{5\times 7} + \frac{1}{9\times 11} + \cdots$$

and

$$\frac{1}{8}\ln 4 = \frac{1}{2\times 4} + \frac{1}{6\times 8} + \frac{1}{10\times 12} + \cdots$$

Joy in looking and comprehending is nature's most beautiful gift.

—Albert Einstein

$$\text{Newton}: \ \frac{\pi}{4\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots.$$

Since Newton routinely integrated series termwise, and $(1+x^2)/(1+x^4) = (1+x^2)(1-x^4+x^8-\cdots) = 1+x^2-x^4-x^6+\cdots$,

$$\int_0^1 \frac{1+x^2}{1+x^4}\,dx = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots.$$

The reader may complete the argument by observing

$$\frac{1+x^2}{1+x^4} = \frac{1}{2}\left[\frac{1}{1-\sqrt{2}x+x^2} + \frac{1}{1+\sqrt{2}x+x^2}\right] \qquad \text{and}$$

evaluate the appropriate integrals with the substitution

$$x + \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\tan\theta.$$

[regarding Newton] Nature to him was an open book, whose letters he could read without effort.

—Albert Einstein

## 1.7 AUGUSTIN-LOUIS CAUCHY (1789–1857)

Cauchy must be considered the founder of integration theory: "Calculation" was henceforth to be replaced by "existence".

In 1823 Cauchy formulated a constructive definition of the integral, reminiscent of Cavalieri's and Fermat's approaches, but with a fundamental difference: In place of specific functions $(x^2, x^{1/3}, \ldots)$ he started with a general function $f$ on $[a, b]$ and formed the sum

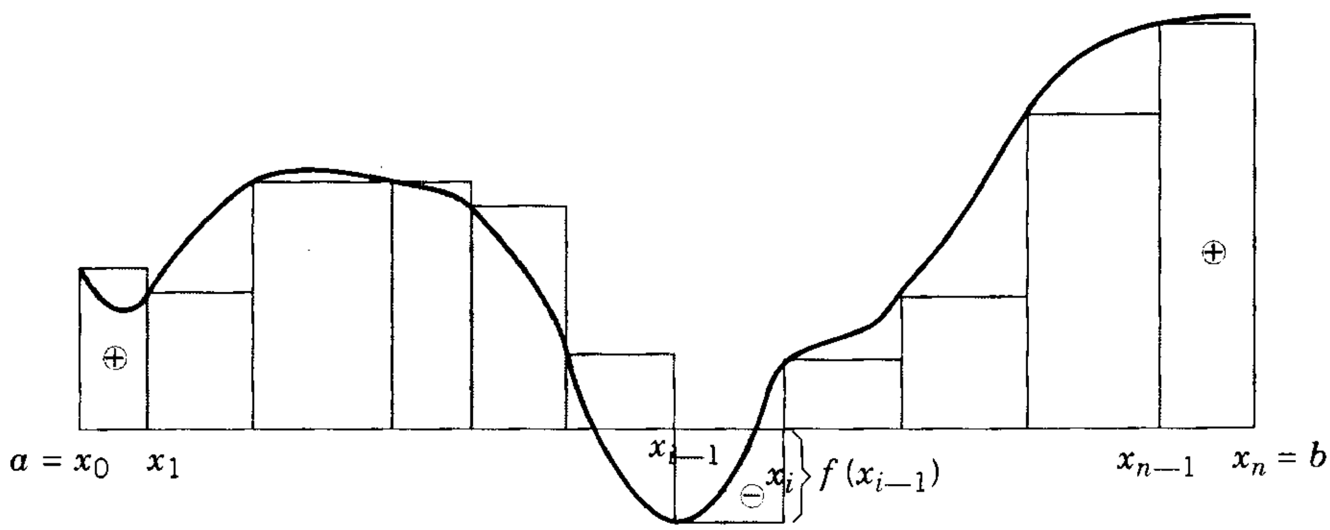
$$S = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_{n-1})(x_n - x_{n-1}),$$

where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ is a partition of $[a, b]$. The integral of Cauchy was to be the limit of such sums as the maximum of the $x_i - x_{i-1}$, $\| \Delta x \|$, approaches zero:

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

The question: what properties of $f$ would guarantee the existence of this limit? Cauchy argued that if $f$ is continuous on $[a, b]$, then this limit would exist. We also mention that Cauchy proved the Fundamental Theorem of Calculus, and thus settled integration for continuous functions on closed, bounded intervals.

Cauchy's Construction:




$$f(x_0)(x_1 - x_0) + \cdots + f(x_{i-1})(x_i - x_{i-1}) + \cdots + f(x_{n-1})(x_n - x_{n-1})$$

> The sole aim of science is the honor of the human mind, and from this point of view a question about numbers is as important as a question about the system of the world.
>
> —C.G.J. Jacobi

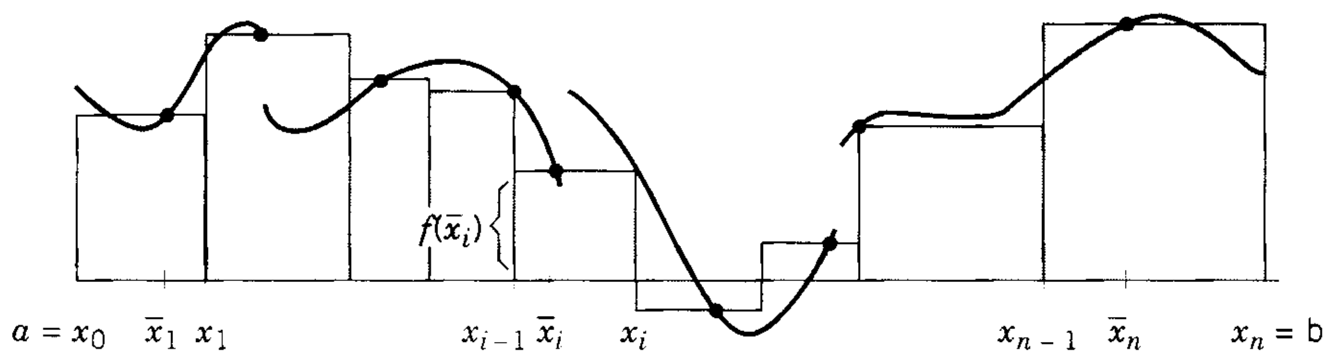
## 1.8 BERNHARD RIEMANN (1826–1866)

While investigating Fourier series and the attendant questions of convergence, Riemann (1854) asked the question: "What is one to understand by $\int_a^b f(x)\,dx$?". His answer (which is the most widely used definition of the integral), the so-called Riemann integral:

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)(x_i - x_{i-1}),$$

where $\bar{x}_i$ is an arbitrary point of $[x_{i-1}, x_i]$ and $f$ is bounded on $[a, b]$.

Riemann's Construction:




$$f(\bar{x}_1)(x_1 - x_0) + \cdots + f(\bar{x}_i)(x_i - x_{i-1}) + \cdots + f(\bar{x}_n)(x_n - x_{n-1})$$

Having given a constructive procedure (much like Cauchy's) Riemann then says: "Let us determine the extent of the validity of this concept," and asks: "In what cases is a function integrable and in what cases is it not?" His subsequent investigation resulted in a weaker requirement than Cauchy's requirement of continuity. He showed that the $\lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)(x_i - x_{i-1})$ exists iff $f$ is bounded on $[a, b]$ and for each pair of positive numbers $\epsilon$ and $\delta$, there exists an $\eta > 0$ such that whenever $P$ is a partition of $[a, b]$ with $\| \Delta x \| < \eta$, then the sum of the lengths of those subintervals $[x_{i-1}, x_i]$ with

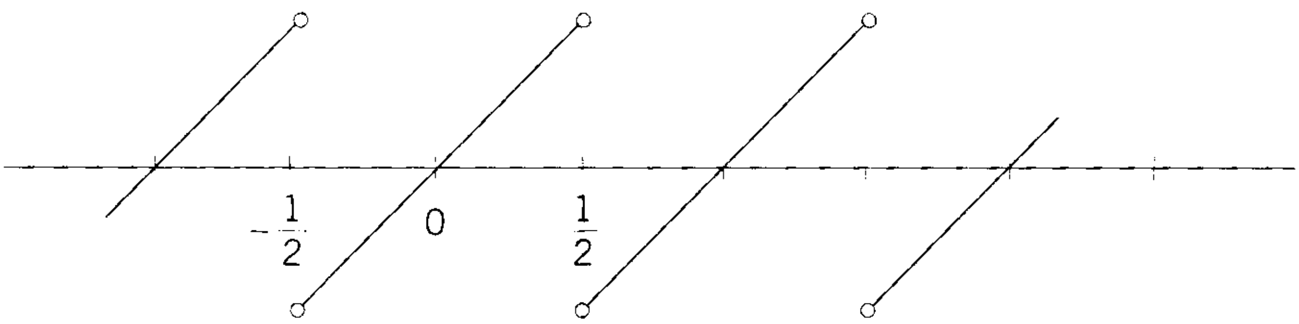
$$\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \geq \delta$$

is less than $\epsilon$. In other words, a bounded function $f$ is Riemann integrable on $[a, b]$ iff large oscillations of $f$ are restricted to "small" sets. Riemann then gives an ingenious example of a function $f$ that is discontinuous on a dense set of points of $R$, but is nevertheless Riemann integrable.

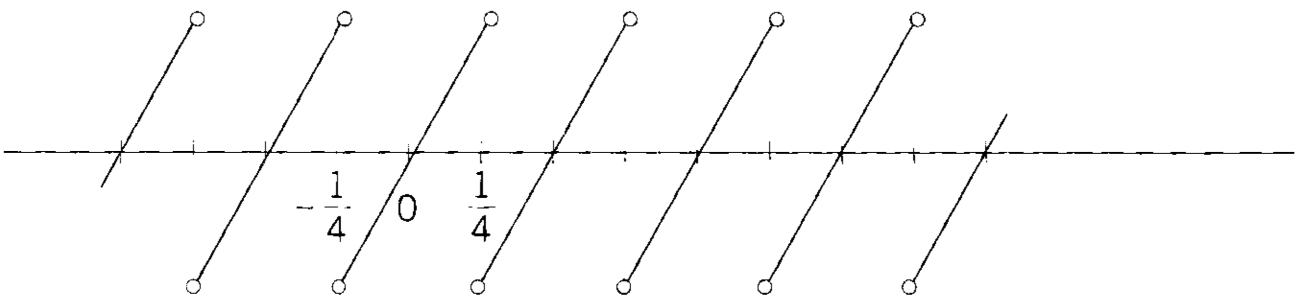
$$\text{Riemann's Function:} \qquad f(x) \equiv \sum_{n=1}^{\infty} \frac{(nx)}{n^2},$$

where $(x)$ denotes the difference between $x$ and the nearest integer if $x$ is not of the form $k + 1/2$, $k$ an integer; otherwise $(k + 1/2) = 0$. We carefully discuss this function. Let $\phi_n(x) = (nx)$, $n = 1, 2, \ldots$.

$n = 1$. $\phi_1$ is discontinuous at $x = (2k + 1)/2$ : $\phi((2k + 1)/2^-) = +1/2$, $\phi((2k + 1)/2) = 0$, $\phi((2k + 1)/2^+) = -1/2$.

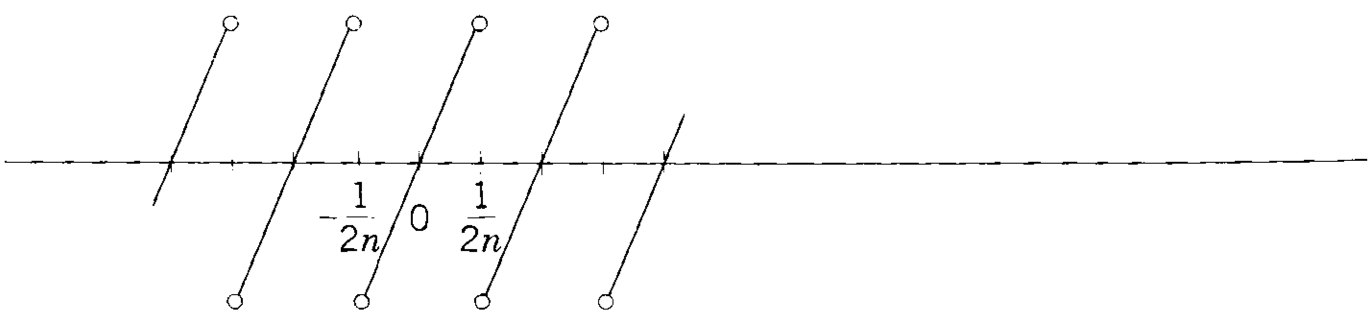



$n = 2$. $\phi_2$ is discontinuous at $x = (2k + 1)/4$ : $\phi((2k + 1)/4^-) = +1/2$, $\phi((2k + 1)/4) = 0$, $\phi((2k + 1)/4^+) = -1/2$.




⋮

$\phi_n$ is discontinuous at $x = (2k + 1)/2n$ : $\phi((2k + 1)/2n^-) = +1/2$, $\phi((2k + 1)/2n) = 0$, $\phi((2k + 1)/2n^+) = -1/2$.




and so on.

Where is $f(x) = \sum_1^\infty (nx)/n^2$ continuous? By the Weierstrass M-Test (Theorem 2.11), the sequence $f_m(x) = \sum_{n=1}^m (nx)/n^2$ converges uniformly to $f$ on $R$. So if $x_0$ is not of the form $(2k+1)/2n;\ n,k = 1,2,\ldots$, then $\phi_n$ is continuous for all $n$, and since uniform limits of continuous functions are continuous (Proposition 2.12), $f$ is continuous at $x_0$. It remains to show $f$ is discontinuous on the dense set of points $\{(2k+1)/2n;\ k,n = 1,2,\ldots\}$. Denote $g(x^-) - g(x^+)$ by $\Delta g(x)$.

$n = 1$. Let $x = (2k+1)/2$; $\Delta\phi_1 = 1,\ \Delta\phi_2 = 0,\ \Delta\phi_3 = 1$, and in general, $\Delta\phi_{2l} = 0,\ \Delta\phi_{2l-1} = 1 : \boxed{1,0}, 1,0,1,0,\ldots.$ Thus $\Delta f((2k+1)/2) = 1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$.

$n = 2$. Let $x = (2k+1)/4$; $\Delta\phi_1 = 0,\ \Delta\phi_2 = 1,\ \Delta\phi_3 = 0,\ \Delta\phi_4 = 0,\ \Delta\phi_5 = 0,\ \Delta\phi_6 = 1,\ \Delta\phi_7 = 0$, etc., repeating in a pattern of $\boxed{0,1,0,0}$, $0,1,0,0,\ldots$. Thus

$$\Delta f\left(\frac{2k+1}{4}\right) = \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \cdots$$
$$= \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$
$$= \frac{1}{2^2} \times \frac{\pi^2}{8}.$$

$\vdots$

$(n)$ $x = (2k+1)/2n$; $\Delta\phi_1 = \Delta\phi_2 = \cdots = \Delta\phi_{n-1} = 0,\ \Delta\phi_n = 1,\ \Delta\phi_{n+1} = \cdots = \Delta\phi_{2n} = 0, \cdots$, repeating in the pattern $\underbrace{0,0,\cdots,0,1}_{n}$, $\underbrace{0,0,\cdots,0,0}_{n}, 0, \cdots$. Hence

$$\Delta f\left(\frac{2k+1}{2n}\right) = \frac{1}{n^2} + \frac{1}{(3n)^2} + \frac{1}{(5n)^2} + \cdots$$
$$= \frac{1}{n^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$
$$= \frac{1}{n^2} \times \frac{\pi^2}{8}.$$

*Note*: In the calculation of $\Delta f$ we have used uniform convergence: "$\lim \sum = \sum \lim$", to show that $f$ is discontinuous on the dense set of points $\{(2k+1)/2n;\ n,k = 1,2,\ldots\}$, where the "jump" at $(2k+1)/2n$ is

$\Delta f((2k+1)/2n) = 1/n^2 \times \pi^2/8$. Otherwise, $f$ is continuous. Since $\pi^2/8n^2 > \delta$ for only a finite number of values of $n$, we can "cover" with a finite number of small intervals, and thus $f$ is Riemann integrable. By the way, what is $\int_0^1 \sum(nx)/n^2\,dx$? Observe that $\int_0^1 \phi_n(x)\,dx = \int_0^1 (nx)\,dx = 0$. Can we conclude

$$\int_0^1 \sum \frac{(nx)}{n^2}\,dx = \sum \frac{1}{n^2} \int_0^1 (nx)\,dx = 0?$$

Yes, because of uniform convergence we may interchange "$\sum$" and "$\int$" (Theorem 5.3). This completes our discussion of Riemann's Function.

During the latter part of the nineteenth century we had several reformulations of the Riemann integral by Peano, Jordan, Volterra, Darboux, and others that we "lump" together under "Darboux sums". Darboux (1875) introduced upper $(\overline{\int}_a^b f(x)\,dx)$ and lower $(\underline{\int}_a^b f(x)\,dx)$ Riemann integrals for a bounded function on $[a,b]$:

$$\underline{\int}_a^b f(x)\,dx = \sup_P \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1}),$$

$$\overline{\int}_a^b f(x)\,dx = \inf_P \sum_{i=1}^n \sup_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1}),$$

over all partitions $P$ of $[a,b]$. $f$ is said to be Riemann integrable iff $\underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx$. This definition is equivalent to Riemann's but is generally easier to apply.

## 1.9 EMILE BOREL (1871–1956), CAMILLE JORDAN (1838–1922), GIUSEPPE PEANO (1858–1932)

Peano tried to connect integrability of a nonnegative function with the "area" of the set $S = \{(x,y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$. This was an idea of the ancients for specific functions, but Peano's contribution (1887) was to mathematically formulate a *definition* of area. He defined the inner area of $S$, $a_i(S)$, as the least upper bound of the areas of all polygons that are contained in $S$, and the outer area of $S$, $a_0(S)$, as the greatest lower bound of the area of all polygons that contain $S$. The set $S$ was defined to have area whenever $a_i(S) = a_0(S)$. Note: If $S = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1,\ x, y \text{ irrational}\}$, then $a_i(S) = 0$, $a_0(S) = 1$.

Jordan (1892) used inner content $c_i(S)$ and outer content $c_0(S)$, determined by using finite covers: Unfortunately, content of rationals of $[0, 1]$ and content of irrationals of $[0, 1]$ is 1. Thus $1 =$ content of $[0, 1] \neq 1 + 1$; not finitely additive on this partitioning of $[0, 1]$.

He then proceeded to define an integral for a bounded function on a Jordan measurable set $E$ as follows:

Let $P$ be a partition of the measurable set $E$ into measurable sets $E_1, E_2, \ldots, E_n$ with $E_i \cap E_j = \phi$, $i \neq j$. Jordan defined upper and lower integrals as:

$$\underline{\int_a^b} f(x)\, dx = \sup_P \sum_{i=1}^{n} \inf_{E_i} f \cdot c(E_i)\,,$$

$$\overline{\int_a^b} f(x)\, dx = \inf_P \sum_{i=1}^{n} \sup_{E_i} f \cdot c(E_i)\,.$$

"Equality" was the Jordan integral. Jordan has partitioned $[a, b]$ into measurable sets whereas Riemann and Cauchy used subintervals.

Borel (1898) stated properties a measure on sets should satisfy:

1. A measure is nonnegative.
2. The measure of a sum of disjoint sets is the sum of the measure of the individual sets.
3. The measure of the difference of two sets when one is a subset of the other is the difference of the measures.
4. Every set whose measure is not zero is uncountable.

Having described a measure, Borel said it should be restricted to sets $E$ which are "constructible" by

1. Forming a union, finite or countable, of disjoint intervals;
2. Forming the complement of any "constructible" set $E$ with respect to any other constructible set $G$ which covers $E$.

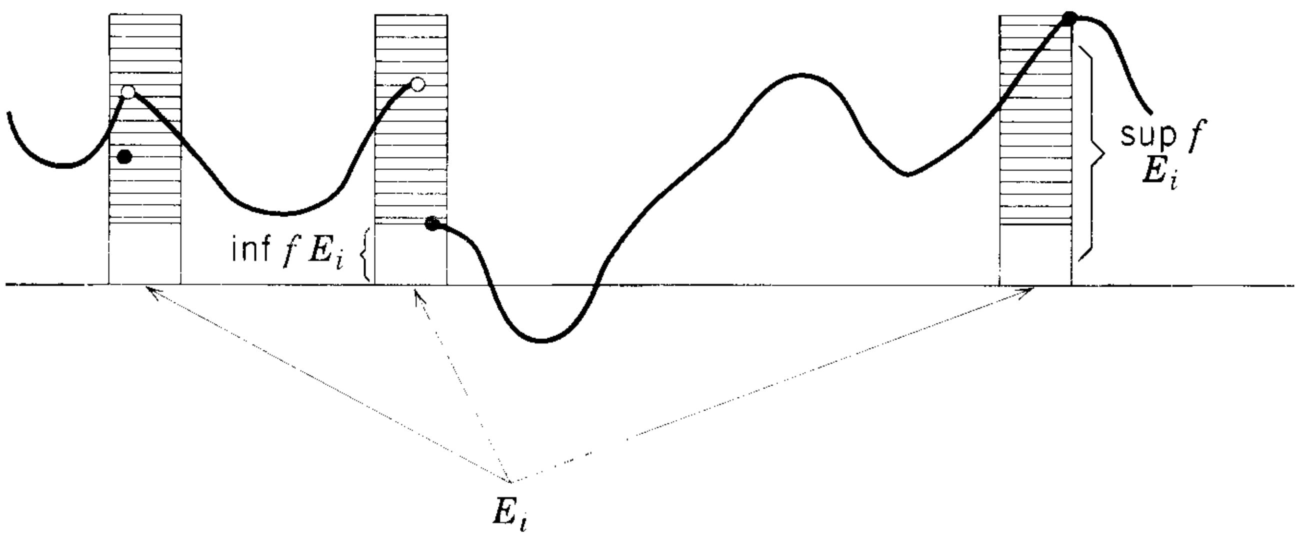
For example, the Borel measure of an interval would be its length, the measure of an open set would be the sum of the lengths of its constituent intervals (Theorem 2.4). Then we could measure a closed set as the complement of an open set, etc. (see Jordan's Construction).

$$\underline{\int_a^b} f(x)\,dx: \quad \inf_{E_1} f \cdot c(E_1) + \cdots + \inf_{E_i} f \cdot c(E_i) + \cdots + \inf_{E_n} f \cdot c(E_n):$$

$$\overline{\int_a^b} f(x)\,dx: \quad \sup_{E_1} f \cdot c(E_1) + \cdots + \sup_{E_i} f \cdot c(E_i) + \cdots + \sup_{E_n} f \cdot c(E_n):$$

The sets $E_i$ are Jordan measurable: finite approximations with intervals.

Jordan's Construction:




## 1.10 HENRI LEBESGUE (1875–1941), WILLIAM YOUNG (1863–1942)

After a few words about Young's contributions, we will concentrate on Lebesgue's. Young developed a measure theory and a theory of integration independently of Lebesgue, but about three or four years later. He did not combine these areas into a coherent whole as did Lebesgue. On the other hand, he was responsible for a Darboux interpretation, upper and lower sums, of the Lebesgue integral, which showed that Lebesgue's approach could be viewed as a very natural generalization of Jordan's integral:

$$l([x_{i-1}, x_i]) \longrightarrow c(E_i) \longrightarrow m(E_i).$$

Young also showed that his measurable sets were the same as Lebesgue's, and raised the question whether nonmeasurable sets exist, a question settled by Vitali in 1905.

Finally we come to Lebesgue. It was Lebesgue (1902) who brought together "measure" and "integration". He generalized the concept of measure by a constructive process based on countably infinite covers: Given a subset $E$ of $R$, he defines the outer measure $m_0(E)$ as the greatest lower bound of the sums $\sum_{i=1}^{\infty} l(I_i)$, where $E \subset \bigcup_{i=1}^{\infty} I_i$, $I_i$ an interval. If $E \subset [a, b]$, then the inner measure of $E$ is defined by $m_i(E) = (b-a) - m_0([a,b] - E)$. The bounded set $E$ is called Lebesgue measurable with measure $m(E)$ provided $m_i(E) = m_0(E) = m(E)$. Since the collection of all finite covers is a subcollection of all countable covers,

$$c_i(E) \leq m_i(E) \leq m_0(E) \leq c_0(E).$$

Every Peano-Jordan measurable set is Lebesque measurable and $c(E) = m(E)$. We will show later (Chapter 3) that Lebesgue measure is countably additive: $m(\bigcup_{i=1}^{\infty} E_k) = \sum_{i=1}^{\infty} m(E_k)$, $E_i \bigcap E_j = \phi$, $E_k$ Lebesgue measurable. Consider the partitioning of $[0, 1]$ into rationals and irrationals: $E = [0, 1]$, $E_1 = \{\text{rationals in } [0, 1]\}$, and $E_2 = \{\text{irrationals in } [0, 1]\}$. Then $c(E_1) = c(E_2) = c(E) = 1$, but $E = E_1 \bigcup E_2$, $E_1 \bigcap E_2 = \phi$. However, $m(E_1) = 0$, $m(E_2) = 1$, $m(E) = m(E_1) + m(E_2)$.

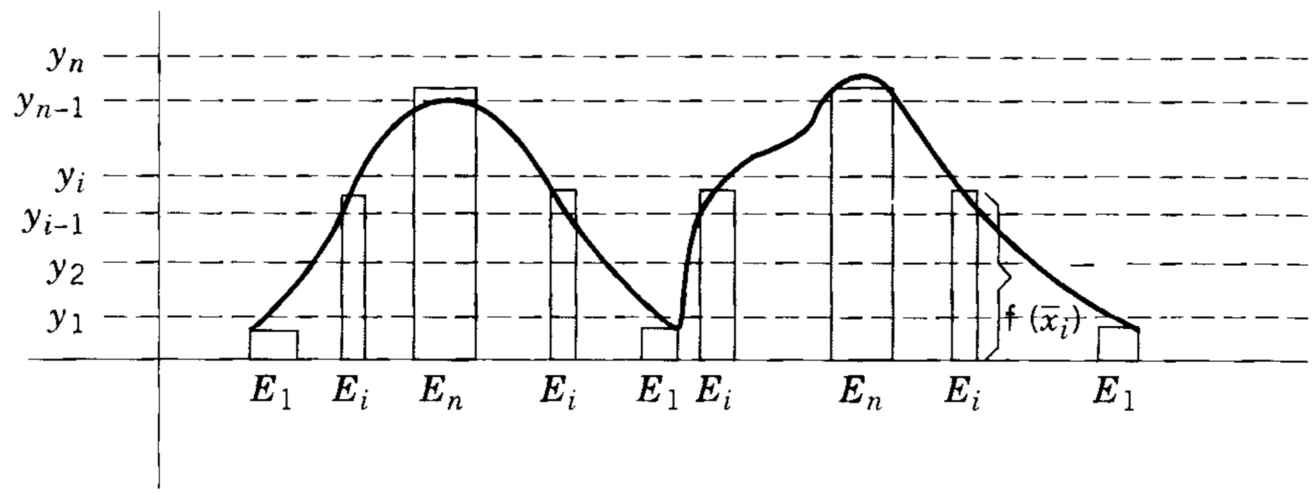
Thus the class of Lebesgue measurable sets is larger than the class of Peano-Jordan measurable sets.

It was a matter of dispute between Borel and Lebesgue whether the class of Lebesgue measurable sets was larger than the class of Borel measurable sets. In 1914 the Russian mathematican Suslin constructed a Lebesgue measurable set that was not Borel measurable: All Borel measurable sets are Lebesgue measurable but the converse is not true.

Lebesgue's construction of the integral was fundamentally different from all his predecessors: His simple, but brilliant idea: partition the **range** of the function rather than the domain!

Assume $A < f < B$ on $[a, b]$. In place of $a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$, we have $A = y_0 < y_1 < \cdots < y_{i-1} < y_i < \cdots < y_n = B$. Specifically, if we let $E_i = \{x \mid y_{i-1} \leq f(x) < y_i\}$, then $E_i \bigcap E_j = \phi$, and $[a, b] = \bigcup_{i=1}^{n} E_i$. Now let $\bar{x}_i$ be any point in $E_i$ and form the Lebesgue sum: $\sum_{i=1}^{n} f(\bar{x}_i) m(E_i)$.

Lebesgue's Sum:




$$f(\bar{x}_1)m(E_1) + \cdots + f(\bar{x}_i)m(E_i) + \cdots + f(\bar{x}_n)m(E_n)$$

The restriction we need to impose on $f$ is that the sets $E_i = \{x \in [a, b] \mid y_{i-1} \leq f(x) < y_i\}$ need to be Lebesgue measurable. Such a function is appropriately said to be Lebesgue measurable: $f^{-1}([y_{i-1}, y_i))$ is a Lebesgue measurable set. It will be shown (Chapter 5) that all functions that are Riemann integrable are Lebesgue integrable, but that the converse is false. We make, however, two observations regarding "Riemann" vs. "Lebesgue:"

1. Since $\bar{x}_i$ may be any point of $[x_{i-1}, x_i]$ in the Riemann sum, as $\bar{x}_i$ "moves" throughout this interval, the Riemann sums cannot vary too much if we are to have a Riemann integral. In other words, as $\bar{x}_i$ "moves" throughout the interval, $f(\bar{x}_i)$ cannot vary too much. No problem if $f$ is continuous or does not have many discontinuities.
2. In the Lebesgue sum, again $\bar{x}_i$ may be any point of $E_i$, but as $\bar{x}_i$ "moves" throughout $E_i$, the Lebesgue sums will not vary too much, because by construction, $y_{i-1}m(E_i) \leq f(\bar{x}_i)m(E_i) < y_i m(E_i)$ and $y_i - y_{i-1}$ is small.

In fact, in light of Darboux's lower and upper Riemann sums, it is natural to form lower and upper Lebesgue sums:

$$\sum_{i=1}^{n} y_{i-1}m(E_i) \leq \sum_{i=1}^{n} f(\bar{x}_i)m(E_i) \leq \sum_{i=1}^{n} y_i m(E_i),$$

that is,

$$\sum_{i=1}^{n} y_i m(E_i) - \sum_{i=1}^{n} y_{i-1} m(E_i) = \sum_{i=1}^{n} (y_i - y_{i-1}) m(E_i)$$

$$\leq \| \Delta y \| \sum_{i=1}^{n} m(E_i) = \| \Delta y \| \, m(E) = \| \Delta y \| \, (b-a) .$$

The key here then; we must be able to "measure" a large variety of sets.

## 1.11 HISTORICAL SUMMARY

Eudoxus (400 B.C.E.) and Archimedes (250 B.C.E.): Method of Exhaustion

Cavalieri (1635) and Fermat (1635): Specific Functions

$$\int_0^1 x^2 \, dx, \quad \int_0^1 x^{p/q} \, dx$$

Newton (1666) and Leibnitz (1684): Antiderivatives for Specific Functions

$$\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad \left(\frac{x^3}{3}\right)' = x^2$$

Cauchy (1823): $f$ Continuous

$$\sum_{i=1}^{n} f(x_{i-1}) \cdot l([x_{i-1}, x_i])$$

Riemann (1854): $f$ Not Too Discontinuous and Bounded

$$\sum_{i=1}^{n} f(\bar{x}_i) \cdot l([x_{i-1}, x_i])$$

Darboux (1875): Equivalent to Riemann

$$\underline{\int_a^b} f(x)\,dx = \sup_P \left\{ \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \cdot l([x_{i-1}, x_i]) \right\}$$

$$\overline{\int_a^b} f(x)\,dx = \inf_P \left\{ \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \cdot l([x_{i-1}, x_i]) \right\}$$

Peano-Jordan (1893) and Borel (1898): $f$ Bounded

$$\underline{\int_a^b} f(x)\,dx = \sup_P \left\{ \sum_{i=1}^{n} \inf_{E_i} f \cdot c(E_i) \right\}$$

$$\overline{\int_a^b} f(x)\,dx = \inf_P \left\{ \sum_{i=1}^{n} \sup_{E_i} f \cdot c(E_i) \right\}$$

Lebesgue (1902) and Young (1905): $f$ Bounded

$$\underline{\int_a^b} f(x)\,dx = \sup_P \left\{ \sum_{i=1}^{n} \inf_{E_i} f \cdot m(E_i) \right\}$$

$$\overline{\int_a^b} f(x)\,dx = \inf_P \left\{ \sum_{i=1}^{n} \sup_{E_i} f \cdot m(E_i) \right\}$$

## 1.12 WHY LEBESGUE?

> The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of that beauty that strikes the senses, the beauty of qualities and appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp.
>
> —Henri Poincaré

Some examples are in order:

**Example 1:** $f(x) = \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1\,. \end{cases}$

1. $f$ is discontinuous at $x = 1/2$ and thus is not Cauchy integrable.
2. $f$ is Riemann integrable because we can "cover" $1/2$ with arbitrarily small intervals, and $\int_0^1 f(x)\,dx = 1/2$.
3. $f$ is not Newton-Leibnitz integrable: We cannot find an antiderivative $F$ so that $F' = f$ on $[a, b]$: The reader may recall from calculus that the derivative has the so-called intermediate-value property ($g(x) \equiv F(x) - 1/2x$, $0 \le x \le 1$; $g'(0) = -1/2$, $g'(1) = 1/2$; $g'(c) = F'(c) - 1/2 = 0$. Therefore $f(c) = 1/2$?).
4. $f$ is Lebesgue integrable: $0 \cdot m([0, 1/2)) + 1 \cdot m([1/2, 1]) = 1/2$.

**Example 2:** $f(x) = \begin{cases} -2/x \cos(1/x^2) + 2x \sin(1/x^2), & 0 < x \le 1 \\ 0, & x = 0\,. \end{cases}$

1. $f$ is discontinuous at $x = 0$ and thus not Cauchy integrable.
2. $f$ is unbounded near $x = 0$ and is not Riemann integrable.
3. $f$ is not Lebesgue integrable (unbounded) even in an extended sense: We show (Chapter 5) $f$ is Lebesgue integrable iff $|f|$ is Lebesgue integrable: $|\,1/x \cos(1/x^2)\,|$ just oscillates too wildly.
4. $f$ is Newton-Leibnitz integrable; it has an antiderivative:

$$F(x) = \begin{cases} x^2 \sin(1/x^2), & 0 < x \le 1 \\ 0, & x = 0\,. \end{cases}$$

We have $F'(x) = f(x)$ and $\int_0^1 f(x)\,dx = F(1) - F(0) = \sin 1$.

In our next example, we consider the Dirichlet function.

**Example 3:** Let $f(x) = \begin{cases} 1, & 0 \le x \le 1,\ x \text{ rational} \\ 0, & 0 \le x \le 1,\ x \text{ irrational}\,. \end{cases}$

The range of $f$ is the set $\{0, 1\}$, so following Lebesgue's approach of partitioning the range, let $A = -1/4 < 3/4 < 5/4 = B$. Thus, $E_1 = \{x \in [0, 1] \mid -1/4 \le f(x) < 3/4\} = \{\text{irrational numbers in } [0, 1]\}$ and $E_2 = \{x \in [0, 1] \mid 3/4 \le f(x) < 5/4\} = \{\text{rational numbers in } [0, 1]\}$.

Then the Lebesgue integral of $f$ over $[0, 1]$ is

$$\int_{[0,1]} f = 0 \cdot m(E_1) + 1 \cdot m(E_2).$$

By enumerating the rational numbers in $[0, 1] : r_1, r_2, \ldots, r_k, \ldots$ and covering $r_k$ with the interval $(r_k - \epsilon/2^k, r_k + \epsilon/2^k)$, Lebesgue concluded (logically and consistently) that

$$m(E_2) \leq \sum m\Big((r_k - \epsilon/2^k, r_k + \epsilon/2^k)\Big) = 2\epsilon$$

and by the arbitrary nature of $\epsilon$, $m(E_2)$, the measure of the rational numbers in $[0, 1]$, must be 0. Thus the Dirichlet function is Lebesgue integrable on $[0, 1]$ and has value zero. That this function is not Riemann integrable is immediate since every "lower sum" is 0 and every "upper sum" is 1. The Dirichlet function is "just too discontinuous" to be susceptible to Riemann integration. The class of Lebesgue integrable functions is "probably" larger (as we show in Chapter 5) than the class of Riemann integrable functions.

In **hindsight**, the foremost advantage of the Lebesgue integral is in facilitation of limit operations; specifically, when is it true that

$$\lim \int f_k = \int \lim f_k\,?$$

The elementary results for the Riemann integral require uniform convergence of the sequence of functions $(f_k)$: If

$$\lim f_k = f \text{ uniformly on } [a, b], \text{ then}$$

$$\lim \int_a^b f_k(x)\,dx = \int_a^b f(x)\,dx = \int_a^b \lim f_k(x)\,dx\,.$$

The requirement of uniform convergence is just too restrictive. We look at some more examples:

**Example 4:** $f_k(x) = \begin{cases} k, & 0 < x \leq 1/k^2 \\ 0, & \text{otherwise}\,. \end{cases}$

Then $\lim f_k(x) \equiv 0$ on $[0, 1]$, and $\lim \int_0^1 f_k(x)\,dx = 0 = \int_0^1 \lim f_k(x)\,dx$: "lim" and "$\int$" may be interchanged, and we do not have uniform convergence. Sometimes we are lucky.

**Example 5:** $f_k(x) = \begin{cases} k, & 0 < x \leq 1/k \\ 0, & \text{otherwise}. \end{cases}$

Then $\lim f_k(x) \equiv 0$ on $[0, 1]$, and

$$\lim \int_0^1 f_k(x)\, dx = 1 \neq \int_0^1 \lim f_k(x)\, dx.$$

It turns out that even the Lebesgue integral does not work in this case. Lebesgue integration is not all-powerful.

**Example 6:** Let $r_1, r_2, \ldots, r_k, \ldots$ be any enumeration of the rational numbers in $[0, 1]$ and define a sequence of functions $(f_k)$ by

$$f_k(x) = \begin{cases} 1, & x = r_1, r_2, \ldots, r_k \\ 0, & \text{otherwise}. \end{cases}$$

Each $f_k$, having only a finite number of discontinuities, is Riemann integrable and $\int_0^1 f_k(x)\, dx = 0$, and so $\lim \int_0^1 f_k(x)\, dx = 0$. But $\lim f_k$ is the Dirichlet function, which is not Riemann integrable: $\int_0^1 \lim f_k(x)\, dx$ does not make sense, let alone equal to 0! However, with the Lebesgue integral,

$$\lim \int_{[0,1]} f_k = 0 = \int_{[0,1]} \lim f_k.$$

**Example 7:** $f_k(x) = e^{-kx}/\sqrt{x}$, $x > 0$. Then $\lim f_n = 0$ (not uniform) on $(0, \infty)$. The Riemann integral is defined for bounded functions. Of course we could discuss the so-called improper Riemann integral, but the Lebesgue integral is defined for bounded or unbounded functions on bounded or unbounded subsets of $R$. Using the so-called Lebesgue Dominated Convergence Theorem, we will be able to show for this function, with Lebesgue integrals, that

$$\lim \int_{(0,\infty)} f_k = 0 = \int_{(0,\infty)} \lim f_k.$$

The Lebesgue integral is not a panacea for every conceivable problem, but its much greater applicability in analysis (probability theory, harmonic analysis, functional analysis, fractals via Hausdorff measure, to name a few areas of research), justify the initial investment of time and effort. In the end, we have something simpler and more powerful, much more powerful!

> Tradition cannot be inherited, and if you want it you must obtain it by great labour.
>
> —T.S. Eliot

> As the drill will not penetrate the granite unless kept to the work hour after hour, so the mind will not penetrate the secrets of mathematics unless held long and vigorously to the work. As the sun's rays burn only when concentrated, so the mind achieves mastery in mathematics, and indeed in every branch of knowledge, only when its possessor hurls all his forces upon it. Mathematics, like all the other sciences, opens its doors to those only who knock long and hard. No more damaging evidence can be adduced to prove the weakness of character than for one to have aversion to mathematics; for whether one wishes so or not, it is nevertheless true; that to have aversion for mathematics means to have aversion to accurate, painstaking, and persistent hard study, and to have aversion to hard study is to fail to secure a liberal education, and thus fail to compete in that fierce and vigorous struggle for the highest and the truest and the best in life which only the strong can hope to secure.
>
> —B.F. Finkel

> The gods did not reveal all things to men at the start; but, as time goes on, by searching, they discover more and more.
>
> —Xenophanes

> Perhaps the most unfortunate fact about mathematics is that it requires us to reason, . . .
>
> —Morris Kline

> In questions of science the authority of a thousand is not worth the humble reasoning of a single individual.
>
> —Galileo

> . . . analysis. This is a rather vague expression for those parts of mathematics in which the ideas of limit, variation, function, and so on, are upper most. . . . An analyst should be able to handle such things as integrals and infinite series just as well as if they were the simple expressions of elementary algebra.
>
> —E.C. Titchmarsh

$$\frac{d}{dx}\left(\int_{a}^{x} f(t)\, dt\right) = f(x)$$

# 2

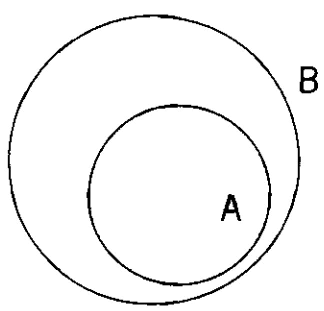
# Preliminaries

We review material useful in developing Lebesgue measure and integration. (If you have completed the typical undergraduate analysis, advanced calculus course, skip to Chapter 3 and refer back as necessary.)

## 2.1 SETS

### 2.1.1 Definitions

A *set* is a collection of objects which are called its members. If $x$ is a member of a set $A$, we write $x \in A$. If $x$ is not a member of a set $A$, we write $x \notin A$. The set which contains no members will be called the *empty set* and will be denoted by $\emptyset$. If a set has at least one member, it is called *nonempty*. If every member of a set $A$ is also a member of a set $B$, we say $A$ is a *subset* of $B$ and write $A \subset B$.

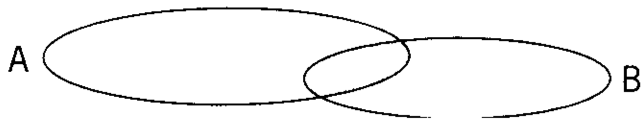



If $A \subset B$ and $B \subset A$, we write $A = B$.

If $A$ and $B$ are subsets of a given set $X$, we may form new subsets of $X$ by three fundamental set-theoretic operations:

1. The *intersection* $A \cap B$ is the set of all members which belong to both $A$ and $B$. Thus

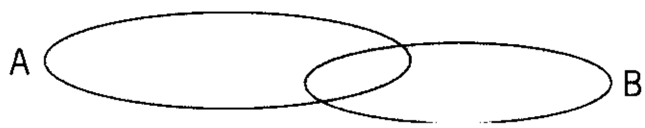
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$




If $A \cap B = \emptyset$, we say $A$ and $B$ are *disjoint*.

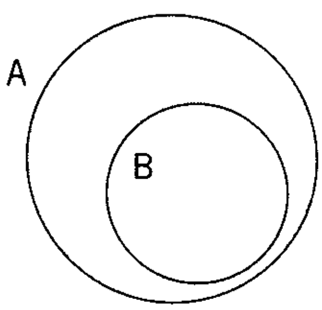

2. The *union* $A \cup B$ is the set of members that are in either $A$ or $B$. Thus

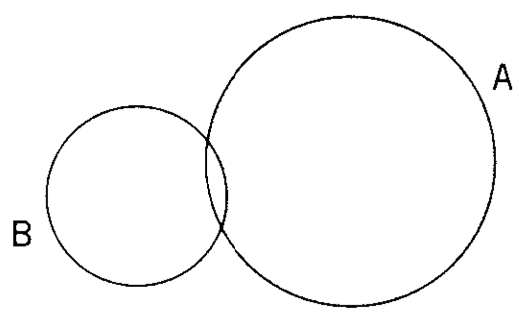
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$



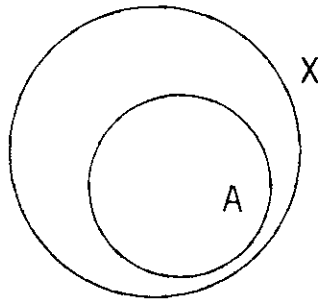

3. The *difference* $A - B$ is the set of members of $A$ that are not members of $B$. Thus

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$




The difference $X - A$ will be called the *complement* of $A$ with respect to $X$, sometimes written $A^c$, whenever $A \subset X$.




The symbol $2^A$ will denote the *collection of all subsets* of $A$.

Let $\{E_\alpha\}$ be any collection of subsets of $X$. We say this collection is *mutually disjoint* if $E_{\alpha_1} \cap E_{\alpha_2} = \emptyset$, $\alpha_1 \neq \alpha_2$, for all pairs in the collection. With $\cup E_\alpha = \{x \mid x \in E_\alpha \text{ for some } \alpha\}$ and $\cap E_\alpha = \{x \mid x \in E_\alpha \text{ for all } \alpha\}$, the reader should prove De Morgan's Laws:

1. "The complement of a union is the intersection of the complements":

$$(\cup E_\alpha)^c = \cap(E_\alpha^c).$$

2. "The complement of an intersection is the union of the complements":

$$(\cap E_\alpha)^c = \cup(E_\alpha^c).$$

## 2.2 SEQUENCES OF SETS

### 2.2.1 Definitions

A sequence of sets $(A_k)$ is said to be *monotonically decreasing* if

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

and *monotonically increasing* if

$$A_1 \subset A_2 \subset A_3 \subset \cdots.$$

Given a sequence of sets $(A_k)$, form sequences $(\underline{A}_k)$ and $(\overline{A}_k)$ by setting

$$\underline{A}_k = \bigcap_{n \geq k} A_n \qquad \text{and} \qquad \overline{A}_k = \bigcup_{n \geq k} A_n.$$

Then $\underline{A}_1 \subset \underline{A}_2 \subset \underline{A}_3 \subset \cdots$ and $\overline{A}_1 \supset \overline{A}_2 \supset \overline{A}_3 \supset \cdots$. The sequences $(\underline{A}_k)$, $(\overline{A}_k)$ are monotonic. Furthermore, $\underline{A}_i \subset \underline{A}_{i+j} \subset \overline{A}_{i+j} \subset \overline{A}_j$: $\underline{A}_i \subset \overline{A}_j$ for natural numbers $i$ and $j$.

Set

$$\underline{A} = \bigcup_{k \geq 1} \underline{A}_k = \bigcup_{k \geq 1} \left( \bigcap_{n \geq k} A_n \right)$$

and

$$\overline{A} = \bigcap_{k \geq 1} \overline{A}_k = \bigcap_{k \geq 1} \left( \bigcup_{n \geq k} A_n \right).$$

$\underline{A}$ is called the *limit inferior* of the sequence $(A_k)$:

$$\underline{A} = \liminf A_k .$$

$\overline{A}$ is called the *limit superior* of the sequence $(A_k)$:

$$\overline{A} = \limsup A_k .$$

If $\underline{A} = \overline{A}$, this common set is denoted by $A$ and the sequence of sets $(A_k)$ is said to have *limit* $A$:

$$\liminf A_k = \lim A_k = \limsup A_k .$$

For a sequence of sets, $(A_k)$, a useful fact:

**Proposition 2.1** *Given a sequence of sets $(A_k)$, we can write $\cup A_k = \cup B_k$ where the sets $B_k$ are mutually disjoint and $B_k \subset A_k$.*

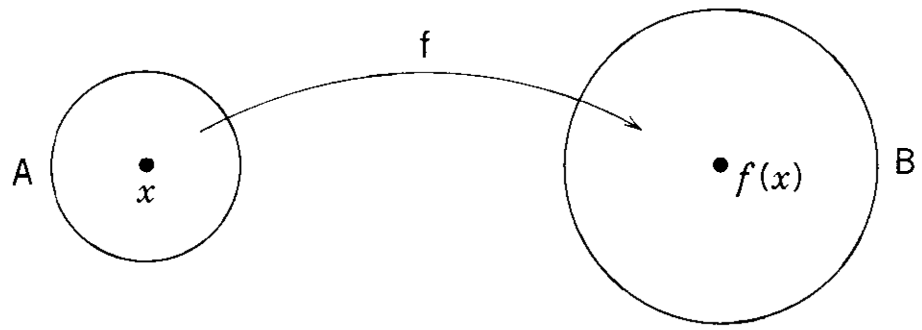
*Proof:* $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - (A_1 \cup A_2)$, and, in general, $B_k = A_k - \bigcup_{i=1}^{k-1} A_i$. Certainly $B_k \subset A_k$ and $\cup B_k \subset \cup A_k$. Suppose $x \in \cup A_k$. Then $x \in A_k$ for some $k$. Let $K$ be the smallest natural number so that $x \in A_K$. Thus, $x \in A_K - \bigcup_{i=1}^{K-1} A_i = B_K$ and the argument is complete. ■

## 2.3 FUNCTIONS

That flower of modern mathematical thought—the notion of a function.
—Thomas McCormack

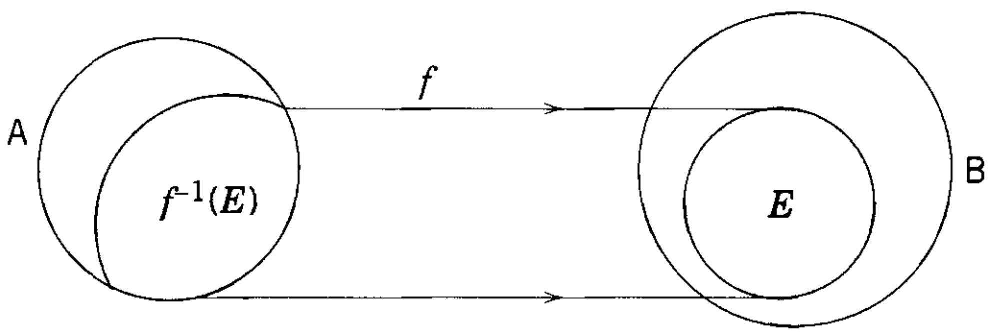
### 2.3.1 Definitions

Consider two sets $A$ and $B$, and suppose that with each member $x$ of $A$ there is associated exactly one member of $B$, denoted by $f(x)$:




Then $f$ is said to be a *function* from $A$ to $B$. $A$ is called the *domain* of $f$ and the set $\{f(x) \mid x \in A\}$ is called the *range* of $f$. We sometimes write $f(A)$. If $f(A) = B$, we say $f$ maps $A$ *onto* $B$. If $E \subset B$, $f^{-1}(E)$ is called the *inverse image* of $E$ under $f$ and is given by:

$$f^{-1}(E) = \{x \in A \mid f(x) \in E\}.$$




If for each $y \in B$, $f^{-1}(y)$ consists of at most one member of $A$, then $f$ is said to be a $1-1$ (*one to one*) mapping of $A$ into $B$; whenever $x_1 \neq x_2$, $x_1, x_2 \in A$, we have $f(x_1) \neq f(x_2)$. The reader should demonstrate the following proposition concerning inverse images and direct images.

**Proposition 2.2** *If $f$ is a function from $A$ into $B$, with $\{A_\alpha\}$ and $\{B_\alpha\}$ collections of subsets of $A$ and $B$, respectively, then*

1. $f(\cup A_\alpha) = \cup f(A_\alpha)$
2. $f^{-1}(\cup B_\beta) = \cup f^{-1}(B_\beta)$
3. $f^{-1}(\cap B_\beta) = \cap f^{-1}(B_\beta)$
4. $f^{-1}(B_\alpha^c) = (f^{-1}(B_\alpha))^c$.

The observant reader may notice the absence of $f(\cap A_\alpha) \stackrel{?}{=} \cap f(A_\alpha)$ and $f(A_\alpha^c) \stackrel{?}{=} (f(A_\alpha))^c$. These statements are false: $A_1 = \{a, c\}$,

$A_2 = \{b, a\}$, $A = \{a, b, c\}$, $B = \{d, e\}$, $f(a) = d$, and $f(b) = f(c) = e$. Then $f(A_1) = B = f(A_2)$ and $f(A_1 \cap A_2) = \{d\} \neq \{d, e\} = f(A_1) \cap f(A_2)$. Also, $f(A_1^c) = \{e\}$ and $(f(A_1))^c = \emptyset$.

Generally speaking, inverse images of unions, complements, and intersections preserve equality.

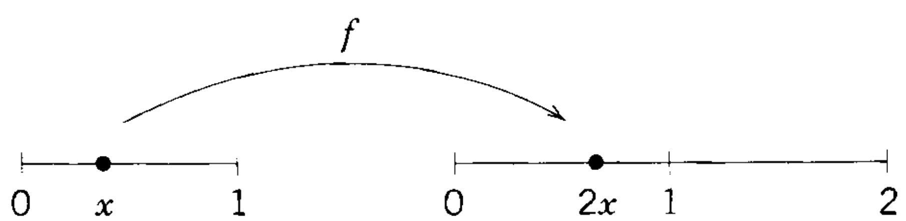
> If you can take away some of the terms of a collection, without diminishing the number of terms, then there is an infinite number of terms in the collection.
>
> —Bertrand Russell

### 2.3.2 Definitions

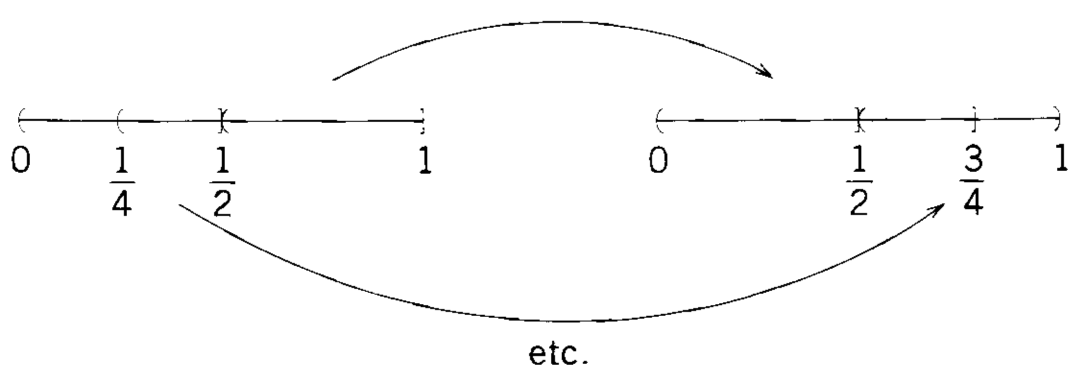
Sets $A$ and $B$ are *equivalent*, $A \sim B$, if there is a $1-1$ mapping from $A$ onto $B$.

1. $A$ is *finite* if $A \sim \{1, 2, \ldots, n\}$ for some $n$.
2. $A$ is *infinite* if $A$ is not finite.
3. $A$ is *countable* if $A$ is equivalent to the set of natural numbers: $A \sim \{1, 2, \ldots\}$ (Cantor's definition). In this case we may write $A = \{a_1, a_2, \ldots\}$.
4. $A$ is *uncountable* if $A$ is neither finite nor countable.

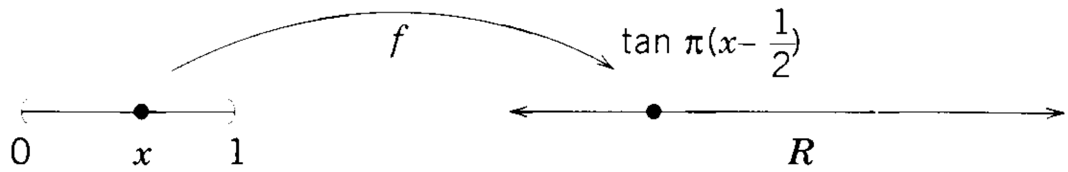
**Example 1:** $[0, 1] \sim [0, 2]$



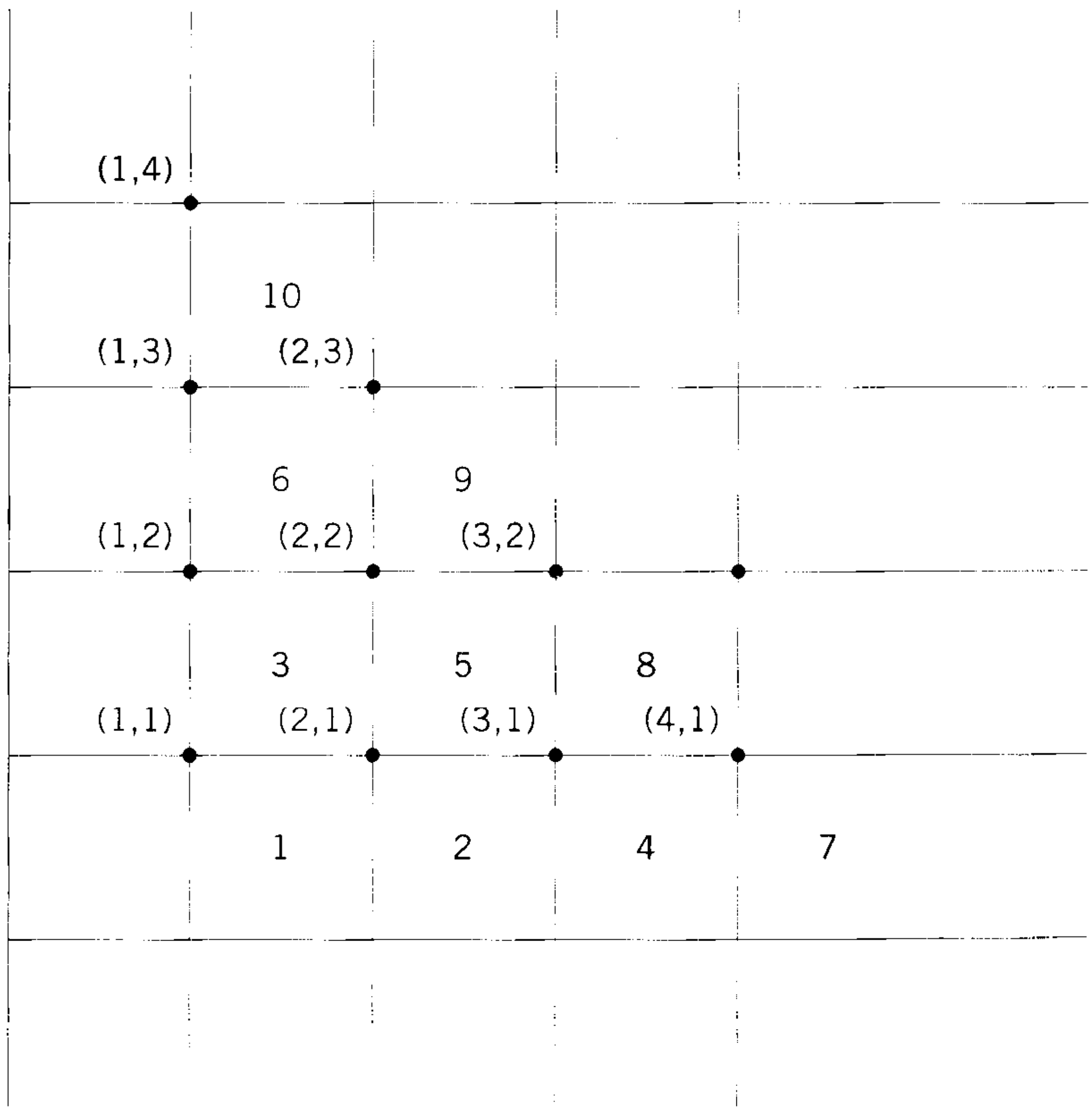

**Example 2:** $(0, 1] \sim (0, 1)$



**Example 3:** $(0, 1) \sim R$




**Example 4:** For the set of natural numbers $\mathbb{N}$, $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$:



**Example 5:** The set of rational numbers is countable. Cantor's argument:

$$\begin{array}{ccccccccc}
0 \longrightarrow 1 & & -1 \longrightarrow 2 & & -2 \longrightarrow 3 & & -3 & 4 & \cdots \\
\downarrow \ \nearrow & \swarrow & & \nearrow & \swarrow & \nearrow & & & \\
\frac{1}{2} & -\frac{1}{2} & \frac{2}{2} & -\frac{2}{2} & \frac{3}{2} & -\frac{3}{2} & & \frac{4}{2} & \cdots \\
\swarrow & \nearrow & \swarrow & \nearrow & \swarrow & & & & \\
\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{3}{3} & -\frac{3}{3} & & \frac{4}{3} & \cdots \\
\downarrow \ \nearrow & \swarrow & \nearrow & \swarrow & \nearrow & & & & \\
\frac{1}{4} & -\frac{1}{4} & \frac{2}{4} & -\frac{2}{4} & \frac{3}{4} & -\frac{3}{4} & & \frac{4}{4} & \cdots \\
\swarrow & \nearrow & \swarrow & \nearrow & \swarrow & & & & \\
\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{3}{5} & & \frac{4}{5} & \cdots
\end{array}$$

$$\begin{array}{rcccccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 9 & 10 \\
\mathbb{Q}: & 0 & 1 & \frac{1}{2} & -1 & 2 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & & -\frac{1}{3} & -2
\end{array}$$

Omit $\frac{2}{2}$. Already counted 1.

It seems reasonable that a line segment has "fewer" points than a square. As an indication of the subtleties involved, this intuitive idea is demonstrably false!

I see it but I do not believe it.

—Cantor

**Example 6:** If $x$ is a real number, $0 < x \leq 1$, then $x = .a_1a_2\ldots$ with $a_i$ either 0 or 1, the so-called dyadic expansion of $x$. For example, $1/2 = .1000\cdots = .0111\cdots$. We agree to use $.0111\cdots$ in this case. If the reader needs a fuller explanation of dyadic expansions instead of decimal expansions please see Appendix A. To any dyadic expansion we assign a sequence of natural numbers, stating the distances between the successive 1's of the dyadic expansion. For example,

$$x = .\underbrace{01}_{2}\underbrace{0001}_{4}\underbrace{001}_{3}\underbrace{0001}_{4}\underbrace{1}_{1}\cdots \longleftrightarrow 24341\cdots.$$

On the other hand, given any natural number, we may construct a real number:

$$34711\cdots \longleftrightarrow .0010001000000111\cdots.$$

Thus the set of all real numbers has the same "number" of elements as the set of all sequences of natural numbers. Now, we decompose any sequence of natural numbers into two sequences in the obvious fashion: take 1st, 3rd, 5th, etc. and 2nd, 4th, 6th, etc.:

$$x = .\underbrace{01}_{2}\underbrace{0001}_{4}\underbrace{001}_{3}\underbrace{0001}_{4}\underbrace{1}_{1} \longleftrightarrow 24341\cdots$$

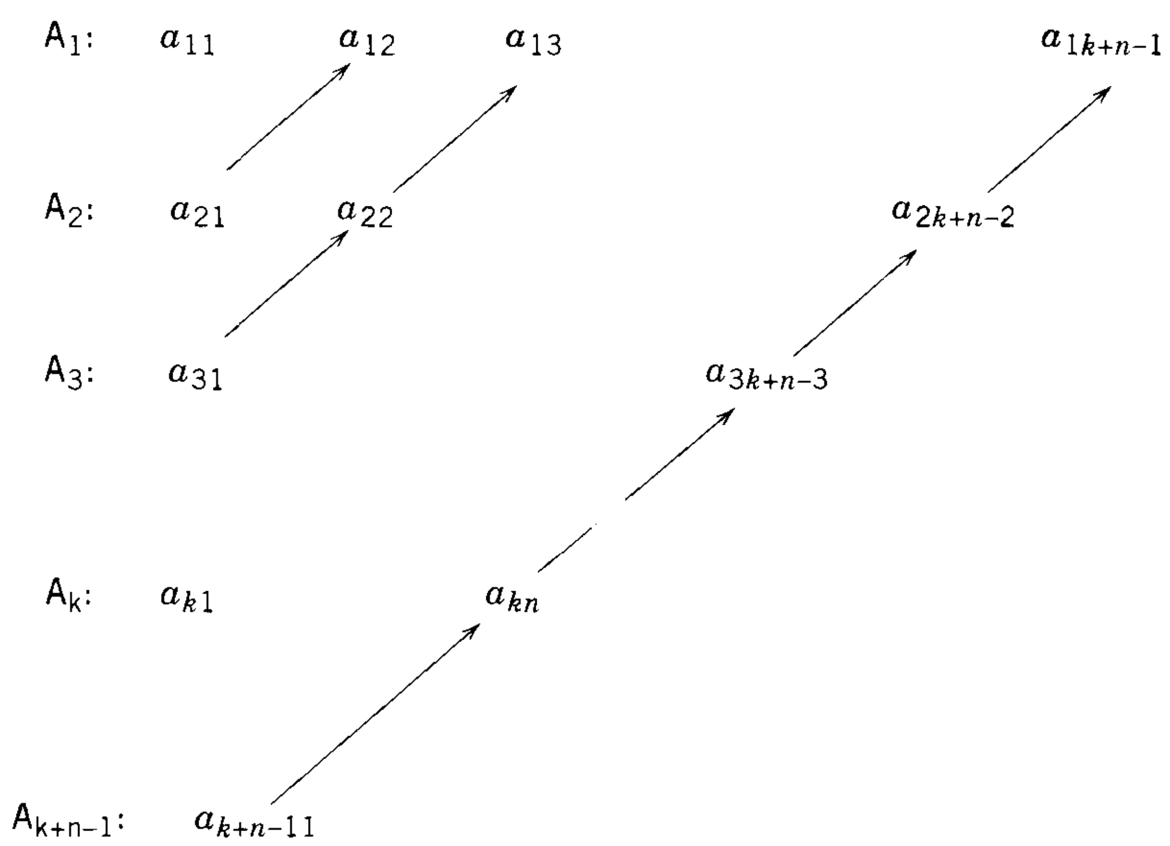
$$\longleftrightarrow(231\cdots,44\cdots,)\longleftrightarrow(.010011\cdots,.00010001\cdots)$$

Thus the numbers of $(0, 1]$ are in one-to-one correspondence with $(x, y)$, $0 < x, y \leq 1$. We must be careful!

> $\cdots$[Regarding Einstein] Common sense, he said, is merely the layer of prejudices which our early training in science has left in our minds.
>
> —H. Margenau

**Proposition 2.3** *The union of a countable collection of countably infinite sets is countable.*

*Proof:* Let $A_k = \{a_{k1}, a_{k2}, \ldots\}$. We will show $\cup A_k$ is countable. A diagram facilitates the argument.




$a_{11}$ is the last term of the first diagonal.

$a_{12}$ is the last term of the second diagonal.

$\vdots$

$a_{1k+n-1}$ is the last term of the $k + n - 1^{\text{th}}$ diagonal.

In other words, if we sum along the diagonals, $a_{1k+n-1} \longleftrightarrow ((k+n-1)(k+n)/2)$. But then the term on the $k+n-1^{\text{th}}$ diagonal preceding $a_{1k+n-1}$, $a_{2k+n-2}$, will correspond with $((k+n-1)(k+n)/2) - 1$, i.e., $a_{2k+n-2} \longleftrightarrow ((k+n-1)(k+n)/2) - 1$. In general, $a_{kn} \longleftrightarrow ((k+n-1)(k+n)/2) - (k-1)$.

Define

$$f((k,n)) = \frac{(k+n-1)(k+n)}{2} - (k-1).$$

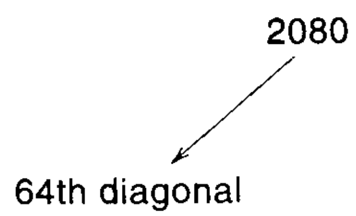
Is $f$ $1-1$?

Suppose $((k_1+n_1-1)(k_1+n_1)/2) - (k_1-1) = ((k_2+n_2-1)(k_2+n_2)/2) - (k_2-1)$. Then

$$n_1 - n_2 = \frac{1}{2}[(k_2+n_2-1)(k_2+n_2-2) - (k_1+n_1-1)(k_1+n_1-2)].$$

If $k_2+n_2 = k_1+n_1$, then $n_1 - n_2 = 0$ and $k_1 = k_2$, $n_1 = n_2$. If $k_2+n_2 > k_1+n_1$, then $k_2+n_2-1 \geq k_1+n_1$ and $k_2+n_2-2 \geq k_1+n_1-1$. Hence $n_1 - n_2 \geq k_1+n_1-1$, or $1 \geq k_1+n_2 \geq 2$. An analogous argument for $k_2+n_2 < k_1+n_1$ completes the argument for showing $f$ is $1-1$.

Is $f$ onto? Given a natural number $N$, can we find $(k,n)$ so that $f((k,n)) = N$? For example, suppose you select $N = 2039$. Again, "diagonals" are the key: Choose $n$ so that $(n(n-1)/2) < 2039 \leq ((n+1)n/2)$. We have $(64 \cdot 63/2) < 2039 \leq (65 \cdot 64/2)$




That is, $(1,64) \longleftrightarrow 2080$. Go "back down" the $64^{\text{th}}$ diagonal 41 "moves". Thus

$$(1+41, 64-41) \longleftrightarrow 2039.$$

The reader may complete the argument for general $N$. ∎

*Note*: What if $g((k,n)) \longleftrightarrow 2^{k-1}(2n-1)$? Is $g$ a $1-1$ mapping of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$?

## 2.4 REAL NUMBERS

We assume the reader is familiar with the real number system.

### 2.4.1 Problem

1. If $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.
2. If $a < c$ for every $c > b$, then $a \leq b$.

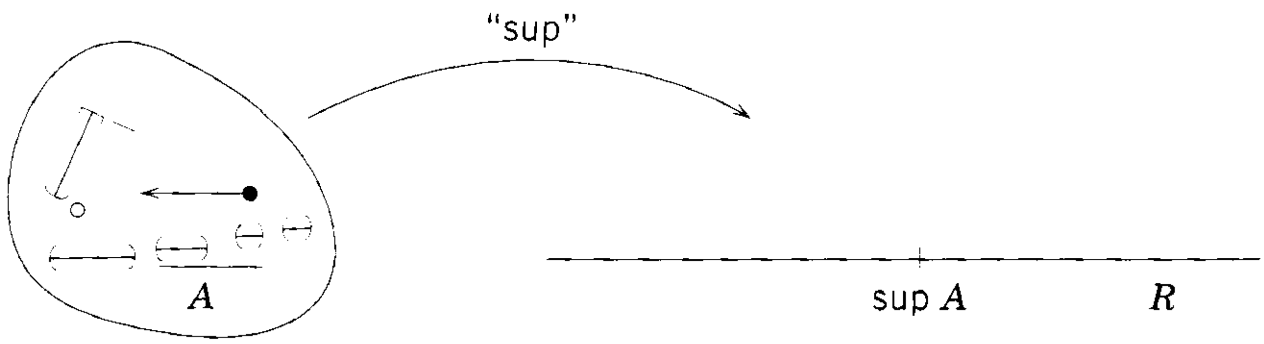
### 2.4.2 Definition

$A$ is a nonempty set of real numbers. We say $A$ is *bounded above* if there is a real number $b$ so that $a \leq b$ for all $a \in A$. The number $b$ is called an *upper bound* for the set $A$. If $A$ is bounded above and below, $A$ is said to be *bounded*. By a *supremum* (least upper bound) of $A$, we mean a real number, written sup $A$, so that:

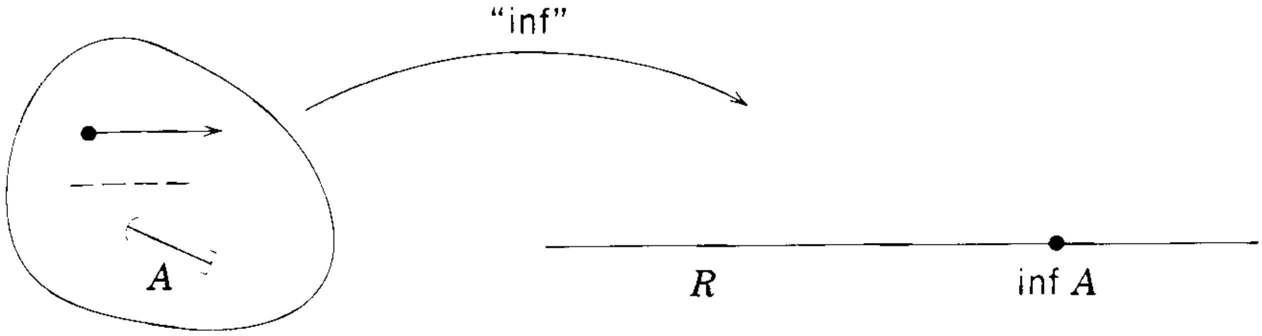
1. $a \leq \sup A$ for all $a \in A$ (sup $A$ is an upper bound of $A$).
2. No number less than sup $A$ is an upper bound of $A$ (sup $A$ is the least, smallest, upper bound of $A$).

We define a *greatest lower bound* (infimum) of $A$, written inf $A$, in the obvious manner.

We have defined sup $A$ for any nonempty set of real numbers that is bounded above and inf $A$ for any nonempty set of real numbers that is bounded below. We may think of "sup" and "inf" as set functions, functions whose domains are collections of sets:




*Collection:* Nonempty subsets of $R$ that are bounded above.




*Collection:* Nonempty subsets of $R$ that are bounded below.

The reader should show: If a set of real numbers has a least upper bound, it has only one. A corresponding statement holds for the "inf". Indeed, we have set functions!!

### 2.4.3 Least Upper Bound Property (LUB)

Every nonempty set of real numbers which is bounded above has a least upper bound called the supremum.

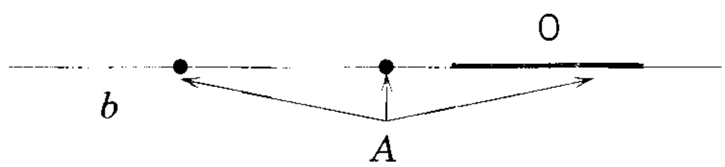
The least upper bound of a set need not be a member of the set: If $S = (0, 1)$, then $\sup S = 1$.

As a consequence of the LUB we have the greatest lower bound property (GLB).

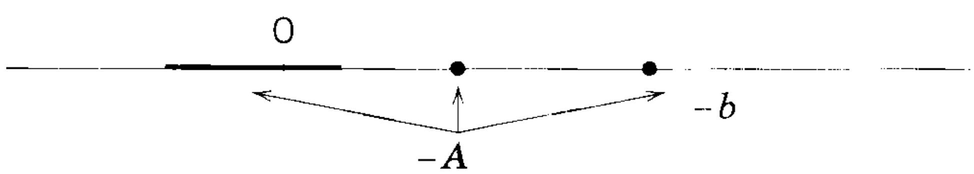
### 2.4.4 Greatest Lower Bound Property (GLB)

Every nonempty set of real numbers which is bounded below has a greatest lower bound called the infimum.

*Proof:* Let $A$ be a nonempty set of real numbers that is bounded below by $b$. We will show $A$ has an infimum. A drawing suggests the argument we will use.




Form a new set, $-A$, the "reflection of $A$": $-A = \{-x \mid x \in A\}$.



Because $b \le a$ for all $a \in A$, $-a \le -b$ for all $a \in A$ and $-b$ is an upper bound for the nonempty set $-A$. By LUB, $-A$ has a supremum, $\sup(-A)$, and because $-b$ is any upper bound, the least upper bound, $\sup(-A)$, must be less than or equal $-b$.

The figure suggests that $\inf(A) = -\sup(-A)$, and this is precisely what we will show. Two requirements must be met:

1. $-\sup(-A)$ is a lower bound of $A$.
2. No number larger than $-\sup(-A)$ is a lower bound of $A$.

We show 1:

If $-\sup(-A)$ is not a lower bound of $A$, then we have $a_1 \in A$ such that $a_1 < -\sup(-A)$. But then $\sup(-A) < -a_1$, that is, we have a member of $-A$, $-a_1$, that is larger than an upper bound of $-A$, $\sup(-A)$. Thus $-\sup(-A)$ is a lower bound for $A$. The reader may argue 2. ∎

### 2.4.5 Problem

Given the set $S$, calculate $\inf S$, $\sup S$:

1. $S = [0, 1) \cup \{2\}$
2. $S = \left\{\dfrac{\sin(x)}{x}, \quad 0 < x < \dfrac{\pi}{2}\right\}$
3. $S = \left\{\left(1 + \dfrac{1}{n}\right)^n; \quad n = 1, 2, \dots\right\}$
4. $S = \left\{\left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}}; \quad n = 1, 2, \dots\right\}$
5. $S = \left\{\sum_{k=0}^{n} \alpha^k; \quad n = 1, 2, \dots \quad \text{and} \quad -1 < \alpha < 1\right\}$
6. $S = \left\{\sum_{1}^{n} \dfrac{(-1)^{k+1}}{k}; \quad n = 1, 2, \dots\right\}$
7. $S = \{$any ordering of rational numbers in $(0, 1)\}$
8. $S = \left\{\dfrac{1}{3} - \dfrac{1}{2} + \dfrac{1}{6}, \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{24}, \dots, \dfrac{1}{3} - \dfrac{1}{2n} + \dfrac{1}{6n^2}, \dots\right\}$
9. $S = \left\{\dfrac{1}{3} + \dfrac{1}{2} + \dfrac{1}{6}, \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{24}, \dots, \dfrac{1}{3} + \dfrac{1}{2n} + \dfrac{1}{6n^2}, \dots\right\}$.

### 2.4.6 Problem

Let $A$ be a nonempty set of real numbers.

1. If $b \le a$ for all $a \in A$, then $b \le \inf A$ because $b$ is a lower bound for $A$, and the inf $A$ is the **greatest** lower bound.
2. If $b \ge a$ for all $a \in A$, then $\sup A \le b$.
3. If $A$ is bounded above, then for every $\epsilon > 0$ there exists an $a \in A$ such that $\sup A - \epsilon < a \le \sup A$. Otherwise, $a \le \sup A - \epsilon \forall a \in A$, that is, $\sup A - \epsilon$ is an upper bound for the set $A$ smaller than the **least** upper bound of $A$.
4. If $A$ is bounded below, then for every $\epsilon > 0$ there exists an $a \in A$ such that $\inf A \le a < \inf A + \epsilon$.

### 2.4.7 Problem

If $A$ is a bounded set of real numbers and $B$ is a nonempty subset of $A$, then

$$\inf A \le \inf B \le \sup B \le \sup A .$$

*Hint*: For $b \in B$, $\inf A \le b \le \sup A$.

### 2.4.8 Problem

$$f(x) = \begin{cases} -1, & x = 0 \\ x, & 0 < x < 1 \\ 2, & x = 1 . \end{cases}$$

$$\inf_{[0,1]} f \le \inf_{(0,1)} f \le \sup_{(0,1)} f \le \sup_{[0,1]} f$$

### 2.4.9 Problem

Let $A, B$ be nonempty sets of real numbers such that $a \le b$ for every $a \in A$ and for every $b \in B$. Show that

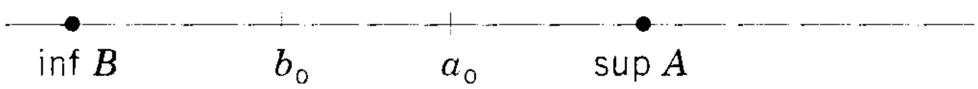
1. $A$ has a supremum and if $B$ has a supremum, $\sup A \le \sup B$.
2. $B$ has an infimum and if $A$ has an infimum, $\inf A \le \inf B$.
3. $\sup A \le \inf B$.

*Hint for* 3: Pick $a \in A$. Then $a \le b$ for all $b \in B$. So $a$ is a lower bound for $B$. Thus $a \le \inf B$. That is, $a \le \inf B$ for all $a \in A$. So $\inf B$ is an upper bound for $A$. Thus $\sup A \le \inf B$. The reader should pick $b \in B$ and follow a similar line of reasoning. Finally, for another argument, assume $\sup A > \inf B$ and arrive at a contradiction.

### 2.4.10 Problem

If $A$ is a nonempty bounded set of real numbers and $B$ is the set of all upper (need this condition: otherwise $A = (0, 1)$, $B = (2, 3]$, $\sup A < \inf B = 2$.) bounds for $A$, then $\sup A = \inf B$.

*Hint*: Problem 2.4.9. Now assume $\inf B < \sup A$.




Then $\inf B$ is not an upper bound for $A$. We have $\inf B < a_0 \le \sup A$, $a_0 \in A$. Then $a_0$ is not a lower bound for $B$. We have $b_0 \in B$ so that $\inf B \le b_0 < a_0$. But every member of $B$ is an upper bound of $A$.

### 2.4.11 Problem

Let $A$ be a nonempty set of real numbers with supremum $\sup A$. Let $c$ be a nonnegative real number. Then $c \sup A = \sup\{ca \mid a \in A\} = \sup(cA)$.

*Hint*: Since $a \le \sup A$ for all $a \in A$, $ca \le c \sup A$ for all $a \in A$. That is, $c \sup A$ is an upper bound for the set $cA$. But $\sup(cA)$ is the smallest upper bound. Hence $\sup(cA) \le c \sup A$. For the reverse inequality, $ca \le \sup(cA)$ for all $a \in A$. That is, $a \le (1/c)\sup(cA)$ for all $a \in A$. But this says $(1/c)\sup(cA)$ is an upper bound for $A$. Since $\sup A$ is the smallest upper bound, $\sup A \le (1/c)\sup(cA)$, or $c \sup A \le \sup(cA)$.

What if $c < 0$?

### 2.4.12 Problem

Let $A, B$ be nonempty sets of real numbers and let $C = \{a + b \mid a \in A,\ b \in B\}$. If $A, B$ have least upper bounds $\sup A$, $\sup B$, then so does $C$ and

$$\sup C = \sup A + \sup B.$$

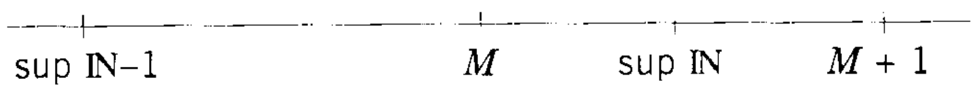
Similarly for the infimums.

*Hint*: Obviously $a + b \leq \sup A + \sup B$ so $\sup C \leq \sup A + \sup B$. Since $a + b \leq \sup C$ for all $a \in A$, $b \in B$, $a \leq \sup C - b$ for all $a \in A$. Thus $\sup A \leq \sup C - b$. But then $b \leq \sup C - \sup A$ for all $b \in B$ so $\sup B \leq \sup C - \sup A$ or $\sup A + \sup B \leq \sup C$.

### 2.4.13 Problem

1. The set $\mathbb{N}$ of natural numbers $1, 2, 3, \ldots$ is unbounded above. (Assume that if $n$ is a natural number then $n + 1$ is a natural number.)

*Hint*:

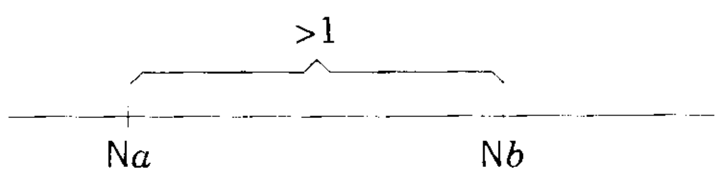



2. For every real number $x$ there exists a natural number $N$ such that $N > x$.

### 2.4.14 Problem

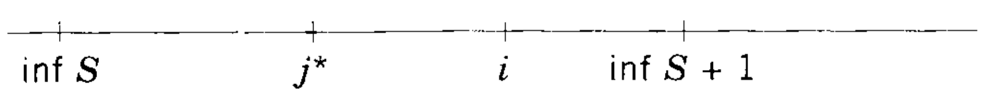
1. If $a, b$ are real numbers, $a < b$, then there is a rational number between $a$ and $b$. (Assume distinct integers are at least 1 unit apart.)

*Hint for* 1: The idea here is that if two numbers are more than one unit apart, we must have an integer between them. Since $b - a > 0$, for some positive integer $N$, $N > 1/(b - a)$ (2.4.13) or




Because $Nb - Na$ is greater than one, we claim there exists an integer $M$ so that $Na < M < Nb$.

Let $S = \{j \mid j \text{ an integer}, Na < j\}$. We claim $S$ has a minimum. Since $\inf S + 1$ is not a lower bound for $S$, we have $i \in S$ so that $\inf S \leq i < \inf S + 1$. If $i$ is the minimum of $S$, we are done. If not, we have $j^* \in S$ so that



Thus the distinct integers $j^*$, $i$ would be less than one unit apart. So $S$ has a minimum, say $M$. Then $M - 1 \le aN$ and

$$M = M - 1 + 1 \le aN + 1 < aN + Nb - Na = Nb.$$

Hence $aN < M < Nb$ or $a < (M/N) < b$ and the argument is complete.

2. If $a, b$ are real numbers, $a < b$, then there is an irrational number between $a$ and $b$.

*Hint*: $a - \sqrt{2} < r < b - \sqrt{2}$.

This problem illustrates the "denseness" of rationals and irrationals. The first demonstration of different "infinities" is due to Greog Cantor:

**THEOREM 2.1** (Cantor, 1874) *The set of all real numbers is uncountable.*

*Proof:* We restrict ourselves to $(0, 1)$ and assume we can "list" all the numbers of $(0, 1)$. The first argument (presentation and historically is Cantor's):

Our list:

$$\begin{aligned} x_1 &= .a_{11}a_{12}a_{13}\cdots \\ x_2 &= .a_{21}a_{22}a_{23}\cdots \\ &\vdots \\ x_k &= .a_{k1}a_{k2}a_{k3}\cdots \\ &\vdots \end{aligned}$$

Let $a^* = .a_1a_2a_3\cdots$ where $a_k = \begin{cases} 5; & a_{kk} \text{ even} \\ 2; & a_{kk} \text{ odd} \end{cases}$.

Since $a^* \ne x_k$ for all $k$, $a^*$ is not on the list, but $a^* \in (0, 1)$.

Another argument: Divide $[0, 1]$ into $[0, 1/3]$, $[1/3, 2/3]$, and $[2/3, 1]$. One of these intervals does not contain $x_1$ (if $x_1$ is $1/3$, pick $[2/3, 1]$).

Call this interval $I_1 = [a_1, b_1]$. Divide $I_1$ into thirds and pick one that does not contain $x_2$. Call this interval $I_2 = [a_2, b_2]$. Repeating the process, we have $I_{k+1} \subset I_k$, and $l(I_k) = 1/3^k$. The set of left-hand endpoints, $\{a_1, a_2, \ldots\}$, is bounded above by 1. Thus by LUB has a supremum $a^*$. But $a^* \le b_k$ for all $k$. Thus $a^* \in [a_k, b_k]$ for all $k$ and $a^* \ne x_k$ for all $k$. $a^*$ is not on our list.

*Note*: For those that know something about Lebesgue measure, the Lebesgue measure of $(0, 1)$ is one and the Lebesgue measure of the countable "list" is zero.

And yet another argument due to Cantor: Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of distinct real numbers.

In any open interval $(c, d)$, there is a number $x$ (and, hence, infinitely many such numbers) which does not occur in the sequence $x_1, x_2, x_3, \ldots$. His argument proceeded as follows:

If no numbers or only one number of the given sequence are in the open interval $(c, d)$, we are done. Otherwise, pick the first two numbers of $x_1, x_2, x_3, \ldots$ that are in $(c, d)$, say $x_{n_1}, x_{n_2}$ (No member of $x_1, x_2, \ldots$ with a smaller subscript than the largest of $n_1, n_2$ is in $(c, d)$.). Consider the open interval formed by $x_{n_1}$ and $x_{n_2}$ and repeat the process. If no members or only one member of $\{x_1, x_2, \ldots\}$ are in this open interval, we are done. Otherwise, pick first two members of $x_1, x_2 \ldots$ that are in this open interval, say $x_{n_3}$ and $x_{n_4}$.

Note that maximum of $n_1, n_2$ is less than minimum of $n_3, n_4$. So, no member of the original sequence with subscripts smaller than the maximum of $n_3$ and $n_4$ belongs to the interval determined by $x_{n_3}$ and $x_{n_4}$. Repeating this process, we have two possibilities:

1. The process terminates after a finite number of steps, say $K$. But then the open interval determined by $x_{n_K}$ and $x_{n_{K+1}}$ contains at most one number of $x_1, x_2 \ldots$ and we are done.
2. There are infinitely many subintervals whose endpoints are from the original set of numbers $x_1, x_2, \ldots$. But, the set of left-hand endpoints form a bounded, increasing sequence, and thus converge to some number $a^*$. Similarly, the right-hand endpoints form a bounded deceasing sequence converging to $b$, $a^* \leq b$.

If $a^* = b$, this number is in every subinterval, and thus $a^*$ is not a member of the sequence $x_1, x_2, x_3, \ldots$ by the selection process ($a^* = x_N$, would not be in an infinite number of the subintervals).

If $a^* < b$, any number in $[a^*, b]$ would be different from every number of $x_1, x_2, \ldots$.

## 2.5 EXTENDED REAL NUMBERS

In measure theory it is convenient to adjoin the two symbols $-\infty$, $+\infty$ to the real numbers $R$, e.g., in discussing the *length* of $(a, \infty)$. These are not

real numbers. Let $R^e = R \cup \{-\infty, +\infty\}$ and call $R^e$ the extended real number system.
The algebraic operations are defined as follows:

$x \in R$: $\quad x + \infty = \infty + x = x - (-\infty) = \infty$

$$x + (-\infty) = -\infty + x = x - \infty = -\infty$$

If $x > 0$, $\quad \infty \cdot x = x \cdot \infty = \infty,$

$$x \cdot (-\infty) = (-\infty) \cdot x = -\infty.$$

If $x < 0$, $\quad \infty \cdot x = x \cdot \infty = -\infty,$

$$x \cdot (-\infty) = (-\infty) \cdot x = \infty.$$

We also define

$$\infty + \infty = \infty \text{ and } (-\infty) + (-\infty) = -\infty,$$
$$\infty \cdot \infty = \infty,$$
$$\infty \cdot 0 = 0 \cdot \infty = (-\infty) \cdot 0 = 0 \cdot (-\infty) = 0.$$

We do not define

$$(\infty) + (-\infty) \; or \; (-\infty) + (\infty),$$
$$\infty \cdot (-\infty), \;\; (-\infty) \cdot \infty, \;\; (-\infty) \cdot (-\infty),$$

nor do we define division by $\infty$ or $-\infty$.

### 2.5.1 Definitions

The *supremum* of a nonempty set $A$ of real numbers which is not bounded above is to be $\infty$. In this case, given any real number $K$, we have a member $a$ of $A$ with $a > K$. Analogously, the *infimum* of a nonempty set of real numbers that is not bounded below is to be $-\infty$. Also, we define $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. In short, every set of real numbers has a supremum and an infimum, although possibly in $R^e$. For every nonempty set of real numbers $A$, $\inf A \leq \sup A$. If we restrict ourselves to nonempty sets of real numbers, then:

1. $A \subset B \Longrightarrow \sup A \leq \sup B$ ("sup" is a monotonic set function)
2. $\sup(cA) = c \sup A$, $c$ nonnegative.

Similar statements may be made for the infimum.

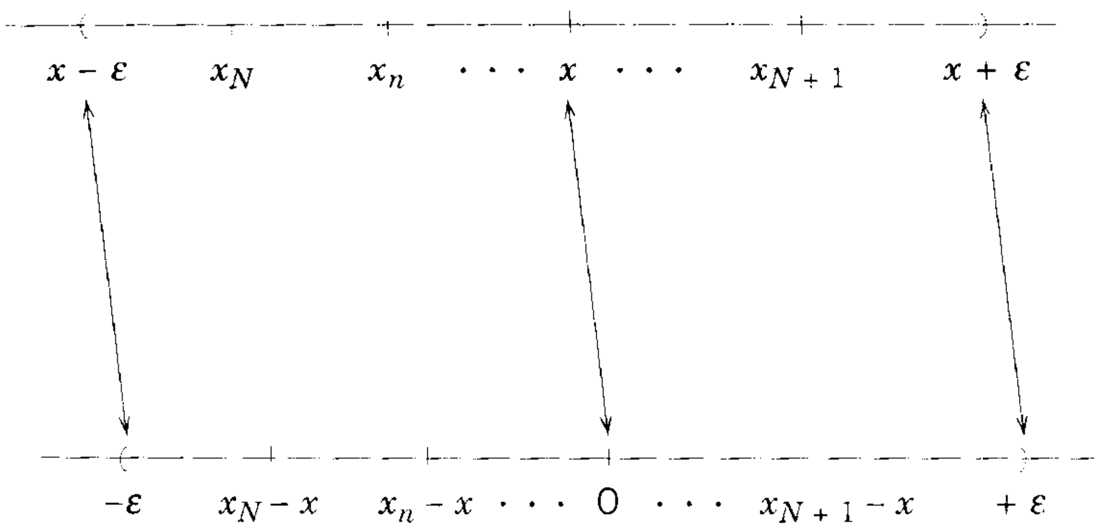
## 2.6 SEQUENCES OF REAL NUMBERS

### 2.6.1 Definition

A *sequence* $(x_n)$ of real numbers *converges* to the real number $x$ if for each $\epsilon > 0$, there is a natural number $N(\epsilon)$ so that for all $n \geq N(\epsilon)$ we have $x - \epsilon < x_n < x + \epsilon$. Equivalently, $-\epsilon < x_n - x < \epsilon$ for all $n \geq N(\epsilon)$. The number $x$ is called the *limit of the sequence* $(x_n)$ and we write

$$x_n \longrightarrow x, \quad \lim x_n = x, \qquad or \qquad \lim_n x_n = x.$$

Otherwise, the sequence is said to *diverge*. Of course, $x_n \longrightarrow x$ means that the points $x_n$, with at most a finite number of exceptions, lie between $x - \epsilon$ and $x + \epsilon$, or $x_n - x$ lies between $-\epsilon$ and $\epsilon$:




If given an arbitrary positive number $C$ we have a natural number $N$ so that $x_n > C$ for all $n \geq N$, we write $\lim x_n = \infty$. Similarly, $\lim x_n = -\infty$ provided for each negative number $C$, we have a natural number $N$ so that $x_n < C$ for all $n \geq N$.

When we use the words "*converges*", "*convergent*", it is understood that $\lim x_n = x$ and $x$ is a real number. If we write $\lim x_n = x$ without mentioning convergence then $x$ may be real, $\infty$, or $-\infty$.

### 2.6.2 Problem

(Squeezing) Given three sequences $(x_n), (y_n)$, and $(z_n)$ such that $x_n \leq y_n \leq z_n$ for all $n$:

1. If $\lim x_n = \lim z_n = L$ a real number, then $\lim y_n = L$.

$$\begin{tikzcd}[column sep=small] x_n \arrow[drr] & \leq & y_n \arrow[d, dashed] & \leq & z_n \arrow[dll] \\ & & L & & \end{tikzcd}$$

2. If $\lim x_n = \infty$, then $\lim y_n = \infty$ ("Forcing").
3. If $\lim y_n = -\infty$, then $\lim x_n = -\infty$ ("Forcing").

One of the most important criteria for covergence of a sequence is given by the next theorem. In contrast to Definition 2.6.1, we do not need to "know" the limit $L$ "a priori".

**THEOREM 2.2** *A nondecreasing (nonincreasing) sequence is convergent if and only if it is bounded above (below). A nondecreasing (nonincreasing) sequence which is not bounded above (below) has limit $\infty$ $(-\infty)$. In all cases, a monotonic sequence has a limit in $R^e$.*

*Proof:* Suppose the sequence $(x_n)$ is nondecreasing and bounded above. We will show the sequence converges, in fact, converges to $\sup\{x_1, x_2, \ldots\}$. The set of real numbers $S = \{x_1, x_2, \ldots\}$ is nonempty and bounded above. By LUB, $S$ has a supremum, a real number, $\sup S$. Let $\epsilon > 0$ be given. Because $\sup S$ is the least upper bound of the set $S$, $\sup S - \epsilon$ is not an upper bound of $S$. Thus, we have a member $x_N$ of $S$ so that $\sup S - \epsilon < x_N \leq \sup S$. Since the sequence $(x_n)$ is nondecreasing by assumption, we have $\sup S - \epsilon < x_n \leq \sup S$ for all $n \geq N$. By definition, $\lim x_n = \sup S$, and this is what we wanted to show. The other parts of the conclusion are left to the reader. ∎

**Example 7:** $x_n = 1 + 1/1! + 1/2! + \cdots + 1/n!$. Obviously $x_n < x_{n+1}$ and since $x_n < 1 + 1 + 1/2^1 + \cdots + 1/2^n < 3$, we have convergence. Of course we recognize the limit as $e$.

**Example 8:** $x_n = (1 + 1/n)^n$. Using the binomial theorem, we may show $x_n < x_{n+1}$ and $x_n < 3$. Again, $\lim x_n = e$.

**Example 9:** $x_n = (1 + 1/n)^{n+1/2}$. Show $x_{n+1} < x_n$ and $0 < x_n$. Again $\lim x_n = e$.

**Example 10:** $x_n = 1 + 1/2 + \cdots + 1/n - \ln n$. $x_{n+1} < x_n$ and $0 < x_n$. $\lim x_n = \gamma$, the Euler-Mascheroni constant.

**Example 11:** $x_n = 1/n + 1/(n+1) + \cdots + 1/(n+n)$. $x_{n+1} < x_n$ and $0 < x_n$.

What is $\lim x_n$?

### 2.6.3 Problem

Let $(x_n)$ be a nondecreasing sequence and $(y_n)$ be a nonincreasing sequence with $x_n \leq y_n$ for all $n$ and $y_n - x_n \longrightarrow 0$. Then we have exactly one real number $L$ so that $x_n \leq L \leq y_n$ for all $n$.

*Hint*: $x_n \longrightarrow x$, $y_n \longrightarrow y$ and argue that $x \leq y$ and in fact, $x = y = L$.

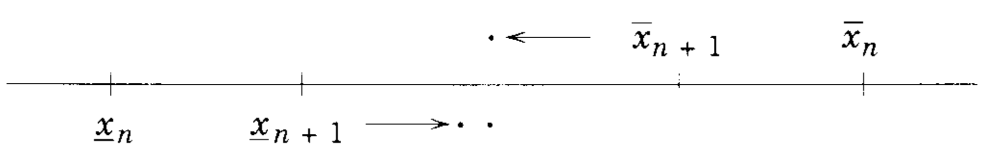
Because monotonicity of sequences is such a useful concept, we construct two very important monotonic sequences from an arbitrary sequence of real numbers.

### 2.6.4 Definition

$(x_n)$ is a sequence of real numbers. Construct sequences $(\underline{x}_n)$, $(\overline{x}_n)$ as follows:

$$\begin{array}{lll}
\underline{x}_1 = \inf\{x_1, x_2, \ldots\} & \leq x_1 \leq \sup\{x_1, x_2, \ldots\} = \overline{x}_1\,. \\
\underline{x}_2 = \inf\{x_2, x_3, \ldots\} & \leq x_2 \leq \sup\{x_2, x_3, \ldots\} = \overline{x}_2\,. \\
\vdots & & \\
\underline{x}_n = \inf\{x_n, x_{n+1}, \cdots\} & \leq x_n \leq \sup\{x_n, x_{n+1}, \ldots\} = \overline{x}_n\,. \\
\vdots & \quad\vdots & \qquad\qquad\vdots
\end{array}$$

Observe that $\underline{x}_n \leq \underline{x}_{n+1}$, $\overline{x}_{n+1} \leq \overline{x}_n$. Roughly:




The sequence $(\underline{x}_n)$ is nondecreasing and thus $\lim \underline{x}_n \in R^e$. This limit (finite or infinite) is called the *limit inferior* of the sequence $(x_n)$ and we write $\liminf x_n = \lim \underline{x}_n$. If $(x_n)$ is not bounded below ($\underline{x}_n = \inf\{x_n, x_{n+1}, \ldots\} = -\infty$) we write $\liminf x_n = -\infty$. Similarly, the sequence $(\overline{x}_n)$ is nonincreasing, $\lim \overline{x}_n \in R^e$, and the *limit superior* of the sequence $(x_n)$, written $\limsup x_n$, is defined to be $\lim \overline{x}_n$. If $(x_n)$ is not bounded above ($\overline{x}_n = \sup\{x_n, x_{n+1}, \ldots\} = \infty$) we write $\limsup x_n = \infty$.

### 2.6.5 Comments

1. $\underline{x}_n \leq x_n \leq \overline{x}_n$ for all $n$ by construction.
2. $\underline{x}_i \leq \underline{x}_{i+j} \leq x_{i+j} \leq \overline{x}_{i+j} \leq \overline{x}_j$ and thus $\liminf x_n \leq \limsup x_n$.
3. If $x_n < \alpha + \epsilon$ for every $\epsilon > 0$, then $\limsup x_n \leq \alpha$.
4. If $x_n > \alpha - \epsilon$ for every $\epsilon > 0$, then $\liminf x_n \geq \alpha$.
5. If $\lim x_n = x$ is a real number, $\infty$, or $-\infty$, then $\liminf x_n = \lim x_n = \limsup x_n$. (If $x$ is a real number, $x - \epsilon < x_n < x + \epsilon$ for all $n \geq N(\epsilon)$ (Apply 3 and 4). The other cases follow easily.)

### 2.6.6 Problem

For the sequences $(x_n)$, calculate $\liminf x_n$, $\limsup x_n$:

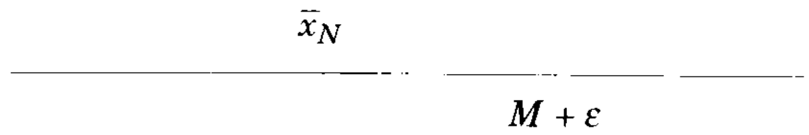
1. $x_n: \quad 1, 2, 3, 1, 2, 3, 1, 2, 3, \cdots.$
2. $x_n: \quad 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \cdots.$
3. $x_n: \quad 1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots.$
4. $x_n: \quad \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \cdots.$
5. $x_n = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}, \qquad n = 1, 2, \ldots.$
6. $x_n = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}, \qquad n = 1, 2, \ldots.$

### 2.6.7 Problem

$(x_n)$ is a bounded sequence of real numbers.

1. If $\limsup x_n = M$, then
   i. for every $\epsilon > 0$, $x_n < M + \epsilon$ for all but a finite number of values of $n$.

*Hint*:




ii. $x_n > M - \epsilon$ for infinitely many values of $n$.

*Hint*: If finite,. . . .

2. If lim inf $x_n = m$, then
   i. for every $\epsilon > 0$, $x_n > m - \epsilon$ for all but a finite number of $n$.
   ii. $x_n < m + \epsilon$ for infinitely many values of $n$.

**2.6.8 Problem**

If $(x_n)$ is any sequence of positive real numbers, then $\liminf x_{n+1}/x_n \leq \liminf x_n{}^{1/n} \leq \limsup x_n{}^{1/n} \leq \limsup x_{n+1}/x_n$.

*Hint*: If $\limsup x_{n+1}/x_n = \infty$, right hand inequality is obvious. So suppose $\limsup x_{n+1}/x_n = M$. Then $x_{n+1}/x_n < M + \epsilon \;\; \forall n \geq N$, i.e., $x_{n+k} \leq (M+\epsilon)^k x_n$ for all $k \geq 0$.

**2.6.9 Problem**

1. If $(x_n)$, $(y_n)$ are sequences of real numbers with $x_n \leq y_n$ for all $n$, then
   i. lim sup $x_n \leq$ lim sup $y_n$
   ii. lim inf $x_n \leq$ lim inf $y_n$.

*Hint*: $\bar{x}_n = \sup\{x_n, x_{n+1}, \ldots\} \leq \sup\{y_n, y_{n+1}, \ldots\} = \bar{y}_n$. Is it true that lim sup $x_n \leq$ lim inf $y_n$?

2. If $(x_n), (y_n)$ are sequences of real numbers then lim inf $x_n+$ lim inf $y_n$ $\leq$ lim inf$(x_n + y_n) \leq$ lim sup$(x_n + y_n) \leq$ lim sup $x_n+$ lim sup $y_n$.

*Hint for the last inequality*: If $(x_n)$ or $(y_n)$ not bounded above, obvious. Otherwise, $x_k + y_k \leq \overline{x}_n + \overline{y}_n$ for all $k \geq n$, i.e., $\overline{x}_n + \overline{y}_n$ is an upper bound for the set $\{x_n + y_n, x_{n+1} + y_{n+1}, \ldots\}$. Hence $\overline{x_n + y_n} \leq \overline{x}_n + \overline{y}_n$.

3. If $x_n$ converges to $x$, then $\lim\inf(x_n + y_n) = x + \lim\inf y_n$.

Besides monotone sequences, another theorem concerns the so called *Cauchy sequences* (Cauchy, 1853). Again, we do not have to know the limit a priori.

### 2.6.10 Definition

A sequence of real numbers is said to be a *Cauchy sequence* if, for every $\epsilon > 0$, we have a natural number $N(\epsilon)$ so that $-\epsilon < x_m - x_n < \epsilon$ whenever $n, m \geq N(\epsilon)$, or, equivalently, $-\epsilon < x_{n+k} - x_n < \epsilon$ whenever $n \geq N(\epsilon)$, $k = 1, 2, \ldots$.

**Example 12:** $x_0 = 0$, $x_1 = 1$, $x_n = (x_{n-1} + x_{n-2})/2$, $n \geq 2$. Then $|x_{n+1} - x_n| = (1/2)^n$. But $x_{n+k}$ lies between $x_{n+1}$ and $x_n$ for all $k > 1$. Thus $|x_{n+k} - x_n| \leq 1/2^n$ for $n \geq 0$ and $k > 0$.

**Example 13:** $x_n = 1 + 1/2 + \cdots + 1/n$. Then $x_{2n} - x_n \geq 1/2$ and not Cauchy.

**Example 14:** $x_n = \sqrt{n}$. Then $|x_{n+k} - x_n| = \sqrt{n+k} - \sqrt{n} > k/2\sqrt{n+k}$.

**Example 15:** $x_n \equiv \int_0^n \sin(t)/t\, dt$. For $m > n$, $x_m - x_n = \int_n^m \sin(t)/t\, dt = -\cos(t)/t\,|_n^m - \int_n^m \cos(t)/t^2$. Thus $|x_m - x_n| \leq 2/n$, $m > n$. This sequence is Cauchy, and converges ($\lim x_n = \pi/2$).

**THEOREM 2.3** *For a sequence $(x_n)$ of real numbers the following conditions are equivalent:*

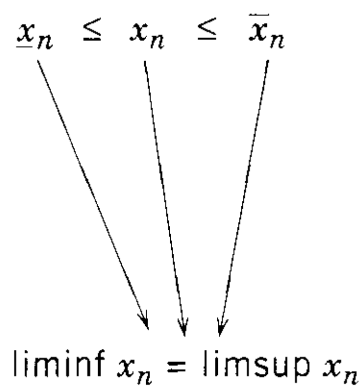
1. *$(x_n)$ converges to $x$:* $\lim x_n = x$, *$x$ a real number.*
2. *Cauchy condition: for every $\epsilon > 0$ there exists a natural number $N(\epsilon)$ so that $-\epsilon < x_n - x_m < \epsilon$ for all $n, m > N(\epsilon)$.*
3. *$-\infty < \lim\inf x_n = \lim\sup x_n < \infty$.*

*Proof:* The arguments are sketched.

$1 \Longrightarrow 2$ $\quad |x_m - x_n| \le |x_m - x| + |x - x_n|$.

$2 \Longrightarrow 3$ $\quad (x_n)$ is bounded and $x_n < x_m + \epsilon$ for all $n \ge N(\epsilon)$. Thus $\overline{x}_n \le x_m + \epsilon$ and $\limsup x_n = \lim \overline{x}_n \le x_m$ for any $m \ge N(\epsilon)$, i.e., $\limsup x_n$ is a lower bound for the set $\{x_m, x_{m+1}, \ldots\}$, that is, $\limsup x_n \le \underline{x}_m \le \liminf x_n$.

$3 \Longrightarrow 1$




## 2.6.11 Comment

If we have a convergent sequence of real numbers $(x_n)$, $x_n \longrightarrow x$ (real), then equivalently $|x_n - x| \longrightarrow 0$. If we let $y_n = |x_n - x|$, and form $(\underline{y}_n)$, $(\overline{y}_n)$, we have $\underline{y}_n = 0$ for all $n$ and $\overline{y}_n$ decreases monotonically to zero. Thus $0 \le y_n \le \overline{y}_n$ and $\overline{y}_{n+1} \le \overline{y}_n$. The usefulness of this result may appear as: "If we are given that a sequence converges, then we may suppose the sequence is nonnegative and monotonically decreasing to zero."

Sequences are frequently given in the form $a_1 + a_2 + \cdots + a_n + \cdots$. The usual terminology is to say we have a series.

## 2.6.12 Definition

A *series* is a sequence of numbers denoted by $\sum a_n$ or $\sum a_k$. A series $\sum a_k$ is said to be *convergent* to the real number $s$ if the sequence of partial sums $(s_n)$; $s_1 = a_1, s_2 = a_1 + a_2, s_n = a_1 + \cdots + a_n$, converges to $s$, and we write

$$s = \sum a_k .$$

Thus $\lim (s - s_n) = 0$ is sometimes written

$$\lim \sum_{n+1}^{\infty} a_k = 0.$$

If $(s_n)$ does not converge, the series is said to be *divergent*. Of course a series $\sum a_k$ with nonnegative terms $a_k \geq 0$ either converges or has limit $\infty$ since the sequence $(s_n)$ of partial sums is a nondecreasing sequence. In other words, a series $\sum a_k$ of nonnegative terms converges *iff* its partial sums are bounded.

### 2.6.13 Comment

Will convergence of a series be affected if we rearrange (change the order of) the terms? And, if we have convergence for the rearranged series, will it be to the same number? For example, the series $1 - 1/2 + 1/3 - 1/4 + \cdots = \ln 2$.

If we "rearrange,"

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots=\frac{1}{2}\ln 2.$$

(If $s_n = 1 - 1/2 + 1/3 - \cdots + (-1)^{n+1}\, 1/n$, then

$$s_{3n}^* = \left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\cdots+\left(\frac{1}{2n-1}-\frac{1}{4n-2}\right)-\frac{1}{4n}$$

$$=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots+\frac{1}{2(2n-1)}-\frac{1}{4n}=\frac{1}{2}s_{2n}$$

and, consequently, $\lim s_{3n}^* = \lim 1/2\, s_{2n} = 1/2 \ln 2$. Since $s_{3n}^* - s_{3n+1}^* \longrightarrow 0$ and $s_{3n}^* - s_{3n+2}^* \longrightarrow 0$, $s_n^* \longrightarrow 1/2 \ln 2$.)

Fortunately, our applications will be restricted to series with nonnegative terms, and in this situation, rearrangements do not affect convergence: Suppose $\sum a_k = S$ and $\sum a_k^* = S^*$, with every term of the original series $(a_k \geq 0)$ occuring exactly once in the new "rearranged" series $\sum a_k^*$. We show $S = S^*$. Let $\alpha < S$. Then $\alpha < s_N$ for some $N$ because the sequence of partial sums $(s_n)$ is nondecreasing. Then $\alpha < s_M^*$ where $M$ is chosen so large so that every term $a_1, a_2, \ldots, a_N$ occurs in $s_M^*$. Thus $S \leq S^*$. An analogous argument shows $S^* \leq S$. In conclusion, the terms of a nonnegative series may be arranged in any order without affecting the sum, finite or infinite.

In our development of Lebesgue measure we will on occasion deal with double series, $\sum_k(\sum_n a_{kn})$, and again questions of convergence and rearrangements will be important. For example, are there conditions so that

$$\begin{array}{ll}
 & a_{11} + a_{12} + \cdots + a_{1n} + \cdots \\
+ & a_{21} + a_{22} + \cdots + a_{2n} + \cdots \\
\vdots & \vdots \quad\ \vdots \qquad\quad \vdots \\
+ & a_{k1} + a_{k2} + \cdots + a_{kn} + \cdots \\
\vdots & \\
= & a_{11} + a_{21} + \cdots + a_{k1} + \cdots \\
+ & a_{12} + a_{22} + \cdots + a_{k2} + \cdots \\
\vdots & \vdots \quad\ \vdots \qquad\quad \vdots \\
+ & a_{1n} + a_{2n} + \cdots + a_{kn} + \cdots \\
\vdots &
\end{array}$$

"Rows then Columns"

"Columns then Rows"

?

We certainly want each term of the original series $a_{kn}$ to appear exactly once in the "rearranged" series. This suggests we need a $1-1$ map of $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. The reader might refer to Proposition 2.3.

Possibilities:

$$\begin{aligned}
\sum_k \left( \sum_n a_{kn} \right) &= (a_{11} + a_{12} + \cdots + a_{1n} + \cdots) \\
&\quad + (a_{21} + a_{22} + \cdots + a_{2n} + \cdots) \\
&\quad \vdots \\
&\quad + (a_{k1} + a_{k2} + \cdots + a_{kn} + \cdots) \\
&\quad \vdots \\
&\quad \downarrow \\
&\stackrel{?}{=} a_{11} + a_{21} + a_{12} + a_{31} + a_{22} + a_{13} + \cdots ,
\end{aligned}$$

that is, correspond the $(k+n-1)(k+n)/2 - (k-1)$ term of the right hand side with $a_{kn}$. This map is $1-1$ from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. If we used the map $(k,n) \longleftrightarrow 2^{k-1}(2n-1)$, then does $\sum_k(\sum_n a_{kn}) = a_{11} + a_{21} + a_{12} + a_{31} + a_{13} + a_{22} + \cdots$? As before, additional restrictions must be imposed, as the next example shows.

$$\begin{array}{cccccccccc}
1 & -1 & 0 & 0 & 0 & \cdots & & \longrightarrow & 0 & \\
0 & 1 & -1 & 0 & 0 & \cdots & & \longrightarrow & 0 & \\
0 & 0 & 1 & -1 & 0 & \cdots & & \longrightarrow & 0 & \text{“Rows then Columns”} \\
\downarrow & \downarrow & \downarrow & \downarrow & & & & & \downarrow & \\
1 & 0 & 0 & 0 & & & \longrightarrow & 1 & 0 &
\end{array}$$

“Columns then Rows”

The rearrangement theorem we are looking for is given in Stromberg. We include the argument for completeness.

**Proposition 2.4** *Suppose that $a_{kn}$ is nonnegative and that $\phi$ is any $1-1$ mapping of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. Then*

$$\sum_n \left( \sum_k a_{kn} \right) = \sum_k \left( \sum_n a_{kn} \right) = \sum_i a_{\phi(i)} \, .$$

*These sums may be finite or infinite.*

*Proof:* Let $\alpha$ be any real number so that $\alpha < \sum_i a_{\phi(i)}$. We will show $\alpha \le \sum_k (\sum_n a_{kn})$ and conclude $\sum a_{\phi(i)} \le \sum_k (\sum_n a_{kn})$. Since $a_{kn}$ are nonnegative, the partial sums of $\sum_i a_{\phi(i)}$ are nondecreasing. Thus we have $I$ so that $\alpha < \sum_{i=1}^{I} a_{\phi(i)}$. Select $K, N$ so large that

$$\left\{ a_{\phi(1)}, a_{\phi(2)}, \ldots, a_{\phi(I)} \right\} \subset \left\{ a_{kn},\ 1 \le k \le K,\ 1 \le n \le N \right\}.$$

Hence

$$\begin{aligned}
\alpha < \sum_{i=1}^{I} a_{\phi(i)} &\le \sum_{k=1}^{K} \left( \sum_{n=1}^{N} a_{kn} \right) \\
&\le \sum_{k=1}^{K} \left( \sum_{n=1}^{\infty} a_{kn} \right) \\
&\le \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} a_{kn} \right),
\end{aligned}$$

and thus $\sum_i a_{\phi(i)} \le \sum_k (\sum_n a_{kn})$.

Now pick $\beta < \sum_k (\sum_n a_{kn})$. Again, we have $K$ so that $\beta < \sum_{k=1}^K (\sum_{n=1}^\infty a_{kn})$ since the terms $\sum_{n=1}^\infty a_{kn}$ are nonnegative. We make the following observations: Since

$$\begin{aligned}
\sum_{k=1}^{2}\left(\sum_{n=1}^{\infty} a_{kn}\right) &= \sum_{n=1}^{\infty} a_{1n} + \sum_{n=1}^{\infty} a_{2n} \\
&= (a_{11} + a_{12} + \cdots + a_{1n} + \cdots) \\
&\quad + \ (a_{21} + a_{22} + \cdots + a_{2n} + \cdots) \\
&= (a_{11} + a_{21}) + (a_{12} + a_{22}) + \cdots + (a_{1n} + a_{2n}) + \cdots \\
&= \sum_{n=1}^{\infty} (a_{1n} + a_{2n}) \\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{2} a_{kn}\right),
\end{aligned}$$

induction on $k$ shows

$$\sum_{k=1}^{K}\left(\sum_{n=1}^{\infty} a_{kn}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{K} a_{kn}\right).$$

Then $\beta < \sum_{n=1}^\infty (\sum_{k=1}^K a_{kn})$ and we choose $N$ so that

$$\beta < \sum_{n=1}^{N}\left(\sum_{k=1}^{K} a_{kn}\right) = \sum_{k=1}^{K}\left(\sum_{n=1}^{N} a_{kn}\right).$$

For $i$ large enough, say, $I$,

$$\{(k,n);\ 1 \le k \le K,\ 1 \le n \le N\} \subset \{\phi(1), \phi(2), \ldots, \phi(I)\}.$$

Then $\beta < \sum_{k=1}^K (\sum_{n=1}^N a_{kn}) \le \sum_{i=1}^I a_{\phi(i)} \le \sum_{i=1}^\infty a_{\phi(i)}$. Thus $\sum_k(\sum_n a_{kn}) \le \sum_i a_{\phi(i)}$. The argument is complete. ∎

## 2.7 TOPOLOGICAL CONCEPTS OF *R*

### 2.7.1 Definition

The *open interval* $(a, b)$ is the set of real numbers between $a$ and $b$. The intervals $(-\infty, a)$, $(b, \infty)$ and $(-\infty, \infty)$ are to be considered open.

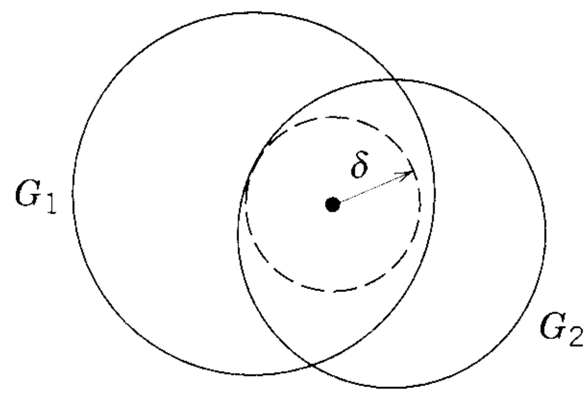
A set $G$ of real numbers is said to be *open* if, for each $p \in G$ we have a positive real number $\delta$, so that $(p - \delta, p + \delta) \subset G$. In other words, $G$ is open if every point of $G$ is in the center of an open interval wholly contained in $G$. $R$ is open and the empty set $\emptyset$ is open. As far as set theoretic operations, "finite" intersections and "all" unions of open sets are open.

**PROPOSITION 2.5**

1. *If $G_1$, $G_2$ are open subsets of $R$, then $G_1 \cap G_2$ is open.*
2. *If $\{G_\alpha\}$ is any collection of open subsets of $R$, then $\cup G_\alpha$ is open.*

*Proof:*

1. Suppose $x \in G_1 \cap G_2$. Then $(x - \delta_i, x + \delta_i) \subset G_i$. Let $\delta = \min\{\delta_1, \delta_2\}$.




2. If $x \in \cup G_\alpha$, then $x \in G_{\alpha_0}$ for some $\alpha_0$. Since $G_{\alpha_0}$ is open, we have $(x - \delta, x + \delta) \subset G_{\alpha_0}$ for some $\delta > 0$. Thus $(x - \delta, x + \delta) \subset G_{\alpha_0} \subset \cup G_\alpha$. ∎

*Note*: We cannot strengthen 1: $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$.

The next theorem reveals the structure of open sets in $R$.

**THEOREM 2.4** *Every nonempty open set of real numbers is the union of a countable collection of mutually disjoint open intervals.*

*Proof:* Let $G$ be a nonempty open set of real numbers and $p$ any point in $G$. Because $G$ is an open set, we have $\delta > 0$ so that $(p - \delta, p + \delta) \subset G$.

Define $a = \inf\{y \mid (y,p) \subset G\}$ and $b = \sup\{z \mid (p,z) \subset G\}$. Then $a,b \in R^e$. If $a$ is a real number, then $a \notin G$ since the openness of $G$ would require $(a - \delta_1, a) \subset G$ for some $\delta_1 > 0$. This contradicts GLB. Similarly for $b$. Regardless, we have an open interval $I(p)$ associated with every point $p \in G$. If $I(p_1), I(p_2)$ are associated with distinct points $p_1, p_2 \in G$, then $I(p_1), I(p_2)$ are identical or disjoint since $a_1 < a_2 < b_1$ would imply $a_1 = a_2$ and $b_2 = b_1$, when $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$. The reader may argue the other cases. Thus $G$ is the union of a collection of disjoint open intervals. Select a single rational number from each interval. This rational number will correspond to exactly one interval because the intervals in our collection are disjoint. Thus the collection is countable and we are done. ■

### 2.7.2 Definitions

A set will be called *closed* if it is the complement of an open set. The closed interval $[a, b]$ is the set of real numbers between and including $a$ and $b$, and is closed (complement of $(-\infty, a) \cup (b, \infty)$).

The sets $\phi$ and $R$ are both open and closed, $[0, 1)$ is neither open nor closed, and the reader may show (using De Morgan's Laws) that "all" intersections and "finite" unions of closed sets are closed. "Finite" unions cannot be strenghtened: $\bigcup_2^\infty [1/n, 1 - 1/n] = (0, 1)$.

### 2.7.3 Definition

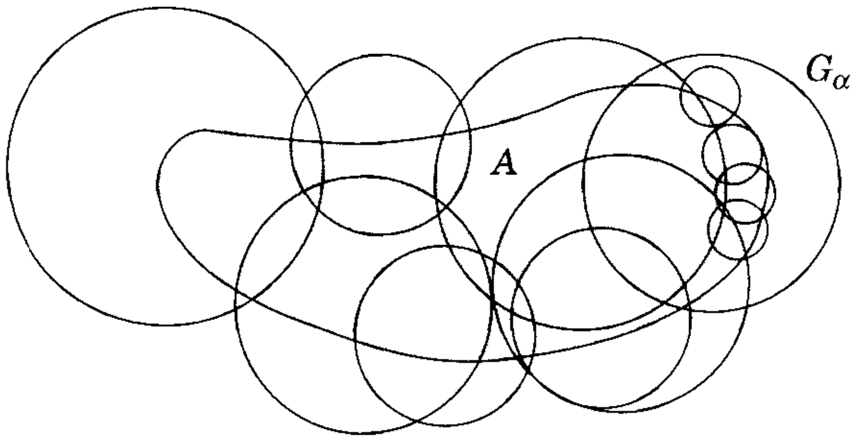
A point $p$ is called a *limit point* of a set $A$ if for every $\delta > 0$, there is an $x \in A$, $x \neq p$, such that $-\delta < x - p < \delta$. In other words, $p$ is a limit point of the set $A$ if every open interval centered at $p$ contains a point of $A$ different from $p$.

**Proposition 2.6** *A nonempty set $F$ of real numbers is closed iff it contains all its limit points.*

*Proof:* Let $F$ be a closed set and $p$ a limit point of $F$. We will show $p \in F$. We argue by contradiction. Assume $p$ is a limit point of the closed set $F$ and $p \notin F$. Then $p$ is a member of the open set $F^c$. Thus we have $\delta > 0$ so that $(p - \delta, p + \delta) \subset F^c$, that is, $(p - \delta, p + \delta) \cap F = \emptyset$. This contradicts the definition of $p$ being a limit point of the set $F$. Now we suppose $F$ contains all its limit points. We show $F$ must be closed, or equivalently, $F^c$ must be open. Let $p \in F^c$. Since $F$ contains all its limit points, and $p \in F^c$, $p$ is not a limit point of $F$, that is, we have a $\delta > 0$ so that $(p - \delta, p + \delta) \cap F = \emptyset$. In other words, given a point in the complement of $F$ we have an open interval centered at that point in the complement of $F$, and hence the complement of $F$ is open. ■

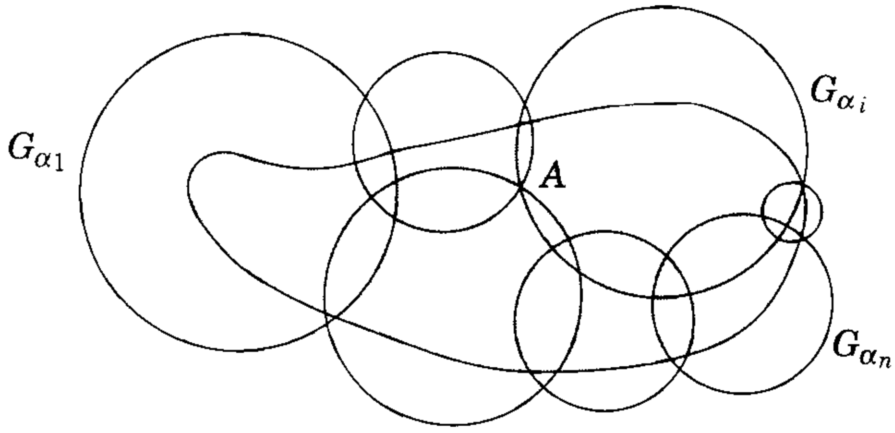
### 2.7.4 Definition

A collection $\{G_\alpha\}$ of open sets *covers* a set $A$ if $A \subset \cup G_\alpha$. In this case, the collection $\{G_\alpha\}$ is called an *open cover* of $A$.




### 2.7.5 Definition

A set of real numbers is *compact* if every open cover of the set contains a finite subcover: If $A \subset \cup G_\alpha$, then $A \subset \cup_{i=1}^{n} G_{\alpha_i}$, $A$ compact, $G_{\alpha_i}$ open.




**PROPOSITION 2.7 (LINDELÖF)** *Any open cover of a set of real numbers contains a countable subcover.*

*Proof:* Let $A$ be any set of real numbers with open cover $\{G_\alpha\}$ : $A \subset \cup G_\alpha$. We want to show

$$A \subset \bigcup_{\text{countable}} G_\alpha.$$

Pick $p \in A$. Then $p \in G_\alpha$ for some $\alpha$ and since $G_\alpha$ is open, $(p-\delta, p+\delta) \subset G_\alpha$ for some $\delta > 0$. Pick rational numbers $r_1, r_2$ so that $p - \delta < r_1 < p < r_2 < p + \delta$. Thus we associate with each point $p$ of $A$ an open interval $I(p) = (r_1, r_2)$ with rational endpoints. Since the set of all possible intervals with rational endpoints is countable, the collection $B = \{I(p) \mid p \in A\}$ is countable. But each member of $B$ may be associated with exactly one member of $\{G_\alpha\}$, say, $G_\alpha^*$. Thus the collection $\{G_\alpha^*\}$ is countable and covers $A$. ■

We now prove one of the most important theorems of analysis.

**THEOREM 2.5 (HEINE-BOREL)** *A set of real numbers is compact iff it is closed and bounded.*

*Proof:* Suppose $K$ is a nonempty compact subset of $R$. We show $K$ is closed and bounded. Since $K \subset \bigcup (-n, n)$, a finite subcollection covers $K$, that is, $K \subset (-N, N)$ for some $N$ and hence $K$ is bounded. We still must show $K$ is closed. Equivalently (Proposition 2.6), will show that any point not in $K$ is not a limit point of $K$. Pick $p \notin K$ and form closed sets $F_n = [p - 1/n, p + 1/n]$, $n = 1, 2, \ldots$. Then $p = \bigcap_n [p - 1/n, p + 1/n]$ and since $p \notin K$, $K \subset (\bigcap F_n)^c = \bigcup F_n^c$. But then $K$ is covered by the collection of open sets $\{F_n^c\}$. Since $K$ is compact, we have $K \subset F_1^c \bigcup F_2^c \bigcup \cdots \bigcup F_N^c$ for some $N$, or $K \subset F_N^c$ since $F_1^c \subset F_2^c \subset \cdots \subset F_N^c$. No point of $K$ is in $[p - 1/N, p + 1/N]$. Thus $p$ is not a limit point of $K$. In $R$, compact sets are bounded and closed. Now suppose $K$ is closed and bounded. To show $K$ is compact, we must show every open cover of $K$ reduces to a finite subcover. Let $\{G_\alpha\}$ be an open cover of $K$. By Proposition 2.7 (Lindelöf), we have a countable subcover: $K \subset \bigcup G_{\alpha_i}$. Define sets $G_n$ and $F_n$ as follows: $G_n = G_{\alpha_1} \bigcup G_{\alpha_2} \bigcup \cdots \bigcup G_{\alpha_n}$ and $F_n = K \bigcap G_n^c$. $G_n$ is open, $F_n$ is closed, $G_n \subset G_{n+1}$ and $F_{n+1} \subset F_n$ for all $n$. If $F_n$ is empty for some $n$, we have a finite subcover. So assume $F_n \neq \emptyset$ for all $n$ and form the closed set $\bigcap F_n$. Each set $F_n$ is a closed, bounded ($F_n \subset K$), and nonempty subset of $K$. If we define $x_n = \sup F_n$, then $x_n \in F_n$ since $F_n$ is closed, the sequence is nonincreasing and bounded below ($F_{n+1} \subset F_n \subset K$). Let $x_0 = \inf\{x_1, x_2 \ldots\}$. Claim: $x_0 \in \bigcap F_n \subset K$. Otherwise, $(x_0 - \delta, x_0 + \delta) \bigcap F_N = \emptyset$ for some $\delta > 0$

and some $N$. Since $F_{n+1} \subset F_n$, $x_0 + \delta$ is a lower bound for $\{x_N, x_{N+1}, \ldots\}$ and hence $\{x_1, x_2, \ldots\}$. But $x_0$ is the greatest lower bound. So $x_0 \in \bigcap F_n \subset K$, which implies $x_0 \in K$, $x_0 \notin G_n$ for all $n$. But the collection $\{G_n\}$ covers $K$. Thus $F_n$ must be empty for some $n$ and the proof is complete. ■

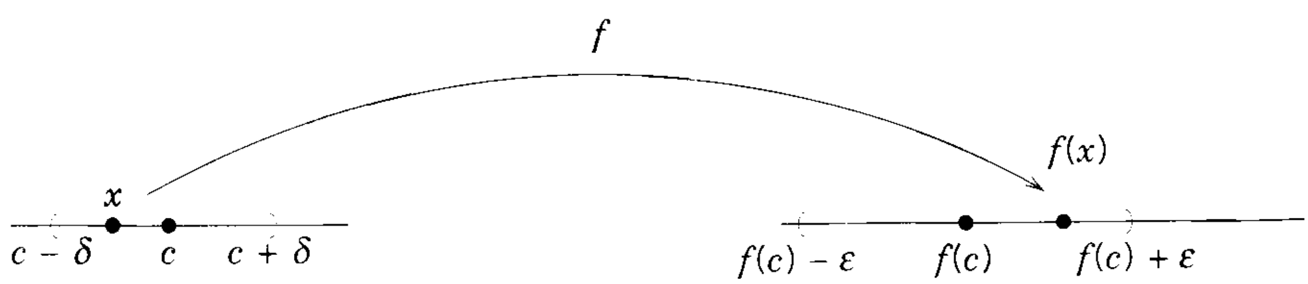
**THEOREM 2.6 BOLZANO-WEIERSTRASS** *Every bounded infinite set of real numbers has a limit point.*

*Proof:* Let $E$ be a bounded infinite set of real numbers. We show $E$ has a limit point. Since $E$ is bounded, we have a closed bounded interval $[a, b]$ that contains $E$. Suppose for every $x \in [a, b]$ we could find an open interval $I(x)$ centered at $x$ that contains only a finite number of points of $E$. The collection of such open intervals would be an open cover of the compact set $[a, b]$. Thus a finite number of such intervals would cover $[a, b]$, and hence $E$. By assumption each of these intervals contains a finite number of points of $E$. So $E$ would be finite. But $E$ is infinite. Thus we have at least one point of $[a, b]$, say $x_0$, so that every open interval $I(x_0)$ centered at $x_0$ contains an infinite number of points of $E$. But then $x_0$ is a limit point of $E$, and this is the conclusion we wanted. ■

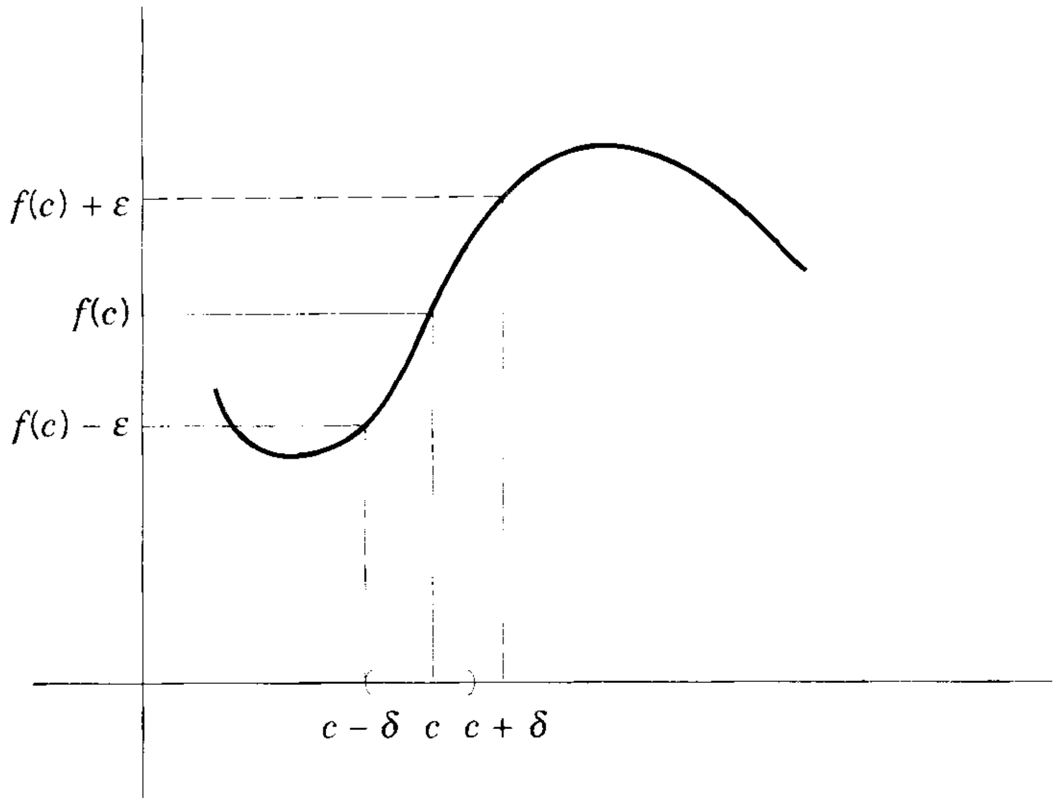
## 2.8 CONTINUOUS FUNCTIONS

### 2.8.1 Definition

A real-valued function $f$ defined on a set $E$ of real numbers is said to be *continuous* at $c \in E$ if for every $\epsilon > 0$, we have a $\delta(c, \epsilon) > 0$ such that $x \in (c - \delta, c + \delta) \bigcap E$ implies $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$. The $\delta$ usually depends on both $c$ and $\epsilon$.

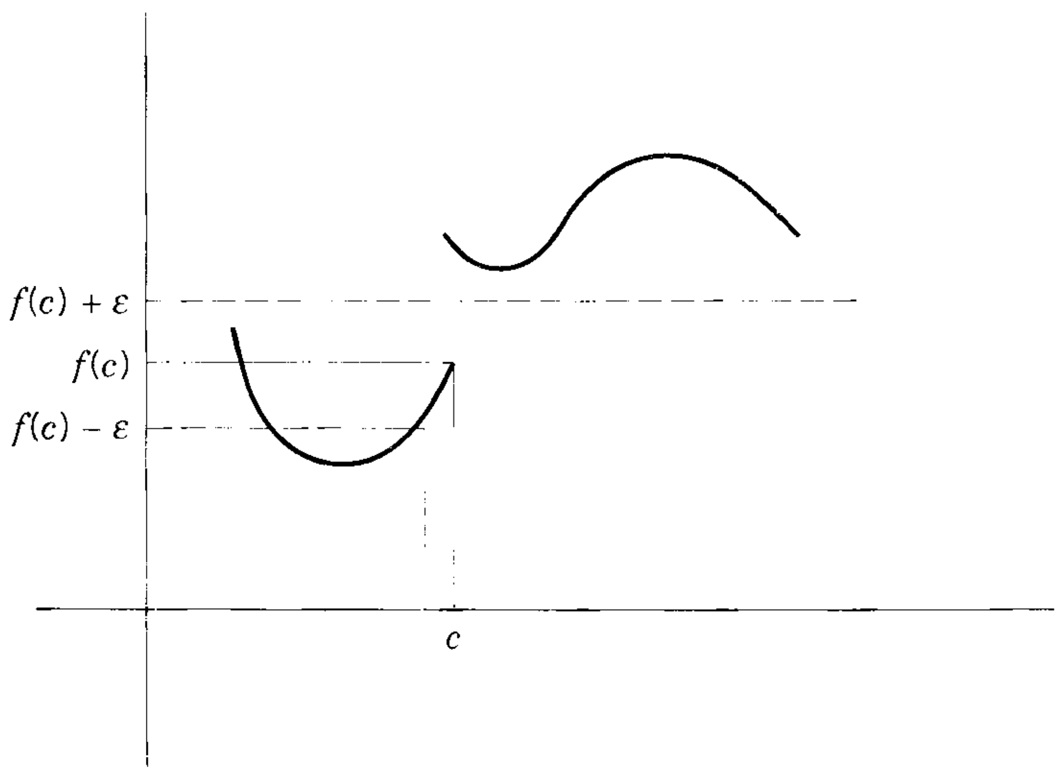



$f$ is continuous at $c$




$f$ is continuous at $c$

If $f$ is not continuous at $c \in E$, then we have $\epsilon > 0$ so that for every $\delta(c, \epsilon) > 0$, there is an $x \in (c - \delta, c + \delta) \cap E$ for which $f(x) > f(c) + \epsilon$ or $f(x) < f(c) - \epsilon$. We note that if $c$ is an isolated point of $E$ then $f$ is automatically continuous at $c$.




$f$ is discontinuous at $c$

**Example 16:** $f$ discontinuous at every point of $E = R$.

$$f(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational}. \end{cases}$$

**Example 17:** $f$ is one on the rationals, $E$, and zero elsewhere. $f$ is discontinuous at every point of $R$, but $f$ is continuous on $E$.

**Example 18:** $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q},\ (p,q) = 1,\ 0 \le x \le 1 \\ 0, & x \text{ irrational},\ 0 \le x \le 1 \end{cases}$;

$E = [0,1]$. $\lim\limits_{x \to c} f(x) = 0$ for all $c \in [0,1]$.

$f$ is continuous at irrationals and discontinuous at rationals.

**Example 19:** $f$ continuous at exactly one point:

$$f(x) = \begin{cases} x, & x \text{ rational} \\ -x, & x \text{ irrational}. \end{cases}$$

$f$ is continuous at $x = 0$.

Continuity is a local property. In Definition 2.8.1 we are given an open interval centered at $f(c)$ : $(f(c) - \epsilon, f(c) + \epsilon)$. We must find an open interval in the domain of $f$ : $(c - \delta, c + \delta) \cap E$, whose image under $f$ is a subset of $(f(c) - \epsilon, f(c) + \epsilon)$. This suggests we look for an inverse image interpretation of continuity:

**Example 20:** $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \le 0 \end{cases}$. $f^{-1}((0,1)) = \emptyset$,

$f^{-1}((\frac{1}{2}, \frac{3}{2})) = (0, \infty)$, and $f^{-1}((-\frac{3}{2}, -\frac{1}{2})) = (-\infty, 0]$. $f$ is discontinuous at $x = 0$ and $(-\infty, 0]$ is not open.

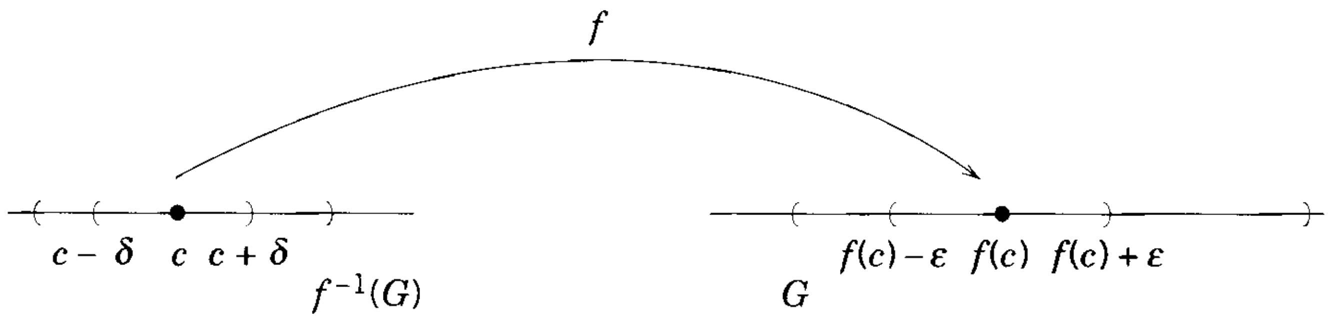
**Example 21:** $f(x) = \begin{cases} x^2, & x \le 1 \\ 3 - x, & x > 1 \end{cases}$. $f^{-1}((\frac{1}{4}, \frac{3}{2})) = \left(-\sqrt{\frac{3}{2}}, -\frac{1}{2}\right)$

$\cup (\frac{1}{2}, 1] \cup (\frac{3}{2}, 3 - \frac{1}{4})$. $f$ is discontinuous at $x = 1$ and $(\frac{1}{2}, 1]$ is not open.

**Example 22:** $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational}, \end{cases}$ and $E$ is the set of rationals. $f^{-1}((\frac{1}{2},\frac{3}{2})) = (\frac{1}{2},\frac{3}{2}) \cap E$, which is open relative to $E$. Inverse images are important.

**PROPOSITION 2.8** *Let $f$ be a real-valued function with domain $E$. Then $f$ is continuous on $E$ iff $f^{-1}(G)$ is open relative to $E$ whenever $G$ is an open subset of $R$.*

*Proof:* Assume $f$ is continuous on $E$, that is, at each point of $E$, and let $G$ be an open subset of $R$. We will show $f^{-1}(G)$ is open relative to $E$. If $f^{-1}(G) = \emptyset$, we are done. So assume $f^{-1}(G) \neq \emptyset$ and let $c \in f^{-1}(G)$. We construct an open interval containing $c$ that lies in $f^{-1}(G)$: Since $c \in f^{-1}(G)$, $f(c) \in G$, and because $G$ is open, $(f(c) - \epsilon, f(c) + \epsilon) \subset G$. By assumption, $f$ is continuous at $c$. So we have a $\delta(c, \epsilon) > 0$ so that $x \in (c - \delta, c + \delta) \cap E$ implies $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$. In other words, $(c - \delta, c + \delta) \cap E \subset f^{-1}(G)$. Thus, to each $c \in f^{-1}(G)$, corresponds $(c - \delta, c + \delta) \cap E$ and since the union of such sets is open relative to $E$, we are done.




Now assume inverse images of open sets are open relative to $E$. We show $f$ is continuous on $E$. Let $c \in E$ and $\epsilon > 0$. Then $(f(c) - \epsilon, f(c) + \epsilon)$ is an open subset of $R$. By assumption, $f^{-1}((f(c) - \epsilon, f(c) + \epsilon))$ is open relative to $E$ and contains the number $c$. Select $\delta > 0$ so that $(c - \delta, c + \delta) \cap E$ is a subset of $f^{-1}((f(c) - \epsilon, f(c) + \epsilon))$, that is,

$$f((c - \delta, c + \delta) \cap E) \subset (f(c) - \epsilon, f(c) + \epsilon).$$

Thus $f$ is continuous at $c$. This completes the argument. ∎

**Example 23:** $f(x) = 1/x$, $0 < x \leq 1$. $f$ is continuous, but $f$ does not preserve boundedness: $f((0, 1]) = [1, \infty)$.

**Example 24:** $f(x) = x^2$, $-1 < x < 1$. $f$ is continuous, but $f$ does not preserve openness: $f((-1, 1)) = [0, 1)$.

**Example 25:** $f(x) = 1/(1 + x^2)$, $1 \leq x < \infty$. $f$ is continuous, but $f$ does not preserve closedness: $f([1, \infty)) = (0, 1/2]$.

Since continuous functions do not necessarily preserve boundedness, openness, or closedness, it is suprising that such functions preserve closed and bounded (compactness).

**PROPOSITION 2.9** *Let $f$ be a real-valued continuous function on $[a, b]$. Then $f$ assumes a maximum and a minimum on $[a, b]$; that is, we have $x_m, x_M \in [a, b]$ such that*

$$f(x_m) \le f(x) \le f(x_M)$$

for all $x \in [a, b]$.

*Proof:* We first show $f$ is bounded on $[a, b]$. Assume $f$ is not bounded on $[a, b]$. Then we may construct a sequence $(x_n)$ in $[a, b]$ with $|f(x_n)| > n$. But by Bolzano-Weierstrass, the sequence must contain a convergent subsequence $(x_{n_k})$ converging to some $c \in [a, b]$. Since $f$ is continuous at $c$, $f(x_{n_k}) \longrightarrow f(c)$. But this contradicts $|f(x_{n_k})| > n_k$. A continuous function on a closed bounded interval must be bounded.

We will now show $f$ assumes a maximum. Let $M = \sup\{f(x) \mid x \in [a, b]\}$. $M$ is finite by the argument above. Since $M - 1/n$ is not an upper bound, we have $z_n$ so that $M - 1/n < f(z_n) \le M$. In this manner we construct a sequence $(z_n)$, and again apply Bolzano-Weierstrass. Thus, we have a subsequence $(z_{n_k})$ with $M - 1/n_k < f(z_{n_k}) \le M$ and $z_{n_k} \longrightarrow x_M$ in $[a, b]$. Continuity of $f$ implies $f(z_{n_k}) \to f(x_M)$.

But $f(z_{n_k}) \longrightarrow M$. That is, $f(x_M) = M$. $f$ assumes its maximum at $x_M$. The reader may argue the "minimum". ∎

**INTERMEDIATE-VALUE THEOREM 2.7** *If $f$ is a real-valued continuous function on an interval $I$, then $f$ has the intermediate-value property on $I$: whenever $a, b \in I$, $a < b$, and $y$ lies between $f(a)$ and $f(b)$, there exists at least one $c \in (a, b)$ so that $f(c) = y$.*

*Proof:* Assume $f(a) < y < f(b)$. Define the set $S = \{x \mid a \le x \le b, f(x) < y\}$. $S$ is not empty ($a \in S$) and is bounded above by $b$. Let $c = \sup S$. We show $f(c) = y$. Since $c - 1/n$ is not an upper bound for $S$, we have $x_n \in S$ so that

$$a < c - \frac{1}{n} < x_n \le c.$$

Thus $x_n \to c$ and $f(x_n) \to f(c)$ by continuity of $f$. But because $x_n \in S$,

$f(x_n) < y$. Hence

$$f(c) \leq y .$$

Let $z_n = \min\{b, c + 1/n\}$. Again, $z_n \to c$, $f(z_n) \to f(c)$, and since $z_n \notin S$, $f(z_n) \geq y$. Thus $f(c) \geq y$. Consequently

$$f(c) = y ,$$

and this is what we wanted to show. The reader may argue $f(a) > y > f(b)$. ∎

We now show continuous functions preserve compactness.

**PROPOSITION 2.10** *If $f$ is a real-valued continuous function on the compact set $K$, then $f(K)$ is compact.*

*Proof:* Sketch: Let $\{G_\alpha\}$ be an open covering of $f(K) : f(K) \subset \cup G_\alpha$. Because $f$ is continuous, $f^{-1}(G_\alpha)$ is open, and $K \subset f^{-1}(\cup G_\alpha) = \cup f^{-1}(G_\alpha)$. In other words, the collection $\{f^{-1}(G_\alpha)\}$ is an open cover of the compact set $K$. By Heine-Borel, a finite number, say $f^{-1}(G_{\alpha_1}), \ldots, f^{-1}(G_{\alpha_N})$ suffices to cover $K$: $K \subset \bigcup_{i=1}^{N} f^{-1}(G_{\alpha_i})$. But then, $f(K) \subset \bigcup_{i=1}^{N} G_{\alpha_i}$. ∎

### 2.8.2 Definition

Let $f$ be a real-valued continuous function with domain $E$. Then $f$ is *uniformly continuous* on $E$ if for every $\epsilon > 0$ there exists $\delta(\epsilon)$ so that $-\epsilon < f(x) - f(y) < \epsilon$ for all $x, y \in E$ for which $-\delta < x - y < \delta$. $\delta$ "works" for the set $E$. Uniform continuity is "global" ("one $\delta$ for all of $E$") whereas continuity is "local" ("a $\delta$ for each $x \in E$") (Heine, 1870).

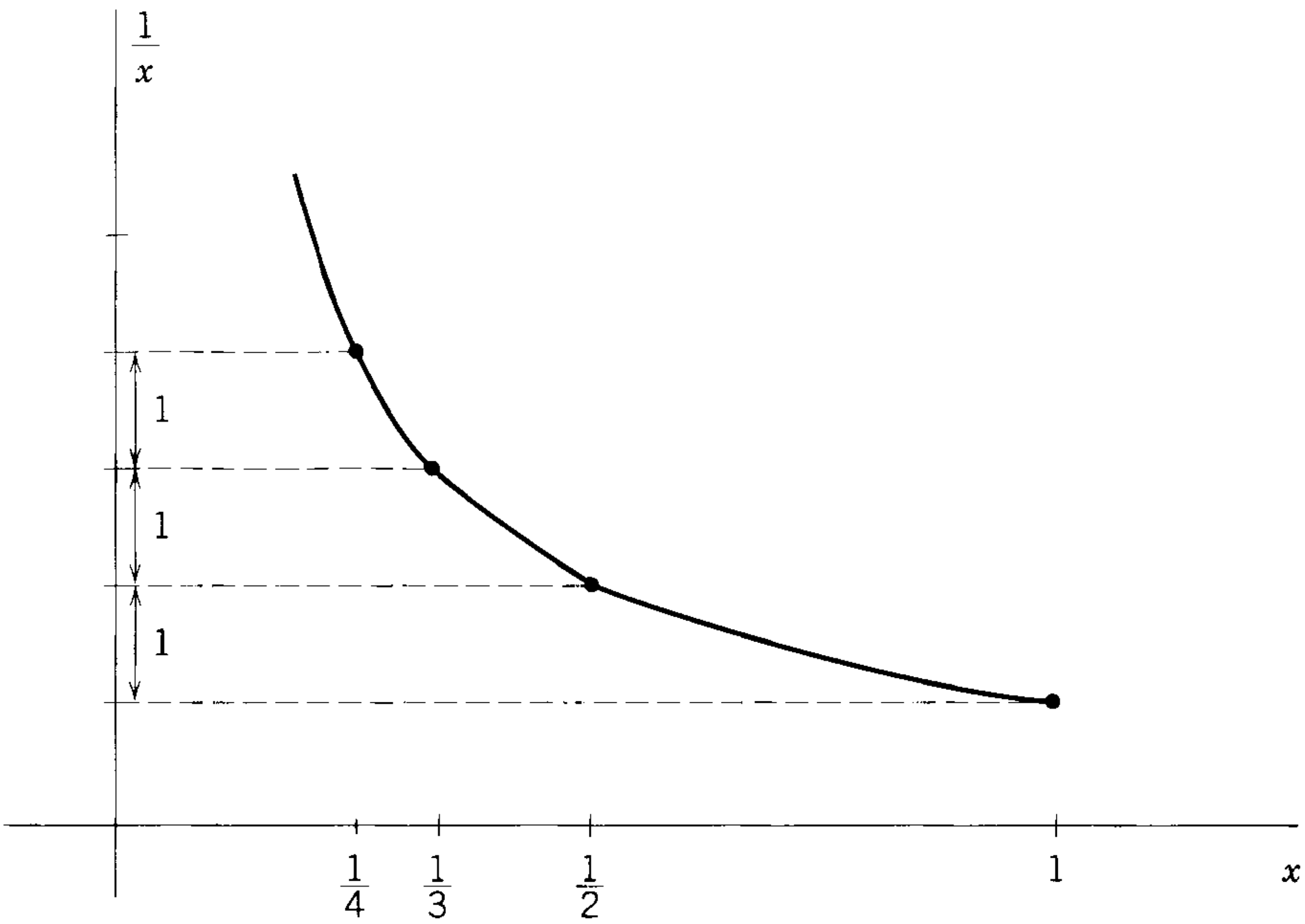
**Example 26:** $f(x) = \sqrt{x}$, $0 \leq x \leq 1$. Since

$$| f(x) - f(y) | = | \sqrt{x} - \sqrt{y} | = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq \frac{|x - y|}{\sqrt{|x - y|}} = \sqrt{|x - y|} ,$$

$x$ and $y$ not both zero, letting $\delta(\epsilon) = \epsilon^2$ shows $f$ is uniformly continuous on $[0, 1]$.

**Example 27:** $f(x) = 1/x$, $0 < x \leq 1$. We claim $f$ is not uniformly continuous on $(0, 1]$. To see this, let $\epsilon = 1$, and form the sequences

$x_n = 1/(n+1)$ and $y_n = 1/n; n = 1, 2, \ldots$. Then $| f(x_n) - f(y_n) | = 1$ and $| x_n - y_n | = 1/n(n+1)$. Since $| x_n - y_n | \to 0$, we cannot find a $\delta(1)$ that "works" for the entire interval $(0, 1]$. The problem is at $x = 0$.




**Theorem 2.8** *A real-valued continuous function on a compact set is uniformly continuous* (Heine, 1872).

*Proof:* Suppose $f$ is a continuous function on a compact set $K$ and let $\epsilon > 0$. We must determine $\delta(\epsilon) > 0$ so that $-\epsilon < f(x) - f(y) < \epsilon$ whenever $x, y$ are any points in $K$ satisfying $-\delta(\epsilon) < x - y < \delta(\epsilon)$.

Select any point $c$ in $K$. Because $f$ is continuous at $c$, we have $\delta(c) > 0$ so that $-\epsilon/2 < f(x) - f(c) < \epsilon/2$ whenever $x \in (c - \delta(c), c + \delta(c)) \cap K$. The open sets $\{(c - \delta(c)/2, c + \delta(c)/2)\}$ form an open cover of the compact set $K$. By Heine-Borel, a finite subcollection of these sets will cover $K$: $K \subset \bigcup_{i=1}^{N} (c_i - \delta(c_i)/2, c_i + \delta(c_i)/2)$. Define $\delta$ as the minimum of the set: $\{\delta(c_1)/2, \delta(c_2)/2, \ldots, \delta(c_N)/2\}$. Pick any points $x, y$ in $K$ satisfying $-\delta < x - y < \delta$. We will show $-\epsilon < f(x) - f(y) < \epsilon$ to complete the proof. Because $x \in (c_i - \delta(c_i)/2, c_i + \delta(c_i)/2)$ for some $i$ (these intervals cover $K$), $-\epsilon/2 < f(c_i) - f(x) < \epsilon/2$. But $| y - c_i | \le | y - x | + | x - c_i | < \delta + 1/2\, \delta(c_i) \le \delta(c_i)$. Hence $-\epsilon/2 < f(y) - f(c_i) < \epsilon/2$. Adding, $-\epsilon < f(y) - f(x) < \epsilon$ whenever $-\delta < y - x < \delta$. ∎

## 2.9 DIFFERENTIABLE FUNCTIONS

### 2.9.1 Definition

Let $f$ be a real-valued function defined on an open interval containing the point $c$. We say that $f$ is *differentiable* at $c$, written $f'(c)$, provided the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite.

**PROPOSITION 2.11** *If $f$ is a real-valued function defined on an open interval $(a, b)$ containing $c$, if $f$ assumes its maximum or minimum at $c$, and if $f$ is differentiable at $c$, then $f'(c) = 0$.*

*Proof:* Assume $f(c)$ is a maximum and that $f'(c) > 0$. So

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0 .$$

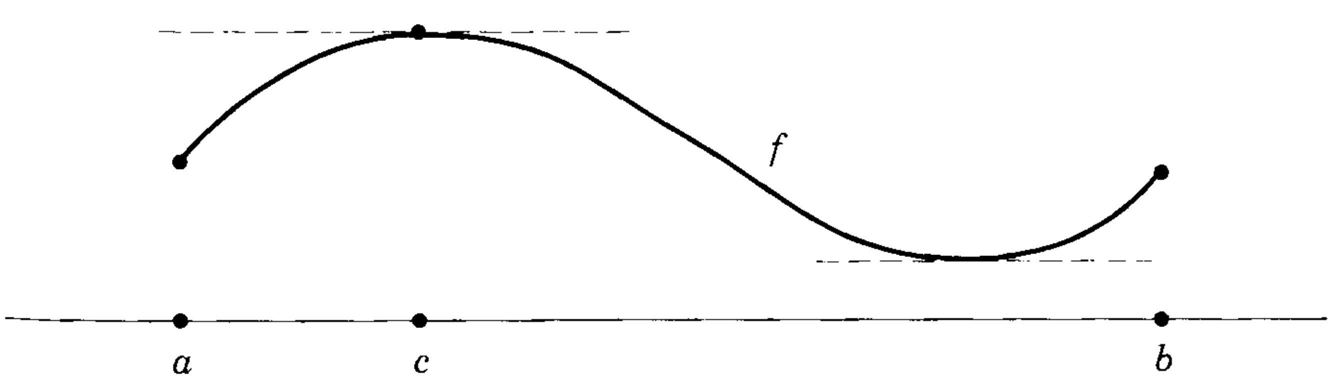
Then we have $\delta > 0$ so that $a < c - \delta < c + \delta < b$ and for $0 < | x - c | < \delta$,

$$\frac{f(x) - f(c)}{x - c} > 0 .$$

Let $x = c + \delta/2$. Then $f(c + \delta/2) > f(c)$. We have a contradiction to $f(c)$ being the maximum. The reader may complete the argument. ■

**ROLLE'S THEOREM 2.9** *Let $f$ be a real-valued continuous function on $[a, b]$, differentiable on $(a, b)$, and satisfy $f(a) = f(b)$. Then there exists $c$, $a < c < b$, with $f'(c) = 0$.*

*Proof:* The figure indicates why such a result might be true.



Since $f$ is continuous on $[a, b]$, $f$ assumes a minimum and maximum on $[a, b]$ (Proposition 2.9),

$$f(x_m) \leq f(x) \leq f(x_M),$$

$a \leq x_M, x_m \leq b$. If $x_m, x_M$ are both endpoints, then since $f(a) = f(b)$, $f$ must be a constant and any point in $(a, b)$ will "work" for $c$. Otherwise, $x_m$ or $x_M$ or both belong to $(a, b)$, and $f$ assumes a maximum or minimum at an interior point. From the previous proposition,

$$f'(x_m) \qquad \text{or} \qquad f'(x_M) = 0,$$

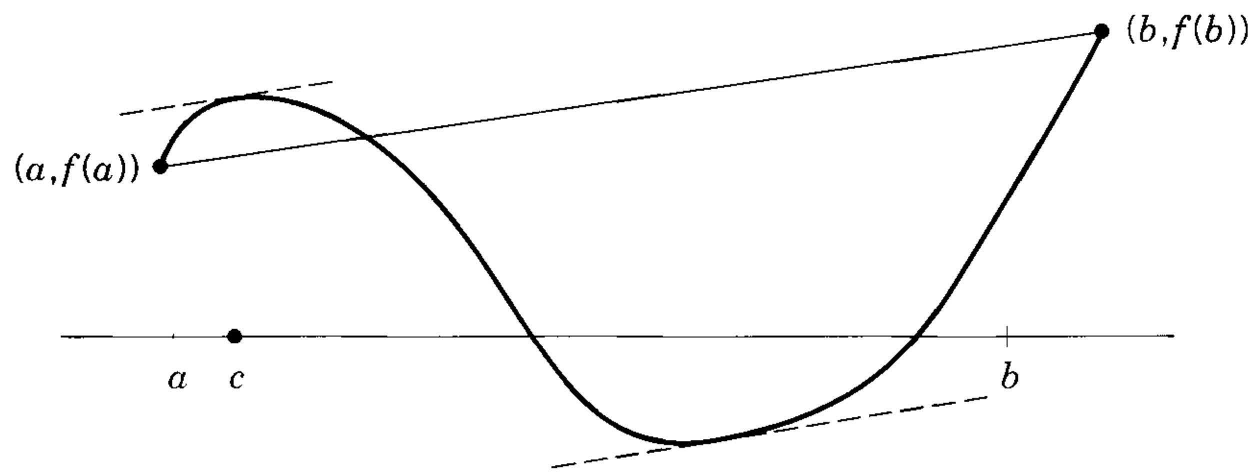
and we are done. ■

We now have a straightforward proof of the Mean Value Theorem.

**Mean Value Theorem 2.10** *If $f$ is a real-valued continuous function on $[a, b]$ and is differentiable on $(a, b)$, then there exists at least one $c \in (a, b)$ so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

*Proof:* A diagram makes the result plausible.




If $f(a) = f(b)$, we have Rolle's Theorem. Otherwise, write the equation of the straight line connecting the points $(a,f(a))$, $(b,f(b))$:

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Define a new function $g$ that measures the separation of "$y$ on the curve" and "$y$ on the straight line":

$$\begin{aligned} g(x) &= f(x) - L(x) \\ &= f(x) - \left[ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]. \end{aligned}$$

The reader may show $g$ satisfies the hypotheses of Rolle's Theorem. The conclusion is immediate. ■

## 2.10 SEQUENCES OF FUNCTIONS

### 2.10.1 Definitions

Suppose $(f_n)$ is a sequence of real-valued functions defined on a common domain $E$. If the sequence of real numbers $(f_n(x))$ converges to a real number for every $x \in E$, we can define a new function $f$ by

$$f(x) = \lim f_n(x), \qquad x \in E.$$

That is, the sequence $(f_n)$ is said to be *convergent* to a limit function $f$ if corresponding to any $\epsilon > 0$ and any $x \in E$, we have a natural number $N(x, \epsilon)$ so that $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ for all $n \geq N(x, \epsilon)$. We frequently say the sequence $(f_n)$ converges pointwise to $f$ on $E$ because, generally, $N(x, \epsilon)$ will depend on the point $x$.

For each $x \in E$, we could form the sequences $(\liminf f_n(x))$ and $(\limsup f_n(x))$, with the attendant functions: $\liminf f_n$ and $\limsup f_n$, respectively. Of course, we would then want to be in $R^e$. Going back then to $\lim f_n(x) = f(x), x \in E$, we want to allow the limit to be $\infty$ or $-\infty$. As with the sequence $(x_n)$ then, "converges" means a "real number" and limit means a "real number", "$\infty$", or "$-\infty$".

The power of the Lebesgue integral and limitations of the Riemann integral are revealed when we ask: under what conditions is it true that

$$\lim \int f_n = \int \lim f_n \;?$$

Some problems are appropriate.

### 2.10.2 Problem

Calculate $\lim f_n$, $\lim \int f_n(x)\,dx$, and $\int (\lim f_n)(x)\,dx$

1. $f_n(x) = x^n, \qquad 0 \le x \le 1.$
2. $f_n(x) = xn^2e^{-nx}, \qquad 0 \le x \le 1.$

3. $f_n(x) = \begin{cases} n^2x, & 0 \le x \le \frac{1}{n} \\ -n^2(x - \frac{2}{n}), & \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1. \end{cases}$

4. $f_n(x) = \begin{cases} \frac{1}{n}, & 0 \le x \le n \\ 0, & n < x. \end{cases}$

5. $f_n(x) = \begin{cases} \frac{1}{n}, & 0 \le x \le n^2 \\ 0, & n^2 < x. \end{cases}$

6. $f_n(x) = \dfrac{nx}{1+nx}, \qquad 0 \le x \le 1.$

7. $f_n(x) = nx(1 - x^2)^n, \qquad 0 \le x \le 1.$

8. $f_n(x) = \dfrac{2n^2x}{1+n^2x^2}, \qquad x \ge 0.$

9. $f_n(x) = \lim\limits_k (\cos n!\, \pi x)^{2k}, \qquad -\infty < x < \infty.$

10. $f_n(x) = n(x^{\frac{1}{n}} - 1), \qquad x > 0.$

11. $f_n(x) = \left(1 + \dfrac{x}{n}\right)^n.$

12. $f_n(x) = \begin{cases} \frac{1}{n}, & x \text{ rational} \\ 0, & x \text{ irrational}. \end{cases}$

For Riemann integrals, a sufficient condition for interchanging the two limiting operations, "lim" and "$\int$", is uniform convergence of the sequence $(f_n)$.

It is in the interchange of limiting operations that we have to be so careful:

**Example 28:** Let $a_{mn} = \dfrac{m}{m+n}; \qquad m, n = 1, 2, 3, \ldots$

Calculate $\lim\limits_n \left(\lim\limits_m a_{mn}\right)$ and $\lim\limits_m \left(\lim\limits_n a_{mn}\right)$.

**Example 29:** $f_n(x) = \dfrac{\sin(nx)}{nx}, \qquad x > 0.$

Calculate $\lim\limits_{n}\left(\lim\limits_{x\to 0}\dfrac{\sin(nx)}{nx}\right)$ and $\lim\limits_{x\to 0}\left(\lim\limits_{n}\dfrac{\sin(nx)}{nx}\right)$.

**Example 30:** $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}.$

Calculate $\lim\limits_{n}\left(\lim\limits_{h\to 0}\dfrac{f_n(x+h) - f_n(x)}{h}\right)$ and

$$\lim_{h\to 0}\left(\frac{\lim\limits_{n} f_n(x+h) - \lim\limits_{n} f_n(x)}{h}\right).$$

"Limit of the derivative" is not necessarily the "derivative of the limit".

**Example 31:** Let $f_n(x) = \displaystyle\sum_{k=0}^{n}\frac{x^2}{(1+x^2)^k}.$

Calculate $\lim\limits_{x\to 0}\left(\lim\limits_{n} f_n(x)\right)$ and $\lim\limits_{n}\left(\lim\limits_{x\to 0} f_n(x)\right)$.

"Limit of the sum" is not necessarily the "sum of the limits".

**Example 32:** Let $f_n(x) = n^2x(1-x^2)^n, \; 0 \leq x \leq 1.$

Calculate $\lim\limits_{n}\left(\lim\limits_{\|\Delta x\|\to 0}\displaystyle\sum_{i=1}^{m} f_n(x_i)\,\Delta x_i\right)$ and

$$\lim_{\|\Delta x\|\to 0}\left(\sum_{i=1}^{m}(\lim_{n} f_n(x_i))\,\Delta x_i\right).$$

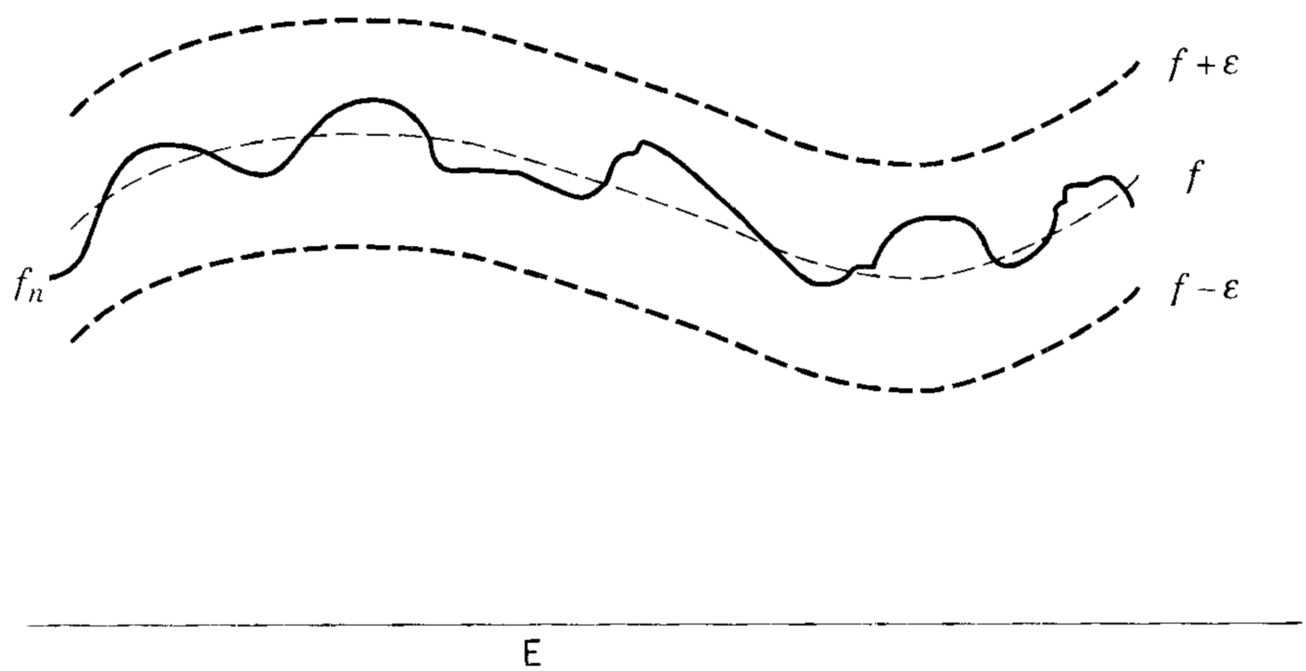
"Limit of the integral is not necessarily the integral of the limit".

### 2.10.3 Definition

A sequence of real-valued functions $(f_n)$ is *uniformly convergent* to a function $f$ on a set $E$ if for each $\epsilon > 0$, we can determine a natural number

$N(\epsilon)$ so that $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ for all $n \geq N(\epsilon)$ and all $x \in E$. We write $\lim f_n = f(unif)$.

The essential difference between pointwise convergence and uniform convergence is the difference between "local" and "global": the $N(x, \epsilon)$ in pointwise convergence generally depends on a particular point $x$ ("an $N$ for each $x$"); the $N(\epsilon)$ of uniform convergence "works" for all points of $E$ ("one $N$ for all $x$"). It is nonsense to talk about uniform convergence at a point. Uniform convergence deals with behavior on a set of points. The figure below illustrates the definition.




The reader should show that an equivalent definition for uniform convergence may be stated as follows:

$$\lim f_n = f(unif) \iff \lim_n \sup_E |f(x) - f_n(x)| = 0 .$$

Generally speaking, uniform convergence preserves good behavior:

**Example 33:** "Bad" $\longrightarrow$ "Good"

$$f_n(x) = \begin{cases} \frac{1}{n}, & x \text{ irrational} \\ 0, & x \text{ rational} \end{cases}, \qquad 0 \leq x \leq 1.$$

$\lim f_n \equiv 0$. $f_n$ is nowhere continuous on $[0,1]$, but the limit function, $f(x) \equiv 0$ on $[0,1]$, is as nice as it gets!

**Example 34:** "Bad" $\longrightarrow$ "Bad"

$$f_n(x) = \begin{cases} 1+\frac{1}{n}, & x \text{ irrational} \\ 0, & x \text{ rational} \end{cases}, \qquad 0 \le x \le 1.$$

$$\lim f_n(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational} \end{cases}, \qquad 0 \le x \le 1 = f(x).$$ $f_n$ and $f$ are

nowhere continuous on $[0,1]$.

As an example of "Good" $\longrightarrow$ "Good" we offer the following:

Uniform convergence preserves continuity and the process of finding the limit of the sequence and the limit at a point may be performed in either order:

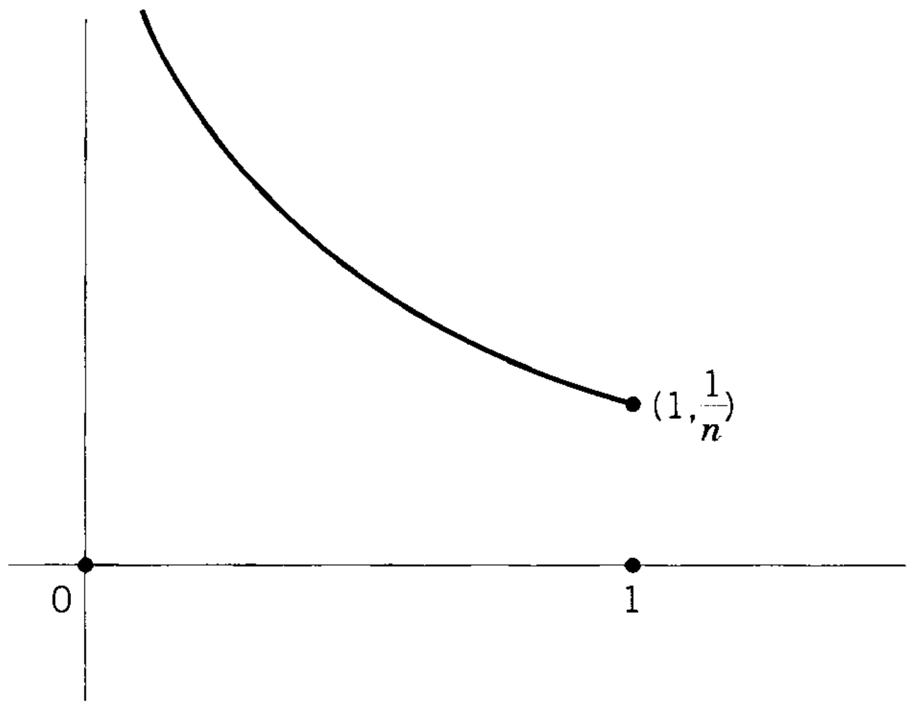
$$\lim_{x\to x_0}\left(\lim_n f_n(x)\right) = \lim_n\left(\lim_{x\to x_0} f_n(x)\right).$$

We show:

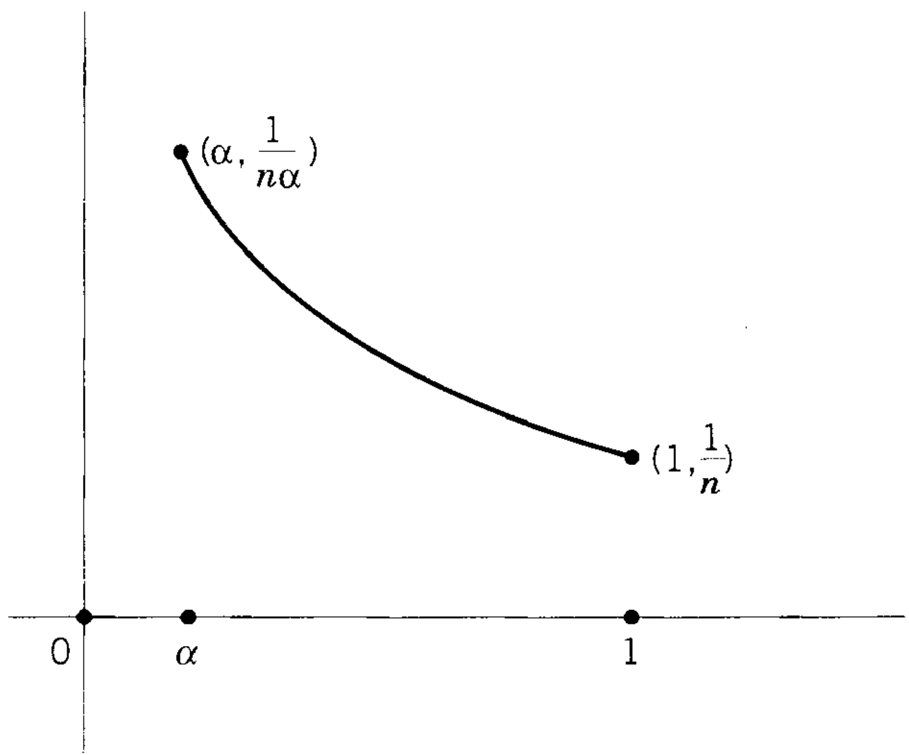
**Proposition 2.12** *Let $(f_n)$ be a sequence of real-valued functions, each of which is continuous at a point $c \in E$. Suppose $\lim f_n = f(unif)$ on $E$. Then $f$ is continuous at $c$.*

*Proof:*
$|f(x)-f(c)| \le |f(x)-f_n(x)| + |f_n(x)-f_n(c)| + |f_n(c)-f(c)|$.
The first and third terms on the right-hand side are small by uniform convergence on $E$ for $n \ge N(\epsilon)$. Once $N(\epsilon)$ is selected, the middle term can be made small under the assumption that $f_{N(\epsilon)}$ is continuous at $c$. ∎

**Example 35:** $f_n(x) = \begin{cases} \dfrac{1}{nx}, & 0 < x \le 1 \\ 0, & x = 0. \end{cases}$

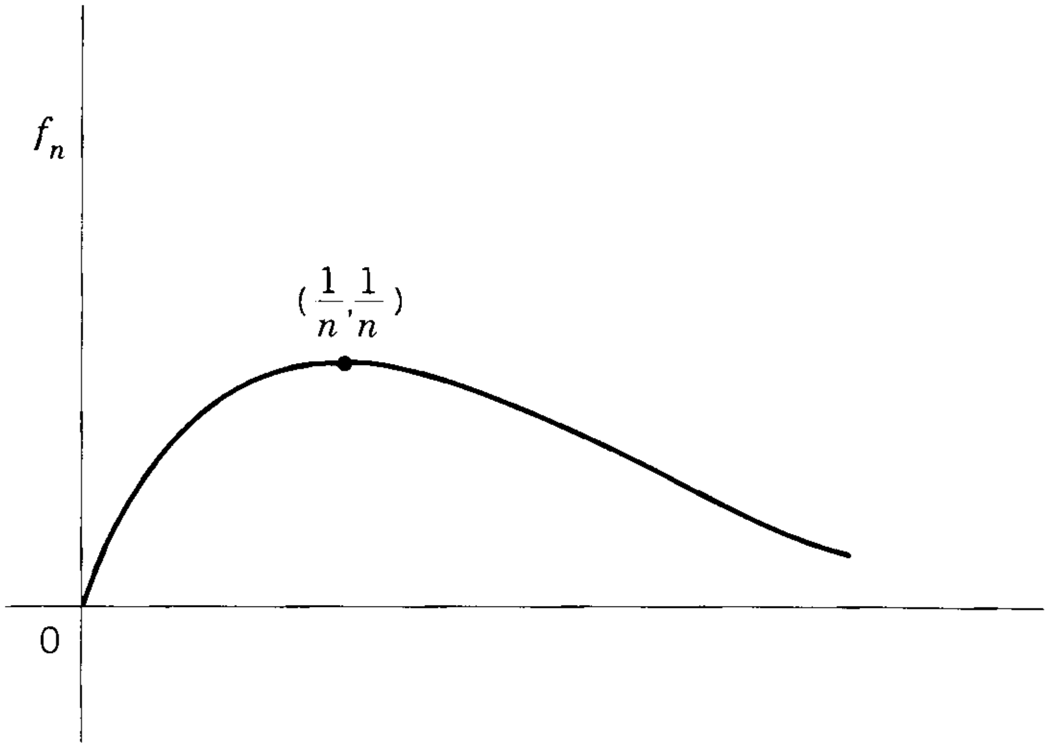



$\lim f_n(x) = 0$, $0 \le x \le 1$; $f(x) \equiv 0$ on $[0, 1]$. Convergence is not uniform: $\epsilon = 1$.



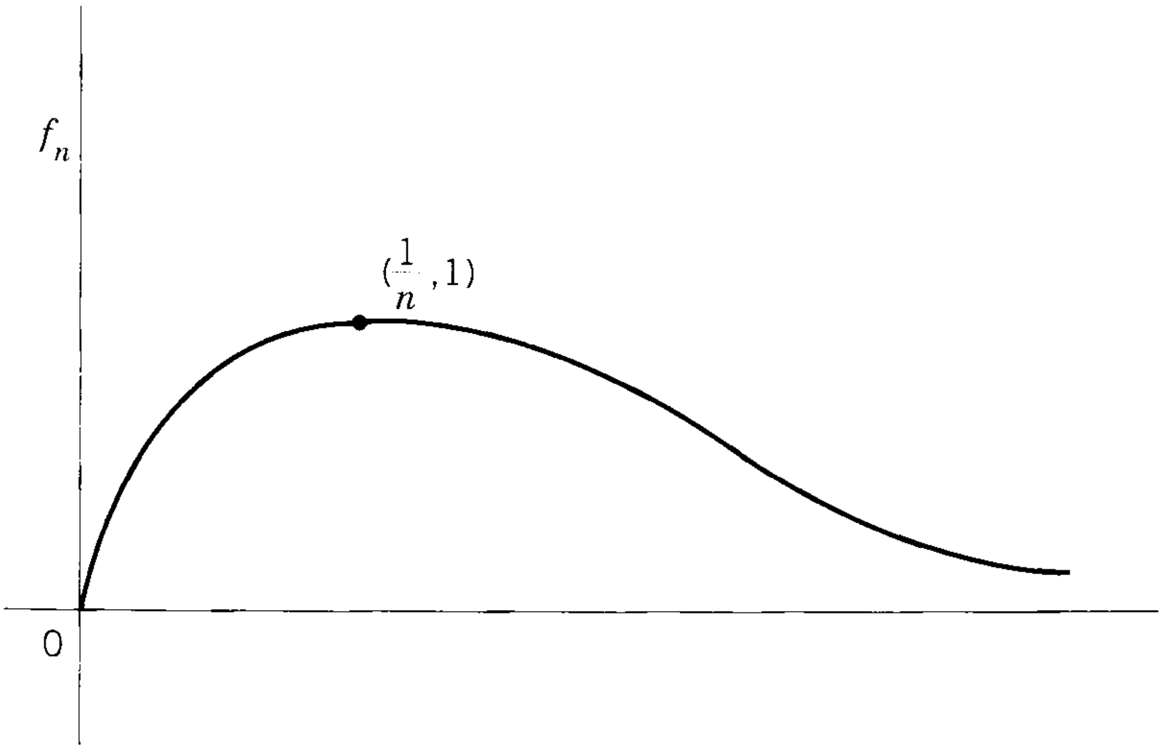

*Note*: Convergence is uniform on $[\alpha, 1]$, whenever $0 < \alpha < 1$.

**Example 36:** $f_n(x) = \dfrac{2x}{1+n^2x^2}$; $x \geq 0$. Then $\lim f_n(x) = f(x) \equiv 0$.



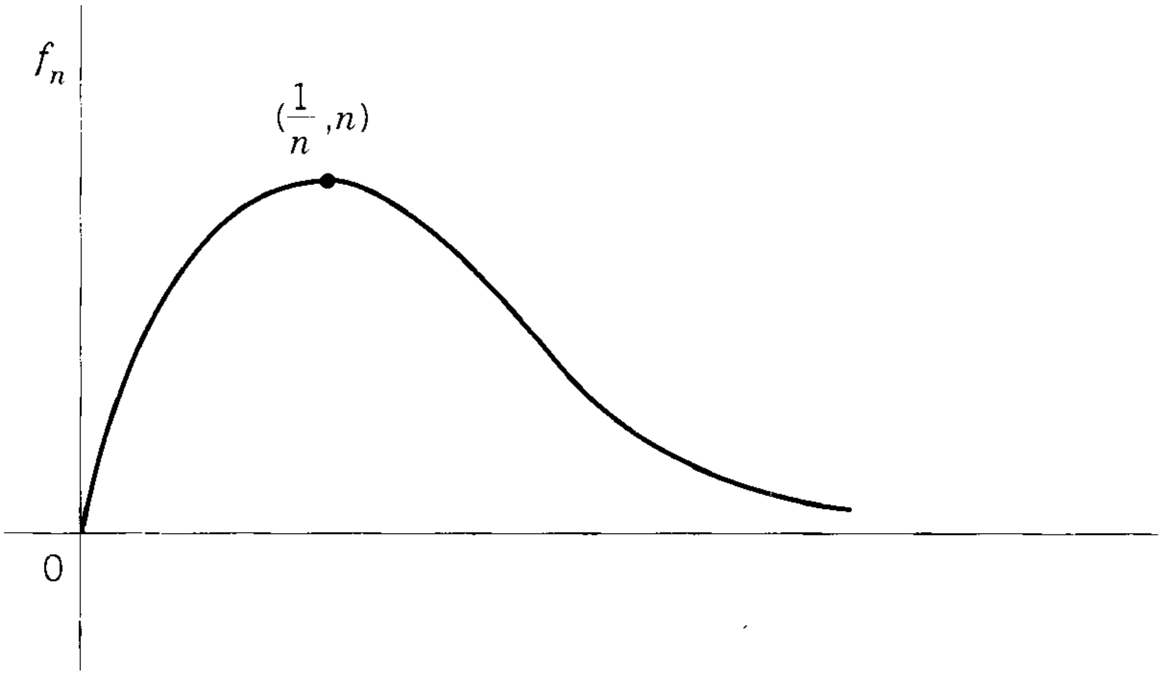

$\lim f_n = f(unif)$ since $\lim\left(\sup_E \mid f_n - f \mid\right) = \lim_n \dfrac{1}{n} = 0$.

**Example 37:** $f_n(x) = \dfrac{2nx}{1+n^2x^2}$; $x \geq 0$. Then $\lim f_n(x) = f(x) \equiv 0$.




Convergence is not uniform: $\lim\left(\sup_E \mid f_n - f \mid\right) = \lim_n 1 = 1$.

**Example 38:** $f_n(x) = \dfrac{2n^2x}{1+n^2x^2}$; $x \geq 0$. Then $\lim f_n(x) = f(x) \equiv 0$.




Convergence is not uniform: $\lim\limits_n(\sup |f_n - f|) = \lim\limits_n n = \infty$.

### 2.10.4 Problem

1. $f_n(x) = x^n$, $\quad 0 \leq x \leq 1$.
2. $f_n(x) = \dfrac{x^n}{(1+x^n)}$, $\quad 0 \leq x \leq 1$.

   Show $(f_n)$ converges uniformly on $[0, 1-\epsilon]$, $0 < \epsilon < 1$. Does $(f_n)$ converge uniformly on $[0, 1]$?
3. $f_n(x) = \dfrac{n^2x}{(1+n^3x^2)}$, $0 \leq x \leq 1$.

   Does $(f_n)$ converge uniformly on $[0, 1]$? How about $[\epsilon, 1], 0 < \epsilon < 1$?
4. $f_n(x) = nxe^{-nx}$, $x \geq 0$.

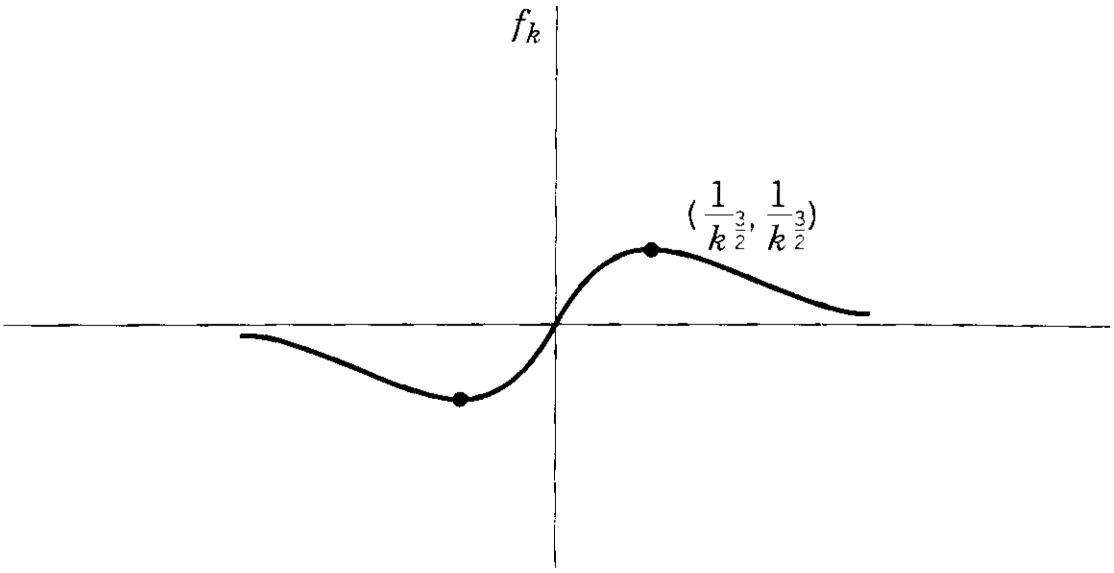
   Does $(f_n)$ converge uniformly on $[0, \infty)$? How about $[\epsilon, \infty)$, $\epsilon > 0$?

A useful test for determining uniform convergence for a series of functions was developed by Weierstrass.

**THEOREM 2.11** (Weierstrass $M$-Test) *A series of real-valued functions $\sum f_k(x)$, $f_k$ defined on a set $E$, converges uniformly on $E$ if $|f_k| \leq M_k$ on $E$, $k = 1, 2, \ldots$ and $\sum M_k$ converges.*

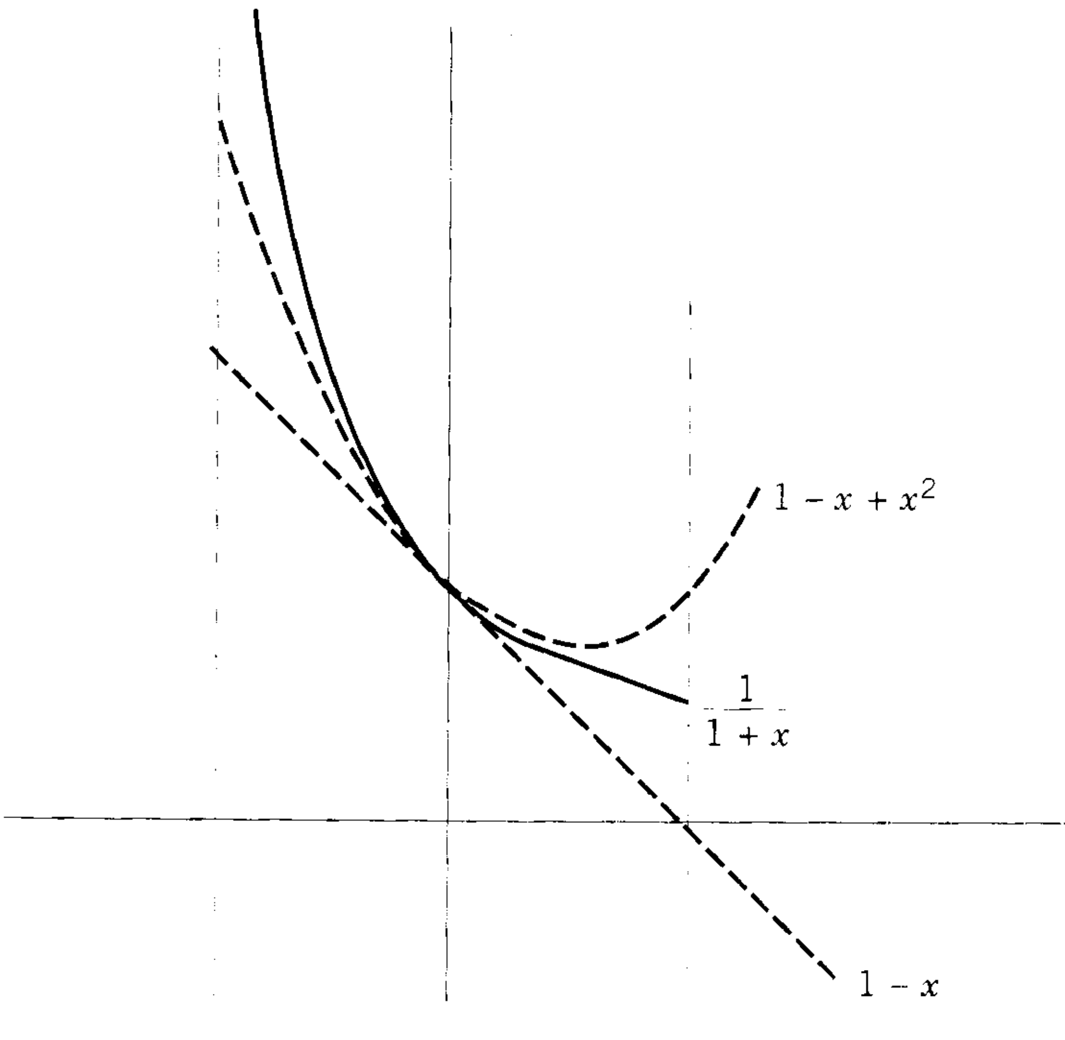
*Proof:* $|\sum_{k=K}^{\infty} f_k(x)| \leq \sum_{k=K}^{\infty} M_k$ for all $x \in E$. ■

**Example 39:** $\sum_k \frac{2x}{1+k^3x^2}$, $-\infty < x < \infty$; $\quad f_k(x) \equiv \frac{2x}{(1+k^3x^2)}$.




Let $M_k = \frac{1}{k^{3/2}}$. We have uniform convergence on $R$.

**Example 40:** $\sum_k (-x)^k$, $-1 < x < 1$; $\quad f_k(x) = (-x)^k$.




Weierstrass isn't helpful here. Define $g_n(x) = 1 - x + x^2 \cdots + (-x)^n$,

$-1 < x < 1$. Then

$$\begin{aligned} g_n(x) &= \frac{1-(-x)^{n+1}}{1+x}, -1 < x < 1, \lim g_n(x) \\ &= \frac{1}{1+x}, -1 < x < 1, \text{and } \lim_n \left( \sup_{-1<x<1} |g_n - g| \right) \\ &= \lim_n \left( \sup_{-1<x<1} \left| \frac{x^{n+1}}{1+x} \right| \right) = \infty. \end{aligned}$$

We do not have uniform convergence on $(-1, 1)$. Let's investigate uniform convergence on $[0, 1)$. Then

$$\lim_n \left( \sup_{0\le x<1} \frac{x^{n+1}}{1+x} \right) = \frac{1}{2}.$$

Again, we do not have uniform convergence on $[0, 1)$. The reader should investigate $[-\alpha, \alpha]$ for uniform convergence where $0 < \alpha < 1$, and recall Proposition 2.12.

### 2.10.5 Problems

1. $\sum_1 \frac{\sin(kx)}{k^2}$, $-\infty < x < \infty$. Is the convergence uniform?

2. $f_n(x) = x + \frac{x}{n}$, $-\infty < x < \infty$. Show we have uniform convergence on $[-\alpha, \alpha]$. How about $(-\infty, \infty)$?

3. $x + \sum_2 \left( x^{\frac{1}{2k-1}} - x^{\frac{1}{2k-3}} \right)$, $-1 < x < 1$.

   Calculate $\lim_n \left[ x + \sum_2^n \left( x^{\frac{1}{2k-1}} - x^{\frac{1}{2k-3}} \right) \right] = g(x)$. Do we have uniform convergence on $(-1, 1)$?

   *Hint*: Proposition 2.12.

   How about $(-1, -\epsilon) \cup (\epsilon, 1)$, $0 < \epsilon < 1$?

4. $\sum_1 \left( x^k - x^{2k} \right)$, $0 \le x \le 1$.

Calculate $\lim_n \left[ \sum_1^n (x^k - x^{2k}) \right]$.

Is this convergence uniform?

*Hint*: Proposition 2.12.

5. $\sum_1 \left(1 - \frac{1}{1+kx}\right)$, $x \geq 0$.

Calculate $\lim_n f_n = \lim_n \left[ \sum_1^n \left(1 - \frac{1}{1+kx}\right) \right]$.

Note that $f_n(0) = 0$. Does $\lim f_n(0) = f(0)$?

**We have completed the review necessary for the succeeding chapters.**

..., there is no study in the world which brings into more harmonious action all the faculties of the mind than the one [mathematics] ...

—J. J. Sylvester

Many arts there are which beautify the mind of man; of all other none do more garnish and beautify it than those arts which are called mathematical.

—H. Billingsley

$$\sum_{-\infty}^{\infty} \frac{\sin^2(k\alpha)}{k^2} = \int_{-\infty}^{\infty} \frac{\sin^2(x\alpha)}{x^2}\, dx, \quad 0 < \alpha < \pi\,.$$

# 3

# *Lebesgue Measure*

The mathematical concept of measure is a generalization of length in $R$, area in $R^2$, and volume in $R^3$. Our goal is to develop a "measure theory" for sets of real numbers, subsets of $R$. This measure theory will be based on lengths of intervals.

We all know what the measure (length) of an interval, say $[0, 1]$, should be. What about $(0, 1)$ or $(0, 1]$? These sets are subsets of $[0, 1]$. Certainly their measures should not exceed that of $[0, 1]$ (monotonic). But, $(1/n, 1 - 1/n) \subset (0, 1)$ for all $n \geq 2$; ok, they must have measure one. How about the interval $(2, \infty)$? Must be "$\infty$". This measure we are trying to develop will be a monotonic mapping from the subsets of $R$ to the nonnegative extended reals (a set function), and the measure of an interval will be its length of course; $[-1/2, 1/2]$ is $[0, 1]$ translated $1/2$ unit to the left and both have measure one (translation invariant). Since $(0, 2] = (0, 1] \cup (1, 2]$, $2 = 1 + 1$, and "the whole is the sum of its parts" (additive). Hold it! The intervals $(0, 1)$ and $(0, 1] = (0, 1) \cup \{1\}$ have the same measure. Thus the set $\{1\}$ has measure zero (points are dimensionless). But then

$$(0, 1] = \bigcup_{x \in (0,1]} \{x\} \text{ and } 1 \neq \sum_{x \in (0,1]} 0.$$

In this case "the whole is not equal to the sum of its parts". For example, suppose the set $S$ consists of the intervals $(1/2, 1]$, $(1/3, 1/2]$, $(1/4, 1/3]$, .... None of the intervals overlap and $S$ is just $(0, 1]$. The measure of $S$, one, is also equal to $(1-1/2)+(1/2-1/3)+(1/3-1/4)+\cdots$, that is, the whole is equal to the sum of its parts in this case. Another example: If $S^* \equiv (0,1)$, then

$$S^* = \left(\bigcup_{k=1}^{\infty}(1/(k+1), 1/k)\right) \cup \{1/2, 1/3, 1/4, \ldots\}.$$

Again, $1 = 1 - 1/2 + 1/2 - 1/3 + \cdots$. This suggests the measure of the set $\{1/2, 1/3, 1/4, \ldots\}$ should be zero. But since the measure of $\{1/k\}$ is zero, again "the whole is the sum of its parts". It appears that if a set is decomposed into a countable number of disjoint sets;

$$(0,1] = \bigcup\left(\frac{1}{k+1}, \frac{1}{k}\right],$$

$$(0,1) = \left(\bigcup\left(\frac{1}{k+1}, \frac{1}{k}\right)\right) \cup \left\{\frac{1}{2}, \frac{1}{3}, \ldots\right\},$$

$$\left\{\frac{1}{2}, \frac{1}{3}, \ldots\right\} = \bigcup\left\{\frac{1}{k}\right\},$$

then "the whole is the sum of its parts" (countable additivity). Unrestricted additivity,

$$(0,1] = \bigcup_{x \in (0,1]} \{x\},$$

causes problems.

The set $\{1/2, 1/3, \ldots\}$, having measure zero, suggests countable sets should have measure zero, i.e., the measure of the set of all rational numbers, a dense set in $R$, would necessarily have measure zero. The irrational numbers in $[0,1]$ would then have measure one. No problem, the irrationals are an uncountable subset of $[0,1]$. Finally, we come to the Cantor set (Appendix A). We remove a countable number of disjoint intervals from $[0,1]$ whose total measure is one. What remains is the Cantor set. However, the Cantor set is uncountable in the same sense as $[0,1]$. The Cantor set is an example of an uncountable set of measure zero!!

It is examples like the Cantor set that demand a careful approach to measure. "We all know", "should", "seems reasonable", "the whole is the sum of its parts", "suggests", etc., are superficial phrases that must be replaced with careful analysis, and this we will do. But the above discussion has been fruitful. Intuition, sprinkled with a little optimism, suggests (it remains to be seen if this is possible logically and consistently) that if we are developing a measure $\mu$ defined on the subsets of $R$, we would hope that these conditions could be met:

1. $\mu(A)$ defined for every set $A$ of real numbers (we can "measure" all sets);
2. $0 \leq \mu(A) \leq \infty$ (nonnegative extended real-valued; length is nonnegative and the "length" of $R$ is $\infty$);
3. $\mu(A) \leq \mu(B)$ provided $A \subset B$ (monotonic);
4. $\mu(\emptyset) = 0$;
5. $\mu(\{a\}) = 0$ (points are dimensionless);
6. $\mu(I) = l(I)$, $I$ an interval (the measure of an interval should be its length);
7. $\mu(c + A) = \mu(A)$ (translation invariance; location doesn't affect length, shouldn't affect the measure);
8. $\mu\left(\bigcup_{1}^{\infty} A_k\right) = \sum_{1}^{\infty} \mu(A_k)$ for any mutually disjoint sequence $(A_k)$ of subsets of real numbers (countable additivity).

The purpose of this chapter is to construct a measure that satisfies as many of the conditions one through eight as possible. This is an ambitious goal. As noted in the first chapter this effort began some millennium ago and has continued to the present day. The mathematics is subtle and exciting.

> "... mathematics: she gives life to our discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes the oblivian and ignorance which are ours by birth."
>
> —Proclus

We hope the reader will share with us this excitement and sense of discovery. A suggestion:

> "Many results must be given of which details are suppressed .... These must not be taken on trust by the student, but must be worked out by his own pen, which must never be out of his own hand while engaged in any mathematical process."
>
> —A. DeMorgan

## 3.1 LENGTH OF INTERVALS

### 3.1.1 Definition

The *length* of an interval $I$ with endpoints $a \leq b$, $a, b \in R^e$, is defined by

$$l(I) = b - a.$$

For example, $l((0, 1]) = l([0, 1]) = 1$, and $l((1, \infty)) = \infty$.

### 3.1.2 Problem

Calculate $l(I)$:

1. $f(x) = x^2, x \geq 0, I = f^{-1}((1/4, 9/16])$.
2. $f(x) = \sin(x), 0 \leq x \leq 2\pi$ and $I = f^{-1}((\sqrt{3}/2, \infty))$.
3. $f(x) = \cos(x), 0 \leq x \leq 2\pi$ and $I = f^{-1}((-\infty, -1])$.
4. $f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 3 - x, & 1 < x < 2 \end{cases}$ and $S \equiv f^{-1}((9/16, 5/4))$.
   What "length" would be reasonable for $S$?
5. $f(x) = x^3$ and $I = f^{-1}((-\infty, -8))$.
6. $f(x) = |x|$ and $I = f^{-1}((-\infty, 0])$.

### 3.1.3 Problem

If $I_1$ and $I_2$ are two intervals with $I_1 \subset I_2$, show $l(I_1) \leq l(I_2)$ (monotonicity).

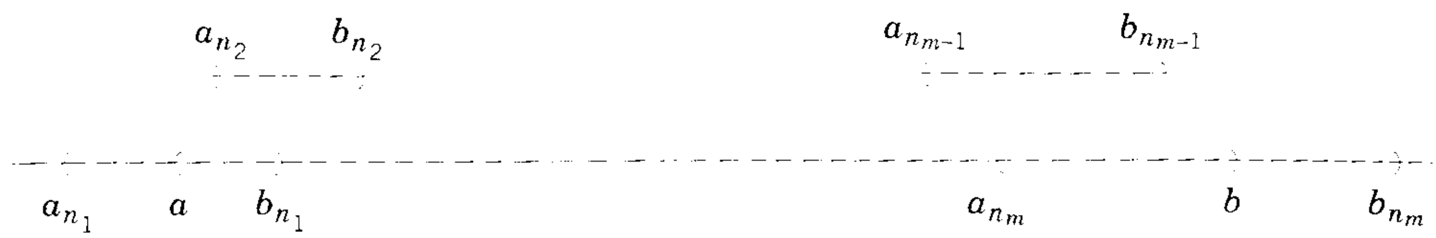
### 3.1.4 Problem

If $I, I_1, I_2, \cdots, I_n$ are bounded open intervals with

$$I \subset \bigcup_1^n I_k, \text{ then } l(I) \leq \sum l(I_k).$$

The length of an interval cannot exceed the length of a "finite cover".

*Hint*:




$$\begin{aligned} b - a = &\quad (b - a_{n_m}) \quad + \quad (a_{n_m} - a_{n_{m-1}}) \quad + \cdots + \quad (a_{n_2} - a) \\ \leq &\quad (b_{n_m} - a_{n_m}) \quad + \quad (b_{n_{m-1}} - a_{n_{m-1}}) \quad + \cdots + \quad (b_{n_1} - a_{n_1}). \end{aligned}$$

The key to this argument is that we can "search through" a finite set of intervals to find $a_{n_1}$ so that $a_{n_1} \leq a$. If we have a countable cover the argument is more subtle. For example,

$$(0, 1) = \bigcup_1^{\infty}(1/(k+1), 1);$$

what would be our choice for $a_{n_1}$?

**Proposition 3.1** *If $I, I_1, I_2, \ldots, I_k, \ldots$ are bounded open intervals with $I \subset \bigcup I_k$, then $l(I) \leq \sum l(I_k)$. In fact, $l(I) \leq inf\{\sum l(I_k) \mid I \subset \bigcup I_k, I_k$ bounded open intervals $\}$.*

*Proof:* Assume $I = (a, b)$ and let $\epsilon > 0$. The intervals $(a - \epsilon, a + \epsilon)$, $(b - \epsilon, b + \epsilon)$, $I_1, I_2, \ldots, I_k, \ldots$ form an open cover of the compact set $[a, b]$. By Heine-Borel (Theorem 2.5), a finite subcollection, $(a - \epsilon, a + \epsilon)$, $(b - \epsilon, b + \epsilon)$, $I_{n_1}, I_{n_2}, \ldots, I_{n_k}$ will cover $[a, b]$, and thus $(a, b)$.

Application of Problems 3.1.3 and 3.1.4 yields

$$\begin{aligned} l(I) &\leq 2\epsilon + 2\epsilon + \sum_{i=1}^{k} l(I_{n_i}) \\ &\leq 4\epsilon + \sum l(I_k). \end{aligned}$$

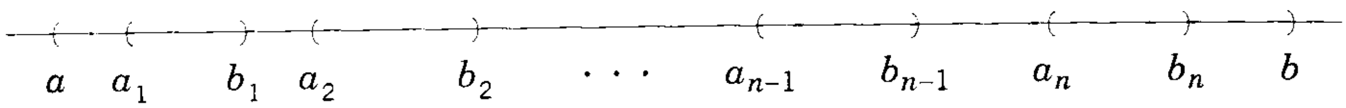
Since this holds for any $\epsilon > 0$, we conclude $l(I) \leq \sum l(I_k)$. ■

### 3.1.5 Problem

1. If $I_1, \ldots, I_n$ is a finite collection of mutually disjoint open intervals, and if $I$ is a bounded open interval with

$$\bigcup_1^n I_k \subset I, \text{ then } \sum_1^n l(I_k) \le l(I).$$

*Hint*:




$$0 \le (b_1 - a) + (a_2 - b_1) + \cdots + (a_n - b_{n-1}) + (b - b_n)$$

2. If $I_1, \ldots, I_k, \ldots$ is a countable collection of mutually disjoint open intervals, and if $I$ is a bounded open interval with

$$\bigcup_1^\infty I_k \subset I, \text{ then } \sum_1^\infty l(I_k) \le l(I).$$

*Hint*:

$$\sum_1^n l(I_k) \le l(I)$$

for every $n$. The nondecreasing sequence

$$\left\{\sum_1^n l(I_k)\right\}$$

is bounded above by $l(I)$, and thus the series

$$\sum_1^\infty l(I_k)$$

converges to a sum not exceeding $l(I)$.

3. If $I = I_1 \cup I_2$, $I$ an open interval, $I_1, I_2$ disjoint intervals, then $l(I) = l(I_1) + l(I_2)$.

### 3.1.6 Problem

Calculate $\sum l(I_k)$:

1. $I_k = (1/(k+1), 1/k)$, $k = 1, 2, \dots$ and $I = (0, 1)$ with $\bigcup I_k \subset I$.
2. $I_k = (1/2k, 1/(2k-1))$, $k = 1, 2, \dots$ .
3. $a_k = 1 + 1/2 + \cdots + 1/k - \int_1^{k+1} 1/x\, dx$ and $I_k = (a_k, a_{k+1})$, $k = 1, 2, \dots$ .

Since every open set $G \subset R$ is the countable union of mutually disjoint open intervals (Theorem 2.4), say $G = \bigcup I_k$, we could define the length of $G$ by

$$l(G) = \sum l(I_k),$$

and proceed via complementation to closed sets, and so on. Instead, we choose an approach initiated by Harnack, Borel, and others.

## 3.2 LEBESGUE OUTER MEASURE

We make the observation that every set of real numbers can be covered with a countable collection of open intervals.

### 3.2.1 Definition

$A$ is any subset of $R$. Form the collection of all countable covers of $A$ by open intervals. The *Lebesgue outer measure* of $A$, $\mu^*(A)$, is given by

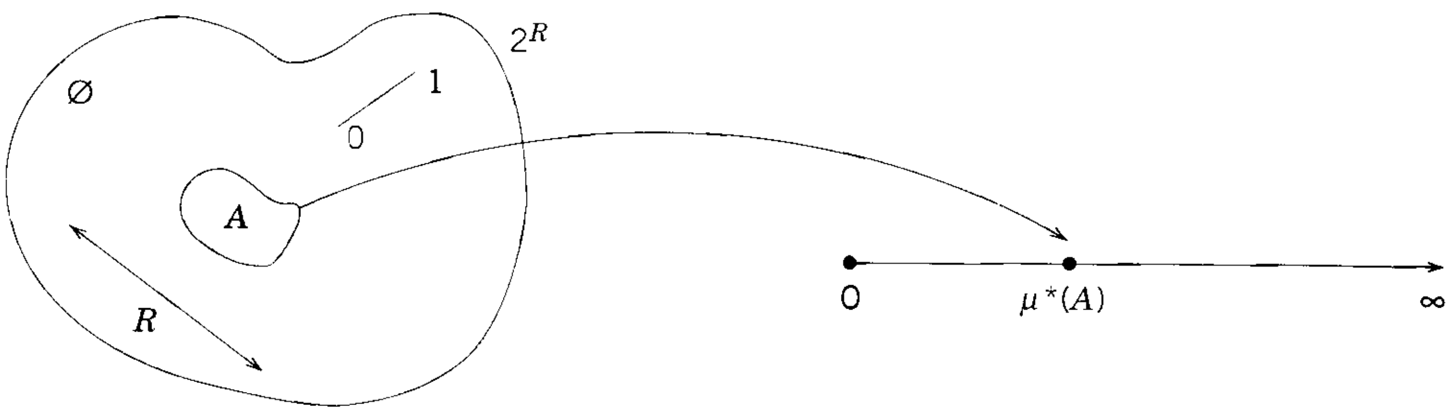
$$\mu^*(A) = \inf\left\{ \sum_1^\infty l(I_k) \mid A \subset \bigcup_1^\infty I_k, \quad I_k \text{ open intervals} \right\}.$$

### 3.2.2 Comments

1. What does $\sum l(I_k)$ mean? For example, suppose $J_1 = I_2$, $J_2 = I_1$, $J_3 = I_4$, $J_4 = I_3$, etc. Is $\sum l(I_k) = \sum l(J_k)$? Does the order make a difference when we sum? Does the labeling matter? Look at Comment 2.6.13.
2. Is it necessary that $I_k$ be open intervals? Could they be closed, or neither open nor closed? By definition, $l(I_k)$ is the same in all cases.

If $I_k$ are closed for example, $I_k = [a_k, b_k] \subset (a_k - \epsilon/2^k, b_k + \epsilon/2^k) = J_k$, and $\sum l(I_k) \leq \sum l(J_k) + 2\epsilon$.

3. The collection of countable open covers is not empty, and since $\sum l(I_k) \geq 0$, it follows (GLB) that the set of all such numbers possesses a nonnegative greatest lower bound.
4. We could require the lengths of the intervals $I_k$ to all be less than a given positive number $\delta$ since each $I_k$ can be subdivided into disjoint subintervals each of length less than $\delta$ without affecting the sum of their lengths.
5. $\mu^*$ is a set function, whose domain is all subsets of $R$, $2^R$, and whose range is $[0, \infty]$, the nonnegative extended reals:

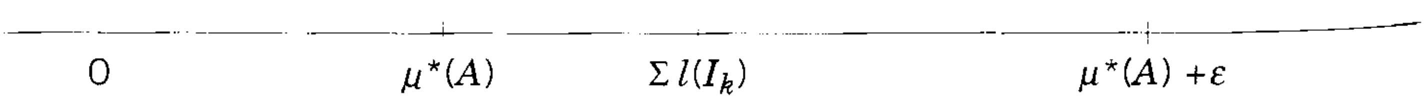



6. If $(I_k)$ is any countable cover of $A$ by open intervals, then since the infimum is a **lower bound,**

$$0 \leq \mu^*(A) \leq \sum l(I_k).$$

7. If $\mu^*(A) < \infty$, then corresponding to each $\epsilon > 0$, there is a countable collection $(I_k)$ of open intervals such that $A \subset \bigcup_1^\infty I_k$ and

$$\mu^*(A) \leq \sum l(I_k) < \mu^*(A) + \epsilon.$$




The infimum is the **greatest** lower bound.

### 3.2.3 Problem

Why is $\mu^*((0,1)) \le 1$? Can we use Proposition 3.1 to conclude $\mu^*((0,1)) = 1$?

### 3.2.4 Problem

Calculate $\mu^*(A)$:

1. $A = \{1, 1/2, 1/3, \ldots\}$.

   *Hint*: $1/k \in (1/k - \epsilon/2^k, 1/k + \epsilon/2^k)$.
2. $A$ is the set of rational numbers.
3. $A$ is any countable set of real numbers.

Is the Lebesgue outer measure our desired measure? Does the Lebesgue outer measure satisfy the eight intuitive conditions mentioned at the beginning of this chapter?

We have the important theorem, which at the risk of being redundant, we state as follows:

**THEOREM 3.1** *The Lebesgue outer measure $\mu^*$ of a set $A$ given by*

$$\mu^*(A) = inf\left\{\sum_1^\infty l(I_k) \mid A \subset \bigcup_1^\infty I_k,\ I_k \text{ open intervals}\right\}$$

*has the following properties:*

1. *$\mu^*$ is defined for every set of real numbers;*
2. *$0 \le \mu^*(A) \le \infty$ (nonnegative and extended real-valued);*
3. *$\mu^*(A) \le \mu^*(B)$ provided $A \subset B$ (monotonic);*
4. *$\mu^*(\emptyset) = 0$;*
5. *$\mu^*(\{a\}) = 0$ (points are dimensionless);*
6. *$\mu^*(I) = l(I)$, $I$ an interval (the Lebesgue outer measure of an interval is its length);*
7. *$\mu^*(c + A) = \mu^*(A)$ (translation invariant);*
8. *$\mu^*\left(\bigcup_1^\infty A_k\right) \le \sum_1^\infty \mu^*(A_k)$ for any sequence of sets of real numbers (countable subadditivity).*

*Proof:* That $\mu^*$ is a nonnegative extended real-valued set function defined on $2^R$ has been discussed (Comment 3.2.2). Monotonicity, property 3, is an immediate consequence of the observation that every open cover of $B$ will be an open cover of $A$ (Problem 2.4.7).

Since the empty set is a subset of every set, $\emptyset \subset \{a\} \subset (a-\epsilon, a+\epsilon)$, we have by monotonicity that $0 \le \mu^*(\emptyset) \le \mu^*(\{a\}) \le \mu^*((a-\epsilon, a+\epsilon)) \le 2\epsilon$ (Comment 3.2.2). Properties 4 and 5 follow by the arbitrary nature of $\epsilon$.

To show the Lebesgue outer measure of an interval is simply its length, property 6, entails several arguments depending on the nature of the interval $I$. We first suppose $I = (a,b)$, $a, b$ real numbers. Since $I$ is any open cover of itself, $\mu^*(I) \le b - a$ (Comment 3.2.2). But $b - a$ is a lower bound (Proposition 3.1).

Since $\mu^*(I)$ is the greatest lower bound we have $\mu^*(I) \ge b - a$. Thus $b - a \le \mu^*(I) \le b - a$: The Lebesgue outer measure of a bounded open interval is its length. If $I = [a,b]$, then $(a,b) \subset [a,b] \subset (a-\epsilon, b+\epsilon)$ for $\epsilon > 0$ and by what we just showed and monotonicity,

$$b - a \le \mu^*([a,b]) \le b - a + 2\epsilon.$$

Thus $\mu^*([a,b]) = b - a$. The cases when $I = (a,b]$, or $[a,b)$ may be completed by the reader. We have then the Lebesgue outer measure of any bounded interval is just its length. Finally, let $I$ be an unbounded interval, for example, $(a,\infty)$. For any positive number $C > a$, $(a,C)$ is contained in $(a,\infty)$. Consequently,

$$C - a = l((a,C)) = \mu^*((a,C)) \le \mu^*(I)$$

since $(a,C)$ is a bounded interval. Thus

$$\mu^*(I) = \infty = l(I).$$

The Lebesgue outer measure of any interval is its length.

Translation invariance, property 7, is based on the fact that length, $l$, is translation invariant: If $I = (a,b)$, then $c + I = (a+c, b+c)$ and $l(I) = l(c+I)$. If $I$ is $(b,\infty)$, $(-\infty,a)$, or $(-\infty,+\infty)$, then $c+I$ is $(b+c,\infty)$, $(-\infty,a+c)$, or $(-\infty,+\infty)$, respectively, and again $l(I) = l(c+I)$. If $A$ is an arbitrary subset of $R$ with $A \subset \bigcup I_k$, then

$$c + A \subset \bigcup (c + I_k),$$

and

$$\mu^*(c + A) \le \sum l(c + I_k) = \sum l(I_k).$$

This tells us that $\mu^*(c + A)$ is a lower bound for the "lengths" of covers of $A$, and because $\mu^*(A)$ is the greatest lower bound of such numbers,

$$\mu^*(c + A) \le \mu^*(A).$$

By starting with a cover $(J_k)$ of $c + A$, we have $A \subset \bigcup(J_k - c)$. The reader may finish the argument.

Countable subadditivity remains. We must show

$$\mu^*\left(\bigcup_1^\infty A_k\right) \le \sum_1^\infty \mu^*(A_k)$$

for any sequence $(A_k)$ of sets of real numbers. Of course if the series $\sum \mu^*(A_k)$ diverges the argument is immediate, so assume $\sum \mu^*(A_k) < \infty$ and let $\epsilon > 0$. For each nonempty $A_k$ choose an open cover $(I_{kn})$ so that

$$A_k \subset \bigcup_{n=1}^\infty I_{kn}$$

and

$$\mu^*(A_k) \le \sum_{n=1}^\infty l(I_{kn}) < \mu^*(A_k) + \frac{\epsilon}{2^k}\,.$$

We may do this by the definition of greatest lower bound (3.2.2). The collection

$$\{I_{11}, I_{12}, \ldots, I_{1n}, \ldots; I_{21}, I_{22}, \ldots, I_{2n}, \ldots; \ldots; I_{k1}, I_{k2}, \ldots, I_{kn}, \ldots\}$$

is a countable collection (Proposition 2.3) of open intervals that cover the set

$$\bigcup_{k=1}^\infty A_k:$$

$$A_1 \subset \bigcup_{n=1}^\infty I_{1n},\ A_2 \subset \bigcup_{n=1}^\infty I_{2n}, \ldots,\ A_k \subset \bigcup_{n=1}^\infty I_{kn}, \ldots$$

and

$$\bigcup_{k=1}^\infty A_k \subset \bigcup_{k=1}^\infty \left(\bigcup_{n=1}^\infty I_{kn}\right).$$

How do we calculate the length of

$$\bigcup_k \bigcup_n I_{kn}?$$

The reader might go back and look at Proposition 2.4. We proceed as follows:

$$\begin{aligned}
\bigcup_k \left( \bigcup_n I_{kn} \right) = \quad & I_{11} \bigcup I_{12} \bigcup \cdots \bigcup I_{1n} \bigcup \cdots \\
\bigcup & \\
& I_{21} \bigcup I_{22} \bigcup \cdots \bigcup I_{2n} \bigcup \cdots \\
\bigcup & \\
& \cdots \\
\bigcup & \\
& I_{k1} \bigcup I_{k2} \bigcup \cdots \bigcup I_{kn} \bigcup \cdots \\
\bigcup & \\
& \cdots \\
= {} & I_{11} \bigcup I_{21} \bigcup I_{12} \bigcup I_{31} \bigcup I_{22} \bigcup I_{13} \bigcup \cdots \\
= {} & \bigcup_{i=1}^{\infty} I_{\phi(i)} \,.
\end{aligned}$$

Since $\bigcup I_{\phi(i)}$ is an open cover of the set $(\bigcup A_k)$,

$$\begin{aligned}
\mu^*\left( \bigcup A_k \right) &\le \sum_i l(I_{\phi(i)}) && (3.2.2) \\
&= \sum_k \left( \sum_n l(I_{kn}) \right) && \text{(Proposition 2.4)} \\
&\le \sum_k \left( \mu^*(A_k) + \frac{\epsilon}{2^k} \right) \\
&= \sum \mu^*(A_k) + \epsilon \,.
\end{aligned}$$

We conclude

$$\mu^*\left(\bigcup A_k\right) \le \sum \mu^*(A_k).$$

The proof is complete. Whew! ∎

After such a long argument, we pause and reflect on just what we have accomplished. Comparison of Theorem 3.1 for the Lebesgue outer measure $\mu^*$ with the "hoped for" properties of a measure discussed in the beginning of this chapter reveals that $\mu^*$ has everything we wanted except possibly the last condition of countable additivity. Is $\mu^*$ our desired measure function? I know, if we could show $\mu^*$ was countably additive we would have done so in Theorem 3.1, but bear with me for a moment. Trying to make $\mu^*$ "work" as our long sought after measure leads naturally to an apparently easier problem: Would $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ for every pair of disjoint subsets $A, B$ of $R$ yield countable additivity? Would finite additivity of the Lebesgue outer measure $\mu^*$ imply countable additivity for $\mu^*$?

### 3.2.5 Problem

1. If the Lebesgue outer measure is finitely additive then it is countably additive.

   *Hint*: $\sum_1^n \mu^*(A_k) = \mu^*\left(\bigcup_1^n A_k\right) \le \mu^*\left(\bigcup_1^\infty A_k\right) \le \sum_1^\infty \mu^*(A_k).$

   *Note*: In fact, the reader may show that if $\mu^*$ is finitely additive, then every subset $E$ of $R$ satisfies Carathéodory's Measurability Criteria (3.3.1) and thus every subset of $R$ is Lebesgue measurable.
2. If $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) = \mu^*(B)$.

   *Hint*: Monotonicity and subadditivity.

The situation appears to have improved. Finite additivity should be easier than countable additivity. If we can show $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$, $A \cap B = \emptyset$, then $\mu^*$ is our desired measure. Alas, such is not to be (as you already suspected). In 1905 Vitali gave the first example of a set of real numbers that could not be decomposed in an additive fashion. The reader may refer to Appendix B for a fuller discussion. What do we do now? We're so "close". Maybe we can "refine" $\mu^*$ to a $\mu^{**}$ in some fashion.

### 3.2.6 Problem

1. Let's define $\mu^{**}(A) = \inf\{\mu^*(B) \mid A \subset B\}$. Do we have anything new here, or is $\mu^{**}(A) = \mu^*(A)$ for all $A \subset R$?

   *Hint*: $A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$ so $\mu^*(A)$ is a lower bound. On the other hand, $A \subset A$ and $\mu^{**}(A) \leq \mu^*(A)$.

2. How about $\mu^{**}(A) = \sup\{\mu^*(B) \mid B \subset A\}$.

   *Hint*: $\mu^*(B) \leq \mu^*(A)$ and $\mu^*(A)$ is an upper bound. On the other hand, $A \subset A$ and $\mu^*(A) \in \{\mu^*(B) \mid A \subset B\}$.

The "obvious improvements?" generate nothing new. We cannot meet all eight of our intuitive demands with the Lebesgue outer measure. Certainly conditions two through seven seem to be indispensable. Apparently we are going to have to modify conditions one or eight or both if we stay with Lebesgue outer measure. If condition one holds, then as will be seen, the fact that we could "measure" all sets would imply that every real-valued function is Lebesgue integrable. Wouldn't that be nice! Unfortunately, Banach and Hausdorff (1923) showed that if we retain condition one and weaken countable additivity to finite additivity, then the general measure problem is solvable in $R^n$ iff $n = 1$ or 2. Another possibility: weaken requirement one and keep countable additivity: Measure fewer sets and retain countable additivity. Maybe we can keep the Lebesgue outer measure, restricting ourselves to those sets that the Lebesgue outer measure deals with in an additive fashion. But how do we "sort out" such sets? And even if we can determine such sets, will the collection of such sets be large enough to build a useful theory of integration?

## 3.3 LEBESGUE MEASURABLE SETS

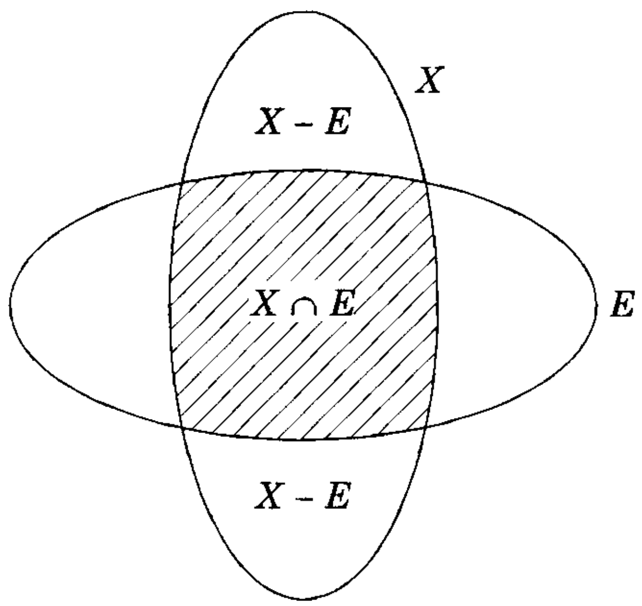
In 1914 Carathéodory formulated a measurability criteria that is widely used today:

### 3.3.1 Carathéodory's Measurability Criteria

*E is any set of real numbers. If*

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X - E)$$

*for every set X of real numbers, then the set E is said to be a* ***Lebesgue measurable*** *set of real numbers. In other words, if the set E "interacts" in an additive fashion with every subset of R, we say the set is Lebesgue measurable.*

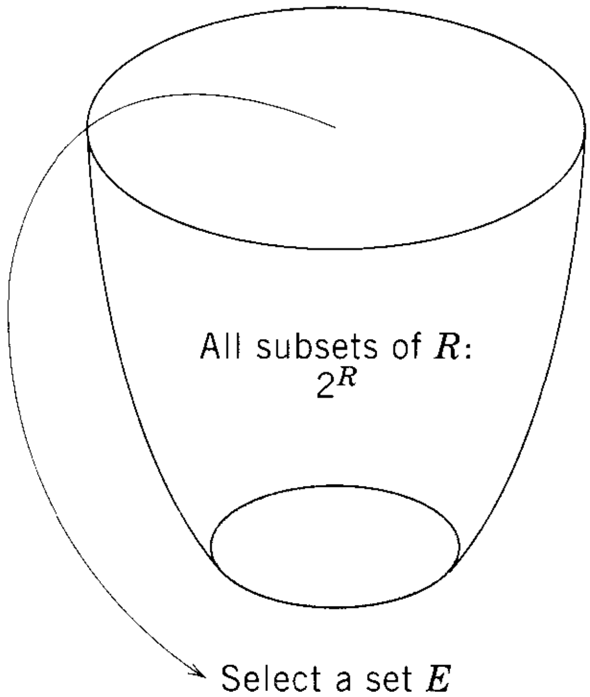



We use the phrase "$E$ splits every subset of $R$ in an additive fashion".

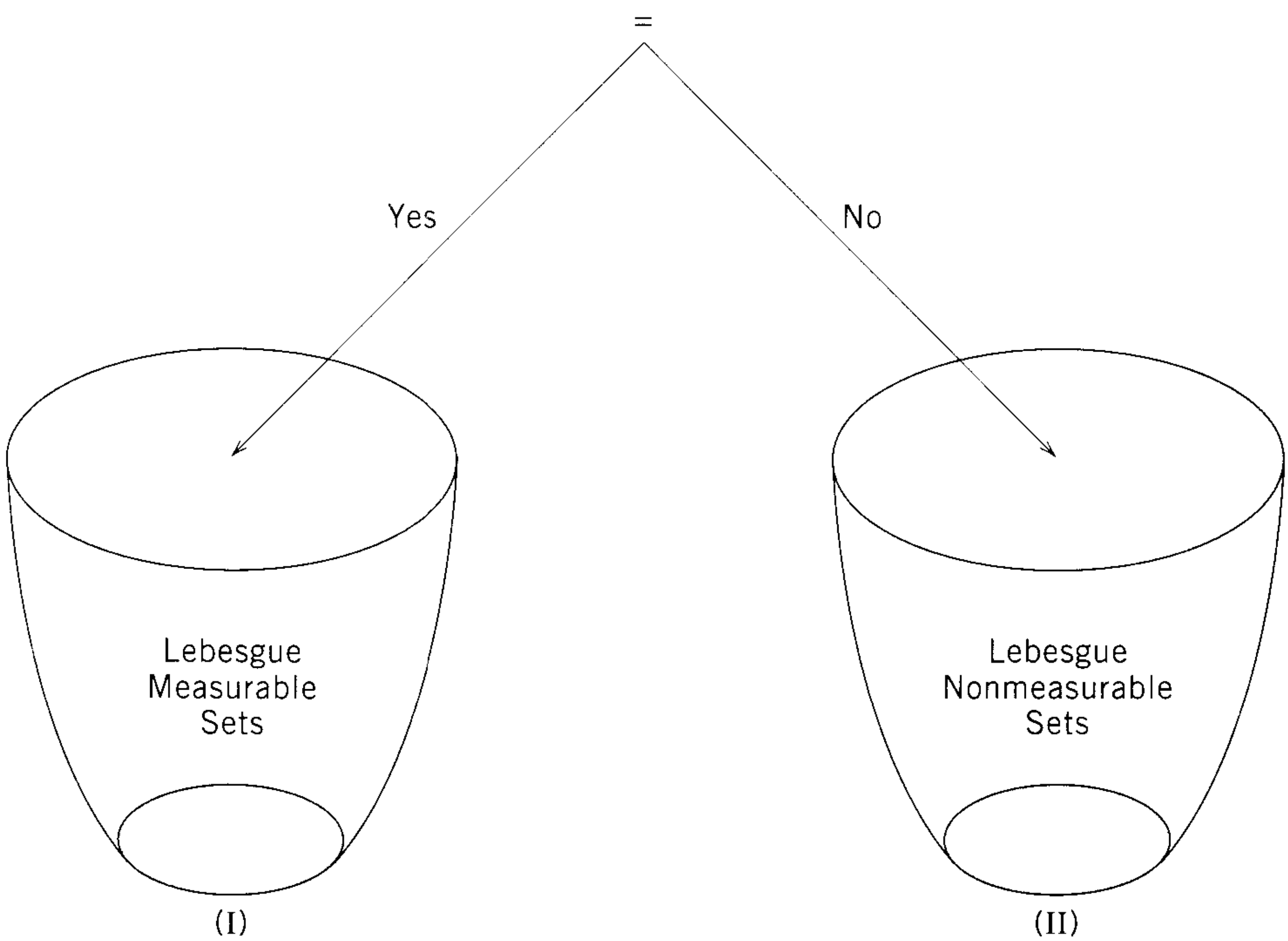
### 3.3.2 Comments

Just how do we use this criteria? Does it yield countable additivity? Does a large class of sets "work"? Consider the collection of all subsets of $R$, $2^R$. From this collection, select a set $E$. We then "go through" $2^R$, checking to see if the selected set $E$ splits every element of $2^R$ in an additive fashion; that is, since $X = (X \cap E) \cup (X - E)$ with $(X \cap E) \cap (X - E) = \emptyset$, we are asking: Is the measure of the whole $(X)$ equal to the sum of the measures of its disjoint parts, $X \cap E$ and $X - E$, for every subset $X$ of real numbers? If we answer yes, then $E$ is said to be Lebesgue measurable, and $E$ is "kept". If there exists even one set $X_1 \in 2^R$ such that $\mu^*(X_1) \neq \mu^*(X_1 \cap E) + \mu^*(X_1 - E)$, then we "discard" $E$ (in fact, $E$ and $X_1$) and say $E$ is a Lebesgue nonmeasurable set of real numbers. In other words, we keep a set $E$ if it splits **every** subset of $R$ is an additive fashion relative to $\mu^*$.

**Schematic:**




Test: $\mu^*(X) \stackrel{?}{=} \mu^*(X \cap E) + \mu^*(X - E)$ for every $X$ in $2^R$.




Because of Vitali's example, we know container (II) is not empty. What kind of sets, if any, are in container (I)? Do we have enough sets in container (I) to build a theory of integration?

### 3.3.3 Comments

1. Since $X = (X \cap E) \cup (X - E)$, we have by subadditivity $\mu^*(X) \leq \mu^*(X \cap E) + \mu^*(X - E)$ always. Thus we need only show $\mu^*(X) \geq \mu^*(X \cap E) + \mu^*(X - E)$ for every subset $X$ of $R$.
2. If $\mu^*(X) = \infty$, then obviously $\mu^*(X) \geq \mu^*(X \cap E) + \mu^*(X - E)$. Carathéodory's Measurability Criteria may be replaced by: $E$ is Lebesgue measurable, provided

$$\mu^*(X) \geq \mu^*(X \cap E) + \mu^*(X - E)$$

for all subsets $X$ of $R$ with $\mu^*(X) < \infty$.
3. If $\mu^*$ is finitely additive, then every subset $E$ of $R$ is Lebesgue measurable: $X = (X \cap E) \cup (X - E)$, $\mu^*(X) = \mu^*(X \cap E) + \mu^*(X - E)$.

### 3.3.4 Problem

1. Show $\emptyset, R$ are in container (I), that is, $\emptyset$ and $R$ are Lebesgue measurable sets: $\emptyset$ and $R$ satisfy Carathéodory's measurability condition.
2. Show that if $E$ is in container (I) (if $E$ is Lebesgue measurable) then its complement, $R - E$, is in container (I) (then $R - E$ is Lebesgue measurable).

Well, we've established that container (I) is not empty; it at least contains $\emptyset$ and $R$. But we certainly cannot base an integration theory on the collection $\{\emptyset, R\}$. It is time to define a sigma algebra ($\sigma$-algebra) and investigate why they are so important.

### 3.3.5 Definition

In a space $\Omega$, a collection $\mathcal{O}$ of subsets of $\Omega$ is said to be a *$\sigma$-algebra*, provided:

1. $\emptyset \in \mathcal{O}$;
2. If $A \in \mathcal{O}$ then $\Omega - A \in \mathcal{O}$;
3. If $(A_k)$ is a sequence of sets in $\mathcal{O}$ then $(\cup A_k) \in \mathcal{O}$.

### 3.3.6 Problem

Show the following collections $\mathcal{O}$ form a $\sigma$-algebra:

1. $\mathcal{O} = \{\emptyset, \Omega\}$.
2. $\mathcal{O}$ is the collection of all subsets of $\Omega$, $2^\Omega$.
3. $\Omega = \{1, 2, 3, \ldots\}$, $\mathcal{O} = \{\emptyset, \{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}, \Omega\}$.
4. $\Omega$ any uncountable set and $\mathcal{O}$ is the collection of all subsets of $\Omega$ which are countable or have countable complements.

### 3.3.7 Problem

Once you are "inside" a $\sigma$-algebra it's hard to get "outside":

1. If $(A_k)$ is a sequence of sets in a $\sigma$-algebra $\mathcal{O}$, then
   i. $\cap A_k$ belongs to the $\sigma$-algebra $\mathcal{O}$;
   ii. $\limsup A_k = \bigcap_{k \geq 1} \left( \bigcup_{n \geq k} A_n \right)$ and

   $\liminf A_k = \bigcup_{k \geq 1} \left( \bigcap_{n \geq k} A_n \right)$ belong to $\mathcal{O}$.
2. Show the collection of open intervals is not a $\sigma$-algebra.

   *Hint*: $[-1, 1] = \cap(-1 - 1/n, 1 + 1/n)$.

### 3.3.8 Problem

1. $\Omega$ is any space, $\mathcal{O}_1$ and $\mathcal{O}_2$ are each $\sigma$-algebras of subsets of $\Omega$. Let $\mathcal{O}_3$ be the collection of sets that belong to both $\mathcal{O}_1$ and $\mathcal{O}_2$, the so-called intersection of $\mathcal{O}_1$ and $\mathcal{O}_2$. Show $\mathcal{O}_3$ is a $\sigma$-algebra of $\Omega$.
2. Let $\mathcal{C}$ be any nonempty family of subsets of $\Omega$. Consider the collection of all $\sigma$-algebras that contain $\mathcal{C}$ ($\mathcal{C} \subset 2^\Omega$ for example). Let $\mathcal{O}$ be the intersection of all such $\sigma$-algebras. Show that $\mathcal{O}$ is a $\sigma$-algebra, the "smallest" $\sigma$-algebra, that contains $\mathcal{C}$. $\mathcal{O}$ is said to be the $\sigma$-algebra generated by $\mathcal{C}$.

### 3.3.9 Problem

Suppose $f$ is a function from $A$ into $B$ and $\mathcal{B}$ is a $\sigma$-algebra in $B$. Show that the collection of inverse images of sets in $\mathcal{B}$ forms a $\sigma$-algebra in $A$. ("Inverse image of a $\sigma$-algebra is a $\sigma$-algebra".)

If we have a $\sigma$-algebra, by performing set operations of countable unions, countable intersections, complements, limits, etc., we will "stay in" the $\sigma$-algebra; mathematically it is closed under the usual set theoretic operations. For this reason we want container (I) to contain a "large" $\sigma$-algebra, certainly something more than $\{\emptyset, R\}$. We know from Vitali's example that container (I) is not $2^R$.

Carathéodory showed that indeed, the collection of subsets of $R$ that satisfy his measurability criteria forms a $\sigma$-algebra, and, the Lebesgue outer measure $\mu^*$ is countably additive on this $\sigma$-algebra. This is the essence of the next theorem, one of the two most important results of Lebesgue measure (The other critical result is Theorem 3.3 that guarantees a "large" $\sigma$-algebra.).

**THEOREM 3.2** (Carathéodory, 1918) *Define the Lebesgue outer measure $\mu^*$ of any subset $E$ of $R$ as follows:*

$$\mu^*(E) = inf\left\{\sum l(I_k) \mid E \subset \cup I_k,\ I_k \text{ open intervals}\right\}$$

*where $l((a,b)) = b - a,\ \ a, b \in R^e$.*
*Then*

1. *The collection of sets $E \subset R$ that satisfy Carathéodory's condition;*

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X - E)$$

*for every subset $X$ of $R$,*

*forms a $\sigma$-algebra, $\mathcal{M}$, and*

2. *The Lebesgue outer measure is countably additive on $\mathcal{M}$, that is,*

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu^*(E_k)$$

*for any mutually disjoint sequence of sets $(E_k)$ in $\mathcal{M}$.*

*The sets of $\mathcal{M}$ are said to be Lebesgue measurable, and when $E \in \mathcal{M}$, we write $\mu(E)$ in place of $\mu^*(E)$.*

*Proof*:

1. We show that the collection of Lebesgue measurable sets $\mathcal{M}$ is a $\sigma$-algebra (3.3.5). This entails three arguments:

i. The empty set is Lebesgue measurable: $\emptyset \in \mathcal{M}$.

$$\mu^*(X \cap \emptyset) + \mu^*(X - \emptyset) = \mu^*(\emptyset) + \mu^*(X) = \mu^*(X).$$

ii. If a set is Lebesgue measurable, then its complement is Lebesgue measurable: If $E \in \mathcal{M}$, then $R - E \in \mathcal{M}$. Carathéodory's criteria is symmetric in $E$ and $R - E$; $\mu^*(X \cap E) + \mu^*(X - E) =$ $\mu^*(X - (R - E)) + \mu^*(X \cap (R - E))$.

iii. Given any sequence $(E_k)$ of Lebesgue measurable sets the union is Lebesgue measurable: If $E_k \in \mathcal{M}$, then $\cup E_k \in \mathcal{M}$. Suppose $E_k \in \mathcal{M}$, $k = 1, 2, \ldots$ . We must show

$$\begin{aligned}\mu^*(X) &= \mu^*(X \cap (\cup E_k)) + \mu^*(X - (\cup E_k))\\ &= \mu^*(X \cap (\cup E_k)) + \mu^*(X \cap (\cup E_k)^c)\end{aligned}$$

for every subset $X$ of $R$.

We begin with an easier problem; showing finite intersections and finite unions of Lebesgue measurable sets are Lebesgue measurable. Let $E_1, E_2$ be Lebesgue measurable sets. We show $E_1 \cap E_2 \in \mathcal{M}$. Since

$$(X \cap (E_1 \cap E_2)) \cup (X - (E_1 \cap E_2)) = X,$$

and the Lebesgue outer measure is subadditive,

$$\mu^*(X) \leq \mu^*(X \cap (E_1 \cap E_2)) + \mu(X - (E_1 \cap E_2)).$$

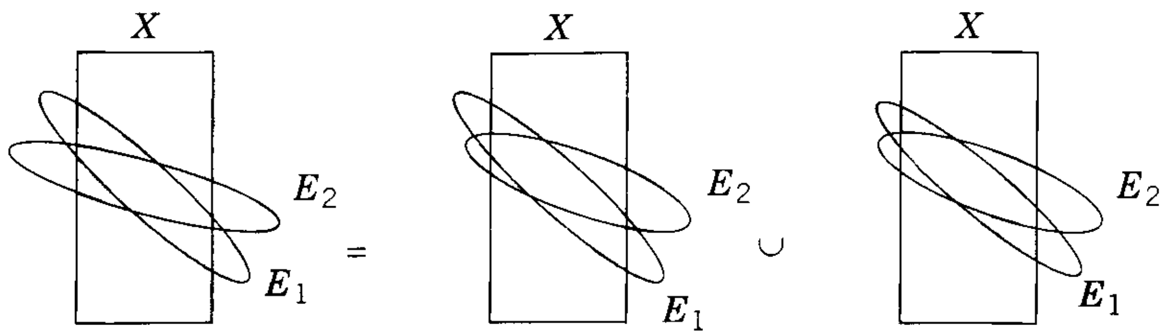
It is sufficient to show

$$\mu^*(X \cap (E_1 \cap E_2)) + \mu^*(X - (E_1 \cap E_2)) \leq \mu^*(X) < \infty.$$

Measurability of $E_2$ implies that

$$\begin{aligned}\mu^*(X \cap (E_1 \cap E_2)) &= \mu^*((X \cap E_1) \cap E_2)\\ &= \mu^*(X \cap E_1) - \mu^*((X \cap E_1) - E_2).\end{aligned}$$

The diagram below may be helpful in "decomposing" the set $X - (E_1 \cap E_2)$.




Since
$X - (E_1 \cap E_2) = (X - E_1) \cup (X - E_2) = (X - E_1) \cup ((X \cap E_1) - E_2)$, and $\mu^*$ is subadditive,

$$\mu^*(X - (E_1 \cap E_2)) \leq \mu^*(X - E_1) + \mu^*((X \cap E_1) - E_2).$$

Adding,

$$\begin{aligned} {}^*(X \cap (E_1 \cap E_2)) + \mu^*(X - (E_1 \cap E_2)) &\leq \mu^*(X \cap E_1) + \mu^*(X - E_1) \\ &= \mu^*(X), \end{aligned}$$

where the last equality follows from Lebesgue measurability of $E_1$. Thus, $E_1, E_2 \in \mathcal{M}$ implies $E_1 \cap E_2 \in \mathcal{M}$. Complementation shows $E_1 \cup E_2 \in \mathcal{M}$: $E_1, E_2 \in \mathcal{M}$ implies $R - E_1, R - E_2 \in \mathcal{M}$ implies $(R - E_1) \cap (R - E_2) \in \mathcal{M}$, that is, $R - (E_1 \cup E_2) \in \mathcal{M}$. But then $E_1 \cup E_2 \in \mathcal{M}$.

Finite unions and intersections follow by induction. Now let $(E_k)$ be any sequence of sets from $\mathcal{M}$. We may construct a new sequence of disjoint sets, all in $\mathcal{M}$, with the same union (Proposition 2.1): Let $\mathcal{M}_1 = E_1$, $\mathcal{M}_2 = E_2 - E_1$, $\mathcal{M}_3 = E_3 - (E_1 \cup E_2), \ldots$ and $\cup E_k = \cup \mathcal{M}_k$, $\mathcal{M}_k \subset E_k$. All these sets belong to $\mathcal{M}$ because of what we have shown above. Consequently, we suppose $(E_k)$ is a mutually disjoint sequence of sets from $\mathcal{M}$. We are trying to show $\cup E_k \in \mathcal{M}$, that is,

$$\mu^*(X) \geq \mu^*(X \cap (\cup E_k)) + \mu^*(X - (\cup E_k)).$$

Again, the idea is to start with "finite unions". We have shown that

$$\bigcup_{k=1}^{n} E_k \in \mathcal{M}, \text{ i.e., } \mu^*(X) = \mu^*\left(X \cap \left(\bigcup_1^n E_k\right)\right) + \mu^*(X - \left(\bigcup_1^n E_k\right).$$

*Claim*: $\mu^*\left(X \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n \mu^*(X \cap E_k)$.

Certainly true for $n = 1$ and we assume true for $n - 1$ sets $E_k$. We split

$$X \cap \left[\bigcup_1^n E_k\right]$$

in an additive manner with $E_n \in \mathcal{M}$.

$$\begin{aligned}
&\mu^*\left(X \cap \left[\bigcup_1^n E_k\right]\right) \\
&= \mu^*\left(\left(X \cap \left[\bigcup_1^n E_k\right]\right) \cap E_n\right) + \mu^*\left(\left(X \cap \left[\bigcup_1^n E_k\right]\right) - E_n\right) \\
&= \mu^*(X \cap E_n) + \mu^*\left(X \cap \left[\bigcup_1^{n-1} E_k\right]\right) \quad (E_k \text{ mutually disjoint}) \\
&= \mu^*(X \cap E_n) + \sum_1^{n-1} \mu^*(X \cap E_k) \quad \text{(induction hypothesis)} \\
&= \sum_1^n \mu^*(X \cap E_k).
\end{aligned}$$

The claim is valid.

Now split $X$ in an additive manner with $\bigcup_{k=1}^n E_k \in \mathcal{M}$:

$$\begin{aligned}
\mu^*(X) &= \mu^*\left(X \cap \left[\bigcup_1^n E_k\right]\right) + \mu^*\left(X - \left[\bigcup_1^n E_k\right]\right) \\
&\geq \mu^*\left(X \cap \left[\bigcup_1^n E_k\right]\right) + \mu^*\left(X - \left[\bigcup_1^\infty E_k\right]\right) \quad \text{(monotonicity)} \\
&= \sum_1^n \mu^*(X \cap E_k) + \mu^*\left(X - \left[\bigcup_1^\infty E_k\right]\right)
\end{aligned}$$

independent of $n$.

Therefore

$$\mu^*(X) \geq \sum_1^\infty \mu^*(X \cap E_k) + \mu^*\left(X - \left[\bigcup_1^\infty E_k\right]\right)$$

$$\geq \mu^*\left(X \cap \left[\bigcup_1^\infty E_k\right]\right) + \mu^*\left(X - \left[\bigcup_1^\infty E_k\right]\right) \quad \text{(subadditivity)}.$$

The reverse inequality follows from subadditivity (Theorem 3.1). We have completed the argument for showing $\mathcal{M}$ is a $\sigma$-algebra of subsets of $R$.

2. We are left the task of showing countable additivity of $\mu^*$ when restricted to members of $\mathcal{M}$. Let $(E_k)$ be a sequence of mutually disjoint sets from $\mathcal{M}$. We must show

$$\mu^*\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty \mu^*(E_k).$$

But in part 1 we showed

$$\mu^*\left(X \cap \left(\bigcup_1^n E_k\right)\right) = \sum_1^n \mu^*(X \cap E_k)$$

for any $X \subset R$. Replacing $X$ with $R$,

$$\mu^*\left(\bigcup_1^n E_k\right) = \sum_1^n \mu^*(E_k).$$

Finite additivity holds.

Then,

$$\sum_1^n \mu^*(E_k) = \mu^*\left(\bigcup_1^n E_k\right)$$

$$\leq \mu^*\left(\bigcup_1^\infty E_k\right) \quad \text{(monotonicity)}$$

$$\leq \sum_1^\infty \mu^*(E_k) \quad \text{(subadditivity)}$$

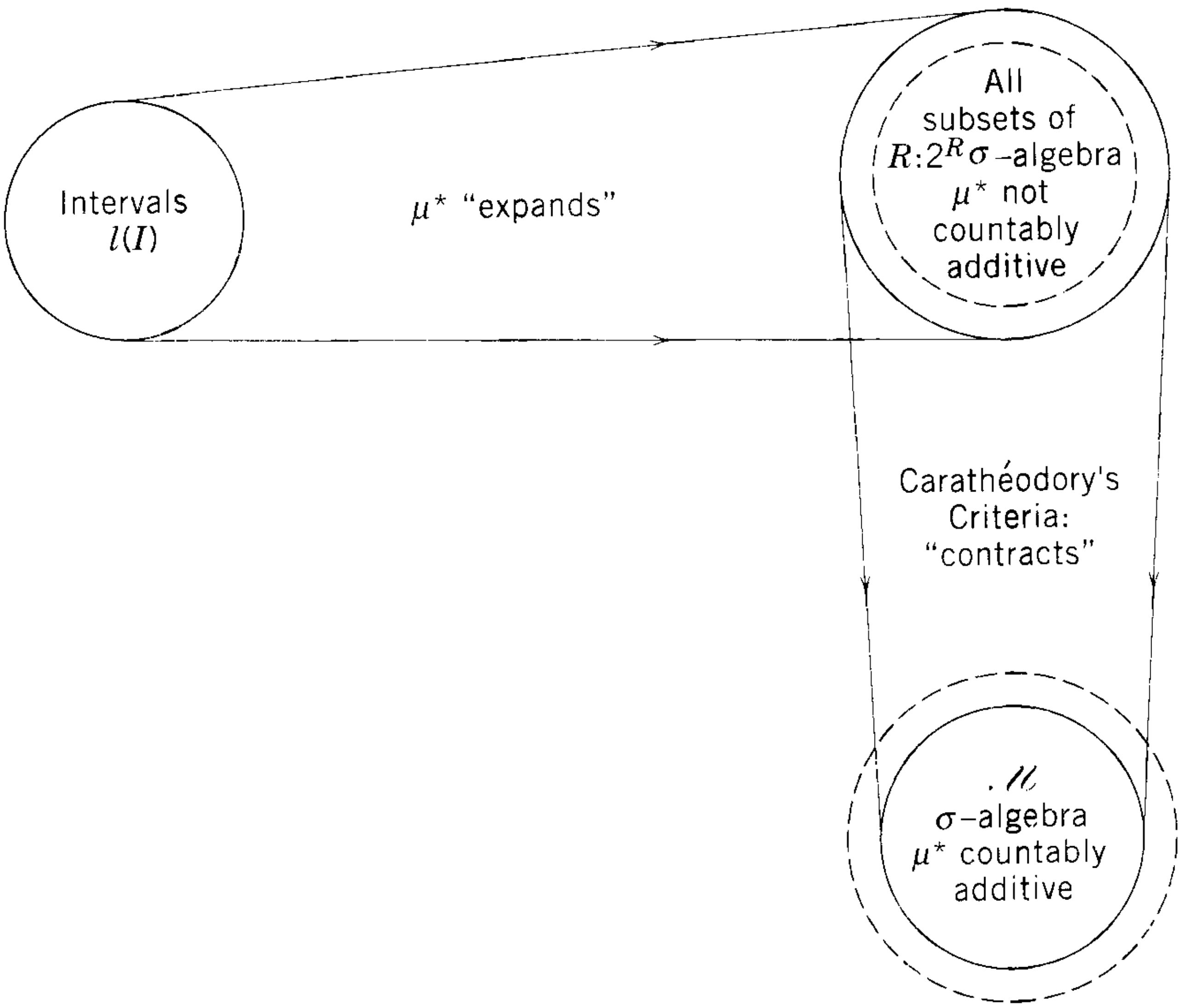
independent of $n$. Thus

$$\sum_{1}^{\infty} \mu^*(E_k) = \mu^*\left(\bigcup_{1}^{\infty} E_k\right)$$

and the conclusion follows. YES! ∎

### 3.3.10 Summary

We have shown that the Lebesgue outer measure $\mu^*$, written $\mu$ when restricted to the $\sigma$-algebra $\mathcal{M}$ of subsets of $R$ satisfying Carathéodory's Condition, is countably additive on $\mathcal{M}$, that is, $\mu$ is countably additive on the $\sigma$-algebra of Lebesgue measurable subsets of $R$.




$\mu$ may be the measure function that we have been seeking for so long. It satisfies our intuitive requirements two through eight, with countable additivity gained at the expense of "measuring" fewer sets: $2^R \longrightarrow \mathcal{M}$. So what is a possible problem? The reader will recall that the only sets that we know for sure are in $\mathcal{M}$ are $\emptyset$ and $R$. Yes, we showed $\mu^*([a, b]) = b - a$, but we haven't shown **any** interval satisfies Carathéodory's Condition. We don't know if $\mathcal{M}$ contains intervals! Are intervals Lebesgue measurable? That is the content of the next proposition: intervals are ok, intervals are Lebesgue measurable.

**PROPOSITION 3.2** *Intervals satisfy Carathéodory's condition; intervals are Lebesgue measurable.*

*Proof:* The main idea of the proof is that for an interval that is the union of two disjoint intervals length, $l$, is additive (3.1.5): $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$, then $l(I) = l(I_1) + l(I_2)$. We must show intervals like $(a, b)$, $(a, \infty)$, etc. satisfy Carathéodory's condition. Our argument will deal with $(a, \infty)$. The reader may show the other intervals are Lebesgue measurable ( $(-\infty, b) \cap (a, \infty) = (a, b)$, $a < b$). So, we must show

$$\mu^*(X) = \mu^*(X \cap (a, \infty)\,) + \mu^*(X - (a, \infty)\,)$$

for every subset $X$ of $R$. Again, because of subadditivity, we need only show $\mu^*(X) \geq \mu^*(X \cap (a, \infty)) + \mu^*(X - (a, \infty))$ for all subsets $X$ of $R$ with $\mu^*(X) < \infty$ (3.3.3).

From the definition of Lebesgue outer measure, we have an open cover $\cup I_k$ of $X$ so that

$$\mu^*(X) \leq \sum l(I_k) < \mu^*(X) + \epsilon\,.$$

Consider $I_k \cap (a, \infty)$ and $I_k - (a, \infty)$.
$I_k \cap (a, \infty)$ is either empty or an open interval and $X \cap (a, \infty) \subset \bigcup (I_k \cap (a, \infty))$. $I_k - (a, \infty)$ is either empty or an interval and $X - (a, \infty) \subset \bigcup (I_k - (a, \infty))$.
Thus

$$\begin{aligned}
&\mu^*(X \cap (a, \infty)) \\
&\qquad + \mu^*(X - (a, \infty)) \leq \mu^*\left(\bigcup (I_k \cap (a, \infty))\right) + \mu^*\left(\bigcup (I_k - (a, \infty))\right) \\
&\qquad \leq \sum l(I_k \cap (a, \infty)) + \sum l(I_k - (a, \infty)) \\
&\qquad = \sum \{ l(I_k \cap (a, \infty)) + l(I_k - (a, \infty)) \} \\
&\qquad = \sum l(I_k) \\
&\qquad < \mu^*(X) + \epsilon,
\end{aligned}$$

and the proof is complete. ∎

But wait! After all this work, are we limited to intervals? Now the power of Theorem 3.2 comes into play. We can "measure" countable unions of countable intersections of complements of... intervals. Such sets can be complicated. It is time to introduce Borel sets.

## 3.4 BOREL SETS

In this section we show $\mathcal{M}$, the $\sigma$-algebra of Lebesgue measurable sets, contains intervals, countable unions and countable intersections of such and their complements, countable..., in short, $\mathcal{M}$ contains all (as yet undefined) Borel sets. Recall that our original measure function $l$ (length) was defined on the family of all intervals of $R$. However, we have a smallest $\sigma$-algebra that contains the collection of open intervals of $R$ (3.3.8). This smallest $\sigma$-algebra is called the family of Borel sets, $\mathcal{B}$, and we say $\mathcal{B}$ is generated by the open intervals in $R$.

### 3.4.1 Definition

The $\sigma$-algebra generated by the collection of all open intervals of $R$ is called the *Borel* $\sigma$-algebra $\mathcal{B}$.

$\mathcal{B}$ contains about every set that "comes up" in analysis. If $\mathcal{M}$ contains $\mathcal{B}$, the Borel sets, we "measure" just about everything of interest.

Because different authors use topologically different generators (open sets, compact sets, etc.) according to their needs, we thought the next proposition useful.

**PROPOSITION 3.3** *The Borel $\sigma$-algebra of subsets of $R$, $\mathcal{B}$, may be described as the $\sigma$-algebra generated by these families of subsets of $R$:*

1. *Open intervals,*
2. *Open sets,*
3. *Closed intervals,*
4. *Closed sets,*
5. *Compact sets,*
6. *Left open, right closed intervals,*
7. *Left closed, right open intervals,*
8. *All intervals.*

*Proof:* Tedious and uninteresting. ■

The foundation has been completed. We show that Borel sets satisfy Carathéodory's Condition, Borel sets are Lebesgue measurable. This is the "other" important theorem. Its importance cannot be overemphasized. It guarantees measurability of about any conceivable set of real numbers.

**Theorem 3.3** *Every Borel set of real numbers is a Lebesgue measurable set.*

*Proof:* The collection of all Lebesgue measurable sets is a $\sigma$-algebra, $\mathcal{M}$, and $\mathcal{B}$, the collection of Borel sets, is the smallest $\sigma$-algebra generated by the open intervals. If we show $\mathcal{M}$ contains the open intervals, that is, open intervals are Lebesgue measurable, we would have $\mathcal{B} \subset \mathcal{M}$. This is Proposition 3.2. For completeness, we review the argument. Since $(a,b) = (a,\infty) \cap (-\infty, b)$, $(-\infty,\infty) = \cup(-k,k)$, and $(a,\infty) = \cup(-\infty, a + 1/k)^c$, we will only concern ourselves with open intervals of the form $(-\infty, b)$. Let $X$ be any subset of $R$. We may assume $\mu^*(X) < \infty$ and show $\mu^*(X) \geq \mu^*(X \cap (-\infty,b)) + \mu^*(X - (-\infty,b))$. Let $\epsilon > 0$ be given. We have a collection of open intervals $\{I_k\}$ containing $X$ so that

$$\mu^*(X) \leq \sum l(I_k) < \mu^*(X) + \epsilon.$$

Since $X \subset \cup I_k$, $X \cap (-\infty,b) \subset \bigcup (I_k \cap (-\infty,b))$ and $X - (-\infty,b) \subset \bigcup(I_k - (-\infty,b))$. The sets $I_k \cap (-\infty,b)$ and $I_k - (-\infty,b)$ are either empty or intervals and $l(I_k) = l(I_k \cap (-\infty,b)) + l(I_k - (-\infty,b))$. Hence

$$\begin{aligned}
&\mu^*(X \cap (-\infty,b)) \\
&\quad + \mu^*(X - (-\infty,b)) \leq \mu^*\left(\bigcup(I_k \cap (-\infty,b))\right) + \mu^*\left(\bigcup(I_k - (-\infty,b))\right) \\
&\qquad \leq \sum \mu^*(I_k \cap (-\infty,b)) + \sum \mu^*(I_k - (-\infty,b)) \\
&\qquad = \sum l(I_k \cap (-\infty,b)) + \sum l(I_k - (-\infty,b)) \\
&\qquad = \sum \{l(I_k \cap (-\infty,b)) + l(I_k - (-\infty,b))\} \\
&\qquad = \sum l(I_k) \\
&\qquad < \mu^*(X) + \epsilon.
\end{aligned}$$

Thus

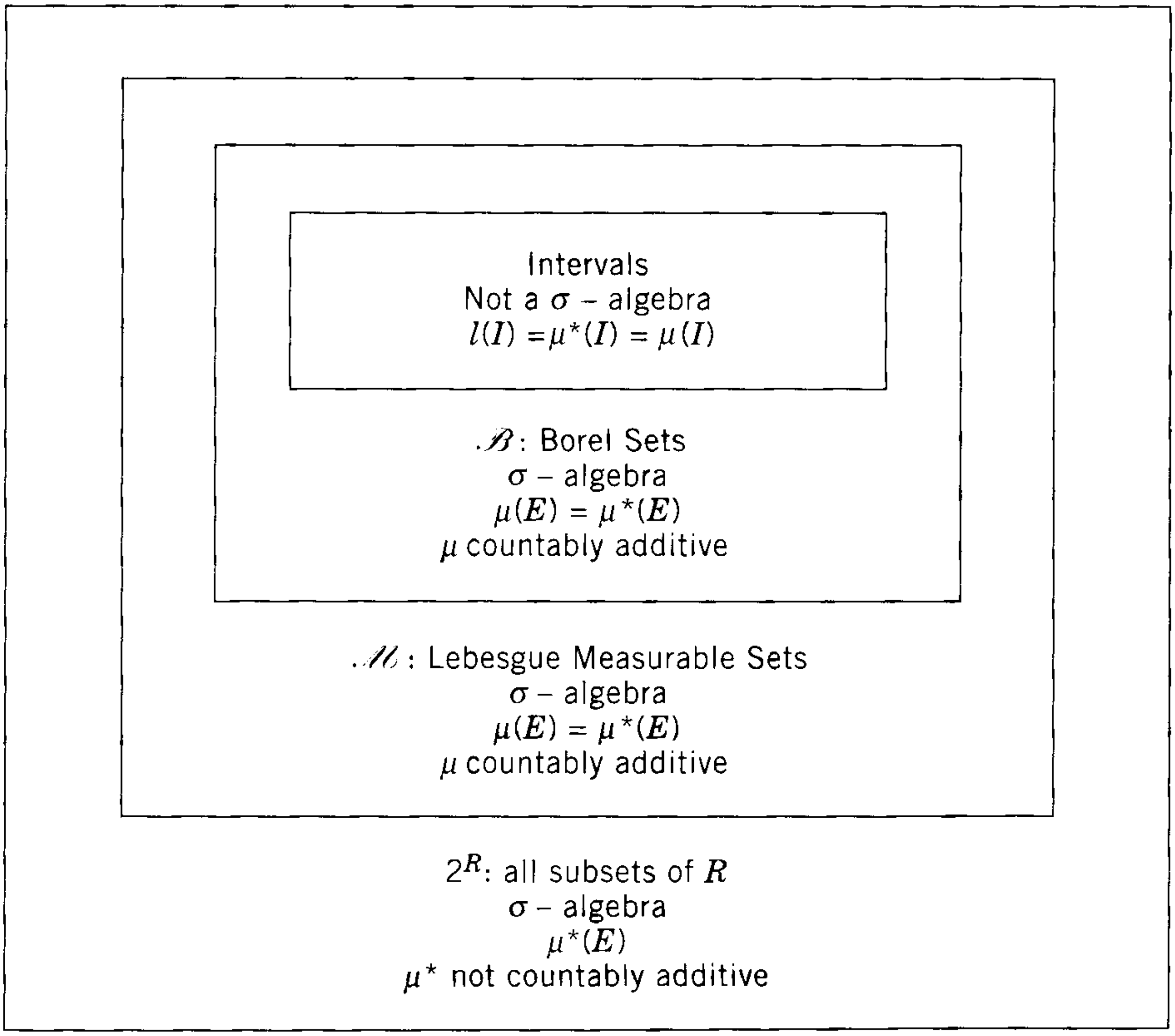
$$\mu^*(X \cap (-\infty, b)) + \mu^*(X - (-\infty, b)) \leq \mu^*(X). \qquad \blacksquare$$

Since every Borel set is Lebesgue measurable, $\mathcal{B} \subset \mathcal{M}$, do we have sets that are Lebesgue measurable but that are not Borel sets? This was an open problem for a number of years. In 1917 the Russian mathematician Suslin constructed such sets. The reader may turn to Appendix C for a fuller discussion. Thus

$$\mathcal{B} \subset \mathcal{M} \subset 2^R,$$

with all the containments being proper.

### 3.4.2 Summary




Now that we've shown the collection of Lebesgue measurable sets contains about every imaginable set of real numbers, we will concentrate on evaluating the measures of various sets.

## 3.5 "MEASURING"

We gather together some previous results, along with some new results to be proved below, that are useful in determining the measure of specific sets of real numbers.

**Theorem 3.4** *In what follows all sets are presumed to be Lebesgue measurable sets of real numbers:*

1. $\mu(\emptyset) = \mu(\{a\}) = 0$.
2. $\mu(I) = l(I)$.
3. $\mu(\textit{countable set}) = 0$.
4. $\mu(\textit{subset of a set of measure zero}) = 0$.
5. $\mu(\cup E_k) \leq \sum \mu(E_k)$ *with* **equality** *whenever the sequence of sets* $(E_k)$ *is mutually disjoint.*
6. $\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2)$.
7. $\mu(E_1) \leq \mu(E_2)$ *if* $E_1 \subset E_2$. *If in addition* $\mu(E_2) < \infty$, *then* $\mu(E_2) - \mu(E_1) = \mu(E_2 - E_1)$.
8. *If* $E_1 \subset E_2 \subset E_3 \subset \cdots$ *then* $\mu(\cup E_k) = \mu(\lim E_k) = \lim \mu(E_k)$.
9. *If* $E_1 \supset E_2 \supset E_3 \supset \cdots$ *and* $\mu(E_1) < \infty$, *then* $\mu(\cap E_k) = \mu(\lim E_k) = \lim \mu(E_k)$. *Need* $\mu(E_1) < \infty$ : $E_k = (k, \infty)$.
10. *If* $\mu(\cup E_k) < \infty$, *then* $\limsup \mu(E_k) \leq \mu(\limsup E_k)$.
11. $\mu(\liminf E_k) \leq \liminf \mu(E_k)$.
12. *If* $\liminf E_k = \limsup E_k$ *and* $\mu(\cup E_k) < \infty$, *then* $\mu(\lim E_k) = \lim \mu(E_k)$.

*Proof:* Parts 1 through 5 have been discussed earlier.

6. $E_1 \cup E_2 = (E_1 - E_2) \cup (E_2 - E_1) \cup (E_1 \cap E_2)$. Thus

$$\begin{aligned} \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) &= \mu(E_1 - E_2) + \mu(E_1 \cap E_2) \\ &\quad + \mu(E_2 - E_1) + \mu(E_2 \cap E_1) \\ &= \mu(E_1) + \mu(E_2)\,. \end{aligned}$$

7. Follows immediately from part 5: Since $E_1 \subset E_2$, $E_2 = (E_2 - E_1) \cup E_1$, and

$$\begin{aligned} \mu(E_2) &= \mu((E_2 - E_1) \cup E_1) \\ &= \mu(E_2 - E_1) + \mu(E_1)\,. \end{aligned}$$

If $\mu(E_2) < \infty$, we have $\mu(E_1) < \infty$ and we may subtract.

8. If $\mu(E_N) = \infty$ for some $N$, then $\mu(E_k) = \infty$ for all $k \geq N$ and $\lim \mu(E_k) = \infty$. Since $E_N \subset \cup E_k$, $\mu(E_N) \leq \mu(\cup E_k)$ and thus $\mu(\cup E_k) = \mu(\lim E_k) = \infty$. So we may suppose $\mu(E_k) < \infty$ for all $k$. Since

$$\cup E_k = E_1 \cup (E_2 - E_1) \cup (E_3 - E_2) \cup \cdots$$

and the sets $E_{k+1} - E_k$ are mutually disjoint, we have

$$\begin{aligned} \mu(\cup E_k) &= \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} - E_k) && (5) \\ &= \mu(E_1) + \sum_k [\mu(E_{k+1}) - \mu(E_k)] && (7) \\ &= \mu(E_1) + \lim_n \sum_{k=1}^{n} [\mu(E_{k+1}) - \mu(E_k)] \\ &= \mu(E_1) + \lim_n \mu(E_{n+1}) - \mu(E_1) \\ &= \lim_k \mu(E_k) \,. \end{aligned}$$

9. Proceed as in part 8: Write a disjoint sequence of sets and use part 5:

$$E_1 - (\cap E_k) = (E_1 - E_2) \cup (E_2 - E_3) \cup \cdots \,.$$

Thus

$$\begin{aligned} \mu(E_1) - \mu(\cap E_k) &= \mu(E_1 - (\cap E_k)) \\ &= \sum_1 \mu(E_k - E_{k+1}) \\ &= \sum_1 [\mu(E_k) - \mu(E_{k+1})] \\ &= \lim_n \sum_1^n [\mu(E_k) - \mu(E_{k+1})] \\ &= \mu(E_1) - \lim_k \mu(E_k) \,. \end{aligned}$$

Since $\mu(E_1) < \infty$, we may subtract and the conclusion follows.

10. We essentially use part 9. Recall

$$\limsup E_k = \bigcap_{m\geq 1}\left(\bigcup_{k\geq m} E_k\right).$$

Then

$$\bigcup_1 E_k \supset \bigcup_2 E_k \supset \cdots \text{ and } E_m \subset \bigcup_{k\geq m} E_k .$$

Thus

$$\mu(E_m) \leq \mu\left(\bigcup_{k\geq m} E_k\right)$$

and, hence,

$$\begin{aligned}\limsup \mu(E_m) &\leq \limsup \mu\left(\bigcup_{k\geq m} E_k\right)\\ &= \lim \mu\left(\bigcup_{k\geq m} E_k\right)\\ &= \mu\left(\bigcap_{m\geq 1}\bigcup_{k\geq m} E_k\right)\\ &= \mu(\limsup E_k) .\end{aligned}$$

11. Again $\liminf E_k = \bigcup_{m\geq 1} (\bigcap_{k\geq m} E_k)$. Then

$$\bigcap_1 E_k \subset \bigcap_2 E_k \subset \cdots \text{ and } \bigcap_1 E_k \subset E_1, \bigcap_2 E_k \subset E_2, \ldots .$$

Thus $\mu(\bigcap_{k\geq m} E_k) \leq \mu(E_m)$. Hence

$$\liminf \mu\left(\bigcap_{k\geq m} E_k\right) \leq \liminf \mu(E_m) .$$

But

$$\liminf \mu\left(\bigcap_{k\geq m} E_k\right) = \lim \mu\left(\bigcap_{k\geq m} E_k\right)$$
$$= \mu\left(\bigcup_{m\geq 1} \bigcap_{k\geq m} E_k\right)$$
$$= \mu(\liminf E_k) \quad \text{by part 8}.$$

12.

$$\limsup \mu(E_k) \leq \mu\left(\bigcap_{m\geq 1}\left(\bigcup_{k\geq m} E_k\right)\right)$$
$$= \mu\left(\bigcup_{m\geq 1}\left(\bigcap_{k\geq m} E_k\right)\right) \leq \liminf \mu(E_k).$$

■

Appropriately, it is time to "calculate"; we take specific sets and compute their measure. Enjoy!

### 3.5.1 Problem

The reader should discuss the Lebesgue measurability of the sets that appear below and calculate their Lebesgue measure when appropriate.

1. $E_k = \left(\frac{1}{(k+1)}, \frac{1}{k}\right); \qquad \bigcup E_k.$

2. $E_k = \left\{x \mid \sin(x) < \frac{1}{k},\ 0 \leq x \leq 2\pi\right\}; \qquad \bigcap E_k.$

3. $E_k = \left(0, 1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln k\right); \qquad \bigcap E_k.$

   $F_k = \left(0, 1 + \frac{1}{2} + \cdots + \frac{1}{k+1} - \ln k\right); \qquad \bigcup F_k.$

4. $E_k = \left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{2k-1},\ \sum_{i=1}^{k} \frac{1}{i(i+1)}\right); \qquad \bigcup_{5} E_k.$

   $F_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k},\ \sum_{i=1}^{k} \frac{1}{i(i+1)}\right); \qquad \bigcap_{2} F_k.$

5. $E_k = \left(\frac{1^\alpha + 2^\alpha + \cdots + (k-1)^\alpha}{k^{\alpha+1}}, 1\right)$, $\alpha > 0$; $\bigcap E_k$.

$F_k = \left(\frac{1^\alpha + 2^\alpha + \cdots + k^\alpha}{k^{\alpha+1}}, 1\right)$, $\alpha > 0$; $\bigcup F_k$.

6. $E_k = (0, x_k)$ where $x_k = \frac{1}{2}\left(x_{k-1} + \frac{a}{x_{k-1}}\right)$ with $x_0 = a > 1$; $\bigcap E_k$.

$F_k = (x_k, a)$; $\bigcup F_k$.

7. $E_k = (0, x_k)$, $x_k = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2k}{2k-1}\right)\left(\frac{2k}{2k+1}\right)$; $\bigcup E_k$.

8. $E_k = \left(ka^{\frac{1}{k}} - 1, kb^{\frac{1}{k}} - 1\right)$, $1 < a < b$; $\liminf E_k, \limsup E_k, \lim E_k$.

9. $E_k = (a_k, b_k)$, $a_k = 1 - \frac{1}{2} + \cdots + \frac{(-1)^{k-1}}{k}$, $b_k = 1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^{k+1}}{2k-1}$; $\liminf E_k, \limsup E_k, \lim E_k$.

10. $E_k = \left(\frac{k}{(k!)^{1/k}}, \left(1 + \frac{1}{k}\right)^{k+1}\right)$; $\bigcap E_k$.

11. Rational numbers; any countable set.
12. Irrational numbers in $(0, 1)$: Positive measure, but does not contain an interval.
13. Cantor set: uncountable set of measure zero.
14. Numbers in $(0, 1)$ without 7 in their decimal expansion.
15. Any set of numbers with Lebesgue outer measure zero (any subset of such a set).
16. The symmetric difference of two measurable sets $E_1$ and $E_2$. $(E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1))$

$$\mu(E_1 \Delta E_2) = 0 \text{ iff } \mu(E_1 - E_2) = \mu(E_2 - E_1) = 0.$$

17. If $E$ Lebesgue measurable, $\mu(E) < \infty$, and $E \subset A$, then $\mu^*(A - E) = \mu^*(A) - \mu(E)$.
18. $\mu^*(A \cup B) + \mu^*(A \cap B) \leq \mu^*(A) + \mu^*(B)$.

## 3.6 STRUCTURE OF LEBESGUE MEASURABLE SETS

We are comfortable with the familiar; closed, open, nowhere dense, countable, etc. Where does Lebesgue measurability enter the picture? Are there relationships between topological properties (open, closed, etc.) and Lebesgue measurability? The next theorem shows that Lebesgue measurable subsets of $R$ are "almost open", "almost closed", and so forth.

**THEOREM 3.5** *For an arbitrary subset $E$ of $R$, the following statements are equivalent:*

1. *$E$ is Lebesgue measurable in the sense of Carathéodory;*
2. *Given $\epsilon > 0$ we can determine an open set $G \subset R$ with $E \subset G$ and $\mu^*(G - E) < \epsilon$ ("exterior" approximation by open sets);*
3. *Given $\epsilon > 0$ we can determine a closed set $F \subset R$ with $F \subset E$ and $\mu^*(E - F) < \epsilon$ ("interior" approximation by closed sets);*
4. *There is $\mathcal{G}_\delta$ set $B_1$ with $E \subset B_1$ and $\mu^*(B_1 - E) = 0$ ($B_1$ is a countable intersection of open sets; if we relax "open", we can obtain a very good approximation by Borel sets);*
5. *There is an $\mathcal{F}_\sigma$ set $B_2$ with $B_2 \subset E$ and $\mu^*(E - B_2) = 0$ ($B_2$ is a countable union of closed sets; if we relax "closed", we have very good approximations from the "inside" by Borel sets).*

*Proof:*
1.$\Longrightarrow$2. Assume $E$ is a Lebesgue measurable subset of $R$ with $\mu(E) < \infty$. From the definition of Lebesgue outer measure, we have an open cover $G = \cup I_k$ so that $E \subset G$ and $\mu^*(G) < \mu(E) + \epsilon$. Since $G = (G - E) \cup E$ and $G$ is Lebesgue measurable (Theorem 3.3), $\mu(G) = \mu(G - E) + \mu(E)$. Because $\mu(E) < \infty$, we may subtract and obtain $\mu(G - E) < \epsilon$. If $\mu(E) = \infty$, let $E_k = E \cap [-k, k]$. $E_k$ is Lebesgue measurable, $\mu(E_k) < \infty$, and by what we just showed we have an open set $G_k$ so that $E_k \subset G_k$ and $\mu(G_k - E_k) < \epsilon/2^k$. Since $E = \cup E_k$ and $E \subset \cup G_k = G$, it follows that

$$\begin{aligned}
\mu^*(G - E) &= \mu(G - E) \\
&\leq \mu\Big(\bigcup (G_k - E_k)\Big) \\
&\leq \sum \mu(G_k - E_k) \\
&< \epsilon .
\end{aligned}$$

We have constructed an open set $G = \cup G_k$ with the desired properties.

2.$\Longrightarrow$3. Follows by "complementation": Apply part 2 to $E^c$. We have $E^c \subset G$ and $\mu^*(G - E^c) < \epsilon$. But then $G^c \subset E$, $\mu^*(E - G^c) = \mu^*(G - E^c) < \epsilon$, and with $F = G^c$, the argument is complete.

3.$\Longrightarrow$4. Let $E \subset R$ and apply part 3 to $E^c$. We have a sequence of closed sets $(F_k)$ so that $F_k \subset E^c$ and $\mu^*(E^c - F_k) < 1/k$. Let $B_1 = \cap F_k^c$. $B_1$ is a $\mathcal{G}_\delta$ set, $E \subset B_1$, and

$$\begin{aligned}\mu^*(B_1 - E) &= \mu^*\left(E^c - \bigcup F_k\right)\\ &\le \mu^*(E^c - F_k)\\ &< 1/k, \qquad k = 1, 2, \dots .\end{aligned}$$

Therefore $\mu^*(B_1 - E) = 0$.

4.$\Longrightarrow$5. Follows by complementation as in 2.$\Longrightarrow$3.

5.$\Longrightarrow$1. We must show $\mu^*(X \cap E) + \mu^*(X - E) = \mu^*(X)$ for every subset $X$ of $R$. Let $E$ be an arbitrary subset of $R$ and $B_2$ the $\mathcal{F}_\sigma$ set guaranteed by part 5: $B_2 \subset E$, $\mu^*(E - B_2) = 0$. Since $B_2$ is Lebesgue measurable (Theorem 3.3), $\mu^*(X \cap E) = \mu^*((X \cap E) \cap B_2) + \mu^*((X \cap E) - B_2)$. Because $(X \cap E) - B_2 \subset E - B_2$, $\mu^*((X \cap E) - B_2) = 0$, and we have $\mu^*(X \cap E) = \mu^*(X \cap B_2)$. On the other hand,

$$\begin{aligned}\mu^*(X - E) &= \mu^*((X - E) \cap B_2) + \mu^*((X - E) - B_2)\\ &= \mu^*(\emptyset) + \mu^*((X - E) - B_2)\\ &\le \mu^*(X - B_2).\end{aligned}$$

Thus,

$$\mu^*(X \cap E) + \mu^*(X - E) \le \mu^*(X \cap B_2) + \mu^*(X - B_2) = \mu^*(X)$$

because $B_2$ is Lebesgue measurable. This yields Carathéodory's Condition on $E$ since $\mu^*(X) \le \mu^*(X \cap E) + \mu^*(X - E)$ by subadditivity. ■

### 3.6.1 Problem

1. $E = \{$ rational numbers in $(0, 1)\}$. What is $F$ of part 3 of Theorem 3.5?

2. $E = \{$irrational numbers in $(0, 1)\}$. How about $F$ of part 3 of Theorem 3.5?

### 3.6.2 Problems

1. Show that $E \subset R$ is Lebesgue measurable iff we have Borel sets $B_1, B_2$ satisfying $B_2 \subset E \subset B_1$ and $\mu^*(B_1 - B_2) = 0$.

   *Hint:* $B_1 - B_2 = (B_1 - E) \cup (E - B_2)$.

2. Let $E$ be a Lebesgue measurable set with $\mu(E) < \infty$. Show that we have a decreasing sequence of open sets $(G_k)$ so that $\lim \mu(G_k) = \mu(E)$.

   *Hint:* $E \subset \mathcal{O}_k, \mu(\mathcal{O}_k - E) < 1/k$, and form $\mathcal{O}_1, \mathcal{O}_1 \cap \mathcal{O}_2, \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3, \ldots$

3. Let $E$ be a Lebesgue measurable set with $\mu(E) < \infty$. Then we have an increasing sequence of closed sets $(F_k)$ so that $\lim \mu(F_k) = \mu(E)$.

Borel sets, being countable unions and intersections, with complementation, of open and closed sets are familiar. The next result relates Borel sets and Lebesgue measurable sets. It is a corollary of Theorem 3.5, but it so succinctly describes Lebesgue measurable sets that we choose to call it a theorem.

**THEOREM 3.6** *Every Lebesgue measurable set of real numbers is the union of a Borel set and a set of Lebesgue measure zero (Lebesgue measure is the completion of Borel measure).*

*Proof:* Let $E$ be a Lebesgue measurable set of real numbers. We then have a Borel set $(\mathcal{F}_\sigma)$ $B$ so that $B \subset E$ and $\mu(E - B) = 0$ (Theorem 3.5). But $E = B \cup (E - B)$; $B$ is our desired Borel set and $E - B$ is the Lebesgue measurable set with Lebesgue measure zero. ∎

The last theorem of this chapter tells us that sets of finite Lebesgue measure are "almost" finite unions of intervals.

**THEOREM 3.7** *Suppose $E$ is any subset of $R$ with $\mu^*(E) < \infty$. Then $E$ is a Lebesgue measurable set of real numbers iff we have a finite union of open intervals $U$ so that*

$$\mu^*(E - U) + \mu^*(U - E) < \epsilon,$$

*for any $\epsilon > 0$.*

*Proof:* We first assume $E$ is Lebesgue measurable and construct a finite union of open intervals $U$ so that

$$\mu^*(E - U) + \mu^*(U - E) < \epsilon .$$

Because $E$ is Lebesgue measurable, we have an open set $G$ so that $E \subset G$ and $\mu(G - E) < \epsilon/2$ (Theorem 3.5). Since every nonempty open set of real numbers is a countable union of disjoint open intervals (Theorem 2.4),

$$E \subset \bigcup_{k=1}^{\infty} I_k = G \text{ and } \mu\left(\left(\bigcup_{1}^{\infty} I_k\right) - E\right) < \epsilon/2.$$

But $\cup I_k = ((\cup I_k) - E) \cup E$, and, consequently,

$$\mu\left(\bigcup_{1}^{\infty} I_k\right) = \sum \mu(I_k) < \mu(E) + \frac{\epsilon}{2} < \infty ,$$

that is, the series $\sum \mu(I_k)$ converges. Choose $N$ so that

$$\sum_{N+1}^{\infty} \mu(I_k) < \epsilon/2 \text{ and define } U = \bigcup_{1}^{N} I_k.$$

*Note*:

i. $U \cap \left(\bigcup_{N+1}^{\infty} I_k\right) = \emptyset.$

ii. $U - E \subset G - E$ and thus $\mu^*(U - E) < \frac{\epsilon}{2}.$

iii. $E - U = E \cap \left(\bigcup_{N+1}^{\infty} I_k\right) \subset \bigcup_{N+1}^{\infty} I_k.$

Then $\mu^*(U - E) + \mu^*(E - U) < \epsilon.$

Conversely, assume we have a finite union of open intervals $U$ so that $\mu^*(E - U) + \mu^*(U - E) < \epsilon/2$. We will construct an open set $G$ so that $E \subset G$, $\mu^*(G - E) < \epsilon$, and then conclude from Theorem 3.5 that $E$ must be Lebesgue measurable.

Consider the set $E - U$. From the definition of Lebesgue outer measure we have an open set $\mathcal{O}_1$ so that $E - U \subset \mathcal{O}_1$ and $\mu^*(E - U) \leq \mu^*(\mathcal{O}_1) < \mu^*(E - U) + \epsilon/2$. Let $G = \mathcal{O}_1 \cup U$. $G$ is an open subset of $R$, $E = U \cup (E - U) \subset U \cup \mathcal{O}_1 = G$, and

$$\begin{aligned} \mu^*(G - E) &= \mu^*((U \cup \mathcal{O}_1) - E) \\ &= \mu^*((U - E) \cup (\mathcal{O}_1 - E)) \\ &\leq \mu^*(U - E) + \mu^*(\mathcal{O}_1) \\ &< \epsilon. \end{aligned}$$ ■

We have completed our development of Lebesgue measure. Knowing what Lebesgue measurable sets are, we are now able to discuss measurable functions. This will be the content of the next chapter.

He who knows not mathematics and the results of recent scientific investigation dies without knowing truth.

—C. H. Schellbach

What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than this of mathematics?

—Benjamin Franklin

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}\ B_{2n}$$

# 4

# *Lebesgue Measurable Functions*

In this chapter we introduce Lebesgue measurable functions. Most of us are familiar with classifications like continuous, Riemann integrable, differentiable, etc. Consequently, we look for relationships between some of these classifications and that of "Lebesgue measurable function". We also show that measurable functions can be approximated by sequences of "simple" functions, and that, most importantly, limiting operations preserve Lebesgue measurability.

The importance of a function being Lebesgue measurable cannot be overemphasized, for it is Lebesgue measurable functions that play the same role in Lebesgue integration (Theorem 5.7) as continuous almost everywhere functions play in Riemann integration (Theorem 5.1).

## 4.1 MEASURABLE FUNCTIONS

The reader will recall that the integrals of Cauchy and his contemporaries were defined for continuous functions. An integration theory was "built" on functions preserving "openness" under inverse images (Proposition 2.8). Maybe Lebesgue measurable functions should have inverse images of open sets as measurable sets. Since every nonempty open subset $G$ of $R$ is the union of a countable collection of mutually disjoint open intervals (Theorem 2.4), say $G = \cup I_k$, and inverse images behave nicely (Proposition 2.2),

$$f^{-1}(G) = f^{-1}(\cup I_k) = \cup f^{-1}(I_k),$$

something along the line of Lebesgue measurability for $f^{-1}(I)$, that is, a requirement of being able to assign a "length" (measure) to inverse images of intervals, might be appropriate.

### 4.1.1 Definition

An extended real-valued function $f$, defined on a Lebesgue measurable set of real numbers $E$, is said to be *Lebesgue measurable* on $E$ if $f^{-1}((c, \infty]) = \{x \in E \mid f(x) > c\}$ is a Lebesgue measurable subset of $E$ for every real number $c$.

*Requirement*: We must be able to "measure" the inverse images of intervals of the type $(c, \infty] = (c, \infty) \cup \{\infty\}$. How about other kinds of intervals, for example, $(c, d)$ and $[c, d)$?

**PROPOSITION 4.1** *Suppose $f$ is an extended real-valued function whose domain is a Lebesgue measurable set of real numbers $E$, and $c$ is any real number. Then the following statements are equivalent:*

1. *$f$ is a Lebesgue measurable function on $E$.*
2. *$f^{-1}((c, \infty]) = \{x \in E \mid f(x) > c\}$ is a Lebesgue measurable subset of $E$.*
3. *$f^{-1}([c, \infty]) = \{x \in E \mid f(x) \geq c\}$ is a Lebesgue measurable subset of $E$.*
4. *$f^{-1}([-\infty, c)) = \{x \in E \mid f(x) < c\}$ is a Lebesgue measurable subset of $E$.*
5. *$f^{-1}([-\infty, c]) = \{x \in E \mid f(x) \leq c\}$ is a Lebesgue measurable subset of $E$.*

*Proof:*

$1. \Longleftrightarrow 2.$

Definition 4.1.1.

$2. \Longrightarrow 3.$

$$f^{-1}([c, \infty]) = f^{-1}\left(\bigcap_k (c - 1/k, \infty]\right) = \bigcap_k f^{-1}((c - 1/k, \infty]).$$

$3. \Longrightarrow 4.$

$$f^{-1}([-\infty, c)) = R - f^{-1}([c, \infty]).$$

4.$\Longrightarrow$5.

$$f^{-1}([-\infty, c]) = \bigcap_k f^{-1}([-\infty, c + 1/k)).$$

5.$\Longrightarrow$2.

$$f^{-1}((c, \infty]) = R - f^{-1}([-\infty, c]).$$

We have inverse images of rather exotic types of intervals are measurable, but again, how about the more common variety, such as $f^{-1}((c, d))$, with $c$ and $d$ real numbers, $c < d$?

### 4.1.2 Problem

Suppose $f$ is a measurable function on the measurable set $E$.

*Show:*

1. $f^{-1}([c, d)) = \{x \in E \mid c \leq f(x) < d\}$ and $f^{-1}((c, d]) = \{x \in E \mid c < f(x) \leq d\}$ are measurable sets.

   *Hint*:

$$f^{-1}([c, d)) = f^{-1}([c, \infty] \cap [-\infty, d))$$
$$= f^{-1}([c, \infty]) \cap f^{-1}([-\infty, d)).$$

   At this time the reader should look at how Lebesgue partitioned the "range" of $f$ for his integral (Chapter 1 or [Ma]).

2. $f^{-1}((c, d)) = \{x \in E \mid c < f(x) < d\}$ is measurable.

   *Hint*:

$$f^{-1}((c, d)) = f^{-1}((c, \infty] \cap [-\infty, d))$$
$$= f^{-1}((c, \infty]) \cap f^{-1}([-\infty, d)).$$

3. $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are measurable sets.

   *Hint*:
$$f^{-1}(\{\infty\}) = \{x \in E \mid f(x) = \infty\} = \bigcap_k \{x \in E \mid f(x) > k\}.$$

4. $f^{-1}(\{c\}) = \{x \in E \mid f(x) = c\}$ is measurable. In particular, the zero's of a measurable function form a measurable set.

*Hint*:

$\{x \in E \mid f(x) = c\} = \{x \in E \mid f(x) \geq c\} \cap \{x \in E \mid f(x) \leq c\}$.

5. Let $G$ be any open set in $R$. Then $f^{-1}(G)$ is a measurable subset of $E$.

   *Hint*:

   $G = \cup I_k$, $I_k$ disjoint open intervals and $f^{-1}(G) = f^{-1}(\cup I_k) = \cup f^{-1}(I_k)$.

Again, continuous functions deal with topological properties of inverse images whereas measurable functions require measurability of inverse images. Different measures could presumably yield different measurable functions. We give two examples (neither necessary for later material) to illustrate that, in general, we must be very careful about the specific measure being discussed.

**Example 1:** Define an outer measure $\alpha^*$ on subsets $A$ of $R$ by

$$\alpha^*(A) = \begin{cases} \text{number of elements in } A \text{ if } A \text{ is finite} \\ \infty, \text{ if } A \text{ is infinite}. \end{cases}$$

Using Carathéodory's Measurability Criteria (3.3.1) the reader may show that every subset of $R$ is measurable. Then, for any extended real-valued function $f$, the set $\{x \mid f(x) > c\}$ is a subset of $R$ and thus measurable. That is, every extended real-valued function is measurable, relative to this specific measure.

**Example 2:** Define an outer measure $\beta^*$ on any set $B$ of real numbers by:

$$\beta^*(B) = \begin{cases} 0, & B = \emptyset \\ 1, & B \neq \emptyset. \end{cases}$$

The reader may show that $\emptyset$, $R$ are the only measurable subsets of $R$. Let $f(x) = x, x \in R$. Then $\{x \mid f(x) > 0\} = (0, \infty)$ is not $\emptyset$ or $R$, that is, the identity function is not measurable. In fact, the reader may show that only constant functions are measurable relative to this measure.

Hopefully Lebesgue measure will give us something between these two extremes, that is, enough interesting functions to build a "useful" theory of integration.

*Caution: In what follows a "measurable set" means a "Lebesgue measurable set of real numbers", a "measurable function" means a*

*"Lebesgue measurable function". For any function, the domain will always be a subset of R and the range will be a subset of R or $R^e$ (real-valued or extended real-valued).*

Using Definition 4.1.1 to show a function is measurable is illustrated by the next example and problem.

**Example 3:**

1. $f$ is constant on a measurable set $E$, say, $f(x) = k$, $x \in E$. Then

$$\{x \mid f(x) > c\} = \begin{cases} \emptyset, & c \geq k \\ E, & c < k\,. \end{cases}$$

The sets $\emptyset$ and $E$ are measurable. Thus $f$ is a measurable function on $E$.

2. $f(x) = \begin{cases} x^2\,, & x < 1 \\ 2\,, & x = 1 \\ 2 - x, & x > 1\,. \end{cases}$

   i. $c < 0$ $\{x \mid f(x) > c\} = (-\infty, 2 - c)$; open intervals are measurable, and $\mu((-\infty, 2 - c)) = \infty$.
   ii. $0 \leq c < 1$ $\{x \mid f(x) > c\} = (-\infty, -\sqrt{c}) \cup (\sqrt{c}, 2 - c)$.
   iii. $1 \leq c < 2$ $\{x \mid f(x) > c\} = (-\infty, -\sqrt{c}) \cup \{1\}$.
   iv. $2 \leq c$ $\{x \mid f(x) > c\} = (-\infty, -\sqrt{c})$. $f$ is a measurable function.

### 4.1.3 Problem

Please show the following functions are measurable on their respective domains and calculate $\mu(f^{-1}((c, \infty]))$.

1. $f(x) = \dfrac{1}{x}, \qquad 0 < x < 1.$

   *Hint*: $c < 0$, $0 \leq c < 1$, $1 \leq c$.

2. $f(x) = \begin{cases} \dfrac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0\,. \end{cases}$

3. $f(x) = \begin{cases} \tan(x), & \dfrac{-\pi}{2} < x < \dfrac{\pi}{2} \\ \infty, & x = \pm\dfrac{\pi}{2}\,. \end{cases}$

4. $f(x) = \begin{cases} 1, & x \text{ rational number in } [0,1] \\ 0, & x \text{ irrational number in } [0,1]. \end{cases}$

This function is continuous nowhere but is a measurable function.

Having labored to show that specific functions are measurable via Definition 4.1.1, we eagerly seek more general results. For example, if we knew that continuous almost everywhere functions were measurable, Example 3 becomes trivial.

That continuous functions are measurable should not be a surprise. After all, Lebesgue was trying to generalize Riemann's integral, which easily handles continuous functions (Problem 5.1.9).

**PROPOSITION 4.2** *Continuous functions defined on measurable sets are measurable functions.*

*Proof:* Let $f$ be a continuous function on the measurable set $E$, and $c$ any real number. We must show $A = \{x \in E \mid f(x) > c\}$ is a measurable subset of $E$. If $A = \emptyset$, the proof is immediate since the empty set is measurable. Otherwise, let $x \in A$. Then $f(x) > c$, and because $f$ is continuous at $x$, we have $\delta(x) > 0$ so that for $z \in (x - \delta(x), x + \delta(x)) \cap E$, $f(z) > c$. Roughly speaking, given any point in $A$, we have an interval centered at that point also in $A$. That is,

$$A = \bigcup_{x \in A} ((x - \delta(x), x + \delta(x)) \cap E)$$

$$= \left( \bigcup_{x \in A} (x - \delta(x), x + \delta(x)) \right) \cap E.$$

In other words, $A$ is the intersection of an open set (measurable) and the measurable set $E$. The set $A$ is measurable and the argument is complete. ∎

We can weaken continuity to continuity except on a set of measure zero, commonly referred to as "continuous almost everywhere." Sets of measure zero do not affect measurability of a function.

### 4.1.4 Definition

A property is said to hold *almost everywhere* on a measurable set if the set of points where it fails to hold has measure zero. In particular, two functions $f$ and $g$ are said to be equal *almost everywhere* if they have the same domain and $\mu(\{x \mid f(x) \neq g(x)\}) = 0$. We sometimes write $f = g$ a.e. on $E$.

**PROPOSITION 4.3** *Suppose $f$ and $g$ are extended real-valued functions defined on a measurable set $E$. If $f$ is a measurable function on $E$ and if $g = f$ except on a set of measure zero, then $g$ is a measurable function on $E$.*

*Proof:* Let $c$ be any real number. We must show $\{x \in E \mid g(x) > c\}$ is a measurable subset of $E$. Define $A = \{x \in E \mid f \neq g\}$. By assumption, $A$ is measurable with measure zero. Then $g = f$ on the measurable set $E - A$, and

$$
\begin{aligned}
\{x \in E \mid g(x) > c\} &= \{x \in E - A \mid g(x) > c\} \cup \{x \in A \mid g(x) > c\} \\
&= \{x \in E - A \mid f(x) > c\} \cup \{x \in A \mid g(x) > c\} \\
&= (\{x \in E \mid f(x) > c\} \cap (E - A)) \\
&\qquad \cup \{x \in A \mid g(x) > c\}.
\end{aligned}
$$

The set $\{x \in A \mid g(x) > c\}$ is measurable since it is a subset of a set of measure zero (Problem 3.5.1). Because $f$ is a measurable function on $E$, $\{x \in E \mid f(x) > c\}$ is a measurable subset of $E$, as is $E - A$. ■

### 4.1.5 Problem

Show that if $f$ is a measurable function on a measurable set $E$ and if $A$ is any measurable subset of $E$, then $f$ is a measurable function on $A$.

*Hint*: $\{x \in A \mid f(x) > c\} = \{x \in E \mid f(x) > c\} \cap A$.

**PROPOSITION 4.4** *Every Riemann integrable function defined on $[a, b]$ is a measurable function on $[a, b]$.*

*Proof:* The reader may recall (Theorem 5.1) that a bounded function $f$ on $[a, b]$ is Riemann integrable iff the set $D$ of its discontinuities has measure zero. Then $f$ is continuous on $[a, b] - D$, hence measurable on $[a, b] - D$. Define $g$ to be $f$ on $[a, b] - D$ and zero on $D$. Then $g$ is a measurable function on $[a,b]$ ( $\{x \in [a,b] \mid g(x) > c\} = \{x \in [a,b] - D \mid f(x) > c\} \cup \{x \in D \mid g(x) > c\}$ and $\{x \in D \mid g(x) > c\} = \emptyset$ if $c \geq 0$ or $D$ if $c < 0$ since $g = 0$ on $D$ ), and $f = g$ except on a set of measure zero. Thus $f$ is measurable by the previous proposition. ■

It is *not* true that measurable functions are Riemann integrable (Problem 4.1.3).

Not all functions are measurable functions, but, since nonmeasurable sets of real numbers are "few and far between," we would expect the same for nonmeasurable functions.

### 4.1.6 Problem

Let $N$ be a nonmeasurable subset of $[0,1]$ and define

$$f(x) = \begin{cases} 1, & x \in N \\ -1, & x \in [0,1] - N. \end{cases}$$

Show $f$ is a nonmeasurable function on $[0,1]$.

*Note*: $|f| = 1$ on $[0,1]$, $|f|$ is a measurable function, $f$ is not.

Most operations performed with measurable functions preserve measurability.

**PROPOSITION 4.5** *Suppose $f$ and $g$ are* **real-valued** *measurable functions, defined on a measurable set $E$, and $k$ is any real number. Then the following functions are measurable functions on $E$:*

$$f + k,\ kf,\ |f|,\ f^2,\ \frac{1}{g}(g \neq 0 \text{ on } E),\ f + g,\ f \cdot g,\ \frac{f}{g}(g \neq 0 \text{ on } E).$$

*Proof*: We sketch the arguments:

i. $\{x \in E \mid f(x) + k > c\} = \{x \in E \mid f(x) > c - k\}$.

ii. $k = 0$, then $kf = 0$ and $\{x \in E \mid kf(x) > c\} = \begin{cases} \emptyset, & c \geq 0 \\ E, & c < 0. \end{cases}$

If $k > 0$, $\{x \in E \mid kf(x) > c\} = \{x \in E \mid f(x) > c/k\}$. Use similar reasoning if $k < 0$.

iii. $\{x \in E \mid \; |f(x)| > c\} = \begin{cases} E, & c < 0 \\ \{x \in E \mid f(x) > c\} \cup \{x \in E \mid f(x) < -c\}, & c \geq 0. \end{cases}$

iv. $\{x \in E | f^2(x) > c\} = \begin{cases} E, & c < 0 \\ \{x | \; |f(x)| > \sqrt{c}\}, & c \geq 0. \end{cases}$

v. $\{x \in E | 1/g(x) > c\} = \begin{cases} \{x \in E | g(x) > 0\}, & c = 0 \\ \{x \in E | g(x) > 0\} \cap \{x \in E | g(x) < 1/c\}, & c > 0 \\ \{x \in E | g(x) > 0\} \cup \{x \in E | g(x) < 1/c\}, & c < 0. \end{cases}$

vi. $\{x | f(x) < g(x)\} = \bigcup_{r_k} (\{x \in E | f(x) < r_k\} \cap \{x \in E | r_k < g(x)\}),$

$r_k$ rational. The denseness and countability of the rationals is convenient isn't it. Then $\{x \in E \mid f(x) + g(x) > c\} = \{x \in E \mid c - g(x) < f(x)\}$.

vii. $f \cdot g = 1/4[(f+g)^2 - (f-g)^2]$ and $f/g = f \cdot (1/g)$. ■

The reader has noticed that we restricted our attention to real-valued functions. Why? Recall that in $R^e$ we did not define $\infty + (-\infty)$ or $(\infty) \cdot (-\infty)$ or division by $\infty$ for example, and this could certainly arise with $f + g$ or $f \cdot g$ or $f/g$. How about the functions $\max(f, g)$, $\min(f, g)$?

**Proposition 4.6** *Suppose $f$ and $g$ are measurable functions defined on a measurable set $E$. Then the following functions are measurable on $E$:*

*1.* $\max(f, g)$, $\min(f, g)$.
*2.* $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$, $|f|$.

*Note:* $f = f^+ - f^-$, $|f| = f^+ + f^-$.

*Proof:*

1. $\{x \in E \mid \max(f(x), g(x)) > c\} = \{x \in E \mid f(x) > c\} \cup \{x \in E \mid g(x) > c\}$ and $\{x \in E \mid \min(f(x), g(x)) > c\} = \{x \in E \mid f(x) > c\} \cap \{x \in E \mid g(x) > c\}$.

2. $\{x \in E \mid \; |f(x)| > c\} = \begin{cases} \{x \in E | f(x) > c\} \cup \{x \in E | f(x) < -c\}, & c \geq 0 \\ E, & c < 0 \end{cases}$

### 4.1.7 Problem

Calculate $f^+, f^-, f^+ - f^-, f^+ + f^-$:

1. $f(x) = \begin{cases} \infty, & x = -\pi/2 \\ \tan(x), & -\pi/2 < x < \pi/2 \\ \infty, & x = \pi/2\,. \end{cases}$

2. $f(x) = \begin{cases} x^2 - 1, & x < 1 \\ 3 - x, & x \geq 1\,. \end{cases}$

3. $f(x) = \sin(x)$.

If we replace measurability of $f$ and $g$ in Proposition 4.5 with continuity, we still have a valid proposition. The operations performed with measurable functions in Proposition 4.5 in no way distinguish measurable functions from continuous functions. It is in limiting operations on sequences of measurable functions that fundamental differences begin to appear.

## 4.2 SEQUENCES OF MEASURABLE FUNCTIONS

Pointwise limits of sequences of continuous functions need not be continuous ($f_k(x) = x^k,\ 0 \leq x \leq 1$ ). In fact, pointwise limits of sequences of Riemann integrable functions need not be Riemann integrable (Chapter 1, Example 6).

For measurable functions, pointwise limits preserve measurability and especially integrability (with some relatively mild additional conditions imposed on the sequence): We will have $\lim \int f_k = \int (\lim f_k)$ under much weaker hypotheses with the Lebesgue integral than the Riemann integral. The next theorem is crucial.

**THEOREM 4.1** *Suppose $(f_k)$ is a sequence of measurable functions defined on a measurable set $E$. Then the following functions are measurable functions on $E$:*

1. *$\bar{f}_k = \sup\{f_k, f_{k+1}, \ldots\}$ and $\underline{f}_k = \inf\{f_k, f_{k+1}, \ldots\}$ for $k = 1, 2, \ldots$ ;*
2. *$\limsup f_k = \lim \bar{f}_k$ and $\liminf f_k = \lim \underline{f}_k$.*
3. *Most importantly, if $\lim f_k$ (finite or infinite) exists for every point of $E$, then the limit function, $(\lim f_k)$, is a measurable function on $E$.*

4. *If $f$ is a function defined on $E$ and $f = \lim f_k$ almost everywhere on $E$, then $f$ is a measurable function on $E$.*

*Proof:* Parts 1 and 2. $\{x \in E \mid \bar{f}_k(x) > c\} = \bigcup_{n=k}^{\infty} \{x \in E \mid f_n(x) > c\}$. Similarly for $\underline{f}_k$.

The sequence $(\bar{f}_k)$ is a nonincreasing sequence of measurable functions and since $\limsup f_k = \lim \bar{f}_k = \inf\{\bar{f}_1, \bar{f}_2, \ldots\}$, measurability follows from part 1. An analogous argument yields measurability for $\liminf f_k$.

3. $\liminf f_k = \lim f_k = \limsup f_k$.
4. Now suppose a function $f$ on $E$ is the almost everywhere limit of $(f_k)$ and let $A = \{x \in E \mid \lim f_k(x) \text{ not defined or } \lim f_k(x) \neq f(x)\}$. The set $A$ has measure zero. Define a new sequence of functions $(g_k)$ on $E$ by

$$g_k(x) = \begin{cases} f_k(x), & x \notin A \\ 0, & x \in A, \end{cases}$$

and let $g$ be given by

$$g(x) = \begin{cases} f(x), & x \notin A \\ 0, & x \in A. \end{cases}$$

Since each function $g_k$ equals a measurable function, $f_k$, almost everywhere on $E$, $g_k$ is measurable (Proposition 4.3). If $x \in A$, $\lim g_k(x) = 0 = g(x)$. If $x \notin A$, $\lim g_k(x) = \lim f_k(x) = f(x) = g(x)$, that is, $\lim g_k = g$ on $E$. By part 3 $g$ is measurable on $E$ and then by applying Proposition 4.3 again, $f$ is measurable on $E$ since it is equal to $g$ almost everywhere on $E$. ■

### 4.2.1 Problem

Calculate $\underline{f}_k$, $\bar{f}_k$, $\limsup f_k$, $\liminf f_k$, and $\lim f_k$ where appropriate, for the given sequences of functions:

1. $f_k(x) = x^k, \qquad 0 \leq x \leq 1.$

2. $f_{2k}(x) = \left(1 + \frac{x}{2k}\right)^{2k}, \quad f_{2k-1}(x) = \left(1 - \frac{x}{2k-1}\right)^{2k-1}, \qquad 0 \leq x < \infty.$

3. $f_{2k}(x) = x - x^{2k}, \qquad f_{2k-1}(x) = x^{2k-1}, \qquad 0 \le x \le 1.$

4. Let $r_1, r_2, \ldots$ be any enumeration of the rational numbers, and let

$$f_k(x) = \begin{cases} 1, & x = r_1, r_2, \ldots, r_k \\ 0, & \text{otherwise}. \end{cases}$$

5. $f_k(x) = \begin{cases} 1, & x = i/j,\ i+j=k,\ 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$

## 4.3 APPROXIMATING MEASURABLE FUNCTIONS

We show that every measurable function is the limit of a sequence of simple functions. In the context of "proof" by "easy $\longrightarrow$ hard", that is, characteristic function $\longrightarrow$ simple functions $\longrightarrow$ measurable functions, Approximation Theorem 4.2 is fundamental. But first, some definitions and problems that will prove useful.

### 4.3.1 Definitions

Let $A$ be any set of real numbers. The *characteristic function* on $A$, denoted by $\mathcal{X}_A$, is defined as follows:

$$\mathcal{X}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Suppose

$$E = \bigcup_1^n E_k,$$

where the sets $E_k$ are measurable, mutually disjoint, subsets of $R$, and $c_1, c_2, \ldots, c_n$ are real numbers. Then a function $\varphi$ defined on $E$ by

$$\varphi(x) = \sum_{k=1}^n c_k \mathcal{X}_{E_k}(x),$$

is called a *simple* function. A simple function assumes a finite number of real values and assumes each of these on a measurable set: $\varphi(x) = c_k$ on $E_k$, $1 \leq k \leq n$.

A simple function is "simply" a linear combination of characteristic functions, and certainly any step function (5.1.1) is a simple function.

### 4.3.2 Problems

Determine if the following functions are "simple" on $[0, 1]$:

1. $\varphi(x) = \begin{cases} 1, & x = 1/n,\ n = 1, 2, \dots \\ -1, & x \neq 1/n. \end{cases}$

2. $\varphi(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational}. \end{cases}$

3. $\varphi(x) = [x]$ (the greatest integer function).
4. Cantor function (Appendix A).

### 4.3.3 Problems

1. If $E$ is a measurable set, then the characteristic function on $E$, $\mathcal{X}_E$, is a measurable function on $E$.
2. If $\varphi$ is a simple function on a measurable set $E$, then $\varphi$ is a measurable function on $E$.

**Approximation Theorem 4.2** *Let $f$ be a measurable function defined on a measurable set $E$. Then there exists a sequence of simple functions $(\varphi_k)$ on $E$, so that*

$$\lim \varphi_k = f \qquad (\textit{finite or infinite})$$

*for all $x \in E$. If $f$ is bounded on $E$, then*

$$\lim \varphi_k = f \qquad (\textit{unif})$$

*on $E$. If $f$ is nonnegative, the sequence $(\varphi_k)$ may be constructed so that it is a monotonically increasing sequence.*

*Proof:* Suppose that $f$ is nonnegative on $E$. We will construct a monotonically increasing sequence $(\varphi_k)$ with $\lim \varphi_k = f$. The idea is essentially Lebesgue's: Divide the range of $f$ and approximate by level curves. Since $f(E) \subset [0, \infty]$, we partition $[0, \infty]$:

*Step 1.* $[0, \infty] = [0, 1) \cup [1, \infty] = [0, 1/2) \cup [1/2, 1) \cup [1, \infty]$. Define $E_{11} = f^{-1}([0, 1/2))$, $E_{12} = f^{-1}([1/2, 1))$, $E_1 = f^{-1}([1, \infty])$, and $\varphi_1 = 0 \cdot \mathcal{X}_{E_{11}} + 1/2 \cdot \mathcal{X}_{E_{12}} + 1 \cdot \mathcal{X}_{E_1}$. Clearly $\varphi_1 \leq f$ on $E$.

*Step 2.* $[0, \infty] = [0, 1) \cup [1, 2) \cup [2, \infty] = [0, 1/4) \cup [1/4, 1/2) \cup [1/2, 3/4) \cup [3/4, 1) \cup [1, 5/4) \cup [5/4, 6/4) \cup [6/4, 7/4) \cup [7/4, 8/4) \cup [2, \infty]$. We have decomposed $[0, \infty]$ into $2^2 + 2^2 + 1$ subintervals at the $2^{\text{nd}}$ step. Form inverse images:

$$E_{21} = f^{-1}\left([0, \frac{1}{4})\right), \qquad E_{22} = f^{-1}\left([\frac{1}{4}, \frac{1}{2})\right), \ldots,$$

$$E_{28} = f^{-1}\left([\frac{7}{4}, \frac{8}{4})\right), \qquad E_2 = f^{-1}([2, \infty]).$$

Define

$$\varphi_2 = 0 \cdot \mathcal{X}_{E_{21}} + \frac{1}{4} \cdot \mathcal{X}_{E_{22}} + \cdots + \frac{7}{4} \cdot \mathcal{X}_{E_{28}} + 2\mathcal{X}_{E_2}, \text{ or}$$

$$\varphi_2 = \sum_{i=1}^{2 \cdot 2^2} \frac{i-1}{2^2} \cdot \mathcal{X}_{E_{2i}} + 2\mathcal{X}_{E_2}.$$

*Note*: $E_{1i} = E_{2\,2i-1} \cup E_{2\,2i}$ for $i = 1, 2$.

$$\vdots$$

*Step k.* $[0, \infty] = [0, 1) \cup [1, 2) \cup \cdots \cup [k-1, k) \cup [k, \infty]$ and partition into $2^k + 2^k + \cdot + 2^k + 1$ subintervals $= k \cdot 2^k + 1$ disjoint subintervals and form inverse images. Thus,

$$\varphi_k = \sum_{i=1}^{k \cdot 2^k} \frac{i-1}{2^k} \, \mathcal{X}_{E_{ki}} + k\mathcal{X}_{E_k}.$$

Note that $E_{ki} = E_{k+1\ 2i-1} \cup E_{k+1\ 2i}$. To construct $\varphi_{k+1}$, divide the intervals $[(i-1)/2^k, i/2^k)$ in half, and then $\varphi_k$ to $\varphi_{k+1}$ at those $x$'s where $\varphi_k$ changes.

Certainly the $\varphi_k$ are nonnegative simple functions. We must show $\varphi_k \le \varphi_{k+1}$ and $\lim \varphi_k = f$ on $E$.

If $f(x_0) = \infty$, then $\varphi_k(x_0) = k \; \forall k$ and $\lim \varphi_k(x_0) = \infty$. If $f(x_0) < \infty$, then for $k > f(x_0)$, $0 \le f(x_0) - \varphi_k(x_0) < 1/2^k$ and $\lim \varphi_k(x_0) = f(x_0)$. All that's left is monotonicity. Recalling that $E_{ki} = E_{k+1\,2i-1} \cup E_{k+1\,2i}$, if $x_0 \in E_{ki}$ for some $i$, then $\varphi_k(x_0) = (i-1)/2^k$ and $\varphi_{k+1}(x_0) = (2i-2)/2^{k+1} = (i-1)/2^k$ or $(2i-1)/2^{k+1}$, and $\varphi_k(x_0) \le \varphi_{k+1}(x_0)$. If $x_0 \notin E_{ki}$, $i = 1, 2, \ldots, k \cdot 2^k$, then $x_0 \in E_k = f^{-1}([k, \infty]) = f^{-1}([k, k+1)) \cup f^{-1}([k+1, \infty])$. So $x_0$ is either in $f^{-1}([k, k+1))$, in which case $\varphi_k(x_0) = k$ and $\varphi_{k+1}(x_0) = j/2^{k+1} > (2k \cdot 2^k)/2^{k+1} = k = \varphi_k(x_0)$, or $x_0 \in f^{-1}([k+1, \infty])$, and then $\varphi_{k+1}(x_0) = k+1 > k = \varphi_k(x_0)$. We have shown that for $f \ge 0$,

$$0 \le \varphi_1 \le \cdots \le \varphi_k \le \varphi_{k+1} \le \cdots$$

and $\lim \varphi_k = f$ on $E$.

If $f$ is nonnegative and bounded on $E$, say, $0 \le f \le M$ on $E$, then for all $k > M$, $0 \le f(x) - \varphi_k(x) < 1/2^k$ for all $x \in E$, that is, $\lim \varphi_k = f\,(unif)$ on $E$.

In the general case ($f$ may be negative), recall that $f = f^+ - f^-$ where $f^+$, $f^-$ are nonnegative measurable functions on $E$ (Proposition 4.6). Apply the above arguments to $f^+$ and $f^-$, noting that the difference of simple functions is again a simple function. This completes the proof. ∎

### 4.3.4 Problems

Determine $\varphi_1, \varphi_2, \varphi_3$ for the following functions:

1. $f(x) = x^2, \qquad 0 \le x \le 1.$

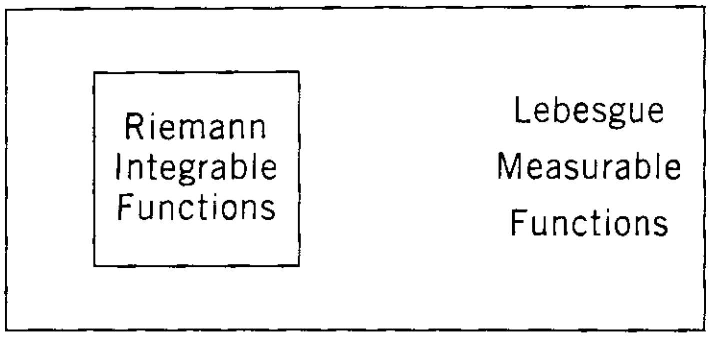
2. $f(x) = \begin{cases} \dfrac{1}{x}, & 0 < x \le 1 \\ \infty, & x = 0\,. \end{cases}$

3. Cantor function (Appendix A).

4. $f(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational}, \end{cases} \qquad 0 \le x \le 1.$

### 4.3.5 Summary

1.




2. If $(f_k)$ are measurable on $E$ and if $f$ is defined on $E$ with $f = \lim f_k$ almost everywhere on $E$, then $f$ is measurable on $E$.
3. Every nonnegative measurable function on $E$ is the limit of a nondecreasing sequence of simple functions on $E$.

## 4.4 ALMOST UNIFORM CONVERGENCE

In this section we prove a remarkable theorem due to Egoroff: If we have pointwise convergence of a sequence of measurable functions on a set of finite measure, then we have uniform convergence on a "large" subset of that set. An example to illustrate this result is useful.

**Example 4:** $f_k(x) = x^k$, $0 \le x \le 1$ and $f(x) = 0$, $0 \le x \le 1$. The convergence is not uniform on $[0, 1]$

$$\left( \sup_{0 \le x \le 1} | f_k(x) - f(x) | = 1 \right),$$

but is uniform on $[0, 1 - \epsilon]$, $0 < \epsilon < 1$, since

$$\sup_{0 \le x \le 1-\epsilon} | f_k(x) - f(x) | = (1 - \epsilon)^k.$$

**Theorem 4.2** (Egoroff, 1911) *Let $E$ be a measurable set of real numbers with finite measure. If $(f_k)$ is a sequence of measurable functions which converge to a real-valued function $f$ almost everywhere on $E$, then, given $\epsilon > 0$, there exists a measurable subset $E_\epsilon$ of $E$ such that $\mu(E - E_\epsilon) < \epsilon$ and the sequence $(f_k)$ converges uniformly to $f$ on $E_\epsilon$.*

*Proof:* Using the sequence $(f_k)$, we will construct a monotonically decreasing sequence of nonnegative measurable functions, $(g_k)$. This new sequence will be shown to converge uniformly to zero on $E_\epsilon$, from which it will immediately follow that $\lim f_k = f(unif)$ on $E_\epsilon$.

Let $A$ denote the subset of $E$ where $f$ is not the limit of the sequence $(f_k)$. Because $A$ has measure zero, $\mu(E) = \mu(E - A)$, and $\lim f_k = f$ on $E - A$. Since the limit of a sequence of measurable functions is measurable (Theorem 4.1), $f$ is a measurable function on $E - A$. Defining $g_k = \sup\{|f_k - f|, |f_{k+1} - f|, \dots\}$ for $k = 1, 2, \dots$, we have from Propositions 4.5, 4.6, and Theorem 4.1 again, that $g_k$ is measurable on $E - A$. Furthermore, $0 \leq g_{k+1} \leq g_k$ and $\lim g_k = 0$ on $E - A$. In other words, the sequence $(g_k)$ is a monotone decreasing sequence of nonnegative measurable functions converging to zero on $E - A$. We will show that, except for a "small" subset of $E - A$, this convergence is uniform. That the original sequence $(f_k)$ would converge uniformly to $f$ on this same set is apparent.

The technical aspect of the argument begins: Let $\epsilon > 0$ be given.

*Stage 1*: We first construct an increasing sequence of measurable subsets of $E - A$ that "fill" $E - A$: $E_k^1 \equiv \{x \in E - A \mid g_k(x) < 1\}$. Clearly $E_1^1 \subset E_2^1 \subset \cdots$, $\cup E_k^1 = E - A$, $E_k^1$ measurable (why?). It follows from Theorem 3.4 that $\lim \mu(E_k^1) = \mu(E - A)$, so for $k$ sufficiently large, say $K_1$, $0 \leq \mu(E - A) - \mu(E_{K_1}^1) < \epsilon/2$. The set $E_{K_1}^1$ is "almost" $E - A$ and on this set, $0 \leq g_k < 1$ for all $k \geq K_1$. But why the sets $E_k^1$? Everything certainly "works as advertised", but ... ?

Pick $u \in E - A$. Since $\lim g_k(u) = 0$, we have for $k$ sufficiently large, say, $k(u)$, that $0 \leq g_k(u) < 1$ for all $k \geq k(u)$.

Pick $v \in E - A$. Since $\lim g_k(v) = 0$, we have for $k$ sufficiently large, say, $k(v)$, that $0 \leq g_k(v) < 1$ for all $k \geq k(v)$.

$\vdots$

Pick $w \in E - A$. Since $\lim g_k(w) = 0$, we have for $k$ sufficiently large, say, $k(w)$, that $0 \leq g_k(w) < 1$ for all $k \geq k(w)$.

$\vdots$

Think about what we are doing: With each point of $E - A$ we are associating a natural number, looking for a "magical" natural number that "works" for all points of $E - A$. (*Idea: How about with each natural number associating a set of points, a subset of $E - A$.*) For example, with the number one, associate the points of $E - A$ such that $g_1 < 1$. With the number two, associate the points of $E - A$ such that $g_2 < 1$. Maybe in this way we can find a "not so magical" natural number that "works" for not all, but "most" of $E - A$. By now you realize that the subset of $E - A$ associated with the number one is $E_1^1$, the subset of $E - A$ associated with the number two is $E_2^1, \ldots$, the subset of $E - A$ associated with $k$ is $E_k^1$, and the "so-called" "not so magical" natural number $K_1$ "works" on "most" of $E - A$: $E_{K_1}^1$. So, after all this, $0 \le g_k < 1$ for all $k \ge K_1$ on $E_{K_1}^1$.

*Stage 2*: Now things become quick and straightforward (thankfully). Understanding Stage 1, we define another sequence of measurable sets in the following manner: $E_k^2 \equiv \{x \in E - A \mid g_k(x) < 1/2\}$. Again, $E_1^2 \subset E_2^2 \subset \cdots$, $\cup E_k^2 = E - A$, $\lim \mu(E_k^2) = \mu(E - A)$, and thus for $k$ sufficiently large, say $K_2$, $0 \le \mu(E - A) - \mu(E_{K_2}^2) < \epsilon/2^2$. Also, $0 \le g_k < 1/2$ for $k \ge K_2$ on $E_{K_2}^2$.

$\vdots$

*Stage n*: We have measurable sets $E_1^n \subset E_2^n \subset \cdots$,

$$\bigcup_k E_k^n = E - A, \ \lim_k \mu(E_k^n) = \mu(E - A),$$

and for $k$ sufficiently large, say $K_n$, $0 \le \mu(E - A) - \mu(E_{K_n}^n) < \epsilon/2^n$ with $0 \le g_k < 1/n$ for $k \ge K_n$ on $E_{K_n}^n$.

Each of the sets $E_{K_1}^1, E_{K_2}^2, \ldots, E_{K_n}^n, \ldots$ is "almost" $E - A$. We will show that

$$E_\epsilon = \bigcap_n E_{K_n}^n$$

is "almost" $E - A$ and that we have uniform convergence on $E_\epsilon$.

$$\begin{aligned}
\mu\Big( (E - A) - \bigcap E_{K_n}^n \Big) &= \mu\Big(\bigcup \big( (E - A) - E_{K_n}^n \big) \Big) \\
&\le \sum \mu\big( (E - A) - E_{K_n}^n \big) \\
&= \sum \big( \mu(E - A) - \mu(E_{K_n}^n) \big) \\
&< \sum \frac{\epsilon}{2^n} = \epsilon .
\end{aligned}$$

Only uniform convergence on

$$E_\epsilon = \bigcap_n E^n_{K_n}$$

remains. Let $\delta > 0$ be given. We show $0 \le g_k < \delta$ on $\bigcap E^n_{K_n}$ for $k$ sufficiently large. Choose $N$ so that $1/N < \delta$. Recall $E^N_{K_N} = \{x \in E - A \mid g_{K_N} < 1/N\}$. But $E^N_{K_N} \supset \bigcap E^n_{K_n}$ and $g_k \le g_{K_N}$ for all $k \ge K_N$, that is, $0 \le g_k < \delta$ for all $k \ge K_N$ and all

$$x \in \bigcap_n E^n_{K_n}.$$

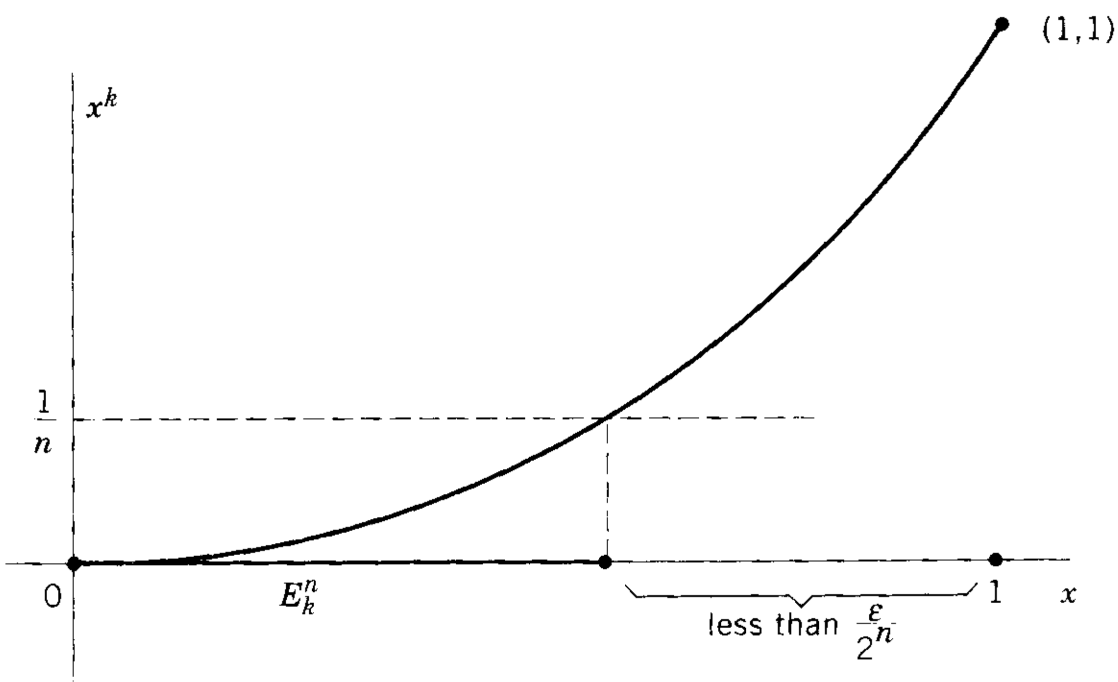
The proof is complete. ∎

By the way, where in the above argument did we use the assumption $\mu(E) < \infty$? The example, $f_k = \mathcal{X}_{[k,k+1]}$ with $f = 0$ shows $\mu(E) < \infty$ is necessary.

#### 4.4.1 Problem

Using Example 4, the reader should repeat the argument of Egoroff's Theorem with $f_k(x) = x^k$, $0 \le x \le 1$ and $f(x) = 0$, $0 \le x \le 1$: $g_k(x) = x^k$, $0 \le x \le 1$ and $\lim g_k = 0$ on $[0, 1)$.

i. Show $E^1_k = [0, 1)$ for all $k$ and $E^1_{K_1} = E^1_1$.
ii. Show $E^2_k = [0, 1/2^{(1/k)})$ and $K_2 \ge 277$ if, for example, $\epsilon = .01$.
iii. Show $E^n_k = [0, 1/n^{(1/k)})$ and $(1 - \epsilon/2^n)^{K_n} < 1/n$.



In 1912 Lusin showed that measurable functions are "almost" continuous. We use Egoroff's Theorem to establish this result.

**THEOREM 4.3** (Lusin, 1912) *If $f$ is a real-valued measurable function, defined on a set $E$ of finite measure, then we may construct a closed subset $E_\epsilon$ of $E$ so that $\mu(E - E_\epsilon) < \epsilon$ and $f$, restricted to $E_\epsilon$, is continuous.*

*Proof:* The idea here is to approximate $f$ with a sequence of simple functions $(\phi_k)$ (Approximation Theorem 4.2), each being continuous except possibly at a finite set of points, and thus the set of discontinuities of all members, being a countable set, has measure zero. Cover this set of discontinuities, with a sequence of open intervals $(I_k)$ so that $\mu(\cup I_k) < \epsilon/2$. On the closed set $E - (\cup I_k)$, the simple functions are continuous and converge to $f$. Applying Egoroff, we have $E_\epsilon$ so that $E_\epsilon \subset E - (\cup I_k)$, $\mu((E - (\cup I_k)) - E_\epsilon) < \epsilon/2$, $\phi_k \longrightarrow f$ uniformly on $E_\epsilon$. Since the uniform limit of a sequence of continuous functions is continuous, the proof is complete. ■

### 4.4.2 Problem

$$f(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational}, \end{cases} \qquad 0 \leq x \leq 1.$$

Looking at the proof of Lusin's Theorem, what will the set $E_\epsilon$ be? How about $(\phi_k)$?

Even though measurable functions are "almost continuous," the reader should realize that "almost continuous" may in fact be nowhere continuous, as problem 4.1.3 indicates.

The life of the individual only has meaning in so far as it aids in making the life of every living thing nobler and beautiful.

—A. Einstein

. . . simplicity in mathematics, like simplicity of character, is an ideal to be achieved only by unremitting toil.

—G. Temple

Great spirits have always encountered violent opposition from mediocre minds.

—A. Einstein

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}}$$

—Ramanujan

# 5

# *Lebesgue Integration*

We introduce the Lebesgue integral, in some respects a generalization of the primitive notion of "area under a curve". This new integral is defined for a class of functions that contains the Riemann integrable functions and permits certain limiting operations to be handled much more easily. The development proceeds as follows:

Section 5.1 reviews the development of the Riemann integral via step functions. The same approach, via simple functions, yields the Lebesgue integral for bounded functions on Lebesgue measurable sets of finite measure. These efforts comprise Section 5.2. Of course, we want to remove the restrictions "bounded" and "finite measure." This is done in Sections 5.3 and 5.4. Finally, Section 5.5 is concerned with applications of $\lim \int f_n = \int \lim f_n$. It is the so called "convergence theorems," which demonstrate the power and usefulness of the Lebesgue integral, a *raison d'être.*

*Caution: Again, "measurable set" means "Lebesgue measurable set of real numbers," "measurable function" means "Lebesgue measurable function". All functions will have domains as subsets of $R$, with ranges in $R$ or $R^e$.*

## 5.1 THE RIEMANN INTEGRAL

We define the Riemann integral of a step function in the obvious way. The extension to more general bounded functions $f$ on $[a, b]$ is via approximation from above and below by step functions.

### 5.1.1 Definition

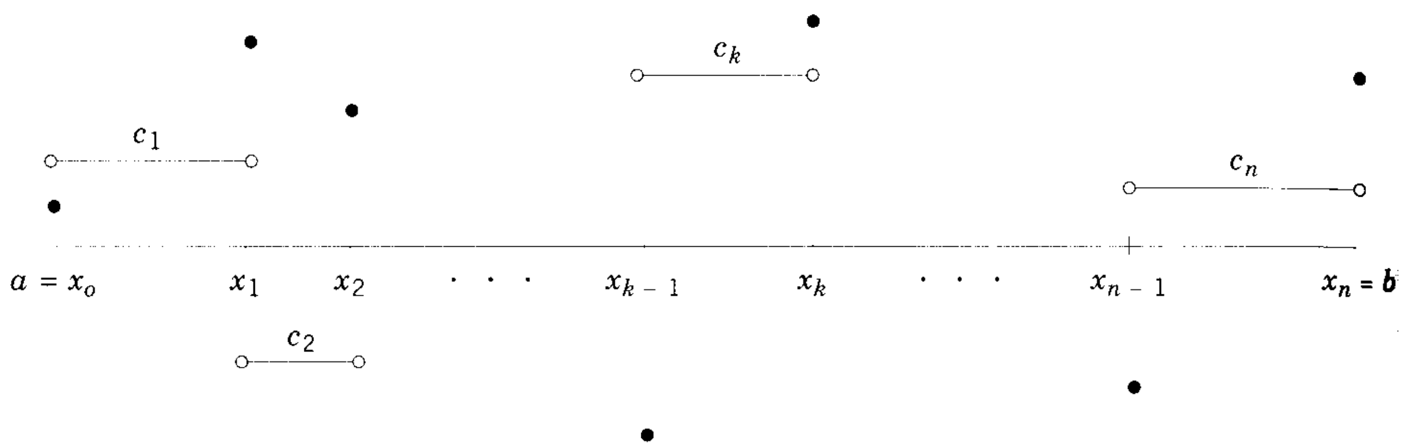
A real-valued function $\phi$ with domain $[a, b]$ is called a *step function* if there is a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of the interval such that $\phi$ is constant on each subinterval $I_k = (x_{k-1}, x_k)$; that is,

$$\phi(x) = c_k \text{ for } x \in I_k, \qquad k = 1, \ldots, n,$$

with $\phi(x_k) = d_k$, $k = 0, 1, \ldots, n$.



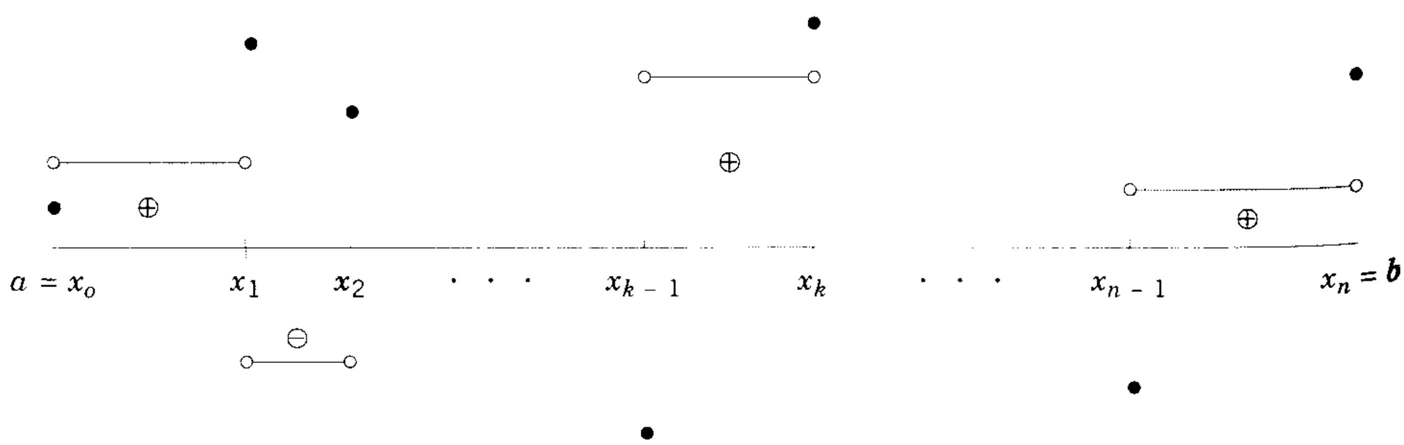

### 5.1.2 Definition

Let $\phi$ be a step function on $[a, b]$:

$$\phi(x) = \begin{cases} c_k, & x_{k-1} < x < x_k, \qquad k = 1, 2, \ldots, n \\ d_k, & k = 0, 1, \ldots, n, \qquad x = x_k. \end{cases}$$

The *Riemann integral* of $\phi$ on $[a, b]$, denoted by $\int_a^b \phi(x)\,dx$, is

$$\int_a^b \phi(x)\,dx = \sum_{k=1}^{n} c_k(x_k - x_{k-1}).$$



### 5.1.3 Comments

1. We could write $\phi = \sum_1^n c_k \mathcal{X}_{(x_{k-1}, x_k)} + \sum_0^n d_k \mathcal{X}_{\{x_k\}}$, and $\int_a^b \phi(x)\, dx = \sum_{k=1}^n c_k \mu((x_{k-1}, x_k)) + \sum_{k=0}^n d_k \mu(\{d_k\}) = \sum_{k=1}^n c_k (x_k - x_{k-1})$.

   The step function's values at the endpoints of the subintervals have no bearing on the existence or value of the Riemann integral of a step function ($d_k$ does not appear in the definition of the integral):

$$\phi_1(x) = \begin{cases} 1, & 0 \le x < 1 \\ 3, & x = 1 \\ 2, & 1 < x \le 2 \end{cases} \quad \text{and} \quad \phi_2(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2, & 1 < x \le 2\,. \end{cases}$$

   $\phi_1 = \phi_2$ for $0 < x < 1$ and $1 < x < 2$, $\phi_1(1) \ne \phi_2(1)$, but

$$\int_0^2 \phi_1(x)\, dx = 1 \cdot 1 + 2 \cdot 1 = \int_0^2 \phi_2(x)\, dx\,.$$

2. The value of the Riemann integral of a step function is independent of the choice of the partition of $[a, b]$ as long as the step function is constant on the open subintervals of the partition, for example,

$$\phi(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2, & 1 < x \le 2\,. \end{cases}$$

   Partition: $\{0, 1, 2\}$, $\int_0^2 \phi(x)\, dx = 1 \cdot 1 + 2 \cdot 1 = 3$.
   Partition:
   $\{0, 1/2, 1, 7/4, 2\}$, $\int_0^2 \phi(x)\, dx = 1 \cdot 1/2 + 1 \cdot 1/2 + 2 \cdot 3/4 + 2 \cdot 1/4 = 3$.

More formally, the Riemann integral of a step function is well defined; it is independent of the particular representation of $\phi$. For example, if $\phi(x) = c_k$, $x_{k-1} < x < x_k$ and we add another partition point $x^*$, $x_{k-1} < x^* < x_k$, we have

$$c_k(x_k - x_{k-1}) = c_k(x_k - x^* + x^* - x_{k-1}) = c_k(x_k - x^*) + c_k(x^* - x_{k-1})\,.$$

**Proposition 5.1** *If $\phi$ and $\psi$ are step functions on $[a, b]$, and $k$ is any real number, then*

1. *$(k\phi)$ is a step function on $[a, b]$, and $\int_a^b (k\phi)(x)\, dx = k \int_a^b \phi(x)\, dx$ (homogeneous);*

2. $(\phi + \psi)$ *is a step function on* $[a, b]$, *and*

$$\int_a^b (\phi + \psi)(x)\, dx = \int_a^b \phi(x)\, dx + \int_a^b \psi(x)\, dx \qquad (\textit{additive});$$

3. $\int_a^b \phi(x)\, dx \leq \int_a^b \psi(x)\, dx$ *if* $\phi \leq \psi$ *on* $[a, b]$ (*monotone*);
4. *If* $a < c < b$, *the integrals* $\int_a^c \phi(x)\, dx$, $\int_c^b \phi(x)\, dx$ *exist, and* $\int_a^c \phi(x)\, dx + \int_c^b \phi(x)\, dx = \int_a^b \phi(x)\, dx$ (*additive on the domain*).

*Proof:* The arguments are straightforward: $\phi$ and $\psi$ each have an associated partition. The union of these partitions will also be a partition of $[a, b]$ for $\phi$, $\psi$, $(k\phi)$, and $(\phi + \psi)$. For part 4, adjoin "$c$" if need be to the partition associated with $\phi$. The reader may provide "rigor." ∎

We now define the Riemann integral for more general functions $f$ on $[a, b]$. Since we will be approximating $f$ from above and below by step functions it is imperative that $f$ be a bounded function on $[a, b]$.

### 5.1.4 Definitions

Let $f$ be a bounded function on $[a, b]$, say, $\alpha \leq f(x) \leq \beta$, for $x \in [a, b]$. Let $\phi, \psi$ denote arbitrary step functions on $[a, b]$ such that $\phi \leq f \leq \psi$.

The *lower Riemann integral* of $f$ on $[a, b]$, $\underline{\int}_a^b f(x)\, dx$, is defined by

$$\underline{\int}_a^b f(x)\, dx = \sup\left\{ \int_a^b \phi(x)\, dx \mid \phi \leq f,\ \phi \text{ a step function} \right\}.$$

The *upper Riemann integral* of $f$ on $[a, b]$, $\overline{\int}_a^b f(x)\, dx$, is defined by

$$\overline{\int}_a^b f(x)\, dx = \inf\left\{ \int_a^b \psi(x)\, dx \mid f \leq \psi,\ \psi \text{ a step function} \right\}.$$

### 5.1.5 Comments

1. Since $\alpha \leq f$, the set $\{\int_a^b \phi(x)\, dx \mid \phi \leq f,\ \phi \text{ a step function}\}$ is not empty (it contains $\alpha$), and because $\phi \leq f \leq \beta$ implies

$$\int_a^b \phi(x)\, dx \leq \int_a^b \beta\, dx = \beta(b - a),$$

the set is bounded above. The least upper bound is a real number: The lower Riemann integral for a bounded function on a closed bounded interval is well-defined ("The" least upper bound, *not* "a" least upper bound). Similarly for the upper Riemann integral.

2. Since $\phi \le \psi$, $\int_a^b \phi(x)\,dx \le \int_a^b \psi(x)\,dx$ by monotonicity for step functions. Because $\phi$ is arbitrary, we may interpret this inequality as saying $\int_a^b \psi(x)\,dx$ is an upper bound for the set
$$\left\{\int_a^b \phi(x)\,dx \mid \phi \le f,\ \phi \text{ a step function}\right\}.$$
But $\underline{\int}_a^b f(x)\,dx$ is the smallest upper bound.

We have
$$\underline{\int}_a^b f(x)\,dx \le \int_a^b \psi(x)\,dx.$$

Again, interpret this last inequality as saying the lower Riemann integral, $\underline{\int}_a^b f(x)\,dx$, is a lower bound for the set
$$\left\{\int_a^b \psi(x)\,dx \mid f \le \psi, \qquad \psi \text{ a step function}\right\}.$$

Since the upper Riemann integral, $\overline{\int}_a^b f(x)\,dx$, is the greatest lower bound,
$$\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx.$$

It follows that a bounded function $f$ on $[a,b]$ satisfies
$$\int_a^b \phi(x)\,dx \le \underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx \le \int_a^b \psi(x)\,dx$$
for any step functions $\phi \le f \le \psi$ on $[a,b]$. Recall Problem 2.4.9.

We would hope that the approximations from "above" and "below" approach a common value, to be called the Riemann integral of $f$ on $[a,b]$.

### 5.1.6 Definition

A bounded function $f$ on $[a,b]$ is *Riemann integrable* on $[a,b]$ whenever $\underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx$. Denote the common value by $\int_a^b f(x)\,dx$; $\underline{\int}_a^b f(x)\,dx = \int_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx$.

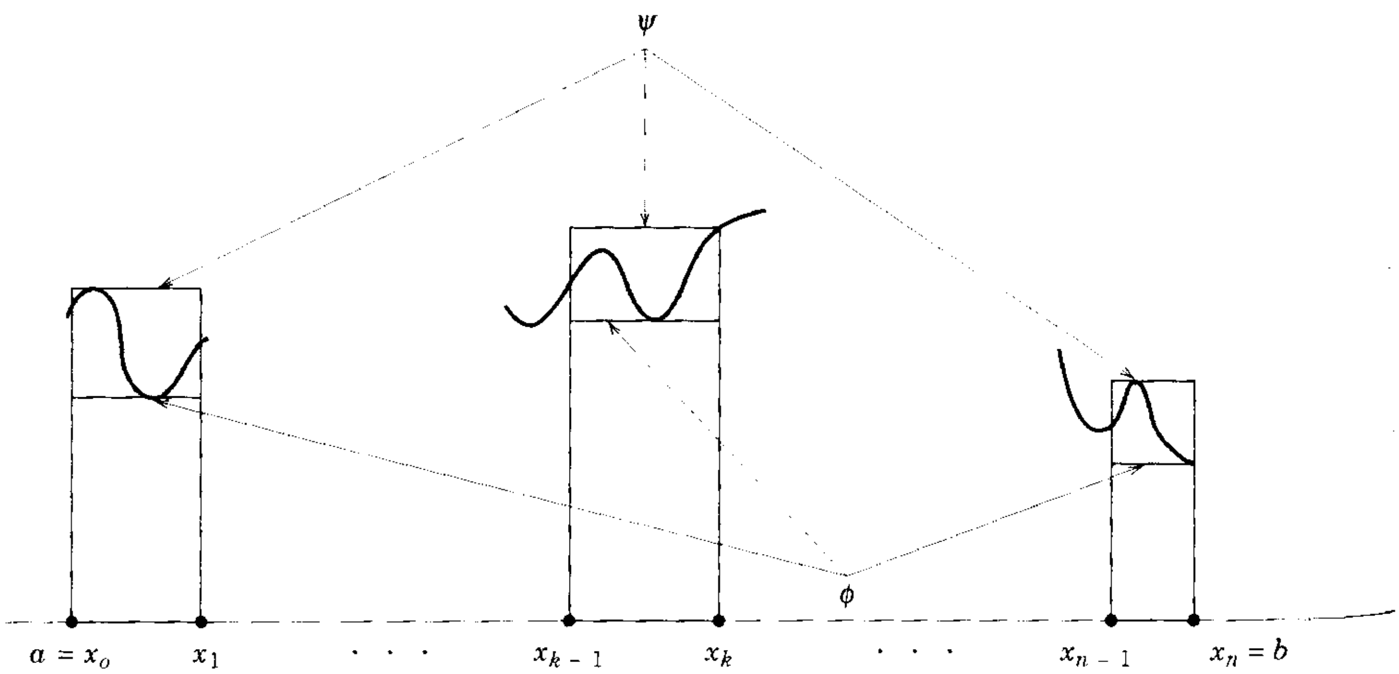
### 5.1.7 Comment

The definition of the Riemann integral for a bounded function $f$ on $[a,b]$ agrees with our original definition 5.1.2 when $f$ is a step function, say, $\hat{\phi} = f = \hat{\psi}$: $\int_a^b \hat{\phi}(x)\,dx \in \{\int_a^b \phi(x)\,dx \mid \phi \leq \hat{\phi},\ \phi$ a step function$\}$ and by monotonicity, $\int_a^b \phi(x)\,dx \leq \int_a^b \hat{\phi}(x)\,dx$, i.e., $\int_a^b \hat{\phi}(x)\,dx$ is an upper bound and a member of this set. Hence $\underline{\int}_a^b f(x)\,dx = \int_a^b \hat{\phi}(x)\,dx = \int_a^b \hat{\psi}(x)\,dx$. Similarly, $\overline{\int}_a^b f(x)\,dx = \int_a^b \hat{\psi}(x)\,dx = \int_a^b \hat{\phi}(x)\,dx$.

We have defined what it means for a bounded function $f$ on $[a,b]$ to be Riemann integrable; a common value for the lower and upper Riemann integrals. An equivalent condition, that is frequently easier to apply, is given by the next result.

**Proposition 5.2** *A bounded function $f$ on $[a,b]$ is Riemann integrable iff for every $\epsilon > 0$, we have step functions $\phi$ and $\psi$, $\phi \leq f \leq \psi$ on $[a,b]$, so that*

$$0 \leq \int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx = \int_a^b [\psi(x) - \phi(x)]dx < \epsilon.$$



$$\int_a^b [\psi(x) - \phi(x)]dx = \boxed{\phantom{x}}$$

*Proof:* Assume the bounded function $f$ is Riemann integrable on $[a, b]$ and let $\epsilon > 0$ be given. From 5.1.5 and the definitions of least upper bound and greatest lower bound we have step functions $\hat{\phi}$ and $\hat{\psi}$, $\hat{\phi} \leq f \leq \hat{\psi}$, so that

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} = \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2} < \int_a^b \hat{\phi}(x)\,dx \leq \underline{\int_a^b} f(x)\,dx$$

$$\leq \overline{\int_a^b} f(x)\,dx \leq \int_a^b \hat{\psi}(x)\,dx < \overline{\int_a^b} f(x) + \frac{\epsilon}{2}$$

$$= \int_a^b f(x)\,dx + \frac{\epsilon}{2}.$$

Thus $0 \leq \int_a^b \hat{\psi}(x)\,dx - \int_a^b \hat{\phi}(x)\,dx = \int_a^b [\hat{\psi}(x) - \hat{\phi}(x)]\,dx < \epsilon$.

Now let $\epsilon > 0$ be given and assume we have step functions $\phi$ and $\psi$, $\phi \leq f \leq \psi$ so that

$$0 \leq \int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx = \int_a^b [\psi(x) - \phi(x)]\,dx < \epsilon.$$

But, for any bounded function $f$ on $[a, b]$,

$$\int_a^b \phi(x)\,dx \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq \int_a^b \psi(x)\,dx.$$

We conclude that $0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon$. By the arbitrary nature of $\epsilon$,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx,$$

that is, $f$, is Riemann integrable on $[a, b]$.

*Note*: When the bounded function $f$ is Riemann integrable on $[a, b]$, we have $\phi \leq f \leq \psi$ on $[a, b]$, $\int_a^b \phi(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b \psi(x)\,dx$, and $\int_a^b [\psi(x) - \phi(x)]\,dx < \epsilon$, for some step functions $\phi$ and $\psi$. ■

The next problems are applications of Proposition 5.2, a revisitation of Cavalieri, Fermat, and so on.

### 5.1.8 Problems

1. Show $f(x) = x^2$, $0 \le x \le 1$ is Riemann integrable on $[0, 1]$.
   i. Form the partition $0 < 1/n < 2/n < \cdots < (n-1)/n < n/n$.

$$\hat{\phi}(x) = \begin{cases} \left(\dfrac{k-1}{n}\right)^2, & x_{k-1} < x < x_k \\ f, & \text{otherwise}, \end{cases}$$

$$\hat{\psi}(x) = \begin{cases} \left(\dfrac{k}{n}\right)^2, & x_{k-1} < x < x_k \\ f, & \text{otherwise}. \end{cases}$$

   ii. Show $\int_0^1 \hat{\phi}(x)\,dx = 1/3 - 1/2n + 1/6n^2$ and thus $1/3 \le \underline{\int}_0^1 x^2\,dx$.

$$\int_0^1 \hat{\psi}(x)\,dx = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \quad \text{and, thus,} \quad \overline{\int}_0^1 x^2\,dx \le \frac{1}{3}$$

   (See Problem 2.4.5.)

   iii. You could conclude $1/3 \le \underline{\int}_0^1 x^2\,dx \le \overline{\int}_0^1 x^2\,dx \le 1/3$ and then $\int_0^1 x^2\,dx$ exists and has value $1/3$. Please show

$$0 \le \int_0^1 \hat{\psi}(x)\,dx - \int_0^1 \hat{\phi}(x)\,dx = \frac{1}{n}.$$

   Thus $\int_0^1 x^2\,dx$ exists by Proposition 5.2. What is its value?

2. Show that $\int_0^1 x^p\,dx$ exists, $p$ any natural number. Is it difficult to evaluate this integral? Look at the next problem.
3. Show $\int_0^1 x^{(p/q)}$ exists and $\int_0^1 x^{(p/q)} = 1/(p/q+1)$, $p, q$ natural numbers.
   i. Form these partitions:

$$P_1: \quad \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\},$$

$$P_2: \quad \left\{0, \left(\frac{1}{n}\right)^q, \left(\frac{2}{n}\right)^q, \ldots, \left(\frac{n}{n}\right)^q\right\},$$

$$P_3: \quad \left\{0, \left(\frac{1}{n}\right)^p, \left(\frac{2}{n}\right)^p, \ldots, \left(\frac{n}{n}\right)^p\right\}.$$

Then

$$\sum_{k=1}^{n} q\left(\frac{k-1}{n}\right)^{p+q-1}\cdot\frac{1}{n}=\sum\left(\frac{k-1}{n}\right)^{p} q\left(\frac{k-1}{n}\right)^{q-1}\cdot\frac{1}{n}$$

$$=\sum\left(\frac{k-1}{n}\right)^{p}\underbrace{\left[\left(\frac{k-1}{n}\right)^{q-1}+\cdots+\left(\frac{k-1}{n}\right)^{q-1}\right]}_{q \text{ terms}}\cdot\frac{1}{n}$$

$$<\sum\left(\frac{k-1}{n}\right)^{p}\left[\left(\frac{k-1}{n}\right)^{q-1}+\left(\frac{k-1}{n}\right)^{q-2}\left(\frac{k}{n}\right)^{1}+\cdots+\left(\frac{k}{n}\right)^{q-1}\right]\cdot\frac{1}{n}$$

$$=\sum\left(\frac{k-1}{n}\right)^{p}\left[\left(\frac{k}{n}\right)^{q}-\left(\frac{k-1}{n}\right)^{q}\right]$$

$$\leq\sum\left(\frac{k}{n}\right)^{p}\left[\left(\frac{k}{n}\right)^{q}-\left(\frac{k-1}{n}\right)^{q}\right]$$

$$<\sum_{k=1}^{n} q\left(\frac{k}{n}\right)^{p+q-1}\cdot\frac{1}{n},$$

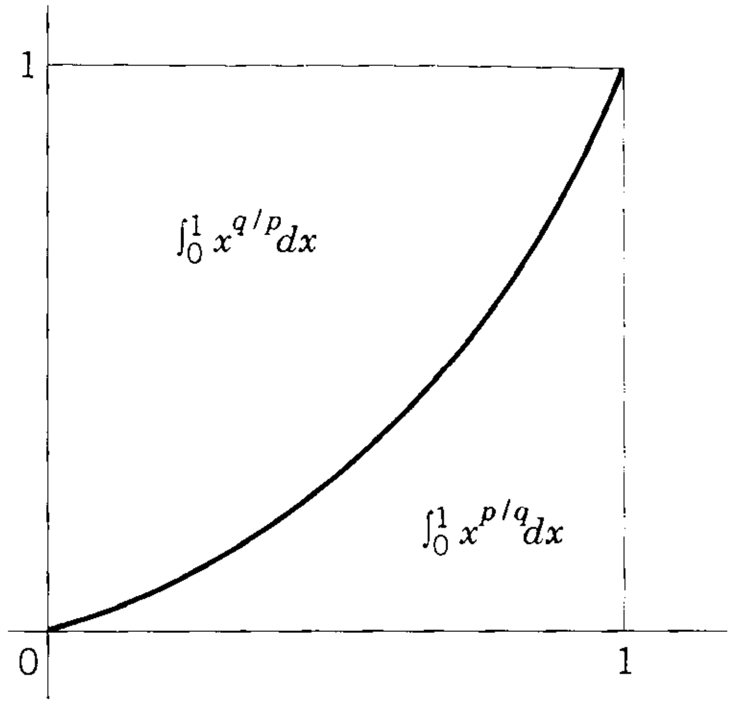
and

$$\sum q\left(\frac{k}{n}\right)^{p+q-1}\cdot\frac{1}{n}-\sum q\left(\frac{k-1}{n}\right)^{p+q-1}\cdot\frac{1}{n}=\frac{q}{n}.$$

Interchange $p$ and $q$ to conclude that $\int_0^1 x^{p+q-1}dx$, $\int_0^1 x^{p/q}dx$, and $\int_0^1 x^{q/p}dx$ exist (Proposition 5.2) and that $\int_0^1 px^{p+q-1}dx = \int_0^1 x^{q/p}dx$, $\int_0^1 qx^{p+q-1}dx = \int_0^1 x^{p/q}dx$.

ii. Since $(p+q)\int_0^1 x^{p+q-1}dx = \int_0^1 x^{q/p}dx + \int_0^1 x^{p/q}dx$, you need only show $\int_0^1 x^{p/q}dx + \int_0^1 x^{q/p}dx = 1$, for then $\int_0^1 x^{p+q-1}dx = 1/(p+q)$ and $\int_0^1 x^{p/q}dx = 1/(1+p/q)$.

This is obvious from the following figure:




or the relationship,

$$1 = \sum \left(\frac{k}{n}\right)^p \left[\left(\frac{k}{n}\right)^q - \left(\frac{k-1}{n}\right)^q\right] + \sum \left(\frac{k-1}{n}\right)^q \left[\left(\frac{k}{n}\right)^p - \left(\frac{k-1}{n}\right)^p\right].$$

iii. Complete problem 2.

Problems like the last three are highly motivational. We desperately seek an alternative method. Generally, we abstract to try and determine what is essential and what is not. We follow in Cauchy's footsteps: Specific functions are to be replaced with more general classifications, such as continuous, monotone, and so forth.

### 5.1.9 Problem

Every continuous function $f$ on $[a, b]$ is Riemann integrable.

i. $f$ is bounded on $[a, b]$ so $\underline{\int_a^b} f(x)\, dx$ and $\overline{\int_a^b} f(x)\, dx$ are well-defined.

ii. $f$ is uniformly continuous on $[a,b]$ by Theorem 2.8. Given $\epsilon > 0$, choose $\delta > 0$, so that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}$$

whenever $|x-y| < \delta,\ x,y \in [a,b]$.

iii. Take any partition $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ so that $x_k - x_{k-1} < \delta$, $k = 1, 2, \ldots, n$ and, thus,

$$-\frac{\epsilon}{b-a} < f(x) - f(y) < \frac{\epsilon}{b-a} \quad \text{for } x,y \in (x_{k-1}, x_k),$$

that is, $f(x) < f(y) + \epsilon/(b-a)$ for all $x \in (x_{k-1}, x_k)$. Conclude $\sup\limits_{(x_{k-1},x_k)} f \le f(y) + \epsilon/(b-a)$, and show

$$\sup_{(x_{k-1},x_k)} f - \inf_{(x_{k-1},x_k)} f \le \frac{\epsilon}{b-a}.$$

iv. Define $\hat{\phi} = \inf f$, $\hat{\psi} = \sup f$ on $(x_{k-1}, x_k)$ and $f$, otherwise.

v. Show $\int_a^b \hat{\psi}(x)\,dx - \int_a^b \hat{\phi}(x)\,dx \le (\epsilon/(b-a))(b-a)$, and then apply Proposition 5.2.

So continuous functions are Riemann integrable (Is existence in Problem 5.1.8 a little easier now?). Can discontinuous functions be Riemann integrable? A little thought should convince you that a finite number of discontinuities will not cause any problems, and if you recall Riemann's example of Chapter 1, in that case, a countable dense set of discontinuities did not cause any difficulties.

### 5.1.10 Problem

Let

$$f(x) = \begin{cases} 1, & x \text{ irrational}, \quad 0 \le x \le 1 \\ 0, & x \text{ rational}, \quad 0 \le x \le 1. \end{cases}$$

Show $\underline{\int_0^1} f(x)\,dx = 0$, $\overline{\int_0^1} f(x)\,dx = 1$ and conclude $f$ is not Riemann integrable. Of course $f$ is nowhere continuous. Maybe "countable" is what we are looking for in the way of a new criteria for Riemann integrability.

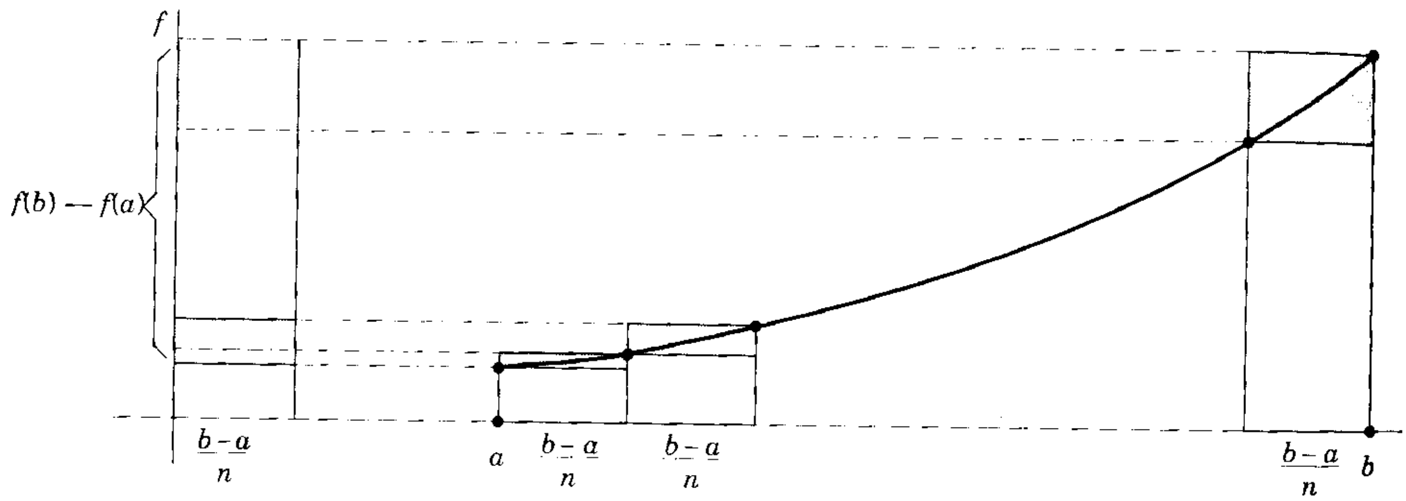
### 5.1.11 Problem

A nondecreasing or nonincreasing real-valued function is said to be monotone.

1. Show every monotone function $f$ on $[a, b]$ is Riemann integrable.
   i. A picture provides the idea. Assume $f$ is nondecreasing. Then $f(a) \leq f(x) \leq f(b)$ and $f$ is bounded. Form the partition $a < a + ((b-a)/n) < a + 2((b-a)/n) < \cdots < a + n((b-a)/n) = b$.

   Define $\hat{\phi}(x) = f(a + (k-1)((b-a)/n))$ and $\hat{\psi}(x) = f(a + k((b-a)/n))$ for $a + (k-1)((b-a)/n) < x < a + k((b-a)/n)$ and $f$, otherwise.

   ii. Show $\int_a^b \hat{\psi}(x)\,dx - \int_a^b \hat{\phi}(x)\,dx \leq ((b-a)/n)[f(b) - f(a)]$




2. Every monotone function $f$ on $[a, b]$ has a countable set of discontinuities. Suppose $f$ is nondecreasing.
   i. If $a < c < b$, show

$$\lim_{x \to c^-} f(x) = \sup_{a \leq x < c} f = f(c^-) \text{ exists,}$$

$$\lim_{x \to c^+} f(x) = \inf_{c < x \leq b} f = f(c^+) \text{ exists,}$$

   and $f(c^-) \leq f(c) \leq f(c^+)$.

   ii. Show $f(a) \leq f(a^+)$ and $f(b^-) \leq f(b)$.
   iii. If $f$ is not continuous at $c$, then $f(c^-) < f(c^+)$.
   iv. If $a < c_1 < c_2 < b$, then $f(c_1^+) \leq f(c_2^-)$.
   v. With each point $c$ of discontinuity associate a rational number $r_c$, $f(c^-) < r_c < f(c^+)$. By (iv) we have distinct rationals for distinct points of discontinuity.

vi. $f$ has a countable set of discontinuities.

Countability of the set of discontinuities is looking better and better.

### 5.1.12 Problem

Define a function $f$ on the Cantor set (Appendix A) by

$$f(x) = \begin{cases} 1, & x \text{ is a member of the Cantor set} \\ 0, & \text{otherwise}. \end{cases}$$

i. Let $\hat{\phi}_0 = 0$ and $\hat{\psi}_0 = 1$, $0 \le x \le 1$. Then $\hat{\phi}_0 \le f \le \hat{\psi}_0$ on $[0, 1]$. Let $\hat{\phi}_1 = 0$ and $\hat{\psi}_1 = 1$ on $[0, 1/3]$, $[2/3, 1]$, and zero otherwise: $\hat{\phi}_1 \le f \le \hat{\psi}_1$, on $[0, 1]$. Continue the obvious way. Then $\hat{\phi}_n \le f \le \hat{\psi}_n$ on $[0, 1]$, and

$$0 \le \underline{\int_0^1} f(x)\, dx \le \overline{\int_0^1} f(x)\, dx \le 1 - \left(\frac{1}{3} + \frac{2}{9} + \cdots\right)$$

i.e., $0 = \underline{\int_0^1} f(x)\, dx = \overline{\int_0^1} f(x)\, dx$. This function is Riemann integrable even though $f$ is discontinuous at each point of the Cantor set, an uncountable set. "Uncountable" is okay sometimes.

Finally, Lebesgue realized that instead of "measuring" discontinuities by "counting" we should be "measuring" discontinuities by "lengths of covers." The next result characterizes Riemann integrability!

**THEOREM 5.1** (Lebesgue, 1902) *A bounded function on a closed bounded interval is Riemann integrable iff the function is continuous almost everywhere. (The set of discontinuities is limited to a set of Lebesgue measure zero.)*

*Proof:* We will use the Riemann integrability criteria (Proposition 5.2): A bounded function $f$ on $[a, b]$ is Riemann integrable iff we can approximate $f$ from below and above by step functions whose integrals can be made arbitrarily close to each other: If we have step functions $\hat{\phi} \le f \le \hat{\psi}$, the difference $\hat{\psi} - \hat{\phi}$ is a measure of how much $f$

may vary on any subinterval of a partition. The standard means of measuring this variation of $f$ on an interval $J$ is by calculating $\sup\{f(x) \mid x \in J \cap [a,b]\} - \inf\{f(x) \mid x \in J \cap [a,b]\}$, traditionally denoted by $\omega_f(J)$. This real number $\omega_f(J)$ ($f$ is bounded on $[a,b]$) is nonnegative. It is natural then to measure the variation of $f$ at a point $x_0 \in [a,b]$, denoted by $\omega_f(x_0)$, as $\omega_f(x_0) = \inf\{\omega_f(I) \mid I$ any open interval containing $x_0\}$. We would expect that for a function continuous at $x_0$, $\omega_f(x_0) = 0$. In fact, for a bounded function $f$ on $[a,b]$, $f$ is continuous at $x_0$ in $[a,b]$ iff $\omega_f(x_0) = 0$. Here is a rough sketch of the argument to establish this result. Suppose $f$ is continuous at $x_0 \in [a,b]$ and $\epsilon > 0$ is given. We have $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$, $-\epsilon < f(x) - f(x_0) < \epsilon$, that is, $f(x) < f(x_0) + \epsilon$, and the $\sup\{f(x) \mid x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]\} \leq f(x_0) + \epsilon$. Using $f(x_0) - \epsilon < f(x)$, $\inf\{f(x) \mid x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]\} \geq f(x_0) - \epsilon$. Subtracting, $\omega_f((x_0 - \delta, x_0 + \delta)) \leq 2\epsilon$. Then $\omega_f(x_0) \leq \omega_f((x_0 - \delta, x_0 + \delta)) \leq 2\epsilon$ for every $\epsilon > 0$. So $\omega_f(x_0) = 0$. Conversely, suppose $\omega_f(x_0) = 0$ for some $x_0 \in [a,b]$ and let $\epsilon > 0$. Then $\inf\{\omega_f(I) \mid I$ any open interval containing $x_0\} = 0$ implies we have an open interval $I^*$ containing $x_0$ so that $0 \leq \omega_f(I^*) < \epsilon$. Since $I^*$ is open, choose $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subset I^*$. Then $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b] \Longrightarrow -\epsilon < f(x) - f(x_0) < \epsilon$, in other words, $f$ is continuous at $x_0$ and the result is established.

At points $x$ of discontinuity of $f$, $\omega_f(x) > 0$. This implies that the set of discontinuities of $f$, say, $D$, may be written as

$$D = \bigcup_n \{x \in [a,b] \mid \omega_f(x) \geq 1/n\}.$$

What do we know about the sets $D_n = \{x \in [a,b] \mid \omega_f(x) \geq 1/n\}$, $n = 1, 2, \ldots$? We claim $D_n$ is a closed set in $[a,b]$, or, equivalently, $D_n^c = \{x \in [a,b] \mid \omega_f(x) < 1/n\}$ is an open set in $[a,b]$. Let $x \in D_n^c$. We will determine a $\delta > 0$ so that $(x - \delta, x + \delta) \cap [a,b] \subset D_n^c$. The argument is very close to what we did above. Since $\omega_f(x) = \inf\{\omega_f(I) \mid I$ any open interval containing $x\} < 1/n$, we have an open interval $I^*$ containing $x$ so that $\omega_f(I^*) < 1/n$. Because $I^*$ is open, there exists a $\delta > 0$ so that $(x - \delta, x + \delta) \cap [a,b] \subset I^* \cap [a,b]$. We are done if we can show $(x - \delta, x + \delta) \cap [a,b] \subset D_n^c$. Let $z \in (x - \delta, x + \delta) \cap [a,b]$. Then $\omega_f(z) \leq \omega_f((x - \delta, x + \delta)) \leq \omega_f(I^*) < 1/n$; $z \in D_n^c$.

The preliminaries have been lengthy, but the proof of the main theorem is now fairly easily constructed if we keep this idea in mind: We can cover the discontinuities of $f$ with a set of small measure. On this "small" set, $f$

may be badly behaved, but $f$ is bounded. On the rest of the interval, a "large" set, $f$ is well-behaved (continuous). Integrability is determined by the behavior of $f$ on the "large" set. It's time to begin in earnest.

We first suppose $f$ is continuous a.e. on $[a, b]$, that is, the set of discontinuities of $f$, $D$, has measure zero, and let $|f| < B$ on $[a, b]$. We will show $f$ is Riemann integrable on $[a, b]$. Because $D_n = \{x \in [a, b] \mid \omega_f(x) \geq 1/n\}$ is a subset of $D$, $D_n$ has measure zero. Thus we have $D_n \subset \cup I_k$, $I_k$ open intervals, $\sum l(I_k) < \epsilon/4B$. But $D_n$ is a closed and bounded subset of $[a, b]$, hence compact. By Heine-Borel, we have a *finite* subcover of $D_n$ by the open intervals, $I_k$, that is,

$$D_n \subset \bigcup_{i=1}^{m} I_{k_i}, I_{k_i}$$

open intervals. Then the set $[a, b] - (I_{k_1} \cup I_{k_2} \cup \cdots \cup I_{k_m})$ is a *finite* union of closed intervals $J_1, J_2, \ldots, J_L$. That is,

$$[a, b] = J_1 \cup J_2 \cup \cdots \cup J_L \cup I_{k_1} \cup I_{k_2} \cup \cdots \cup I_{k_m}.$$

Recall that all points of $D_n$ are in $I_{k_1} \cup I_{k_2} \cup \cdots \cup I_{k_m}$. Thus $\omega_f(x) < 1/n$ on $J_1 \cup J_2 \cup \cdots \cup J_L$. Can we now define step functions $\phi$, $\psi$ in the obvious way:

$$\phi(x) \equiv \inf f \text{ on the intervals } J_1, \ldots, J_L, I_{k_1}, \ldots, I_{k_m},$$

$$\psi(x) \equiv \sup f \text{ on the intervals } J_1, \ldots, J_L, I_{k_1}, \ldots, I_{k_m}?$$

The problem here is that we need some kind of estimate for

$$\sup_{J_i} f(x) - \inf_{J_i} f(x),$$

having only the information that $\omega_f(x) < 1/n$ on each $J_i$. Does "pointwise" small, $\omega_f(x) < 1/n$, imply "global" small, $\omega_f(J_i) < 1/n$, or something like this? Again, we call on Heine-Borel. Since $\omega_f(x) < 1/n$ for each $x \in J_i$, we have an open interval containing $I_x$ so that $\omega_f(I_x) < 1/n$. But then the collection $\{I_x\}$ is an open cover of the closed, bounded, and hence compact set, $J_i$. A finite subcover must cover $J_i$. We have a partition of $J_i$, and the "sup of $f$"-"inf of $f$" over any of the subintervals of this partition is less than $1/n$. This is exactly what we need! (Clearly this does not imply $\omega_f(J_i) < 1/n$!) Do this for each $J_i$, $i = 1, 2, \ldots, L$. We have a *finite* collection of subintervals whose union is $J_1 \cup J_2 \cup \cdots \cup J_L$, and on any one of these subintervals, say $J^*$,

$$\omega_f(J^*) = \sup\{f(x) \mid x \in J^* \cap [a,b]\} - \inf\{f(x) \mid x \in J^* \cap [a,b]\} < \frac{1}{n}.$$

**Now define** step functions $\hat{\phi}$, $\hat{\psi}$ in the obvious way (finite number of subintervals is critical to being a *step* function):

$$\hat{\phi} = \inf f \text{ on the } \textit{subintervals} \text{ of } J_1, J_2, \ldots, J_L \text{ and } I_{k_1}, \ldots, I_{k_m},$$

$$\hat{\psi} = \sup f \text{ on the } \textit{subintervals} \text{ of } J_1, J_2, \ldots, J_L \text{ and } I_{k_1}, \ldots, I_{k_m}.$$

Then

$$\begin{aligned}\int_a^b [\hat{\psi}(x) - \hat{\phi}(x)]dx &\leq \frac{1}{n}\sum l(\text{subintervals of } J_i) + 2B\sum l(I_{k_n}) \\ &\leq \frac{1}{n}(b-a) + 2B\left(\frac{\epsilon}{4B}\right) \\ &< \epsilon \text{ for } n \text{ sufficiently large.}\end{aligned}$$

So $f$ is Riemann integrable on $[a, b]$ (Proposition 5.2).

We next suppose $f$ is Riemann integrable on $[a, b]$. We want to show the bounded function $f$ is continuous a.e. on $[a, b]$. Because the set of discontinuities of $f$, $D$, is given by $D = \cup D_n = \cup\{x \in [a,b] \mid \omega_f(x) \geq 1/n\}$, if we can show $\mu(D_n) = 0$, we would be done, since a countable union of sets of measure zero is a set of measure zero. Fix an $n$, say, $N$, and consider the set

$$D_N = \{x \in [a,b] \mid \omega_f(x) \geq 1/N\}, \qquad \text{with } \epsilon > 0.$$

Because $f$ is Riemann integrable on $[a, b]$ by assumption, we have step functions $\hat{\phi} \leq f \leq \hat{\psi}$ with $\int_a^b [\hat{\psi}(x) - \hat{\phi}(x)]dx < \epsilon/N$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition associated with $\hat{\phi}$ and $\hat{\psi}$, and split the collection $\{(x_0, x_1), \ldots, (x_{n-1}, x_n)\}$ into two subcollections $\mathcal{D}_1$ and $\mathcal{D}_2$, as follows: If $(x_{k-1}, x_k) \cap D_N \neq \emptyset$, put it in $\mathcal{D}_1$. Otherwise, put it in $\mathcal{D}_2$. Then every point of $D_N$ is in $\mathcal{D}_1$ or a member of the set $\{a, x_1, x_2, \ldots, x_{n-1}, b\}$ So

$$\int_a^b [\hat{\psi}(x) - \hat{\phi}(x)]dx = \sum_{\mathcal{D}_1} + \sum_{\mathcal{D}_2} < \frac{\epsilon}{N}\,.$$

For $\sum_{\mathcal{D}_1}$, some point of $D_N$ is in each subinterval, i.e., each subinterval of $\mathcal{D}_1$ contains a point $x$ with $\omega_f(x) \geq 1/N$. But then $\hat{\psi} - \hat{\phi} \geq 1/N$ on this subinterval. Consequently,

$$\frac{\epsilon}{N} > \sum_{\mathcal{D}_1} \geq \frac{1}{N}\sum \quad \text{(lengths of subintervals that contain points of } D_N\text{)},$$

so $\sum_{\mathcal{D}_1}(x_k - x_{k-1}) < \epsilon$. The intervals of $\mathcal{D}_1$, along with the finite set $\{a, x_1, \ldots, x_{n-1}, b\}$ contain all points of $D_N$. So $\mu(D_N) < \epsilon$, and the proof of this theorem is complete. Congratulations! ■

We have characterized Riemann integrable functions. We now show that the integral properties that hold for step functions (Proposition 5.1) remain valid for Riemann integrable functions. This theme, step functions first (easiest), and then more general functions as combinations or limits of easier functions, is repeated throughout this chapter.

**Theorem 5.2** *If bounded functions $f$ and $g$ are Riemann integrable on $[a, b]$, and $k$ is any real number, then*

1. *$(kf)$ is Riemann integrable on $[a, b]$, and*

$$\int_a^b (kf)(x)\,dx = k\int_a^b f(x)\,dx \qquad (\textit{homogeneous});$$

2. *$(f + g)$ is Riemann integrable on $[a, b]$, and*

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad (\textit{additive});$$

3.

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \text{ if } f \leq g \text{ on } [a, b] \qquad (\textit{monotone});$$

4. *If $a < c < b$, $f$ is Riemann integrable on $[a, c]$ and $[c, b]$, and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad (\textit{additive on the domain});$$

5. *If $\alpha \leq f(x) \leq \beta$ on $[a, b]$,*

$$\alpha(b-a) \leq \int_a^b f(x)\,dx \leq \beta(b-a) \qquad (\textit{mean value}).$$

*Proof:* Theorem 5.1 quickly yields integrability of the appropriate functions. A more direct argument, based on Proposition 5.2, is of interest.

1. If $k = 0$, obvious. Assume $k > 0$ and $f$ is Riemann integrable on $[a, b]$. Let $\epsilon > 0$ be given. From Proposition 5.2 we have step functions $\hat{\phi}, \hat{\psi}$ on $[a, b]$ so that $\hat{\phi} \leq f \leq \hat{\psi}$,

$$\int_a^b \hat{\phi}(x)dx \leq \int_a^b f(x)dx \leq \int_a^b \hat{\psi}(x)dx,$$

and

$$\int_a^b [\hat{\psi}(x) - \hat{\phi}(x)]dx < \epsilon/k.$$

Since $\int_a^b (k\hat{\phi})(x)dx = k\int_a^b \hat{\phi}(x)dx$ and $\int_a^b (k\hat{\psi})(x)dx = k\int_a^b \hat{\psi}(x)dx$ (Proposition 5.1),

$$\begin{aligned}
\int_a^b \left(k\hat{\phi}\right)(x)dx &= k\int_a^b \hat{\phi}(x)dx \\
&\leq k\int_a^b f(x)dx \\
&\leq k\int_a^b \hat{\psi}(x)dx \\
&= \int_a^b \left(k\hat{\psi}\right)(x)dx
\end{aligned}$$

and

$$\int_a^b (k\hat{\psi})(x)dx - \int_a^b (k\hat{\phi})(x)dx < \epsilon :$$

But,

$$k\hat{\phi} \le kf \le k\hat{\psi}$$

on $[a,b]$. That is, we have step functions $(k\hat{\phi}), (k\hat{\psi})$ bracketing $(kf)$ so that $\int_a^b (k\hat{\psi})(x)dx - \int_a^b (k\hat{\phi})(x)dx < \epsilon$. But then $(kf)$ is Riemann integrable on $[a,b]$ by Proposition 5.2. Since $k\int_a^b f(x)dx$ and $\int_a^b (kf)(x)dx$ lie between $k\int_a^b \hat{\phi}(x)dx$ and $k\int_a^b \hat{\psi}(x)dx$, they are equal by the arbitrary nature of $\epsilon$. The reader may argue the case when $k < 0$.

2. From Proposition 5.2,

$$\hat{\phi}_f \le f \le \hat{\psi}_f \qquad \text{and} \qquad \int_a^b \hat{\phi}_f(x)dx \le \int_a^b f(x)dx \le \int_a^b \hat{\psi}_f(x)dx,$$

and

$$\hat{\phi}_g \le g \le \hat{\psi}_g \qquad \text{and} \qquad \int_a^b \hat{\phi}_g(x)dx \le \int_a^b g(x)dx \le \int_a^b \hat{\psi}_g(x)dx.$$

Adding, $\hat{\phi}_f + \hat{\phi}_g \le f + g \le \hat{\psi}_f + \hat{\psi}_g$, and application of Proposition 5.1 yields

$$\begin{aligned}
\int_a^b \hat{\phi}_f(x)dx + \int_a^b \hat{\phi}_g(x)dx &= \int_a^b \left(\hat{\phi}_f + \hat{\phi}_g\right)(x)dx \\
&\le \underline{\int_a^b} (f+g)(x)dx \\
&\le \overline{\int_a^b} (f+g)(x)dx \\
&\le \int_a^b \left(\hat{\psi}_f + \hat{\psi}_g\right)(x)dx \\
&= \int_a^b \hat{\psi}_f(x)dx + \int_a^b \hat{\psi}_g(x)dx.
\end{aligned}$$

Application of Proposition 5.2 again yields existence of $\int_a^b (f+g)(x)dx$, and it along with $\int_a^b f(x)dx + \int_a^b g(x)dx$, lies between $\int_a^b \hat{\psi}_f(x)dx + \int_a^b \hat{\psi}_g(x)dx$ and $\int_a^b \hat{\phi}_f(x)dx + \int_a^b \hat{\phi}_g(x)dx$.

3.

$$\begin{aligned}\int_a^b f(x)dx &= \sup\left\{\int_a^b \phi(x)dx \mid \phi \le f, \quad \phi \text{ a step function}\right\}\\ &\le \sup\left\{\int_a^b \phi(x)dx \mid \phi \le g, \quad \phi \text{ a step function}\right\},\\ &\qquad \text{since } f \le g\\ &= \int_a^b g(x)dx.\end{aligned}$$

4. Let $a < c < b$. Since $f$ is Riemann integrable on $[a, b]$, we have step functions $\hat{\phi}, \hat{\psi}$, a common partition including $c$, so that $\hat{\phi} \le f \le \hat{\psi}$ on $[a, b]$ and

$$\int_a^b \hat{\psi}(x)dx - \int_a^b \hat{\phi}(x)dx < \epsilon$$

by Proposition 5.2. But

$$\int_a^c \hat{\phi}(x)dx \le \underline{\int_a^c} f(x)dx \le \overline{\int_a^c} f(x)dx \le \int_a^c \hat{\psi}(x)dx$$

and

$$\begin{aligned}0 &\le \int_a^c \hat{\psi}(x)dx - \int_a^c \hat{\phi}(x)dx\\ &= \int_a^c (\hat{\psi} - \hat{\phi})(x)dx\\ &\le \int_a^b (\hat{\psi} - \hat{\phi})(x)dx\\ &< \epsilon\end{aligned}$$

by Proposition 5.1 and thus $\int_a^c f(x)dx$ makes sense by Proposition

5.2. An analogous arguments yields existence of $\int_c^b f(x)dx$. Thus

$$\int_a^b \hat{\phi}(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx \le \int_a^b \hat{\psi}(x)dx$$

and

$$\int_a^b \hat{\phi}(x)dx \le \int_a^b f(x)dx \le \int_a^b \hat{\psi}(x)dx.$$

This completes the argument for 4.

5. The constant functions $\alpha$ and $\beta$ are step functions satisfying $\alpha \le f \le \beta$ on $[a, b]$. From part 3,

$$\alpha(b-a) = \int_a^b \alpha\, dx \le \int_a^b f(x)dx \le \int_a^b \beta\, dx = \beta(b-a). \qquad \blacksquare$$

We conclude our brief treatment of the Riemann integral with some examples, a convergence theorem pertaining to:

$$\int_a^b (\lim f_k)(x)dx \stackrel{?}{=} \lim \int_a^b f_k(x)dx,$$

and The Fundamental Theorem of Calculus.

**Example 1:**

1. $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}, \qquad 0 \le x \le 1.$ If $r_1, r_2, \ldots$ is any enumeration of the rationals in $[0, 1]$, we may define a uniformly bounded, monotonic sequence of functions $(f_k)$ as follows:

$$f_k(x) = \begin{cases} 1, & x = r_1, r_2, \ldots, r_k \\ 0, & \text{otherwise} \end{cases}, \qquad 0 \le x \le 1.$$

Then $\lim f_k = f$, $\int_0^1 f_k(x)dx = 0$, and as a consequence, $\lim \int_0^1 f_k(x)dx = 0$. However, $\int_0^1 (\lim f_k)(x)dx$ is not even defined!

2. $f(x) = 0,\ 0 \le x \le 1,\ f_k(x) = \begin{cases} k^2 x, & 0 \le x \le 1/k \\ -k^2(x - 2/k), & 1/k \le x \le 2/k \\ 0, & 2/k \le x \le 1\,. \end{cases}$

Then $\lim f_k = f$(not uniformly), the sequence $(f_k)$ is not uniformly bounded, $\int_0^1 f_k(x)dx = 1$, and

$$\lim \int_0^1 f_k(x)dx = 1 \neq 0 = \int_0^1 (\lim f_k)(x)dx.$$

Uniform boundedness is not enough, nor monotonicity. The concept we are looking for with the Riemann integral is uniform convergence: The uniform limit of a sequence of Riemann integrable functions is Riemann integrable and the integral of the limit is the limit of the integrals.

**THEOREM 5.3** *Suppose we have a sequence $(f_k)$ of Riemann integrable functions on $[a, b]$. If $\lim f_k = f$ (unif) on $[a, b]$ then*

1. *$(\lim f_k)$ is Riemann integrable on $[a, b]$;*
2. *$\lim \int_a^b f_k(x)dx = \int_a^b f(x)dx = \int_a^b (\lim f_k)(x)dx$.*

*Proof*:

1. $f$ is Riemann integrable on $[a, b]$. Let $\epsilon > 0$. From uniform convergence of $(f_k)$ on $[a, b]$, we have a natural number $K$ so that

$$f_k(x) - \frac{\epsilon}{4(b-a)} \leq f(x) \leq f_k(x) + \frac{\epsilon}{4(b-a)} \tag{1}$$

for all $x \in [a, b], k \geq K$. So $f$ is bounded on $[a, b]$, and because $f_k$ is Riemann integrable on $[a, b]$, we have step functions $\phi_k$ and $\psi_k$, $\phi_k \leq f_k \leq \psi_k$ on $[a, b]$, and $\int_a^b [\psi_k(x) - \phi_k(x)]dx < \epsilon/2, k \geq K$. But then,

$$\begin{aligned}
\phi_k(x) - \frac{\epsilon}{4(b-a)} &\leq f_k(x) - \frac{\epsilon}{4(b-a)} \\
&\leq f(x) \\
&\leq f_k(x) + \frac{\epsilon}{4(b-a)} \\
&\leq \psi_k(x) + \frac{\epsilon}{4(b-a)}
\end{aligned}$$

on $[a,b]$, $k \geq K$: We have step functions $\phi_k - (\epsilon/4(b-a))$ and $\psi_k + (\epsilon/4(b-a))$, bracketing $f$, and

$$\int_a^b \left[\left(\psi_k(x) + \frac{\epsilon}{4(b-a)}\right) - \left(\phi_k(x) - \frac{\epsilon}{4(b-a)}\right)\right] dx$$

$$= \int_a^b [\psi_k(x) - \phi_k(x)]\, dx + \frac{\epsilon}{2} < \epsilon,$$

$k \geq K$. Thus $f$ is Riemann integrable on $[a,b]$.

2. We may integrate (1) by Proposition 5.2:

$$\int_a^b f_k(x)dx - \frac{\epsilon}{4} \leq \int_a^b f(x)dx \leq \int_a^b f_k(x)dx + \frac{\epsilon}{4},$$

$k \geq K$, that is,

$$-\frac{\epsilon}{4} \leq \int_a^b f(x)dx - \int_a^b f_k(x)dx \leq \frac{\epsilon}{4},$$

$k \geq K$, and the proof is complete. ∎

### 5.1.13 Problem

Suppose we have a sequence of Riemann integrable functions $(f_k)$ on $[a,b]$. If $f(x) = \sum_{k=1}^{\infty} f_k(x)$, $a \leq x \leq b$, the series converging uniformly on $[a,b]$ to $f$, then

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x)\right) dx = \int_a^b f(x)dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

In other words, the series may be integrated term by term.

*Hint*: $F_n(x) \equiv \sum_{k=1}^{n} f_k(x)$. Then $F_n(x)$ is Riemann integrable on $[a,b]$, $\lim F_n = f$ uniformly on $[a,b]$, and application of Theorem 5.3 completes the argument.

We may weaken some of these requirements of Theorem 5.3 when we deal with the Lebesgue integral.

And now, one of the most celebrated theorems in mathematics:

"The Fundamental Theorem of Calculus"!

**Theorem 5.4** *The Fundamental Theorem of Calculus: Let $f$ be a bounded function on $[a,b]$. If $f$ is Riemann integrable on $[a,b]$ and $F(x) \equiv \int_a^x f(t)dt$, then $F$ is continuous on $[a,b]$. Additionally, if $f$ is continuous at $x_0 \in (a,b)$, then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.*

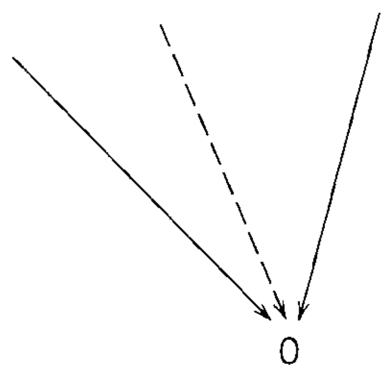
*Proof:* We show that $F$ is continuous with the assumption of $f$ being Riemann integrable. Equivalently, $\lim_{x \to x_0}[F(x) - F(x_0)] = 0$ for $x_0 \in [a,b]$.

We begin: Since $f$ is bounded on $[a,b]$

$$-B \leq f(t) \leq B, \qquad a \leq t \leq b.$$

We may integrate because $f$ is Riemann integrable by assumption, and thus $\int_{x_0}^x -B\,dt \leq \int_{x_0}^x f(t)\,dt \leq \int_{x_0}^x B\,dt$, $a \leq x_0 \leq x \leq b$ (Theorem 5.2), that is,

$$-B(x - x_0) \leq F(x) - F(x_0) \leq B(x - x_0), \qquad a \leq x_0 \leq x \leq b.$$




We have $\lim_{x \to x_0^+} [F(x) - F(x_0)] = 0$. A similar argument shows $\lim_{x \to x_0^-}[F(x) - F(x_0)] = 0$. $F$ is continuous on $[a,b]$.

Now, assume $f$ is continuous at $x_0$, $a < x_0 < b$. We show $F$ is differentiable at $x_0$, and $F'(x_0) = f(x_0)$. Equivalently,

$$\lim_{x \to x_0} \left[\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right] = 0, \qquad x_0 \in (a,b).$$

Since $f$ is continuous at $x_0$, given $\epsilon > 0$, we have $\delta > 0$ so that

$$f(x_0) - \epsilon < f(t) < f(x_0) + \epsilon, \qquad t \in (x_0 - \delta, x_0 + \delta) \cap [a,b].$$

We may integrate ($f$ is continuous; the Riemann integrable exists by

Theorem 5.2 or Problem 5.1.9)

$$\int_{x_0}^{x} [f(x_0) - \epsilon]\, dt \le \int_{x_0}^{x} f(t)\, dt \le \int_{x_0}^{x} [f(x_0) + \epsilon]\, dt,$$

$x \in [x_0, x_0 + \delta) \cap [a, b]$. That is,

$$[f(x_0) - \epsilon](x - x_0) \le F(x) - F(x_0) \le [f(x_0) + \epsilon](x - x_0),$$

$x \in [x_0, x_0 + \delta) \cap [a, b]$. Thus

$$-\epsilon \le \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \le \epsilon, \qquad x \in (x_0, x_0 + \delta) \cap [a, b].$$

Hence $\lim_{x \to x_0^+}[(F(x) - F(x_0))/(x - x_0) - f(x_0)] = 0$. The reader may argue $\lim_{x \to x_0^-}$. ■

### 5.1.14 Comment

If $f$ is continuous on $[a, b]$,

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x), \qquad a < x < b.$$

### 5.1.15 Problems

1. Suppose $f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 0, & 0 < x < 1 \\ 1, & 1 \le x \le 2 \end{cases}$. Graph $f$.

   Calculate $\int_{-1}^{x} f(t)dt$, $\int_0^x f(t)dt$ and graph.

   Investigate the continuity of these three functions at $x = 0$ and $x = 1$ using the definition.

   Calculate $\frac{d}{dx}\left(\int_{-1}^{x} f(t)dt\right)$, $\frac{d}{dx}\left(\int_0^x f(t)dt\right)$ where appropriate, and graph. Compare with the graph of $f$.

2. Suppose $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 < x < 2 \\ 3 - x, & 2 \le x \le 3 \end{cases}$. Graph $f$.

Calculate and graph $\int_0^x f(t)dt$, $\int_{1/2}^x f(t)dt$. Investigate the continuity and differentiability of these three functions at $x = 1$ and $x = 2$ using the definition.

Calculate $\frac{d}{dx}\left(\int_0^x f(t)dt\right)$, $\frac{d}{dx}\left(\int_{1/2}^x f(t)dt\right)$

where appropriate, and graph. Compare with the graph of $f$.

The "other ?" Fundamental Theorem of Calculus is what we frequently use for evaluating integrals.

**Theorem 5.5** *If $f$ is continuous on $[a, b]$, differentiable on $(a, b)$, and if the derivative, $f'$, is Riemann integrable on $[a, b]$, then*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

*Proof:* Since $f'$ is Riemann integrable on $[a, b]$ by assumption, we have step functions $\phi \le f' \le \psi$ on $[a, b]$ with $\int_a^b \phi(x)\,dx \le \int_a^b f'(x)dx \le \int_a^b \psi(x)\,dx$, and $\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \epsilon$ for any $\epsilon > 0$. Take the common partition formed by $\phi$ and $\psi$, say $\{a, x_1, x_2, \ldots, x_{n-1}, b\}$. Suppose $\phi = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)}$ and $\psi = \sum_{i=1}^n d_i \chi_{(x_{i-1}, x_i)}$. Thus $c_i \le f' \le d_i$ on $(x_{i-1}, x_i)$.

Since

$$f(b) - f(a) = \sum_{i=1}^n [f(x_i) - f(x_{i-1})]$$

$$= \sum_{i=1}^n f'(\xi_i)(x_i - x_{i-1}), \qquad x_{i-1} < \xi_i < x_i,$$

by the Mean Value Theorem (Theorem 2.10), we have

$$\sum c_i(x_i - x_{i-1}) \le f(b) - f(a) \le \sum d_i(x_i - x_{i-1}),$$

that is, $\int_a^b \phi(x)dx \leq f(b) - f(a) \leq \int_a^b \psi(x)dx$. So $f(b) - f(a)$ and $\int_a^b f'(x)dx$ are between $\int_a^b \phi(x)dx$, $\int_a^b \psi(x)dx$. Thus

$$|f(b) - f(a) - \int_a^b f'(x)\,dx| < \epsilon.$$

The argument is complete. ■

Looking back at Problem 5.1.8, existence is via Problem 5.1.9 and evaluation follows from this last result, Theorem 5.5. Isn't mathematics powerful:

> For it is unworthy of excellent men to lose hours like slaves in the labor of computation.
>
> —Gottfried W. Leibnitz

This concludes our treatment of the Riemann integral.

## 5.2 THE LEBESGUE INTEGRAL FOR BOUNDED FUNCTIONS ON SETS OF FINITE MEASURE

We develop the Lebesgue integral for bounded functions $f$ on a set $E$ of finite Lebesgue measure. The treatment parallels that of the Riemann integral, replacing step functions with simple functions. The reader might review 5.1 at this time.

### 5.2.1 Definition

A function $\phi$, whose domain is a measurable set $E$ with finite measure, is called a *simple function*, provided

$$\phi(x) = \sum_{k=1}^{n} c_k \, \chi_{E_k}(x),$$

where the $c_k$ are real numbers (not necessarily distinct) and $\bigcup E_k = E$, $E_k$ mutually disjoint measurable subsets of $E$. (We may assume

$$\bigcup_1^n E_k = E \text{ since } E_0 = E - \bigcup_1^n E_k$$

is a measurable set and $0 \cdot \mu(E_0) = 0$.)

Every step function is a simple function. The converse is false.

**Example 2:** A simple function that is not a step function:

$$\phi(x) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational}. \end{cases}$$

$$\phi = 1 \cdot \chi_{\{\text{irrationals}\}} + 0 \cdot \chi_{\{\text{rationals}\}} = \chi_{\{\text{irrationals}\}}.$$

A simple function is "simply" a finite linear combination of characteristic functions and is easily shown to be a measurable function (Problem 4.3.3).

### 5.2.2 Comment

A simple function may have many representations.

$$\phi(x) = \begin{cases} -1, & 0 < x \leq 1 \\ 2, & 1 < x \leq 2 \\ 0, & 2 < x \leq 3. \end{cases}$$

Let $E_1 = (0, 1]$, $E_2 = (1, 2]$ $E_3 = (2, 3]$ and $D_1 = (0, 1/2)$, $D_2 = [1/2, 1]$, $D_3 = (1, 2]$, $D_4 = (2, 3]$. Then

$$\begin{aligned} \phi &= -1\chi_{E_1} + 2\chi_{E_2} + 0\chi_{E_3} \\ &= -1\chi_{D_1} + -1\chi_{D_2} + 2\chi_{D_3} + 0\chi_{D_4} \\ &= -1\chi_{D_1} + -1\chi_{D_2} + 2\chi_{D_3}. \end{aligned}$$

We define the Lebesgue integral of a simple function $\phi$ on $E$.

### 5.2.3 Definition

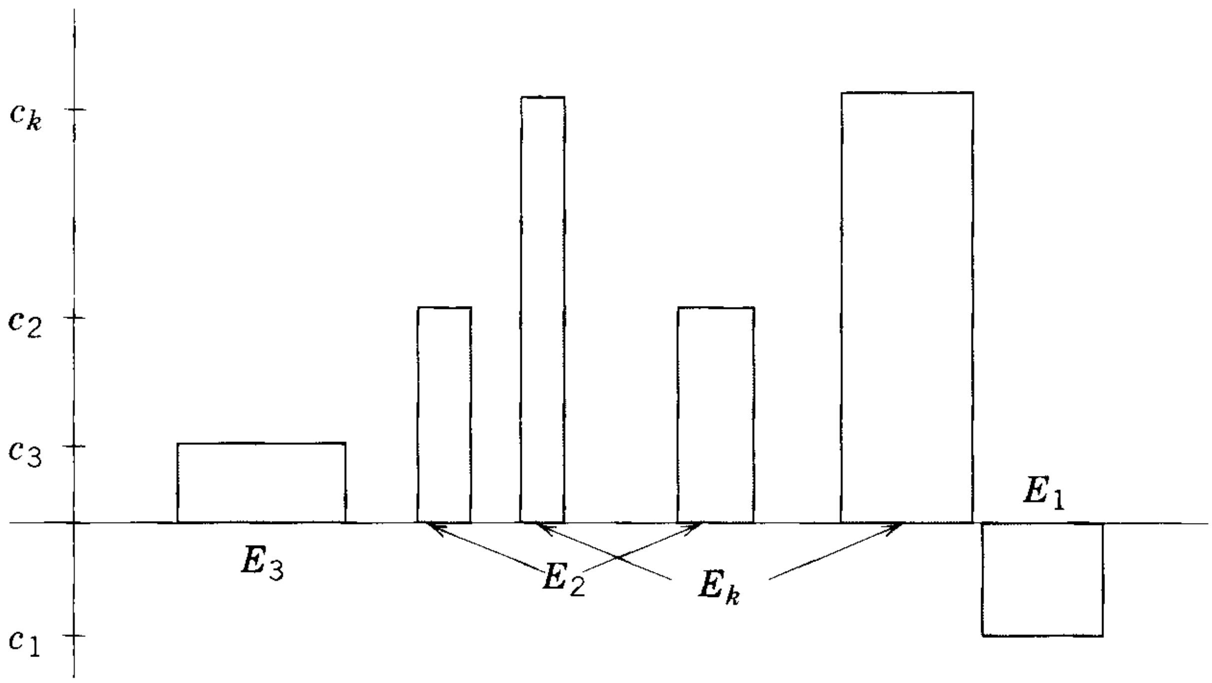
Suppose $\phi$ is a simple function defined on a measurable set $E$, that is,

$$\phi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x),$$

with $\bigcup_1^n E_k = E$, $E_k$ mutually disjoint, $\mu(E) < \infty$, $c_k$ real.

The *Lebesgue integral* of $\phi$ on $E$, $\int_E \phi$, is defined as

$$\int_E \phi = \sum_{k=1}^{n} c_k \, \mu(E_k) .$$




### 5.2.4 Comments

1. If $\phi$ is a step function on $[a, b]$, $\phi$ is a simple function, and

$$\int_E \phi = \sum_{k=1}^{n} c_k \, \mu(E_k) = \sum_{k=1}^{n} c_k \, \mu((x_{k-1}, x_k)) + \sum_{k=1}^{n+1} d_k \, \mu(x_{k-1})$$
$$= \sum_{k=1}^{n} c_k (x_k - x_{k-1}) = \int_a^b \phi(x) \, dx .$$

The Lebesgue integral of a step function agrees with the Riemann integral of a step function.

2. Because of the many possible representations of $\phi$ we must check to see that our definition is not ambiguous. Suppose

$$E = \bigcup_{i=1}^{n} E_i = \bigcup_{j=1}^{m} F_j .$$

Somehow we want to refine both of these decompositions of $E$ into a "common" decomposition. Each set in this "common" decomposition would be a subset of $E_i$ *and* $F_j$. "And" suggests intersection:

$$E = \bigcup_i \bigcup_j (E_i \cap F_j) = \bigcup_j \bigcup_i (E_i \cap F_j)\,.$$

Now suppose $\phi = \sum_i c_i \chi_{E_i} = \sum_j d_j \chi_{F_j}$, $\{E_i\}$ and $\{F_j\}$ mutually disjoint collections of Lebesgue measurable sets, $\mu(E) < \infty$. Then we claim

$$\sum_i c_i\, \mu(E_i) = \sum_j d_j\, \mu(F_j)\,,$$

the Lebesgue integral is independent of the representation. We know the nonempty $E_i \cap F_j$ are measurable and mutually disjoint. Because

$$\bigcup_{i=1}^{n} E_i = E = \bigcup_{j=1}^{m} F_j,$$

$$\begin{aligned}
\sum_i c_i\, \mu(E_i) &= \sum_i c_i \sum_j \mu(E_i \cap F_j) = \sum_i \sum_j c_i \mu(E_i \cap F_j) \\
&= \sum_j \sum_i d_j \mu(E_i \cap F_j) = \sum_j d_j \sum_i \mu(E_i \cap F_j) \\
&= \sum_j d_j \mu(F_j)\,,
\end{aligned}$$

since if $E_i \cap F_j \neq \emptyset$, then $c_i = c_i \chi_{E_i} = \phi = d_j \chi_{F_j} = d_j$, and the argument is complete.

**Example 3:** We calculate some Lebesgue integrals of simple functions.

1. $$\phi(x) = \begin{cases} -1, & 0 < x \le 1 \\ 2, & 1 < x \le 2 \\ 0\;, & 2 < x \le 3\,. \end{cases}$$

$$\phi = -1\chi_{(0,1]} + 2\chi_{(1,2]}$$

$$\int_{(0,3]} \phi = -1 \cdot 1 + 2 \cdot 1 = 1 = \int_0^3 \phi(x)\, dx$$

2. $\phi(x) = \begin{cases} 0, & x \text{ rational}, \ 0 \le x \le 1 \\ 1, & x \text{ irrational}, \ 0 \le x \le 1 . \end{cases}$

$$\phi = \chi_{[0,1]\cap\{\text{irrationals}\}}$$

$$\int_{[0,1]} \phi = 1\,\mu([0,1] \cap \{\text{irrationals}\}) = 1 .$$

3. $\phi(x) = \begin{cases} 1, & x \in C \cap \{\text{rationals}\} \\ 2, & x \in C \cap \{\text{irrationals}\}, \\ 3, & x \in [0,1] - C \end{cases}$ where $C$ is the Cantor set.

Then $\int_{[0,1]} \phi = 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1 = 3$ because $\mu(C) = 0$.

**PROPOSITION 5.3** *If $\phi, \psi$ are simple functions defined on a set $E$ with finite measure, and $k$ is any real number, then*

1. *$(k\phi)$ is a simple function on $E$, and $\int_E (k\phi) = k \int_E \phi$ (homogeneous);*
2. *$(\phi + \psi)$ is a simple function on $E$, and $\int_E(\phi + \psi) = \int_E \phi + \int_E \psi$ (additive);*
3. *$\int_E \phi \le \int_E \psi$ if $\phi \le \psi$ on $E$ (monotone);*
4. *If $E_1$ and $E_2$ are disjoint measurable subsets of $E$ with $E = E_1 \cup E_2$, the integrals $\int_{E_1} \psi$ and $\int_{E_2} \psi$ exist, and $\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi$ (additive on the domain).*

*Proof:* Suppose $\phi = \sum_{i=1}^{n} c_i \chi_{E_i}$, $\psi = \sum_{j=1}^{m} d_j \chi_{F_j}$, where

$$E = \bigcup_{i=1}^{n} E_i = \bigcup_{j=1}^{m} F_j ,$$

$\{E_i\}$ and $\{F_j\}$ are mutually disjoint collections of measurable subsets of $E$.

1. $\int_E(k\phi) = \sum_{i=1}^n (kc_i)\chi_{E_i} = k\sum_{i=1}^n c_i\chi_{E_i} = k\int_E \phi$.
2. $A_{ij} \equiv E_i \cap F_j$. The nonempty sets in the collection of $A_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$, are mutually disjoint measurable sets whose union is $E$.

   Then

$$(\phi + \psi) = \sum_{i=1}^{n}\sum_{j=1}^{m}(c_i + d_j)\chi_{A_{ij}}, \qquad \text{and}$$

$$\begin{aligned}\int_E(\phi + \psi) &= \sum_{i=1}^{n}\sum_{j=1}^{m}(c_i + d_j)\mu(A_{ij}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\mu(E_i \cap F_j) + \sum_{j=1}^{m}\sum_{i=1}^{n} d_j\mu(E_i \cap F_j) \\ &= \sum_{i=1}^{n} c_i \sum_{j=1}^{m}\mu(E_i \cap F_j) + \sum_{j=1}^{m} d_j \sum_{i=1}^{n}\mu(E_i \cap F_j) \\ &= \sum_{i=1}^{n} c_i\mu(E_i) + \sum_{j=1}^{m} d_j\mu(F_j) \\ &= \int_E \phi + \int_E \psi.\end{aligned}$$

3. If $\phi \leq \psi$, then $\psi - \phi$ is a nonnegative simple function on $E$, whose integral will be nonnegative by definition of the integral, and then from parts 1 and 2 we have

$$0 \leq \int_E(\psi - \phi) = \int_E \psi + \int_E -\phi = \int_E \psi - \int_E \phi \qquad \text{and, thus,}$$

$$\int_E \phi \leq \int_E \psi.$$

4. We first observe $\chi_E = \chi_{E_1} + \chi_{E_2}$, $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$.

$$\int_E \psi = \sum_{j=1}^{m} d_j\, \mu(F_j) = \sum_{j=1}^{m} d_j\, \mu\big((F_j \cap E_1) \cup (F_j \cap E_2)\big)$$

$$= \sum_{j=1}^{m} d_j\, \mu(F_j \cap E_1) + \sum_{j=1}^{m} d_j\, \mu(F_j \cap E_2)\,.$$

But $\{F_j \cap E_1\}$, $\{F_j \cap E_2\}$ are collections of mutually disjoint measurable subsets of $E_1, E_2$, respectively, with

$$E_1 = \bigcup_{j=1}^{m} (F_j \cap E_1), \qquad E_2 = \bigcup_{j=1}^{m} (F_j \cap E_2), \qquad \text{and, since}$$

the integral is independent of the "decomposition" used, we have

$$\int_{E_1} \psi = \sum_{j=1}^{m} d_j\, \mu(F_j \cap E_1), \qquad \int_{E_2} \psi = \sum_{j=1}^{m} d_j\, \mu(F_j \cap E_2),$$

and thus

$$\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi\,. \qquad \blacksquare$$

We have completed the preliminaries for extending the definition of the Lebesgue integral from simple functions to bounded functions.

### 5.2.5 Definitions

Suppose $f$ is a bounded function defined on a measurable set $E$ with finite measure; say $\alpha \le f \le \beta$ on $E$, $\mu(E) < \infty$. Let $\phi$ and $\psi$ denote simple functions such that $\phi \le f \le \psi$ on $E$.

The *lower Lebesgue integral* of $f$ on $E$, $\underline{\int}_E f$, is given by

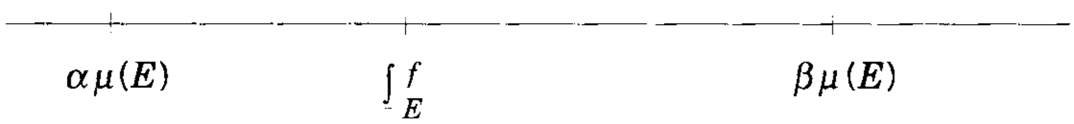
$$\underline{\int}_E f \equiv \sup\left\{ \int_E \phi \mid \phi \le f, \qquad \phi \text{ a simple function} \right\}.$$

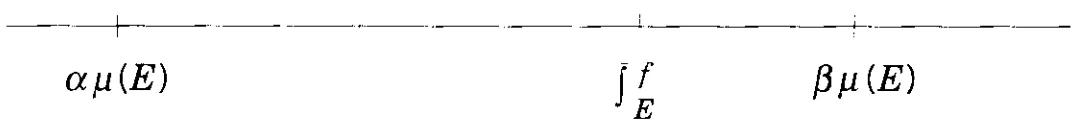
The *upper Lebesgue integral* of $f$ on $E$, $\overline{\int}_E f$, is

$$\overline{\int_E} f \equiv \inf\left\{ \int_E \psi \mid f \leq \psi, \qquad \psi \text{ a simple function} \right\}.$$

### 5.2.6 Comments

1. The constant simple functions $\alpha, \beta$ assure us that the lower and upper Lebesgue integrals are well-defined, since the appropriate sets will be nonempty and bounded above and below.




2. If $\phi \leq f \leq \psi$, $\phi$ and $\psi$ simple functions, then by monotonicity (Proposition 5.3),

$$\int_E \phi \leq \int_E \psi.$$

Because $\psi$ is arbitrary, we interpret this inequality as $\int_E \phi$ is a lower bound for the set $\{\int_E \psi \mid f \leq \psi, \psi \text{ a simple function}\}$. Because $\overline{\int}_E f$ is the greatest lower bound, $\int_E \phi \leq \overline{\int}_E f$. Similarly, $\int_E \phi \leq \overline{\int}_E f$ for $\phi \leq f$; view $\overline{\int}_E f$ as an upper bound. Since $\underline{\int}_E f$ is the least upper bound, we may conclude: $\underline{\int}_E f \leq \overline{\int}_E f$.

We have established that a bounded function $f$ on a set $E$ of finite measure has lower and upper integrals, satisfying

$$\int_E \phi \le \underline{\int_E} f \le \overline{\int_E} f \le \int_E \psi$$

for any simple functions $\phi \le f \le \psi$ on $E$.

### 5.2.7 Definition

A bounded function $f$, defined on a measurable set $E$ with finite measure, is *Lebesgue integrable* on $E$ whenever the lower and upper Lebesgue integrals are the same. Denote the common value by $\int_E f$:

$$\underline{\int_E} f = \int_E f = \overline{\int_E} f.$$

In this case, $\phi \le f \le \psi$ on $E$ implies $\int_E \phi \le \int_E f \le \int_E \psi$ for any simple functions $\phi, \psi$.

### 5.2.8 Comment

The definition of the Lebesgue integral given in 5.2.7 agrees with the original Definition 5.2.3 when $f$ is a simple function: If $f = \hat{\phi}$ a simple function, then $\int_E \hat{\phi} \in \{\int_E \phi \mid \phi \le \hat{\phi}\}$ and by monotonicity $\int_E \phi \le \int_E \hat{\phi}$, i.e., $\int_E \hat{\phi}$ is an upper bound and a member of the set $\{\int_E \phi \mid \phi \le \hat{\phi}\}$. Hence $\underline{\int_E} f = \int_E \hat{\phi}$. Similarly $\overline{\int_E} f = \int_E \hat{\phi}$, and the upper and lower Lebesgue integrals have the common value $\int_E \hat{\phi}$.

The reader should recall that the Riemann integral of a step function agrees with the Lebesgue integral of the same step function (5.2.4) and that every step function is a simple function. The next theorem goes much, much farther. It tells us that the Lebesgue integral exists for any Riemann integrable function and, in fact, the Lebesgue integral and Riemann integral have the same value. The next theorem will also be used in conjunction with Theorem 5.5 to calculate Lebesgue integrals in Sections 5.3, 5.4, and 5.5.

**THEOREM 5.6** *Let $f$ be a bounded function on the interval $[a, b]$. If $f$ is*

*Riemann integrable on* $[a,b]$, *then* $f$ *is Lebesgue integrable on* $[a,b]$, *and*

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,.$$

*Proof:*

$$\begin{aligned}
\underline{\int_a^b} f(x)\,dx &= \sup\left\{\int_a^b \phi(x)\,dx \mid \phi \le f, \qquad \phi \text{ a step function}\right\} \\
&= \sup\left\{\int_{[a,b]} \phi \mid \phi \le f, \qquad \phi \text{ a step function}\right\} \\
&\le \sup\left\{\int_{[a,b]} \phi \mid \phi \le f, \qquad \phi \text{ a simple function}\right\} \\
&= \underline{\int_{[a,b]}} f \\
&\le \overline{\int_{[a,b]}} f \\
&= \inf\left\{\int_{[a,b]} \psi \mid f \le \psi, \qquad \psi \text{ a simple function}\right\} \\
&\le \inf\left\{\int_{[a,b]} \psi \mid f \le \psi, \qquad \psi \text{ a step function}\right\} \\
&= \inf\left\{\int_a^b \psi(x)\,dx \mid f \le \psi, \qquad \psi \text{ a step function}\right\} \\
&= \overline{\int_a^b} f(x)\,dx\,.
\end{aligned}$$

Since $f$ is Riemann integrable, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ and the conclusion follows. ■

So Riemann integrability implies Lebesgue integrability. That the Lebesgue integral generalizes the Riemann integral, that there are functions that we can integrate in the Lebesgue sense and not in the Riemann sense, is revealed by

**Example 4:** $f(x) = \begin{cases} 1, & x \text{ rational}, \ 0 \le x \le 1 \\ 0, & x \text{ irrational}, \ 0 \le x \le 1 \, . \end{cases}$

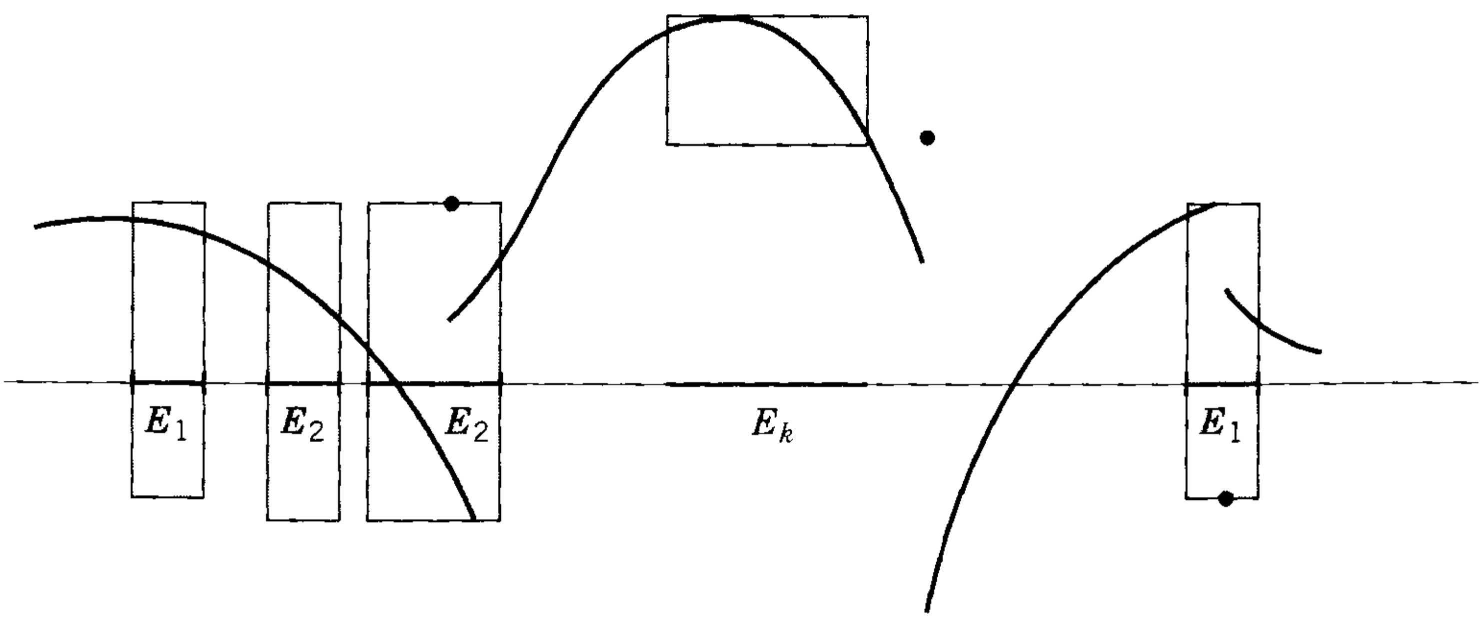
$\underline{\int}_0^1 f(x)dx = 0$, $\overline{\int}_0^1 f(x)dx = 1$. Thus $f$ is not Riemann integrable. However it is trivally Lebesgue integrable. It is a simple function after all, and

$$\int_{[0,1]} f = 1\,\mu(\{\text{rationals}\} \cap [0,1]) = 0\,.$$

Lebesgue characterized the types of bounded functions that are Riemann integrable (Theorem 5.1): continuous almost everywhere. We will eventually show that bounded Lebesgue measurable functions play an analogous role for Lebesgue integration. Before beginning that undertaking, we have a criteria for Lebesgue integrability similar to Proposition 5.2:

**PROPOSITION 5.4** *A bounded function $f$, defined on a set $E$ with finite measure, is Lebesgue integrable iff for every $\epsilon > 0$ we have simple functions $\phi$ and $\psi$, $\phi \le f \le \psi$ on $E$, so that*

$$0 \le \int_E \psi - \int_E \phi = \int_E (\psi - \phi) < \epsilon\,.$$



The reader should compare this figure with that of Proposition 5.2. In particular, $E_1$ and $E_2$ are not necessarily intervals.

*Proof:* Assume the bounded function $f$ is Lebesgue integrable on the measurable set $E$, $\mu(E) < \infty$, and let $\epsilon > 0$. From the definitions of greatest lower bound and least upper bound we have simple functions $\hat{\phi}$ and $\hat{\psi}$, $\hat{\phi} \le f \le \hat{\psi}$ on $E$, so that

$$\int_E f - \frac{\epsilon}{2} = \underline{\int}_E f - \frac{\epsilon}{2} < \int_E \hat{\phi} \le \underline{\int}_E f$$

$$\le \overline{\int}_E f \le \int_E \hat{\psi} < \overline{\int}_E f + \frac{\epsilon}{2} = \int_E f + \frac{\epsilon}{2}.$$

Thus $0 \le \int_E \hat{\psi} - \int_E \hat{\phi} = \int_E (\hat{\psi} - \hat{\phi}) < \epsilon$.

For the other direction, let $\epsilon > 0$ be given along with simple functions $\phi$ and $\psi$, $\phi \le f \le \psi$, so that $0 \le \int_E \psi - \int_E \phi = \int_E (\psi - \phi) < \epsilon$. Then again, from greatest lower bound and least upper bound properties (5.2.6)

$$\int_E \phi \le \underline{\int}_E f \le \overline{\int}_E f \le \int_E \psi.$$

Hence $0 \le \overline{\int}_E f - \underline{\int}_E f < \epsilon$ and the conclusion follows from the arbitrary nature of $\epsilon$.

*Note:* When the bounded function $f$ is Lebesgue integrable on $E$, $\mu(E) < \infty$, we have $\phi \le f \le \psi$ on $E$, $\int_E \phi \le \int_E f \le \int_E \psi$, and $\int_E (\psi - \phi) < \epsilon$, for some simple functions $\phi$ and $\psi$. ∎

We now develop some examples, using the Fundamental Idea of Lebesgue: Partition the range. Then the domain of $f$ will be decomposed, hopefully, into measurable sets if $f$ is not too badly behaved.

In these examples $\int_E f = \int_a^b f(x)dx$. Calculate first the Riemann integral and then the Lebesgue integral as indicated.

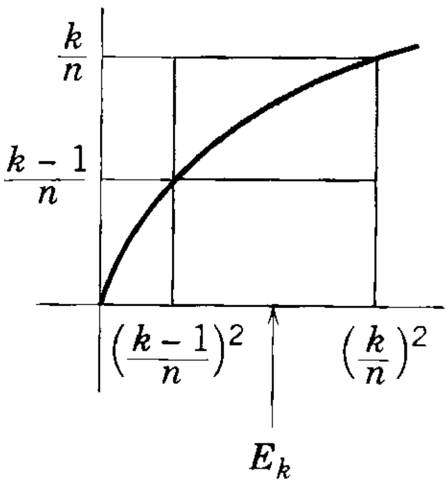
**Example 5:**

1. Let

$f(x) = \sqrt{x}, \qquad 0 \le x \le 1.$

$$\hat{\phi}(x) = \frac{k-1}{n}, \qquad \left(\frac{k-1}{n}\right)^2 \le x < \left(\frac{k}{n}\right)^2, \qquad 1 \le k \le n, \ \hat{\phi}(1) = 1;$$

$$\hat{\psi}(x) = \frac{k}{n}, \qquad \left(\frac{k-1}{n}\right)^2 \le x < \left(\frac{k}{n}\right)^2, \qquad 1 \le k \le n, \ \hat{\psi}(1) = 1.$$




$$\int_{[0,1]} (\hat{\psi} - \hat{\phi}) = \frac{1}{n}\sum_1^n \left[\left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2\right] = \frac{1}{n}$$

Thus $\int_{[0,1]} \sqrt{x}$ exists.

Also,

$$2\sum\left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n} < \sum \frac{(k-1)(2k-1)}{n^3} = \sum\left(\frac{k-1}{n}\right)\left[\left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2\right]$$
$$< \sum\left(\frac{k}{n}\right)\left[\left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2\right] < 2\sum\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}.$$

Thus

$$\frac{2}{3} = 2 \cdot \int_0^1 x^2\,dx = \int_{[0,1]} \sqrt{x}.$$

2.

$f(x) = e^x, \quad 0 \le x \le \ln 2$

$$\hat{\phi}(x) = 1 + \frac{k-1}{n}, \quad \ln\left(1 + \frac{k-1}{n}\right) \le x < \ln\left(1 + \frac{k}{n}\right), \quad f \text{ otherwise;}$$

$$\hat{\psi}(x) = 1 + \frac{k}{n}, \quad \ln\left(1 + \frac{k-1}{n}\right) \le x < \ln\left(1 + \frac{k}{n}\right), \quad f \text{ otherwise.}$$

Then

$$\int_{[0,\ln 2]} (\hat{\psi} - \hat{\phi}) = \frac{1}{n} (\ln 2 - 0)$$

$$= \frac{1}{n} \ln 2 < \frac{1}{n}.$$

Thus $\int_{[0,\ln 2]} e^x$ exists.
Show

$$1 - \frac{1}{n+1} < n \ln\left(1 + \frac{1}{n}\right)$$

$$< (n+1) \ln\left(1 + \frac{1}{n}\right) < 1 + \frac{1}{n} \quad \text{i.e.,}$$

$$1 - \frac{1}{n+1} < \underline{\int}_{[0,\ln 2]} e^x \leq \overline{\int}_{[0,\ln 2]} e^x < 1 + \frac{1}{n},$$

and conclude $\int_{[0,\ln 2]} e^x = 1 = \int_0^{\ln 2} e^x \, dx$.

3.

$$f(x) = \begin{cases} 1, & x \in [\frac{1}{2}, 1] \cup [\frac{1}{4}, \frac{1}{3}] \cup \cdots \\ 0, & \text{otherwise} \end{cases}, \qquad 0 \leq x \leq 1.$$

$$\int_{[0,1]} f = \sum \mu\left( \left[\frac{1}{2k}, \frac{1}{2k-1}\right] \right) = \ln 2.$$

4. $f(x) = \begin{cases} x^{1/3}, & 0 \leq x < 1 \\ 2, & 1 < x \leq 2 \end{cases}.$

Let $E_k = \{x \mid (k-1)/n \leq f(x) < k/n\}$, with $\hat{\phi}, \hat{\psi}$ defined in the obvious way. Calculate $\int_{[0,2]} f$.

5. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$ $E_k$ as in 4.

$$\int_E (\hat{\psi} - \hat{\phi}) = \frac{1}{n} \sum \mu(E_k) = \frac{2}{n}.$$

6. Suppose $f$ is continuous on $[a, b]$. Thus $m < f < M$ on $[a, b]$. Let $E_k = \{x \mid (k-1)((M-m)/n) \leq f(x) < k((M-m)/n)\}$. Is $E_k$ measurable?

   Let

$$\hat{\phi}(x) = (k-1)\frac{M-m}{n}, \qquad x \in E_k \qquad \text{and}$$

$$\hat{\psi}(x) = k\frac{M-m}{n}, \qquad x \in E_k.$$

   Then

$$\int_{[a,b]} (\hat{\psi} - \hat{\phi}) = \frac{M-m}{n} \sum \mu(E_k) = \frac{M-m}{n}(b-a).$$

   Compare with 5.1.9.

We are now ready to prove the Lebesgue integral counterpart of Theorem 5.1; a characterization of Lebesgue integrability in terms of Lebesgue measurable functions.

**THEOREM 5.7** *Let $f$ be a bounded function on a set $E$ with finite measure. Then $f$ is Lebesgue integrable on $E$ iff $f$ is measurable on $E$.*

*Proof:* Let $|f| \leq M$ on $E$ and assume $f$ is measurable on $E$. We will show $f$ is Lebesgue integrable by constructing simple functions $\hat{\phi}$ and $\hat{\psi}$, $\hat{\phi} \leq f \leq \hat{\psi}$ on $E$, so that

$$0 \leq \int_E (\hat{\psi} - \hat{\phi}) < \epsilon.$$

Let $E_k = \{x \mid ((k-1)/n)M < f(x) \leq (k/n)M\}$, $-n \leq k \leq n$. Then $E = \bigcup E_k$, $E_k$ mutually disjoint measurable sets. (why?) We define $\hat{\phi}, \hat{\psi}$:

$$\hat{\phi} = \frac{M}{n} \sum_{-n}^{n} (k-1)\chi_{E_k} \qquad \text{and} \qquad \hat{\psi} = \frac{M}{n} \sum_{-n}^{n} k\chi_{E_k}.$$

Obviously $\hat{\phi} \leq f \leq \hat{\psi}$ and $0 \leq \int_E (\hat{\psi} - \hat{\phi}) = (M/n)\mu(E)$. Thus $f$ is Lebesgue integrable on $E$ by Proposition 5.4. Bounded, measurable functions are integrable on sets of finite measure.

Now we assume $f$ is Lebesgue integrable and bounded on the set $E$, $\mu(E) < \infty$. We will show $f$ is a measurable function on $E$ by showing that $f$ equals almost everywhere the "inf" of a sequence of simple functions. Then the result will follow by Theorem 4.1.

Because $f$ is bounded and Lebesgue integrable on $E$, we have simple functions $\phi_n$ and $\psi_n$ so that $\phi_n \le f \le \psi_n$ on $E$, $\int_E \phi_n \le \int_E f \le \int_E \psi_n$, and

$$\int_E (\psi_n - \phi_n) < \frac{1}{n}, \qquad n = 1, 2, 3, \ldots .$$

Define two measurable functions (Theorem 4.1):

$$\phi^* = \sup\{\phi_1, \phi_2, \ldots\} \qquad \text{and} \qquad \psi^* = \inf\{\psi_1, \psi_2, \ldots\}.$$

Certainly $\phi_n \le \phi^* \le f \le \psi^* \le \psi_n$ on $E$ for all $n \ge 1$. We want to show $\phi^* = \psi^*$ almost everywhere on $E$ and thus conclude $f = \psi^*$ a.e. By Proposition 4.3 $f$ will be measurable on $E$. Consider the set

$$\{x \in E \mid \psi^*(x) - \phi^*(x) > 0\} = \bigcup_m \left\{x \in E \mid \psi^*(x) - \phi^*(x) > \frac{1}{m}\right\}$$

$$\subset \bigcup_m \left\{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\right\}$$

for all $n \ge 1$. If we show $\{x \in E \mid \psi^*(x) - \phi^*(x) > 1/m\}$ has measure zero we would be finished. We know this set is measurable because $\psi^* - \phi^*$ is a measurable function.

By construction, $\int_E (\psi_n - \phi_n) < 1/n$. But then $1/n > \int_E (\psi_n - \phi_n) > (1/m)\mu(\{x \mid \psi_n(x) - \phi_n(x) > 1/m\})$ by application of Proposition 5.3 with $E_1 = \{x \mid \psi_n - \phi_n > 1/m\}$ and $E_2 = \{x \mid \psi_n - \phi_n \le 1/m\}$. Thus

$$\mu\left(\{x \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\}\right) < \frac{m}{n} \qquad \text{for all } n \ge 1.$$

i.e., $\mu(\{x \mid \psi^*(x) - \phi^*(x) > 1/m\}) = 0$ and the proof is completed by recalling that a countable union of sets of measure zero is a measurable set of measure zero. ∎

We have characterized Lebesgue integrability. We now show that the Lebesgue integral has the important properties of linearity and monotonicity.

**THEOREM 5.8** *If the bounded functions $f$ and $g$ are measurable on the set $E$, $\mu(E) < \infty$, and $k$ is any real number, then $f$ and $g$ are Lebesgue integrable on $E$ and*

1. *$(kf)$ is Lebesgue integrable on $E$, and $\int_E (kf) = k \int_E f$ (homogeneous);*
2. *$(f+g)$ is Lebesgue integrable on $E$, and $\int_E (f+g) = \int_E f + \int_E g$ (additive);*
3. *$\int_E f \leq \int_E g$ if $f \leq g$ on $E$ (monotone);*
4. *If $E_1$ and $E_2$ are disjoint measurable subsets of $E$ with $E = E_1 \cup E_2$, $f$ is Lebesgue integrable on $E_1$ and $E_2$, and*

$$\int_E f = \int_{E_1} f + \int_{E_2} f \qquad \textit{(additive on the domain)};$$

5. *If $\alpha \leq f \leq \beta$ on $E$, $\alpha\mu(E) \leq \int_E f \leq \beta\mu(E)$ (mean value).*

*Proof:* Existence of Lebesgue integrability for the various parts 1 through 5 follows immediately from the previous theorem. However, the following arguments are of interest:

1. Let $k > 0$. Because $f$ is Lebesgue integrable, we have simple functions $\phi$ and $\psi$ so that $\phi \leq f \leq \psi$, $\int_E \phi \leq \int_E f \leq \int_E \psi$ and $\int_E (\psi - \phi) < \epsilon/k$. But then $k\phi \leq kf \leq k\psi$, $k \int_E \phi \leq k \int_E f \leq k \int_E \psi$ and $k \int_E (\psi - \phi) = \int_E (k\psi - k\phi) < \epsilon$. The last inequality implies $kf$ is Lebesgue integrable. Thus $k \int_E \phi = \int_E k\phi \leq \int_E kf \leq \int_E k\psi = k \int_E \psi$. So $k \int_E f$ and $\int_E kf$ are between $\int_E k\phi$ and $\int_E k\psi$, with $\int_E k(\psi - \phi) < \epsilon$. The reader may complete the argument when $k = 0$ and $k < 0$.
2. $f$ is Lebesgue integrable: $\phi_f \leq f \leq \psi_f$, $\int_E \phi_f \leq \int_E f \leq \int_E \psi_f$ with

$$\int_E (\psi_f - \phi_f) < \frac{\epsilon}{2}.$$

$g$ is Lebesgue integrable: $\phi_g \leq g \leq \psi_g$, $\int_E \phi_g \leq \int_E f \leq \int_E \psi_g$ with $\int_E (\psi_g - \phi_g) < \epsilon/2$. Adding, we have $\phi_f + \phi_g \leq f + g \leq \psi_f + \psi_g$, $\int_E (\phi_f + \phi_g) \leq \int_E f + \int_E g \leq \int_E (\psi_f + \psi_g)$ with $\int_E [(\psi_f + \psi_g) - (\phi_f + \phi_g)] < \epsilon$. Again this last inequality implies $f + g$ is Lebesgue integrable on $E$ and

$$\int_E (\phi_f + \phi_g) \leq \int_E (f + g) \leq \int_E (\psi_f + \psi_g).$$

Thus $\int_E f + \int_E g$ and $\int_E (f+g)$ are between $\int_E(\phi_f + \phi_g)$ and $\int_E(\psi_f + \psi_g)$, where $\int_E[(\psi_f + \psi_g) - (\phi_f + \phi_g)] < \epsilon$.

3. $\int_E g - \int_E f = \int_E (g-f)$ by parts 1 and 2. Since $g - f \geq 0$, let $\hat{\phi} \equiv 0$. Then $0 = \int_E \hat{\phi} \leq \int_E (g-f)$.
4. Because $f$ is Lebesgue integrable on $E$, we have simple functions $\phi \leq f \leq \psi$ on $E$ so that

$$\int_E \phi \leq \int_E f \leq \int_E \psi$$

and $\int_E (\psi - \phi) < \epsilon$.

Since $\int_E \phi = \int_{E_1} \phi + \int_{E_2} \phi$ and $\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi$ (Proposition 5.3), we conclude $\phi \leq f \leq \psi$ on $E_1$ and $\int_{E_1}(\psi - \phi) < \epsilon$ and $\phi \leq f \leq \psi$ on $E_2$ and $\int_{E_2}(\psi - \phi) < \epsilon$. Thus $f$ is Lebesgue integrable on $E_1$ and $E_2$ and

$$\int_E \phi = \int_{E_1} \phi + \int_{E_2} \phi \leq \int_{E_1} f + \int_{E_2} f \leq \int_{E_1} \psi + \int_{E_2} \psi = \int_E \psi .$$

Again, $\int_{E_1} f + \int_{E_2} f$ and $\int_E f$ are between $\int_E \phi$ and $\int_E \psi$. Thus $\int_E f = \int_{E_1} f + \int_{E_2} f$ by the arbitrary nature of $\epsilon$.
5. Immediate from part 3:

$$\alpha\mu(E) = \int_E \alpha \leq \int_E f \leq \int_E \beta = \beta\mu(E) .$$

The argument is complete. ∎

The next result shows one of the important advantages of the Lebesgue integral over the Riemann integral: If we change the values of a Riemann integrable function on a set of measure zero we may destroy integrability. This is illustrated by the functions $f(x) = 0$, $0 \leq x \leq 1$,

$$g(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} . \end{cases}$$

Here $\int_0^1 f(x)\,dx = 0$, $g$ differs from $f$ on a countable set (rationals, measure zero), but $\int_0^1 g(x)\,dx$ does not exist (Problem 5.1.10).

On the other hand, Lebesgue integrability is independent of unpleasant behavior on a set of measure zero.

**THEOREM 5.9** *If $f$ is a bounded, Lebesgue integrable function on a set $E$ of finite measure, and $g$ is a bounded function on $E$ such that $g = f$ a.e. on $E$, then $g$ is Lebesgue integrable on $E$ and*

$$\int_E g = \int_E f .$$

*Proof:* The function $f$ is measurable by Theorem 5.7 and application of Proposition 4.3 yields measurability for $g$, and thus integrability for $g$ (Theorem 5.7). Let $A = \{x \mid f(x) \neq g(x)\}$. The set $A$ has measure zero, thus $E - A$ is measurable and, by Theorem 5.8,

$$\int_E f = \int_{E-A} f + \int_A f = \int_{E-A} g = \int_{E-A} g + \int_A g = \int_E g . \qquad \blacksquare$$

**Example 6:**

1.

$$g(x) = \begin{cases} 2x, & x \text{ rational} \\ 1-x, & x \text{ irrational} \end{cases}, \qquad 0 \leq x \leq 1$$

and $f(x) = 1 - x,\ 0 \leq x \leq 1$. Thus

$$\int_{[0,1]} f = \int_{[0,1]} (1-x) = \int_0^1 (1-x)dx = \frac{1}{2} = \int_{[0,1]} g .$$

$\int_0^1 g(x)dx$ does not exist since $g$ is continuous at only one point, $x = 1/3$.

2.

$$g(x) = \begin{cases} x^2, & x \in C \cap \{\text{irrationals}\} \\ x, & x \in C \cap \{\text{rationals}\} \\ 1, & x \notin C \end{cases}, \qquad 0 \leq x \leq 1 ,$$

$C$ the Cantor set, and $f(x) = 1,\ 0 \leq x \leq 1$. Then $f$ is Riemann integrable, and

$$\int_{[0,1]} f = \int_0^1 1\,dx = 1 = \int_{[0,1]} g .$$

3.

$$g(x) = \begin{cases} \frac{1}{q}, & x = p/q; \quad p, q \text{ integers}, \ (p,q) = 1 \\ 0, & \text{otherwise}, \end{cases} \quad , \quad 0 \le x \le 1$$

and $f(x) = 0, \ 0 \le x \le 1$. Thus

$$\int_{[0,1]} f = \int_0^1 f(x)dx = 0 = \int_{[0,1]} g = \int_0^1 g(x)dx$$

since $g$ is continuous a.e. and thus Riemann integrable.

Sometimes Riemann integrals are easier to calculate because of Theorem 5.9.

### 5.2.9 SUMMARY

***Riemann Integral***

1. $\phi, \psi$ step functions on $[a, b]$:

$$\phi(x) = \begin{cases} c_k, & x_{k-1} < x < x_k, \qquad k = 1, 2, \ldots, n \\ d_k, & x = x_k, \qquad k = 0, 1, \ldots, n, \end{cases}$$

where

$$[a, b] = \bigcup (x_{k-1}, x_k) \cup \{x_0, x_1, \ldots, x_n\}.$$

2. $\displaystyle\int_a^b \phi(x)dx \equiv \sum_1^n c_k(x_k - x_{k-1})$

3. $\phi \le f \le \psi$; $\phi, \psi$ step functions, $f$ bounded:

$$\underline{\int_a^b} f(x)dx \equiv \sup\left\{\int_a^b \phi(x)dx \mid \phi \le f\right\}$$

and

$$\overline{\int_a^b} f(x)dx \equiv \inf\left\{\int_a^b \psi(x)dx \mid f \le \psi\right\}.$$

4. If $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$ for the bounded function $f$, $f$ is Riemann integrable on $[a, b]$. Denoted by $\int_a^b f(x)dx$.

5. A bounded function $f$ is Riemann integrable on $[a,b]$ iff $f$ is continuous a.e. on $[a,b]$.

### Lebesgue Integral

1. $\phi, \psi$ simple functions on $E$, $\mu(E) < \infty$:

$$\phi(x) = \sum_1^n c_k \chi_{E_k}(x)$$

where $E = \bigcup E_k$, $E_k$ mutually disjoint Lebesgue measurable sets.

2. $$\int_E \phi \equiv \sum_1^n c_k \mu(E_k).$$

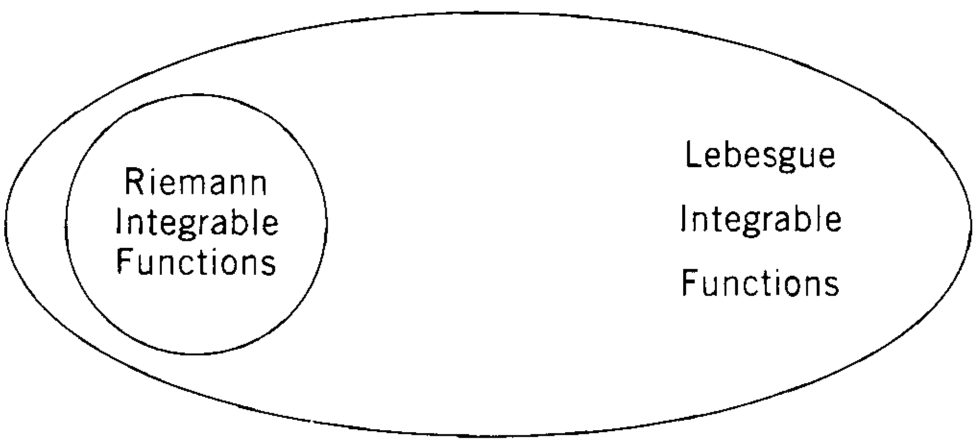
3. $\phi \le f \le \psi$; $\phi, \psi$ simple functions, $f$ bounded:

$$\underline{\int}_E f \equiv \sup\left\{\int_E \phi \mid \phi \le f\right\}$$

and

$$\overline{\int}_E f \equiv \inf\left\{\int_E \psi \mid f \le \psi\right\}.$$

4. If $\underline{\int}_E f = \overline{\int}_E f$ for the bounded function $f$, $f$ is Lebesgue integrable on $E$. Denoted by $\int_E f$.
5. A bounded function $f$ is Lebesgue integrable on $E$, $\mu(E) < \infty$, iff $f$ is Lebesgue measurable on $E$.




We now remove the restrictions that $\mu(E) < \infty$ and $f$ be bounded.

## 5.3 THE LEBESGUE INTEGRAL FOR NONNEGATIVE MEASURABLE FUNCTIONS

This section is the starting point for many treatments of the Lebesgue integral, and the reader who has omitted the previous two sections may treat it as such. For those who have studied the previous two sections, this material may be viewed as a natural generalization in the following progression, or evolution, of ideas:

Riemann integral for bounded functions on closed, bounded intervals via step functions $\longrightarrow$

Lebesgue integral for bounded functions on sets of finite measure via simple functions $\longrightarrow$

Lebesgue integral for nonnegative measurable functions on measurable sets via approximation by nonnegative monotone sequences of simple functions $\longrightarrow$

Lebesgue integral for measurable functions via: $f = f^+ - f^-$, with $f^+$ and $f^-$ nonnegative measurable functions;
Define $\int_E f = \int_E f^+ - \int_E f^-$.

Examples will clarify some difficulties and problems faced when dealing with measurable (sometimes unbounded) functions that are both positive and negative.

**Example 7:**

1. $f(x) = \begin{cases} 1/x, & 0 < x \leq 1 \\ 1/(x-2), & 1 < x < 2. \end{cases}$ $f$ is unbounded (positive and negative).

How to define $\int_{(0,2)} f$? Maybe

$$\int_{(0,2)} f \stackrel{?}{=} \int_{(0,1]} f + \int_{(1,2)} f$$

$$\stackrel{?}{=} \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{x}\, dx + \lim_{\epsilon \to 0} \int_1^{2-\epsilon} \frac{1}{x-2}\, dx$$

$$\stackrel{?}{=} \lim_{\epsilon \to 0} \int_{(\epsilon,1]} \frac{1}{x} + \lim_{\epsilon \to 0} \int_{(1,2-\epsilon)} \frac{1}{x-2}$$

$$= \infty - \infty \text{ (not defined).}$$

2. $f(x) = \begin{cases} x^{-1/2}, & 0 < x \le 1 \\ 1/(x-2), & 1 < x < 2. \end{cases}$

$$\int_{(0,2)} f \stackrel{?}{=} \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-\frac{1}{2}} dx + \lim_{\epsilon \to 0} \int_1^{2-\epsilon} \frac{1}{x-2} dx$$

$$= 2 - \infty = -\infty .$$

3. $f(x) = \begin{cases} x^{-1/2}, & 0 < x \le 1 \\ -(2-x)^{-1/2}, & 1 < x < 2. \end{cases}$

$$\int_{(0,2)} f \stackrel{?}{=} \int_{(0,1]} f + \int_{(1,2)} f$$

$$\stackrel{?}{=} \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-\frac{1}{2}} dx + \lim_{\epsilon \to 0} \int_1^{2-\epsilon} \frac{-1}{\sqrt{2-x}} dx$$

$$= 2 - 2 = 0 .$$

4. $f(x) = x^{-1/2},\ 0 < x \le 1$. $f$ is nonnegative and unbounded, and

$$\int_{(0,1]} f = \int_{(0,1]} x^{-\frac{1}{2}} \stackrel{?}{=} \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-\frac{1}{2}} dx = 2 .$$

5. $f(x) = 1/x^2,\ x \ge 1$. $f$ is nonnegative and bounded on a set of infinite measure, with

$$\int_{[1,\infty)} \frac{1}{x^2} \stackrel{?}{=} \lim_{R \to \infty} \int_1^R \frac{1}{x^2} dx = 1.$$

6. $f(x) = \begin{cases} (-1)^n (n+1)^{-1}, & n\pi < x < (n+1)\pi \\ 0, & \text{otherwise} \end{cases}; \quad n = 0, 1, 2, \ldots .$

$$\int_{[0,\infty)} f \stackrel{?}{=} \pi\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \pi \ln 2 .$$

7. $f(x) = \begin{cases} \sin(x)/x, & x \neq 0 \\ 1, & x = 0\,. \end{cases}$

$$\int_{[0,\infty)} f \stackrel{?}{=} \lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}\,.$$

In these examples, the functions, being unbounded, causes problems with integral interpretations, since the Riemann integral is only defined for bounded functions, as has been the Lebesgue integral (so far). The functions, being both positive and negative, add further complications.

One thing at a time. Recall (Proposition 4.6) that any measurable function can be written as the difference of two **nonnegative** measurable functions: $f = f^+ - f^-$. As a result of this decomposition, it is natural to define the Lebesgue integral of $f$, whether $f$ is positive, negative, or both, as

$$\int_E f = \int_E f^+ - \int_E f^-\,,$$

and this will be meaningful if we have meaning for $\int_E f^+$ and $\int_E f^-$ in some sense.

This will be our approach then: restrict our attention to nonnegative measurable functions whose domains need not have finite measure, and then in the next section remove the condition that $f$ be nonnegative.

**Example 8:** $f$ nonnegative and unbounded:

1. $f(t) = \begin{cases} t^{-1}, & 0 < t \leq 1 \\ (2-t)^{-1/2}, & 1 \leq t < 2\,. \end{cases}$

$$\int_{(0,2)} f \stackrel{?}{=} \int_{(0,1]} f + \int_{(1,2)} f \stackrel{?}{=} \infty + 2 = \infty\,.$$

2. $f(t) = 1/t, \qquad t \geq 1.$

$$\int_{[1,\infty)} f \stackrel{?}{=} \lim_{R\to\infty} \int_{[1,R]} \frac{1}{t} \stackrel{?}{=} \lim_{R\to\infty} \int_1^R \frac{1}{t}\,dt = \infty\,.$$

3. $f(t) = 1/t^2, \qquad t \geq 1.$

$$\int_{[1,\infty)} f \stackrel{?}{=} \lim_{R\to\infty} \int_{[1,R]} \frac{1}{t^2} = \lim_{R\to\infty} \int_1^R \frac{1}{t^2}\,dt = 1\,.$$

4. $f(t) = 1/t, \qquad t > 0.$

$$\int_{(0,\infty)} \frac{1}{t} \overset{?}{=} \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{t}\,dt + \lim_{\epsilon \to 0^+} \int_1^{\frac{1}{\epsilon}} \frac{1}{t}\,dt$$

$$\overset{?}{=} \lim_{\epsilon \to 0^+} \int_{(\epsilon,1)} \frac{1}{t} + \lim_{\epsilon \to 0^+} \int_{(1,\frac{1}{\epsilon})} \frac{1}{t}$$

$$\overset{?}{=} \infty + \infty = \infty\,.$$

5. $f(x) = \begin{cases} t^{-1/2}, & 0 < t < 1 \\ t^{-2}, & t \geq 1 \end{cases}.$

$$\int_{(0,\infty)} f \overset{?}{=} \int_{(0,1)} t^{-\frac{1}{2}} + \int_{[1,\infty)} t^{-2}$$

$$\overset{?}{=} \lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} t^{-\frac{1}{2}}dt + \lim_{R \to \infty} \int_1^R t^{-2}dt$$

$$= 2 + 1\,.$$

How do we define the Lebesgue integral for a nonnegative, not necessarily bounded, measurable function? That is the problem! The idea that "works", is inherent in the Approximation Theorem 4.2: We can approximate any nonnegative measurable function $f$ with a monotone increasing sequence of nonnegative simple functions; $0 \leq \phi_n \leq \phi_{n+1} \leq \cdots$, $\lim \phi_n = f$ on $E$. It is natural to define the Lebesgue integral of $f$ on $E$, $\int_E f$, by

$$\int_E f = \int_E (\lim \phi_n) = \lim \int_E \phi_n\,.$$

Roughly speaking, we must make sense of $\int_E \phi_n$, that is, the Lebesgue integral of a nonnegative simple function. But that is easy to do (if you've read the previous section some of the material that follows will be repetitious), and since simple functions are by definition bounded, boundedness will cause no difficulties. We begin with simple functions as linear combinations of characteristic functions.

### 5.3.1 Definition

Let $\phi$ be a *nonnegative simple function* on $R$, that is,

$$\phi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x),$$

where $E_k$ are mutually disjoint Lebesgue measurable subsets of $R$,

$$R = \bigcup_{k=1}^{n} E_k,$$

and $c_k$ are nonnegative real numbers (*Note*: Nothing lost in assuming

$$\bigcup_{k=1}^{n} E_k = R,$$

for if not, then

$$E_0 \equiv R - \bigcup_{k=1}^{n} E_k$$

is a measurable set and

$$R = \bigcup_{k=0}^{n} E_k.).$$

Thus $\mu(E_k)$ will be infinite for at least one $k$, $1 \le k \le n$.

Obviously a simple function $\phi$ has many representations as linear combinations of characteristic functions.

### 5.3.2 Definition

The *Lebesgue integral* of a nonnegative simple function $\phi$, on a measurable set $E$, written $\int_E \phi$, is defined by

$$\int_E \phi = \sum_{1}^{n} c_k \mu(E \cap E_k),$$

where $\phi = \sum_{1}^{n} c_k \chi_{E_k}$, $E_k$ mutually disjoint, $R = \bigcup_{1}^{n} E_k$, $c_k \ge 0$.

We employ the usual convention of defining $\int_E \phi = 0$ whenever $\phi = 0$ on $E$, even if $\mu(E) = \infty$ $(0 \cdot \infty = 0)$. Also, if $\mu(E) = 0$, then $\mu(E \cap E_k) = 0$ for all $k$ and $\int_E \phi = 0$. The problem, $\infty - \infty$, does not arise because $c_k \geq 0$.

### 5.3.3 Comments

1. For nonnegative simple functions on $E$, $\mu(E) < \infty$, we are in agreement with our previous Definition 5.2.3.
2. Even though a given simple function $\phi$ has many representations, the Lebesgue integral of $\phi$ is well-defined: Suppose

$$\phi = \sum_1^n c_k \chi_{E_k}, \quad c_k \geq 0$$

and

$$\phi = \sum_1^m d_j \chi_{D_j}, \quad d_j \geq 0$$

with

$$R = \bigcup_1^n E_k = \bigcup_1^m D_j,$$

$E_k$ and $D_j$ mutually disjoint measurable subsets of $R$. We show that

$$\sum_1^n c_k \mu(E \cap E_k) = \sum_1^m d_j \mu(E \cap D_j),$$

that is, the integral as we have defined it is independent of the representation of $\phi$.

Note that

$$E_k = E_k \cap \left( \bigcup_1^m F_j \right) = \bigcup_1^m (E_k \cap F_j)$$

and

$$F_j = F_j \cap \left( \bigcup_1^n E_k \right) = \bigcup_1^n (E_k \cap F_j).$$

Thus

$$\begin{aligned}
\sum_1^n c_k\mu(E\cap E_k) &= \sum_1^n c_k\mu\left(E\cap\left(\bigcup_1^m (E_k\cap F_j)\right)\right)\\
&= \sum_1^n c_k\mu\left(\bigcup_1^m (E\cap E_k\cap F_j)\right)\\
&= \sum_1^n c_k\sum_1^m \mu(E\cap E_k\cap F_j)\\
&= \sum_1^n\sum_1^m d_j\mu(E\cap E_k\cap F_j)\\
&= \sum_1^m\sum_1^n d_j\mu(E\cap E_k\cap F_j)\\
&= \sum_1^m d_j\sum_1^n \mu(E\cap E_k\cap F_j)\\
&= \sum_1^m d_j\mu\left(E\cap\left(\bigcup_1^n (E_k\cap F_j)\right)\right)\\
&= \sum_1^m d_j\mu(E\cap F_j)
\end{aligned}$$

since, for $E_k\cap E\cap F_j\neq\emptyset$, $c_k = c_k\chi_{E_k} = \phi = d_j\chi_{F_j} = d_j$, and if $E_k\cap E\cap F_j = \emptyset$, no contribution because $\mu(\emptyset) = 0$.

3. The Lebesgue integral of a nonnegative simple function is a nonnegative real number or $\infty$.

**PROPOSITION 5.5** *If $\phi, \psi$ are nonnegative simple functions on $R$, if $E$ is any measurable subset of $R$, and $k$ is any nonnegative real number, then*

*1. $(k\phi)$ is a nonnegative simple function on $E$, and*

$$\int_E (k\phi) = k\int_E \phi \qquad (\textit{homogeneous});$$

2. *$(\phi + \psi)$ is a nonnegative simple function on $E$, and*

$$\int_E (\phi + \psi) = \int_E \phi + \int_E \psi \qquad (\textit{additive});$$

3. $$\int_E \phi \leq \int_E \psi \textit{ if } 0 \leq \phi \leq \psi \textit{ on } E \qquad (\textit{monotone});$$

4. *If $E_1$ and $E_2$ are disjoint measurable subsets of $E$ with $E = E_1 \cup E_2$, the integrals $\int_{E_1} \psi$ and $\int_{E_2} \psi$ exist, and*

$$\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi \qquad (\textit{additive on the domain}).$$

*Proof:*

1. Suppose $\phi = \sum_1^n c_i \chi_{E_i}$. Then $k\phi = \sum k c_i \chi_{E_i}$ and

$$\int_E (k\phi) = \sum_1^n (kc_i)\mu(E \cap E_i) = k \sum_1^n c_i \mu(E \cap E_i) = k \int_E \phi .$$

2. Let $\phi = \sum_1^n c_k \chi_{E_k}$ and $\psi = \sum_1^m d_j \chi_{F_j}$, $0 \leq c_k, d_j$. The idea is to form the $n \cdot m$ sets;

$$\begin{array}{l} E_1 \cap F_1,\ E_1 \cap F_2, \ldots,\ E_1 \cap F_m \\ E_2 \cap F_1,\ E_2 \cap F_2, \ldots,\ E_2 \cap F_m \\ \vdots \\ E_n \cap F_1,\ E_n \cap F_2, \ldots,\ E_n \cap F_m . \end{array}$$

If $E_k \cap F_j \neq \emptyset$, define $\phi + \psi$ as $c_k + d_j$. The nonempty $E_k \cap F_j$ are mutually disjoint measurable subsets of $R$,

$$R = \bigcup_{k,j} (E_k \cap F_j),$$

and

$$\phi + \psi = \sum_{k,j} (c_k + d_j)\chi_{E_k \cap F_j}.$$

Thus

$$\int_E (\phi + \psi) = \sum_{k,j} (c_k + d_j)\mu(E_k \cap F_j \cap E)$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} (c_k + d_j)\mu(E_k \cap F_j \cap E)$$

$$= \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu(E_k \cap F_j \cap E) + \sum_{j=1}^{m} d_j \sum_{k=1}^{n} \mu(E_k \cap F_j \cap E)$$

$$= \sum_{k=1}^{n} c_k \mu(E_k \cap E) + \sum_{j=1}^{m} d_j \mu(F_j \cap E)$$

$$= \int_E \phi + \int_E \psi$$

3. Suppose $\phi = \sum_{k=1}^{n} c_k \chi_{E_k}$, $E_k$ mutually disjoint, and $\psi = \sum_{j=1}^{m} d_j \chi_{F_j}$, $F_j$ mutually disjoint, where

$$\bigcup_1^n E_k = R = \bigcup_1^m F_j.$$

Since $0 \leq \phi \leq \psi$, $0 \leq c_k \leq d_j$ on nonempty $E_k \cap F_j$ and thus

$$\int_E \phi = \sum_{k=1}^{n} c_k \mu(E_k \cap E) = \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu(E_k \cap F_j \cap E)$$

$$\leq \sum_{j=1}^{m} d_j \sum_{k=1}^{n} \mu(E_k \cap F_j \cap E) = \sum_{j=1}^{m} d_j \mu(F_j \cap E)$$

$$= \int_E \psi.$$

4.

$$\int_E \psi = \sum d_j \mu(F_j \cap E) = \sum d_j \mu\big(F_j \cap (E_1 \cup E_2)\big)$$

$$= \sum d_j \big[\mu(F_j \cap E_1) + \mu(F_j \cap E_2)\big]$$

$$= \sum d_j \mu(F_j \cap E_1) + \sum d_j \mu(F_j \cap E_2)$$

$$= \int_{E_1} \psi + \int_{E_2} \psi. \qquad \blacksquare$$

We now define the Lebesgue integral of a nonnegative measurable function. In fact, we give two commonly used definitions and show their equivalence.

### 5.3.4 Definitions

#### Definition A

If $f$ is a nonnegative, measurable function, defined on a measurable set $E$, the *Lebesgue integral* of $f$ over $E$, $\int_E f$, is given by

$$\int_E f \equiv \sup\left\{\int_E \phi \mid \phi \leq f, \qquad \phi \text{ nonnegative and simple}\right\}.$$

#### Definition B

If $f$ is a nonnegative, measurable function, defined on a Lebesgue measurable set $E$, and $\phi_n$ is a nonnegative monotone sequence of simple functions, $0 \leq \phi_n \leq \phi_{n+1}$ on $E$, with

$$\lim \phi_n(x) = f(x) \qquad \text{(finite or infinite)}$$

for all $x \in E$, the *Lebesgue integral* of $f$ over $E$, $\int_E f$, is given by

$$\int_E f \equiv \lim \int_E \phi_n = \int_E (\lim \phi_n).$$

Some comments are in order before we show the equivalence of these definitions, hereafter referred to as $A$ and $B$.

### 5.3.5 Comments

1. Suppose $f$ is nonnegative, bounded and measurable on a set $E$ of finite measure. Does Definition A agree with Definition 5.2.5? Yes! Theorem 5.7 tell us that $\underline{\int}_E f = \overline{\int}_E f$ (integrable by 5.2.5). But then,

$$\underline{\int}_E f = \sup\left\{\int_E \phi \mid \phi \leq f, \qquad \phi \text{ simple}\right\}$$

$$= \sup\left\{\int_E \phi \mid \phi \leq f, \qquad \phi \text{ simple and nonnegative}\right\},$$

since $f$ is nonnegative, and this is 5.3.4.

2. Because $f$ is nonnegative, we always have simple functions below $f(\phi = 0)$. Thus the set $\{\int_E \phi \mid \phi \le f\}$ is nonempty and the "sup" is a nonnegative member of the extended reals.
3. If $f$ is nonnegative and simple, say, $f = \hat{\phi}$, then $\int_E \hat{\phi} \in \{\int_E \phi \mid \phi \le \hat{\phi}\}$ and $\int_E \phi \le \int_E \hat{\phi}$ by Proposition 5.5; $\int_E \hat{\phi}$ is a member of and an upper bound of the set $\{\int_E \phi \mid \phi \le f\}$. Thus $\int_E \hat{\phi} = \sup\{\int_E \phi \mid \phi \le \hat{\phi}\}$. Definition 5.3.2 and A are in agreement.
4. By the Approximation Theorem 4.2, we always have a monotone sequence $(\hat{\phi}_m)$ of nonnegative simple functions, $0 \le \hat{\phi}_m \le \hat{\phi}_{m+1}$ on $E$ with

$$\lim \hat{\phi}_m = f \qquad \text{on } E\,.$$

The sequence $(\int_E \hat{\phi}_m)$ is a nondecreasing sequence in the extended reals (Proposition 5.5), so the limit is defined in the extended reals: $\lim \int_E \hat{\phi}_m$ is a nonnegative real number or $\infty$.
5. $\int_E f$, as given by B, is well-defined:

Suppose we have sequences of simple functions $(\phi_n)$, $(\hat{\phi}_m)$, $0 \le \phi_n \le \phi_{n+1}$ and $0 \le \hat{\phi}_m \le \hat{\phi}_{m+1}$ on $E$ with $\lim \phi_n = \lim \hat{\phi}_m = f$. We claim

$$\lim \int_E \phi_n = \lim \int_E \hat{\phi}_m\,.$$

The argument is as follows: Pick $0 \le \phi_n \le f$. Then $\lim \hat{\phi}_m \ge \phi_n$ on $E$. We will show

$$\lim \int_E \hat{\phi}_m \ge \int_E \phi_n\,.$$

Because $\phi_n$ is nonnegative and simple, we have

$$\int_E \phi_n = \sum_{k=1}^{N} c_k \mu(E \cap E_k), \qquad c_k \ge 0,$$

where $E = \bigcup_{k=1}^{N} (E \cap E_k)$ and the nonempty $E \cap E_k$ are mutually disjoint measurable subsets of $E$. So we must show

$$\lim \int_E \hat{\phi}_m \ge \sum_{k=1}^{N} c_k \mu(E \cap E_k)\,.$$

But the integral is additive on the domain (Proposition 5.5). Thus

$$\int_E \hat{\phi}_m = \sum_{k=1}^{N} \int_{E\cap E_k} \hat{\phi}_m .$$

Our claim will be justified provided we can show

$$\lim_m \int_{E\cap E_k} \hat{\phi}_m \geq c_k \mu(E \cap E_k), \qquad c_k \geq 0 .$$

If $c_k = 0$, immediate. Suppose $c_k > 0$. Let $\alpha$ be any number between zero and one. (This idea of $0 < \alpha < 1$ has been attributed to W. Rudin.) We construct a sequence of sets $(B_m)$ as follows:

$$B_m = \left\{ x \in E \cap E_k \mid \hat{\phi}_m(x) \geq \alpha c_k \right\} .$$

$B_m$ is measurable, $B_m \subseteq B_{m+1}$ since $\hat{\phi}_m \leq \hat{\phi}_{m+1}$, and $E \cap E_k = \cup B_m$ (If $p \in E \cap E_k$, $\phi_n(p) = c_k$ and consequently $\lim_m \hat{\phi}_m(p) \geq c_k > \alpha c_k$, i.e., $\hat{\phi}_m(p) > \alpha c_k$ for $m$ sufficiently large, in other words, $p \in B_m$ for $m$ sufficiently large.). So the sequence $B_m$ is an expanding sequence of measurable sets that "fills" $E \cap E_k$. Then the sequence $(\mu(B_m))$ is monotone increasing with

$$\lim \mu(B_m) = \mu(E \cap E_k) \qquad \text{(Theorem 3.4)}.$$

But the sequence $(\int_{E\cap E_k} \hat{\phi}_m)$ is also monotone increasing with

$$\int_{E\cap E_k} \hat{\phi}_m \geq \int_{B_m} \hat{\phi}_m \geq \alpha c_k \mu(B_m) .$$

Therefore,

$$\lim_m \int_{E\cap E_k} \hat{\phi}_m \geq \lim_m \alpha c_k \mu(B_m) = \alpha c_k \mu(E \cap E_k) .$$

But this holds for any $\alpha$ between 0 and 1. Thus

$$\lim_m \int_{E\cap E_k} \hat{\phi}_m \geq c_k \mu(E \cap E_k) ,$$

and the argument is complete. The reverse inequality is obtained by interchanging $\hat{\phi}_m$ and $\phi_n$.

**PROPOSITION 5.6** *The Definitions A and B of the Lebesgue integral of a nonnegative measurable function are equivalent.*

*Proof:* By the Approximation Theorem 4.2, we have a monotone sequence of nonnegative simple functions, $0 \leq \hat{\phi}_n \leq \hat{\phi}_{n+1}$ on $E$, with

$$\lim \hat{\phi}_n = f \text{ on } E, \text{ and } \int_E \hat{\phi}_n \leq \int_E \hat{\phi}_{n+1}.$$

We show

$$\lim \int_E \hat{\phi}_n = \sup\left\{\int_E \phi \mid \phi \leq f\right\}.$$

Suppose $0 \leq \phi^* \leq f$. Then $\lim_n \hat{\phi}_n = f \geq \phi^*$ on $E$ and the argument in Comment 5.3.5 yields

$$\int_E \phi^* \leq \lim \int_E \hat{\phi}_n \qquad \text{(finite or infinite).}$$

In other words, $\lim \int_E \hat{\phi}_n$ is an upper bound for the set $\{\int_E \phi \mid \phi \leq f\}$. Therefore,

$$\sup\left\{\int_E \phi \mid \phi \leq f\right\} \leq \lim \int_E \hat{\phi}_n.$$

On the other hand, $\int_E \hat{\phi}_n \in \{\int_E \phi \mid \phi \leq f\}$ for all $n$, and the sequence $(\int_E \hat{\phi}_n)$ is nondecreasing (Proposition 5.5). So $\lim \int_E \hat{\phi}_n \leq \sup\{\int_E \phi \mid \phi \leq f\}$. Combining, $\lim \int_E \hat{\phi}_n = \sup\{\int_E \phi \mid \phi \leq f\}$. ■

We have the familiar properties of the integral.

**PROPOSITION 5.7** *If $f$ and $g$ are nonnegative measurable functions, defined on a measurable set $E$, and $k$ is any nonnegative real number, then*

1. *$(kf)$ is nonnegative, measurable, and $\int_E (kf) = k \int_E f$ (homogeneous);*
2. *$(f + g)$ is nonnegative, measurable, and $\int_E (f+g) = \int_E f + \int_E g$ (additive);*
3. *$\int_E f \leq \int_E g$ if $0 \leq f \leq g$ (monotone);*
4. *if $E_1$ and $E_2$ are disjoint measurable subsets of $E$ with $E = E_1 \cup E_2$, the integrals $\int_{E_1} f$ and $\int_{E_2} f$ exist in $R^e$, and*

$$\int_E f = \int_{E_1} f + \int_{E_2} f \qquad \textit{(additive on domain).}$$

*Proof:* Measurability of the appropriate functions follows from Proposition 4.5.

1. From the Approximation Theorem 4.2, we have a sequence $(\hat{\phi}_n)$ of simple functions satisfying $0 \leq \hat{\phi}_n \leq \hat{\phi}_{n+1}$ with $\lim \hat{\phi}_n = f$. But then, $0 \leq k\hat{\phi}_n \leq k\hat{\phi}_{n+1}$ and $\lim_n(k\hat{\phi}_n) = (kf)$. Using Definition B,
$$k \int_E f = k \lim \int_E \hat{\phi}_n = \lim \int_E k\hat{\phi}_n = \int_E kf\,.$$
2. $\lim \phi_n = f$, $\lim \psi_n = g$ implies $\lim(\phi_n + \psi_n) = f + g$. Thus, $\lim \int_E \phi_n = \int_E f$, $\lim \int_E \psi_n = \int_E g$ implies $\lim \int_E(\phi_n + \psi_n) = \int_E(f+g)$, etc.
3. If $0 \leq \phi \leq f$, then $\phi \leq g$. Thus $\{\phi \mid \phi \leq f\} \subset \{\phi \mid \phi \leq g\}$. Hence
$$\sup\left\{\int_E \phi \mid \phi \leq f\right\} \leq \sup\left\{\int_E \phi \mid \phi \leq g\right\},$$
that is, $\int_E f \leq \int_E g$.
4. $\int_E \phi_n = \int_{E_1} \phi_n + \int_{E_2} \phi_n$ by Proposition 5.5. The sequences $(\int_E \phi_n)$, $(\int_{E_1} \phi_n)$, and $(\int_{E_2} \phi_n)$ are monotone increasing, limits are defined and nonnegative, possibly in the extended reals. The result follows by taking limits.

This completes the proof. ∎

We showed (Approximation Theorem 4.2) that every nonnegative measurable function is the monotone limit of a sequence of nonnegative simple functions. It was very natural then to use this result to define the Lebesgue integral for a nonnegative (possibly unbounded) measurable function in terms of the Lebesgue integral for nonnegative simple functions. How does this work for specific functions?

**Example 9:**

1. $f(x) = 1/x^2$, $x \geq 1$. Calculate $\int_{[1,\infty)} 1/x^2$.

Define a sequence of simple functions as follows:

$$\phi_1(x) = \begin{cases} 1, & x = 1 \\ \dfrac{1}{2^2}, & 1 < x \leq 2 \\ 0, & 2 < x\,. \end{cases}$$

*Note*: $\int_{[1,\infty)} \phi_1 = \int_{[1,2)} \phi_1 = 1 \cdot \frac{1}{2^2}$.

$$\phi_2(x) = \begin{cases} 1, & x = 1 \\ 1/\left(\frac{3}{2}\right)^2, & \frac{2}{2} < x \le \frac{3}{2} \\ 1/\left(\frac{4}{2}\right)^2, & \frac{3}{2} < x \le \frac{4}{2} \\ 1/\left(\frac{5}{2}\right)^2, & \frac{4}{2} < x \le \frac{5}{2} \\ 1/\left(\frac{6}{2}\right)^2, & \frac{5}{2} < x \le \frac{6}{2} \\ 0, & 3 < x. \end{cases}$$

*Note*: $\int_{[1,\infty)} \phi_2 = 2\left[\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2}\right]$.

In general,

$$\phi_n(x) = \begin{cases} 1, & x = 1 \\ 1/\left(1 + \frac{k}{2^{n-1}}\right)^2, & 1 + \frac{(k-1)}{2^{n-1}} < x \le 1 + \frac{k}{2^{n-1}}, \quad 1 \le k \le n2^{n-1} \\ 0, & n + 1 < x, \end{cases}$$

with $\int_{[1,\infty)} \phi_n = 2^{n-1}\left[\frac{1}{(2^{n-1}+1)^2} + \cdots + \frac{1}{((n+1)\,2^{n-1})^2}\right]$.

It is straightforward to show $0 \le \phi_n \le \phi_{n+1}$ and $\lim \phi_n = f$ on $[1, \infty)$.

With the observation $1/((k+1)k) < 1/(k \cdot k) < 1/((k-1) \cdot k)$, the reader may show

$$2^{n-1}\left[\frac{1}{(2^{n-1}+2)(2^{n-1}+1)} + \cdots + \frac{1}{((n+1)2^{n-1}+1)(n+1)2^{n-1}}\right] < \int_{[1,\infty)} \phi_n$$
$$< 2^{n-1}\left[\frac{1}{2^{n-1}(2^{n-1}+1)} + \cdots + \frac{1}{((n+1)2^{n-1}-1)(n+1)2^{n-1}}\right],$$

that is,

$$\frac{1}{1 + \frac{1}{2^{n-1}}} - \frac{1}{n + 1 + \frac{1}{2^{n-1}}} < \int_{[1,\infty)} \phi_n < 1 - \frac{1}{n+1}.$$

Thus $\lim \int_{[1,\infty)} \phi_n = 1$. By Definition B,

$$\int_{[1,\infty)} \frac{1}{x^2} \equiv \lim \int_{[1,\infty)} \phi_n = 1.$$

2. Calculate $\int_{(0,1]} 1/\sqrt{x}$.

$$\phi_1(x) = 1, \;\; 0 < x \leq 1. \qquad \int_{(0,1]} \phi_1 = 1\,.$$

$$\phi_2(x) = \begin{cases} \frac{4}{2}, & 0 < x \leq \frac{4}{16} \\ \frac{3}{2}, & \frac{4}{16} < x \leq \frac{4}{9} \\ \frac{2}{2}, & \frac{4}{9} < x \leq 1\,. \end{cases}$$

$$\begin{aligned} \int_{(0,1]} \phi_2 &= \tfrac{4}{2}\left[\left(\tfrac{2}{4}\right)^2 - 0^2\right] \\ &+ \tfrac{3}{2}\left[\left(\tfrac{2}{3}\right)^2 - \left(\tfrac{2}{4}\right)^2\right] \\ &+ \tfrac{2}{2}\left[\left(\tfrac{2}{2}\right)^2 - \left(\tfrac{2}{3}\right)^2\right] \\ &= 1 + 2\left(\tfrac{1}{3^2} + \tfrac{1}{4^2}\right). \end{aligned}$$

$$\phi_3(x) = \begin{cases} \frac{12}{4}, & 0 < x \leq \left(\frac{4}{12}\right)^2 \\ \frac{11}{4}, & \left(\frac{4}{12}\right)^2 < x \leq \left(\frac{4}{11}\right)^2 \\ \frac{10}{4}, & \left(\frac{4}{11}\right)^2 < x \leq \left(\frac{4}{10}\right)^2 \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \frac{5}{4}, & \left(\frac{4}{6}\right)^2 < x \leq \left(\frac{4}{5}\right)^2 \\ \frac{4}{4}, & \left(\frac{4}{5}\right)^2 < x \leq \left(\frac{4}{4}\right)^2\,. \end{cases}$$

$$\int_{(0,1]} \phi_3 = 1 + 4\left(\frac{1}{5^2} + \frac{1}{6^2} + \cdots + \frac{1}{12^2}\right).$$

The reader should determine $\phi_n$, show $0 \leq \phi_n \leq \phi_{n+1}$, $\lim \phi_n = f$, and that

$$\int_{(0,1]} \phi_n = 1 + 2^{n-1}\left(\frac{1}{(2^{n-1}+1)^2} + \cdots + \frac{1}{(n2^{n-1})^2}\right).$$

Hence,

$$\lim \int_{(0,1]} \phi_n = 1 + 1 = 2,$$

and

$$\int_{(0,1]} \frac{1}{\sqrt{x}} = 2.$$

After a few calculations like the last two examples, we look for an easier way to calculate Lebesgue integrals. The reader has probably realized that in these examples we "really" approximated $f$ by a monotone sequence $(f_n)$ of nonnegative measurable functions and then approximated each $f_n$ by a sequence of simple functions. Specifically:

1*. For calculating $\int_{[1,\infty)} 1/x^2$, let $f_n(x) = 1/x^2$, $1 \leq x \leq n$, and 0 otherwise. Then $\int_{[1,\infty)} f_n = \int_{[1,n]} f_n = \int_1^n 1/x^2 dx = 1 - 1/n$, and $\lim \int_{[1,\infty)} f_n = 1$. Wouldn't it be "convenient" if $\lim \int_{[1,\infty)} f_n = \int_{[1,\infty)} \lim f_n = \int_{[1,\infty)} 1/x^2$.

   Did you use Theorem 5.6?

2*. For calculating $\int_{(0,1]} 1/\sqrt{x}$, let

$$f_n(x) = \begin{cases} 1/\sqrt{x}, & 1/n^2 < x \leq 1 \\ n, & 0 < x \leq 1/n^2. \end{cases}$$

   Then $\int_{(0,1]} f_n = \int_0^{1/n^2} n\,dx + \int_{1/n^2}^1 1/\sqrt{x}dx = 2 - 2/n$ and $\lim \int_{(0.1]} f_n = 2$.

   Did you use Theorem 5.6?

In these calculations, many simplifications arise if $\lim \int f_n = \int(\lim f_n)$ is valid for monotone sequences of nonnegative measurable functions, not just monotone sequences of nonnegative simple functions.

Again, Definition B requires approximation of $f$ by monotone sequences of simple functions. Possible, but technically cumbersome.

Recall that the limit of a sequence of measurable functions is again a measurable function. Wouldn't it be "nice" if simple functions could be replaced by measurable functions, i.e., since $\phi_n \uparrow f$, $\int \phi_n \uparrow \int f$, $\phi_n$ simple, can we show $f_n \uparrow f$, $\int f_n \uparrow \int f$, $f_n$ measurable? This is the essence of the next theorem.

**THEOREM 5.10** *Lebesgue Monotone Convergence Theorem (LMCT), Beppo Levi, 1906* Let $(f_k)$ be a monotone increasing sequence of nonnegative measurable functions on a measurable set $E$: $0 \leq f_1 \leq f_2 \leq \cdots$ on $E$. Then

$$\lim \int_E f_k = \int_E (\lim f_k) .$$

*Proof:* We give two arguments. The first is based on Definition A for the integral of a nonnegative measurable function. To that end, we observe that $(\lim f_k)$ is nonnegative $(0 \leq f_k)$ and measurable on $E$ (The limit of a sequence of measurable functions is measurable by Theorem 4.1.). Thus

$$\int_E (\lim f_k) = \sup\left\{ \int_E \phi \mid \phi \leq (\lim f_k), \quad \phi \text{ nonnegative and simple} \right\}.$$

Since the integral preserves monotonicity for nonnegative measurable functions, and $0 \leq f_k \leq f_{k+1} \leq \cdots \leq (\lim f_k)$ on $E$, we have $0 \leq \int_E f_k \leq \int_E f_{k+1} \leq \cdots \int_E (\lim f_k)$, that is, $\lim \int_E f_k \leq \int_E (\lim f_k)$.

We will show $\int_E (\lim f_k) \leq \lim \int_E f_k$ to complete the argument. Let $\phi$ be any simple function satisfying $0 \leq \phi \leq (\lim f_k)$. If we can show $\lim \int_E f_k \geq \int_E \phi$, then this would say $\lim \int_E f_k$ is an upper bound for the set $\{\int_E \phi \mid \phi \leq (\lim f_k), \phi \text{ nonnegative and simple}\}$. But the least upper bound of this set, $\int_E (\lim f_k)$, would be less than or equal the upper bound, $\lim \int_E f_k$, and the conclusion would follow. We now show $\lim \int_E f_k \geq \int_E \phi$, $\phi \leq (\lim f_k)$, $\phi$ nonnegative and simple. Since $\phi$ is nonnegative and simple, $\int_E \phi = \sum_1^N c_i \mu(E \cap E_i)$, $c_i \geq 0$ and $\bigcup_1^N (E \cap E_i) = E$ where $E \cap E_i$ are mutually disjoint measurable subsets of $E$. Because $\int_E f_k$ is additive on $E$ for nonnegative measurable functions (Proposition 5.7), it is sufficient to show

$$\lim_k \left( \sum_1^N \int_{E \cap E_i} f_k \right) \geq \sum_1^N c_i \mu(E \cap E_i) ,$$

and this will be accomplished, if we show

$$\lim_k \int_{E\cap E_i} f_k \geq c_i\mu(E\cap E_i)\,.$$

If $c_i = 0$, done. Assume $c_i > 0$. The idea is to construct an increasing sequence of measurable sets $(B_k)$ which "fill" $E \cap E_i$, on each of which $f_k$ is "close" to $c_i$. The ingenious idea that follows is apparently due to W. Rudin. Let $0 < \alpha < 1$, and define $B_k = \{x \in E \cap E_i \mid f_k(x) \geq \alpha c_i\}$. These sets are measurable, $B_k \subseteq B_{k+1}$, since $f_k \leq f_{k+1}$, and $\cup B_k = E \cap E_i$ because $(\lim f_k) \geq \phi = c_i$ on $E \cap E_i$. The sequence $(\mu(B_k))$ is nondecreasing, and $\lim \mu(B_k) = \mu(E \cap E_i)$ by Theorem 3.4. But $\int_{E\cap E_i} f_k \geq \int_{B_k} f_k \geq \alpha c_i \mu(B_k)$. Therefore,

$$\lim_k \int_{E\cap E_i} f_k \geq \alpha c_i \lim_k \mu(B_k) = \alpha c_i \mu(E\cap E_i), \qquad 0 < \alpha < 1.$$

Thus $\lim_k \int_{E\cap E_i} f_k \geq c_i\mu(E\cap E_i)$, and the theorem is proved using Definition A.

The second argument is based on Definition B, the Approximation Theorem 4.2, and extensive use of Proposition 5.7. From the Approximation Theorem we have:

$$0 \leq \phi_{11} \leq \phi_{12} \leq \cdots \leq \phi_{1n} \leq \cdots \leq f_1, \quad \lim_n \phi_{1n} = f_1 \text{ on } E, \quad \text{and } \lim_n \int_E \phi_{1n} = \int_E f_1;$$

$$0 \leq \phi_{21} \leq \phi_{22} \leq \cdots \leq \phi_{2n} \leq \cdots \leq f_2, \quad \lim_n \phi_{2n} = f_2 \text{ on } E, \quad \text{and } \lim_n \int_E \phi_{2n} = \int_E f_2;$$

$$\vdots$$

$$0 \leq \phi_{k1} \leq \phi_{k2} \leq \cdots \leq \phi_{kn} \leq \cdots \leq f_k, \quad \lim_n \phi_{kn} = f_k \text{ on } E, \quad \text{and } \lim_n \int_E \phi_{kn} = \int_E f_k;$$

etc.

Construct a new sequence of simple functions, $(\hat{\phi}_k)$, with $\lim \hat{\phi}_k = (\lim f_k)$ as follows:

$$\hat{\phi}_1 = \phi_{11};$$

$$\hat{\phi}_2 = \max\{\phi_{12}, \phi_{22}\} \geq \phi_{12} \geq \phi_{11} = \hat{\phi}_1;$$

$$\vdots$$

$$\hat{\phi}_k = \max\{\phi_{1k}, \phi_{2k}, \ldots, \phi_{k-1k}, \phi_{kk}\} \geq \hat{\phi}_{k-1};$$

etc.

The $\hat{\phi}_k$ are simple and $0 \le \hat{\phi}_1 \le \hat{\phi}_2 \le \cdots \le \hat{\phi}_k \le \cdots$, with $\lim \hat{\phi}_k = (\lim f_k)$. By Definition B, $\lim \int_E \hat{\phi}_k = \int_E (\lim f_k)$. However, $0 \le \hat{\phi}_k \le f_k \le (\lim f_k)$ and, since the integral preserves monotonicity for nonnegative measurable functions (Proposition 5.8), we have $\int_E \hat{\phi}_k \le \int_E f_k \le \int_E (\lim f_k)$. Taking limits,

$$\lim \int_E \hat{\phi}_k \le \lim \int_E f_k \le \int_E (\lim f_k).$$

Recalling $\int_E (\lim f_k) = \lim \int_E \hat{\phi}_k$ from above, the conclusion follows. ■

### 5.3.6 Comments

1. What if we initially have a monotone increasing sequence of measurable functions $(g_k)$, not necessarily nonnegative: $g_1 \le g_2 \le g_3 \le \cdots$? We could try subtracting $g_1 : 0 \le g_2 - g_1 \le g_3 - g_1 \le \cdots$ $(\infty - \infty?)$. The sequence $(g_k - g_1)$ satisfies the hypotheses of the LMCT. So,

$$\lim \int_E (g_k - g_1) = \int_E \lim (g_k - g_1) = \int_E [(\lim g_k) - g_1].$$

   We would have to be able to subtract. No problem if all integrals are finite, and we would need $\int_E (f - g) = \int_E f - \int_E g$. These conditions may not be available to us. Consider this example:

$$g_k(x) = \begin{cases} -1, & k \le x \\ 0, & \text{otherwise} \end{cases}.$$

   Then $g_k \le g_{k+1}$, $\lim g_k = 0$, $-\infty = \lim \int_R g_k \ne 0 = \int_R (\lim g_k)$.

2. It's natural to consider monotone decreasing sequences. Can we draw any useful conclusions in this setting. Suppose $f_1 \ge f_2 \ge \cdots$. Then $-f_1 \le -f_2 \le \cdots$, or $0 \le f_1 - f_2 \le f_1 - f_3 \le \cdots$ $(\infty - \infty?)$. The sequence $(f_1 - f_k)$ is nonnegative and monotone increasing. This looks very much like Comment 1, with

$$f_k(x) = -g_k(x) = \begin{cases} 1, & k \le x \\ 0, & \text{otherwise} \end{cases}$$

   Then $f_1 \ge f_2 \ge \cdots \ge 0$, $\lim f_k = 0$, $\lim \int_R f_k = \infty \ne 0 = \int_R \lim f_k$.

### 5.3.7 Problems

Approximate unbounded measurable functions (whose domains have finite measure) with bounded measurable functions via "truncation" of the range. Note the frequent use of Theorem 5.6.

1. Calculate $\int_{(0,1]} t^{-1/2}$.

$$\textit{Hint}: \quad f_k(t) = \begin{cases} 0, & 0 < t < 1/k^2 \\ t^{-1/2}, & 1/k^2 \leq t \leq 1 . \end{cases}$$

Show $0 \leq f_k \leq f_{k+1}$. Then,

$$\int_{(0,1]} t^{-\frac{1}{2}} = \int_{(0,1]} (\lim f_k) = \lim \int_{(0,1]} f_k = \lim \int_{[\frac{1}{k^2},1]} f_k$$

$$= \lim \int_{\frac{1}{k^2}}^{1} t^{-\frac{1}{2}} dt = 2 .$$

Calculate $\int_{(0,1]} t^{\alpha}$, $-1 < \alpha < 0$.

2. Calculate $\int_{[0,1)} (1 - t^2)^{-1/2}$.

$$\textit{Hint}: \quad f_k(t) = \begin{cases} (1 - t^2)^{-1/2}, & 0 \leq t \leq 1 - 1/k \\ 0, & 1 - 1/k < t < 1 . \end{cases}$$

Show $0 \leq f_k \leq f_{k+1}$ and $\int_{[0,1)} (1 - t^2)^{-1/2} = \lim \int_{[0,1)} f_k = \lim \int_0^{1-1/k} (1 - t^2)^{-1/2} dt = \pi/2$.

3. Suppose $f(t) = \begin{cases} t^{-1}, & 0 < t \leq 1 \\ (2 - t)^{-1/2}, & 1 \leq t < 2 \end{cases}$ Calculate $\int_{(0,2)} f$.

$$\textit{Hint:} \quad f_k(t) = \begin{cases} 0, & 0 < t < 1/k \\ t^{-1}, & 1/k \leq t \leq 1 - 1/k \\ 0, & 1 - 1/k < t < 1 + 1/k \\ (2 - t)^{-1/2}, & 1 + 1/k \leq t \leq 2 - 1/k^2 \\ 0, & 2 - 1/k^2 < t < 2 . \end{cases}$$

Thus
$\int_{(0,2)} f = \lim \int_{(0,2)} f_k = \lim[\int_{1/k}^{1-1/k} t^{-1} dt + \int_{1+1/k}^{2-1/k} (2-t)^{-1/2} dt] = \infty.$

### 5.3.8 Problems

All but number 9 deal with bounded measurable functions on sets of infinite measure. Essentially, "truncate" the domain, and notice how often Theorem 5.6 is used.

1. Calculate $\displaystyle\int_{[0,\infty)} e^{-t}$.

   *Hint*: $$f_k(t) = \begin{cases} e^{-t}, & 0 \le t \le k \\ 0, & k < t \end{cases}.$$

   Show $0 \le f_k \le f_{k+1}$ and thus $\int_{[0,\infty)} e^{-t} = \int_{[0,\infty)} \lim f_k = \lim \int_{[0,\infty)} f_k = \lim \int_0^k e^{-t} dt = 1$.

2. Calculate $\displaystyle\int_{(-\infty,\infty)} (1+t^2)^{-1}$.

   *Hint*: $$f_k(t) = \begin{cases} (1+t^2)^{-1}, & |t| \le k \\ 0, & |t| > k \end{cases}.$$

3. Calculate $\displaystyle\int_{[1,\infty)} t^{-1}$.

   *Hint*: $$f_k(t) = \begin{cases} t^{-1}, & 1 \le t \le k \\ 0, & k < t \end{cases}.$$

   Calculate $\displaystyle\int_{[1,\infty)} t^{-\alpha},\ \alpha > 1$.

4. Show $\displaystyle\int_{(0,1]} -t \ln t = \frac{1}{4}$.

   *Hint*: $$f_k(t) = \begin{cases} -t \ln t, & 1/k \le t \le 1 \\ 0, & 0 < t < 1/k \end{cases}.$$

5. $f_k(t) = \begin{cases} 1, & 0 \le t \le k \\ 0, & k < t. \end{cases}$

Show $\int_{[0,\infty)} \lim f_k = \lim \int_{[0,\infty)} f_k$. Does LMCT apply?

6. $f_k(t) = \begin{cases} k^{-1}, & 0 \le t \le k \\ 0, & k < t \end{cases}.$

Show $\lim f_k = 0$ uniformly on $[0, \infty)$.

Calculate $\displaystyle\int_{[0,\infty)} \lim f_k$ and $\displaystyle\lim \int_{[0,\infty)} f_k$. Are they the same? What is the problem?

7. $f_k(t) = \begin{cases} 0, & 0 \le t < k \\ k^{-1}, & k \le t \le k^2 \\ 0, & k^2 < t \end{cases}.$

Show $\lim f_k = 0$ uniformly on $[0, \infty)$.

Calculate $\displaystyle\int_{[0,\infty)} \lim f_k$ and $\displaystyle\lim \int_{[0,\infty)} f_k$. Any problem?

8. $f_k(t) = \begin{cases} 0, & 0 \le t < k \\ 1, & k \le t \end{cases}.$ Can we use LMCT? Why?

Calculate $\displaystyle\int_{[0,\infty)} \lim f_k$ and $\displaystyle\lim \int_{[0,\infty)} f_k$.

9. An unbounded measurable function on a domain of infinite measure whose integral may be evaluated by a combination of techniques used above.

$$f(t) = \begin{cases} t^{-1/2}, & 0 < t \le 1 \\ t^{-2}, & 1 < t \end{cases}.$$

Calculate $\displaystyle\int_{(0,\infty)} f$.

**PROPOSITION 5.8** *If $(g_k)$ is a sequence of nonnegative measurable functions defined on a measurable set $E$, then*

$$\int_E \sum g_k = \sum \int_E g_k .$$

*Proof:* Let $f_n = g_1 + \cdots + g_n$ and use LMCT:

$$\int_E \sum g_k = \int_E (\lim f_n) = \lim \int_E f_n = \lim \int_E \sum_1^n g_k$$

$$= \lim \sum_1^n \int_E g_k = \sum \int_E g_k . \qquad \blacksquare$$

### 5.3.9 Problems

1. $\displaystyle\int_{[0,1]} \sum_0 x^n(1-x) \stackrel{?}{=} \sum_0 \int_{[0,1]} x^n(1-x).$

   Show $\displaystyle\sum_0 x^n(1-x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x = 1 . \end{cases}$

   Is the convergence uniform on $[0, 1]$? Calculate $\displaystyle\sum_0 \int_{[0,1]} x^n(1-x)$.

2. $\displaystyle\int_{[0,1]} \sum_0 x(1-x)^n \stackrel{?}{=} \sum_0 \int_{[0,1]} x(1-x)^n.$

   Show $\displaystyle\sum_0 x(1-x)^n = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 . \end{cases}$

   Is the convergence uniform on $[0, 1]$? Calculate $\displaystyle\sum_0 \int_{[0,1]} x(1-x)^n$.

3. $\displaystyle\int_{[0,1]} \sum_0 \frac{x}{(1+x)^n} \stackrel{?}{=} \sum_0 \int_{[0,1]} \frac{x}{(1+x)^n}.$

   Show $\displaystyle\sum_0 \frac{x}{(1+x)^n} = \begin{cases} 0, & x = 0 \\ x+1, & 0 < x \le 1 . \end{cases}$

   Is the convergence uniform on $[0, 1]$? Calculate $\displaystyle\sum_0 \int_{[0,1]} \frac{x}{(1+x)^n}$.

4. $\displaystyle\int_{[0,1]} \sum_0 \frac{x^2}{(1+x^2)^n} \stackrel{?}{=} \sum_0 \int_{[0,1]} \frac{x^2}{(1+x^2)^n}.$

Show $\displaystyle\sum_0 \frac{x^2}{(1+x^2)^n} = \begin{cases} 0, & x=0 \\ 1+x^2, & 0<x\le 1. \end{cases}$

Is the convergence uniform on [0, 1]? Calculate $\displaystyle\sum_0 \int_{[0,1]} \frac{x^2}{(1+x^2)^n}.$

5. $\displaystyle\int_{[0,1]} \sum_0 \frac{x^{3/2}}{(1+x^2)^n} \stackrel{?}{=} \sum_0 \int_{[0,1]} \frac{x^{3/2}}{(1+x^2)^n}.$

Show $\displaystyle\sum_0 \frac{x^{3/2}}{(1+x^2)^n} = \begin{cases} 0, & x=0 \\ x^{-1/2}+x^{3/2}, & 0<x\le 1. \end{cases}$

Is the convergence uniform on [0, 1]? Calculate $\displaystyle\sum_0 \int_{[0,1]} \frac{x^{3/2}}{(1+x^2)^n}.$

6. Consider the integral $\int_E e^{-x}$:

Suppose $E=[0,1]$. Of course $\int_{[0,1]} e^{-x} = \int_0^1 e^{-x}dx = 1-e^{-1}$. On the other hand, $e^{-x}=\sum((-x)^k/k!)$, $-\infty<x<\infty$, and the question arises, does $\int_{[0,1]} e^{-x} = \int_{[0,1]} \sum((-x)^k/k!)$ equal $\sum \int_{[0,1]}((-x)^k/k!)$? That is, $1-e^{-1} \stackrel{?}{=} \sum((-1)^{k+1}/(k+1)!)$. But is Proposition 5.8 applicable? What if we define $f_n(x) = 1+((-x)/1!)+((-x)^2/2!)+\cdots+((-x)^n/n!)$ for $0\le x\le 1$. Is $f_n(x)\ge 0$ on $[0,1]$? Is $0\le f_n\le f_{n+1}$?

Another possibility: We know $\int\sum = \sum\int$ holds for the Riemann integral if we have uniform convergence (Problem 5.1.14). Does $\sum((-x)^k/k!)$ converge uniformly on [0, 1]?

And yet another possibility: $f_k(x)\equiv(1-x/k)^k$, $0\le x\le 1$. Show $0\le f_k\le f_{k+1}$, $\lim f_k = f$ and by LMCT,

$$\int_{[0,1]} e^{-x} = \lim \int_{[0,1]} \left(1-\frac{x}{k}\right)^k = \lim \int_0^1 \left(1-\frac{x}{k}\right)^k dx$$

$$= \lim \frac{k}{k+1}\left(1-\left(1-\frac{1}{k}\right)^{k+1}\right)$$

$$= 1-e^{-1}.$$

Suppose now $E = [0, \infty)$. We cannot use uniform convergence on this set $E$. We can, however, truncate the domain:

$$f_k(x) = \begin{cases} f(x), & 0 \le x \le k \\ 0, & k < x. \end{cases}$$

Then $0 \le f_k \le f_{k+1}$ and use LMCT.

*Note*: If $f_k(x) = \begin{cases} (1 - x/k)^k, & 0 \le x \le k \\ 0, & k < x \end{cases}$,

then $0 \le f_k \le f_{k+1}$
and using LMCT,

$$\int_{[0,\infty)} e^{-x} = \lim \int_{[0,\infty)} f_k = \lim \int_{[0,k]} f_k = \lim \int_0^k \left(1 - \frac{x}{k}\right)^k dx$$

$$= \lim \frac{k}{k+1} = 1\,.$$

7.

i. Show

$$\int_{[0,\infty)} \sum_1 (e^{-nx} - e^{-2nx}) = \sum_1 \int_{[0,\infty)} (e^{-nx} - e^{-2nx}) = \sum \frac{1}{2n} = \infty.$$

ii. Does

$$\int_{[\ln 2,\infty)} \sum_1 (e^{-nx} - 2e^{-2nx}) = \sum_1 \int_{[\ln 2,\infty)} (e^{-nx} - 2e^{-2nx})?$$

Is Proposition 5.8 applicable?

iii. Show

$$\int_{[0,\infty)} \sum_1 (e^{-nx} - 2e^{-2nx}) \neq \sum_1 \int_{[0,\infty)} (e^{-nx} - 2e^{-2nx}) = 0.$$

### 5.3.10 Comment

A very interesting application of the LMCT is the following: Suppose $f$ is any nonnegative Lebesgue measurable function on $R$. Then for any

Lebesgue measurable set $E$, we may calculate $\int_E f$. That is, on the sigma algebra $\mathcal{M}$ of Lebesgue measurable sets, we have a set function, say $\lambda$:

$$E \in \mathcal{M}, \qquad \lambda(E) \equiv \int_E f.$$

What properties, if any, does this set function $\lambda$ have in common with the set function $\mu$, the Lebesgue measure set function? Review Proposition 5.7 at this time.

i. Clearly $\lambda(\emptyset) = 0$.

ii. If $A$ and $B$ are Lebesgue measurable sets, $A \subseteq B$, then $\mu(B) \geq \mu(A)$ (monotone). But,

$$\lambda(B) = \int_B f = \int_{B-A} f + \int_A f \geq \int_A f = \lambda(A).$$

The set function $\lambda$ is also monotone. In particular, $\lambda(E) \geq 0$ for all $E \in \mathcal{M}$.

iii. If

$$E = \bigcup_1^\infty E_k, \; E_i \cap E_j = \emptyset$$

when $i \neq j$, then

$$\mu(E) = \mu\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty \mu(E_k):$$

The Lebesgue measure $\mu$ is countably additive. Can we show

$$\lambda(E) = \lambda\left(\bigcup_1^\infty E_k\right) = \int_E f = \sum_1^\infty \int_{E_k} f = \sum_1^\infty \lambda(E_k)?$$

Actually, Proposition 5.7 and induction yields

$$\int_{\bigcup_1^n E_k} f = \sum_1^n \int_{E_k} f.$$

Finite additivity is not a problem. Countable additivity will follow from the LMCT.

We define a new sequence of functions on $E$ as follows:

$$g_1(x) = \begin{cases} f(x), & x \in E_1 \\ 0, & x \in E - E_1 . \end{cases}$$

$$g_2(x) = \begin{cases} f(x), & x \in E_1 \cup E_2 \\ 0, & x \in E - (E_1 \cup E_2) . \end{cases}$$

$$\vdots$$

$$g_n(x) = \begin{cases} f(x), & x \in \bigcup_1^n E_k \\ 0, & x \in E - \bigcup_1^n E_k . \end{cases}$$

$$\vdots$$

We have a nonnegative, monotone increasing sequence of Lebesgue measurable functions $(g_n)$ on $E$ with $\lim g_n = f$ on $E$. Application of the LMCT yields

$$\lambda\left(\bigcup_1^\infty E_k\right) = \int_E f = \int_E \lim g_n = \lim \int_E g_n = \lim \int_{\bigcup_1^n E_k} f$$

$$= \lim \sum_1^n \int_{E_k} f = \sum_1^\infty \int_{E_k} f = \sum_1^\infty \lambda(E_k).$$

In summary, $\lambda$ is a nonnegative extended real-valued set function defined on the sigma algebra of Lebesgue measurable sets, $\mathcal{M}$, such that

i. $\lambda(\emptyset) = 0$;
ii. $\lambda(E) \geq 0$ for all $E \in \mathcal{M}$;
iii. $\lambda\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty \lambda(E_k)$, $E_i \cap E_j = \emptyset$, $i \neq j$ and $E_k \in \mathcal{M}$.

These three properties are the most important properties of any "measure" and they are the defining properties for measures in general. Given a measure, we can construct another measure relatively easily.

For sequences that are not monotonic but nonnegative we have the following theorem;

**THEOREM 5.11** *(Fatou, 1906) If $(f_k)$ is a sequence of nonnegative measurable functions defined on a measurable set $E$, then*

$$\int_E (\liminf f_k) \leq \liminf \int_E f_k .$$

*Proof:* The idea here is that "lim inf" yields monotone increasing sequences that are then amenable to the LMCT. We begin: $\underline{f}_1 \equiv \inf\{f_1, f_2, \ldots\}$. $\underline{f}_1$ is measurable, $0 \leq \underline{f}_1 \leq f_n$ for all $n$, and $\int_E \underline{f}_1 \leq \int_E f_n$ for all $n \geq 1$. That is,

$$\int_E \underline{f}_1 \leq \left\{\int_E f_1, \int_E f_2, \ldots, \int_E f_n, \ldots\right\}.$$

In other words, $\int_E \underline{f}_1$ is a lower bound for the set $\{\int_E f_1, \int_E f_2, \ldots, \int_E f_n, \ldots\}$. We have

$$\int_E \underline{f}_1 \leq \underline{\int_E f_1} \equiv \inf\left\{\int_E f_1, \int_E f_2, \ldots, \int_E f_n, \ldots\right\}.$$

Define $\underline{f}_2 = \inf\{f_2, f_3, \ldots\}$. $\underline{f}_2$ is measurable, $0 \leq \underline{f}_1 \leq \underline{f}_2$,

$$\int_E \underline{f}_1 \leq \int_E \underline{f}_2 \text{ and } \underline{\int_E f_1} \leq \underline{\int_E f_2} = \inf\left\{\int_E f_2, \int_E f_3, \ldots\right\}.$$

In general, if

$$\underline{f}_m \equiv \inf\{f_m, f_{m+1}, \ldots\},$$

then $0 \leq \underline{f}_1 \leq \underline{f}_2 \leq \cdots \leq \underline{f}_m \leq \cdots$, $\int_E \underline{f}_m \leq \underline{\int_E f_m} = \inf\{\int_E f_m, \int_E f_{m+1}, \ldots\}$. The sequences $(\int_E \underline{f}_m)$, $(\underline{\int_E f_m})$ are nonnegative, monotone increasing sequences of perhaps extended-real numbers, and hence have limits in the extended reals:

$$\lim_m \int_E \underline{f}_m \leq \lim_m \underline{\int_E f_m} .$$

But, application of the LMCT tells us that

$$\lim_m \int_E \underline{f}_m = \int_E \left(\lim_m \underline{f}_m\right) = \int_E (\liminf f_k) .$$

Thus

$$\int_E (\liminf f_k) \leq \lim_m \underline{\int_E f_m} = \liminf \int_E f_k,$$

and this is what we wanted to show. ∎

**Example 10:**

1. $f_n = 1/n\,\chi_{[0,n]}$. $\lim \int_{[0,\infty)} f_n = 1 \neq 0 = \int_{[0,\infty)} \lim f_n$.
   But $\int_{[0,\infty)} \liminf f_n = \int_{[0,\infty)} \lim f_n = 0 \leq 1 = \liminf \int_{[0,\infty)} f_n$.
2. $g_n = n\,\chi_{[1/n,2/n]}$. $\lim g_n = 0$, and
   i. $\lim \int_{[0,\infty)} g_n = 1 \neq \int_{[0,\infty)} \lim g_n = 0$;
   ii. $g_n \to 0$ on $[0, \infty)$;
   iii. LMCT? (not monotone);
   iv. $\int_{[0,\infty)} \liminf f_n = 0 \leq 1 = \liminf \int_{[0,\infty)} f_n$.
   v. What if $g_n = -n\,\chi_{[1/n,2/n]}$?
3. Nonnegative is necessary, even with uniform convergence!
   $f_n = -1/n\,\chi_{[0,n]}$. $f_n \to 0$ (unif) on $[0, \infty)$.

$$\lim \int_{[0,\infty)} f_n = -1 \neq \int_{[0,\infty)} \lim f_n = 0,$$

   but

$$\int_E \liminf f_n = 0 > -1 = \liminf \int_E f_n.$$

4. $f_n = \chi_{[n,n+1]}$

$$\int_E \liminf f_n = 0 < 1 = \liminf \int f_n.$$

(Inequality may be strict.)

This concludes our treatment of the Lebesgue integral for nonnegative measurable functions defined on arbitrary measurable sets of real numbers.

## 5.4 THE LEBESGUE INTEGRAL AND LEBESGUE INTEGRABILITY

Men pass away but their deeds abide.

—Cauchy

The requirement that $f$ be nonnegative is eliminated. We discuss measurable functions defined on any measurable set of real numbers.

### 5.4.1 Definitions

Let $f$ be a measurable function defined on a measurable set $E$. We know that $f$ can be written as the difference of nonnegative measurable functions: $f = f^+ - f^-$ (Proposition 4.6). Calculate $\int_E f^+$, $\int_E f^-$ according to Definition A or B.

If both $\int_E f^+$ and $\int_E f^-$ are $\infty$, we say that the Lebesgue integral of $f$ on $E$ is not defined ($\infty - \infty$ is not defined in $R^e$).

If either $\int_E f^+$ or $\int_E f^-$ (but not both) are finite, we define the *Lebesgue integral* of $f$ by

$$\int_E f = \int_E f^+ - \int_E f^- .$$

If both $\int_E f^+$ and $\int_E f^-$ are finite, we say $f$ is *Lebesgue integrable* on $E$ and

$$\int_E f = \int_E f^+ - \int_E f^- .$$

In this case, $\int_E f \in R$.

*Caution:* Hereafter, "Lebesgue integrable" will be shortened to "integrable".

### 5.4.2 Comments

1. How does the new definition (5.4.1), applied to a nonnegative, measurable function $f$, compare with the previous definition for nonnegative, measurable functions (5.3.4)? Because $f$ is nonnegative, $f^+ = f, f^- = 0$, and $\int_E f^- = 0$. The Lebesgue integral of $f$ on $E$ exists (5.4.1) and $\int_E f = \int_E f^+$. We have consistency.
2. How does the new definition (5.4.1), applied to a bounded, measur-

able function $f$ on a set of finite measure $E$, compare with our original definition for bounded functions on a set of finite measure (5.2.5), or an equivalent condition, Proposition 5.4? Because $f$ is bounded, measurable, and $f=f^+-f^-$, the functions $f^+$, $f^-$ are nonnegative, bounded, measurable functions on a set of finite measure $E$. Integrability follows from Theorem 5.7, that is, $\underline{\int}_E f^- = \overline{\int}_E f^-$ and $\underline{\int}_E f^+ = \overline{\int}_E f^+$. But,

$$\underline{\int}_E f^+ = \sup\left\{\int_E \phi \mid \phi \le f^+,\ \phi \text{ simple}\right\}$$

$$= \sup\left\{\int_E \phi \mid \phi \le f^+,\ \phi \text{ simple and nonnegative}\right\}$$

$$= \int_E f^+,$$

and similarly for $f^-$. We have: $f=f^+-f^-$, $f$ bounded and measurable on a set of finite measure $E$, $f^+$ and $f^-$ are integrable according to the "old" Definition 5.2.5, and $\underline{\int}_E f^+ = \overline{\int}_E f^+ = \int_E f^+$, $\underline{\int}_E f^- = \overline{\int}_E f^- = \int_E f^-$. We want to show $\underline{\int}_E f = \overline{\int}_E f = \int_E f^+ - \int_E f^-$. We apply Proposition 5.4 to $f^+$ and $f^-$. Let $\epsilon > 0$ be given. We have $\phi^+ \le f^+ \le \psi^+$, $\int_E \psi^+ - \int_E \phi^+ < \epsilon/2$ and $\phi^- \le f^- \le \psi^-$, $\int_E \psi^- - \int_E \phi^- < \epsilon/2$. Combining, $\phi^+ - \psi^- \le f \le \psi^+ - \phi^-$, and $\int_E(\psi^+ - \phi^-) - \int_E(\phi^+ - \psi^-) < \epsilon$ by Proposition 5.3. So, $f$ is integrable according to the old definition (5.2.5) by Proposition 5.4.

3. When we say $f$ is integrable on $E$, it is understood that $f$ is measurable on $E$.
4. If $f$ is integrable on $E$, $\int_E f = \int_E (f^+ - f^-) = \int_E f^+ - \int_E f^-$.
5. Suppose $f$ is integrable on $E$ and $f = f_1 - f_2$, where $f_1$ and $f_2$ are nonnegative measurable functions such that $\int_E f_1$ and $\int_E f_2$ are both finite. Is it true that $\int_E f = \int_E f_1 - \int_E f_2$? Because $f$ is integrable, $\int_E f = \int_E f^+ - \int_E f^-$. Since $f^+ - f^- = f = f_1 - f_2$, we have $f^+ + f_2 = f_1 + f^-$, and because $(f^+ + f_2)$ and $(f_1 + f^-)$ are nonnegative measurable functions, application of Proposition 5.7 yields

$$\int_E f^+ + \int_E f_2 = \int_E (f^+ + f_2) = \int_E (f_1 + f^-) = \int_E f_1 + \int_E f^-.$$

We may subtract since all terms are finite and conclude

$$\int_E f^+ - \int_E f^- = \int_E f_1 - \int_E f_2 .$$

In other words, if $f$ is integrable, and we have any representation for $f$ as the difference of two nonnegative measurable functions having finite integrals, the difference of their integrals may be used to calculate the integral of $f$.

**Example 11:** Recall Example 6.

1. $f(x) = \begin{cases} 1/x, & 0 < x \leq 1 \\ 1/(x-2), & 1 < x < 2 . \end{cases}$

$$\int_{(0,2)} f^+ = \int_{(0,1]} \frac{1}{x} = \infty \text{ and } \int_{(0,2)} f^- = \int_{(1,2)} \frac{-1}{x-2} = \infty ,$$

The Lebesgue integral of $f$ on $(0, 2)$ is not defined, and we can't have integrability without an integral.

2. $f(x) = \begin{cases} x^{-1/2}, & 0 < x \leq 1 \\ 1/(x-2), & 1 < x < 2 . \end{cases}$

$$\int_{(0,2)} f^+ = \int_{(0,1]} x^{-\frac{1}{2}} = 2 \text{ and } \int_{(0,2)} f^- = \int_{(1,2)} \frac{-1}{x-2} = \infty .$$

The Lebesgue integral of $f$ is defined, we may write $\int_{(0,2)} f = -\infty$, but $f$ is not integrable on $(0, 2)$.

3. $f(x) = \begin{cases} x^{-1/2}, & 0 < x \leq 1 \\ -(2-x)^{-1/2}, & 1 < x < 2 . \end{cases}$

$$\int_{(0,2)} f^+ = \int_{(0,1]} x^{-\frac{1}{2}} = 2 \text{ and } \int_{(0,2)} f^- = \int_{(1,2)} (2-x)^{-\frac{1}{2}} = 2 .$$

The Lebesgue integral of $f$ is defined, we may write $\int_{(0,2)} f = 2 - 2 = 0$, and thus $f$ is integrable on $(0, 2)$.

4. $f(x) = \begin{cases} x^{-1/2}, & 0 < x \le 1 \\ 1/x, & x > 1\,. \end{cases}$

$$\int_E f = \int_E f^+ = \int_{(0,1]} x^{-\frac{1}{2}} + \int_{(1,\infty)} \frac{1}{x} = 2 + \infty = \infty$$

by Proposition 5.7.

The Lebesgue integral of $f$ is defined, $\int_{(0,\infty)} f = \infty$, and $f$ is not integrable on $(0, \infty)$.

5. $f(x) = \begin{cases} -x^{-1/2}, & 0 < x \le 1 \\ x^{-2}, & x > 1\,. \end{cases}$

$$\int_{(0,\infty)} f^+ = \int_{(1,\infty)} x^{-2} = 1 \text{ and } \int_{(0,\infty)} f^- = \int_{(0,1]} x^{-\frac{1}{2}} = 2.$$

The Lebesgue integral of $f$ is defined, $\int_{(0,\infty)} f = 1 - 2 = -1$, and $f$ is integrable on $(0, \infty)$.

The usual properties (homogeneous, monotonic, etc.) for integrable functions and their integrals are valid (Proposition 5.12), and could be proven now. We defer these arguments at this time. We would rather discuss relationships between $\int_E f$, $\int_E |f|$, and $\int_E g$ when $g = f$ almost everywhere on $E$.

**Proposition 5.9** *Suppose $f$ is a measurable function defined on a measurable set $E$. Then $f$ is integrable on $E$ iff $|f|$ is integrable on $E$. Furthermore,*

$$\left| \int_E f \right| \le \int_E |f|\,.$$

*Proof:* Assume $f$ is integrable on $E$. We want to show $|f|$ is measurable and $\int_E |f|^+$, $\int_E |f|^- < \infty$. But since $f$ is measurable, $|f|$ is measurable (Proposition 4.6), $\int_E |f|^- = 0$, and $\int_E |f|^+ = \int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^- < \infty$ because $f$ is integrable on $E$. Thus $|f|$ is integrable on $E$.

Assume $|f|$ is integrable on $E$. We show $f$ is integrable on $E$. Since $f$ is measurable by hypothesis, and $\int_E f^+ + \int_E f^- = \int_E (f^+ + f^-) = \int_E |f| = \int_E |f|^+ < \infty$, the nonnegative integrals $\int_E f^+$, $\int_E f^-$ are both finite. Consequently, $f$ is integrable on $E$.

$$\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right| \le \left|\int_E f^+\right| + \left|\int_E f^-\right|$$
$$= \int_E f^+ + \int_E f^- = \int_E (f^+ + f^-) = \int_E |f|\,.$$

### 5.4.3 Comments

1. This proposition is false if we are dealing with the Riemann integral:

   Let $f(x) = \begin{cases} 1, & x \text{ rational}, \quad 0 \le x \le 1 \\ -1, & x \text{ irrational}, \quad 0 \le x \le 1. \end{cases}$

   Then $f$ is measurable on $[0, 1]$, $\int_0^1 |f(x)|\,dx = 1$, but $\underline{\int}_0^1 f(x)dx = -1 < 1 = \overline{\int}_0^1 f(x)dx$.
2. If $f$ is integrable on $E$, then $f$ is real-valued almost everywhere on $E$ ("If $f$ is integrable, $f$ is finite a.e."):

   $$\{x \in E \mid f(x) = \pm\infty\} = \{x \in E \mid f^+(x) = +\infty\} \cup \{x \in E \mid f^-(x) = +\infty\}.$$

   Then

   $$\infty > \int_E f^+ \ge \int_{\{x \in E \mid f^+(x) = +\infty\}} f^+ \ge n\mu(\{x \in E \mid f^+(x) = +\infty\})$$

   for all $n$. We have a contradiction unless $\mu(\{x \in E \mid f^+(x) = +\infty\}) = 0$. The reader may show $\mu(\{x \in E \mid f^-(x) = +\infty\}) = 0$.

   *Note*: $f(x) = 1/x$, $0 < x \le 1$. Then $f$ is real-valued everywhere, but is not integrable.
3. If $f, g$ are integrable on $E$, then $|f|, |g|$ are integrable on $E$, and since $|f+g| \le |f| + |g|$, $\int_E |f+g| = \int_E |f+g|^+ \le \int_E |f| + \int_E |g| < \infty$ (Proposition 5.7). Thus $|f+g|$, and hence $(f+g)$, is integrable on $E$.

### 5.4.4 Problems

1. Suppose

$$f(x) = \begin{cases} (-1)^{n+1} n^{.49}, & (n+1)^{-1/2} < x < n^{-1/2},\ n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Show $f$ is integrable on $[0, 1]$.

2. Suppose

$$f(x) = \begin{cases} (-1)^{n+1} n^{.5}, & (n+1)^{-1/2} < x < n^{-1/2},\ n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

i. Show $f$ is not integrable on $[0, 1]$.
ii. Does an "improper" Riemann integral on $[0, 1]$ exist?

3. Suppose $f(x) = -\frac{2}{x}\cos\left(\frac{1}{x^2}\right) + 2x\sin\left(\frac{1}{x^2}\right), \qquad 0 < x \leq 1.$

i. Show $f$ is not integrable over $(0, 1]$.

*Hint*: $|f(x)| \geq 2/x\ |\cos(1/x^2)|\ -2x \geq 1/x - 2x$ on the interval $((2n + 1/3)\pi)^{-1/2} \leq x \leq ((2n - 1/3)\pi)^{-1/2}$.

4. Suppose $f(x) = \begin{cases} \frac{1}{x}\cos\left(\frac{\pi}{x^2}\right) - \frac{x}{\pi}\sin\left(\frac{\pi}{x^2}\right), & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$

i. Show $f$ is not integrable on $[0, 1]$.
ii. Show an "improper" Riemann integral exists on $[0, 1]$.
iii. Let $F(x) = \begin{cases} -\frac{1}{2\pi}x^2\sin\left(\frac{\pi}{x^2}\right), & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$

Then $F'(x) = f(x)$ and the Newton-Leibnitz integral is $F(1) - F(0) = 0$.

5. Suppose

$$f(x) = \begin{cases} (-1)^n (n+1)^{-2}, & n\pi < x < (n+1)\pi,\ n = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Show $f$ is integrable on $[0, \infty)$. In fact,

$$\int_{[0,\infty)} f = \pi\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) = \frac{\pi^3}{12}.$$

6. Suppose

$$f(x) = \begin{cases} (-1)^n(n+1)^{-1}, & n\pi < x < (n+1)\pi,\ n = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

   i. Show $f$ is not integrable on $[0, \infty)$.
   ii. Show an "improper" Riemann integral of $f$ makes sense: In fact, $\int_0^\infty f(x)dx = \pi \ln 2$.

7. Suppose $f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & 0 < x, \\ 1, & x = 0. \end{cases}$

   i. Show $f$ is not integrable on $[0, \infty)$.
   ii. An "improper" Riemann integral of $f$ makes sense: In fact, it can be shown that $\int_0^\infty \sin(x)/x\, dx = \pi/2$.

**Proposition 5.10** *If $f$ is a measurable function defined on a measurable set $E$, and $g$ is integrable on $E$ with $|f| \le |g|$, then $\int_E |f| \le \int_E |g|$ and $f$ is integrable on $E$.*

*Proof:* We have $\int_E |f| \le \int_E |g| < \infty$ from Propositions 5.7 and 5.9. To show $f$ is integrable on $E$ requires: $f$ measurable on $E$ (given) and showing $\int_E f^+$, $\int_E f^-$ are both finite.

But $0 \le \int_E f^+ + \int_E f^- = \int_E |f| < \infty$, and the argument is complete. ■

### 5.4.5 Problems

1. $f(x) = \begin{cases} \dfrac{1}{2}, & x = 0 \\ \dfrac{1 - \cos(x)}{x^2}, & 0 < x. \end{cases}$

   Show $0 \le f(x) \le 1/2$, $0 \le x \le \pi/2$, and $f(x) \le 2/x^2$, $\pi/2 \le x$. Conclude $f$ is integrable on $[0, \infty)$.

Let $f_k(x) = \begin{cases} f(x), & 0 \le x \le 2k\pi \\ 0, & x > 2k\pi \end{cases}$.

Then $0 \le f_k \le f_{k+1}$, $\lim f_k = f$, and

$$\begin{aligned}
\int_{[0,\infty)} f &= \int_{[0,\infty)} \lim f_k \\
&= \lim \int_{[0,\infty)} f_k = \lim \int_{[0,2k\pi]} f_k = \lim \int_0^{2k\pi} \frac{1-\cos(x)}{x^2}\,dx \\
&= \lim \left\{ \frac{\cos(x)-1}{x}\bigg|_0^{2k\pi} + \int_0^{2k\pi} \frac{\sin(x)}{x}\,dx \right\} \\
&= \lim \int_0^{2k\pi} \frac{\sin(x)}{x}\,dx .
\end{aligned}$$

Thus $\displaystyle\int_{[0,\infty)} \frac{1-\cos(x)}{x^2} = \int_0^\infty \frac{\sin(x)}{x}\,dx.$

. Is $(\sin(x)/x)^2$ integrable on $[0,\infty)$? Calculate $\int_{[0,\infty)} (\sin(x)/x)^2$.

*Hint*:

$$\begin{aligned}
\int_{[0,\infty)} \left(\frac{\sin(x)}{x}\right)^2 &= \lim \int_0^{k\pi} \frac{1-\cos(2x)}{2x^2}\,dx \\
&= \lim \int_0^{2k\pi} \frac{1-\cos(x)}{x^2}\,dx = \int_0^\infty \frac{\sin(x)}{x}\,dx.
\end{aligned}$$

*Note*: From problems 1 and 2,

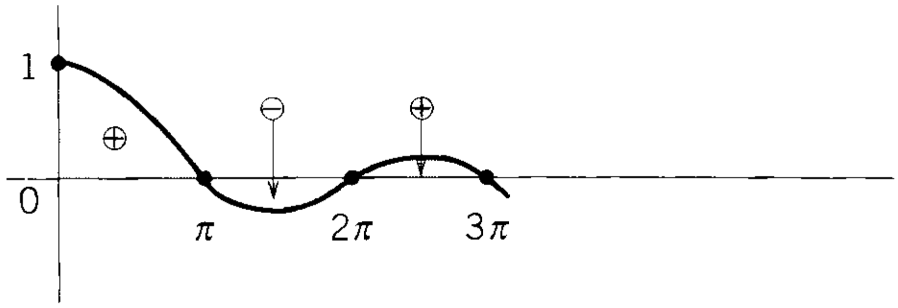
$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \int_{[0,\infty)} \frac{1-\cos(x)}{x^2} = \int_0^\infty \left(\frac{\sin(x)}{x}\right)^2$$

that is,

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx\ !$$

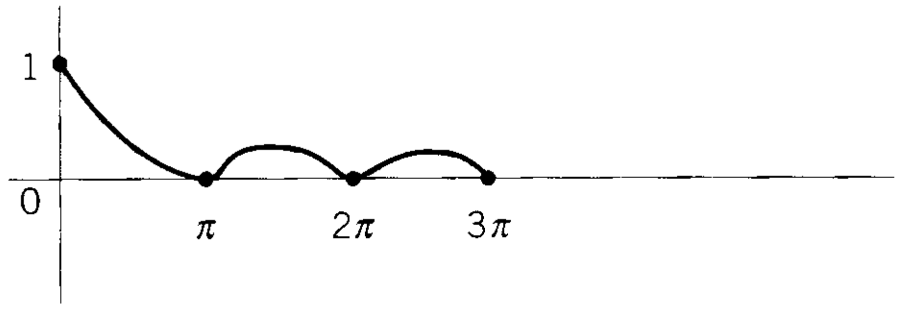
Isn't this amazing!

$\dfrac{\sin(x)}{x}$:




here

$\left(\dfrac{\sin(x)}{x}\right)^2$:



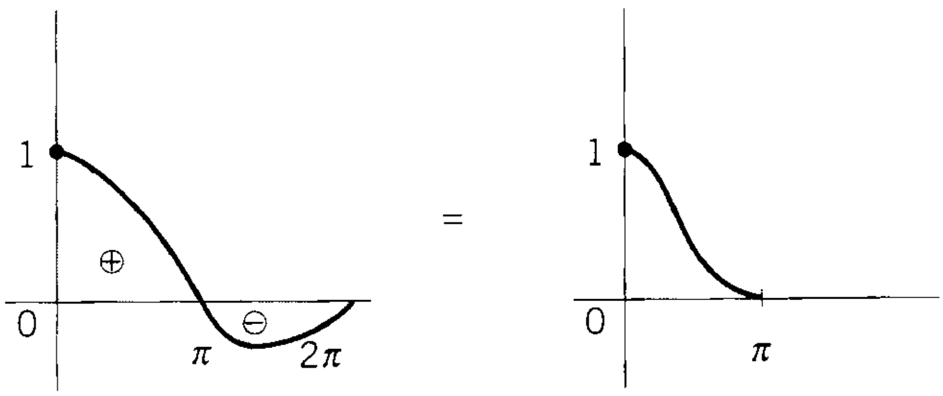

On closer inspection, show

$$\int_{2\alpha}^{2\beta} \frac{\sin(x)}{x}\, dx = \int_{\alpha}^{\beta} \left(\frac{\sin(x)}{x}\right)^2 dx, \quad \alpha, \beta = n\pi.$$

Oh,

$$\int_0^{2\pi} \frac{\sin(x)}{x}\, dx = \int_0^{\pi} \left(\frac{\sin(x)}{x}\right)^2 dx,$$

that is,




Then

$$\int_0^{2n\pi} \frac{\sin(x)}{x}\, dx = \int_0^{n\pi} \left(\frac{\sin(x)}{x}\right)^2 dx$$

and now it is reasonable that

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx.$$

3. Are the following functions integrable on the sets $E$?

   i. $\dfrac{\tan^{-1}(x)}{x^{3/2}}$ on $(0, \infty)$.

   ii. $\dfrac{\ln x}{x^{1+\alpha}}$ on $(1, \infty)$, $\alpha > 0$.

   iii. $\dfrac{1 - \cos(x)}{x^{2+\alpha}}$ on $(0, \infty)$, $0 < \alpha < 1$.

   iv. $\dfrac{\sin(x)}{x^{1+\alpha}}$ on $(0, \infty)$, $0 < \alpha < 1$.

Sets of measure zero *do not* affect Lebesgue integrability.

**PROPOSITION 5.11** *If $f = g$ a.e. on a measurable set $E$, and if $g$ is integrable on $E$, then $f$ is integrable on $E$ and*

$$\int_E f = \int_E g.$$

*Proof:* The function $g$ is measurable on $E$ by the assumption of being Lebesgue integrable on $E$. Since $f$ is equal almost everywhere to a measurable function $g$, $f$ is measurable on $E$ (Proposition 4.3). Application of Proposition 4.6 yields measurability of $f^+$ and $f^-$ on $E$. Let $A = \{x \in E \mid f(x) \neq g(x)\}$. Then $f^+ = g^+$ and $f^- = g^-$ on $E - A$, and $\int_{E-A} f^+ = \int_{E-A} g^+$ and $\int_{E-A} f^- = \int_{E-A} g^-$, that is, $f$ is measurable on $E - A$, $\int_{E-A} f^+, \int_{E-A} f^- < \infty$: $f$ is integrable on $E - A$. Because $A$ is a measurable subset of $E$, $f$ is measurable on $A$ (Problem 4.1.4), $\mu(A) = 0$, and hence $\int_A f^+, \int_A f^- = 0$. But then $\int_E f^+ = \int_{E-A} f^+ + \int_A f^+ < \infty$ and $\int_E f^- = \int_{E-A} f^- + \int_A f^- < \infty$: The function $f$ is integrable on $E$.

Then

$$\begin{aligned}\int_E g &= \int_E g^+ - \int_E g^- = \int_{E-A} g^+ + \int_A g^+ - \int_{E-A} g^- - \int_A g^- \\ &= \int_{E-A} f^+ + \int_A f^+ - \int_{E-A} f^- - \int_A f^- = \int_E f^+ - \int_E f^- \\ &= \int_E f.\end{aligned}$$

**Example 12:**

1. $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases},$

   $g \equiv 0$. $f = g$ a.e. and $\int_{(-\infty,\infty)} f = \int_{(-\infty,\infty)} g = 0.$

2. $f(x) = \begin{cases} x, & x \text{ rational}, \quad 0 \le x \le 1 \\ 1-x, & x \text{ irrational}, \quad 0 \le x \le 1 \end{cases},$

   $g(x) = 1 - x, 0 \le x \le 1.$

   $f = g$ a.e. on $[0, 1]$ and $\int_{[0,1]} f = \int_{[0,1]} g = \int_0^1 (1-x)dx = \frac{1}{2}$ .

3. $f(x) = \begin{cases} x^2, & x \text{ rational} \\ e^{-|x|}, & x \text{ irrational} \end{cases}, \qquad g(x) = e^{-|x|}.$

   $\int_{(-\infty,\infty)} f = \int_{(-\infty,\infty)} g = 2.$

4. $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \quad 0 \le x \le 1, \ (p, q) = 1 \\ 0, & x \text{ irrational} \end{cases}, \qquad g(x) \equiv 0.$

   $\int_{[0,1]} f = \int_{[0,1]} g = 0.$

5. $f(x) = \begin{cases} x^2, & x \text{ in the Cantor set} \\ \sin(x), & \text{otherwise} \end{cases}, \qquad g(x) \equiv \sin(x).$

$$\int_{[0,\pi]} f = \int_{[0,\pi]} g = 2.$$

**PROPOSITION 5.12** *If $f, g$ are integrable on a measurable set $E$, and $k$ is any real number, then*

1. *$(kf)$ is integrable on $E$, and $\int_E (kf) = k \int_E f$ (homogeneous);*
2. *$(f + g)$ is integrable on $E$, and $\int_E (f + g) = \int_E f + \int_E g$ (additive);*
3. *$\int_E f \le \int_E g$ if $f \le g$ on $E$ (monotone);*
4. *If $E_1$ and $E_2$ are disjoint measurable subsets of $E$ with $E = E_1 \cup E_2$, $f$ is integrable on $E_1$ and $E_2$, and*

$$\int_E f = \int_{E_1} f + \int_{E_2} f \qquad \textit{(additive on the domain)}.$$

*Proof:*

1. If $k \ge 0$, $\int_E (kf)^+ = \int_E kf^+ = k \int_E f^+ < \infty$ and $\int_E (kf)^- = k \int_E f^- < \infty$ because $kf^+, kf^-$ are nonnegative measurable functions (Proposition 5.7). By definition, $(kf)$ is integrable on $E$. Furthermore, $\int_E (kf) = \int_E (kf)^+ - \int_E (kf)^- = k \int_E f^+ - k \int_E f^- = k \int_E f$, where the last equality is the definition of $f$ being integrable on $E$. If $k < 0$, $(kf)^+ = (-k) f^-$, $(kf)^- = (-k) f^+$, $\int_E (kf)^+ = -k \int_E f^- < \infty$, and $\int_E (kf)^- = -k \int_E f^+ < \infty$, that is, $(kf)$ is integrable on $E$. Again,

$$\int_E (kf) = \int_E (kf)^+ - \int_E (kf)^- = \int_E (-k) f^- - \int_E (-k) f^+$$
$$= k\left[\int_E f^+ - \int_E f^-\right] = k \int_E f.$$

2. Since $f, g$ are integrable on $E$, $|f|, |g|$ are integrable on $E$ (Proposition 5.9). Because $\int_E |f| = \int_E |f|^+ < \infty$, $\int_E |g| =$

$\int_E |g|^+ < \infty$, and $|f+g| \le |f| + |g|$, $\int_E |f+g| \le \int_E(|f| + |g|) \le \int_E |f| + \int_E |g| < \infty$ (Proposition 5.7). But $|f+g|^+ = |f+g|$ and $|f+g|^- = 0$, $\int_E |f+g|^+ < \infty$, $\int_E |f+g|^- < \infty$, that is, $|f+g|$ is integrable on $E$, but then (Proposition 5.9) $f+g$ is integrable on $E$. Look at comment 5.4.3.

Now, $f+g = (f^+ + g^+) - (f^- + g^-)$, that is, the integrable function $(f+g)$ has been written as the difference of two nonnegative measurable functions, $(f^+ + g^+)$ and $(f^- + g^-)$, whose integrals are finite. Comment 5.4.2 reveals

$$\begin{aligned}\int_E (f+g) &= \int_E (f^+ + g^+) - \int_E (f^- + g^-)\\ &= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-\\ &= \int_E f + \int_E g.\end{aligned}$$

3. Since $f \le g$ on $E$, $f^+ - f^- \le g^+ - g^-$, i.e., $f^+ + g^- \le g^+ + f^-$. Because $(f^+ + g^-)$, $(g^+ + f^-)$ are nonnegative measurable functions we may apply Proposition 5.7 to conclude

$$\int_E f^+ + \int_E g^- = \int_E (f^+ + g^-) \le \int_E (g^+ + f^-) = \int_E g^+ + \int_E f^-.$$

Because all terms are finite, we may subtract: $\int_E f \le \int_E g$.

4.

$$\begin{aligned}\int_E f &= \int_E f^+ - \int_E f^-\\ &= \int_{E_1} f^+ + \int_{E_2} f^+ - \int_{E_1} f^- - \int_{E_2} f^-\\ &= \int_{E_1} f + \int_{E_2} f. \qquad \text{(Proposition 5.7)}\end{aligned}$$

In the next section we prove the major convergence theorem of Lebesgue integration, the so-called Lebesgue Dominated Convergence Theorem (LDCT).

## 5.5 CONVERGENCE THEOREMS

We continue our discussion on the question:

$$\int_E (\lim f_k) \stackrel{?}{=} \lim \int_E f_k .$$

The reader may recall the results that have been established:

**THEOREM 5.3** *If $(f_k)$ is a sequence of Riemann integrable functions on $[a,b]$, and if $\lim f_k = f$ uniformly on $[a,b]$, then*

1. $(\lim f_k)$ *is Riemann integrable on $[a,b]$;*
2. $\lim \int_a^b f_k(x)dx = \int_a^b (\lim f_k)(x)dx.$

**THEOREM 5.10** (LMCT) *If $(f_k)$ is a monotone increasing sequence of nonnegative measurable functions on a measurable set $E$, then*

$$\lim \int_E f_k = \int_E (\lim f_k) .$$

**THEOREM 5.11** (*Fatou*) *If $(f_k)$ is a sequence of nonnegative measurable functions on a measurable set $E$, then*

$$\int_E (\liminf f_k) \leq \liminf \int_E f_k .$$

**PROPOSITION 5.8** *If $(g_k)$ is a sequence of nonnegative measurable functions defined on a measurable set $E$, then*

$$\int_E \sum g_k = \sum \int_E g_k .$$

**Example 13:**

1. $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

   If $r_1, r_2, \ldots$ is any enumeration of the rationals, define

$$f_k(x) = \begin{cases} 1, & x = r_1, r_2, \ldots, r_k \\ 0, & \text{otherwise.} \end{cases}$$

Then $0 \le f_k \le f_{k+1}$, $\lim f_k = f$, $\int_{[0,\infty)} f = \int_{[0,\infty)} \lim f_k = \lim \int_{[0,\infty)} f_k$ $= 0$ via LMCT.

2.

i.

$$\sum_0 \int_{[0,1]} \frac{2x^3}{(1+x^2)^n} = \int_{[0,1]} \sum_0 \frac{2x^3}{(1+x^2)^n} = \int_{[0,1]} (1+x^2)2x = \frac{3}{2}.$$

ii.

$$\sum_0 \int_{[0,1]} \frac{x^\alpha}{(1+x^\beta)^n} = \int_{[0,1]} \sum_0 \frac{x^\alpha}{(1+x^\beta)^n} = \int_{(0,1]} (x^{\alpha-\beta} + x^\alpha).$$

iii.

$$\sum_0 \int_{[0,1]} x^\alpha (1-x^\beta)^n = \int_{[0,1]} \sum_0 x^\alpha (1-x^\beta)^n$$

$$= \int_{(0,1]} \left\{ \begin{matrix} x^{\alpha-\beta}, & 0 < x < 1 \\ 0, & 1 = x \end{matrix} \right\}.$$

iv.

$$\sum_1 \int_{[0,1]} \frac{x}{[1+(n-1)x][1+nx]} = \int_{[0,1]} \sum_1 \frac{x}{[1+(n-1)x][1+nx]}$$

$$= \int_{[0,1]} \left\{ \begin{matrix} 1, & 0 < x \le 1 \\ 0, & x = 0 \end{matrix} \right\} = 1.$$

3.

$$\int_{[0,\pi]} \sum_1 \frac{\sin(n^2 x)}{n^2} = \sum_1 \int_{[0,\pi]} \frac{\sin(n^2 x)}{n^2} = 2 \sum \frac{1}{(2n-1)^4} = \frac{\pi^4}{48}$$

via Theorem 5.3. Can we use Proposition 5.8?

The reader may have noticed the frequent occurence of "monotone" and "nonnegative" in our "limiting" discussions. It is these restrictions that we seek to modify or eliminate, although, other requirements must be imposed. The new requirements do not severely restrict applications of the Lebesgue integral, and in fact result in a very powerful tool for analysis, the so-called Lebesgue Dominated Convergence Theorem (LDCT).

**THEOREM 5.12** (*Lebesgue's Dominated Convergence Theorem, 1910*) *Let* $(f_k)$ *be a sequence of measurable functions, defined on a measurable set* $E$, *such that*

$$\lim f_k = f \text{ a.e. on } E.$$

*Suppose we have an integrable function* $g$ *on* $E$ *such that* $|f_k| \leq g$ *on* $E$. *Then* $f$ *is integrable on* $E$ *and*

$$\int_E (\lim f_k) = \int_E f = \lim \int_E f_k.$$

*Proof:* Since sets of measure zero do not affect Lebesgue integrability, and $\mu(\{x \in E \mid g(x) = \infty\}) = 0$ because $g$ is integrable on $E$, we may as well assume $\lim f_k = f$ (real-valued) on $E$. Furthermore, $f$, as the limit of a sequence of measurable functions, is measurable (Theorem 4.1), and the reader may argue (Proposition 5.10) that $f$, and for that matter, all other functions appearing below, are integrable on $E$.

Our first argument will be based on the Lebesgue Monotone Convergence Theorem: monotone increasing and nonnegative. Monotonicity is not a problem: Recall that for a sequence of functions $(f_k)$ we may construct two related monotone sequences $(\underline{f}_k)$ and $(\overline{f}_k)$ where $\underline{f}_k = \inf\{f_k, f_{k+1}, \ldots\}$ and $\overline{f}_k = \sup\{f_k, f_{k+1}, \ldots\}$ respectively. So, we have the following relationships between $g, f$, and terms of the sequences $(\underline{f}_k)$ and $(\overline{f}_k)$:

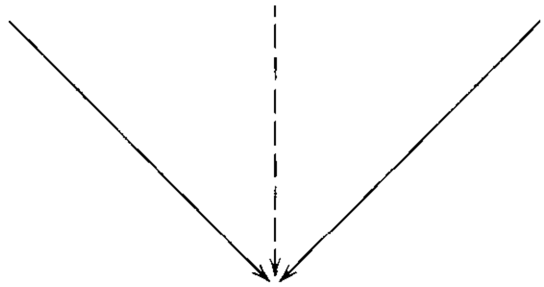
$$-g \leq \underline{f}_k \leq \underline{f}_{k+1} \leq f_{k+1}, f \leq \overline{f}_{k+1} \leq \overline{f}_k \leq g \text{ on } E. \tag{1}$$

Everything follows from this string of inequalities. Initially, all functions being integrable, along with monotonicity, yield (Proposition 5.12)

$$-\infty < \int_E -g \leq \int_E \underline{f}_{k+1} \leq \int_E f_{k+1}, \int_E f \leq \int_E \overline{f}_{k+1} \leq \int_E g < +\infty.$$

We want to use the "squeezing" principle:

$$-\infty < \int_E \underline{f}_{k+1} \le \int_E f_{k+1}, \int_E f \le \int_E \overline{f}_{k+1} < +\infty$$



? Common Limit?

If we can show a common limit, that is, $\lim \int_E \underline{f}_k = \lim \int_E \overline{f}_k$, then $\lim \int_E f_k$ exists and $\lim \int_E f_k = \int_E f = \int_E (\lim f_k)$, which is the conclusion we want. Returning to (1), we have

$$0 \le g + \underline{f}_k \le g + \underline{f}_{k+1} \le 2g \text{ and } 0 \le g - \overline{f}_k \le g - \overline{f}_{k+1} \le 2g.$$

The sequences $(g + \underline{f}_k)$ and $(g - \overline{f}_k)$ are nonnegative, monotone increasing, and have limits $g + f$ and $g - f$ respectively. Application of the monotone convergence theorem yields

$$\int_E g + \lim \int_E \underline{f}_k = \int_E \lim(g + \underline{f}_k) = \int_E (g + f) = \int_E g + \int_E f$$

and

$$\int_E g - \lim \int_E \overline{f}_k = \int_E \lim(g - \overline{f}_k) = \int_E (g - f) = \int_E g - \int_E f,$$

that is,

$$\lim \int_E \underline{f}_k = \int_E f = \lim \int_E \overline{f}_k,$$

and this argument is complete.

The next argument is more direct; an application of Fatou's Theorem. "Fatou" for $(g + f_k)$:

$$\int_E g + \int_E f = \int_E (g + f) = \int_E \liminf(g + f_k) \le \liminf \int_E (g + f_k)$$

$$= \liminf\left(\int_E g + \int_E f_k\right) = \int_E g + \liminf \int_E f_k,$$

so $\int_E f \leq \liminf \int_E f_k$. "Fatou" for $(g - f_k)$:

$$\int_E g - \int_E f = \int_E (g - f) = \int_E \liminf(g - f_k) \leq \liminf \int_E (g - f_k)$$
$$= \liminf\left(\int_E g - \int_E f_k\right) = \int_E g - \limsup \int_E f_k,$$

so $\limsup \int_E f_k \leq \int_E f$. Combining, $\limsup \int_E f_k \leq \int_E f \leq \liminf \int_E f_k$. ■

### 5.5.1 Comments

1. We need "domination" on $E$: If $f_k = k\mathcal{X}_{(0,1/k)}$, then $\lim f_k = 0$, and $\lim \int_{[0,1]} f_k = 1 \neq 0 = \int_{[0,1]}(\lim f_k)$.
2. We need "integrable domination" on $E$. If $f_k = \mathcal{X}_{[k,2k]}$, then $\lim f_k = 0$ and $\lim \int_R f_k = \infty \neq 0 = \int_R(\lim f_k)$.

The following problems illustrate some applications of the convergence theorems that have appeared in this course. They partially answer "Why Lebesgue?"

### 5.5.2 Problems

(Theorem 5.6 should be used quite often.)

1.

i. Show $\lim \displaystyle\int_{[0,1]} \frac{kx}{1 + k^2x^2} = 0.$

We see that $\int_{[0,1]} (kx)/(1 + k^2x^2) = \int_0^1 (kx)/(1 + k^2x^2)dx = (\ln(1 + k^2x^2))/(2k)\,|_0^1 = (\ln(1 + k^2))/(2k)$, and thus $\lim \int_{[0,1]} (kx)/(1 + k^2x^2) = \lim(\ln(1 + k^2))/(2k) = 0$. No problem, but, in anticipation of exercises to come, we try other approaches. For example, $\lim(kx)/(1 + k^2x^2) = 0$, $0 \leq x \leq 1$. But we do not have uniform convergence on $[0, 1]$ : $f_k(1/k) = 1/2$. How about $0 \leq f_k \leq f_{k+1}$? Not true, so LMCT doesn't apply. But, we may use LDCT with $g(x) = 1/2$ on $[0, 1]$. Conclude that $\lim \int = \int \lim = 0$.

ii Evaluate $\lim \int_{[0,1]} \frac{k^{3/2}x}{1+k^2x^2}$ two ways.

*Hint*: $g(x) = \begin{cases} 0, & x = 0 \\ x^{-1/2}, & 0 < x \le 1. \end{cases}$

2.

i. Evaluate $\lim \int_{[0,\infty)} e^{-kx}$

*Hint*: $g(x) = e^{-x}$.

ii. Evaluate $\lim \int_{[0,\infty)} e^{-kx^2}$.

iii. Evaluate $\lim \int_{[0,\infty)} k^{3/2}xe^{-kx}$.

3. Show $\lim \int_{[0,\pi/2]} (\sin(x))^k = 0$.

4. Show $\lim \int_{[-2,2]} \frac{x^{2k}}{1+x^{2k}} = 2$.

*Hint*: $\lim(x^{2k}/(1+x^{2k})) = \begin{cases} 0, & 0 \le x < 1 \\ 1/2, & x = 1 \\ 1, & 1 < x \le 2 \end{cases}$.

5. $f_k(x) = \frac{1}{1+x^{2k}}$, $0 \le x$. Show $\lim f_k(x) = \begin{cases} 1, & 0 \le x < 1 \\ 1/2, & x = 1 \\ 0, & 1 < x \end{cases}$.

Evaluate $\lim \int_{[0,1]} f_k$, $\lim \int_{(1,\infty)} f_k$, $\lim \int_{[0,\infty)} f_k$.

6. Evaluate $\int_{[0,1]} \sum \frac{x^n}{n}$.

7.

i. Evaluate $\displaystyle\int_{(0,\infty)} xe^{-kx}, \qquad k = 1, 2, \ldots.$

*Hint*: $g_n(x) = \begin{cases} xe^{-kx}, & 0 \le x \le n \\ 0, & n < x \end{cases},$

$0 \le g_n \le g_{n+1}$, and use LMCT to show $\int_{(0,\infty)} xe^{-kx} = \lim \int_{[0,n]} g_n = \lim_n \int_0^n xe^{-kx}dx = 1/k^2$, where the last equality follows by integration by parts for the Riemann integral.

ii. Show $\displaystyle\int_{[0,\infty)} \frac{x}{e^x - 1} = \sum \frac{1}{k^2}.$

*Hint*: $x(e^x - 1)^{-1} = x(\sum_1 e^{-kx}) = \sum_1 xe^{-kx}$, $x > 0$, and $\int_{[0,\infty)} x/(e^x - 1) = \int_{(0,\infty)} x/(e^x - 1) = \int_{(0,\infty)} \sum_1 xe^{-kx}$, where the first equality follows from, "sets of measure zero do not affect the Lebesgue integral".

iii. Show $\displaystyle\int_{[0,\infty)} \frac{x^2}{e^x - 1} = \frac{1}{2}\sum_1 \frac{1}{k^3}.$

iv. Show $\displaystyle\int_{[0,\infty)} \frac{x^\alpha}{e^x - 1} = \zeta(\alpha + 1)\,\gamma(\alpha + 1), \quad \alpha > 0.$

(Zeta-Gamma)

8. Show that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots.$

Recall $\tan^{-1}(x) = \displaystyle\int_0^x \frac{1}{1 + t^2}\, dt$

and $\dfrac{1}{1 + t^2} = \begin{cases} \sum_0 (-t^2)^n, & -1 < t < 1 \\ 1/2, & t = 1 \end{cases}.$

Can we conclude $\int \sum = \sum \int$ ?

*Hint*: $f_k(t) = \begin{cases} \sum_0^k (-t^2)^n, & 0 \le t < 1 \\ 1/2, & t = 1 \end{cases}, \qquad g(t) = 1,$

and LDCT.

Another approach: $\dfrac{1}{1+t^2} = \begin{cases} \sum_0 (t^{4n} - t^{4n+2}), & 0 \le t < 1 \\ 1/2, & t = 1 \end{cases}$

Then $t^{4n} - t^{4n+2} \ge 0$, $0 \le t \le 1$, and we may conclude

$$\begin{aligned} \frac{\pi}{4} = \tan^{-1}(1) &= \int_0^1 \frac{1}{1+t^2}\,dt = \int_{[0,1]} \frac{1}{1+t^2} = \int_{[0,1)} \frac{1}{1+t^2} \\ &= \int_{[0,1)} \sum_0 (t^{4n} - t^{4n+2}) = \sum_0 \int_{[0,1)} (t^{4n} - t^{4n+2}) \\ &= \sum_0 \int_{[0,1]} (t^{4n} - t^{4n+2}) = \sum_0 \left(\frac{1}{4n+1} - \frac{1}{4n+3}\right) \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots , \end{aligned}$$

without constructing a dominating function.

9. Show that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$.

Recall $\ln 2 = \displaystyle\int_0^1 \frac{1}{1+t}\,dt = \int_{[0,1]} \left\{ \begin{matrix} \sum_0 (-t)^n, & 0 \le t < 1 \\ 1/2, & t = 1 \end{matrix} \right\}.$

*Hint*: $f_k(t) = \begin{cases} \sum_0^k (-t)^n, & 0 \le t < 1 \\ 1/2, & 1 = t \end{cases}, \qquad g(t) = 1, 0 \le t \le 1.$

Again, we could observe that

$$\frac{1}{1+t} = \begin{cases} \sum_0 (t^{2n} - t^{2n+1}), & 0 \le t < 1 \\ \frac{1}{2}, & t = 1 \end{cases},$$

$\int \sum = \sum \int$, since $t^{2n} - t^{2n+1} \ge 0, \qquad 0 \le t \le 1.$

Then

$$\ln 2 = \sum_0 \left( \frac{1}{2n+1} - \frac{1}{2n+2} \right)$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots .$$

10. Suppose $f(t) = t^{-1}$, $t > 0$ and $f_k(t) = \begin{cases} t^{-1+1/k}, & 0 < t \le 1 \\ t^{-1-1/k}, & 1 < t. \end{cases}$

Show

$$0 \le f_k \le f_{k+1}$$

and

$$\int_{[1,x]} f_k = \begin{cases} k(x^{1/k} - 1), & 0 < x \le 1 \\ \dfrac{k(x^{1/k} - 1)}{x^{1/k}}, & 1 < x. \end{cases}$$

Then

$$\ln x = \int_1^x \frac{1}{t}\,dt = \int_{[1,x]} \frac{1}{t} = \int_{[1,x]} \lim f_k = \lim \int_{[1,x]} f_k .$$

$$\ln x \overset{?}{=} \lim k(x^{1/k} - 1), \qquad 0 < x .$$

11. If $f(t) = \sum a_k t^k$, $-\infty < t < \infty$, $a_k \ge 0$, then show

$$\int_{[0,\infty)} e^{-t} f(t) = \sum a_k k!$$

*Hint*: $\int_{[0,\infty)} e^{-t} \sum a_k t^k = \sum a_k \int_{[0,\infty)} t^k e^{-t}$,

$$g_n(t) = \begin{cases} t^k e^{-t}, & 0 \le t \le n \\ 0, & n < t \end{cases}, \quad 0 \le g_n \le g_{n+1}, \quad \lim g_n(t) = t^k e^{-t},$$

$0 \le t < \infty$, and $\int_{[0,\infty)} t^k e^{-t} = \lim_n \int_{[0,n]} t^k e^{-t}$.

12.

i. Evaluate $\int_{[0,\infty)} e^{-t^2}\cos(t)$.

*Hint*: $e^{-t^2}\cos(t) = \lim_n \sum_0^n ((-1)^k/(2k)!)t^{2k}e^{-t^2}$,

$$f_n(t) = \begin{cases} \sum_0^n ((-1)^k/(2k)!)t^{2k}e^{-t^2}, & 0 \le t \le n \\ 0, & n < t \end{cases},$$

$$\lim f_n(t) = e^{-t^2}\cos(t),\ 0 \le t < \infty,\ |f_n(t)| \le e^{-t^2} \cdot e^t.$$

Apply LDCT:

$$\begin{aligned}
\int_{[0,\infty)} e^{-t^2}\cos(t) &= \lim \int_{[0,\infty)} f_n(t) \\
&= \lim_n \int_{[0,n]} \sum_0^n \frac{(-1)^k}{(2k)!} t^{2k}e^{-t^2} \\
&= \lim_n \sum_0^n \frac{(-1)^k}{(2k)!} \int_{[0,n]} t^{2k}e^{-t^2} \\
&= \lim_n \left(1 - \frac{1}{2^2 \cdot 1!} + \frac{1}{2^4 \cdot 2!} - \cdot + \frac{(-1)^n}{2^{2n}n!}\right) \int_{[0,\infty)} e^{-t^2} \\
&= e^{-1/4} \cdot \frac{\sqrt{\pi}}{2} \quad \text{(See problem 17.).}
\end{aligned}$$

ii. Show $\int_{(0,\infty)} e^{-t^2}\cos(xt) = \frac{\sqrt{\pi}}{2} e^{-x^2/4}$.

13. Show $\int_{(0,\infty)} \frac{\sin \alpha t}{e^t - 1} = \sum \frac{\alpha}{k^2 + \alpha^2}, \qquad \alpha > 0.$

*Hint*: $\int_{(0,\infty)} \frac{\sin \alpha t}{e^t - 1} = \int_{(0,\infty)} \sin \alpha t(e^{-t} + e^{-2t} + \cdot).$

Let $f_n(t) = \sum_1^n e^{-kt}\sin \alpha t$, and define

$$g(t) = \begin{cases} ?, & 0 \le t \le 1 \\ 2e^{-t}, & 1 < t. \end{cases}$$

Apply LDCT to conclude

$$\int_{(0,\infty)} \frac{\sin \alpha t}{e^t - 1} = \int_{(0,\infty)} \lim f_n$$

$$= \lim_n \int_{(0,\infty)} \sum_{k=1}^{n} e^{-kt} \sin \alpha t$$

$$= \lim_n \sum_{k=1}^{n} \int_{(0,\infty)} e^{-kt} \sin \alpha t\,.$$

Show $\displaystyle\int_{[0,\infty)} e^{-kt} \sin \alpha t = \frac{\alpha}{\alpha^2 + k^2}$.

*Hint*: $g_n(t) \equiv \begin{cases} e^{-kt} \sin \alpha t, & 0 \le t \le n \\ 0, & n < t\,. \end{cases}$

14.
i. Show $\displaystyle\int_{(0,1)} \frac{1}{1-t} \ln\left(\frac{1}{t}\right) = \sum_1 \frac{1}{k^2} = \frac{\pi^2}{6}$.

*Hint*: $\displaystyle\frac{1}{1-t} \ln\left(\frac{1}{t}\right) = \sum_1 \frac{(1-t)^{k-1}}{k}, \qquad 0 < t < 1.$

ii. Show $\displaystyle\int_{(1,\infty)} t^{-1}(t-1)^{-1} \ln t = \sum \frac{1}{k^2} = \frac{\pi^2}{6}$.

*Hint*: $t^{-1} \ln t (t-1)^{-1} = t^{-1} \ln t \sum_1 t^{-k}$, $t > 1$.

Evaluate $\sum_1 \int_{(1,\infty)} t^{-(k+1)} \ln t$.

*Hint*: $g_n(t) \equiv \begin{cases} t^{-(k+1)} \ln t, & 1 \le t \le n \\ 0, & n < t\,. \end{cases}$

Use LMCT to show $\int_{(1,\infty)} t^{-(k+1)} \ln t = \dfrac{1}{k^2}$, and complete the argument.

15. Show $\displaystyle\int_{(0,1)} \frac{t^{\alpha-1}}{1+t^\beta} = \frac{1}{\alpha} - \frac{1}{\alpha+\beta} + \frac{1}{\alpha+2\beta} - \frac{1}{\alpha+3\beta} + \cdots$ where $\alpha, \beta > 0$.

*Hint*: $(1+t^{\beta})^{-1} = (1 - t^{\beta} + t^{2\beta} - \cdots),\ 0 \le t < 1.$

Form $t^{\alpha-1}(1-t^{\beta}) + t^{\alpha-1}(t^{2\beta} - t^{3\beta}) + t^{\alpha-1}(t^{4\beta} - t^{5\beta}) + \cdots$ and let $f_k(t) \equiv t^{\alpha-1}(t^{2k\beta} - t^{(2k+1)\beta})$, $0 < t < 1$ and $k = 0, 1, 2, \cdots$ . Thus

$$\int_{(0,1)} \sum f_k = \sum \int_{(0,1)} f_k$$
$$= \sum \left( \frac{1}{\alpha + 2k\beta} - \frac{1}{\alpha + (2k+1)\beta} \right).$$

Can you use LDCT?

16.

i. Recall $1 + \frac{t}{k} \le e^{t/k}$, $(1+\frac{t}{k})^k \le e^t$, $-k \le t$, and $\lim(1+\frac{t}{k})^k = e^t$.

ii. Evaluate $\lim\limits_k \int_{(0,\infty)} (1+\frac{t}{k})^k e^{-\alpha t}$, $\alpha > 1$.

*Hint*: $(1+\frac{t}{k})^k e^{-\alpha t} \le e^t e^{-\alpha t} = e^{(1-\alpha)t}$, $t \ge 0$.

Next show $\displaystyle\int_{[0,\infty)} e^{(1-\alpha)t} = \frac{1}{\alpha - 1}, \qquad \alpha > 1.$

*Hint*: $g_k(t) \equiv \begin{cases} e^{(1-\alpha)t}, & 0 \le t \le k \\ 0, & k < t. \end{cases}$

iii. Show $\displaystyle\lim \int_{[0,k]} \left(1 - \frac{t}{k}\right)^k t^{\alpha-1} = \int_{(0,\infty)} e^{-t} t^{\alpha-1}, \qquad \alpha > 0.$

*Hint*: Show $(1 - \frac{t}{k}) \le e^{-t/k}$, $0 \le t \le k$ and define

$$f_k(t) = \begin{cases} (1 - t/k)^k, & 0 \le t \le k \\ 0 & , \ k < t. \end{cases}$$

iv. Evaluate $\displaystyle\lim_k \int_{[0,k]} \left(1 - \frac{t}{k}\right)^k e^{\alpha t}$ if $\alpha < 1$.

v. Evaluate $\displaystyle\lim \int_{[0,1]} \left(1 + \frac{t}{k}\right)^{-k-1}.$

vi. Show $\displaystyle\lim \int_{(0,\infty)} \frac{1}{(1 + t/k)^k} = 1.$

Look for a dominate function $g$:

If $0 < t \leq 1$,

$$\left(1+\frac{t}{k}\right)^{-k} \leq 1\,.$$

If $1 < t$,

$$\left(1+\frac{t}{k}\right)^{-k} = \left[\sum \binom{k}{i}\left(\frac{t}{k}\right)^{i}\right]^{-1}$$

$$\leq \left[\frac{k(k-1)}{2}\cdot\frac{t^2}{k}\right]^{-1}$$

$$\leq \frac{4}{t^2}\,, \qquad k \geq 2.$$

17.

Show $\displaystyle\int_{[0,\infty)} e^{-x^2} = \frac{\sqrt{\pi}}{2}$.

Let

$$f_k(t) = \begin{cases} \left(1-\dfrac{t^2}{k}\right)^k, & 0 \leq t \leq \sqrt{k} \\ 0, & t > \sqrt{k} \end{cases}$$

and

$$g_k(t) = f_k(t)\cdot\left(1-\frac{t^2}{k}\right)^{1/2}.$$

i. $f_k(t) \leq e^{-t^2}$, $\quad t \geq 0$ and $\lim f_k(t) = e^{-t^2}$, $\quad t \geq 0$.

ii. $g_k(t) \leq e^{-t^2}$, $\quad t \geq 0$ and $\lim g_k(t) = e^{-t^2}$, $\quad t \geq 0$.

iii.

$$\int_{[0,\infty)} f_k = \sqrt{k}\int_0^{\pi/2} \cos^{2k+1}(t)dt$$

$$= \sqrt{k}\;\frac{(2k)(2k-2)\cdots(2)}{(2k+1)(2k-1)\cdots(3)}$$

and

$$\int_{[0,\infty)} g_k = \sqrt{k} \int_0^{\pi/2} \cos^{2k+2}(t)dt$$

$$= \sqrt{k}\ \frac{(2k+1)(2k-1)\cdots(1)}{(2k+2)(2k)\cdots(2)} \cdot \frac{\pi}{2} .$$

iv. Since $f_k, g_k$ are dominated by the integrable function $e^{-t^2}$, $t \geq 0$, we have

$$\lim \int_{[0,\infty)} f_k = \int_{[0,k]} \lim f_k$$

$$= \int_{[0,\infty)} e^{-t^2}$$

$$= \lim \int_{[0,\infty)} g_k \, .$$

v.

$$\left(\int_{[0,\infty)} e^{-t^2}\right)^2 = \left(\lim \int_{[0,\infty)} f_k\right)\left(\lim \int_{[0,\infty)} g_k\right)$$

$$= \lim\left(\int_{[0,\infty)} f_k \cdot \int_{[0,\infty)} g_k\right)$$

$$= \frac{\pi}{4} .$$

vi.

$$\int_{[0,\infty)} e^{-t^2} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2} .$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2 = \frac{a_0^2}{2} + \sum_{1}^{\infty}\left(a_k^2 + b_k^2\right)$$

# Appendix A

# Cantor's Set

When you have eliminated the impossible, whatever remains, however improbable, must be the truth.

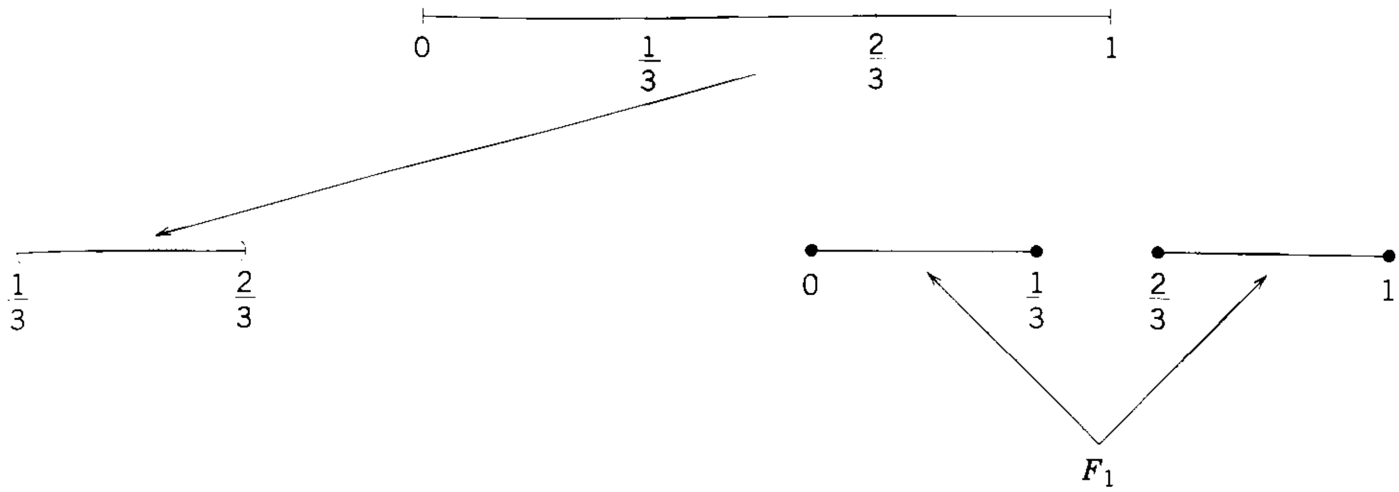
—Sir Arthur Conan Doyle

## A.1 CANTOR'S SET

The German mathematician Georg Cantor (1845–1918) is regarded as the creator of set theory. Earlier (Chapter 2) we used the "Cantor diagonalization argument" to show the set of real numbers is uncountable. In this appendix, the beautiful Cantor set, a source of so many examples and counterexamples in mathematics, is developed.

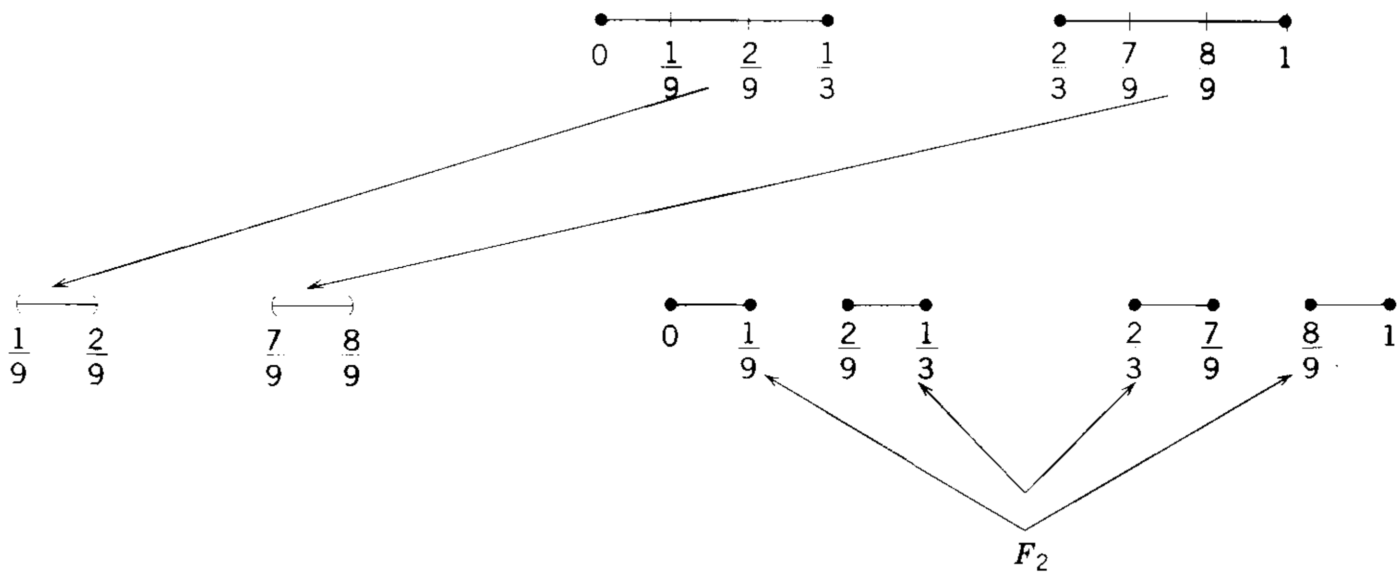
We begin with a constructive (geometrical) process applied to the interval $[0, 1]$:

**Step 1.** Divide $[0, 1]$ into three equal parts and remove the "middle third", the open interval $(1/3, 2/3)$:




The closed set $F_1 = [0, 1/3] \cup [2/3, 1]$ is left.

**Step 2.** Divide the two closed subintervals of $F_1$ into three equal parts and remove the "middle thirds," the open intervals $(1/9, 2/9)$, and $(7/9, 8/9)$:




The closed set $F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ is left.

Continuing in this way, after $n - 1$ steps we have deleted $1 + 2^1 + 2^2 + \cdots + 2^{n-2} = 2^{n-1} - 1$ disjoint open intervals, and have left the closed set $F_{n-1}$ consisting of $2^{n-1}$ closed intervals, each of length $1/3^{(n-1)}$.

**Step n.** Divide the remaining $2^{n-1}$ closed intervals into three equal parts and remove the "middle thirds." The closed set that is left, $F_n$, consists of $2^n$ closed subintervals, each of length $1/3^n$.

$\vdots$

**DEFINITION A**

The *Cantor set*, $C$, is what's left in $[0, 1]$ after deleting "middle thirds," namely,

$$C = \cap F_n .$$

The constructive approach is complete.

We now give an analytic definition of $C$.

**DEFINITION B**

The *Cantor set*, $C$, is the points of $[0, 1]$ which have a base three (ternary) expansion without the digit 1. That is, for $x \in C$,

$$x \underset{3}{=} .a_1 a_2 \cdots a_n \cdots \qquad , \quad a_n = 0 \text{ or } 2 .$$

Why two definitions? The idea here is that depending on the property to be discussed, Definition A will be preferable to Definition B and conversely. It's like the statement; Prove $0 = \pi - \pi^3/3! + \pi^5/5! - \cdots$. Who believes it, but when we recognize this is $\sin(\pi)$, with the "unit circle" interpretation of sine and the "power series" interpretation of sine—no problem. Our task at hand: reconcile Definition A and Definition B, the constructive versus descriptive interpretations of the Cantor set.

Some discussion of ternary expansions is appropriate. We will show that every number in the interval $(0, 1)$ has at least one, and at most two, ternary expansions. Recall $x \underset{3}{=} .a_1 a_2 a_3 \cdots a_n \cdots$ means $a_n = 0, 1,$ or $2$ and $x = \sum_1^\infty a_n/3^n$. The integers $0, 1, 2$ are called the ternary digits and $x \underset{3}{=} .a_1 a_2 \cdots a_n \cdots$ a ternary expansion of $x$.

*Claim*: Every number in $(0, 1)$ has a ternary expansion.

**Step 1.** Since $0 < x < 1$, $x$ will be in one of the "thirds": $(0, 1/3)$ or $[1/3, 2/3)$ or $[2/3, 1)$. So $0 < 3x < 1$ or $1 \le 3x < 2$ or $2 \le 3x < 3$. Let $a_1 = [3x]$, where $[\ ]$ is the greatest integer function. Thus $a_1 = 0$ or $1$ or $2$

and $a_1 \le 3x < a_1 + 1$, that is,

$$\frac{a_1}{3^1} \le x < \frac{a_1}{3^1} + \frac{1}{3^1}\,.$$

Since $a_1 \le 3x < a_1 + 1$, $0 \le 3x - a_1 < 1$, and we have a number, $3x - a_1$, between 0 and 1 again. Hence this number falls in one of the "thirds": $[0, 1/3)$ or $[1/3, 2/3)$ or $[2/3, 1)$, and $0 \le 3(3x - a_1) < 3$.

**Step 2.** Let $a_2 = [3(3x - a_1)]$. Thus $a_2 = 0$ or 1 or 2 and $a_2 \le 3(3x - a_1) < a_2 + 1$, that is,

$$\frac{a_1}{3^1} + \frac{a_2}{3^2} \le x < \frac{a_1}{3^1} + \frac{a_2}{3^2} + \frac{1}{3^2}\,.$$

Since $a_2 \le 3(3x - a_1) < a_2 + 1, 0 \le 3(3x - a_1) - a_2 < 1$, and we have a number, $3(3x - a_1) - a_2$, between 0 and 1 again. Hence this number falls in one of the "thirds": $[0, 1/3)$ or $[1/3, 2/3)$ or $[2/3, 1)$, and $0 \le 3(3(3x - a_1) - a_2) < 3$.

$\vdots$

**Step n.** Let $a_n = [3^n x - 3^{n-1}a_1 - \cdots - 3a_{n-1}]$. Thus $a_n = 0$ or 1 or 2 and $a_n \le 3^n x - 3^{n-1}a_1 - \cdots - 3a_{n-1} < a_n + 1$, that is,

$$\frac{a_1}{3^1} + \frac{a_2}{3^2} + \cdots + \frac{a_n}{3^n} \le x < \frac{a_1}{3^1} + \frac{a_2}{3^2} + \cdots + \frac{a_n}{3^n} + \frac{1}{3^n}\,.$$

Continuing the process, we have shown that, given $0 < x < 1$,

$$\frac{a_1}{3^1} + \cdots + \frac{a_n}{3^n} \le x < \frac{a_1}{3^1} + \cdots + \frac{a_n}{3^n} + \frac{1}{3^n}, \qquad n = 1, 2, \cdots\,.$$

Every number in $(0, 1)$ has a ternary expansion.

*Claim*: Every number in $(0, 1)$ has at most two ternary expansions. Suppose $x \underset{3}{=} .a_1a_2 \cdots a_n \cdots \underset{3}{=} .b_1b_2 \cdots b_n \cdots$ are two different ternary expansions of $x$ and $N$ is the first value of $n$ where $b_n \ne a_n$, that is, $x \underset{3}{=} .a_1a_2 \cdots a_{N-1}a_Na_{N+1} \cdots \underset{3}{=} .a_1a_2 \cdots a_{N-1}b_Nb_{N+1} \cdots$ with $a_N < b_N$. Because $a_N, b_N = 0, 1$, or 2, and $a_N < b_N$, we have these possibilities:

$a_N = 0$, $b_N = 1$ or 2, and $a_N = 1$, in which case $b_N = 2$. But

$$\begin{aligned} x &\leq \sum_{1}^{N-1} \frac{a_k}{3^k} + \frac{a_N}{3^N} + \sum_{N+1}^{\infty} \frac{2}{3^k} \\ &= \sum_{1}^{N-1} \frac{b_k}{3^k} + \frac{a_N + 1}{3^N} \\ &\leq \sum_{1}^{N-1} \frac{b_k}{3^k} + \frac{b_N}{3^N} \\ &\leq \sum_{1}^{\infty} \frac{b_k}{3^k} \\ &= x . \end{aligned}$$

Thus $b_N = a_N + 1$, $a_{N+1} = a_{N+2} = \cdots = 2$ and $b_{N+1} = b_{N+2} = \cdots = 0$. We have $x \underset{3}{=} .a_1 a_2 \cdots a_{N-1} a_N 222 \cdots$ or $x \underset{3}{=} .a_1 a_2 \cdots a_{N-1}(a_N + 1)000 \cdots$.

Either the ternary expansion is unique or we have exactly two such representations, one ending in "2's" and the other in "0's", starting from the same location.

We return our attention to the Cantor set. We show that Definition A implies Definition B; removal of "middle thirds" implies every point of the Cantor set has a ternary expansion without the digit 1.

In Step 1 of the constructive process we removed $(1/3, 2/3)$. What can we say about the ternary expansions $.a_1 a_2 \cdots a_n \cdots$ of such numbers? We claim $a_1 = 1$ and among $a_2, a_3, \cdots$, some $a_i$ is not zero and some $a_i$ is not two. If $a_1 = 0$, then $x = 0/3 + a_2/3^2 + \cdots \leq 2/3^2 + 2/3^3 + \cdots = 1/3$, but $1/3 < x < 2/3$.

If $a_1 = 2$, then $x = 2/3 + a_2/3^2 + \cdots \geq 2/3$. We have $a_1 = 1$. If $a_2 = a_3 = \cdots = 0$, $x = 1/3$ and if $a_2 = a_3 = \cdots = 2$, then $x = 2/3$. Thus the ternary expansion of every number in $(1/3, 2/3)$ starts with 1, and contains a nonzero entry and a "non" two entry. That is, every number in $(1/3, 2/3)$ must have a one in its ternary expansion.

$$\frac{1}{3} \underset{3}{=} .\underline{1}00 \cdots \underset{3}{=} .\underline{0}222 \cdots ;$$

*Note:*

$$\frac{2}{3} \underset{3}{=} .\underline{2}00 \cdots \underset{3}{=} .\underline{1}222 \cdots .$$

The open interval $(a_1, b_1)$ removed at the first step can be written with $a_1 = .1, b_1 = .2$.

In Step 2 of the constructive process we removed $(1/9, 2/9)$ and $(7/9, 8/9)$. What can we say about the ternary expansions of such numbers? We claim $a_2 = 1$ and among $a_3, a_4, \cdots$ , some $a_i$ is not zero and some $a_i$ is not two: If $a_2 = 0$, then $x = a_1/3^1 + 0/3^2 + \cdots$ . Several cases must be considered: Suppose $a_1 = 0$. Then $x = 0/3^1 + 0/3^2 + \cdots \leq 2/3^3 + 2/3^4 + \cdots = 1/9$. Can't happen. Suppose $a_1 = 1$. Then $1/3 \leq x = 1/3 + 0/3^2 + \cdots \leq 1/3 + 0/3^2 + \; 2/3^3 + \cdots = 1/3 + 1/9$. Can't happen. Suppose $a_1 = 2$. Then $2/3 \leq x = 2/3 + 0/3^2 + \cdots \leq 7/9$. Can't happen. The other possibility is that $a_2 = 2$, that is, $x = a_1/3^1 + 2/3^2 + \cdots$ . Again, suppose $a_1 = 0$. Then $2/9 \leq x = 0/3^1 + 2/3^2 + \cdots \leq 1/3$. Can't happen. Suppose $a_1 = 1$. Then $5/9 \leq x = 1/3^1 + 2/3^2 + \cdots = 6/9$. Can't happen. Finally, what if $a_1 = 2$. Then $8/9 \leq x = 2/3^1 + 2/3^2 + \cdots = 1$. We have $a_2 = 1$, that is, $x = .a_1 1 a_3 a_4 \cdots$ . If $a_3 = a_4 = \cdots = 0$, then $x = 1/9$ or $7/9$ ($4/9$ was removed at step 1). If $a_3 = a_4 = \cdots = 2$, then $x = 2/9$ or $8/9$ ($5/9$ was removed at step 1). Thus every number in $(1/9, 2/9)$ and $(7/9, 8/9)$ has a ternary expansion that must have a 1 in the second entry, and contains at least one nonzero entry and at least one "non" two entry.

*Note*:

$$\frac{1}{9} \underset{3}{=} .0\underline{1}000 \cdots \underset{3}{=} .0\underline{0}222 \cdots ;$$

$$\frac{2}{9} \underset{3}{=} .0\underline{2}000 \cdots \underset{3}{=} .0\underline{1}222 \cdots ;$$

$$\frac{7}{9} \underset{3}{=} .2\underline{1}00 \cdots \underset{3}{=} .2\underline{0}222 \cdots ;$$

$$\frac{8}{9} \underset{3}{=} .2\underline{2}00 \cdots \underset{3}{=} .2\underline{1}22 \cdots .$$

The open intervals $(a_2, b_2)$ removed at the second stage can be written with $a_2 \underset{3}{=} .(2\alpha_1)1, b_2 \underset{3}{=} .(2\alpha_1)2$, $\alpha_1 = 0$ or $1$.

In general, at step n of the construction, the open interval $(a_n, b_n)$ is removed if $a_n \underset{3}{=} .(2\alpha_1)(2\alpha_2) \cdots (2\alpha_{n-1})1$ and $b_n \underset{3}{=} .(2\alpha_1)(2\alpha_2) \cdots (2\alpha_{n-1})2$. To see this, the idea is that the left-hand endpoint of an interval removed at the n$^{th}$ step is "constructed" from a right-hand endpoint $b_{n-1}$

of an interval removed at the preceding step ($a_n = b_{n-1} + 1/3^n$) or "constructed" from 0 ($a_n = 0 + 1/3^n$). For example, at the first step we removed $(1/3, 2/3) = (0 + 1/3, 0 + 2/3) \underset{3}{=} (.1, .2)$: "Constructed" from 0. At the second step, we removed $(1/9, 2/9) = (0 + 1/9, 0 + 2/9) \underset{3}{=} (.01, .02)$, "Constructed" from 0, and $(7/9, 8/9) \underset{3}{=} (.2 + 1/3^2, .2 + 2/3^2) \underset{3}{=} (.21, .22)$, "Constructed" from right-hand endpoint, .2, of $(.1, .2)$. Induction provides "rigor". In general,

$$a_n = b_{n-1} + \frac{1}{3^n} \qquad \text{or} \qquad a_n = \frac{1}{3^n}, \qquad \text{that is,}$$

$$a_n \underset{3}{=} .(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})1, \qquad \alpha_k = 0 \quad \text{or} \quad 1 \quad \text{and}$$

$$b_n = a_n + \frac{1}{3^n} = .(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})2\,.$$

We've shown that if $x$ is removed in the geometric process, then $x$ must have 1 in its ternary expansion and a nonzero and "non" two appearing thereafter. That is, Definition A implies Definition B.

The converse remains: If $x \underset{3}{=} .a_1 a_2 \cdots a_n \cdots$ has a one in its ternary expansion with a 0 and a 2 appearing thereafter, then it was removed by the geometrical process. Suppose $x \underset{3}{=} .1 \cdots 0 \cdots 2 \cdots$. Then $1/3 < x < 2/3$ and $x$ was removed at step 1. Suppose $x \underset{3}{=} .a_1 1 \cdots 2 \cdots 0 \cdots$. Then $a_1 = 0$ or $2$ ($a_1 = 1$ was removed at Step 1), and $1/9 < x < 2/9$ or $7/9 < x < 8/9$. Thus $x$ was removed at Step 2. In general, $x \underset{3}{=} .(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})1 \cdots 2 \cdots 0$, $a_k = 0$ or $1$, that is, $.(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})1 < x < .(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})2$, and $a_n < x < b_n$. The number $x$ was removed at the n$^{th}$ step. (Induction provides "rigor".)

We have completed the arguments for showing the equivalence of the constructive and descriptive definitions of the Cantor set. We now investigate some remarkable properties of the Cantor set.

1. *The Cantor set is uncountable.*

This statement may seem false. After all, the only obvious numbers are $0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \cdots$, the countable set of endpoints of the open intervals that we removed. So, what now? We try some other points: $1/4 = 1/3 - 1/9 + 1/27 - \cdots \underset{3}{=} .02020\cdots$ ,

$3/4 \underset{3}{=} .2020\cdots$ . At least we are convinced something other than "endpoints" are in $C$. It is appropriate that the "Cantor diagonalization process", along with Definition B, shows uncountability: Assume $C$ is countable and make a "list":

$$x_1 \underset{3}{=} .a_{11}a_{12}a_{13}\cdots$$

$$x_2 \underset{3}{=} .a_{21}a_{22}a_{23}\cdots$$

$$x_3 \underset{3}{=} .a_{31}a_{32}a_{33}\cdots$$

$$\vdots$$

with $a_{nm} = 0$ or 2.

Let $x \underset{3}{=} .a_1a_2a_3\cdots$ where $a_n = \begin{cases} 0 & \text{if } a_{nn} = 2 \\ 2 & \text{if } a_{nn} = 0 \end{cases}$.

The number $x$ does not appear on our list.

2. The Cantor set is measurable with measure zero. (After all, what is the probability that a ternary expansion does not have a "1"?)

Measurability is obvious. As for measure zero, given $\epsilon > 0$, choose $n$ so large, say, $N$, such that $(2/3)^N < \epsilon$. Recall $F_N$ consists of $2^N$ closed intervals each of length $1/3^N$, that is, $\mu(F_N) < \epsilon$. Since $C = \cap F_n \subset F_N$, $\mu(C) < \epsilon$. From the arbitrary nature of $\epsilon$, $\mu(C) = 0$. Definition A works great!

The Cantor set is an uncountable set of measure zero.

3. The Cantor set contains no open intervals: The Cantor set is nowhere dense.

   Since the length of the intervals removed is one $(1/3 + 2(1/9) + 4(1/27) + \cdots = 1)$, it is not surprising that $C$ does not contain any intervals: In the construction of the Cantor set, intervals of the form $((3k+1)/3^n, (3k+2)/3^n)$, $0 \le k \le 3^{n-1} - 1$, $n = 1, 2, \ldots$, are removed, and thus given any interval $(a, b) \subset (0, 1)$, we have $((3k+1)/3^n, (3k+2)/3^n) \subset (a, b)$ for $n$ sufficiently large.
4. Every point of the Cantor set is a limit of points of the Cantor set (perfect).

Let $x_0 \in C$ and $(a, b)$ any open interval containing $x_0$. Since $x_0 \in C = \cap F_n$, $x_0 \subset F_n$ for all $n$. Let $I_n$ be a closed interval of $F_n$ that contains $x_0$. Because the length of any $I_n$ is $1/3^n$, we may choose $n$ so large, say $N$, such that $I_N \subset (a, b)$. At least one of the endpoints of $I_N$ is different from $x_0$. That is, given any open interval about $x_0$, it contains a point of the Cantor set different from $x_0$, and a point not in the Cantor set by part 3. Thus $x_0$ is a limit point of the points of $C$.

A point in $C$ is a limit point of points in $C$ and points not in $C$.

5. It is possible to construct Cantor type sets (nowhere dense, perfect, etc.) of positive measure $\alpha$, $0 < \alpha < 1$. These are the so-called, generalized Cantor sets. A very nice discussion may be found in Stromberg (1981) [St].

We will now discuss a function associated with the Cantor set, the so-called Cantor function, that has very surprising properties. Specifically, we construct a function $\Phi$ such that:

i. $\Phi$ is continuous and nondecreasing on $[0, 1]$;
ii. $\Phi$ maps the Cantor set (a set of measure zero) onto $[0, 1]$ continuously;
iii. $\Phi' = 0$ for almost all $x$ in $[0, 1]$: We start at elevation zero, end at elevation one, while moving horizontally most of the time!

Curiouser and curiouser!

—Lewis Carroll

We need some results regarding monotone functions. Recall a function $f$ is said to be nondecreasing on $(a, b)$ if $a < x < y < b$ implies $f(x) \leq f(y)$.

**Proposition A.1** *Let $f : (a, b) \longrightarrow R$ be nondecreasing and suppose $a < c < b$. Then*

1. *$lim_{x \to c^-} f(x) = f(c^-)$ and $lim_{x \to c^+} f(x) = f(c^+)$ both exist, in fact, $f(c^-) = sup\{f(x) \mid a < x < c\}$ and $f(c^+) = inf\{f(x) \mid c < x < b\}$;*
2. *$-\infty < f(c^-) \leq f(c) \leq f(c^+) < \infty$;*
3. *$a < c < d < b$ implies $f(c^+) \leq f(d^-)$.*

*Proof*: $a < x < c$ implies $f(x) \leq f(c)$, that is, $\sup\{f(x) \mid a < x < c\} \leq f(c)$. Similiarly $f(c) \leq \inf\{f(x) \mid c < x < b\}$. We must show $\lim_{x \to c^-} f(x) = f(c^-) = \sup\{f(x) \mid a < x < c\}$. Let $\epsilon > 0$ be given. From the definition of "sup" we have $a < x_1 < c$ with

$$\sup\{f(x) \mid a < x < c\} - \epsilon < f(x_1) \leq \sup\{f(x) \mid a < x < c\}.$$

Since $f$ is nondecreasing, $a < x_1 < x < c$ implies

$$\sup\{f(x) \mid a < x < c\} - \epsilon < f(x) \leq \sup\{f(x) \mid a < x < c\}.$$

That is, $\lim_{x \to c^-} f(x) = \sup\{f(x) \mid a < x < c\}$.

The reader may argue "inf" and "right-hand limit". If $c < d$, choose $x$ so that $c < x < d$. Then $f(c^+) \leq f(x) \leq f(d^-)$. ■

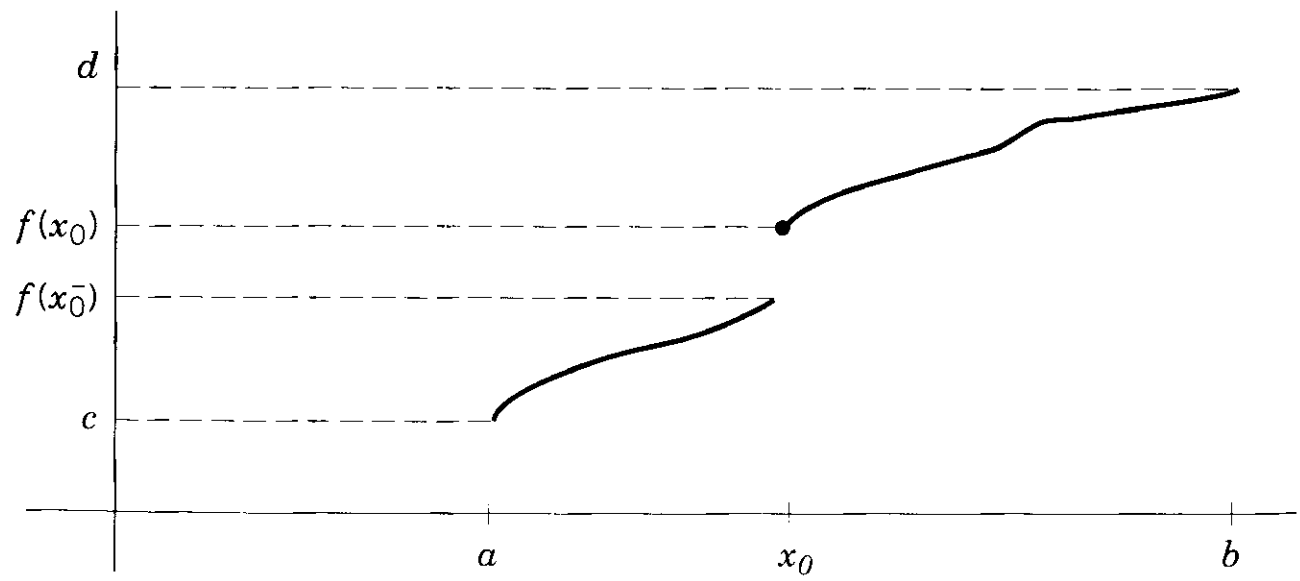
**PROPOSITION A.2** *Let* $f$ : $[a, b]$ **onto** $[c, d]$ *be nondecreasing. Then* $f$ *is continuous on* $[a, b]$.

Let $\epsilon > 0$ be given and suppose $a < x_0 < b$. We will show $f$ is continuous at $x_0$. Observe $f(a) = c$, $f(b) = d$.

By the previous proposition, $f(x_0^-), f(x_0^+)$ both exist and $f(x_0^-) \leq f(x_0) \leq f(x_0^+)$.

We want to show $f(x_0^-) = \lim_{x \to x_0^-} f(x) = f(x_0)$.

Suppose $f(x_0^-) < f(x_0)$:




Then $c < \dfrac{f(x_0^-) + f(x_0)}{2} < d$. What maps to $\dfrac{f(x_0^-) + f(x_0)}{2}$?

Something must, because $f$ is onto! A similar argument shows $f(x_0) = f(x_0^+)$, $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to b^-} f(x) = f(b)$. ∎

We are now in a position to start construction of the Cantor function $\Phi$.

1. First define a function $\phi$ that maps the Cantor set onto $[0, 1]$.

   Let $x$ be any point in the Cantor set: $x \underset{3}{=} .a_1 a_2 \cdots a_n \cdots = .a_1 a_2 \cdots a_n \cdots_{(3)}$, with $a_k = 0$ or 2. We map such a point to the point $.(a_1/2)(a_2/2) \cdots (a_n/2) \cdots_{(2)}$ (binary expansion), that is,

$$\phi\left(\sum_1 \frac{a_n}{3^n}\right) = \sum_1 \left(\frac{a_n}{2}\right)\left(\frac{1}{2^n}\right),$$

   where $a_n$ is either 0 or 2. For example,

$$\phi\left(\frac{1}{3}\right) = \phi(.0222 \cdots_{(3)}) = .0111 \cdots_{(2)} = \frac{1}{2},$$

$$\phi\left(\frac{2}{3}\right) = \phi(.2000 \cdots_{(3)}) = .1000 \cdots_{(2)} = \frac{1}{2},$$

$$\phi\left(\frac{1}{9}\right) = \phi\left(\frac{2}{9}\right) = \frac{1}{4}, \quad \phi\left(\frac{7}{9}\right) = \phi\left(\frac{8}{9}\right) = \frac{3}{4},$$

$$\phi\left(\frac{1}{4}\right) = \frac{1}{3}, \quad \phi\left(\frac{3}{4}\right) = \frac{2}{3}, \quad \text{etc}.$$

   Obviously $\phi$ maps the Cantor set into $[0, 1]$. Is the mapping onto? Certainly any number in $[0, 1]$ has a binary expansion, say $.b_1 b_2 \cdots b_n \cdots_{(2)}$, $b_k = 0$ or 1. Just "double" each digit. Then

$$\phi(.(2b_1)(2b_2) \cdots (2b_n) \cdots_{(3)}) = .b_1 b_2 \cdots b_n \cdots_{(2)},$$

   and $.(2b_1)(2b_2) \cdots (2b_n) \cdots_{(3)}$ is a member of the Cantor set. So, $\phi$ maps the Cantor set onto $[0, 1]$.

2. We claim $\phi$ is a nondecreasing function on the Cantor set. Let $x, y \in C$, $x < y$;

$$x \underset{3}{=} .a_1 a_2 \cdots a_{N-1} a_N a_{N+1} \cdots, \qquad a_n = 0 \text{ or } 2,$$

$$y \underset{3}{=} .a_1 a_2 \cdots a_{N-1} b_N b_{N+1} \cdots, \qquad b_n = 0 \text{ or } 2,$$

with $a_n = b_n$ for $n < N$ and $a_N = 0$, $b_N = 2$, that is,

$$x \underset{3}{=} .a_1 a_2 \cdots a_{N-1} 0 a_{N+1} \cdots, \qquad a_n = 0 \text{ or } 2,$$

$$y \underset{3}{=} .a_1 a_2 \cdots a_{N-1} 2 b_{N+1} \cdots, \qquad b_n = 0 \text{ or } 2.$$

Then

$$\begin{aligned}
\phi(x) &= \sum_{1}^{N-1} \frac{a_n}{2^{n+1}} + \frac{0}{2^{N+1}} + \sum_{N+1}^{\infty} \frac{a_n}{2^{n+1}} \\
&\leq \sum_{1}^{N-1} \frac{a_n}{2^{n+1}} + \sum_{N+1}^{\infty} \frac{2}{2^{n+1}} \\
&= \sum_{1}^{N-1} \frac{a_n}{2^{n+1}} + \frac{2}{2^{N+1}} \\
&= .\left(\frac{a_1}{2}\right)\left(\frac{a_2}{2}\right) \cdots \left(\frac{a_{N-1}}{2}\right)\left(\frac{2}{2}\right)\left(\frac{0}{2}\right) \cdots \quad (2) \\
&\leq .\left(\frac{a_1}{2}\right)\left(\frac{a_2}{2}\right) \cdots \left(\frac{a_{N-1}}{2}\right)\left(\frac{2}{2}\right)\left(\frac{b_{N+1}}{2}\right) \cdots \quad (2) \\
&= \phi(y) .
\end{aligned}$$

The function $\phi$ is nondecreasing on the Cantor set.

3. We extend $\phi$ to $[0,1]$, and this extension, $\Phi$, will be our desired function.

Here is the idea: If a point is not in the Cantor set, it is in an open interval that was removed during the construction, for example, take a point in $(1/3, 2/3)$, say, $1/2$. We already know $\phi(1/3) = \phi(2/3) = 1/2$, that is, $\Phi(1/3) = \Phi(2/3) = 1/2$, and $\Phi$ is to be nondecreasing. Thus $\Phi(1/2)$ should be $1/2$. How about .4? Oh, $\Phi(.4) = 1/2$ would be reasonable. Apparently, we want

$$\left[\frac{1}{3}, \frac{2}{3}\right] \xrightarrow{\Phi} \left\{\frac{1}{2}\right\}, \quad \left[\frac{1}{9}, \frac{2}{9}\right] \xrightarrow{\Phi} \left\{\frac{1}{4}\right\}, \quad \left[\frac{7}{9}, \frac{8}{9}\right] \xrightarrow{\Phi} \left\{\frac{3}{4}\right\}, \text{ etc.}$$

A nice way to encompass these ideas and see that we have an extension of $\phi$ is to define $\Phi$ as:

$$\Phi(x) \equiv \sup\{\phi(y) \mid y \leq x, y \in C\}.$$

If $x \in C$, then $\Phi(x) = \phi(x)$.

If $x \notin C$, then $x$ is in an interval removed, say $x \in (a_n, b_n)$, where $a_n \in C$. Then $\Phi(x) = \phi(a_n) = \phi(b_n)$. For example, $4/27 = .010222\cdots_{(3)}$, and $.002222\cdots_{(3)} < .010222\cdots_{(3)} < .02000\cdots_{(3)}$ and thus $\Phi(4/27) = \phi(.002222\cdots_{(3)}) = 1/4$.

$\Phi$ maps $[0, 1]$ onto $[0, 1]$.

4. $\Phi$ is nondecreasing and continuous on $[0, 1]$.

That $\Phi$ is nondecreasing is an immediate consequence of the property that if $A$ is a nonempty subset of $B$, then $\sup A \leq \sup B$. As for continuity, $\Phi$ is a nondecreasing map of $[0, 1]$ onto $[0, 1]$, and thus continuous by the previous propositions.

$\Phi$ maps a set of measure zero, the Cantor set, continuously onto a set of measure 1, $[0, 1]$.

5. We claim $\Phi$ is differentiable, with derivative zero, almost everywhere on $[0, 1]$.

Recall that in the construction of the Cantor set, an open interval $(a_n, b_n)$ was removed iff

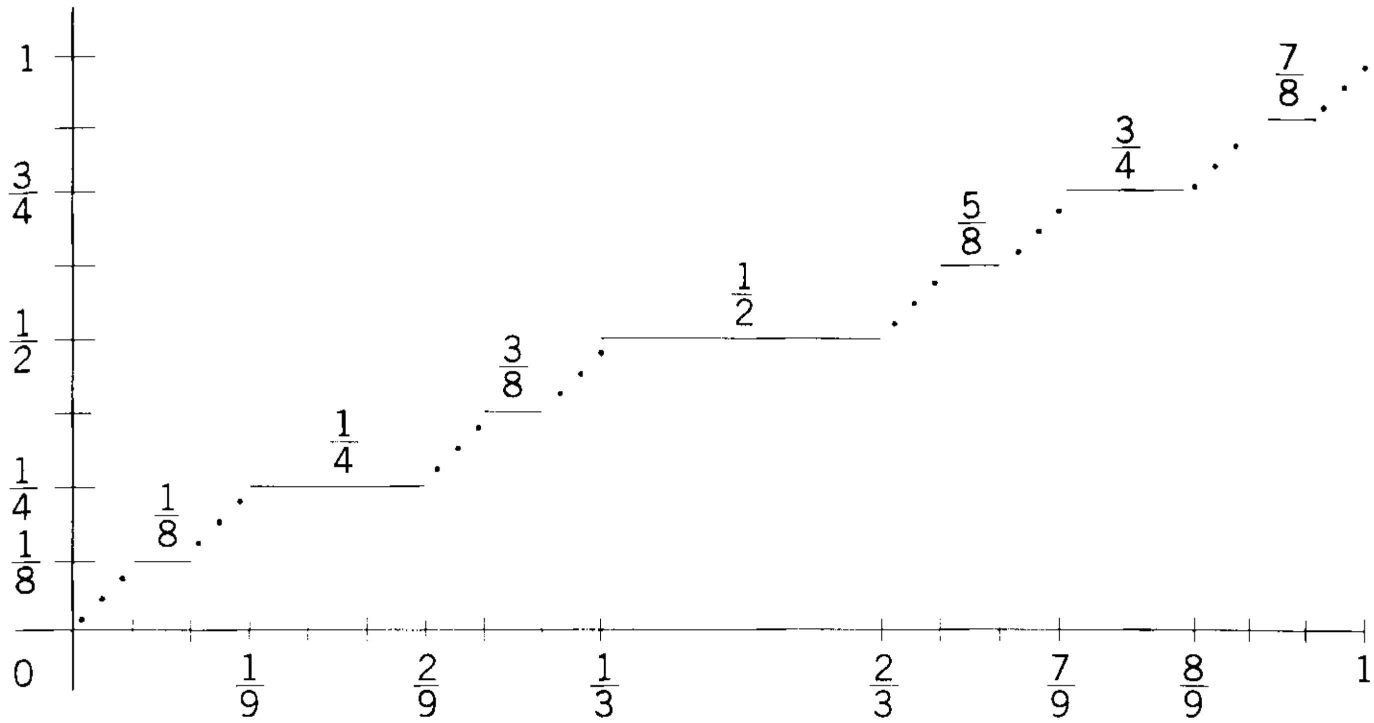
$$a_n \underset{3}{=} .(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})1 \quad \text{and} \quad b_n \underset{3}{=} .(2\alpha_1)(2\alpha_2)\cdots(2\alpha_{n-1})2$$

with $\alpha_k = 0$ or $1$. But,

$$\begin{aligned}\Phi(a_n) &= \Phi\big(.(2\alpha_1)\cdots(2\alpha_{n-1})022\cdots\ _{(3)}\big)\\ &= .\alpha_1\cdots\alpha_{n-1}100\cdots\ _{(2)}\\ &= \Phi(.(2\alpha_1)\cdots(2\alpha_{n-1})2).\end{aligned}$$

Thus $\Phi$ is constant on the intervals removed; $\Phi'$ is zero on the intervals removed. The function $\Phi'$ is zero almost everywhere.

A graph of $\Phi$ is in order:




By the way, what is $\int_{[0,1]} \Phi$ ? $\int_{[0,1]} \Phi'$ ?

An interesting article on this material [F1] appears in the April 1994 issue of Mathematics Magazine.

*Appendix B*

# *A Lebesgue Nonmeasurable Set*

> We are nature's unique experiment to make the rational intelligence prove itself sounder than the reflex. Knowledge is our destiny.
>
> —J. Bronowski

## B.1 A LEBESGUE NONMEASURABLE SET

The first example of a Lebesgue nonmeasurable set of real numbers was discovered by Vitali in 1905. In the next few years several mathematicians (Van Vleck (1908) and F. Bernstein (1908) among others) discovered such sets. All of their constructions used the Axiom of Choice: for any nonempty collection $C$ of sets there is a choice function $f$ such that $f(A) \in A$ for each $A \in C$. The natural question then became: Is it possible to construct a Lebesgue nonmeasurable set of real numbers without the Axiom of Choice? In 1970 Solovay showed that the Axiom of Choice was required; uncountably many choices are needed:

It is not possible to construct a Lebesgue nonmeasurable set of real numbers without using the Axiom of Choice!

We now construct a Lebesgue nonmeasurable set of real numbers. This construction involves the notions of "equivalent relations" and "equivalence classes". Examples serve to illustrate the concepts involved.

**Example 1.** Let $I$ be the set of integers: $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Two integers are said to be equivalent if they differ by a multiple of three: We write $n_1 \sim n_2$ provided $n_1 - n_2 = 3 \cdot k, k = 0, \pm 1, \pm 2, \ldots$. What do these "classes" look like?

"Starting with 0": Add and subtract multiples of three:

$$I_0 = \{\ldots, -6, -3, \underline{0}, 3, 6, \ldots\}.$$

"Starting with 1": Add and subtract multiples of three:

$$I_1 = \{\ldots, -5, -2, \underline{1}, 4, 7, \ldots\}.$$

Continuing,

$$I_2 = \{\ldots, -4, -1, \underline{2}, 5, 8, \ldots\},$$

$$I_{-3} = \{\ldots, -6, -\underline{3}, 0, 3, 6, \ldots\},$$

$$I_5 = \{\ldots, -1, 2, \underline{5}, 8, \ldots\},$$

and so forth.

We notice that

$$I_0 = I_3 = I_{-30} = \ldots = I_{3k} = \ldots,$$

$$I_1 = I_4 = I_{-5} = \ldots = I_{3k+1} = \ldots,$$

$$I_2 = I_{-4} = I_{302} = \ldots = I_{3k+2} = \ldots,$$

and that $I = I_0 \cup I_1 \cup I_2$ with $I_i \cap I_j = \emptyset$ for $i \neq j$, $0 \leq i, j \leq 2$. We have decomposed the set of integers, with the relation, $n_1 \sim n_2$ provided $n_1 - n_2 = 3k$, into three disjoint sets ("equivalence classes"):

i. Every integer belongs to one of the sets $I_0$, $I_1$, or $I_2$;
ii. If $n_1 \sim n_2$, $I_{n_1} = I_{n_2}$;
iii. If $I_{n_1} \cap I_{n_2} \neq \emptyset$, $\quad I_{n_1} = I_{n_2}$.

**Example 2.** Let $Q$ denote the set of rational numbers: $Q = \{p/q \mid p, q \text{ integers}, \ q \neq 0\}$. Two rational numbers $p/q, r/s$ are said to be equivalent, $p/q \sim r/s$, if $ps = qr$. What do these "classes" look like?

"Starting with 0": $Q_{0/1} = \left\{0, \frac{0}{1}, \frac{0}{-1}, \frac{0}{2}, \frac{0}{-2}, \ldots\right\}.$

"Starting with 1": $Q_{1/1} = \left\{\frac{1}{1}, \frac{-1}{-1}, \frac{2}{2}, \frac{-2}{-2}, \ldots\right\}.$

Continuing,

$$Q_{1/2} = \left\{\frac{1}{2}, \frac{-1}{-2}, \frac{2}{4}, \frac{-2}{-4}, \frac{3}{6}, \frac{-3}{-6}, \ldots\right\},$$

$$Q_{-3/7} = \left\{\frac{-3}{7}, \frac{3}{-7}, \frac{-6}{14}, \frac{6}{-14}, \ldots\right\},$$

and so forth. We notice that

$$Q_{1/2} = Q_{-500/-1000} = Q_{3/6} = \cdots,$$

$$Q_{-3/4} = Q_{-300/400} = Q_{33/-44} = \cdots,$$

and that

$$Q = \cup Q_{p/q}.$$

We form the so-called equivalence class:

$$Q_{p/q} = \left\{\frac{r}{s} \in Q \mid ps = rq\right\}.$$

The collection $\{Q_{p/q} \mid p/q \in Q\}$ of all such equivalence classes is a countable collection of pairwise disjoint subsets of $Q$ whose union is $Q$. In fact, the reader may show:

i. Every rational number belongs to one of the sets $Q_{p/q}$;
ii. If $p/q \sim r/s$, $Q_{p/q} = Q_{r/s}$;
iii. If $Q_{p/q} \cap Q_{r/s} \neq \emptyset$, $\quad Q_{p/q} = Q_{r/s}$.

**Example 3.** Let $R$ be the set of real numbers. Two real numbers $x, y$ are said to be equivalent, $x \sim y$, if their difference is a rational number: $x \sim y$ if $x - y = r$, $r$ rational. What do these "classes" look like?

"Starting with 0":

$$R_0 = \{0 + r \mid \text{ rational}\} = Q,$$

$$R_{\sqrt{2}} = \{\sqrt{2} + r \mid r \text{ rational}\},$$

$$R_{3/4} = \left\{\frac{3}{4} + r \mid r \text{ rational}\right\} = R_0 = Q,$$

$$R_\pi = \{\pi + r \mid r \text{ rational}\},$$

and so forth.
We notice that

$$R_0 = R_{3/4} = \cdots = R_{p/q} = \cdots = Q,$$

$$R_\pi = R_{\pi+r}, \quad r \text{ rational},$$

$$R_{\sqrt{2}} = R_{\sqrt{2}+3/4} = R_{\sqrt{2}+r},$$

and so forth.

We have $R = \cup R_x$. Form the so-called equivalence classes:

$$R_x = \{y \in R \mid x - y \text{ is rational}\}.$$

Then the reader should show:

i. Every real number belongs to one of the sets $R_x$;
ii. If $x_1 \sim x_2$, then $R_{x_1} = R_{x_2}$;
ii. If $R_{x_1} \cap R_{x_2} \neq \emptyset$, then $R_{x_1} = R_{x_2}$.

In fact, the collection $\{R_x \mid x \in R\}$ of all such equivalence classes is an uncountable collection of pairwise disjoint subsets of $R$, in fact, each is dense in $R$, whose union is $R$.

We are ready to construct a Lebesgue nonmeasurable set of real numbers in the interval $(-1, 1)$.

Our first task is to decompose this interval in essentially the same manner as $R$ of Example 3:

For $x \in (-1, 1)$, define $A_x = \{y \in (-1, 1) \mid y - x = r,\ r \text{ rational}\}$. For example, $A_{1/2} = \{y \in (-1, 1) \mid y - 1/2 \text{ is rational}\} = \{\text{rationals in } (-1, 1)\}$, $A_{-1/\sqrt{2}} = \{y \in (-1, 1) \mid y + 1/\sqrt{2} \text{ is rational}\}$, and so forth.

Again, $A_{1/2} = A_{-3/4} = \cdots = \{\text{rationals in } (-1,1)\}$ and $(-1,1) = \cup A_x$, The reader may show:

i. Each set $A_x$ is countable;
ii. If $x$ is rational, $A_x$ is the set of rationals of $(-1,1)$;
iii. If $x$ is irrational, then each member of $A_x$ is an irrational number;
iv. If $x_1$ and $x_2$ are members of $(-1,1)$ with $x_1 - x_2$ a rational number, then $A_{x_1} = A_{x_2}$;
v. If $x_1$ and $x_2$ are members of $(-1,1)$ with $x_1 - x_2$ an irrational number, then $A_{x_1} \cap A_{x_2} = \emptyset$.
vi. The collection of distinct sets $A_x$ is uncountable.

In summary, we have decomposed the interval $(-1,1)$ into an uncountable collection of pairwise disjoint sets; each of these sets is itself countable, one such set consisting of the rationals in $(-1,1)$, and each of the others consisting only of irrational numbers.

From each of these disjoint subsets $A_x$, pick a point (We have used the Axiom of Choice, picking a point from each of an uncountable number of disjoints sets). Call this set $\mathcal{N}$. $\mathcal{N}$ is an uncountable set, a subset of $(-1,1)$, and $\mathcal{N} \cap A_x$ is a single point. That $\mathcal{N}$ is nonmeasurable will result from the countable additivity and translation invariance requirements for Lebesgue measure.

Enumerate the rationals in $(-2,2): r_1, r_2, r_3 \ldots$. Define $\mathcal{N} + r_n = \{x + r_n \mid x \in \mathcal{N},\ -2 < r_n < 2,\ r_n \text{ rational}\}$, and note that since $\mathcal{N} \subset (-1,1)$, $\mathcal{N} + r_n \subset (-3,3)$. We claim $(\mathcal{N} + r_n) \cap (\mathcal{N} + r_m) = \emptyset$ if $r_n \neq r_m$: Suppose the intersection is nonempty; $z \in (\mathcal{N} + r_n) \cap (\mathcal{N} + r_m)$. Then $z = x + r_n = y + r_m$, or $x - y$ is rational, with $x, y \in \mathcal{N}$. But this means that $x$ and $y$ are in the same equivalence class. However, $\mathcal{N}$ was constructed by taking points from mutually disjoint sets, so $x$ must equal $y$. But then $r_n = r_m$, and this is a contradiction to $r_n \neq r_m$. So, the collection $\{\mathcal{N} + r_n\}$ forms a countable, mutually disjoint cover of $(-1,1)$, each member of which is a subset of $(-3,3)$:

$$(-1,1) \subset \bigcup_{n=1}^{\infty} (\mathcal{N} + r_n) \subset (-3,3).$$

If $\mathcal{N}$ is measurable, $\mathcal{N} + r_n$ is measurable, and because Lebesgue measure is translation invariant $(\mu(\mathcal{N}) = \mu(\mathcal{N} + r_n))$, and countably additive

$$\left(\mu\left(\bigcup_1^{\infty} (\mathcal{N} + r_n)\right)\right) = \sum_1^{\infty} \mu(\mathcal{N} + r_n)),$$

we have

$$2 = \mu(\,(-1,1)\,) \le \mu\left(\bigcup_{n=1}^{\infty}(\mathcal{N}+r_n)\right) = \sum_{n=1}^{\infty}\mu(\mathcal{N}+r_n) = \sum_{n=1}^{\infty}\mu(\mathcal{N})$$
$$\le \mu(\,(-3,3)\,) = 6\,.$$

So,

$$2 \le \sum_{n=1}^{\infty}\mu(\mathcal{N}) \le 6\,.$$

The left-hand inequality, $2 \le \sum_{n=1}^{\infty}\mu(\mathcal{N})$, implies $\mu(\mathcal{N}) > 0$. The right-hand inequality, $\sum_{n=1}^{\infty}\mu(\mathcal{N}) \le 6$ implies $\mu(\mathcal{N}) = 0$. We can't have it both ways. The set $\mathcal{N}$ must be nonmeasurable.

**Summary of Construction:**

I. Partition $(-1,1)$ into an uncountable collection of mutually disjoint countable sets;

II. Use the Axiom of Choice to pick a point from each of these sets. This set $\mathcal{N}$ is the set we show to be nonmeasurable;

III. Form a countable collection of mutually disjoint translates of $\mathcal{N}$ : $\{\mathcal{N}+r_n \mid r_1, r_2, \ldots$ an enumeration of the rational numbers of $(-2,2)\}$;

IV. Observe $(-1,1) \subset \cup(\mathcal{N}+r_m) \subset (-3,3)$. Because Lebesgue measure is translation invariant and countably additive, measurability of $\mathcal{N}$ implies $\mu(\mathcal{N}) = 0$ and $\mu(\mathcal{N}) > 0$.

What if $A$ is any set of positive Lebesgue measure: $\mu(A) > 0$? (Every subset of a set of Lebesgue measure zero is Lebesgue measurable and has Lebesgue measure zero.) Since

$$A \subset \bigcup_{n=1}^{\infty}(-n,n),\ \ \mu(A \cap (-N,N)\,) > 0$$

for some natural number $N$, and thus $A \cap (-N,N)$ must be uncountable since any countable set is Lebesgue measurable with measure zero. Select any countable subset of $A \cap (-N,N)$. This countable set will play the role of the "rationals" in the proceeding construction. Mimic the previous arguments to establish that every set of real numbers with positive Lebesgue measure contains a Lebesgue nonmeasurable subset.

Over the years attempts having been made to enlarge the class of Lebesgue measurable sets [SS]. The "idea" would be to find a measure $\lambda$ so that

1. $\lambda(E)$ defined for all $E \subset R$ (all subsets of $R$ are "measurable");
2. $\lambda([0,1]) = 1$;
3. $\lambda\left(\bigcup_{i=1}^{\infty} E_n\right) = \sum_{i=1}^{\infty} \lambda(E_n)$,
   $E_n$ mutually disjoint (countable additivity);
4. $\lambda(E + a) = \lambda(E)$ if $E$ measurable (translation invariant).

As has been shown, Lebesgue measure satisfies parts 2,3, and 4, with requirements 3 and 4 forcing abandonment of 1. How about replacing 3 with

$$3' : \lambda\left(\bigcup_{i=1}^{K} E_n\right) = \sum_{i=1}^{K} \lambda(E_n),$$

finite additivity, in an attempt to "measure" all subsets of $R$? We have the following results [Ba]. There exists a set function $\lambda$ defined for all subsets of $R$ such that

1′. $0 \leq \lambda(E) \leq \infty$ for all $E \subset R$
2′. $\lambda([0,1]) = 1$
3′. $\lambda$ is finitely additive
4′. $\lambda$ is translation invariant.

Then Banach, Hausdorff showed the above result holds in $R^n$ iff $n = 1, 2$.

In 1933 Kolmogorov [Ko] developed probability theory from an axiomatic standpoint with countable additivity being of fundamental importance: We want countable additivity, even at the expense of not being able to "measure" all sets of real numbers.

# Appendix C

# Lebesgue, Not Borel

The unexamined life is not worth living.

—Socrates

## C.1 LEBESGUE, NOT BOREL

We show that there exist Lebesgue measurable sets that are not Borel sets: the class of Borel sets, $\mathcal{B}$, is properly contained in the class of Lebesgue measurable sets, $\mathcal{M}$.

We will use the

**PROPOSITION C.1** *If $f$ is a continuous, strictly increasing mapping of $R$ onto $R$, then $f$ maps Borel sets to Borel sets.*

*Proof:* The argument is indirect. Let $\mathcal{C}$ be the collection of all subsets of $R$ whose image under $f$ is a Borel set. That is, if $A \in \mathcal{C}$, then $f(A)$ is a Borel set. We will show that this collection $\mathcal{C}$ is a sigma algebra that contains the closed intervals, i.e., $\mathcal{B}$. We claim $\mathcal{C}$ is a sigma algebra of subsets of $R$.

Suppose $A \in \mathcal{C}$. We claim $A^C \in \mathcal{C}$.

Since $f$ is strictly increasing, $f$ is $1-1$, and consequently $f(A^C) \cap f(A) = \emptyset$. Thus $R = f(R) = f(A \cup A^C) = f(A) \cup f(A^C)$, that is, $R - f(A) = f(A^C)$. Because $A \in \mathcal{C}$, $f(A)$ is a Borel set. Thus $R - f(A) = f(A^C)$ is a Borel set, that is, $A^C \in \mathcal{C}$.

Next, suppose $(A_k)$ is a sequence of sets in $\mathcal{C}$. Since $f(\cup A_k) = \cup f(A_k)$, and each $f(A_k)$ is a Borel set (since by assumption, $A_k \in \mathcal{C}$), the countable union of Borel sets is a Borel set, that is, $f(\cup A_k)$ is a Borel set. In other words, $\cup A_k \in \mathcal{C}$. $\mathcal{C}$ is a sigma algebra. We now show closed intervals are in $\mathcal{C}$. Suppose $I$ is a closed interval. Then $f(I)$ is an closed interval: $f([a, b]) = [f(a), f(b)]$ (Intermediate Value Theorem, increasing).

Thus $\mathcal{C}$ is a sigma algebra containing all closed intervals. Because $\mathcal{B}$ is the smallest such sigma algebra, $\mathcal{B} \subset \mathcal{C}$. Borel sets are mapped to Borel sets.

We are ready to prove the

**THEOREM C.1** *There is a Lebesgue measurable set of real numbers that is not a Borel set.*

*Proof:* Let $\Phi$ be the Cantor function. Extend $\Phi$ to $R$, $\hat{\Phi}$, by defining $\hat{\Phi}(x) = 0$, $x < 0$ and $\hat{\Phi}(x) = 1$, $x > 1$. Form a new function $f$ by defining $f(x) = x + \hat{\Phi}(x)$, $x \in R$. Then because $\hat{\Phi}$ is continuous and nondecreasing on $[0, 1]$, $f$ is a continuous, strictly increasing mapping of $R$ onto $R$: the function $f$ maps Borel sets to Borel sets by the proposition we just proved.

Let $C$ denote the Cantor set. We show $f(C)$ is a Borel set and $\mu(f(C)) = 1$.

1. $f(C)$ is a Borel set.

   $C$ is a closed set, $C$ is a Borel set, and since $f$ maps Borel sets to Borel sets, $f(C)$ is a Borel set. In particular, $f(C)$ is a Lebesgue measurable set and $[0, 1] - f(C)$ is a Lebesgue measurable set. What is the Lebesgue measure of $f(C)$?
2. $\mu(f(C)) = 1$.

$$\begin{aligned}\mu(f([0,1] - C)) &= \mu(f(\cup I_k)) = \mu(\cup f(I_k)) \\ &= \sum \mu(f(I_k)) = \sum \mu((a_k + \Phi(a_k), b_k + \Phi(b_k)) \\ &= \sum \mu((a_k, b_k)) \text{ since } \Phi(a_k) = \Phi(b_k).\end{aligned}$$

Thus $\mu(f([0,1]-C)) = \sum \mu((a_k, b_k)) = 1$. Finally,

$$\begin{aligned} 2 = \mu([0,2]) = \mu(f([0,1])) &= \mu(f(C \cup ([0,1]-C))) \\ &= \mu(f(C) \cup f([0,1]-C)) \\ &= \mu(f(C)) + \mu(f([0,1]-C)) \\ &= \mu(f(C)) + 1, \end{aligned}$$

that is,

$$\mu(f(C)) = 1.$$

Because every set of positive Lebesgue measure contains a nonmeasurable set (Appendix B), $f(C)$ contains a nonmeasurable set $\mathcal{N}$. But since $f$ is $1-1$, $f^{-1}$ makes sense and thus $f^{-1}(\mathcal{N}) \subset C$. Since the Cantor set $C$ has Lebesgue measure zero, and every subset of a set of Lebesgue measure zero is Lebesgue measurable with measure zero, $f^{-1}(\mathcal{N})$ is a Lebesgue measurable set and $\mu(f^{-1}(\mathcal{N})) = 0$. However, $f^{-1}(\mathcal{N})$ is **not** a Borel set, since then $f(f^{-1}(\mathcal{N})) = \mathcal{N}$ must be a Borel set by the previous proposition: $f^{-1}(\mathcal{N})$ is our desired set: Lebesgue measurable, not Borel.

# *Appendix D*

# *A Space-Filling Curve*

*There are things which seem incredible to most men who have not studied mathematics.*

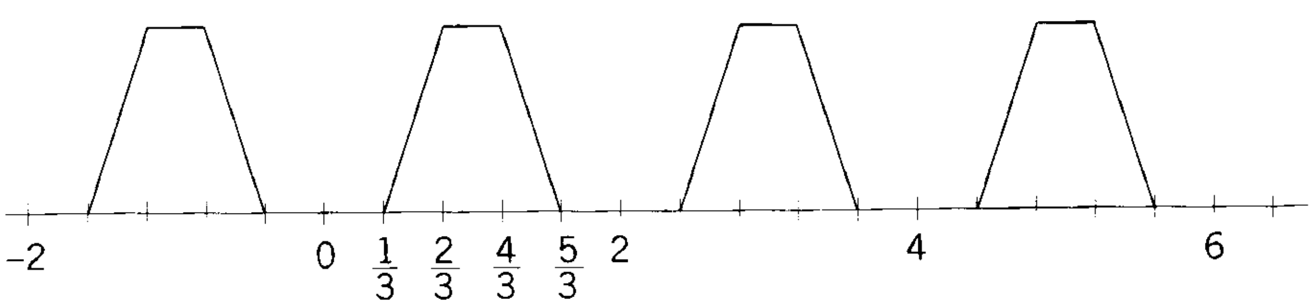
—Archimedes

And, I might add, even to those of us who have studied mathematics.

## D.1 A SPACE-FILLING CURVE

We will present an example of a continuous curve that passes through every point of the unit square. This example is due to I. J. Schoenberg [Scho], and appeared in the *Bulletin of the American Mathematical Society*, 1936.

Define a function $f$ on $[0,2]$ as zero on $[0,1/3]$, linear on $[1/3,2/3]$, one on $[2/3,4/3]$, linear on $[4/3,5/3]$, and zero on $[5/3,2]$. Extend periodically:



Define two functions $x$ and $y$ on [0,1] as follows:

$$x(t) = \frac{1}{2}f(t) + \frac{1}{2^2}f(3^2t) + \frac{1}{2^3}f(3^4t) + \cdots = \sum_0^\infty \frac{1}{2^{n+1}}f(3^{2n}t),$$

$$y(t) = \frac{1}{2}f(3t) + \frac{1}{2^2}f(3^3t) + \frac{1}{2^3}f(3^5t) + \cdots = \sum_0^\infty \frac{1}{2^{n+1}}f(3^{2n+1}t),$$

$0 \le t \le 1$.

The Weierstrass $M$-Test shows that $x$ and $y$ are continuous on [0,1], and obviously, $0 \le x(t), y(t) \le 1$. What needs to be shown is that given any point in the unit square, say $(x_0, y_0)$ where $0 \le x_0, y_0 \le 1$, we have $t_0$ in [0,1] so that $x(t_0) = x_0$ and $y(t_0) = y_0$. The "idea" is to write $x_0$ and $y_0$ as binary expansions:

$$x_0 \underset{2}{=} .a_0a_2a_4\cdots$$

$$y_0 \underset{2}{=} .a_1a_3a_5\cdots, \qquad a_n = 0 \text{ or } 1.$$

Let $t_0$ be this particular ternary expansion:

$$t_0 \underset{3}{=} .(2a_0)(2a_1)(2a_2)(2a_3)\cdots = \sum_0^\infty \frac{2a_n}{3^{n+1}}, \qquad a_n = 0 \text{ or } 1.$$

*Note*: $0 \le t_0 \underset{3}{\le} .222\cdots = 1$, so $t_0 \in [0,1]$.

Actually, $t_0$ is a member of the Cantor set. We will show that $x(t_0) = x_0$ and $y(t_0) = y_0$.

What makes everything work is the observation that $\sum_0^\infty (2a_n/3^{n+1}) = t_0 = \sum_0^\infty (2f(3^nt_0)/3^{n+1})$ if $t_0$ is a member of the Cantor set. This observation will follow from $f(3^nt_0) = a_n$:

Observe that

$$3^nt_0 = 3^n\left(\frac{2a_0}{3^1} + \frac{2a_1}{3^2} + \cdots + \frac{2a_{n-1}}{3^n}\right) + 3^n\left(\frac{2a_n}{3^{n+1}} + \frac{2a_{n+1}}{3^{n+2}} + \cdots\right)$$

$$= 2I_n + \frac{2a_n}{3^1} + \frac{2a_{n+1}}{3^2} + \cdots, \qquad I_n \text{ an integer}.$$

If $a_n = 0$, then $2I_n \le 3^nt_0 = 2I_n + (2a_{n+1}/3^2) + \cdots \le 2I_n + 1/3$, that is,

$3^n t_0$ is in an interval of the form $[2I_n, 2I_n + 1/3]$. But $f$ vanishes on such intervals. Thus $f(3^n t_0) = a_n$ if $a_n = 0$. If $a_n = 1$, then $2I_n + 2/3 \leq 3^n t_0 \leq 2I_n + 1$. But $f$ is one on such intervals. Thus $f(3^n t_0) = a_n$ if $a_n = 1$. But this is exactly what we are trying to show:

$$\sum_0^\infty \frac{2a_n}{3^{n+1}} = t_0 = \sum_0^\infty \frac{2f(3^n t_0)}{3^{n+1}}, \qquad f(3^n t_0) = a_n .$$

Thus,

$$x(t_0) = \sum_0^\infty \frac{1}{2^{n+1}} f(3^{2n} t_0) = \sum_0^\infty \frac{a_{2n}}{2^{n+1}} = x_0 ,$$

and

$$y(t_0) = \sum_0^\infty \frac{1}{2^{n+1}} f(3^{2n+1} t_0) = \sum_0^\infty \frac{a_{2n+1}}{2^{n+1}} = y_0 .$$

The mapping $t \longrightarrow (x(t), y(t))$ maps the Cantor set onto the unit square!

# Appendix E

# An Everywhere Continuous, Nowhere Differentiable Function

## E.1 AN EVERYWHERE CONTINUOUS, NOWHERE DIFFERENTIABLE, FUNCTION

Probably the first function that students encounter that is continuous and nondifferentiable at a point, is $f(x) = | x |$, the nondifferentiability occuring at $x = 0$. A "corner" is the intuitive explanation. Naturally, this leads to two problem points, say $r_1$ and $r_2$. No problem; just let $f(x) = | x - r_1 | + | x - r_2 |$. Let's see how the argument might proceed: Since $r_1 \neq r_2$, choose $h$ so that $0 < | h | < | r_1 - r_2 |$. Form difference quotients:

$$\frac{f(r_i + h) - f(r_i)}{h} = \frac{| r_1 - r_2 + h | - | r_1 - r_2 |}{h} + \frac{| h |}{h} = \pm 1 + \frac{| h |}{h} .$$

The first term has a limit depending only on whether $r_1 < r_2$ or $r_2 < r_1$, and the second term depends on whether $h$ is positive or negative. Thus we do not have a limit for the difference quotient since $h$ may be positive or negative.

How about a finite number, say $N$, of corners. Proceed as above, where

$0 < | h | < | r_i - r_j |$, $i \neq j$, $1 \leq i,j \leq N$. Pick $r_M$, and then

$$\frac{f(r_M + h) - f(r_M)}{h} = \sum_{\substack{n=1 \\ n \neq M}}^{N} \frac{| r_M + h - r_n | - | r_M - r_n |}{h} + \frac{| h |}{h}$$

$$= \sum_{\substack{n=1 \\ n \neq M}}^{N} \pm 1 + \frac{| h |}{h} .$$

The term $| h | /h$ causes the problem. This function is not differentiable at $r_1, r_2, \ldots, r_N$.

Can we construct a function that has a countable number of corners, say, at $r_1, r_2, \ldots$ ? Maybe even countably dense "corners" if we are talking about the rationals. There is a method for doing this, called "condensation of singularities," due to Hankel (1870), and later refined by Cantor (1882). It is Cantor's approach that we will use. We obviously try $f(x) = \sum_1^\infty | x - r_n |$, but what does this mean when $x \neq r_i$, $i = 1, 2, \ldots$? We need convergence, and because of the Weierstrass $M$-Test, let's modify $f$, say,

$$f(x) = \sum_1^\infty \frac{| x - r_n |}{3^n}, \qquad 0 \leq x \leq 1,$$

where $r_1, r_2, \ldots$ is any enumeration of the rationals in [0, 1]. Now, not only is $f$ meaningful on [0,1], $f$ is continuous. We claim $f$ is not differentiable at any rational, say $r_M$.

The argument follows the lines of what we did before for the finite case:

$$\frac{f(r_M + h) - f(r_M)}{h} = \sum_{n=1}^{M-1} \frac{| r_M - r_n + h | - | r_M - r_n |}{h \cdot 3^n} + \frac{| h |}{h \cdot 3^M}$$

$$+ \sum_{n=M+1}^{\infty} \frac{| r_M - r_n + h | - | r_M - r_n |}{h \cdot 3^n} ,$$

where $0 < | h | < \min | r_i - r_M |$, $i = 1, 2, \ldots, M - 1$. The first term has a definite limit:

$$\sum_{n=1}^{M-1} \frac{| r_M - r_n + h | - | r_M - r_n |}{h \cdot 3^n} = \sum_{n=1}^{M-1} \frac{\pm 1}{3^n} = L .$$

The second term, $|h|/h \cdot 3^M$, obviously depends on the sign of $h$. The last term is dominated by $\sum_{M+1} 1/3^n = 1/(2 \cdot 3^M)$. So, the difference quotient, $(f(r_M + h) - f(r_M))/h$, is greater than $L + (1 - 1/2) \cdot 1/3^M$ if $h$ is positive and less than $L - (1 - 1/2) \cdot 1/3^M$ if $h$ is negative. We can't have it both ways. The function $f$ is continuous on [0,1] and not differentiable on the rationals, a countable dense set. Is $f$ differentiable at the irrationals? If $x$ is irrational, $x - r_n \neq 0$ for all $n$, and the term that caused difficulty above ($|h|/h \cdot 3^M$) does not appear.

The construction of a function that is continuous and nowhere differentiable requires something other than "condensation of singularities." We give an example due to Weierstrass (1875), although apparently Bolzano (1834) and Cellérier (approximately 1830) had earlier examples. Weierstrass' Everywhere Continuous, Nowhere Differentiable, Function: $f(x) = \sum_0 b^n(\cos a^n\pi x)$, $0 < b < 1$, $a$ an odd natural number, with $ab > 1 + 3\pi/2$.

The requirement that $0 < b < 1$ ensures that $f$ is everywhere continuous by, what else, the Weierstrass $M$-Test. As for differentiability, fix $x$ and look at difference quotients: $(f(x + h) - f(x))/h$. A sequence $(h_K)$ of positive numbers converging to zero will be exhibited for which $|(f(x + h_K) - f(x))/h_K|$ becomes arbitrarily large.

Choose an integer $I_K$ so that $I_K \leq a^K x + 1/2 < I_K + 1$, $K = 1, 2, \ldots$, and define $h_K = (I_K + 1 - a^K x)/a^K$. Note that $1/2a^K < h_K \leq 3/2a^K$. The reason for choosing this $h_K$ will be apparent in a moment. Now,

$$\frac{f(x + h_K) - f(x)}{h_K} = \sum_{n=0}^{K-1} b^n \frac{\cos(a^n\pi(x + h_K)) - \cos(a^n\pi x)}{h_K} + \sum_{n=K}^{\infty} b^n \frac{\cos(a^n\pi(x + h_K)) - \cos(a^n\pi x)}{h_K}.$$

For the first sum, we use the mean-value theorem for derivatives to obtain

$$\sum_{n=0}^{K-1} b^n(-\sin(a^n\pi(x + \eta_K))a^n\pi),$$

and thus

$$\left|\sum_{n=0}^{K-1} b^n(-\sin(a^n\pi(x + \eta_K))a^n\pi)\right| < \frac{a^K b^K}{ab - 1} \cdot \pi.$$

For the other sum, we easily simplify due to the particular choice of $h_K$. Since $n \geq K$ in the second sum, $a^n\pi(x+h_K) = a^{n-K}\pi a^K(x+h_K) = a^{n-K}\pi(I_K+1)$ and $\cos(a^n\pi(x+h_K)) = (-1)^{I_K+1}$ for all $n \geq K$. Also, $a^n\pi x = a^{n-K}\pi a^K x = a^{n-K}\pi(I_K + \varsigma_K)$, $-1/2 \leq \varsigma_K < 1/2$. Thus $\cos(a^n\pi x) = \cos(a^{n-K}\pi I_K + a^{n-K}\pi\varsigma_K) = (-1)^{I_K}\cos(a^{n-K}\pi\varsigma_K)$. Now, the second sum becomes

$$(-1)^{I_K+1}\sum_{n=K}^{\infty} b^n\left(\frac{1+\cos(a^{n-K}\pi\varsigma_K)}{h_K}\right).$$

Because $b^n((1+\cos(a^{n-K}\pi\varsigma_K))/h_K)$ is positive for $n \geq K$, and $1/h_K \geq 2a^K/3$, we have

$$\left|\sum_{n=K}^{\infty} b^n \frac{\cos(a^n\pi(x+h_K)) - \cos(a^n\pi x)}{h_K}\right| \geq \frac{2}{3}a^K b^K.$$

But then,

$$\left|\frac{f(x+h_K)-f(x)}{h_K}\right| \geq \frac{2}{3}a^K b^K - \frac{a^K b^K}{ab-1}\cdot\pi = a^K b^K\left(\frac{2}{3} - \frac{\pi}{ab-1}\right),$$

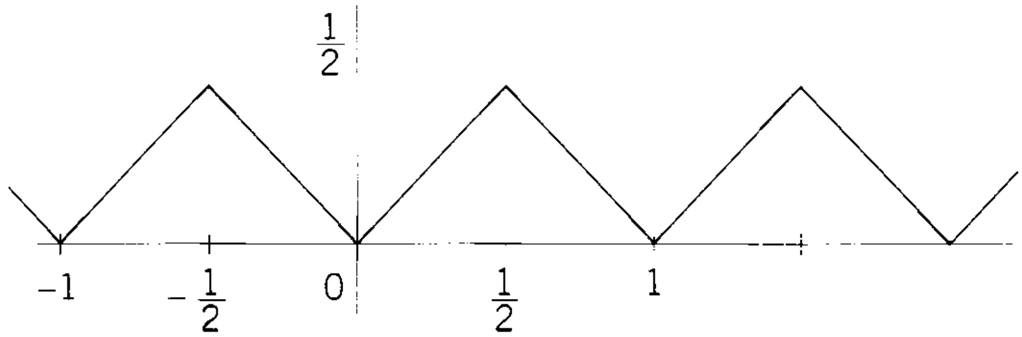
and since $ab > 1 + 3\pi/2$, $2/3 - \pi/(ab-1) > 0$. We have then that as $K \longrightarrow \infty$, $h_K \longrightarrow 0$ and $|(f(x+h_K)-f(x))/h_K|$ increases without bound. The function $f$ is not differentiable at $x$.

The last example we present is due to Patrick Billingsley (*American Mathematical Monthly*, 1982). His argument is developed around the following observation: If $f$ is differentiable at $x$, and if $\alpha_K \leq x \leq \beta_K$, $0 < \beta_K - \alpha_K$, where $\beta_K - \alpha_K \longrightarrow 0$, then $(f(\beta_K)-f(\alpha_K))/(\beta_K-\alpha_K) \longrightarrow f'(x)$. He then proceeds to exhibit sequences $(\alpha_K)$, $(\beta_K)$ such that $(f(\beta_K)-f(\alpha_K))/(\beta_K-\alpha_K)$ does not have a limit.

*Note*: The observation follows from the equation:

$$\begin{aligned}\frac{f(\beta_K)-f(\alpha_K)}{\beta_K-\alpha_K} - f'(x) &= \left[\frac{f(\beta_K)-f(x)}{\beta_K - x} - f'(x)\right]\left(\frac{\beta_K - x}{\beta_K-\alpha_K}\right)\\ &\quad + \left[\frac{f(x)-f(\alpha_K)}{x-\alpha_K} - f'(x)\right]\left(\frac{x-\alpha_K}{\beta_K-\alpha_K}\right).\end{aligned}$$

Now, suppose $f_0$ is defined by $f_0(x)$ is the distance from $x$ to the nearest integer:




This function, $f_0$, is periodic on [0,1) with period one and vanishes at the integers. Most importantly, $f_0$ is linear on the intervals $[m/2^1, (m+1)/2^1], [m/2^2, (m+1)/2^2], \ldots, [m/2^n, (m+1)/2^n], \ldots$ where $m = 0, \pm 1, \pm 2, \ldots$ and $n = 1, 2, \ldots$. We say $f_0$ is linear on any interval whose endpoints are successive dyadic rationals.

Billingsley's Function:

$$f(x) = \sum_{n=0}^{\infty} \frac{f_0(2^n x)}{2^n}.$$

This function is continuous by Weierstrass $M$-Test. Fix $x$. Since we have an integer $N$ so that $N \le x < N+1, 0 \le x - N < 1$, we may as well assume $x \in [0, 1)$. Considering dyadic expansions, we have:

$$\frac{a_1}{2} \le x < \frac{a_1}{2} + \frac{1}{2}, \qquad a_1 = 0 \text{ or } 1.$$

$$\frac{a_1}{2} + \frac{a_2}{2^2} \le x < \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{1}{2^2}, \qquad a_1, a_2 = 0 \text{ or } 1.$$

$$\vdots$$

$$\frac{a_1}{2} + \cdots + \frac{a_K}{2^K} \le x < \frac{a_1}{2} + \cdots + \frac{a_K}{2^K} + \frac{1}{2^K}, \qquad a_i = 0 \text{ or } 1.$$

So, let

$$\alpha_K = \frac{a_1}{2} + \cdots + \frac{a_K}{2^K} \le x < \frac{a_1}{2} + \cdots + \frac{a_K}{2^K} + \frac{1}{2^K} = \beta_K.$$

Now form the difference quotient

$$\frac{f(\beta_K)-f(\alpha_K)}{\beta_K-\alpha_K}=\sum_{n=0}^{K-1}\frac{f_0(2^n\beta_K)-f_0(2^n\alpha_K)}{2^n\beta_K-2^n\alpha_K}$$

$$+\sum_{n=K}^{\infty}\frac{f_0(2^n\beta_K)-f_0(2^n\alpha_K)}{2^n\beta_K-2^n\alpha_K}\,.$$

Consider the first sum: $0 \le n \le K-1$.

$$2^n\beta_K=2^n\left(\frac{2^{K-1}a_1+\cdots+a_K+1}{2^K}\right)=\frac{m+1}{2^{K-n}}$$

and

$$2^n\alpha_K=2^n\left(\frac{2^{K-1}a_1+\cdots+a_K}{2^K}\right)=\frac{m}{2^{K-n}}\,.$$

But $f_0$ is linear on $[m/2^{K-n}, (m+1)/2^{K-n}]$ as long as $K-n \ge 1$, that is, $0 \le n \le K-1$. Thus

$$\sum_{n=0}^{K-1}\frac{f(2^n\beta_K)-f_0(2^n\alpha_K)}{2^n(\beta_K-\alpha_K)}=\sum_{n=0}^{K-1}\pm 1\,.$$

The second sum vanishes because $2^n\beta_K=(m+1)/2^{K-n}$ and $2^n\alpha_K=(m/2^{K-n})$ are integers for $K \le n$, i.e., $n \ge K$. So, if differentiable at $x$, we would have the nonsense that $f'(x)=\sum_0^\infty \pm 1$! This concludes our treatment of continuous, nowhere differentiable, functions.

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