

Optimal control on a vaccine metapopulation SIR model

July 29, 2022

Abstract

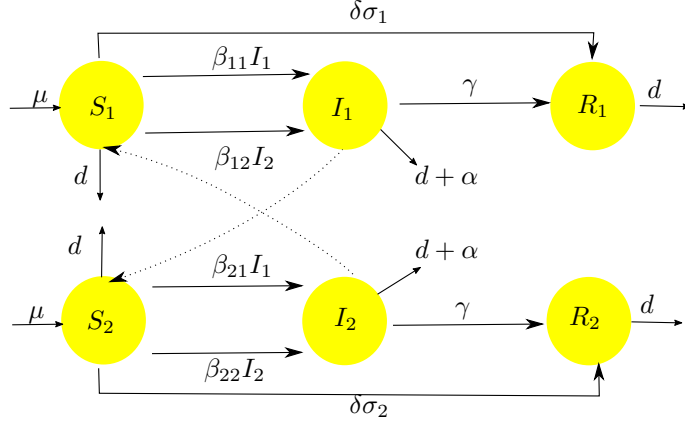
1 Introduction

We consider the following SIR model of metapopulations with migrations:

$$\begin{cases} \frac{dS_i}{dt} = \mu N_i - S_i(t) \sum_{j=1}^n \beta_{ij} I_j(t) - \delta \sigma_i S_i(t) - d S_i(t) \\ \frac{dI_i}{dt} = S_i(t) \sum_{j=1}^n \beta_{ij} I_j(t) - (d + \alpha + \gamma) I_i(t) \\ \frac{dR_i}{dt} = \gamma I_i(t) + \delta \sigma_i S_i(t) - d R_i(t) \\ S(0) = S_{i0}; \quad I_i(0) = I_{i0}; \quad R_i(0) = R_{i0} \end{cases} \quad (1)$$

where

- β_{ij} is the transmission rate per capita per time unit of individuals of the class i with the class j .
- μ is the birth rate.
- σ_i is the vaccination rate at the node i (control variable).
- γ is the recovery rate.
- d is the natural death rate.
- α is the death rate from the disease (case fatality).
- δ is the vaccine effectiveness.



Experimento método directo, minimiz

$\beta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$N = [10000]$
 $I(0) = [1 \ 1]$

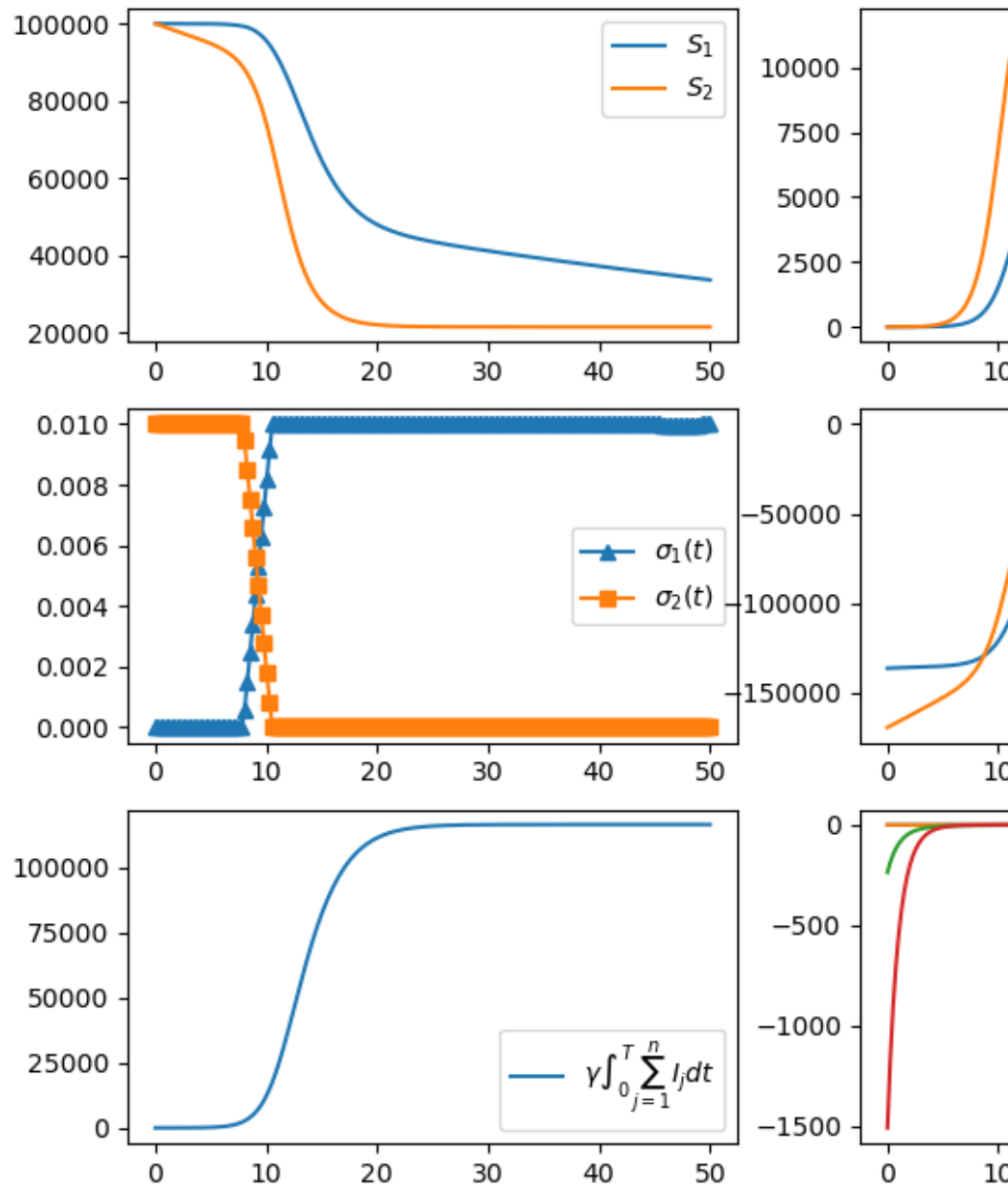


Figure 1: Caption

We suppose that $\sigma_T : [0, T] \rightarrow \mathbb{R}_+$ is a non-negative function. We consider the admissible control set

$$U_{ad} = \{\sigma = (\sigma_1, \dots, \sigma_n) : \sigma_i \text{ measurable}, \sigma_i \geq 0, \sigma_1 + \dots + \sigma_n = \sigma_T(t)\}$$

Nota 26/07/2022:

$\sigma_i = \#$ Vacunados nodo $i \times$ unidad de tiempo \times susceptible del nodo i .

La cantidad total de vacunas suministradas en el nodo i es $\sigma_i S_i$. Estamos considerando igual de admisibles distribuciones de vacunas que en total pueden resultar en cantidades de vacunados muy diferentes. Parece más razonable que el conjunto admisible sea algo del tipo:

$$\sigma_1 S_1 + \dots + \sigma_n S_n \leq \sigma_T(t)$$

Ahora $\sigma_T(t)$ representan cantidad de vacunas por unidad de tiempo en la población total. Quizás que la cantidad de vacunados no se exprese como $\sigma_i S_i$. Discutir un poco esto.

The objective function given by

$$J(\sigma) = \int_0^T \sum_{i=1}^n I_i(t) dt$$

we formulate the optimal control problem

$$\text{find } \sigma^* \in U_{ad} \text{ such that } J(\sigma^*) = \min_{\sigma \in U_{ad}} J(\sigma) \quad (2)$$

2 Some observations on dynamics of vaccine metapopulation model

We assume $\mu = d = \alpha = 0$ and $\delta = 1$. In this situation $N'_i := (S_i + I_i + R_i)' = 0$, i.e. the total population of node i remain constant. Cosequently $R_i = N - S_i - I_i$ and we can drop the equation for R_i of the system. Therefore, we can study the SIR metapopulation model with vaccination (SIRmv) system.

$$\begin{cases} S'_i = -S_i(t) \sum_{j=1}^n \beta_{ij} I_j(t) - \sigma_i(t) S_i(t) & (\text{SIRmv1}) \\ I'_i = S_i(t) \sum_{j=1}^n \beta_{ij} I_j(t) - \gamma I_i(t) & (\text{SIRmv2}) \end{cases}$$

We note that the system of equations (SIRmv1), (SIRmv2) possibly has a discontinuous right hand side. In this case, in this case, following to [Filippov, 1988]. **Poner el significado de solución**

We will use the following well known and elementary formula for the solution of a linear scalar equation of first order $x'(t) + p(t)x(t) = q(t)$:

$$x(t) = e^{-\int_0^t p(s) ds} \left\{ x(0) + \int_0^t e^{\int_0^s p(r) dr} q(s) ds \right\} \quad (3)$$

From (SIRmv1) and (3) we obtain that

$$S_i(t) = S_i(0) \exp \left(- \int_0^t \sum_{j=1}^n \beta_{ij} I_j(s) + \sigma(s) ds \right)$$

The following conjecture establish that if is a irreducible matrix (see [Meyer, 2000]) the epidemic is transmitted from any node to the rest with infinite speed.

Proposition 2.1. *Suppose that β is a irreducible matrix. If there exists i such that $I_i(0) > 0$ then for every j and $t > 0$ we have $I_j(t) > 0$.*

Proof. Agregar + □

Lemma 2.2. *Let $h \in L^\infty([0, +\infty))$ be a function. Suppose that there exists $\lim_{t \rightarrow \infty} h(t)$ and denote it by h_∞ . Then*

$$h_\infty = \lim_{t \rightarrow \infty} \gamma \int_0^t h(s) e^{\gamma(s-t)} ds. \quad (4)$$

Proof. Given $\varepsilon > 0$ there exists $t_0 \geq 0$ such that

$$t \geq t_0 \Rightarrow |h(t) - h_\infty| < \varepsilon.$$

Using the identity

$$\gamma \int_0^t e^{\gamma(s-t)} ds = 1 - e^{-\gamma t},$$

we can deduce

$$\begin{aligned} \left| h_\infty - \gamma \int_0^t h(s) e^{\gamma(s-t)} ds \right| &= \left| e^{-\gamma t} h_\infty + \gamma \int_0^t (h_\infty - h(s)) e^{\gamma(s-t)} ds \right| \\ &\leq e^{-\gamma t} |h_\infty| + \gamma \int_0^{t_0} |h_\infty - h(s)| e^{\gamma(s-t)} ds + \gamma \int_{t_0}^t |h_\infty - h(s)| e^{\gamma(s-t)} ds \\ &\quad e^{-\gamma t} |h_\infty| + 2\gamma \|h\|_{L^\infty} e^{\gamma(t_0-t)} t_0 + \varepsilon. \end{aligned}$$

This inequality implies the result in lemma. □

Remark 1. Of course, the existence of the limit on the right hand side in equation (4) does not guarantee the existence of limit of $h(t)$ for $t \rightarrow \infty$. An example of this fact is obtained as follow. Let

$$h = \sum_{n=0}^{\infty} \mathbb{1}_{[n, n+\frac{1}{2}]},$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A . Then $h(s) + h(s - \frac{1}{2}) = 1$, for $s \in [0, +\infty]$. We define $\varphi(s) = h(e^s - 1)$. Then

$$\begin{aligned} 1 &= \lim_{t \rightarrow \infty} \gamma \int_0^t \left(\varphi(s) + \varphi\left(s - \frac{1}{2}\right) \right) e^{\gamma(s-t)} ds \\ &= \lim_{t \rightarrow \infty} \gamma \int_0^t \varphi(s) e^{\gamma(s-t)} ds + \gamma \int_0^t \varphi\left(s - \frac{1}{2}\right) e^{\gamma(s-t)} ds \quad (5) \end{aligned}$$

On the other hand

$$\begin{aligned}\gamma \int_0^t \varphi\left(s - \frac{1}{2}\right) e^{\gamma(s-t)} ds &= \gamma e^{\frac{\gamma}{2}} \int_0^{t-\frac{1}{2}} \varphi(r) e^{\gamma(r-t)} dr \\ &= \gamma e^{\frac{\gamma}{2}} \int_0^t \varphi(r) e^{\gamma(r-t)} dr - \gamma e^{\frac{\gamma}{2}} \int_{t-\frac{1}{2}}^t \varphi(r) e^{\gamma(r-t)} dr \quad (6)\end{aligned}$$

In my understanding

$$\lim_{t \rightarrow \infty} \int_{t-\frac{1}{2}}^t \varphi(r) e^{\gamma(r-t)} dr = \frac{1}{2\gamma} (1 - e^{-\gamma/2}).$$

Taking account of (5), (6) we infer that

$$\lim_{t \rightarrow \infty} \gamma \int_0^t \varphi(s) e^{\gamma(s-t)} ds = \frac{e^{\gamma/2}}{2(1 + e^{\gamma/2})}$$

The following proposition expresses the fact that epidemic is extinguished when $t \rightarrow \infty$ and, in certain sense, the total quantity of applied vaccines $\sigma_i S_i$ goes to zero when $t \rightarrow \infty$.

Proposition 2.3.

$$\lim_{t \rightarrow \infty} \gamma \int_0^t \sigma_i(s) S_i(s) e^{\gamma(s-t)} ds = I_i(\infty) = 0.$$

Proof. Adding equations (SIRmv1) and (SIRmv2) we obtain

$$(S_i + I_i)' = -\sigma_i S_i - \gamma I_i \leq 0.$$

Therefore $S_i + I_i$ is a monotone non increasing function. Hence $\lim_{t \rightarrow \infty} (S_i + I_i)$ there exists. From (SIRmv1) the same considerations are true for function S_i . Consequently there exists $\lim_{t \rightarrow \infty} S_i(t) =: S_i(\infty)$. We deduce that there exists $\lim_{t \rightarrow \infty} I_i(t) =: I_i(\infty)$. If $I(\infty) > 0$, we could choose t_0 large enough for that $t \geq t_0$ implies $I_i(t) > I_i(\infty)/2 =: a > 0$. Then $(S_i(t) + I_i(t))' \leq -\gamma I_i(t) \leq -\gamma a$. This inequality implies that $S_i(t) + I_i(t) \rightarrow -\infty$, when $t \rightarrow \infty$, which is a contradiction. Consequently $I_i(\infty) = 0$.

From (SIRmv2) and (3) we obtain that

$$\begin{aligned}I_i(t) &= e^{-\gamma t} \left\{ I_i(0) + \int_0^t e^{\gamma s} S_i(s) \sum_{j=1}^n \beta_{ij} I_j(s) ds \right\} \\ &= e^{-\gamma t} \left\{ I_i(0) - \int_0^t e^{\gamma s} [S_i'(s) + \sigma_i(s) S_i(s)] ds \right\} \\ &= e^{-\gamma t} \left\{ I_i(0) - \int_0^t e^{\gamma s} \sigma_i(s) S_i(s) ds - e^{\gamma t} S_i(t) + S_i(0) - \gamma \int_0^t e^{\gamma s} S_i(s) ds \right\} \\ &= e^{-\gamma t} (S_i(0) + I_i(0)) - S_i(t) + \gamma \int_0^t e^{\gamma(s-t)} S_i(s) ds - \int_0^t e^{\gamma s} \sigma_i(s) S_i(s) ds\end{aligned}$$

Taking the limit for $t \rightarrow \infty$ in previous identities and using Lemma 2.2 \square

Conjecture 2.4.

$$\lim_{t \rightarrow \infty} \sigma_i(s) S_i(s) = 0.$$

3 Existence minimizers

In this Section, we prove that the optimal control problem (2) has a solution. That is, we prove that the hypothesis of the Filippov-Cesari Theorem are satisfied (see [Seierstad and Sydsæter, 1987]). In what follows, we will use the following notation

$$x = (S_1, \dots, S_n, I_1, \dots, I_n, R_1, \dots, R_n)$$

$$f_0(x_i, \sigma_i, t) = f_0(S_1, \dots, S_n, I_1, \dots, I_n, R_1, \dots, R_n; \sigma_1, \dots, \sigma_n; t) = \sum_{i=1}^n I_i(t).$$

For $i = 1, 2, \dots, n$, we denote by

$$f_i = (f_{0i}, f_{2i}, f_{3i}), \quad f = (f_1, \dots, f_n)$$

with

$$f_{1i} = \mu N_i - S_i(t) \sum_{j=1}^n \beta_i I_j(t) - \delta \sigma_i S_i(t) - d S_i(t)$$

$$f_{2i} = S_i(t) \sum_{j=1}^n \beta_i I_j(t) - (d + \alpha + \gamma) I_i(t)$$

$$f_{3i} = \gamma I_i(t) + \delta \sigma_i S_i(t) - d R_i(t)$$

and we define

$$N(x, U_{ad}, t) = \{(f_0 + \gamma, f) : \gamma \leq 0, \sigma \in U_{ad}\}.$$

Now, we will be in conditions to prove the following result.

Theorem 3.1. *The optimal control problem (2) has a solution $\sigma^* \in U_{ad}$.*

Proof. We will see that $N(x, U_{ad}, t)$ is a convex set, for every (x, t) .

Let $(a_1, b_1), (a_2, b_2) \in N(x, U_{ad}, t)$ be, then there exist $\gamma_1, r_2 \leq 0$ and $\sigma_1, \sigma_2 \in U_{ad}$ such that

$$(f_0(x, y_1, t) + \gamma_1; f(x, \sigma_2, t)) = (a_1, b_1)$$

and

$$(f_0(x, \sigma_2, t) + \gamma_2, f(x, \sigma_2, t)) = (a_2, b_2)$$

then

$$\lambda(a_1, b_1) + (1 - \lambda)(a_2, b_2) = (\lambda a_1 + (1 - \lambda)a_2, \lambda b_1 + (1 - \lambda)b_2)$$

$$= (\lambda(f_0(x, \sigma_1, t) + \gamma_1) + (1 - \lambda)(f_0(x_1, \sigma_2, t) + r_2), \lambda f(x_1, r_1, t) + (1 - \lambda)f(x_1 \sigma_2, t)).$$

Now, we consider the second component

$$\lambda f(x, \sigma_1, t) + (1 - \lambda)f(x, \sigma_2, t)$$

and from the linearity of f with respect σ , we have

$$\lambda f(x, \sigma_1, t) + (1 - \lambda)f(x, \sigma_2, t) = f(x, \lambda \sigma_1 + (1 - \lambda)\sigma_2, t)$$

Moreover, $\bar{\sigma} = \lambda\sigma_1 + (1-\lambda)\sigma_2 \in U_{ad}$. In fact, $\lambda\sigma_1 + (1-\lambda)\sigma_2$ is measurable, $\lambda\sigma_1 + (1-\lambda)\sigma_2 \geq 0$ and if we consider $\sigma_1 = (\sigma_{11}, \dots, \sigma_{1n})$ and $\sigma_2 = (\sigma_{21}, \dots, \sigma_{2n})$ in U_{ad} , then we have

$$\begin{aligned}\lambda\sigma_1 + (1-\lambda)\sigma_2 &= (\lambda\sigma_{11} + (1-\lambda)\sigma_{21}, \dots, \lambda\sigma_{1n} + (1-\lambda)\sigma_{2n}) \\ \lambda\sigma_{11} + (1-\lambda)\sigma_{21} + \dots + \lambda\sigma_{nr} + (1-\lambda)\sigma_{2n} &= \\ \lambda(\sigma_{11} + \dots + \sigma_{1n}) + (1-\lambda)(\sigma_{21} + \dots + \sigma_{2n}) &= \sigma_{Tot}.\end{aligned}$$

Next, we will see that there exists $\gamma \leq 0$ such that

$$f_0(x, \bar{\sigma}, t) + \gamma = \lambda a_1 + (1-\lambda)a_2.$$

We note that f_0 is constant with respect to the control variable, then

$$f_0(x, \lambda\sigma_1 + (1-\lambda)\sigma_2, t) = \lambda f_0(x_1, \sigma_1, t) + (1-\lambda)f_0(x, \sigma_2, t)$$

If we define $\gamma = \lambda\gamma_1 + (1-\lambda)\gamma_2 \leq 0$, we have that

$$\begin{aligned}f_0(x_1\lambda\sigma_1 + (1-\lambda)\sigma_2, t) + \gamma &= [\lambda f_0(x_1\sigma_1, t) + (1-\lambda)f_0(x, \sigma_2, t)] + (\lambda\gamma_1 + (1-\lambda)\gamma_2) \\ &= \lambda f_0(x_1\sigma_1, t) + \lambda\gamma_1 + (1-\lambda)f_0(x, \sigma_2, t) + (1-\lambda)\gamma_2 \\ &= \lambda(f_0(x, \sigma_1, t) + \gamma_1) + (1-\lambda)(f_0(x, \sigma_2, t) + \gamma_2) \\ &= \lambda a_1 + (1-\lambda)a_2.\end{aligned}$$

Therefore, it was proved that there exists $\gamma = \lambda\gamma_1 + (1-\lambda)\gamma_2 \leq 0$ and there exists $\bar{\sigma} = \lambda\sigma_1 + (1-\lambda)\sigma_2 \in U_{ad}$ such that

$$(\lambda a_1 + (1-\lambda)a_2, \lambda b_1 + (1-\lambda)b_2) = (f_0(x, \bar{\sigma}, t) + \gamma, f(x, \bar{\sigma}, t)),$$

i.e.

$$(\lambda a_1 + (1-\lambda)a_2, \lambda b_1 + (1-\lambda)b_2) \in N(x, U_{ad}, t)$$

and $N(x, U_{ad}, t)$ is a convex set, for all fixed (x, t) .

Moreover, U_{ad} is a compact set, since $0 \leq \sigma_i \leq \sigma_{Tot}, \forall i = 1, \dots, n$. Finally, taking into account that the number of susceptible, infected and removed individuals are bounded by the total quantity of individuals, we have that $\|x(t)\| \leq b$. Therefore, we have verified the hypothesis of Filippov-Cesari Existence Theorem and the thesis holds. \square

4 Optimality system

In this Section, we obtain the optimality system, which is derived of the Pontryagin Maximum Principle. We define the Lagrangian by

$$L = \sum_{i=1}^n I_i(t)$$

and the Hamiltonian by

$$H = L + \sum_{i=1}^n \left[\lambda_{1i} \frac{dS_i}{dt} + \lambda_{2i} \frac{dI_i}{dt} + \lambda_{3i} \frac{dR_i}{dt} \right]$$

where λ_{1i} , λ_{2i} and λ_{3i} are the adjoint variables to be determined suitably.

Theorem 4.1. Let (I^*, σ^*) be a optimal solution for the optimal control problem (2), where $I^*(t) = (I_1^*(t), \dots, I_n^*(t))$ and $\sigma^*(t) = (\sigma_1^*(t), \dots, \sigma_n^*(t))$. Then there exist adjoint variables λ_{1i} , λ_{2i} and λ_{3i} for $i = 1, \dots, n$, that satisfy

$$\frac{d\lambda_{1i}}{dt} = -\lambda_{1i} \left(\sum_{j=1}^n \beta_{ij} I_j(t) - \delta \sigma_i - d \right) - \lambda_{2i} \left(\sum_{j=1}^n \beta_{ij} I_j(t) \right) - \lambda_{3i} \delta \sigma_i \quad (7)$$

$$\frac{d\lambda_{2i}}{dt} = -1 - \lambda_{2i} (d + \alpha + \gamma) - \lambda_{3i} \gamma \quad (8)$$

$$\frac{d\lambda_{3i}}{dt} = \lambda_{3i} d \quad (9)$$

with transversality conditions

$$\lambda_{1i}(T) = \lambda_{2i}(T) = \lambda_{3i}(T) = 0 \quad (10)$$

Furthermore, the optimality equation is given by

$$(\lambda_{1i} - \lambda_{1i}) \delta S_i^* = 0. \quad (11)$$

Proof. We know, from the Pontryagin Maximum Principle (see [?]), that if (I^*, σ^*) is a optimal solution for the optimal control problem (2), then there exists a vectorial function $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$ satisfying the following equalities

$$\begin{cases} \frac{d\lambda}{dt} = - \frac{\partial H(t, I^*(t), \sigma^*(t), \lambda(t))}{\partial \lambda} \\ \frac{\partial H(t, I^*(t), \sigma^*(t), \lambda(t))}{\partial \sigma} = 0 \\ \frac{dI}{dt} = - \frac{\partial H(t, I^*(t), \sigma^*(t), \lambda(t))}{\partial I} \end{cases}$$

Here, taking into account that

$$H = \sum_{i=1}^n I_i(t) + \sum_{i=1}^n \left[\lambda_{1i} \frac{dS_i}{dt} + \lambda_{2i} \frac{dI_i}{dt} + \lambda_{3i} \frac{dR_i}{dt} \right]$$

we have

$$\begin{aligned} \frac{d\lambda_{1i}}{dt} &= - \frac{dH}{dS_i} = - \frac{dI_i}{dS_i} - \left[\lambda_{1i} \frac{\partial}{\partial S_i} \left(\frac{dS_i}{dt} \right) + \lambda_{2i} \frac{\partial}{\partial S_i} \left(\frac{dI_i}{dt} \right) + \lambda_{3i} \frac{\partial}{\partial S_i} \left(\frac{dR_i}{dt} \right) \right] \\ \frac{d\lambda_{2i}}{dt} &= - \frac{dH}{dI_i} = - \frac{dI_i}{dI_i} - \left[\lambda_{1i} \frac{\partial}{\partial I_i} \left(\frac{dS_i}{dt} \right) + \lambda_{2i} \frac{\partial}{\partial I_i} \left(\frac{dI_i}{dt} \right) + \lambda_{3i} \frac{\partial}{\partial I_i} \left(\frac{dR_i}{dt} \right) \right] \\ \frac{d\lambda_{3i}}{dt} &= - \frac{dH}{dR_i} = - \frac{dI_i}{dR_i} - \left[\lambda_{1i} \frac{\partial}{\partial R_i} \left(\frac{dS_i}{dt} \right) + \lambda_{2i} \frac{\partial}{\partial R_i} \left(\frac{dI_i}{dt} \right) + \lambda_{3i} \frac{\partial}{\partial R_i} \left(\frac{dR_i}{dt} \right) \right] \end{aligned}$$

that is, we obtain the equations (7), (8) and (9), with the transversality conditions (10). Moreover, the optimality equation is given by

$$\frac{\partial H(t, I^*(t), \sigma^*(t), \lambda(t))}{\partial \sigma} = 0$$

i.e.

$$(\lambda_{1i} - \lambda_{1i}) \delta S_i^* = 0.$$

and the thesis holds. \square

Remark 4.2. We note that the optimal control σ^* is not explicit in the optimality equation (11).

5 Numerical results

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References

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