Optimal control on a vaccine metapopulation SIR model

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Abstract

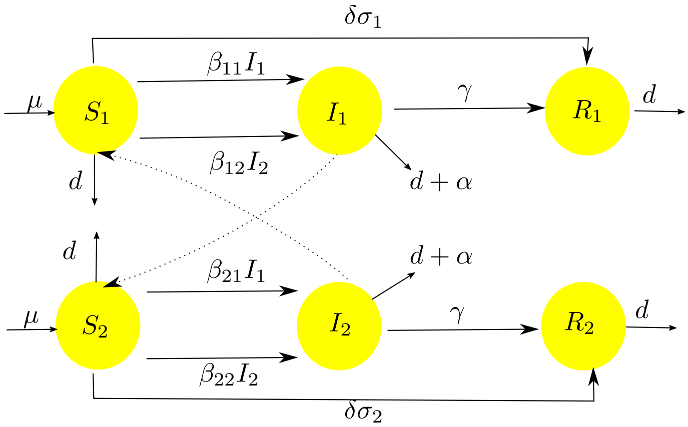
1 Introduction

We consider the following SIR model of metapopulations with migrations:

$$\begin{cases} \frac{dS_i}{dt} = \mu N_i - S_i(t)\sum_{j=1}^{n}\beta_{ij}I_j(t) - \delta\sigma_i S_i(t) - dS_i(t) \\ \frac{dI_i}{dt} = S_i(t)\sum_{j=1}^{n}\beta_{ij}I_j(t) - (d+\alpha+\gamma)I_i(t) \\ \frac{dR_i}{dt} = \gamma I_i(t) + \delta\sigma_i S_i(t) - dR_i(t) \\ S(0) = S_{i0}; \quad I_i(0) = I_{i0}; \quad R_i(0) = R_{i0} \end{cases} \tag{1}$$

where

- β_{ij} is the transmission rate per capita per time unit of individuals of the class i with the class j.
- μ is the birth rate.
- σ_i is the vaccination rate at the node i (control variable).
- γ is the recovery rate.
- d is the natural death rate.
- α is the death rate from the disease (case fatality).
- δ is the vaccine effectiveness.



We suupose that $\sigma_T : [0,T] \to \mathbb{R}_+$ is a non-negative function. We consider the admissible control set

$$U_{ad} = \{\sigma = (\sigma_1, \ldots, \sigma_n) : \sigma_i \;\text{ measurable }, \sigma_i \geqslant 0, \sigma_1 + \cdots + \sigma_n = \sigma_T(t)\}$$

Nota 26/07/2022:

$\sigma_i =$# Vacunados nodo i $\times$ unidad de tiempo $\times$ suceptible del nodo i.

La cantidad total de vacunas suministradas en el nodo i es $\sigma_i S_i$. Estamos considerando igual de admisibles distribuciones de vacunas que en total pueden resultar en cantidades de vacunados muy diferentes. Parece más razonable que el conjunto admisible sea algo del tipo:

$$\sigma_1 S_1 + \cdots + \sigma_n S_n \leq \sigma_T(t)$$

Ahora $\sigma_T(t)$ representan cantidad de vacunas por unidad de tiempo en la población total. Quizás que la cantidad de vacunados no se exprese como $\sigma_i S_i$. Discutir un poco esto.

The objective function given by

$$J(\sigma) = \int_0^T \sum_{i=1}^n I_i(t)dt$$

we formulate the optimal control problem

$$\text{find} \quad \sigma^* \in U_{ad} \quad \text{such that} \quad J(\sigma^*) = \min_{\sigma \in U_{ad}} J(\sigma) \tag{2}$$

2 Some observations on dynamics of vaccine metapopulation model

We assume $\mu = d = \alpha = 0$ and $\delta = 1$. In this situation $N_i' := (S_i + I_i + R_i)' = 0$, i.e. the total population of node i remain constant. Cosequently $R_i = N - S_i - I_i$ and we can drop the equation for R_i of the system. Therefore, we can study the SIR metapopulation model with vaccination (SIRmv) system.

$$\begin{cases} S_i' = -S_i(t) \displaystyle\sum_{j=1}^n \beta_{ij} I_j(t) - \sigma_i(t) S_i(t) & \text{(SIRmv1)} \\ I_i' = S_i(t) \displaystyle\sum_{j=1}^n \beta_{ij} I_j(t) - \gamma I_i(t) & \text{(SIRmv2)} \end{cases}$$

We note that the system of equations (SIRmv1), (SIRmv1) possibly has a discontinuous right hand side. I this case, I this case, following to [**?**]. Poner el significado de solución

We will use the following well known and elementary formula for the solution of a linear scalar eqution of first order $x'(t) + p(t)x(t) = q(t)$:

$$x(t) = e^{-\int_0^t p(s)ds} \left\{ x(0) + \int_0^t e^{\int_0^s p(r)dr} q(s) ds \right\} \tag{3}$$

From (SIRmv1) and (3) we obtain that

$$S_i(t) = S_i(0)\exp\left(-\int_0^t \sum_{j=1}^n \beta_{ij} I_j(s) + \sigma(s) ds\right)$$

The following conjecture establish that if is a irreducible matrix (see [**?**]) the epidemic is transmitted from any node to the rest with infinite speed.

Proposition 2.1. *Suppose that β is a irreducible matrix. If there exists i such that $I_i(0) > 0$ then for evey j and $t > 0$ we have $I_j(t) > 0$.*

Proof. Agregar + □

Lemma 2.2. *Let $h \in L^\infty([0, +\infty))$ be a function. Suppose that there exists $\lim_{t\to\infty} h(t)$ and denote it by h_∞. Then*

$$h_\infty = \lim_{t\to\infty} \gamma \int_0^t h(s) e^{\gamma(s-t)} ds. \tag{4}$$

Proof. Given $\varepsilon > 0$ there exists $t_0 \geq 0$ such that

$$t \geq t_0 \Rightarrow |h(t) - h_\infty| < \varepsilon.$$

Using the identity

$$\gamma \int_0^t e^{\gamma(s-t)} ds = 1 - e^{-\gamma t},$$

we can deduce

$$\begin{aligned}
\left| h_\infty - \gamma \int_0^t h(s) e^{\gamma(s-t)} ds \right| &= \left| e^{-\gamma t} h_\infty + \gamma \int_0^t (h_\infty - h(s)) e^{\gamma(s-t)} ds \right| \\
&\leq e^{-\gamma t} |h_\infty| + \gamma \int_0^{t_0} |h_\infty - h(s)| e^{\gamma(s-t)} ds + \gamma \int_{t_0}^t |h_\infty - h(s)| e^{\gamma(s-t)} ds \\
&\qquad e^{-\gamma t} |h_\infty| + 2\gamma \|h\|_{L^\infty} e^{\gamma(t_0 - t)} t_0 + \varepsilon.
\end{aligned}$$

This inequality implies the result in lemma. □

Remark 1. Of course, the existence of the limit on the right hand side in equation (4) does not guarantee the existence of limit of $h(t)$ for $t \to \infty$. An example of this fact is obtained as follow. Let

$$h = \sum_{n=0}^\infty \mathbb{1}_{[n, n+\frac{1}{2}]},$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A. Then $h(s) + h(s - \frac{1}{2}) = 1$, for $s \in [0, +\infty]$. We define $\varphi(s) = h(e^s - 1)$. Then

$$\begin{aligned}
1 &= \lim_{t\to\infty} \gamma \int_0^t \left(\varphi(s) + \varphi\left(s - \frac{1}{2}\right)\right) e^{\gamma(s-t)} ds \\
&= \lim_{t\to\infty} \gamma \int_0^t \varphi(s) e^{\gamma(s-t)} ds + \gamma \int_0^t \varphi\left(s - \frac{1}{2}\right) e^{\gamma(s-t)} ds
\end{aligned} \tag{5}$$

On the other hand

$$\gamma \int_0^t \varphi\left(s - \frac{1}{2}\right) e^{\gamma(s-t)} ds = \gamma e^{\frac{\gamma}{2}} \int_0^{t-\frac{1}{2}} \varphi(r)\, e^{\gamma(r-t)} dr$$
$$= \gamma e^{\frac{\gamma}{2}} \int_0^t \varphi(r)\, e^{\gamma(r-t)} dr - \gamma e^{\frac{\gamma}{2}} \int_{t-\frac{1}{2}}^t \varphi(r)\, e^{\gamma(r-t)} dr \quad (6)$$

In my understanding

$$\lim_{t\to\infty} \int_{t-\frac{1}{2}}^t \varphi(r)\, e^{\gamma(r-t)} dr = \frac{1}{2\gamma}(1 - e^{-\gamma/2}).$$

Taking account of (5), (6) we infer that

$$\lim_{t\to\infty} \gamma \int_0^t \varphi(s)\, e^{\gamma(s-t)} ds = \frac{e^{\gamma/2}}{2(1 + e^{\gamma/2})}$$

The following proposition to expresses the fact that epidemic is extinguished when $t \to \infty$ and, in certain sense, the total quantity of applied vaccines $\sigma_i S_i$ goes to zero when $t \to \infty$.

Proposition 2.3.

$$\lim_{t\to\infty} \gamma \int_0^t \sigma_i(s) S_i(s) e^{\gamma(s-t)} ds = I_i(\infty) = 0.$$

Proof. Adding equations (SIRmv1) and (SIRmv2) we obtain

$$(S_i + I_i)' = -\sigma_i S_i - \gamma I_i \le 0.$$

Therefore $S_i + I_i$ is a monotone non increasing function. Hence $\lim_{t\to\infty}(S_i + I_i)$ there exists. From (SIRmv1) the same considerations are true for function S_i. Consequently there exists $\lim_{t\to\infty} S_i(t) =: S_i(\infty)$. We deduce that there exists $\lim_{t\to\infty} I_i(t) =: I_i(\infty)$. If $I(\infty) > 0$, we could choose t_0 large enough for that $t \ge t_0$ implies $I_i(t) > I_i(\infty)/2 =: a > 0$. Then $(S_i(t) + I_i(t))' \le -\gamma I_i(t) \le -\gamma a$. This inequality implies that $S_i(t) + I_i(t) \to -\infty$, when $t \to \infty$, which is a contradiction. Consequently $I_i(\infty) = 0$.

From (SIRmv2) and (3) we obtain that

$$I_i(t) = e^{-\gamma t} \left\{ I_i(0) + \int_0^t e^{\gamma s} S_i(s) \sum_{j=1}^n \beta_{ij} I_j(s) ds \right\}$$
$$= e^{-\gamma t} \left\{ I_i(0) - \int_0^t e^{\gamma s} \left[S_i'(s) + \sigma_i(s) S_i\right] ds \right\}$$
$$= e^{-\gamma t} \left\{ I_i(0) - \int_0^t e^{\gamma s} \sigma_i(s) S_i(s) ds - e^{\gamma t} S_i(t) + S_i(0) - \gamma \int_0^t e^{\gamma s} S_i(s) ds \right\}$$
$$= e^{-\gamma t} \left(S_i(0) + I_i(0)\right) - S_i(t) + \gamma \int_0^t e^{\gamma(s-t)} S_i(s) ds - \int_0^t e^{\gamma s} \sigma_i(s) S_i(s) ds$$

Taking the limit for $t \to \infty$ in previous identities and using Lemma 2.2 □

Conjecture 2.4.

$$\lim_{t\to\infty} \sigma_i(s) S_i(s) = 0.$$

3 Existence minimizers

In this Section, we prove that the optimal control problem (2) has a solution. That is, we prove that the hypothesis of the Filippov-Cesari Theorem are satisfied (see [**?**]). In what follows, we will use the following notation

$$x = (S_1, \ldots, S_n, I_1, \ldots, I_n, R_1, \ldots, R_n)$$

$$f_0\left(x_i, \sigma_i, t\right) = f_0\left(S_1, \ldots, S_1, I_1 \ldots, I_n, R_1, \cdots, R_n; \sigma_1, \cdots, \sigma_n; t\right) = \sum_{i=1}^{n} I_i(t).$$

For $i = 1, 2, \cdots, n$, we denote by

$$f_i = \left(f_{0i}, f_{2i}, f_{3i}\right), \quad f = (f_1, \ldots, f_n)$$

with

$$f_{1i} = \mu N_i - S_i(t) \sum_{j=1}^{n} \beta_i I_j(t) - \delta \sigma_i S_i(t) - d S_i(t)$$

$$f_{2i} = S_i(t) \sum_{j=1}^{n} \beta_i I_j(t) - (d + \alpha + \gamma) I_i(t)$$

$$f_{3i} = \gamma I_i(t) + \delta \sigma_i S_i(t) - d R_i(t)$$

and we define

$$N\left(x, U_{ad}, t\right) = \left\{\left(f_0 + \gamma, f\right) : \, \gamma \leqslant 0, \, \sigma \in U_{ad}\right\}.$$

Now, we will are in conditions to prove the following result.

Theorem 3.1. *The optimal control problem (2) has a solution $\sigma^* \in U_{ad}$.*

Proof. We will see that $N(x, U_{ad}, t)$ is a convex set, for every (x, t).

Let $(a_1, b_1), (a_2, b_2) \in N\left(x, U_{ad}, t\right)$ be, then there exist $\gamma_1, r_2 \leq 0$ and $\sigma_1, \sigma_2 \in U_{ad}$ such that

$$\left(f_0\left(x, y_1, t\right) + \gamma_1; f\left(x, \sigma_2, t\right)\right) = (a_1, b_1)$$

and

$$\left(f_0\left(x, \sigma_2, t\right) + \gamma_2, f\left(x, \sigma_2, t\right)\right) = (a_2, b_2)$$

then

$$\lambda\left(a_1, b_1\right) + (1 - \lambda)\left(a_2, b_2\right) = \left(\lambda a_1 + (1 - \lambda) a_2, \lambda b_1 + (1 - \lambda) b_2\right)$$

$$= \left(\lambda\left(f_0\left(x, \sigma_1, t\right) + \gamma_1\right) + (1 - \lambda)\left(f_0\left(x_1, \sigma_2, t\right) + r_2\right), \lambda f\left(x_1, r_1, t\right) + (1 - \lambda) f\left(x_1 \sigma_2, t\right)\right).$$

Now, we consider the second component

$$\lambda f\left(x, \sigma_1, t\right) + (j - \lambda) f\left(x, \sigma_2, t\right)$$

and from the linearity of f with respect σ, we have

$$\lambda f\left(x, \sigma_1, t\right) + (1 - \lambda) f\left(x, \sigma_2, t\right) = f\left(x, \lambda \sigma_1 + (1 - \lambda) \sigma_2, t\right)$$

Moreover, $\overline{\sigma} = \lambda\sigma_1 + (1-\lambda)\sigma_2 \in U_{ad}$. In fact, $\lambda\sigma_1 + (1-\lambda)\sigma_2$ is measurable, $\lambda\sigma_1 + (1-\lambda)\sigma_2 \geqslant 0$ and if we consider $\sigma_1 = (\sigma_{11}, \ldots\ldots, \sigma_{1n})$ and $\sigma_2 = (\sigma_{21} \ldots\ldots \sigma_{2n})$ in U_{ad}, then we have

$$\begin{aligned}
&\lambda\sigma_1 + (1-\lambda)\sigma_2 = (\lambda\sigma_{11} + (1-\lambda)\sigma_{21}, \ldots \lambda\sigma_{1n} + (1-\lambda)\sigma_{2n}) \\
&\lambda\sigma_{11} + (1-\lambda)\sigma_{21} + \cdots + \lambda\sigma_{nr+} + (1-\lambda)\sigma_{2n} = \\
&\lambda(\sigma_{11} + \cdots + \sigma_{1n}) + (1-\lambda)(\sigma_{21} + \cdots + \tau_{2n}) = \sigma_{Tot}.
\end{aligned}$$

Next, we will see that there exists $\gamma \leqslant 0$ such that

$$f_0(x, \overline{\sigma}, t) + \gamma = \lambda a_1 + (1-\lambda)a_2.$$

We note that f_0 is constant with respect to the control variable, then

$$f_0(x, \lambda\sigma_1 + (1-\lambda)\sigma_2, t) = \lambda f_0(x_1, \sigma_1, t) + (1-\lambda) f_0(x, \sigma_2, t)$$

If we define $\gamma = \lambda\gamma_1 + (1-\lambda)\gamma_2 \leqslant 0$, we have that

$$\begin{aligned}
f_0(x_1 \lambda\sigma_1 + (1-\lambda)\sigma_2, t) + \gamma &= [\lambda f_0(x_1 \sigma_1, t) + (1-\lambda) f_0(x, \sigma_2, t)] + (\lambda\gamma_1 + (1-\lambda)\gamma_2] \\
&= \lambda f_0(x_1\sigma_1, t) + \lambda\gamma_1 + (1-\lambda) f_0(x, \sigma_2 t) + (1-\lambda)\gamma_2 \\
&= \lambda(f_0(x, \sigma_1, t) + \gamma_1) + (1-\lambda)(f_0(x, \sigma_2, t) + \gamma_2) \\
&= \lambda a_1 + (1-\lambda)a_2.
\end{aligned}$$

Therefore, it was proved that there exists $\gamma = \lambda\gamma_1 + (1-\lambda)\gamma_2 \leqslant 0$ and there exists $\overline{\sigma} = \lambda\sigma_1 + (1-\lambda)\sigma_2 \in U_{ad}$ such that

$$(\lambda a_1 + (1-\lambda)a_2, \lambda b_1 + (1-\lambda)b_2) = (f_0(x, \overline{\sigma}, t) + \gamma, f(x, \overline{\sigma}, t)),$$

i.e.

$$(\lambda a_1 + (1-\lambda)a_2, \lambda b_1 + (1-\lambda)b_2) \in N(x, U_{ad}, t)$$

and $N(x, U_{ad}, t)$ is a convex set, for all fixed (x, t).

Moreover, U_{ad} is a compact set, since $0 \leqslant \sigma_i \leqslant \sigma_{\text{Tot}}, \forall i = 1, \ldots, n$. Finally, taking into account that the number of susceptible, infected and removed individuals are bounded by the total quantity of individuals, we have that $\|x(t)\| \leqslant b$. Therefore, we have verified the hypothesis of Filippov-Cesari Existence Theorem and the thesis holds. □

4 Optimality system

In this Section, we obtain the optimality system, which is derived of the Pontryagin Maximum Principle. We define the Lagrangian by

$$L = \sum_{i=1}^{n} I_i(t)$$

and the Hamiltonian by

$$H = L + \sum_{i=1}^{n} \left[\lambda_{1i} \frac{dS_i}{dt} + \lambda_{2i} \frac{dI_i}{dt} + \lambda_{3i} \frac{dR_i}{dt} \right]$$

where λ_{1i}, λ_{2i} and λ_{3i} are the adjoint variables to be determined suitably.

Theorem 4.1. *Let (I^*, σ^*) be a optimal solution for the optimal control problem (2), where $I^*(t) = (I_1^*(t), \cdots, I_n^*(t))$ and $\sigma^*(t) = (\sigma_1^*(t), \cdots, \sigma_n^*(t))$. Then there exist adjoint variables λ_{1i}, λ_{2i} and λ_{3i} for $i = 1, \cdots, n$, that satisfy*

$$\frac{d\lambda_{1i}}{dt} = -\lambda_{1i}\left(\sum_{j=1}^{n} \beta_{ij} I_j(t) - \delta\sigma_i - d\right) - \lambda_{2i}\left(\sum_{j=1}^{n} \beta_{ij} I_j(t)\right) - \lambda_{3i}\delta\sigma_i \tag{7}$$

$$\frac{d\lambda_{2i}}{dt} = -1 - \lambda_{2i}\left(d + \alpha + \gamma\right) - \lambda_{3i}\gamma \tag{8}$$

$$\frac{d\lambda_{3i}}{dt} = \lambda_{3i} d \tag{9}$$

with transversality conditions

$$\lambda_{1i}(T) = \lambda_{2i}(T) = \lambda_{3i}(T) = 0 \tag{10}$$

Furthermore, the optimality equation is given by

$$(\lambda_{1i} - \lambda_{1i})\delta S_i^* = 0. \tag{11}$$

Proof. We know, from the Pontryagin Maximum Principle (see [**?**]), that if (I^*, σ^*) is a optimal solution for the optimal control problem (2), then there exists a vectorial function $\lambda(t) = (\lambda_1(t), \cdots, \lambda_n(t))$ satisfying the following equalities

$$\begin{cases} \frac{d\lambda}{dt} = -\frac{\partial H(t, I^*(t)), \sigma^*(t), \lambda(t)}{\partial I} \\ \frac{\partial H(t, I^*(t)), \sigma^*(t), \lambda(t)}{\partial \sigma} = 0 \\ \frac{dI}{dt} = -\frac{\partial H(t, I^*(t)), \sigma^*(t), \lambda(t)}{\partial\partial\lambda}. \end{cases}$$

Here, taking into account that

$$H = \sum_{i=1}^{n} I_i(t) + \sum_{i=1}^{n}\left[\lambda_{1i}\frac{dS_i}{dt} + \lambda_{2i}\frac{dI_i}{dt} + \lambda_{3i}\frac{dR_i}{dt}\right]$$

we have

$$\frac{d\lambda_{1i}}{dt} = -\frac{dH}{dS_i} = -\frac{dI_i}{dS_i} - \left[\lambda_{1i}\frac{\partial}{\partial S_i}\left(\frac{dS_i}{dt}\right) + \lambda_{2i}\frac{\partial}{\partial S_i}\left(\frac{dI_i}{dt}\right) + \lambda_{3i}\frac{\partial}{\partial S_i}\left(\frac{dR_i}{dt}\right)\right]$$

$$\frac{d\lambda_{2i}}{dt} = -\frac{dH}{dI_i} = -\frac{dI_i}{dI_i} - \left[\lambda_{1i}\frac{\partial}{\partial I_i}\left(\frac{dS_i}{dt}\right) + \lambda_{2i}\frac{\partial}{\partial I_i}\left(\frac{dI_i}{dt}\right) + \lambda_{3i}\frac{\partial}{\partial I_i}\left(\frac{dR_i}{dt}\right)\right]$$

$$\frac{d\lambda_{3i}}{dt} = -\frac{dH}{dR_i} = -\frac{dI_i}{dR_i} - \left[\lambda_{1i}\frac{\partial}{\partial R_i}\left(\frac{dS_i}{dt}\right) + \lambda_{2i}\frac{\partial}{\partial R_i}\left(\frac{dI_i}{dt}\right) + \lambda_{3i}\frac{\partial}{\partial R_i}\left(\frac{dR_i}{dt}\right)\right]$$

that is, we obtain the equations (7), (8) and (9), with the transversality conditions (10). Moreover, the optimality equation is given by

$$\frac{\partial H(t, I^*(t)), \sigma^*(t), \lambda(t))}{\partial \sigma} = 0$$

i.e.

$$(\lambda_{1i} - \lambda_{1i})\delta S_i^* = 0.$$

and the thesis holds. □

Remark 4.2. *We note that the optimal control σ^* is not explicit in the optimality equation (11).*

5 Numerical results

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