Masters Theorem Proof

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$$T(n) = aT(n) + O(n^d)$$

$$T(n) = \begin{cases} O(n^{d\log(n)}) \ , \quad \text{if} \quad a = b^d \\ O(n^d) \ , \quad \text{if} \quad a < b^d \\ O(n^{\log_b(a)}) \ , \quad \text{if} \quad a > b^d \end{cases}$$

$$\text{number of problems} = a^j$$

$$\text{work} = a^j c(n/b^j)$$

$$\text{total work} = \sum_{j=0}^{log_b(n)} a^j \cdot c(\frac{n}{b^j})^d = \sum_{j=0}^{log_2(n)} cn^d (\frac{a}{b^d})^j = cn^d \sum_{j=0}^{log_2(n)} (\frac{a}{b^d})^j$$

0.1 Case: $a < b^d$

$$\text{Suppose } \frac{a}{b^d} < 1$$

$$\text{then } cn^d \sum_{j=0}^{log_2(n)} \left(\frac{a}{b^d}\right) = cn^d \left(\frac{1}{1 - \frac{a}{b^d}}\right) = cn^d \left(\frac{b^d}{b^d - a}\right) = O(n^d)$$

Given that a and b are constant

0.2 Case: $a > b^d$

$$\text{Suppose } \frac{a}{b^d} > 1$$

$$\sum_{j=0}^{x} r^j = \frac{r^{x+1} - 1}{r - 1}$$

$$T(n) = cn^d \left(\frac{\frac{a}{b^d}^{log_b(n)+1} - 1}{\frac{a}{b^d} - 1}\right) = O\left(n^d \frac{a^{log_b(n)}}{b^{d \cdot log(n^d)}}\right)$$

$$O\left(n^d \frac{a^{log_b(n)}}{b^{log_b(n^d)}}\right) = O\left(n^d \frac{a^{log_b(n)}}{n^d}\right) = O\left(a^{log_b(n)}\right) = O\left(n^{log_b(a)}\right)$$

0.3 Case $a = b^d$

$$\text{Suppose} a = b^d$$

$$\text{then } cn^d \sum_{j=0}^{log_b(n)} \left(\frac{a}{b^d}\right) = cn^d \left(log_b(n) + 1\right) = cn^d log_b(n) + cn^d$$

$$O\left(n^d log_b(n)\right)$$