ThinkDSP

This notebook contains code examples from Chapter 7: Discrete Fourier Transform

In [1]:

```
# Get thinkdsp.py

import os

if not os.path.exists('thinkdsp.py'):
    !wget https://github.com/AllenDowney/ThinkDSP/raw/master/code/thinkdsp.py
```

```
--2022-04-27 09:00:46--  https://github.com/AllenDowney/ThinkDSP/raw/master/code/thinkdsp
.py
Resolving github.com (github.com)... 192.30.255.112
Connecting to github.com (github.com)|192.30.255.112|:443... connected.
HTTP request sent, awaiting response... 302 Found
Location: https://raw.githubusercontent.com/AllenDowney/ThinkDSP/master/code/thinkdsp.py
[following]
--2022-04-27 09:00:46--  https://raw.githubusercontent.com/AllenDowney/ThinkDSP/master/co
de/thinkdsp.py
Resolving raw.githubusercontent.com (raw.githubusercontent.com)... 185.199.108.133, 185.1
99.109.133, 185.199.110.133, ...
Connecting to raw.githubusercontent.com (raw.githubusercontent.com)|185.199.108.133|:443.
.. connected.
HTTP request sent, awaiting response... 200 OK
Length: 48687 (48K) [text/plain]
Saving to: 'thinkdsp.py'

thinkdsp.py         100%[===================>]  47.55K  --.-KB/s    in 0.005s

2022-04-27 09:00:46 (8.45 MB/s) - 'thinkdsp.py' saved [48687/48687]
```

In [2]:

```
import numpy as np
import matplotlib.pyplot as plt

from thinkdsp import decorate
PI2 = 2 * np.pi
```

In [3]:

```
# suppress scientific notation for small numbers
np.set_printoptions(precision=3, suppress=True)
```

Complex sinusoid

Here's the definition of ComplexSinusoid, with print statements to display intermediate results.

In [4]:

```
from thinkdsp import Sinusoid

class ComplexSinusoid(Sinusoid):
    """Represents a complex exponential signal."""

    def evaluate(self, ts):
        """Evaluates the signal at the given times.
```

```
        ts: float array of times

        returns: float wave array
        """
        print(ts)
        phases = PI2 * self.freq * ts + self.offset
        print(phases)
        ys = self.amp * np.exp(1j * phases)
        return ys
```

Here's an example:

In [5]:

```
signal = ComplexSinusoid(freq=1, amp=0.6, offset=1)
wave = signal.make_wave(duration=1, framerate=4)
print(wave.ys)
```

```
[0.   0.25 0.5  0.75]
[1.    2.571 4.142 5.712]
[ 0.324+0.505j -0.505+0.324j -0.324-0.505j  0.505-0.324j]
```

The simplest way to synthesize a mixture of signals is to evaluate the signals and add them up.

In [6]:

```
from thinkdsp import SumSignal

def synthesize1(amps, freqs, ts):
    components = [ComplexSinusoid(freq, amp)
                  for amp, freq in zip(amps, freqs)]
    signal = SumSignal(*components)
    ys = signal.evaluate(ts)
    return ys
```

Here's an example that's a mixture of 4 components.

In [7]:

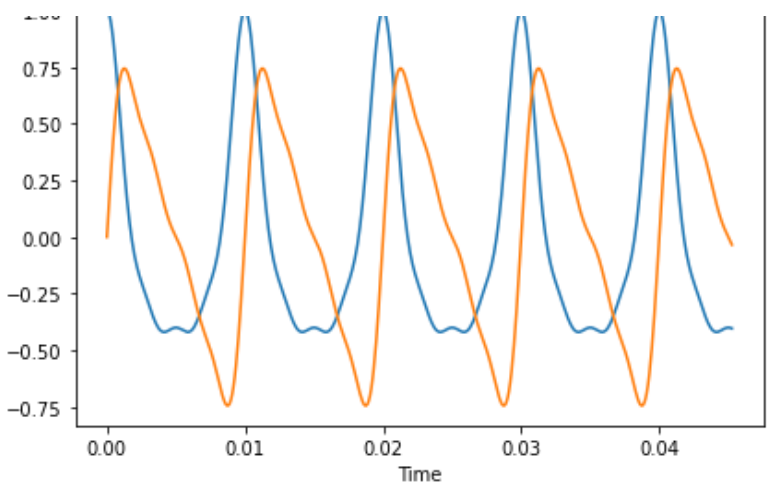
```
amps = np.array([0.6, 0.25, 0.1, 0.05])
freqs = [100, 200, 300, 400]
framerate = 11025

ts = np.linspace(0, 1, framerate, endpoint=False)
ys = synthesize1(amps, freqs, ts)
print(ys)
```

```
[0. 0. 0. ... 1. 1. 1.]
[  0.      0.057   0.114 ... 628.148 628.205 628.262]
[0. 0. 0. ... 1. 1. 1.]
[   0.      0.114   0.228 ... 1256.295 1256.409 1256.523]
[0. 0. 0. ... 1. 1. 1.]
[   0.      0.171   0.342 ... 1884.443 1884.614 1884.785]
[0. 0. 0. ... 1. 1. 1.]
[   0.      0.228   0.456 ... 2512.59  2512.818 2513.046]
[1.   +0.j    0.995+0.091j 0.979+0.18j  ... 0.953-0.267j 0.979-0.18j
 0.995-0.091j]
```

Now we can plot the real and imaginary parts:

In [8]:

```
n = 500
plt.plot(ts[:n], ys[:n].real)
plt.plot(ts[:n], ys[:n].imag)
decorate(xlabel='Time')
```



The real part is a mixture of cosines; the imaginary part is a mixture of sines. They contain the same frequency components with the same amplitudes, so they sound the same to us:

In [9]:

```
from thinkdsp import Wave

wave = Wave(ys.real, framerate)
wave.apodize()
wave.make_audio()
```

Out[9]:

Your browser does not support the audio element.

In [10]:

```
wave = Wave(ys.imag, framerate)
wave.apodize()
wave.make_audio()
```

Out[10]:

Your browser does not support the audio element.

We can express the same process using matrix multiplication.

In [11]:

```
def synthesize2(amps, freqs, ts):
    args = np.outer(ts, freqs)
    M = np.exp(1j * PI2 * args)
    ys = np.dot(M, amps)
    return ys
```

And it should sound the same.

In [12]:

```
amps = np.array([0.6, 0.25, 0.1, 0.05])
ys = synthesize2(amps, freqs, ts)
print(ys)
```

```
[1.   +0.j    0.995+0.091j 0.979+0.18j  ... 0.953-0.267j 0.979-0.18j
 0.995-0.091j]
```

In [13]:

```
wave = Wave(ys.real, framerate)
wave.apodize()
wave.make_audio()
```

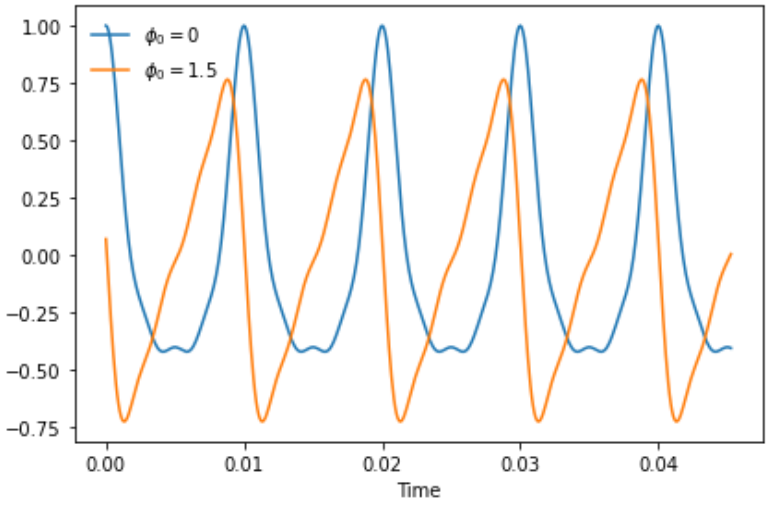
Out[13]:

Your browser does not support the audio element.

To see the effect of a complex amplitude, we can rotate the amplitudes by 1.5 radian:

In [14]:

```
phi = 1.5
amps2 = amps * np.exp(1j * phi)
ys2 = synthesize2(amps2, freqs, ts)

n = 500
plt.plot(ts[:n], ys.real[:n], label=r'$\phi_0 = 0$')
plt.plot(ts[:n], ys2.real[:n], label=r'$\phi_0 = 1.5$')
decorate(xlabel='Time')
```



Rotating all components by the same phase offset changes the shape of the waveform because the components have different periods, so the same offset has a different effect on each component.

Analysis

The simplest way to analyze a signal---that is, find the amplitude for each component---is to create the same matrix we used for synthesis and then solve the system of linear equations.

In [15]:

```
def analyze1(ys, freqs, ts):
    args = np.outer(ts, freqs)
    M = np.exp(1j * PI2 * args)
    amps = np.linalg.solve(M, ys)
    return amps
```

Using the first 4 values from the wave array, we can recover the amplitudes.

In [16]:

```
n = len(freqs)
amps2 = analyze1(ys[:n], freqs, ts[:n])
print(amps2)
```

```
[0.6 -0.j 0.25+0.j 0.1 -0.j 0.05+0.j]
```

If we define the `freqs` from 0 to N-1 and `ts` from 0 to (N-1)/N, we get a unitary matrix.

In [17]:

```
N = 4
ts = np.arange(N) / N
freqs = np.arange(N)
args = np.outer(ts, freqs)
```

```
M = np.exp(1j * PI2 * args)
print(M)
```

```
[[ 1.+0.j  1.+0.j  1.+0.j  1.+0.j]
 [ 1.+0.j  0.+1.j -1.+0.j -0.-1.j]
 [ 1.+0.j -1.+0.j  1.-0.j -1.+0.j]
 [ 1.+0.j -0.-1.j -1.+0.j  0.+1.j]]
```

To check whether a matrix is unitary, we can compute $M^* M$, which should be the identity matrix:

In [18]:

```
MstarM = M.conj().transpose().dot(M)
print(MstarM.real)
```

```
[[ 4. -0.  0.  0.]
 [-0.  4. -0.  0.]
 [ 0. -0.  4. -0.]
 [ 0.  0. -0.  4.]]
```

The result is actually $4I$, so in general we have an extra factor of N to deal with, but that's a minor problem.

We can use this result to write a faster version of `analyze1`:

In [19]:

```
def analyze2(ys, freqs, ts):
    args = np.outer(ts, freqs)
    M = np.exp(1j * PI2 * args)
    amps = M.conj().transpose().dot(ys) / N
    return amps
```

In [20]:

```
N = 4
amps = np.array([0.6, 0.25, 0.1, 0.05])
freqs = np.arange(N)
ts = np.arange(N) / N
ys = synthesize2(amps, freqs, ts)

amps3 = analyze2(ys, freqs, ts)
print(amps3)
```

```
[0.6 +0.j 0.25+0.j 0.1 -0.j 0.05-0.j]
```

Now we can write our own version of DFT:

In [21]:

```
def synthesis_matrix(N):
    ts = np.arange(N) / N
    freqs = np.arange(N)
    args = np.outer(ts, freqs)
    M = np.exp(1j * PI2 * args)
    return M
```

In [22]:

```
def dft(ys):
    N = len(ys)
    M = synthesis_matrix(N)
    amps = M.conj().transpose().dot(ys)
    return amps
```

And compare it to analyze2:

In [23]:

```
print(dft(ys))
```

```
print(dft(ys))
```

```
[2.4+0.j  1. +0.j  0.4-0.j  0.2-0.j]
```

The result is close to `amps * 4`.

We can also compare it to `np.fft.fft`. FFT stands for Fast Fourier Transform, which is an even faster implementation of DFT.

In [24]:

```
print(np.fft.fft(ys))
```

```
[2.4+0.j  1. -0.j  0.4-0.j  0.2-0.j]
```

The inverse DFT is almost the same, except we don't have to transpose M and we have to divide through by N.

In [25]:

```
def idft(amps):
    N = len(amps)
    M = synthesis_matrix(N)
    ys = M.dot(amps) / N
    return ys
```

We can confirm that `dft(idft(amps))` yields `amps`:

In [26]:

```
ys = idft(amps)
print(dft(ys))
```

```
[0.6 +0.j  0.25+0.j  0.1 -0.j  0.05-0.j]
```

Real signals

Let's see what happens when we apply DFT to a real-valued signal.

In [27]:

```
from thinkdsp import SawtoothSignal

framerate = 10000
signal = SawtoothSignal(freq=500)
wave = signal.make_wave(duration=0.1, framerate=framerate)
wave.make_audio()
```

Out[27]:

Your browser does not support the audio element.

`wave` is a 500 Hz sawtooth signal sampled at 10 kHz.

In [28]:

```
hs = dft(wave.ys)
len(wave.ys), len(hs)
```

Out[28]:

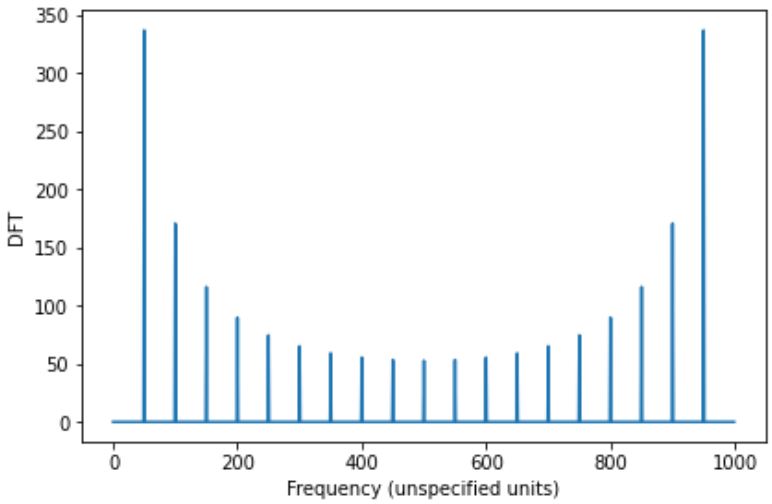
```
(1000, 1000)
```

`hs` is the DFT of this wave, and `amps` contains the amplitudes.

In [29]:

```
amps = np.abs(hs)
```

```
plt.plot(amps)
decorate(xlabel='Frequency (unspecified units)', ylabel='DFT')
```



The DFT assumes that the sampling rate is N per time unit, for an arbitrary time unit. We have to convert to actual units -- seconds -- like this:

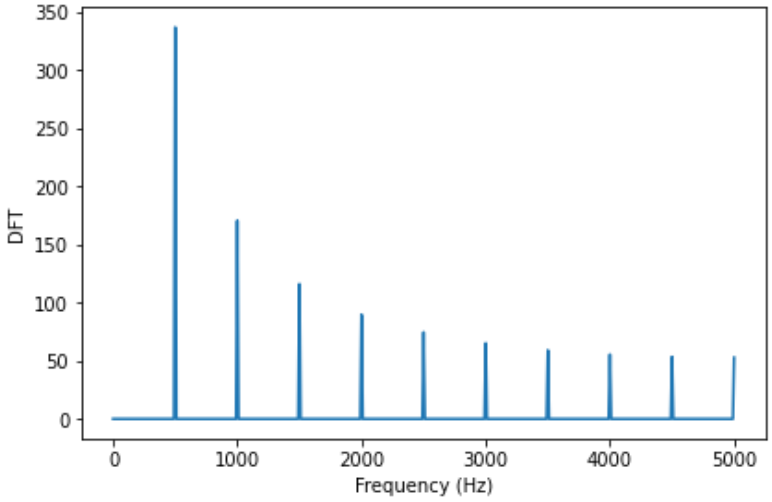
In [30]:

```
N = len(hs)
fs = np.arange(N) * framerate / N
```

Also, the DFT of a real signal is symmetric, so the right side is redundant. Normally, we only compute and plot the first half:

In [31]:

```
plt.plot(fs[:N//2+1], amps[:N//2+1])
decorate(xlabel='Frequency (Hz)', ylabel='DFT')
```



Let's get a better sense for why the DFT of a real signal is symmetric. I'll start by making the inverse DFT matrix for $N = 8$.

In [32]:

```
M = synthesis_matrix(N=8)
```

And the DFT matrix:

In [33]:

```
Mstar = M.conj().transpose()
```

And a triangle wave with 8 elements:

And a triangle wave with 8 elements:

In [34]:

```
from thinkdsp import TriangleSignal

wave = TriangleSignal(freq=1).make_wave(duration=1, framerate=8)
wave.ys
```

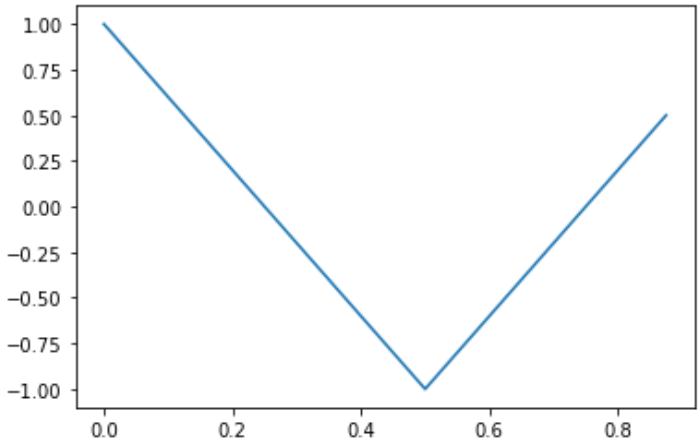
Out[34]:

```
array([ 1. ,  0.5,  0. , -0.5, -1. , -0.5,  0. ,  0.5])
```

Here's what the wave looks like.

In [35]:

```
wave.plot()
```



Now let's look at rows 3 and 5 of the DFT matrix:

In [36]:

```
row3 = Mstar[3, :]
print(row3)
```

```
[ 1.   -0.j    -0.707-0.707j -0.   +1.j     0.707-0.707j -1.   -0.j
  0.707+0.707j  0.   -1.j    -0.707+0.707j]
```

In [37]:

```
row5 = Mstar[5, :]
row5
```

Out[37]:

```
array([ 1.   -0.j   , -0.707+0.707j,  0.   -1.j   ,  0.707+0.707j,
       -1.   -0.j   ,  0.707-0.707j, -0.   +1.j   , -0.707-0.707j])
```

They are almost the same, but row5 is the complex conjugate of row3.

In [38]:

```
def approx_equal(a, b, tol=1e-10):
    return np.sum(np.abs(a-b)) < tol
```

In [39]:

```
approx_equal(row3, row5.conj())
```

Out[39]:

```
True
```

When we multiply the DFT matrix and the wave array, the element with index 3 is:

In [40]:

```
X3 = row3.dot(wave.ys)
X3
```

Out[40]:

```
(0.5857864376269055-1.1102230246251565e-16j)
```

And the element with index 5 is:

In [41]:

```
X5 = row5.dot(wave.ys)
X5
```

Out[41]:

```
(0.5857864376269062-5.551115123125783e-16j)
```

And they are the same, within floating point error.

In [42]:

```
abs(X3 - X5)
```

Out[42]:

```
8.005932084973442e-16
```

Let's try the same thing with a complex signal:

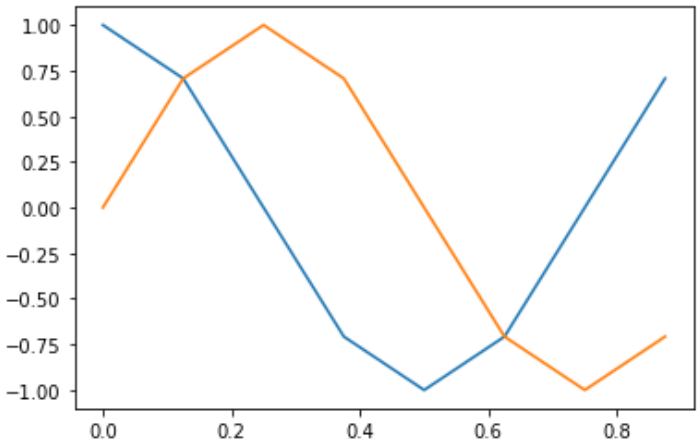
In [43]:

```
wave2 = ComplexSinusoid(freq=1).make_wave(duration=1, framerate=8)
plt.plot(wave2.ts, wave2.ys.real)
plt.plot(wave2.ts, wave2.ys.imag)
```

```
[0.    0.125 0.25  0.375 0.5   0.625 0.75  0.875]
[0.    0.785 1.571 2.356 3.142 3.927 4.712 5.498]
```

Out[43]:

```
[<matplotlib.lines.Line2D at 0x7f9a223cdd50>]
```



Now the elements with indices 3 and 5 are different:

In [44]:

```
X3 = row3.dot(wave2.ys)
X3
```

Out[44]:

```
(1.4432899320127035e-15-2.7755575615628914e-16j)
```

(1.4432899320127035e-15-2.7755575615628914e-16j)

In [45]:

```
X5 = row5.dot(wave2.ys)
X5
```

Out[45]:

```
3.3306690738754696e-16j
```

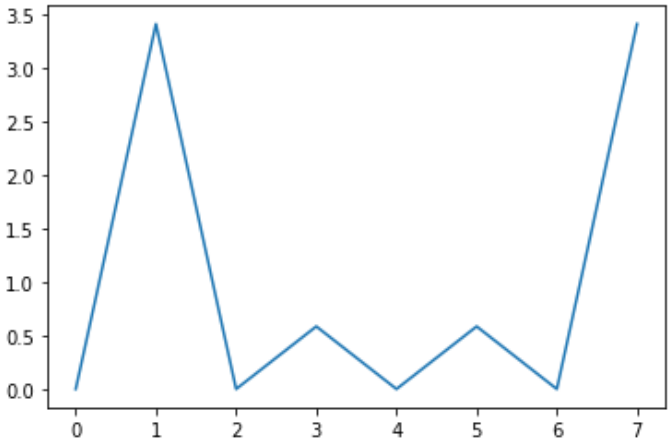
Visually we can confirm that the FFT of the real signal is symmetric:

In [46]:

```
hs = np.fft.fft(wave.ys)
plt.plot(abs(hs))
```

Out[46]:

```
[<matplotlib.lines.Line2D at 0x7f9a2233f350>]
```



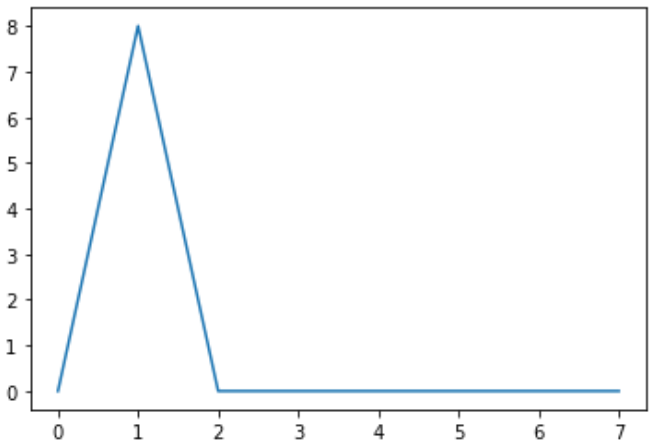
And the FFT of the complex signal is not.

In [47]:

```
hs = np.fft.fft(wave2.ys)
plt.plot(abs(hs))
```

Out[47]:

```
[<matplotlib.lines.Line2D at 0x7f9a222b6890>]
```



Another way to think about all of this is to evaluate the DFT matrix for different frequencies. Instead of 0 through $N-1$, let's try $0, 1, 2, 3, 4, -3, -2, -1$.

In [48]:

```
N = 8
ts = np.arange(N) / N
freqs = np.arange(N)
freqs = [0, 1, 2, 3, 4, -3, -2, -1]
args = np.outer(ts, freqs)
M2 = np.exp(1j * PI2 * args)
```

In [49]:

```
approx_equal(M, M2)
```

Out[49]:

```
True
```

So you can think of the second half of the DFT as positive frequencies that get aliased (which is how I explained them), or as negative frequencies (which is the more conventional way to explain them). But the DFT doesn't care either way.

The `thinkdsp` library provides support for computing the "full" FFT instead of the real FFT.

In [50]:

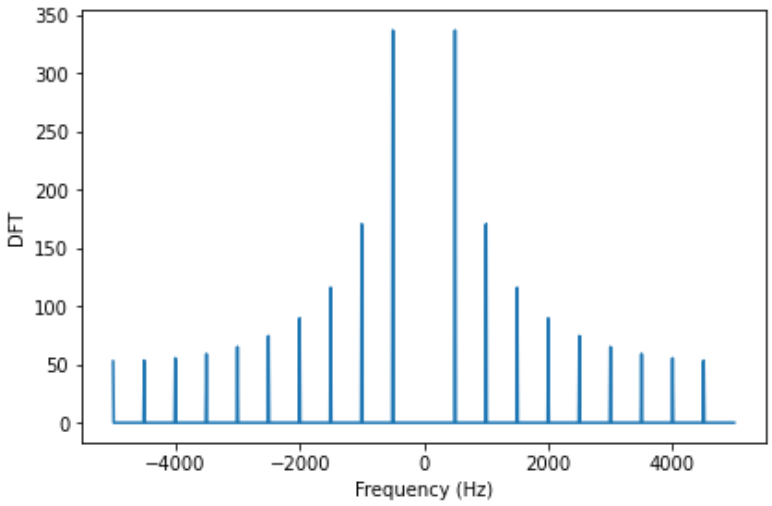
```
framerate = 10000
signal = SawtoothSignal(freq=500)
wave = signal.make_wave(duration=0.1, framerate=framerate)
```

In [51]:

```
spectrum = wave.make_spectrum(full=True)
```

In [52]:

```
spectrum.plot()
decorate(xlabel='Frequency (Hz)', ylabel='DFT')
```



In [52]:

```

```

Exercise 02

In [96]:

```
# Get thinkdsp.py

import os

if not os.path.exists('thinkdsp.py'):
    !wget https://github.com/AllenDowney/ThinkDSP/raw/master/code/thinkdsp.py
```

In [97]:

```
import numpy as np
import matplotlib.pyplot as plt

from thinkdsp import decorate
PI2 = 2 * np.pi
```

In [98]:

```
# suppress scientific notation for small numbers
np.set_printoptions(precision=3, suppress=True)
```

(1) 生成複合訊號

In [99]:

```
def synthesize2(amps, freqs, ts):
    args = np.outer(ts, freqs)
    M = np.exp(1j * PI2 * args)
    ys = np.dot(M, amps)
    return ys
```

In [100]:

```
amps = np.array([0.5, 0.5, 0.5, 0.5])
freqs = np.array([261,329,391,523])
framerate = 16384
ts = np.linspace(0, 1, framerate, endpoint=False)
ys = synthesize2(amps, freqs, ts)
print(ys)
print(len(ys))
```

```
[2.   +0.j    1.978+0.287j 1.912+0.567j ... 1.805-0.833j 1.912-0.567j
 1.978-0.287j]
16384
```

In [101]:

```
from thinkdsp import Wave

wave = Wave(ys, framerate)
wave.apodize()
wave.make_audio()
```

Out[101]:

Your browser does not support the audio element.

(2) DFT實作

In [102]:

```
def dft2(ys,k,N):
```

```
  res = 0
  for n in range(N):
    res += ys[n]*np.exp(-2*np.pi*1j*n*k/N)
  return res
```

In [103]:

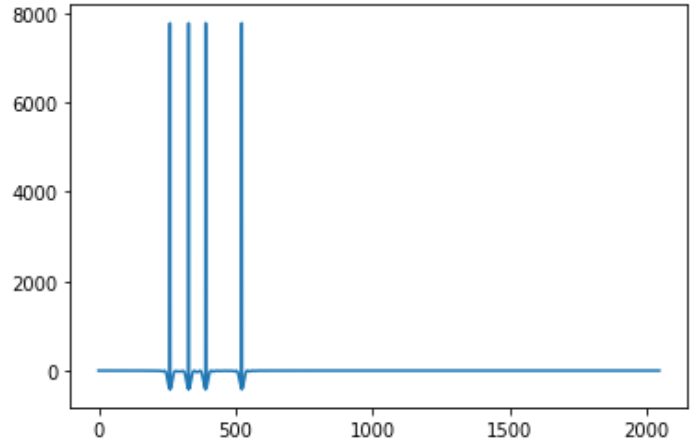
```
N = framerate
amps = [dft2(wave.ys,k,N) for k in range(N)]
print(amps[:10])
plt.plot(ts[:len(ts)//8]*framerate,amps[:len(amps)//8])
```

```
[(-0.26706550802703877+0.01844729796598157j), (-0.24759459858563543+0.017017196160982597j
), (-0.23154711797433442+0.015685893915070605j), (-0.22075728425109406+0.0145905367704229
1j), (-0.21656165848327918+0.013845846890898388j), (-0.21965862553436982+0.01353269388691
5456j), (-0.2300295469451884+0.01368993269651507j), (-0.24693034360380778+0.0143103313772
23259j), (-0.26895537368763095+0.015341014180420 59j), (-0.2941682916755873+0.016688403004
095624j)]
```

```
/usr/local/lib/python3.7/dist-packages/matplotlib/cbook/__init__.py:1317: ComplexWarning:
Casting complex values to real discards the imaginary part
  return np.asarray(x, float)
```

Out[103]:

```
[<matplotlib.lines.Line2D at 0x7f9383234950>]
```



上面這段code跑超久= =，可以去泡杯咖啡。

(3) FFT實作

In [104]:

```
def fft(ys,k,N):
  # if len(ys)<=10:
  #   return dft2(ys,k,N)
  if len(ys)==1:
    return ys[0]
  ys_odd = ys[::2]
  ys_even = ys[1::2]

  k_half = k
  if k>=N//2:
    k_half = k-N//2

  return fft(ys_even,k_half,N//2) + np.exp(-2*np.pi*1j*k/N)*fft(ys_odd,k_half,N//2)
```

In [105]:

```
N = framerate
# amps = [fft(wave.ys,k,N) for k in range(N)]
print(len(wave.ys))
amps = FFT(wave.ys)
print(amps[:10])
plt.plot(ts[:len(ts)//8]*framerate,amps[:len(amps)//8])
```

```
16384
```

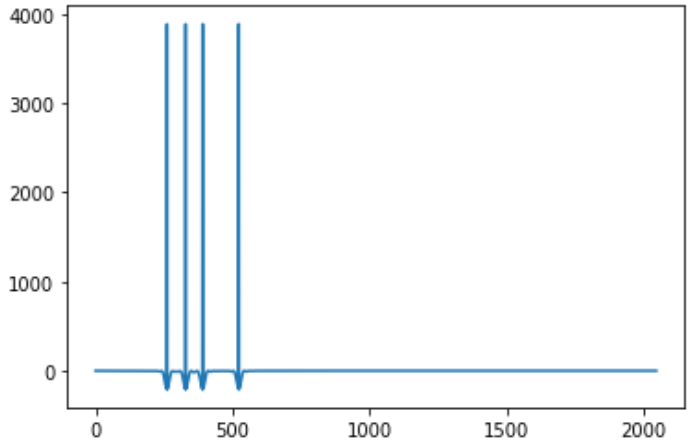
```
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:3: ComplexWarning: Casting c
omplex values to real discards the imaginary part
  This is separate from the ipykernel package so we can avoid doing imports until
```

```
[-0.267+0.j    -0.268-0.001j -0.27 -0.003j -0.273-0.004j -0.276-0.004j
 -0.281-0.004j -0.285-0.004j -0.29 -0.003j -0.294-0.002j -0.297-0.001j]
```

```
/usr/local/lib/python3.7/dist-packages/matplotlib/cbook/__init__.py:1317: ComplexWarning:
Casting complex values to real discards the imaginary part
  return np.asarray(x, float)
```

Out[105]:

```
[<matplotlib.lines.Line2D at 0x7f9383214590>]
```



以上為自行實作之FFT，依照課本理論公式計算得出，可以看到，與np.fft大致相同。也與dft相同。

In [106]:

```
def FFT(x):
    """A recursive implementation of the 1D Cooley-Tukey FFT"""
    x = np.asarray(x, dtype=float)
    N = x.shape[0]
    # print(x.shape)

    if N <= 1:  # this cutoff should be optimized
        return x[0]

    elif N % 2 > 0:
        raise ValueError("size of x must be a power of 2")

    else:
        X_even = FFT(x[::2])
        X_odd = FFT(x[1::2])
        factor = np.exp(-2j * np.pi * np.arange(N) / N)
        return np.concatenate([X_even + factor[:int(N / 2)] * X_odd,
                               X_even + factor[int(N / 2):] * X_odd])
```

In [107]:

```
N = framerate
# amps = [fft(wave.ys,k,N) for k in range(N)]
print(len(wave.ys))
amps = FFT(wave.ys)
print(amps[:10])
plt.plot(ts[:len(ts)//8]*framerate,amps[:len(amps)//8])
```

```
16384
```

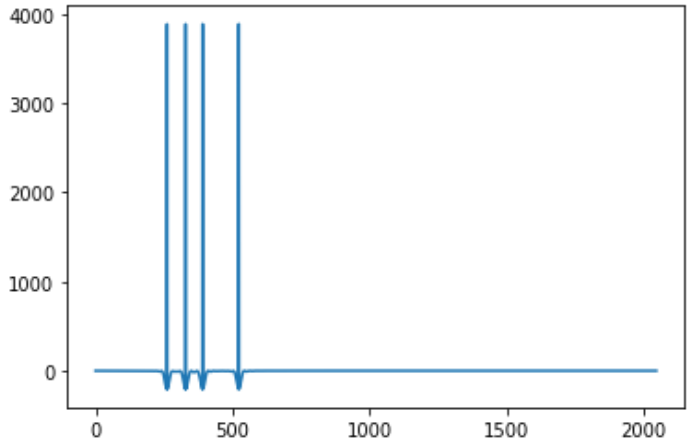
```
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:3: ComplexWarning: Casting c
omplex values to real discards the imaginary part
  This is separate from the ipykernel package so we can avoid doing imports until
```

```
[-0.267+0.j    -0.268-0.001j -0.27 -0.003j -0.273-0.004j -0.276-0.004j
 -0.281-0.004j -0.285-0.004j -0.29 -0.003j -0.294-0.002j -0.297-0.001j]
```

```
/usr/local/lib/python3.7/dist-packages/matplotlib/cbook/__init__.py:1317: ComplexWarning:
Casting complex values to real discards the imaginary part
  return np.asarray(x, float)
```

Out[107]:

```
[<matplotlib.lines.Line2D at 0x7f9383174d10>]
```



以上為參考網路教學對於FFT的實作，與np.fft大致。與我自行依造公式撰寫的也相同。

In [108]:

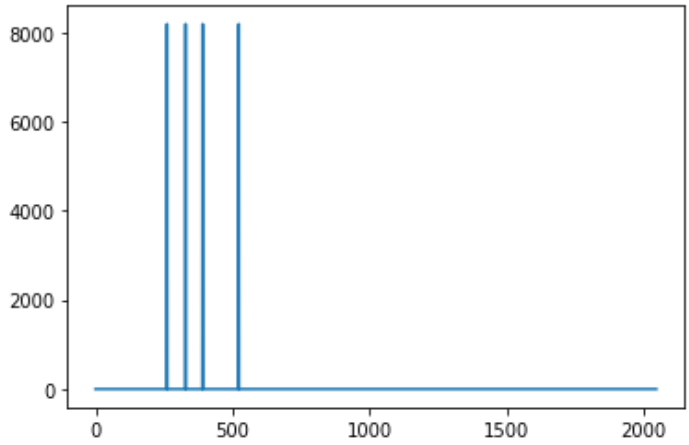
```
amps = np.fft.fft(ys)
print(amps[:10])
plt.plot(ts[:len(ts)//8]*framerate,amps[:len(amps)//8])
```

```
[-0.-0.j -0.-0.j -0.-0.j -0.-0.j -0.+0.j -0.-0.j -0.-0.j  0.-0.j -0.-0.j
  0.-0.j]
```

```
/usr/local/lib/python3.7/dist-packages/matplotlib/cbook/__init__.py:1317: ComplexWarning:
Casting complex values to real discards the imaginary part
  return np.asarray(x, float)
```

Out[108]:

```
[<matplotlib.lines.Line2D at 0x7f93831580d0>]
```



最後是np.fft的結果，與dft的計算結果都相同。