Floating Point

Adapted from CMU course 15-213: Introduction to Computer Systems

Instructor:

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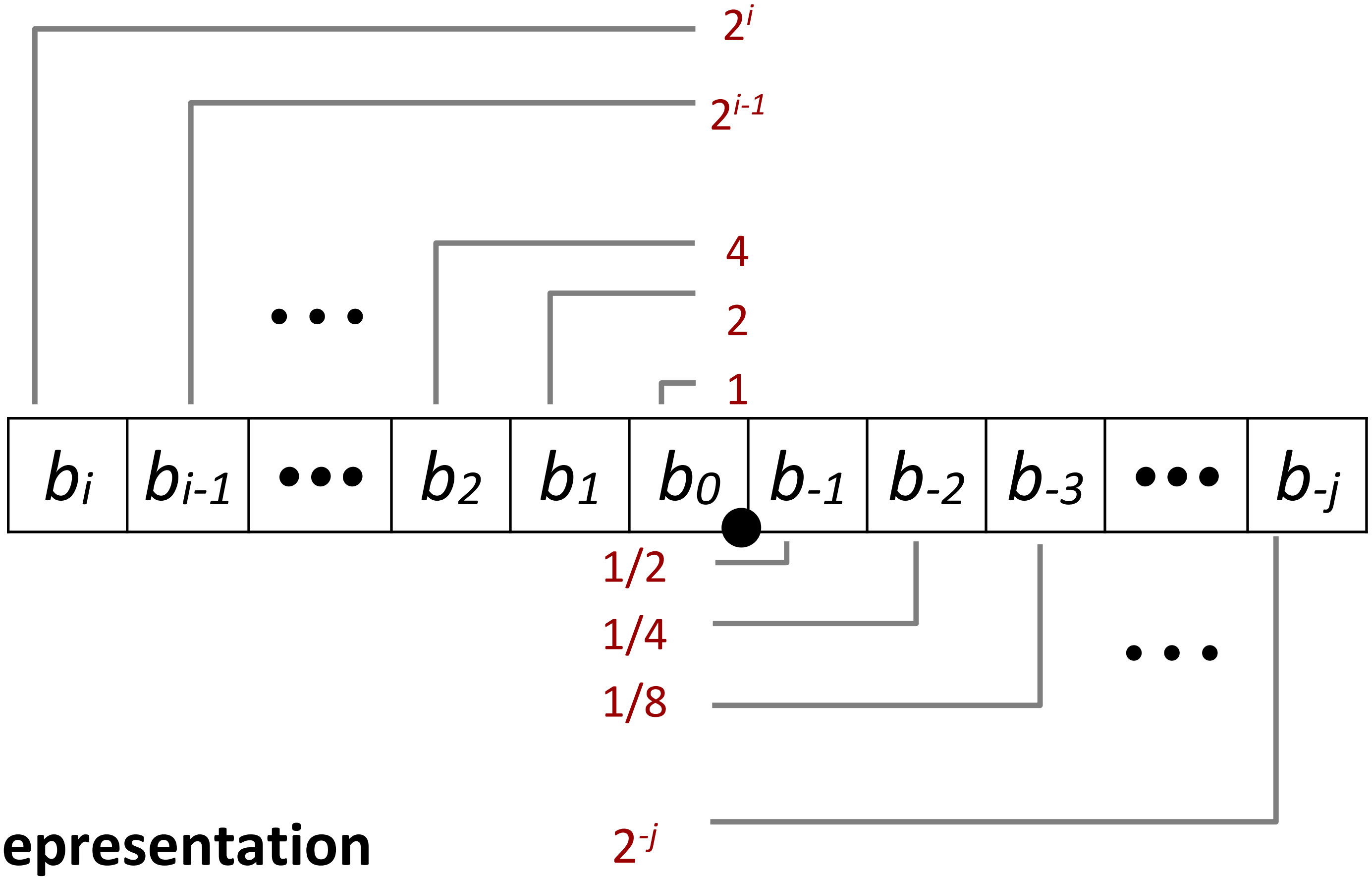
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



Representation

- Bits to right of "binary point" represent fractional powers of 2
- Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

Value		Representation	
5 3/4	= 23/4	101.11_2	= 4 + 1 + 1/2 + 1/4
2 7/8	= 23/8	10.111_2	= 2 + 1/2 + 1/4 + 1/8
1 7/16	= 23/16	1.0111_2	= 1 + 1/4 + 1/8 + 1/16

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation #1**
  - Can only exactly represent numbers of the form x/2^k
    - Other rational numbers have repeating bit representations
  - Value / Representation

Value	Representation
1/3	`0.0101010101[01]`…$_2$
1/5	`0.001100110011[0011]`…$_2$
1/10	`0.0001100110011[0011]`…$_2$

- **Limitation #2**
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
  - Some CPUs don’t implement IEEE 754 in full
    e.g., early GPUs
- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

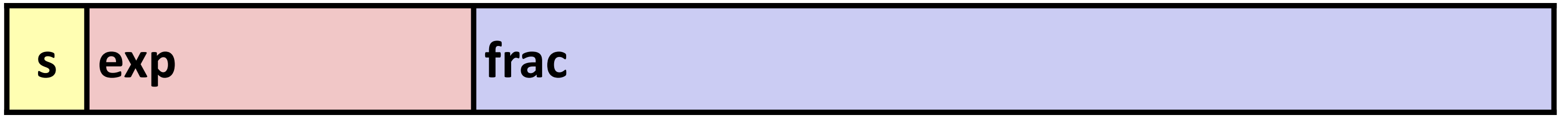
Example:
15213_{10} = $(-1)^0$ x 1.1101101101101_2 x 2^{13}

- **Numerical Form:**

$$(-1)^s\, M\;\; 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two

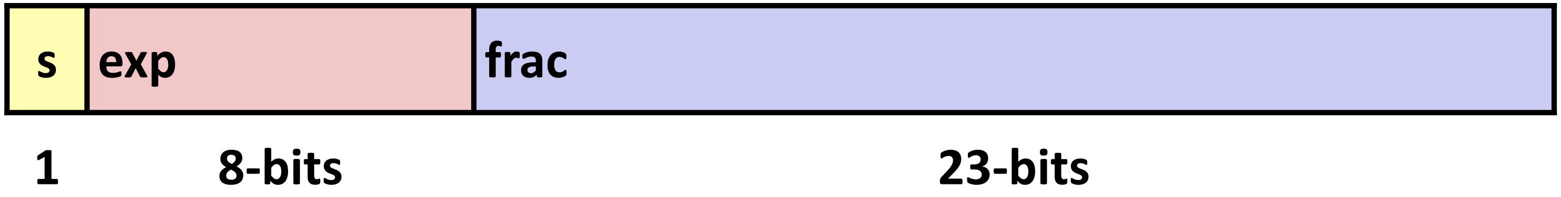
- **Encoding**
  - MSB `s` is sign bit ***s***
  - **exp** field encodes ***E*** (but is not equal to E)
  - **frac** field encodes ***M*** (but is not equal to M)



Precision options

- **Single precision: 32 bits**

  ≈ 7 decimal digits, $10^{\pm 38}$

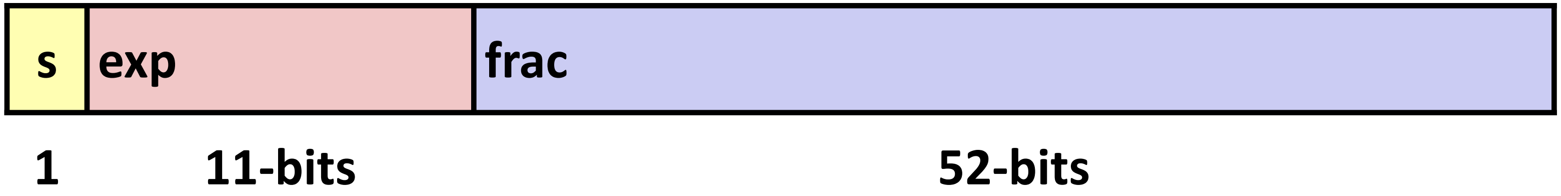


s exp frac

1 8-bits 23-bits

- **Double precision: 64 bits**

  ≈ 16 decimal digits, $10^{\pm 308}$



s exp frac

1 11-bits 52-bits

- **Other formats: half precision, quad precision**

"Normalized" Values

$v = (-1)^s M 2^E$

- **When: exp ≠ 000…0 and exp ≠ 111…1**

- **Exponent coded as a *biased* value: *E* = *Exp* – *Bias***
  - *Exp*: unsigned value of exp field
  - *Bias* = 2^{k-1} - 1, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- **Significand coded with implied leading 1: *M* =** $1.xxx…x_2$
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s \, M \, 2^E$

$E = Exp - Bias$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  $M \quad = \quad 1.\underline{1101101101101}_2$

  `frac=` $\quad \underline{1101101101101}0000000000_2$

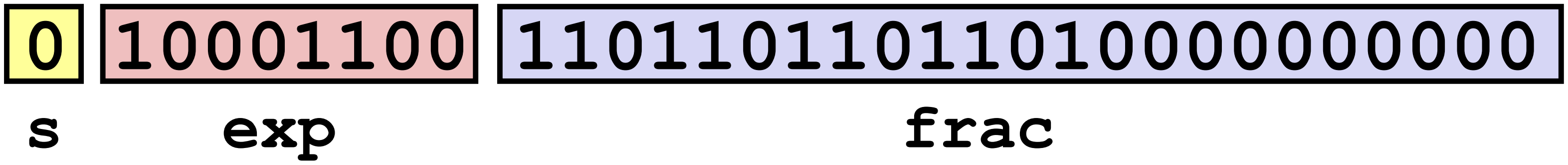
- **Exponent**

  $E \quad = \quad 13$

  $Bias \quad = \quad 127$

  $Exp \quad = \quad 140 \quad = \quad 10001100_2$

- **Result:**



s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s \, M \, 2^E$

$E = 1 - Bias$

- **Condition:** exp = 000…0

- **Exponent value:** ***E*** **= 1 – Bias (instead of 0 –** ***Bias, why?*****)**
- **Significand coded with implied leading 0:** ***M*** **= 0.xxx…x**$_2$
  - `xxx…x`: bits of `frac`
- **Cases**
  - `exp` = 000…0, `frac` = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers closest to 0.0
    - Equispaced

Special Values

- **Condition: `exp` = `111…1`**

- **Case: `exp` = `111…1`, `frac` = `000…0`**
  - **Represents value ∞ (infinity)**
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = `111…1`, `frac` ≠ `000…0`**
  - **Not-a-Number (NaN)**
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

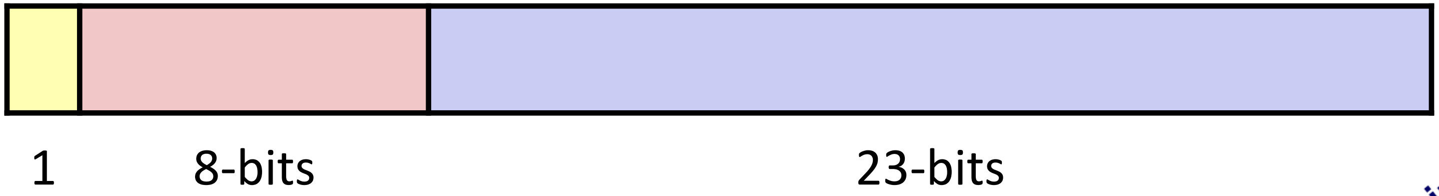
C float Decoding Example

$$v = (-1)^s \, M \, 2^E$$
$$E = Exp - Bias$$

$Bias = 2^{k-1} - 1 = 127$

float: 0xC0A00000

binary: ____ ____ ____ ____ ____ ____ ____ ____



E = **-> Exp =** (decimal)

S =

M =

$v = (-1)^s \, M \, 2^E =$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

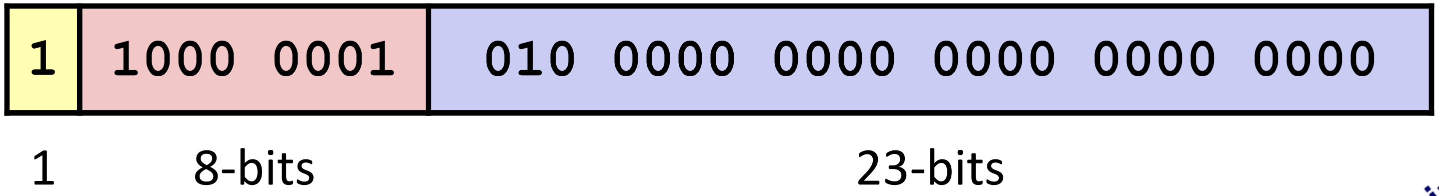
C float Decoding Example

$v = (-1)^s M 2^E$

$E = Exp - Bias$

float: `0xC0A00000`

binary: 1100 0000 1010 0000 0000 0000 0000 0000



1 | 1000 0001 | 010 0000 0000 0000 0000 0000

1 | 8-bits | 23-bits

E = -> Exp = (decimal)

S =

M = 1 .

$v = (-1)^s M 2^E =$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

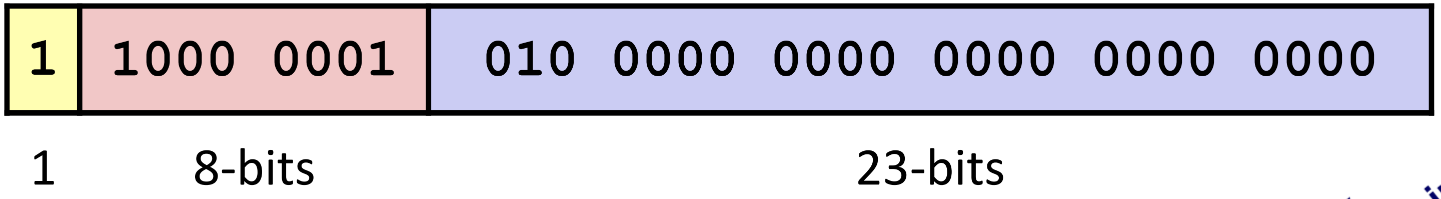
C float Decoding Example

$$v = (-1)^s \, M \, 2^E$$
$$E = Exp - Bias$$

Bias = $2^{k-1} - 1 = 127$

float: `0xC0A00000`

binary: 1100 0000 1010 0000 0000 0000 0000 0000



Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

E = 129 -> Exp = 129 – 127 = 2 (decimal)

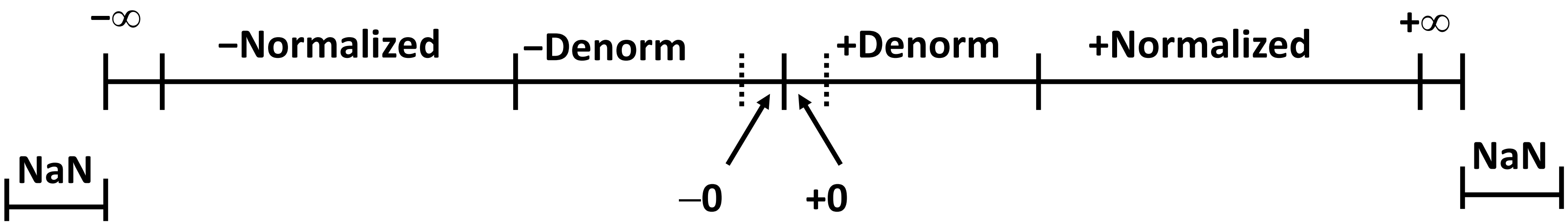
S = 1 -> negative number

M = 1.010 0000 0000 0000 0000 0000

= 1 + 1/4 = 1.25

$v = (-1)^s \, M \, 2^E = (-1)^1 * 1.25 * 2^2 = -5$

Visualization: Floating Point Encodings



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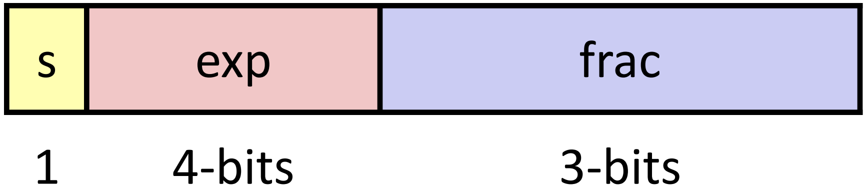
Tiny Floating Point Example



- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

$v = (-1)^s\, M\, 2^E$

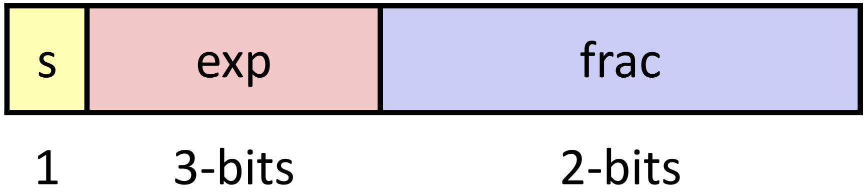
n: E = Exp – Bias

d: E = 1 – Bias

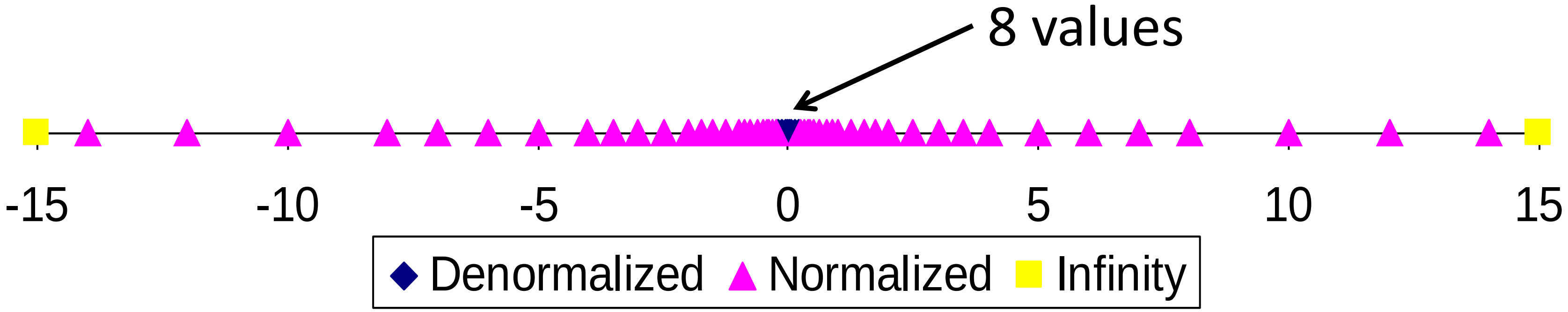
	s	exp	frac	E	Value	
	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized numbers	0	0000	010	-6	2/8*1/64 = 2/512	$(-1)^0(0+1/4)*2^{-6}$
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	$(-1)^0(1+1/8)*2^{-6}$
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized numbers	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

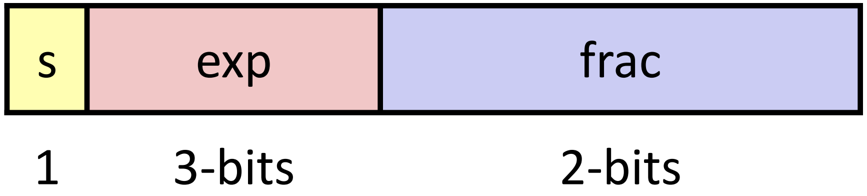


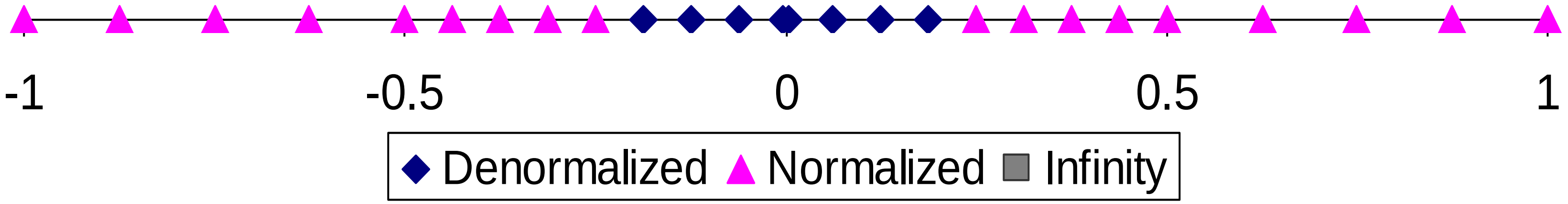
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Special Properties of the IEEE Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield? The answer is complicated.
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- $x +_f y = \mathrm{Round}(x + y)$

- $x \times_f y = \mathrm{Round}(x \times y)$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
Towards zero	$1 ↓	$1 ↓	$1 ↓	$2 ↓	–$1 ↑
Round down (−∞)	$1 ↓	$1 ↓	$1 ↓	$2 ↓	–$2 ↓
Round up (+∞)	$2 ↑	$2 ↑	$2 ↑	$3 ↑	–$1 ↑
Nearest Even (default)	$1 ↓	$2 ↑	$2 ↑	$2 ↓	–$2 ↓

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - C99 has support for rounding mode management
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is **0**
  - "Half way" when bits to right of rounding position = **100**…$_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

- **$(-1)^{s1}\ M1\ \ 2^{E1}\ \ \text{x}\ \ (-1)^{s2}\ M2\ \ 2^{E2}$**
- **Exact Result: $(-1)^{s}\ M\ \ 2^{E}$**
  - Sign *s*: *s1 ^ s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1* + *E2*
- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit **`frac`** precision
- **Implementation**
  - Biggest chore is multiplying significands

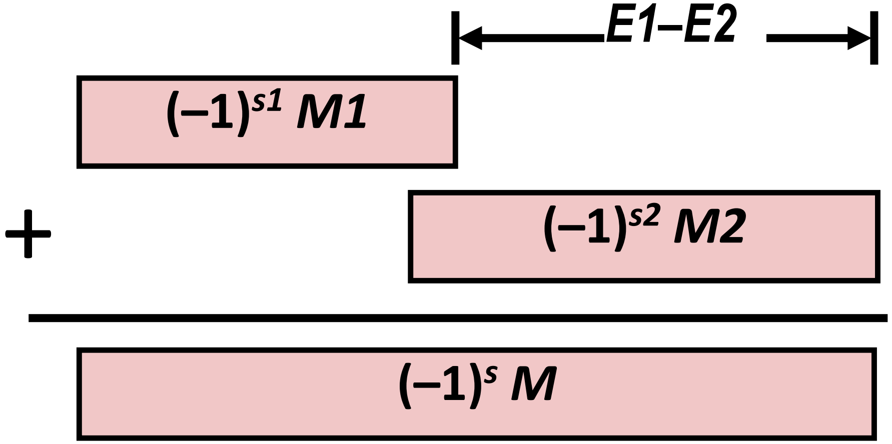
4 bit mantissa: $1.010*2^{2}$ **x** $1.110*2^{3} = 10.0011*2^{5}$

$= 1.00011*2^{6} = 1.001*2^{6}$

Floating Point Addition

- $(-1)^{s1}\, M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\, M2\ \ 2^{E2}$
  - Assume $E1 > E2$
- **Exact Result:** $(-1)^{s}\, M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: $E1$
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Get binary points lined up



$$1.010*2^{2} + 1.110*2^{3} = (0.1010 + 1.1100)*2^{3}$$
$$= 10.0110 * 2^{3} = 1.00110 * 2^{4} = 1.010 * 2^{4}$$

Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? *Yes*
    - But may generate infinity or NaN
  - Commutative? *Yes*
  - Associative? *No*
    - Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0 is additive identity? *Yes*
  - Every element has additive inverse? *Almost*
    - Yes, except for infinities & NaNs
- **Monotonicity**
  - a ≥ b ⇒ a+c ≥ b+c? *Almost*
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

Compare to Commutative Ring

- Closed under multiplication? ***Yes***
  - But may generate infinity or NaN
- Multiplication Commutative? ***Yes***
- Multiplication is Associative? ***No***
  - Possibility of overflow, inexactness of rounding
  - Ex: `(1e20*1e20)*1e-20= inf`, `1e20*(1e20*1e-20)= 1e20`
- 1 is multiplicative identity? ***Yes***
- Multiplication distributes over addition? ***No***
  - Possibility of overflow, inexactness of rounding
  - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 = NaN`

Monotonicity

- $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? ***Almost***
  - Except for infinities & NaNs

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Floating Point in C

- **C Guarantees Two Levels**
  - **`float`** single precision
  - **`double`** double precision
- **Conversions/Casting**
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

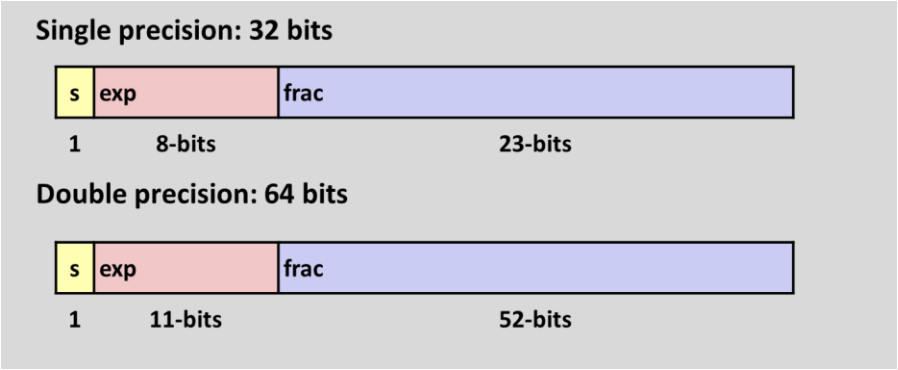
```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

- `x == (int)(float) x` ✗
- `x == (int)(double) x` ✓
- `f == (float)(double) f` ✓
- `d == (double)(float) d` ✗
- `f == -(-f);` ✓
- `2/3 == 2/3.0` ✗
- `d < 0.0      ⇒   ((d*2) < 0.0)` ✓
- `d > f        ⇒   -f > -d` ✓
- `d * d >= 0.0` ✓
- `(d+f)-d == f` ✗

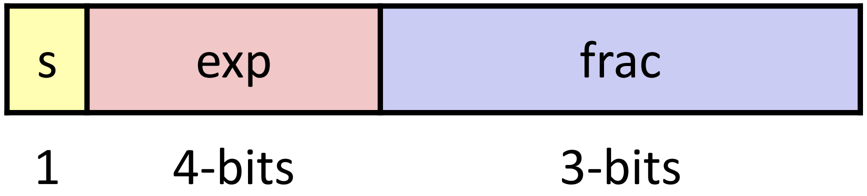
Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers



HOW ROUNDING WORKS

Creating Floating Point Number



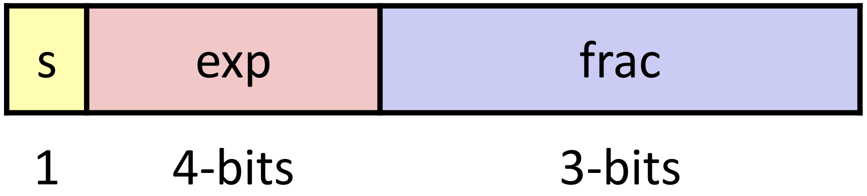
- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

`128`	`10000000`
`15`	`00001101`
`33`	`00010001`
`35`	`00010011`
`138`	`10001010`
`63`	`00111111`

Normalize



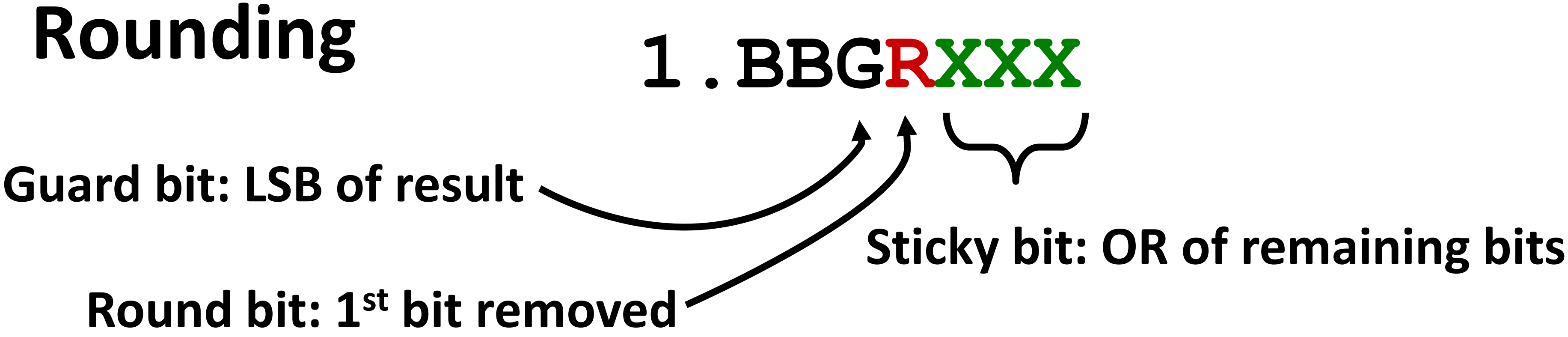
s exp frac

1 4-bits 3-bits

- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
`128`	`10000000`	`1.0000000`	`7`
`15`	`00001101`	`1.1010000`	`3`
`17`	`00010001`	`1.0001000`	`4`
`19`	`00010011`	`1.0011000`	`4`
`138`	`10001010`	`1.0001010`	`7`
`63`	`00111111`	`1.1111100`	`5`

Rounding



1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round up conditions**
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Numeric Result*
`128`	`1.000`	`7`		`128`
`15`	`1.101`	`3`		`15`
`17`	`1.000`	`4`		`16`
`19`	`1.010`	`4`		`20`
`138`	`1.001`	`7`		`134`
`63`	`10.000`	`5`	`1.000/6`	`64`

Additional Slides

Interesting Numbers `{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	**00…00**	**00…00**	**0.0**
Smallest Pos. Denorm.	**00…00**	**00…01**	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Single ≈ 1.4 x 10^{-45}			
Double ≈ 4.9 x 10^{-324}			
Largest Denormalized	**00…00**	**11…11**	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Single ≈ 1.18 x 10^{-38}			
Double ≈ 2.2 x 10^{-308}			
Smallest Pos. Normalized	**00…01**	**00…00**	$1.0 \times 2^{-\{126,1022\}}$
Just larger than largest denormalized			
One	**01…11**	**00…00**	**1.0**
Largest Normalized	**11…10**	**11…11**	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
Single ≈ 3.4 x 10^{38}			
Double ≈ 1.8 x 10^{308}			