Course Notes

COMP130023.01 Theory of Computation

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What do we study?

- What is computation, i.e., computation model
- Finite automaton, context-free grammar
- Turing machine (= algorithm)
- Computability
- Complexity class (P, NP, PSPACE, EXP, L, NL, ...)
- NP completeness, reduction

Contents

1 Introduction

1.1 Big-O Notation

***def* 1.1.** *Let* $f, g : \mathbb{N} \to \mathbb{R}$

- *Write* $f = O(g)$ *if* $(\exists c > 0)(\exists N)(\forall n \geq N)(|f(n)| \leq cg(n))$.
- *Write* $f = \Omega(g)$ *if* $(\exists c > 0)(\exists N)(\forall n \geq N)(|f(n)| \geq cg(n))$.
- *Write* $f = \Theta(g)$ *if* $(\exists c_1, c_2 > 0)(\exists N)(\forall n \geq N)(c_1 g(n) \leq |f(n)| \leq c_2 g(n))$.
- *Write* $f = o(g)$ *if* $(\forall \epsilon > 0)(\exists N)(\forall n \geq N)(|f(n)| \leq \epsilon g(n))$.

Big-O Notation is the most commonly used one among the four.

***e.g.* 1.1.** $f(n) = 6n^4 - 3n^3 + 5 \Rightarrow f(n) = O(n^4)$

Proof. $|6n^4 - 3n^3 + 5| \leq 6n^4 + 3n^4 + 5n^4 = 13n^4$ □

***e.g.* 1.2.** $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = O(n^3)$

***exercises* 1.1.** *Write in big-O notation:*

1. $5 + 0.001n^3 + 0.25n$
2. $500n + 100n^{1.5} + 50n \log_{10} n$
3. $n^2 \log_2 n + n(\log_2 n)^2$
4. $3 \log_8 n + \log_2(\log_2 n)$

solution $O(n^3); O(n^1.5); O(n^2 \log n); O(\log n)$ □

***prop* 1.1.**

- $f_1 = O(g_1), f_2 = O(g_2) \Rightarrow f_1 f_2 = O(g_1 g_2)$
- $f(O(g)) = O(fg)$
- $f_1 = O(g_1), f_2 = O(g_2) \Rightarrow f_1 + f_2 = O(max(g_1, g_2))$
- $f_1 = O(g), f_2 = O(g) \Rightarrow f_1 + f_2 = O(g)$
- $f = O(g) \Rightarrow kf = O(g)$

often encountered:

- *constant:* $O(1)$
  - $\sum\limits_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$
  - $\sum\limits_{k=1}^{n} \frac{1}{k^2} = O(1)$
  - $\sum\limits_{k=1}^{n} \frac{1}{k \ln k} = \ln \ln n + O(1)$
- *double logarithmic:* $O(\log \log n)$
- *logarithmic:* $O(\log n)$
- *polylogarithmic:* $O((\log n)^c), c > 0$
- *linear:* $O(n)$
- *quasilinear:* $O(n \log^c n), O(n \log^{O(1)} n)$
- *quadratic:* $O(n^2)$

***def* 1.2.** $\omega(g), \theta(g)$

- $f = \omega(g)\ if (\forall c > 0)(\exists N)(\forall n \geq N)(f(n) \geq cg(n))$
- $g = \theta(g)$ *(or equivalently* $f \sim g$*)* $if\ (\forall \epsilon > 0)(\exists N)(\forall n \geq N)(|f(n) - g(n)| < \epsilon g(n))$

1.2 Alphabets and Languages

***def* 1.3.** (*alphabet*) *An alphabet is a set of symbols*

- *Roman alphabet* $: a, b, c, d, \cdots, z$
- *binary alphabet* $: 0, 1$

***def* 1.4.** *(string and its length)*

*A **string** (over an alphabet) is a finite sequence of symbols from the alphabet.*
***Empty string** is string of no symbols, denoted by* ε.

The set of all string is denoted by Σ^*. *Denote by* Σ^n *the set of all **string of length*** n.
So, $\Sigma^* = \cup_{n \geq 0} \Sigma^n$.

Denote the length of a string w *by* $|w|$

***e.g.* 1.3.** $\{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, \cdots\}, |\varepsilon| = 0, |0110| = 4$

***def* 1.5.** *(concatenation) Two strings over the same alphabet can be combined by the operation of **concatenation**. The concatenation of* x *and* y *is denoted by* xy.

***def* 1.6.** *(substring, suffix, prefix)*

A string v is a substring of w if $\exists$ strings x and y such that $w = xvy$.

*If $w = xv$ for some x, then v is a **suffix** of w.*

*If $w = vy$ for some y, then v is a **prefix** of w.*

***def* 1.7.** *("power") The string w^i is defined: $w^0 = \varepsilon, w^{i+1} = w^i w, i \in \mathbb{N}$*

***e.g.* 1.4.** $01^0 = \varepsilon, 01^1 = 01, 01^2 = 0101$

***def* 1.8.** *(reversal) The reversal of a string w,denoted by w^R,is the string "spelled backwards"*

A formal definition can be given by induction on length:

1. *If $w = \varepsilon, w^R = w = \varepsilon$*

2. *If $|w| = n + 1$,where $w = ua, a \in \Sigma$,then $w^R = au^R$*

***def* 1.9.** *(language) **Language** is a set of strings over an alphabet, That is, $L \subseteq \Sigma^*$.*

For example,$\emptyset, \Sigma^*, \Sigma$ are all languages

- $\sigma = \{0, 1\}$
- $Even = \{0, 10, 100, 110, \cdots, \}$
- $Odd = \{1, 11, 101, \cdots\}$
- $Prime = \{10, 11, 101, 111, \cdots\}$
- $Palindrome = \{w | w^R = w\} = \{\varepsilon, 0, 1, 00, 11, \cdots\}$

***def* 1.10.** *(complement, binary language operations)*

*Let L be a language. The **complement** of L, denoted by $\overline{L}$, is $\Sigma^* - \overline{L}$. So $\overline{\overline{L}} = L$.*

*Note that since L is a set, we can define **union**($\cup$), **interchapter***($\cap$) and difference.*

*The **concatenation** of L_1 and L_2 is defined by $L_1 L_2 = w \in \Sigma^*, w = xy, \exists x \in L_1, y \in L_2$*

1.3 Encoding of Problems

1.3.1 Examples

1. *(**Integer multiplication**) Given two non-negative integers x, y, compute xy.*

2. *(**Primality testing**) Given $n \in \mathbb{N}$, decide if n is a prime.*

3. *(**Hamiltonian cycle**) Given an undirected graph G, test if G has a Hamiltonian cycle.*

1.3.2 Analysis

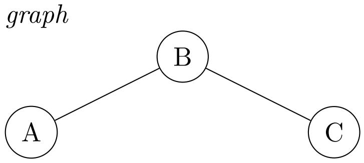
- ***Decision*** *problem: 2,3*
- ***Computation*** *problem: 1*

1.3.3 Conclusion

- *By* ***encoding****, decision problem is language. Any computation problem is a function from* Σ^* *to* Σ^**. Our course only concerns decision problem, namely language.*
- *By* ***preprocessing****, one can switch between encodings.*

e.g. **1.5.**

- *graph*

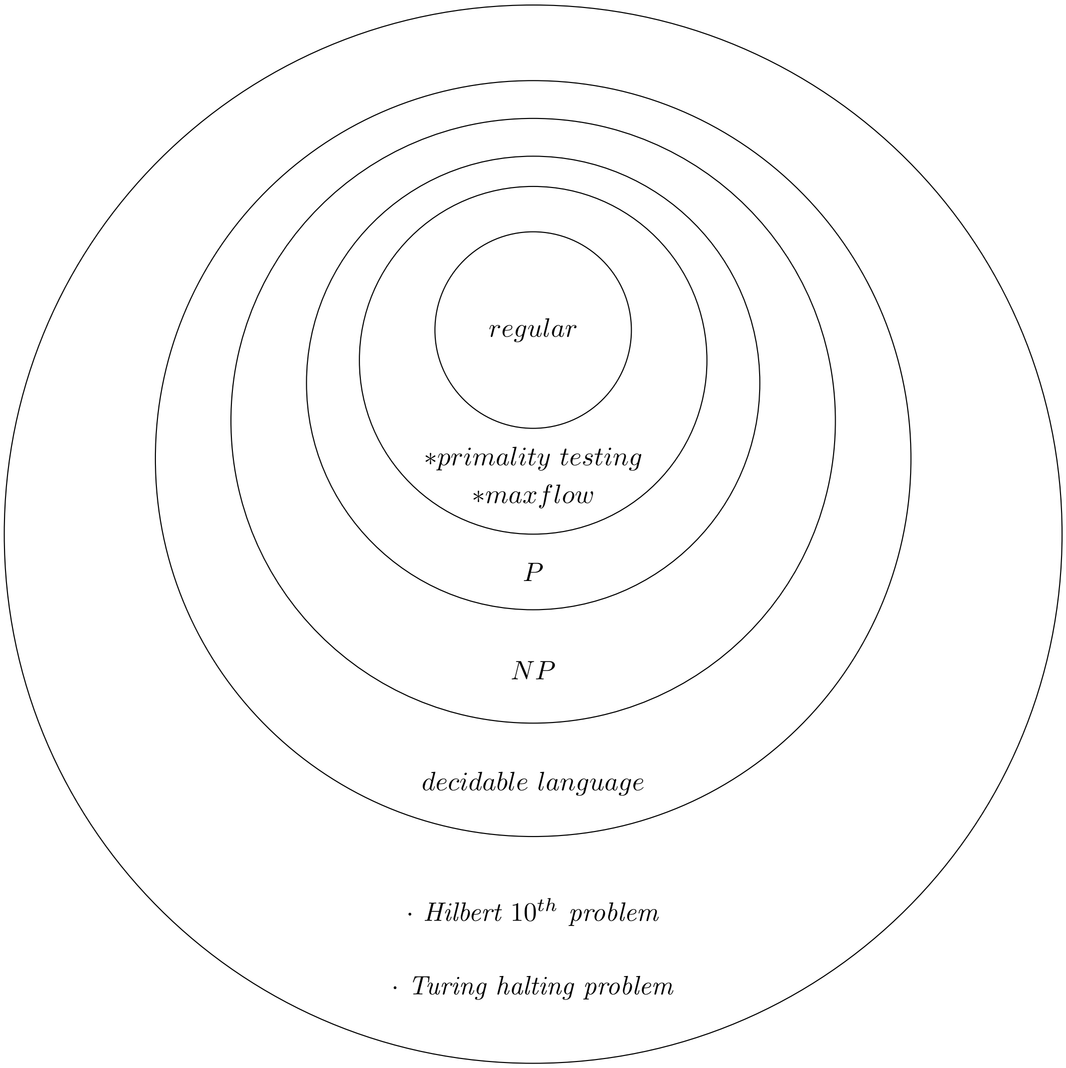


- *adjacency matrix*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- *adjacency list*

  (1,2),(1,3),$\cdots$



all languages $L \subseteq \Sigma^* = \{\varepsilon, 0, 1, 00, 01, \cdots\}$

remark.

NP: *problems that are efficiently verifiable*

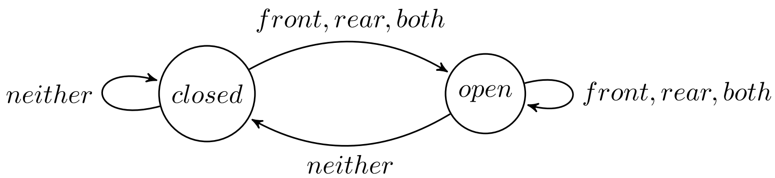
P: *problems that are efficiently solvable, i.e. solvable in* $n^{O(1)}$ *time*

regular language: *problems that are solvable without memory,i.e. solvable by finite automation*

2 Automata and Language

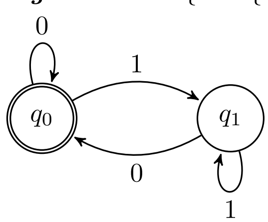
2.1 Finite Automaton

***e.g.* 2.1.** *Automatic Door*



	$front$	$rear$	$both$	$neither$
$front$	✓	×	✓	×
$rear$	×	✓	✓	×

***e.g.* 2.2.** $L = \{w \in \{0,1\}^* | w = w_1 w_2 \cdots w_n, w_n = 0\}$



***remark*.** q_0 : *accepted state*

$Q = \{q_0, q_1\}, \Sigma = \{0,1\}, \delta : Q \times \Sigma \to Q$

	0	1
q_0	q_0	q_1
q_1	q_0	q_1

***def* 2.1.** *(finite automaton)*

*A **finite automaton** is a 5-tuple* $(Q, \Sigma, \delta, q_0, F)$*, where:*

1. Q *is a finite set called the states*

2. Σ *is the alphabet*

3. $\delta : Q \times \Sigma \to Q$ *is the transition function*

4. q_0 *is the start state*

5. $F \subseteq Q$ *is the set of accept states*

***def* 2.2.** *Let* $M = (Q, \Sigma, \delta, q_0, F)$ *be a finite automaton, let* $w = w_1 w_2 \cdots w_n$ *be a string, where each* $w_i \in \Sigma$.

Then M accept w if there is a sequence of states $r_0, r_1, \cdots, r_n \in Q$, *such that:*

1. $r_0 = q_0$

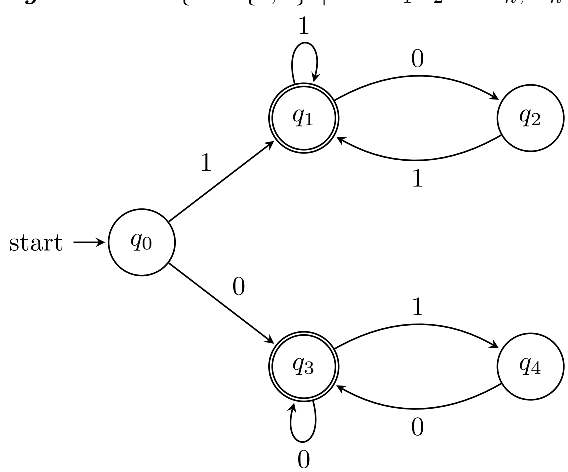
2. $\delta(r_i, w_{i+1}) = r_{i+1}$, *for* $i = 0, 1, 2, \cdots, n-1$

3. $r_n \in F$

***def* 2.3.** *If L is the set of strings that M accepts, we say L is the language of M, and write L(M) = L, we say M recognizes/decides/accepts L.*

If M accepts no string, it recognizes one language namely, the empty language.

***e.g.* 2.3.** $L = \{w \in \{0,1\}^* | w = w_1 w_2 \cdots w_n, w_n = w_1\}$



2.2 Regular Language

def* 2.4.** *(regular language)* $L \subseteq \Sigma^*$ *is a* ***regular language *if there is a finite automaton that accepts L*

Let $A, B \subseteq \Sigma^*$, *define:*

- *(union)* $A \cup B = \{x \in \Sigma^* | x \in A \text{ or } x \in B\}$

- *(concatenation)* $AB = \{xy | x \in A, y \in B\}$
- *(star)* $A^* = \{x_1 x_2 \cdots x_k | k \geq 0, x_1, x_2, \cdots, x_k \in A\}$

***thm* 2.5.** *If A_1, A_2 are regular languages, so is $A_1 \cup A_2$*

Proof. Let $M_1 = (Q_1, \Sigma_1, \delta_1, q_{10}, F_1)$ accepts A_1, $M_2 = (Q_2, \Sigma_2, \delta_2, q_{20}, F_2)$ accepts A_2, construct $M = (Q, \Sigma, \delta, q_0, F)$:

1. $Q = Q_1 \times Q_2 = \{(r_1, r_2) | r_1 \in Q_1, r_2 \in Q_2\}$
2. $\delta : Q \times \Sigma \to Q$ is defined as for each $(r_1, r_2) \in Q$, and each $a \in \Sigma$, let $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
3. $q_0 = (q_{10}, q_{20})$
4. $F = \{(r_1, r_2) | r_1 \in F_1 \text{ or } r_2 \in F_2\}$

□

***remark*.** *so is* $A \cap B = \overline{\overline{A} \cup \overline{B}}$[1]

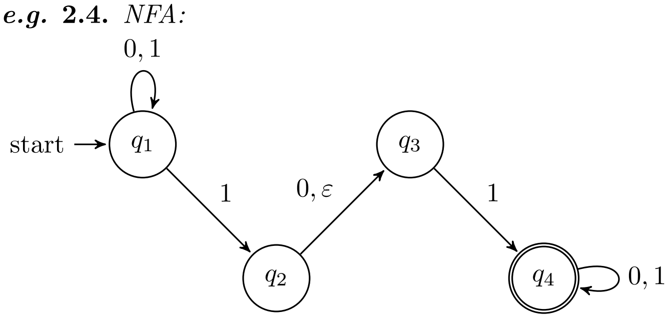
2.3 DFA and NFA

***thm* 2.6.** *If A_1, A_2 are regular languages, so is $A_1 A_2$*

- ***DFA***: *deterministic finite automaton*
- ***NFA***: *nondeterministic...*

If at least one of these processes accepts, then the entire computation accepts.

***e.g.* 2.4.** *NFA:*



input: 010110

[1] *proof of the closure under complement will be mentioned later*

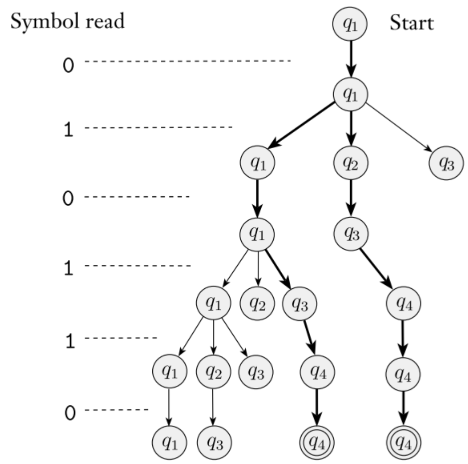


Figure 1: *The computation of NFA on input 010110*

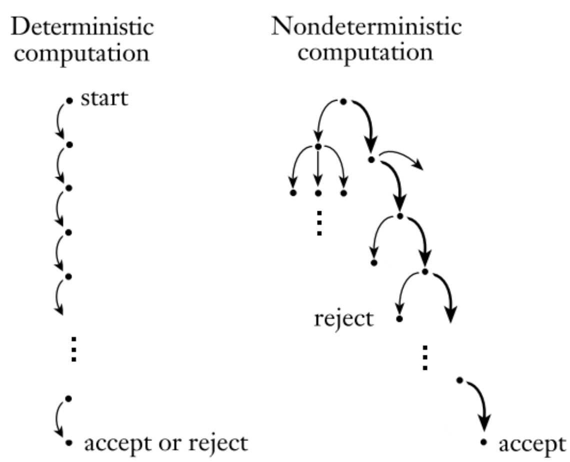
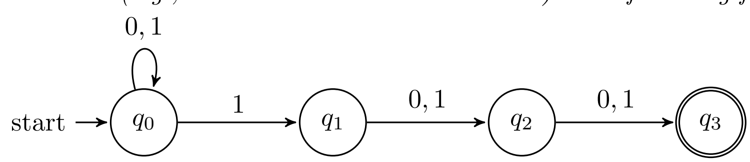


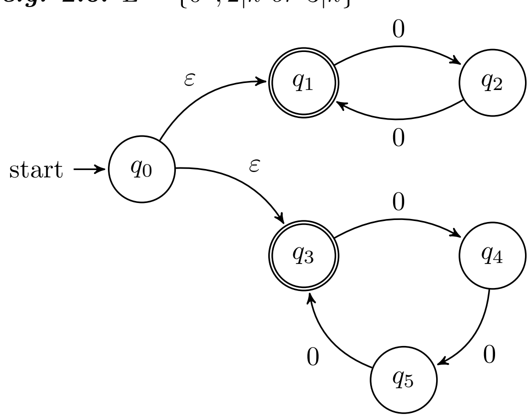
Figure 2: *Deterministic and nondeterministic computations with an accepting branch*

remark. *If a language can be accepted by DFA, then its time complexity is O(n), space complexity is O(1).*

***e.g.* 2.5.** *Let A be the language consisting of all strings over $\{0, 1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA recognizes A.*



***e.g.* 2.6.** $L = \{0^k, 2|k \text{ or } 3|k\}$



***def* 2.7.** *(NFA)*

*An **NFA** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$,where:*

1. *Q is a finite set of states*
2. *Σ is the alphabet*
3. *$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to P(Q)$ is the transive function*
4. *$q_0 \in Q$ is the start state*
5. *$F \subseteq Q$ is the set of accept states*

remark. *$P(Q)$ is the collection of all the subsets of Q (power set)*

- *variant1: $\delta : Q \times \Sigma^* \to Q$*

  We guarantee there is at most one applicable transition.

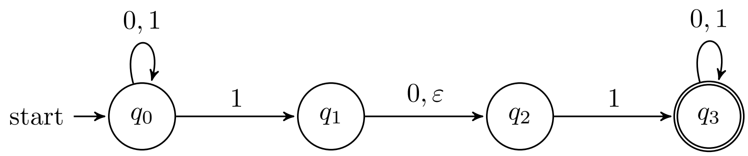
- *variant2: If there are multiple applicable transitions,non-deterministically choose one.*

***def* 2.8.**

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA, and let $w \in \Sigma^*$. Say N accepts w if we can write $w = y_1 y_2 \cdots y_m$, where $y_i \in \Sigma \cup \{\varepsilon\}$, and there exist $r_0, r_1, \cdots, r_m \in Q$, such that:

1. $r_0 = q_0$
2. $r_{i+1} \in \delta(r_i, y_{i+1})$ *for* $i = 0, 1, \cdots, m-1$
3. $r_m \in F$

***e.g.* 2.7.** $Q = \{q_0, q_1, q_2, q_3\}, \Sigma = \{0, 1\}, F = \{q_3\}, \delta : Q \times \Sigma \to Q$



$q \backslash \Sigma$	0	1	ε
q_0	$\{q_0\}$	$\{q_0, q_1\}$	$\emptyset$
q_1	$\{q_2\}$	$\emptyset$	$\{q_2\}$
q_2	$\emptyset$	$\{q_3\}$	$\emptyset$
q_3	$\{q_3\}$	$\{q_3\}$	$\emptyset$

***thm* 2.9.** *Every NFA has an equivalent DFA*

Proof. *Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing A. Construct a DFA $M = (Q', \Sigma, \delta', q_0', F)$ recognizing A:*

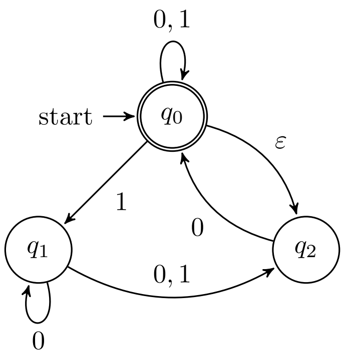
Let $E(R) = \{q \in Q |$ q can be reached from R by traveling along zero or more ε arrows$\}$

Define M as follows:

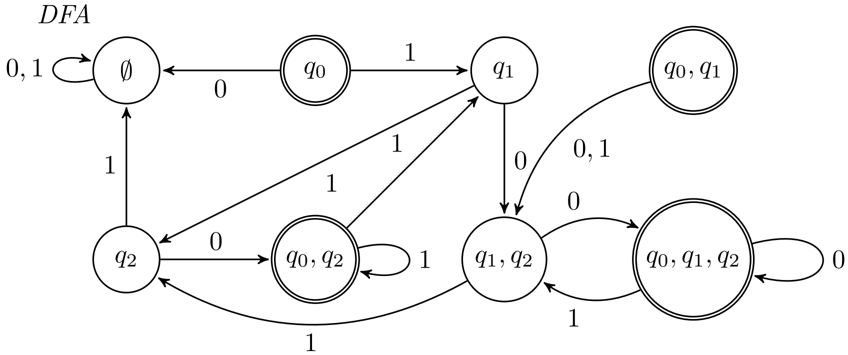
1. $Q' = P(Q)$
2. *For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \{q \in Q : q \in E(\delta(r, a))$ for some $r \in R\} = \bigcup_{r \in R} E(\delta(r, a))$*
3. $q_0' = E(\{q_0\})$
4. $F' = \{R \in Q', R \cap F \neq \emptyset\}$

$\square$

***e.g.* 2.8.** *NFA*



DFA



***corollary* 2.10.** *A language is a regular language iff an NFA recognizes it.*

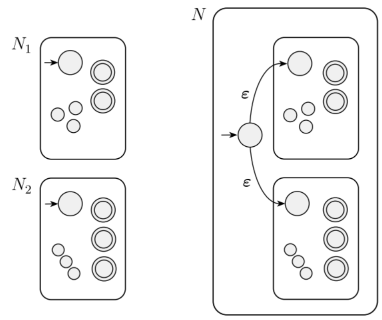


Figure 3: *Construction of an NFA N to recognize $A_1 \cup A_2$*

***thm* 2.5.** *The class of regular languages is closed under union.*

Second proof

Proof. *See figure 3.*

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recignize A_1 and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

Construct $N = (Q, \Sigma, \delta, q, F)$ as follows:

1. $Q = Q_1 \cup Q_2 \cup \{q_0\}$
2. q_0 *is the start state*
3. $F = F_1 \cup F_2$
4. *For $q \in Q, a \in \Sigma \cup \{\varepsilon\}$.*

$$\text{Let } \delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q \\ \delta_2(q,a) & q \in Q \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

□

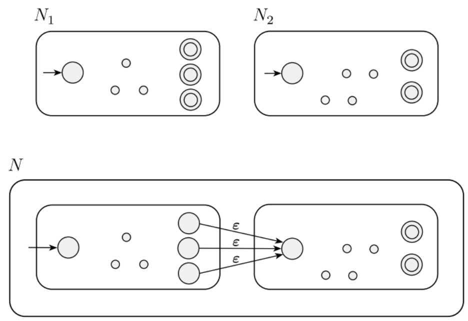


Figure 4: *Construction of an NFA N to recognize A_1A_2*

***thm* 2.11.** *The class of regular languages is closed under concatenation.*

Proof. *See figure 4. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recignize A_1 and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2*

Construct $N = (Q, \Sigma, \delta, q, F)$ as follows:

1. $Q = Q_1 \cup Q_2$
2. *q_1 is the start state*
3. $F = F_2$
4. *For $q \in Q, a \in \Sigma \cup \{\varepsilon\}$.*

$$\text{Let } \delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

□

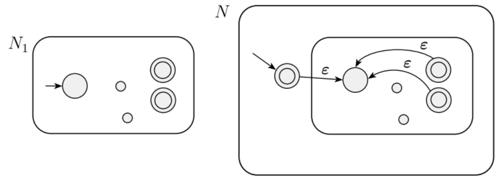


Figure 5: *Construction of an NFA N to recognize A^**

***thm* 2.12.** *The class of regular languages is closed under star.*

Proof. *See figure 5. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recignize A_1*

Construct $N = (Q, \Sigma, \delta, q, F)$ as follows:

1. $Q = Q_1 \cup \{q_0\}$
2. q_0 *is the start state*
3. $F = \{q_0\} \cup F_1$
4. *For $q \in Q, a \in \Sigma \cup \{\varepsilon\}$.*

$$\text{Let } \delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

□

***thm* 2.13.** *The class of regular languages is closed under complement.*

Proof. *Let DFA $M = (Q, \Sigma, \delta, q_0, Q_{accept})$ recognizing A, then $\overline{A} = \Sigma^* - A$, construct $M' = (Q, \Sigma, \delta, q_0, Q'_{accept}), Q'_{accept} = Q - Q_{accept}$, it's easy to prove $L(Q') = \overline{A}$.* □

***def* 2.14.** *(regular expression)*

*R is a **regular expression** if R is:*

1. *a for some $a \in \Sigma$*
2. ε

3. $\emptyset$
4. $R_1 \cup R_2$, *where* R_1, R_2 *are regular expressions*
5. $R_1 R_2$, *where* R_1, R_2 *are regular expressions*
6. R^*, *where* R *is a regular expression*

***e.g.* 2.9.**

1. 0^*10^*
2. $\Sigma^*1\Sigma^* = \{w | w = \cdots 1 \cdots\}$
3. $\Sigma^*001\Sigma^* = \{w | w = \cdots 001 \cdots\}$
4. $1^*(01^+)^* = \{w \in \{0,1\}$, *every 0 in w is followed by at least 1(there's no succesive 0)*$\}$
5. $(\Sigma\Sigma)^* = \{w|$*the length of w is even*$\}$

***e.g.* 2.10.**

***thm* 2.15.** *A language is regular iff some regular expression described it*

***lemma* 2.16.** ($\Leftarrow$) *If a language is decribed by a regular expression, then it's regular*

Proof.

1. $R = a, a \in \Sigma$
2. $R = \varepsilon$
3. $R = \emptyset$
4. $R = R_1 \cup R_2$
5. $R = R_1 R_2$
6. $R = R_1^*$

$\square$

3 Lesson 3