1 Introduction

1.1 Big-O Notation

def 1.1. *Let $f, g : \mathbb{N} \to \mathbb{R}$*

- *Write $f = O(g)$ if $(\exists c > 0)(\exists N)(\forall n \geq N)(|f(n)| \leq cg(n))$.*

- *Write $f = \Omega(g)$ if $(\exists c > 0)(\exists N)(\forall n \geq N)(|f(n)| \geq cg(n))$.*

- *Write $f = \Theta(g)$ if $(\exists c_1, c_2 > 0)(\exists N)(\forall n \geq N)(c_1 g(n) \leq |f(n)| \leq c_2 g(n))$.*

- *Write $f = o(g)$ if $(\forall \epsilon > 0)(\exists N)(\forall n \geq N)(|f(n)| \leq \epsilon g(n))$.*

Big-O Notation is the most commonly used one among the four.

e.g. 1.1. $f(n) = 6n^4 - 3n^3 + 5 \Rightarrow f(n) = O(n^4)$

Proof. $|6n^4 - 3n^3 + 5| \leq 6n^4 + 3n^4 + 5n^4 = 13n^4$ □

e.g. 1.2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = O(n^3)$

exercises 1.1. *Write in big-O notation:*

1. $5 + 0.001n^3 + 0.25n$

2. $500n + 100n^{1.5} + 50n \log_{10} n$

3. $n^2 \log_2 n + n(\log_2 n)^2$

4. $3 \log_8 n + \log_2(\log_2 n)$

   solution $O(n^3); O(n^1.5); O(n^2 \log n); O(\log n)$ □

prop 1.1.

- $f_1 = O(g_1), f_2 = O(g_2) \Rightarrow f_1 f_2 = O(g_1 g_2)$

- $f(O(g)) = O(fg)$

- $f_1 = O(g_1), f_2 = O(g_2) \Rightarrow f_1 + f_2 = O(max(g_1, g_2))$

- $f_1 = O(g), f_2 = O(g) \Rightarrow f_1 + f_2 = O(g)$

- $f = O(g) \Rightarrow kf = O(g)$

often encountered:

- *constant:* $O(1)$

  - $\sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$

  - $\sum_{k=1}^{n} \frac{1}{k^2} = O(1)$

  - $\sum_{k=1}^{n} \frac{1}{k \ln k} = \ln \ln n + O(1)$

- *double logarithmic:* $O(\log \log n)$

- *logarithmic:* $O(\log n)$

- *polylogarithmic:* $O((\log n)^c), c > 0$

- *linear:* $O(n)$

- *quasilinear:* $O(n \log^c n), O(n \log^{O(1)} n)$

- *quadratic:* $O(n^2)$

***def* 1.2.** $\omega(g), \theta(g)$

- $f = \omega(g) \ if (\forall c > 0)(\exists N)(\forall n \geq N)(f(n) \geq cg(n))$

- $g = \theta(g)$ *(or equivalently $f \sim g$) if* $(\forall \epsilon > 0)(\exists N)(\forall n \geq N)(|f(n) - g(n)| < \epsilon g(n))$

1.2 Alphabets and Languages

***def* 1.3.** *(alphabet) An alphabet is a set of symbols*

- *Roman alphabet* : $a, b, c, d, \cdots, z$

- *binary alphabet* : $0, 1$

***def* 1.4.** *(string and its length)*

A **string** *(over an alphabet) is a finite sequence of symbols from the alphabet.*

Empty string *is string of no symbols, denoted by ε.*

The set of all string is denoted by Σ^*. Denote by Σ^n the set of all **string of length** *n.*

So, $\Sigma^* = \cup_{n \geq 0} \Sigma^n$.

Denote the length of a string w by $|w|$

***e.g.* 1.3.** $\{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, \cdots\}, |\varepsilon| = 0, |0110| = 4$

def* 1.5.** *(concatenation) Two strings over the same alphabet can be combined by the operation of* **concatenation. The concatenation of x and y is denoted by xy.*

def 1.6. *(substring, suffix, prefix)*

A string v is a substring of w if $\exists$ strings x and y such that $w = xvy$.

   *If $w = xv$ for some x, then v is a **suffix** of w.*

   *If $w = vy$ for some y, then v is a **prefix** of w.*

def 1.7. *("power") The string w^i is defined: $w^0 = \varepsilon, w^{i+1} = w^i w, i \in \mathbb{N}$*

e.g. 1.4. $01^0 = \varepsilon, 01^1 = 01, 01^2 = 0101$

def 1.8. *(reversal) The reversal of a string w, denoted by w^R, is the string "spelled backwards"*

   A formal definition can be given by induction on length:

  1. *If $w = \varepsilon, w^R = w = \varepsilon$*

  2. *If $|w| = n + 1$, where $w = ua, a \in \Sigma$, then $w^R = au^R$*

def 1.9. *(language) **Language** is a set of strings over an alphabet, That is, $L \subseteq \Sigma^*$.*

   For example, $\emptyset, \Sigma^*, \Sigma$ are all languages

  • *$\sigma = \{0, 1\}$*

  • *$Even = \{0, 10, 100, 110, \cdots, \}$*

  • *$Odd = \{1, 11, 101, \cdots\}$*

  • *$Prime = \{10, 11, 101, 111, \cdots\}$*

  • *$Palindrome = \{w | w^R = w\} = \{\varepsilon, 0, 1, 00, 11, \cdots\}$*

def 1.10. *(complement, binary language operations)*

   *Let L be a language. The **complement** of L, denoted by $\overline{L}$, is $\Sigma^* - \overline{L}$. So $\overline{\overline{L}} = L$.*

   *Note that since L is a set, we can define **union**($\cup$), **interchapter***($\cap$) and difference.*

   *The **concatenation** of L_1 and L_2 is defined by $L_1 L_2 = w \in \Sigma^*, w = xy, \exists x \in L_1, y \in L_2$*

1.3 Encoding of Problems

1.3.1 Examples

  1. *(**Integer multiplication**) Given two non-negative integers x, y, compute xy.*

  2. *(**Primality testing**) Given $n \in \mathbb{N}$, decide if n is a prime.*

  3. *(**Hamiltonian cycle**) Given an undirected graph G, test if G has a Hamiltonian cycle.*
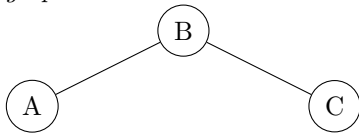
1.3.2 Analysis

- **Decision** problem: 2,3

- **Computation** problem: 1

1.3.3 Conclusion

- By **encoding**, decision problem is language. Any computation problem is a function from Σ^* to Σ^*. Our course only concerns decision problem, namely language.

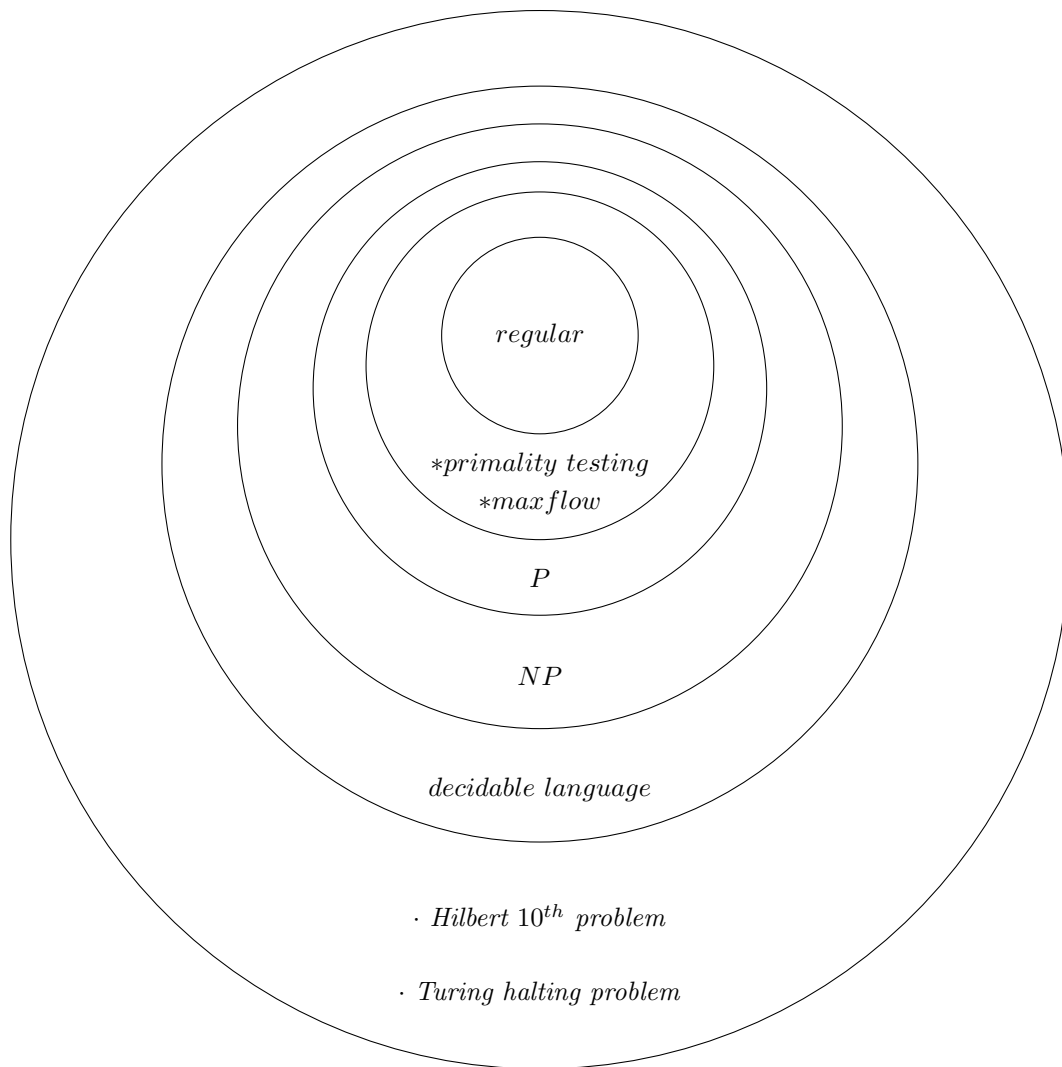- By **preprocessing**, one can switch between encodings.

e.g. **1.5.**

- *graph*

- *adjacency matrix*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- *adjacency list*

  (1,2),(1,3),$\cdots$

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Concentric circles from inner to outer:

regular

*primality testing
*maxflow

P

NP

decidable language

· Hilbert 10th problem

· Turing halting problem

all languages $L \subseteq \Sigma^* = \{\varepsilon, 0, 1, 00, 01, \cdots\}$

remark.

NP: *problems that are efficiently verifiable*

P: *problems that are efficiently solvable, i.e. solvable in $n^{O(1)}$ time*

regular language: *problems that are solvable without memory, i.e. solvable by finite automation*

    number of regular languages: $\aleph_0$

    number of all languages: $\aleph_1$