Numerical solutions of the Navier-Stokes equation using A.D.I methods

1 Primitive variables formulation

The mass and momentum equations for an incompressible fluid in the Boussinesq approximation, with a constant viscosity are:

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla}\vec{u} = -Pr\vec{\nabla}P + Pr\nabla^2\vec{u} - PrRa\theta\vec{e_z} \tag{1a}$$
$$\vec{\nabla} \cdot \vec{u} = 0 \tag{1b}$$

with the Prandtl number $Pr = \frac{\nu}{\kappa}$ and the Rayleigh number $Ra = \frac{\alpha\rho g\Delta T d^3}{\kappa\eta}$.

We used free slip boundary conditions on the top and bottom boundaries $\sigma_{xy}(x, z = 0, l_z) = 0$. We deduce a boundary condition for v_x:

$$\sigma_{xy}(x, z = 0, l_z) = 0 \Longleftrightarrow \partial_z v_x(x, z = 0, l_z) = 0$$

The top and bottom boundaries are impermeable $v_z(x, z = 0, l_z) = 0$. The free-slip boundary condition gives an other boundary condition for v_z:

$$\begin{aligned} \sigma_{xy}(x, z = 0, l_z) = 0 &\Longrightarrow \partial_x\sigma_{xy}(x, z = 0, l_z) = 0 \\ &\Longrightarrow \partial_z^2 v_z(x, z = 0, l_z) = 0 \end{aligned}$$

The z component of the Navier-Stokes equation gives a boundary condition for the pressure:

$$\partial_z P(x, z = 0, l_z) = -Ra\theta(x, z = 0, l_z) \tag{2}$$

Periodic boundary conditions are imposed on the vertical boundary conditions $(0, l_x)$

2 The projection method

We apply a first order splitting method to the Navier-Stokes equation (eq.1b)

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla}\vec{u} = Pr\nabla^2\vec{u} \tag{3a}$$
$$\partial_t \vec{u} = -Pr\vec{\nabla}P - PrRa\theta\vec{e_z} \tag{3b}$$
$$\vec{\nabla} \cdot \vec{u} = 0 \tag{3c}$$

2.1 Kim and Moin's scheme

The scheme

In this section, we present the fractional-step method proposed by Kim and Moin [4]:

$$\frac{\vec{u}^{n+\frac{1}{2}} - \vec{u}^n}{\Delta t} + \frac{1}{2}\left(3\vec{u}^n \cdot \vec{\nabla}\vec{u}^n - \vec{u}^{n-1} \cdot \vec{\nabla}\vec{u}^{n-1}\right) = \frac{Pr}{2}\nabla^2\left(\vec{u}^{n+\frac{1}{2}} + \vec{u}^n\right) \tag{4a}$$
$$\frac{\vec{u}^{n+1} - \vec{u}^{n+\frac{1}{2}}}{\Delta t} = -Pr\vec{\nabla}\phi^{n+1} \tag{4b}$$

We report eq.4b in eq.4a:

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} + \frac{1}{2}\left(3\vec{u}^n \cdot \vec{\nabla}\vec{u}^n - \vec{u}^{n-1} \cdot \vec{\nabla}\vec{u}^{n-1}\right) = -Pr\vec{\nabla}\left(1 - \frac{Pr\Delta t}{2}\nabla^2\right)\phi^{n+1} + \frac{Pr}{2}\nabla^2\left(\vec{u}^{n+1} + \vec{u}^n\right)$$

From this equation, we deduce the expression[1] for $P^{n+1} = \left(1 - \frac{Pr\Delta t}{2}\nabla^2\right)\phi^{n+1}$, and the second order time precision. By taking the divergence of equation 4b, we find an equation for ϕ^{n+1}:

$$\nabla^2\phi^{n+1} = \frac{1}{Pr\Delta t}\vec{\nabla}\cdot\vec{u}^{n+\frac{1}{2}}$$

In conclusion, we solve the following system of equations:

$$\frac{\vec{u}^{n+\frac{1}{2}} - \vec{u}^n}{\Delta t} + \frac{1}{2}\left(3\vec{u}^n\cdot\vec{\nabla}\vec{u}^n - \vec{u}^{n-1}\cdot\vec{\nabla}\vec{u}^{n-1}\right) = \frac{Pr}{2}\nabla^2\left(\vec{u}^{n+\frac{1}{2}} + \vec{u}^n\right) \tag{5a}$$

$$\nabla^2\phi^{n+1} = \frac{1}{Pr\Delta t}\vec{\nabla}\cdot\vec{u}^{n+\frac{1}{2}} \tag{5b}$$

$$\vec{u}^{n+1} = \vec{u}^{n+\frac{1}{2}} - \Delta t Pr\vec{\nabla}\phi^{n+1} \tag{5c}$$

Boundary condition for $\vec{u}^{n+\frac{1}{2}}$

$\vec{u}^{n+\frac{1}{2}}$ is the solution of 3a.

$$\vec{u}^{n+\frac{1}{2}} = \vec{u^*}^n + \Delta t\frac{\partial\vec{u^*}^n}{\partial t} + O(\Delta t^2)$$

with:

$$\vec{u^*}^n = \vec{u}^n$$
$$\frac{\partial\vec{u^*}^n}{\partial t} = \frac{\partial\vec{u}^n}{\partial t} + Pr\vec{\nabla}P$$

then

$$\begin{aligned}\vec{u}^{n+\frac{1}{2}} &= \vec{u}^n + \Delta t\left(\frac{\partial\vec{u}^n}{\partial t} + Pr\vec{\nabla}P\right) + O(\Delta t^2)\\ &= \vec{u}^{n+1} + Pr\Delta t\vec{\nabla}P + O(\Delta t^2)\end{aligned}$$

from the expression of P in function of ϕ, we deduce $P = \phi + O(\Delta t)$. Finally, we find the boundary condition for $\vec{u}^{n+\frac{1}{2}}$:

$$\vec{u}^{n+\frac{1}{2}} = \vec{u}^{n+1} + Pr\Delta t\vec{\nabla}\phi^n + O(\Delta t^2) \tag{6}$$

In 2 dimensions

The equation 5a can be rewritten as:

$$\left[1 - \frac{Pr\Delta t}{2}\left(\partial_x^2 + \partial_z^2\right)\right]\vec{u}^{n+1} = \left[1 + \frac{Pr\Delta t}{2}\left(\partial_x^2 + \partial_z^2\right)\right]\vec{u}^n - \frac{\Delta t}{2}\left(3\vec{u}^n\cdot\vec{\nabla}\vec{u}^n - \vec{u}^{n-1}\cdot\vec{\nabla}\vec{u}^{n-1}\right)$$

The advection term is treated as a source/sink term. This equation can be factorized as:

$$\begin{aligned}\left(1 - \frac{Pr\Delta t}{2}\partial_x^2\right)\left(1 - \frac{Pr\Delta t}{2}\partial_z^2\right)\vec{u}^{n+1} = &\left(1 + \frac{Pr\Delta t}{2}\partial_x^2\right)\left(1 + \frac{Pr\Delta t}{2}\partial_z^2\right)\vec{u}^n\\ &- \frac{\Delta t}{2}\left(3\vec{u}^n\cdot\vec{\nabla}\vec{u}^n - \vec{u}^{n-1}\cdot\vec{\nabla}\vec{u}^{n-1}\right)\end{aligned}$$

This equation can be solved with an A.D.I. scheme [2, 5, 6].

The equation 5b is solved with Fourier transform in the horizontal plane and 2nd order finite differences in the z direction:

$$\widehat{\phi}_{j+1}^{n+1}(k) - (2 + k^2\Delta z^2)\widehat{\phi}_j^{n+1}(k) + \widehat{\phi}_{j-1}^{n+1}(k) = \frac{\Delta z^2}{Pr\Delta t}\widehat{\vec{\nabla}\cdot\vec{u}_j}^{n+\frac{1}{2}}(k)$$

The equation 5c is explicit.

[1] The sign is different from Kim and Moin [4] : $P^{n+1} = \left(1 + \frac{Pr\Delta t}{2}\nabla^2\right)\phi^{n+1}$ (page 310)

2.2 Scheme 1

A first order discretisation is used for the time derivatives:

$$\frac{\vec{u}^{n+\frac{1}{2}} - \vec{u}^n}{\Delta t} + \vec{u}^n \cdot \vec{\nabla} \vec{u}^{n+\frac{1}{2}} = Pr \nabla^2 \vec{u}^{n+\frac{1}{2}} \tag{7a}$$

$$\frac{\vec{u}^{n+1} - \vec{u}^{n+\frac{1}{2}}}{\Delta t} = -Pr \vec{\nabla} P^{n+1} - Pr Ra \theta^{n+1} \vec{e_z} \tag{7b}$$

$$\vec{\nabla} \cdot \vec{u}^{n+1} = 0 \tag{7c}$$

$\vec{u}^n$ is divergence free. $\vec{u}^{n+\frac{1}{2}}$ (such as $\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}} \neq 0$) is an approximation of $\vec{u}^{n+1}$ (such as $\vec{\nabla} \cdot \vec{u}^{n+1} = 0$). The pressure equation for P^{n+1} is the Laplace's equation given by taking the divergence of the equation 7b:

$$\nabla^2 P^{n+1} = \frac{\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}}}{Pr \Delta t} - Ra \partial_z \theta^{n+1}$$

Finally, the projection method applied to Navier-Stokes equation requires to solve the following set of equations:

$$\vec{u}^{n+\frac{1}{2}} + \Delta t \vec{u}^n \cdot \vec{\nabla} \vec{u}^{n+\frac{1}{2}} - \Delta t Pr \nabla^2 \vec{u}^{n+\frac{1}{2}} = \vec{u}^n \tag{8a}$$

$$\nabla^2 P^{n+1} = \frac{\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}}}{Pr \Delta t} - Ra \partial_z \theta^{n+1} \tag{8b}$$

$$\vec{u}^{n+1} = \vec{u}^{n+\frac{1}{2}} - \Delta t Pr \vec{\nabla} P^{n+1} - \Delta t Pr Ra \theta^{n+1} \vec{e_z} \tag{8c}$$

Variables are located on the staggered grid 1(a).

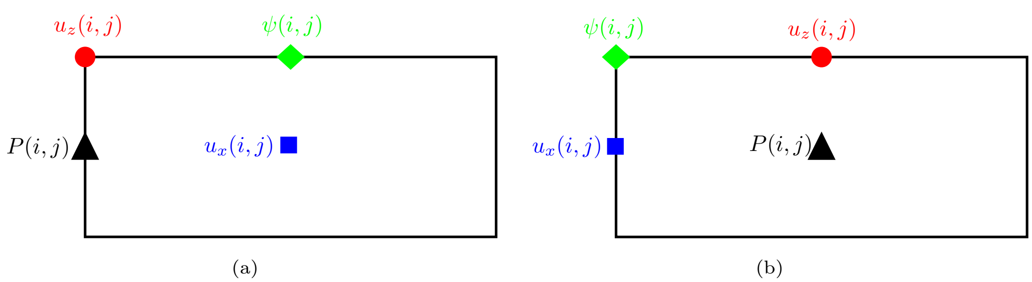


Figure 1: Staggered grids

Resolution of Equation 8a

We use a Peaceman and Rashford scheme [2, 5, 6] to solve both the x, y component of the equation 8a. This A.D.I scheme is first order time precision. Second order centered finite differences are used to express the spatial derivatives.

Resolution of Equation 8b

First of all, we have used an iterative schemes to solve the equation 8b, replaced by:

$$\partial_\tau P^{n+1} = \nabla^2 P^{n+1} - \frac{\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}}}{Pr \Delta t} + Ra \partial_z \theta^{n+1}$$

We use a Peaceman and Rashford scheme [2, 5] and second order centered finite differences to solve this equation. This solution have been iterated until $\|\delta P\| = \left\| \frac{P_{\nu+1}^{n+1} - P_\nu^{n+1}}{P_\nu^{n+1}} \right\| < 10^{-9}$ with $\tau = \nu \Delta \tau$. This iterative scheme is very slow at high resolution.

We have solved the equation 8a using a pseudo-spectral method: 1-D Fourier transform along x and second order finite differences along z:

$$\left(\partial_z^2 - k^2\right) \widehat{P}_k^{n+1}(z) = \widehat{F}_k^{n+1}(z) \quad \text{with} \quad F^{n+1}(x,z) = \frac{\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}}}{Pr \Delta t} - Ra \partial_z \theta^{n+1}$$

The solution of this equation is obtained by inverting a tridiagonal matrix for each k mode. For the fundamental mode $k = 0$ the matrix is singular. We use a Green's function g to solve the equation $\partial_z^2 \widehat{P}_0^{n+1}(z) = \widehat{F}_0^{n+1}(z)$. The solution is:

$$\widehat{P}_0^{n+1}(z) = \int_0^1 g(z - z')\widehat{F}_0^{n+1}(z')dz' \quad \text{with} \quad g(z) = \frac{|z|}{2}$$

We have tried different methods to calculate this integral (trapezoidal rule,rectangular rule, Simpson's rule). They all give comparable results.

Resolution of Equation 8c

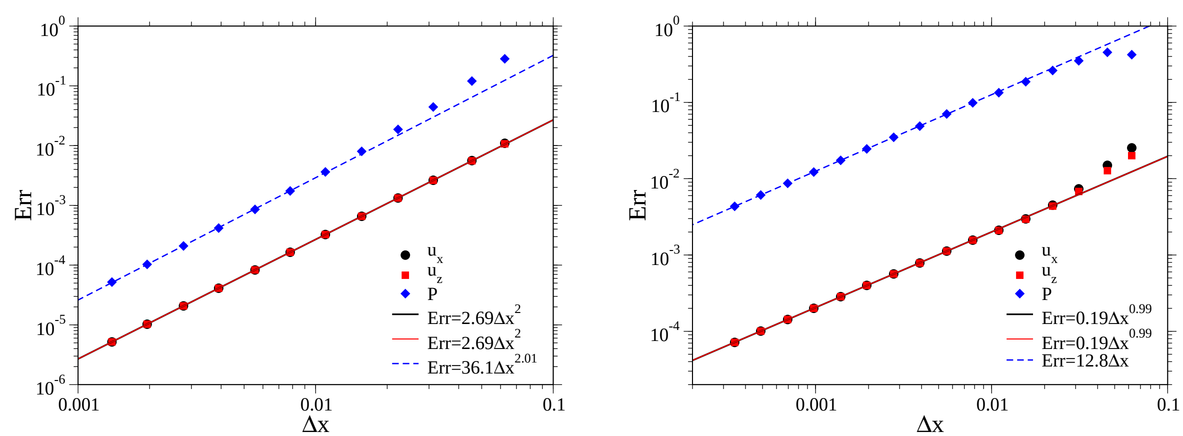
The expression of $\vec{u}^{n+1}$ is explicitly obtained from the solutions of equations 8a and 8b.

Estimation of the precision

We use second order centered finite differences for the spatial derivatives in the previous scheme. We check the time and spatial precisions given by this scheme with the Taylor-Green vortex (cf. appendix B). We compare the numerical solution ($\tilde{f}$ being $\tilde{u}_x$ or $\tilde{u}_z$ or $\tilde{P}$) and the analytical solution (f being u_x or u_z or P). We compute the errors

$$Err_f = \sqrt{\frac{1}{l_x l_z T} \sum_{i=1}^{n_x+1} \sum_{k=1}^{n_z+1} \sum_{n=1}^{n_t+1} \left(\tilde{f}_{ik}^n - f_{ik}^n\right)^2 \Delta x \Delta z \Delta t}$$

with $l_x = n_x \Delta x = 1$ and $l_z = n_z \Delta z = 1$ the size of box. In the following, we have fixed $\Delta x = \Delta z$ and we have computed the solution during a time $T = n_t \Delta t$.



(a) $\Delta t = 0.1\Delta x^2$ (b) $\Delta t = 0.1\Delta x$

Figure 2: Scheme 1 - Errors in function of de grid mesh size Δx

$\Delta t \propto \Delta x^2$

To check the spatial resolution, we impose $dt = 0.1\Delta x^2$ so that the errors should be $O(\Delta x^2)$. We have computed the solution during a time $T = 1/\left[(k_x^2 + k_z^2)\nu\right]$ so that the amplitude of the velocities are divided by e at the end of the run. The figure 2(a) shows that this scheme is first order time precision $O(\Delta t)$.

$\Delta t \propto \Delta x$

To check the time resolution, we impose $dt = 0.1\Delta x$ so that the errors should be $O(\Delta x)$. We have computed the solution during a time $T = \ln(10)/\left[(k_x^2 + k_z^2)\nu\right]$ so that the amplitude of the velocities are divided by 10 at the end of the run. The figure 2(b) shows that this scheme is first order time precision $O(\Delta t)$.

2.3 Scheme 2

We use the semi-discrete system of equations (equations 3a-d) given by Karniadakis et al. [3] to solve equations 1a and 1b:

$$\frac{\hat{\vec{u}} - \vec{u}^n}{\Delta t} = -\vec{u}^n \cdot \vec{\nabla}\vec{u}^n \tag{9a}$$

$$\frac{\hat{\hat{\vec{u}}} - \hat{\vec{u}}}{\Delta t} = -Pr\vec{\nabla}P^{n+1} \tag{9b}$$

$$\frac{\vec{u}^{n+1} - \hat{\hat{\vec{u}}}}{\Delta t} = Pr\nabla^2\frac{\vec{u}^{n+1} + \vec{u}^n}{2} \tag{9c}$$

We have used an explicit first order Euler scheme for equation 9a. We take the divergence of the equation 9b to obtain a Poisson equation for the pressure (the divergence of equation 9c gives $\vec{\nabla} \cdot \hat{\hat{\vec{u}}} = 0$):

$$\nabla^2 P^{n+1} = \frac{\vec{\nabla} \cdot \hat{\vec{u}}}{Pr\Delta t} \tag{10}$$

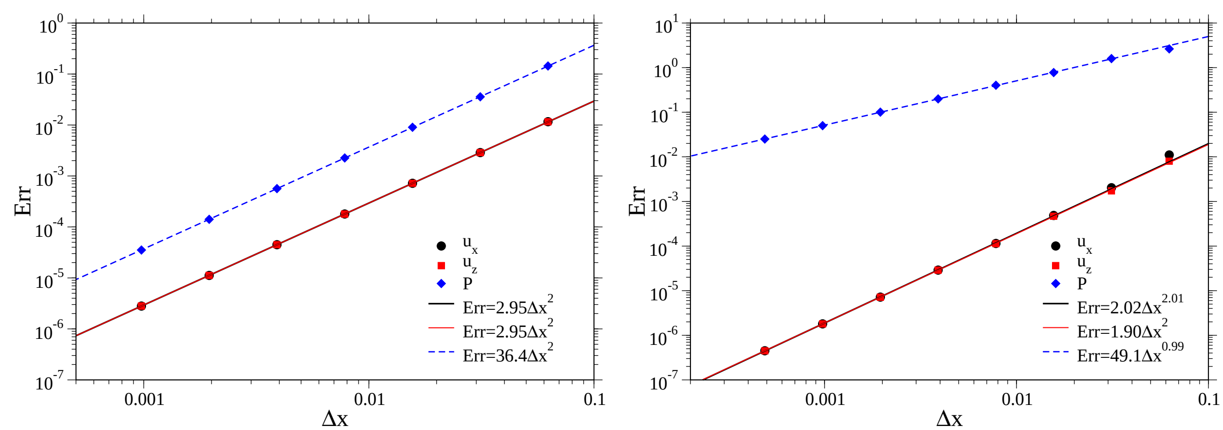
This equation is solved using a pseudo-spectral method: 1D Fourier transform for x and finite differences for z (cf. Resolution of Equation 8b #2.2). We have used a semi-implicit second order scheme for equation 9c. This scheme can be slightly modified:

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} = Pr\nabla^2\frac{\vec{u}^{n+1} + \vec{u}^n}{2} + \frac{\hat{\hat{\vec{u}}} - \vec{u}^n}{\Delta t} \tag{11}$$

Scheme 11 is a Crank-Nicolson scheme, and $\frac{\hat{\hat{\vec{u}}} - \vec{u}^n}{\Delta t}$ acts as a source term. This equation can be easily factorised, and solved using a Peaceman and Rashford scheme [2, 5]. To ensure the conservation of momentum, we have used a finite volume formulation to solve the set of equations 9a, 10, and 11.

Estimation of the precision

As for the scheme 1, we check the time and spatial resolutions given by these scheme with the Taylor-Green vortex (cf. appendix B). We consider two cases (cf. #2.2) a) $\Delta t \propto 0.1\Delta x^2$ and b) $\Delta t \propto 0.1\Delta x$.



(a) $\Delta t = 0.1\Delta x^2$ (b) $\Delta t = 0.1\Delta x$

Figure 3: Scheme 2 - Errors in function of de grid mesh size Δx

3 Stream function formulation

In a 2D cartesian geometry:

$$\vec{\nabla} \cdot \vec{u} = 0 \Longrightarrow \vec{u} = \vec{\nabla} \wedge (\psi\vec{e_y})$$

ψ is the stream function. We impose horizontal periodic conditions on vertical walls $x = 0$ and $x = l_x$, and free-slip boundary conditions on horizontal walls $z = 0$ and $z = l_z$: $\psi(z = 0, l_z) = 0$ and $\partial_z^2 \psi(z = 0, l_z) = 0$.

We take the curl of the Navier-Stokes equation (eq.1a) :

$$\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) = Pr \nabla^4 \psi \quad \text{with} \quad J(f, g) = \partial_x f \partial_z g - \partial_z f \partial_x g \tag{12}$$

We use the following time splitting:

$$\begin{aligned}
\partial_t \nabla^2 \psi_{(1)} &= Pr \nabla^4 \psi_{(1)} \quad \text{for } t^n \leq t \leq t^{n+\frac{1}{2}}, \psi_{(1)}^n = \psi^n \\
\partial_t \nabla^2 \psi_{(2)} &= -J(\psi_{(2)}, \nabla^2 \psi_{(2)}) \quad \text{for } t^n \leq t \leq t^{n+1}, \psi_{(2)}^n = \psi_{(1)}^{n+\frac{1}{2}} \\
\partial_t \nabla^2 \psi_{(3)} &= Pr \nabla^4 \psi_{(3)} \quad \text{for } t^n \leq t \leq t^{n+\frac{1}{2}}, \psi_{(3)}^n = \psi_{(2)}^{n+1}
\end{aligned}$$

3.1 Scheme 3

We use a pseudo-spectral method to solve this set of equations: Fourier transforms along x and finite differences along z:

$$\partial_t \left(\partial_z^2 - k^2\right) \widehat{\psi}_{(1)} = \left(\partial_z^4 - 2k^2 \partial_z^2 + k^4\right) \widehat{\psi}_{(1)} \tag{13a}$$

$$\partial_t \left(\partial_z^2 - k^2\right) \widehat{\psi}_{(2)} = -\widehat{J}(\psi_{(2)}, \nabla^2 \psi_{(2)}) \tag{13b}$$

$$\partial_t \left(\partial_z^2 - k^2\right) \widehat{\psi}_{(3)} = \left(\partial_z^4 - 2k^2 \partial_z^2 + k^4\right) \widehat{\psi}_{(3)} \tag{13c}$$

Equations 13a and 13c can be solved using a second order semi-implicit scheme (Crank-Nicolson) with a time step $\frac{\Delta t}{2}$:

$$\left[\left(\partial_z^2 - k^2\right) - \frac{\Delta t}{4} \left(\partial_z^4 - 2k^2 \partial_z^2 + k^4\right)\right] \widehat{\psi}_{(i)}^{n+\frac{1}{2}} = \left[\left(\partial_z^2 - k^2\right) + \frac{\Delta t}{4} \left(\partial_z^4 - 2k^2 \partial_z^2 + k^4\right)\right] \widehat{\psi}_{(i)}^n \quad \text{i=1,3}$$

When using second order finite differences, this scheme leads to invert pentadiagonal matrices.

The equation 13b is solved with an explicit scheme because of the non linear term J. With a first order explicit Euler scheme:

$$\left(\partial_z^2 - k^2\right) \widehat{\psi}_{(2)}^{n+1} = \left(\partial_z^2 - k^2\right) \widehat{\psi}_{(2)}^n - \Delta t \widehat{J}(\psi_{(2)}^n, \nabla^2 \psi_{(2)}^n) \tag{14}$$

When using second order finite differences, this scheme leads to invert tridiagonal matrices.

4 Rayleigh-Bénard convection

We have to solve the mass equation, the momentum equation and the heat equation:

$$\partial_t \theta + \vec{u} \cdot \vec{\nabla} \theta = \nabla^2 \theta$$

with periodic boundary conditions on the vertical planes $x = 0, l_x$. The temperature is fixed on the horizontal top ($z = 0$) and bottom boundaries ($z = l_z$): $T(x, z = 0) = -0.5$ and $T(x, z = l_z) = 0.5$. We solve the mass and Navier-Stokes equations using the previous schemes. A Peaceman and Rashford scheme [2, 5] is used to solve the heat equation.

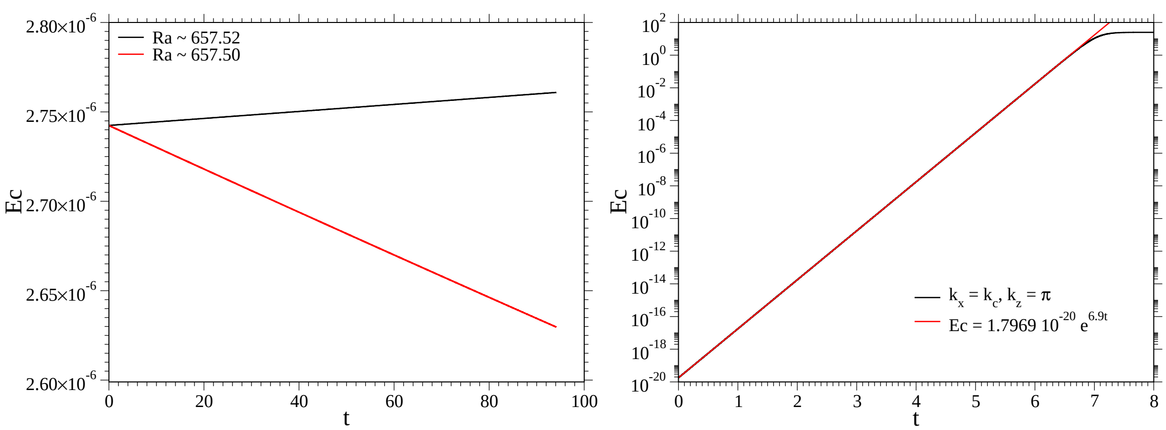
4.0.1 Growth rate

The equation 19 (cf. Apprendix A) gives the values of the growth rate σ of convective modes k_x and k_z in function of the Prandtl number Pr and the Rayleigh number Ra.

We can use the growth rate to fix the numerical critical Rayleigh Number. For wavenumbers of $k_x = \pi/\sqrt{2}$ and $k_z = \pi$, the analytical Rayleigh number is $Ra_c = 27\pi^4/4 \approx 657.51$. For a given Rayleigh number and a given convective mode $\vec{k}$, if the growth rate is positive then Rayleigh number is greater than the critical value. The figure 4(a) displays the time evolution of the kinetic energy for two values of the Rayleigh number close to the critical value. We deduce that :

$$657.50 < Ra_c < 657.52$$

We can verify if the numerical model is able to give approximate values of σ. The figure 4(b) displays the time evolution of the kinetic energy given by a numerical model with $Ra = 6 \times 13$, $Pr = 1$ and an aspect ratio $2\pi/k_c$. Each components of the velocity growth in $e^{\sigma t}$. Then, the kinetic energy growth in $e^{2\sigma t}$. We deduce the value of σ from the figure (red line). The figure 5 displays the values of σ given by the numerical model and the equation 19 for different values of Ra and k_x. The numerical values are in good agreement with the analytical ones.



(a) Critical Rayleigh number (b) Growth rate

Figure 4: Time evolution of the kinetic energy with $k_c = \pi/\sqrt{2}$, $k_z = \pi$ a) for two different Rayleigh numbers close to the critical value, and b) $Ra = 10^3$ (black line), the red line is a scaling.

4.1 Second order time discretisation

We use a Crank-Nicolson scheme to solve the equation 3a:

$$\left[1 - \frac{\Delta t}{2}\left(Pr\nabla^2 - \vec{u}^{n+\frac{1}{2}} \cdot \vec{\nabla}\right)\right] \vec{u}^{n+\frac{1}{2}} = \left[1 + \frac{\Delta t}{2}\left(Pr\nabla^2 - \vec{u}^{n} \cdot \vec{\nabla}\right)\right] \vec{u}^{n}$$

Remark This equation can be factorized. Then, we obtain the Peaceman & Rashford scheme for each component x and z of this equation.

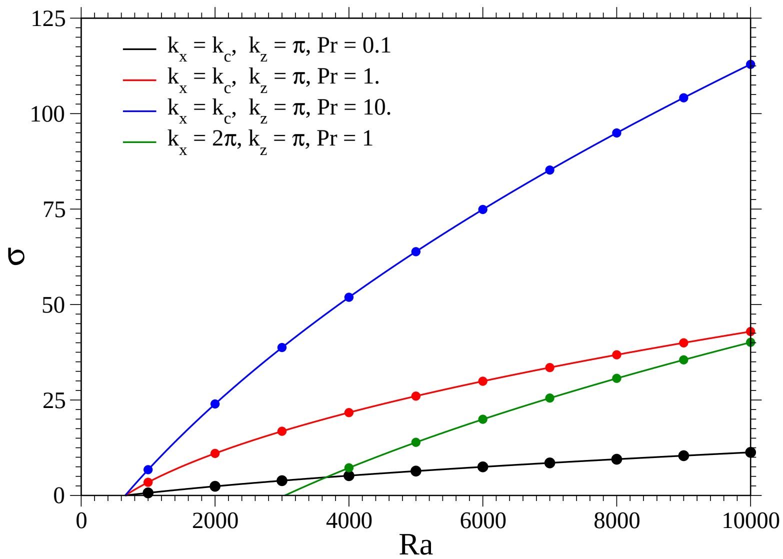


Figure 5: Rayleigh Bénard convection. Growth rate σ as a function of the Rayleigh Number Ra for different modes k_x and different Prandtl number Pr. The straight lines show the analytical solutions and the dots are given by the numerical model.

We discretize the equation 3a at instant $n + \frac{1}{2}$, then:

$$\vec{u}^{n+1} = \vec{u}^{n+\frac{1}{2}} - \Delta t Pr \vec{\nabla} \frac{P^{n+\frac{1}{2}} + P^{n+1}}{2} - \Delta t Pr Ra \frac{\theta^{n+\frac{1}{2}} + \theta^{n+1}}{2} \vec{e_z}$$

We take the divergence of this last equation:

$$\nabla^2 P^{n+1} = 2 \frac{\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}}}{\Delta t Pr} - \nabla^2 P^{n+\frac{1}{2}} - Ra \partial_z \left(\theta^{n+\frac{1}{2}} + \theta^{n+1} \right)$$

Question: $\theta^{n+\frac{1}{2}} = \theta^n$ and $P^{n+\frac{1}{2}} = P^n$?

Finally, we have to solve the following set of equations:

$$\left[1 - \frac{\Delta t}{2} \left(Pr \nabla^2 - \vec{u}^{n+\frac{1}{2}} \cdot \vec{\nabla} \right) \right] \vec{u}^{n+\frac{1}{2}} = \left[1 + \frac{\Delta t}{2} \left(Pr \nabla^2 - \vec{u}^{n} \cdot \vec{\nabla} \right) \right] \vec{u}^{n} \tag{15a}$$

$$\nabla^2 P^{n+1} = 2 \frac{\vec{\nabla} \cdot \vec{u}^{n+\frac{1}{2}}}{\Delta t Pr} - \nabla^2 P^{n+\frac{1}{2}} - Ra \partial_z \left(\theta^{n+\frac{1}{2}} + \theta^{n+1} \right) \tag{15b}$$

$$\vec{u}^{n+1} = \vec{u}^{n+\frac{1}{2}} - \Delta t Pr \vec{\nabla} \frac{P^{n+\frac{1}{2}} + P^{n+1}}{2} - \Delta t Pr Ra \frac{\theta^{n+\frac{1}{2}} + \theta^{n+1}}{2} \vec{e_z} \tag{15c}$$

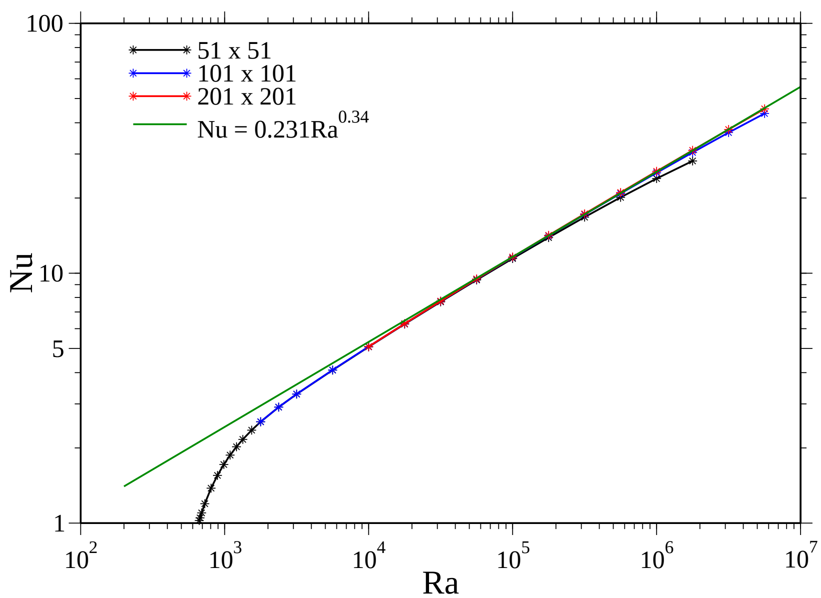


Figure 6: Nusselt number Nu versus Rayleigh number Ra for models of Rayleigh-Bénard convection with $Pr = 1$

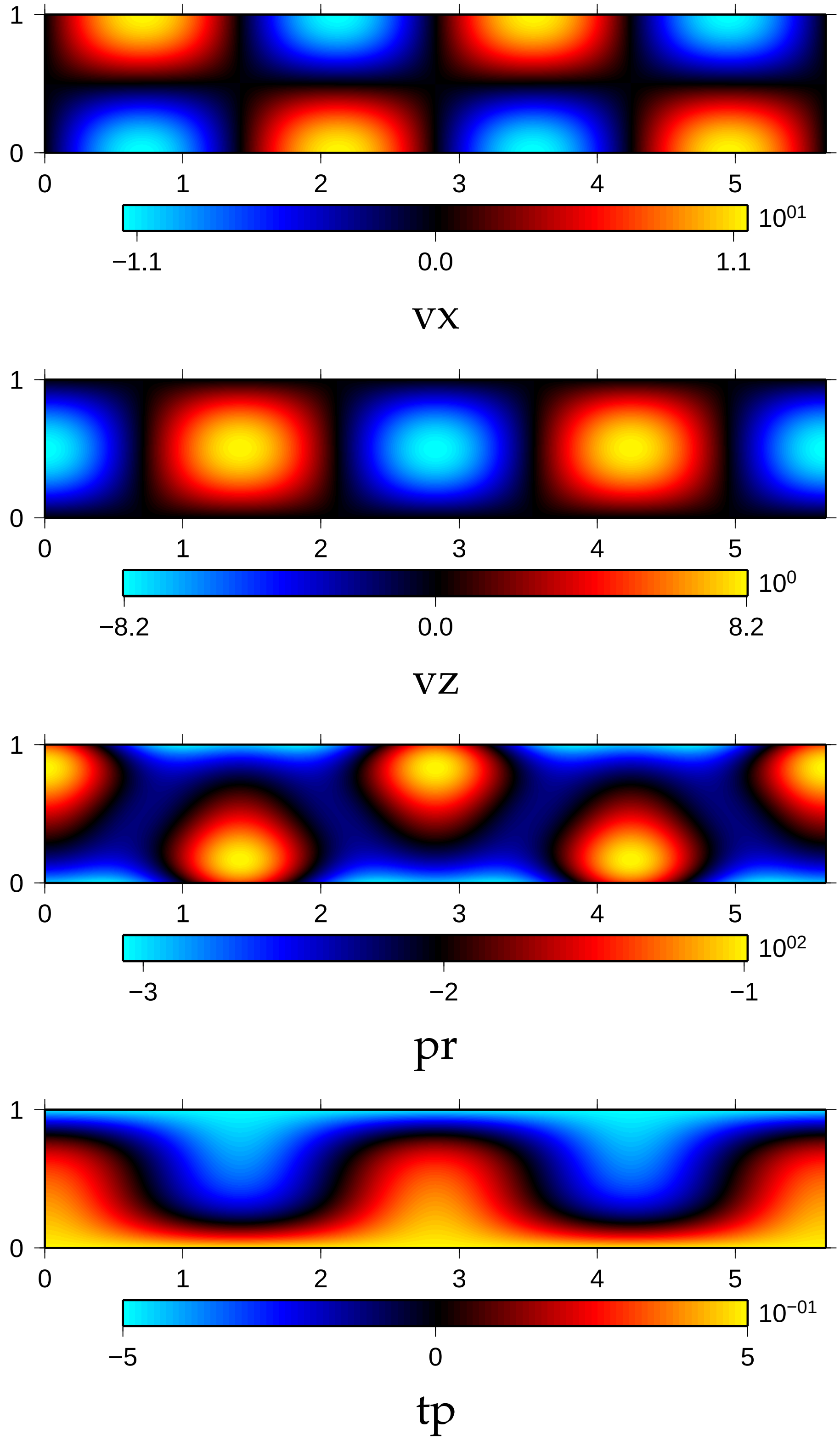


Figure 7: Rayleigh Bénard convection, $Pr = 1$, $Ra = 10^3$ and 301×51 grid points.

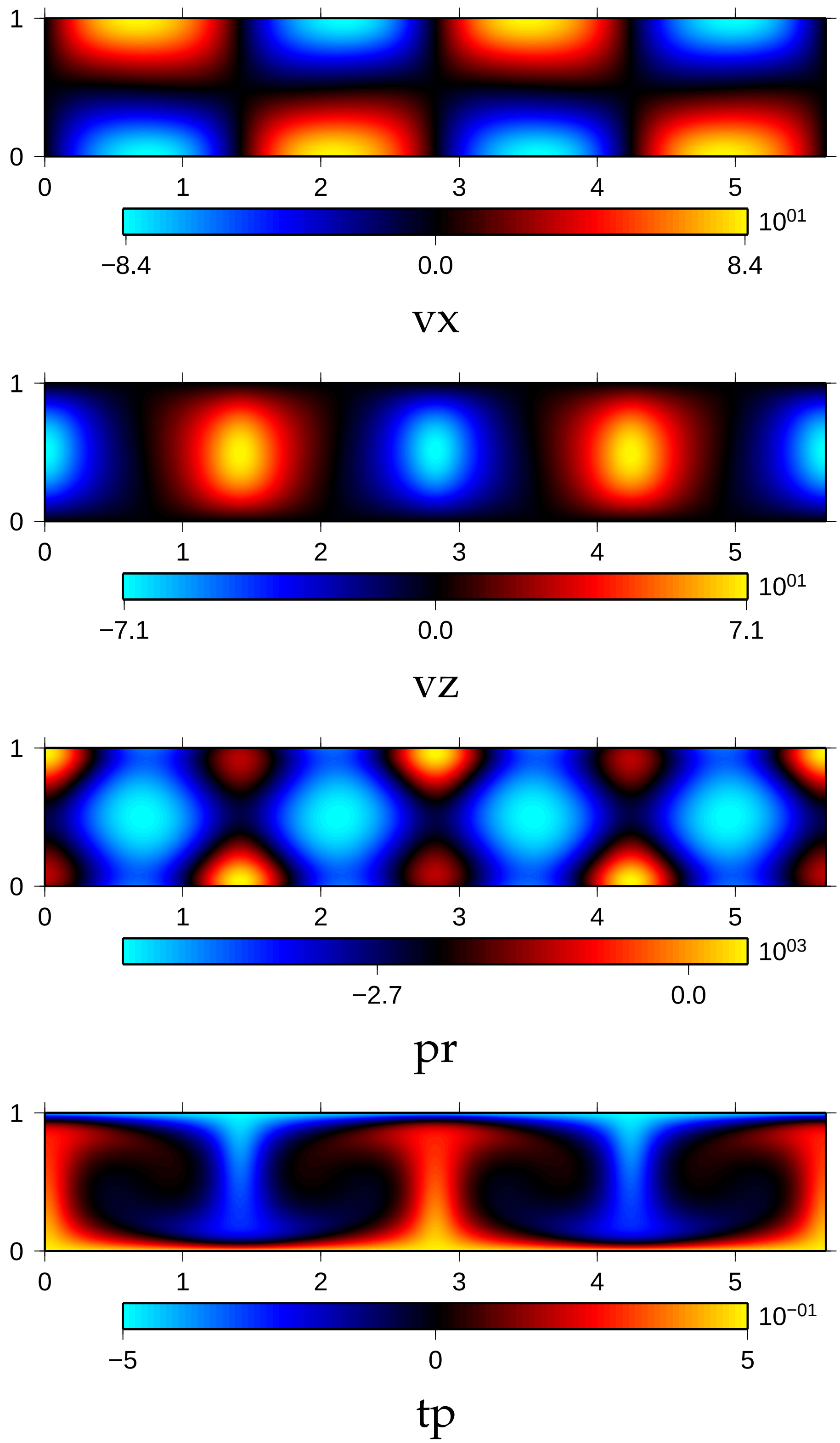


Figure 8: Rayleigh Bénard convection, $Pr = 1$, $Ra = 10^4$ and 301×51 grid points.

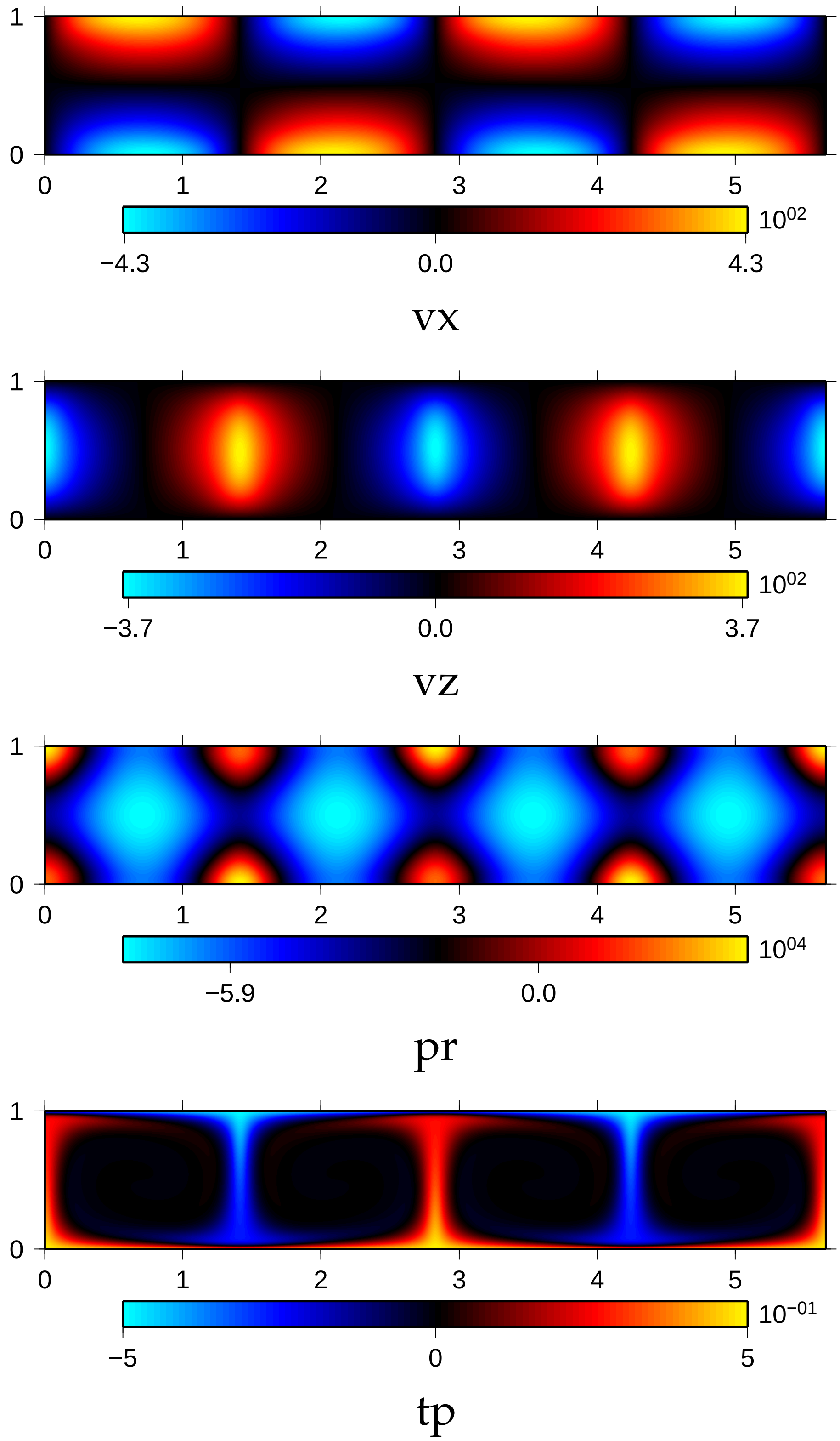


Figure 9: Rayleigh Bénard convection, $Pr = 1$, $Ra = 10^5$ and 601×101 grid points.

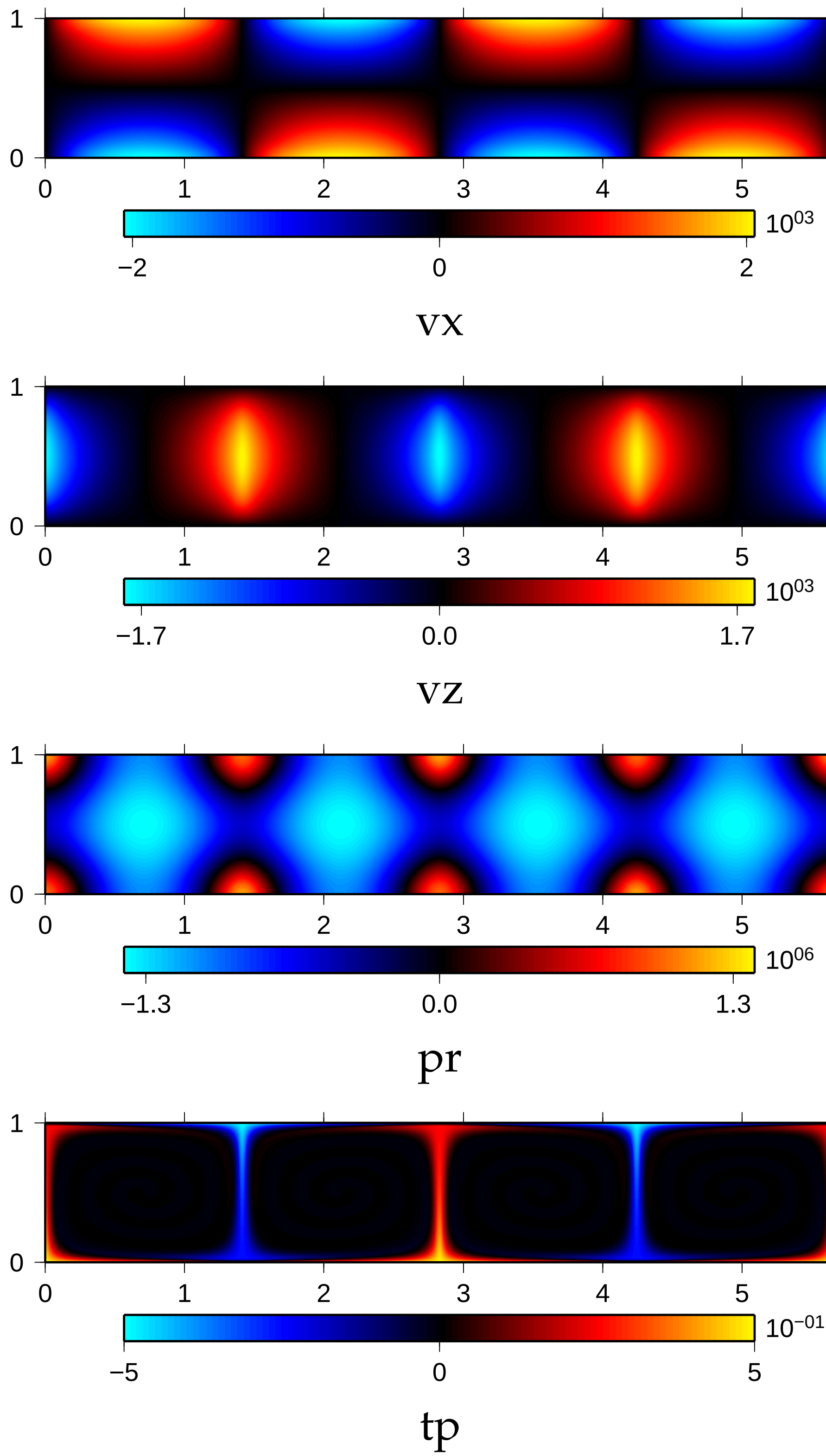


Figure 10: Rayleigh Bénard convection, $Pr = 1$, $Ra = 10^6$ and 601×101 grid points.

A Rayleigh-Bénard convection: a linear stability analysis

from S. Chandrasekhar [1].

Equations

The equations of convection of a viscous fluid are:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{16a}$$

$$\rho \left(\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} P + \eta \nabla^2 \vec{v} + \rho \vec{g} \tag{16b}$$

$$\rho Cp \left(\partial_t T + \vec{v} \cdot \vec{\nabla} T \right) = k \nabla^2 T \tag{16c}$$

In the Boussinesq approximation: $\rho = \rho_0 \left[1 - \alpha(T - T_s)\right]$.

Equilibrium

The equilibrium is defined as:

$$\begin{aligned} &\rho_0 \\ &\vec{V_0} = \vec{0} \\ &T_0 = \Delta T \frac{z}{d} + T_S \\ &\frac{dP_0}{dz} = \rho_0 \left[1 - \alpha(T_0 - T_s)\right] g \end{aligned}$$

Linearised equations

We pertubate the equilibrium with infinitesimal perturbations:

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \\ \vec{v} &= \vec{v_0} + \delta\vec{v} \\ P &= P_0 + \delta P \\ T &= T_0 + \delta T \end{aligned}$$

In the Boussinesq approximation: $\delta\rho = \alpha\rho_0 \left(T - T_S\right)$. The equations for the perturbations are:

$$\vec{\nabla} \cdot \delta\vec{v} = 0 \tag{17a}$$

$$\rho_0 \partial_t \delta\vec{v} = -\vec{\nabla} \delta P + \eta \nabla^2 \delta\vec{v} - \alpha\rho_0 \delta T \vec{g} \tag{17b}$$

$$\partial_t \delta T + \frac{\Delta T}{d} \delta v_z = \kappa \nabla^2 \delta T \tag{17c}$$

Dimensionless equations

The dimensionless variables are defined as:

$$\begin{aligned} &\vec{r} = d\vec{r^*}, \quad t = \tfrac{d^2}{\kappa} \\ &\delta P = \tfrac{\kappa\eta}{d^2}\Pi, \quad \delta\vec{v} = \tfrac{\kappa}{d}\vec{u}, \quad \delta T = \Delta T \theta \end{aligned}$$

The dimensionless linearised equations are:

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{18a}$$

$$\frac{1}{Pr} \partial_\tau \vec{u} = -\vec{\nabla}\Pi + \nabla^2 \vec{u} - Ra\theta\vec{e_z} \tag{18b}$$

$$\partial_\tau \theta + u_z = \nabla^2 \theta \tag{18c}$$

with the Prandtl number $Pr = \frac{\nu}{\kappa}$ and the Rayleigh number $Ra = \frac{\alpha g \delta T d^3}{\nu\kappa}$

A.1 2D geometry

We define the stream function $\vec{u} = \vec{\nabla} \wedge (\psi \vec{e_y})$. We take the divergence of the Navier-Stokes equation and finally we obtain:

$$\frac{1}{Pr}\partial_\tau \nabla^2 \psi = \nabla^4 \psi - Ra \partial_x \theta$$
$$\partial_\tau \theta + \partial_x \psi = \nabla^2 \theta$$

We develop the solution in normal modes

$$(\psi, \theta) = (\Psi_k, \Theta_k)\, e^{\imath \vec{k}\cdot\vec{r} + \sigma\tau}$$

and

$$-\sigma k^2 \Psi_k = Pr k^4 \Psi_k - \imath Pr Ra k_x \Theta_k$$
$$\sigma \Theta_k + \imath k_x \Psi_k = -k^2 \Theta_k$$

equivalent to:

$$\imath Pr Ra k_x \Theta_k = (\sigma + Pr k^2) k^2 \Psi_k$$
$$(\sigma + k^2)\Theta_k = -\imath k_x \Psi_k$$

We find the relation

$$\sigma^2 + (Pr + 1)\, k^2 \sigma + \frac{Pr}{k^2}\left(k^6 - Ra k_x^2\right) = 0 \tag{19}$$

To find the critical wave number, we impose $\sigma = 0$ and we look for the minimum value of Ra:

$$\frac{dRa}{dkx^2} = 0 \Longleftrightarrow {k_{x}}_c^2 = \frac{k_z^2}{2} \tag{20}$$

For these values of k_{xc}, we find the critical Rayleigh number Ra_c:

$$Ra_c = \frac{27}{4} k_z^4 \tag{21}$$

Free-slip boundary conditions

We impose $k_z = n\pi$ with $n \in \mathbb{N}$ to verify free-slip conditions on top and bottom boundaries. For $n = 1$, the equations 20 and 21 give:

$$k_{xc} = \frac{\pi}{\sqrt{2}} \quad \text{and} \quad Ra_c = \frac{27}{4}\pi^4 \tag{22}$$

B The Taylor–Green vortex

To test the previous schemes, we solve the momentum and mass equations in a 2D box of dimensions $[0,1]\times[0,1]$, with $Pr = 1$ and without the gravity forcing:

$$
\begin{aligned}
&\partial_t \vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}P + \nu\nabla^2\vec{u}\\
&\vec{\nabla}\cdot\vec{u} = 0
\end{aligned}
$$

We impose impermeable and free-slip boundary conditions on all surfaces. The initial conditions are given by the analytical solution at time $t = 0$:

$$
\begin{aligned}
u_x &= -k_z V \sin(k_x x)\cos(k_z z)e^{-\left(k_x^2+k_z^2\right)\nu t}\\
u_z &= +k_x V \cos(k_x x)\sin(k_z z)e^{-\left(k_x^2+k_z^2\right)\nu t}\\
P &= \left[\cos(2k_x x)\left(\frac{Vk_z}{2}\right)^2 + \cos(2k_z z)\left(\frac{Vk_x}{2}\right)^2\right] e^{-2\left(k_x^2+k_z^2\right)\nu t}
\end{aligned}
$$

We can define the stream function $\vec{u} = \vec{\nabla}\wedge(\psi\vec{e}_y)$ and the vorticity $\vec{\omega} = \vec{\nabla}\wedge\vec{u}$. The analytical solutions are:

$$
\begin{aligned}
\psi &= V\sin(k_x x)\sin(k_z z)e^{-\left(k_x^2+k_z^2\right)\nu t}\\
\vec{\omega} &= (k_x^2+k_z^2)V\sin(k_x x)\sin(k_z z)e^{-\left(k_x^2+k_z^2\right)\nu t}\vec{e}_y
\end{aligned}
$$

The figure 11 shows an example of the initial condition for $k_x = k_z = 2\pi$

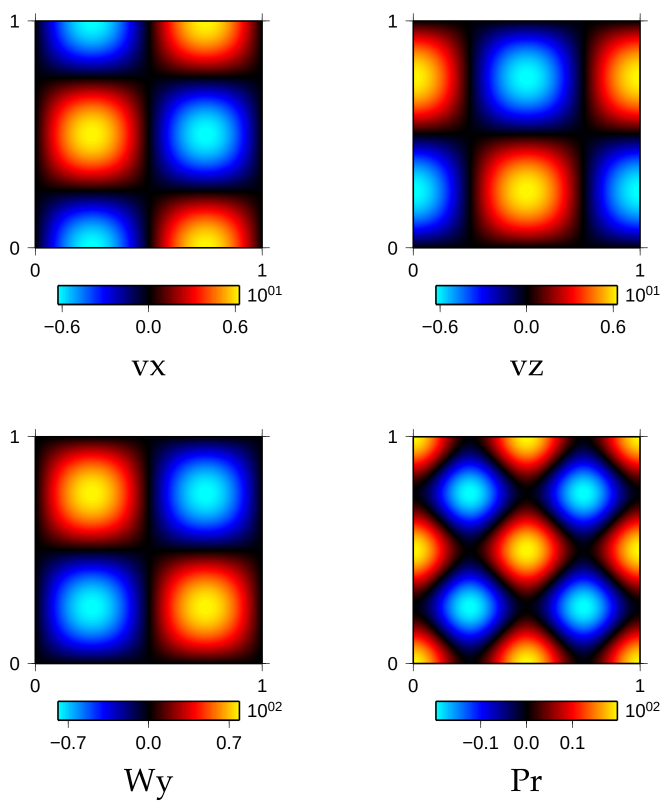


Figure 11: Taylor-Green vortex. The velocity (v_x, v_z), the vorticity w_y and the pressure Pr for $k_x = k_z = 2\pi$.

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