

PHYC90045 Introduction to Quantum Computing

## Week 4



**Lecture 7**  
Reversible computation, one qubit adder, the Deutsch-Josza algorithm, Universality in quantum computing.

**Lecture 8**  
Two basic quantum algorithms: Bernstein-Vazirani and Simon's Algorithms

**Lab 4**  
Logical statements, Reversible logic, Adder, Deutsch-Josza algorithm

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## Overview



In this lecture we will discuss reversible logic and our first quantum algorithm – the Deutsch-Josza algorithm,

1. Reversible (classical) logic
2. The Deutsch-Josza algorithm
3. Aside: Universality in quantum computing

Along the way we will encounter common patterns often turn up in quantum algorithms, and will highlight them because they will help make sense of what of future quantum circuits.

See:

- Kaye, 1.5, 6.1-6.4
- Nielsen and Chuang, 1.4, 3.1
- Reiffel, 6, 7.3-7.5

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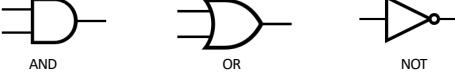
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## Universality in classical logic



A set of (classical) gates is said to be *functionally complete* (or “universal”) if every possible truth table (ie. Boolean function) can be expressed using members of the set.

For example, in classical logic: {AND, OR, NOT} is functionally complete.



Every logical circuit can be made from combinations of these gates. In fact, the NAND gate alone is universal. However, we cannot implement these gates directly in a quantum computer. AND/OR are **not reversible**.

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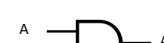
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# Irreversible Functions

We cannot implement AND or OR because these functions are irreversible.



A	B	A B
0	0	0
0	1	0
1	0	0
1	1	1

We cannot determine the inputs from the output.

Irreversible functions are *not unitary*.

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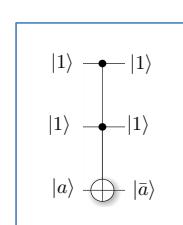
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## Reversible Logic

Classically, if we would like to calculate some Boolean function,  $f$ , we can construct a circuit out of AND, OR and NOT gates. However, AND and OR gates are **not reversible**, and so can't be implemented in a quantum computer.

But by use of reversible circuits and additional bits/qubits, we can express everything in terms of reversible gates (such as Toffoli):



The diagram shows a quantum circuit enclosed in a blue border. It consists of three horizontal lines representing qubits. The top two qubits are labeled  $|1\rangle$  and the bottom qubit is labeled  $|a\rangle$ . A Toffoli gate (a three-qubit gate with control on the top two and a target on the bottom) is applied between the first and second qubits. This is followed by a NOT gate (a single-qubit gate) applied to the third qubit. The output state is  $|\bar{a}\rangle$ .

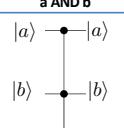
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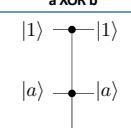

  
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# Reversible Logic and the Toffoli Gate

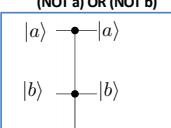
a AND b



a XOR b



(NOT a) OR (NOT b)



a	b	a $\wedge$ b
0	0	0
0	1	0
1	0	0
1	1	1

a	b	a $\wedge$ b
0	0	0
0	1	0
1	0	0
1	1	1

a	b	a $\oplus$ b
0	0	0
0	1	1
1	0	1
1	1	0

a	b	$\neg a \vee \neg b$
0	0	1
0	1	1
1	0	1
1	1	0

The Toffoli gate is **universal and reversible**. In principle every classical boolean function can be written in terms of reversible gates (such as Toffoli) which can be implemented on a quantum computer.

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# One Bit Adder

We can, for example, implement a one bit adder using only reversible gates:

We will now explain this circuit and in the lab we will extend this to a two bit adder.

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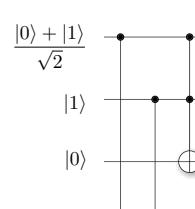
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# 1+1 Quantum Style

Here is what happens when we add together numbers in superposition:

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad “+” \quad |1\rangle$$



A quantum circuit diagram illustrating the addition of two qubits. The top row shows the input state  $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ . The bottom row shows the output state  $|0\rangle$ . The circuit consists of three horizontal lines representing qubits. The first qubit starts at the bottom with a circle containing a plus sign ( $+$ ). The second qubit starts at the bottom with a circle containing a plus sign ( $+$ ). The third qubit starts at the top with a circle containing a plus sign ( $+$ ). Vertical lines connect the first and second qubits. Horizontal lines connect the second and third qubits. The third qubit ends at the bottom with a circle containing a plus sign ( $+$ ).

Let's do the walkthrough...

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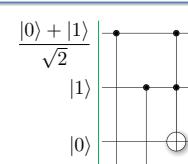
# One Qubit adder - example

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|1\rangle$

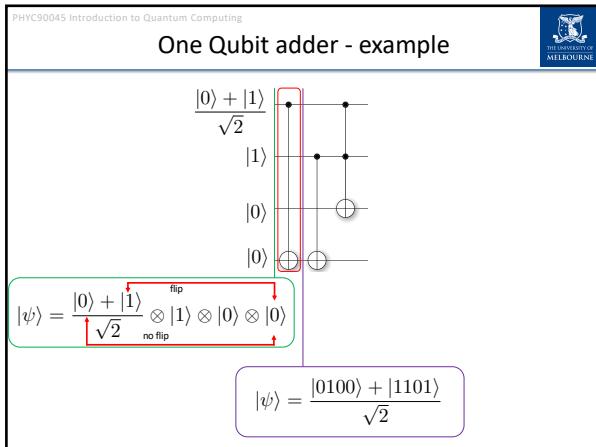
$|0\rangle$

$|0\rangle$

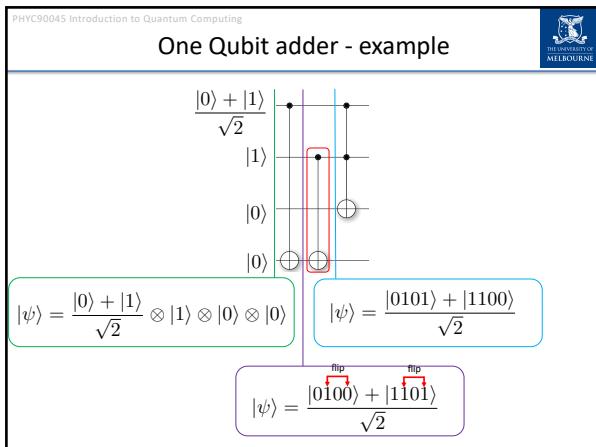


$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle$$

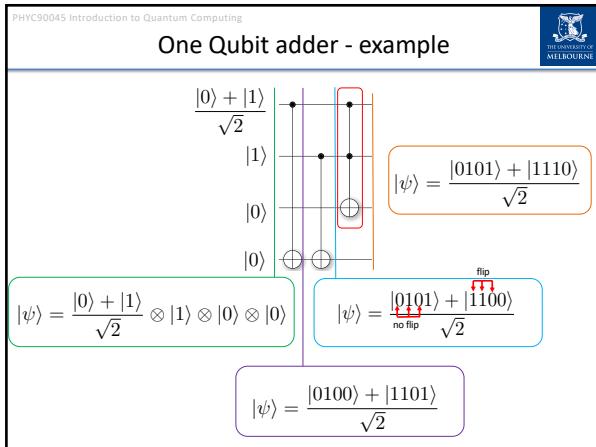
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## One Qubit adder - example

$$|\psi\rangle = \frac{|0101\rangle + |1110\rangle}{\sqrt{2}}$$

Or, considering the last two registers as a two-qubit register (binary notation):

$$|\psi\rangle = \frac{|0101\rangle + |1110\rangle}{\sqrt{2}}$$

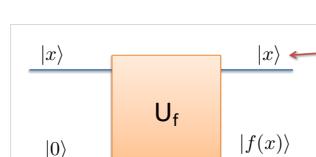
$$|\psi\rangle = \frac{|0\rangle|1\rangle|1\rangle + |1\rangle|1\rangle|2\rangle}{\sqrt{2}}$$

0+1=1      1+1=2

Note: this is an entangled state.

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We cannot compute **irreversible** functions directly (not unitary). One strategy which you often see to make an irreversible function reversible is simply to propagate the input to the output:



```

    graph LR
        In((|x>)) --> Uf[U_f]
        In --> Out((|f(x)>))
        Uf --> Out
        Uf --> PropOut((|x>))
        PropOut --> Out
    
```

The diagram illustrates a quantum circuit. An input state  $|x\rangle$  enters from the left. It passes through a unitary gate labeled  $U_f$ , which produces the output state  $|f(x)\rangle$ . Simultaneously, the input state  $|x\rangle$  is also passed through another path, indicated by a red arrow, which results in the calculated output  $|f(x)\rangle$ .

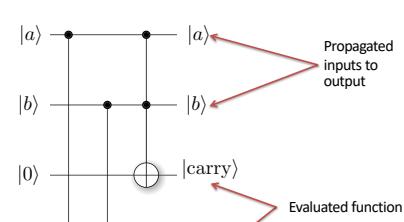
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# One Bit Adder

The adder is an example. Given  $a + b$ , we can't uniquely determine  $a$  and  $b$ . So we used this trick:



Propagated inputs to output

Evaluated function

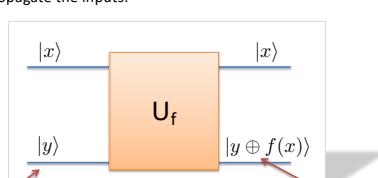
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# Implementing irreversible functions

One strategy which you often see to make an irreversible function reversible is simply to propagate the inputs:



Must deal with all possible inputs, not just 0.

Add (bit-by-bit modulo 2) input and calculated  $f(x)$

You will see this pattern in several of the quantum algorithms we will study.

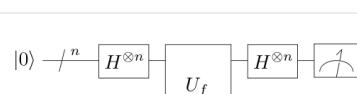
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# Deutsch-Josza algorithm

- Given a boolean function,  $f$ , determine if:  
 $f$  is constant (always gives the same result), or  
 $f$  is balanced (gives equal numbers of 0s and 1s)
- Classical algorithm (worst case)** needs  $2^n/2+1$  queries
- Quantum algorithm** needs just 1 query.



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Deutsch-Josza algorithm (2)

Let's take the example with just one bit/qubit input. There are 4 choices of function:

CONSTANT	$f_1(0) = 0$	
	$f_1(1) = 0$	
BALANCED	$f_2(0) = 1$	
	$f_2(1) = 1$	
	$f_3(0) = 0$	
	$f_3(1) = 1$	
$f_4(0) = 1$		
	$f_4(1) = 0$	

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## Example of a constant function



$$\left| \begin{array}{c} |x\rangle \xrightarrow{\text{---}} |x\rangle \\ |y\rangle \xrightarrow{\text{---} \quad X \quad \text{---}} |y \oplus 1\rangle \end{array} \right.$$

$$\left. \begin{array}{l} f(0) = 1 \\ f(1) = 1 \end{array} \right\}$$

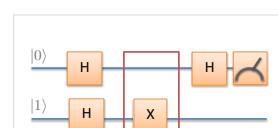
The circuit always flips (ie. NOT) the second qubit, regardless of the input. This function is **constant**, since the output is always 1.

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# Deutsch algorithm: constant function



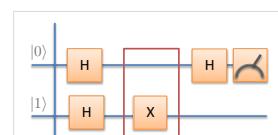
The function,  $U$ ,  
is implemented  
by the gates  
inside this box

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# Deutsch algorithm: walkthrough



$$|\psi\rangle = |0\rangle \otimes |1\rangle$$

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## Deutsch algorithm: walkthrough



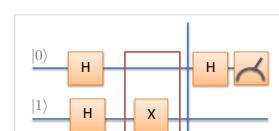
$$\begin{aligned}
 |\psi\rangle &= H|0\rangle \otimes H|1\rangle \\
 &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
 &= |+\rangle \otimes |-\rangle
 \end{aligned}$$

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# Deutsch algorithm: walkthrough



$$\begin{aligned}
 |\psi\rangle &= |+\rangle \otimes X |-\rangle \\
 &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes X \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
 &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|1\rangle - |0\rangle}{\sqrt{2}} \\
 &= -|+\rangle \otimes |-\rangle
 \end{aligned}$$

Global phase is unmeasurable

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## Deutsch algorithm: walkthrough



$$\begin{aligned} |\psi\rangle &= -H|+\rangle \otimes |- \rangle \\ &= -|0\rangle \otimes |- \rangle \end{aligned}$$

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## Deutsch algorithm: walkthrough

$$|\psi\rangle = -|0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= -|0\rangle \otimes |- \rangle$$

We will measure “0” with 100% probability. This indicates (with a single evaluation of the function) that the function is **constant**.

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## Example of a balanced function


$$\left\{ \begin{array}{l} f(0) = 0 \\ f(1) = 1 \end{array} \right.$$

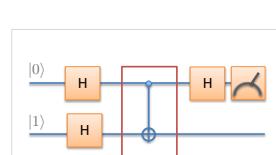
The circuit only flips (ie. NOT) the second qubit, if the input is a 1. This function is **balanced**, since the output has equal numbers of 0 and 1 output.

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## Deutsch algorithm: balanced function



A quantum circuit diagram illustrating the Deutsch algorithm for a balanced function. The circuit has two input wires, labeled  $|0\rangle$  and  $|1\rangle$ . Each wire passes through a Hadamard gate ( $H$ ). The wires then enter a red rectangular box representing the function  $U$ . Inside this box, there is a single control point on the top wire and a blue dashed circle on the bottom wire, indicating a CNOT gate between them. After exiting the box, both wires pass through another Hadamard gate ( $H$ ). Finally, the two wires are measured, represented by meter symbols at the end of each wire.

The function,  $U$ ,  
is implemented  
by the gates  
inside this box

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## Deutsch algorithm: walkthrough



$$|\psi\rangle = |0\rangle \otimes |1\rangle$$

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## Deutsch algorithm: walkthrough



$$\begin{aligned}
 |\psi\rangle &= H|0\rangle \otimes H|1\rangle \\
 &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
 &= |+\rangle \otimes |-\rangle
 \end{aligned}$$

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## Deutsch algorithm: walkthrough

$$|\psi\rangle = CNOT|+\rangle|-\rangle$$

This is a common pattern called “phase kickback”

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## Phase kickback

Consider X applied to the output register:

$$\begin{aligned} X |-\rangle &= X \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= \frac{|1\rangle - |0\rangle}{\sqrt{2}} \\ &= -|-\rangle \end{aligned}$$

If we were to apply an X gate to the target qubit we get a *global phase change*. Otherwise the state is unchanged.

What would happen when apply a *control-X* gate?

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## Phase kickback

Now with a control-X gate:

$$\begin{aligned} \text{CNOT } |+\rangle |-\rangle &= \text{CNOT} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|0\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}} \frac{|1\rangle - |0\rangle}{\sqrt{2}} \\ \text{CNOT } |+\rangle |-\rangle &= \frac{|0\rangle |-\rangle - |1\rangle |-\rangle}{\sqrt{2}} \end{aligned}$$

If we were to apply a control-X gate then any control states which apply the X-gate receive a phase change. The “1” state receives the (relative) phase change in this case.

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## Phase kickback

$$\begin{aligned} \text{CNOT } |+\rangle |-\rangle &= \frac{|0\rangle |-\rangle - |1\rangle |-\rangle}{\sqrt{2}} \\ &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} |-\rangle \\ &= |-| |-\rangle \end{aligned}$$

This causes the **phase** to be applied to the *control qubit*. This is known as phase kickback.

The state of the target qubit remains unchanged.

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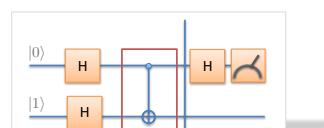
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# Deutsch algorithm: walkthrough



$$|\psi\rangle = CNOT|+\rangle|- \rangle = |-\rangle|- \rangle$$

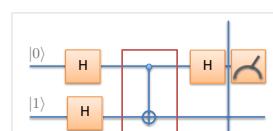
Phase kickback changes the state of the *control* qubit.

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# Deutsch algorithm: walkthrough



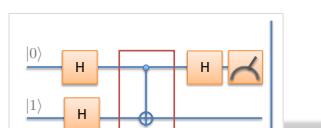
$$\begin{aligned} |\psi\rangle &= H|-\rangle \otimes |-\rangle \\ &= |1\rangle \otimes |-\rangle \end{aligned}$$

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# Deutsch algorithm: walkthrough



$$|\psi\rangle = |1\rangle \otimes |- \rangle$$

We will measure “1” with 100% probability. This indicates (with a single evaluation of the function) that the function is **balanced**.

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Deutsch-Josza (3)

			Measured
CONSTANT	$f_2(0) = 0$	$f_2(1) = 0$	0
	$f_2(0) = 1$	$f_2(1) = 1$	0
BALANCED	$f_4(0) = 0$	$f_4(1) = 1$	1
	$f_4(0) = 1$	$f_4(1) = 0$	1
	Classically, this would require <u>two queries</u> .		

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Phase kickback for balanced functions

Constant functions

Balanced functions

The control qubit is unchanged by the **constant** functions.

The same phase kickback pattern that we just saw applies to both **balanced** functions

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Multiple qubits: Deutsch-Josza

This means that there are multiple qubits in the register

There are multiple Hadamard gates here

Only a single qubit here

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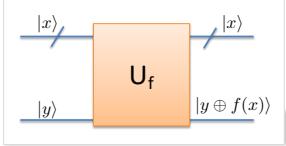
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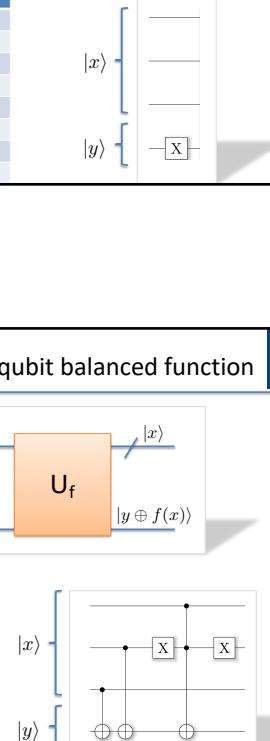
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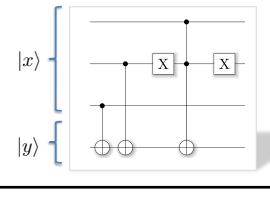
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Example of multi-qubit constant function



x	f(x)
000	1
001	1
010	1
011	1
100	1
101	1
110	1
111	1





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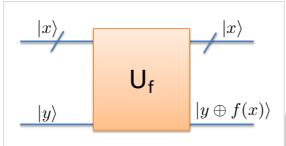
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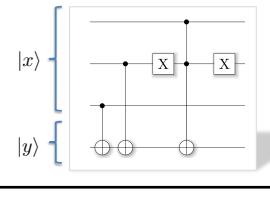
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Example of multi-qubit balanced function



x	f(x)
000	0
001	1
010	1
011	0
100	1
101	0
110	1
111	0





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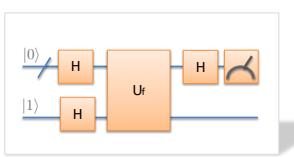
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Multiple qubits: Deutsch-Josza



Let's walkthrough this circuit...

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# Recap: binary and decimal representations



n qubits

$|0\rangle$     $H$

shorthand notation

$|0\rangle \xrightarrow{H^{\otimes n}} |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \dots \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|\psi\rangle = \left[ \frac{1}{\sqrt{2}} \right]^n (|00\dots0\rangle + \dots + |11\dots1\rangle)$

i.e. even superposition over binary rep of integers:  $i = 0$  to  $2^n - 1$

In general we use two representations in the QUI ( $N = 2^n$ ):

"binary"  
 $|\psi\rangle = a_{0\dots00} |0\dots00\rangle + a_{0\dots01} |0\dots01\rangle + a_{0\dots10} |0\dots10\rangle + \dots + a_{1\dots11} |1\dots1\rangle$

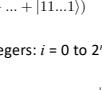
"decimal"  
 $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle + \dots + a_{N-1} |N-1\rangle$

e.g.  $a_{101} |101\rangle$

$|\psi\rangle = \sum_{\phi} a_{\phi} |\phi\rangle$

$a_{\phi} = |a_{\phi}| e^{i\theta_{\phi}}$

Binary  
 DECIMAL  
 EXPONENT (i.e.  $a_{\phi}$ )  
 MAGNITUDE :  $a_{\phi}$   
 PHASE ANGLE :  
 Decimals : Radians  
 0° 45° 90° 135° 180° 225° 270° 315°

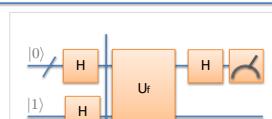


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# Deutsch-Josza Walkthrough



After the initial Hadamard gates, the state is ( $n$  qubits,  $N = 2^n$ ):

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Equal (even) superposition of states

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## General Function Phase Kickback

Using phase kickback, after the function has been applied:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_x (-1)^{f(x)} |x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If the function evaluates to "1" then the target qubit is flipped, and we pick up a phase. Otherwise, there is no phase applied. This is a simple way to write that.

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Hadamard applied to a general state

Amplitude  $a_z \rightarrow$  how many times does the binary representation of  $z$  and  $x$  have 1's in the same location?

$$|x\rangle \xrightarrow{H^{\otimes n}} |x\rangle = \frac{1}{\sqrt{N}} \sum_{z=0}^{N-1} a_z |z\rangle \quad x_0 z_0 + x_1 z_1 + x_2 z_2 + \dots + x_n z_n$$

Shorthand for the bitwise dot product is:  $x \cdot z = \sum_{j=0}^n x_j z_j$

When 1's in the same location, we get a sign change  $\rightarrow (-1)^{x \cdot z}$

Hadamards applied to a general state ( $n$  qubits,  $N = 2^n$ ):

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{N}} \sum_{z=0}^{N-1} (-1)^{x \cdot z} |z\rangle$$


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e.g. Hadamard applied to a general state

$H^{\otimes 3} |000\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$H^{\otimes 3} |000\rangle = \frac{1}{\sqrt{2^3}} \underbrace{(|000\rangle + |001\rangle + \dots + |111\rangle)}_{\text{bitwise: } x \cdot z = 0}$

$H^{\otimes 3} |x=0\rangle = \frac{1}{\sqrt{2^3}} \sum_{z=0}^{2^3-1} (-1)^{x.z} |z\rangle$

$H^{\otimes 3} |x=4\rangle = \frac{1}{\sqrt{2^3}} \sum_{z=0}^{2^3-1} (-1)^{x.z} |z\rangle$

$H^{\otimes 3} |x\rangle = \frac{1}{\sqrt{2^3}} \sum_{z=0}^{2^3-1} (-1)^{x.z} |z\rangle$

General for  $n=3$

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$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{N}} \sum_{z=0}^{N-1} (-1)^{x \cdot z} |z\rangle$$

Before and after the final Hadamard gates:

$$|\psi\rangle = H^{\otimes n} \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{f(x)} |x\rangle$$

$$|\psi\rangle = \frac{1}{N} \sum_{x=0}^{N-1} (-1)^{f(x)} \sum_{z=0}^{N-1} (-1)^{x \cdot z} |z\rangle$$

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Constant function

For a constant function ( $f(x) = 0$  for all  $x$ , or  $f(x) = 1$  for all  $x$ ):

$$|\psi\rangle = \frac{1}{N} \sum_{x=0}^{N-1} (-1)^{f(x)} \sum_{z=0}^{N-1} (-1)^{x \cdot z} |z\rangle$$

$$= \frac{(-1)^{f(0)}}{N} \sum_{x=0}^{N-1} \sum_{z=0}^{N-1} (-1)^{x \cdot z} |z\rangle$$

$$= \frac{(-1)^{f(0)}}{N} \sum_{z=0}^{N-1} \left( \sum_{x=0}^{N-1} (-1)^{x \cdot z} \right) |z\rangle$$

$$= (-1)^{f(0)} |z=0\rangle$$

So for a constant function "0" will always be measured (global phase is unimportant).

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Balanced Function

$$|\psi\rangle = H^{\otimes n} \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{f(x)} |x\rangle$$

$$|\psi\rangle = \frac{1}{N} \sum_{x=0}^{N-1} (-1)^{f(x)} \sum_{z=0}^{N-1} (-1)^{x \cdot z} |z\rangle$$

For a balanced function (equal number of  $f(x) = 0$  and  $f(x) = 1$ ):

$$|\psi\rangle = \frac{1}{N} \sum_{z=0}^{N-1} \left( \sum_{x, f(x)=0} (-1)^{x \cdot z} - \sum_{x, f(x)=1} (-1)^{x \cdot z} \right) |z\rangle$$

Which has zero amplitude for the  $|z=0\rangle$  state, and non-zero for other states.

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If 0 is measured, then the function is constant.  
If any other value is measured, then the function is balanced.

The Deutsch-Josza algorithm evaluates if a function is constant or balanced with a single query. Classically we would require  $O(2^n)$  queries.

Of course, there are classical probabilistic algorithms with establish with high probability in few queries, but only with high probability of success not with certainty.

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Deutsch-Josza algorithm

- Given a Boolean function,  $f$ , determine if:
  - $f$  is constant (always gives the same result)
  - $f$  is balanced (gives equal numbers of 0s and 1s)
- Classical algorithm** (worst case) needs  $2^n/2+1$  queries
- Quantum algorithm** needs just 1 query.

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Aside: Universality in Quantum Computing

In classical computing the NAND gate is universal: every Boolean function can be implemented as a sequence of NAND (NOT AND) gates

In quantum computing every quantum circuit can be expressed as a sequence of:

CNOT      Single Qubit Rotations

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**Aside: Universality in Quantum Computing**

In classical computing the NAND gate is universal: every Boolean function can be implemented as a sequence of NAND (NOT AND) gates



In quantum computing every quantum circuit can be approximated as a sequence of:



CNOT                      Hadamard                      T Gate  
 $(Z \text{ axis}, \pi/4)$

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## Aside: Outline of Universal Gate Set Proof

**(1)** The following sequence of gates creates an irrational fraction of  $2\pi$  angle rotation gate:

$$THTH$$

Specifically, you can show by direct multiplication that it is a rotation around

$$\vec{n} = \left( \cos \frac{\pi}{8}, \sin \frac{\pi}{8}, \cos \frac{\pi}{8} \right)$$

By an angle defined by:  $\cos \frac{\theta}{2} = \cos^2 \frac{\pi}{8}$

**(2)** You can use this to approximate (within some error,  $\delta$ ) **every rotation around  $n$** . This takes  $\sim 2n/\delta$  applications of THTH.



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## Aside: Outline of Universal Gate Set Proof

**(3)** The following sequence of gates creates an irrational fraction of  $2\pi$

$$HTHT$$

Around a different axis:

$$\vec{m} = \left( \cos \frac{\pi}{8}, -\sin \frac{\pi}{8}, \cos \frac{\pi}{8} \right)$$

$$\text{By an angle defined by: } \cos \frac{\theta}{2} = \cos^2 \frac{\pi}{8}$$

**(4)** You can use this to approximate (within some error,  $\epsilon$ ) **every rotation around  $\mathbf{m}$** . This takes  $\sim 2\pi/\epsilon$  applications of HTHT.

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### Aside: Outline of Universal Gate Set Proof



(5) These gates can (approximately) implement any rotation around two different axes, you can use this to approximate any single qubit rotation.

(6) Single qubit rotations plus the CNOT gate is universal

(7) Therefore {H, T, CNOT} is universal.

Full proof (carefully keeping track of approximations)  
is found in Nielsen and Chuang 4.5.3

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### Week 4



**Lecture 7**  
Universality in quantum computing, Reversible computation, one qubit adder, the Deutsch-Josza algorithm

**Lecture 8**  
Two basic quantum algorithms: Bernstein-Vazirani and Simon's Algorithms

**Lab 3**  
Logical statements, Reversible logic, Adder, Deutsch-Josza algorithm

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