PHYC90045 Introduction to Quantum Computing

This Week

Lecture 9
Quantum search – introduction to Grover's algorithm for amplitude amplification, geometric interpretation

Lecture 10
Optimality, Succeeding with Certainty, Quantum Counting

Lab
Grover's algorithm

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Grover's Algorithm

Physics 90045
Lecture 9

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Introduction to Grover's algorithm

- This lecture: Grover's search algorithm
  - Grover's algorithm
  - Worked Example
  - Geometric interpretation

References:

Reiffel, Chapter 9.1-9.2
Kaye, Chapter 8.1-8.2
Nielsen and Chuang, Chapter 6.1-6.2

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Reminder: Outer Product

For two quantum states $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

We can define an outer product between them:

$$|\psi\rangle\langle\phi| = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c^* & d^* \end{bmatrix}$$

$$= \begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$|1\rangle\langle 2|$

For number basis states, this specifies a matrix with a single "1" in the location 1,2. In general:

$|\text{row}\rangle\langle\text{column}|$

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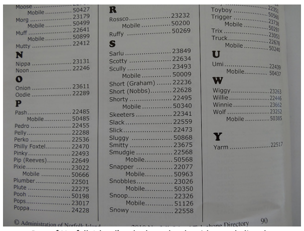
Unordered Search

Grover's algorithm performs a similar* problem to this: You are given *a telephone book*

And a phone number: *23675*

Your task:
Find the name which goes with that number…



Part of Norfolk Island's telephone book, with people listed by nickname (Photo: Wikicommons)

* Not all that similar, better examples later….

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Quantum search – Grover's problem

Given an black box (oracle), U_f, which computes the function:

$$f: \{0,1\}^n \rightarrow \{0,1\}$$

Find an x s.t. $f(x) = 1$

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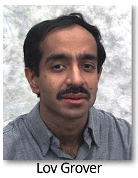
Grover's Algorithm (1996)

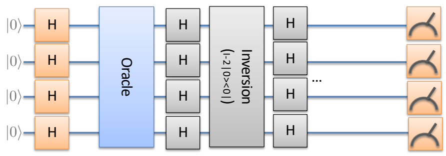
- Unordered search, find one marked item among many
- Classically, this requires N/2 queries to the oracle
- Quantum mechanically, requires only O(√N) queries.



Lov Grover

Simple problem = search for one integer marked by the oracle.

High level structure:



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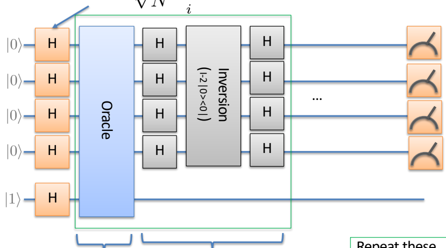
Two basic steps in Grover's algorithm

Quantum database: $|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$ (i.e. all integers 0 to N-1)



The "oracle" Identifies a particular marked state, m

$I - 2|\Phi\rangle\langle\Phi|$ "Inversion about the mean"

Repeat these two operations O(√N) times

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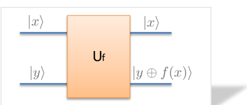
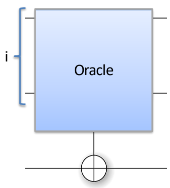
The Oracle

The task of recognizing the correct solution goes to the "oracle".



Binary function, or "oracle" Identifying a marked state, m



Designed to flip the last bit if the input, i, is a solution

The oracle is just a Boolean function (as seen in previous lectures)

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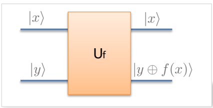
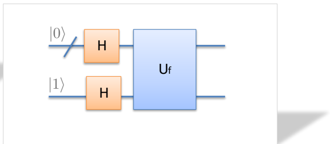
Phase kickback for Boolean function



Binary function, or "oracle"



After the function has been applied:

Target qubit remains the same

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} (-1)^{f(i)} |i\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If the oracle function evaluates to "1" then the target qubit is flipped, and we pick up a phase (associated with the control qubit state). Otherwise, there is no phase applied. This is a simple way to write that.

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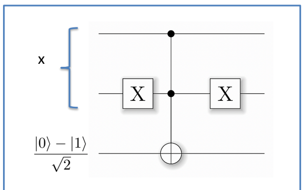
Example: Oracle recognizing the state "2 = |10>"



Phase kickback

$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to -|10\rangle$$
$$|11\rangle \to |11\rangle$$

The effect on each of the 4 states in the 2-qubit control register, x:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I - 2|10\rangle\langle 10|$$

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The marked state

Initially in Grover's algorithm, we will be searching for a *single (integer) solution*, ***m***. In that case the effect of the oracle on the control register is:

$$I - 2|m\rangle\langle m| \quad \text{(in decimal ket notation)}$$

As a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

-1 in the mth position

Here, as in future slides, we are only writing out the control qubits (in this case 2 qubits only).

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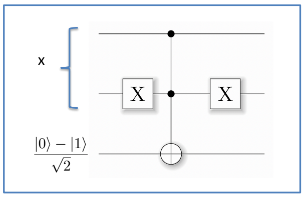
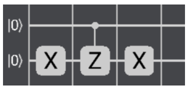
Example: Oracle recognizing the state "2 = |10>"



Phase kickback

$$
\begin{aligned}
|00\rangle &\to |00\rangle \\
|01\rangle &\to |01\rangle \\
|10\rangle &\to -|10\rangle \\
|11\rangle &\to |11\rangle
\end{aligned}
$$

In practice we can implement the oracle without the check qubit using a controlled-Z gate (ex. Show the circuit right marks the state |10>, i.e. |m=2>).



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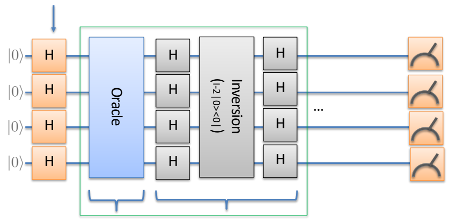
Two steps to Grover's algorithm

Set up "data base"



$I - 2|m\rangle\langle m|$ The oracle

$I - 2|\Phi\rangle\langle\Phi|$ "Inversion about the mean"

Repeat these two operations O(√N) times

One iteration of Grover where $|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$

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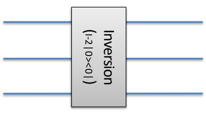
Unpicking the details: "Inversion" operation

The "Inversion" part is just applying a phase to the zero state:



$$I - 2|0\rangle\langle 0| = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

How? Recall outer product etc: $|\psi\rangle\langle\phi| = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c^* & d^* \end{bmatrix} = \begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes (1 0 \dots 0) = \begin{pmatrix} 10\dots0 \\ 00\dots0 \\ \vdots \end{pmatrix} \qquad I = \begin{pmatrix} 100\dots0 \\ 010\dots0 \\ 001\dots0 \\ \vdots \end{pmatrix}$$

$$I - 2|0\rangle\langle 0| = \begin{pmatrix} 100\dots0 \\ 010\dots0 \\ 001\dots0 \\ \vdots \end{pmatrix} - 2\begin{pmatrix} 10\dots0 \\ 00\dots0 \\ \vdots \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

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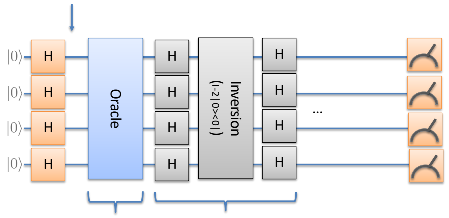
Inversion about the mean

$$|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$$ Set up "data base"



$$I - 2|\Phi\rangle\langle\Phi|$$

"Inversion about the mean"…let's see how that works.

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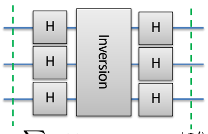
Apply inversion about the mean to general state



$|\Psi\rangle = \sum_i a_i|i\rangle$ $|\Psi'\rangle$

Applying Hadamards both sides:

$$I - 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n}$$
$$= I - 2|\Phi\rangle\langle\Phi|$$

$|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$ Equal superposition

$$|\Psi\rangle = \sum_i a_i|i\rangle \to |\Psi'\rangle = (I - 2|\Phi\rangle\langle\Phi|)\sum_i a_i|i\rangle$$

General state

$$= \sum_i a_i|i\rangle - 2\frac{1}{\sqrt{N}}\sum_k |k\rangle\frac{1}{\sqrt{N}}\sum_j\langle j|\sum_i a_i|i\rangle$$
$$= \sum_i a_i|i\rangle - 2\sum_k |k\rangle\left(\frac{1}{N}\sum_j a_j\right)$$
$$= \sum_i (a_i - 2A)|i\rangle$$

$A \equiv \left(\frac{1}{N}\sum_j a_j\right)$ Average amplitude in state $|\Psi\rangle$

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Inversion about the mean

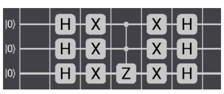
Consider a general state. The resulting amplitude from the "Inversion about the mean" step is:

$$\sum_i a_i|i\rangle \to \sum_i (a_i - 2A)|i\rangle$$

Original amplitude

Average amplitude

In practice on the QUI…



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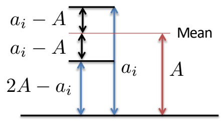
Inversion about the mean

Amplitudes of the state, before and after:



When the state undergoes this transformation:

$$\sum_i a_i |i\rangle \rightarrow -\sum_i (2A - a_i)|i\rangle$$

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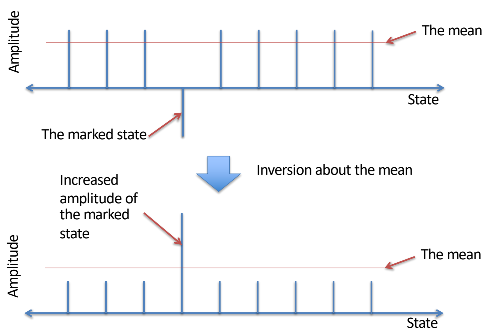
Effect of inversion about the mean



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Interactive Example

https://codepen.io/samtonetto/full/BVOGmW

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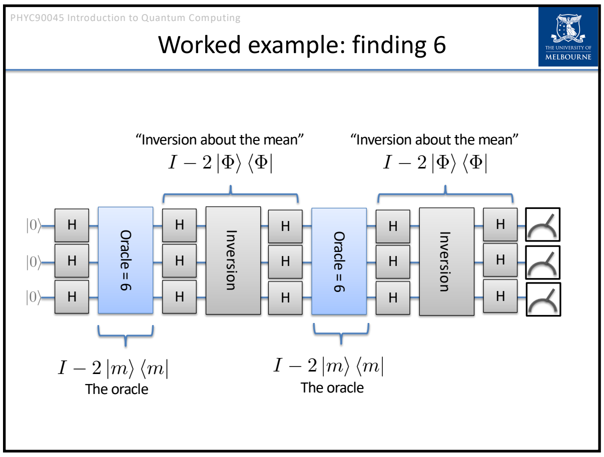
PHYC90045 Introduction to Quantum Computing
Worked example: finding 6
"Inversion about the mean"
$I - 2|\Phi\rangle\langle\Phi|$
"Inversion about the mean"
$I - 2|\Phi\rangle\langle\Phi|$
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
Oracle = 6
Inversion
Oracle = 6
Inversion
$I - 2|m\rangle\langle m|$
The oracle
$I - 2|m\rangle\langle m|$
The oracle

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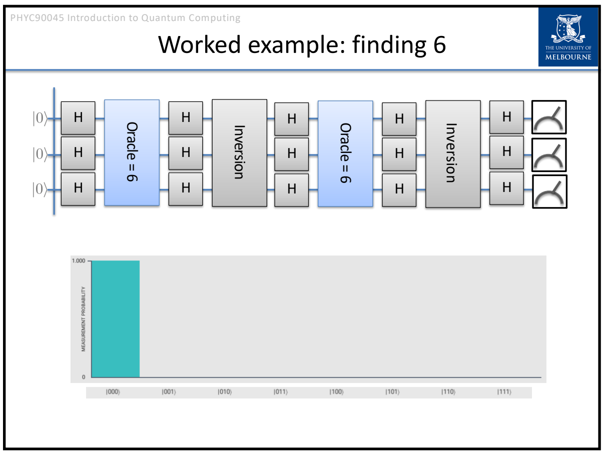
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Worked example: finding 6
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
Oracle = 6
Inversion
Oracle = 6
Inversion

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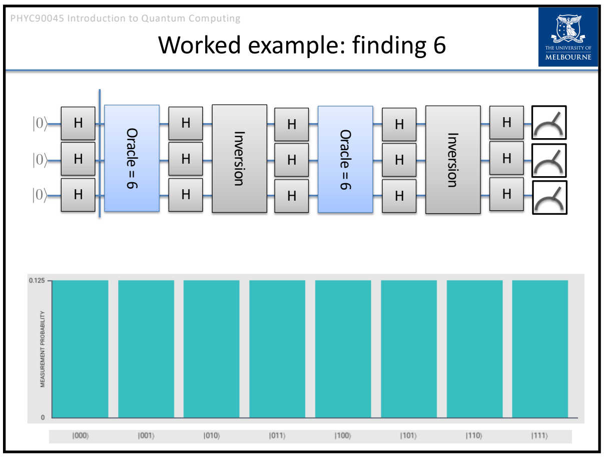
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Worked example: finding 6
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
Oracle = 6
Inversion
Oracle = 6
Inversion

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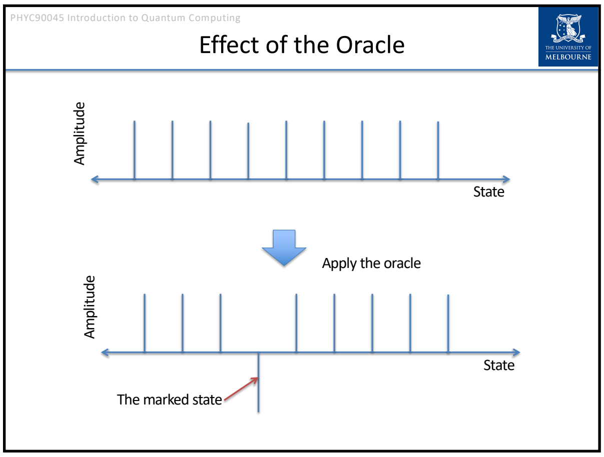
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Effect of the Oracle
Amplitude
State
Apply the oracle
Amplitude
State
The marked state

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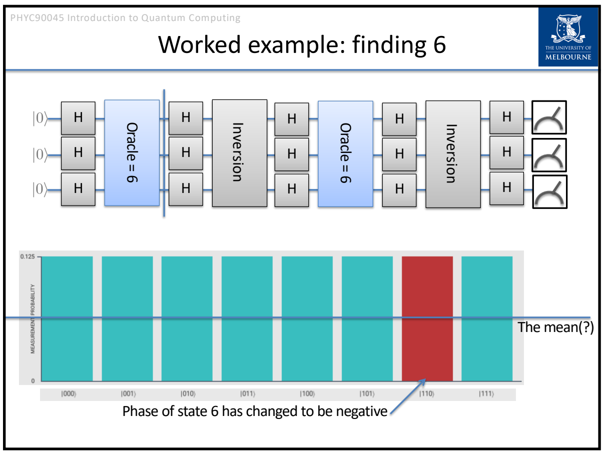
PHYC90045 Introduction to Quantum Computing
Worked example: finding 6
|0⟩
|0⟩
|0⟩
H
Oracle = 6
Inversion
Oracle = 6
Inversion
The mean(?)
Phase of state 6 has changed to be negative

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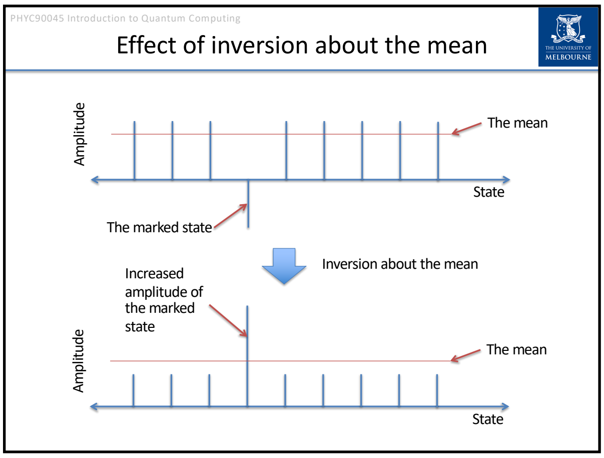
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Effect of inversion about the mean
Amplitude
The mean
State
The marked state
Inversion about the mean
Increased amplitude of the marked state
Amplitude
The mean
State

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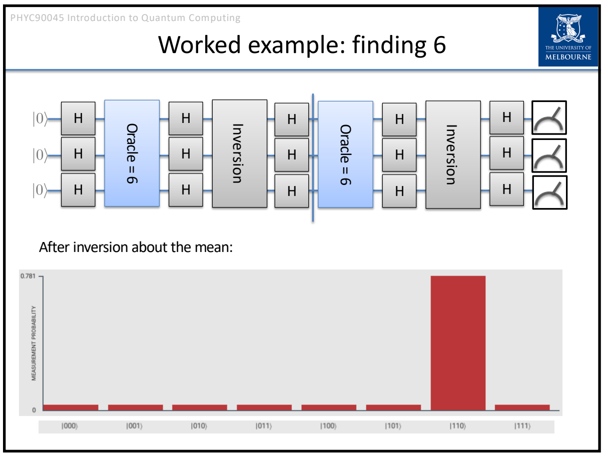
PHYC90045 Introduction to Quantum Computing
Worked example: finding 6
|0⟩
H
Oracle = 6
Inversion
After inversion about the mean:
0.781
0
|000⟩ |001⟩ |010⟩ |011⟩ |100⟩ |101⟩ |110⟩ |111⟩

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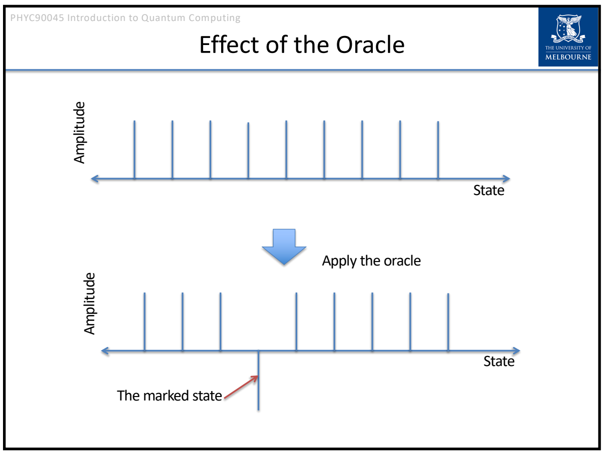
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Effect of the Oracle
Amplitude
State
Apply the oracle
Amplitude
State
The marked state

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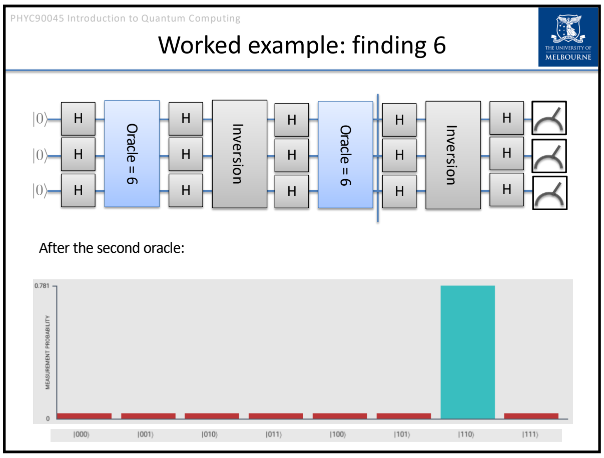
PHYC90045 Introduction to Quantum Computing
Worked example: finding 6
|0⟩
H
Oracle = 6
Inversion
After the second oracle:
0.781
0
|000⟩ |001⟩ |010⟩ |011⟩ |100⟩ |101⟩ |110⟩ |111⟩

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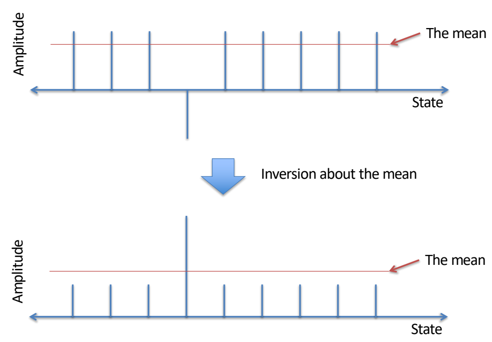
Effect of inversion about the mean



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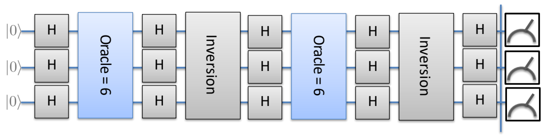
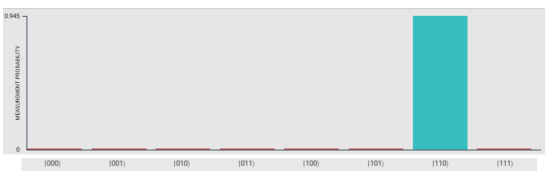
Worked example: finding 6



After the second oracle:



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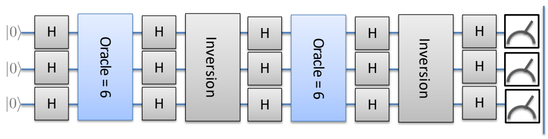
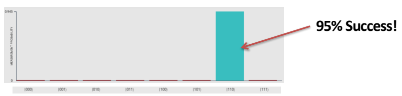
Grover's algorithm





Finally, we measure the marked state after 2 applications of the oracle, and find the marked state, 6 with 95% probability

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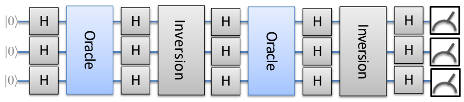
Geometric interpretation of Grover's algorithm



A very useful basis:

$$|a\rangle = |m\rangle$$ Solution!

$$|b\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \notin \text{solutions}} |i\rangle$$ Non-solutions...

We only need to consider the amplitude of these two states in Grover's algorithm.
Every operation is also real, so we can plot on a circle.

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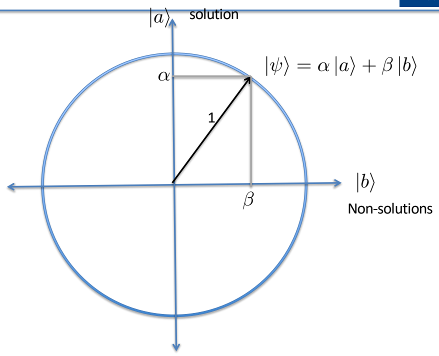
Geometric Interpretation



Every state in Grover's algorithm can be expressed as a superposition of these vectors

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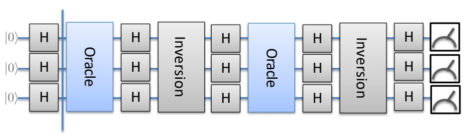
Equal superposition



Equal superposition state:

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$$
$$= \frac{1}{\sqrt{N}} |a\rangle + \frac{\sqrt{N-1}}{\sqrt{N}} |b\rangle$$

$$|a\rangle = |m\rangle \qquad |b\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \notin \text{solutions}} |i\rangle$$

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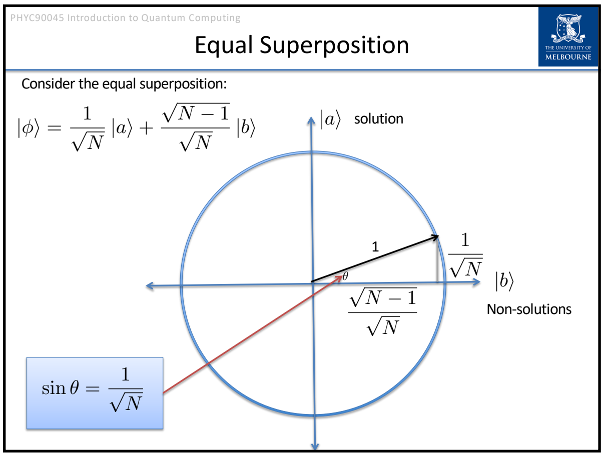
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Equal Superposition
Consider the equal superposition:
$|\phi\rangle = \frac{1}{\sqrt{N}}|a\rangle + \frac{\sqrt{N-1}}{\sqrt{N}}|b\rangle$
$|a\rangle$ solution
1
$\frac{1}{\sqrt{N}}$
θ
$|b\rangle$
$\frac{\sqrt{N-1}}{\sqrt{N}}$
Non-solutions
$\sin\theta = \frac{1}{\sqrt{N}}$

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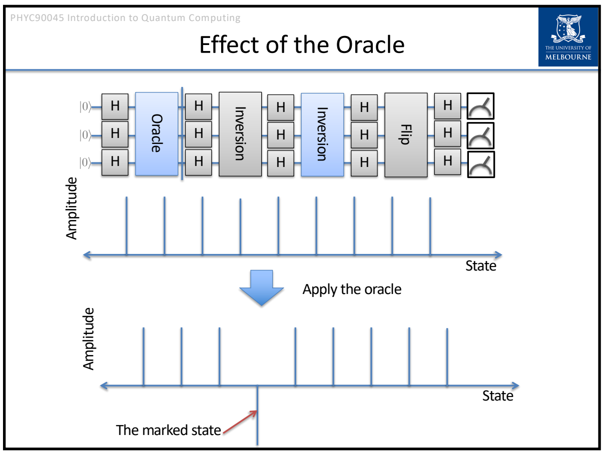
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Effect of the Oracle
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
Oracle
Inversion
Inversion
Flip
Amplitude
State
Apply the oracle
Amplitude
State
The marked state

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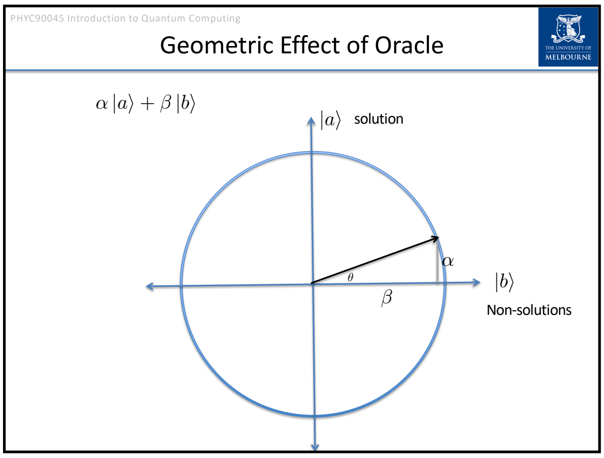
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Geometric Effect of Oracle
$\alpha|a\rangle + \beta|b\rangle$
$|a\rangle$ solution
α
θ
β
$|b\rangle$
Non-solutions

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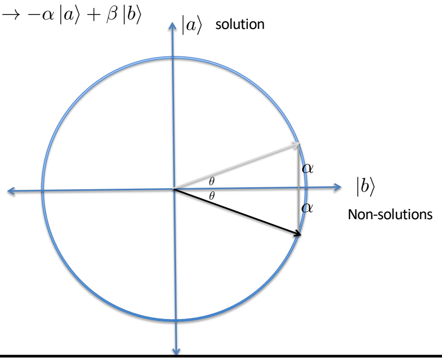
Geometric Effect of Oracle

$$\alpha\,|a\rangle + \beta\,|b\rangle \rightarrow -\alpha\,|a\rangle + \beta\,|b\rangle$$



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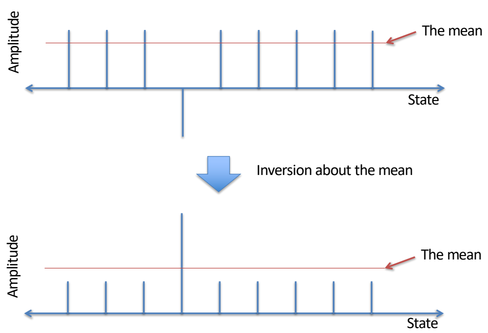
Effect of inversion about the mean



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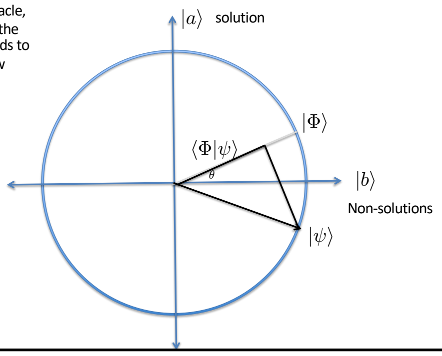
Inversion about the mean

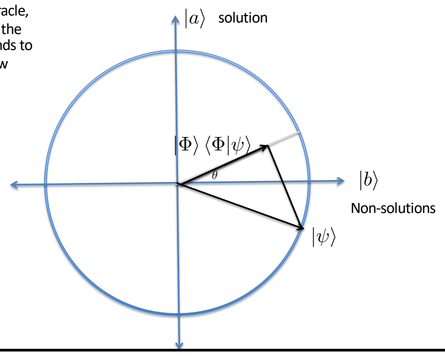
Similar to the oracle, inversion about the mean corresponds to a reflection. Now about **equal superposition**.



42

Inversion about the mean

Similar to the oracle, inversion about the mean corresponds to a reflection. Now about **equal superposition**.



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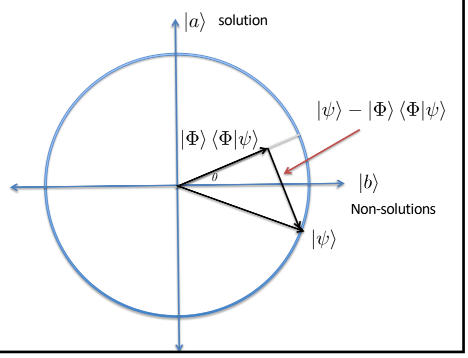
Inversion about the mean



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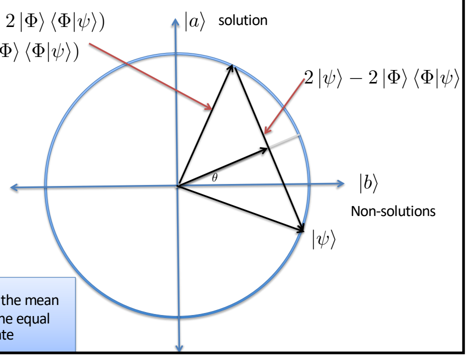
Inversion about the mean

$$|\psi\rangle - (2|\psi\rangle - 2|\Phi\rangle\langle\Phi|\psi\rangle)$$
$$= -(|\psi\rangle - 2|\Phi\rangle\langle\Phi|\psi\rangle)$$



Reflection about the mean reflects around the equal superposition state

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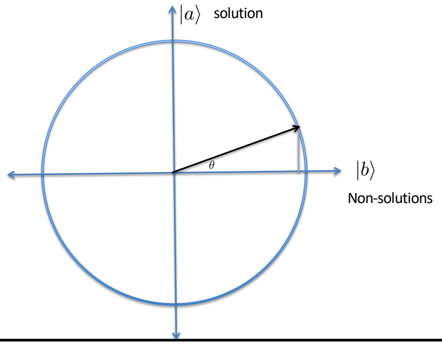
Geometric Effect of both Oracle and Inversion

Combining both effects:



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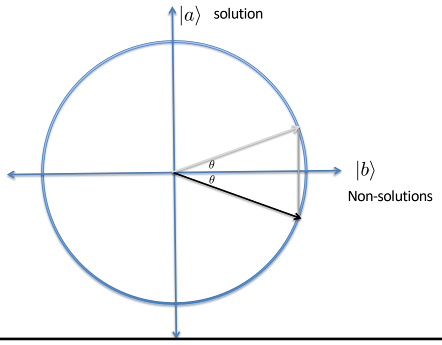
Geometric Effect of both Oracle and Inversion

Combining both effects:



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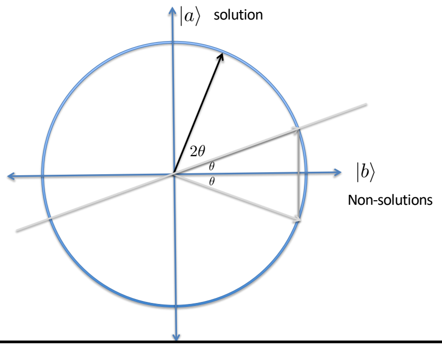
Geometric Effect of both Oracle and Inversion

Combining both effects:



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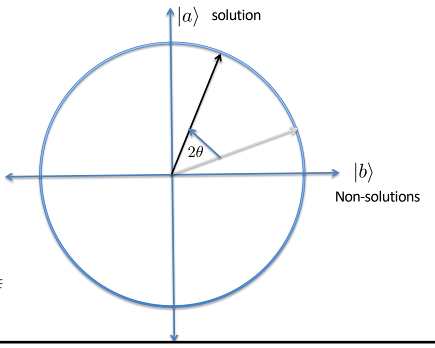
Total effect of one Grover iteration

Product of two reflections is a rotation.



$$\sin\theta = \frac{1}{\sqrt{N}}$$

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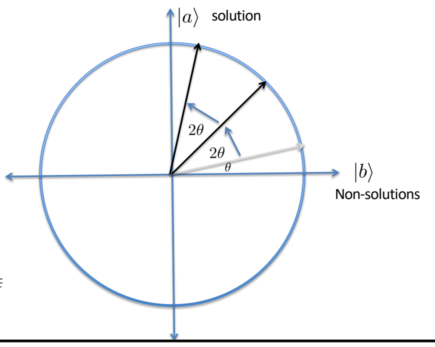
Many Grover iterations

Product of two reflections is a rotation.



$$\sin\theta = \frac{1}{\sqrt{N}}$$

50

How many iterations required?

$$\sin\theta = \frac{1}{\sqrt{N}}$$

For small angles,

$$\theta \approx \frac{1}{\sqrt{N}}$$

After n iterations, we rotate to have only marked solutions:

$$(2n+1)\theta = \frac{\pi}{2}$$

$$n \approx \frac{\pi}{4}\sqrt{N}$$

The number of steps, *n*, required scales as O(√N), and not with N as it would classically.

This is a “polynomial” rather than an “exponential” speedup.

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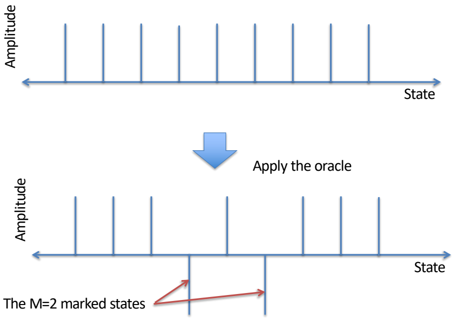
Multiple Solutions

There can be more than one solution to a problem.



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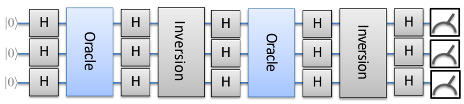
Geometric interpretation of Grover's algorithm



A very useful basis:

$$|a\rangle = \frac{1}{\sqrt{M}} \sum_{i \in \text{solutions}} |i\rangle$$ Solutions!

$$|b\rangle = \frac{1}{\sqrt{N-M}} \sum_{i \notin \text{solutions}} |i\rangle$$ Non-solutions...

We only need to consider the amplitude of these two states in Grover's algorithm.
Every operation is also real, so we can plot on a circle.

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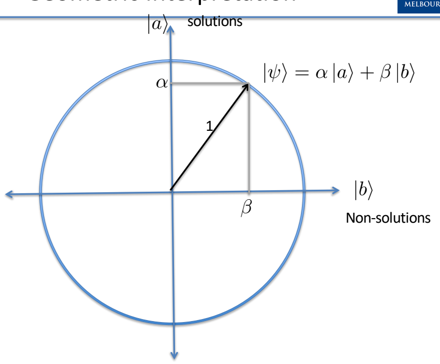
Geometric Interpretation



Every state in Grover's algorithm can be expressed as a superposition of these vectors

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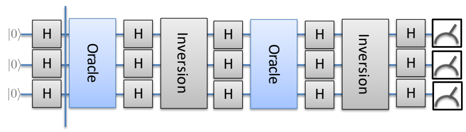
Equal superposition



Equal superposition state:

$$|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$$

$$|\Phi\rangle = \frac{\sqrt{M}}{\sqrt{N}}|a\rangle + \frac{\sqrt{N-M}}{\sqrt{N}}|b\rangle$$

$$|a\rangle = \frac{1}{\sqrt{M}}\sum_{i\in\text{solutions}} |i\rangle$$

$$|b\rangle = \frac{1}{\sqrt{N-M}}\sum_{i\notin\text{solutions}} |i\rangle$$

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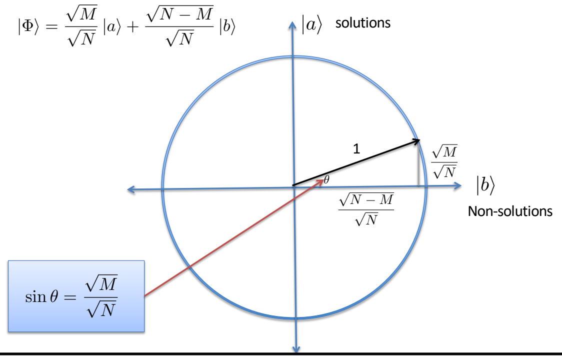
Equal Superposition

Consider the equal superposition:

$$|\Phi\rangle = \frac{\sqrt{M}}{\sqrt{N}}|a\rangle + \frac{\sqrt{N-M}}{\sqrt{N}}|b\rangle$$



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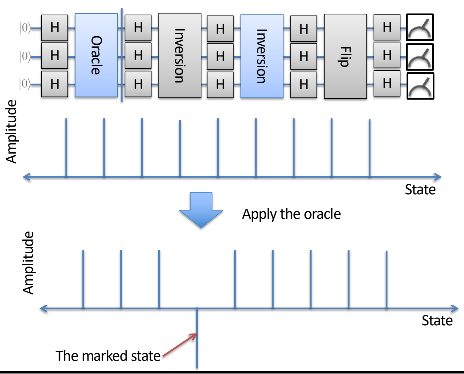
Effect of the Oracle



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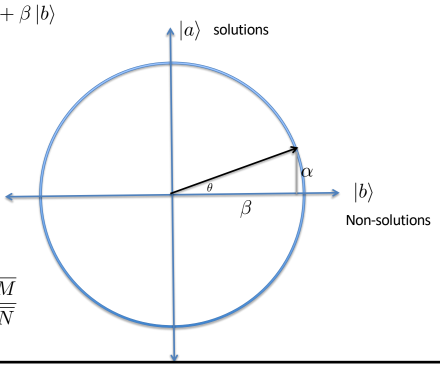
Geometric Effect of Oracle

$\alpha\,|a\rangle + \beta\,|b\rangle$



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

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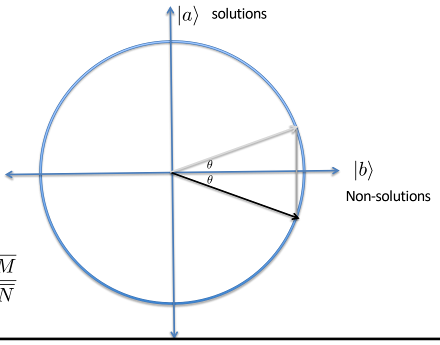
Geometric Effect of both Oracle and Inversion

Combining both effects:



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

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Geometric Effect of both Oracle and Inversion

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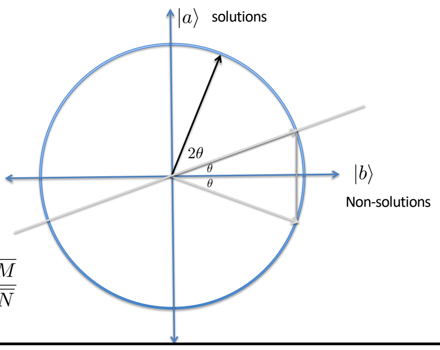

$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

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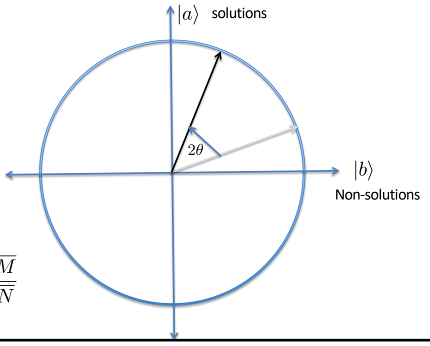
Total effect of one Grover iteration

Product of two reflections is a rotation.



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

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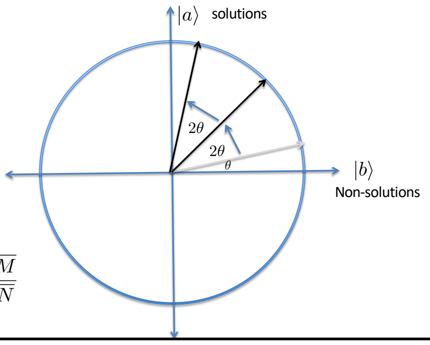
Many Grover iterations

Product of two reflections is a rotation.



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

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How many iterations required?

$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

For small angles,

$$\theta \approx \frac{\sqrt{M}}{\sqrt{N}}$$

After n iterations, we rotate to have only marked solutions:

$$(2n+1)\theta = \frac{\pi}{2}$$

$$n \approx \frac{\pi\sqrt{N}}{4\sqrt{M}}$$

Having multiple solutions is faster than searching for a single marked solution.

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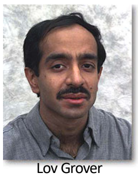
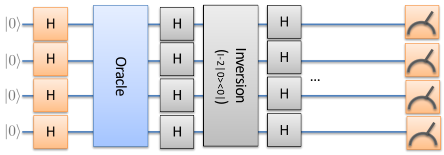
Grover's Algorithm

- Unordered search, find one marked item among many
- Classically, this requires N/2 uses of the oracle
- Quantum mechanically, requires only O(√N).



Lov Grover



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