

Week 8

Lecture 15

Simple classical error correction codes, Quantum error correction codes, stabilizer formalism, 5-qubit code, 7-qubit Steane code

Lecture 16

The more advanced quantum error correction codes, Fault Tolerance, QEC threshold, surface code.

Lab 8

Quantum error correction

Fault Tolerance and Topological Error Correction

Lecture 16

Overveiw

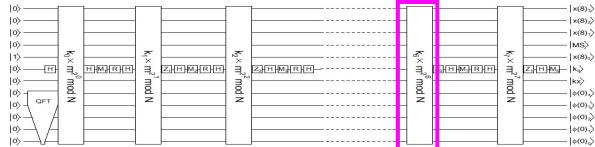
This lecture we will introduce more advanced error correction for quantum computers:

- Review some of the concepts from last lecture
- Fault Tolerance
- Concatenating quantum error correction codes
- The “threshold”
- Topological quantum error correction: The surface code

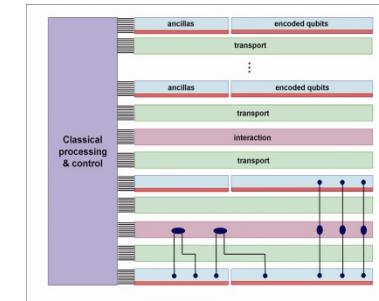
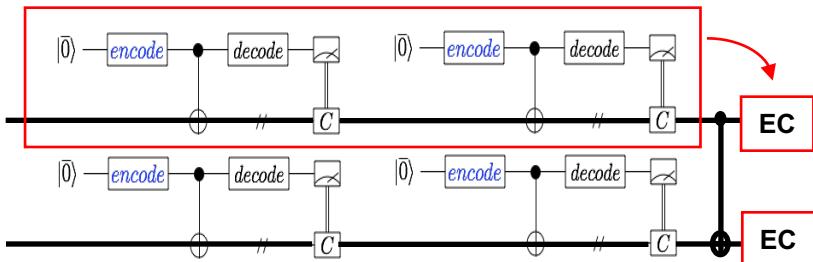
Rieffel, Chapter 11
Kaye, Chapter 10
Nielsen and Chuang, Chapter 10

Quantum computing hierarchy

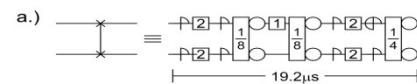
**algorithms
regime:
 $>>10^3$ qubits**

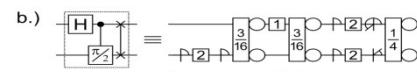


**QEC
architecture
regime:
1000's qubits**

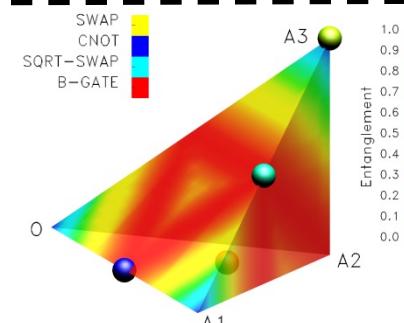


**quantum
control
& QEC regime:
10's qubits**

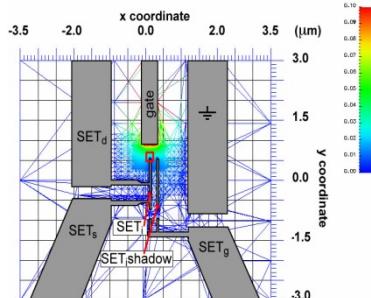
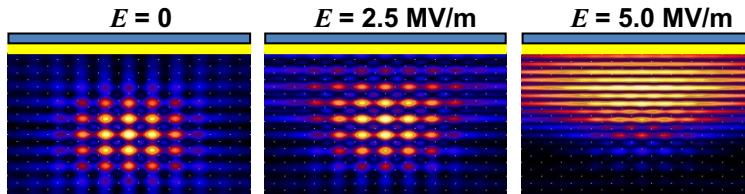
a.) 

b.) 

$\boxed{n} = e^{i\alpha\pi(X \otimes X + Y \otimes Y)}$ $\boxed{2} = R_x(\pi/2)$ $\boxed{1} = R_x(\pi)$
 $\oplus = R_z(\pi/4)$ $\oplus = R_z(5\pi/4)$ $\circ = R_z(0)$



**Physics
regime:
1-2 qubits**



Three qubit phase-flip code

Encode one qubit as three (redundant information)

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|+++ \rangle + \beta|--- \rangle$$

A rogue error occurs (*phase flip* of first qubit):


$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|-++ \rangle + \beta|\textcolor{red}{+}- - \rangle$$

Measure correlations (“stabilizers”) between qubits:

$$X_1 X_2 = -1$$

$$X_2 X_3 = +1$$

Correct errors which have occurred (in this case, phase flip the first qubit):


$$\alpha|\textcolor{red}{-}++ \rangle + \beta|\textcolor{red}{+}- - \rangle \rightarrow \alpha|+++ \rangle + \beta|--- \rangle$$

Correcting Both Bit and Phase Errors

$IXZZX$

$XIXZZ$

$ZXIXZ$

$ZZXIX$

Five qubit code is the smallest QEC which corrects both bit and phase flips.

Seven qubit “Steane” code is a CSS code which allows encoded operations to be easily applied.

$XXXIII$

$XXIIXXI$

$XIXIXIX$

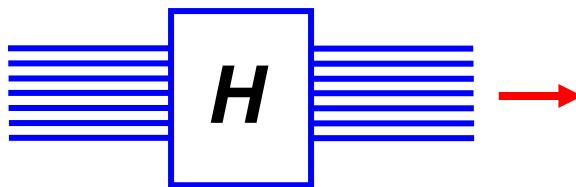
$ZZZZII$ I

$ZZIIZZI$ I

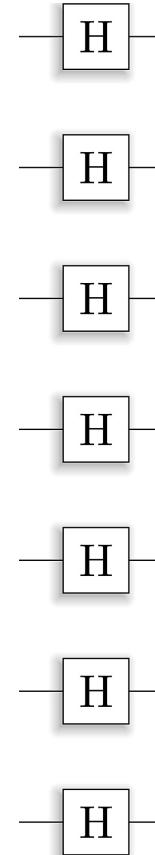
$ZIZIZIZ$

Logical Gates

Transversal gates:

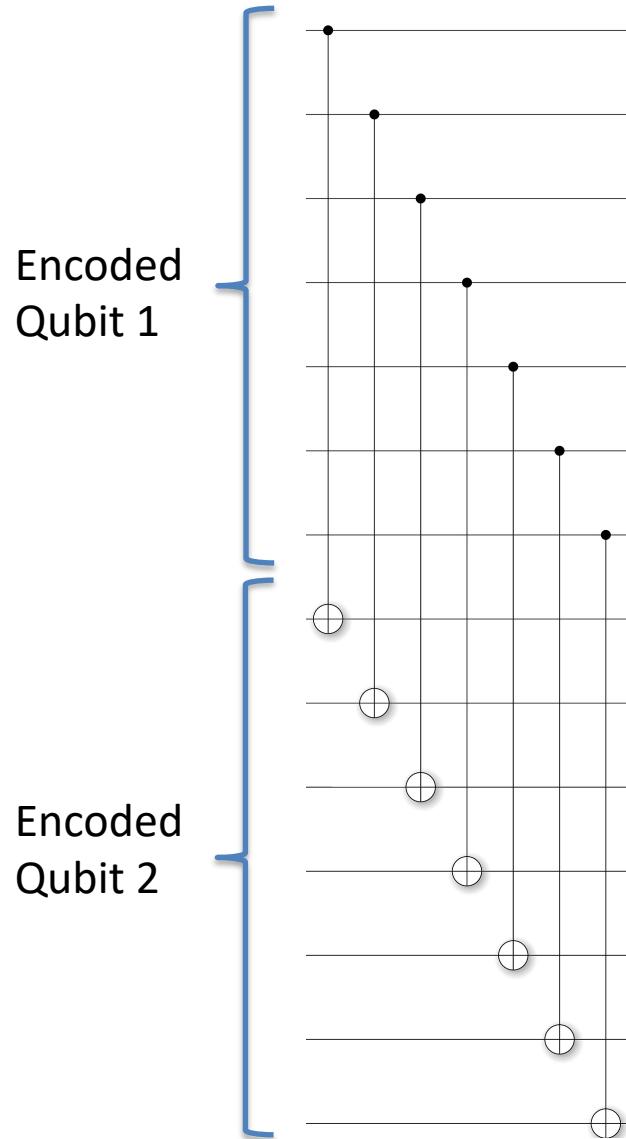


Hadamard on a
single logical qubit



This gate can be operated while leaving the logical qubit encoded, protected by the QEC code.

Logical CNOT



Can also implement CZ,
Swap transversally

Danger! CNOTs can propagate errors. We need to make sure this happens in a controlled way.

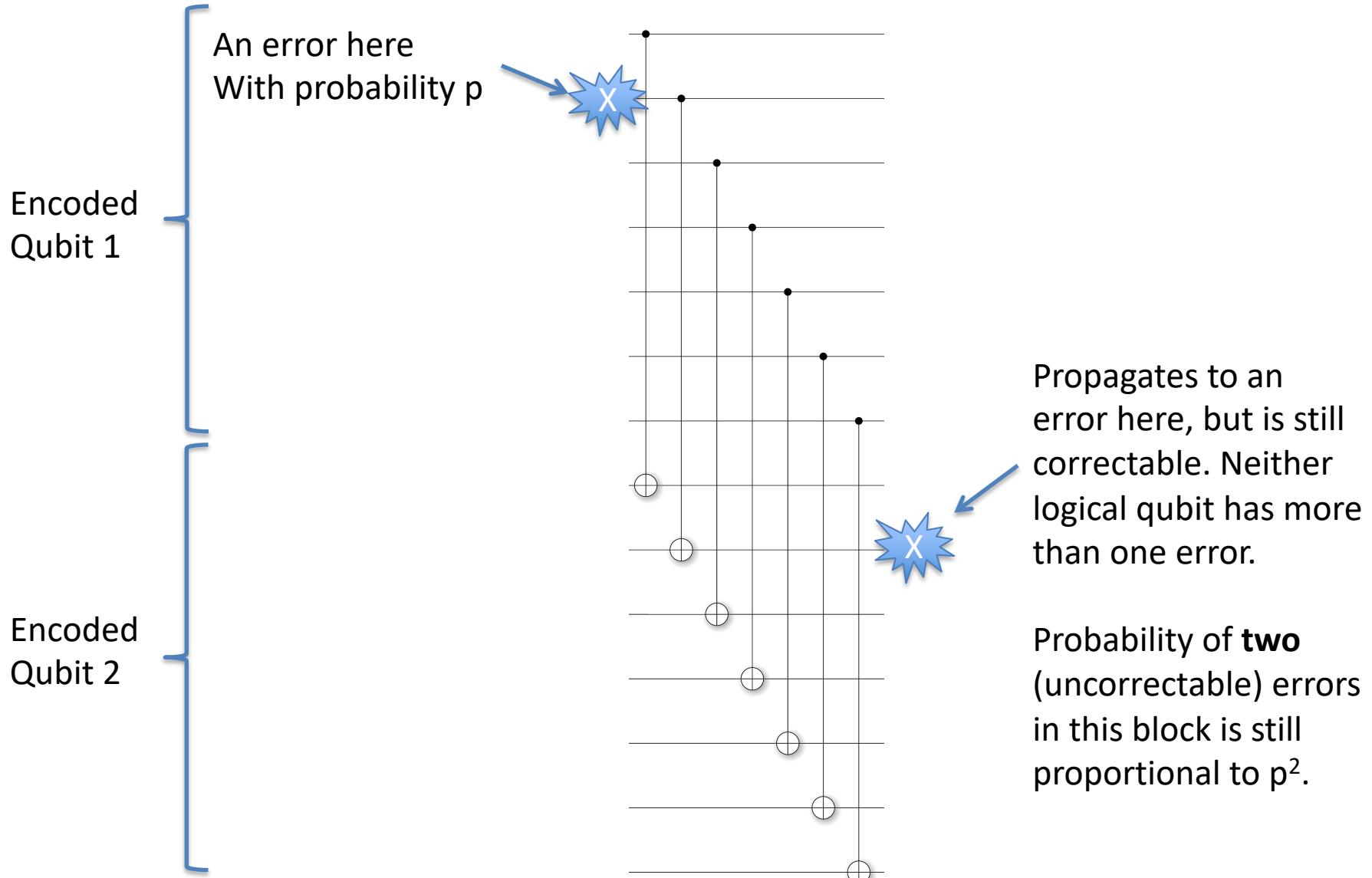
Fault Tolerance

Strategy: Take the original circuit and replace it with the *logical* version. In doing so we need to control the **spread** of errors. Doing this in a way which controls the spread of errors is known as fault tolerance:

Fault tolerant: a single error in any of the QEC procedures causes at most one error in the block of encoded qubits (which can be corrected)

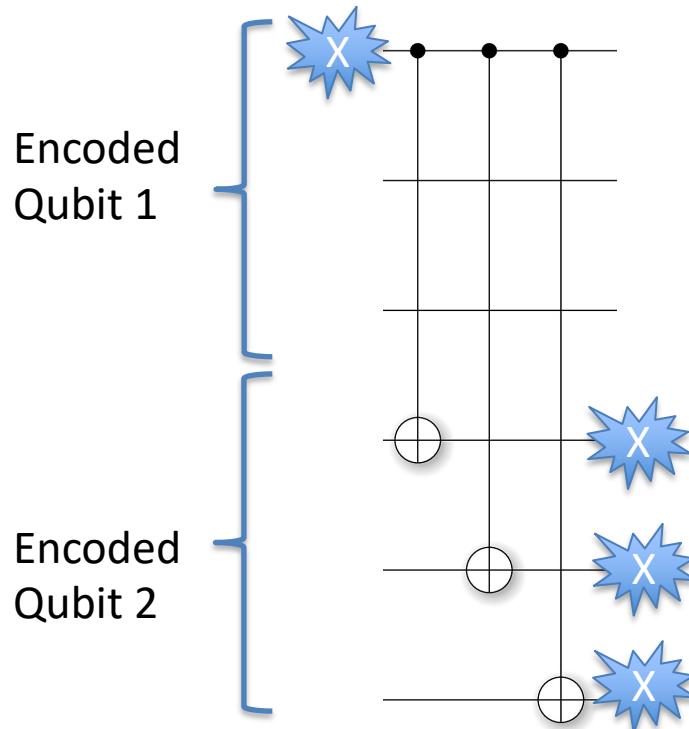
A single error (on a physical qubit) should not propagate to two errors on the same logical qubit, otherwise we would not be able to correct that qubit.

Transversal CNOT is Fault Tolerant



NOT Fault Tolerant

Consider the following CNOT gate for the 3-qubit bit flip code (000/111)

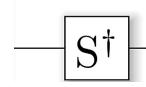
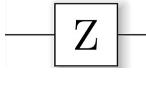
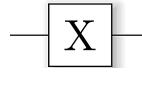
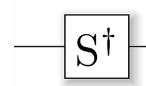
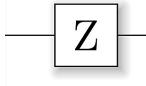
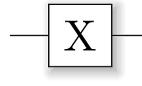
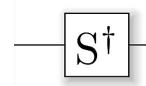
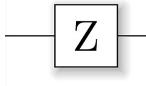
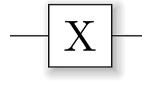
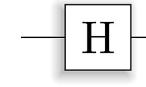
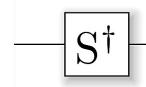
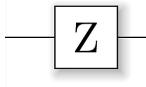
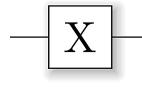
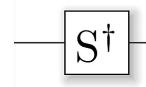
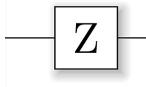
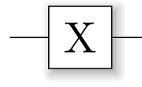
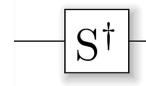
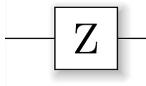
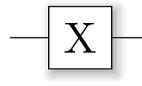
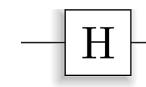
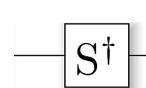
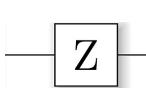
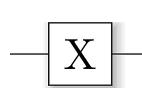


Although this circuit is “correct” (the operation it performs – assuming no error is an encoded CNOT)...

.. a physical single error can cause the second encoded qubit to be uncorrectable. It is **not** fault tolerant!

Care needed! Every operation (including measurement of syndromes) can have errors. Not only do X errors propagate, so do Z errors.

Transversal gates are Fault Tolerant



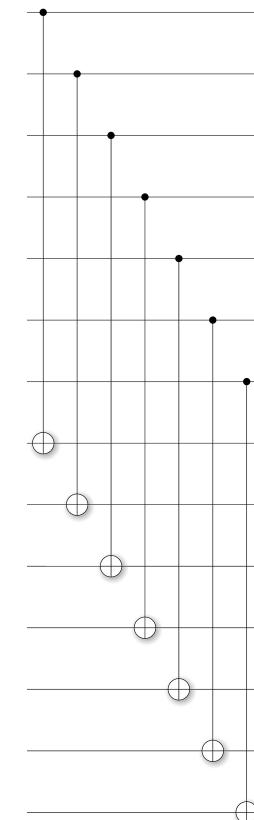
Logical X

Logical Z

Logical S

Logical H

Logical CNOT



Not the only way to achieve fault tolerance, but a very useful one!

Larger distance codes

We have seen some simple error correction codes which correct one error (distance 3 codes). How can we construct quantum error correction codes which correct more than one error?

$$\begin{aligned}|0_L\rangle &\rightarrow |00000\rangle \\|1_L\rangle &\rightarrow |11111\rangle\end{aligned}$$

Distance 5 bit flip code

More errors needed before uncorrectable, leading to a logical error.
More physical qubits give more locations for potential errors.

Concatenated codes

Systematic way to increase the distance of a code. Feed the code back into itself:

$$|0_{L2}\rangle = \frac{1}{\sqrt{8}}(|0_L0_L0_L0_L0_L0_L0_L\rangle + |1_L0_L1_L0_L1_L0_L1_L\rangle + |0_L1_L1_L0_L0_L1_L1_L\rangle + |1_L1_L0_L0_L1_L1_L0_L\rangle + |0_L0_L0_L1_L1_L1_L1_L\rangle + |1_L0_L1_L1_L0_L1_L0_L\rangle + |0_L1_L1_L1_L1_L0_L0_L\rangle + |1_L1_L0_L1_L0_L0_L1_L\rangle)$$

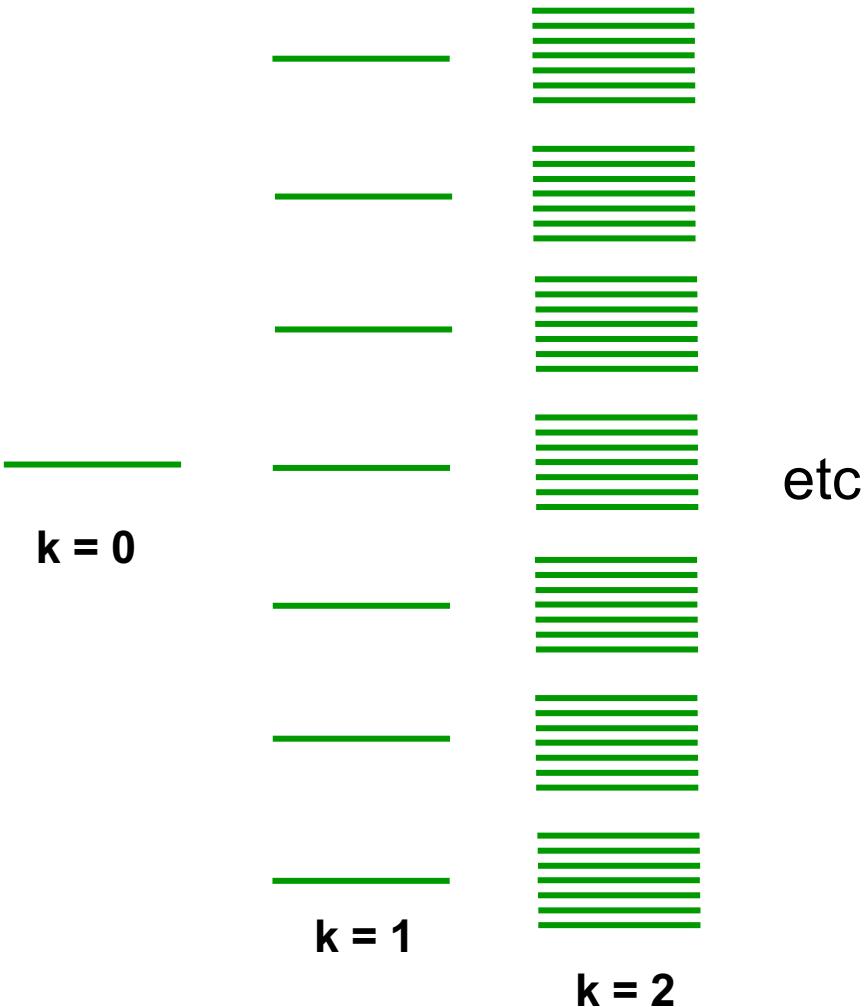
$$|1_{L2}\rangle = \frac{1}{\sqrt{8}}(|1_L1_L1_L1_L1_L1_L1_L\rangle + |0_L1_L0_L1_L0_L1_L0_L\rangle + |1_L0_L0_L1_L1_L0_L0_L\rangle + |0_L0_L1_L1_L0_L0_L1_L\rangle + |1_L1_L1_L0_L0_L0_L\rangle + |0_L1_L0_L0_L1_L0_L1_L\rangle + |1_L0_L0_L0_L0_L1_L1_L\rangle + |0_L0_L1_L0_L1_L1_L0_L\rangle)$$

$$|0_L\rangle = \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$$

$$|1_L\rangle = \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |111000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle)$$

This method is known as “concatenation” of error correction codes.

Error after different levels of encoding



Logical error rate achieved

$$p_{\text{fail}} = p_{\text{th}}(p/p_{\text{th}})^{2k}$$

$$p_{\text{th}} = 10^{-5}, p = 10^{-6}$$

$$k=1: p_{\text{fail}} = 10^{-7}$$

$$k=2: p_{\text{fail}} = 10^{-9}$$

$$k=3: p_{\text{fail}} = 10^{-13}$$

Error Correction Threshold

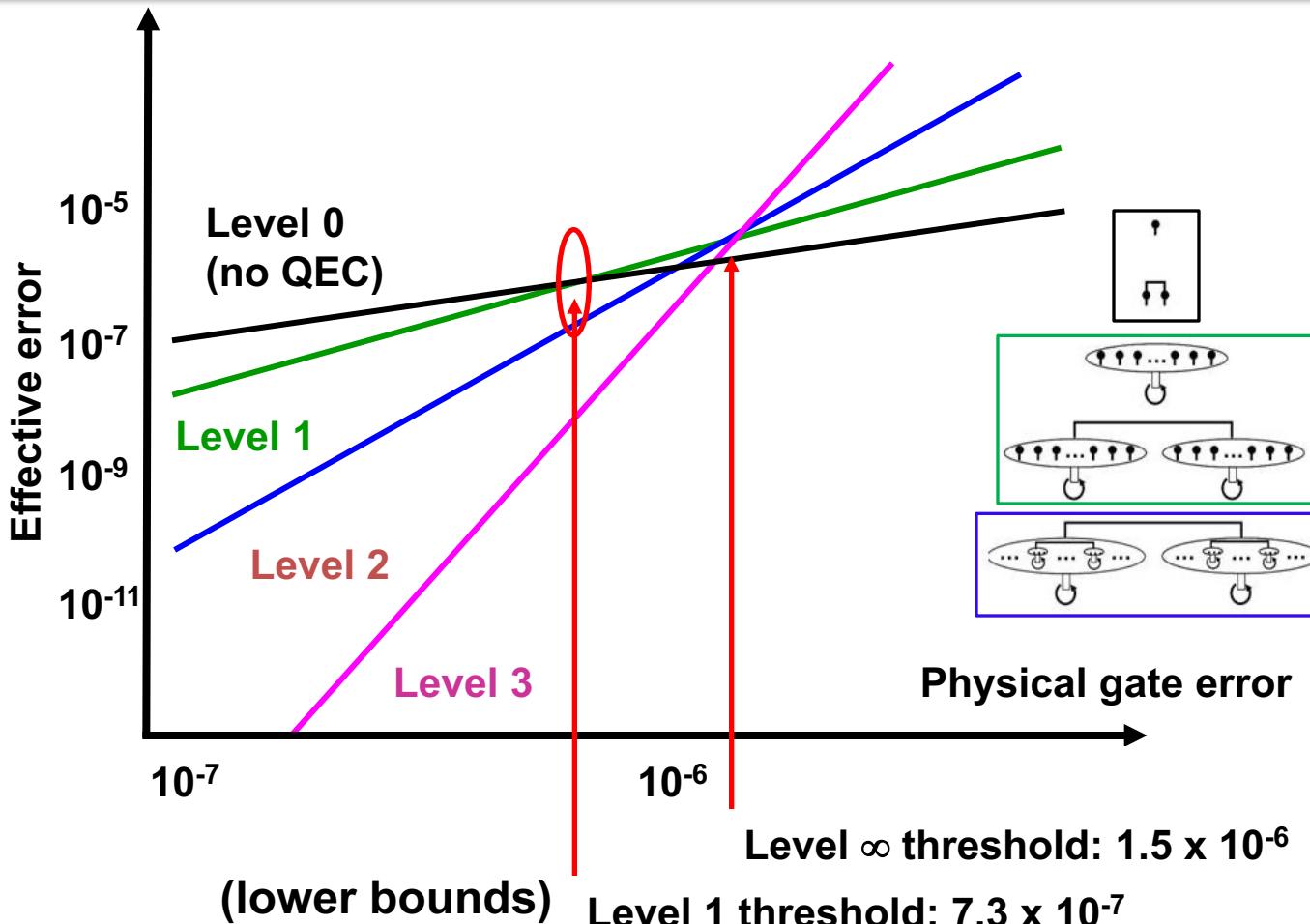
Inputs:

$$p_{\text{trans}} = p_{\text{gate}}/10$$

$$p_{\text{mem}} = p_{\text{gate}}/10$$

$$T_{\text{trans}} = T_{\text{gate}}/10$$

$$T_{\text{meas}} = 100 T_{\text{gate}}$$



From physical qubits to logical qubits

Hierarchy of qubits, quantum error correction (QEC) and concatenation

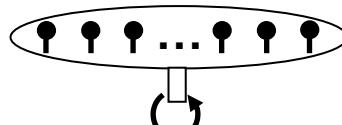
1. Individual physical qubit with external control and measurement



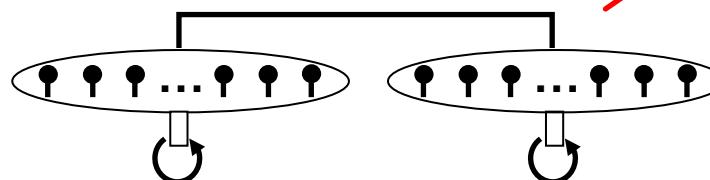
2. Coupling physical qubits to perform logic gates



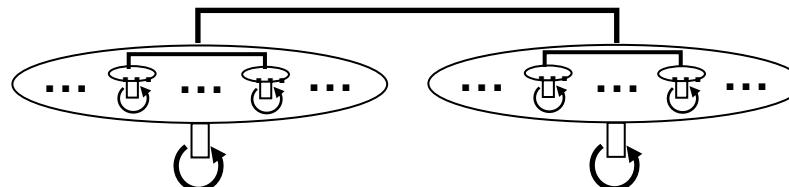
3. Encoded logical qubit and QEC including classical feed-forward processing



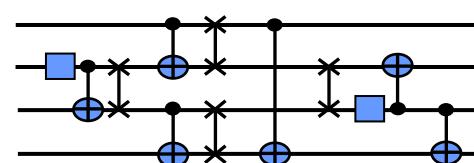
4. Coupling encoded qubits to perform logic gates protected by QEC



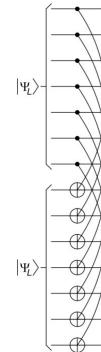
5. Recursive concatenation of encoding and QEC for fault-tolerant operation



6. Fault-tolerant implementation of quantum circuits over encoded logical qubits



Transversal CNOT



$$p_{\text{fail}} \approx p_{\text{th}} \left(\frac{p}{p_{\text{th}}} \right)^{2^k}$$

Concatenation: to gain protection $p \ll p_{\text{th}}$

For a given physical implementation:

How is this achieved?

What is the threshold p_{th} ?

The Surface Code

- A topological code suited to solid state
 (Kitaev 1997,
 Raussendorff 2007)
- A remarkably high threshold of $\sim 1\%$
 (Wang et al, 2011)
- ◆ Parallel and synchronous control required

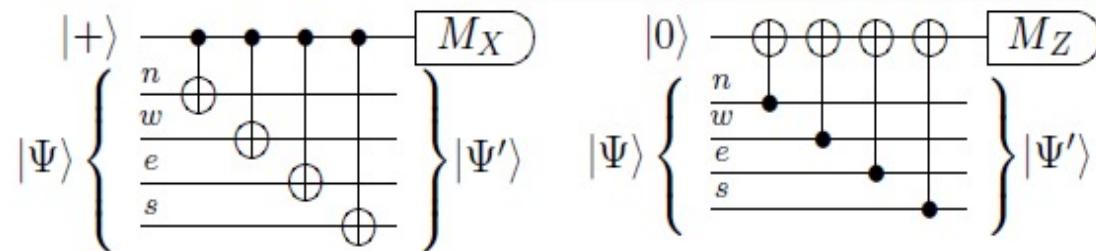
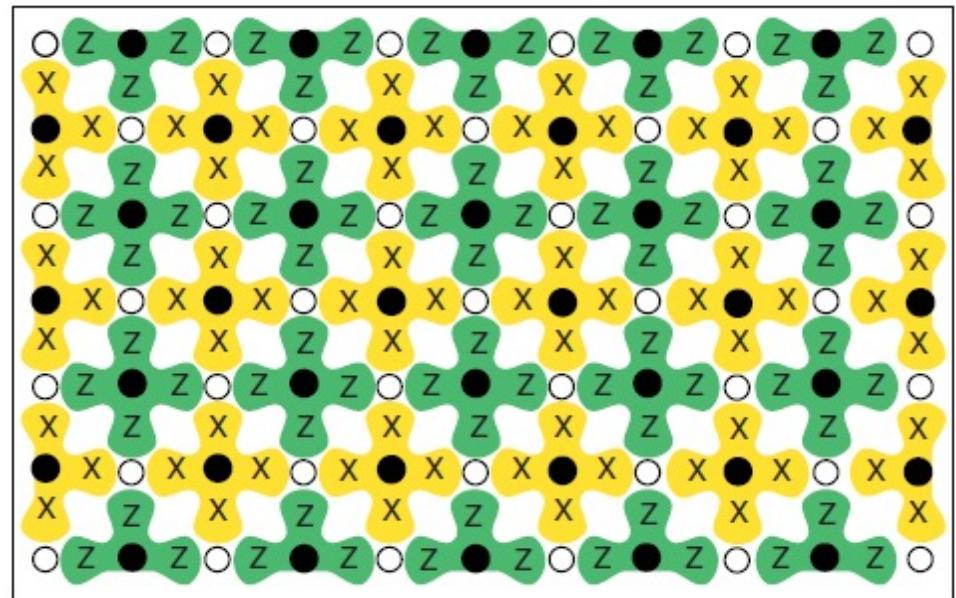
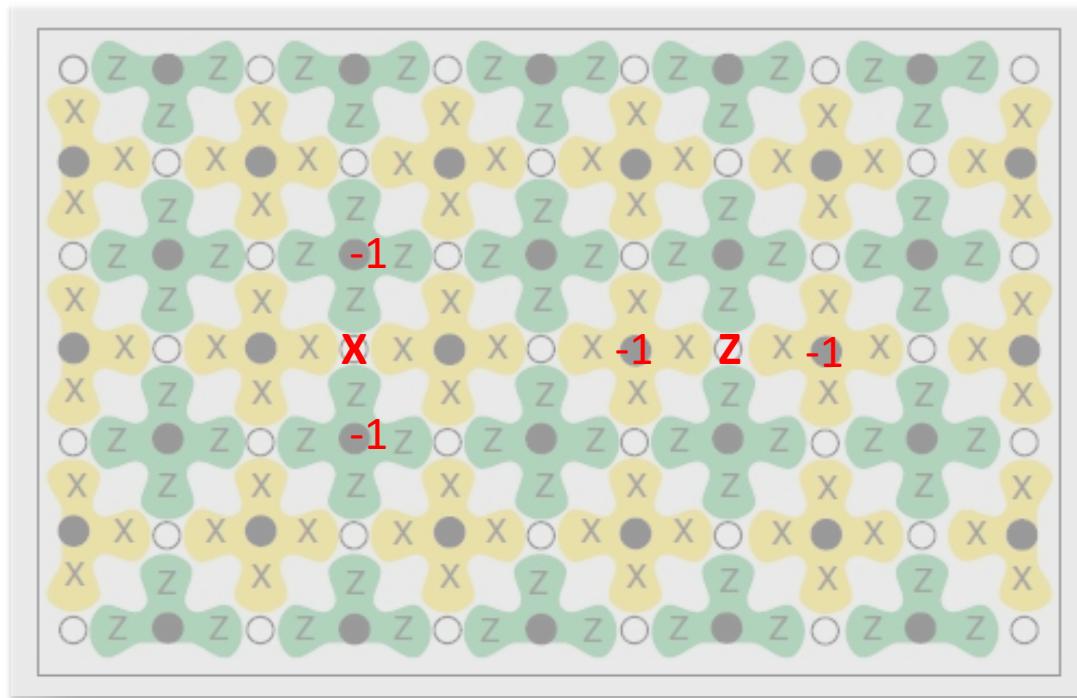
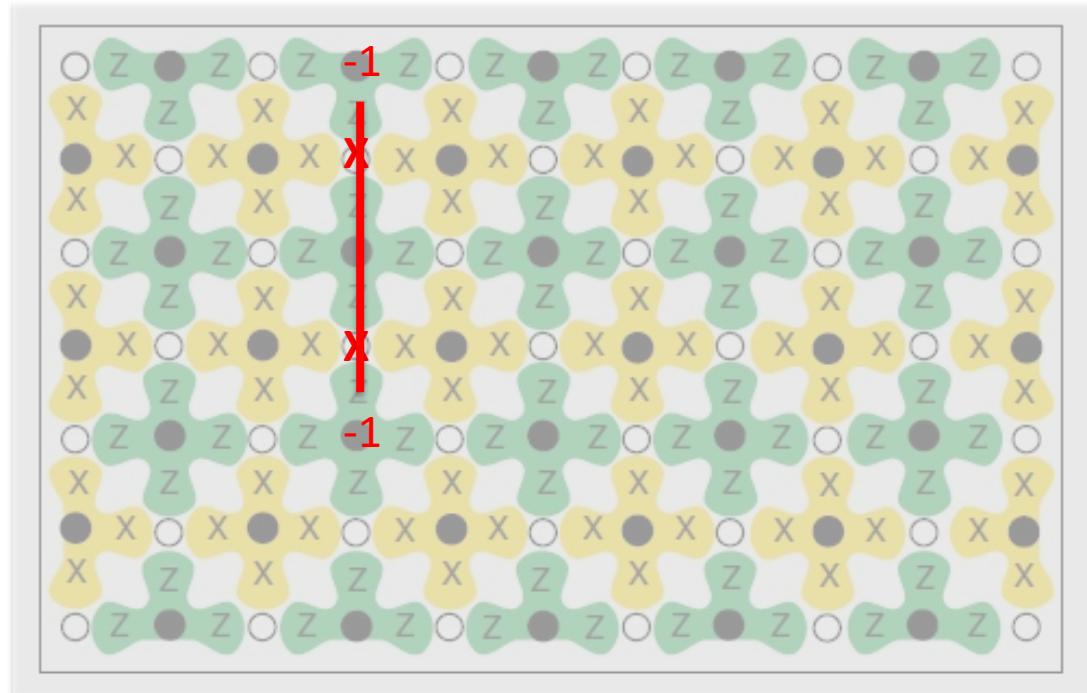


Fig. 5. Circuits used to measure X -stabilizers (left) and Z -stabilizers (right).

Errors on the surface code

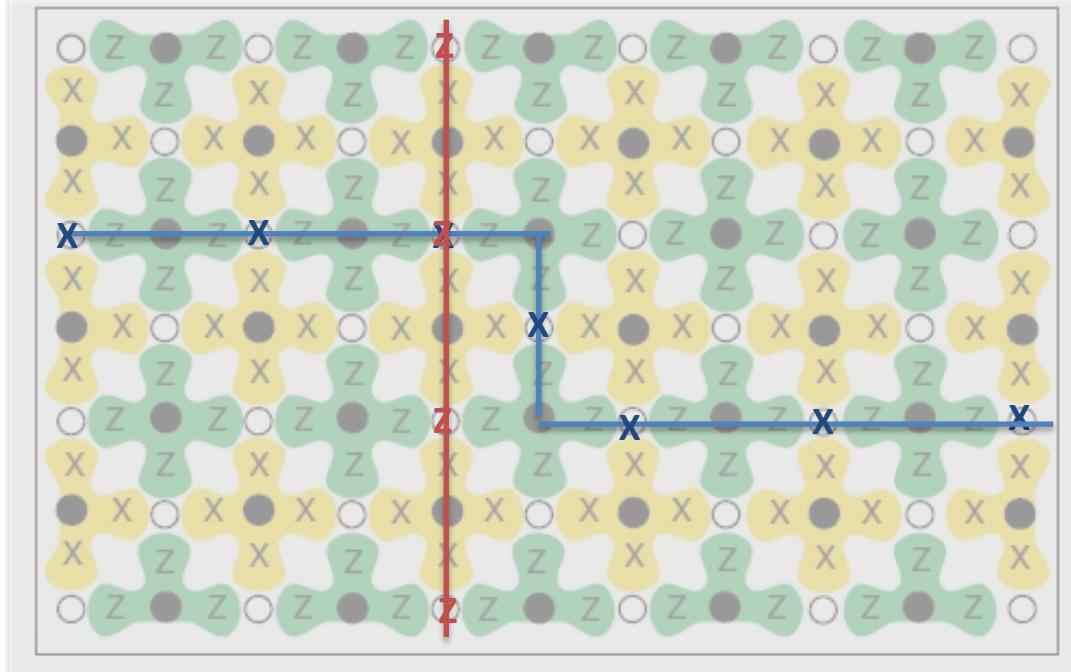


Chains of Errors



- Errors form chains, can only see syndrome changes (-1) at the ends.
- **Minimum weight matching** determines the most likely errors.
- Chains greater than half way across the surface can cause failure.

Logical Operators on the surface code

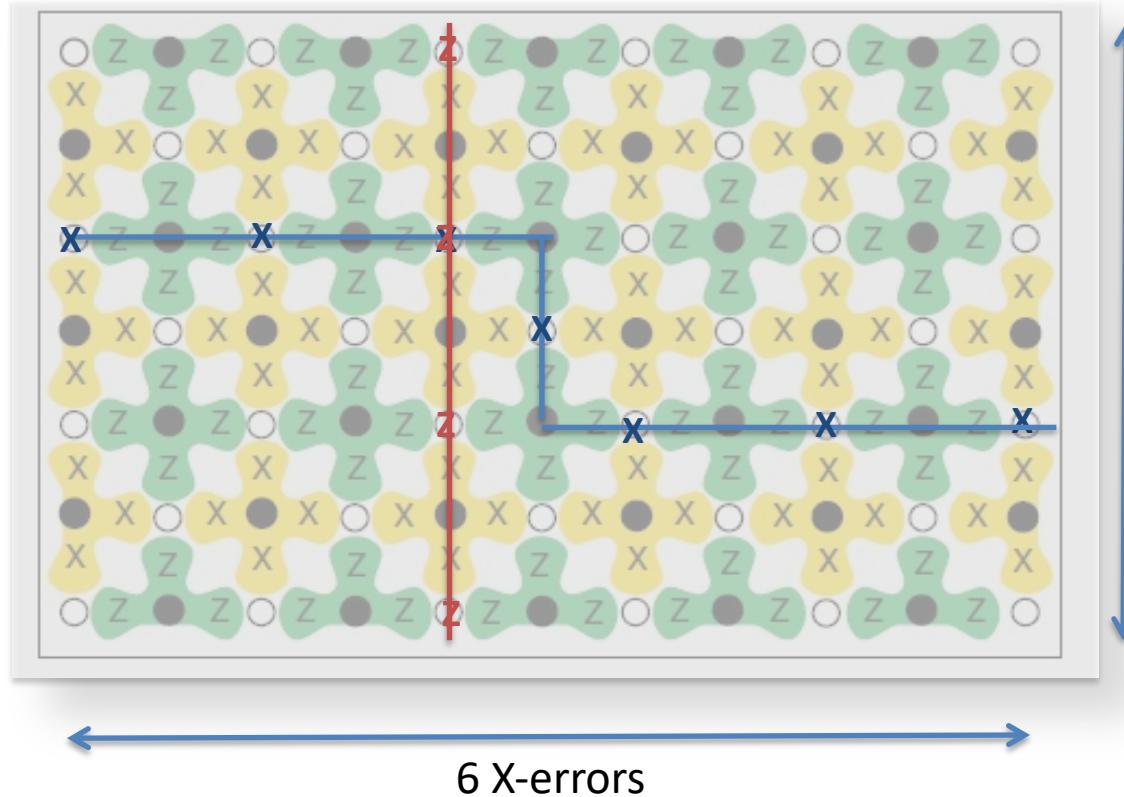


A logical X operation is a chain of X operations, left to right

A logical Z operation is a chain of Z operations, top to bottom

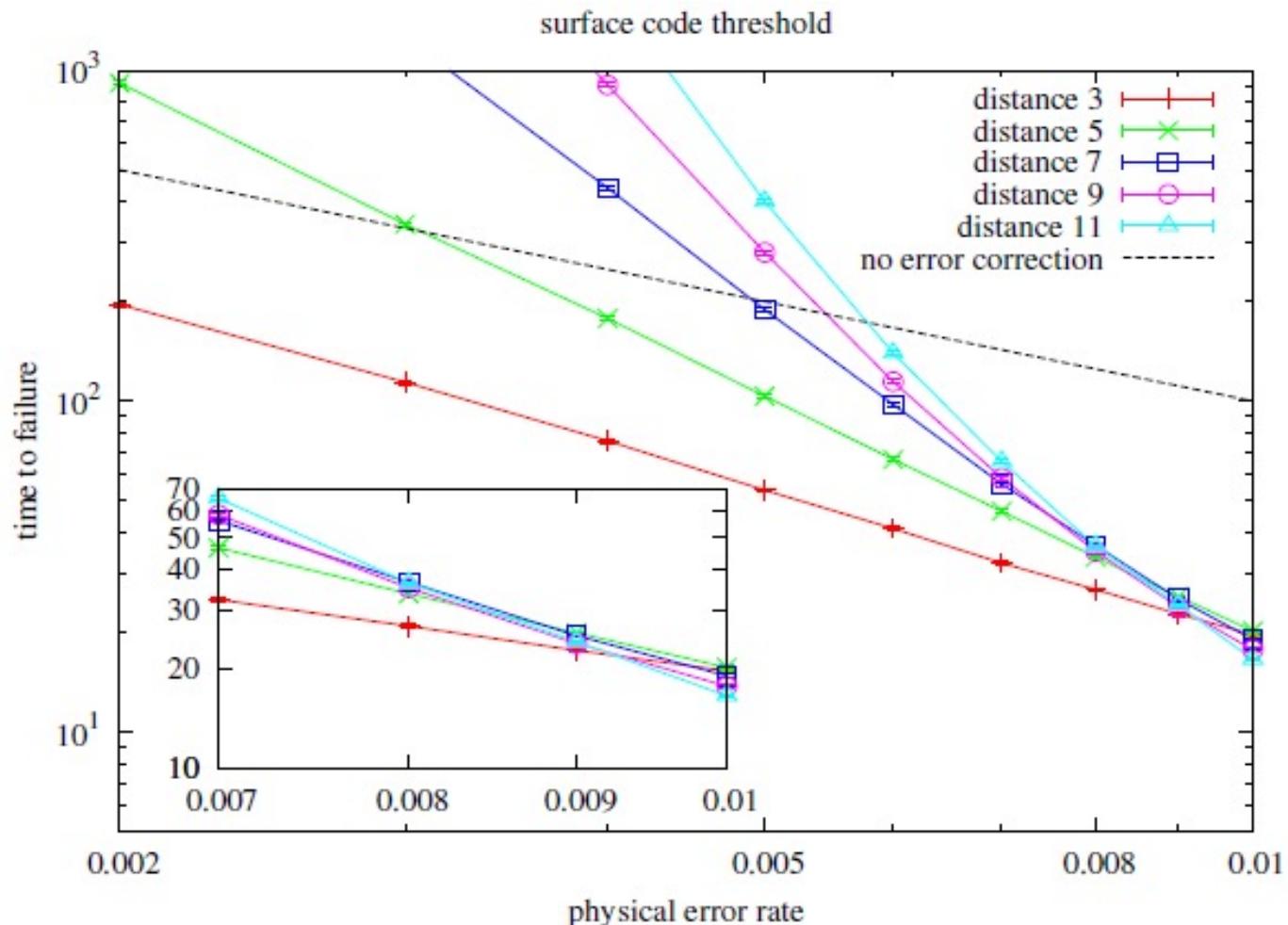
Logical operations anti-commute (as they should)

Distance of the Surface Code



- Distance of the code is equal to the length of a side.
- Scale up by simply making larger patch of surface code (concatenation not required)
- Topologically defined, so easy to map onto physical architectures

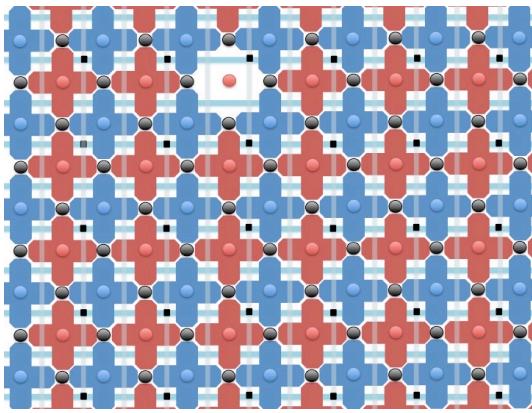
The Threshold



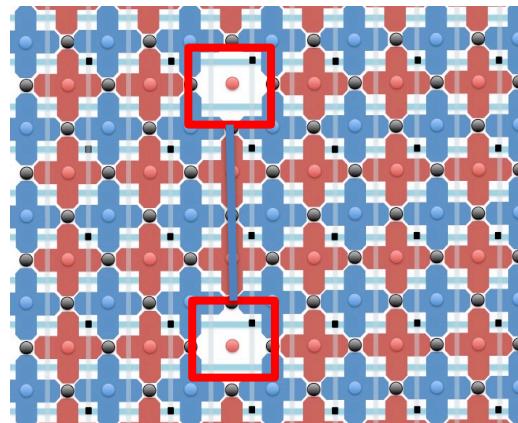
DS Wang et al

Mulitple qubits

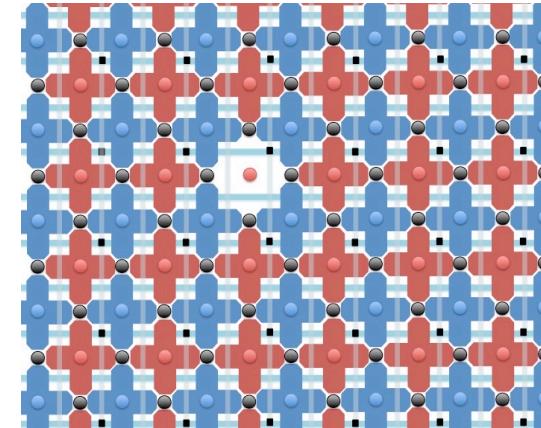
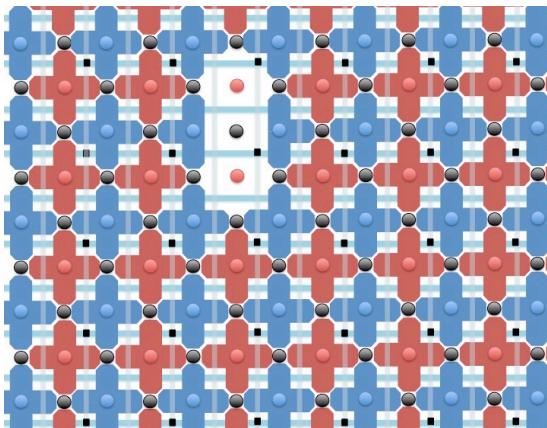
Code qubits



Logical Ops



Moving qubits



Defects are
artificial
boundaries

Requirements for an Error Corrected Shor

PHYSICAL REVIEW A 86, 032324 (2012)

Surface codes: Towards practical large-scale quantum computation

Austin G. Fowler

Centre for Quantum Computation and Communication Technology, School of Physics, The University of Melbourne, Victoria 3010, Australia

Matteo Mariantoni, John M. Martinis, and Andrew N. Cleland

*Department of Physics, University of California, Santa Barbara, California 93106-9530, USA
and California Nanosystems Institute, University of California, Santa Barbara, California 93106-9530, USA*

(Received 2 August 2012; published 18 September 2012)

Bits in factored number	2000
Number of Logical qubits required	4000
Number of qubits in surface code	~20 million qubits
Time for one measurement	100 ns
Total time required	26 hours

Research topic: Bring these requirements down!

Experimental proposal

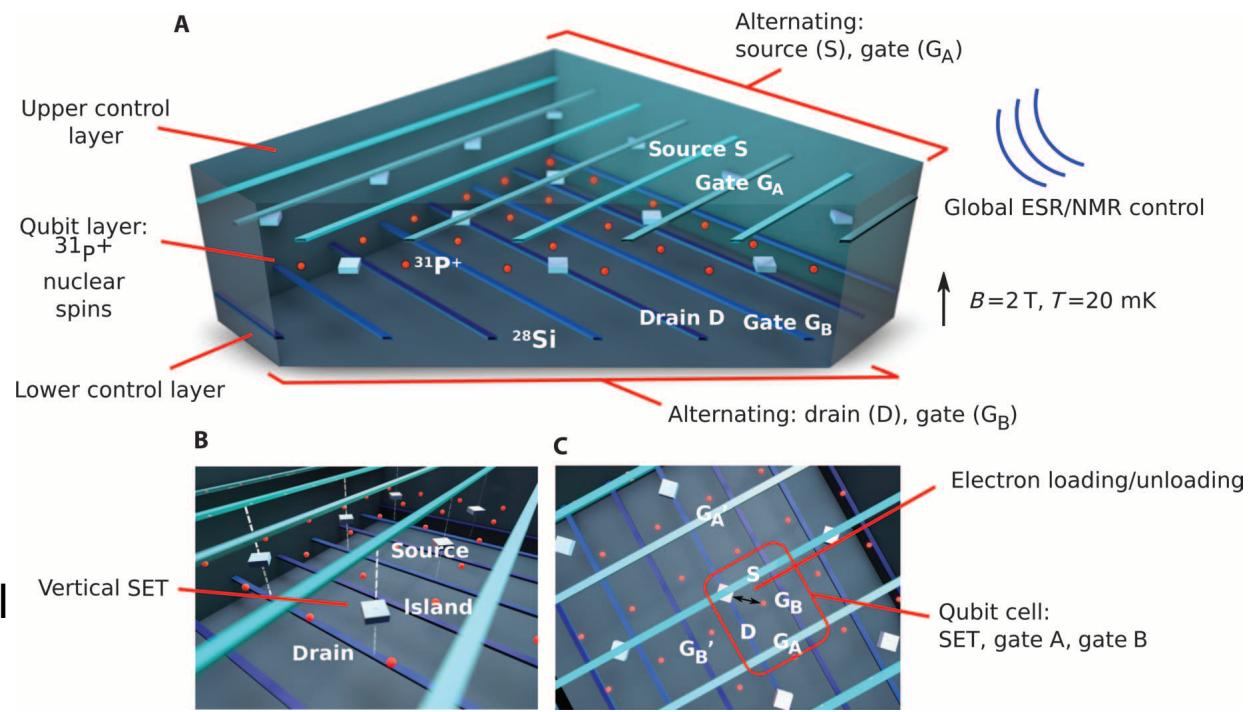
High-level:

Qubit: uniform nuclear spin

Addressing: electron load
(not gate defined wavefn)

Gates: electron load and
global ESR/NMR control

Operation: parallel, 60 MHz
loading pulses, robust to local
variations



C. Hill et al, *Science Advances* 2015

Criss-cross gate array → parallel shared control of qubit addressing (robust)

For N qubits, # control lines scales as \sqrt{N}

Established ESR/NMR spin control (Morton et al Nature 2008, Pla et al Nature 2013)

3D STM fabrication of array (McKibbin Nanotechnology 2013)

Initial proposal: CNOT dipole coupling (slow) → developing faster gates (MHz regime)

Lecture 15

Simple classical error correction codes, Quantum error correction codes, stabilizer formalism, 5-qubit code, 7-qubit Steane code

Lecture 16

The more advanced quantum error correction codes, Fault Tolerance, QEC threshold, surface code.

Lab 8

Quantum error correction