# MULT90063 Introduction to Quantum Computing

## Assignment 1

**Due: 5pm Friday, April 16, Week 6, 2021**

Assignment 1 for MULT90063 *Introduction to Quantum Computing*.

Work on your own, attempt all questions, and hand in your completed written work on or before the due date as per instructions above. The circuits you create for this assignment should be created in QUI and be saved with a filename which records your student number and the question number. The circuits should also be “shared”, and a link to the relevant circuit should be included in the written assignment. Submit this assignment online via LMS.

**Total marks = 60**

---

**Question 1** [6 marks]

**(a)** Starting from the $|0\rangle$ state, what state do you end up in if you apply a Hadamard gate?

**(b)** Starting from the $|0\rangle$ state, find an operation you can apply resulting in the state:

$$\frac{|0\rangle + \ i\,|1\rangle}{\sqrt{2}}$$

**(c)** Find the Cartesian co-ordinates and plot the state you found in part **(b)** on the Bloch sphere.

**(d)** Starting from the $|0\rangle$ state, find an operation you can apply to result in the state,

$$\frac{|0\rangle - \sqrt{3}e^{-i\frac{\pi}{4}}|1\rangle}{2}$$

**(e)** Find the Cartesian co-ordinates, and plot the state found in part **(d)** on the Bloch sphere.

**(f)** Starting from the state

$$\frac{|0\rangle - \sqrt{3}e^{-i\frac{\pi}{4}}|1\rangle}{2}$$

find an operation which results in the state,

$$\frac{|0\rangle + \ i\,|1\rangle}{\sqrt{2}}$$

**Question 2** [6 marks]

Consider a system with an even number of qubits, $n$. $n$ Hadamard gates are applied to an arbitrary $n$-qubit state $|x\rangle$ (where $x$ is an $n$-bit integer). Find (and simplify as much as possible) an expression for the *product* of:

**(a)** the first four amplitudes: $a_0, a_1, a_2, a_3$,

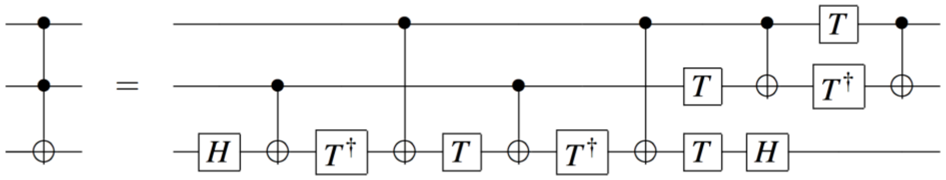
**(b)** all of the amplitudes,

of the resulting state. Please show your working.

---

**Question 3** [10 marks]

Below is shown the decomposition of the 3-qubit Toffoli gate into an equivalent circuit comprising only 1-qubit and 2-qubit gates.




**(a)** Write down the action of $H$, $T$ and $T^\dagger$ on an arbitrary single qubit state in ket notation.

**(b)** Starting from the initial 3-qubit state $|110\rangle$ (ordering: top-middle-lower), and staying in ket notation, trace through each step of the circuit to verify the output produced.

**(c)** Explain why such a decomposed circuit might be important in physical quantum computers.

**Question 4** [8 marks]

A four-qubit GHZ state is given by:

$$|GHZ\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}$$

**(a)** Demonstrate a circuit which constructs the GHZ-state, using only single qubit and two-qubit operations. Optimize the circuit as much as possible.

A four qubit W-state is given by:

$$|W\rangle = \frac{|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle}{2}$$

**(b)** Find a circuit which constructs the four qubit W-state, using only single qubit and two-qubit operations. Optimise this circuit as much as possible.

---

**Question 5** [10 marks]

**(a)** Construct an oracle with five qubits, which identifies (ie. applies a phase to) numbers which are squares (0, 1, 4, 9, 16, and 25) less than 32. You may use multiply controlled operations.

Optimize your circuit to use as few operations as possible. Briefly describe your implementation of this circuit including any optimizations which you have made.

**(b)** Use the oracle which you constructed in part (a) to implement a single iteration of Grover's algorithm, starting from an equal superposition, and calculate the probability of measuring a number which is a square at the output. Show your working.

**(c)** How many iterations of Grover's algorithm are required to give a high probability of success using your oracle? Use the geometric picture and show your working.

**(d)** The question is changed to find squares (strictly) less than $2^{20}$ (using 20 qubits). How many iterations of Grover's algorithm would be required?

**Question 6** [10 marks]

In this question we will investigate the CHSH inequality, closely related to Bell's Theorem, something that has been claimed to be *the most profound discovery of science,* leading to fierce debate about our everyday perception of *locality* and *realism*.

Consider the two parties Alice and Bob, each having their own measurement apparatus. Alice can measure either one of two observables $Q$ or $R$, while Bob can measure either of the observables $S$ or $T$.

**(a)** Suppose that the observables $Q, R, S, T$ take values $\{\pm 1\}$. Consider the quantity $QS + RS + RT - QT$. Note that there is an implied tensor product in these combined operators. In this case, what is the upper bound on the expectation (mean value) of the quantity $QS + RS + RT - QT$? This is called Bell's inequality.

**(b)** Next we consider measuring the expectation value of the quantity $QS + RS + RT - QT$ when Alice and Bob have access the quantum states. Consider the case when Alice and Bob have access to the observables

$$Q = Z \qquad S = -\frac{Z + X}{\sqrt{2}}$$

$$R = X \qquad T = \frac{Z - X}{\sqrt{2}}$$

where Alice controls the first qubit and Bob controls the second. The states $|\psi_{\mathrm{Alice}}\rangle = cos(\theta_A)|0\rangle + sin(\theta_A)|1\rangle$, $|\psi_{\mathrm{Bob}}\rangle = cos(\theta_B)|0\rangle - sin(\theta_B)|1\rangle$ . What are the eigenvalues for each of the observables $Q, R, S$ & $T$? What is the quantity $\langle QS + RS + RT - QT\rangle$ for the product state $|\psi_{\mathrm{Alice}}\rangle \otimes |\psi_{\mathrm{Bob}}\rangle$? What is the maximum possible value?

**(c)** Now consider the case that Alice and Bob share the state $|\psi_{\mathrm{AB}}\rangle = cos(\phi)|01\rangle - sin(\phi)|10\rangle$). For what values of $\phi$ is the quantity $\langle QS + RS + RT - QT\rangle$ maximum (given the observables in part (b))? What is the significance of this $\phi$ for $|\psi_{\mathrm{AB}}\rangle$? How does this value for $\langle QS + RS + RT - QT\rangle$ compare to the value in part (a)? This is called the CHSH (Clauser-Horne-Shimony-Holt) inequality.

---

**Question 7** [10 marks]

Consider a function $f(x)$ where $x$ is a $n$-bit input. The function returns "1" if there are strictly more bits set to zero in the input than there are bits set to one, otherwise it returns "0".

**(a)** Design a quantum circuit that implements $f(x)$ for $x$ represented by $n$=3 qubits. Draw the quantum circuit using one-qubit gates, two-qubit gates and as few Toffoli gates as possible. Explain its working. You may use ancilla qubits.

**(b)** Work out a 5-bit quantum implementation of $f(x)$ – if there are more 0's than 1's in the input, your function should return 1, otherwise returns 0. Draw a schematic quantum circuit which demonstrates the working of this function. You do not need to draw the full quantum circuit, a clear strategy that demonstrates the working for 5-qubit inputs is sufficient. You must use only one, two and three-qubit gates. Hint: You may use your 3-bit implementation from part (b) as a black box.