MULT90063 Introduction to Quantum Computing

Lab Session 3

Introduction

Welcome to Lab 3 of MULT90063 Introduction to Quantum Computing.

In this lab session, you will learn about multi-qubit systems in the QUI, implement several two-qubit quantum gates, and generate entanglement. In the latter half of the lab session, students will implement two simple protocols that exploit quantum entanglement that we covered in the Lectures – teleportation and dense coding.

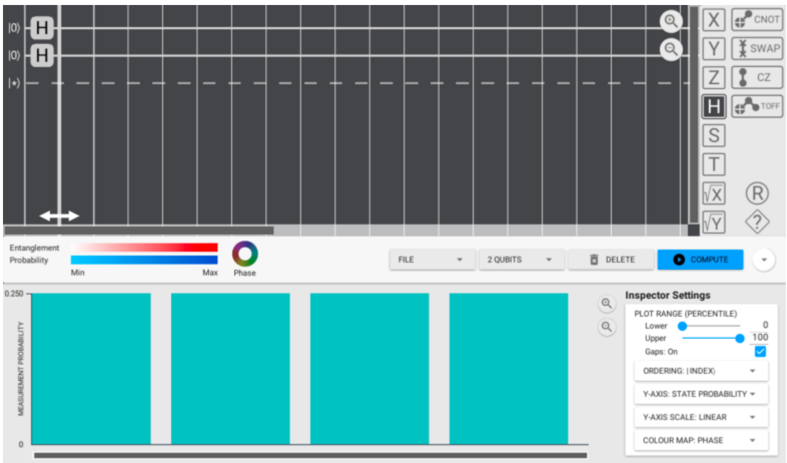
The purpose of this lab session is to:

- implement multi-qubit circuits, and understand the binary state representation, single qubit operations, and measurement on these systems
- implement two-qubit gates on multi-qubit systems
- investigate and understand entanglement generated in simple cases
- implement dense coding and teleportation

Single qubit gates and measurement on multi-qubit Systems

In quantum computing we need to become familiar with the binary representation of information encoded on qubit systems, and the effect of logic operations and measurement on these "binary" states.

Exercise 2.2.1 Set up a two-qubit system in the QUI, place a Hadamard in the first slot for each qubit, and hit compute, i.e:



Mathematically, the circuit has performed the operation:

$$|0\rangle \otimes |0\rangle \rightarrow (H \otimes H)\,|0\rangle \otimes |0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

We will use the more convenient shorthand notation :

$$|00\rangle \rightarrow H_1 H_2 |00\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Based on this two-qubit state answer the following (don't forget to renormalise as required):

What's the probability of measuring any of the four basis states in the final state? _____
What's the probability of measuring $|0\rangle$ in the first qubit? _____
What's the probability of measuring $|0\rangle$ in the second qubit? _____

If you measure $|0\rangle$ in qubit-1, what does the state collapse to? ________________________

If you measure $|1\rangle$ in qubit-1, what does the state collapse to? ________________________

If you measure $|0\rangle$ in qubit-2, what does the state collapse to? ________________________

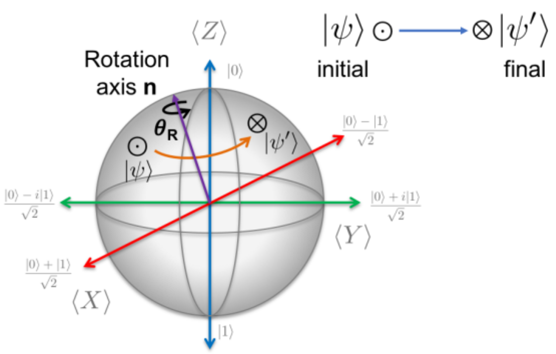
If you measure $|1\rangle$ in qubit-2, what does the state collapse to? ________________________

Add a measurement gate for qubit-1 after the Hadamard and investigate what happens (hit compute a number of times), consulting the State Info Cards (SICs) for each basis state. Delete the measurement gate on qubit-1 and add a measurement gate to qubit-2, repeat.

Exercise 2.2.2 Now we will repeat, but create a more general two-qubit state. Consider the following operation:

$$|00\rangle \rightarrow |\psi\rangle = R_X(\theta_R)_1\, H_2\, |00\rangle$$

Recall the definition of the R-gate in terms of the Pauli operators σ = (*X*, *Y*, *Z*):



$$|\psi'\rangle = R_{\hat{\mathbf{n}}}(\theta_{\mathrm{R}})\,|\psi\rangle$$

$$R_{\hat{\mathbf{n}}}(\theta_R) = e^{i\theta_g}\left(I\cos\frac{\theta_R}{2} - i\,\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\sin\frac{\theta_R}{2}\right)$$

$$\boldsymbol{\sigma} = (X, Y, Z) \qquad \hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$$

Construct the $R_X(\theta_R)$ operator in the matrix representation (global phase zero) and convert to ket representation to show that it transforms qubit-1 as:

$$R_X(\theta_R)_1\,|0\rangle = \cos\frac{\theta_R}{2}\,|0\rangle - i\sin\frac{\theta_R}{2}\,|1\rangle$$

Hence, express the final state of the following independent operations on qubit-1 and qubit-2 in the ket basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$|00\rangle \rightarrow |\psi\rangle = R_X(\theta_R)_1\, H_2\, |00\rangle =$ ____

Based on this two-qubit state answer the following (don't forget to renormalise as required):

What's the probability of measuring <u>any</u> of the four basis states in the final state? _____
What's the probability of measuring $|0\rangle$ in the first qubit? _____
What's the probability of measuring $|0\rangle$ in the second qubit? _____

If you measure $|0\rangle$ in qubit-1, what does the state collapse to? ________________________

If you measure $|1\rangle$ in qubit-1, what does the state collapse to? ________________________

If you measure $|0\rangle$ in qubit-2, what does the state collapse to? ________________________

If you measure $|1\rangle$ in qubit-2, what does the state collapse to? ________________________

Verify the above by examining this state in separable form:

$$|\psi\rangle = \left(\cos\frac{\theta_R}{2}|0\rangle - i\sin\frac{\theta_R}{2}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

Now program the corresponding circuit in the QUI (R-gate: X-axis, global phase zero) and examine the output state probability distribution and SICs for various choices of θ_R. Perform measurements on qubit-1 and qubit-2 (as per questions above) and make sure you understand the outputs by moving the time scrubber through the circuit (ask a tutor if need be).

Exercise 2.2.3 Generalise Ex 2.2.1 and Ex 2.2.2 to more than 2 qubits, and measurements on subsets of qubits, and make sure you understand the QUI results.

Two-qubit and three-qubit gates

A two-qubit gate acts on two qubits and performs a certain operation either on both qubits, or on one of the qubits conditioned with the state of the other qubit.

CNOT Gate: Target qubit flips when control qubit is in "1" state.

On a general superposition (assuming qubit-1 is the control) we have:

$$\mathrm{CNOT}(a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle) \rightarrow a\,|00\rangle + b\,|01\rangle + c\,|11\rangle + d\,|10\rangle$$

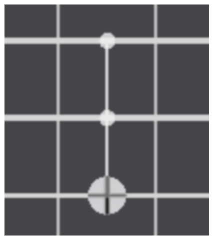
NB. Another way to view CNOT here is that the amplitudes (c, d) of the $|10\rangle$ and $|11\rangle$ states have been switched.

SWAP Gate: States of qubit-1 and qubit-2 are interchanged.

On a general superposition we have:

$$\mathrm{SWAP}(a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle) \rightarrow a\,|00\rangle + b\,|10\rangle + c\,|01\rangle + d\,|11\rangle$$

NB. Amplitudes (b, c) of the $|01\rangle$ and $|10\rangle$ states have been switched. Also, a SWAP gate can be constructed from 3 CNOTs as:



Toffoli Gate: Flip the target if <u>both</u> controls are in the "1" state.

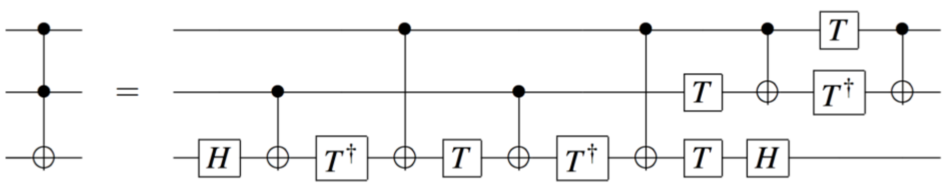
$$\begin{aligned} & a\,|000\rangle + b\,|001\rangle + c\,|010\rangle + d\,|011\rangle \\ & e\,|100\rangle + f\,|101\rangle + g\,|110\rangle + h\,|111\rangle \\ & \rightarrow a\,|000\rangle + b\,|001\rangle + c\,|010\rangle + d\,|011\rangle \\ & \quad e\,|100\rangle + f\,|101\rangle + h\,|110\rangle + g\,|111\rangle \end{aligned}$$

Controlled Rotations: Perform given rotation R on target qubit if control/s is/are in the "1" state.

The QUI provides a default controlled-Z gate in the library, and you can program a general controlled-rotation by adding control(s) to a single qubit R-gate once it is in the circuit (right click).

Exercise 2.3.1 Program each of the above gates in the QUI, using a non-trivial input state (e.g. constructed as per Ex 2.2.2). Examine the outputs carefully and make sure you understand how the states have been transformed.

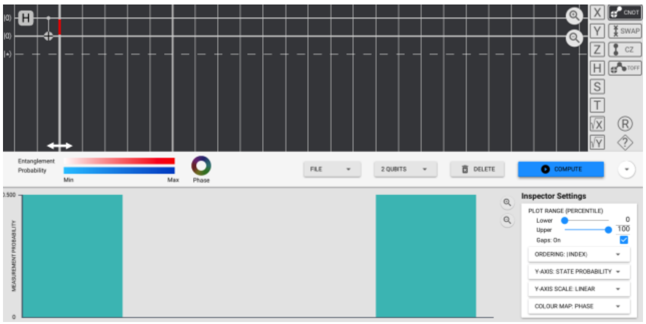
Exercise 2.3.2 As we saw in lectures, the Toffoli gate can be decomposed into CNOTs and single qubit gates (see below).



Note: from T gate definition, show from the expression for $R_{\hat{n}}(\theta_R)$ that to implement T† in terms of R gate you need to reverse angle and global phase settings. Program T† in the QUI and verify it acts as the Toffoli gate (including for superposition states as input). Program the circuit above for the Toffoli gate in the QUI and verify using various inputs that it works correctly.

Entanglement generation

A key aspect of quantum behaviour exploited in a quantum computer is entanglement. In the QUI we indicate the level of entanglement in the overall state at any given time point by plotting the "bi-partite" entanglement (i.e. between qubits above and below) vertically on the time scrubber bar itself (white = no entanglement, red = max entanglement).



Exercise 2.4.1 Program a circuit to create a maximally entangled Bell-State:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Note that final state has maximal entanglement, as indicated by the scale shown in the Control panel. If you move the time-scrubber back one time step to just after the qubit-1 Hadamard you will see the time-scrubber indicates zero entanglement (since the state is separable at this point).

In general, (for more than 2 qubits) hover over the time-scrubber to bring up the quantitative value at a given qubit bi-partition point.

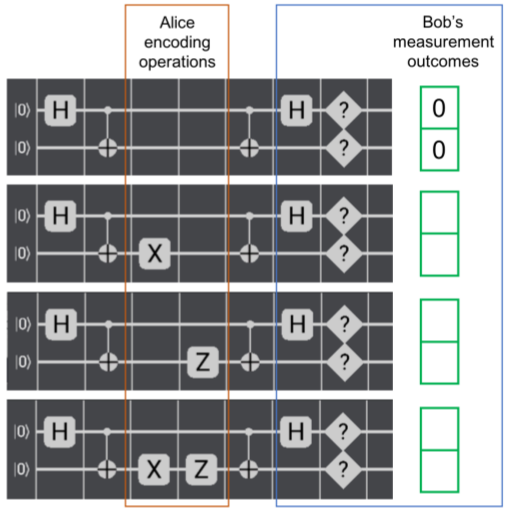
Exercise 2.4.2 Replace the Hadamard on qubit-1 in the above circuit with a $R_X(\theta_R)$ gate and examine the change in entanglement as a function of θ_R and fill in the table.

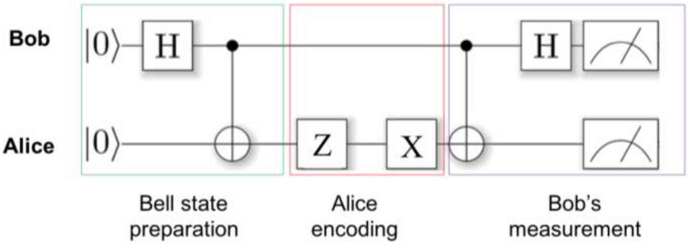
Rotation angle, θ_R (multiples of π)	Prob[|00>]	Prob[|11>]	Entanglement level
0			
1/4			
1/2			
3/4			
1			

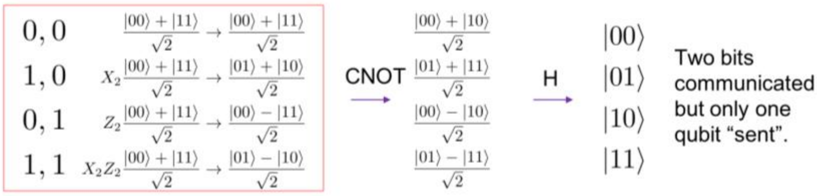
Optional: From the definition of entanglement entropy in the lectures ($S = -\sum_i p_i \log p_i$) can you quantitatively understand the results?

Dense Coding

As we saw in lectures, in classical communication if Alice wants to send two bits of information to Bob she will have to transmit the two bits over the channel. However, by using quantum entanglement, Alice can send one qubit over a quantum channel and transmit two bits of classical information. This is called dense coding – see below.

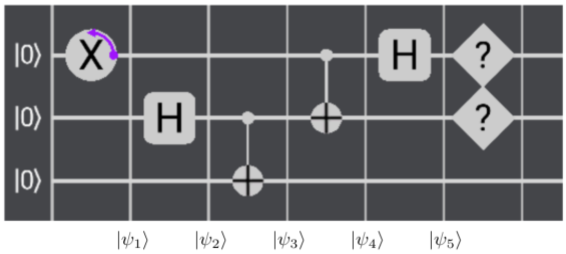






Exercise 2.5.1 Program the basic dense coding circuit in the QUI. Based on Alice's encoding table of operations, code each of the four possible two-bit strings and run the circuit for each case. Move the time-scrubber through the circuit to understand/verify the evolution of the state, and the final state prior to Bob's measurement of both qubits. Fill in the table (above right) verifying the data received by Bob. For this circuit, why do we not really need to have the measurement gates?

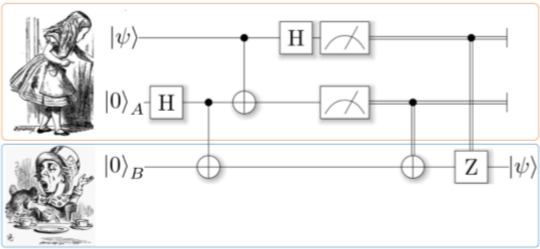
Quantum Teleportation



Alice has a state $|\psi\rangle$ that she wishes to transfer to Bob's qubit. They can't copy because of the no-cloning theorem, so they use the spooky properties of quantum entanglement. Initially they share two qubits (A = Alice, B = Bob) that are entangled, and then Alice makes measurements on her two qubits. Those measurements cause the original state $|\psi\rangle$ to be instantaneously imprinted on Bob's qubit. Incredible, but true (verified in labs).

You might wonder: now that we have a SWAP gate why doesn't Alice just use that? Yes, she could, and that is how we would normally do it in a quantum computer when everything is proximal, but imagine Alice and Bob physically separate after setting up the initial shared entanglement…the teleport procedure will still work no matter where they are!

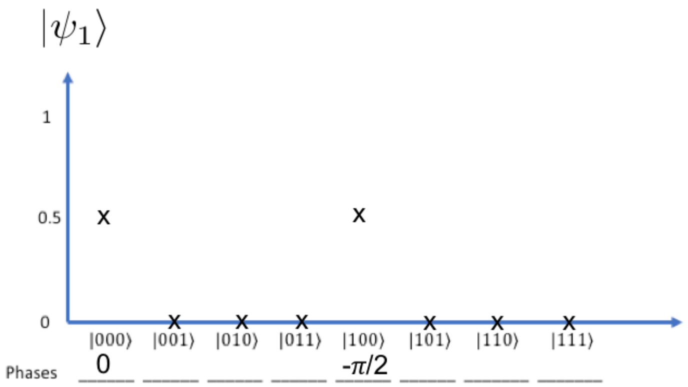
In lectures, we saw how teleportation is implemented through the following circuit:

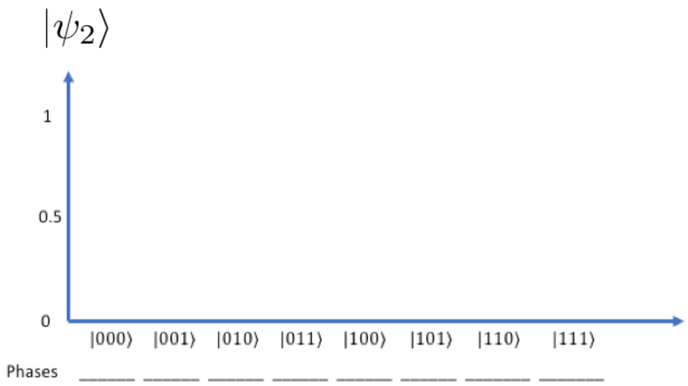


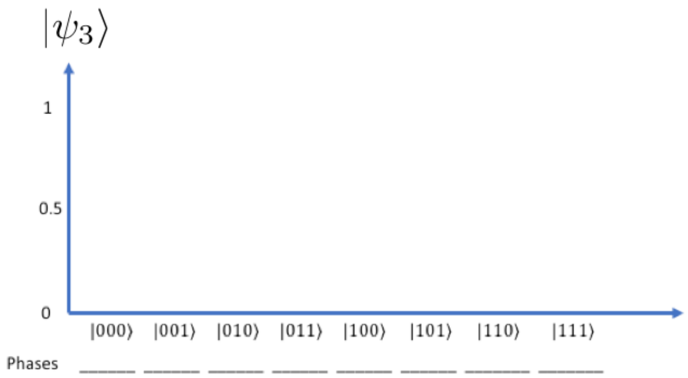
Alice measures	Bob receives	Alice directs Bob to apply to his qubit	Bob corrects to successfully reconstruct Alice's original state.
$0,0$	$\alpha\|0\rangle + \beta\|1\rangle$	No operation	$\alpha\|0\rangle + \beta\|1\rangle \rightarrow \alpha\|0\rangle + \beta\|1\rangle$
$0,1$	$\alpha\|1\rangle + \beta\|0\rangle$	X	$X(\alpha\|1\rangle + \beta\|0\rangle) \rightarrow \alpha\|0\rangle + \beta\|1\rangle$
$1,0$	$\alpha\|0\rangle - \beta\|1\rangle$	Z	$Z(\alpha\|0\rangle - \beta\|1\rangle) \rightarrow \alpha\|0\rangle + \beta\|1\rangle$
$1,1$	$\alpha\|1\rangle - \beta\|0\rangle$	ZX	$ZX(\alpha\|1\rangle - \beta\|0\rangle) \rightarrow \alpha\|0\rangle + \beta\|1\rangle$

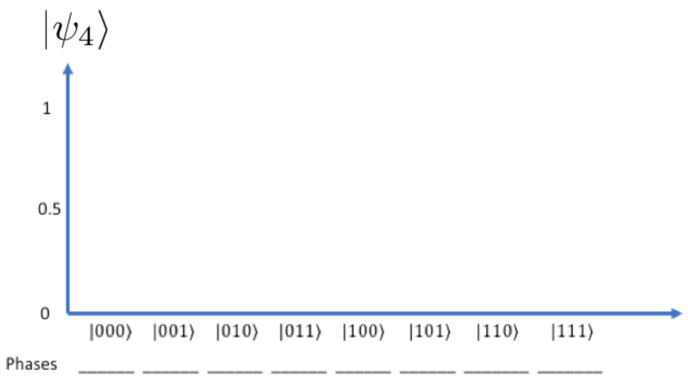
Exercise 2.6.1 In the QUI, set the teleportation circuit (up to and including the measurement gates) using a $R_X(\theta_R)$ gate to create the state $|\psi\rangle$ that Alice will teleport to Bob. Save circuit as "Lab-2 Teleport" (example below has $\theta_R = \pi/2$), and global phase zero)

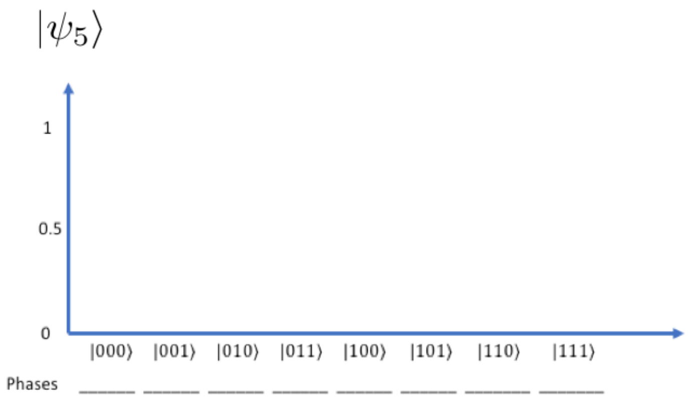
Before you run the QUI simulation, go through the circuit by hand with $\theta_R = \pi/2$ and fill in the following (vertical axes = basis state probability):





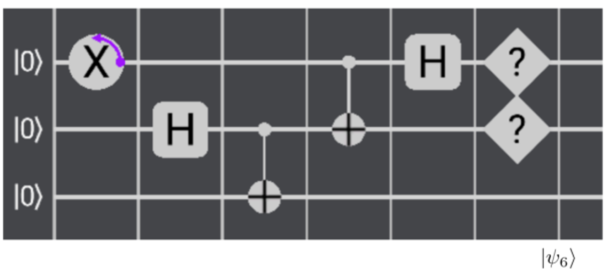






Exercise 2.6.2 Now run the circuit "Lab-2 Teleport" and for each time step (1-5) compare the results to those above.

Exercise 2.6.3 Run the circuit "Lab-2 Teleport" enough times to result in all four of the measurement outcomes on Alice's qubits and note the state of the system at time step 6 (as per below):



From the QUI output fill in the table below for the state of the 3rd qubit (Bob's qubit) in each scenario.

Alice measures	System state $\|\psi_6\rangle$	Bob's state (t = 6)	Correction to obtain Alice's state
0 , 0	$\frac{1}{\sqrt{2}}\|000\rangle + \frac{-i}{\sqrt{2}}\|001\rangle$	$\frac{1}{\sqrt{2}}\|0\rangle + \frac{-i}{\sqrt{2}}\|1\rangle$	No correction
0 , 1			
1 , 0			
1 , 1			

Exercise 2.6.4 Exploration. Run the teleportation circuit a number of times for different values of θ_R. For each instance examine the output SICs and see if you can confirm by inspection the teleport of Alice's state as a result of the required correction.