This Week

Lecture 11

Quantum Phase Estimation, Fourier Transformations, Quantum Fourier Transform, QUI examples, Inverse QFT

Lecture 12

Shor’s Quantum Factoring algorithm, Shor’s algorithm for factoring and discrete logarithm, HSP Problem

Lab 6

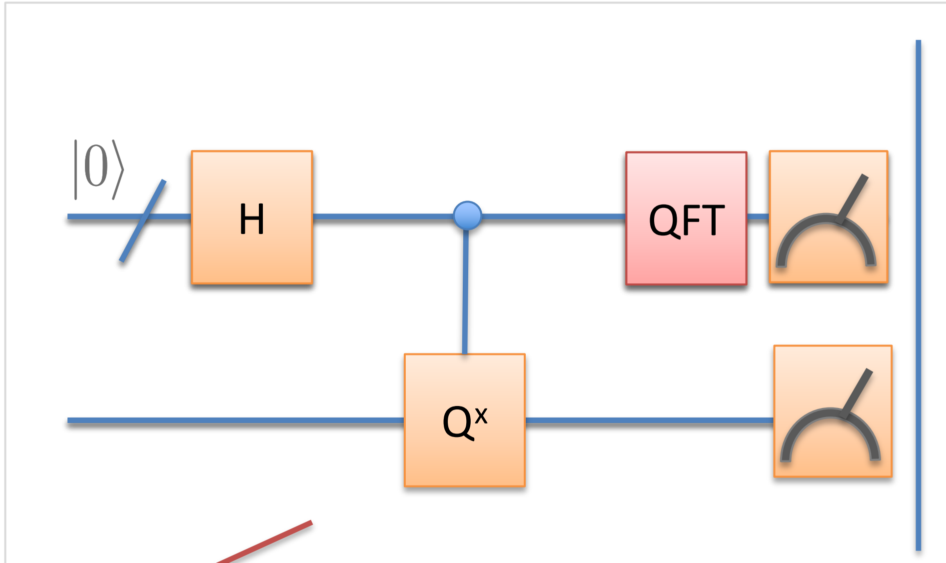
QFT and Shor’s algorithm

Lecture 11 overview

- Fourier Transformations
  - Quantum Phase estimation
  - Regular Fourier Transform
  - Fourier Transform as a matrix
  - Quantum Fourier Transform (QFT)
  - QUI examples
  - Inverse QFT

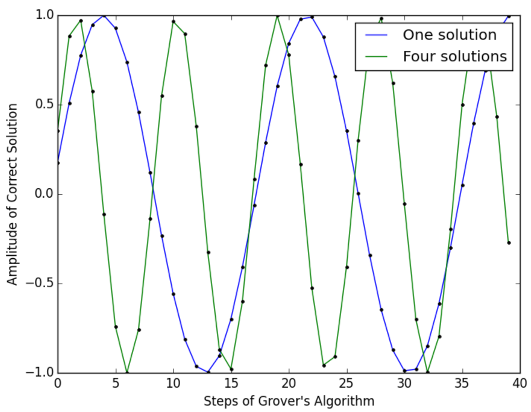
Reiffel, Chapter 8
Kaye, Chapter 7
Nielsen and Chuang, Chapter 5

Last lecture: Quantum Counting



Dimension: N’

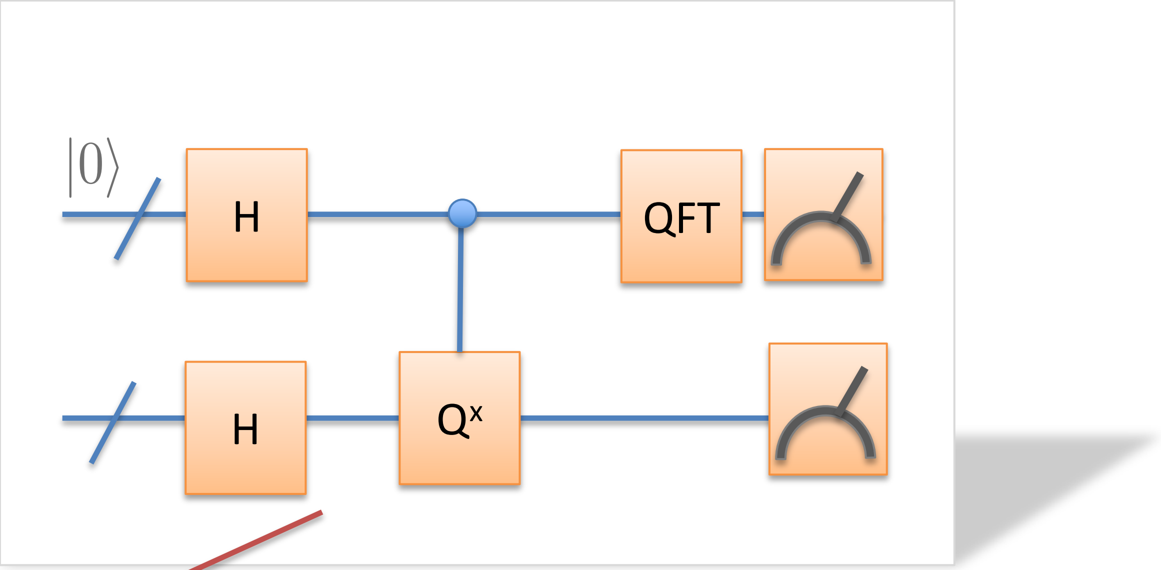
Dimension: N

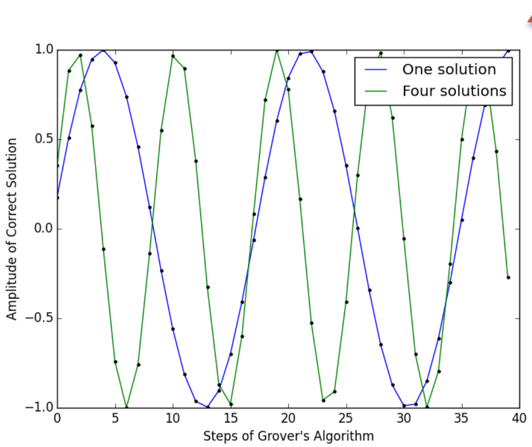


$$|\psi\rangle = \sum_x \sin(2x+1)\theta\,|x\rangle \otimes |\psi_g\rangle$$

After Fourier transforming a periodic function, we get a good approximation to frequency.

Last Lecture: Quantum Counting





$$|\psi\rangle = \sum_x \sin(2x+1)\theta\,|x\rangle \otimes |\psi_g\rangle$$

After **Fourier transform** of a periodic function, we get a good approximation to frequency. Example of **quantum phase estimation**.

Quantum Phase Estimation

The Problem

Consider an eigenvector $|\psi\rangle$ of a unitary matrix (an operation which you could implement on a quantum computer) U:

$$\boxed{U|\psi\rangle = \underbrace{\exp(2\pi i\theta)}_{\text{Eigenvalue}}\underbrace{|\psi\rangle}_{\text{Eigenvector}}}$$

The Quantum Phase Estimation algorithm estimates the angle θ. Notice that since U is unitary, all eigenvalues of U will be of this form.

The T gate

For example, consider the T gate:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\frac{\pi}{4}) \end{bmatrix}$$

An eigenvector of the T gate is

$$\boxed{T\left|1\right\rangle = \underbrace{\exp\left(i\frac{\pi}{4}\right)}_{\text{Eigenvalue}} \underbrace{\left|1\right\rangle}_{\text{Eigenvector}}}$$

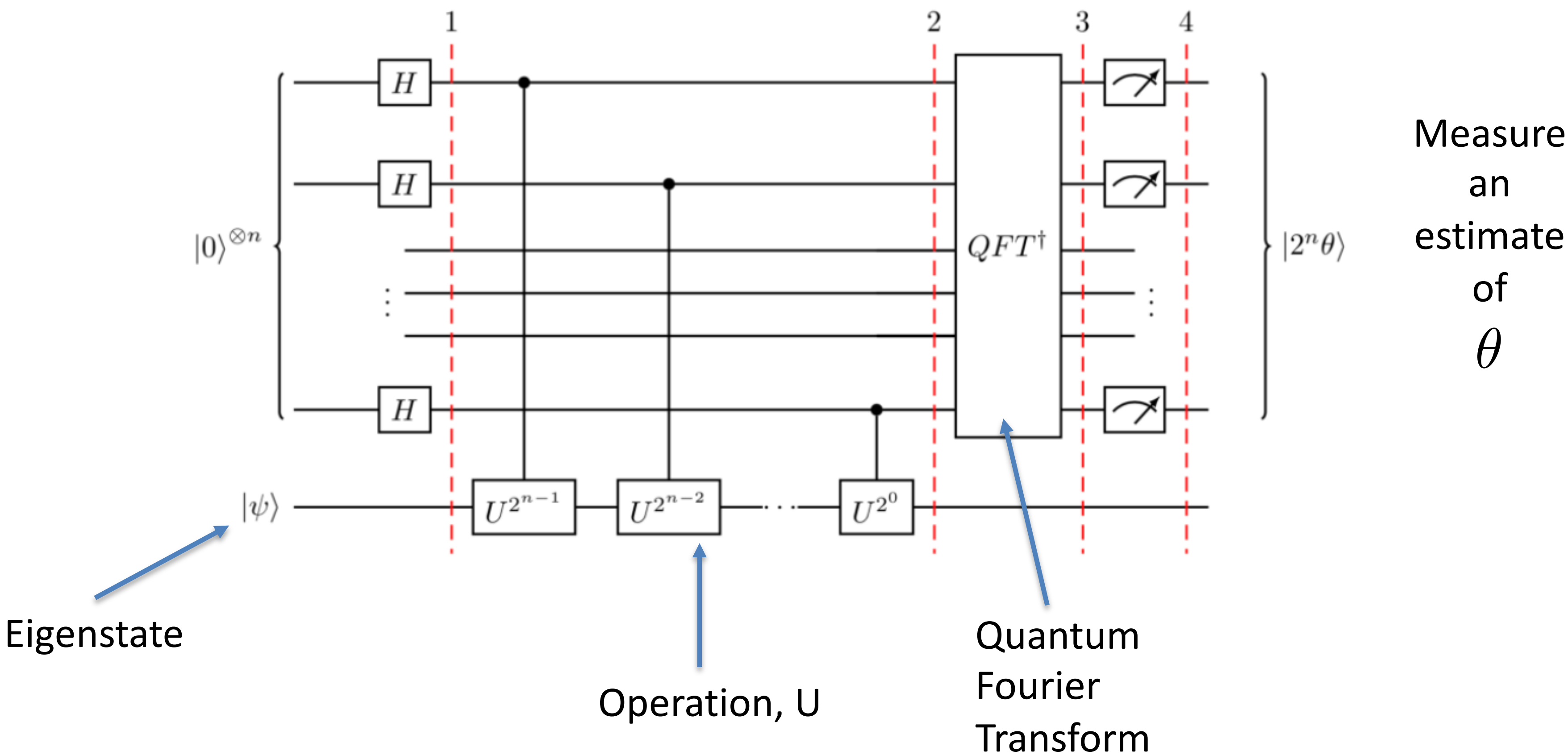
$$\boxed{U\left|\psi\right\rangle = \exp(2\pi i\theta)\left|\psi\right\rangle}$$

So in this case, we want to find:

$$\theta = \frac{1}{8}$$

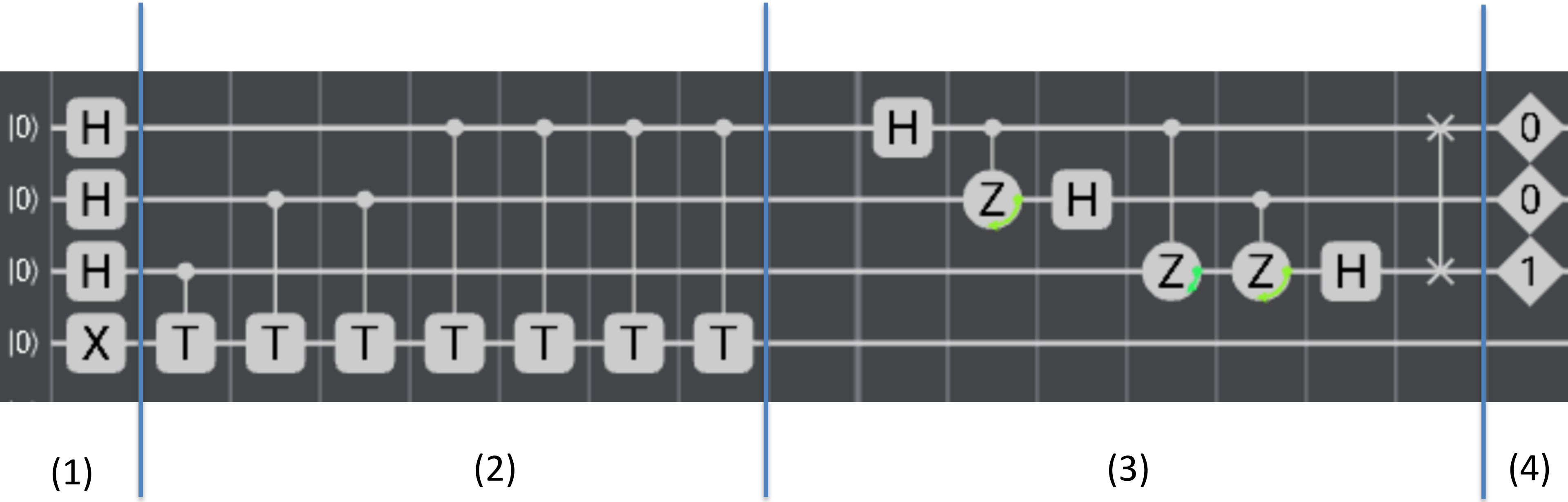
Quantum Phase Estimation gives a way to do this on a quantum computer.

Quantum Phase Estimation Circuit



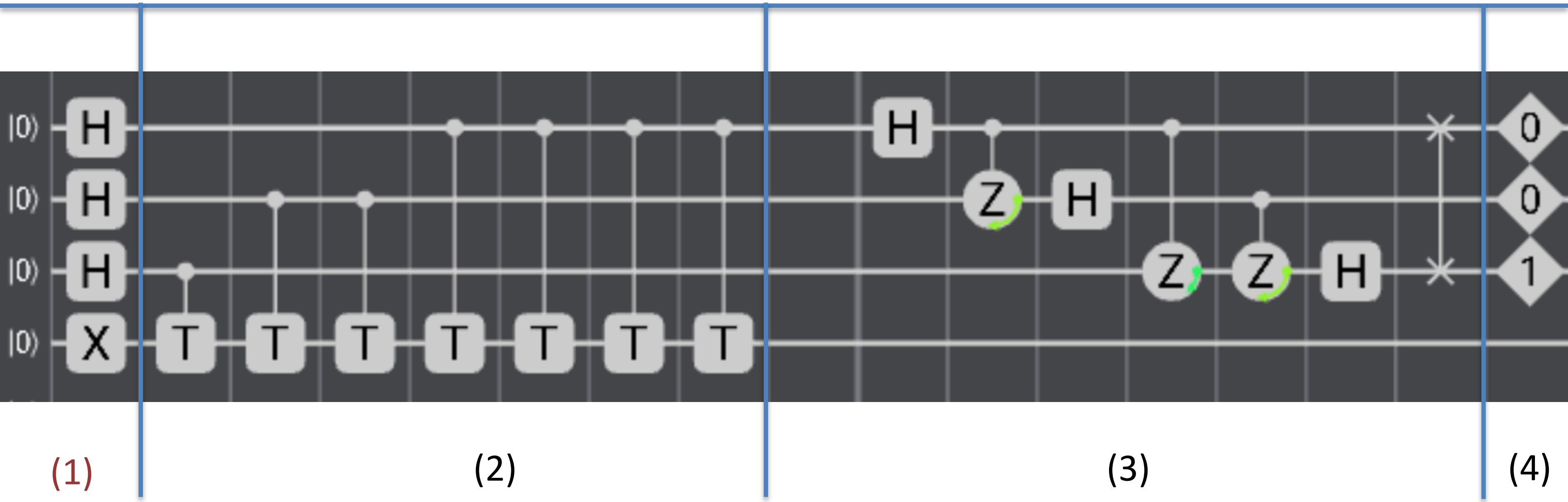
(1) Prepare equal superposition, x in the upper register x and eigenstate $|\psi\rangle$.
(2) Apply U^x to lower register
(3) Apply (inverse) Quantum Fourier Transform to the upper register
(4) Measure to obtain an estimate equal to $2^n\theta$.

QPE Circuit in the QUI



(1) Prepare equal superposition, x and eigenstate $|\psi\rangle$.
(2) Apply U^x to lower register
(3) Apply (inverse) Quantum Fourier Transform to the x register
(4) Measure to obtain an estimate equal to $2^n\theta$.

Step 1: Equal Superposition

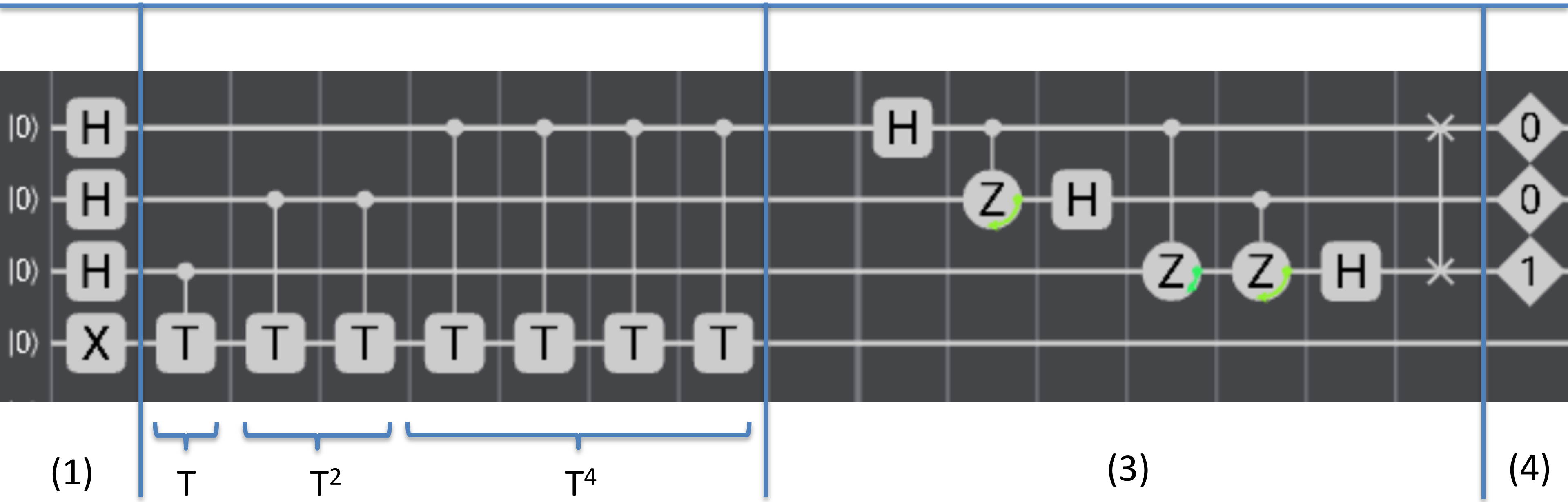


(1) (2) (3) (4)

After the first step of the algorithm

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle\, |1\rangle$$

Step 2: Applying U^x

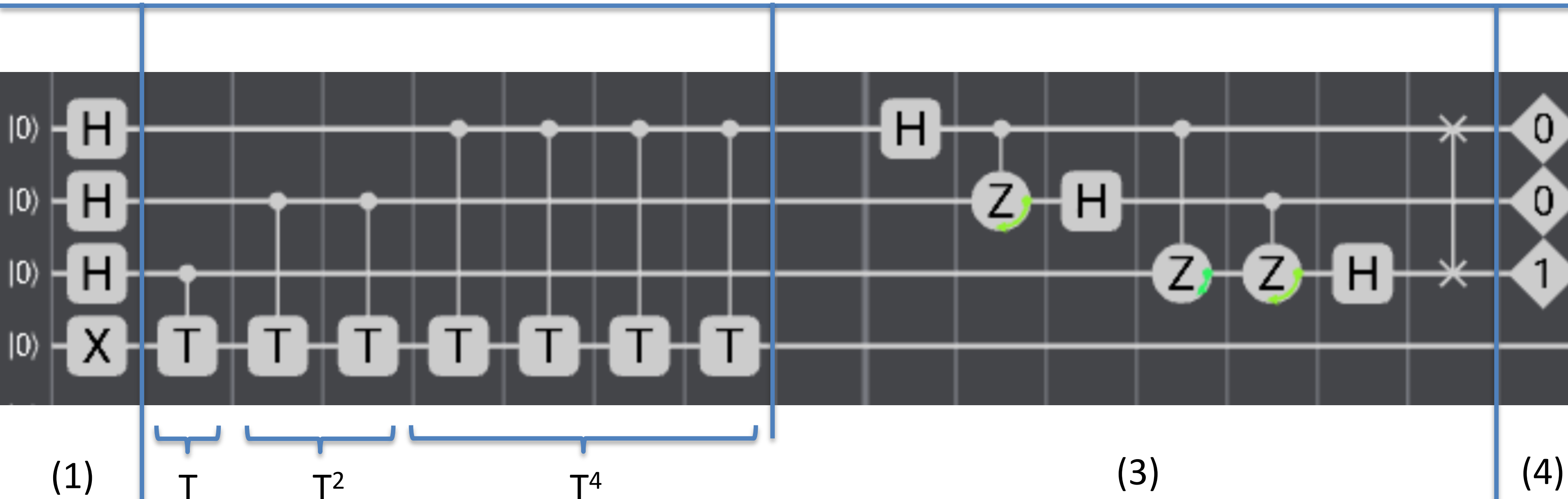


After step 2:

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle T^x |1\rangle$$

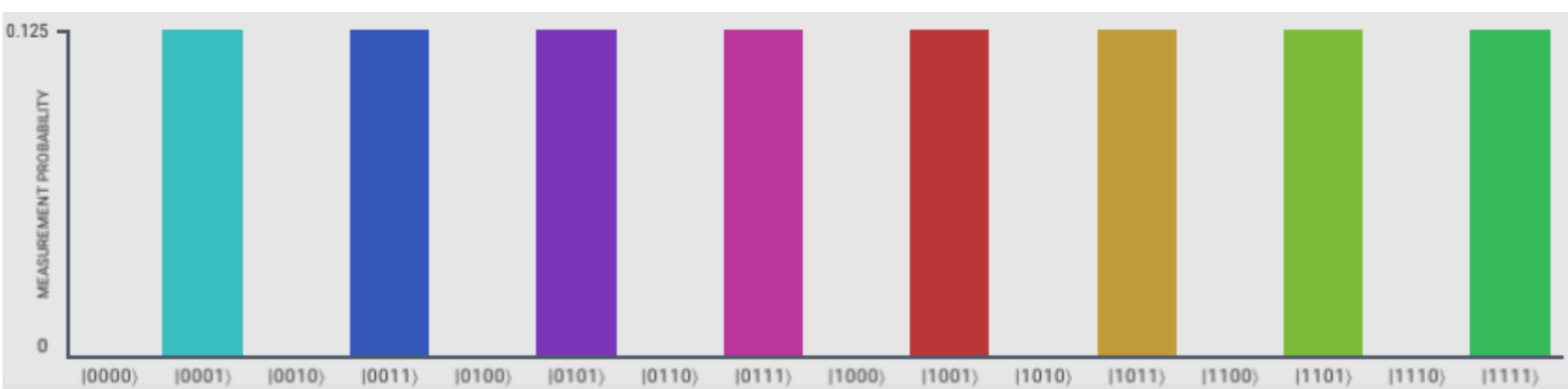
$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \exp\left(i\frac{\pi}{4}x\right) |1\rangle$$

Step 2: Applying U^x

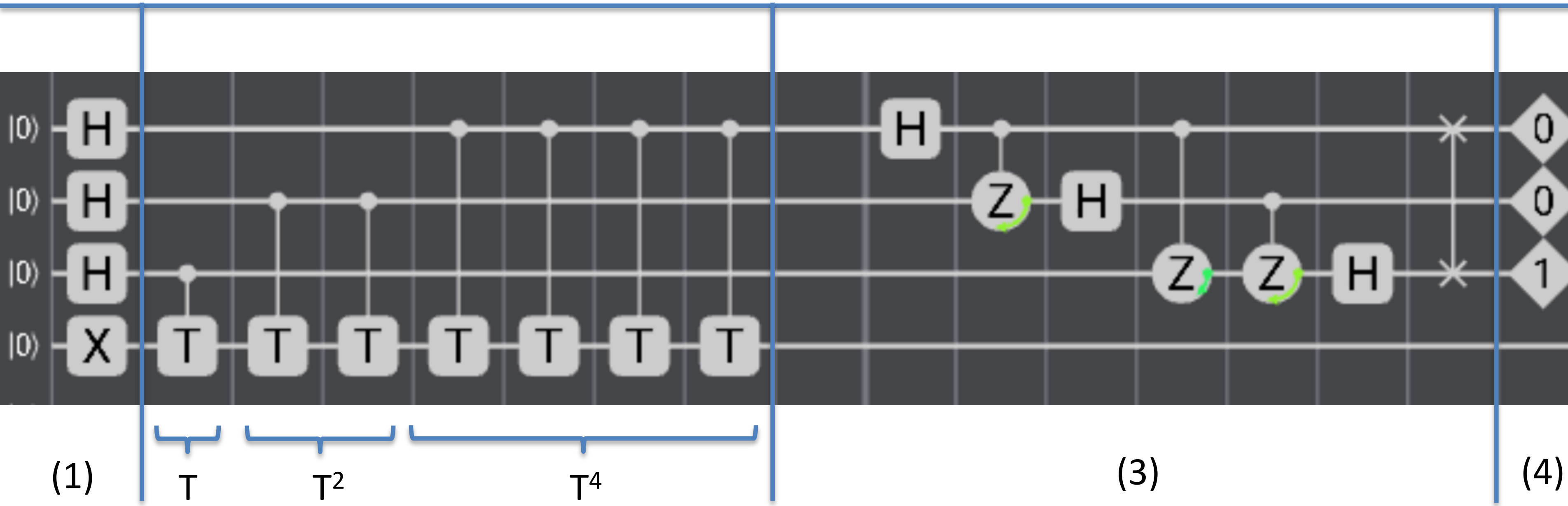


$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \exp\left(i\frac{\pi}{4}x\right) |x\rangle\, |1\rangle$$

Equal superposition, where each state has a phase proportional to x:



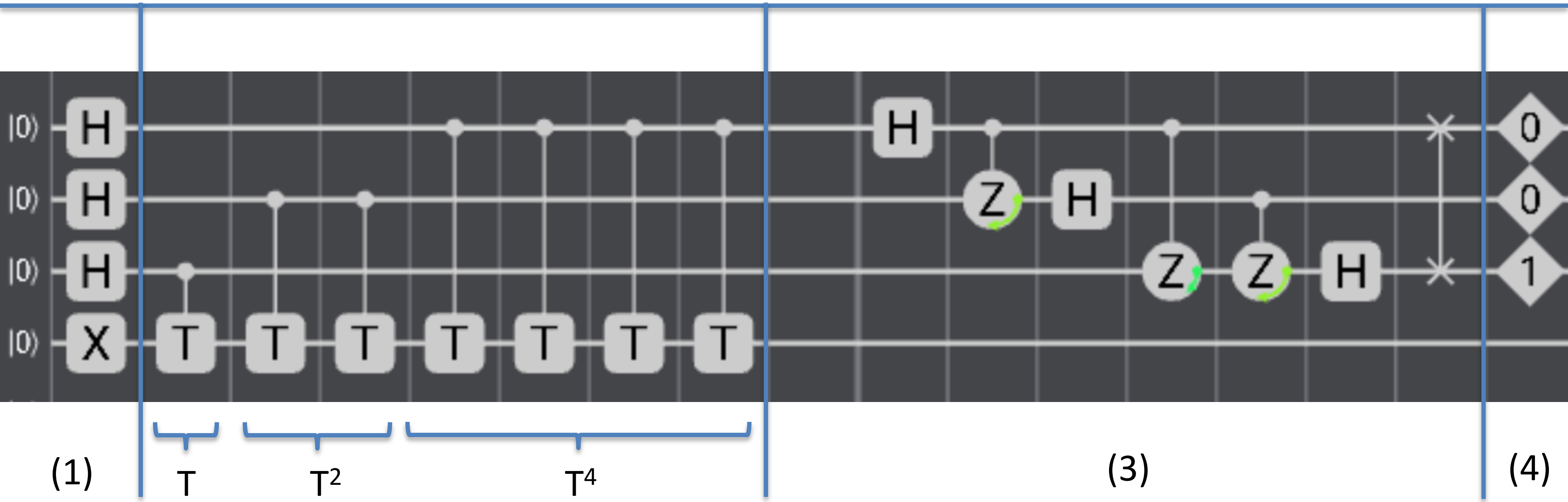
Step 3: Quantum Fourier Transform



The QFT determines the period of the function, in this case it will exactly find the answer:

$$|\psi\rangle = |001\rangle\,|1\rangle$$

Step 4: Measure



Finally we measure an obtain the result 001_2=1, so in this case:

$$2^n \theta = 1$$

$$\theta = \frac{1}{8}$$

As we expected!

$$U\left|\psi\right\rangle = \exp(2\pi i \theta)\left|\psi\right\rangle$$

$$T\left|1\right\rangle = \exp\left(i\frac{\pi}{4}\right)\left|1\right\rangle$$

Quantum Phase Estimation Circuit

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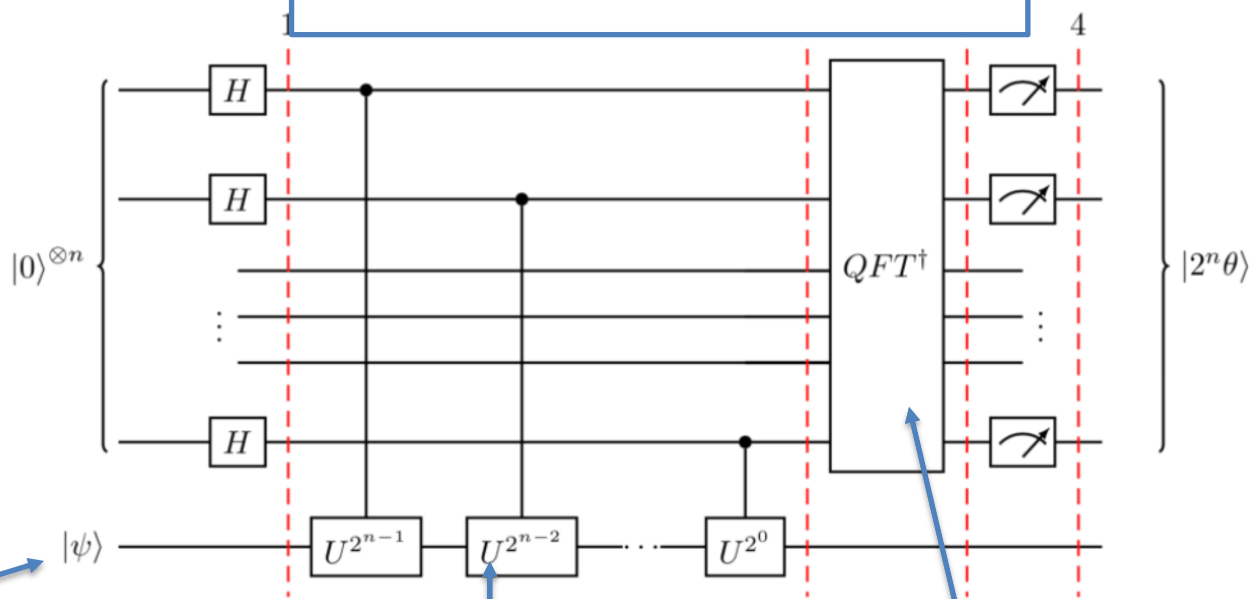
$$U\,|\psi\rangle = \exp(2\pi i\theta)\,|\psi\rangle$$



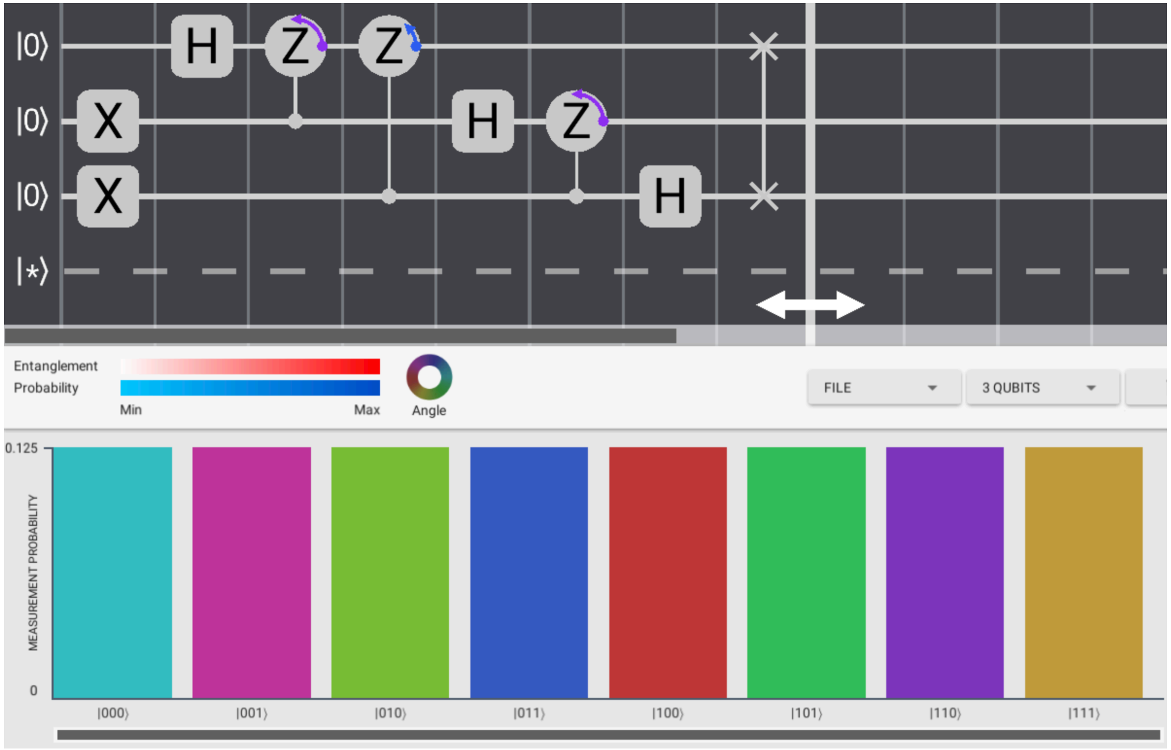
(1) Prepare equal superposition, x and eigenstate $|\psi\rangle$.
(2) Apply U^x to lower register
(3) Apply (inverse) Quantum Fourier Transform to the x register
(4) Measure to obtain an estimate equal to $2^n\theta$.

Quantum Fourier Transform

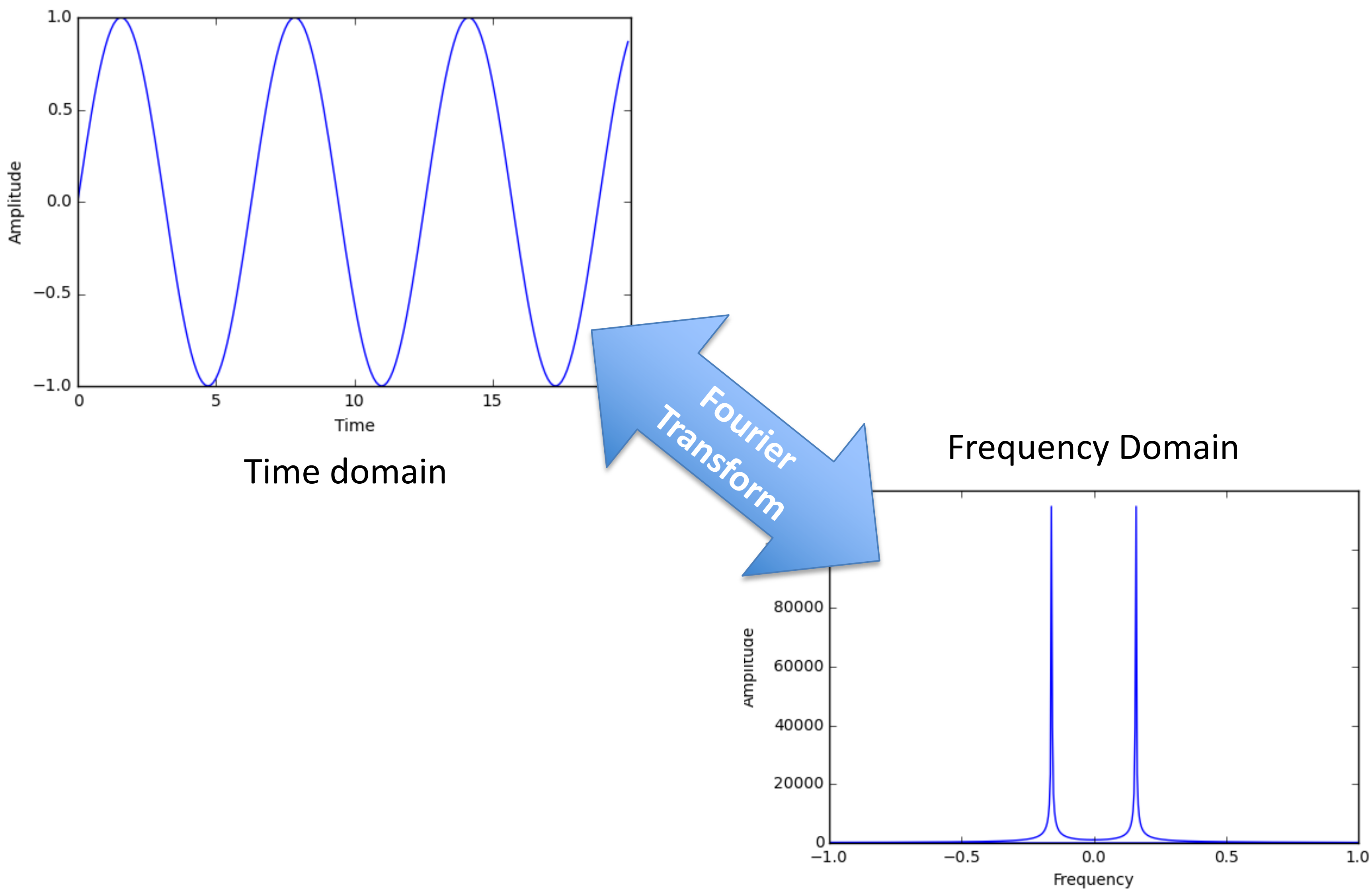
Fourier Transform in Quantum Computing

In QC the equivalent of the Fourier Transform – quantum Fourier Transform (QFT) – is important in a number of algorithms, most notably Shor's Factoring algorithm…

Hence, before we can cover Shor's algorithm we need to understand the QFT and how to implement it in a QC (and on the QUI)…



Introduction to Fourier Transform



Discrete Fourier Transform

Maps a vector: $(x_0, x_1, \ldots, x_{N-1}) \in \mathcal{C}^N$ to a vector: $(y_0, y_1, \ldots, y_{N-1}) \in \mathcal{C}^N$

According to:

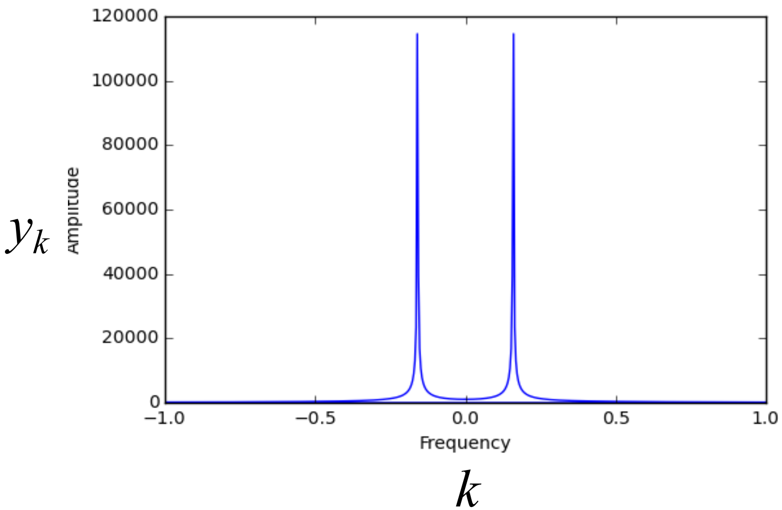
$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

NB.

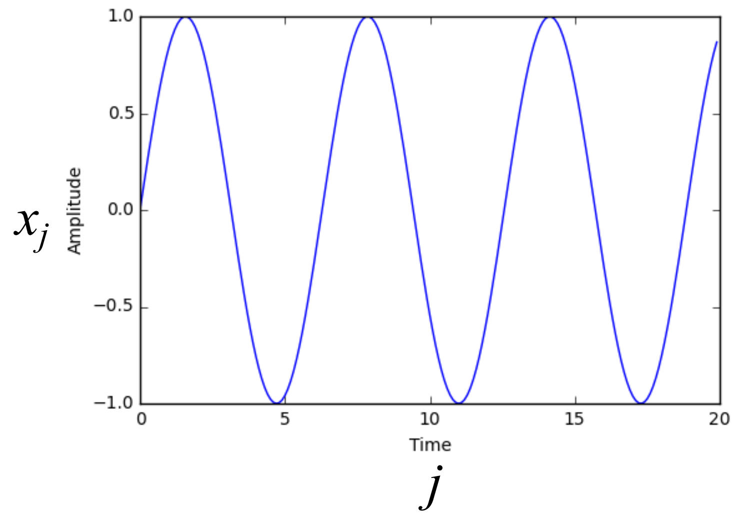
i = sqrt(-1)

j and k are integers

e.g. Frequency Domain



e.g. Time Domain



Example: Fourier transform of periodic function

Imagine that we had a periodic function:

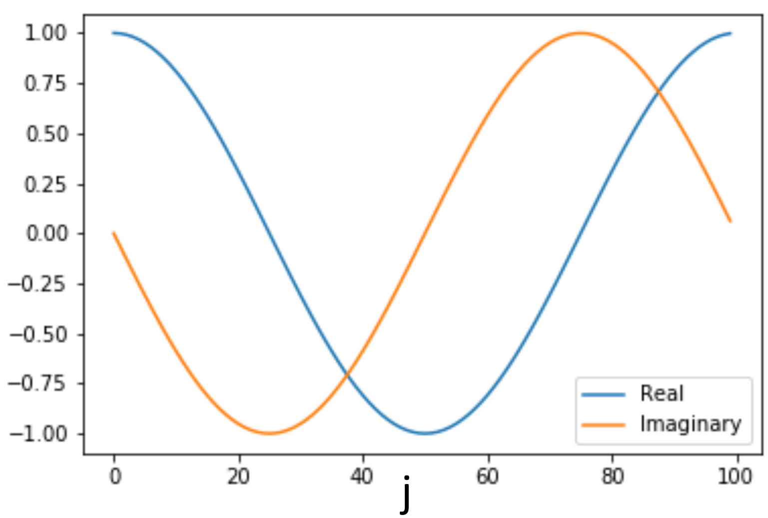
The frequency, u

$$x_j = \exp\left(-2\pi i \frac{uj}{N}\right)$$

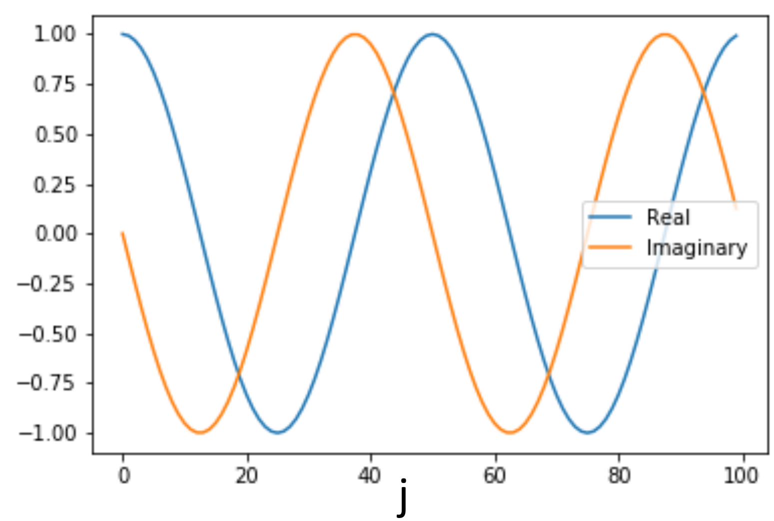
0<=j<N

Complex number, i^2=-1

u=1



u=2



Example: Periodic function

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \exp\left(2\pi i \frac{jk}{N}\right)$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(-2\pi i \frac{uj}{N}\right) \exp\left(2\pi i \frac{jk}{N}\right)$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(-2\pi i \frac{j(k-u)}{N}\right)$$

If k=u then

$$\boxed{y_u = \sqrt{N}}$$

Example: Periodic function

For any other value of k,

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(-2\pi i \frac{j(k-u)}{N}\right)$$

Recall, for a geometric series,

$$1 + r + r^2 + \ldots + r^{N-1} = \frac{1-r^N}{1-r}$$

Where for us,

$$r = \exp\left(-2\pi i \frac{k-u}{N}\right)$$

And therefore

$$r^N = 1$$

Except for k=u,

$$\boxed{y_k = 0}$$

Fourier Transform as a Matrix

We define the Fourier transformation matrix as follows:

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

$$y_k = \sum_j F_{kj} x_j \quad \text{where} \quad F_{kj} = \frac{1}{\sqrt{N}} e^{2\pi i jk/N}$$

For example:

N=2: $F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

N=4: $F = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

We will see that the quantum Fourier transform for one qubit is a Hadamard gate!

Quantum Fourier Transform (QFT)

The Fourier transform, written in this matrix form is unitary. It can make a valid quantum operation:

$$|\psi\rangle = \sum_{j=0}^{N-1} x_j |j\rangle \xrightarrow{\text{QFT}} |\psi'\rangle = \sum_{j=0}^{N-1} y_j |j\rangle$$

with

$$y_k = \sum_{j=0}^{N-1} F_{kj} x_j$$

$$F_{kj} = \frac{1}{\sqrt{N}} e^{2\pi i jk/N}$$

On an individual basis state $|a\rangle$ (i.e. j = a only non-zero x_j) we have:

$$|a\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k |k\rangle\,,\ y_k = \sum_{j=0}^{N-1} F_{kj} x_j = F_{ka} = \frac{1}{\sqrt{N}} e^{2\pi i ka/N}$$

i.e. $$\text{QFT}\,|a\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} ka} |k\rangle$$

(more familiar form relating variables a and k by Fourier transform)

Question: How can we systematically make this operation using quantum gates?

Product Form of QFT

The Fourier transform can be expressed in a product notation:

$$|j_1,\ldots,j_n\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_1 j_2\ldots j_{n-1}j_n}|1\rangle}{\sqrt{2}}$$

(this is not obvious – see appendix at end)

Where the notation $$0.j_1 j_2 \ldots j_n = \frac{j_1}{2} + \frac{j_2}{2^2} + \ldots + \frac{j_{n-1}}{2^{n-1}} + \frac{j_n}{2^n}$$

is shorthand for writing a fraction in binary notation. That is,

$$0.1 = \frac{1}{2}$$
$$0.11 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$$
$$0.101 = \frac{1}{2} + \frac{1}{2^3} = \frac{5}{8}$$

etc

Product Form: One Qubit

$$|j_1,\ldots,j_n\rangle \to \frac{|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_{n-1} j_n}|1\rangle}{\sqrt{2}}$$

For one qubit (ie. n=1, N=2): $|j_1\rangle \to \dfrac{|0\rangle + e^{2\pi i\,0.j_1}|1\rangle}{\sqrt{2}}$ $\quad j_1 = 0,\,1$

$$|0\rangle \to \frac{|0\rangle + e^{2\pi i\ 0.0}|1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Beware binary fraction!
0.1 = 1/2 etc

$$|1\rangle \to \frac{|0\rangle + e^{2\pi i\ \underline{0.1}}|1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{\pi i}|1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

(i.e. a Hadamard)

As before, we get:

$$F = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Product Form: Two Qubits

$$|j_1, \ldots, j_n\rangle \rightarrow \frac{|0\rangle + e^{2\pi i \; 0.j_n}\,|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i \; 0.j_{n-1}j_n}\,|1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i \; 0.j_1 j_2 \ldots j_{n-1}j_n}\,|1\rangle}{\sqrt{2}}$$

$$\boxed{|j_1 j_2\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\, 0.j_2}\,|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i\, 0.j_1 j_2}\,|1\rangle}{\sqrt{2}}}$$

$$|00\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$|01\rangle \rightarrow \frac{|0\rangle + e^{i2\pi \;\, 0.1}\,|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi \;\, 0.01}\,|1\rangle}{\sqrt{2}} = \frac{|00\rangle + i\,|01\rangle - |10\rangle - i\,|11\rangle}{2}$$

$$|10\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi \;\, 0.1}\,|1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

$$|11\rangle \rightarrow \frac{|0\rangle + e^{i2\pi \;\, 0.1}\,|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi \;\, 0.11}\,|1\rangle}{\sqrt{2}} = \frac{|00\rangle - i\,|01\rangle - |10\rangle + i\,|11\rangle}{2}$$

Product Notation: Two Qubits

$$|j_1,\ldots,j_n\rangle \to \frac{|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_{n-1} j_n}|1\rangle}{\sqrt{2}}$$

$$|00\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$|01\rangle \to \frac{|0\rangle + e^{i2\pi\ 0.1}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi\ 0.01}|1\rangle}{\sqrt{2}} = \frac{|00\rangle + i|01\rangle - |10\rangle - i|11\rangle}{2}$$

$$|10\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi\ 0.1}|1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

$$|11\rangle \to \frac{|0\rangle + e^{i2\pi\ 0.1}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi\ 0.11}|1\rangle}{\sqrt{2}} = \frac{|00\rangle - i|01\rangle - |10\rangle + i|11\rangle}{2}$$

As before: $F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

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Pick it apart…

Look a little bit more closely:

$$|j_1,\ldots,j_n\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_{n-1} j_n}|1\rangle}{\sqrt{2}}$$

Very similar to equal superposition. All qubits have an equal amplitude, just not an equal phase.

Each qubit acquires a phase dependent on (the original state of) all prior qubits.

$$\frac{|0\rangle + e^{2\pi i\, 0.j_1 j_2 \ldots j_n}|1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{2\pi i\,[\frac{j_1}{2} + \frac{j_2}{2^2} + \ldots + \frac{j_n}{2^n}]}|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + e^{2\pi i\,\frac{j_1}{2}} e^{2\pi i\,\frac{j_2}{2^2}} \ldots e^{2\pi i\,\frac{j_n}{2^n}}|1\rangle}{\sqrt{2}}$$

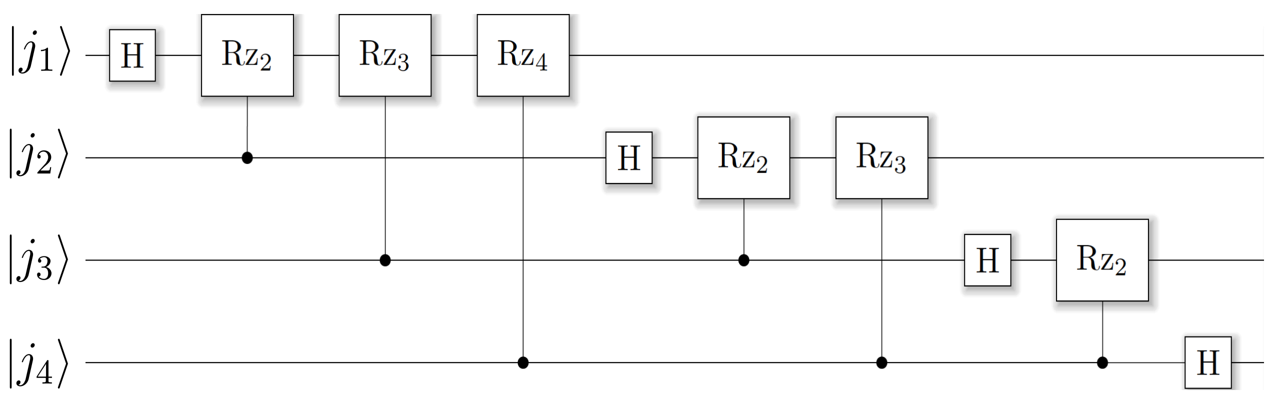
Product of phases applied, i.e. of the form $\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i j_k/2^k} \end{pmatrix}$ e.g. rotation by $\theta = \frac{2\pi}{2^k}$ controlled by j_k

Circuit for QFT

Look carefully at the product form:

$$|j_1,\ldots,j_n\rangle \rightarrow \underbrace{\frac{|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle}{\sqrt{2}}}_{|j_1\rangle} \otimes \underbrace{\frac{|0\rangle + e^{2\pi i\ 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}}}_{|j_2\rangle} \otimes \ldots \otimes \underbrace{\frac{|0\rangle + e^{2\pi i\ 0.j_1 j_2\ldots j_{n-1}j_n}|1\rangle}{\sqrt{2}}}_{|j_n\rangle}$$

Suggests an efficient circuit implementation – e.g. for n=4:



$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2 j_3 j_4}|1\rangle\right) \rightarrow |j_4\rangle$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_2 j_3 j_4}|1\rangle\right) \rightarrow |j_3\rangle$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_3 j_4}|1\rangle\right) \rightarrow |j_2\rangle$$

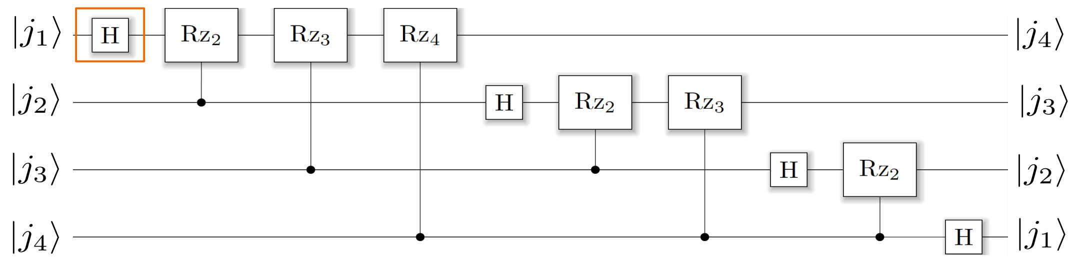
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_4}|1\rangle\right) \rightarrow |j_1\rangle$$

Controlled rotations with: $$R_{z_k} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

Notice how the required QFT form is recovered by re-labelling qubits

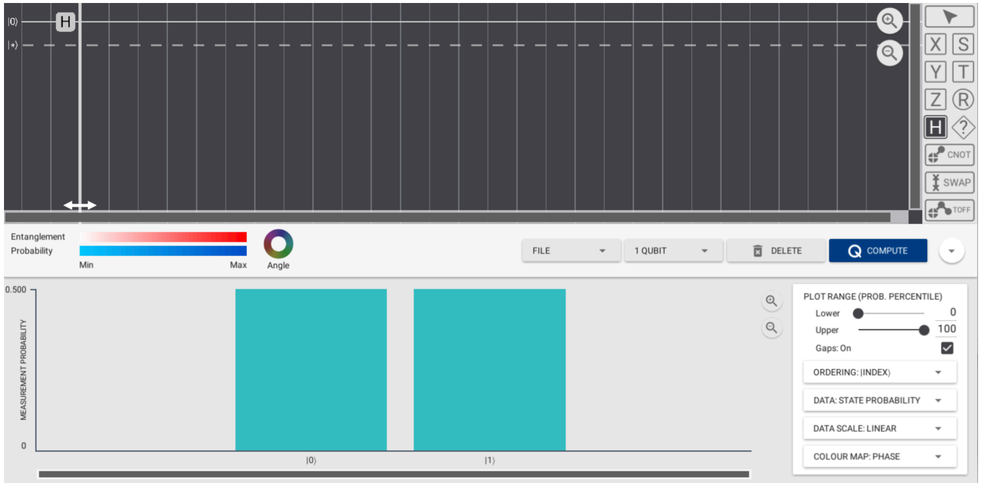
One qubit QFT circuit

Look at the pattern of the circuit:



For one qubit we have just a H-gate:

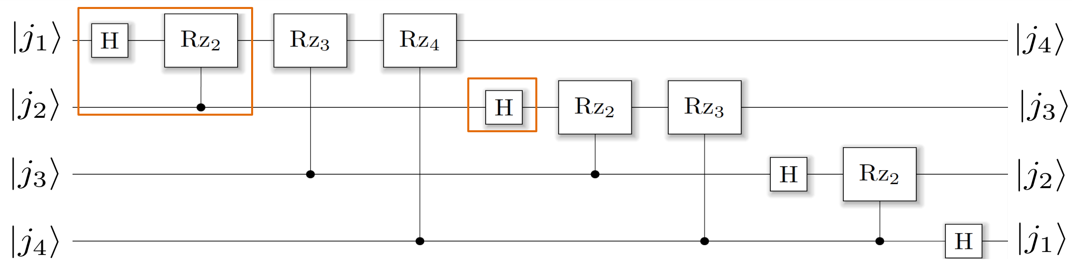
$$|j_1\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\, 0.j_1}\,|1\rangle}{\sqrt{2}}$$



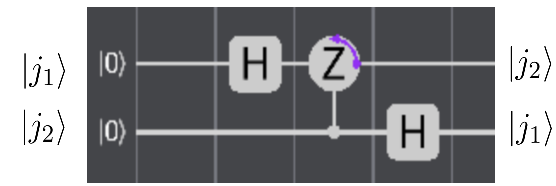
$$|0\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\ 0.0}\,|1\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\ 0.1}\,|1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{\pi i}\,|1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Two Qubit QFT circuit



QUI



$$R_{z_k} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix} \qquad R_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

QUI gates:

$$\begin{aligned} R_Z(\theta_R) = e^{i\theta_g}\left[I\cos\frac{\theta_R}{2} - iZ\sin\frac{\theta_R}{2}\right] &= e^{i\theta_g}\left[\begin{pmatrix} \cos\frac{\theta_R}{2} & 0 \\ 0 & \cos\frac{\theta_R}{2} \end{pmatrix} - i\begin{pmatrix} \sin\frac{\theta_R}{2} & 0 \\ 0 & -\sin\frac{\theta_R}{2} \end{pmatrix}\right] \\ &= e^{i\theta_g}\begin{pmatrix} \cos\frac{\theta_R}{2} - i\sin\frac{\theta_R}{2} & 0 \\ 0 & \cos\frac{\theta_R}{2} + i\sin\frac{\theta_R}{2} \end{pmatrix} \\ &= e^{i\theta_g}\begin{pmatrix} e^{-i\theta_R/2} & 0 \\ 0 & e^{+i\theta_R/2} \end{pmatrix} \\ &= e^{i\theta_g}e^{-i\theta_R/2}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_R} \end{pmatrix} \end{aligned}$$

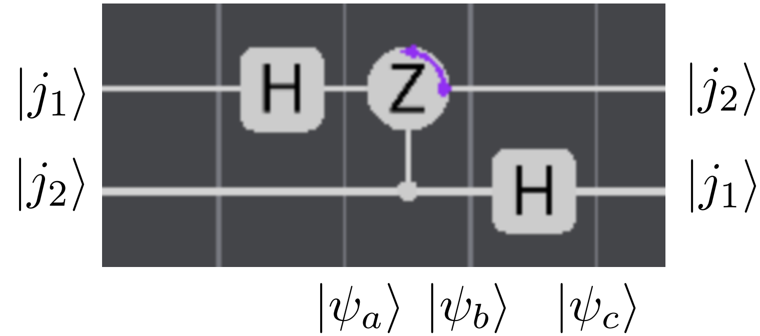
$$R_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \equiv R_Z\left(\frac{\pi}{2}\right) \qquad \text{with} \quad \theta_g = \frac{\pi}{4}$$

Global phase cancels prefactor

Two Qubit QFT circuit - walkthrough

$$|j_1, \ldots, j_n\rangle \rightarrow \frac{|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_{n-1} j_n}|1\rangle}{\sqrt{2}}$$

$$R_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$



Check it gives the product form:

$$|\psi_a\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi j_1}|1\rangle\right) \otimes |j_2\rangle$$

Hadamard has negative sign on |1> if $j_1 = 1$

$$|\psi_b\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi j_1} e^{i(\pi/2) j_2}|1\rangle\right) \otimes |j_2\rangle$$

R_{z2} applied only when $j_2 = 1$

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi j_1} e^{i(\pi/2) j_2}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi j_2}|1\rangle\right)$$

Hadamard on $|j_2>$

Binary fractions: $e^{i\pi j_1} e^{i(\pi/2) j_2} = e^{2\pi i (j_1/2 + j_2/4)} = e^{2\pi i\, 0.j_1 j_2}$

$$\boxed{|\psi_c\rangle = \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2}|1\rangle\right)}_{|j_2\rangle} \otimes \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.j_2}|1\rangle\right)}_{|j_1\rangle}}$$

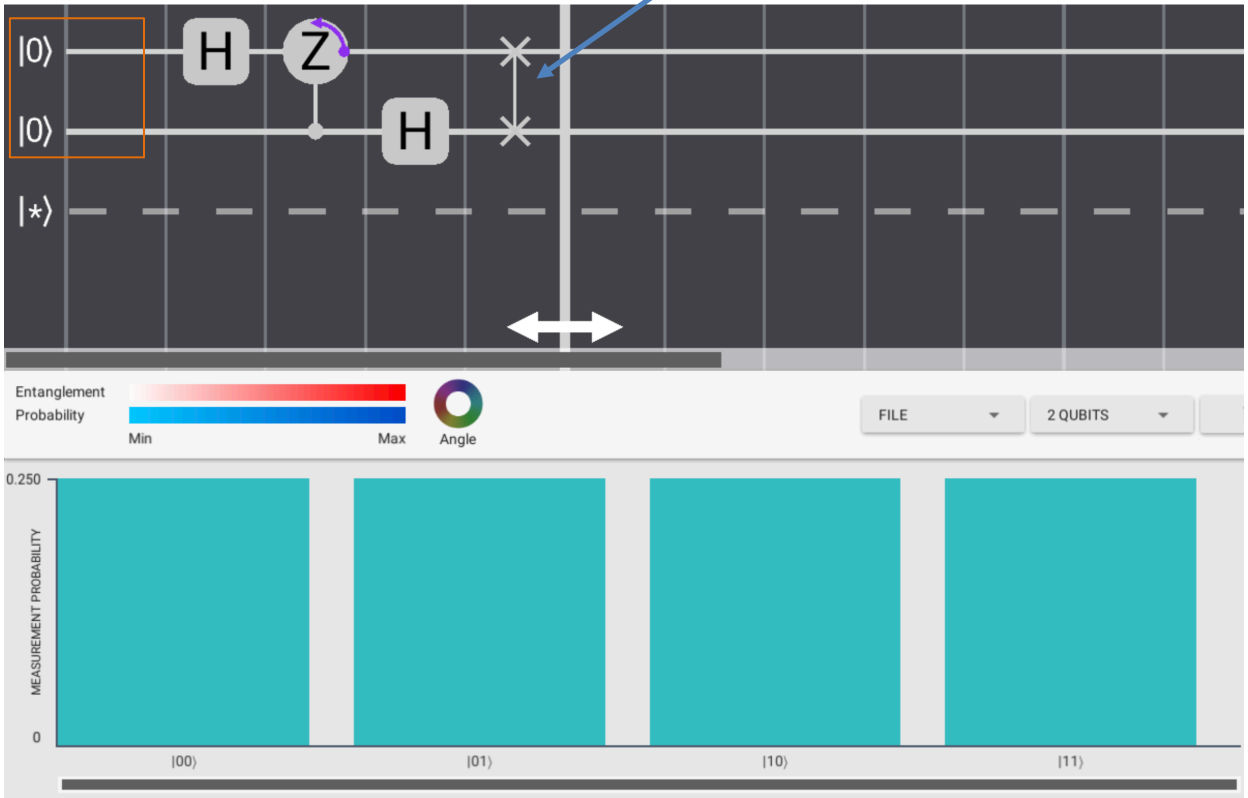
i.e. circuit gives product form with j1 and j2 order reversed

Two Qubit QFT circuit - QUI

Note: inserted SWAP gate so ordering is same c/f ket expression

$|j_1\rangle$

$|j_2\rangle$



$|j_1\rangle$

$|j_2\rangle$

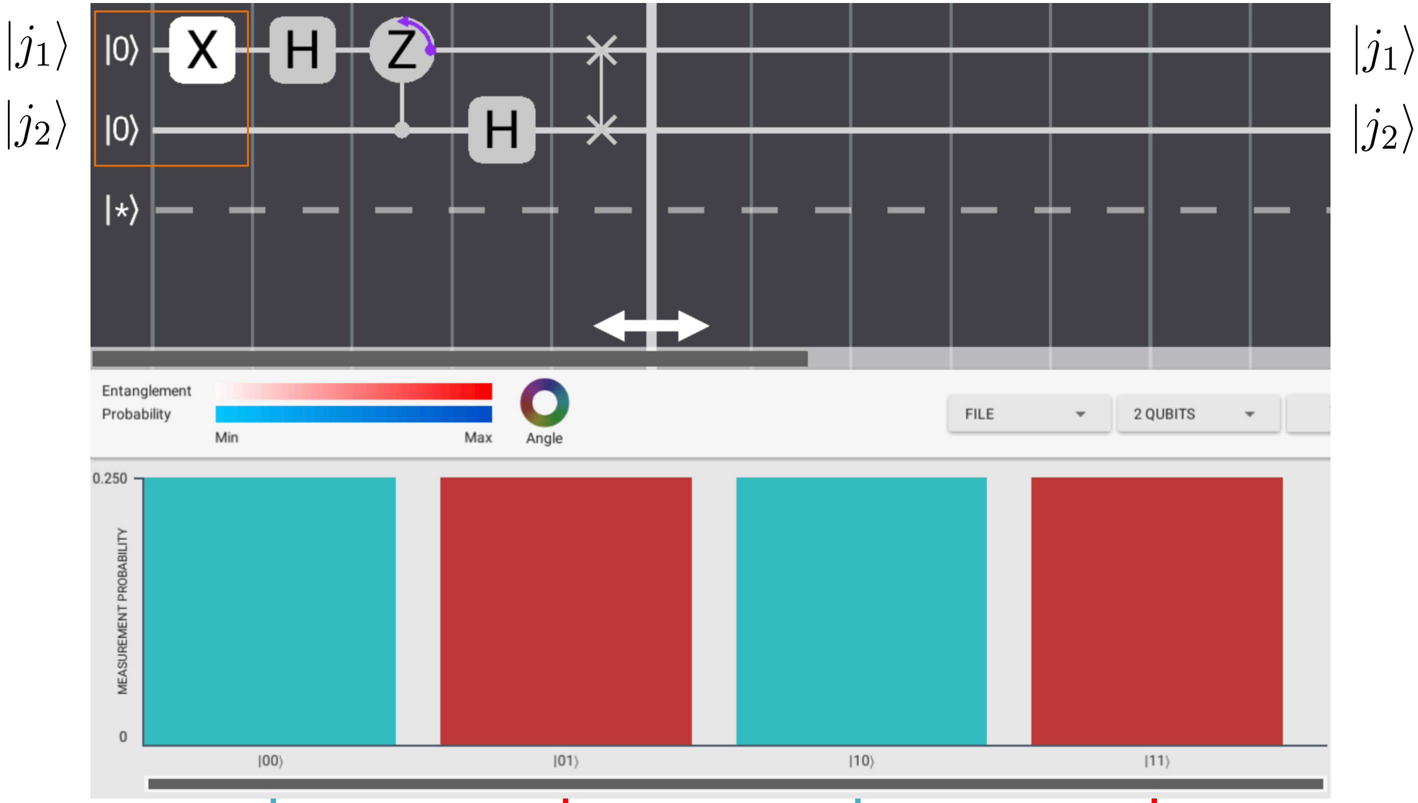
All phases zero

$$\boxed{|00\rangle} \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

Two Qubit QFT circuit - QUI

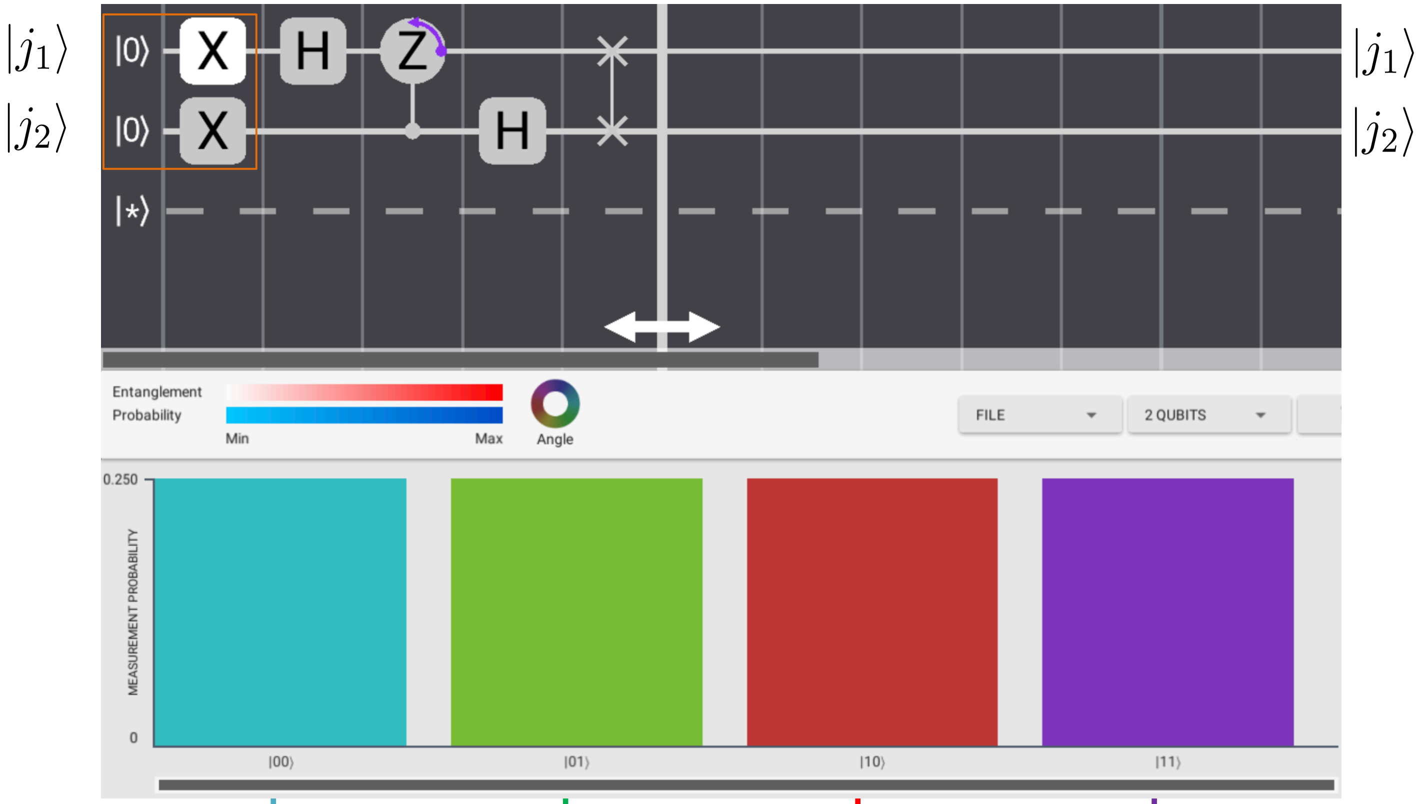
$$\boxed{|01\rangle} \to \frac{|0\rangle + e^{i2\pi\ 0.1}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi\ 0.01}|1\rangle}{\sqrt{2}} = \frac{|00\rangle + i\,|01\rangle - |10\rangle - i\,|11\rangle}{2}$$

Two Qubit QFT circuit - QUI



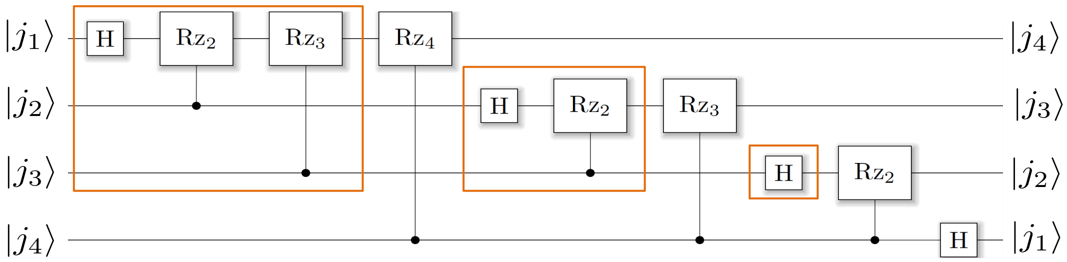
$$\boxed{|10\rangle} \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi\ 0.1}|1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

Two Qubit QFT circuit - QUI



$$\boxed{|11\rangle} \rightarrow \frac{|0\rangle + e^{i2\pi\ 0.1}|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + e^{i2\pi\ 0.11}|1\rangle}{\sqrt{2}} = \frac{|00\rangle - i\,|01\rangle - |10\rangle + i\,|11\rangle}{2}$$

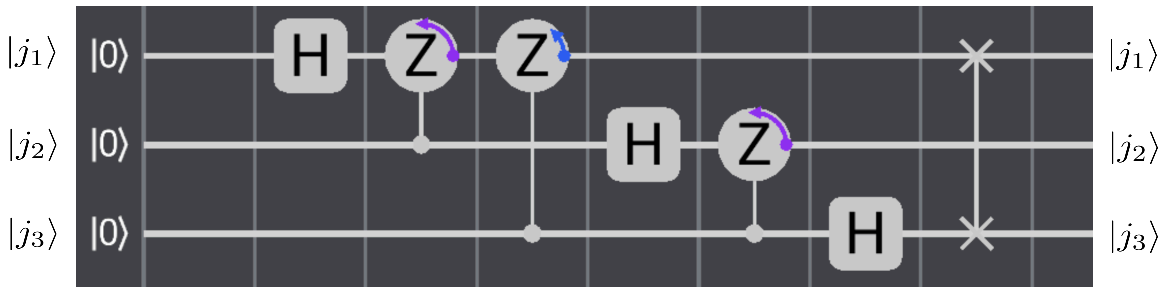
Three Qubit QFT circuit



Rotation gates in the QUI:

$$R_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \equiv R_Z\left(\frac{\pi}{2}\right) \qquad \text{with} \quad \theta_g = \frac{\pi}{4}$$

$$R_{z_3} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \equiv R_Z\left(\frac{\pi}{4}\right) \qquad \text{with} \quad \theta_g = \frac{\pi}{8}$$



SWAP gate reverses order, so same as input

Three Qubit QFT - QUI

Example: |011>

NB. same as $\pi/4$ etc

$$\boxed{|011\rangle} \rightarrow \left(\frac{1}{\sqrt{2}}\right)^3 \Big(|000\rangle + e^{3\pi i/4}|001\rangle + e^{3\pi i/2}|010\rangle + e^{9\pi i/4}|011\rangle$$
$$+e^{i\pi}|100\rangle + e^{7\pi i/4}|101\rangle + e^{5\pi i/2}|110\rangle + e^{13\pi i/4}|111\rangle\Big)$$

Step back for a moment

After all that, let's check on what we were trying to achieve:

On a single basis state $$\mathrm{QFT}\,|a\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} ka} \,|k\rangle$$

e.g. $$|011\rangle \rightarrow \left(\frac{1}{\sqrt{2}}\right)^3 \Big(|000\rangle + e^{3\pi i/4}\,|001\rangle + e^{3\pi i/2}\,|010\rangle + e^{9\pi i/4}\,|011\rangle$$
$$+e^{i\pi}\,|100\rangle + e^{7\pi i/4}\,|101\rangle + e^{5\pi i/2}\,|110\rangle + e^{13\pi i/4}\,|111\rangle\Big)$$

i.e. 3=101 $$|3\rangle \rightarrow \left(\frac{1}{\sqrt{2}}\right)^3 \Big(|0\rangle + e^{3\pi i/4}\,|1\rangle + e^{3\pi i/2}\,|2\rangle + e^{9\pi i/4}\,|3\rangle$$
$$+e^{i\pi}\,|4\rangle + e^{7\pi i/4}\,|5\rangle + e^{5\pi i/2}\,|6\rangle + e^{13\pi i/4}\,|7\rangle\Big)$$

It obeys: $$\mathrm{QFT}\,|3\rangle = \frac{1}{\sqrt{8}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{8} 3k} \,|k\rangle$$ (check it!)

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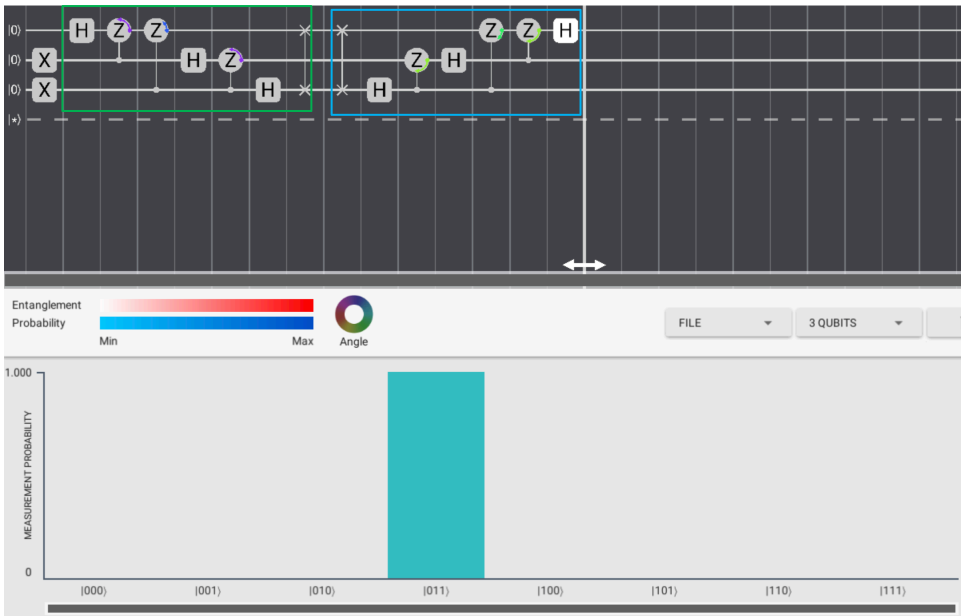
Programing the Inverse QFT

As with any circuit: invert the QFT by inverting every gate and reversing the order:



$$R_{z_2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \equiv R_Z\left(\frac{\pi}{2}\right) \qquad \text{with} \quad \theta_g = \frac{\pi}{4}$$

$$R_{z_3} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \equiv R_Z\left(\frac{\pi}{4}\right) \qquad \text{with} \quad \theta_g = \frac{\pi}{8}$$

$$R_Z(\theta_R) = e^{i\theta_g}\left[I\cos\frac{\theta_R}{2} - iZ\sin\frac{\theta_R}{2}\right]$$

$$= e^{i\theta_g} e^{-i\theta_R/2} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_R} \end{pmatrix}$$

$$R_Z^\dagger(\theta_R) = e^{-i\theta_g} e^{+i\theta_R/2} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta_R} \end{pmatrix}$$

i.e. Reverse signs of θ_R and θ_g

e.g. |011>

$$|011\rangle \boxed{\xrightarrow{\text{QFT}}} \left(\frac{1}{\sqrt{2}}\right)^3 \Big(|000\rangle + e^{3\pi i/4}|001\rangle + e^{3\pi i/2}|010\rangle + e^{9\pi i/4}|011\rangle$$

$$+ e^{i\pi}|100\rangle + e^{7\pi i/4}|101\rangle + e^{5\pi i/2}|110\rangle + e^{13\pi i/4}|111\rangle\Big) \boxed{\xrightarrow{\text{QFT}^\dagger}} |011\rangle$$

Appendix: proof of the product form

In case you want to go through it at your leisure

$$
\begin{aligned}
|j\rangle &\rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i\, jk/N} \, |k\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i\, j \sum_l k_l 2^{-l}} \, |k_1 \ldots k_n\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \otimes_l e^{2\pi i\, j k_l 2^{-l}} \, |k_l\rangle \\
&= \frac{1}{\sqrt{N}} \otimes_l \left[|0\rangle + e^{2\pi i\, j 2^{-l}} \, |1\rangle \right] \\
&= \frac{|0\rangle + e^{2\pi i\, 0.j_n} \, |1\rangle}{\sqrt{2}} \otimes \ldots \otimes \frac{|0\rangle + e^{2\pi i\, 0.j_1 j_2 \ldots j_n} \, |1\rangle}{\sqrt{2}}
\end{aligned}
$$

This Week

Lecture 11

Fourier Transformations, Regular Fourier Transform, Fourier Transform as a matrix, Quantum Fourier Transform, QUI examples, Inverse QFT

Lecture 12

Shor's Quantum Factoring algorithm, Shor's algorithm for factoring and discrete logarithm, HSP Problem

Lab 6

QFT and Shor's algorithm