MULT90063 Introduction to Quantum Computing

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MULT 90063 Assignment 1

Q1: (a) $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

So the state we get is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

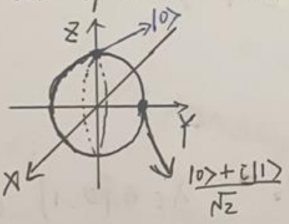
(b) A rotation by $-\frac{\pi}{2}$ around axis X.

The matrix of this operation is $R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

Then, $R_1|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

So this operation meet the requirements.

(c)

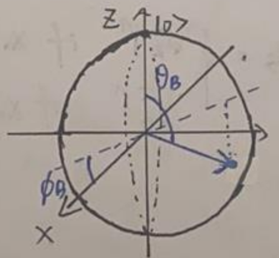


(d) The operation is a rotation by $-\frac{2\pi}{3}$ around axis $(X+Y)$

The matrix of this operation is: $R_2 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2}e^{\frac{i\pi}{4}} \\ -\frac{\sqrt{3}}{2}e^{-\frac{i\pi}{4}} & \frac{1}{2} \end{pmatrix}$

So. $R_2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2}e^{-\frac{i\pi}{4}} \end{pmatrix} = \frac{|0\rangle - \sqrt{3}e^{-\frac{i\pi}{4}}|1\rangle}{2}$

(e)



$\phi_B = -\frac{\pi}{4}$

$\theta_B = -\frac{2\pi}{3}$

(f). From two previous question, we know that.

$$R_2^{\dagger}\cdot\begin{pmatrix}\frac{1}{2}\\ -\frac{\sqrt{3}}{2}e^{-\frac{i\pi}{4}}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\quad R_1\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$$

So $(R_1\cdot R_2^{\dagger})\begin{pmatrix}\frac{1}{2}\\ -\frac{\sqrt{3}}{2}e^{-\frac{i\pi}{4}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$

$$R_1R_2^{\dagger}=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}e^{\frac{-i\pi}{4}} & \frac{i}{2}-\frac{\sqrt{3}}{2}e^{\frac{i\pi}{4}}\\ \frac{i}{2}+\frac{\sqrt{3}}{2}e^{\frac{-i\pi}{4}} & \frac{1}{2}-\frac{\sqrt{3}}{2}ie^{\frac{i\pi}{4}}\end{pmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{6}}{4}i & \frac{1}{2}i-\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i\\ \frac{1}{2}i+\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i & \frac{\sqrt{6}}{4}+\frac{1}{2}-\frac{\sqrt{6}}{4}i\end{pmatrix}$$

assume axis is $(a.b.c)$, angle is θ

so Rotation matrix should be

$$R=I\cdot\cos\frac{\theta}{2}-i\cdot\hat{n}\sigma\sin\frac{\theta}{2}$$

$$=I\cdot\cos\frac{\theta}{2}-i(aX+bY+cZ)\sin\frac{\theta}{2}$$

$$=I\cdot\cos\frac{\theta}{2}-i\cdot\sin\frac{\theta}{2}\begin{bmatrix}c & a-ib\\ a+ib & -c\end{bmatrix}$$

so

$$\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\cdot c=\frac{1}{\sqrt{2}}\left(\frac{1}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{6}}{4}i\right)\quad ①$$

$$-i\sin\frac{\theta}{2}(a-ib)=\frac{1}{\sqrt{2}}\left(\frac{1}{2}i-\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i\right)\quad ②$$

$$-i\cdot\sin\frac{\theta}{2}(a+ib)=\frac{1}{\sqrt{2}}\left(\frac{1}{2}i+\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i\right)\quad ③$$

$$\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}c=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{4}+\frac{1}{2}-\frac{\sqrt{6}}{4}i\right)\quad ④$$

so we can get:

$$\cos\frac{\theta}{2}=\frac{1}{\sqrt{2}}\left(\frac{1}{2}+\frac{\sqrt{6}}{4}\right),\quad \cos^2\frac{\theta}{2}=\frac{5+2\sqrt{6}}{16}$$

$$\sin^2\frac{\theta}{2}=\frac{11-2\sqrt{6}}{16}\quad \sin\frac{\theta}{2}=\frac{\sqrt{11-2\sqrt{6}}}{4},\quad \theta=2\arcsin\frac{\sqrt{11-2\sqrt{6}}}{4}$$

From ②③,

$$-i\sin\frac{\theta}{2}\cdot a=\frac{1}{\sqrt{2}}\left(\frac{i}{2}-\frac{\sqrt{6}}{4}i\right)$$

$$\sin\frac{\theta}{2}\cdot a=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{4}-\frac{1}{2}\right)=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{6}-2}{4}=\frac{\sqrt{3}-\sqrt{2}}{4}$$

so $a=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{11-2\sqrt{6}}}$

we can also get $b=\frac{\sqrt{3}}{\sqrt{11-2\sqrt{6}}}$, $c=\frac{-\sqrt{3}}{\sqrt{11-2\sqrt{6}}}$

so rotation matrix is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{2}+\frac{\sqrt{6}}{4}+\frac{\sqrt{6}}{4}i & \frac{1}{2}i-\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i\\ \frac{1}{2}i+\frac{\sqrt{6}}{4}-\frac{\sqrt{6}}{4}i & \frac{\sqrt{6}}{4}+\frac{1}{2}-\frac{\sqrt{6}}{4}i\end{pmatrix}$$

axis is $\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{11-2\sqrt{6}}},\ \frac{\sqrt{3}}{\sqrt{11-2\sqrt{6}}},\ \frac{-\sqrt{3}}{\sqrt{11-2\sqrt{6}}}\right)$

angle is $2\arcsin\frac{\sqrt{11-2\sqrt{6}}}{4}$

Q2. We know that $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{N}} \sum_{z=0}^{N-1} (-1)^{x\cdot z} |z\rangle$, $N=2^n$

(a) so, $a_0 = \frac{1}{\sqrt{N}} (-1)^{x\cdot 0}$, $a_2 = \frac{1}{\sqrt{N}} (-1)^{x\cdot 2}$

$a_1 = \frac{1}{\sqrt{N}} (-1)^{x\cdot 1}$, $a_3 = \frac{1}{\sqrt{N}} (-1)^{x\cdot 3}$

$\Rightarrow a_0 \cdot a_1 \cdot a_2 \cdot a_3 = (\frac{1}{\sqrt{N}})^4 \cdot (-1)^{6x}$

$= \frac{1}{N^2} = \frac{1}{4^n}$

(b) $a_i = \frac{1}{\sqrt{N}} (-1)^{x\cdot i}$

so $\prod_{i=0}^{N-1} a_i = \prod_{i=0}^{N-1} \frac{1}{\sqrt{N}} (-1)^{x\cdot i}$

$= (\frac{1}{\sqrt{N}})^N \cdot (-1)^{x\cdot \frac{N(N-1)}{2}}$

Because n is even number , $N=2^n \geq 2^2$

so $(-1)^{x\cdot \frac{N(N-1)}{2}} = 1$

$\Rightarrow \prod_{i=0}^{N-1} a_i = (\frac{1}{\sqrt{N}})^N = (\frac{1}{\sqrt{2^n}})^{2^n} = (\frac{1}{\sqrt{2}})^{n\cdot 2^n}$

3. (a) $H(a_0|0\rangle + a_1|1\rangle) = \frac{a_0+a_1}{\sqrt{2}}|0\rangle + \frac{a_0-a_1}{\sqrt{2}}|1\rangle$

$T(a_0|0\rangle + a_1|1\rangle) = a_0|0\rangle + a_1 \cdot e^{\frac{i\pi}{4}}|1\rangle$

$T^{+}(a_0|0\rangle + a_1|1\rangle) = a_0|0\rangle + e^{\frac{-i\pi}{4}} a_1|1\rangle$

(b) $S_0 = |110\rangle$, $S_1 = |11\rangle \cdot H|0\rangle = \frac{1}{\sqrt{2}}(|110\rangle + |111\rangle)$

$S_2 = \frac{1}{\sqrt{2}}(|111\rangle + |110\rangle)$, $S_3 = \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{4}}|111\rangle$

$S_4 = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}|111\rangle$

$S_5 = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\pi}{4}}|111\rangle$

$S_6 = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{4}}|111\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\pi}{4}}|110\rangle$

$S_7 = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{2}}|111\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\pi}{4}}|110\rangle$

$S_8 = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{2}}|110\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\pi}{4}}|111\rangle$

$S_9 = \frac{1}{\sqrt{2}}e^{-\frac{i\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}e^{\frac{3i\pi}{4}}|111\rangle$

$S_{10} = \frac{1}{2}e^{-\frac{i\pi}{4}}|100\rangle + \frac{1}{2}e^{-\frac{i\pi}{4}}|101\rangle + \frac{1}{2}e^{\frac{3i\pi}{4}}|100\rangle - \frac{1}{2}e^{\frac{3i\pi}{4}}|101\rangle$

$= e^{\frac{-i\pi}{4}}|101\rangle$

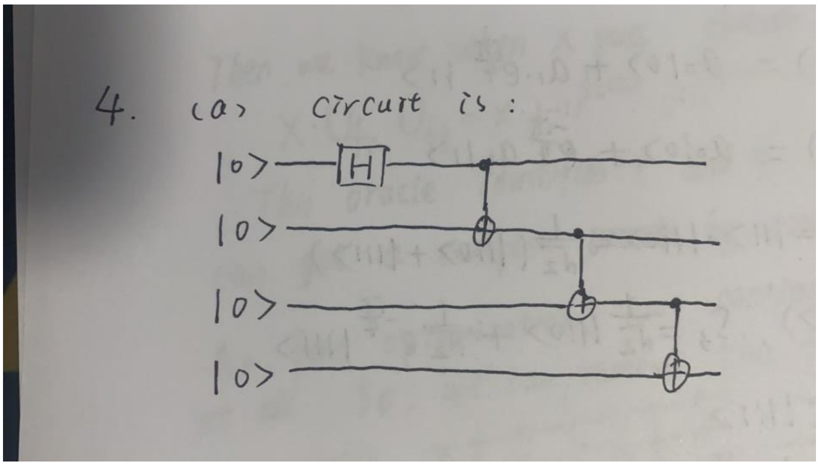
$S_{11} = e^{\frac{i\pi}{4}} \cdot e^{-\frac{i\pi}{4}}|101\rangle = |101\rangle$

$S_{12} = |111\rangle$

$S_{12} = \text{Toffoli } |110\rangle = |111\rangle$

3.(c). ① . Toffoli gate is a universal reversible logic gate. which means that any reversible circuit can be constructed from Toffoli gates. So the decomposed circuit make sure we can implement toffoli gates on quantum computer with basic gates. And we can also construct any reversible circuit on quantum computer by using this circuit.

② Decomposed circuit may save computing resources on a quantum computer Since it use basic operation. 3-qubits gates cost more than 2-qubits gates



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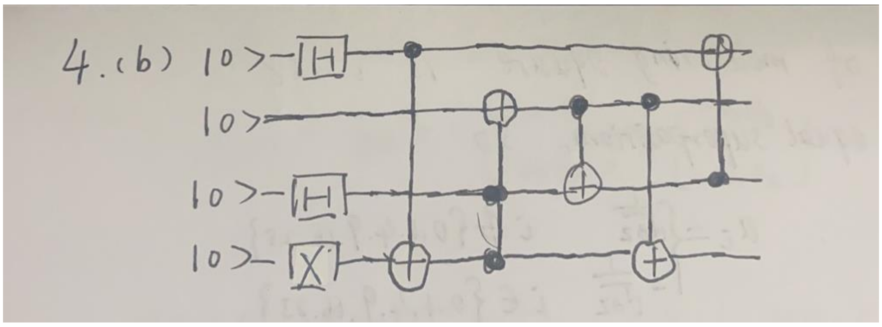

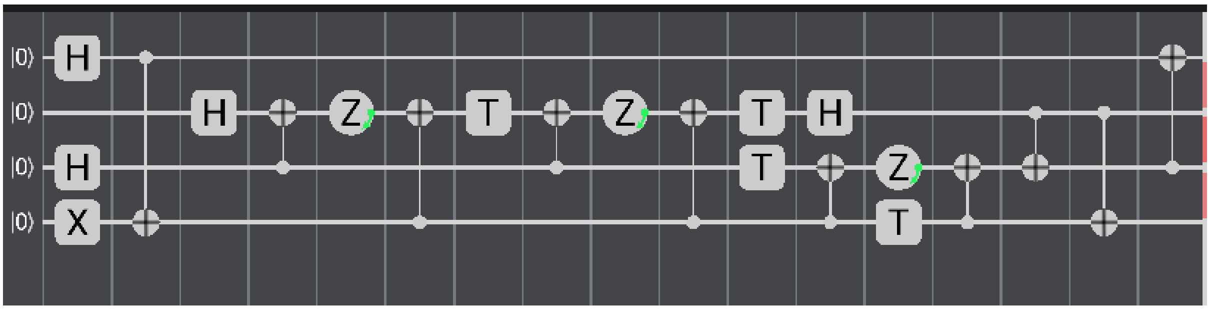

The below circuit is the implementation of above circuit by only using single qubit and two qubit operations.

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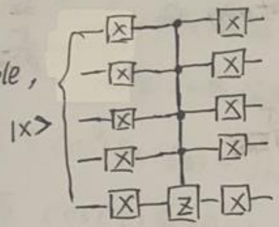
5. (1) Firstly, we can divide the oracle into several parts. Each part is a smaller oracle which identifies 0.1.4.9.16.25 seperately.

For example,



is an oracle which identifies |0>

Because by passing through an oracle, we get.

$$X \cdot U_{f_i} = X \cdot (-1)^{f_i(x)} \quad \text{where } f_i(x) = \begin{cases} 0 & x \neq i \\ 1 & x = i \end{cases}$$

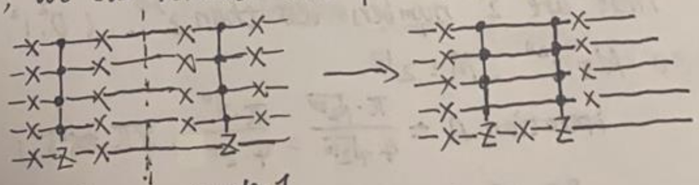
Then we know when X pass through two oracles $U_{f_i} . U_{f_j}$

$$X \cdot U_{f_i} \cdot U_{f_j} = X \cdot (-1)^{f_i(x)} \cdot (-1)^{f_j(x)}.$$

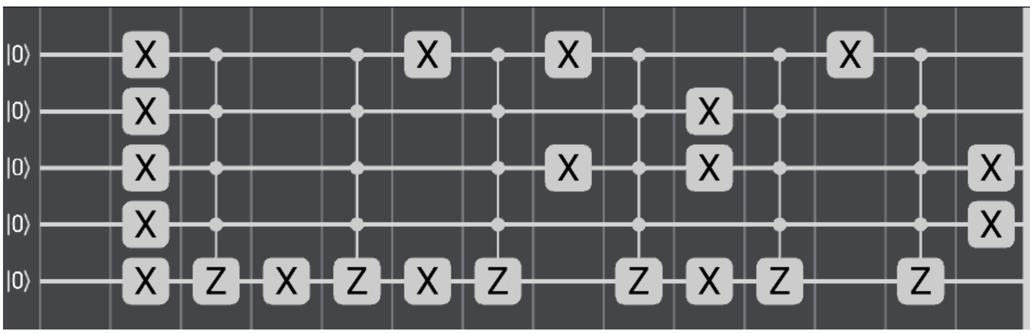
This oracle identifies i and j. Then we know that we can get a complex oracle by combine smaller oracles.

As for optimization, two continuous X gate means no change at all. So, we can remove this from our oracle.

For example



Oracle: 0 Oracle: 1



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5(b) The probability of measuring square is 0.948

Starting from an equal superposition, So

$|\Psi\rangle = \sum_i a_i |i\rangle$, $a_i = \begin{cases} \frac{1}{\sqrt{32}} & i \notin \{0.1.4.9.16.25\} \\ -\frac{1}{\sqrt{32}} & i \in \{0.1.4.9.16.25\} \end{cases}$

So $|\Psi'\rangle = (I - 2|\Phi\rangle\langle\Phi|) \sum a_i |i\rangle$

$= \sum (a_i - 2A)|i\rangle$, $A = \frac{1}{N}\sum_j a_j$

$= \frac{1}{32} \cdot 20 \cdot \frac{1}{\sqrt{32}}$

$= \frac{5}{32\sqrt{2}}$

So the magnitude of measuring 0

is $|a| = a_0 - 2A$

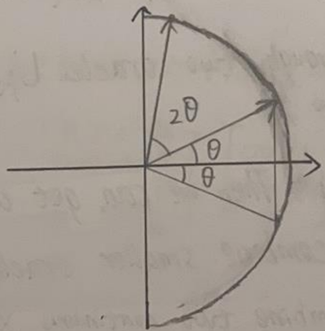
$= -\frac{1}{\sqrt{32}} - 2 \cdot \frac{5}{32\sqrt{2}}$

$= -\frac{9}{16\sqrt{2}}$ $P(|0\rangle) = |a|^2 = 0.158$

So probability is $P = 0.158 \times 6 = 0.948$

(c) One iteration is required.

As $N = 32$, $M = 6$. $n \approx \frac{\pi \cdot \sqrt{32}}{4\sqrt{6}} = \frac{\pi}{\sqrt{3}} = 1.81 \approx 1$



$\sin\theta = \frac{\sqrt{m}}{\sqrt{N}} = \frac{\sqrt{3}}{4}$

(d). There are 2^{10} squares less than 2^{20}. $(0^2, 1^2, \cdots (2^{10}-1)^2)$

so $N = 2^{20}$, $M = 2^{10}$

iteration $n \doteq \frac{\pi \cdot \sqrt{2^{20}}}{4 \cdot \sqrt{2^{10}}} = \frac{\pi \cdot 2^{10}}{4 \cdot 2^5} = 8\pi \approx 25.1$

So 25 iterations are required

5(b): https://qui.science.unimelb.edu.au/circuits/60787943088e4a008b2560d8

Q6:

(a): $QS+RS+RT-QT=(Q+R)S+(R-Q)T$

As $R, Q \in \{\pm 1\}$, we know $(Q+R)S=0$ or $(R-Q)T=0$

So. $(Q+R)S+(R-Q)T = \pm 2$

$$E(QS+RS+RT-QT) = \sum_{q,r,s,t} p(q,r,s,t)(qs+rs+rt-qt)$$

$$\le \sum_{q,r,s,t} p(q,r,s,t)\cdot 2 = 2$$

(suppose $p(q,r,s,t)$ is the probability of $Q=q$, $R=r$, $S=s$, $T=t$)

(b) $Q=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ so eigenvaletes are $\lambda_1=1, \lambda_2=-1$

$R=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so eigenvalues are $\lambda_1=1 . \lambda_2=-1$

$S= -\frac{Z+X}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda^2-1=0.$

So eigenvalues are. $\lambda_1=1 . \lambda_2=-1..$

$T=\frac{Z-X}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda^2-1=0$

so eigenvalues are $\lambda_1=1$ $\lambda_2=-1$

6 (b) $|\psi_{Alice} \otimes \psi_{Bob}\rangle = \begin{bmatrix} \cos\theta_A \\ \sin\theta_A \end{bmatrix} \otimes \begin{bmatrix} \cos\theta_B \\ -\sin\theta_B \end{bmatrix} = \begin{bmatrix} \cos\theta_A \cos\theta_B \\ -\cos\theta_A \sin\theta_B \\ \sin\theta_A \cos\theta_B \\ -\sin\theta_A \sin\theta_B \end{bmatrix}$

so $\langle QS+RS+RT-QT \rangle = \langle \psi_{AB} | \langle QS+RS+RT-QT \rangle | \psi_{AB} \rangle$

$$= \begin{bmatrix} \cos\theta_A \cos\theta_B \\ -\cos\theta_A \sin\theta_B \\ \sin\theta_A \cos\theta_B \\ -\sin\theta_A \sin\theta_B \end{bmatrix}^T \cdot \begin{bmatrix} -\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \cos\theta_A \cos\theta_B \\ -\cos\theta_A \sin\theta_B \\ \sin\theta_A \cos\theta_B \\ -\sin\theta_A \sin\theta_B \end{bmatrix}$$

$= -\sqrt{2}\cos(\theta_A+\theta_B)\cos\theta_A\cos\theta_B + \sqrt{2}\sin(\theta_A+\theta_B)\sin\theta_A\cos\theta_B + \sqrt{2}\sin(\theta_A+\theta_B)\sin\theta_B\cos\theta_A$
$+ \sqrt{2}\cos(\theta_A+\theta_B)\sin\theta_A\sin\theta_B$

$= -\sqrt{2}\left(\cos^2(\theta_A+\theta_B) - \sin^2(\theta_A+\theta_B)\right)$

$= -\sqrt{2}\cos(2(\theta_A+\theta_B))$

so the maximum values is $\sqrt{2}$.
quantity is $-\sqrt{2}\cos(2(\theta_A+\theta_B))$

(c) $\langle QS+RS+RT-QT \rangle = \langle \psi_{AB} | \langle QS+RS+RT-QT \rangle | \psi_{AB} \rangle$

$$= \begin{bmatrix} 0 \\ \cos(\phi) \\ -\sin(\phi) \\ 0 \end{bmatrix}^T \begin{bmatrix} -\sqrt{2} & 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ \cos(\phi) \\ -\sin(\phi) \\ 0 \end{bmatrix}$$

$$= [0, \sqrt{2}\cos\phi + \sqrt{2}\sin\phi, -\sqrt{2}\cos\phi - \sqrt{2}\sin\phi, 0] \begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \\ 0 \end{bmatrix}$$

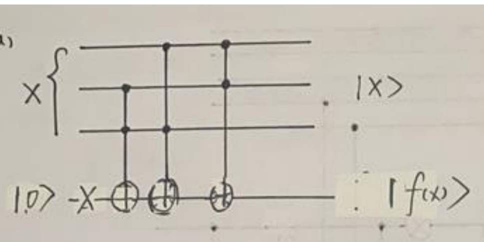
$= \sqrt{2}(\sin\phi + \cos\phi)^2 = 2\sqrt{2}\sin^2(\phi + \frac{\pi}{4})$

so when $\phi = \frac{\pi}{4}$, the quantity is $2\sqrt{2}$.

(c) ϕ firstly can decide the probability and magnitude of
② each state $|01\rangle, |10\rangle$.
Then ϕ can influence the the entanglement of $|\psi_{Alice}\rangle$
and will influence the quality of $\langle QS+RS+RT-QT \rangle$

③ The value for $\langle QS+RS+RT-QT \rangle$ is larger than the value in (a). $2\sqrt{2} > 2$.
So it does not follow bell's inequlity

7(a)



Only when $X = |111\rangle, |110\rangle, |101\rangle, |011\rangle$, $f(x) = 0$

Therefore. these toffoli will act once if there are 2 bit "1"
or act 3 times if there are 3 bit "1"
or don't act if there is less or equal to one "1"

So. if "1" is more than "0". it means at least two "1"
and toffoli will act once or 3 times, which will get
0 at last
if "1" is fewer, it means all these three toffoli gate won't act and the result will get 1

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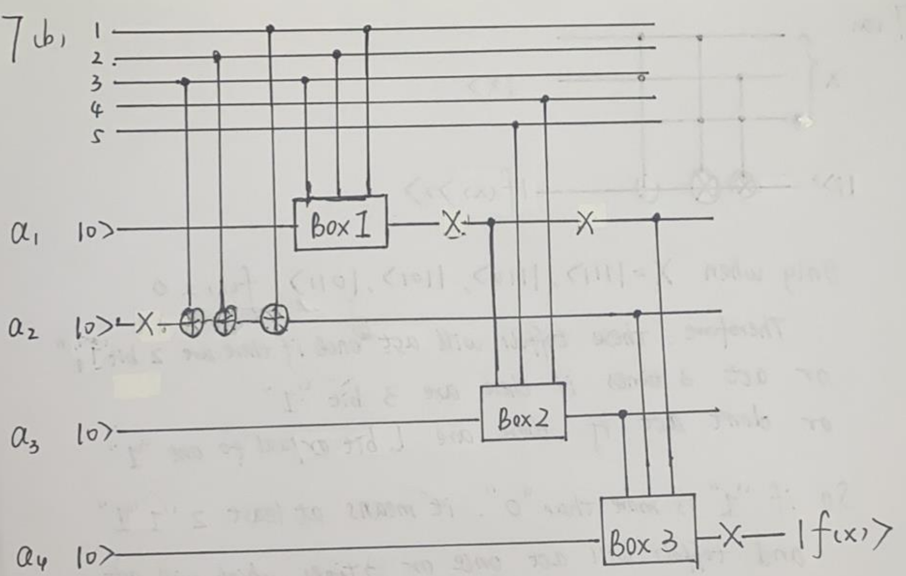
7(b)



As shown in figure., box 1.2.3 are 3-bit implementation.

Box I input is qubit 1.2.3

Box 2 input is a_1. qubit 4.5

Box 3 input is a_1, a_2, a_3

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