Week by week

(1) Introduction to quantum computing
(2) Single qubit representation and operations
(3) Two and more qubits
(4) Simple quantum algorithms
(5) Quantum search (Grover's algorithm)
(6) Quantum factorization (Shor's algorithm)
(7) Quantum supremacy and noise
(8) Programming real quantum computers (IBM Q)
(9) Quantum error correction (QEC)
(10) QUBO problems and Adiabatic Quantum Computation (AQC)
(11) Variational/hybrid quantum algorithms (QAOA and VQE)
(12) Solving linear equations, QC computing hardware

Week 1

Lecture 1

1.1 A very brief history of computing

1.2 The quantum world and quantum computing

Lecture 2

2.1 The mathematics of quantum states

2.2 Complex numbers and quantum amplitudes

2.3 Basic linear algebra: ket and matrix notation

2.4 State representation in the QUI

Practice class 1

The Quantum User Interface (QUI), lecture 1 & 2 review exercises

Today: The Mathematics of Quantum Computing

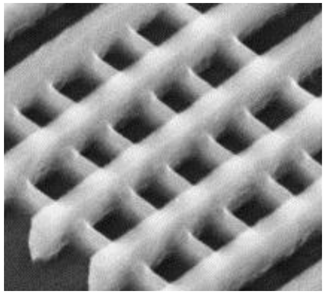
At first the mathematics of quantum computing can look daunting, it largely comes down to two familiar areas:

(1) Complex Numbers
(2) Linear algebra (ie. matrices and vectors)

Physicists have developed a slightly different notation for expressing Linear Algebra, known as Dirac notation, which is very useful for expressing quantum states and operations.

In today's lecture we will review these areas, and put what we have learnt so far (about quantum computing) in this context.

Lecture 1 recap



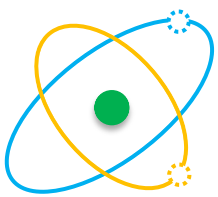
Intel.com

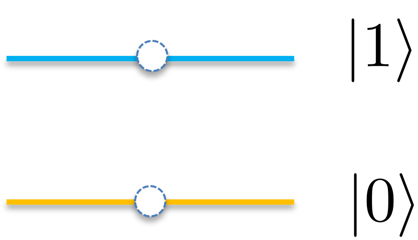
on “1”

off “0”

Only “0” <u>or</u> “1” at a time in any given physical representation (transistor)

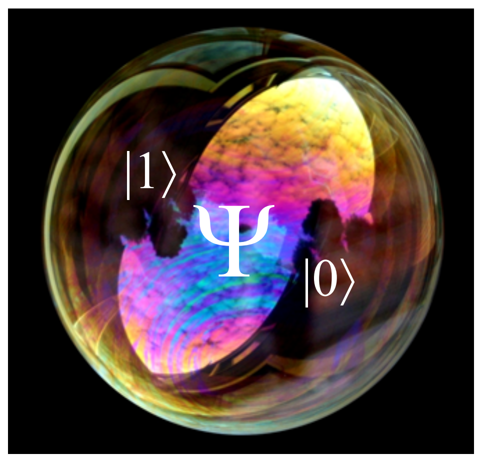
Notation for bits:
Classical $\rightarrow$ 0 and 1
Quantum $\rightarrow$ $|0\rangle$ and $|1\rangle$





One electron in quantum superposition.

quantum superposition



Superposition

measurement/observation

state collapse $\rightarrow$ random outcome

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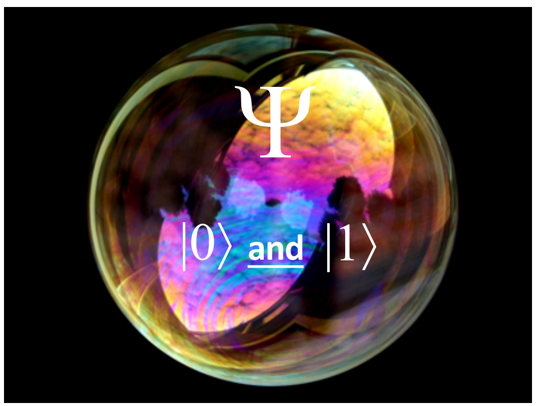
2.1 The mathematics of quantum states

Quantum bits and probabilities

Prepare

A qubit in a particular quantum superposition, say with more 0 than 1

(as many times as we like)



Measure

quantum superposition collapse

or

?

measurement

repeat the process…

Results

Analyse all outcomes

“0” outcomes

Say 60 times out of 100 trials

Probability: Prob[“0”] ~ 60/100 = 0.6

“1” outcomes

Say 40 times out of 100 trials

Probability: Prob[“1”] ~ 40/100 = 0.4

Quantum systems are stochastic

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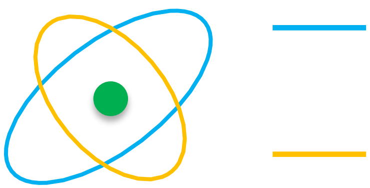
Qubits – how we describe them mathematically

A qubit, e.g. an electron in an atom



$|1\rangle$

$|0\rangle$

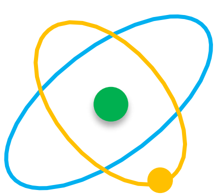
The qubit can be in $|0\rangle$ or $|1\rangle$ or both at the same time.

The overall quantum state is denoted by $|\psi\rangle$

First let's look at the case of no quantum superposition: $|\psi\rangle = |0\rangle$ or $|\psi\rangle = |1\rangle$

i.e. definitive scenarios, one state or the other (no surprises)

$|\psi\rangle = |0\rangle$ The electron is in the ground state



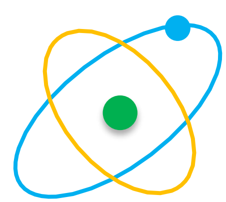
Before: $|\psi\rangle = |0\rangle$

measurement

Collapses to: $|\psi\rangle = |0\rangle$

Outcome always "0" $\rightarrow$ probability: Prob["0"] = 1.0

$|\psi\rangle = |1\rangle$ The electron is in the excited state



Before: $|\psi\rangle = |1\rangle$

measurement

Collapses to: $|\psi\rangle = |1\rangle$

Outcome always "1" $\rightarrow$ probability: Prob["1"] = 1.0

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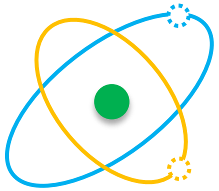
Qubit maths – quantum superposition



Qubit in both $|0\rangle$ and $|1\rangle$ states

$\rightarrow$ i.e. $|\psi\rangle$ is a quantum superposition of $|0\rangle$ and $|1\rangle$

Maths: we describe $|\psi\rangle$ as a linear combination of basis states over the linear space $(|0\rangle,|1\rangle)$:

$$\boxed{|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle}$$

- a_0 and a_1 are the "quantum state amplitudes" (over $\mathbb{C}$ -> complex numbers)

  $\rightarrow$ tell us how much of $|0\rangle$ and $|1\rangle$ are in the superposition $|\psi\rangle$

- modulus squared of the amplitudes, $|a_0|^2$ and $|a_1|^2$ $\rightarrow$ probabilities of measuring "0" or "1"

- total probability must be 1, so we have the "normalization" condition: $|a_0|^2 + |a_1|^2 = 1$

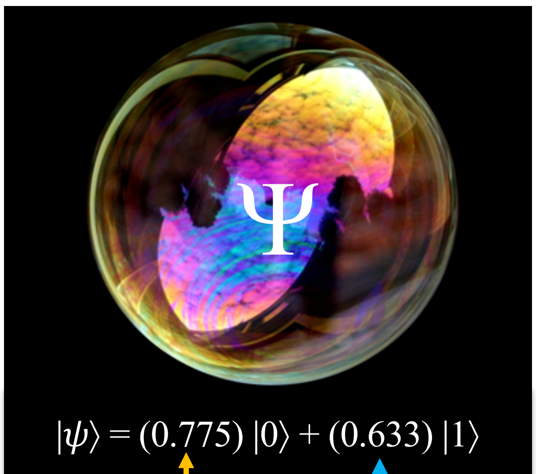
Measurement summary: Born's Rule

The probability of measuring a quantum computer in a state $|i\rangle$, with amplitude a_i is given by $|a_i|^2$.

Examples

Assume for now the amplitudes are <u>real</u>:

Let's prepare a qubit in that particular "60:40" superposition



a_0 a_1

?

measurement

Measurement

or

The probability of getting a "0" outcome is given by:

Prob["0"] = $(0.775)^2$ = 0.6

The probability of getting a "1" outcome is given by:

Prob["1"] = $(0.633)^2$ = 0.4

NB. In any given prepare-measure shot, the outcome will appear random!

But, by repeating the prepare-measurement process many times

the probabilities emerge from $|\psi\rangle = (0.775)\,|0\rangle + (0.633)\,|1\rangle$

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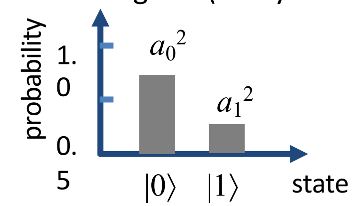
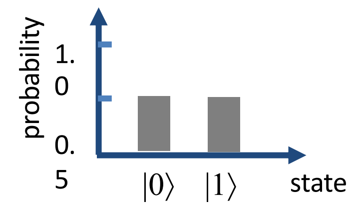
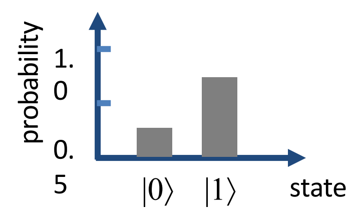
Examples

Assume the amplitudes a_0 and a_1 are real.

Quantum state (before)	Probability measuring 0	Probability measuring 1	Histogram (many shots)
$\|\psi\rangle = a_0 \|0\rangle + a_1 \|1\rangle$	${a_0}^2$	${a_1}^2$	
$\|\psi\rangle = \frac{1}{\sqrt{2}} \|0\rangle + \frac{1}{\sqrt{2}} \|1\rangle$			
$\|\psi\rangle = \frac{1}{2} \|0\rangle + \frac{\sqrt{3}}{2} \|1\rangle$			

${a_0}^2 + {a_1}^2 = 1$

Examples – cont.

Complex amplitudes:

$$|\psi\rangle = 0.775e^{i\pi}\,|0\rangle + 0.633\,e^{-i\pi/3}|1\rangle$$

$|a_0|$ θ_0 $|a_1|$ θ_1

?

measurement

Measurement

or

The probability of getting a

"0" outcome is given by:

$$|a_0|^2 = (0.775)^2 = 0.6$$

Prob["0"] = (0.775)^2 = 0.6

The probability of getting a

"1" outcome is given by:

$$|a_0|^2 = (0.633)^2 = 0.4$$

Prob["1"] = (0.633)^2 = 0.4

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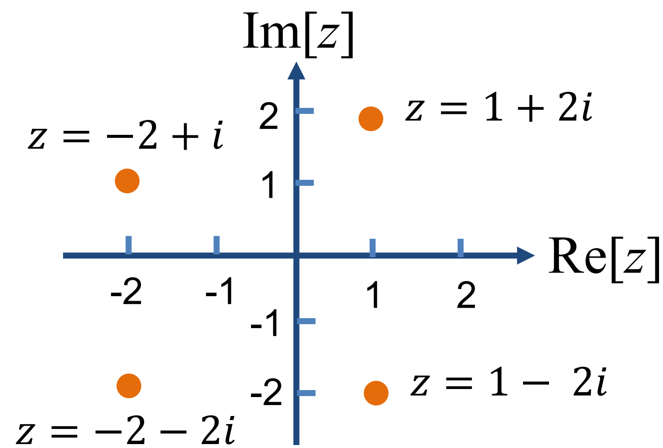
2.2 Complex numbers and quantum amplitudes

Complex numbers: basics

$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ where a_0 and a_1 are complex numbers.

Complex numbers recap: $\boxed{i = \sqrt{-1}\text{, and so } i^2 = -1}$



$z = x + iy$

"Real" part is denoted by Re[z] = x

"Imaginary" part is denoted by Im[z] = y

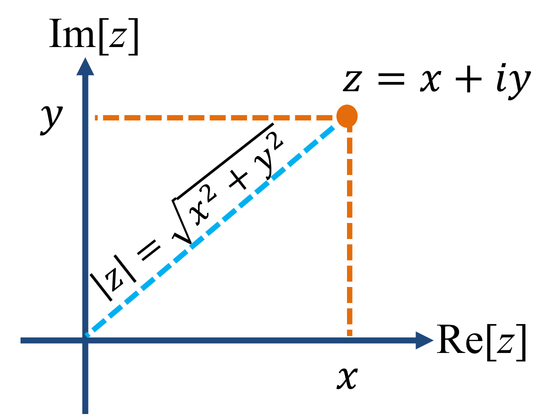
Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$

Conjugate: $z = x + iy \rightarrow z^* = x - iy$

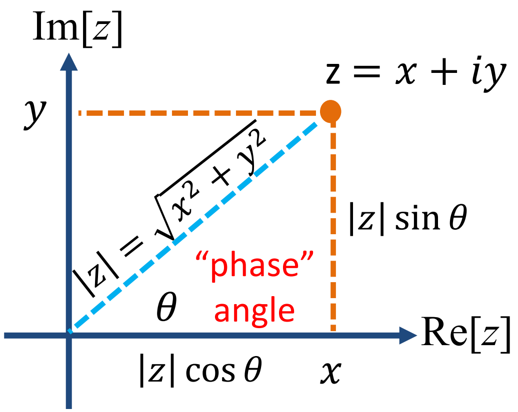
$z\,z^* = (x + iy)(x - iy) = x^2 - ixy + iyx + y^2 = x^2 + y^2$

i.e. $z\,z^* = x^2 + y^2 = |z|^2$ i.e. $|z| = \sqrt{x^2 + y^2}$



$|z|$ is referred to as the "modulus" or "magnitude"

Complex numbers: polar notation



Cartesian: $z = x + iy$ in terms of x=Re[z] and y=Im[z]

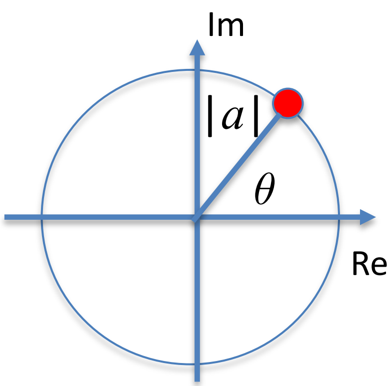
Polar: $z = r\,(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

NB identity: $e^{i\theta} = \cos\theta + i\sin\theta, r = |z| = \sqrt{x^2 + y^2}$

In the QUI, we use "polar notation". e.g. for amplitudes in the state: $|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle$

Ignoring subscripts, consider a complex amplitude *a*

$$a = \mathrm{Re}[a] + i\,\mathrm{Im}[a] = |a|e^{i\theta} \rightarrow \begin{array}{l} |a| = \sqrt{\mathrm{Re}[a]^2 + \mathrm{Im}[a]^2} \\ \theta = \tan^{-1}\left(\mathrm{Im}[a]/\mathrm{Re}[a]\right) \end{array}$$



For each complex amplitude we have a magnitude and phase angle:

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle \;\rightarrow |\psi\rangle = |a_0|e^{i\theta_0}\,|0\rangle + |a_1|e^{i\theta_1}\,|1\rangle$$

2.3 Basic linear algebra: ket and matrix notation

Operations and states as matrices and vectors

As we will see:

- Quantum states are represented as (complex) vectors.
- Quantum operations are represented as (complex) matrices.

To perform an operation on a quantum state, multiply a matrix by a vector. More on this next lecture.

Dirac's "ket" notation

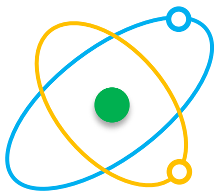
- A lot of quantum mechanics comes down to linear algebra (matrices and vectors), but uses a slightly different notation introduced by Dirac:

$|\psi\rangle$ ← A "ket" is a element of a **linear vector space** over $\mathbb{C}$ which represents *the state* of a qubit.

We write the general state of a qubit in "ket" notation as:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$$

Where the quantum amplitudes a_0 and a_1 are complex numbers.



The wave-like attributes of quantum systems are encapsulated by amplitudes represented as complex numbers…

Ignoring subscripts: $a = x + iy$ where x=Re[a] and y=Im[a]

Linear Algebra and Dirac notation

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle$$

For qubits we can use column vectors to represent a convenient basis for kets:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Computational basis states

$$a_0\,|0\rangle + a_1\,|1\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$a_0, a_1 \in \mathbb{C}$$

General qubit state

a_0 and a_1 are “amplitudes”

Dual vectors

$\langle\psi|$ A "bra" is a **row vector** .

For a qubit state,

$$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \quad a_0, a_1 \in \mathbb{C}$$

we define the corresponding *dual vector* to be:

$$\langle\psi| = \begin{bmatrix} a_0^* & a_1^* \end{bmatrix}$$

Inner Product

$\langle\psi|\phi\rangle$

A "braket" is an **inner product**
(analogous to dot product for vectors in 3D)

For two quantum states $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

We can define an inner product between them

$$\langle\psi|\phi\rangle \equiv \langle\psi||\phi\rangle$$

$$= \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$= a^* c + b^* d$$

Orthogonality

Two states are *orthogonal* if their inner product is zero

$$\langle\psi|\phi\rangle = 0$$

“Z-basis” (computational basis)

For $|0\rangle$ and $|1\rangle$

$$\langle 0|1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 0$$

Computational basis states are orthogonal

“X-basis” (+/- states)

For

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\langle +|-\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 0$$

These states are also orthogonal

Outer Product

$|\psi\rangle\langle\phi|$ is an **outer product**

For two quantum states $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

We can define an outer (tensor) product between them:

$$|\psi\rangle\langle\phi| = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c^* & d^* \end{bmatrix}$$

$$= \begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}$$

…in case we need it later…

Matrix transpose

To find the transpose of a matrix, exchange the rows and the columns. In terms of matrix elements:

$$A^T_{ij} = A_{ji}$$

For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

First row has become the first column, second row the second column.

Matrix Adjoint

To find the "Hermitian adjoint" of a matrix (also known as the conjugate transpose), exchange the rows and the columns and also take the complex conjugate. In terms of matrix elements:

$$\mathrm{A}^{\dagger}_{\mathrm{ij}} = A^{*}_{ji}$$

For example:

$$\begin{bmatrix} 1 & 2i \\ 3+i & 4 \end{bmatrix}^{\dagger} = \begin{bmatrix} 1 & 3-i \\ -2i & 4 \end{bmatrix}$$

First row has become the first column, second row the second column, and we've taken the complex conjugate.

Unitary Matrices

A matrix, U, is called a "unitary" matrix if $U^\dagger U = UU^\dagger = I$.

Knowing a matrix is unitary is makes finding the **inverse matrix** very easy.

All of the **operations** on a quantum computer (except for measurement) will turn out to be unitary.

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^\dagger \\
=&\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \\
=&\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Hermitian Matrices

A matrix, A, is called a “Hermitian” matrix if $A^\dagger = A$.

An interesting fact about Hermitian matrices is that their **eigenvalues are real**.

Measurements on quantum computers are closely related to Hermitian matrices (as we will see in coming lectures).

Three examples of Hermitian matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Bra-ket Notation

ket, $\|\psi\rangle$	Column vector, $\begin{bmatrix} a \\ b \end{bmatrix}$
$\|0\rangle$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\|1\rangle$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
bra, $\langle\psi\|$	Row vector, $\begin{bmatrix} a^* & b^* \end{bmatrix}$
Operator, U	Unitary matrix, $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$

2.4 State representation in the Quantum User Interface (QUI)

QUI: registration

The QUI is a quantum computer programming and simulation tool developed at the University of Melbourne, used in research and teaching.

The QUI is accessed through a web-based interface.

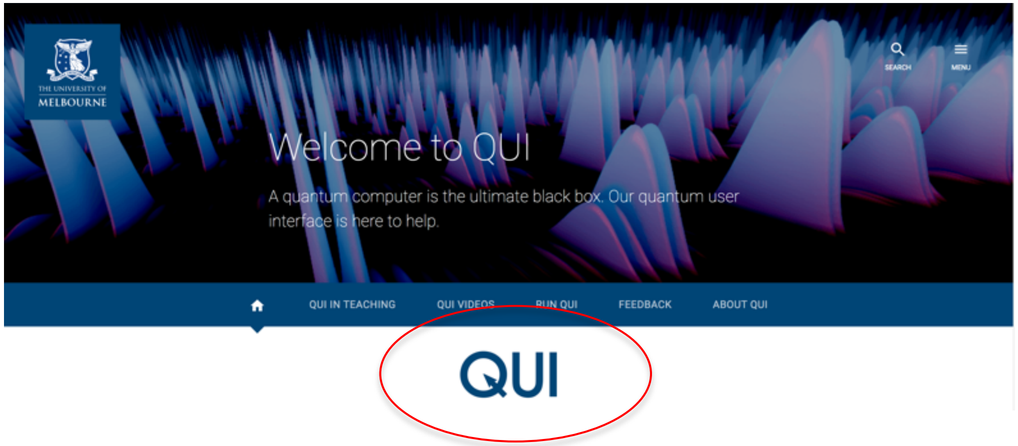
Step 1: Open a web browser (preferably Google Chrome or Firefox), and go to QUIspace.org.

Step 2: Click on QUI

Step 3: Sign up. You will need to create an account to access QUI for the first time.

Follow the steps to create your account (email address and answering a few simple questions).

Step 4: Once you have signed-up, start the QUI!



Prior to practice class 1: register for the QUI as per above (**you must use your UoM email**)

In class 1: switch on expanded capabilities (number of qubits,…)

The QUI structure

- you create a quantum program (“circuit”)
- execute (“compute”) on the UoM quantum computer simulator
- the results are sent back to your QUI session to display

Program editor – place logic operations on the qubits to create a “circuit”

qubits, initialised in the state $|0\rangle$, with time lines left to right

Library of logical operations

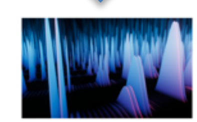
Compute – runs program

State information card (SIC) showing more detail from the plot

Plotting controls

quantum state visualization at slider

You: local QUI session



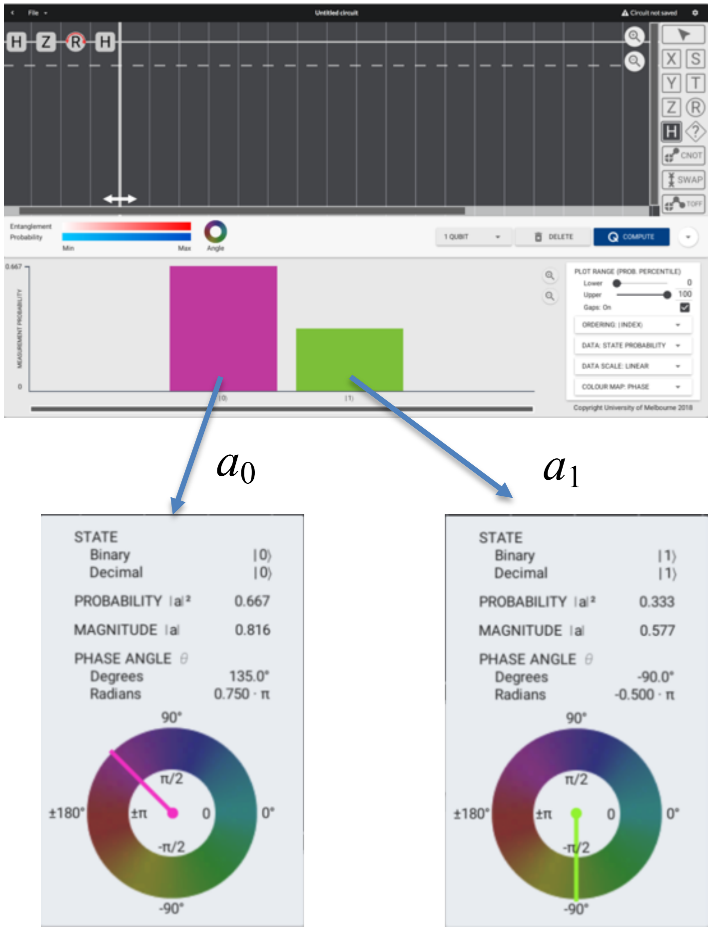
UoM: quantum engine

quantum computer simulator at UoM

More details: QUIspace.org

Complex numbers in the QUI

Quantum User Interface (QUI) – UoM programming and simulation environment.



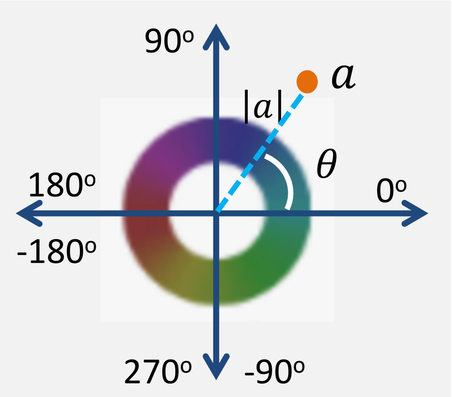
Quantum program in upper panel

Lower panel gives the mathematical representation of the quantum state – i.e. the complex amplitudes (at the time-step corresponding to the vertical slider)

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle$$

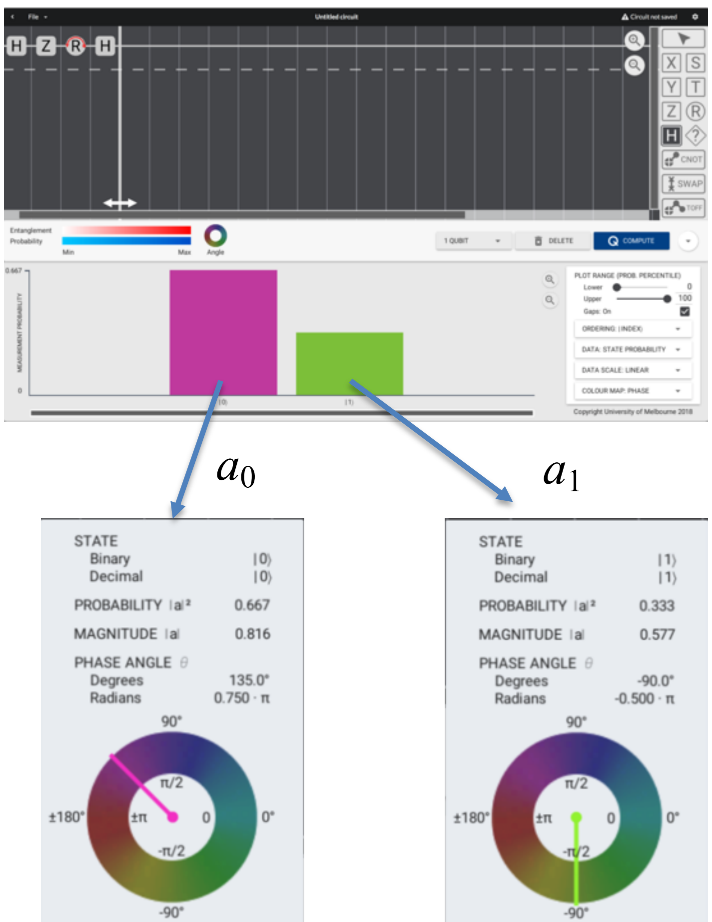
$$\rightarrow |\psi\rangle = |a_0|e^{i\theta_0}\,|0\rangle + |a_1|e^{i\theta_1}\,|1\rangle$$

QUI



“Ket” and “Matrix” representations of quantum states

Consider the following state in the QUI environment:



“ket” notation (amplitudes in QUI polar form)

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle$$

$$\rightarrow |\psi\rangle = |a_0|e^{i\theta_0}\,|0\rangle + |a_1|e^{i\theta_1}\,|1\rangle$$

$$a_0 = 0.816 \times e^{i(0.750\,\pi)}$$
$$a_1 = 0.577 \times e^{i(-0.500\,\pi)}$$

“matrix” notation (amplitudes in QUI polar form)

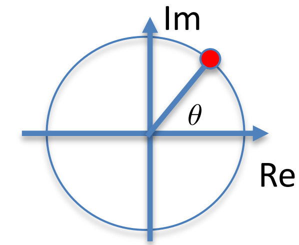
$$a_0|0\rangle + a_1|1\rangle \rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$a_0|0\rangle + a_1|1\rangle \rightarrow \begin{bmatrix} 0.816 \times e^{i(0.750\,\pi)} \\ 0.577 \times e^{i(-0.500\,\pi)} \end{bmatrix}$$

Amplitudes in the QUI

Quantum systems actually have *wave-like* properties. The complex state amplitudes a_0 and a_1 represent the magnitude and *phase* of this wave.

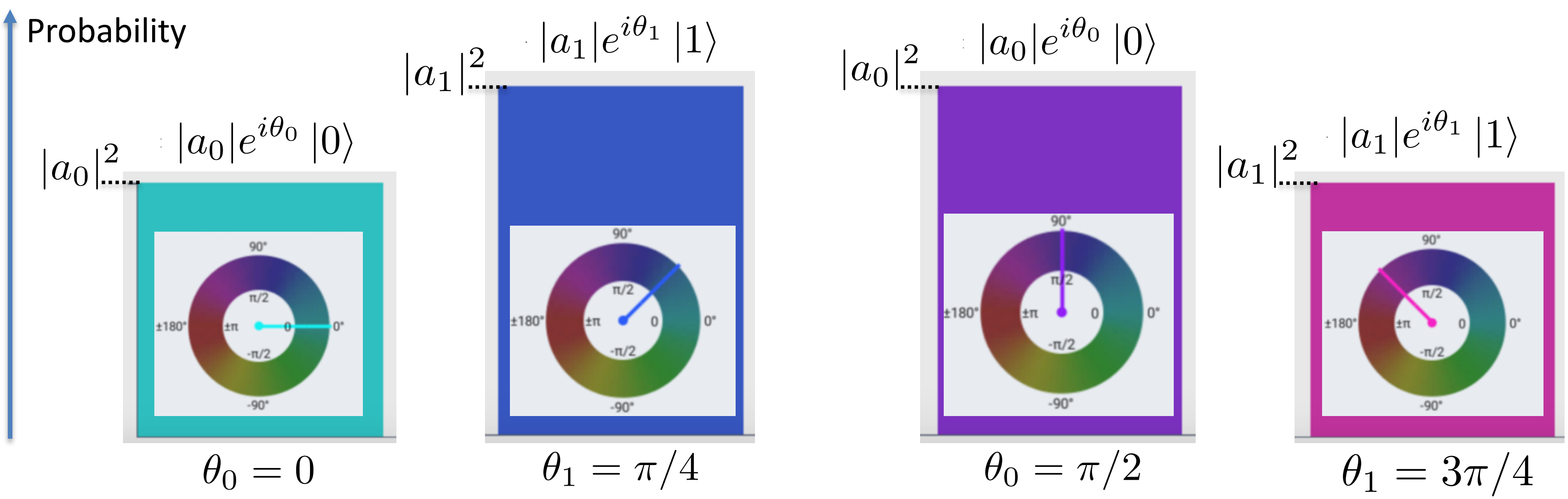
$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle \;\rightarrow |\psi\rangle = |a_0|e^{i\theta_0}\,|0\rangle + |a_1|e^{i\theta_1}\,|1\rangle$$



Recall: $a = \mathrm{Re}[a] + i\,\mathrm{Im}[a] = |a|e^{i\theta} \rightarrow$ $\begin{array}{l} |a| = \sqrt{\mathrm{Re}[a]^2 + \mathrm{Im}[a]^2} \\ \theta = \tan^{-1}\left(\mathrm{Im}[a]/\mathrm{Re}[a]\right) \end{array}$

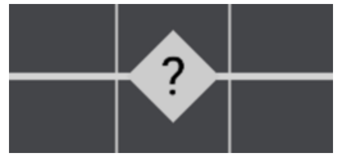
$$e^{i\theta} = \cos\theta + i\sin\theta$$

In the QUI, phase is represented using an angular colour map, and probability by histogram, e.g. two different single-qubit states:



QUI: measurement operation

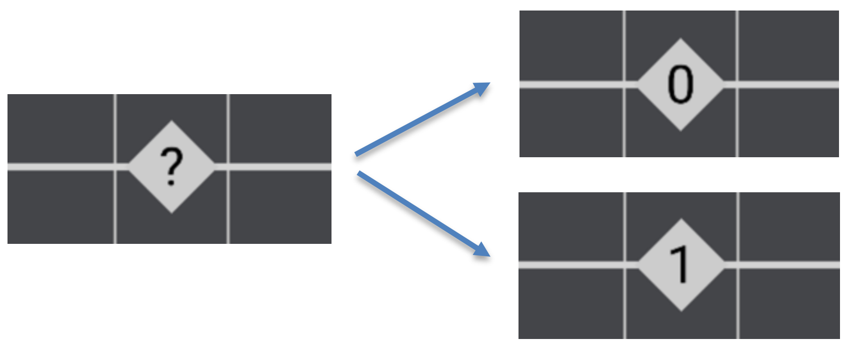
In the QUI the measurement operation looks like this:



By default, measurements are made in the “computational basis” (i.e. 0 or 1).

When you run the circuit, the QUI will randomly select a measurement outcome based on the amplitudes of the state at that point:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$$



Prob = $|a_0|^2$

Prob = $|a_1|^2$

In some circuit diagrams notation is:  (but we like QUI’s spinning lottery symbol)

Practice labs (aka “tutorials”)

Review and practice lecture concepts/content through QUI exercises.

Lab sheets -> download from LMS and bring hard copy, and/or access in tutorial

Work through the exercises individually and/or groups

2 demonstrators per class (ask them lots of questions!)

Week 1

Lecture 1

1.1 A very brief history of computing

1.2 The quantum world and quantum computing

Lecture 2

2.1 The mathematics of quantum states

2.2 Complex numbers and quantum amplitudes

2.3 Basic linear algebra: ket and matrix notation

2.4 State representation in the QUI

Practice class 1

The Quantum User Interface (QUI), lecture 1 & 2 review exercises