Week by week

(1) Introduction to quantum computing
(2) Single qubit representation and operations
(3) Multi-qubit systems
(4) Simple quantum algorithms
(5) Quantum search (Grover's algorithm)
(6) Quantum factorization (Shor's algorithm)
(7) Quantum supremacy and noise
(8) Programming real quantum computers (IBM Q)
(9) Quantum error correction (QEC)
(10) QUBO problems and Adiabatic Quantum Computation (AQC)
(11) Variational/hybrid quantum algorithms (QAOA and VQE)
(12) Solving linear equations, QC computing hardware

Week 3

Lecture 5

5.1 Two qubit systems and operations

5.2 Entanglement

Lecture 6

6.1 Dense coding

6.2 Teleportation

Lab 3

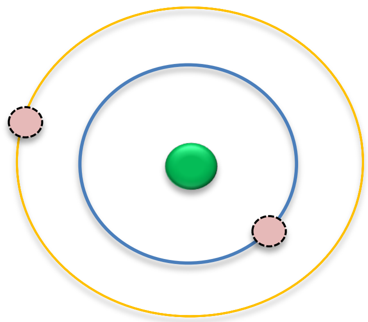
Two qubit operations, entanglement, dense coding, teleportation

5.2 Entanglement

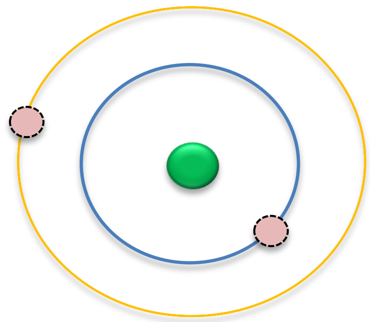
PHYC90045

Lecture 5 (continued)

Separable states



$$|\psi\rangle = a|0\rangle + b|1\rangle$$



$$|\phi\rangle = c|0\rangle + d|1\rangle$$

A separable state is one which can be written as

$$|\Phi\rangle = |\psi\rangle \otimes |\phi\rangle$$

All separable states (of two qubits) can be written as:

$$|\psi\rangle = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle$$

Examples of separable states

Consider the state: $$|\psi\rangle = \frac{|00\rangle + |01\rangle}{\sqrt{2}}$$

It is *separable* because: $$|\psi\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Consider the state: $$|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

It is also *separable* because: $$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Constructing a Bell state

This is one of four states named after the physicist John Bell (who figured out how to experimentally explore reality of entanglement).

Consider the following circuit in the QUI:

Execution:

$$|00\rangle \xrightarrow[\text{H}]{} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow[\text{CNOT}]{} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Question: Is $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ separable?

Entanglement

Answer: No! We can never find a, b, c, d, i.e.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \neq (a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle)$$

A state which is not separable is called an **entangled** state.

Entanglement is a uniquely quantum mechanical property, with no direct classical analogue.

Entanglement Measure

We would like to have a measure of *how much* entanglement a state has. Some states are more entangled than others:

$\|00\rangle$	Not entangled, separable
$\sqrt{0.99}\,\|00\rangle + \sqrt{0.01}\,\|11\rangle$	Entangled, but close to a separable state
$\frac{1}{\sqrt{2}}\,\|00\rangle + \frac{1}{\sqrt{2}}\,\|11\rangle$	Maximally entangled

One simple measure is to ask: How many Bell states (asymptotically) do would Alice and Bob need to share to construct this state, given they are allowed to do Local Operations and Classical Communication *(LOCC)*, but not interact their qubits directly. For pure states, this is equivalent to the *Entropy of Entanglement*.

Entropy of entanglement

Entanglement is a type of correlation between two systems, say A and B.

To see how much correlation there is between A and B: We will measure B, throw away the result, and ask how many bits information of information do we need to determine the state of A?

For example, taking the state (with Alice controlling the first qubit, Bob the second):

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

We measure the state of the second (Bob's) qubit, and forget the result.

50% of the time, the first qubit (Alice's) collapses to the state $|0\rangle$
50% of the time, the first qubit (Alice's) collapses to the state $|1\rangle$

Entropy of Entanglement

Bob needs some information to determine Alice's state. How much? That's measured by the *entropy*.

Entanglement entropy is given by:

$$S = -\sum_i p_i \log p_i$$

where p_i is the probability of measuring ith state of Alice's qubit.

For this case of a Bell state, p_0=50%, p_1=50%,

$$S = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

Here, and throughout this subject, logarithms are taken base 2.

Therefore, a Bell State has 1 bit of entanglement (max possible).

Entropy measure of a separable state

For the separable state: $|00\rangle$

We measure the state of the second (Bob's) qubit.
100% of the time, the first qubit (Alice's) collapses to the state $|0\rangle$

The entropy of entanglement is therefore: $S = -1 \times \log 1 = 0$

All separable states have an entropy of entanglement of 0.

For the state: $\sqrt{0.99}\,|00\rangle + \sqrt{0.01}\,|11\rangle$ $S = 0.08$

This measure of entanglement generalizes between two subsystems of qubits A and B, and is how the measure of entanglement is calculated in the QUI.

Schmidt Decomposition

How much entanglement is present in a general state?

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

Can be hard to tell. It's not in anything like product form. For that we will use SVD. Arrange as a matrix:

$$A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$

Taking Singular Value Decomposition (SVD):

$$A = \sum \lambda_i |u_i\rangle\langle v_i|$$

Allows us to express the state in this convenient form:

$$|\psi\rangle = \sum \lambda_i |u_i\rangle |v_i\rangle$$

This from is known as the "Schmidt Decomposition"

Schmidt Rank

$$|\psi\rangle = \sum \lambda_i |u_i\rangle |v_i\rangle$$

Several terms might have a singular value of 0. The number of non-zero terms is called the **Schmidt rank**.

If a state has a Schmidt rank of 1:

$$|\psi\rangle = |u_0\rangle \otimes |v_0\rangle$$

Then the state is separable, and not entangled.

If a state has a Schmidt rank greater than 1, then the state is entangled. Schmidt rank is a very coarse measure of entanglement. We would like a finer measure.

Entanglement Entropy

$$|\psi\rangle = \sum \lambda_i |u_i\rangle |v_i\rangle$$

A more fine-grained measure of entanglement is the **entanglement entropy**. Form a probability distribution:

$$p_i = \lambda_i^2$$

From which you can calculate the entanglement entropy:

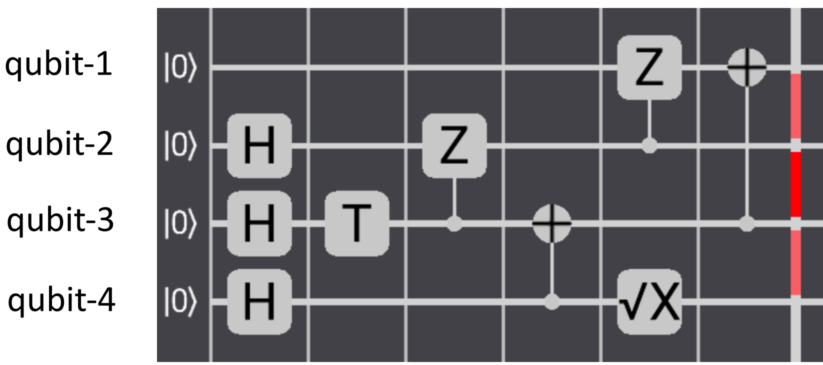
$$S = -\sum_i p_i \log p_i$$

This is a measure of entanglement. The higher the entanglement entropy, the more entanglement.

Entanglement in the QUI

The time scrubber is the vertical bar which moves left and right to show the quantum state at each time step.

The entropy of entanglement is shown in a red colour scale between min and max values possible. Each segment corresponds to the entropy between the system of qubits above and below for that particular bi-partition.



Entanglement entropy between qubit 1 and qubits {2 & 3 & 4} partition

Entanglement entropy between qubits {1 &2} and qubits {3 & 4} partitions

Entanglement entropy between qubit 4 and qubits {1 & 2 & 3} partition

QUBIT ENTANGLEMENT

Entanglement level: 2.000

Quantum entanglement can occur between each qubit and every other qubit. This entanglement segment represents the total entanglement between qubits above and qubits below.

Entanglement and quantum computing

A state which is *not separable* is **entangled**. For example:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

In this lecture we will see how entangled states can be critical in various quantum computing tasks and apply these in the Lab to gain experience in how entangled states work.

In particular we will discuss

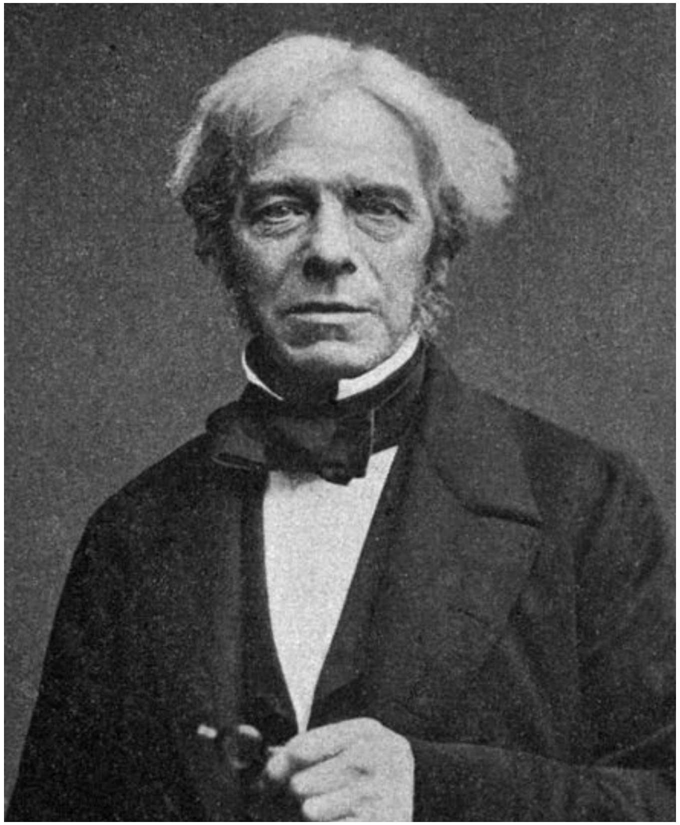
1. Dense Coding
2. No-cloning theorem
3. Quantum teleportation

Entanglement as a resource

When asked what practical use electricity was, Faraday reportedly replied:

"Why sir, there is every probability that you will be able to tax it"

Entanglement is similar, a ***resource*** useful for many quantum information tasks.



Faraday

6.1 Dense coding

Dense Coding

Alice would like to send **two classical bits** to Bob.

01

Wants to send

01

Alice

Bob

Alice and Bob can use a **quantum** NBN, and share some initial entanglement – can they get any advantage?

Dense Coding

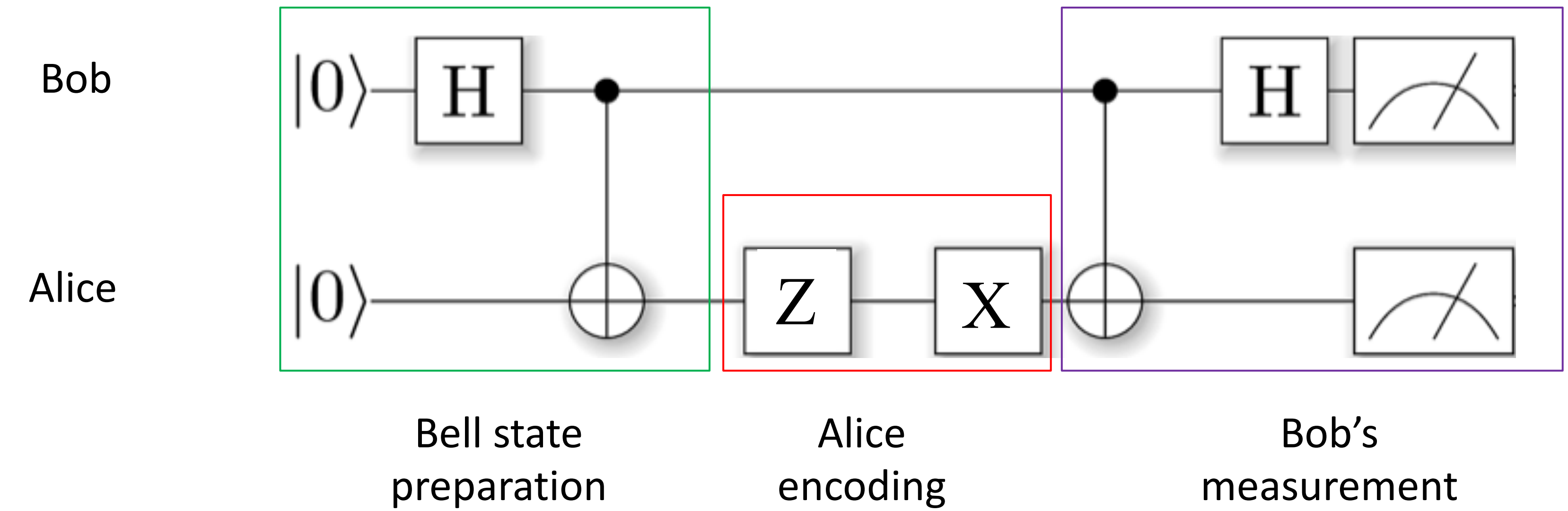
Entanglement makes it possible.

(1) Alice and Bob share an entangled state
(2) Alice flips her qubit one of four ways, based on the state she wants to send
(3) Alice sends her qubit to Bob
(4) Bob measures correlations between the qubits, to reveal which of the four (ie. two bits) operations Alice applied

Alice

Bob

Dense Coding Circuit



Bell state preparation:

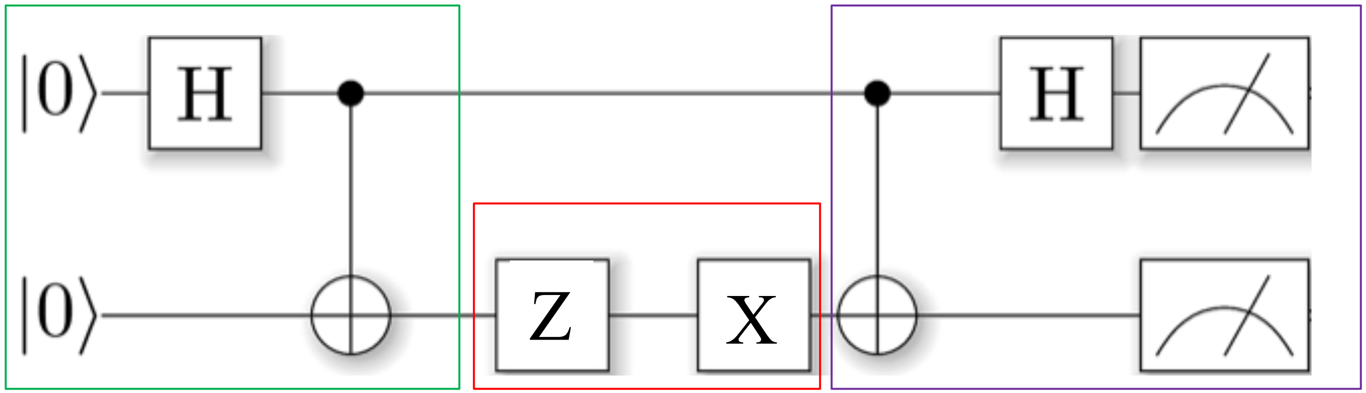
$$|00\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\rangle$$

$$\text{CNOT} \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Dense Coding Circuit

Bob

Alice



Bell state preparation

Alice encoding

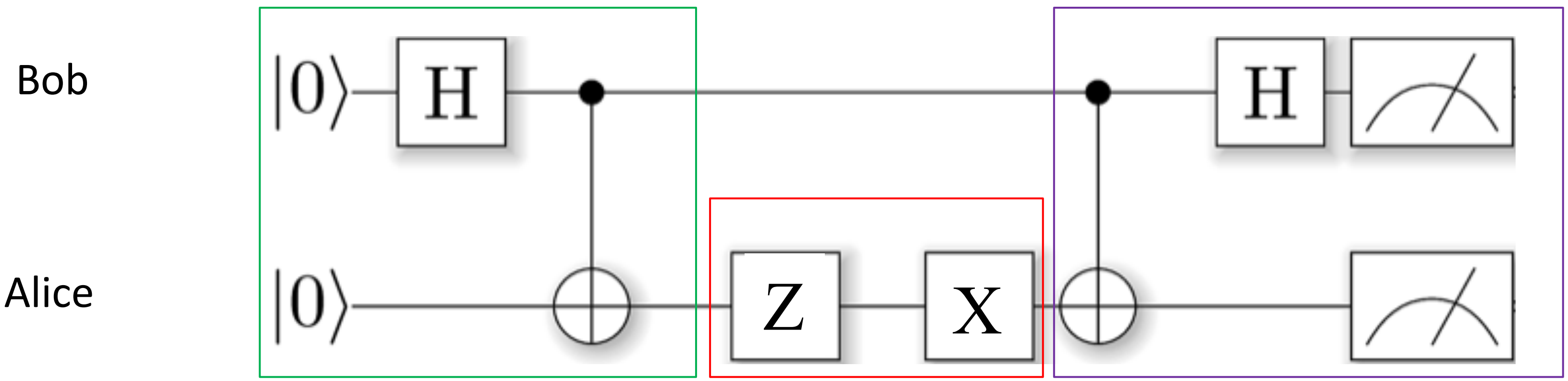
Bob's measurement

$$|00\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle$$
$$\rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$0,0 \qquad \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$0,1 \qquad X_2\frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$1,0 \qquad Z_2\frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$1,1 \qquad X_2 Z_2\frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Alice applies one of four different operations to her qubit, based on the classical information she would like to send.

Dense Coding Circuit



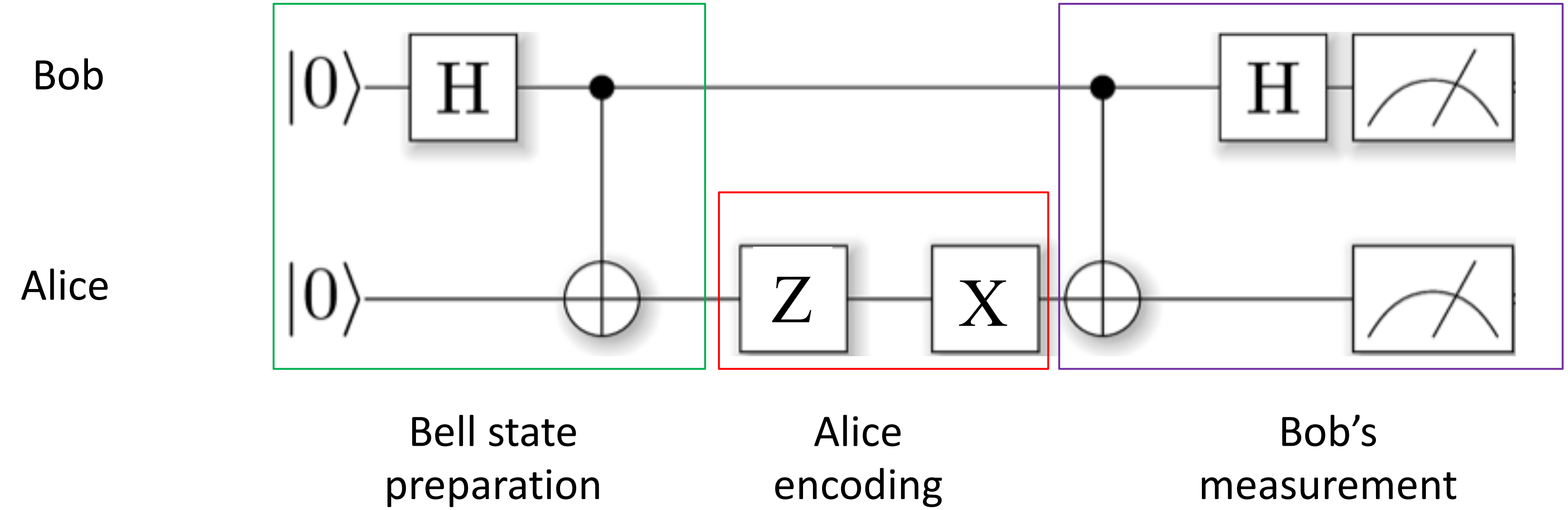
Bell state preparation

Alice encoding

Bob's measurement

			CNOT $\rightarrow$		H $\rightarrow$
$0,0$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$		$\frac{\lvert 00\rangle + \lvert 10\rangle}{\sqrt{2}}$		$\lvert 00\rangle$
$0,1$	$X_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 01\rangle + \lvert 10\rangle}{\sqrt{2}}$		$\frac{\lvert 01\rangle + \lvert 11\rangle}{\sqrt{2}}$		$\lvert 01\rangle$
$1,0$	$Z_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 00\rangle - \lvert 11\rangle}{\sqrt{2}}$		$\frac{\lvert 00\rangle - \lvert 10\rangle}{\sqrt{2}}$		$\lvert 10\rangle$
$1,1$	$X_2Z_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 01\rangle - \lvert 10\rangle}{\sqrt{2}}$		$\frac{\lvert 01\rangle - \lvert 11\rangle}{\sqrt{2}}$		$\lvert 11\rangle$

Dense Coding Circuit



$$H|+\rangle = |0\rangle, H|-\rangle = |1\rangle$$

$0,0$	$\frac{|00\rangle+|11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle+|11\rangle}{\sqrt{2}}$	
$0,1$	$X_2\frac{|00\rangle+|11\rangle}{\sqrt{2}} \rightarrow \frac{|01\rangle+|10\rangle}{\sqrt{2}}$	
$1,0$	$Z_2\frac{|00\rangle+|11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle-|11\rangle}{\sqrt{2}}$	
$1,1$	$X_2Z_2\frac{|00\rangle+|11\rangle}{\sqrt{2}} \rightarrow \frac{|01\rangle-|10\rangle}{\sqrt{2}}$	

CNOT →

$$\frac{|00\rangle+|10\rangle}{\sqrt{2}} = \frac{|0\rangle+|1\rangle}{\sqrt{2}}|0\rangle = |+\rangle|0\rangle$$
$$\frac{|01\rangle+|11\rangle}{\sqrt{2}} = \frac{|0\rangle+|1\rangle}{\sqrt{2}}|1\rangle = |+\rangle|1\rangle$$
$$\frac{|00\rangle-|10\rangle}{\sqrt{2}} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}|0\rangle = |-\rangle|0\rangle$$
$$\frac{|01\rangle-|11\rangle}{\sqrt{2}} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}|1\rangle = |-\rangle|1\rangle$$

H →

$$|00\rangle$$
$$|01\rangle$$
$$|10\rangle$$
$$|11\rangle$$

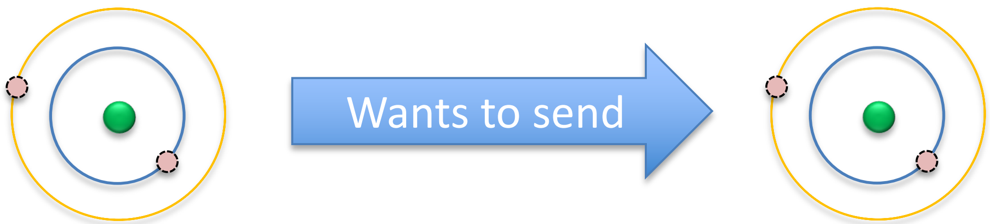
Two bits communicated but only one qubit "sent".

Makes use of pre-existing entanglement.

6.2 Teleportation

A Quantum Computing Bus?

To understand the role entanglement can play in quantum information processing, we will consider how it can be to transmit quantum information around our quantum computer (and potentially between quantum computers)



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\qquad$ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Alice $\qquad$ Bob

Communication around the quantum computer is an important primitive. We could physically move quantum systems, but there is a (potentially) better way: **teleportation**

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Sending classical information

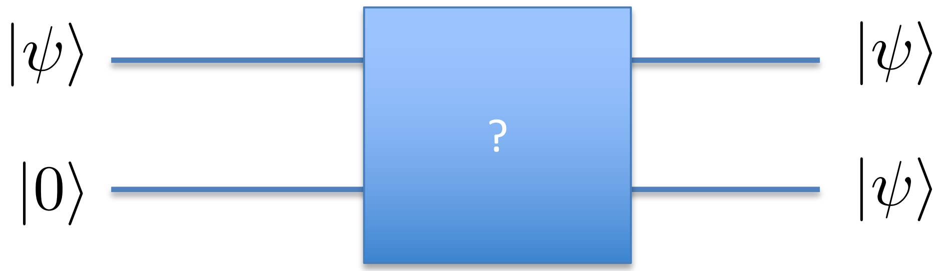
How would we do this classically? Measure everything about the state, then send that information down (classical) bus and recreate a perfect copy elsewhere.

Image by ChtiTux, Used here under CC-by-SA 1.0 license.

Problem: we can’t do this in quantum mechanics because classical measurement (1) *collapses the system*, and (2) *this clones the system* which we can’t do in quantum mechanics.

No-cloning theorem

Can we make a circuit which clones the input state?



That is, we ask if it is possible to make a *unitary* transformation s.t.

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \to (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

No-cloning theorem: the answer is **no**.

Proof of no-cloning theorem

If we had a cloning circuit, we could use it on two arbitrary states, $|\psi\rangle$ and $|\phi\rangle$

$$U|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle \qquad U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

Inner product on LHS: $\langle 0| \langle\phi| U^\dagger U |\psi\rangle |0\rangle = \langle\phi|\psi\rangle$

Inner product on RHS: $\langle\psi|\langle\psi|\phi\rangle|\phi\rangle = \langle\psi|\phi\rangle^2$

But the only solutions to $x^2=x$ are $x=0$ or $x=1$. We can only have a circuit clone states *which are orthogonal ($x=0$ case), not arbitrary states*.

There can be <u>no unitary transformation which clones two arbitrary states</u>.

Teleportation

Entanglement makes it possible.

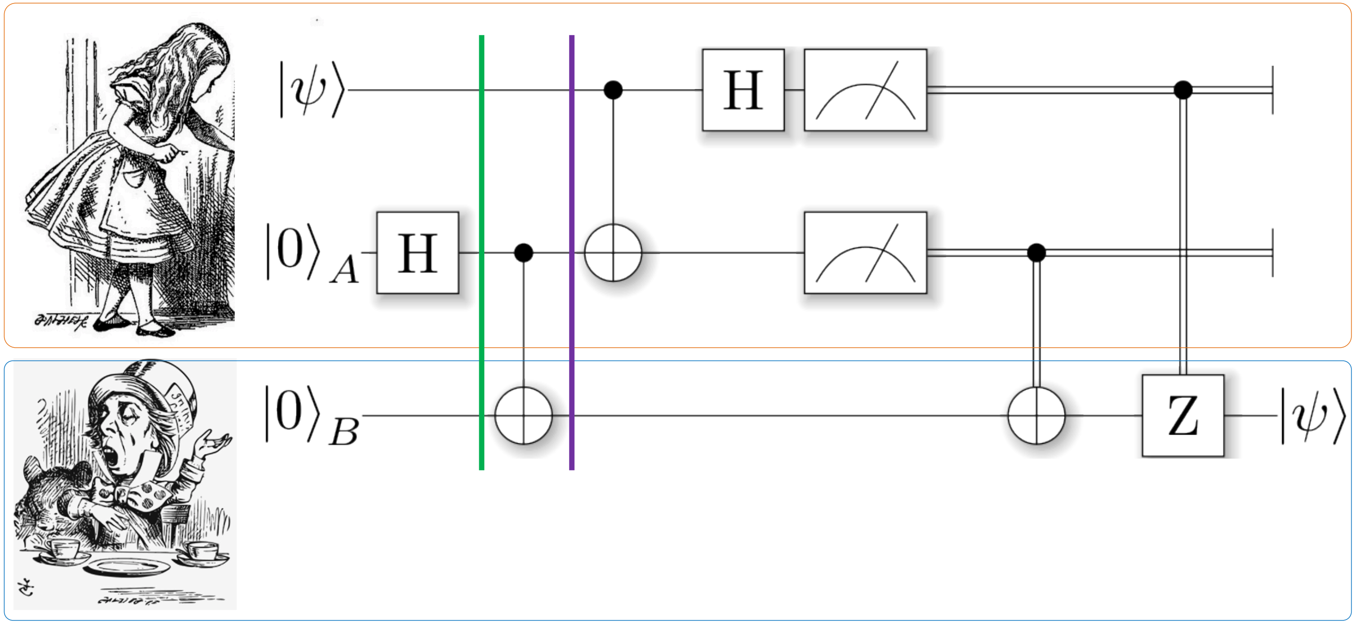
(1) Alice has a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
(2) Alice and Bob share an entangled state
(3) Alice measures correlations between her qubit and half of the entangled state
(4) Alice sends the results of the measurements to Bob
(5) Bob uses them to reconstruct the original state in his qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Alice

Bob

Teleportation



Double lines indicate classical operations

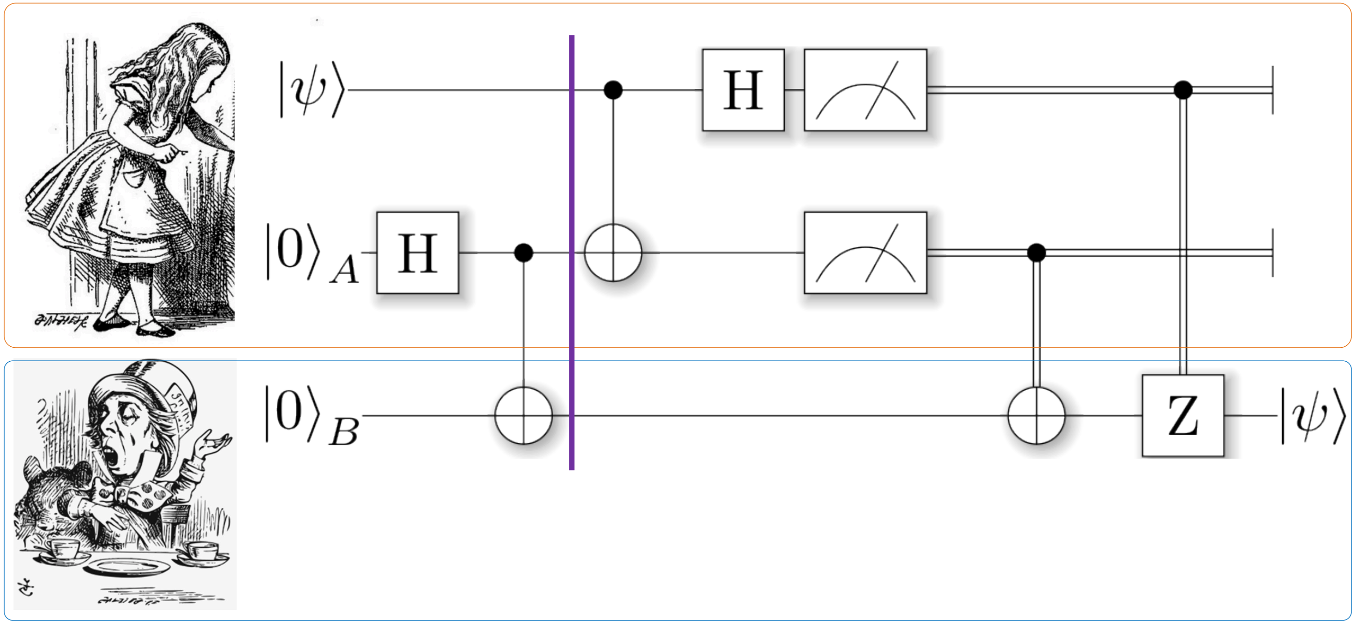
$$(\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle$$

Hadamard (A) $$\longrightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$$

CNOT(A-B) $$\longrightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice's state Bell state preparation (shared)

Teleportation

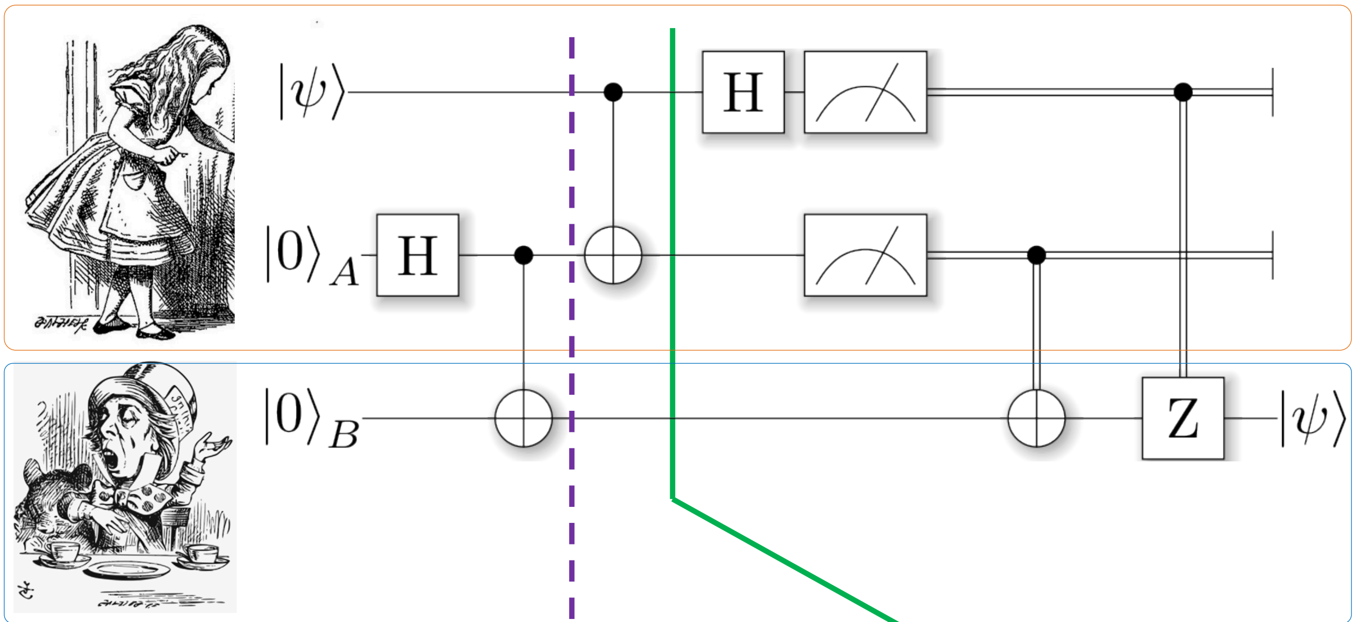


Total system state: $(\alpha|0\rangle + \beta|1\rangle)\,\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Alice's state $|\psi\rangle$ Shared entangled state A & B

Expand: $\longrightarrow$ $\dfrac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right)$

Teleportation



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right)$$

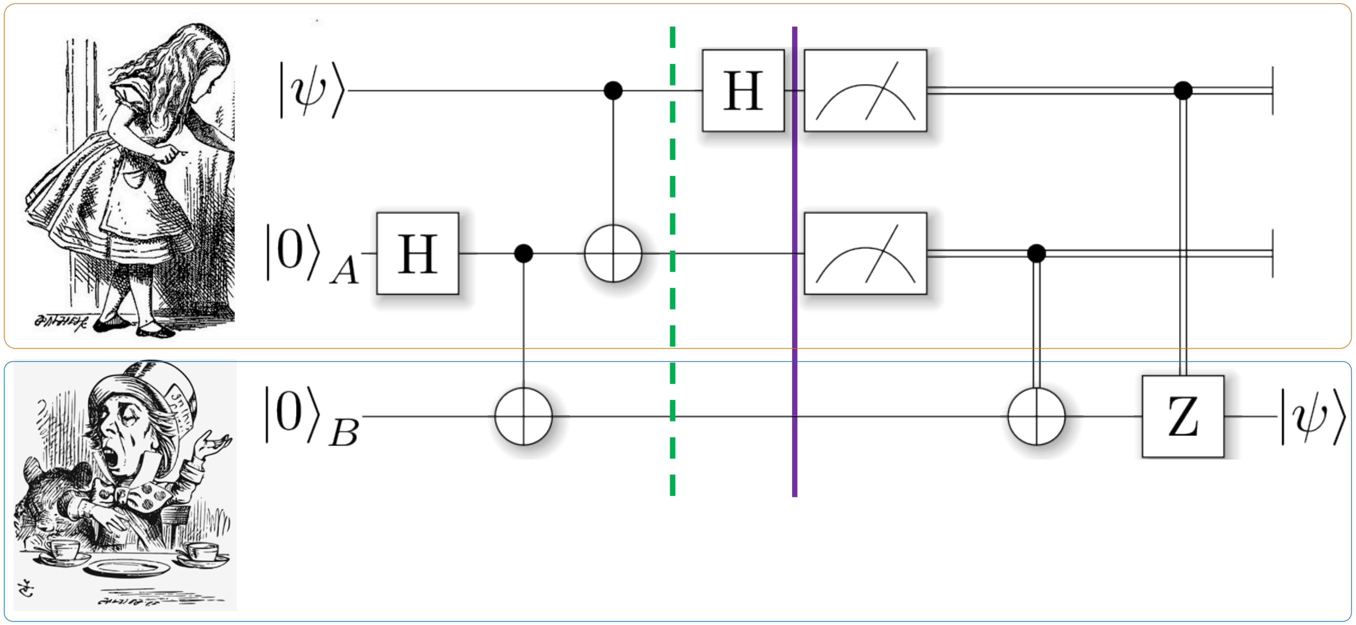
CNOT[1,2]

$$\xrightarrow{} \alpha\frac{|000\rangle + |011\rangle}{\sqrt{2}} + \beta\frac{|110\rangle + |101\rangle}{\sqrt{2}}$$

Rewrite (ex):

$$\begin{aligned} &\frac{|+\rangle|0\rangle}{2}(\alpha|0\rangle + \beta|1\rangle) \\ +&\frac{|+\rangle|1\rangle}{2}(\alpha|1\rangle + \beta|0\rangle) \\ +&\frac{|-\rangle|0\rangle}{2}(\alpha|0\rangle - \beta|1\rangle) \\ +&\frac{|-\rangle|1\rangle}{2}(\alpha|1\rangle - \beta|0\rangle) \end{aligned}$$

Teleportation

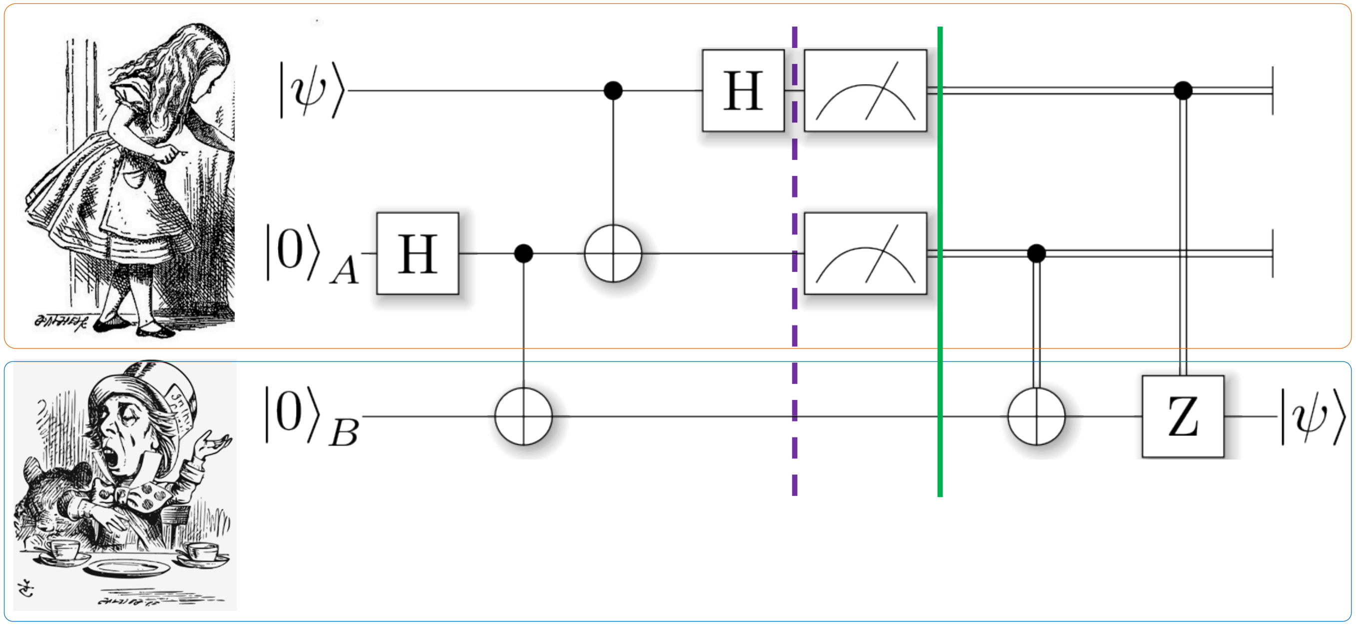


$$\frac{|+\rangle|0\rangle}{2}(\alpha|0\rangle+\beta|1\rangle)$$
$$+\frac{|+\rangle|1\rangle}{2}(\alpha|1\rangle+\beta|0\rangle)$$
$$+\frac{|-\rangle|0\rangle}{2}(\alpha|0\rangle-\beta|1\rangle)$$
$$+\frac{|-\rangle|1\rangle}{2}(\alpha|1\rangle-\beta|0\rangle)$$

Hadamard

$$\frac{|0\rangle|0\rangle}{2}(\alpha|0\rangle+\beta|1\rangle)$$
$$+\frac{|0\rangle|1\rangle}{2}(\alpha|1\rangle+\beta|0\rangle)$$
$$+\frac{|1\rangle|0\rangle}{2}(\alpha|0\rangle-\beta|1\rangle)$$
$$+\frac{|1\rangle|1\rangle}{2}(\alpha|1\rangle-\beta|0\rangle)$$

Teleportation



$$
\frac{|0\rangle|0\rangle}{2}(\alpha|0\rangle + \beta|1\rangle) + \frac{|0\rangle|1\rangle}{2}(\alpha|1\rangle + \beta|0\rangle) + \frac{|1\rangle|0\rangle}{2}(\alpha|0\rangle - \beta|1\rangle) + \frac{|1\rangle|1\rangle}{2}(\alpha|1\rangle - \beta|0\rangle)
$$

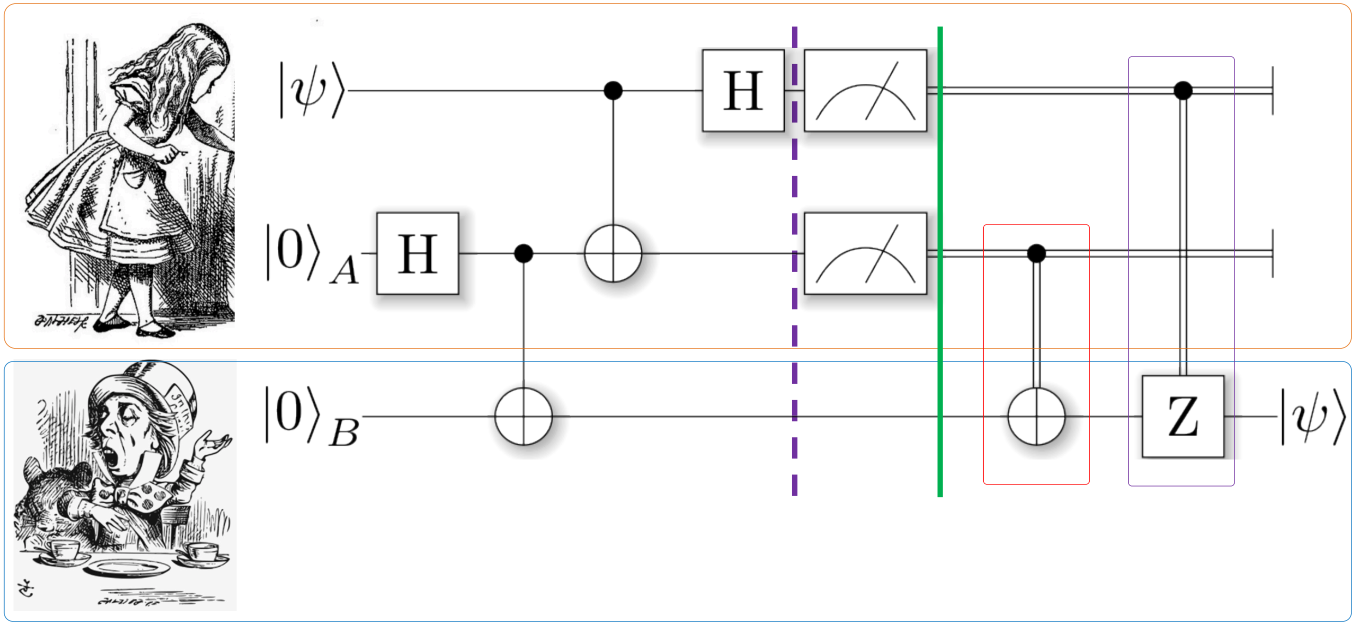
Alice measures her two qubits.

Bob's qubit collapses to one of the four possibilities.

Alice now <u>tells</u> Bob her outcomes (double lines indicate classical communication).

Bob will perform simple corrections shown.

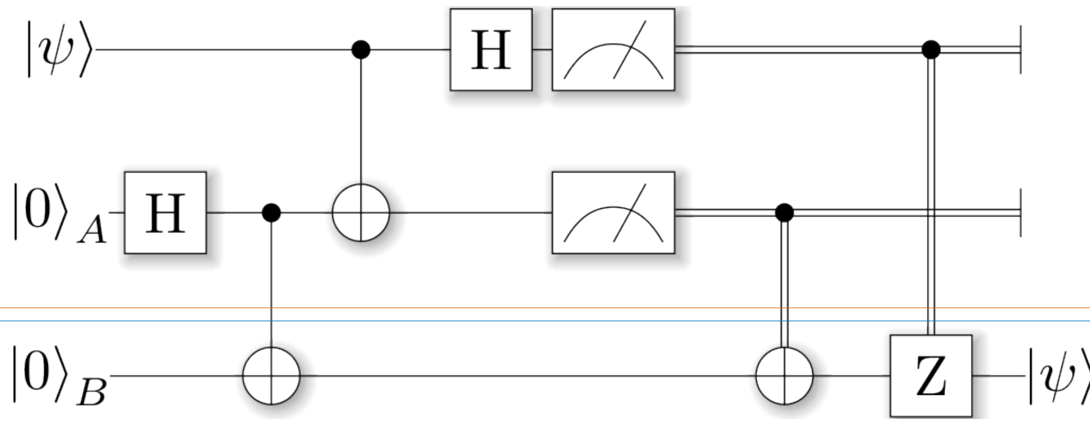
Teleportation



$$
\frac{|0\rangle|0\rangle}{2}(\alpha|0\rangle+\beta|1\rangle)
+\frac{|0\rangle|1\rangle}{2}(\alpha|1\rangle+\beta|0\rangle)
+\frac{|1\rangle|0\rangle}{2}(\alpha|0\rangle-\beta|1\rangle)
+\frac{|1\rangle|1\rangle}{2}(\alpha|1\rangle-\beta|0\rangle)
$$

Alice measures	Bob's qubit
$0,0$	$\alpha\|0\rangle+\beta\|1\rangle$
$0,1$	$\alpha\|1\rangle+\beta\|0\rangle \rightarrow \alpha\|0\rangle+\beta\|1\rangle$
$1,0$	$\alpha\|0\rangle-\beta\|1\rangle \rightarrow \alpha\|0\rangle+\beta\|1\rangle$
$1,1$	$\alpha\|1\rangle-\beta\|0\rangle \rightarrow \alpha\|0\rangle-\beta\|1\rangle \rightarrow \alpha\|0\rangle+\beta\|1\rangle$

Teleportation



$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

Alice measures	Bob's qubit
$0,0$	$\alpha\|0\rangle + \beta\|1\rangle$
$0,1$	$\alpha\|1\rangle + \beta\|0\rangle$
$1,0$	$\alpha\|0\rangle - \beta\|1\rangle$
$1,1$	$\alpha\|1\rangle - \beta\|0\rangle$

i.e. after correction Bob has successfully reconstructed Alice's original state.

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle$$

$$X(\alpha|1\rangle + \beta|0\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle$$

$$Z(\alpha|0\rangle - \beta|1\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle$$

$$ZX(\alpha|1\rangle - \beta|0\rangle) \rightarrow \alpha|0\rangle + \beta|1\rangle$$

Teleportation

Entanglement makes it possible.

Alice

(1) Alice has a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
(2) Alice and Bob share an entangled state
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$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Bob

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Lecture 5

5.1 Two qubit systems and operations

5.2 Entanglement

Lecture 6

6.1 Dense coding

6.2 Teleportation

Lab 3

Two qubit operations, entanglement, dense coding, teleportation