MULT90063 Introduction to Quantum Computing

Lab Class 1

Welcome to Lab Class 1 of MULT90063 Introduction to Quantum Computing.

The purpose of this first tutorial class is to:

- Review your knowledge of complex numbers.
- Start using the Quantum User Interface (QUI).
- Understand complex amplitudes and their representation in the QUI system.
- Review matrices and vectors

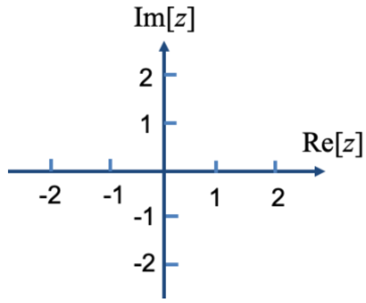
Refresher on complex numbers

Consider complex numbers $z = x + iy$ where x and y are ordinary real numbers, and $i^2 = -1$. The real and imaginary parts of z are Re[z] = x and Im[z] = y respectively.

Question 1.1 For the complex number $z = 3 + 2i$ determine the real and imaginary parts:

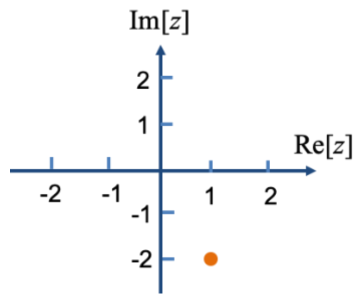
Re[z] = ______ and Im[z] = ______ .

Question 1.2 Plot the complex number $z = -1.5 + i$ in the complex plane.



Question 1.3 Write out $z = x + iy$ for the point shown (right):

z = ______ + i _______



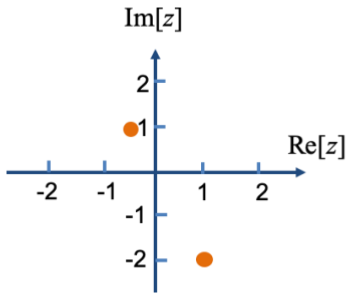
Consider two complex numbers: $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$ where $\{x_1, y_1, x_2, y_2\}$ are real numbers. Addition and multiplication is as follows:

$$z_1 + z_2 = x_1 + iy_1 + x_1 + iy_1 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2$$

Question 1.4 Given $z_1 = -2 + i$ and $z_2 = 3 - 2i$, determine $z = z_1 + z_2$ and plot:

$z =$ ______ $+ i$ _______

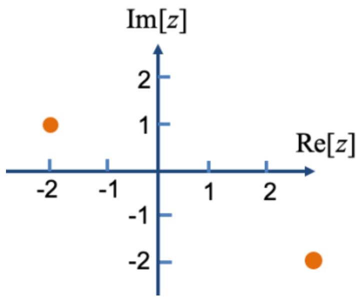


Question 1.5 Given $z_1 = -1/2 + i$ and $z_2 = 1 - 2i$, determine $z = z_1 z_2$ and plot:

$z =$ ______ $+ i$ _______

The "conjugate" z^* of a complex number z is formed by changing the sign of i, i.e. $z = x + iy \rightarrow z^* = x - iy$. The product $z\,z^* = (x+iy)(x-iy) = x^2 - ixy + iyx + y^2 = x^2 + y^2$. The magnitude of a complex number is denoted $|z|$ and is given by $|z| = \sqrt{x^2 + y^2}$. The magnitude squared of the complex number is given by: $z\,z^* = x^2 + y^2 = |z|^2$.

Question 1.6 For $z = -2 + i$ determine the magnitude $|z|$ and $|z|^2$.

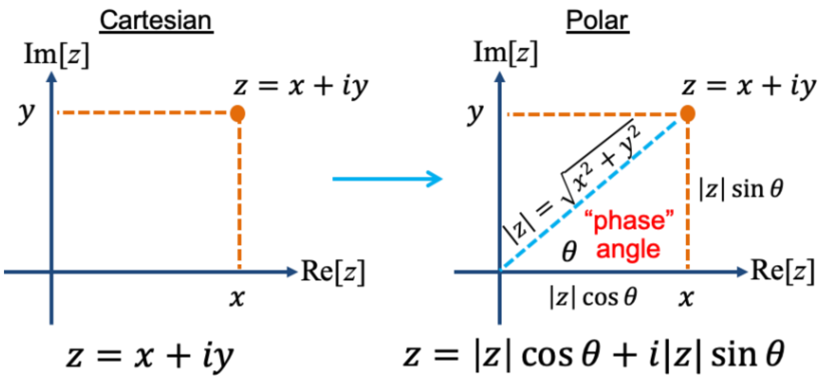


$|z| =$ ___ , $|z|^2 =$ ___ .

Question 1.7 Given $z = 3 - 2\,i$ determine the magnitude $|z|$ and $|z|^2$.

$|z| =$ ______ , $|z|^2 =$ ______ .

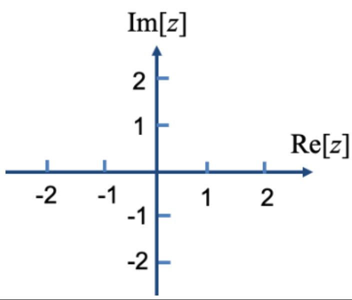
In the QUI, we use "polar notation" (angle) as a compact way to visualise complex numbers.



$$z = |z|\cos\theta + i\,|z|\sin\theta = |z|(\cos\theta + i\sin\theta)$$

$$\rightarrow z = |z|e^{i\theta}$$

1.8 Plot $z = 1 + i$ on the complex plane, convert to polar notation, and label plot with magnitude and angle:



In QC we often measure angles in radians rather than degrees.

1.9 Use the conversions to fill in the table.

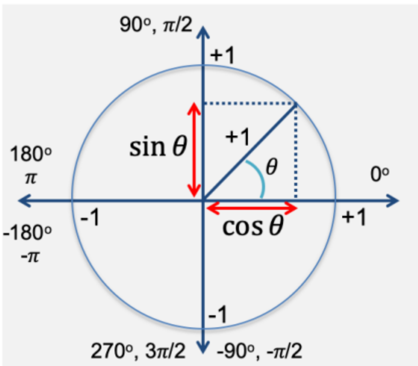
degrees	radians
18	
30	
	$\pi/3$
	$-\pi/4$

Conversion:

θ in rad = $[\pi/180]$ x (θ in deg)

θ in deg = $[180/\pi]$ x (θ in rad)

To understand the representation of quantum amplitudes, it's very useful to know how to compute sin and cos quickly around the unit circle (unit means radius 1):



Angle θ (radians)	$\cos\theta$	$\sin\theta$
0	1	0
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/2$	0	1
π	-1	0
$3\pi/2$	0	-1
2π	1	0
$-\pi/2$	0	-1
$-\pi$	-1	0

The Quantum User Interface (QUI)

The QUI is a web-based graphical user interface (developed by the Hollenberg group at the University of Melbourne) to program, simulate and analyse quantum circuits. The QUI allows the users to specify qubit number, build quantum circuits, simulate and examine the quantum state at every time step in the circuit/program. The latter feature is critical to understanding QC, and distinguishes QUI from other on-line programming/simulation tools.

The QUI is accessed through a web-based interface (quispace.org – click on blue QUI logo). See the notes "QUI Intro" on LMS for quick guide. *If you haven't signed up already please let a demonstrator know and follow these steps.*

Step 1: Open a web browser (preferably Google Chrome or Firefox), go to quispace.org.

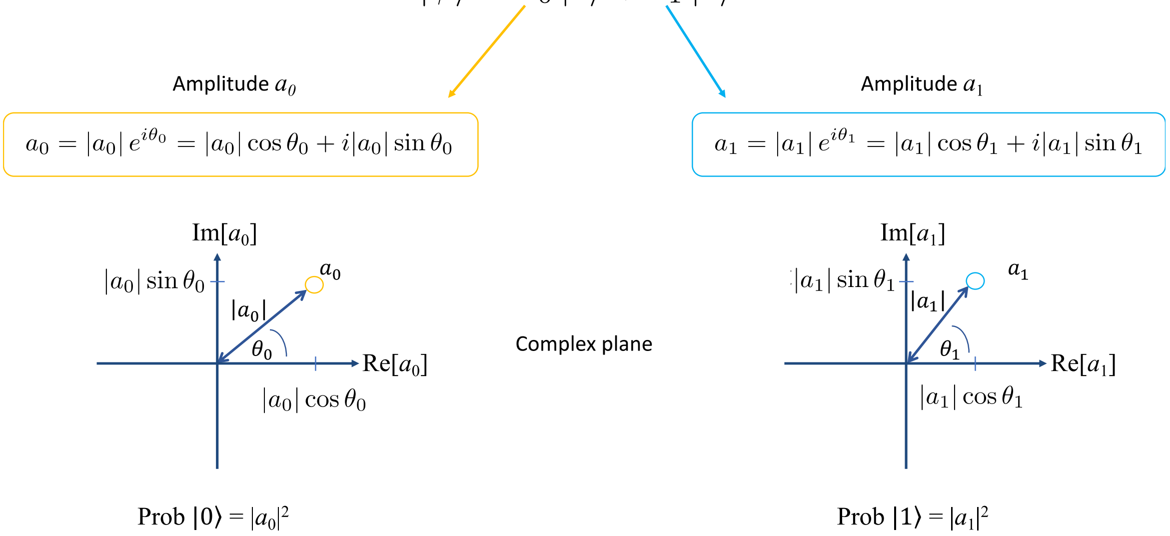
Step 2: You will need to create an account to access QUI for the first time. In order to access expanded capabilities of the QUI you must use your **University of Melbourne email address** as your login name. Follow the steps to create your account.

Step 3: Once you have signed-up, start the QUI.

Quantum States, Amplitude, Probability and Phase

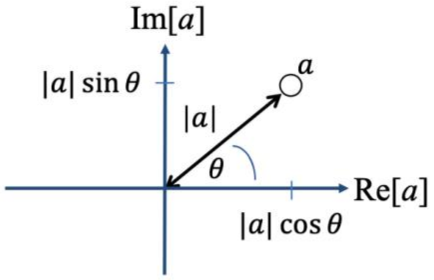
In general, a quantum superposition for a single qubit is written as $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$. The "state amplitudes" a_0 and a_1 are complex numbers. In the QUI we use polar notation, $|a|e^{i\theta}$, to describe the magnitude ($|a|$) and phase (θ) of the state amplitudes:

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle$$

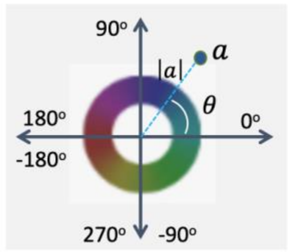


For a given amplitude $a = |a|e^{i\theta}$, QUI uses a colour wheel for the phase angle θ.

$\mathrm{a} = \mathrm{Re}[\mathrm{a}] + i\,\mathrm{Im}[\mathrm{a}] = |a|e^{i\theta}$



QUI



degrees	radians
0	0
90	$\pi/2$
180	π
360	2π

As per lecture notes, the State Info Card (SIC) in the QUI gives the value of the complex amplitude a_i of a given state component $|i\rangle$ is given by $a_i = |a_i|e^{i\theta_i}$ where $|a_i|$ is the magnitude, and θ_i is the phase angle (colour wheel scale). The probability of measuring the state $|i\rangle$ is $|a_i|^2$.

Exercise 3.1 Consider the computational states $|0\rangle$ and $|1\rangle$ in the QUI.

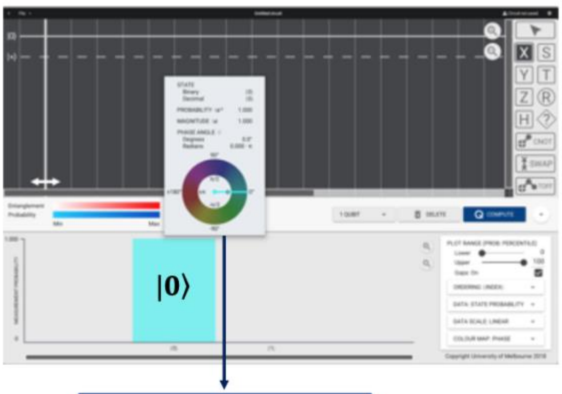
Select 1 qubit, and press compute (nothing in the circuit as per the schematic below). The system has been initialised in the $|0\rangle$ state. Mouse-over the histogram in the results panel (panel below the circuit) to bring up the State Info Cards (SICs) with the amplitude phase and magnitude values:

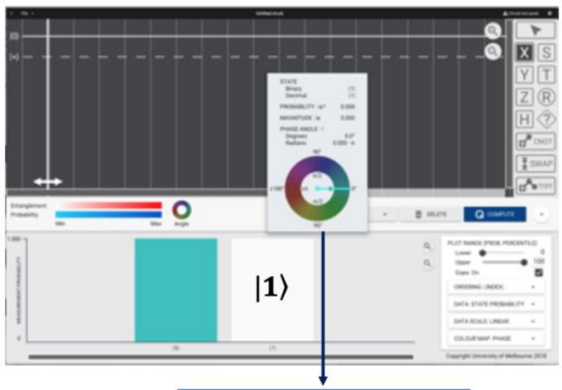
$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \rightarrow \text{initialized in } |0\rangle$$

$$\text{i.e. } a_0 = 1,\ a_1 = 0$$

$$a_0 = |a_0|\cos\theta_0 + i\,|a_0|\sin\theta_0 = 1 \rightarrow \begin{cases} \theta_0 = 0 \\ |a_0| = 1 \end{cases}$$

$$a_1 = |a_1|\cos\theta_1 + i\,|a_1|\sin\theta_1 = 0 \rightarrow \begin{cases} \theta_1 = 0 \\ |a_1| = 0 \end{cases}$$





SIC info for a_0

$\theta_0 = 0$

$|a_0| = 1$

$|a_0|^2 = 1$

SIC info for a_1

$\theta_1 = 0$

$|a_1| = 0$

$|a_1|^2 = 0$

Question: What are the probabilities of measuring the result "0" and "1" respectively?

Prob["0"] = ____________ : Prob["1"] = ____________

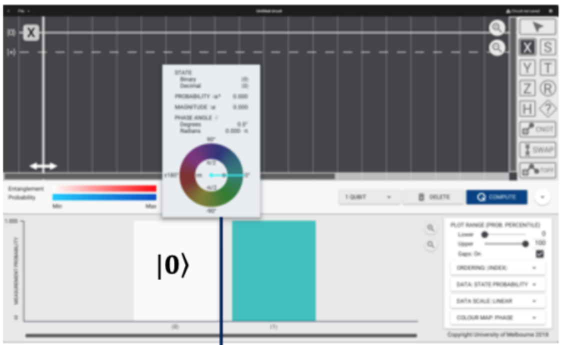
Now we will put the system into the $|1\rangle$ state. Click on the X-gate in the gate library on the right, and click it into the first time block as shown in the schematic below. Press compute. The X-gate flips the state of the qubit from $|0\rangle$ (as per the above – QUI qubits always initialised in $|0\rangle$) so qubit has been put into the $|1\rangle$ state. Mouse-over the histogram in the results panel (panel below the circuit) to bring up the State Info Cards (SICs) with the amplitude phase and magnitude values:

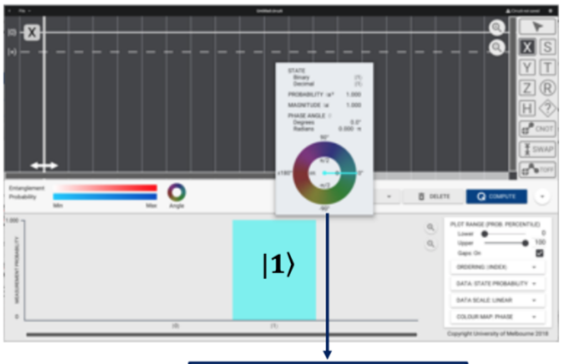
$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \rightarrow \text{initialized in } |1\rangle$$

i.e. $a_0 = 0$, $a_1 = 1$

$$a_0 = |a_0|\cos\theta_0 + i\,|a_0|\sin\theta_0 = 0 \rightarrow \begin{cases} \theta_0 = 0 \\ |a_0| = 0 \end{cases}$$

$$a_1 = |a_1|\cos\theta_1 + i\,|a_1|\sin\theta_1 = 1 \rightarrow \begin{cases} \theta_1 = 0 \\ |a_1| = 1 \end{cases}$$





SIC info for a_0

$\theta_0 = 0$
$|a_0| = 0$
$|a_0|^2 = 0$

SIC info for a_1

$\theta_1 = 0$
$|a_1| = 1$
$|a_1|^2 = 1$

Question: What are the probabilities of measuring the result "0" and "1" respectively?

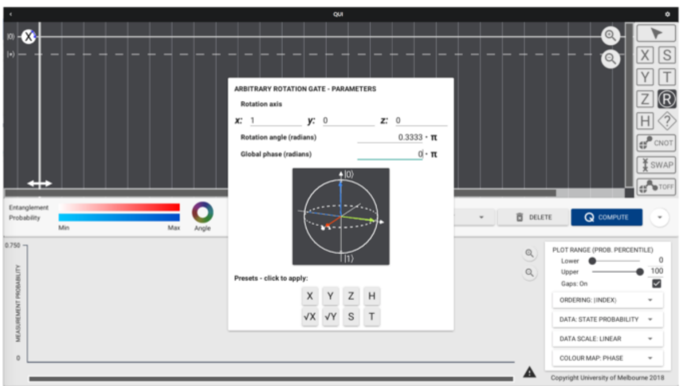
Prob["0"] = ____________ : Prob["1"] = ____________

Now we will set up a non-trivial quantum superposition state in the QUI. Set to 1-qubit (if not already) and clear any existing circuit.

Exercise 3.3 To construct our example state, start from a blank 1-qubit circuit and choose the R-gate from the gate library (for now, don't worry about what it is or how it works). Click the R-gate into the first time block. Once placed in the circuit **right click** on the R-gate in the circuit to bring up the editable gate menu:

Edit Parameters -> set axis to X, rotation angle to $\theta_R = \frac{\pi}{3}$, and global phase to zero.

Press compute and the QUI output will look like the following (bottom right):



Don't worry how this gate works for now, we will just look at the final output state to further illustrate the QUI notation for the individual complex amplitudes.

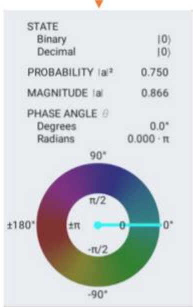
The overall quantum state at the end of this circuit is a superposition $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$. We will now inspect the individual state amplitudes in the SICs.

Question: In the following, convert from the QUI polar notation for the complex amplitudes to cartesian as indicated.

Overall quantum state: $|\psi\rangle = \boxed{a_0\,|0\rangle} + a_1\,|1\rangle$

QUI representation of complex amplitude a_0 in polar form:

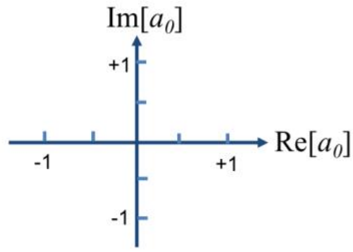
$|a_0|e^{i\theta_0}$



Convert amplitude a_0 to cartesian form and plot:

$$a_0 = |a_0|e^{i\theta_0} = \mathrm{Re}[a_0] + i\mathrm{Im}[a_0]$$

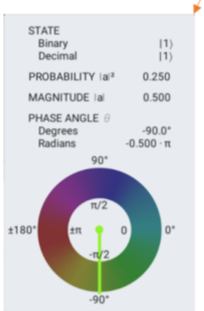
$a_0 =$ _______ $+\,i$ _______



Overall quantum state: $|\psi\rangle = a_0\,|0\rangle + \boxed{a_1\,|1\rangle}$

QUI representation of complex amplitude a_1 in polar form:

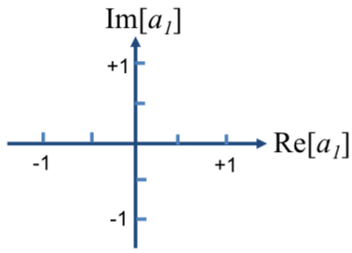
$|a_1|e^{i\theta_1}$



Convert amplitude a_1 to cartesian form and plot:

$$a_1 = |a_1|e^{i\theta_1} = \mathrm{Re}[a_1] + i\mathrm{Im}[a_1]$$

$a_1 =$ _______ $+\,i$ _______



Question: Verify that the state created is:

$$|0\rangle \rightarrow \frac{\sqrt{3}}{2}\,|0\rangle + \frac{-i}{2}\,|1\rangle = \qquad |0\rangle + \qquad |1\rangle$$

In above space provided write the complex amplitudes in polar form, $|a_0|e^{i\theta_0}|0\rangle + |a_1|e^{i\theta_1}|1\rangle$.

Question: What are the respective probabilities of measuring "0" and "1" states?

Prob["0"] = ____________ : Prob["1"] = ____________

Exercise 3.4 Measurement on the QUI. On the same circuit in the above exercise 3.3, add a measurement from the gate library ("?" in the diamond symbol) in the time block after the R-gate. Press compute and you will see the measurement symbol spin and settle on a random outcome "0" or "1" according to the associated probabilities in the quantum state.

Every time you press compute you will get a new measurement outcome. You can move the vertical slider bar to inspect the quantum state before and after measurement.

Question: Do you see how the state effectively collapses after any given measurement?

Question: Press the compute button many times (say N = 100) and record the number of 0 and 1 outcomes and fill in the table below. Compare the estimated probabilities with those expected.

Quantum superposition component	Probability	Measurement record	# outcomes, n	Estimated Prob = n/N
$\vert 0\rangle$	0.75	𝍸 𝍸 $\cdots$		
$\vert 1\rangle$	0.25	𝍸 $\cdots$		

Exercise 3.5 To construct another example superposition state, right click on the R-gate in the circuit to bring up the rotation gate menu again. Program some random parameters for axis and rotation angle. Press compute repeat the above analyses of the amplitudes and measurement outcome probabilities. Generate other superpositions to make sure you understand the complex polar notation used in QUI, and the physical interpretation of the quantum states produced in terms of measurement outcomes.

Next week we will look at how the qubit operations work, and the Bloch Sphere representation (those little animations that keep popping up).

Matrices and Vectors

Here we will review some common concepts in linear algebra. If you have any trouble, be sure to ask the tutor to help, as getting these concepts clear now will be extremely useful in the coming weeks.

Exercise 4.1 Perform the following matrix multiplications.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} =$$

In the lectures we saw that the qubit states could be expressed as:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Express your answers for Exercise 4.1 as a linear combination of these two vectors.

Exercise 4.2 Write

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

explicitly as a column vector.

Exercise 4.3 Write

$$\frac{\sqrt{3}i}{2}|0\rangle - \frac{1}{2}|1\rangle$$

explicitly as a column vector.

Exercise 4.4 To be correctly normalised, the probabilities of a state have to add to 1. The probabilities are equal to the (modulus of the) amplitude squared. Check that the state given in Exercise 4.3 is correctly normalised.

Also in lectures, we saw how to find the Hermitian adjoint of a matrix. To do this find the matrix transpose, and take the complex conjugate of all elements.

Exercise 4.5 Find the following

$$\begin{bmatrix} 3 & 1+2i \\ 0 & 1 \end{bmatrix}^{\dagger} =$$

$$\begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}^{\dagger} =$$

Exercise 4.6 Which of the two matrices in Exercise 4.5 is Hermitian? Why?

Exercise 4.7 A matrix, U, is unitary if $U^{\dagger}U = UU^{\dagger} = I$. Is the matrix Y, given below, unitary?

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Exercise 4.8 Show that the following matrix is unitary,

$$U = e^{i\varphi/2}\begin{bmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{-i\varphi_2}\sin\theta & e^{-i\varphi_1}\cos\theta \end{bmatrix}$$

Show that (for appropriate choices of angles) that:

$$U = e^{i\varphi/2}\begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{bmatrix}$$