This Week

Lecture 9

Quantum search – introduction to Grover’s algorithm for amplitude amplification, geometric interpretation

Lecture 10

Optimality, Succeeding with Certainty, Quantum Counting

Lab

Grover’s algorithm

Grover’s Algorithm

Lecture 9

Introduction to Grover’s algorithm

- This lecture: Grover’s search algorithm
  - Grover’s algorithm
  - Worked Example
  - Geometric interpretation

References:

Reiffel, Chapter 9.1-9.2
Kaye, Chapter 8.1-8.2
Nielsen and Chuang, Chapter 6.1-6.2

Reminder: Outer Product

THE UNIVERSITY OF MELBOURNE

For two quantum states $$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

We can define an outer product between them:

$$|\psi\rangle\langle\phi| = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c^* & d^* \end{bmatrix}$$

$$= \begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 2|$$

For number basis states, this specifies a matrix with a single “1” in the location 1,2. In general:

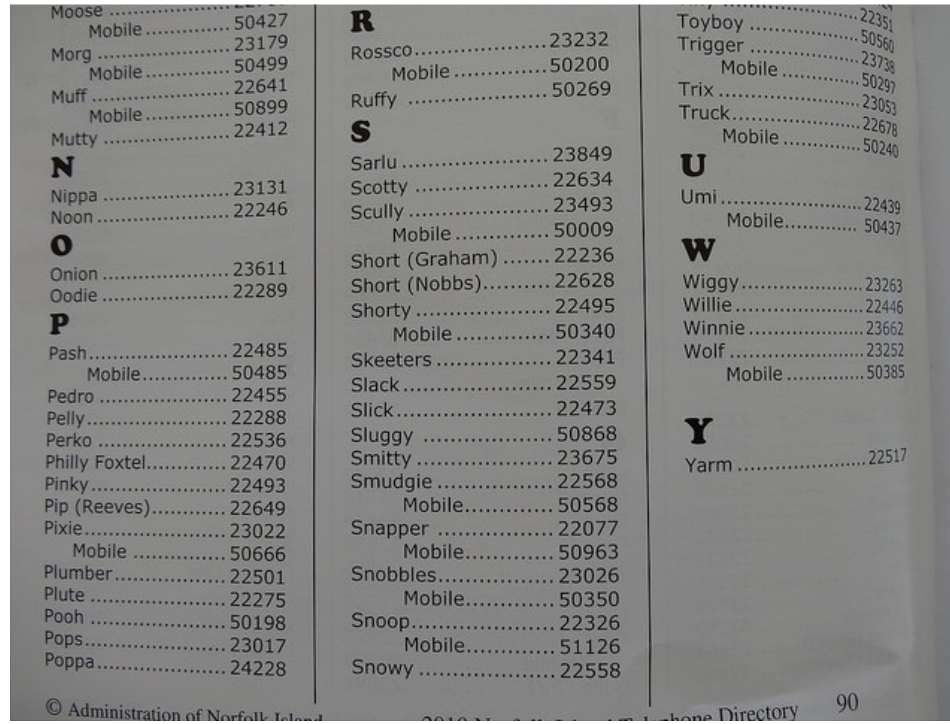
$$|\mathrm{row}\rangle\langle\mathrm{column}|$$

Unordered Search

Grover's algorithm performs a similar* problem to this: You are given *a telephone book*

And a phone number: *23675*

Your task:
Find the name which goes with that number…


Moose
Mobile 50427
Morg 23179
Mobile 50499
Muff 22641
Mobile 50899
Mutty 22412
N
Nippa 23131
Noon 22246
O
Onion 23611
Oodie 22289
P
Pash 22485
Mobile 50485
Pedro 22455
Pelly 22288
Perko 22536
Philly Foxtel 22470
Pinky 22493
Pip (Reeves) 22649
Pixie 23022
Mobile 50666
Plumber 22501
Plute 22275
Pooh 50198
Pops 23017
Poppa 24228
R
Rossco 23232
Mobile 50200
Ruffy 50269
S
Sarlu 23849
Scotty 22634
Scully 23493
Mobile 50009
Short (Graham) 22236
Short (Nobbs) 22628
Shorty 22495
Mobile 50340
Skeeters 22341
Slack 22559
Slick 22473
Sluggy 50868
Smitty 23675
Smudgie 22568
Mobile 50568
Snapper 22077
Mobile 50963
Snobbles 23026
Mobile 50350
Snoop 22326
Mobile 51126
Snowy 22558
Toyboy 22351
.... 50560
Trigger 23738
Mobile 50297
Trix 23053
Truck 22678
Mobile 50240
U
Umi 22439
Mobile 50437
W
Wiggy 23263
Willie 22446
Winnie 23662
Wolf 23252
Mobile 50385
Y
Yarm 22517
© Administration of Norfolk Island
Telephone Directory 90

Part of Norfolk Island's telephone book, with people listed by nickname (Photo: Wikicommons)

* Not all that similar, better examples later….

Quantum search – Grover's problem

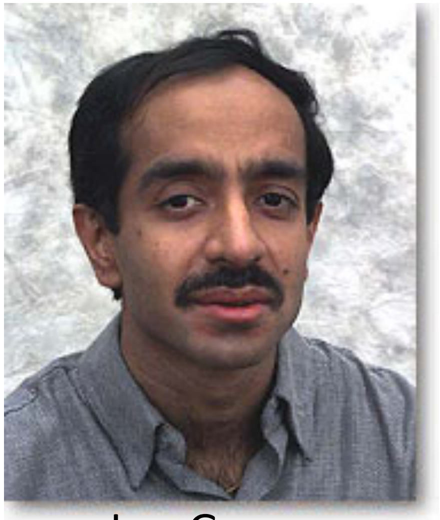
Given an black box (oracle), U_f, which computes the function:

$$f: \{0,1\}^n \rightarrow \{0,1\}$$

Find an x s.t. $f(x) = 1$

Grover's Algorithm (1996)

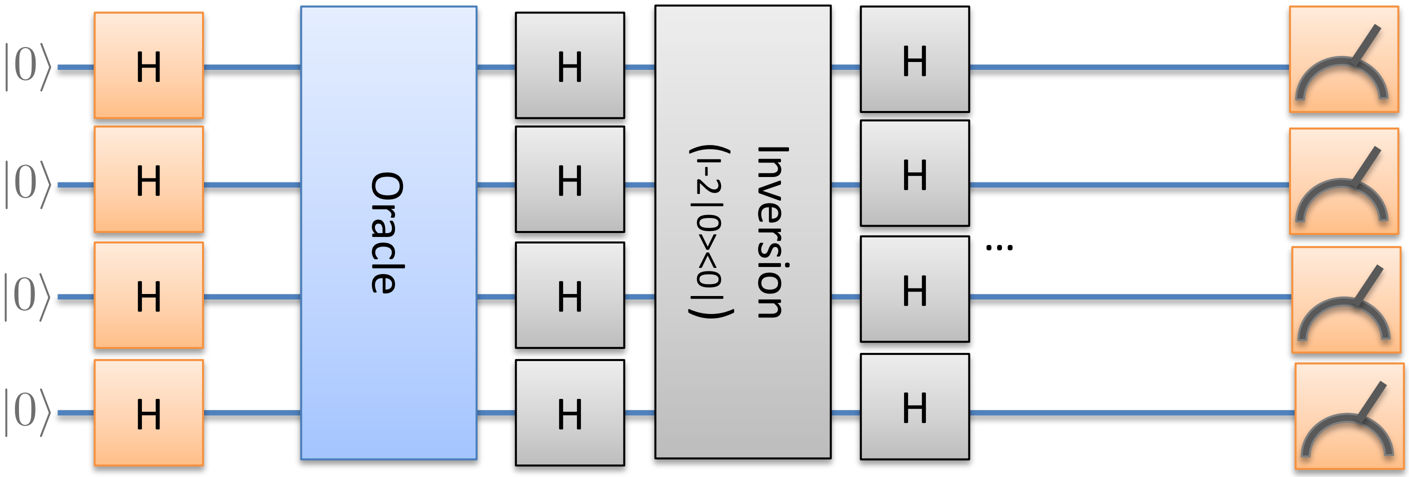
- Unordered search, find one marked item among many
- Classically, this requires N/2 queries to the oracle
- Quantum mechanically, requires only O(√N) queries.



Lov Grover

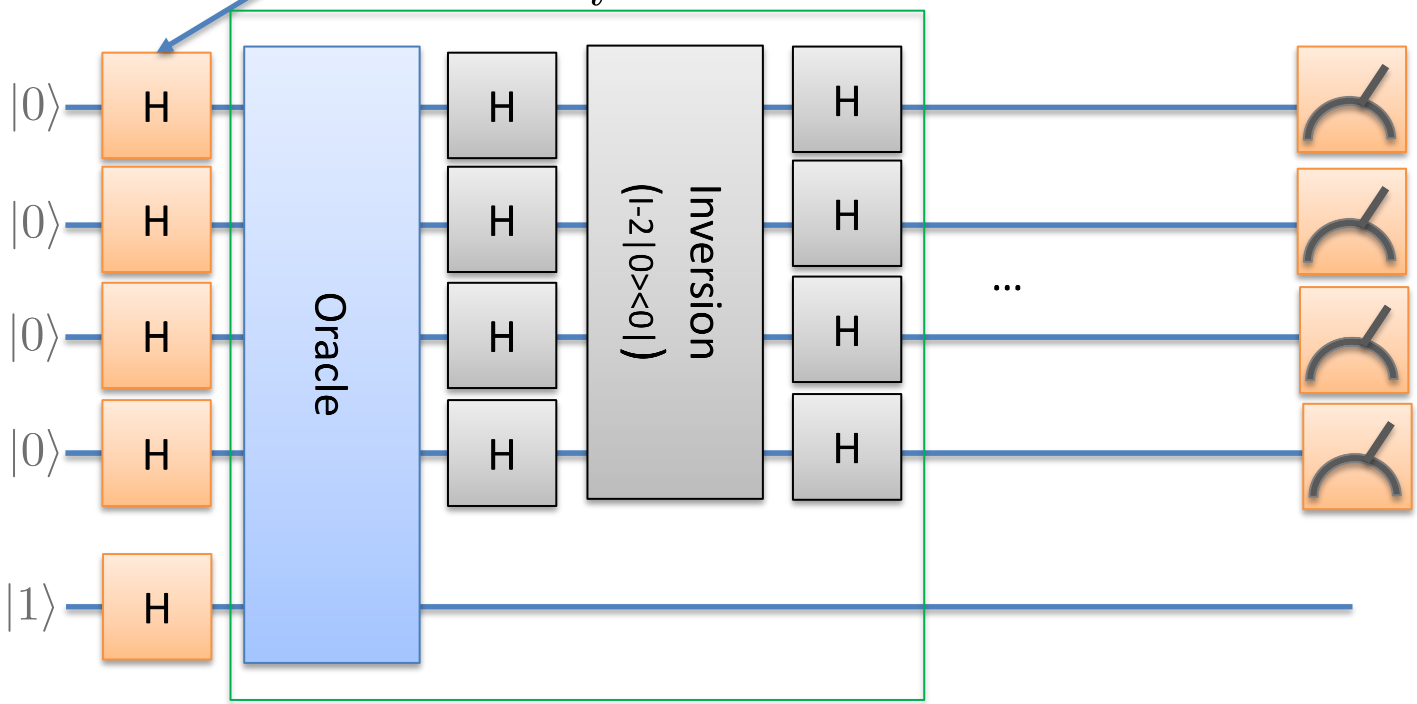
Simple problem = search for one integer marked by the oracle.

High level structure:



Two basic steps in Grover's algorithm

Quantum database: $|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$ (i.e. all integers 0 to N-1)



The "oracle"
Identifies a particular marked state, m

$$I - 2\,|\Phi\rangle\,\langle\Phi|$$

"Inversion about the mean"

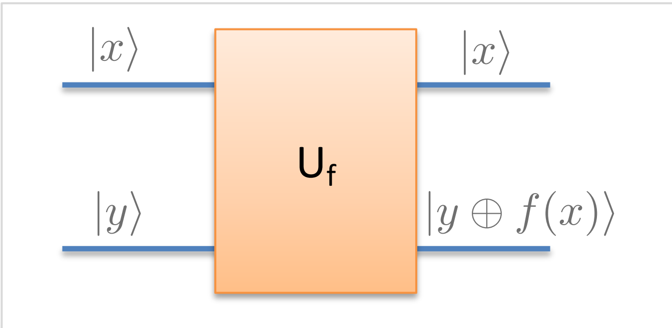
Repeat these two operations O(√N) times

The Oracle

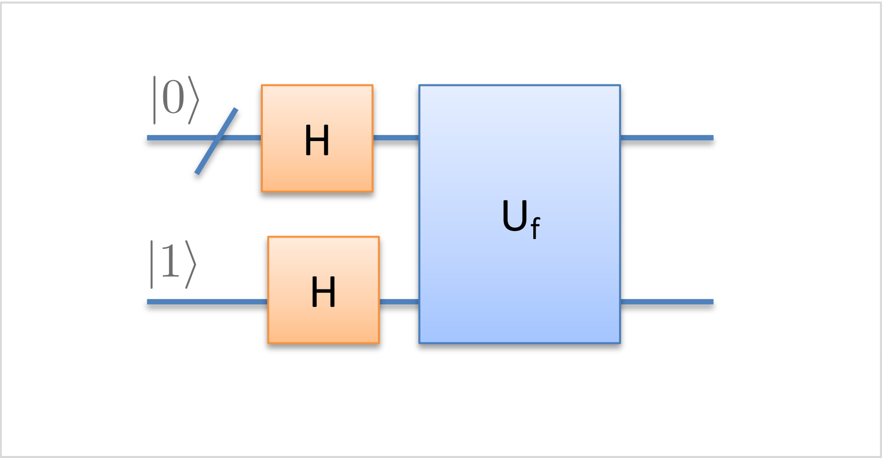
The task of recognizing the correct solution goes to the “oracle”.

Designed to flip the last bit if the input, i, is a solution

Phase kickback for Boolean function



Binary function, or "oracle"



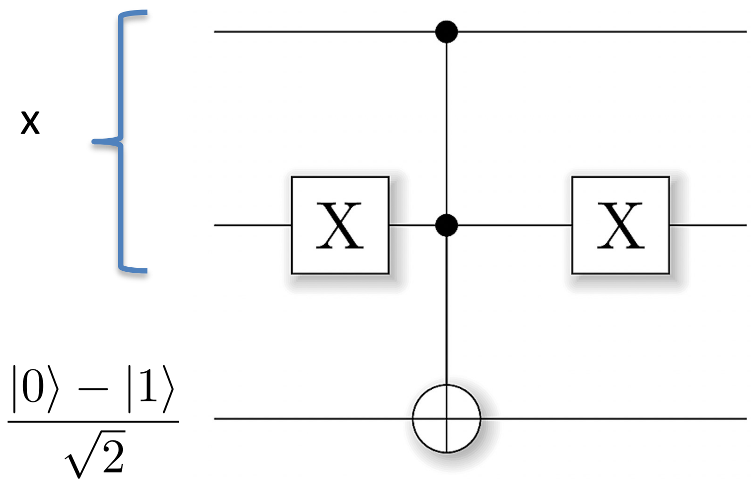
After the function has been applied:

Target qubit remains the same

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} (-1)^{f(i)} |i\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If the oracle function evaluates to "1" then the target qubit is flipped, and we pick up a phase (associated with the control qubit state). Otherwise, there is no phase applied. This is a simple way to write that.

Example: Oracle recognizing the state “2 = |10>”



Phase kickback

$$\begin{aligned} |00\rangle &\rightarrow |00\rangle \\ |01\rangle &\rightarrow |01\rangle \\ |10\rangle &\rightarrow -|10\rangle \\ |11\rangle &\rightarrow |11\rangle \end{aligned}$$

The effect on each of the 4 states in the 2-qubit control register, x:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I - 2\,|10\rangle\langle 10|$$

The marked state

Initially in Grover's algorithm, we will be searching for a *single (integer) solution*, ***m***. In that case the effect of the oracle on the control register is:

$$I - 2\left|m\right\rangle\left\langle m\right|$$

(in decimal ket notation)

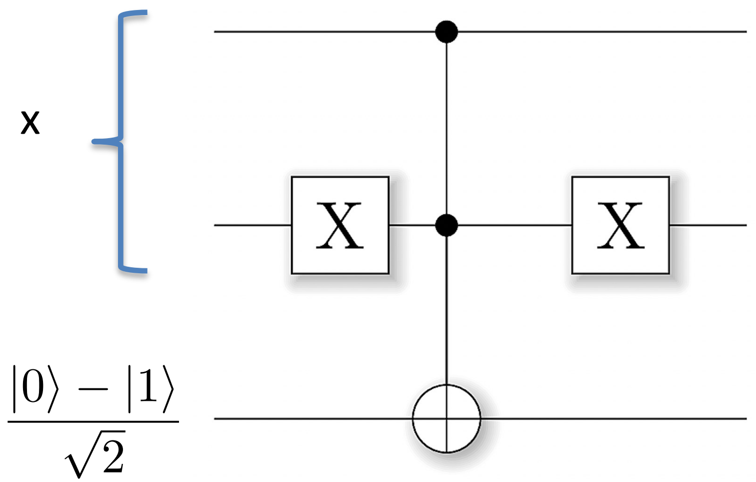
As a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

-1 in the mth position

Here, as in future slides, we are only writing out the control qubits (in this case 2 qubits only).

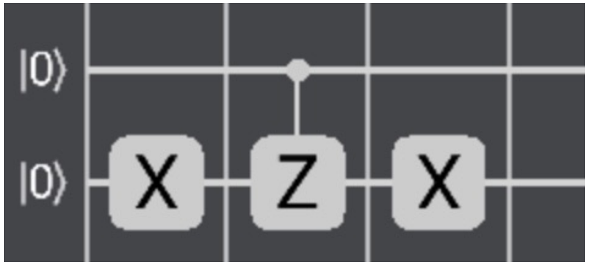
Example: Oracle recognizing the state “2 = |10>”



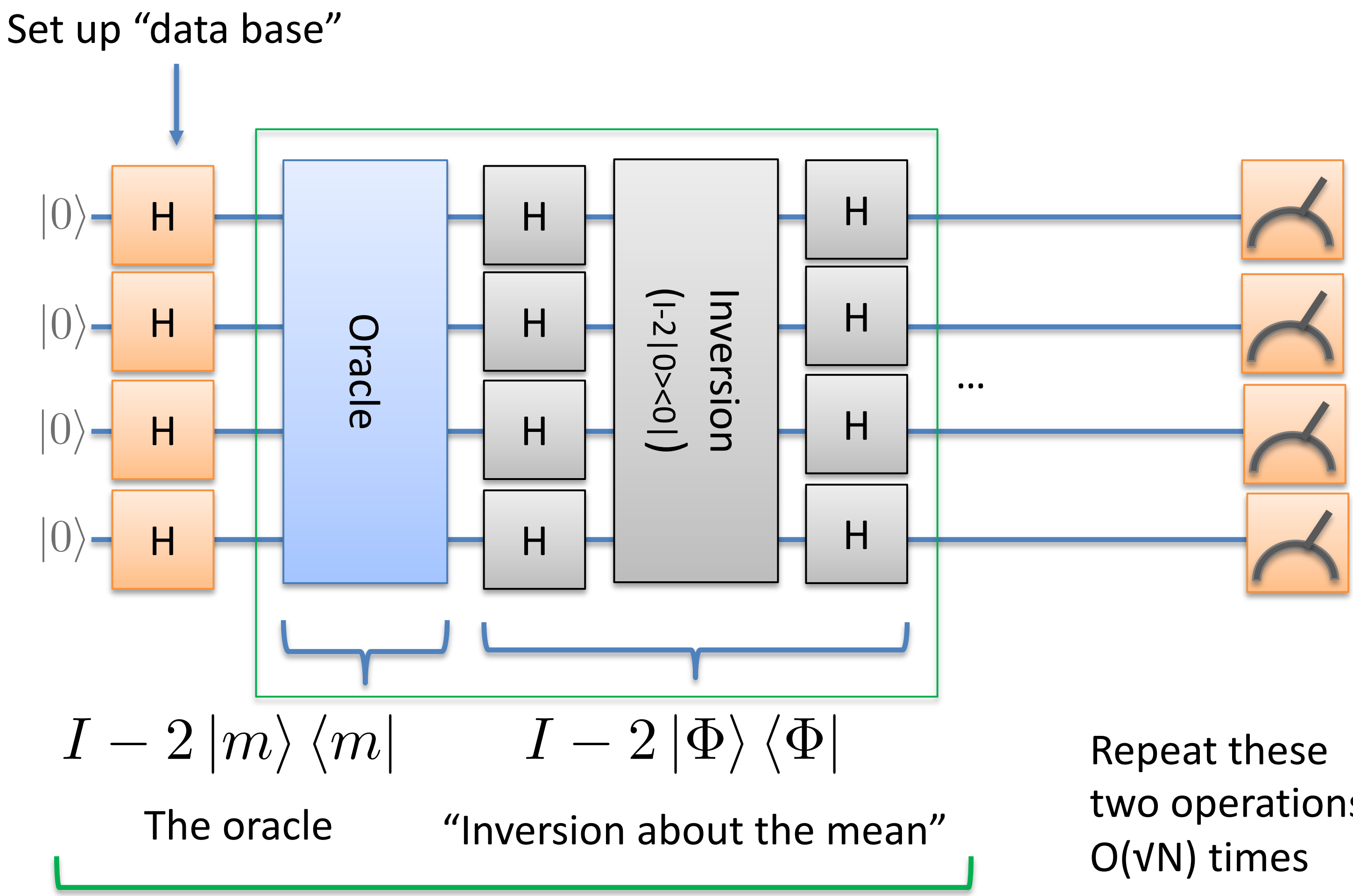
Phase kickback

$$
\begin{aligned}
|00\rangle &\rightarrow \quad |00\rangle \\
|01\rangle &\rightarrow \quad |01\rangle \\
|10\rangle &\rightarrow -|10\rangle \\
|11\rangle &\rightarrow \quad |11\rangle
\end{aligned}
$$

In practice we can implement the oracle without the check qubit using a controlled-Z gate (ex. Show the circuit right marks the state |10>, i.e. |m=2>).



Two steps to Grover's algorithm

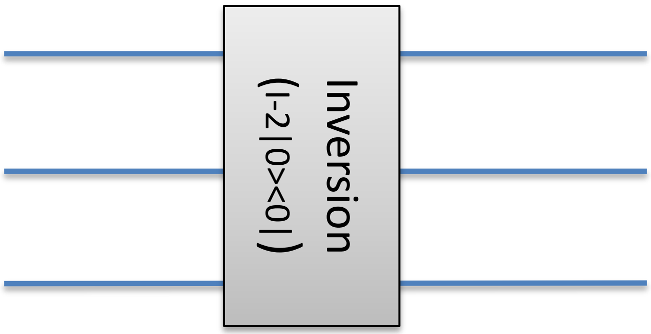


Repeat these two operations O(√N) times

One iteration of Grover where $|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$

Unpicking the details: “Inversion” operation

The “Inversion” part is just applying a phase to the zero state:



$$I - 2\left|0\right\rangle\left\langle 0\right| = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

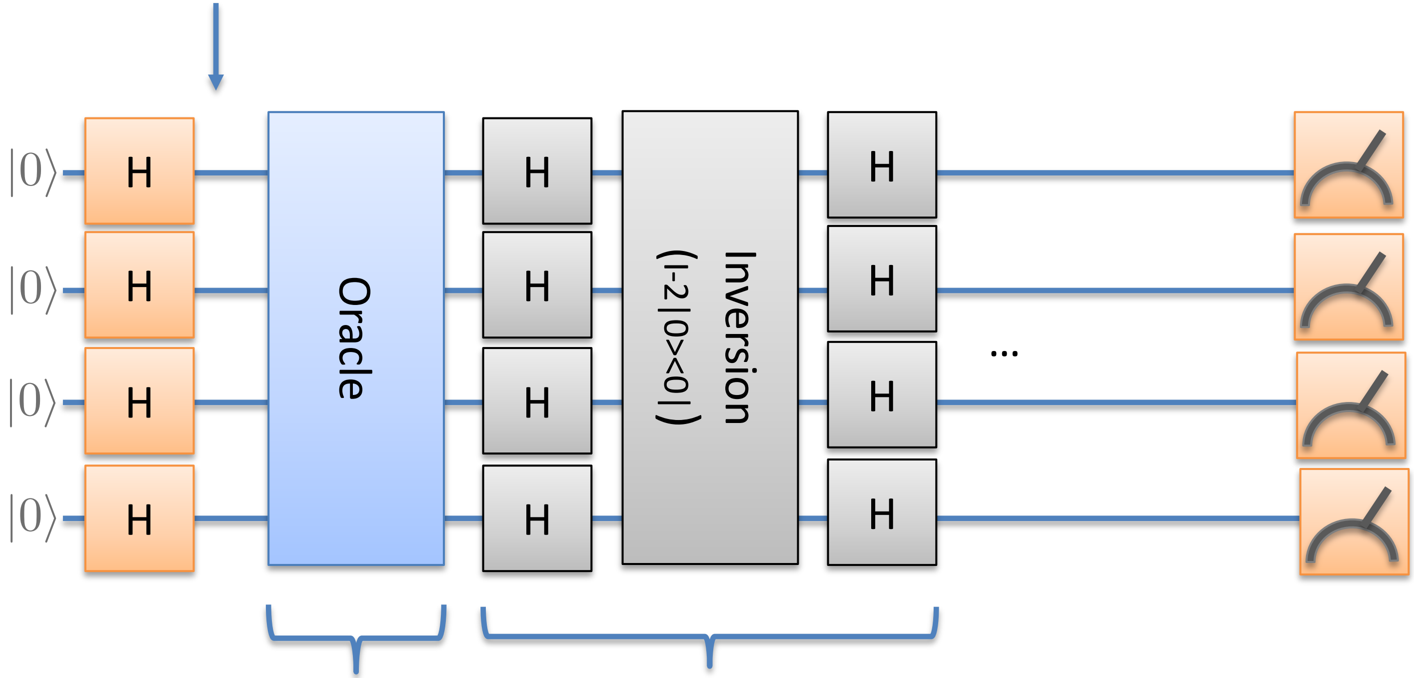
How? Recall outer product etc: $\left|\psi\right\rangle\left\langle\phi\right| = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c^* & d^* \end{bmatrix} = \begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}$

$$\left|0\right\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \left|0\right\rangle\left\langle 0\right| = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes (1\,0\dots0) = \begin{pmatrix} 1\,0\dots0 \\ 0\,0\dots0 \\ \vdots \end{pmatrix} \qquad I = \begin{pmatrix} 1\,0\,0\dots0 \\ 0\,1\,0\dots0 \\ 0\,0\,1\dots0 \\ \vdots \end{pmatrix}$$

$$I - 2\left|0\right\rangle\left\langle 0\right| = \begin{pmatrix} 1\,0\,0\dots0 \\ 0\,1\,0\dots0 \\ 0\,0\,1\dots0 \\ \vdots \end{pmatrix} - 2\begin{pmatrix} 1\,0\dots0 \\ 0\,0\dots0 \\ \vdots \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Inversion about the mean

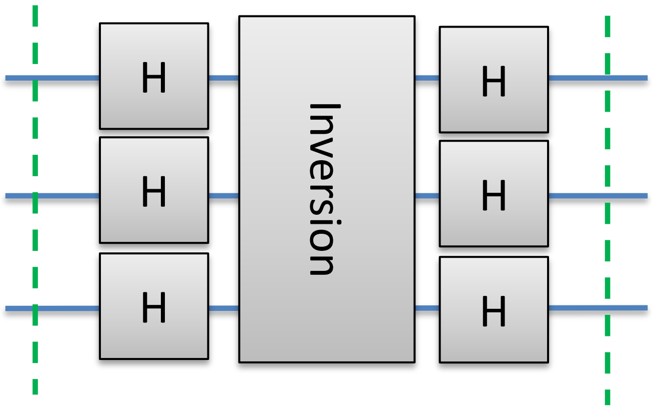
$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$$ Set up “data base”



$$I - 2\,|\Phi\rangle\,\langle\Phi|$$

“Inversion about the mean”…let’s see how that works.

Apply inversion about the mean to general state



$$|\Psi\rangle = \sum_i a_i |i\rangle \qquad |\Psi'\rangle$$

Applying Hadamards both sides:

$$I - 2H^{\otimes n} |0\rangle \langle 0| H^{\otimes n}$$
$$= I - 2 |\Phi\rangle \langle \Phi|$$

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$$

Equal superposition

$$|\Psi\rangle = \sum_i a_i |i\rangle \rightarrow |\Psi'\rangle = (I - 2|\Phi\rangle\langle\Phi|) \sum_i a_i |i\rangle$$

$$= \sum_i a_i |i\rangle - 2\frac{1}{\sqrt{N}} \sum_k |k\rangle \frac{1}{\sqrt{N}} \sum_j \langle j| \sum_i a_i |i\rangle$$

$$= \sum_i a_i |i\rangle - 2\sum_k |k\rangle \left(\frac{1}{N} \sum_j a_j \right)$$

$$= \sum_i (a_i - 2A) |i\rangle$$

General state

$$A \equiv \left(\frac{1}{N} \sum_j a_j \right)$$

Average amplitude in state $|\Psi\rangle$

Inversion about the mean

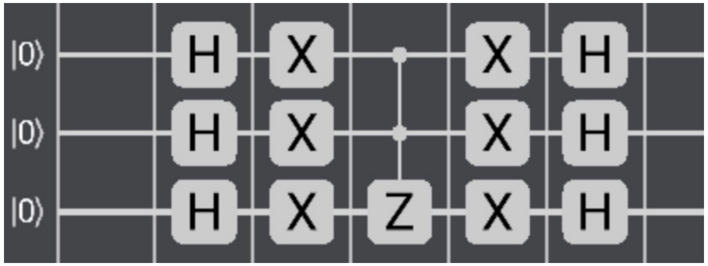
Consider a general state. The resulting amplitude from the "Inversion about the mean" step is:

$$\sum_i a_i |i\rangle \rightarrow \sum_i (a_i - 2A) |i\rangle$$

Original amplitude

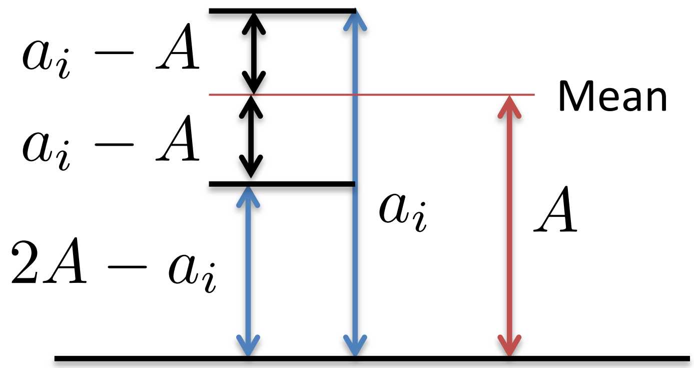
Average amplitude

In practice on the QUI…



Inversion about the mean

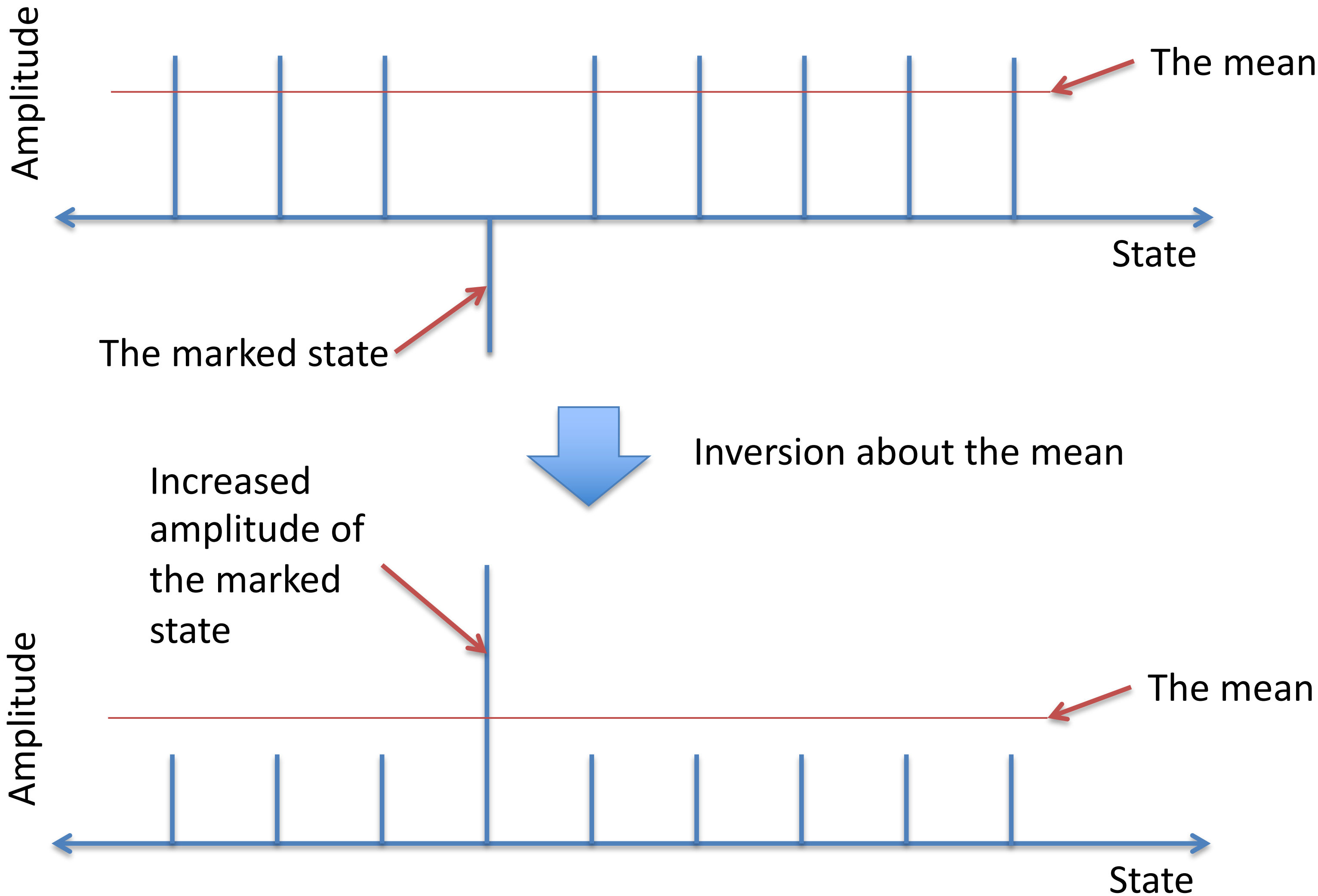
Amplitudes of the state, before and after:



When the state undergoes this transformation:

$$\sum_i a_i |i\rangle \rightarrow -\sum_i (2A - a_i) |i\rangle$$

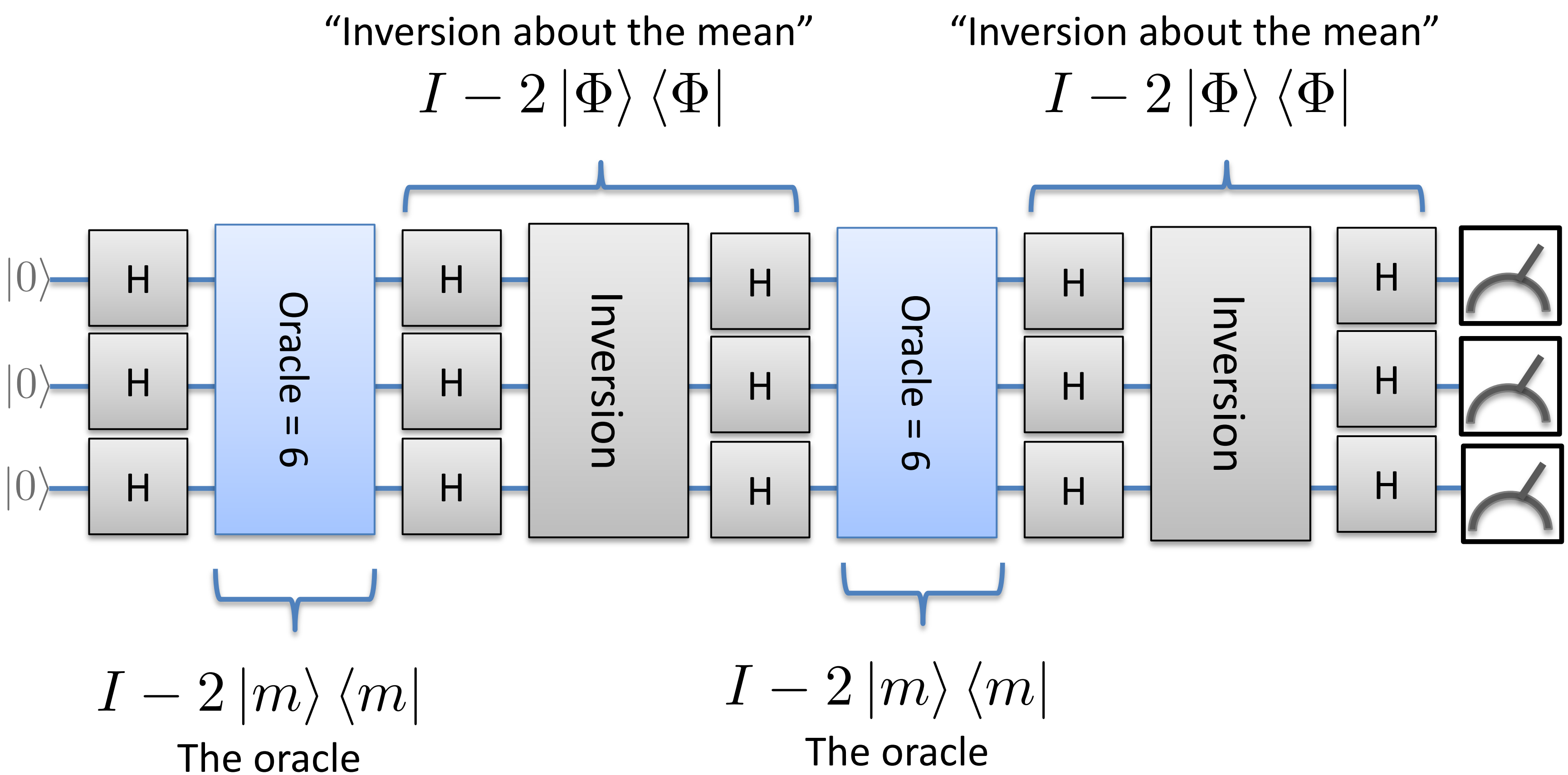
Effect of inversion about the mean



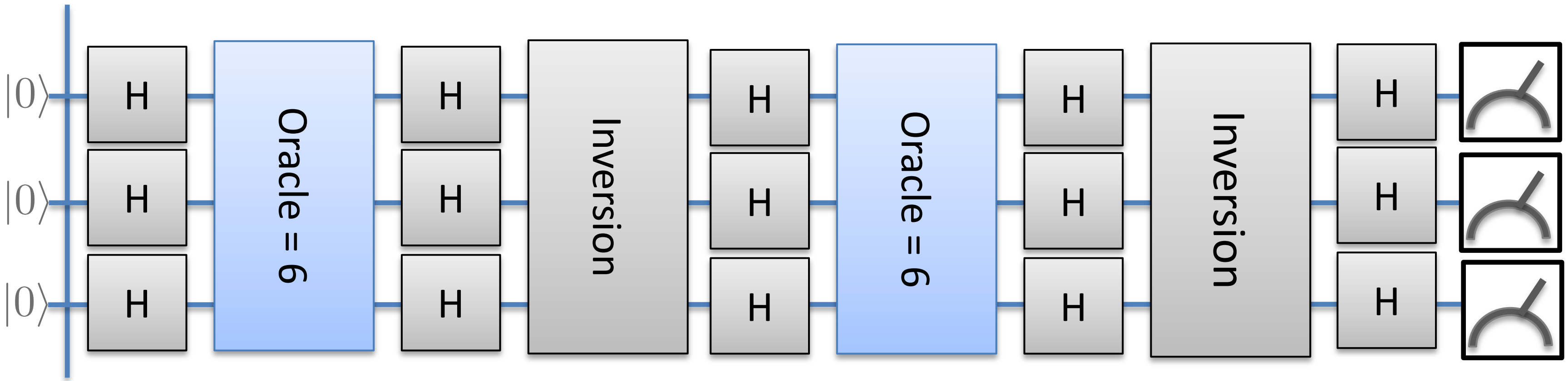
Interactive Example

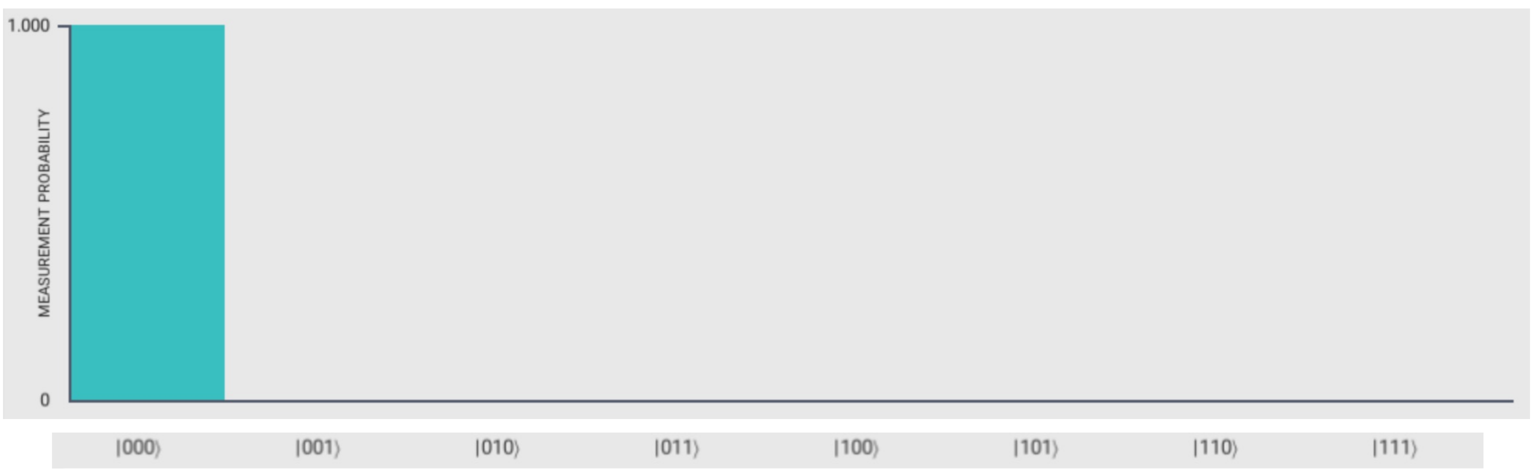
https://codepen.io/samtonetto/full/BVOGmW

Worked example: finding 6

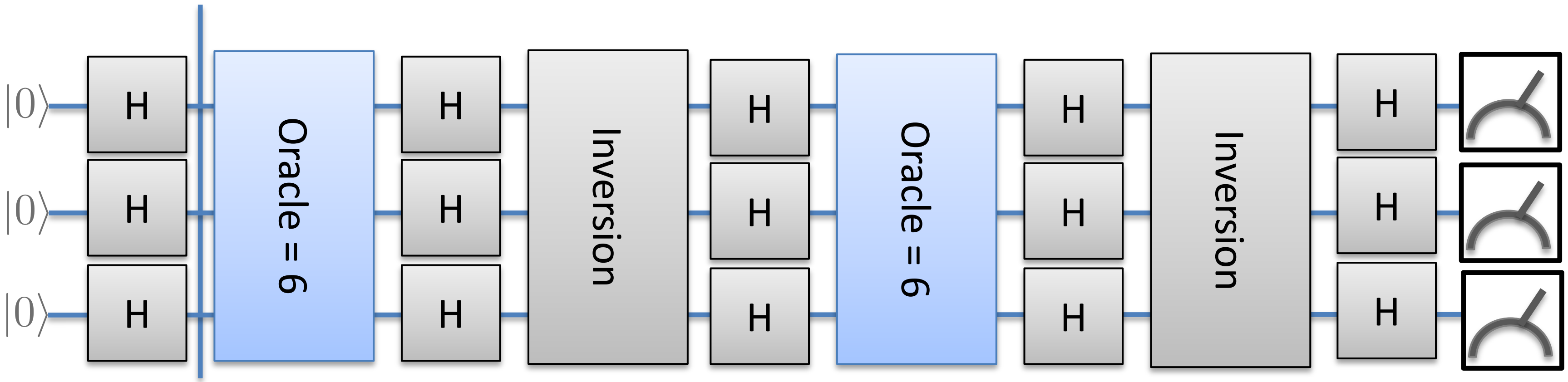


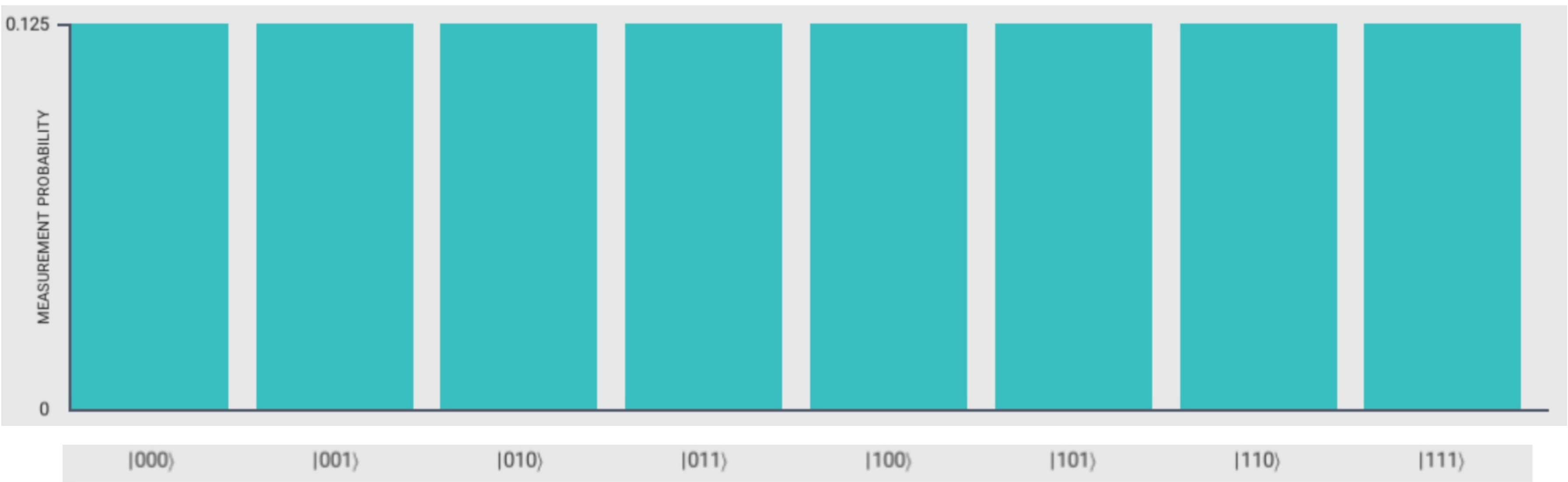
Worked example: finding 6



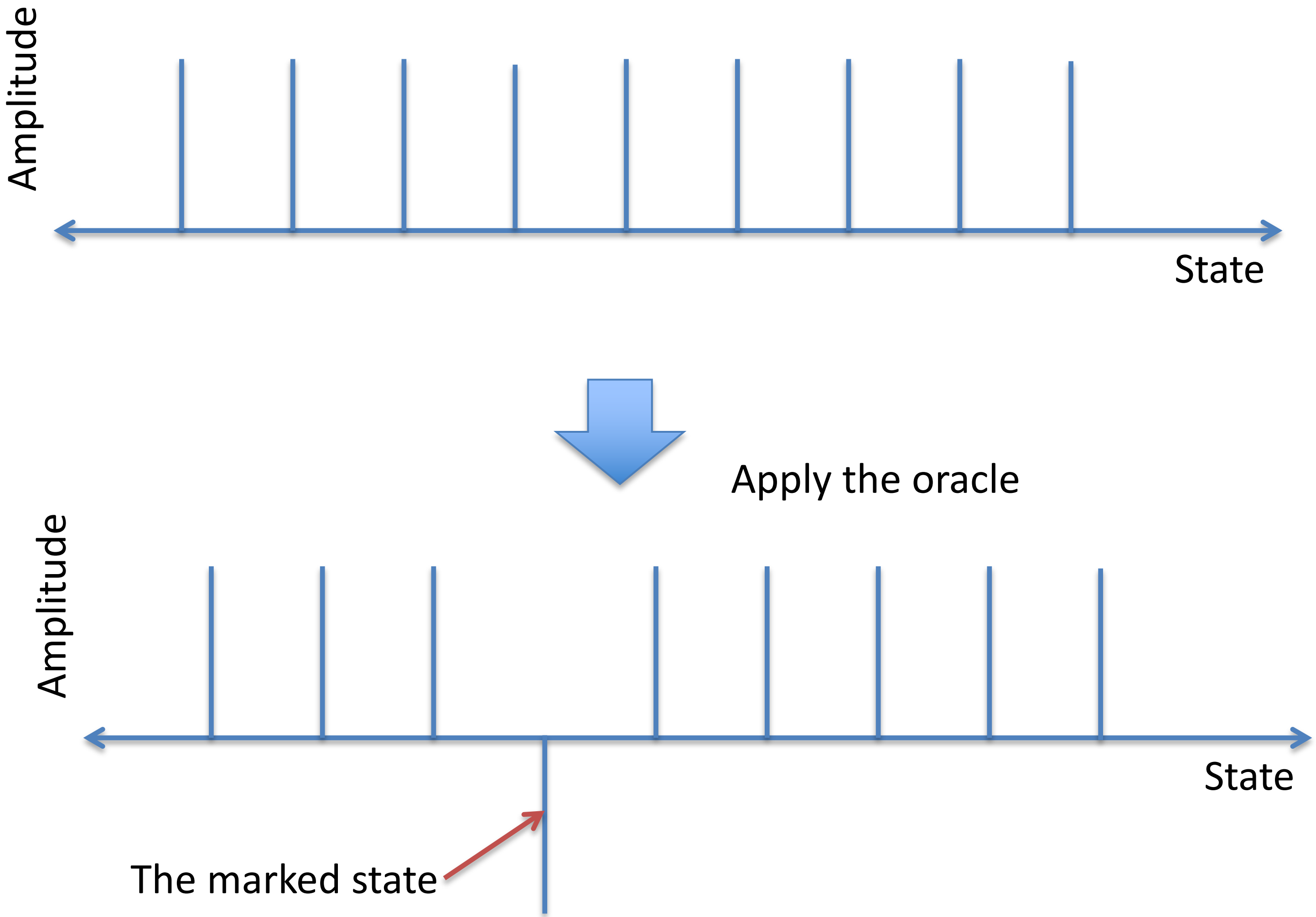


Worked example: finding 6

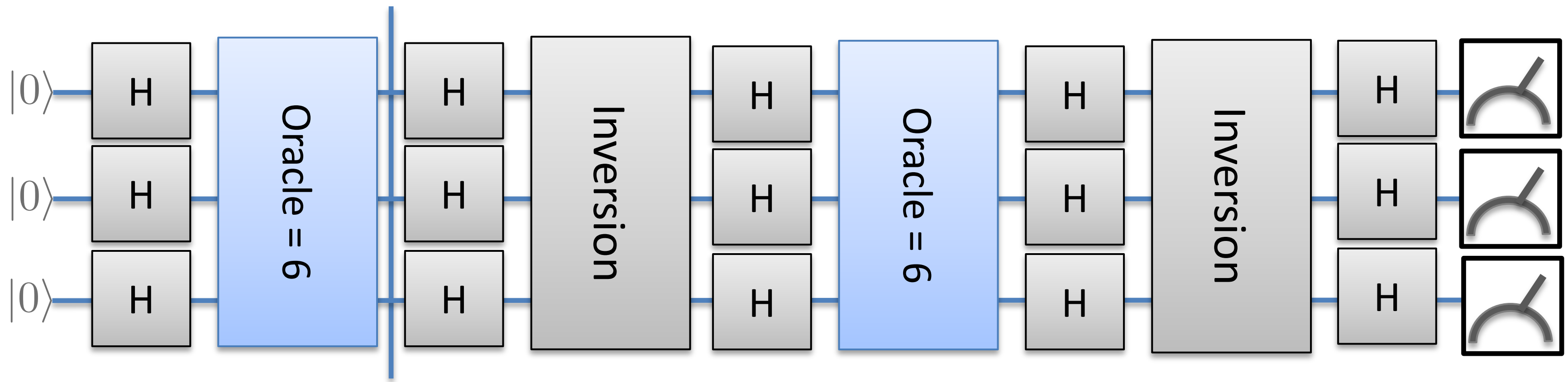


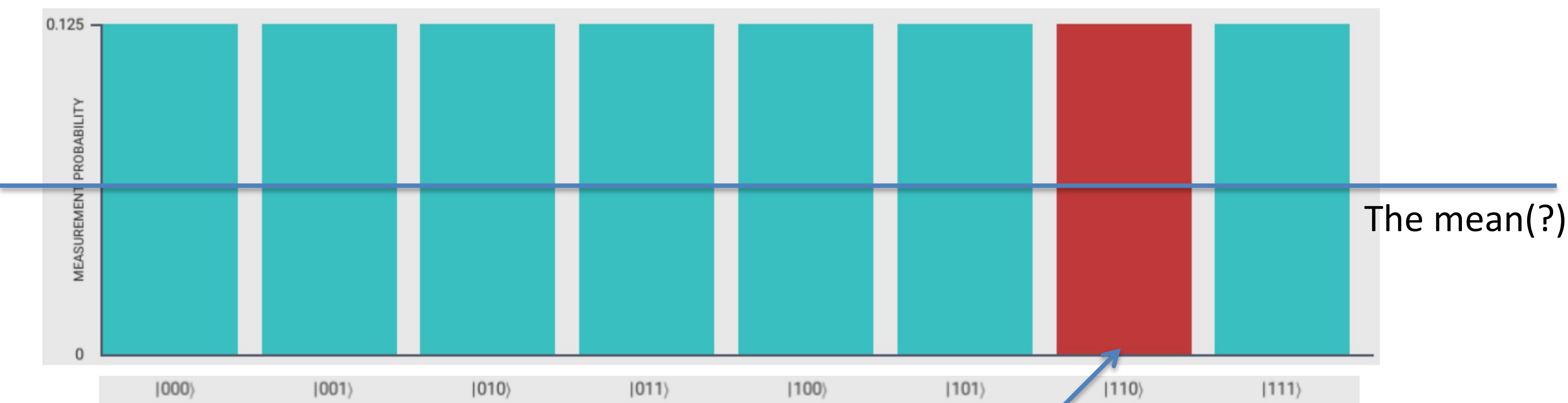


Effect of the Oracle



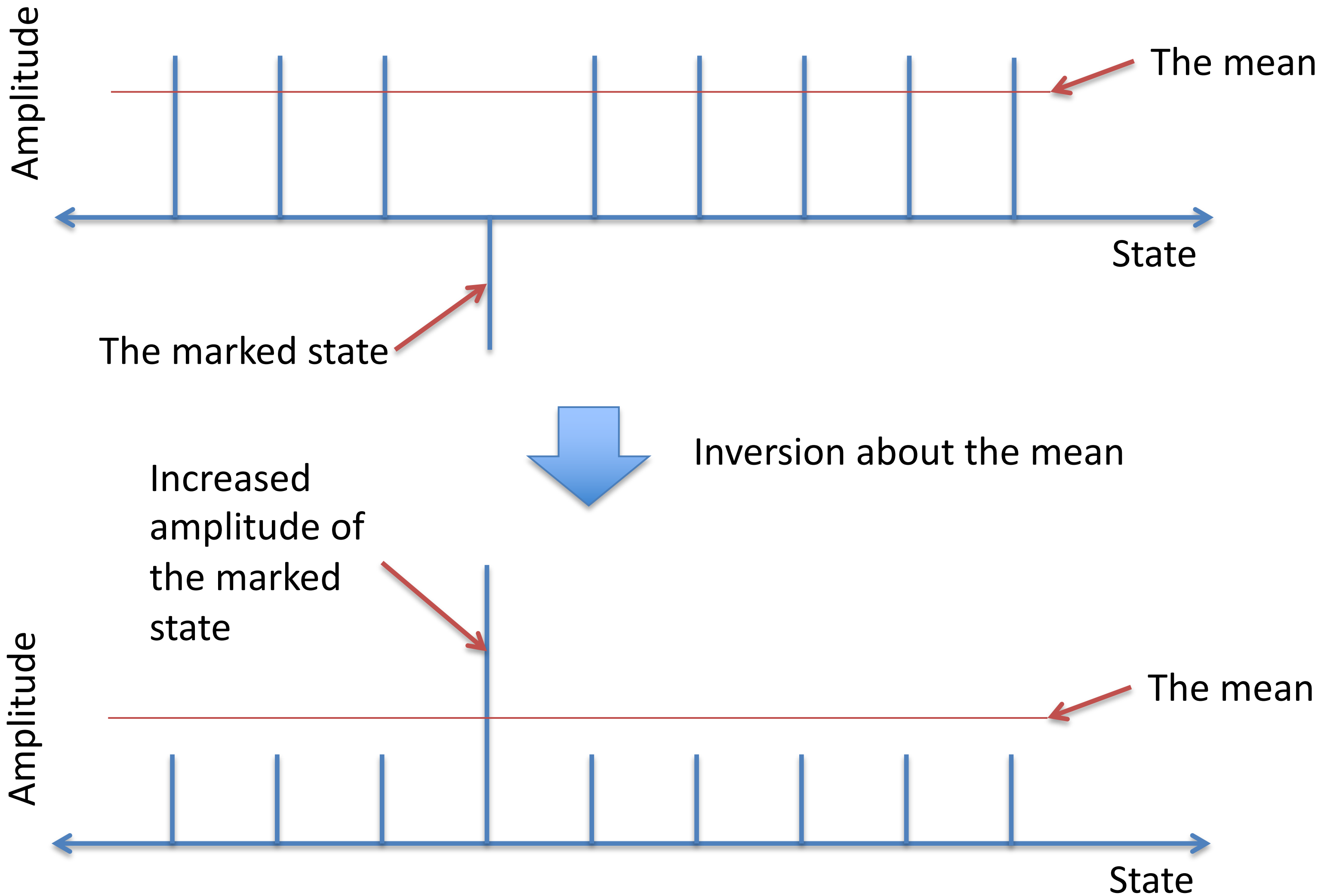
Worked example: finding 6



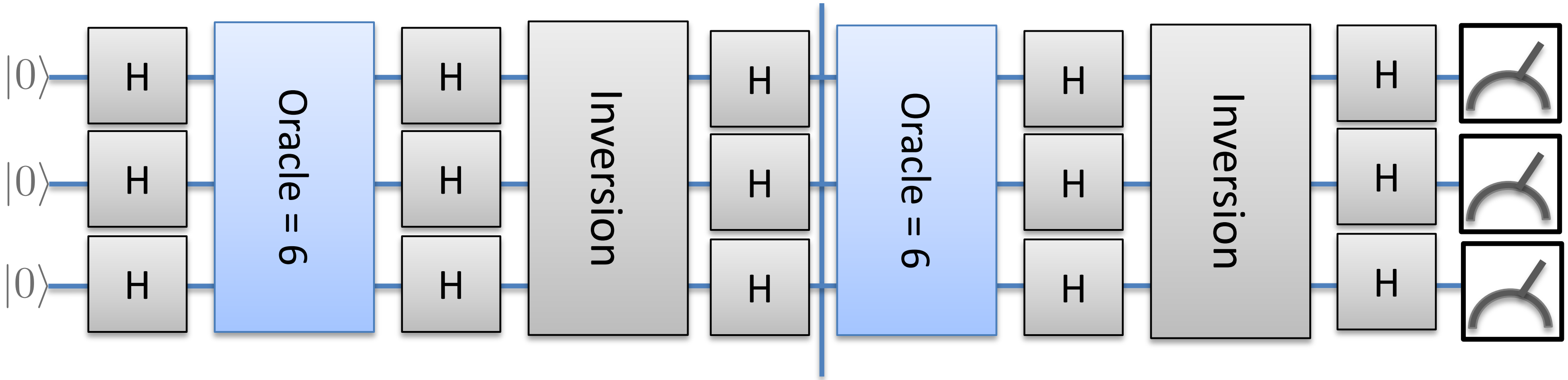


Phase of state 6 has changed to be negative

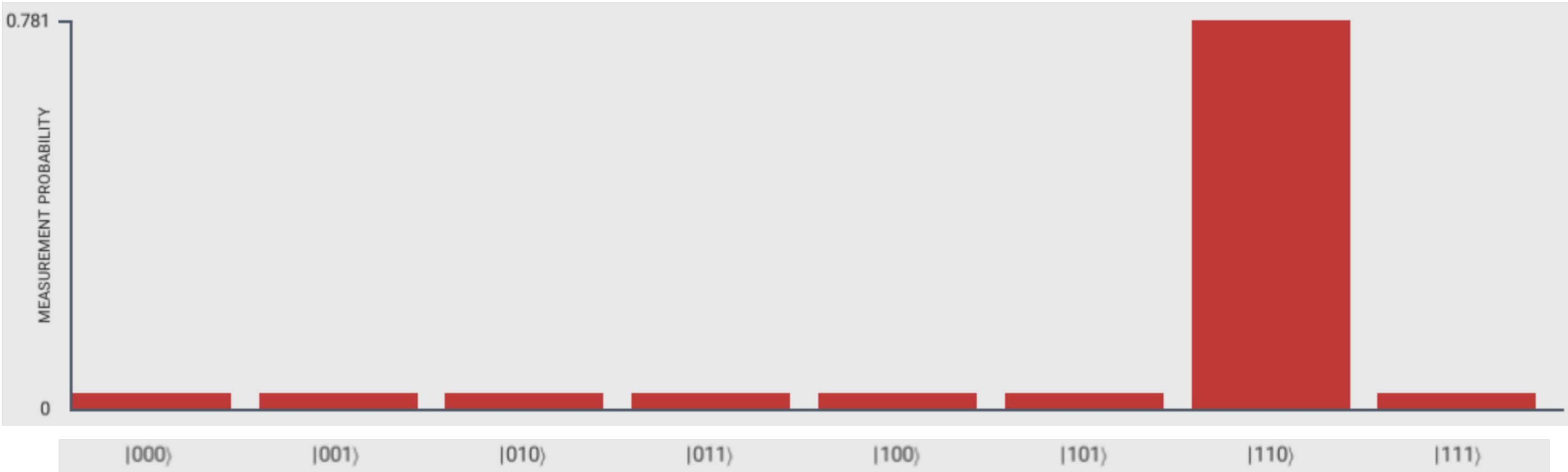
Effect of inversion about the mean



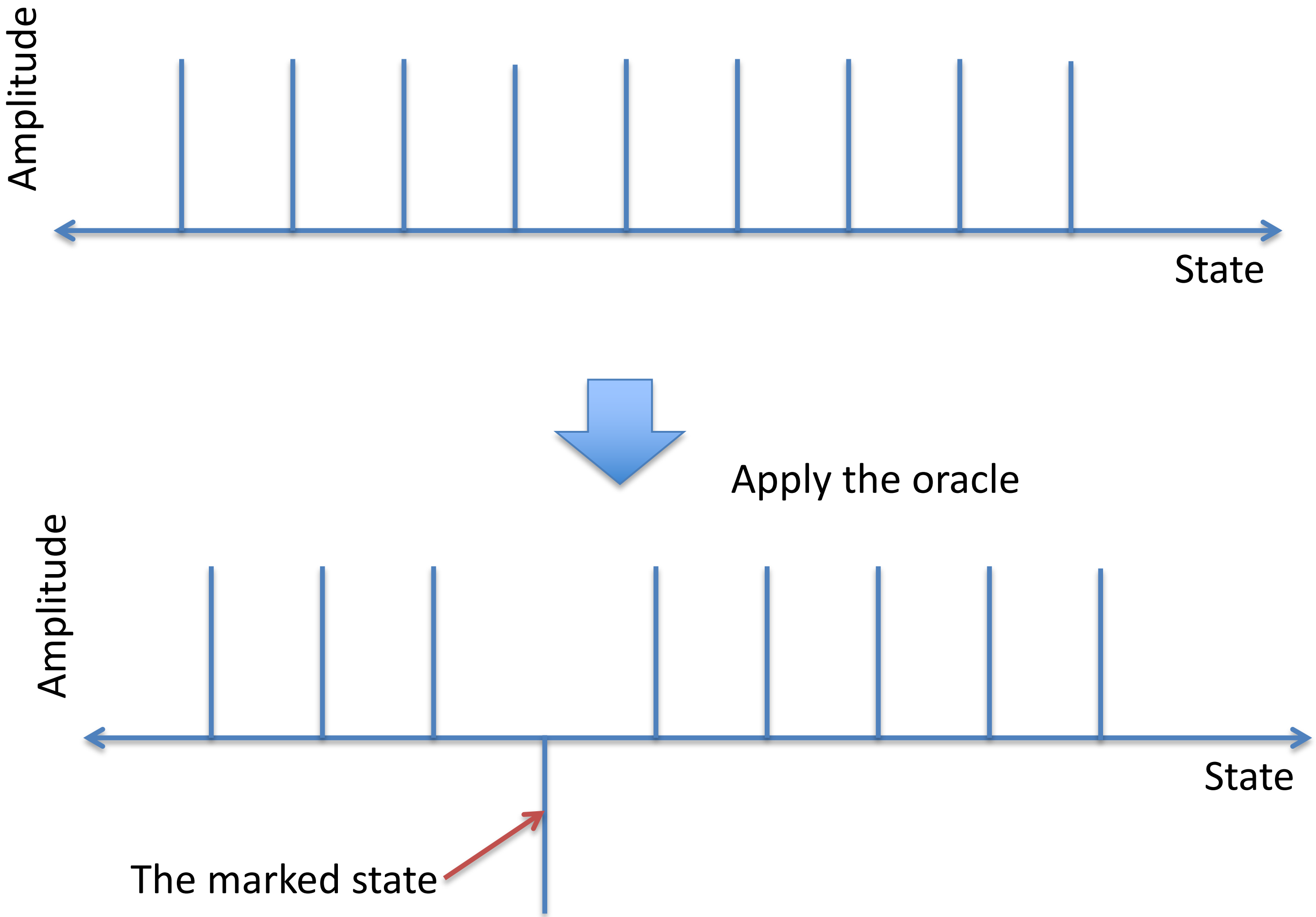
Worked example: finding 6



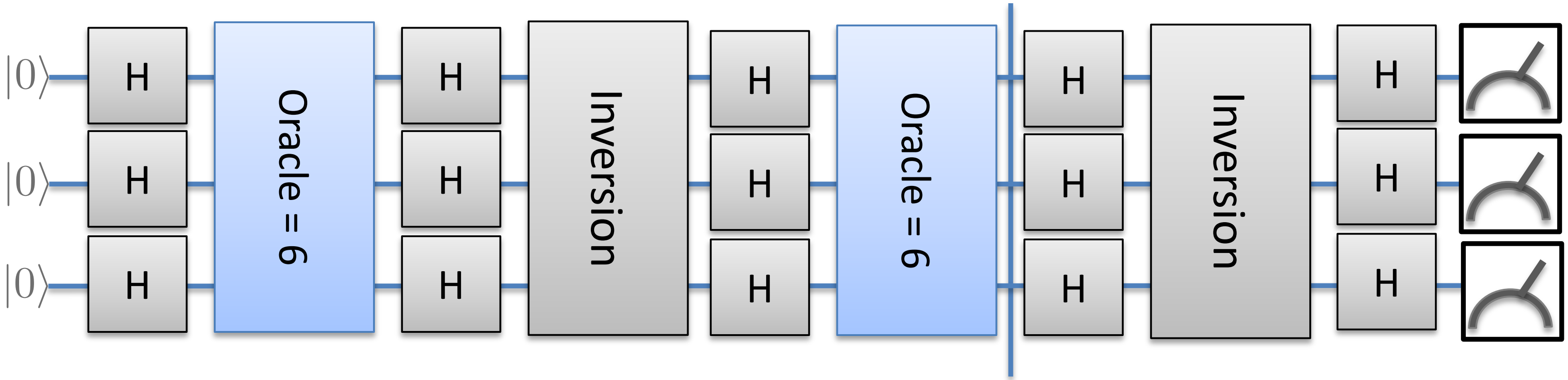
After inversion about the mean:



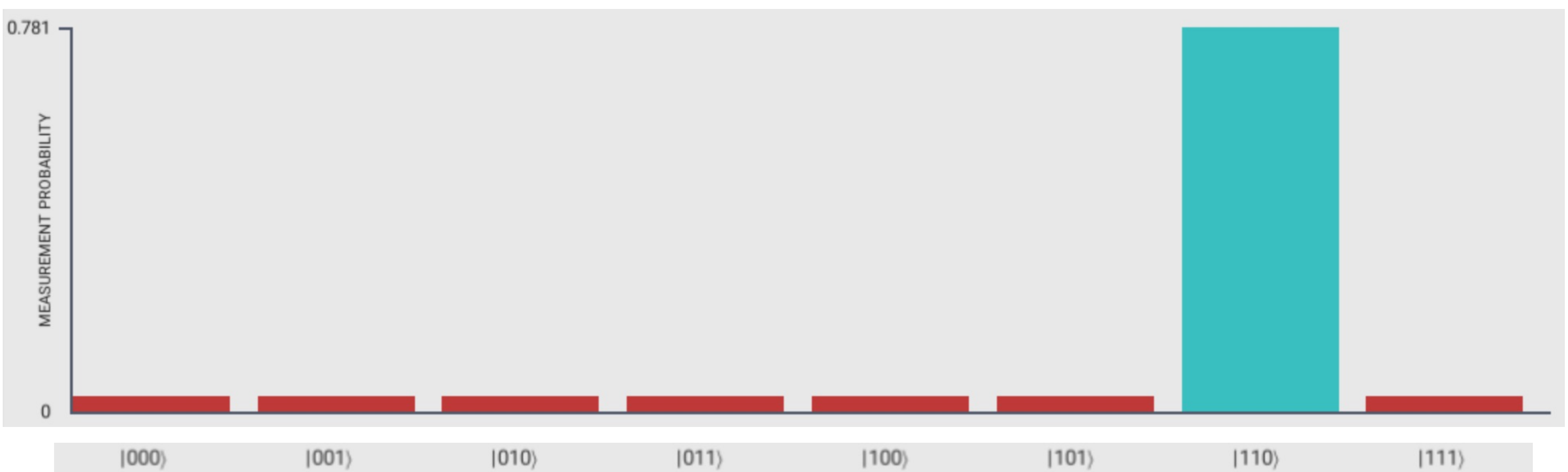
Effect of the Oracle



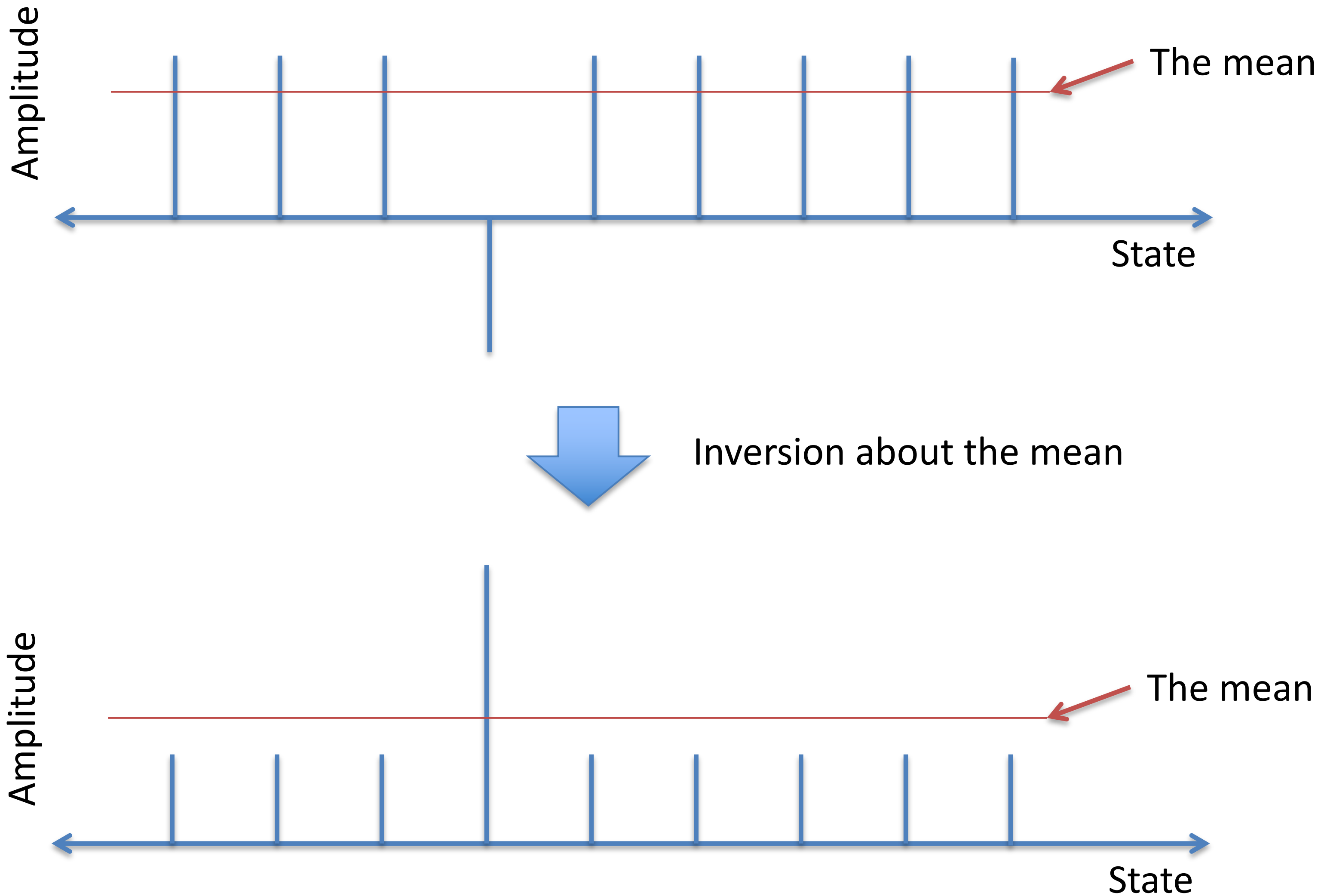
Worked example: finding 6



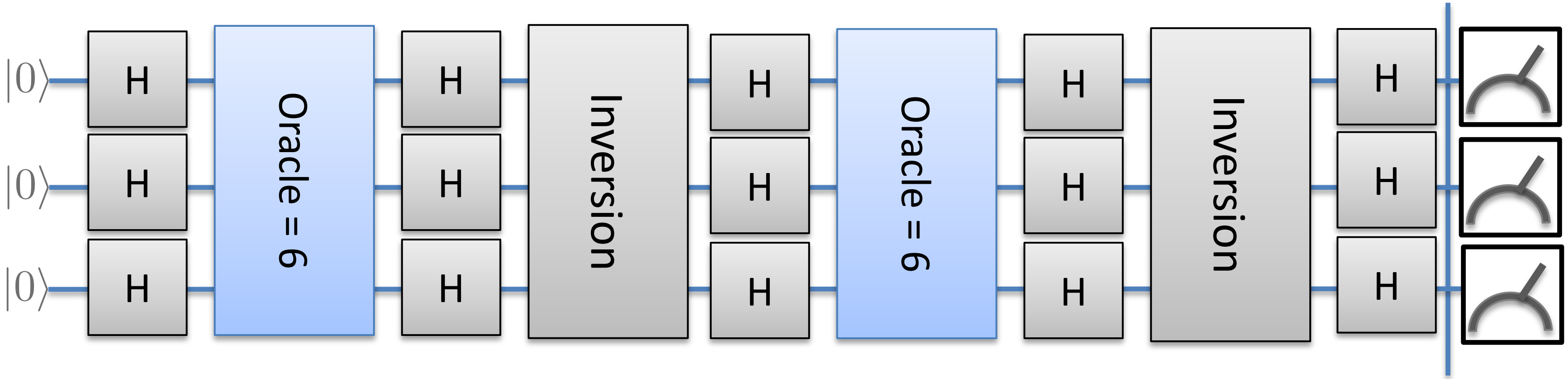
After the second oracle:



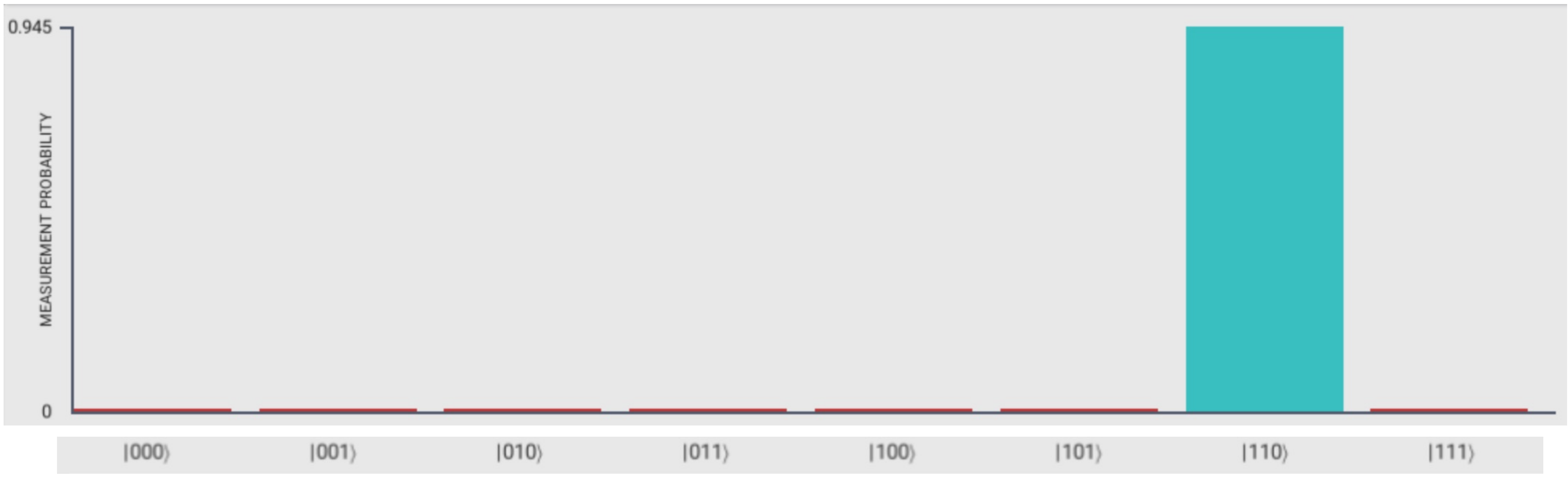
Effect of inversion about the mean



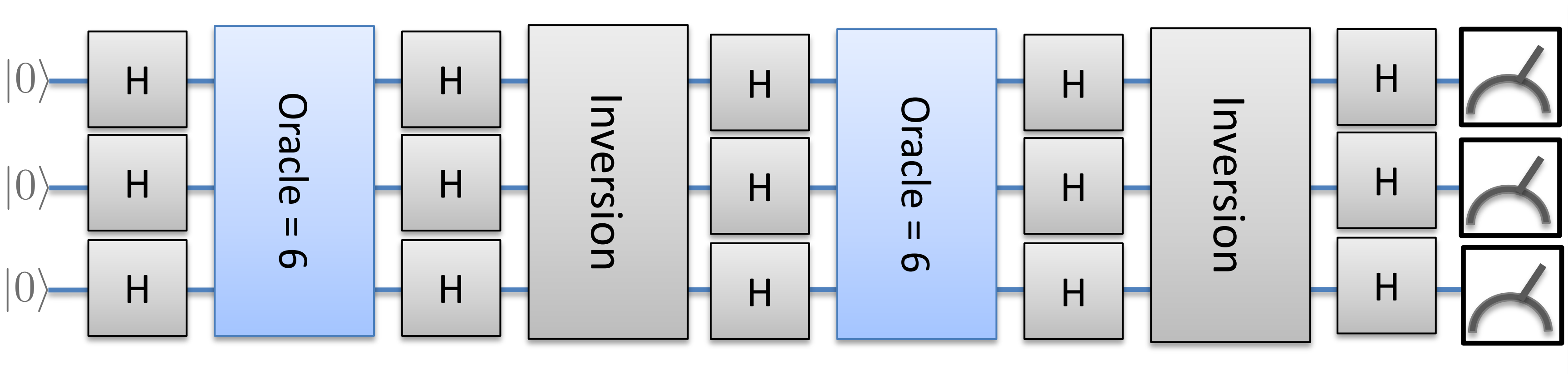
Worked example: finding 6



After the second oracle:



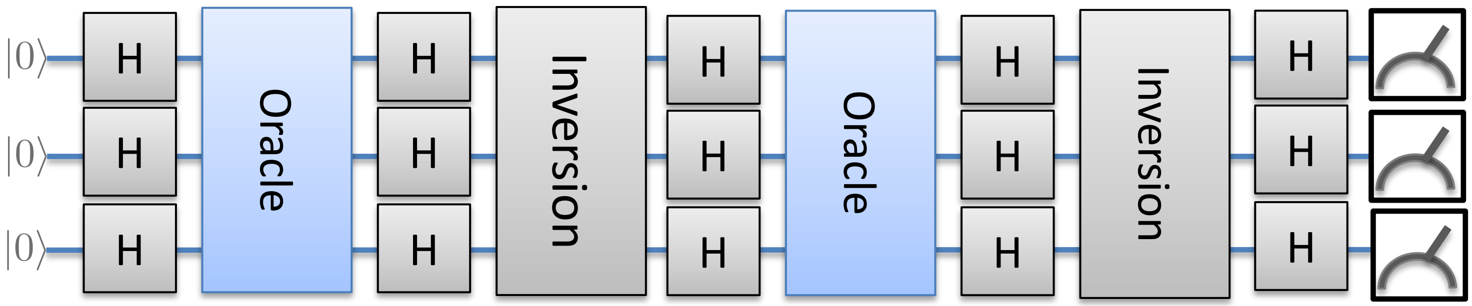
Grover's algorithm



95% Success!

Finally, we measure the marked state after 2 applications of the oracle, and find the marked state, 6 with 95% probability

Geometric interpretation of Grover's algorithm



A very useful basis:

$$|a\rangle = |m\rangle$$

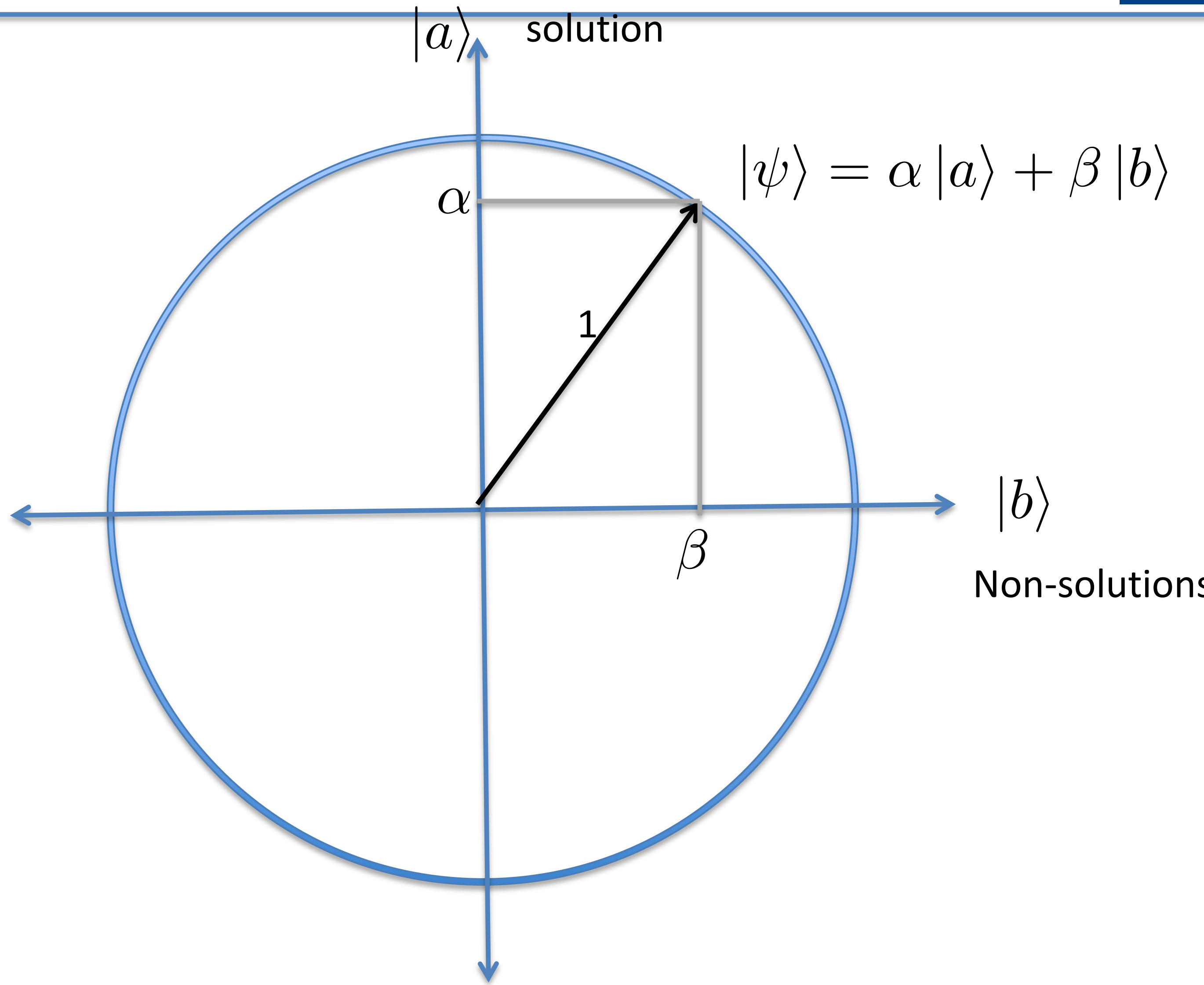
Solution!

$$|b\rangle = \frac{1}{\sqrt{N-1}} \sum_{i \notin \text{solutions}} |i\rangle$$

Non-solutions…

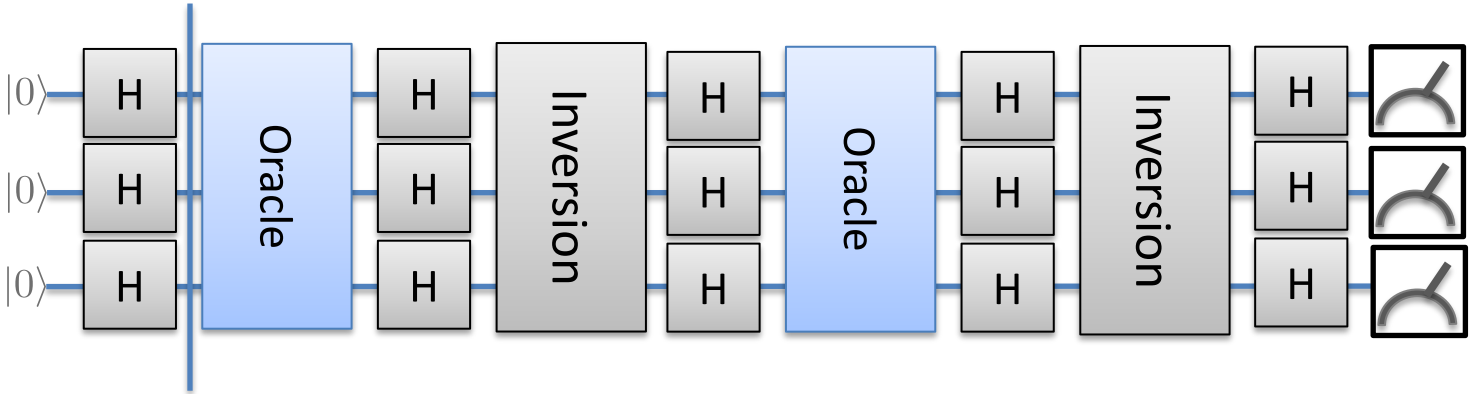
We only need to consider the amplitude of these two states in Grover's algorithm. Every operation is also real, so we can plot on a circle.

Geometric Interpretation



Every state in Grover's algorithm can be expressed as a superposition of these vectors

Equal superposition



Equal superposition state:

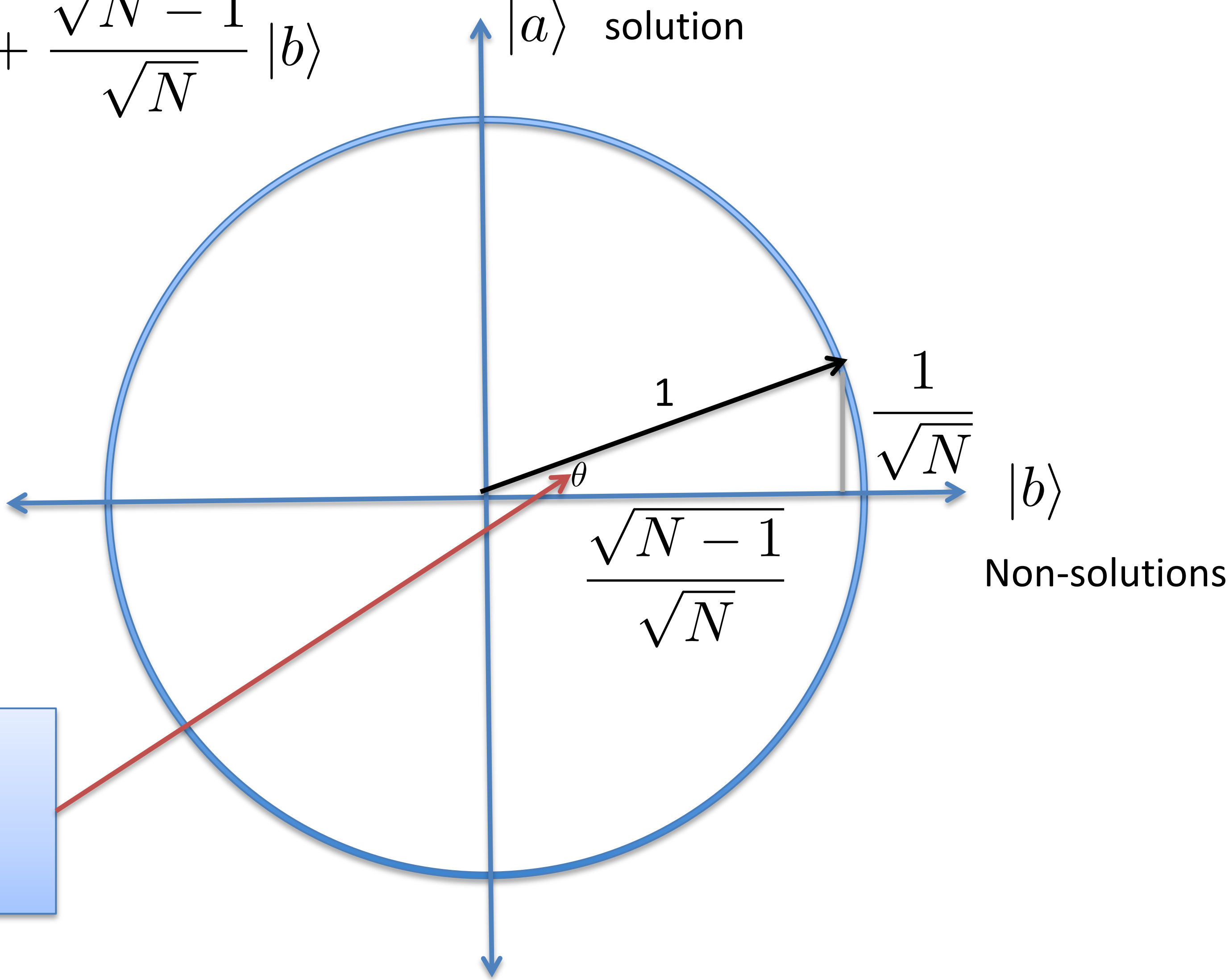
$$|\Phi\rangle = \frac{1}{\sqrt{N}}\sum_i |i\rangle$$
$$= \frac{1}{\sqrt{N}}|a\rangle + \frac{\sqrt{N-1}}{\sqrt{N}}|b\rangle$$

$$|a\rangle = |m\rangle \qquad |b\rangle = \frac{1}{\sqrt{N-1}}\sum_{i \notin \text{solutions}} |i\rangle$$

Equal Superposition

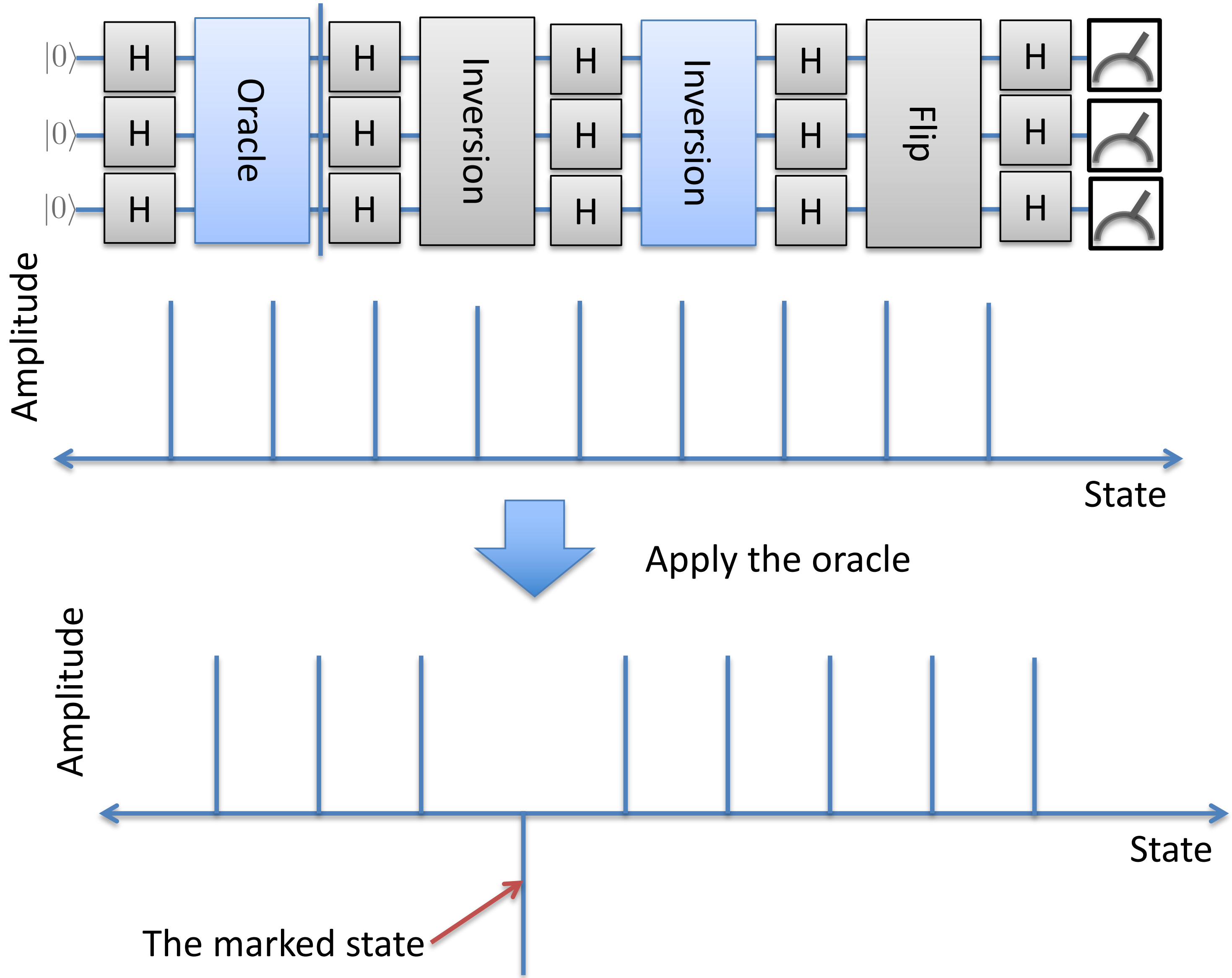
Consider the equal superposition:

$$|\phi\rangle = \frac{1}{\sqrt{N}}|a\rangle + \frac{\sqrt{N-1}}{\sqrt{N}}|b\rangle$$



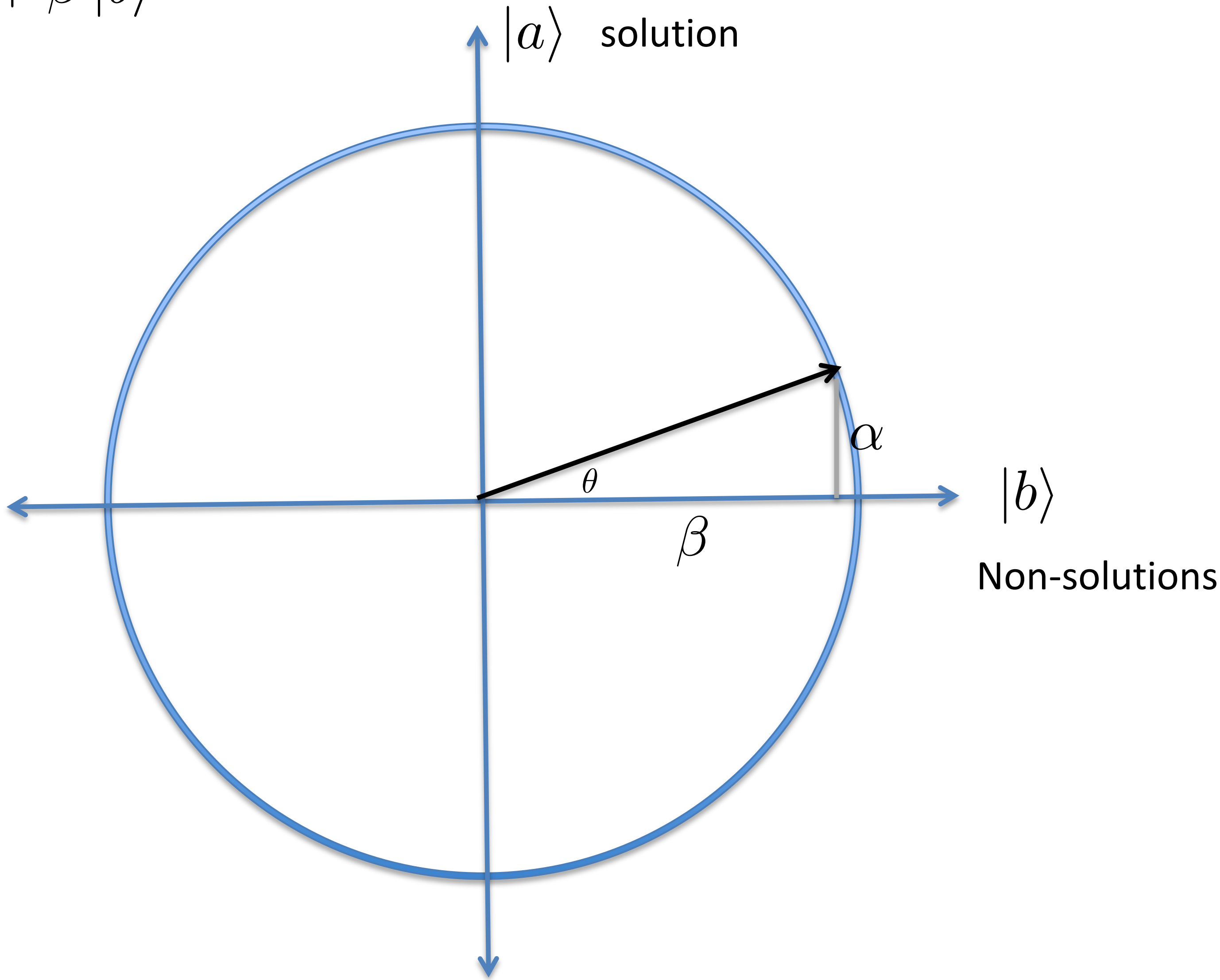
$$\sin\theta = \frac{1}{\sqrt{N}}$$

Effect of the Oracle



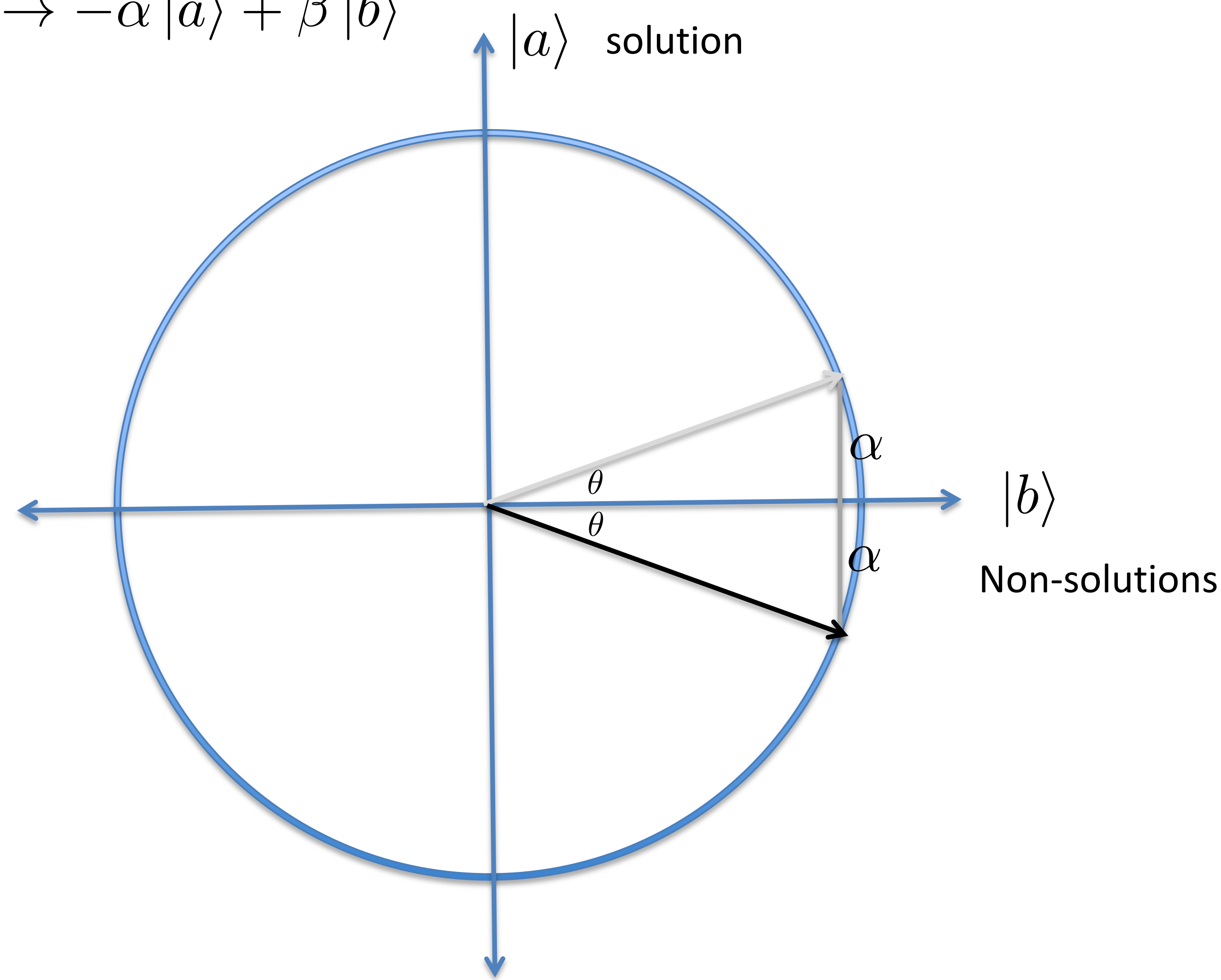
Geometric Effect of Oracle

$$\alpha |a\rangle + \beta |b\rangle$$

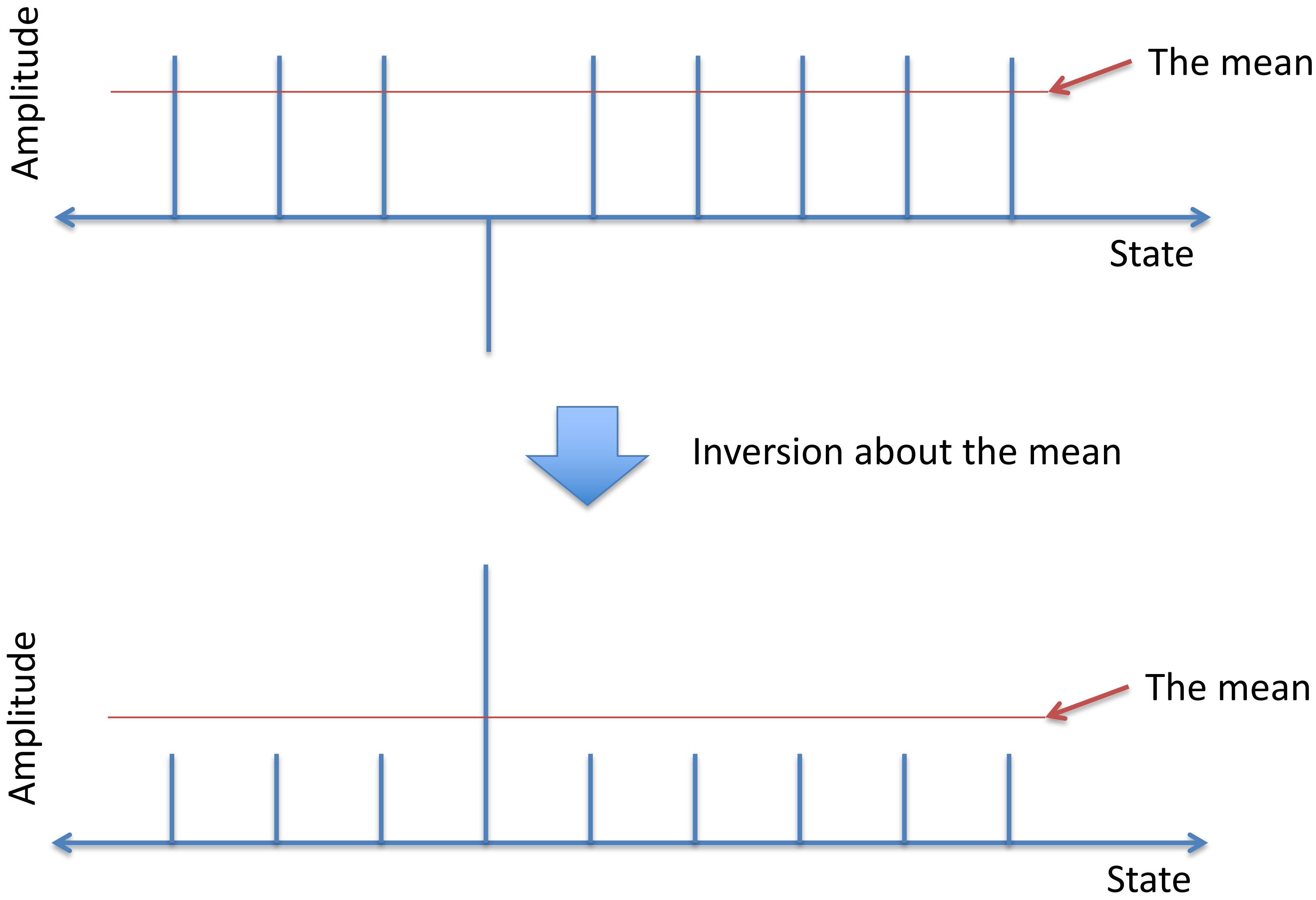


Geometric Effect of Oracle

$$\alpha\,|a\rangle + \beta\,|b\rangle \rightarrow -\alpha\,|a\rangle + \beta\,|b\rangle$$

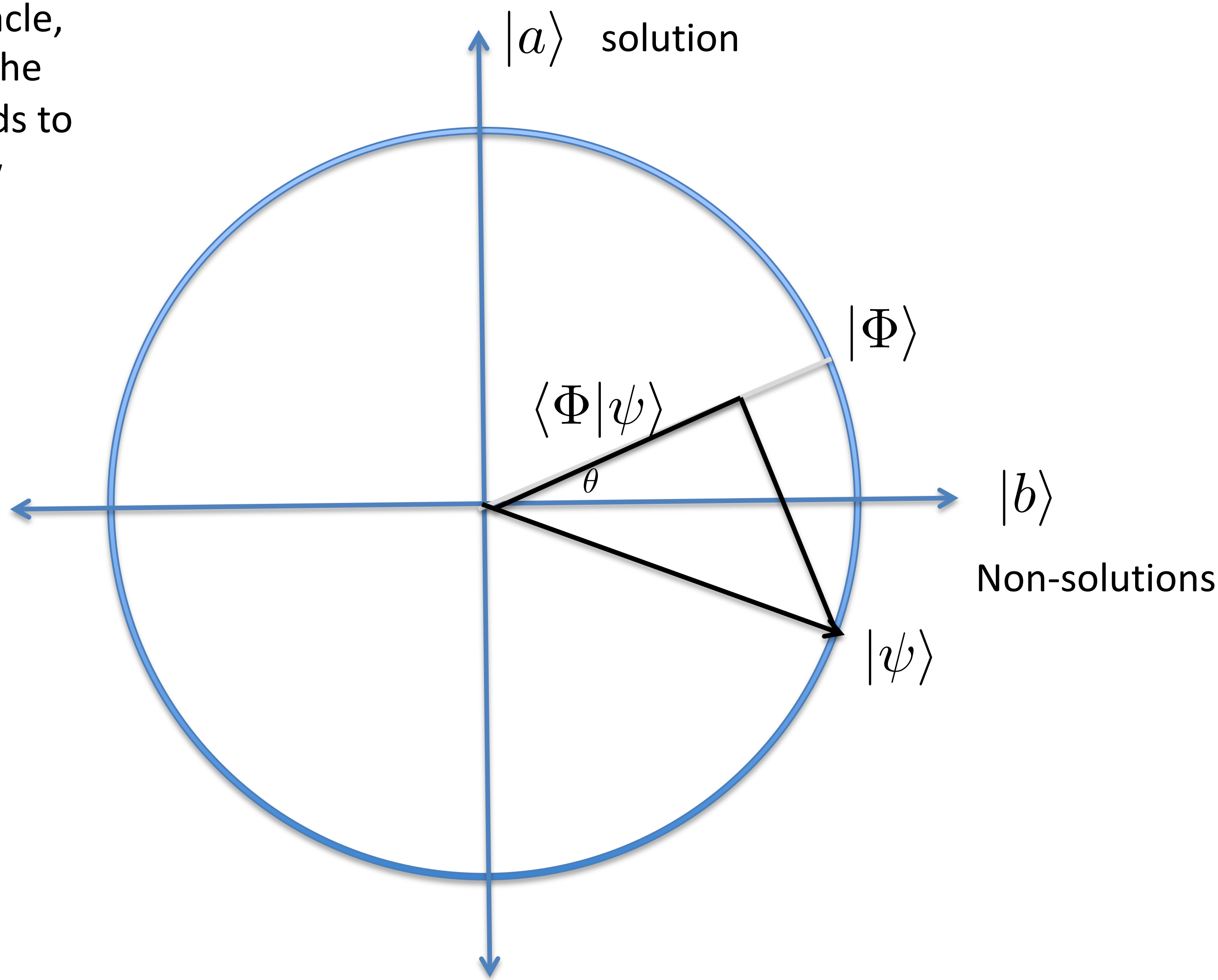


Effect of inversion about the mean



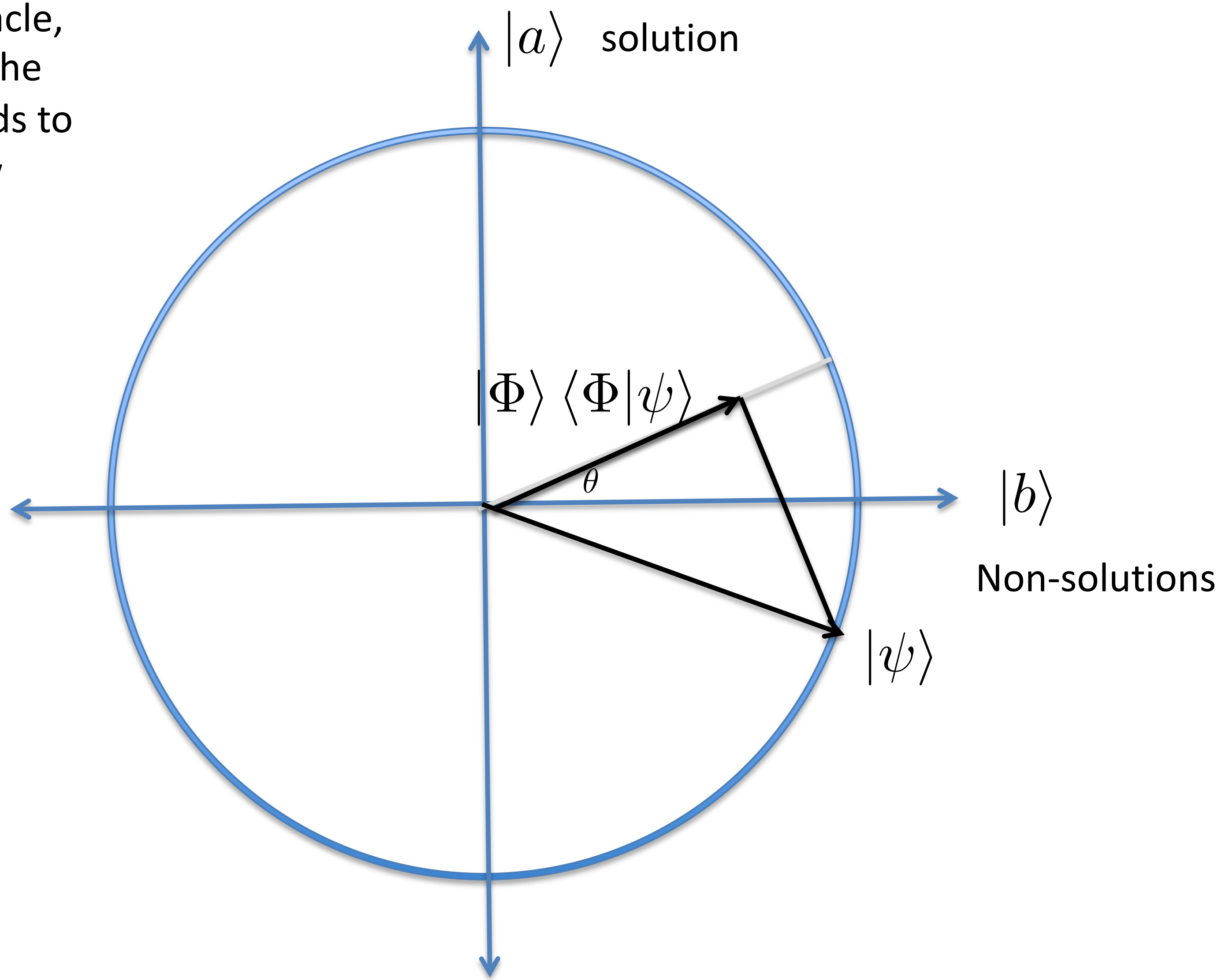
Inversion about the mean

Similar to the oracle, inversion about the mean corresponds to a reflection. Now about **equal superposition**.

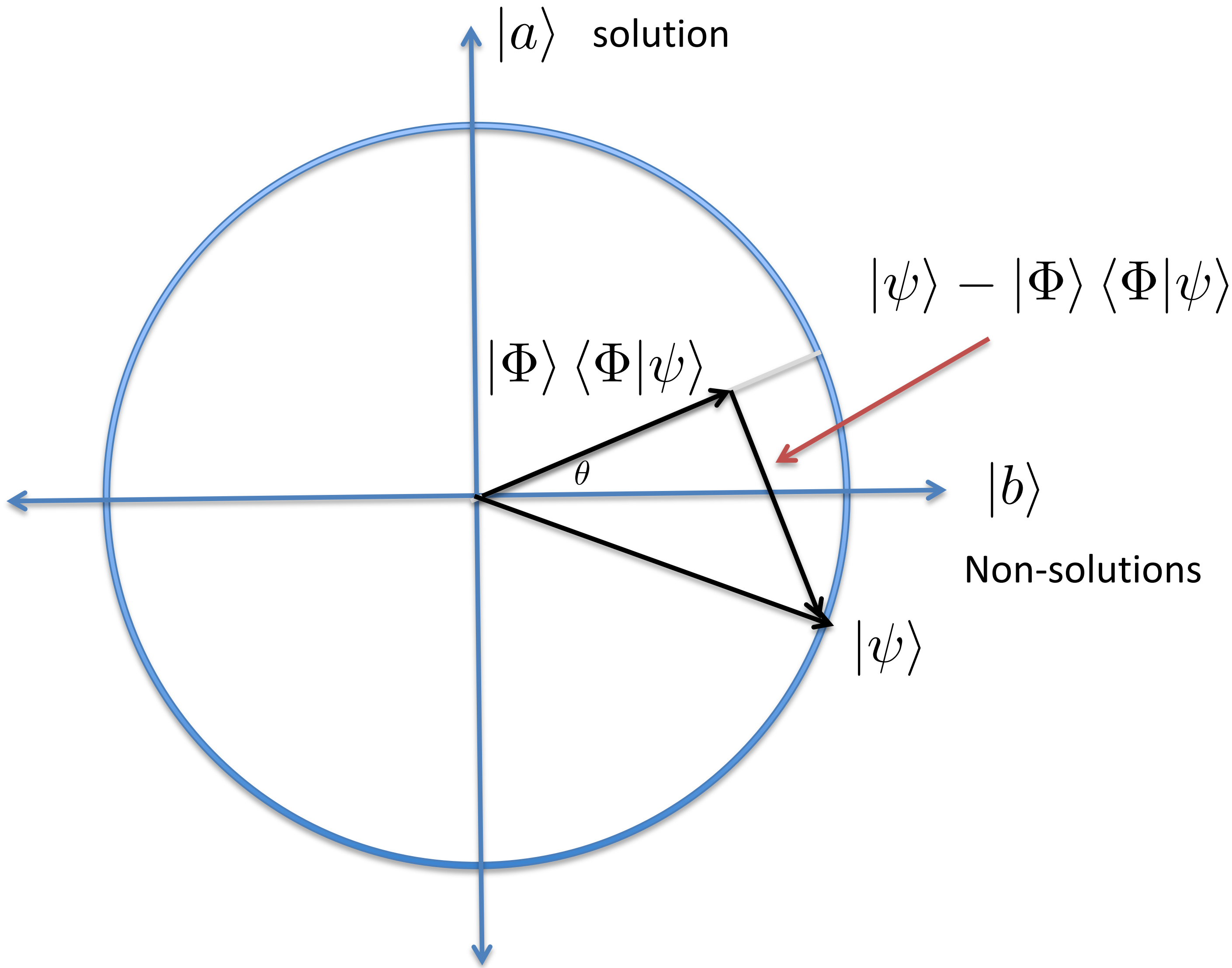


Inversion about the mean

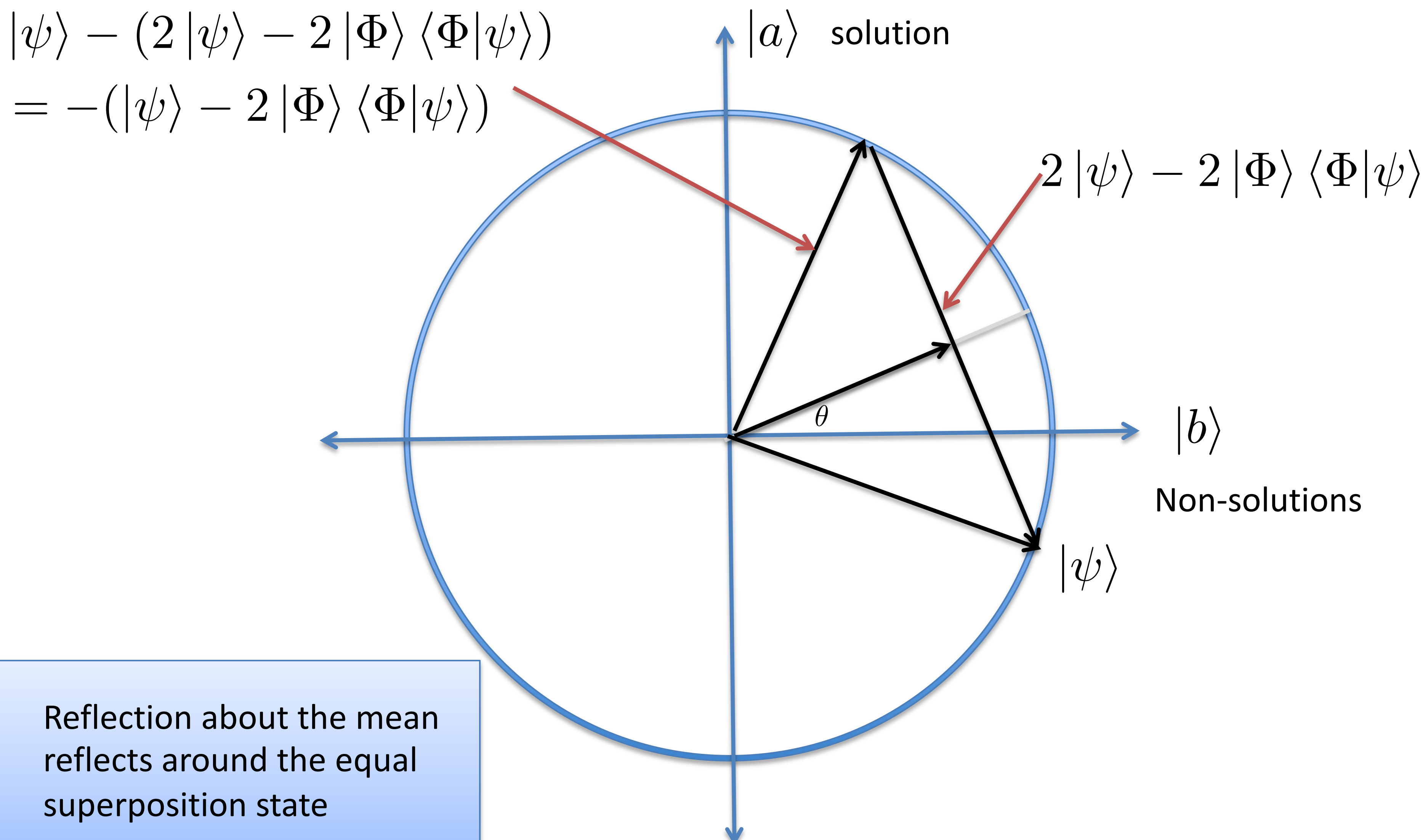
Similar to the oracle, inversion about the mean corresponds to a reflection. Now about **equal superposition**.



Inversion about the mean



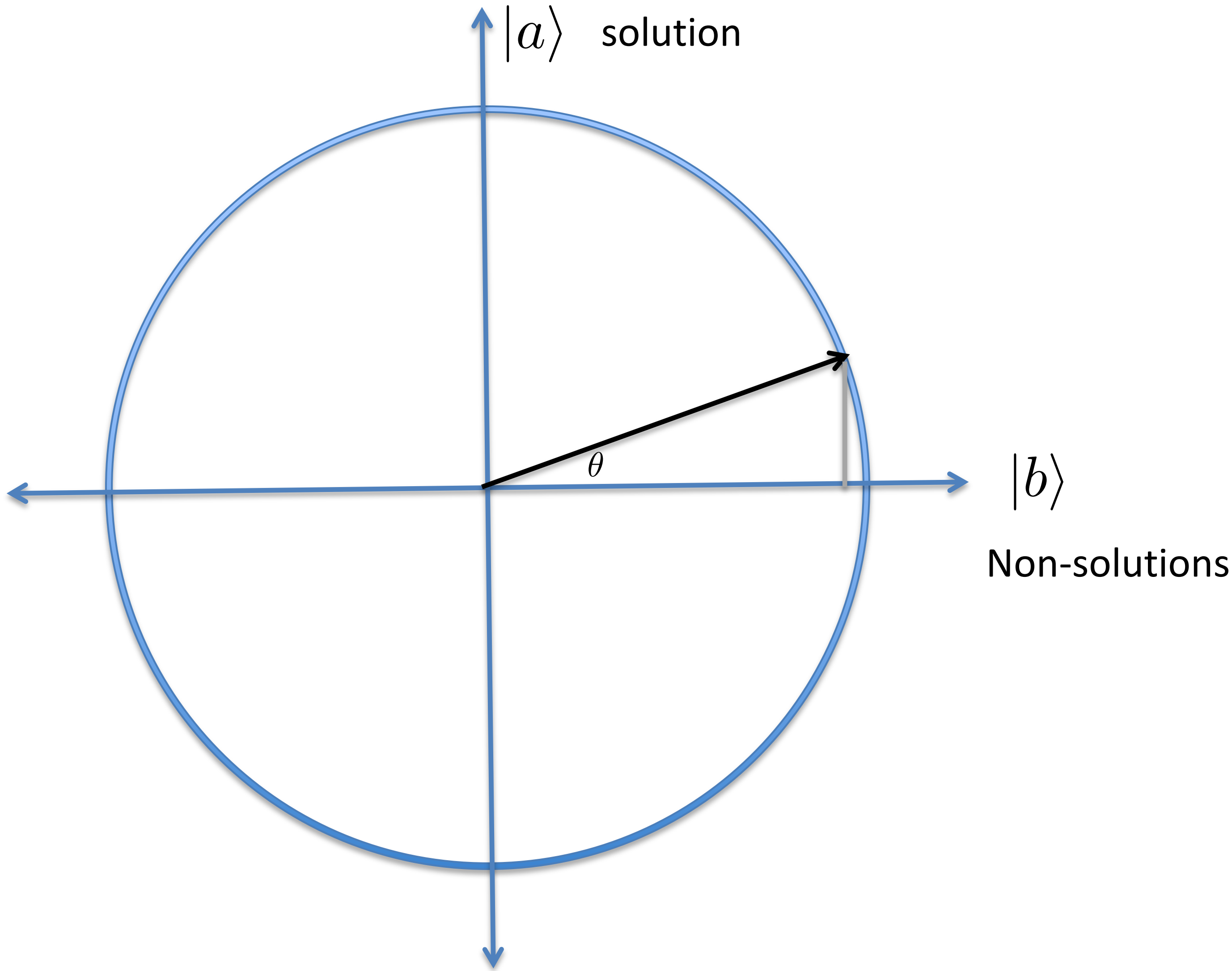
Inversion about the mean



Reflection about the mean reflects around the equal superposition state

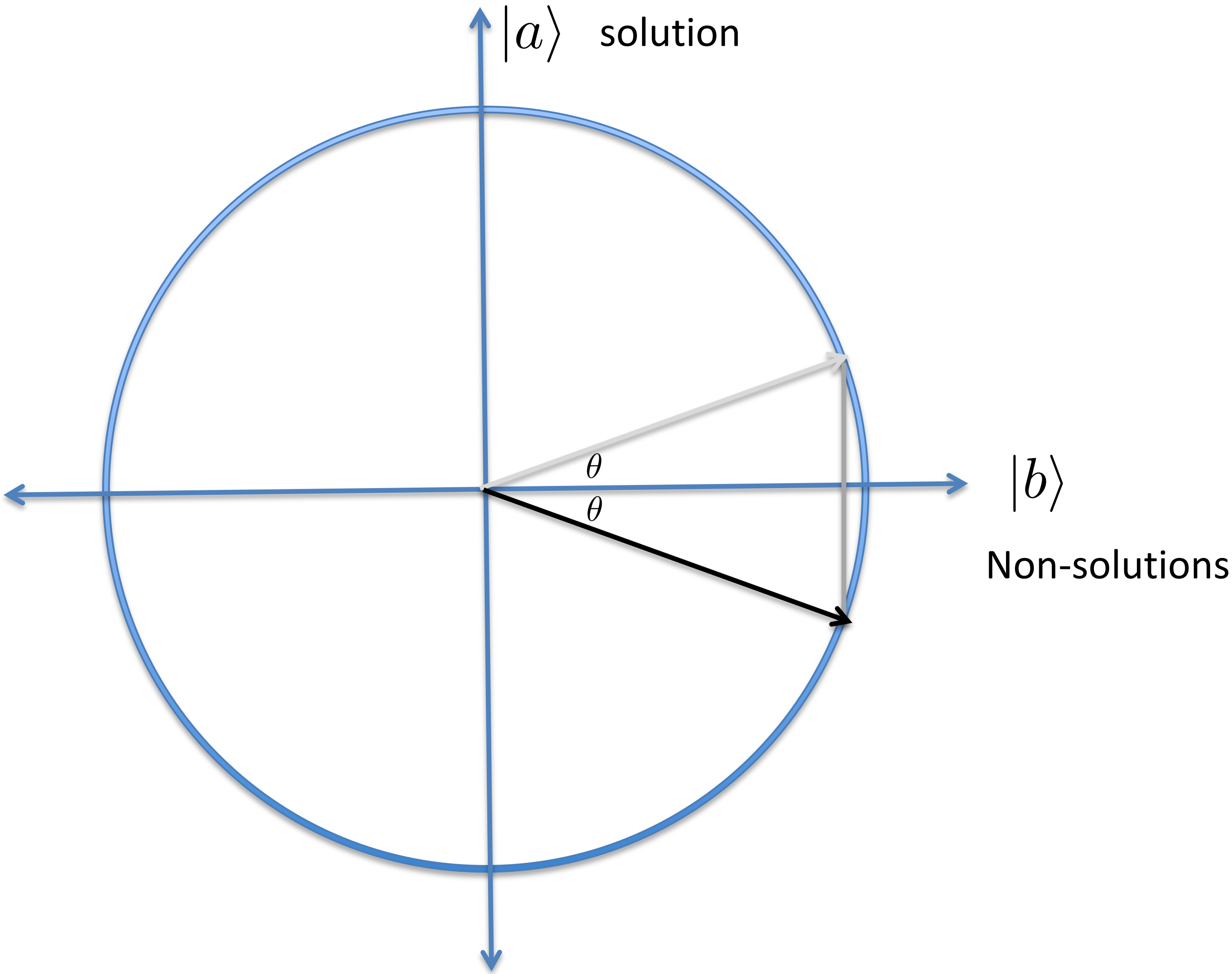
Geometric Effect of both Oracle and Inversion

Combining both effects:



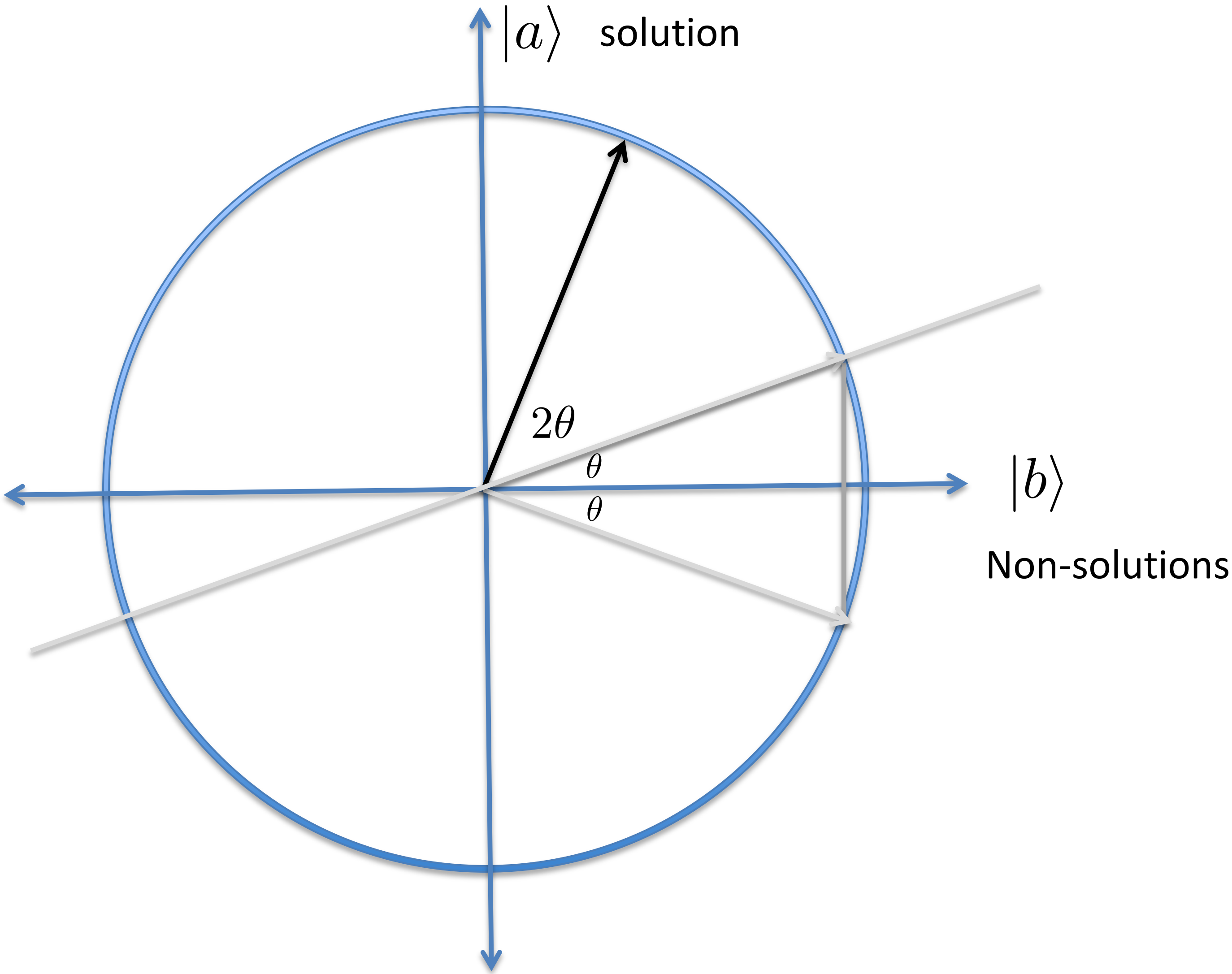
Geometric Effect of both Oracle and Inversion

Combining both effects:



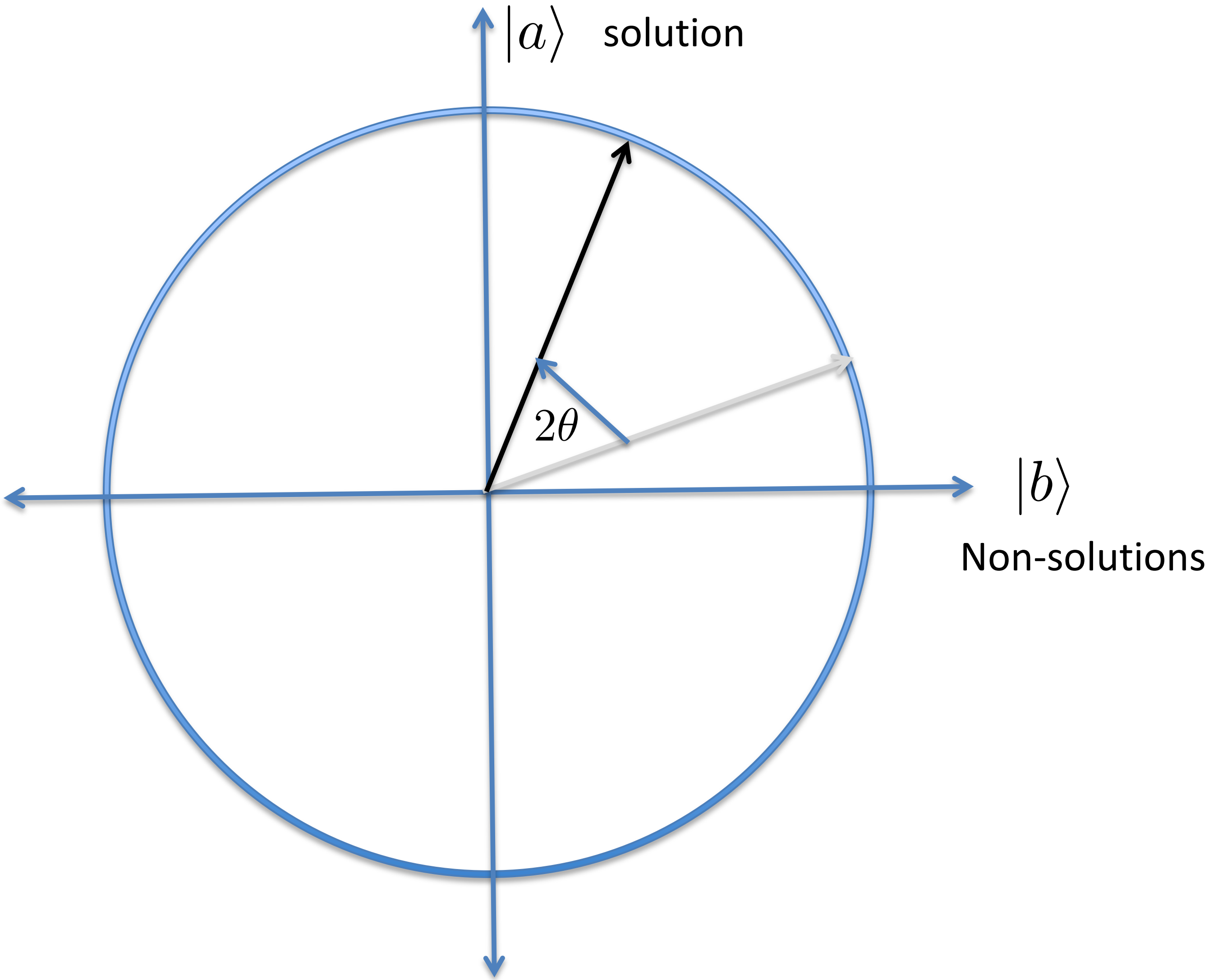
Geometric Effect of both Oracle and Inversion

Combining both effects:



Total effect of one Grover iteration

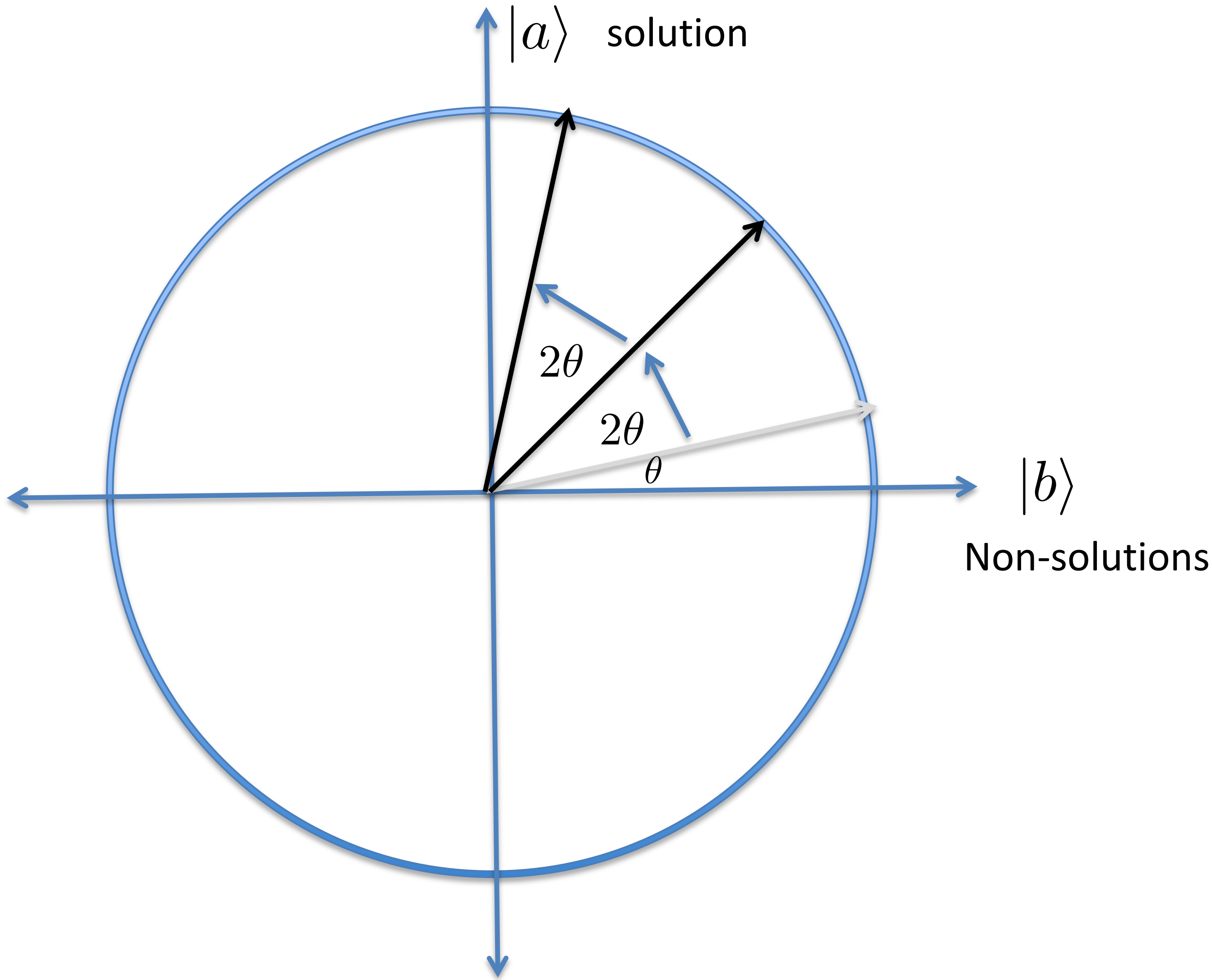
Product of two reflections is a rotation.



$$\sin\theta = \frac{1}{\sqrt{N}}$$

Many Grover iterations

Product of two reflections is a rotation.



$$\sin\theta = \frac{1}{\sqrt{N}}$$

How many iterations required?

$$\sin\theta = \frac{1}{\sqrt{N}}$$

For small angles,

$$\theta \approx \frac{1}{\sqrt{N}}$$

After n iterations, we rotate to have only marked solutions:

$$(2n+1)\theta = \frac{\pi}{2}$$

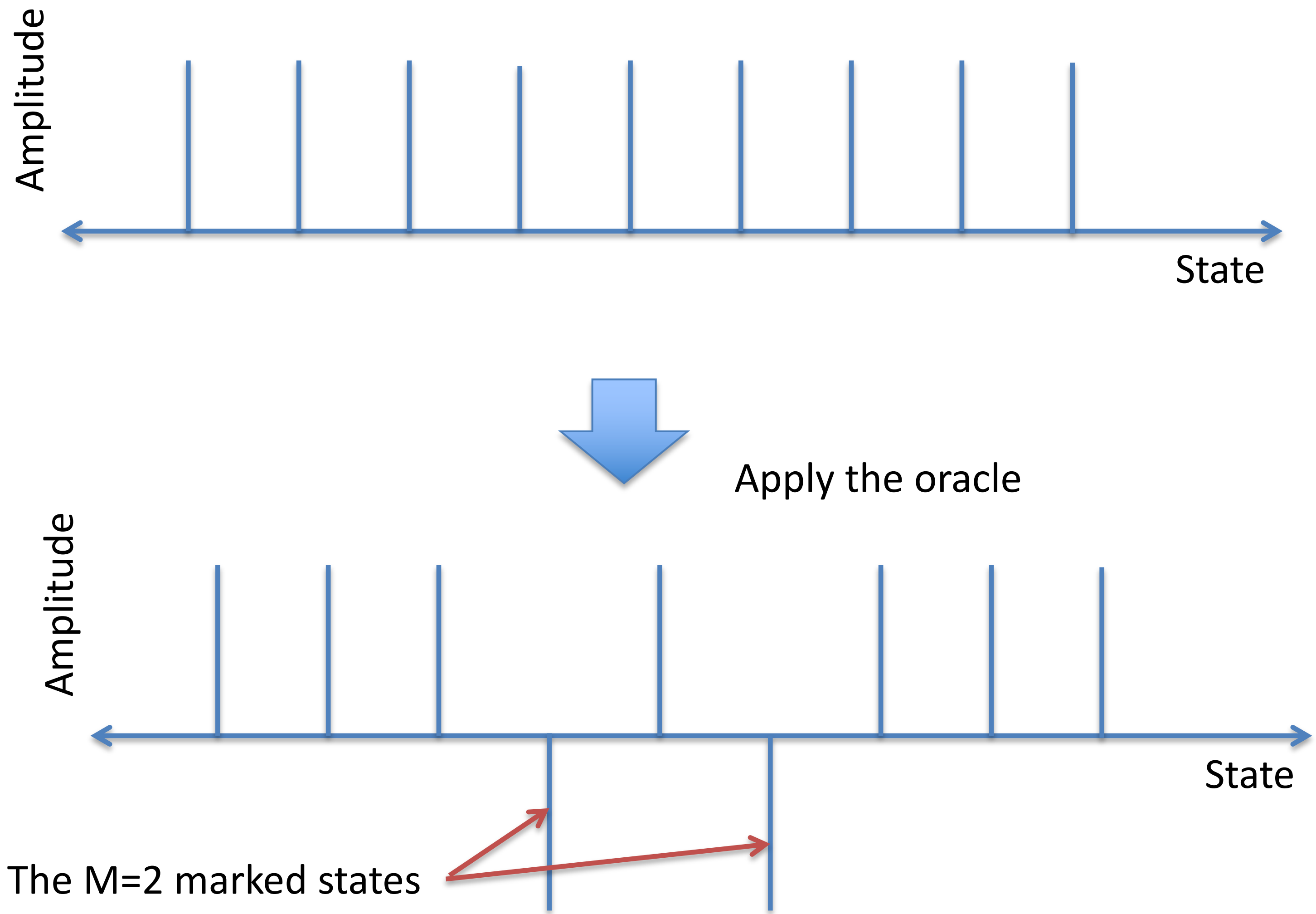
$$n \approx \frac{\pi}{4}\sqrt{N}$$

The number of steps, *n*, required scales as O(√N), and not with N as it would classically.

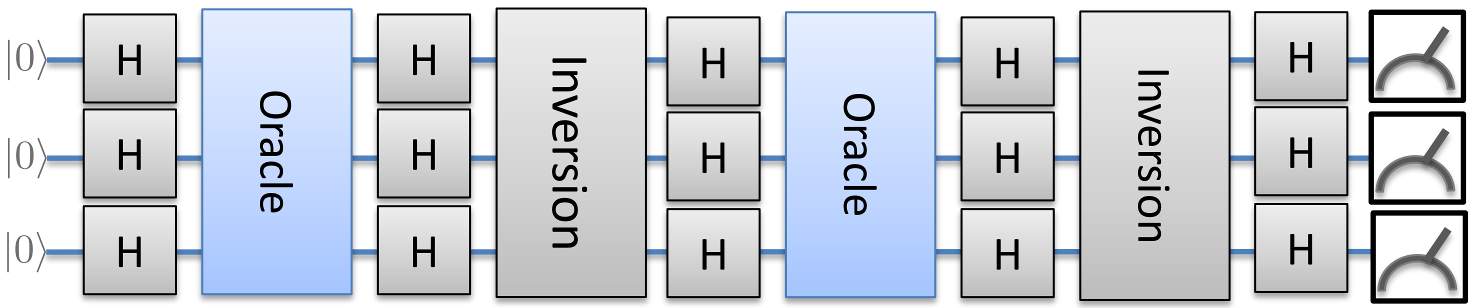
This is a “polynomial” rather than an “exponential” speedup.

Multiple Solutions

There can be more than one solution to a problem.



Geometric interpretation of Grover's algorithm



A very useful basis:

$$|a\rangle = \frac{1}{\sqrt{M}} \sum_{i \in \text{solutions}} |i\rangle$$

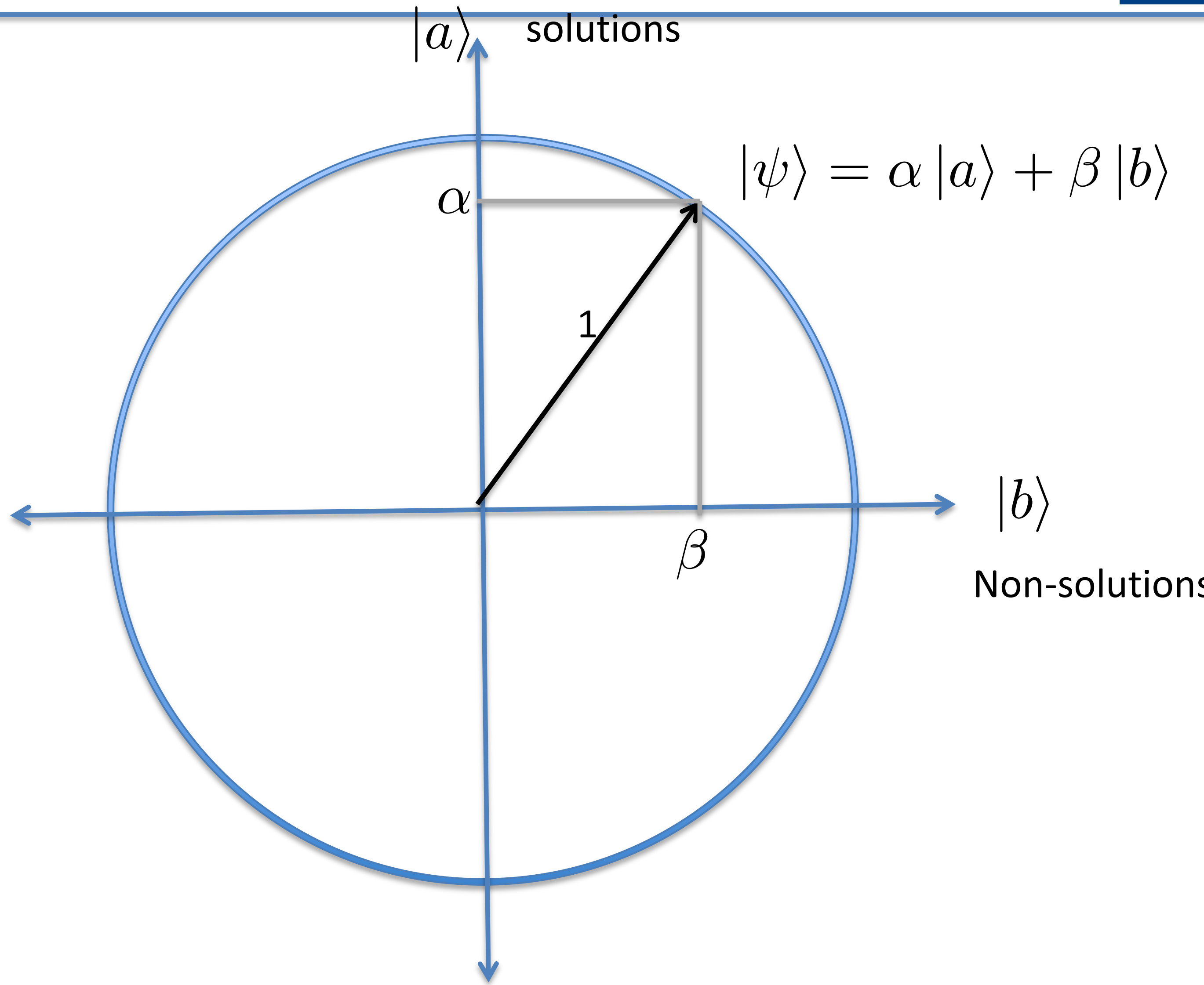
Solutions!

$$|b\rangle = \frac{1}{\sqrt{N-M}} \sum_{i \notin \text{solutions}} |i\rangle$$

Non-solutions…

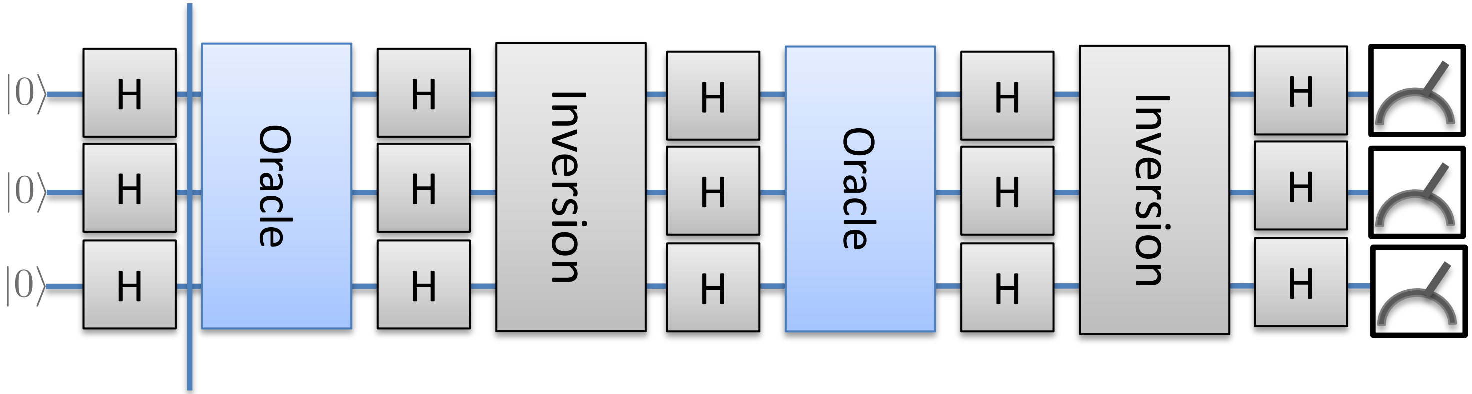
We only need to consider the amplitude of these two states in Grover's algorithm. Every operation is also real, so we can plot on a circle.

Geometric Interpretation



Every state in Grover's algorithm can be expressed as a superposition of these vectors

Equal superposition



Equal superposition state:

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$$

$$|\Phi\rangle = \frac{\sqrt{M}}{\sqrt{N}} |a\rangle + \frac{\sqrt{N-M}}{\sqrt{N}} |b\rangle$$

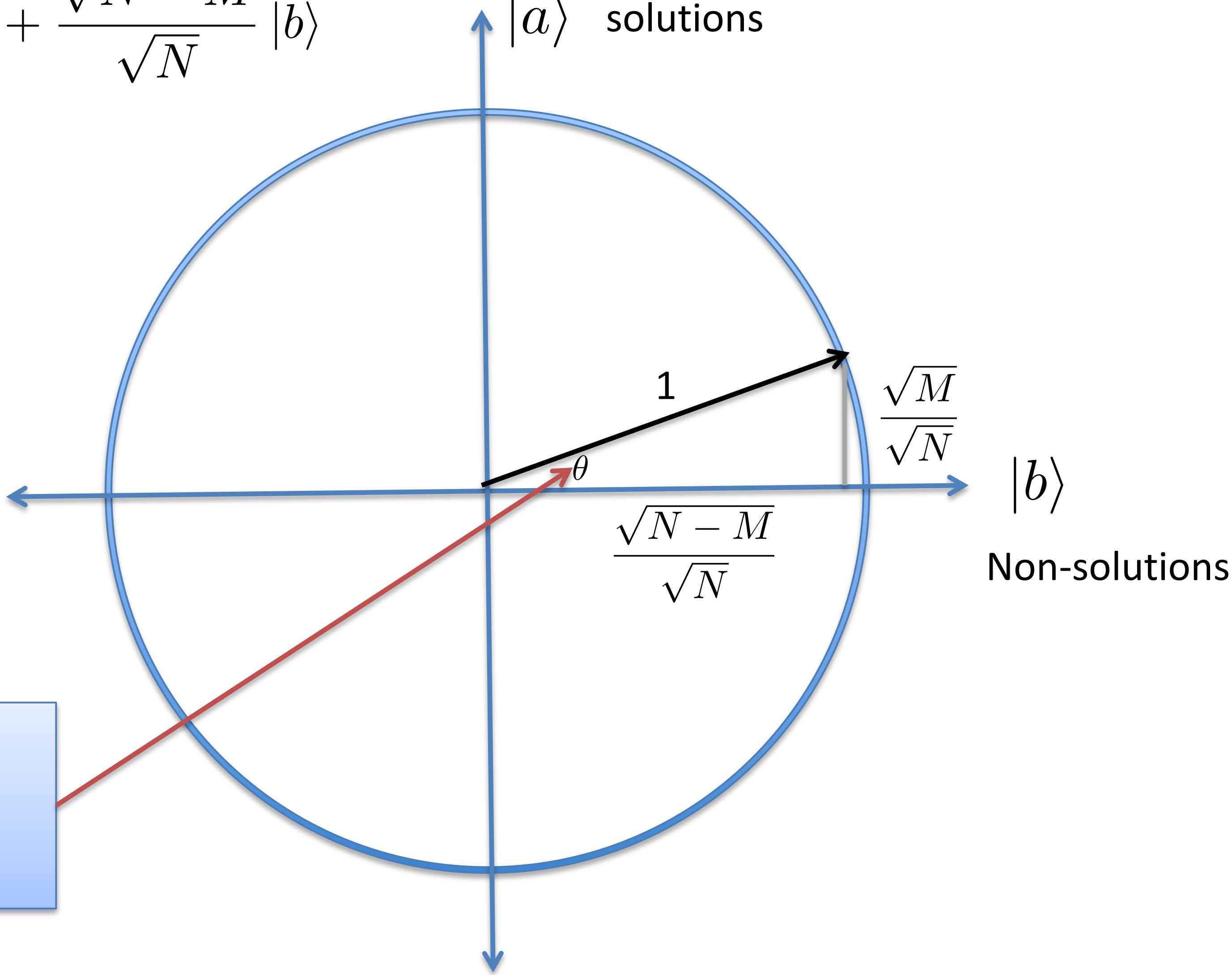
$$|a\rangle = \frac{1}{\sqrt{M}} \sum_{i \in \text{solutions}} |i\rangle$$

$$|b\rangle = \frac{1}{\sqrt{N-M}} \sum_{i \notin \text{solutions}} |i\rangle$$

Equal Superposition

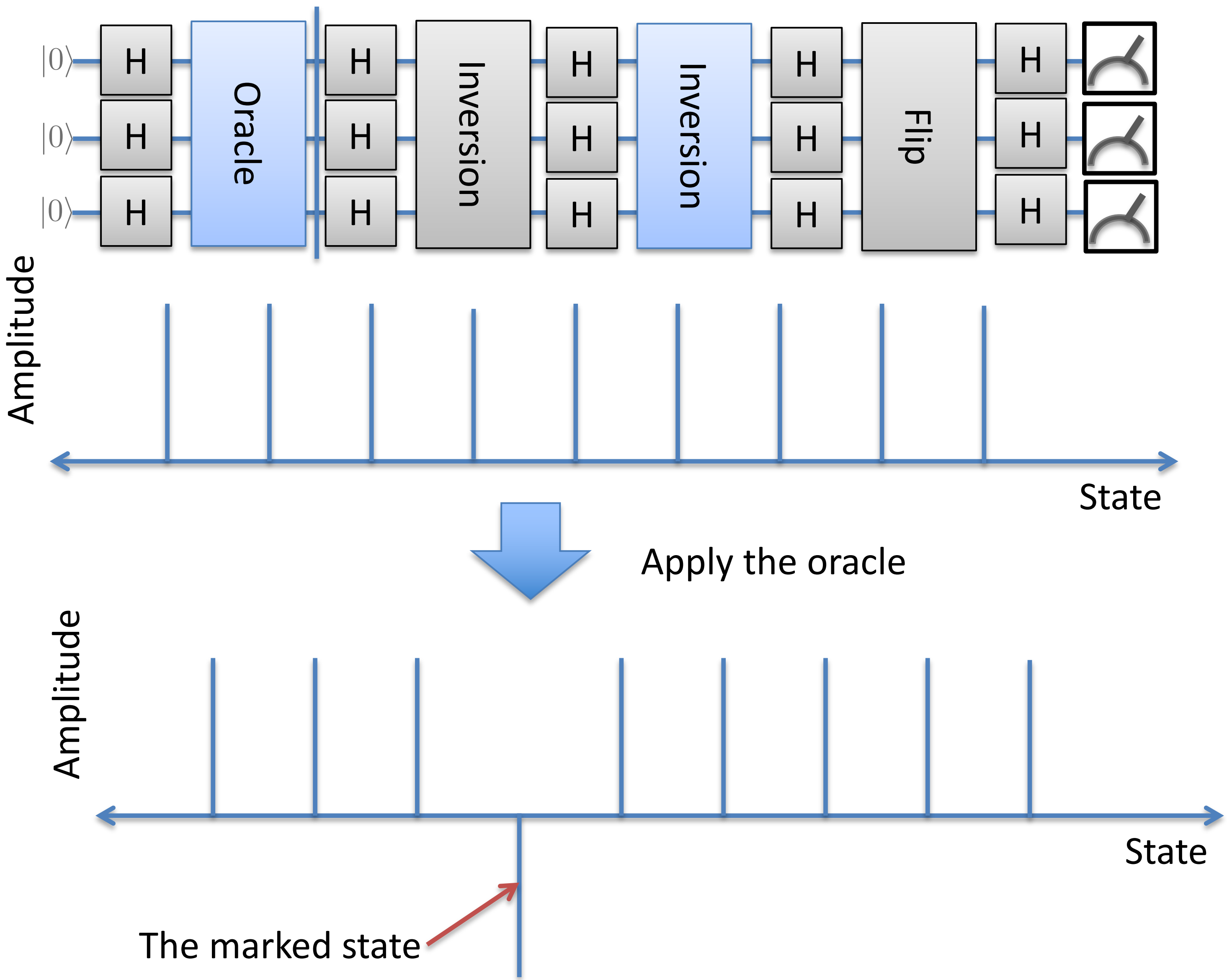
Consider the equal superposition:

$$|\Phi\rangle = \frac{\sqrt{M}}{\sqrt{N}}|a\rangle + \frac{\sqrt{N-M}}{\sqrt{N}}|b\rangle$$



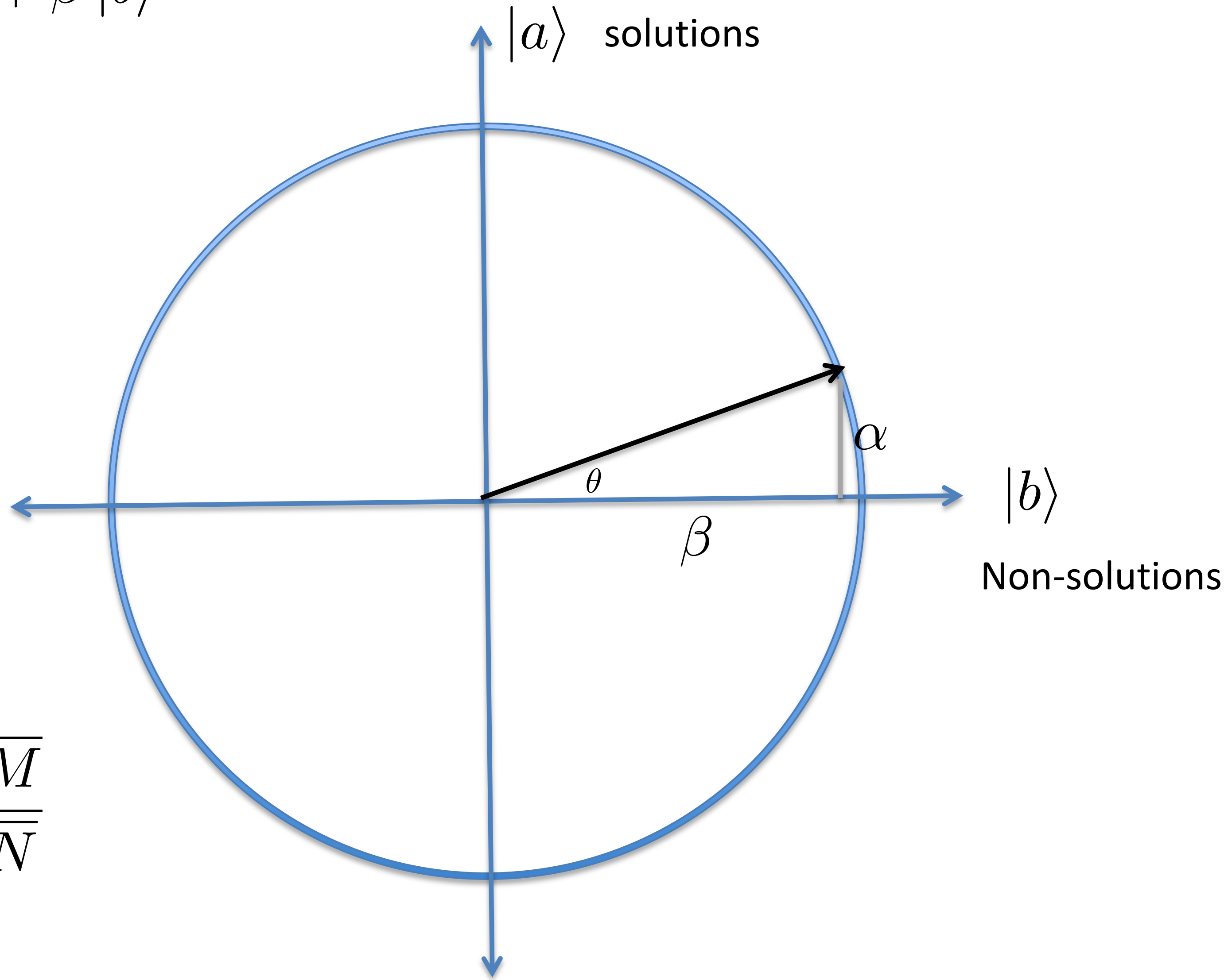
$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Effect of the Oracle



Geometric Effect of Oracle

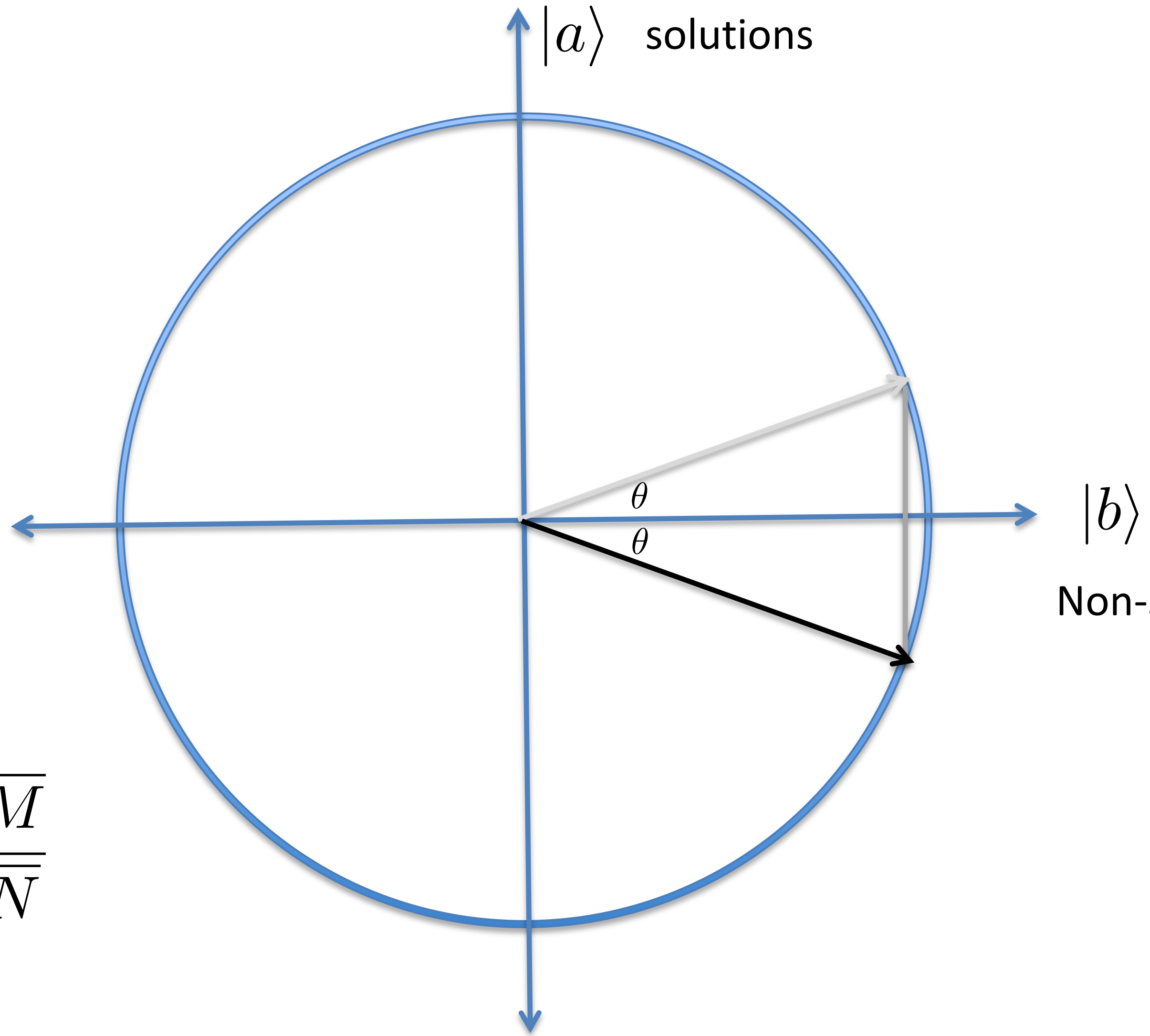
$$\alpha\left|a\right\rangle + \beta\left|b\right\rangle$$



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Geometric Effect of both Oracle and Inversion

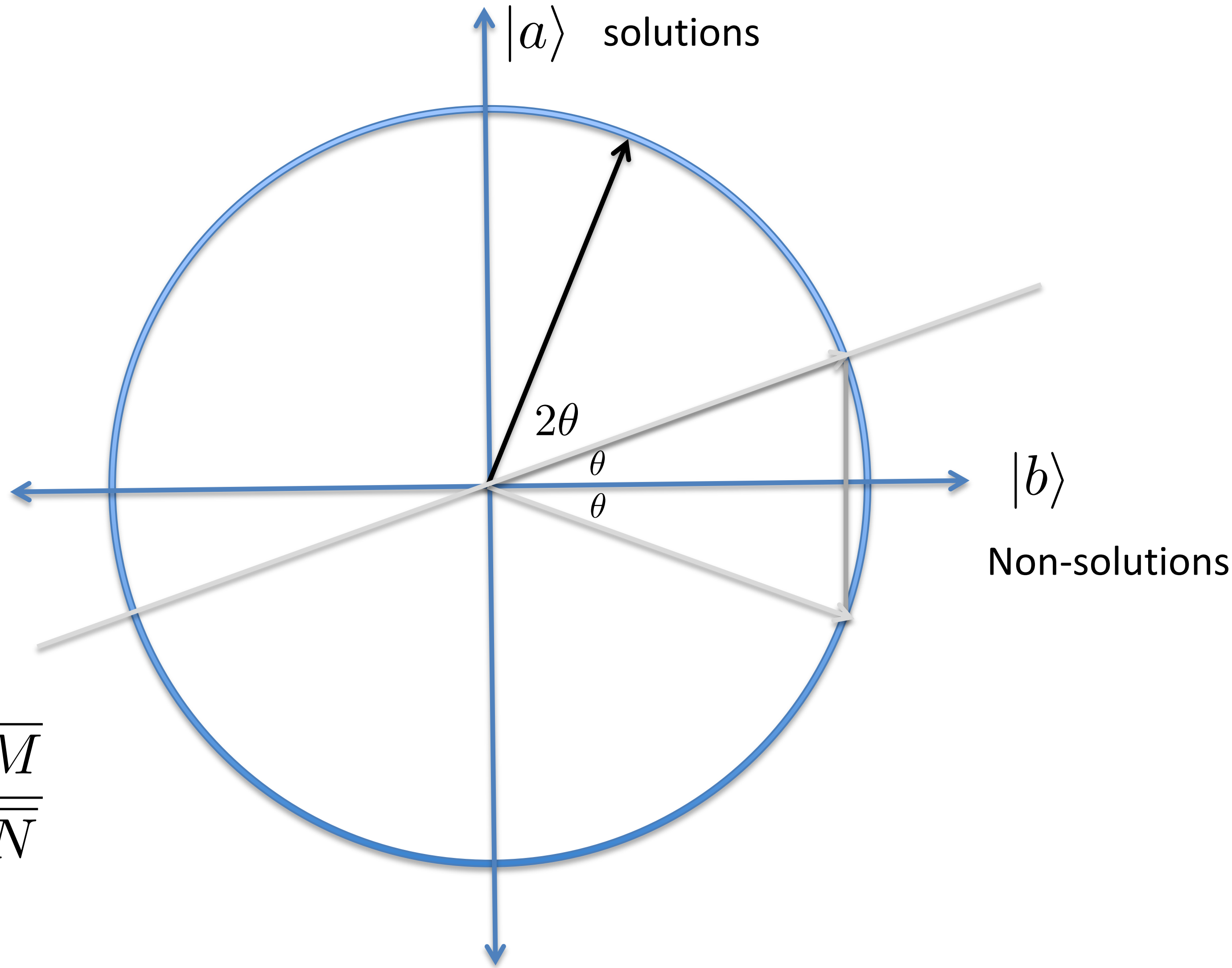
Combining both effects:



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Geometric Effect of both Oracle and Inversion

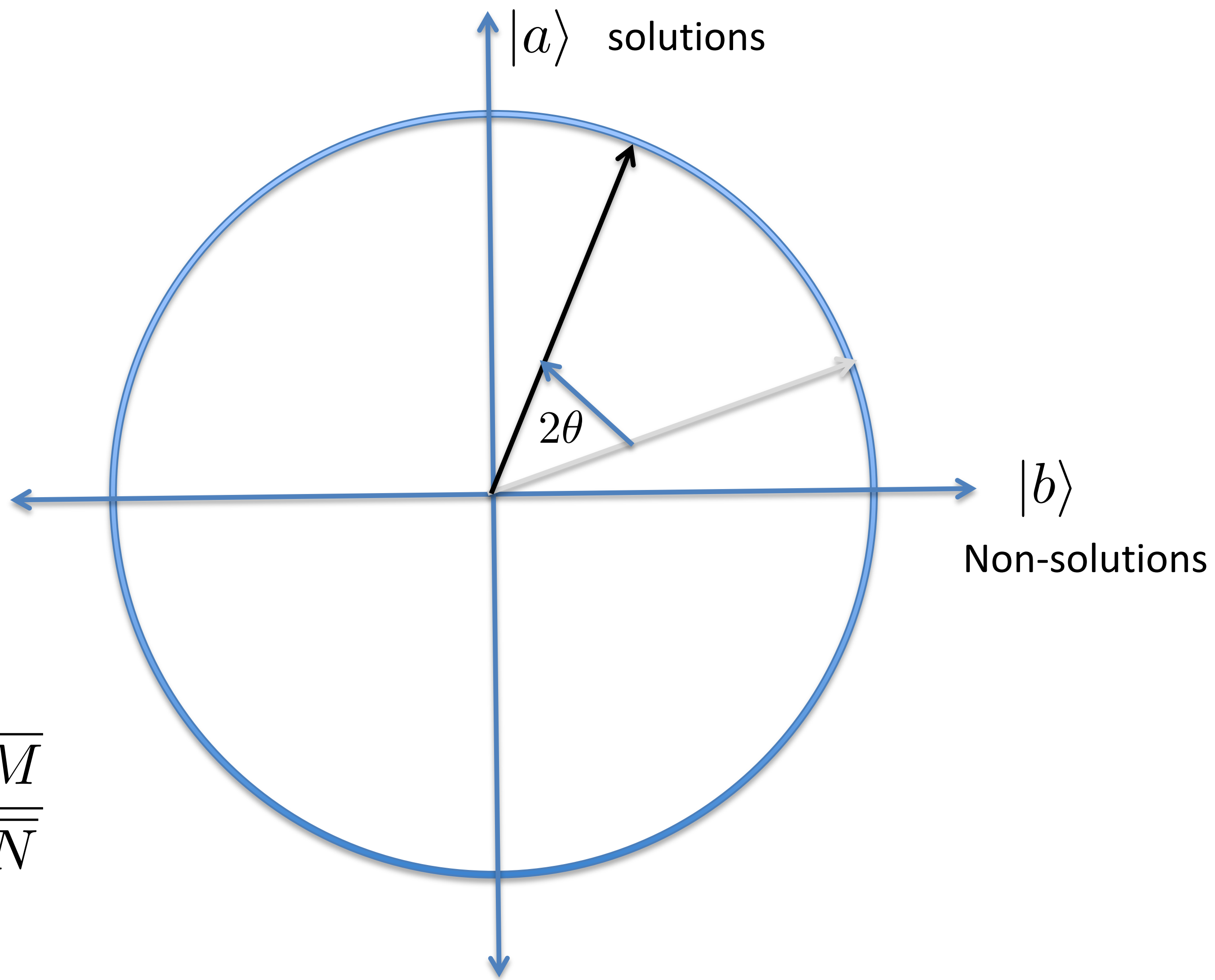
Combining both effects:



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Total effect of one Grover iteration

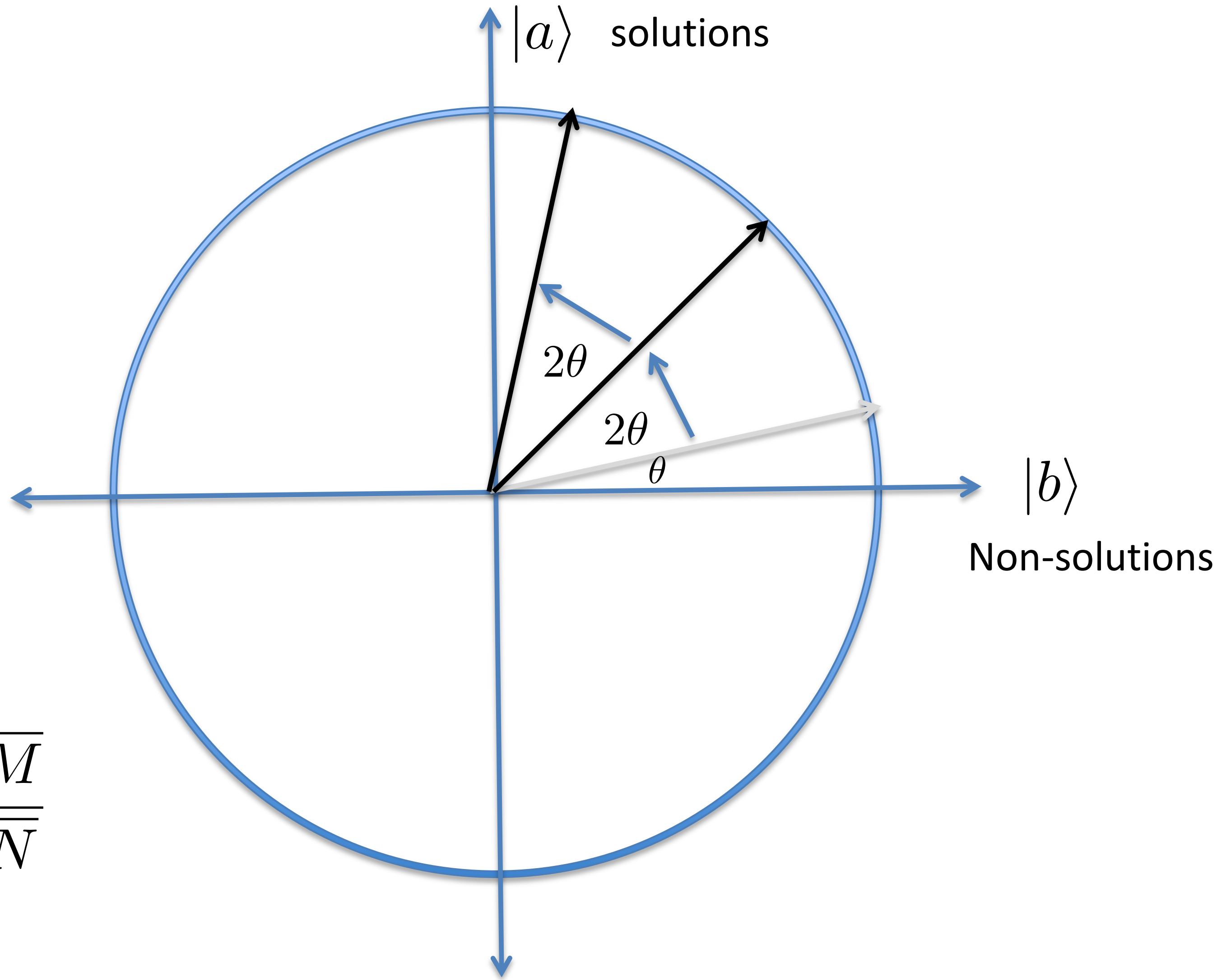
Product of two reflections is a rotation.



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Many Grover iterations

Product of two reflections is a rotation.



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

How many iterations required?

$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

For small angles,

$$\theta \approx \frac{\sqrt{M}}{\sqrt{N}}$$

After n iterations, we rotate to have only marked solutions:

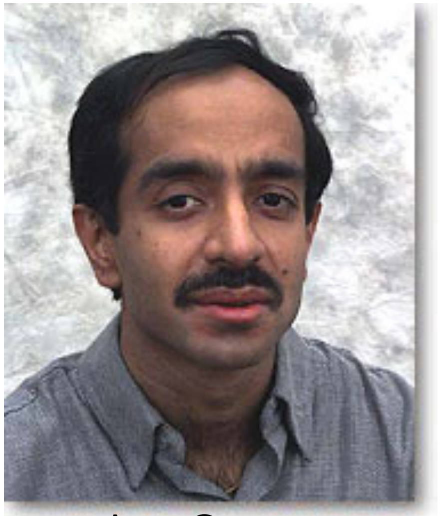
$$(2n+1)\theta = \frac{\pi}{2}$$

$$n \approx \frac{\pi\sqrt{N}}{4\sqrt{M}}$$

Having multiple solutions is faster than searching for a single marked solution.

Grover's Algorithm

- Unordered search, find one marked item among many
- Classically, this requires N/2 uses of the oracle
- Quantum mechanically, requires only O(√N).



Lov Grover